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Chapter 1

Introduction

This book will start you on the long and difficult trek to becoming a modern problem solver. Along the path, you will learn how to use the tools of math, computers, and science.

Why should you bother? There are big problems in this world that will require expert problem solvers. Those people will make the world a better place while enjoying interesting and lucrative careers. We are talking about engineers, scientists, doctors, computer programmers, architects, actuaries, and mathematicians. Right now, those occupations represent about 6% of all the jobs in the United States. Soon, that number is expected to rise above 10%. On average, people in that 10% of the population are expected to have salaries twice that of their non-technical counterparts.

Solving problems is difficult. At some point on this journey, you will see people who are better at solving problems than you are. You, like every other person who has gone on this journey, will think "I have worked so hard on this, but that person is better at it than I am. I should quit." Don't.

First, solving problems is like a muscle. The more you do, the better you get at it. It is OK to say "I am not good at this yet." That just means you need more practice.

Second, you don't need to be the best in the world. 10 million people your age can be better at solving problems than you, *and you can still be in the top 10% of the world*. If you complete this journey, there will be problems for you to solve and a job where your problem-solving skills will be appreciated.

So where do we start?

1.1 Atoms

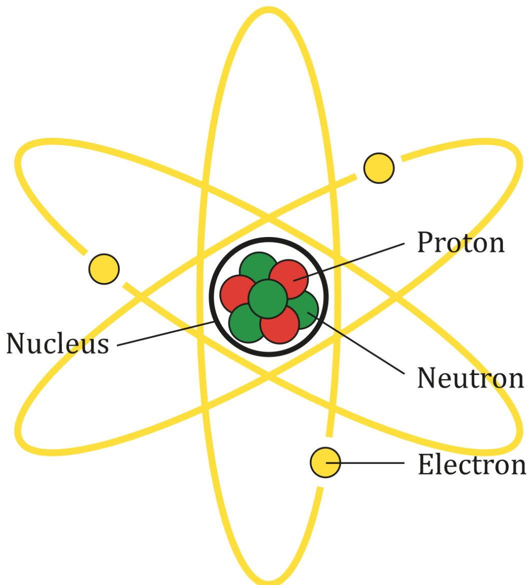
The famous physicist Richard Feynman once asked this question: "If, in some cataclysm, all of scientific knowledge were to be destroyed, and only one sentence was passed on to the next generation of creatures, what statement would contain the most information in the fewest words?"

His answer was "All things are made of atoms—little particles that move around in perpetual motion, attracting each other when they are a little distance apart, but repelling upon being squeezed into one another."

That seems like a good place to start.

All things (including the air around you) are made of atoms. Atoms are very tiny – there are more atoms in a drop of water than there are drops of water in all the oceans.

Every atom has a nucleus that contains protons and neutrons. There is also a cloud of electrons flying around the nucleus. However, the mass of the atom comes mainly from the protons and neutrons, which are exponentially heavier than electrons.



Watch **Elements and atoms** from Khan Academy at https://youtu.be/IFKnq9QM6_A.

We classify atoms by the numbers of protons they have. An atom with one proton is a hydrogen atom, an atom with two protons is a helium atom, and so forth (refer to periodic table on pg..). We say that hydrogen and helium are

elements because the classification of elements is based on proton number. And we give each element an atomic symbol. Hydrogen gets H. Helium gets He Oxygen gets O. Carbon gets C, etc.

Often two hydrogen atoms will attach to an oxygen atom. The result is a water molecule. Why do they cluster together? because they share electrons in their clouds.

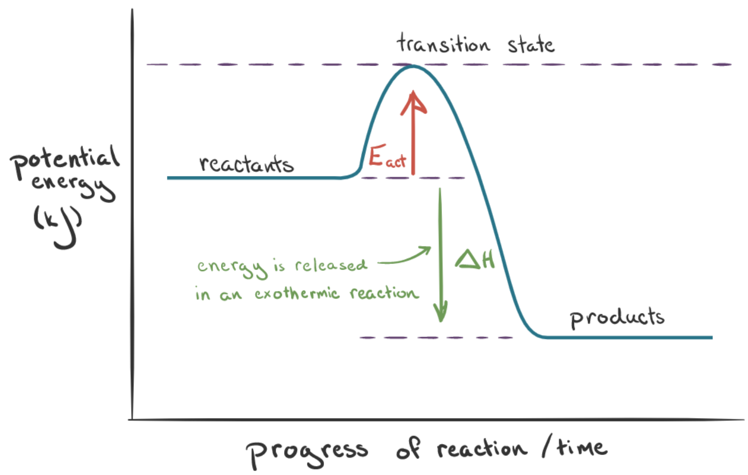
A molecule is described by the elements it contains. Water is H_2O because it has two hydrogen atoms and one oxygen atom.

There are many kinds of molecules. You know a few:

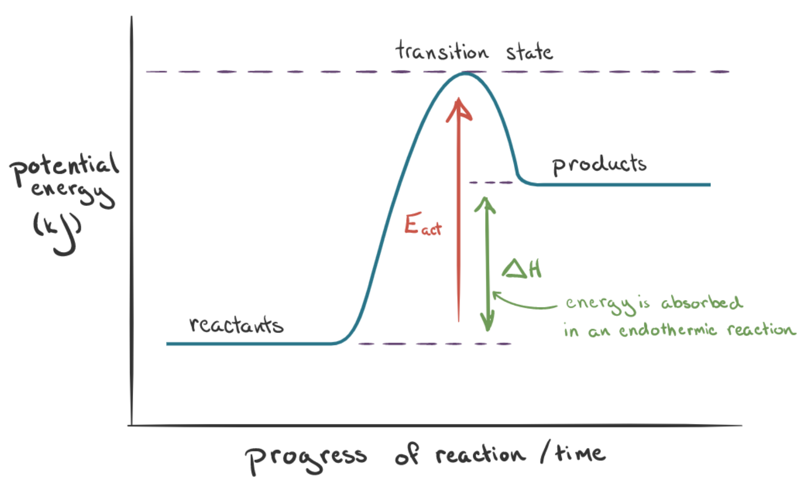
- Table salt is crystals made of $NaCl$ molecules: a sodium atom attached to a chlorine atom.
- Baking soda, or sodium bicarbonate, is $NaHCO_3$.
- Vinegar is a solution including acetic acid (CH_3COOH).
- O_2 is the oxygen molecules that you breathe out of the air (Air, a blend of gases, is mostly N_2.).

Sometimes two hydrogen atoms form a molecule (H_2). Sometimes two oxygen atoms form a molecule (O_2). If you mix these together and light a match, they will rearrange themselves into water molecules. This is called a *chemical reaction*. In any chemical reaction, the atoms are rearranged into new molecules.

Some chemical reactions (like the burning of hydrogen gas described above) are *exothermic* – that is, they give off energy. Burning hydrogen gas happens quickly and gives off a lot of energy. If you have enough, it will make quite an explosion.



Other chemical reactions are *endothermic* – that is they consume energy. Photosynthesis, the process by which plants consume energy from the sun to make sugar from CO_2 and H_2O requires an endothermic chemical reaction.



1.2 Mass and Acceleration

Each atom has a mass, so everything that is made up of atoms has a mass, which is pretty much everything. We measure mass in grams. A paper clip is about 1 gram of steel. An adult human can weigh 70,000 grams, so for larger things we often talk about kilograms. A kilogram is 1000 grams.

The first interesting thing about mass is that objects with more mass require more force to accelerate. For example, pushing a bicycle so that it accelerates from a standstill to jogging speed in 2 seconds requires a lot less force than pushing a train so that it accelerates at the same rate.

You will probably find it useful to watch Khan Academy's summary of Newton's second law of motion: https://youtu.be/ou9YMWlJgkE

Newton's Second Law of Motion

The force necessary to accelerate an object of mass m is given by:

$$F = ma$$

That is the force is equal to the mass times the acceleration.

What are the units here? We already know that mass is measured in kilograms. We can measure velocity in meters per second, but that is different from acceleration. Acceleration is the rate of change in velocity. So if we want to go from 0 to 5 meters per second (that's jogging speed) in two seconds. That is a change in velocity of 2.5 meters per second every second. We would say this acceleration is $2.5m/s^2$.

What about measuring force? Newton decided to name the unit after himself: The force necessary to accelerate one kilogram at $1m/s^2$ is known as *a newton*.

Exercise 1 Acceleration

Working Space

While driving a bulldozer, you come across a train car (with no brakes and no locomotive) on a track in the middle of a city. The train car has a label telling you that it weighs 2,400 kg. There is a bomb welded to the interior of the train car, and the timer tells you that you can safely push the train car for 120 seconds. To get the train car to where it can explode safely, you need to accelerate it to 20 meters per second. Fortunately, the track is level and the train car's wheels have almost no rolling resistance.
With what force, in newtons, do you need to push the train for those 120 seconds?

Answer on Page 389

1.3 Mass and Gravity

The second interesting thing about mass is that masses are attracted to each other by the force we call *gravity*. The force of attraction between two objects is proportional to the product of their masses. As objects get farther away, the force decreases. That is why you are more attracted to the earth than you are to distant stars, which have much more mass than the earth.

Newton's Law of Universal Gravitation

Two masses (m_1 and m_2) that are a distance of r from each other, are attracted toward each other with a force of magnitude:

$$F = G\frac{m_1 m_2}{r^2}$$

where G is the universal gravitational constant. If you measure the mass in kilograms and the distance in meters. G is about 6.674×10^{-11}. That will get you the force of the attraction in newtons.

Exercise 2 Gravity

Working Space

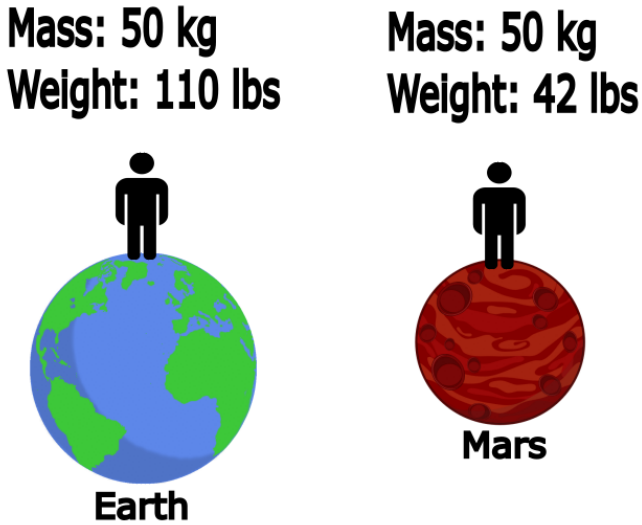
The earth's mass is about 6×10^{24} kilograms.
Your spacecraft's mass is 6,800 kilograms.
Your spacecraft is also about 100,000 km from the center of the earth. (For reference, the moon is about 400,000 km from the center of the earth.)
What is the force of gravity that is pulling your spacecraft and the earth toward each other?

Answer on Page 389

1.4 Mass and Weight

Gravity pulls on things proportional to their mass, so we often ignore the difference between mass and weight.

The weight of an object is the force due to the object's mass and gravity. When we say, "This potato weighs 1 pound," we actually mean "This potato weighs 1 pound on earth." That same potato would weigh about one-fifth of a pound on the moon.



But that potato has a mass of 0.45 kg anywhere in the universe.

Chapter 2

Atomic and Molecular Mass

A proton and a neutron have about the same mass. An electron, on the other hand, has much less mass: One neutron weighs about the same amount as 2000 electrons. Thus, the mass of any object comes mostly from the protons and neutrons in the nucleus of its atoms.

We know how many protons an atom has by what element it is, but how do we know the number neutrons?

If you buy a balloon filled with helium, it will have two different kinds of helium atoms: Most of the helium atoms will have 2 neutrons, but a few will have only 1 neutron. We say that these are two different *isotopes* of helium. We call them helium-4 (or ^{4}He) and helium-3 (or ^{3}He). Isotopes are named for the sum of protons and neutrons the atom has: helium-3 has 2 protons and 1 neutron.

Watch Khan Academy's **Atomic mass, number, and isotopes** at `https://www.khanacademy.org/science/chemistry/atomic-structure-and-properties/introduction-to-the-atom/v/atomic-number-mass-number-and-isotopes`

A hydrogen atom nearly always has just 1 proton and no neutrons. A helium atom nearly always has 2 protons and 2 neutrons. So, if you have a 100 hydrogen atoms and 100 helium atoms, the helium will have about 4 times more mass than the hydrogen. We say "Hydrogen is about 1 atomic mass unit(amu), and helium-4 is about 4 atomic mass units."

What, precisely, is an atomic mass unit? It is defined as 1/12 of the mass of a carbon-12 atom. Scientists have measured the mass of helium-4, and it is about 4.0026 atomic mass units. (By the way, an atomic mass unit is also called a *dalton*.)

Now you are ready to take a good look at the periodic table of elements. Here is the version from Wikipedia:

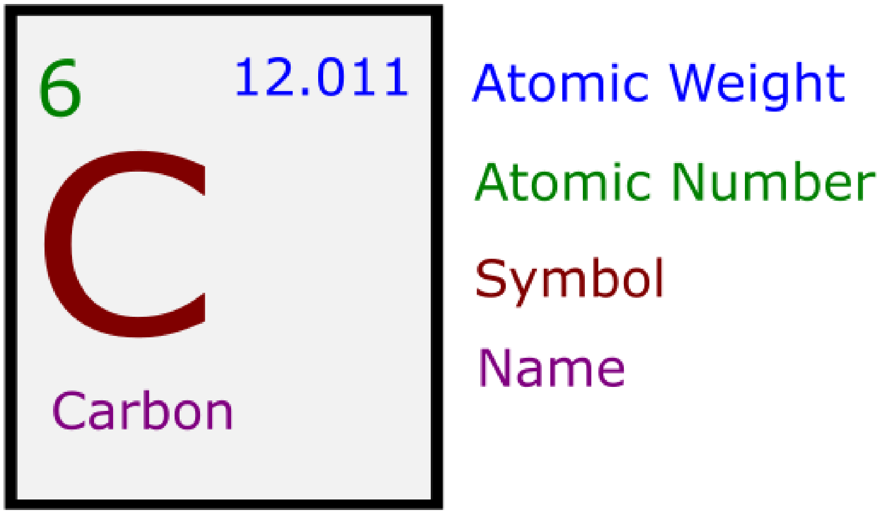
Periodic Table of Elements

IA	IIA	IIIB	IVB	VB	VIB	VIIB	VIIIB	VIIIB	VIIIB	IB	IIB	IIIA	IVA	VA	VIA	VIIA	VIIIA
1 **H** Hydrogen 1.01																	2 **He** Helium 4.00
3 **Li** Lithium 6.94	4 **Be** Beryllium 9.01											5 **B** Boron 10.81	6 **C** Carbon 12.01	7 **N** Nitrogen 14.01	8 **O** Oxygen 16.00	9 **F** Fluorine 19.00	10 **Ne** Neon 20.18
11 **Na** Sodium 22.99	12 **Mg** Magnesium 24.31											13 **Al** Aluminum 26.98	14 **Si** Silicon 28.09	15 **P** Phosphorus 30.97	16 **S** Sulfur 32.06	17 **Cl** Chlorine 35.45	18 **Ar** Argon 39.95
19 **K** Potassium 39.10	20 **Ca** Calcium 40.08	21 **Sc** Scandium 44.96	22 **Ti** Titanium 47.87	23 **V** Vanadium 50.94	24 **Cr** Chromium 52.00	25 **Mn** Manganese 54.94	26 **Fe** Iron 55.85	27 **Co** Cobalt 58.93	28 **Ni** Nickel 58.69	29 **Cu** Copper 63.55	30 **Zn** Zinc 65.38	31 **Ga** Gallium 69.72	32 **Ge** Germanium 72.63	33 **As** Arsenic 74.92	34 **Se** Selenium 78.97	35 **Br** Bromine 79.90	36 **Kr** Krypton 83.80
37 **Rb** Rubidium 85.47	38 **Sr** Strontium 87.62	39 **Y** Yttrium 88.91	40 **Zr** Zirconium 91.22	41 **Nb** Niobium 92.91	42 **Mo** Molybdenum 95.95	43 **Tc** Technetium (98)	44 **Ru** Ruthenium 101.07	45 **Rh** Rhodium 102.91	46 **Pd** Paladium 106.42	47 **Ag** Silver 107.87	48 **Cd** Cadmium 112.41	49 **In** Indium 114.82	50 **Sn** Tin 118.71	51 **Sb** Antimony 121.76	52 **Te** Tellurium 127.60	53 **I** Iodine 126.90	54 **Xe** Xenon 131.29
55 **Cs** Cesium 132.91	56 **Ba** Barium 137.33	57 - 71 **Lanthanides**	72 **Hf** Hafnium 178.49	73 **Ta** Tantalum 180.95	74 **W** Tungsten 183.84	75 **Re** Rhenium 186.21	76 **Os** Osmium 190.23	77 **Ir** Iridium 192.22	78 **Pt** Platinum 195.08	79 **Au** Gold 196.97	80 **Hg** Mercury 200.59	81 **Tl** Thallium 204.38	82 **Pb** Lead 207.20	83 **Bi** Bismuth 208.98	84 **Po** Polonium (209)	85 **At** Astatine (210)	86 **Rn** Radon (222)
87 **Fr** Francium (223)	88 **Ra** Radium (226)	89 - 103 **Actinides**	104 **Rf** Rutherfordium (265)	105 **Db** Dubnium (268)	106 **Sg** Seaborgium (271)	107 **Bh** Bohrium (270)	108 **Hs** Hassium (277)	109 **Mt** Meitnerium (276)	110 **Ds** Darmstadtium (281)	111 **Rg** Roentgenium (280)	112 **Cn** Copernicium (285)	113 **Nh** Nihonium (284)	114 **Fl** Flerovium 289	115 **Mc** Moscovium (288)	116 **Lv** Livermorium (293)	117 **Ts** Tennessine (294)	118 **Og** Oganesson (294)

57 **La** Lanthanum 138.91	58 **Ce** Cerium 140.12	59 **Pr** Praseodymium 140.91	60 **Nd** Neodymium 144.24	61 **Pm** Promethium (145)	62 **Sm** Samarium 150.36	63 **Eu** Europium 151.96	64 **Gd** Gadolium 157.25	65 **Tb** Terbium 158.93	66 **Dy** Dysprosium 162.50	67 **Ho** Holmium 164.93	68 **Er** Erbium 167.26	69 **Tm** Thulium 168.93	70 **Yb** Ytterbium 173.05	71 **Lu** Lutetium 174.97
89 **Ac** Actinium (227)	90 **Th** Thorium 232.04	91 **Pa** Protactinium 231.04	92 **U** Uranium 238.03	93 **Np** Neptunium (237)	94 **Pu** Plutonium (244)	95 **Am** Americium (243)	96 **Cm** Curium (247)	97 **Bk** Berkelium (247)	98 **Cf** Californium (251)	99 **Es** Einsteinium (252)	100 **Fm** Fermium (257)	101 **Md** Mendelevium (258)	102 **No** Nobelium (259)	103 **Lr** Lawrencium (262)

There is a square for each element. In the middle, you see the atomic symbol and the name of the element. In the upper right corner is the atomic number – the number of protons in the atom.

In the upper left corner is the atomic mass in atomic mass units.



Look at the atomic mass of boron. About 80% of all boron atoms have six neutrons. The other 20% have only 5 neutrons. So most boron atoms have a mass of about 11 atomic mass units, but some have a mass of about 10 atomic mass units. The atomic mass of boron is equivalent to the average mass of a boron atom: 10.811.

Exercise 3 Mass of a Water Molecule

Working Space

Using the periodic table, what is the average mass of one water molecule in atomic mass units?

Answer on Page 390

2.1 Molar Mass

An atomic mass unit is a very, very, very small unit; we would much rather work in grams. It turns out that $6.02214076 \times 10^{23}$ atoms equal 1 mole(a standard measure for chemistry). Scientists use this number so much that they gave it a name: *the Avogadro constant* or *Avogadro's number*.

Watch Khan Academy's discussion of the mole at https://www.khanacademy.org/science/ap-chemistry-beta/x2eef969c74e0d802:atomic-structure-and-properties/x2eef969c74e0d802:moles-and-molar-mass/v/the-mole-and-avogadro-s-number

If you have 12 doughnuts, that's a dozen doughnuts. If you have $6.02214076 \times 10^{23}$ doughnuts, you have a *mole* of doughnuts. (Note: it isn't practical to measure doughnuts this way: A mole of doughnuts would be about the size of the earth. We use moles for small things like molecules.)

Let's say you want to know how much a mole of $NaCl$ weighs. From the periodic table, you see that Na has an atomic mass of 22.98976 atomic mass units. And Cl has 35.453 atomic mass units. One atom of $NaCl$ has a mass of $22.98976 + 35.453 = 58.44276$ atomic mass units. Then a mole of $NaCl$ has a mass of 58.44276 grams. Handy, right?

Exercise 4 Burning Methane

Natural gas is mostly methane (CH_4). When one molecule of methane burns, two oxygen molecules (O_2) are consumed. One molecule of H_2O and one molecule of CO_2 are produced.
If I need 200 grams of water, how many grams of methane do I need to burn?
(This is how the hero in "The Martian" made water for his garden.)

Working Space

Answer on Page 390

2.2 Heavy atoms aren't stable

When you look at the periodic table, there are a surprisingly large number of elements. You might be told to "Drink milk so that you can get the calcium you need." However, no one has told you "You should eat kale so that you get enough copernicium in your diet."

Copernicium, with 112 protons and 173 neutrons, has only been observed in a lab. It is highly radioactive and unstable(meaning it decays): a copernicium atom usually lives for less than a minute before decaying.

The largest stable element is lead, which has 82 protons and between 122 and 126 neutrons. Elements with lower atomic numbers than lead, have at least one stable isotope. Elements with higher atomic numbers than lead don't.

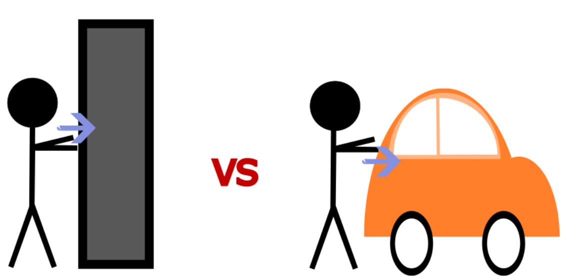
Bismuth, with an atomic number of 83, is *almost* stable. In fact, most bismuth atoms will live for billions of years before decaying.

Chapter 3

Work and Energy

In this chapter, we are going to talk about how engineers define work and energy. We have already talked about force. Force is measured in newtons, and one newton is equal to the force necessary to accelerate one kilogram at a rate of $1 m/s^2$.

When you lean on a wall, you are exerting a force on the wall, but you aren't doing any work. On the other hand, if you push a car for a mile, you are clearly doing work. Work, to an engineer, is the force you apply to something, as well as the distance that it moves, in the direction of the applied force. We measure work in *joules*. A joule is one newton of force over one meter.



For example, if you push a car uphill with a force of 10 newtons for 12 meters, you have done 120 joules of work.

Work is how energy is transferred from one thing to another. When you push the car, you also burn sugars(energy of the body) in your blood. That energy is then transferred to the car: after it has been pushed uphill.

Thus, we measure the energy something consumes or generates in units of work: joules, kilowatt-hours, horsepower-hours, foot-pounds, BTUs(British Thermal Unit), and calories.

Let's go over a few different forms that energy can take.

Watch Khan Academy's **Changes in energy** at https://www.khanacademy.org/science/ms-physics/x1baed5db7c1bb50b:energy/x1baed5db7c1bb50b:changes-in-energy/a/changes-in-energy

3.1 Heat

When you heat something, you are transferring energy to it. The BTU is a common unit for heat: One BTU is the amount of heat required to raise the temperature of one pound of water, by one degree. One BTU is about 1,055 joules. In fact, when you buy and sell natural gas as fuel, it is priced by the BTU.

3.2 Electricity

Electricity is the movement of electrons. When you push electrons through a space that resists their passage (like a light bulb), energy is transferred from the power source (a battery) into the source of the resistance.

Let's say your lightbulb consumes 60 watts of electricity, and you leave it on for 24 hours. We would say that you have consumed 1.44 kilowatt hours or 3,600,000 joules.

Watch Khan Academy's **Introduction to charge** at https://www.khanacademy.org/science/in-in-class10th-physics/in-in-electricity/in-in-electric-current-circuit/v/intro-to-charge

3.3 Chemical Energy

As mentioned early, some chemical reactions consume energy and some produce energy. Thus, energy can be stored in the structure of a molecule. When a plant uses photosynthesis to rearrange water and carbon dioxide into a sugar molecule, it converts the energy in the sunlight(solar energy) into chemical energy. Remember photosythesis is a process that releases energy. Therefore, the sugar molecule has more chemical energy than the carbon dioxide and water molecules that were used in its creation.

In our diet, we measure this energy in *kilocalories*. A calorie is the energy necessary to raise one gram of water one degree Celsius: it is about 4.19 joules. This is a very small unit: an apple has about 100,000 calories(100 kilocalories), so people working with food started measuring everything in kilocalories.

Here is where things get confusing: People who work with food got tired of saying "kilocalories", so they just started using "Calorie" to mean 1,000 calories. This has created terrible confusion over the years. So if the C is capitalized, "Calorie" probably means kilocalorie.

3.4 Kinetic Energy

A mass in motion has energy. For example, if you are in a moving car and you slam on the breaks, the energy from the motion of the car will be converted into heat in the breaks and under the tires.

How much energy does the car have?

Formula for Kinetic Energy

$$E = \frac{1}{2}mv^2$$

where E is the energy in joules, m is the mass in kilograms, and v is the speed in meters per second.

3.5 Gravitational Potential Energy

Watch Khan Academy's **Potential energy** at `https://youtu.be/oGzwVYPxKjg`

When you lift something heavy onto a shelf, you are giving it *potential energy*. The amount of energy that you transferred to it is proportional to its weight and the height that you lifted it.

On the surface of the earth, gravity will accelerate a heavy object downward at a rate of $9.8m/s^2$.

Formula for Gravitational Potential Energy

On earth, then, gravitational potential energy is given by

$$E = (9.8)mh$$

where E is the energy in joules, m is the mass of the object you lifted, and h is the height that you lifted it.

There are other kinds of potential energy. For example, when you draw a bow, you have given that bow potential energy. When you release it, the potential energy is transferred to the arrow, which expresses it as kinetic energy.

3.6 Conservation of Energy

The first law of thermodynamics says "Energy is neither created nor destroyed."

Energy can change forms: Your cells consume chemical energy to give gravitational potential energy to a car you push up a hill. However, the total amount of energy in a closed system stays constant.

