PHYS 598 SDA RECITATION 3 - PROBLEM SET

(Starred problems to be done only after you're done with the unstarred problems.)

1. Let $\mu \in \mathbb{R}$. Let X_1 and X_2 be independent random variables with distributions $N(\mu, 1)$ and $N(\mu, 4)$ respectively. [The notation $N(\mu, \sigma)$ is standard, where μ is the mean and σ is the variance.] Let the variables T_1, T_2 and T_3 be defined by

$$T_1 = \frac{X_1 + X_2}{2}, \qquad T_2 = 2X_1 - X_2, \qquad T_3 = \frac{4X_1 + X_2}{5}. \tag{1}$$

Find the mean and variance of T_1, T_2 and T_3. Which of $\mathbb{E}[T_1]$, $\mathbb{E}[T_2]$ and $\mathbb{E}[T_3]$ would you prefer to have an estimator for μ? [$\mathbb{E}[Y]$ is standard notation for expectation/mean of the distribution Y.]

P.S. If you're feeling especially math-inclined : For what (α, β), with $\alpha + \beta = 1$ would $T = \alpha X_1 + \beta X_2$ be the best estimator for μ?

2.* Let X be a $N(0, 1)$ random variable. Use integration by parts to show that

$$\mathbb{E}[X^{n+2}] = (n+1)\ \mathbb{E}[X^n] \quad \forall n \in \mathbb{Z}^+. \tag{2}$$

Reason why $\mathbb{E}[X^n] = 0$ when n is odd; what is the general answer if n is even? (Hint : probably a rare occasion to use "!!")

3. Let (X, Y) be a random vector with joint probability density function

$$f_{X,Y}(x, y) = ke^{-(x+y)} \quad \text{if } 0 < y < x, \qquad 0 \text{ otherwise.} \tag{3}$$

- Using that $\mathbb{P}[(X, Y) \in \mathbb{R}^2] = 1$, find the value of k.
- For the regions S and T in $\mathbb{R}^2$:

  $$S = \{(x, y)\ :\ x \in [0, 1],\ y \in [0, 1]\}, \qquad T = \{(x, y)\ :\ 0 < x < y\},$$

  calculate the probability that (X, Y) is inside the given region
- Find the marginal *probability distribution function* (PDF) of Y, and hence identify the distribution of Y. (Google might be your best friend here!)

4. Let (X, Y) be a random vector with joint probability density function.

$$f_{X,Y}(x, y) = \begin{cases} \frac{y-x}{2}, & x \in [-1, 0], y \in [0, 1]; \\ \frac{x+y}{2}, & x \in [0, 1], y \in [0, 1]; \\ 0, & \text{otherwise}; \end{cases} \tag{4}$$

Find the marginal PDF of X. Show that the correlation coefficient X and Y is zero. Show/infer that X and Y are not independent.

5.* Let X be a random variable. Also, let Z be a random variable, independent of X, such that $\mathbb{P}[Z = 1] = \mathbb{P}[Z = -1] = 1/2$. Let $Y = XZ$. Show that X and Y are uncorrelated.

Solutions

1. We have

$$\mathbb{E}[T_1] = \frac{1}{2}(\mathbb{E}[X_1] + \mathbb{E}[X_2]) = \mathbb{E}[T_2] = \mu.$$

Similarly, $\mathbb{E}[T_2] = \mathbb{E}[T_3] = \mu$, so all are unbiased when used as estimators of μ. We have

$$\mathrm{Var}(T_1) = \left(\frac{1}{2}\right)^2 (\mathrm{Var}(X_1) + 2\mathrm{Cov}(X_1, X_2) + \mathrm{Var}(X_2)) = \frac{1}{4}(1 + 0 + 4) = 5,$$

and similarly $\mathrm{Var}(T_2) = 8$, $\mathrm{Var}(T_3) = 4/5$. That is, we prefer $\mathbb{E}[T_3]$ as an estimator of μ, because T_3 has the smallest variance and so is likely to be closest to its mean.

2. Since $\mathbb{E}[X] = 0$, induction gives that $\mathbb{E}[X^n] = 0$ for all odd n. Similarly, $\mathbb{E}[X^n] = 1 \times 3 \times 5 \times \ldots \times (n-1)$ when n is even.

3. We just need to compute a bunch of integrals here...

- Equate
$$\mathbb{P}[(X,Y) \in \mathbb{R}^2] \equiv \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x,y)\ dxdy = 1.$$
The value of the integral is $k/2$, that is $k = 2$.

- We have
$$\mathbb{P}[(X,Y) \in S] = \int_0^1 \int_0^x 2e^{-(x+y)} = \frac{1}{e^2} - \frac{2}{e} + 1,$$
Since $f_{X,Y}(x,y) = 0\ \forall (x,y) \in T$, $\mathbb{P}[(X,Y) \in T] = 0$.

- We integrate x out, giving for $y > 0$,
$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y) dx = 2e^{-2y}.$$
So Y has an *exponential distribution* with parameter $\lambda = 2$ (or, equivalently, a *Gamma distribution* with parameters $\alpha = 1$ and $\beta = 2$).

4. As above, the marginal PDF is

$$f_X(x) = \int_0^1 f_{X,Y}(x,y) dy$$

which can be computed in the three cases as

$$f_X(x) = \begin{cases} \int_0^1 \frac{y-x}{2} dx = \frac{1-2x}{4}; & \\ \int_0^1 \frac{x+y}{2} = \frac{1+2x}{4}; & \\ 0, & \text{otherwise}; \end{cases}$$

This gives $\mathbb{E}[X] = 0$, $\because f_X(x) = f_X(-x)$. To find $\mathbb{E}[XY]$, we calculate

$$\mathbb{E}[XY] = \int_0^1 \int_{-1}^1 xy f_{X,Y}(x,y) dx dy = 0.$$

Hence, the covariance is $\mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] = 0$ (note that we do not need to know the value of $\mathbb{E}[Y]$). Hence the correlation coefficient is also zero.

To show dependence, we will show that the joint PDF of X and Y does not factorize into the marginal densities. For example, dividing $f_{X,Y}(x,y)$ by $f_X(x)$ gives $2(y+x)/(1+2x)$ if $x \in [0,1]$ and $2(y-x)/(1-2x)$ if $x \in [-1,0]$, which does not depend only on y, so cannot be equal to $f_Y(y)$. Hence, X and Y are not independent.

5. All we need to show is that $\mathrm{Cov}(X,Y) = 0$, using the fact that $\mathbb{E}(XZ) = \mathbb{E}(X)\mathbb{E}(Z)$.