PHYS 598 SDA RECITATION 5 - PROBLEM SET

(Starred problems to be done only after you're done with the unstarred problems.)

1. We say that a sequence of real numbers $\{a_n\}$ is a *Cauchy sequence* provided that for every $\varepsilon > 0$, there is a natural number N so that when $n, m \geq N$, we have that $|a_n - a_m| \leq \varepsilon$. All Cauchy sequences are, by definition, convergent.

- Show that **if** $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so is $\{a_n^2\}_{n=1}^{\infty}$. What about $\{a_n^p\}_{n=1}^{\infty}$, where $p \in \mathbb{R}^+$?
- *Give an example of a Cauchy sequence $\{a_n^2\}_{n=1}^{\infty}$ such that $\{a_n\}_{n=1}^{\infty}$ is *not* Cauchy.

2. Define a differential operator $L = D_x - x^2$. Find the kernel $\ker(L)$.

3. Consider the initial value problem on the real line

$$u_t = ku_{xx}, \quad u(x,0) = \varphi(x). \tag{1}$$

If φ is bounded and continuous, then the unique solution is given by

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)\, dy \tag{2}$$

where $S(x-y,t)$ is the heat kernel given by

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-x^2/4kt}. \tag{3}$$

- As $t \to 0$, something remarkable happens to $S(x,t)$ - it starts to look more and more like a "function" you may be (very?) familiar with. To see this, on whatever software you have access to, plot $S(x,t)$ for $t = \{1, 0.1, 0.01, 0.001, 0.0001\}$. Why is this expected?

- For a generic initial value problem

$$u_t = ku_{xx}, \quad u(x,t_0) = \varphi(x), \tag{4}$$

  what condition should you impose on $S(x,t)$? The hint is in the previous sub-part.

- *Why is the condition "bounded and continuous" necessary in writing the heat kernel solution?

Motivation : In QFT, initial value problems are everywhere. In the path integral language, $\varphi(x)$ would be called the source and $S(x,t)$, the ....

4. Consider the most general Dirichlet problem

$$u_t - ku_{xx} = f(x,t), \quad u(x,0) = \varphi(x), \quad u(0,t) = g(t), \quad u(L,t) = h(t). \tag{5}$$

on the closed interval $[0, L]$. Convince yourself that the problem admits *at most* one solution.

5.* If φ is bounded and continuous, then the initial value problem (1) has a unique solution which satisfies the estimate

$$||u(x,t)||_p \leq ||\varphi(x)||_p \tag{6}$$

for each $p \geq 1$. In particular, the heat equation (1) is well-posed in L^p for each $p \geq 1$.

6. Look up the write-ups on `https://www.maths.tcd.ie/ pete/pde/wave1.pdf` and `.../wave2.pdf`. Discuss the **definitions** and **lemmas**. You do not need to remember any of the explicit formulae, just get a hang of what's happening.