Exercise 5 The Energy of Falling

A 5 kg cannonball falls off the top of a 3 meter ladder. Just before it hits the floor, all of its gravitational potential energy has been converted into kinetic energy. How fast is the cannonball going when it hits the floor?

Working Space

Answer on Page 390

3.7 Efficiency

Watch Khan Academy's **Laws of thermodynamics** at `https://www.khanacademy.org/science/ap-biology/cellular-energetics/cellular-energy/a/the-laws-of-thermodynamics`

Although energy is always conserved as it moves through different forms, scientists aren't always that good at controlling it.

For example, a car engine consumes the chemical energy in gasoline. Only about 20% of the energy consumed is used to turn the wheels. Most of the energy is actually lost as heat. If you run a car for a while, the engine gets very hot and the exhaust going out the tailpipe turns hot.

A human is about 25% efficient. Most of the loss is in the heat produced during the chemical reactions that turns food into motion.

In general, if you are trying to increase efficiency in any system, the solution is usually easy to identify because heat is produced. Reduce heat, Increase efficiency.

Light bulbs are an interesting case. To get the light of a 60 watt incandescent bulb, you can use an 8 watt LED or a 16 watt fluorescent light. Thus, we say that the LED light is much more efficient: If you run both, the incandescent bulb will consume 1.44 kilowatt-hours. The LED will consume only 0.192 kilowatt-hours.

Besides light, the incandescent bulb is producing a lot of heat. If it is inside your house, what happens to the heat? It warms your house.

In the winter, when you want light and heat, the incandescent bulb is 100% efficient!

In the summer, if you are running the air conditioner, the incandescent bulb is worse than just "inefficient at making light" – it is actually counteracting the air conditioner!

Chapter 4

Units and Conversions

At this point, you are working with a lot of units: grams for weight, joules for energy, newtons for force, meters for distance, seconds for time, etc. For each type of measurement, there are several different units; for example, distance can be measured in feet, miles, and light-years.

Some Equalencies

Distance	
1 mile	1.6093 kilometers
1 foot	0.3048 meters
1 inch	2.54 centimeters
1 light-year	9.461×10^{12} kilometers
Volume	
1 milliliter	1 cubic centimeter
1 quart	0.9461 liters
1 gallon	3.7854 liters
1 fluid ounce	29.6 milliliters
Mass	
1 pound	0.4535924 kilograms
1 ounce	0.4535924 grams
1 metric ton	1000 kilograms
Force	
1 newton	1 kilogram meter per sec^2
Pressure	
1 pascal	1 newton per square meter
1 bar	0.98692 atmosphere
1 pound per square inch	6897 pascals
Energy	
1 joule	1 newton meter
1 calorie	4.184 joules
1 kilowatt-hour	3.6×10^6 joules

(You don't need to memorize these! Just remember that this page is here.)

In the metric system, prefixes are often used to express a multiple. Here are the common prefixes:

Common Prefixes for Metric Units

giga	$\times 10^9$
mega	$\times 10^6$
kilo	$\times 10^3$
milli	$\div 10^3$
micro	$\div 10^6$
nano	$\div 10^9$

(These are worth memorizing. Here's a mnemonic: "King Henery Doesn't Usually Drink Chocolate Milk.")

4.1 Conversion Factors

Here is a really handy trick to remembering how to do conversions between units.

Often, you will be given a table like the one above, and someone will ask you "How many miles are in 0.23 light-years?" You know that 1 mile = 1.6093 kilometers and that 1 light-year is 9.461×10^{12} kilometers. How do you do the conversion?

The trick is to treat the two parts of the equality as a fraction that equals 1. That is, you think:

$$\frac{1 \text{ miles}}{1.6093 \text{ km}} = \frac{1.6093 \text{ km}}{1 \text{ miles}} = 1$$

and

$$\frac{1 \text{ light-years}}{9.461 \times 10^{12} \text{ km}} = \frac{9.461 \times 10^{12} \text{ km}}{1 \text{ light-years}} = 1$$

We call these fractions *conversion factors*.

Now, your problem is

$$0.23 \text{ light-years} \times \textit{ Some conversion factors} = ? \text{ miles}$$

Note that when you multiply fractions together, things in the numerators can cancel with things in the denominator:

$$\left(\frac{31\pi}{47}\right)\left(\frac{11}{37\pi}\right) = \left(\frac{31\cancel{\pi}}{47}\right)\left(\frac{11}{37\cancel{\pi}}\right) = \left(\frac{31}{47}\right)\left(\frac{11}{37}\right)$$

When working with conversion factors, you will do the same with the units:

$$0.23 \text{ light-years}\left(\frac{9.461 \times 10^{12} \text{ km}}{1 \text{ light-years}}\right)\left(\frac{1 \text{ miles}}{1.6093 \text{ km}}\right) =$$

$$0.23 \text{ }\cancel{\text{light-years}}\left(\times\frac{9.461 \times 10^{12} \text{ }\cancel{\text{km}}}{1 \text{ }\cancel{\text{light-years}}}\right)\left(\frac{1 \text{ miles}}{1.6093 \text{ }\cancel{\text{km}}}\right) = \frac{(0.23)(9.461 \times 10^{12})}{1.6093} \text{ miles}$$

Exercise 6 Simple Conversion Factors

How many calories are in 4.5 kilowatt-hours?

Working Space

Answer on Page 390

4.2 Conversion Factors and Ratios

Conversion factors also work on ratios. For example, if you are told that a bug is moving 0.5 feet every 120 milliseconds. What is that in meters per second?

The problem then is

$$\frac{0.5 \text{ feet}}{120 \text{ milliseconds}} = \frac{? \text{ m}}{\text{second}}$$

So you will need conversion factors to replace the "feet" with "meters" and to replace "milliseconds" with "seconds":

$$\left(\frac{0.5 \cancel{\text{feet}}}{120 \cancel{\text{milliseconds}}}\right)\left(\frac{0.3048 \text{ meters}}{1 \cancel{\text{feet}}}\right)\left(\frac{1000 \cancel{\text{milliseconds}}}{1 \text{ second}}\right) = \frac{(0.5)(0.3048)(1000)}{120} \text{ m/second}$$

Exercise 7 Conversion Factors

The hole in the bottom of the boat lets in 0.1 gallons every 2 minutes. How many milliliters per second is that?

Working Space

Answer on Page 391

4.3 When Conversion Factors Don't Work

Conversion factors only work when the units being converted are proportional to each other. Gallons and liters, for example, are proportional to each other: If you have n gallons, you have $n \times 3.7854$ liters.

Degrees celsius and degrees farenheit are *not* proportional to each other. If your food is n degrees celsius, it is $n \times \frac{9}{5} + 32$ degrees farenheit. You can't use conversion factors to convert celsius to farenheit.

Watch Khan Academy's video on this at https://www.khanacademy.org/test-prep/sat/x0a8c2e5f:untitled-652/x0a8c2e5f:problem-solving-and-data-analysis-lessons-by-skill/a/gtp--sat-math--article--units--lesson

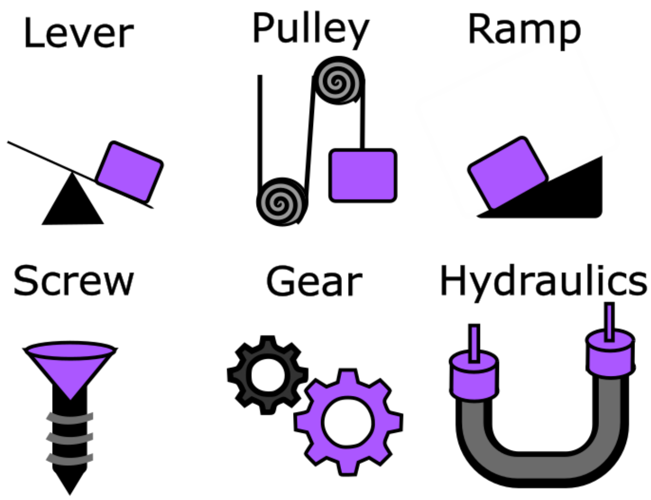
Chapter 5

Simple Machines

As mentioned earlier, physicists define work to be the force applied times the distance it is applied over. So, if you pushed your car 100 meters with 17 newtons of force, you have done 1700 joules of work.

Humans have always had to move really heavy things, so many centuries ago we developed simple machines to decrease the amount of force necessary to execute those tasks. These include things like:

- Levers
- Pulleys
- Ramps
- Gears
- Hydraulics
- Screws



While these machines can decrease the force needed, they don't change the amount of work that must be done. So if the force is decreased to a third, the distance that you must apply the force is increased by a factor of three.

"Mechanical gain" is what we call the increase in force.

5.1 Levers

A lever rotates on a fulcrum. To decrease the necessary force, the load is placed nearer to the fulcrum than where the force is applied.

In particular, physicists talk about the *torque* created by a force. When you push on a lever, the torque is the product of the force you exert and the distance from the point of rotation.

Torque is typically measured in newton-meters.

To balance two torques, the products must be the same. So, assuming that the forces are applied in the proper direction,

$$R_L F_L = R_A F_A$$

where R_L and R_A are the distance from the fulcrum to the where the load's force and the applied force (respectively) are applied, and F_L and F_A are the amounts of the forces.

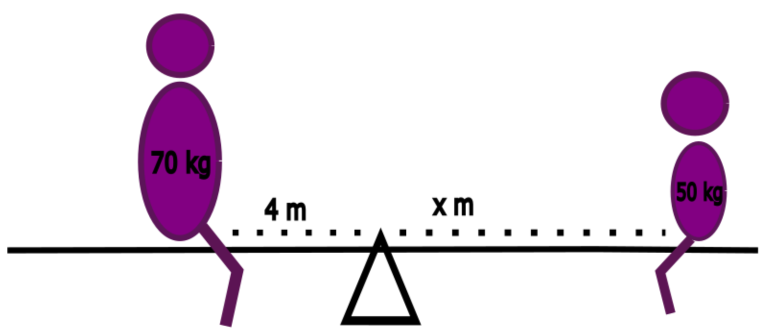
Exercise 8 Lever

Paul, who weighs 70 kilograms, sits on a see-saw 4 meters from the fulcrum. Jan, who weighs 50 kilograms, wants to balance. How far should Jan sit from the fulcrum?

Working Space

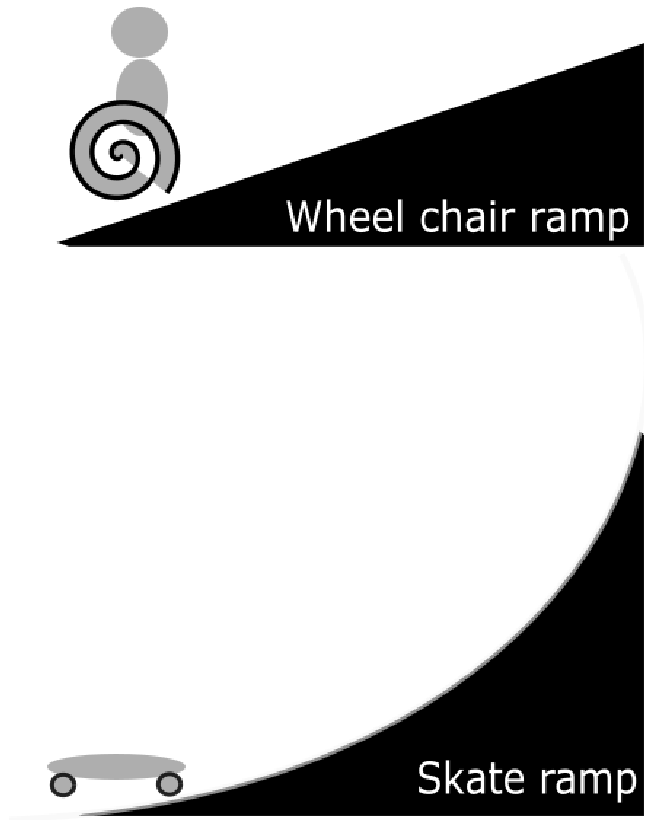
Answer on Page 391

Watch Khan Academy's video on levers: https://www.khanacademy.org/science/physics/discoveries/simple-machines-explorations/a/lever



5.2 Ramps

Ramps, or incline planes, let you roll or slide objects up to a higher level. Steeper ramps give you less mechanical gain. For example, it is much easier to roll a ball up a wheelchair



ramp than on a skateboard ramp.

Assuming the ramp has a constant steepness, the mechanical gain is equal to the ratio of the length of the ramp divided by the amount that it rises.

If you assume there is no friction, the force that you push a weight up the ramp will be:

$$F_A = \frac{V}{L} F_G$$

Where F_A is the force you need to push. L is the length of the ramp, V is the amount of vertical gain and F_G is the force of gravity on the mass.

(We haven't talked about the sine function yet, but in case you already know about it: Note that

$$\frac{V}{L} = \sin\theta$$

where θ is the angle between the ramp and level.)

Exercise 9 Ramp

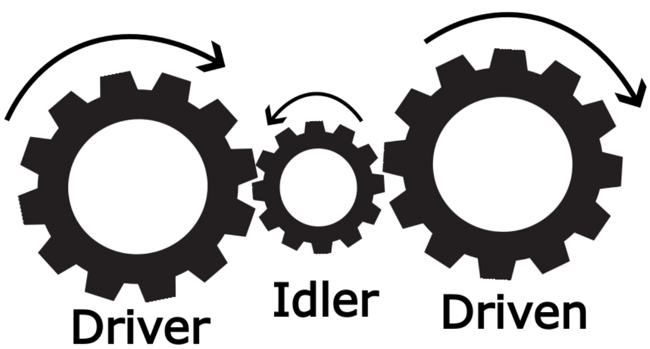
A barrel of oil weighs 136 kilograms. You can push with a force of up to 300 newtons. You have to get the barrel onto a platform that is 2 meters. What is the shortest board that you can use as a ramp?

Working Space

Answer on Page 391

5.3 Gears

Gears (which might have a chain connecting them like on a bicycle) have teeth and come in pairs. You apply torque to one gear, and it applies torque to another. The torque is increased or decreased based on the ratio between the teeth on the gears.



If N_A is the number of teeth on the gear you are turning with a torque of T_A, and N_L is the number of teeth on the gear it is turning, the resulting torque is:

$$T_L = \frac{N_A}{N_L} T_A$$

Exercise 10 Gears

The bicycle is an interesting case because we are not trying to get mechanical gain. We want to spin the pedals slower with more force.
You like to pedal your bike at 70 revolutions per minute. The chainring that is connected to your pedals has 53 teeth. The circumference of your tire is 2.2 meters. You wish to ride a 583 meters per minute.
How many teeth should the rear sprocket have?

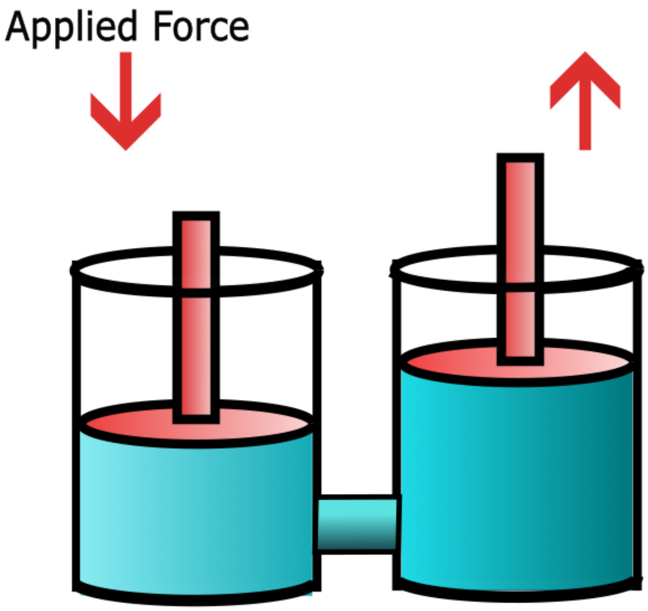
Working Space

Answer on Page 391

Watch Khan Academy's introduction to simple machines here: `https://www.khanacademy.org/science/physics/discoveries/simple-machines-explorations/a/simple-machines-and-how-to-use-`

5.4 Hydraulics

In a hydraulic system, like the braking system of a car, you exert force on a piston filled with fluid. The fluid carries that pressure into another cylinder. The pressure of the fluid pushes the piston in that cylinder out.



The pressure in the hose can be measured in pounds per square inch (PSI) or newtons per square meter (Pascals or Pa). We will use Pascals.

To figure out how much pressure you create, you divide the force by the area of the piston head you are pushing.

To figure out how much force that creates on the other end, you multiply the pressure times the area of the piston head that is pushing the load.

Exercise 11 Hydraulics

Working Space

Your car has disc brakes. When you put 2,500,000 pascals of pressure on the brake fluid, the car stops quickly. As the car designer, you would like that to require 12 newtons of force from the driver's foot. What should the radius of the master cylinder (the one the driver is pushing on) be?

Answer on Page 392

Chapter 6

Cognitive Biases 1

Our brains were designed over millions of years by the evolutionary process. The resulting mind is an amazing and powerful tool, however not flawless. The human brain has tendencies (or biases) that nudge us toward bad judgment and poor decisions.

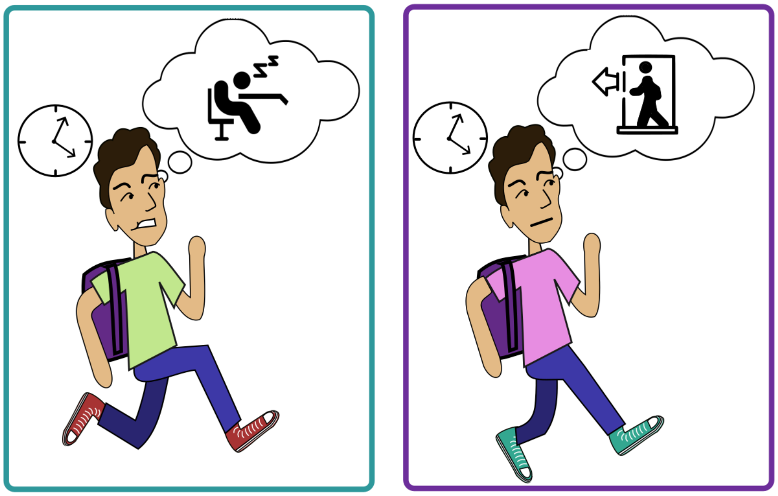
It would be irresponsible to teach you powerful ideas without also teaching you about the cognitive biases that follow them. There are about 50 that you should know about, but let's start with only a few.

6.1 Fundamental Attribution Error

You tend to attribute the mistakes of another person to their character, but attribute your own mistakes to the situation.

Let's say you are at lunch and someone asks you "Why was Larry late for class ?" You are likely to say "Larry is lazy and disorganized."

If someone asks you "Why were you late for class last week?" You are likely to say, "I don't remember; The class before it must have run long."



The solution? Cut people some slack. You probably don't know the whole story, so assume that their character is as strong as yours.

Or maybe you also need to hold yourself to a higher standard? Do you find yourself frequently rationalizing your bad judgment, lateness, or rudeness? This could be an opportunity for you to become a better person whose character is stronger regardless of the situation.

6.2 Self-Serving Bias

Self-serving bias is when you blame the situation for your failures, but attribute your successes to your strengths.

For example, when asked "Why did you lose the match?" you are likely to answer "The referee wasn't fair." When you are asked "Why did you win the match?" you are likely to answer "Because I have been training for weeks, and I was very focused."

This bias tends to make us feel better about ourselves, but it makes it difficult for us to be objective about our strengths and weaknesses.

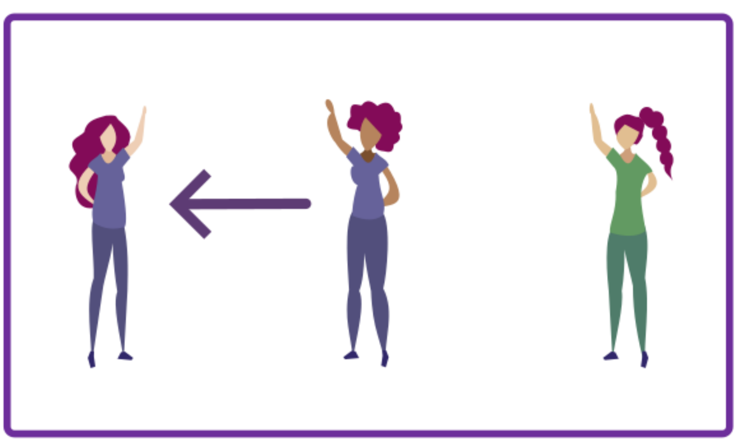
6.3 In-group favoritism

In-group favoritism: We tend to favor people who are in a group with us over people who are not in groups with us.

When asked "Who is the better goalie, Ted or John?"

If Ted is a Star Trek fan like you, you are likely to think he is also a good goalie.

As you might imagine, this unconscious tendency is the source of a lot of subtle discrimination based on race, gender, age, and religion.



6.4 The Bandwagon Effect and Groupthink

The bandwagon effect is our tendency to believe the same things that the people around us believe. This is how fads spread so quickly: one person buys in, and then the people they know have a strong tencdency to buy in as well.

Groupthink is similar: To create harmony with the people around us, we go along with things we disagree with.

It takes a lot of perspective to recognize when those around us are wrong. And it takes even more courage to openly disagree with them.

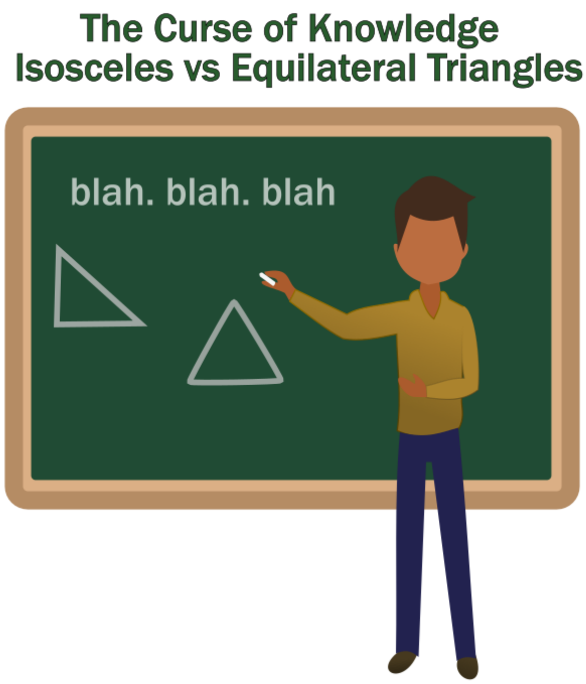
6.5 The Curse of Knowledge

Once you know something, you tend to assume everyone else knows it too.

This is why teaching is sometimes difficult: a teacher will assume that everyone in the audience already knows the same things the teacher knows.

Also, when we learn that a friend doesn't know something that we know, we are often very surprised. This surprise can sometimes manifest as hurtful behavior.

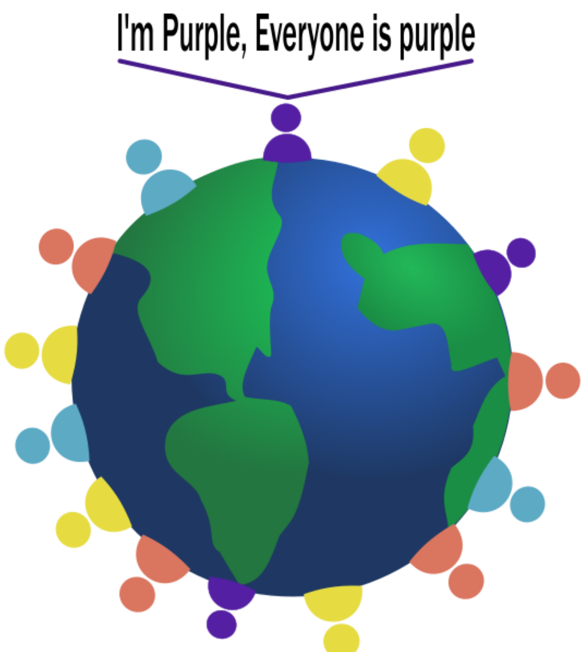
When I find a gap in a friend's knowledge, I try to remind myself that the friend certainly knows many things that I don't. I also try to imagine how it would feel if they teased me for my ignorance.



6.6 False Consensus

We tend to believe that more people agree with us than is actually the case. For example, if you are a member of a particular religion, you tend to overestimate the percentage of people in the world who are members of that religion.

When people vote in elections, they are often surprised when their preferred candidate loses. “Everyone, and I mean EVERYONE, voted for Smith!” they yell. “There must have been a mistake in counting the votes.”



6.7 The Spotlight Effect

You tend to overestimate how much other people are paying attention to your behavior and appearance.

Think of six people that you talked to today. Can you even remember what shoes most of them were wearing? Do you care? Do you think any of them remember which shoes you wore today?

There is an old saying “You would worry a lot less about how people think of you, if you realized how rarely they do.”

6.8 The Dunning-Kruger Effect

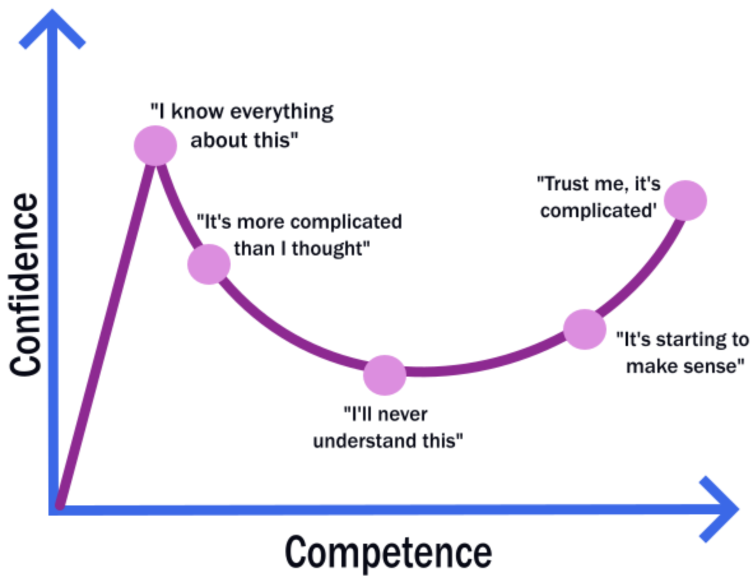
The less you know, the more confident you are.

When a person doesn't know all the nuance and context in which a question is asked, the question seems simple. Thus the person tends to be confident in their answer. As they learn more about the complexity of the space in which the question lives, they often realize the answer is not nearly so obvious.

