PHYS 598 SDA RECITATION 4 - PROBLEM SET

(Starred problems to be done only after you're done with the unstarred problems.)

1. Recall that a real-valued matrix $\mathcal{O}$ is orthogonal if $\mathcal{O}\mathcal{O}^t = \mathbb{I}$. Also, a real symmetric matrix $\mathcal{R}$ is defined by $R = R^t$. Suppose that you are given an invertible $n \times n$ matrix M.

- *Polar decomposition*: Prove that $\mathcal{R} = \sqrt{MM^t}$ is (positive) symmetric and $\mathcal{O} = \mathcal{R}^{-1}M$ is orthogonal. This is called the polar decomposition of M. (Why?)
- **Conformal matrices*: If $MM^t = \lambda\mathbb{I}$ for some $\lambda \in \mathbb{R}$. Determine the set of all 2×2 conformal matrices.

  (In physics, orthogonal transformations appear everywhere, and symmetries of physical systems are described by groups $O(n)$, which have orthogonal matrices as a representation. In special relativity, the Lorentz group is *isomorphic* to the orthogonal group $O(3, 1)$ that mixes space and time. If you ever decide to read about AdS/CFT, conformal matrices are of prime relevance.)

2. Let Ω be a convex domain in $\mathbb{R}^n$. We consider the Hilbert space $L^2(\Omega)$ of square integrable functions with scalar product

$$\langle f, g\rangle = \int_\Omega f(x)g^*(x)dx.$$

- Consider the Hilbert space $L^2[0, 1]$. Find a non-trivial function

  $$f(x) = ax^3 + bx^2 + cx + d,$$

  such that

  $$\langle f(x), x\rangle = 0, \qquad \langle f(x), x^2\rangle = 0, \qquad \langle f(x), x^3\rangle = 0,$$

  where $\langle \cdot, \cdot\rangle$ denotes the scalar product.
- Consider the Hilbert space $L^2([-1, 1])$. The *Chebyshev polynomials* are defined by

  $$T_n(x) := \cos(n \cos^{-1} x), \quad n \in \mathbb{Z}^+.$$

  Using a computer or otherwise, convince yourself that $T_n(x) \in L^2([-1, 1])$.

  (Chebyshev polynomials are used in study of nonlinear systems, a notable example being the system of two independent oscillators which plot the *Lissajous curves*.)

3. Can one construct an orthonormal basis in the Hilbert space $L^2(\mathbb{R})$ starting from ($\sigma > 0$)

$$e^{-|x|/\sigma},\ xe^{-|x|/\sigma},\ x^2e^{-|x|/\sigma}, \ldots ?$$

A note on notation : Typically, ℓ_p is used to indicate a p-summable discrete set of values. For example, $\ell_p(\mathbb{Z}^+)$ is the set of complex-valued sequences $\{a_n\}$ such that $\sum_{n\in\mathbb{Z}^+} |a_n|^p < \infty$. L^p is usually used to indicate p-summable functions (with respect to some measure) on a non-discrete measure space, such as the usual $L^p(\mathbb{R})$, the set of functions $f\colon \mathbb{R} \to \mathbb{C}$ such that $\int_\mathbb{R} |f(x)|^p dx < \infty$. The main point is that they are mathematically different notation for the same concept.

For more problems, you can consult `http://issc.uj.ac.za/downloads/problems/hilbert.pdf`.