PHYS 598 SDA RECITATION 6 & 7 - PROBLEM SET

Starred problems to be done at the end. An indispensable reference is `Problems and Solutions for Groups, Lie Groups, Lie Algebras With Applications, Steeb et al.`

I. Practice with Lie Groups: In mathematics, a **Lie group** (pronounced "Lee") is a group that is also a *differentiable manifold*, with the property that the group operations are compatible with the smooth structure. The standard Wiki-level example are 2×2 real invertible matrices under multiplication, denoted by $\mathrm{GL}(2,\mathbb{R})$. This group is an open subset of $\mathbb{R}^4$, and is disconnected - it has two connected components corresponding to the positive and negative values of the determinant.

If these statements are unclear, please ask me before proceeding! For a more formal treatment, consult Ch.2 of `Homogeneous Finsler Spaces, Springer Monographs in Mathematics`.

I-1. A complex $2n \times 2n$ matrix S is called *symplectic* if $S^t J S = J$, where

$$J = \begin{pmatrix} \mathbb{O}_n & \mathbb{I}_n \\ -\mathbb{I}_n & \mathbb{O}_n \end{pmatrix}$$

and $\mathbb{I}_n$ is the $n\times n$ identity matrix, etc. Prove that the set of $2n\times 2n$ complex symplectic matrices, denoted by $\mathrm{Sp}(n,\mathbb{C})$, is a matrix Lie group[1] [i.e., it is a topologically closed[2] subgroup of $\mathrm{GL}(2n,\mathbb{C})$]. *To show this, first prove closure, then show inclusion of the identity and finally, the existence of an inverse.*

I-2. The Lie group $\mathrm{SU}(1,1)$ is defined as the group of 2×2 matrices V that satisfy

$$V\sigma_3 V^\dagger = \sigma_3 \quad \text{and} \quad \det V = 1.$$

The Lie group $\mathrm{SO}(2,1)$ is the group of transformations on vectors $\vec{x} \in \mathbb{R}^3$ (with determinant $=1$) that preserves $x_1^2 + x_2^2 - x_3^3$. Display the *homomorphism* from $\mathrm{SU}(1,1)$ onto $\mathrm{SO}(2,1)$. *All that is required is to show that the group multiplication law is preserved.*

I-3.* Let $\alpha, \beta \in \mathbb{R}$. Let T be the 2×2 unitary diagonal matrix

$$T(\alpha,\beta) = \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{i\beta} \end{pmatrix}.$$

Find the condition on a unitary 2×2 matrix U s.t. $UT(\alpha,\beta)U^{-1} = T(\alpha',\beta')$. *Think of all possible cases.*

II. Homogenous spaces: A smooth manifold M endowed with a transitive, smooth action by a Lie group G is called a Homogeneous G-space. *If any of these terms are unfamiliar to you, open to* §2 `here`. Your task is to understand the examples of Isometry groups in the Wiki article on `Homogeneous spaces`.

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[1] Some people denote the group of $2n\times 2n$ complex symplectic matrices by $\mathrm{Sp}(2n,\mathbb{C})$. But I don't like those people.

[2] Topologically, not necessarily smooth, as defined above!

III. A Permutation group... is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G (which are thought of as bijective functions from the set M to itself). Since permutations are bijections of a set, they can be represented by Cauchy's two-line notation. This notation lists each of the elements of M in the first row, and for each element, its image under the permutation below it in the second row. If σ is a permutation of the set $M = \{x_1, x_2, \ldots, x_n\}$ then,

$$\sigma = \begin{pmatrix} x_1 & x_2 & x_3 & \ldots & x_n \\ \sigma(x_1) & \sigma(x_2) & \sigma(x_3) & \ldots & \sigma(x_n) \end{pmatrix}.$$

III-1. The group of all permutations of a set M is the symmetric group $\mathrm{Sym}(M)$. Often, the notation S_n is interchangeably used (since set M has n elements). The term *permutation group* thus means a subgroup of the symmetric group. *Convince yourself of this last statement.*

III-2. Consider the permutation group S_3, such that

$$S_3 = \{\sigma_1 = (),\ \sigma_2 = (12),\ \sigma_3 = (13),\ \sigma_4 = (23),\ \sigma_5 = (123),\ \sigma_6 = (132)\}.$$

Compute/look up the *Cayley table* of S_3. *Convince yourself that S_3 is isomorphic to the dihedral group D_3. This is very relevant to students working in crystallography - and you might just know it already!*

III-3.* Show that the inverse of a permutation matrix is its transpose.

IV. Reading exercise: We shall want to understand Lie groups and Homogeneous spaces in the context of *Grassmannian manifolds*. Our aim is to get through as much as possible of §2 in this PDF.

V.* **Further readings:** The link here explains *Schubert varieties* very simply. It's worth a read!