For example, a lot of people will confidently proclaim "Taxes are too high! We need to lower taxes." An economist who has studied government budgets, deficits, history, and monetary policy, might say something like "Maybe taxes *are* too high. Or maybe they are too low. Or maybe we are taxing the wrong things. It is a complex question."

When I am talking with people about a particular topic, I do my best to defer to the person in the conversation who I think has the most knowledge in the area. If I disagree with the person, I try to figure out why our opinions are different.

Similarly, you should assume that any opinion that is voiced, specifically, in an internet discussion is wildly over-simplified. If you really care about the subject, read a book by a respected expert. Yes, a whole book – there are few interesting topics that can be legitimately explained in less than 100 pages.



6.9 Confirmation Bias

You tend to find and remember information that supports beliefs you already have. You tend to avoid and dismiss information that contradicts your beliefs.

If you believe that intelligent creatures have visited from other planets, you will tend to look for data to support your beliefs. When you find data that shows that it is just too far for any creature to travel, you will try to find a reason why the data is incorrect.

Confirmation bias is one reason why people don't change their beliefs more often.

Confirmation bias wrecks many, many studies. The person doing the study often has a hypothesis that they believe and very much want to prove true. It is very tempting to discard data that doesn't support the hypothesis. Or maybe the person throws all the data away and experiments again and again until they get the result they want.

When you design an experiment, you must describe it explicitly before you start. You must tell someone: "If the hypothesis I love is incorrect, the results will look like this. If the hypothesis I love is correct, the results will look like that. And if the results look any other way, I have neither proved nor disproved the hypothesis."

Once the experiment is underway, you must not change the plan and you must not discard any data.

This is scientific integrity. You should demand it from yourself, and you should expect it from others.

6.10 Survivorship bias

You will pay more attention to those that survived a process than those who failed.

After looking at a lot of old houses, you might say "In the 1880s, they built great houses." However, you haven't seen the houses that were built in the 1880s and didn't survive. Which houses tended to survive for a long time? Only the great houses – you are basing your opinion on a very skewed sample.

Chapter 7

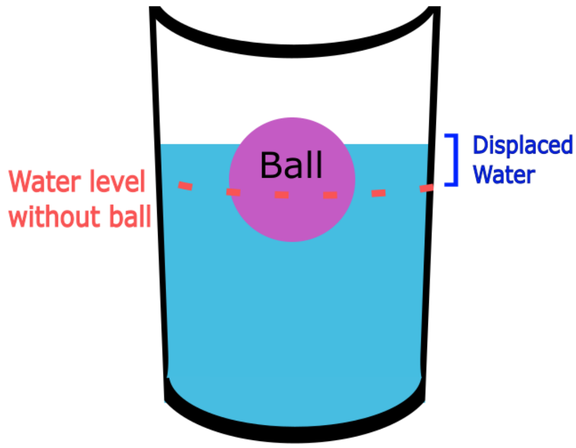
Buoyancy

When you put a boat into water, it will sink into the water until the mass of the water it displaces is equal to the mass of the boat. We think of this in terms of forces. Gravity pulls the mass of the boat down. The *buoyant force* pushes the boat up. A boat dropped into the water will bob up and down a bit before reaching an equilibrium where the two forces are equal.

Watch Khan Academy's introduction to buoyance at https://www.khanacademy.org/science/in-in-class9th-physics-india/in-in-gravity/in-in-pressure-in-liquids-archimedes-princip v/archimedes-principle-buoyancy-fluids-physics-khan-academy

The buoyant force pushes things up – against the force of gravity. The force is equal to the weight of the fluid being replaced. So, for example, a cubic meter of freshwater has a mass of about 1000kg. If you submerge anything with a volume of one meter in freshwater on earth, the buoyant force will be about 9800 newtons.

For some things, like a block of styrofoam, this buoyant force will be sufficient to carry it to the surface. Once it reaches the surface, it will continue to rise (displacing less water) until the mass of the water it displaces is equal to its mass. And then we say "It floats!"



For some things, like a block of lead, the buoyant force is not sufficient to lift it to the surface, and thus we say "It sinks!"

This is why a helium balloon floats through the air. The air that it displaces weighs more than the balloon and the helium itself. (It is easy to forget that air has a mass, but it does.)

Exercise 12 Buoyancy

You have an aluminum box that has a heavy base, so it will always float upright. The box and its contents weigh 10 kg. Its base is 0.3 m x 0.4 m. It is 1m tall. When you drop it into freshwater ($1000 \mathrm{kg/m}^3$), how far will it sink before it reaches equilibrium.

Working Space

Answer on Page 392

7.1 The Mechanism of Buoyancy

As you dive down in the ocean, you will experience greater and greater pressure from the water. And if you take a balloon with you, you will gradually see it get smaller as the water pressure compresses the air in the balloon.

Let's say you are 3 meters below the surface of the water. What is the pressure in Pascals (newtons per square meter)? You can think of the water as a column of water crushing down upon you. The pressure over a square meter is the weight of 3 cubic meters of water pressing down.

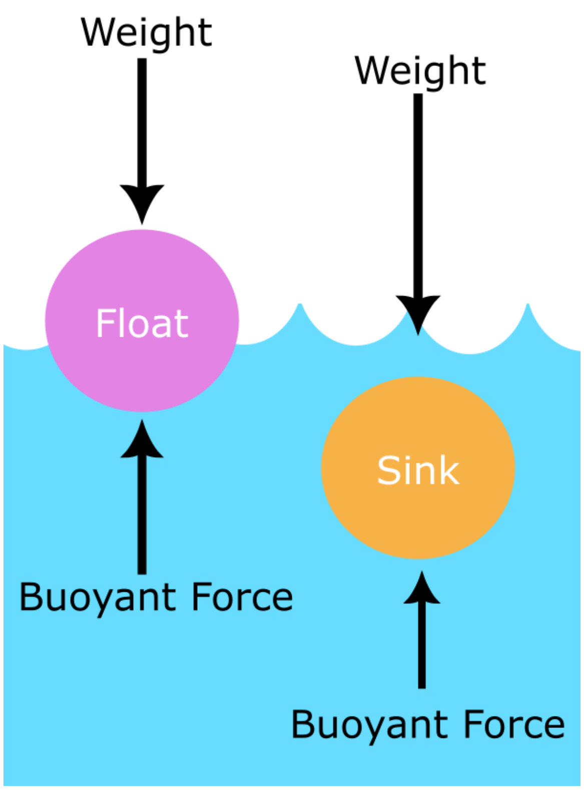
$$p = (3)(1000)(9.8) = 29,400 \text{ Pa}$$

This is called *hydrostatic pressure*. The general rule for hydrostatic pressure in Pascals p is

$p = dgh$

Where d is the density of the fluid in kg per cubic meter, g is the acceleration due to gravity in m/s^2, and h is the height of the column of fluid above you.

So, where does buoyant force come from? Basically, the pressure pushing up on the deepest part of the object is higher than the pressure pushing down on the shallowest part of the object. That is where bouyancy comes from.



Exercise 13 Hydrostatic Pressure

Working Space

You dive into a tank of olive oil on Mars. How much more hydrostatic pressure does your body experience at 5 meters deep than it did at the surface?
The density of olive oil is about 900 kg per square meter. The acceleration due to gravity on Mars is 3.721 m/s^2.

Answer on Page 392

Notice that although the pressure is increasing as you go deeper, the buoyant force will *not increase* because the buoyant force is always equal to the weight of the fluid that is displaced, regardless if that is 1 meter or 100 meters underwater.

Also, saltwater is denser then freshwater. That is why people float better in the sea than they do in a river.

And, lipids, like fats and oils, are less dense than water. That is why people with a lot of body fat tend to float better than people with less body fat. And why oil floats in a glass of water.

Chapter 8

Heat

Let's say you put a 1 kg aluminum pan that is 80° C into 3 liters of water that is 20° C. Energy, in the form of heat, will be transferred from the pan to the water until they are at the same temperature. (We call this "thermal equilibrium.")

What will the temperature of the water be?

8.1 Specific Heat Capacity

If you are heating something, the amount of energy you need to transfer to it depends on three things: the mass of the thing you are heating, the amount of temperature change you want, and the *specific heat capacity* of that substance.

Energy in Heat Transfer

The energy moved in a heat transfer is given by

$$E = mc\Delta_T$$

where m is the mass, Δ_T is the change in temperature, and c is the specific heat capacity of the substance.

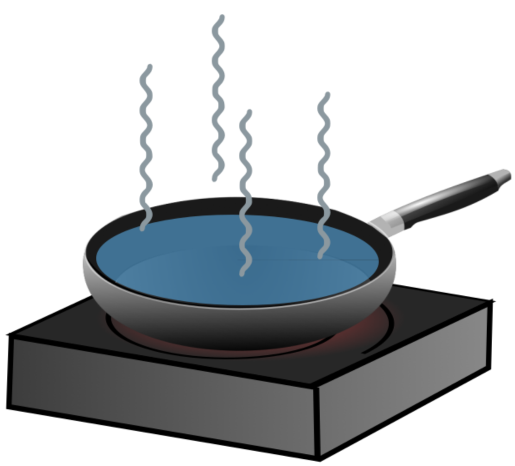
(Note that this assumes no phase change. For example, this formula works nicely on warming liquid water, but it gets more complicated if you warm the water past its boiling point.)

Can we guess the specific heat capacity of a substance? It is very, very difficult to guess the specific heat of a substance, so we determine it by experimentation.

For example, someone determined that it took about 0.9 joules to raise the temperature of solid aluminum one degree Celsius. So we say "The specific heat capacity of aluminum is 0.9 J/g °C."

The specific heat capacity of liquid water is about 4.2 J/g °C.

To answer the question, then, the amount of energy given off by the pan must equal the amount of energy absorbed by the water. And they need to be the same temperature at the end. Let T be the final temperature of both.



Watch Khan Academy's discussion of heat capacity at https://www.khanacademy.org/science/ap-chemistry-beta/x2eef969c74e0d802:thermodynamics/x2eef969c74e0d802:heat-capacity-and-calorimetry/v/heat-capacity

Three liters of water weighs 3,000 grams, so the change in energy in the water will be:

$$E_W = mc\Delta_T = (3000)(4.2)(T - 20) = 12600T - 252000 \text{ joules}$$

The pan weighs 1000 grams, so the change in energy in the pan will be::

$$E_P = mc\Delta_T = (1000)(0.9)(T - 80) = 900T - 72000 \text{ joules}$$

Total energy stays the same so $E_W + E_P = 0$. So you need to solve

$$(12600T - 252000) + (900T - 72000) = 0$$

And find that the temperature at equilibrium will be

$$T = 24°C$$

Exercise 14 Thermal Equilibrium

Working Space

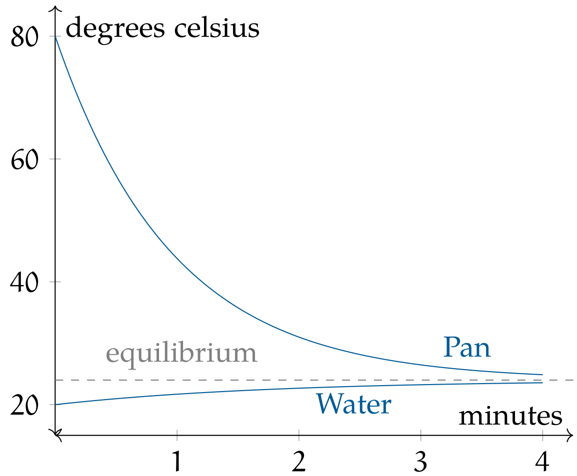
Just as you put the aluminium pan in the water as described above, someone also puts a 1.2 kg block of copper cooled to 10 ° C. The specific heat of solid copper is about 0.4 J/g °C.
What is the new temperature at equilibrium?

Answer on Page 392

8.2 Getting to Equilibrium

When two objects with different temperatures are touching, the speed at which they exchange heat is proportional to the differences in their temperatures. Thus, as their temperatures get closer together, the heat exchange slows down.

In our example, the pan and the water will get close to equilibrium quickly, but they may never actually reach equilibrium.



Exercise 15 Cooling Your Coffee

Working Space

You have been given a ridiculously hot cup of coffee and a small pitcher of chilled milk.
You need to start chugging your coffee in three minutes, and you want it as cool as possible at that time. When should you add the milk to the coffee?

Answer on Page 393

8.3 Specific Heat Capacity Details

For any given substance, the specific heat capacity often changes a lot when the substance changes state. For example, ice is 2.1 J/g °C, whereas liquid water is 4.2 J/g°C.

Watch Khan Academy's discussion of the specific heat of water: `https://www.khanacademy.org/science/biology/water-acids-and-bases/water-as-a-solid-liquid-and-gas/v/specific-heat-of-water`

Even within a given state, the specific heat capacity varies a bit based on the temperature and pressure. If you are trying to do these sorts of calculations with great accuracy, you will want to find the specific heat capacity that matches your situation. For example, I might look for the specific heat capacity for water at 22°C at 1 atmosphere of pressure(

atm).

Chapter 9

Basic Statistics

You live near a freeway, and someone asks you, “How fast do cars on that freeway drive?”

You say “Pretty fast.”

And they say, “Can you be more specific?”

And you point your radar gun at a car, and say “That one is going 32.131 meters per second.”

And they say, “I don’t want to know about that specific car. I want to know about all the cars.”

So, you spend the day beside the freeway measuring the speed of every car that goes by. And you get a list of a thousand numbers. Here is part of the list:

30.462 m/s	29.550 m/s	29.227 m/s
37.661 m/s	27.899 m/s	28.113 m/s
24.382 m/s	35.668 m/s	43.797 m/s
31.312 m/s	37.637 m/s	30.891 m/s

There are 12 numbers here. We say that there are 12 *samples*.

9.1 Mean

We often talk about the *average* of a set of samples, which is the same as the *mean*. To get the mean, sum up the samples and divide that number by the number of samples.

The numbers in that table sum to 388.599. If you divide that by 12, you find that the mean of those samples is 32.217 m/s.

We typically use the greek letter μ ("mu") to represent the mean.

Definition of Mean

If you have a set of samples $x_1, x_2, \ldots, x_n$, the mean is:

$$\mu = \frac{1}{n} \sum_{i=1}^{n} x_i$$

This may be the first time you are seeing a summation ($\sum$). The equation above is equivalent to:

$$\mu = \frac{1}{n}(x_1 + x_2 + \ldots + x_n)$$

Exercise 16 Mean Grade

Teachers often use the mean for grading. For example, if you took six quizzes in a class, your final grade might be the mean of the six scores. Find the mean of these six grades: 87, 91, 98, 65, 87, 100.

Working Space

Answer on Page 393

If you tell your friend "I measured the speed of 1000 cars, and the mean is 31.71 m/s", your friend will wonder "Are most of the speeds clustered around 31.71? Or are they all over the place and just happen to have a mean of 31.71?" To answer this question we use variance.

9.2 Variance

Definition of Variance

If you have n samples $x_1, x_2, \ldots, x_n$ that have a mean of μ, the *variance* is defined to be:

$$v = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu)^2$$

That is, you figure out how far each sample is from the median, you square that, and then you take the mean of all those squared distances.

x	$x - \mu$	$(x - \mu)^2$
30.462	-1.755	3.079
29.550	-2.667	7.111
29.227	-2.990	8.938
37.661	5.444	29.642
27.899	-4.318	18.642
28.113	-4.104	16.839
24.382	-7.835	61.381
35.668	3.451	11.912
43.797	11.580	134.106
31.312	-0.905	0.818
37.637	5.420	29.381
30.891	-1.326	1.757
$\sum x = 386.599$		$\sum (x - \mu)^2 = 323.605$
mean $= 32.217$		variance $= 26.967$

Thus, the variance of the 12 samples is 26.967. The bigger the variances, the farther the samples are spread apart; the smaller the variances, the closer samples are clustered around the mean.

Notice that most of the data points deviate from the mu by 1 to 5 m/s. Isn't it odd that the variance is a big number like 26.967? Remember that it represents the average of the squares. Sometimes, to get a better feel for how far the samples are from the mean, we use the square root of the variance, which is called *the standard deviation*.

The standard deviation of your 12 samples would be $\sqrt{26.9677} = 5.193$ m/s.

The standard deviation is used to figure out a data point is an outlier. For example, if you are asked"That car that just sped past. Was it going freakishly fast?" You might respond, "No, it was within a standard deviation of the mean." or "Yes, it's speed was 2 standard deviations more than the mean. They will probably get a ticket."

A singular μ usually represents the mean. σ usually represents the standard deviation. So σ^2 represents the variance.

Exercise 17 Variance of Grades

Now find the variance for your six grades.
As a reminder, they were: 87, 91, 98, 65, 87, 100.
What is your standard deviation?

Working Space

Answer on Page 393

9.3 Median

Sometimes you want to know where the middle is. For example, you want to know the speed at which half the cars are going faster and half are going slower. To get the median, you sort your samples from smallest to largest. If you have an odd number of samples, the one in the middle is the median. If you have an even number of samples, we take the mean of the two numbers in the middle.

In our example, you would sort your numbers and find the two in the middle:

24.382
27.899
28.113
29.227
29.550
30.462
30.891
31.312
35.668
37.637
37.661
43.797

You take the mean of the two middle numbers: $(30.462 + 30.891)/2 = 30.692$. The median speed would be 30.692 m/s.

Medians are often used when a small number of outliers majorly skew the mean. For example, income statistics usually use the median income because a few hundred billionares raise the mean a lot.

Exercise 18 Median Grade

Find the median of your six grades: 87, 91, 98, 65, 87, 100.

Working Space

Answer on Page 393

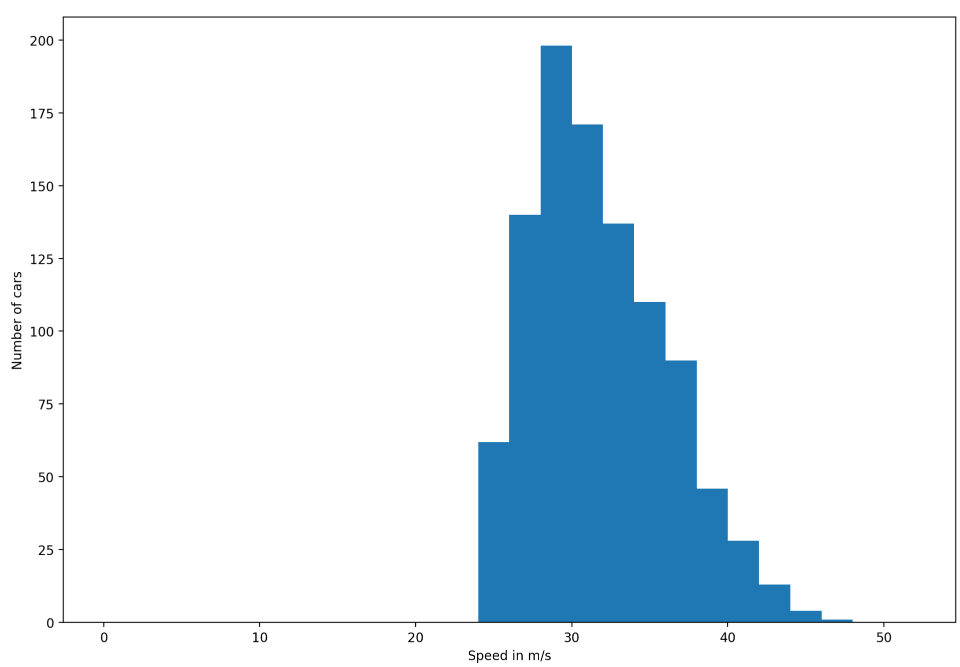
9.4 Histograms

A histogram is a bar chart that shows how many samples are in each group. In our example, we group cars by speed. Maybe we count the number of cars going between 30 and 32 m/s. And then we count the cars going between 32 and 34 m/2. And then we make a bar chart from that data.

Your 1000 cars would break up into these groups:

0 - 2 m/s	0 cars
2 - 4 m/s	0 cars
4 - 6 m/s	0 cars
...	...
20 - 22 m/s	0 cars
22 - 24 m/s	0 cars
24 - 26 m/s	65 cars
26 - 28 m/s	160 cars
28 - 30 m/s	175 cars
30 - 32 m/s	168 cars
32 - 34 m/s	150 cars
34 - 36 m/s	114 cars
36 - 38 m/s	79 cars
38 - 40 m/s	52 cars
40 - 42 m/s	20 cars
42 - 44 m/s	12 cars
44 - 46 m/s	4 cars
46 - 48 m/s	1 cars
48 - 50 m/s	0 cars

Now we make a bar chart from that:



Often a histogram will tell the story of the data. Here, you can see that no one is going less than 24 m/s, but a lot of people travel at 30 m/s. There are a few people who travel over 40 m/s, but there are also a couple of people who drive a lot faster than anyone else.

9.5 Root-Mean-Squared

Scientists have a mean-like statistic that they love. It is named quadratic mean, but most just calls it Root-Mean-Squared or RMS.

Definition of RMS

If you have a list of numbers $x_1, x_2, \ldots, x_n$, their RMS is

$$\sqrt{\frac{1}{n}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)}$$

You are taking the square root of the mean of squares of the samples, thus the name Root-Mean-Squared.

Using your 12 samples:

x	x^2
30.462	927.933
29.550	873.203
29.227	854.218
37.661	1418.351
27.899	778.354
28.113	790.341
24.382	594.482
35.668	1272.206
43.797	1918.177
31.312	980.441
37.637	1416.544
30.891	954.254
Mean of x^2	1064.875
RMS	32.632

Why is RMS useful? Let's say that all cars had the same mass m, and you need to know what the average kinetic energy per car is. If you know the RMS of the speeds of the cars is v_{rms}, the average kinetic energy for each car is

$$k = \frac{1}{2} m v_{rms}^2$$

(You don't believe me? Let's prove it. Substitute in the RMS:

$$k = \frac{1}{2}m\sqrt{\frac{1}{n}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)}^2$$

The square root and the square cancel each other out:

$$k = \frac{1}{2}m\frac{1}{n}\left(x_1^2 + x_2^2 + \ldots + x_n^2\right)$$

Use the distributive property:

$$k = \frac{1}{n}\left(\frac{1}{2}mx_1^2 + \frac{1}{2}mx_2^2 + \ldots + \frac{1}{2}mx_n^2\right)$$

That is all the kinetic energy divided by the number of cars, which is the mean kinetic enegy per car. Quod erat demonstrandum! (That is a Latin phrase that means "which is what I was trying to demonstrate". You will sometimes see "QED" at the end of a long mathematic proof.))

Now you are ready for the punchline: kinetic energy and heat are the same thing. Instead of cars, heat is the kinetic energy of molecules moving around. More on this soon.

Video: Mean, Median, Mode: https://www.youtube.com/watch?v=5C9LBF3b65s

Chapter 10

Basic Statistics in Spreadsheets

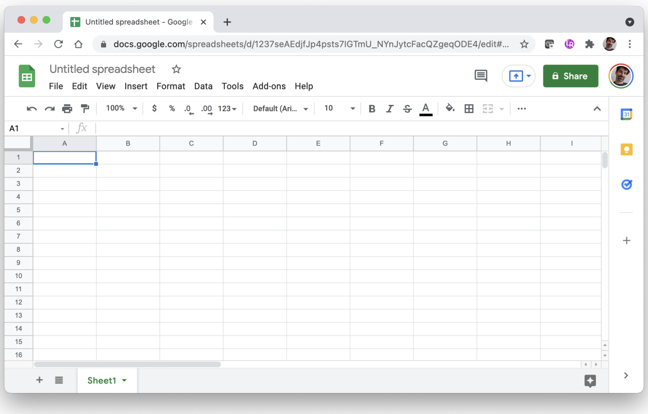
When you completed the problems in the last section, you probably noticed how long it took to compute statistics like the mean, the median, and variance by hand. Luckily, computers were designed to free us from these sorts of tedious tasks. The most basic tool for automating calculations is the spreadsheet program.

There are lots of spreadsheet programs including Microsoft's Excel and Apple's Numbers. Any spreadsheet program will work; they are all very similar. The instructions and screenshots here will be from Google Sheets – a free spreadsheet program you use through your web browser.

10.1 Your First Spreadsheet

In whatever spreadsheet program you are using, create a new spreadsheet document.

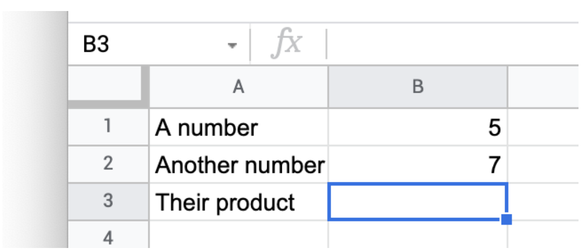
A spreadsheet is essentially a grid of cells. In each cell you can put data (like numbers or text) and formulas.



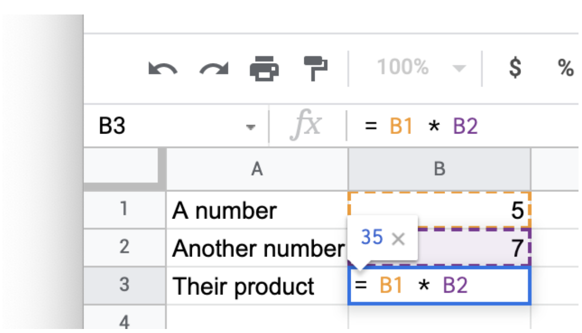
Let's put some labels in the column:

- Select the first cell (A1) and type "A number".
- Select the cell below it (A2) and type "Another number".
- Select the cell below that one (A3) and type "Their product".
- In the next column, type the number 5 in B1 and 7 in B2.

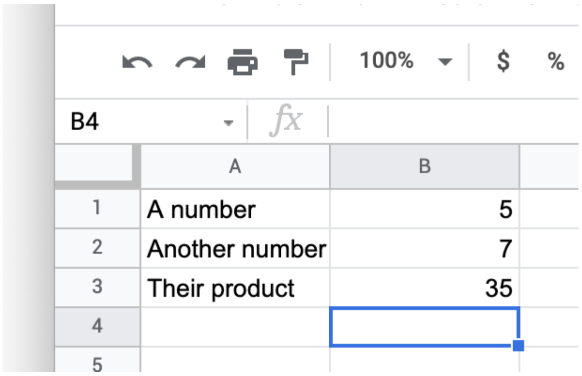
It should look like this:



Now put a formula in cell B3. Select B3, and type "= B1 + B2". The spreadsheet knows this is a formula because it starts with '='. It will look like this as you type:



When you press Return or Tab, the spreadsheet will remember the formula, but display its value:



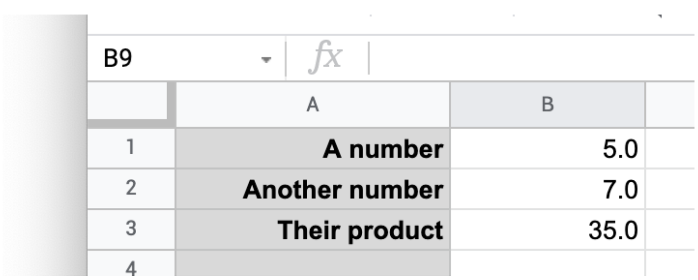
If you change the values of cell B1 or B2, the cell B3 will automatically be recalculated. Try it.

