PHYS 598 SDA RECITATION 1 - PROBLEM SET

1. Even though every norm gives rise to a metric, not every metric arises from a norm. Let d be any metric on a vector space V. Show that

$$\widetilde{d}(v,w) = \frac{d(v,w)}{1+d(v,w)}$$

defines a new metric that *cannot* arise from a norm. (Quick way to see this: assume $d(v,w)$ does arise from a norm.)

2. A sequence of vector spaces and linear maps

$$\dots \longrightarrow V_{n-1} \xrightarrow{j_n} V_n \xrightarrow{j_{n+1}} V_{n+1} \longrightarrow \dots$$

is said to be exact at V_n if im j_n = ker j_{n+1}. A sequence is EXACT if it is exact at each of its constituent vector spaces. A sequence of vector spaces and linear maps of the form

$$\mathbf{0} \longrightarrow U \xrightarrow{j} V \xrightarrow{k} W \longrightarrow \mathbf{0} \tag{1}$$

is a SHORT EXACT sequence. (Here $\mathbf{0}$ denotes the trivial 0-dimensional vector space, and the unlabeled arrows are the obvious linear maps.) In differential geometry, especially relevant for work on the Maxwell's equations, we have the short exact sequence

$$\mathbb{H}_1 \xrightarrow{\text{grad}} \mathbb{H}_{\text{curl}} \xrightarrow{\text{curl}} \mathbb{H}_{\text{div}} \xrightarrow{\text{div}} \mathbb{L}_2.$$

Convince yourself of the following: ((d) is optional)

(a) The sequence (1) is exact at U **iff** j is injective (one-one).

(b) The sequence (1) is exact at W **iff** k is surjective (onto).

(c) Let U and V be vector spaces. Then the following sequence is short exact:

$$\mathbf{0} \longrightarrow U \xrightarrow{\imath_1} U \times V \xrightarrow{\pi_2} V \longrightarrow \mathbf{0}$$

The indicated linear maps are defined by

$$\imath_1 : U \to U \times V : a \to (a,0), \quad \pi_2 : U \times V \to V : (a,b) \to b.$$

(d) Suppose that the following sequence of $\{V_i\}$ and linear maps $\{f_i\}$ is LONG exact,

$$\mathbf{0} \longrightarrow V_n \xrightarrow{f_n} V_{n-1} \xrightarrow{f_{n-1}} \dots \xrightarrow{f_2} V_1 \xrightarrow{f_1} V_0 \longrightarrow \mathbf{0}.$$

Show that

$$\sum_{k=0}^{n} (-1)^k \dim(V_k) = 0.$$

3. Define $K : C[0,1] \to C[0,1]$ by

$$Kf(x) = \int_0^1 k(x,y)f(y)dy$$

where $k : [0,1] \times [0,1] \to \mathbb{R}$ is continuous. Prove that K is bounded.

4. Find the kernel and range of the linear operator $K : C[0,1] \to C[0,1]$ defined by

$$Kf(x) = \int_0^1 \sin \pi(x-y)f(y)dy.$$

5. Let V be an n-dimensional vector space. If $x \in V$ has components $x = (x_1, x_2, ..., x_n)$ in an orthonormal basis. Show that the ℓ_p-norm $||x||_p := \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}$ arises from an inner product only when $p = 2$.