10.2 Formatting

Every spreadsheet lets you change the formatting of your columns and cells. They are all a little different, so play with your spreadsheet a little now. Try to do the following:

- Set the background of the first column to light gray.
- Right-justify the text in the first column.
- Make the text in the first column bold.
- Make the numbers in the second column have one digit after the decimal point.

It should look something like this:



That's a spreadsheet. You have a grid of cells. Each cell can hold a value or a formula that uses values from other cells. The cells with formulas automatically update as you edit the values in the other cells.

10.3 Comma-Separated Values

A lot of data is exchanged in a file format called *Comma-Separated Values* or just CSV. Each CSV file holds one table of data. It is a text file, and each line of text corresponds to one row of data in the table. The data in each column is separated by a comma. The first line of a CSV is usually the names of the columns. A CSV might look like this:

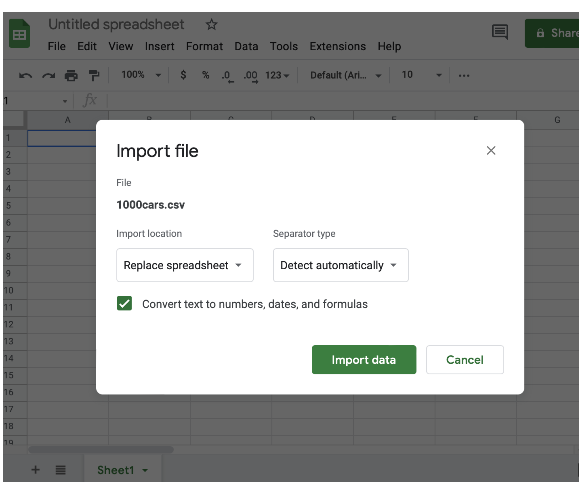
```
studentID,firstName,lastName,height,weight
1,Marvin,Sumner,260,45.3
2,Lucy,Harris,242,42.2
3,James,Boyd,261,44.2
```

In your digital resources for this module, you should have a file called `1000cars.csv`. It is a CSV with only one column called "speed". The first few lines look like this:

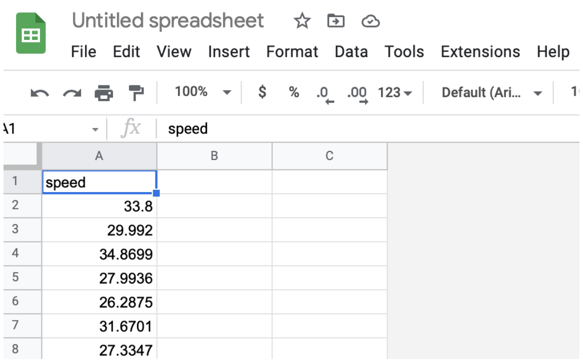
```
speed
33.8000
29.9920
34.8699
27.9936
```

There is a title line and 1000 data lines.

Import this CSV into your spreadsheet program. In Google Sheets, it looks like this:



You should see a long, long column of data appear. (Mine goes from cell A2 through A1001.)



10.4 Statistics in Spreadsheets

Let's take the mean of all 1000 numbers. In cell B2, type in a label: "Mean". (Feel free to format your labels as you wish. Bolding is recommended.)

In cell C2, enter the formula "=AVERAGE(A2:A1001)". When you press return, the cell will show the mean: 31.70441, if done correctly.

	A	B	C
1	speed		
2	33.8	**Mean**	31.7044106
3	29.992		
4	34.8699		
5	27.9936		

Notice that by specifying that the function `AVERAGE` was to be performed on a range of cells: cells A2 through A1001.

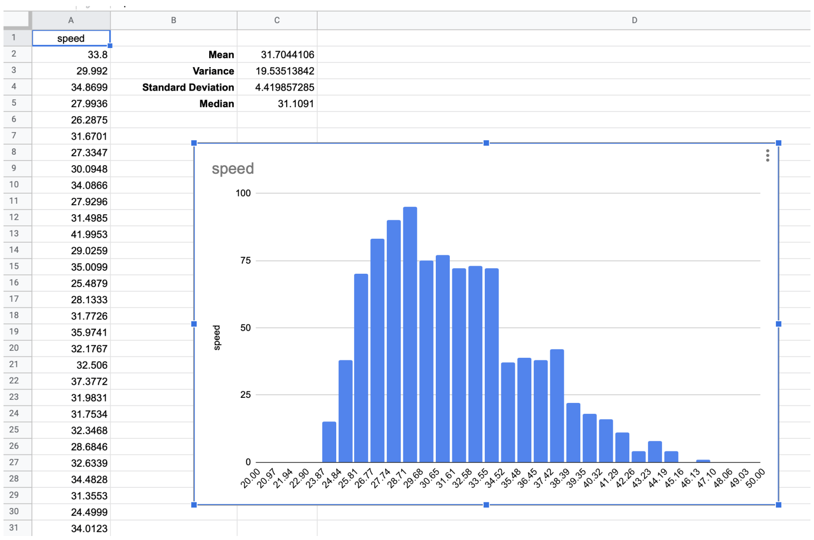
Do the calculations for variance, standard deviation, and median.

- The function for variance is `VAR`.
- The function for standard deviation is `STDEV`.
- The function for median is `MEDIAN`.

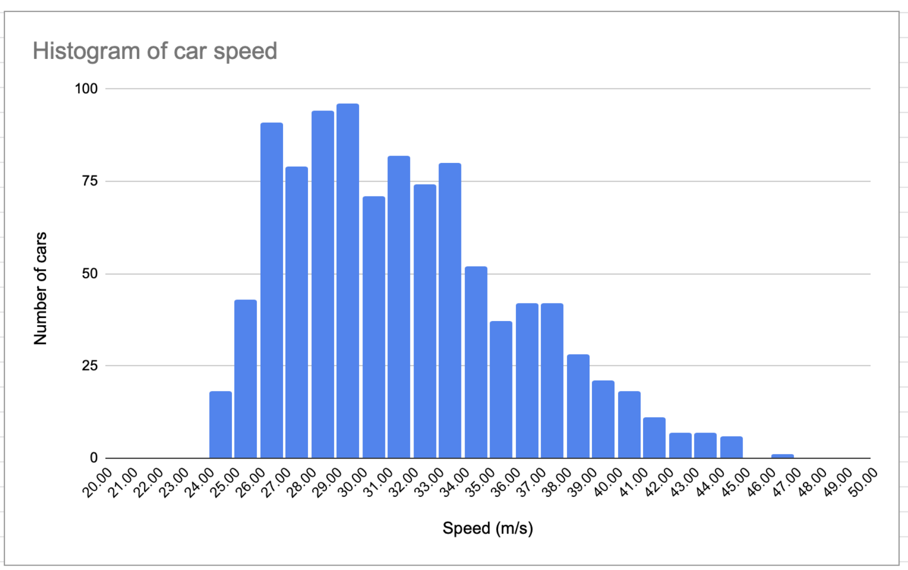
	A	B	C
1	speed		
2	33.8	**Mean**	31.7044106
3	29.992	**Variance**	19.53513842
4	34.8699	**Standard Deviation**	4.419857285
5	27.9936	**Median**	31.1091
6	26.2875		
7	31.6701		

10.5 Histogram

Most spreadsheets have the ability to create a histogram. In Google Sheets, you select the entire range A2:A1001 by selecting the first cell and then shift-clicking the last. Then you choose Insert→Chart. In the inspector, change the type of the chart to a histogram. This will get you a basic histogram.



Play with the formatting to see how unique you can make data. Here is an example:



Exercise 19 RMS

In your spreadsheet, calculate the quadratic mean (the root-mean-squared) of the speeds. You will need the following three functions:

- `SUMSQ` returns the sum of the squares of a range of cells.
- `COUNT` returns the number of cells in a range that contains numbers.
- `SQRT` returns the square root of a number.

Working Space

Answer on Page 393

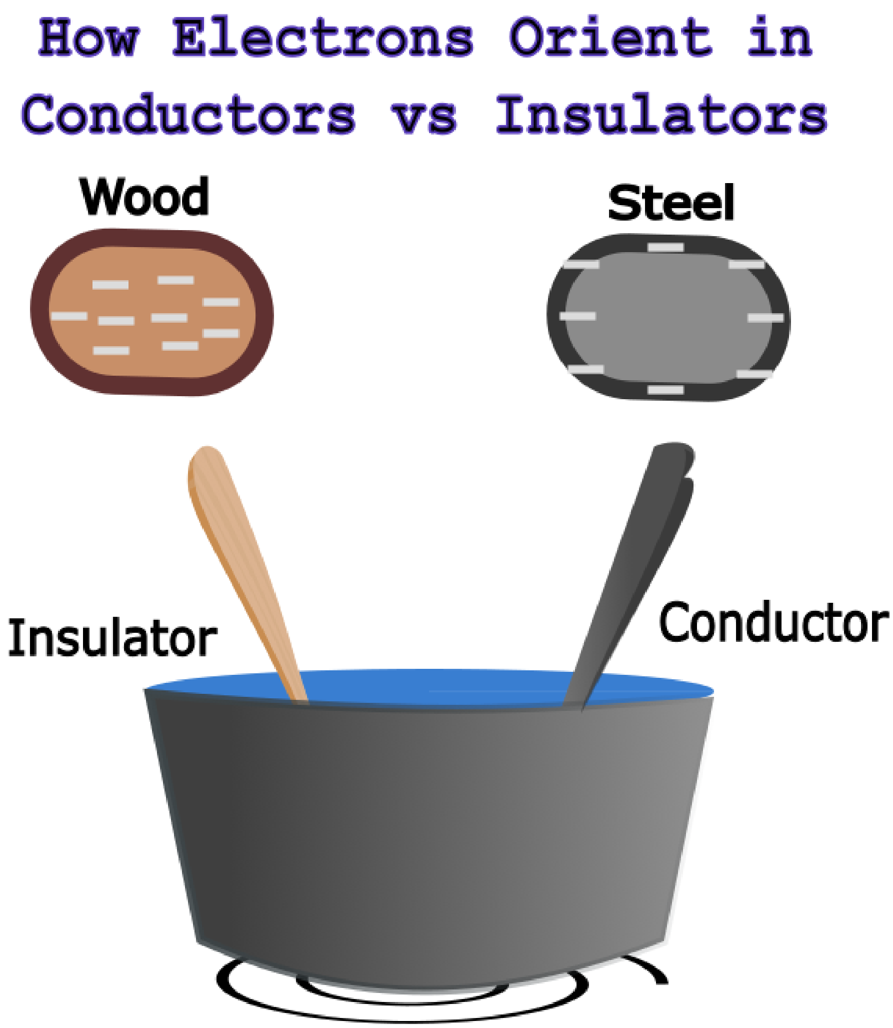
Chapter 11

Introduction to Electricity

What happens when you turn on a flashlight? The battery in the flashlight acts as an electron pump. The electrons flow through the wires to the lightbulb (or LED). As the electrons pass through the lightbulb, they excite the molecules within, which gives off light and heat. (LEDs also give off light and heat, but they give off a lot less heat.) Then the electrons return to the battery to be pumped around again.

When electricity is flowing through a copper wire, the protons and neutrons of the copper stay put while the electrons jump between the atoms on their way from the battery to the lightbulb and back again.

In some materials, like copper and iron, electrons are loosely bound to their nuclei, forming a sea of electrons, which allows energy to flow. These are good *electrical conductors*. In other materials, like glass and plastic, electrons don't leave their nuclei easily. Thus, they are terrible electrical conductors – we call them *electrical insulators*. For example, the plastic around a wire is electrical insulation.



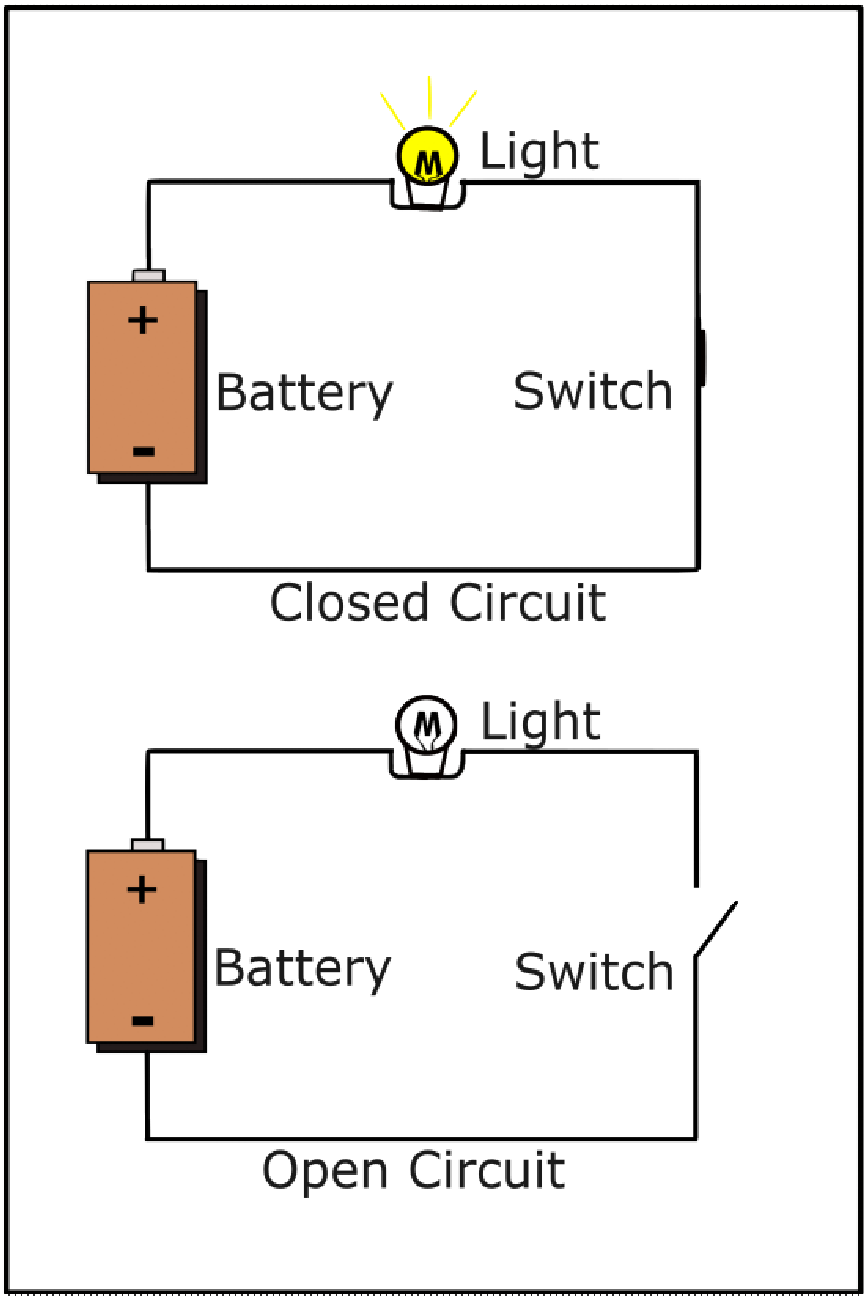
11.1 Units

Electrons are very small, so to study them, scientists came up with a unit that represents *a lot* of electrons. 1 *coulomb* is about 6,241,509,074,460,762,608 electrons. When 5 coulombs enter one end of the wire every second (and simultaneously 5 coulombs exit the other end), we say "This wire is carrying 5 amperes of current."

(Truthfully, we usually shorten ampere to just "amp". This is sometimes a little awkward because we often shorten the word "amplifier" to "amp". You should be able to tell which is which from the context.)

If you look at the circuit breakers or fuses for your home's electrical system, you'll see that each one is rated in amps. For example, maybe the circuit that supplies power to your kitchen has a 10 amp circuit breaker. If for some reason, more than 10 amps tries to pass through that wire, the circuit breaker will turn off the whole circuit.

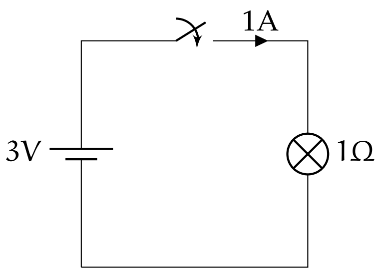
When it is on, your flashlight pushes about 1 amp of current through the lightbulb(When it is off, there is no current in the lightbulb).



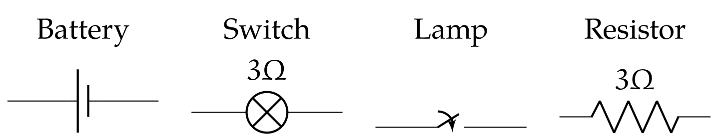
The lightbulb creates *Resistance* that the current pushes through. Think of it like plumbing: The current is the amount of water passing through a pipe. The resistance is something that tries to stop the current – like a ball of hair. The battery is what allows the current to push through the resistance; we call that pressure *voltage*.

11.2 Circuit Diagrams

Here is a circuit diagram of your flashlight:



The lines are wires. The symbols that we will use:



The battery pushes the electrons from one end and pulls them back in at the other, so the circuit must go around in a circle for the current to flow. This is why the current stops flowing when the switch breaks the circuit.

You can think of a switch as having zero resistance when it is closed and infinite resistance when it is open.

For our purposes, a lamp is just a resistor that gives off light.

11.3 Ohm's Law

Resistance is measured in *ohms*, and we use a Greek capital omega for that: Ω

Voltage is measured in *volts*.

Ohm's Law

Whenever a voltage V is pushing a current I through a resistance of I, the following is true:

$$V = IR$$

where V is in volts, I is in amps, and R is in ohms.

11.4 Power and Watts

Joule's Law

When a current I is passing through a resistance R, the power consumed is

$$W = I^2R$$

where W is in watts, I is in amps, and R is in ohms.

Of course $V = IR$, so we can extend this to:

$$W = I^2R = IV = \frac{V^2}{R}$$

Your flashlight's batteries provide about 3 volts. How much battery power is the flashlight using when it is on? The power (in watts) produced by the battery is the product of the voltage (in volts) and the current (in amps). So your flashlight is giving off $3volts \times 1amp = 3watts$ of power. Some of that power is given off as light, some as heat.

A watt is 1 joule of energy per second. We say that a watt is a measure of *power*.

When we talk about how much energy is stored in a battery, we use a unit like a kilowatt-hour. A kilowatt-hour is equivalent to 3.6 million joules.

11.5 Another great use of RMS

In many electrical problems, the voltage fluctuates a lot. For example, the fluctuations in voltage makes the sound that comes out of an audio speaker.

You can use the root-mean-squared of the voltage to figure out the average power your speaker is consuming.

Let's say that the RMS of the voltage you are sending to the speaker is V_{rms} and the resistance of the speaker is R ohms, then the power consumed by the speaker is:

$$P = \frac{V_{rms}^2}{R}$$

Similarly, if you know the RMS of the current you are pushing through the speaker is I_{rms}, then the power consumed by the speaker is:

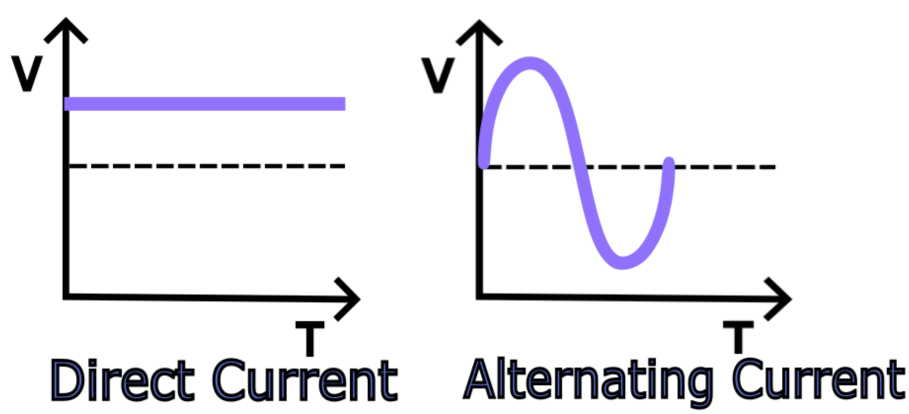
$$P = I_{rms}R$$

11.6 Electricity Dangers

Large amounts of electricity moving through your body can hurt or even kill you. You must be careful around electricity.

However, your body is not a very good conductor, so low-voltage systems (like a flashlight) don't have enough voltage to move significant amounts of current through your body.

However, the electricity in a power outlet has much more voltage. The voltage in these outlets is fluctuating between positive and negative, so we call it *Alternating Current* or AC.

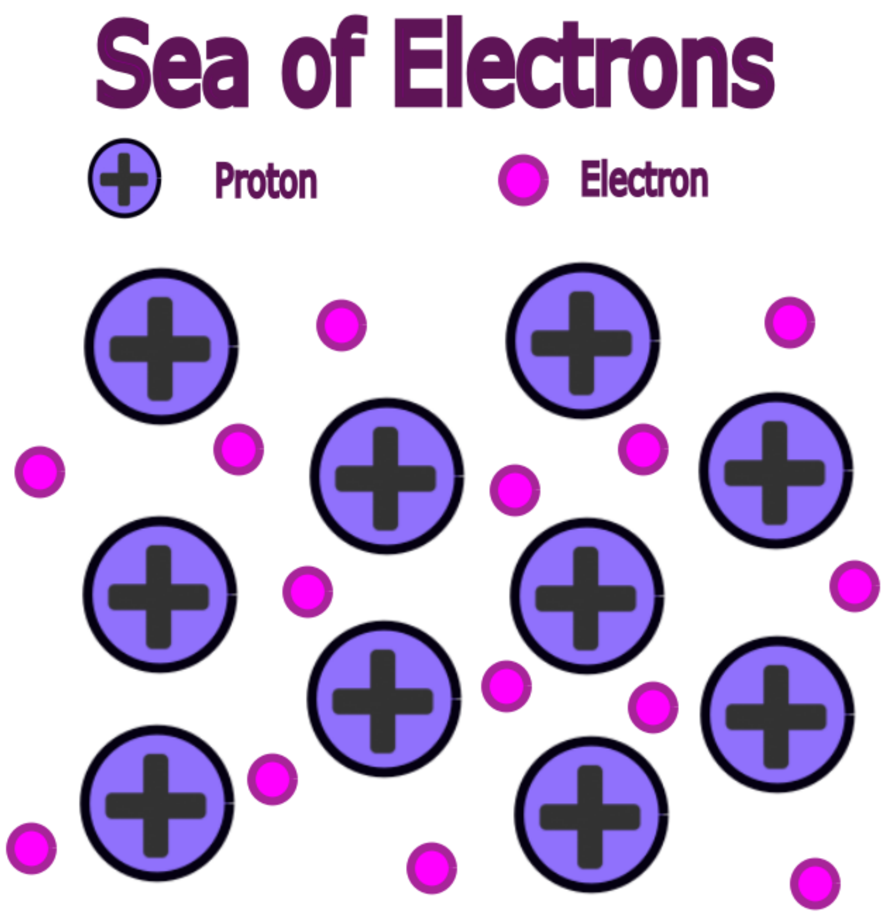


In most countries, the RMS of the voltage between 110 and 240 V. (The peak voltage is always $\sqrt{2}$ times the RMS value. In the US, for example, people say "Our outlets supply 120 V." They mean that the RMS of the voltage difference between the wire and the earth is 120V. The peak voltage is almost 170V.)

How much current can a human handle? Not much. You can barely feel 1 mA moving through your body, but at 16 mA, your muscles will clench and you won't be able to relax them – many people die from electrocution because they grab a wire which pushes enough current through their body to prevent them from letting go of the wire. At 20 mA, a human's respiratory muscles become paralyzed.

The fuse breaker in a house will often allow 20 A to flow through the circuit before it shuts off the power: Always, always, always shut off the power before touching any of the wiring in your house.

While water is actually a mediocre conductor, it can still deliver enough current to kill you. If you see a wire in a puddle, you should not touch the puddle. Interestingly, because of the salt, sea water is more than 100 times better at conducting electricity than the water you drink.

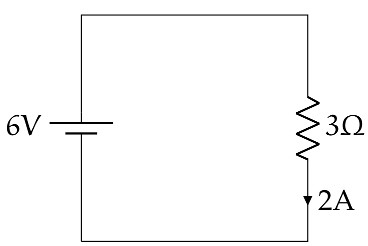


If you hold a wire in each hand, how many Ohms of resistance will your body have? Once it gets past your skin, you will look like a bag of salt water to the electricity. After the skin, your body will have a resistance of about 300Ω. However, the skin is a pretty good insulator. If you have dry, calloused hands, your skin may add a 100,000Ω to the resistance.

Chapter 12

DC Circuit Analysis

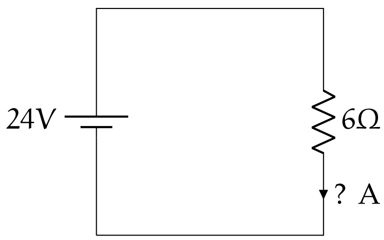
In the most basic circuit, you have only a battery and a resistor:



In this case, you only need Ohm's Law: $V = IR$. In this case, $6V = 3\Omega \times 2A$.

Exercise 20 Ohm's Law

How many amps are going around the circuit?



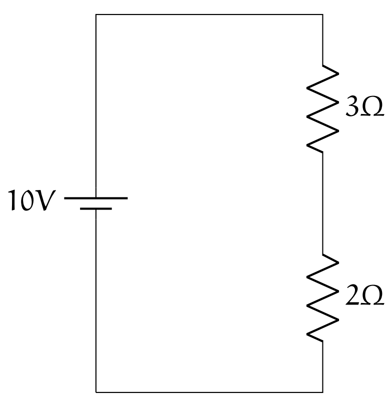
Working Space

Answer on Page 394

12.1 Resistors in Series

When you have two resistors wired together in a long line, we say they are "in series". If you have two resistors R_1 and R_2 wired in series, the total resistance is $R_1 + R_2$.

In this diagram, for example, the total resistance is 5Ω.



The current flowing through the circuit, then, is $10/4 = 2A$.

By Ohm's law, the voltage drop across the upper resistor is $IR = 2A \times 3\Omega = 6V$.

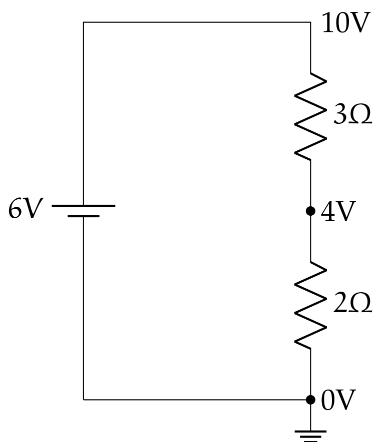
The voltage drop across the lower resistor is $IR = 2A \times 2\Omega = 4V$.

Notice that the battery pumps the voltage up to 10V, then the two resistors drop it by exactly 10V. This is known as "Kirchhoff's Voltage Law":

Kirchhoff's Voltage Law

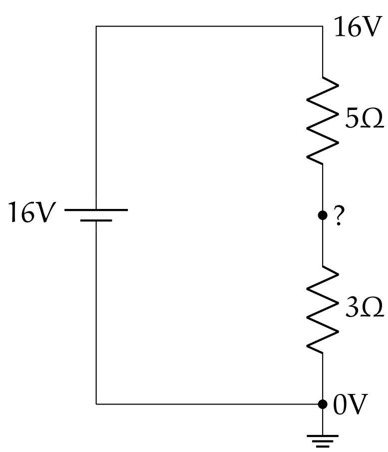
As you make a loop around a circuit, the sum of the voltage increase must equal the sum of the voltage decrease.

The negative end of the battery is connected to "ground" (it has zero voltage), then we can draw a diagram with the voltages(That symbol in the lower right represents a connection to ground).



Exercise 21 Resistors In Series

What is the current going around the circuit?
What is the voltage drop across each resistor?

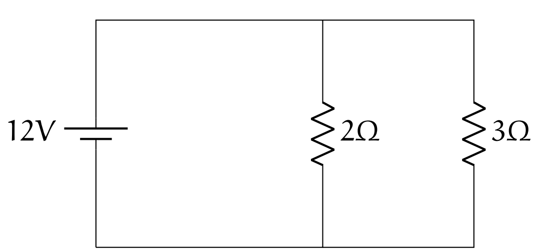


Working Space

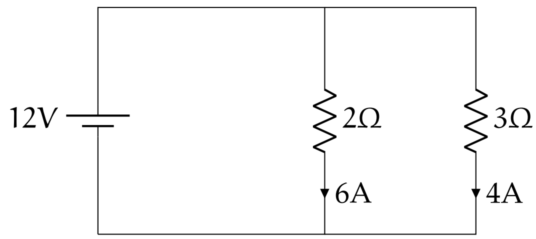
Answer on Page 394

12.2 Resistors in Parallel

Look at this circuit. Note that the current can go two different paths.



There is 12 volts pushing current through both resistors. So 6A will go through the 2Ω resistor and 4A will go through the 3Ω resistor.



Thus, a total of 10 A will be going through the battery.

Imagine you are a battery. You can't see that you have two resistors. What does it feel like to you? $\frac{V}{I} = R$, and $V = 12$ and $I = 10$. So the effective resistance of the two resistors in parallel is $\frac{12}{10}$ or $\frac{6}{5}\Omega$.

Resistance in Parallel

If you have several resistances $R_1, R_2, \ldots, R_n$ wired in parallel, their effective resistance R_t is given by

$$\frac{1}{R_t} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots + \frac{1}{R_n}$$

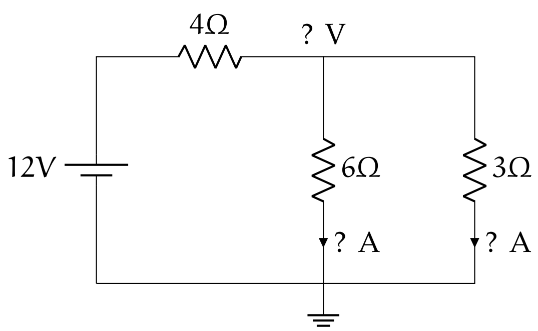
In our example:

$$\frac{1}{R_t} = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Thus $R_t = \frac{6}{5}\Omega$.

Exercise 22 Resistors In Parallel

What is the current going through the battery? What is the drop over the 4Ω resistor? What is the current in each branch?



Working Space

Answer on Page 394

Chapter 13

Charge

If you rub a balloon against your hair and then place it next to a wall it will stick. We say that it has gotten an *electrical charge*. It stole some electrons from your hair, and now the ballon has slightly more electrons than protons. We say that it has a negative electrical charge.

Objects with slightly more protons than electrons have a positive charge.

This charge is measured in coulombs. The charge of a single proton is about 1.6×10^{-19} coulombs.

An object with a negative charge and an object with a positive charge will be attracted to each other. Two objects with the same charge will be repelled by each other.

Coulomb's Law

If two objects with charge q_1 and q_2 (in coulombs) are r meters from each other, the force of attraction or repulsion is given by

$$F = K\frac{|q_1 q_2|}{r^2}$$

where F is in newtons and K is Coulomb's constant: about 8.988×10^9.

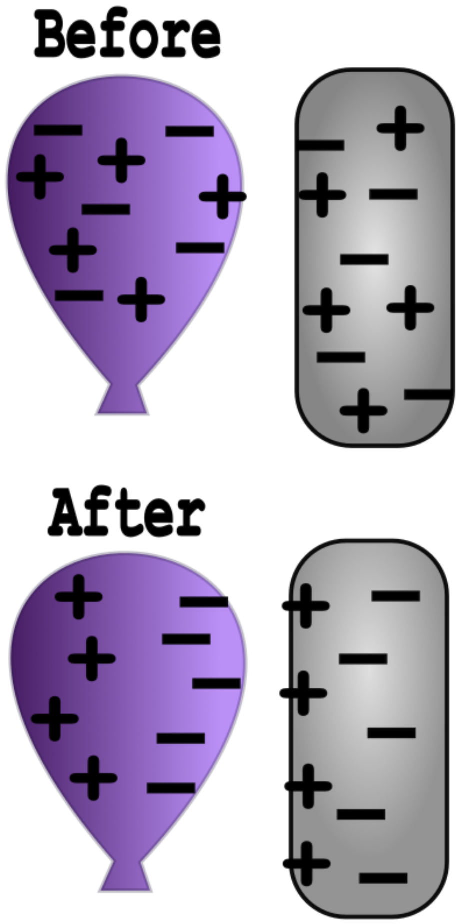
Exercise 23 Coulomb's Law

Two balloons are charged with an identical quantity and type of charge: -5×10^{-9} coulombs. They are held apart at a separation distance of 12 cm. Determine the magnitude of the electrical force of repulsion between them.

Working Space

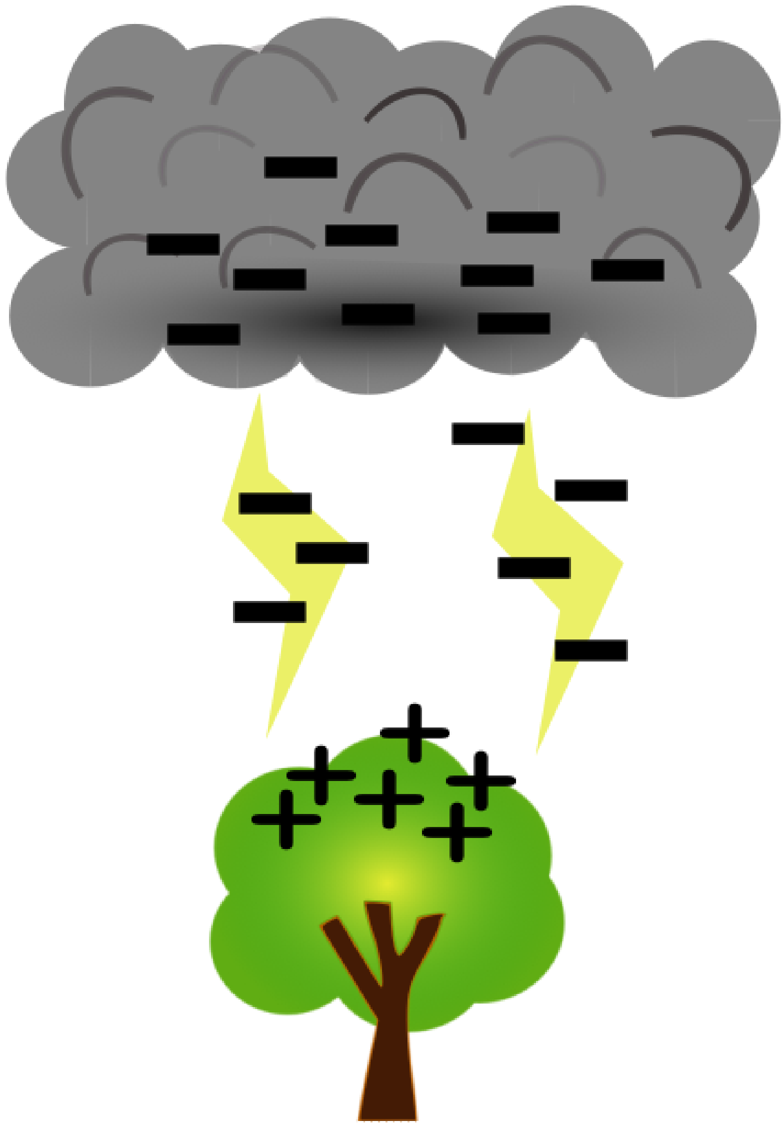
Answer on Page 395

At this point, you might ask "If the wall has zero charge, why is the balloon attracted to it?" The answer: the electrons in the wall move away from the balloon. The negative charge on the balloon pushes electrons into the wall, so the surface of the wall gets a mild positive charge. The surface is close to the balloon, so the attraction is stronger than the repulsion.



13.1 Lightning

A cloud is a cluster of water droplets and ice particles. These droplets and ice particles are always moving up and down through the cloud. In this process, electrons get stripped off and end up on the water droplets at the bottom of the cloud(water droplets collect at the bottom because they are denser). The air between the droplets is a pretty good insulator, and thus the electrons are reluctant to jump anywhere. However, eventually, the charge gets so strong that even the insulating properties of the air is not enough to prevent the jump, causing lightning.



A lot of lightning moves within a cloud or between clouds. However, a few jump to the earth. These bolts of lightning vary in the amount of electrons they carry, but the average is about 15 coulombs.

And thunder occurs because the electrons heat the air they pass through, causing the air to expand suddenly, and the resulting shockwave is the sound we know as thunder.

13.2 But...

This idea that opposite charges attract creates some heavy questions that you do not yet have the tools to work with. So the answer is basically “Don’t ask that question now!”

However, you probably have these questions, so I will point you in the direction of the answers.

The first is “In any atom bigger than hydrogen, there are multiple protons in the nucleus. Why don’t the protons push each other out of the nucleus?”

We aren’t ready to talk about it, but there is a force called *the nuclear force* which pulls the protons and neutrons in the nucleus of the atom toward each other. At very, very small distances it is strong enough to overpower the repulsive force due to the protons’ charges.

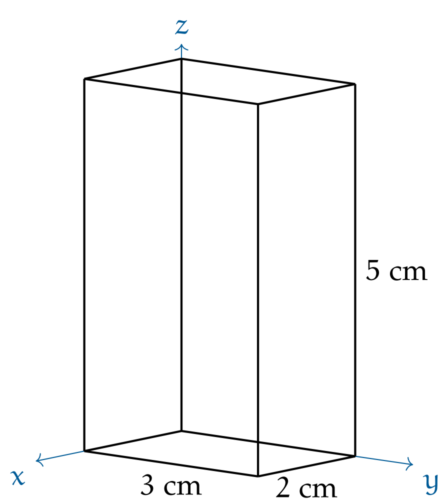
Another question is “Why do the electrons whiz around in a cloud so far from the nucleus of the atom? Negatively charged electrons should cling to the protons in the center, right?”

We aren’t ready to talk about it, but quantum mechanics tells us that electrons like to live in a certain specific energy level. Hugging protons isn’t one of those levels.

Chapter 14

Volumes of Common Solids

The volume of a rectangular solid is the product of its three dimensions. So if a block of ice is 5 cm tall, 3 cm wide and, 2 cm deep, it's volume is $5 \times 3 \times 2 = 30$ cubic centimeters.



A cubic centimeter is the same as a milliliter. A milliliter of ice weighs about 0.92 grams.

So the block of ice would have a mass of $30 \times 0.92 = 27.6$ grams.

A sphere with a radius of r has a volume of

$$v = \frac{4}{3}\pi r^3$$

(For completeness, the surface area of that sphere would be

$$a = 4\pi r^2$$

Note that a circle of radius r is one quarter of ths: πr^2.)

Exercise 24 Flying Sphere

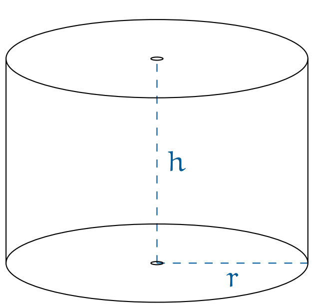
An iron sphere is traveling at 5 m/s. (It is not spinning.) The sphere has a radius of 1.5 m. Iron has a density of 7,800 kg per cubic meter. How much kinetic energy does the sphere have?

Working Space

Answer on Page 395

14.1 Cylinders

The base and the top of a right cylinder are identical circles. The circles are on parallel planes. The sides are perpendicular to those planes.

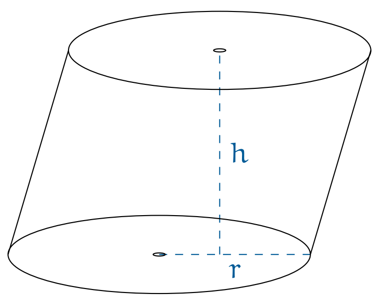


The volume of the a right cylinder of radius r and height h is given by:

$$v = \pi r^2 h$$

That is, it is the area of the base times the height.

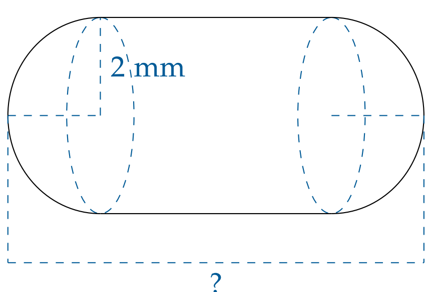
What if the base and top are identical, but the sides aren't perpendicular to the base? This is called *oblique cylinder*.



The volume is still the height times the area of the base. Note, however, that the height is measured perpendicular to the bottom and top.

Exercise 25 Tablet

A drug company has to create a tablet with volume of 90 cubic millimeters. The tablet will be a cylinder with half spheres on each end. The radius will be 2mm.
How long do they need to make the tablet to be?



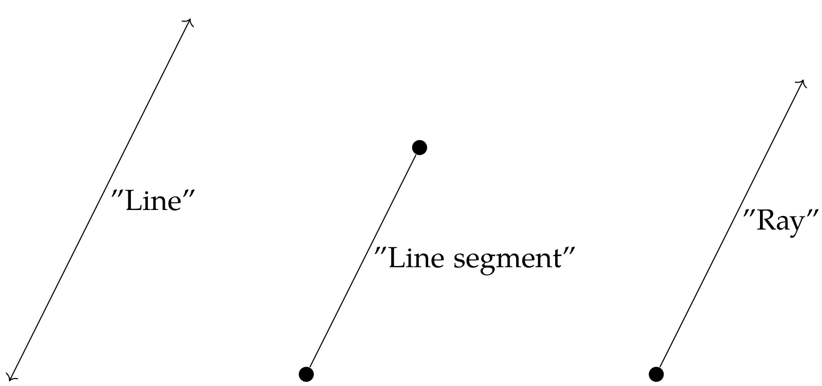
Working Space

Answer on Page 395

Chapter 15

Angles

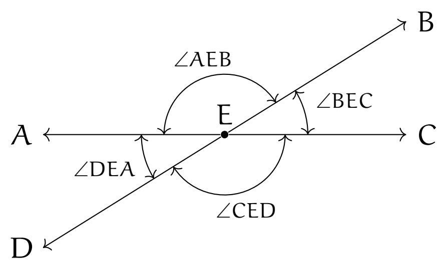
In the following recommend videos, the narrator talks about lines, line segments, and rays. When mathematicians talk about *lines*, they mean a straight line that goes forever in two directions. And if you pick any two points on that line; the space between them is a *line segment*. If you take any line, pick a point on that line and discard all the points on one side of the point, that is a *ray*. All three have no width.



Watch the following videos from Khan Academy:

- Introduction to angles: https://youtu.be/H-de6Tkxej8
- Measuring angles in degrees: https://youtu.be/92aLiyeQj0w

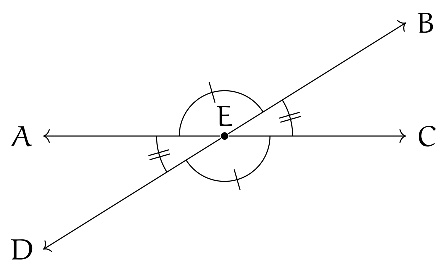
When two lines cross, they form four angles:



What do we know about those angles?

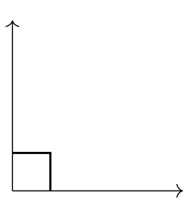
- The sum of any two adjacent angles add to be $180°$. So, for example, $m\angle AEB + m\angle BEC = 180°$. We use the phrase "add to be $180°$" so often that we have a special word for it: *supplementary*.
- The sum of all four angles is $360°$.
- Angles opposite each other are equal. So, for example, $m\angle AEB = m\angle CED$.

In a diagram, to indicate that two angles are equal we often put hash marks in the angle:



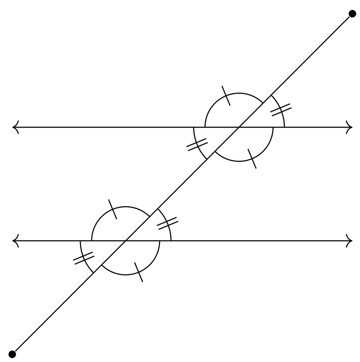
Here the two angles with a single hash mark are equal and the two angles with double hash marks are equal.

When two lines are perpendicular, the angle between them is $90°$ and we say they meet at a *right angle*. When drawing diagrams, we indicate right angles with an elbow:



When an angle is less than $90°$, it is said to be *acute*. When an angle is more than $90°$, it is said to be *obtuse*.

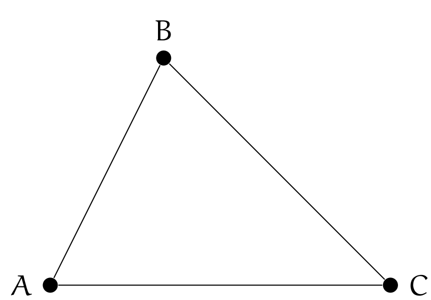
If two lines are parallel, line segments that intersect both lines, form the same angles with each line:



Chapter 16

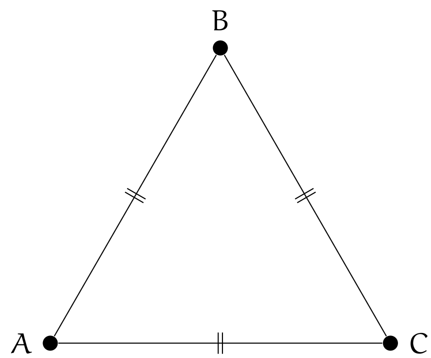
Introduction to Triangles

Connecting any three points with three line segments will get you a triangle. Here is the triangle ABC which was created by connecting three points A, B, and C:

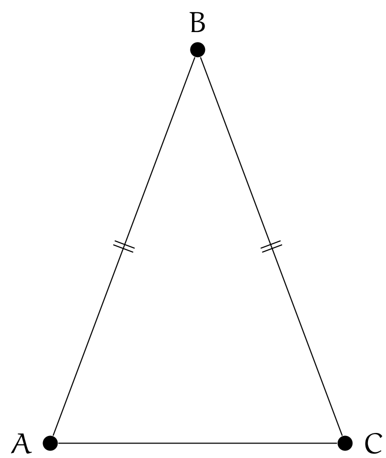


16.1 Equilateral and Isosceles Triangles

We talk a lot about the length of the sides of triangles. If all three sides of the triangle are the same length, we say it is an *equilateral triangle*:

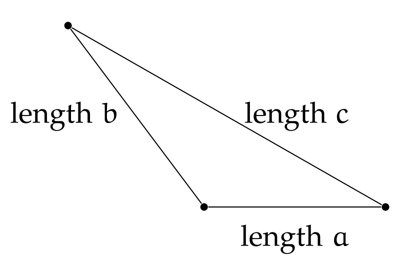


If only two sides of the triangle are the same length, we say it is an *isosceles triangle*:



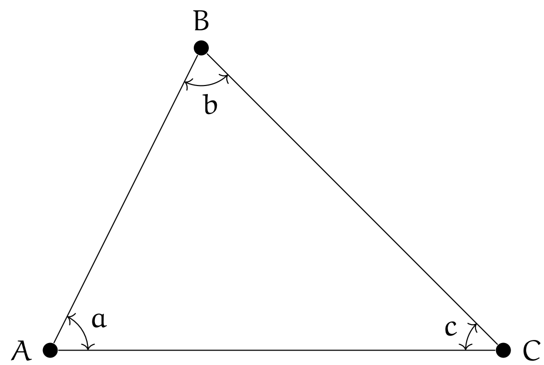
The shortest distance between two points is always the straight line between them. Thus, you can be certain that the length of one side will *always* be less than the sum of the lengths of the remaining two sides. This is known as the *triangle inequality*.

For example, in this diagram c must be less than $a + b$.

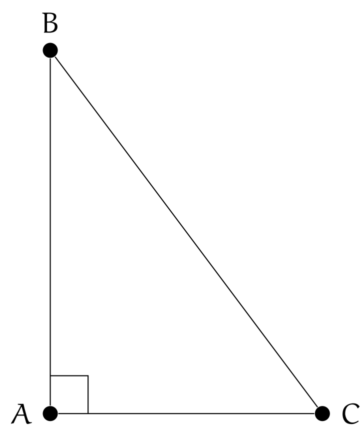


16.2 Interior Angles of a Triangle

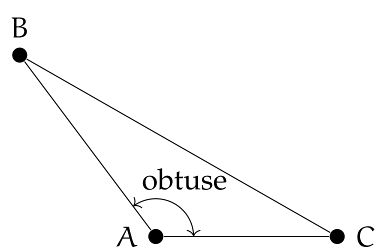
We also talk a lot about the interior angles of a triangle:



A triangle where one of the interior angles is a right angle is said to be a *right triangle*:



If a triangle has an obtuse interior angle, it is said to be an *obtuse triangle*:



If all three interior angles of a triangle are less than 90°, it is said to be an *acute triangle*.

The measures of the interior angles of a triangle always add up to 180°. For example, if we know that a triangle has interior angles of 37° and 56°, we know that the third interior angle is 87°.

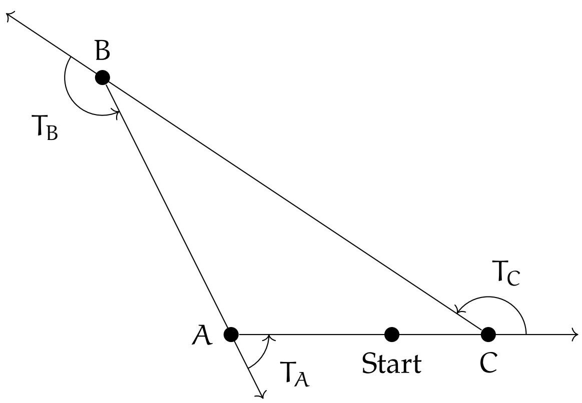
Exercise 26 Missing Angle

One interior angle of a triangle is 92°. The second angle is 42°. What is the measure of the third interior angle?

Working Space

Answer on Page 396

How can you know that the sum of the interior angles is 180°? Imagine that you started on the edge of a triangle and walked all the way around to where you started. (going counter-clockwise.) You would turn three times to the left:



After these three turns, you would be facing the same direction that you started in. Thus $T_A + T_B + T_C = 360°$. The measures of the interior angles are a, b, and c. Notice that a and T_A are supplementary. So we know that:

- $T_A = 180 - a$
- $T_B = 180 - b$
- $T_C = 180 - c$

So we can rewrite the equation above as

$$(180 - a) + (180 - b) + (180 - c) = 360°$$

Which is equivalent to

$$a + b + c = 360°$$

Exercise 27 Interior Angles of a Quadrilateral

Any four-sided polygon is a *quadrilateral*. Using the same "walk around the edge" logic, what is the sum of the interior angles of any quadrilateral?

Working Space

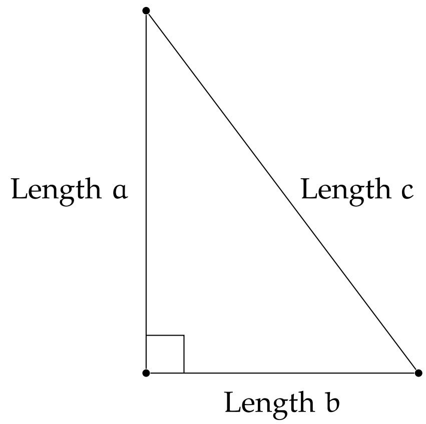
Answer on Page 396

Chapter 17

Pythagorean Theorem

Watch's Khan Academy's Intro to the Pythagorean Theorem video at `https://youtu.be/AA6RfgP-AHU`.

If you have a right triangle, the edges that touch the right angle are called *the legs*. The third edge, which is always the longest, is known as *the hypotenuse*. The Pythagorean Theorem gives us the relationship between the length of the legs and the length of the hypotenuse.



The Pythagorean Theorem tells us that $a^2 + b^2 = c^2$.

For example, if one leg has a length of 3 and the other has a length of 4, then $a^2 + b^2 = 3^2 + 4^2 = 25$. Thus c^2 must equal 25. So you know the hypotenuse must be of length 5.

(In reality, it rarely works out to be such a tidy number. For example, what is the length of the hypotenuse if the two legs are 3 and 6? $a^2 + b^2 = 3^2 + 6^2 = 45$. The length of the hypotenuse is the square root of that: $\sqrt{45} = \sqrt{9 \times 5} = 3\sqrt{5}$, which is approximately 6.708203932499369.)

Exercise 28 Find the Missing Length

What is the missing measure?

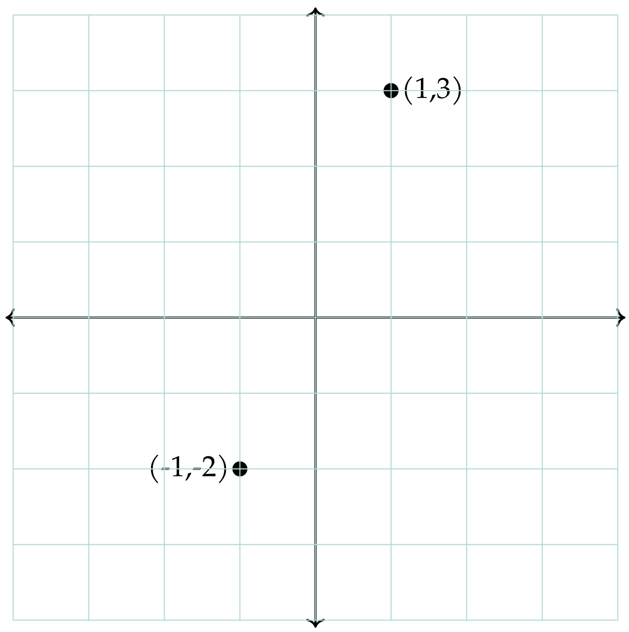
Leg 1 = 6, Leg 2 = 8, Hypotenuse = ?
(It should be a whole number.)
Leg 1 = 5, Leg 2 = ?, Hypotenuse = 13
(It should be a whole number.)
Leg 1 = ?, Leg 2 = 15, Hypotenuse = 17
(It should be a whole number.)
Leg 1 = 3, Leg 2 = 3, Hypotenuse = ?
(It is an irrational number. Give the exact answer and then use a calculator to get an approximation.)

Working Space

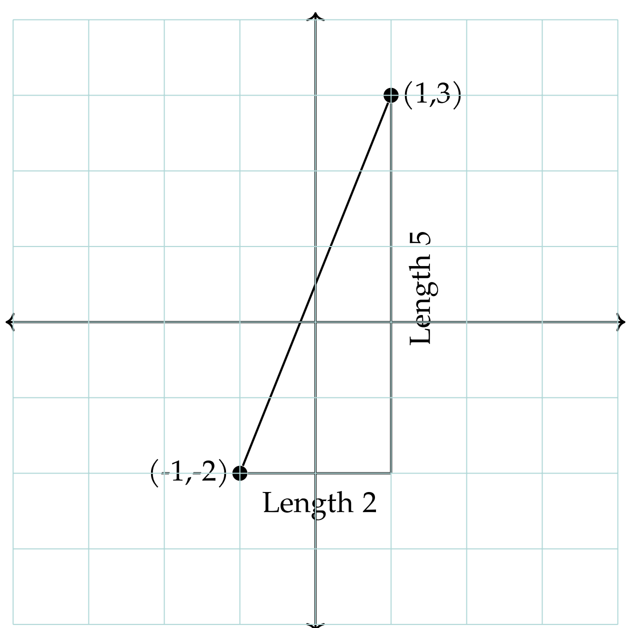
Answer on Page 396

17.1 Distance between Points

What is the distance between these two points?



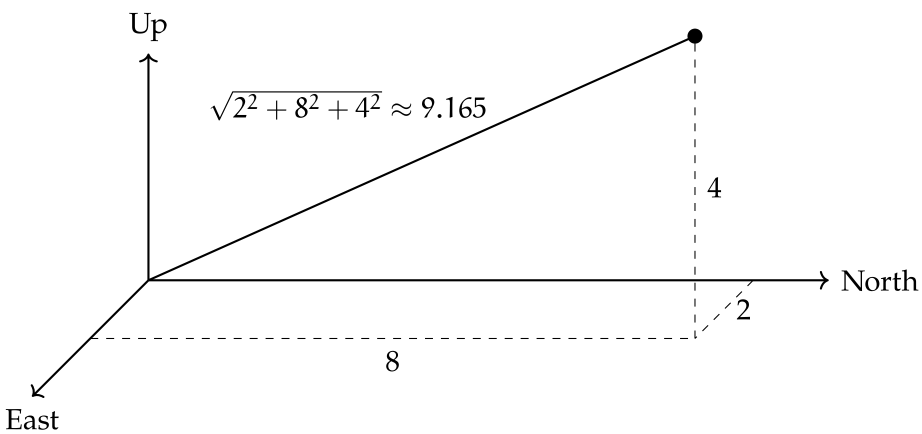
We can draw a right triangle and use the Pythagorean Theorem:



The distance between the two points is $\sqrt{2^2+5^2} = \sqrt{29} \approx 5.385165$. That is, you square the change in x and add it to the square of the change in y. The distance is the square root of that sum.

17.2 Distance in 3 Dimensions

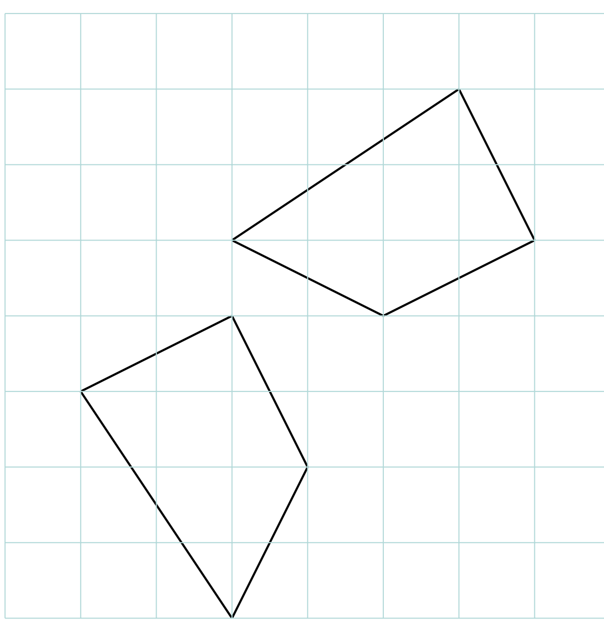
What if the point is in three-dimensional space? That is, you move 2 meters East, 8 meters North, and 4 meters up in the air. How far are you from where you started? You just square each, sum them, and take the square root: $\sqrt{2^2 + 8^2 + 4^2} = \sqrt{84} = 2\sqrt{21} \approx 9.165$ meters.



Chapter 18

Congruence

Look at this picture of two geometric figures.



They are the same shape, right? If you cut one out with scissors, it would lay perfectly on top of the other. In geometry, we say they are *congruent*.

What is the official definition of "congruent"? Two geometric figures are congruent if you can transform one into the other using only rigid transformations.

You might be wondering now, what are rigid transformations? A transformation is *Rigid* if it doesn't change the distances between the points or the measure of the angles between the lines, they form. These are all rigid transformations:

- Translations
- Rotations
- Reflections

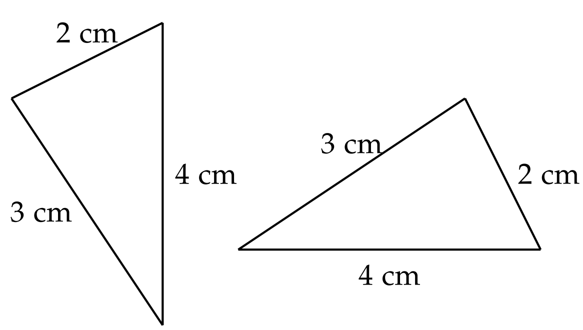
Once again imagine cutting out one figure with scissors and trying to match it with the second figure, your actions are rigid transformations:

- Translations - sliding the cutout left and right and up and down
- Rotations - rotating the cutout clockwise and counterclockwise
- Reflection - flipping the piece of paper over

A transformation is rigid if it is some combination of translations, rotations, and reflections.

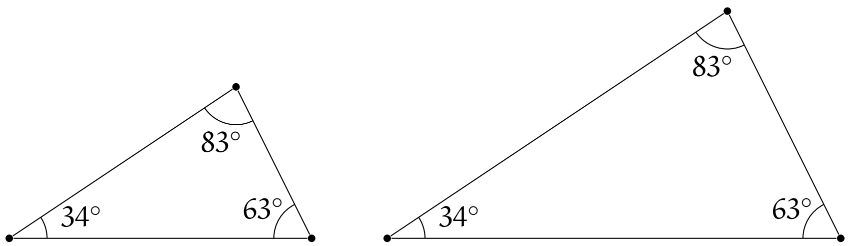
18.1 Triangle Congruency

If the sides of two triangles have the same length, the triangles must be congruent:



To be precise, the Side-Side-Side Congruency Test says that two triangles are congruent if three sides in one triangle are the same length as the corresponding sides in the other. We usually refer to this as the SSS test.

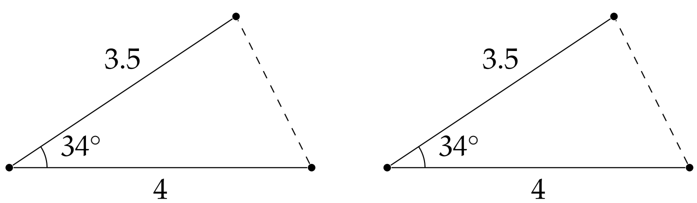
Note that two triangles with all three angles equal are not necessarily congruent. For example, here are two triangles with the same interior angles, but they are different sizes:



These triangles are not congruent, but they are *similar*. Meaning they have the same shape, but are not necessarily the same size.

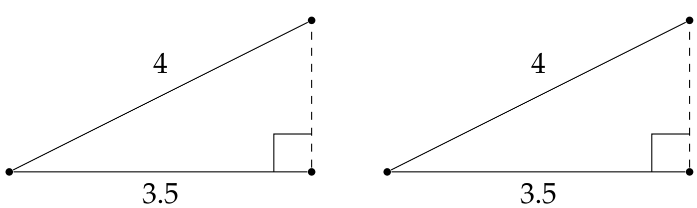
Therefore, if you know two angles of a triangle, you can calculate the third. So it makes sense to say "If two triangles have two angles that are equal, they are similar triangles." And if two similar triangles have one side that is equal in length, they must be the same size – so they are congruent. Thus, the Side-Angle-Angle Congruency Test says that two triangles are congruent if two angles and one side match.

What if you know that two triangles have two sides that are the same length and that the angle between them is also equal?



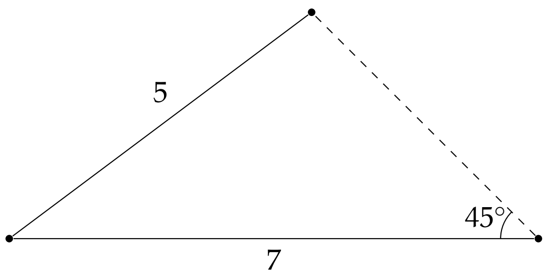
Yes, they must be congruent. This is the Side-Angle-Size Congruency Test.

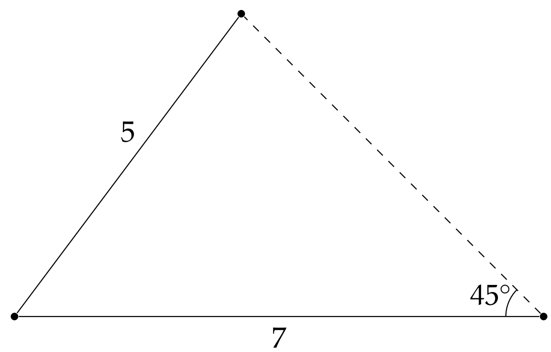
What if the angle isn't the one between the two known sides? If it is a right angle, you can be certain the two triangles are congruent. (How do I know? Because the Pythagorean Theorem tells us that we can calculate the length of the third side. There is only one possibility, thus all three sides must be the same length.)



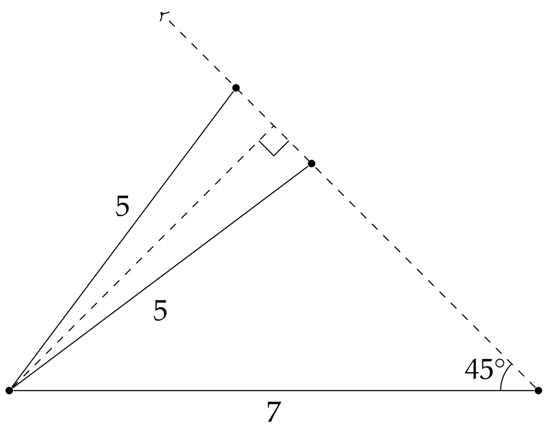
In this case, the third side of each triangle must be $\sqrt{4^2 - 3.5^2} \approx 1.9$.

What if the know angle is less than 90°? *The triangles are not necessarily congruent.* For example, let's say that there are two triangles with sides of length 5 and 7 and that the corresponding angle (at the end of the side of length 7) on each is 45°. Two different triangles satisfy this:





Let's see this another way by laying one triangle on top of the other:



So there is *not* a general Side-Side-Angle Congruency Test.

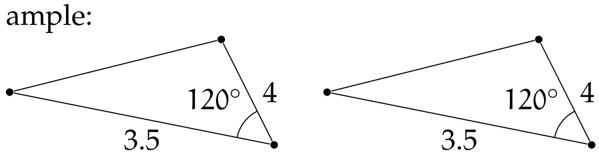
Here, then, is the list of common congruency tests:

- Side-Side-Side: All three sides have the same measure
- Side-Angle-Angle: Two angles and one side have the same measure
- Side-Angle-Side: Two sides and the angle between them have the same measure
- Side-Side-Right: They are right triangles and have two sides have the same measure

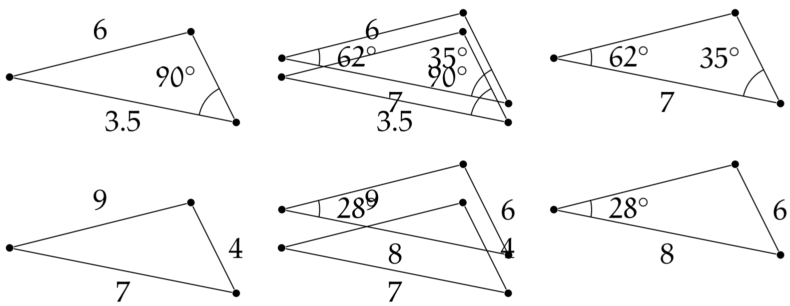
Exercise 29 Congruent Triangles

Working Space

Ted is terrible at drawing triangles: he always draws them exactly the same. Fortunately, he has marked these diagrams with the sides and angles that he measured. For each pair of triangles, write if you know them to be congruent and which congruency test proves it. For example:



(These drawings are clearly not accurate, but you are told the measurements are.) The answer is "Congruent by the Side-Angle-Side test."



Answer on Page 396

Chapter 19

Vectors

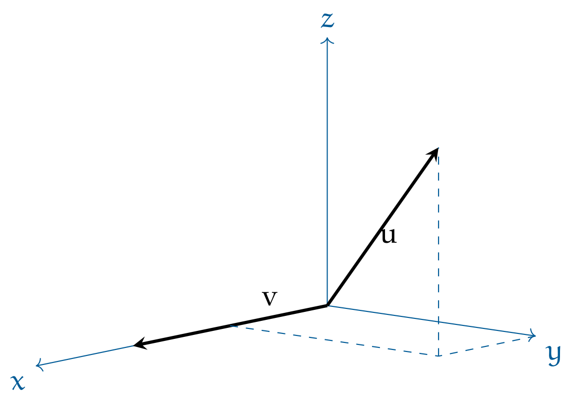
We have talked a some about forces, but in the calculations that we have done, we have only talked about the magnitude of a force. It is equally important to talk about its direction. To do the math on things with a magnitude and a direction (like forces), we need vectors.

For example, if you jump out of a plane (hopefully with a parachute), several forces with different magnitudes and directions will be acting upon you. Gravity will push you straight down. That force will be proportional to your weight. If there were a wind from the west, it would push you toward the east. That force will be proportional to the square of the speed of the wind and approximately proportional to your size. Once you are falling, there will be resistance from the air that you are pushing through – that force will point in the opposite direction from the direction you are moving and will be proportional to the square of your speed.

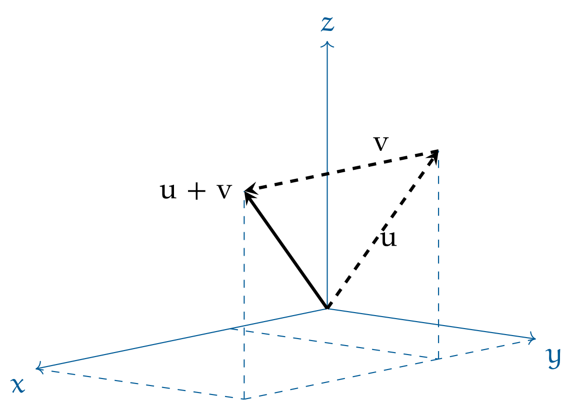
To figure out the net force (which will tell us how we will accelerate), we will need to add these forces together. So we need to learn to do math with vectors.

19.1 Adding Vectors

A vector is typically represented as a list of numbers, with each number representing a particular dimension. For example, if I am creating a 3-dimensional vector representing a force, it will have three numbers representing the amount of force in each of the three axes. For example, if a force of one newton is in the direction of the x-axis, I might represent the vector as $v = [1, 0, 0]$. Another vector might be $u = [0.5, 0.9, 0.7]$



Thinking visually, when we add to vectors, we put the starting point second vector at the ending point of the first vector.



If you know the vectors, you will just add them element-wise:

$$u + v = [0.5, 0.9, 0.7] + [1.0, 0.0, 0.0] = [1.5, 0.9.0.7]$$

These vectors have 3 components, so we say they are *3-dimensional*. Vectors can have any number of components. For example, the vector $[-12.2, 3, \pi, 10000]$ is 4-dimensional.

You can only add two vectors if they have the same dimension.

$$[12, -4] + [-1, 5] = [11, 1]$$

Addition is commutative: If you have two vectors a and b, then $a + b$ is the same as $b + a$.

Addition is also associative: If you have three vectors a, b, and c, it doesn't matter which order you add them in. That is, $a + (b + c) = (a + b) + c$.

A 1-dimensional vector is just a number. We say it is a *scalar*, not a vector.

Exercise 30 Adding vectors

Add the following vectors:

- $[1, 2, 3] + [4, 5, 6]$
- $[-1, -2, -3, -4] + [4, 5, 6, 7]$
- $[\pi, 0, 0] + [0, \pi, 0] + [0, 0, \pi]$

Working Space

Answer on Page 397

Exercise 31 Adding Forces

You are adrift in space. You are near two different stars. The gravity of one star is pulling you towards it with a force of $[4.2, 5.6, 9.0]$ newtons. The gravity of the other star is pulling you towards it with a force of $[-100.2, 30.2, -9.0]$ newtons. What is the net force?

Working Space

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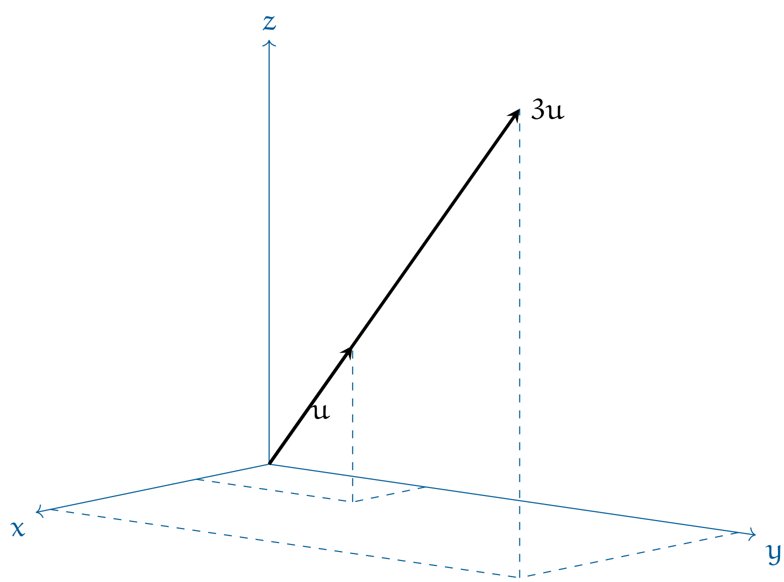
19.2 Multiplying a vector with a scalar

It is not uncommon to multiply a vector by a scalar. For example, a rocket engine might have a force vector v. If you fire 9 engines in the exact same direction, the resulting force vector would be $9v$.

Visually, when we multiply a vector u by a scalar a, we get a new vector that goes in the same direction as u but has a magnitude a times as long as u.



When you multiply a vector by a scalar, you just multiply each of the components by the scalar:

$$3 \times [0.5, 0.9, 0.7] = [1.5, 2.7, 3.6]$$

Exercise 32 Multiplying a vector and a scalar

Simplify the following expressions:

- $2 \times [1, 2, 3]$
- $[-1, -2, -3, -4] \times -2$
- $\pi[\pi, 2\pi, 3\pi]$

Working Space

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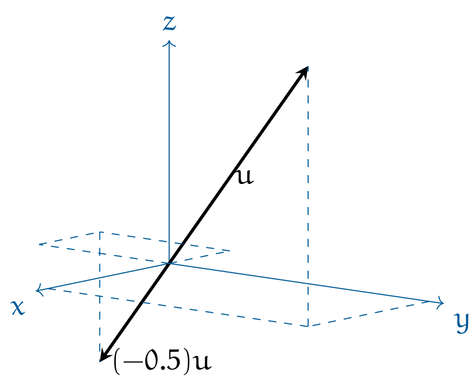
Note that when you multiply a vector times a negative number, the new vector points in the opposite direction.



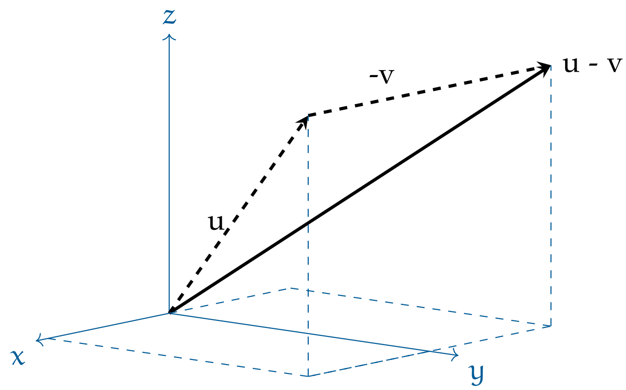
19.3 Vector Subtraction

As you might guess, when you subtract one vector from another, you just do element-wise subtraction:

$$[4, 2, 0] - [3, -2, 9] = [1, 4, -9]$$

So, $u - v = u + (-1v)$.

So visually, you reverse the one that is being subtracted:



19.4 Magnitude of a Vector

The *magnitude* of a vector is just its length. We write the magnitude of a vector v as $|v|$.

We compute the magnitude using the pythagorean theorem. If $v = [3, 4, 5]$, then

$$|v| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{50} \approx 7.07$$

(You might notice that the notation for the magnitude is exactly like the notation for absolute value. If you think of a scalar as a 1-dimensional vector, the absolute value and the magnitude are the same. For example, the absolute value of -5 is 5. If you take the magnitude of the one-dimenional vector $[-5]$, you get $\sqrt{25} = 5$.)

Notice that if you scale up a vector, its magnitude scales by the same amount. For example:

$$|7[3, 4, 5]| = 7\sqrt{50} \approx 7 \times 7.07$$

The rule then is: If you have any vector v and any scalar a:

$$|av| = |a||v|$$

Exercise 33 Magnitude of a Vector

Find the magnitude of the following vectors:

- $[1, 1, 1]$
- $[-5, -5, -5]$ (that is the same as $-5 \times [1, 1, 1]$)
- $[3, 4, -4] + [-2, -3, 5]$

Working Space

Answer on Page 397

19.5 Vectors in Python

NumPy is a library that allows you to work with vectors in Python. You might need to install it on your computer. This is done with `pip`. `pip3` installs things specifically for Python 3.

```
pip3 install NumPy
```

We can think of a vector as a list of numbers. There are also grids of numbers known as *matrices*. NumPy deals with both in the same way, so it refers to both of them as arrays.

The study of vectors and matrices is known as *Linear Algebra*. Some of the functions we need are in a sublibrary of NumPy called `linalg`.

As a convention, everyone who uses NumPy, imports it as *np*.

Create a file called `first_vectors.py`:

```
import NumPy as np

# Create two vectors
v = np.array([2,3,4])
u = np.array([-1,-2,3])
print(f"u = {u}, v = {v}")
```

```
# Add them
w = v + u
print(f"u + v = {w}")

# Multiply by a scalar
w = v * 3
print(f"v * 3 = {w}")

# Get the magnitude
# Get the magnitude
mv = np.linalg.norm(v)
mu = np.linalg.norm(u)
print(f"|v| = {mv}, |u| = {mu}")
```

When you run it, you should see:

```
> python3 first_vectors.py
u = [-1 -2  3], v = [2 3 4]
u + v = [1 1 7]
v * 3 = [ 6  9 12]
|v| = 5.385164807134504, |u| = 3.7416573867739413
```

19.5.1 Formatting Floats

The numbers 5.385164807134504 and 3.7416573867739413 are pretty long. You probably want it rounded off after a couple of decimal places.

Numbers with decimal places are called *floats*. In the placeholder for your float, you can specify how you want it formatted, including the number of decimal places.

Change the last line to look like this:

```
    print(f"|v| = {mv:.2f}, |u| = {mu:.2f}")
```

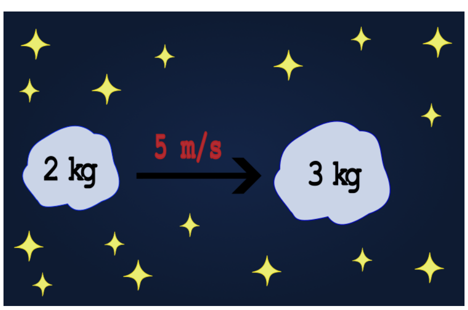
When you run the code, it will be neatly rounded off to two decimal places:

```
|v| = 5.39, |u| = 3.74
```

Chapter 20

Momentum

Let's say a 2 kg block of putty is flying through space at 5 meters per second, and it collides with a larger 3 kg block of putty that is not moving at all. When the two blocks deform and stick to each other, how fast will the resulting big block be moving?



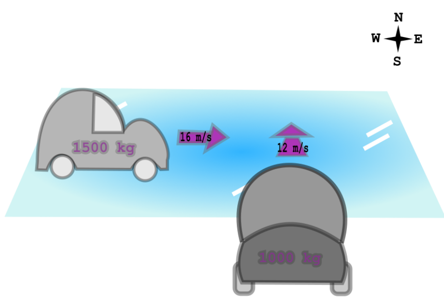
Every object has *momentum*. The momentum is a vector quantity: It points in the direction that the object is moving and has a magnitude equal to its mass times its speed.

Given a set of objects that are interacting, we can sum all their momentum vectors to get the total momentum. In such a set, the total momentum will stay constant.

So, in our example, one object has a momentum vector of magnitude of 10 kg m/s, the other has a momentum of magnitude 0. Once they have merged, they have a combined mass of 5 kg. Thus, the velocity vector must have magnitude 2 m/s and pointing in the same direction that the first mass was moving.

Exercise 34 Cars on Ice

A car weighing 1000 kg is going north at 12 m/s. Another car weighing 1500 kg is going east at 16 m/s. They both hit a patch of ice (with zero friction) and collide. Steel is bent and the two objects become one. How what is the velocity vector (direction and magnitude) of the new object sliding across the ice?



Working Space

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Notice that kinetic energy ($1/2mv^2$) is *not* conserved here. Before the collision, the moving putty block has $(1/2)(2)(5^2) = 25$ joules of kinetic energy. Afterward, the big block has $(1/2)(5)(2^2) = 10$ joules of kinetic energy. What happened to the energy that was lost? It was used up deforming the putty.

What if the blocks were marble instead of putty? Then there would be very little deforming, so kinetic energy *and* momentum would be conserved. The two blocks would end up having different velocity vectors.

Let's assume for a moment that they strike each other straight on, so there is motion in only one direction, both before and after the collision. Can we solve for the speeds of the first block (v_1) and the second block (v_2)?

We end up with two equations. Conservation of momentum says:

$$2v_1 + 3v_2 = 10$$

Conservation of kinetic energy says:

$$(1/2)(2)(v_1^2) + (1/2)(3)(v_2^2) = 25$$

Using the first equation, we can solve for v_1 in terms of v_2:

$$v_1 = \frac{10 - 3v_2}{2}$$

Substituting this into the second equation, we get:

$$\left(\frac{10 - 3v_2}{2}\right)^2 + \frac{3v_2^2}{2} = 25$$

Simplifying, we get:

$$v_2^2 - 4v_2 + 0 = 0$$

This quadratic has two solutions: $v_2 = 0$ and $v_2 = 4$. $v_2 = 0$ represents the situation before the collision. Substituting in $v_2 = 4$:

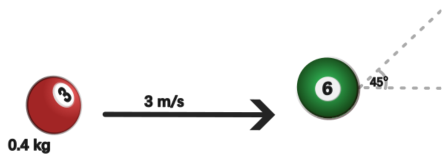
$$v_1 = \frac{10 - 3(4)}{2} = -1$$

Thus, if the blocks are hard enough that kinetic energy is conserved, after the collision, the smaller block will be heading in the opposite direction at 1 m/s. The larger block will be moving at 4 m/s in the direction of the original motion.

Exercise 35 Billiard Balls

Working Space

A billiard ball weighing 0.4 kg and traveling at 3 m/s hits a billiard ball (same weight) at rest. It strikes obliquely so that the ball at rest starts to move at a 45 degree angle from the path of the ball that hit it.
Assuming all kinetic energy is conserved. How what is the velocity vector of each ball after the collision?



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Chapter 21

The Dot Product

If you have two vectors $u = [u_1, u_2, \ldots, w_n]$ and $v = [v_1, v_2, \ldots, v_n]$, we define the *dot product* $u \cdot v$ as

$$u \cdot v = (u_1 \times v_1) + (u_2 \times v_2) + \cdots + (u_n \times v_n)$$

So, for example,

$$[2, 4, -3] \cdot [5, -1, 1] = 2 \times 5 + 4 \times -1 + -3 \times 1 = 3$$

This may not seem like a very powerful idea, but dot products are *incredibly* useful. The enormous GPUs(Graphics Processing Unit) that let video games render scenes so quickly? They primarily function by computing huge numbers of dot products at mind-boggling speeds.

Exercise 36 Basic dot products

Compute the dot product of each pair of vectors:

- $[1, 2, 3]$, $[4, 5, -6]$
- $[\pi, 2\pi]$, $[2, -1]$
- $[0, 0, 0, 0]$, $[10, 10, 10, 10]$

Working Space

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21.1 Properties of the dot product

Sometimes we need an easy way to say "The vector of appropriate length is filled with zeros." We use the notation $\vec{0}$ to represent this. Then, for any vector v, this is true:

$$v \cdot \vec{0} = 0$$

The dot product is commutative:

$$v \cdot u = u \cdot v$$

The dot product of a vector with itself is its magnitude squared:

$$v \cdot v = |v|^2$$

If you have a scalar a then:

$$(v) \cdot (au) = a(v \cdot u)$$

So, if v and w are vectors that go in the same direction,

$$v \cdot w = |v||w|$$

If v and w are vectors that go in opposite directions,

$$v \cdot w = -|v||w|$$

21.2 Cosines and dot products

Furthermore, dot products' interaction with cosine makes them even more useful is what makes them so useful: If you have two vectors v and u,

$$v \cdot u = |v||u| \cos \theta$$

where θ is the angle between them.

So, for example, if two vectors v and u are perpendicular, the angle between them is $\pi/2$. The cosine of $\pi/2$ is 0: The dot product of any two perpendicular vectors is always 0. In fact, if the dot product of two non-zero vectors is 0, the vectors *must be* perpendicular.

Exercise 37 Using dot products

What is the angle between these each pair of vectors:

- $[1, 0]$, $[0, 1]$
- $[3, 4]$, $[4, 3]$

Working Space

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If you have two non-zero vectors v and u, you can always compute the angle between them:

$$\theta = \arccos(\frac{v \cdot u}{|v||u|})$$

21.3 Dot products in Python

NumPy will let you do dot products using the the symbol @. Open `first_vectors.py` and add the following to the end of the script:

```
# Take the dot product
d = v @ u
print("v @ u =", d)

# Get the angle between the vectors
a = np.arccos(d / (mv * mu))
print(f"The angle between u and v is {a * 180 / np.pi:.2f} degrees")
```

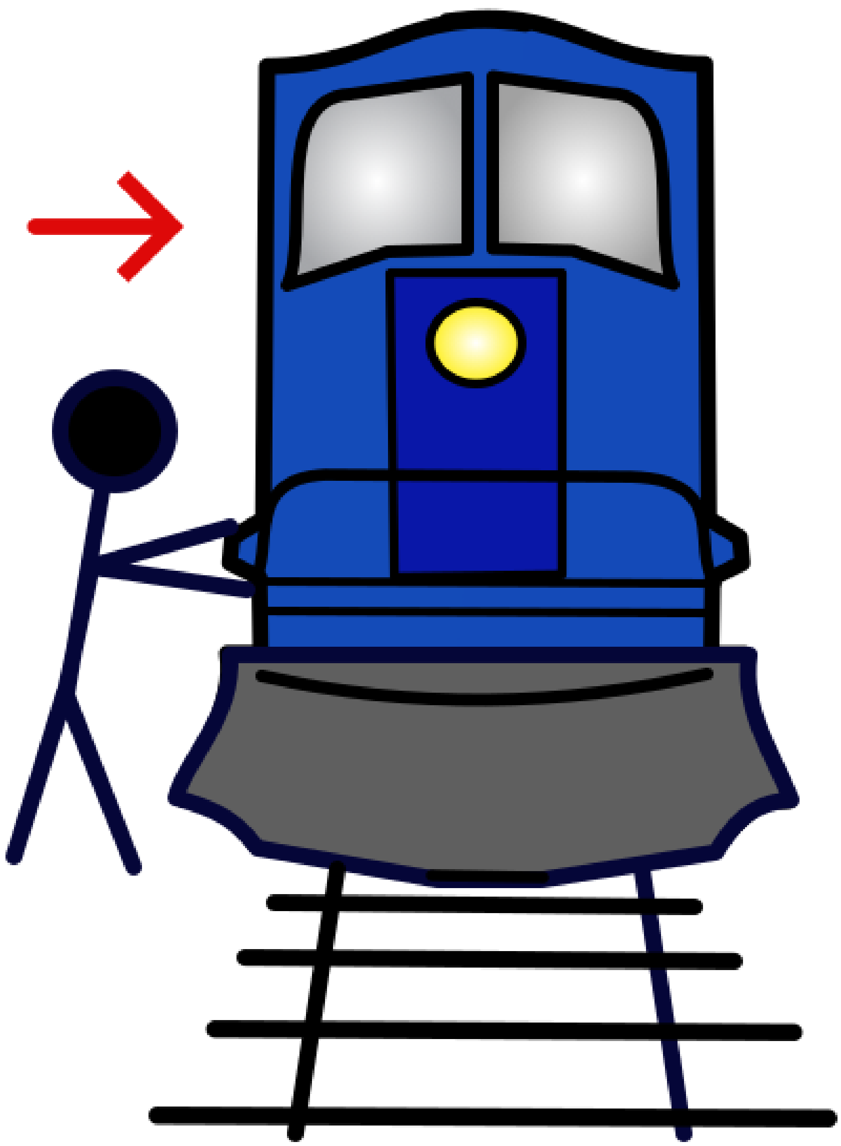
When you run it you should get:

```
v @ u = 4
The angle between u and v is 78.55 degrees
```

21.4 Work and Power

Earlier, we mentioned that mechanical work is the product of the force you apply to something and the amount it moves. For example, if you push a train with a force of 10 newtons as it moves 5 meters, you have done 50 joules of work.

What if you try to push the train sideways? That is, it moves down the track 5 meters, but you push it as if you were trying to derail it – perpendicular to its motion. You have done no work because the train didn't move at all in the direction you were pushing.



Now that you know about dot products: The work you do is the dot product of the force vector you apply and the displacement vector of the train. (The displacement vector is the vector that tells how the train moved while you pushed it.)

Similarly, we mentioned that power is the product of the force you apply and the velocity of the mass you are applying it to. It is actually the dot product of the force vector and the velocity vector.

For example, if you are pushing a sled with a force of 10 newtons and it is moving 2 meters per second, but your push is 20 degrees off, you aren't transferring 20 watts of power to the sled. You are transferring $10 \times 2 \times \cos(20 \text{ degrees}) \approx 18.8$ watts of power.

Chapter 22

Functions and Their Graphs

You can think of a function as a machine: you put something into the machine, it processes it, and out comes something else, a product. Just as we often use the variable x to stand in for a number, we often use the variable f to stand in for a function.

For example, we might ask, “Let the function f be defined like this:

$$f(x) = -5x^2 + 12x + 2$$

What is the value of $f(3)$?”

You would run the number 3 through “the machine”: $-5(3^2)+12(3)+2 = -7$. The answer would be “$f(3)$ is 7”.

However, Some functions are not defined for every possible input. For example:

$$f(x) = \frac{1}{x}$$

This is defined for any x except 0, because you can't divide 1 by 0. The set of values that a function can process is called its *domain*.

Exercise 38 Domain of a function

Let the function f be given by $f(x) = \sqrt{x-3}$. What is its domain?

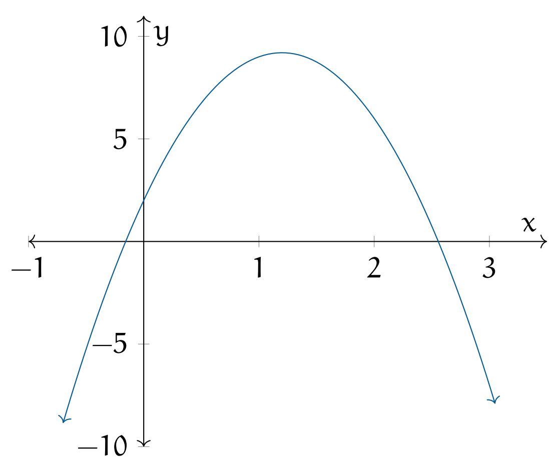
Working Space

Answer on Page 400

22.1 Graphs of Functions

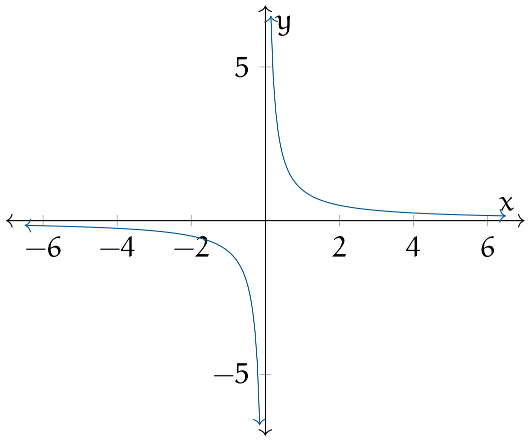
If you have a function, f, its graph is the set of pairs (x, y) such that $y = f(x)$. We usually draw a picture of this set, called a *graph*. The graph not only includes the picture, but also the values of x and y used to create it.

Here is the graph of the function $f(x) = -5x^2 + 12x + 2$:



(Note this is just part of the graph: it goes infinitely in both directions, remember your vectors.)

Here is the graph of the function $f(x) = \frac{1}{x}$:



Exercise 39 **Draw a graph**

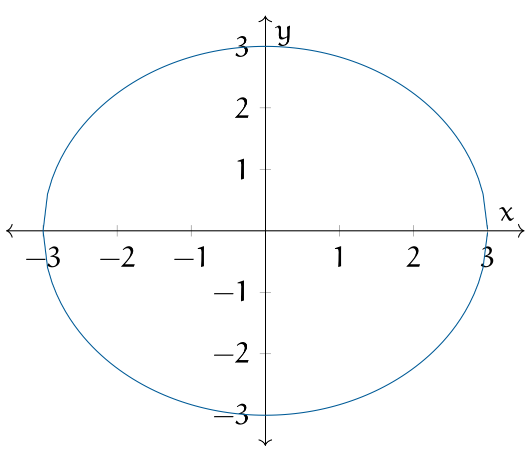
Let the function f be given by $f(x) = -3x + 3$. Sketch its graph.

Working Space

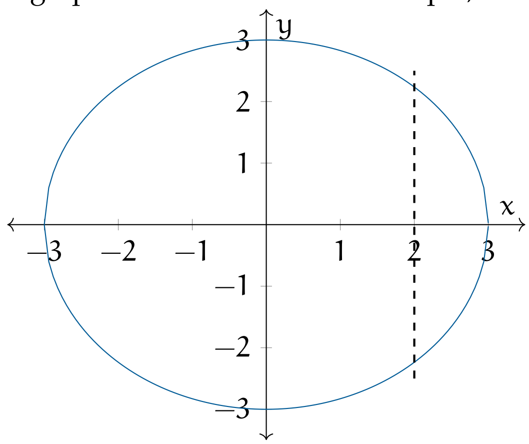
Answer on Page 400

22.2 Can this be expressed as a function?

Note that not all sets can be expressed as graphs of functions. For example, here is the set of points (x, y) such that $x^2 + y^2 = 9$:



This cannot be the graph of a function because what would $f(0)$ be? 3 or -3? This set fails what we call "the vertical line test": If any vertical line contains more than one point from the set, it isn't the graph of a function. For example, the vertical line $x = 2$ would cross the graph twice:

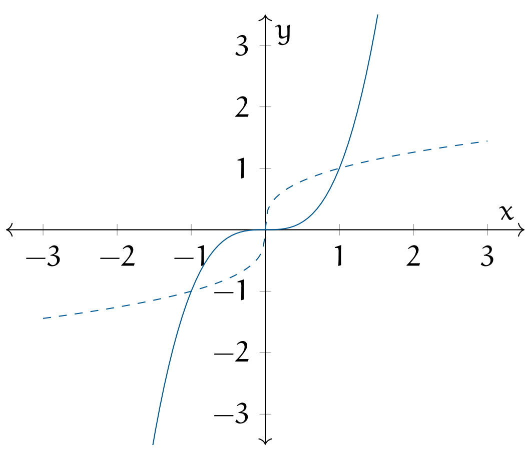


22.3 Inverses

Some functions have inverse functions. If a function f is a machine that turns number x into y, the inverse (usually denoted f^{-1}) is the machine that turns y back into x.

For example, let $f(x) = 5x + 1$. Its inverse is $f^{-1}(x) = (x - 1)/5$. (Spot check it: $f(3) = 16$ and $f^{-1}(16) = 3$)

Does the function $f(x) = x^3$ have an inverse? Yes, $f^{-1}(x) = \sqrt[3]{x}$. Let's plot the function (solid line) and its inverse (dashed):

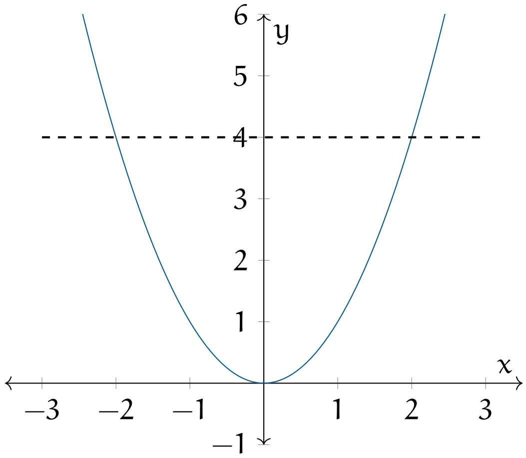


The inverse is the same as the function, just with its axes swapped. This tells us how to solve for an inverse: We swap x and y and solve for y.

For example, if you are given the function $f(x) = 5x + 1$, its graph is all (x, y) such that $y = 5x + 1$. The graph of its inverse is all (x, y) such that $x = 5y + 1$. So you solve for y:

$y = (x - 1)/5$.

Not every function has an inverse. For example, $f(x) = x^2$. Note that $f(2) = f(-2) = 4$. What would $f^{-1}(4)$ be? 2 or -2? This implies the "horizontal line test": If any horizontal line contains more than one point of a function's graph, that function has no inverse.



In some problems, you need an inverse and you don't need the whole domain, so you trim the domain to a set you can define an inverse on. This allow you to make claims such as "If we restrict the domain to the nonnegative numbers, the function $f(x) = x^2 - 5$ has an inverse: $f^{-1}(x) = \sqrt{x + 5}$.

This begs the question: What is the domain of the inverse function f^{-1}?

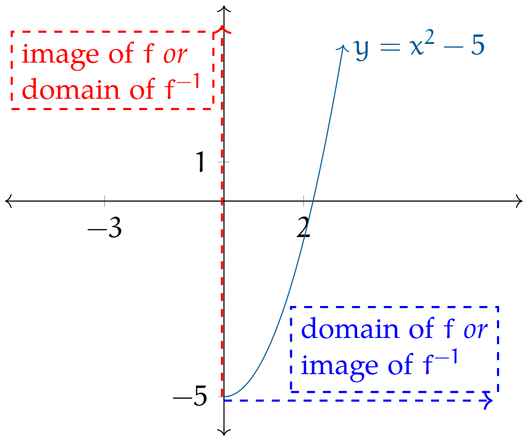
If we let X be the domain of f, we can run every member of X through "the machine" and gather them in a set on the other side. This set would be the *image* of the f "machine". (This is the *range* of f.)

What is the image of $f(x) = x^2 - 5$? It is the set of all real numbers greater than or equal to -5. We write this

$$\{x \in \mathbb{R} | x \geq -5\}$$

Now we can say: **The image of the function is the domain of the inverse function.**

In our example, we can use any number greater than or equal to -5 as input into the inverse function.



Exercise 40 Find the inverse

Let $f(x) = (x-3)^2 + 2$. Sketch the graph.
Using all the real numbers as a domain, does this function have an inverse?
How would you restrict the domain to make the function invertible?
What is the inverse of that restricted function?
What is the domain of the inverse?

Working Space

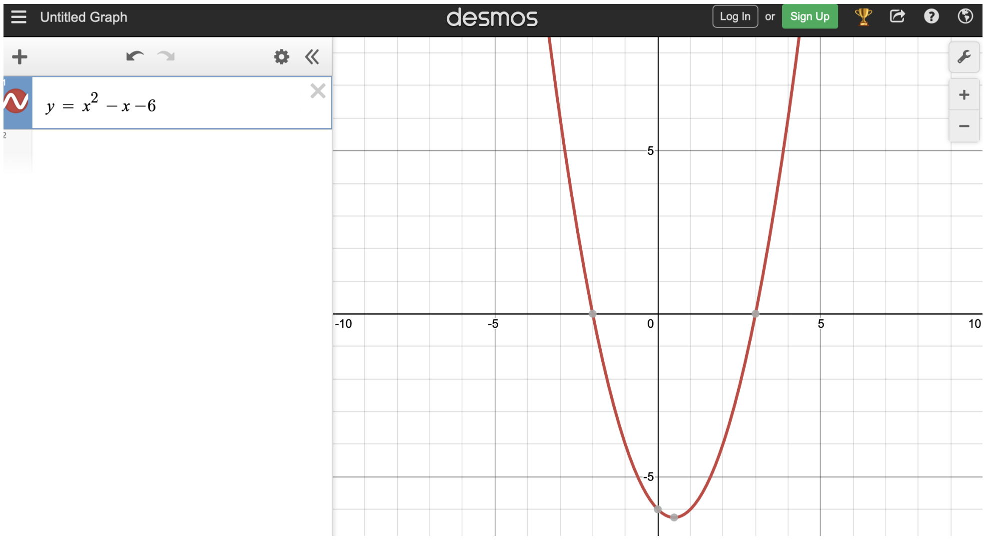
Answer on Page 400

22.4 Graphing Calculators

One really easy way to understand your function better is to use a graphing calculator. Desmos is a great, free online graphing calculator.

In a web browser, go to Desmos: `https://www.desmos.com/calculator`

In the field on the left, enter the function $y = x^2 - x - 6$. (For the exponent, just prefix it with a caret symbol: "x2".)



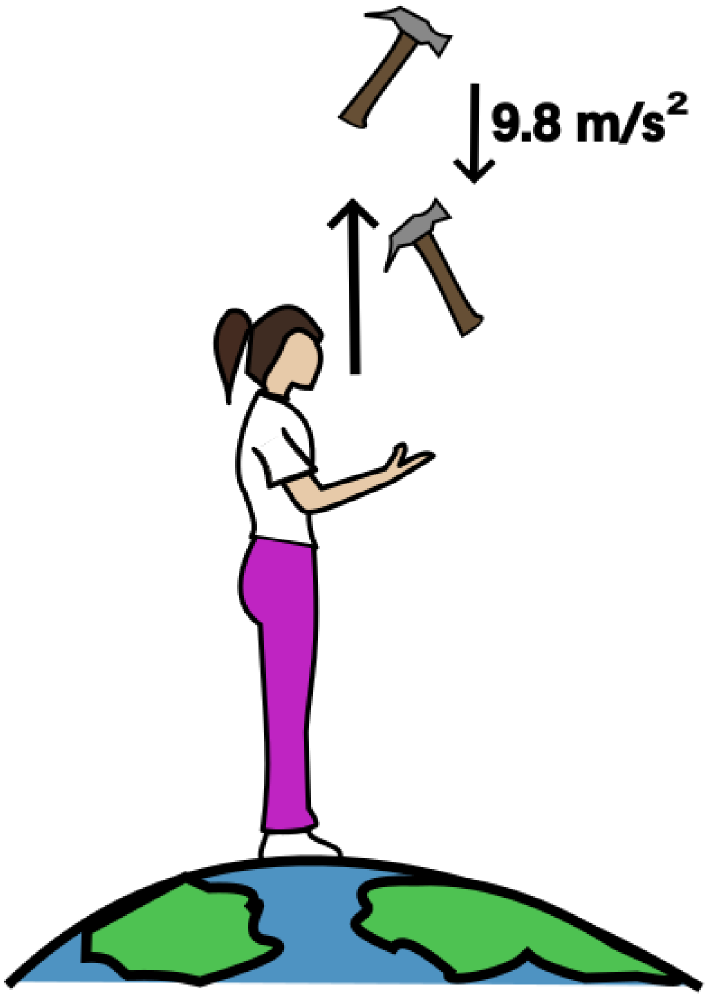
Untitled Graph
desmos
Log In
or
Sign Up
$y = x^2 - x - 6$
5
-10
-5
0
5
10
-5

Chapter 23

Falling Bodies

Because of gravity, if you throw a hammer straight up in the air, from the moment it leaves your hand until it hits the ground, it is accelerating toward the center of the earth at a constant rate.

Acceleration can be defined as change in velocity. If the hammer leaves your hand with a velocity of 12 meters per second upward, one second later it will be rising, and its velocity will have slowed to 2.2 meters per second. One second after that, the hammer will be falling at a rate of 7.6 meters per second. Every second the hammer's velocity is changing by 9.8 meters per second, and that change is always toward the center of the earth. When the hammer is going up, gravity is slowing it down by 9.8 meters per second, each second it is in the air. When the hammer is coming down, gravity is speeding it up by 9.8 meters



per second.

Acceleration due to gravity on earth is a constant negative 9.8 meters per second per second:

$$a = -9.8$$

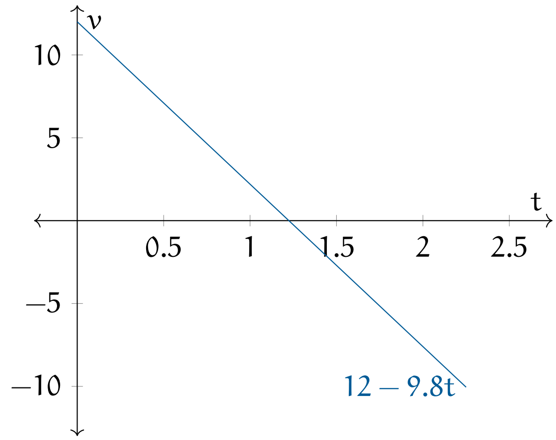
(Why is it negative? We are talking about height, which increases as you go away from the center of the earth. Acceleration is changing the velocity in the opposite direction.)

23.1 Calculating the Velocity

Given that the acceleration is constant, it makes sense that the velocity is a straight line. Assuming once again that the hammer leaves your hand at 12 meters per second, then the upwards velocity at time t is given by:

$$v = 12 - 9.8t$$

Note that the velocity of the hammer is being given as a function. Here is its graph:



Exercise 41 When is the apex of flight?

Given the hammer's velocity is given by $12-9.8t$, at what time (in seconds) does it stop rising and begin to fall?

Working Space

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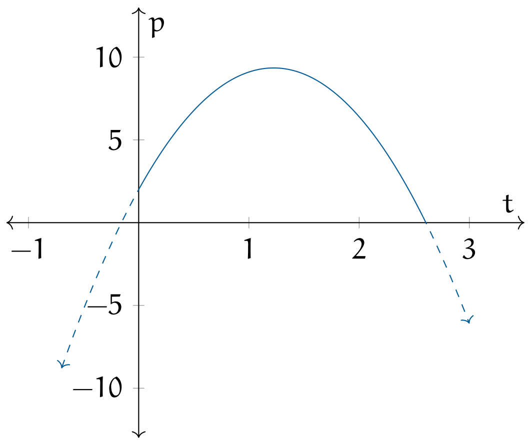
At this point, we need to acknowledge air resistance. Gravity is not the only force on the hammer; as it travels through the air, the air tries to slow it down. This force is called *air resistance*, and for a large, fast-moving object (like an airplane) it is GIGANTIC force. For a dense object (like a hammer) moving at a slow speed (what you generate with your hand), air resistance doesn't significantly affect acceleration.

23.2 Calculating Position

If you let go of the hammer when it is 2 meters above the ground, the height of the hammer is given by:

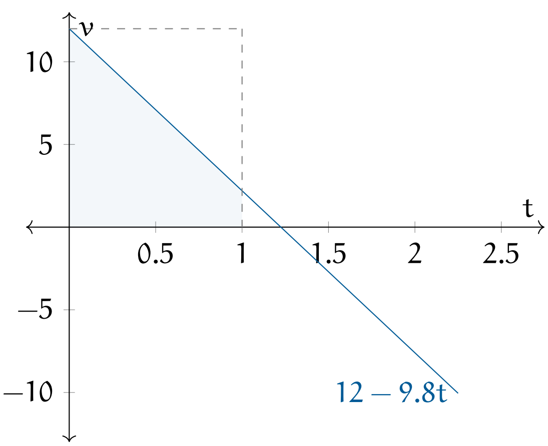
$$p = -\frac{9.8}{2}t^2 + 12t + 2$$

Here is a graph of this function:



How do we know? **The change in position between time 0 and any time t is equal to the area under the velocity graph between $x = 0$ and $x = t$.**

Let's use the velocity graph to figure out how much the position has changed in the first second of the hammer's flight. Here's the velocity graph with the area under the graph for the first second filled in:



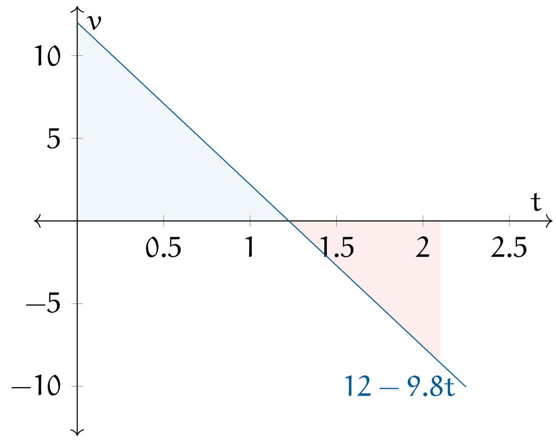
The blue filled region is the area of the dashed rectangle minus that empty triangle in its upper left. The height of the rectangle is twelve and its width is the amount of time the hammer has been in flight (t). The triangle is t wide and $9.8t$ tall. Thus, the area of the blue region is given by $12t - \frac{1}{2}9.8t^2$.

That's the change in position. Where was it originally? 2 meters off the ground. So the height is given by $p = 2 + 12t - \frac{1}{2}9.8t^2$. We usually write terms so that the exponent decreases, so:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

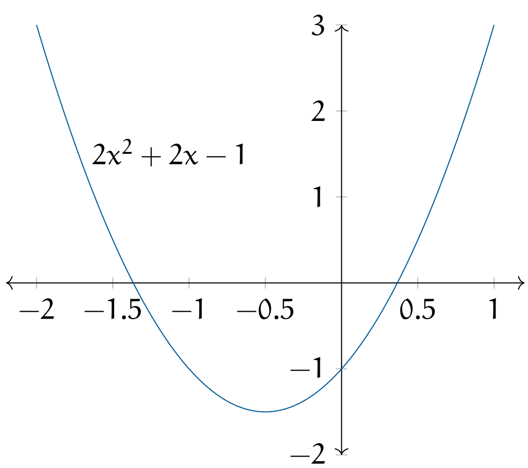
Finding the area under the curve like this is called *integration*. We say "To find a function that gives the change in position, we just integrate the velocity function." A lot of the study of calculus is learning to integrate different sorts of functions.

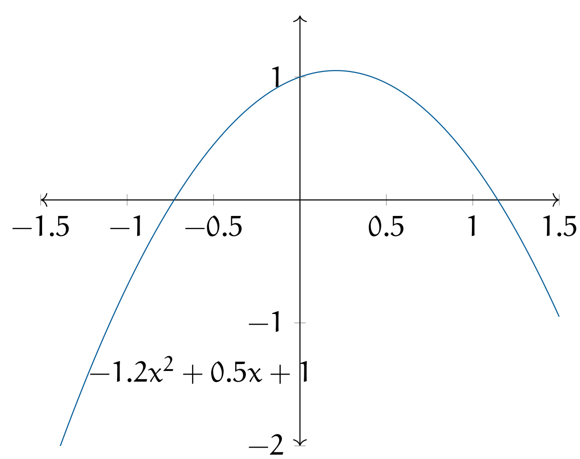
One important note about integration: Any time the curve drops under the x-axis, the area is considered negative. (Which makes sense, right? If the velocity is negative, the hammer's position is decreasing.)



23.3 Quadratic functions

Functions of the form $f(x) = ax^2 + bx + c$ are called *quadratic functions*. If $a > 0$, the ends go up. If $a < 0$, the ends go down.





The graph of a quadratic function is a *parabola*.

23.4 Simulating a falling body in Python

Now you are going to write some Python code that simulates the flying hammer. First, we are just going to print out the position, speed, and acceleration of the hammer for every 1/100th of a second after it leaves your hand. (Later we will make a graph.)

Create a file called `falling.py` and type this into it:

```
# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Is the hammer still aloft?
while height > 0.0:

    # Show the values
    print(f"{current_time:.2f} s:")
    print(f"\tacceleration: {acceleration:.2f} m/s/s")
    print(f"\tspeed: {speed:.2f} m/s")
    print(f"\theight: {height:.2f} m")

    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step


print(f"Hit the ground: Complete")
```

When you run it, you will see something like this:

```
0.00 s:
        acceleration: -9.80 m/s/s
```

```
        speed: 12.00 m/s
        height: 2.00 m
0.01 s:
        acceleration: -9.80 m/s/s
        speed: 11.90 m/s
        height: 2.12 m
0.02 s:
        acceleration: -9.80 m/s/s
        speed: 11.80 m/s
        height: 2.24 m
0.03 s:
        acceleration: -9.80 m/s/s
        speed: 11.71 m/s
        height: 2.36 m
...
2.60 s:
        acceleration: -9.80 m/s/s
        speed: -13.48 m/s
        height: 0.20 m
2.61 s:
        acceleration: -9.80 m/s/s
        speed: -13.58 m/s
        height: 0.07 m
Hit the ground: Complete
```

Note that the acceleration isn't changing at all, but it is changing the speed, and the speed is changing the height.

We can see that the hammer in our simulation hits the ground just after 2.61 seconds.

23.4.1 Graphs and Lists

Now, we are going to graph the acceleration, speed, and height using a library called matplotlib. However, to make the graphs, we need to gather all the data into lists.

For example, we will have a list of speeds, and the first three entries will be 12.0, 11.9, and 11.8.

We create an empty list and assign it to a variable like this:

```
x = []
```

Then we can add items like this:

```
x.append(3.14)
```

To get the first time back, we can ask for the object at index 0.

```
y = x[0]
```

Note that the list starts at 0. So if you have 32 items in the list, the first item is at index 0. The last item is at index 31.

Duplicate the file `falling.py` and name the new copy `falling_graph.py`

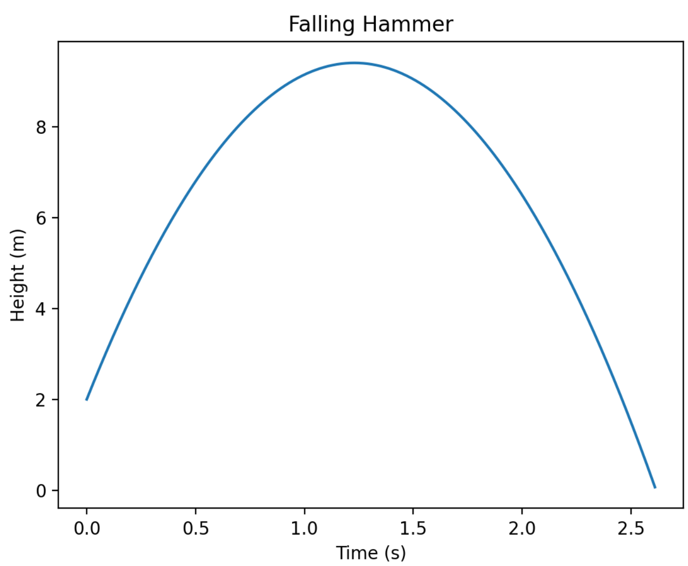
We are going to make a plot of the height over time. At the start of the program, you will import the matplotlib library. At the end of the program, you will create a plot and show it to the user.

In `falling_graph.py`, add the bold code:

```
import matplotlib.pyplot as plt

# Acceleration on earth
acceleration = -9.8 # m/s/s

# Size of time step
time_step = 0.01 # seconds

# Initial values
speed = 12  # m/s upward
height = 2  # m above the ground
current_time = 0.0  # seconds after release

# Create empty lists
accelerations = []
speeds = []
heights = []
times = []

# Is the hammer still aloft?
while height > 0.0:

    # Add the data to the lists
    times.append(current_time)
    accelerations.append(acceleration)
    speeds.append(speed)
    heights.append(height)
```

```
    # Update height
    height = height + time_step * speed

    # Update speed
    speed = speed + time_step * acceleration

    # Update time
    current_time = current_time + time_step

# Make a plot
fig, ax = plt.subplots()
fig.suptitle("Falling Hammer")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Height (m)")
ax.plot(times, heights)
plt.show()
```

When you run the program, you should see a graph of the height over time.

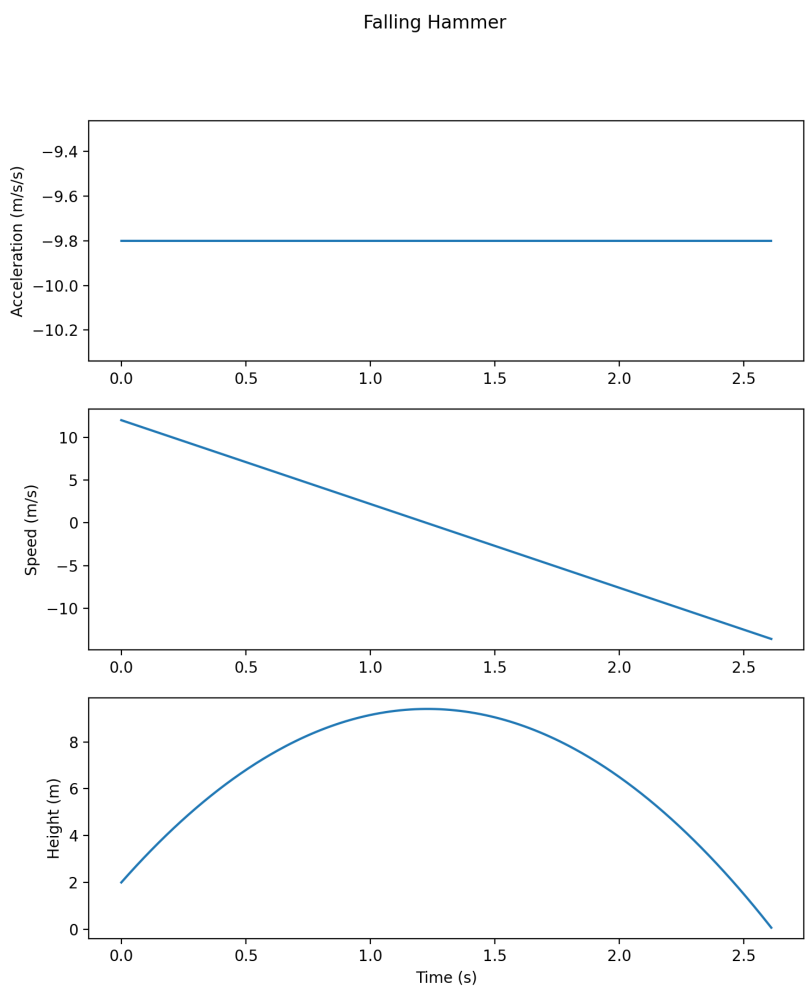


It is more interesting if we can see all three: acceleration, speed, and height. So lets make three stacked plots. Change the plotting code in `falling_graph.py` to:

```
# Make a plot with three subplots
fig, ax = plt.subplots(3,1)
fig.suptitle("Falling Hammer")
```

```
# The first subplot is acceleration
ax[0].set_ylabel("Acceleration (m/s/s)")
ax[0].plot(times, accelerations)

# Second subplot is speed
ax[1].set_ylabel("Speed (m/s)")
ax[1].plot(times, speeds)

# Third subplot is height
ax[2].set_xlabel("Time (s)")
ax[2].set_ylabel("Height (m)")
ax[2].plot(times, heights)
plt.show()
```

Now you will get plots of all three variables:



This is what we expected, right? The acceleration is a constant negative number. The speed is a straight line with a negative slope. The height is a parabola.

A natural question at this point is "When exactly will the hammer hit the ground?" That is, when does $\mathtt{height} = 0$? The values of $\mathtt{t}$ where a function is zero are known as its *roots*. Height is given by a quadratic function. In the next chapter, you will get the trick for finding the roots of any quadratic function.

Chapter 24

Solving Quadratics

A quadratic function has three terms: $ax^2 + bx + c$. a, b, and c are known as the *coeffcients*. The coefficients can be any constant, except that a can never be zero. (If a is zero, it is a linear function, not a quadratic.)

When you have an equation with a quadratic function on one side and a zero on the other, you have a quadratic equation. For example:

$$72x^2 - 12x + 1.2 = 0$$

How can you find the values of x that will make this equation true?

You can always reduce a quadratic equation so that the first coefficient is 1, so that your equation looks like this:

$$x^2 + bx + c = 0$$

For example, if you are asked to solve $4x^2 + 8x - 19 = -2x^2 - 7$

$$4x^2 + 8x - 19 = -2x^2 - 7$$

$$6x^2 + 8x - 12 = 0$$

$$x^2 + \frac{4}{3}x - 2 = 0$$

Here, $b = \frac{4}{3}$ and $c = -2$.

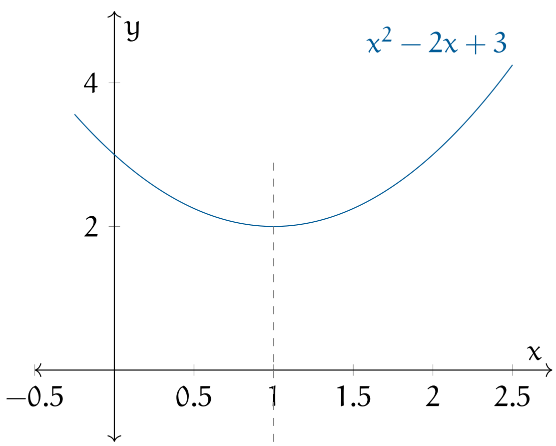
$x^2 + bx + c = 0$ when

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2}$$

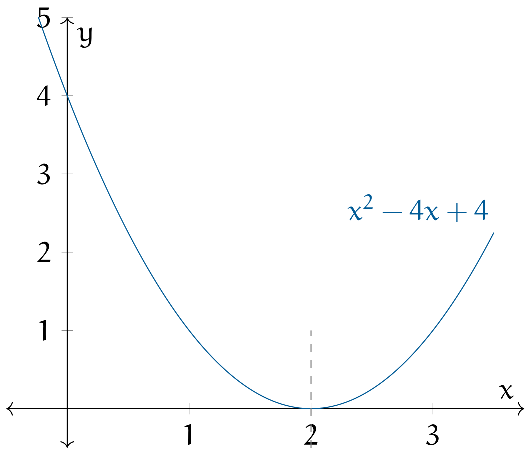
What does this mean?

For any b and c, the graph of $x^2 + bx + c$ is a parabola that goes up on each end. Its low point is at $x = -\frac{b}{2}$.

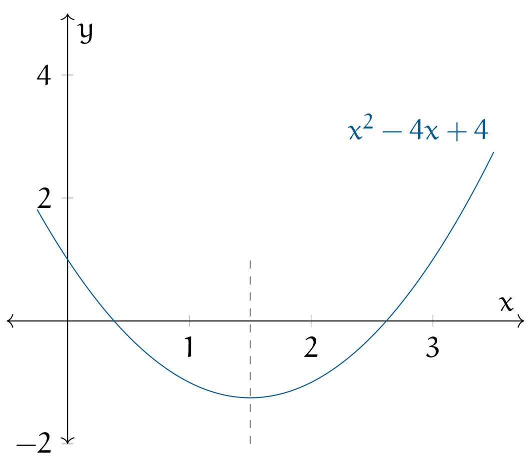
If there are no real roots ($b^2 - 4c < 0$), which means the parabola never gets low enough to cross the x-axis:



If there is one real root ($b^2 - 4c = 0$), it means that the parabola just touches the x-axis.



If there are two real roots ($b^2 - 4c > 0$), it means that the parabola crosses the x-axis twice as it dips below and then returns:



Exercise 42 Roots of a Quadratic

In the last chapter, you found that the function for the height of your flying hammer is:

$$p = -\frac{1}{2}9.8t^2 + 12t + 2$$

At what time will the hammer hit the ground?

Working Space

Answer on Page 401

24.1 The Traditional Quadratic Formula

If the last explanation was a little tricky to understand the quadratic formula is a nifty tool.

The Quadratic Formula

$ax^2 + bx + c = 0$ when

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

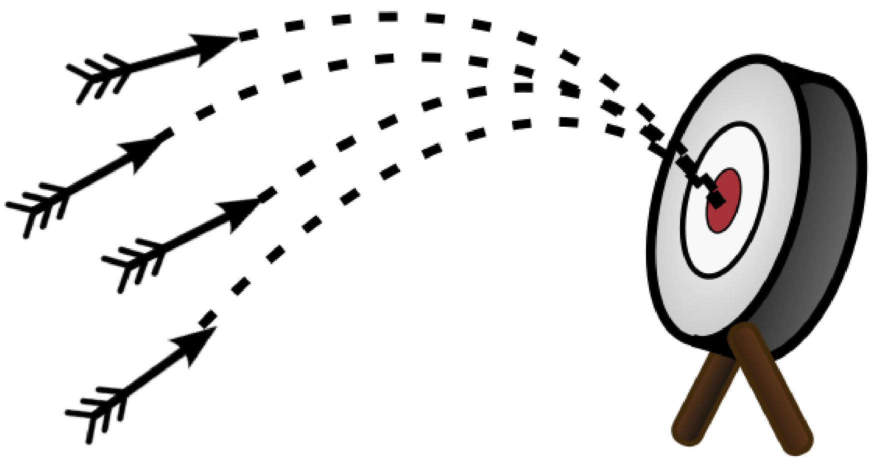
Chapter 25

Drag

The very first computers were created to do calculations of how artillery would fly when shot at different angles. The calculations were similar to the ones you just did for the flying hammer with two important differences:

- They were interested in two dimensions: the height and the distance across the ground.

- However, artillery flies a lot faster than a hammer, so they had to worry about drag from the air.



25.1 Wind resistance

The first thing they did was put one of the shells in a wind tunnel. They measured how much force was created when they pushed 1 m/s of wind over the shell. Let's say it was 0.1 newtons.

One of the interesting things about the drag from the air (often called *wind resistance*) is that it increases with the *square* of the speed. Thus, if the wind pushing on the shell is 3 m/s, instead of 1 m/s, the resistance is $3^2 \times 0.1 = 0.9$ newtons.

(Why? Intuitively, three times as many air molecules are hitting the shell and each molecule is hitting it three times harder.)

So, if a shell is moving with the velocity vector v, the force vector of the drag points in the exact opposite direction. If μ is the force of wind resistance of the shell at 1 m/s, then the magnitude of the drag vector is $\mu |v|^2$.

25.2 Initial velocity and acceleration due to gravity

Let's say a shell is shot out of a tube at s m/s, and let's say the tube is tilted θ radians above level. Then, the initial velocity will be given by the vector $[s\cos(\theta), s\sin(\theta)]$

(The velocity of the shell is actually a 3-dimensional vector, but we are only going to worry about height and horizontal distance; we are assuming that the operator pointed it in the right direction.)

To figure out the path of the shell, we need to compute its acceleration. We remember that

$$F = ma$$

(Note that F and a are vectors.) Dividing both sides by m we get:

$$a = \frac{F}{m}$$

So let's figure out the net force on the shell so that we can calculate the acceleration vector.

If the shell has a mass of b, the force due to gravity will be in the downward direction with a magnitude of $9.8b$ newtons.

To get the net force, we will need to add the force due to gravity with the force due to wind resistance.

25.3 Simulating artillery in Python

Create a file called `artillery.py`.

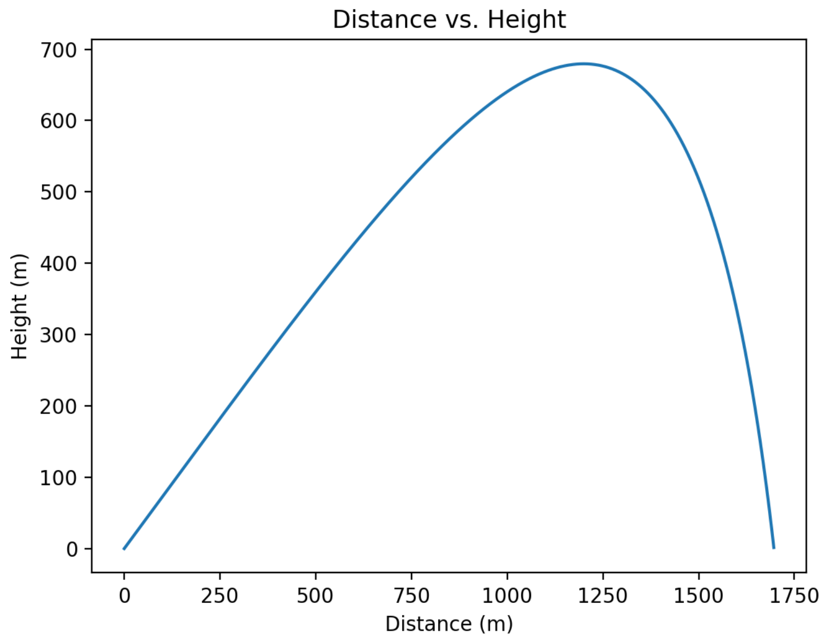
```
import numpy as np
import matplotlib.pyplot as plt

# Constants
mass = 45 # kg
start_speed = 300.0 # m/s
theta = np.pi/5 # radians (36 degrees above level)
time_step = 0.01 # s
wind_resistance = 0.05 # newtons in 1 m/s wind
force_of_gravity = np.array([0.0, -9.8 * mass]) # newtons

# Initial state
position = np.array([0.0, 0.0]) # [distance, height] in meters
velocity = np.array([start_speed * np.cos(theta), start_speed * np.sin(theta)])
time = 0.0 # seconds

# Lists to gather data
distances = []
heights = []
times = []
```

```
    # While shell is aloft
    while position[1] >= 0:
        # Record data
        distances.append(position[0])
        heights.append(position[1])
        times.append(time)

        # Calculate the next state
        time += time_step
        position += time_step * velocity

        # Calculate the net force vector
        force = force_of_gravity - wind_resistance * velocity**2

        # Calculate the current acceleration vector
        acceleration = force / mass

        # Update the velocity vector
        velocity += time_step * acceleration

    print(f"Hit the ground {position[0]:.2f} meters away at {time:.2f} seconds.")

    # Plot the data
    fig, ax = plt.subplots()
    ax.plot(distances, heights)
    ax.set_title("Distance vs. Height")
    ax.set_xlabel("Distance (m)")
    ax.set_ylabel("Height (m)")
    plt.show()
```

When you run it, you should get a message like:

```
Hit the ground 1696.70 meters away at 20.73 seconds.
```

You should also see a plot of the shell's path:



25.4 Terminal velocity

If you shot the shell very, very high in the sky, it would keep accelerating toward the ground until the force of gravity and the force of the wind resistance were equal. The speed at which this happens is called the *terminal velocity*. The terminal velocity of a falling human is about 53 m/s.

Exercise 43 Terminal velocity

What is the terminal velocity of shell described in our example?

Working Space

Answer on Page 402

Chapter 26

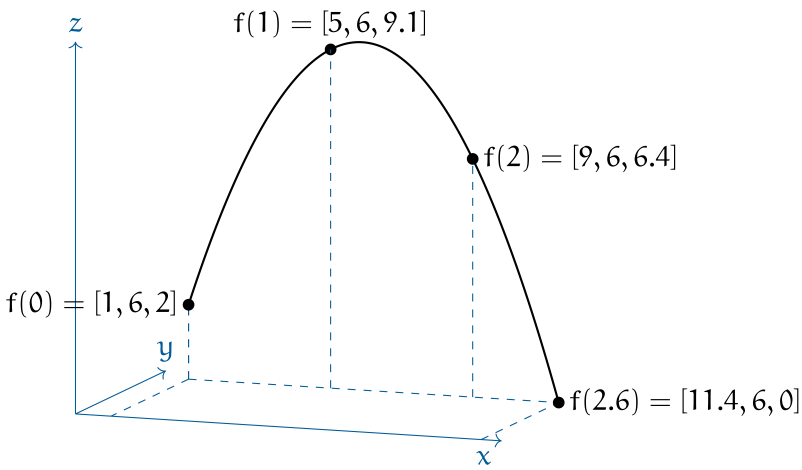
Vector-valued Functions

In the last chapter, you calculated the flight of the shell. For any time t, you could find a vector $[\mathtt{distance}, \mathtt{height}]$. This can be thought of as a function f that takes a number and returns a 2-dimensional vector. We call this a *vector-valued* function from $\mathbb{R} \to \mathbb{R}^2$.

We often make a vector-valued function by defining several real-valued functions. For example, if you threw a hammer with an initial upward speed of 12 m/2 and a horizontal speed of 4 m/s along the x axis from the point $(1, 6, 2)$, its position at time t (during its flight) would be given by:

$$f(t) = [4t + 1, 6, -4.8t^2 + 12t + 2]$$

That is, x is increasing with t, y is constant, and z is a parabola.

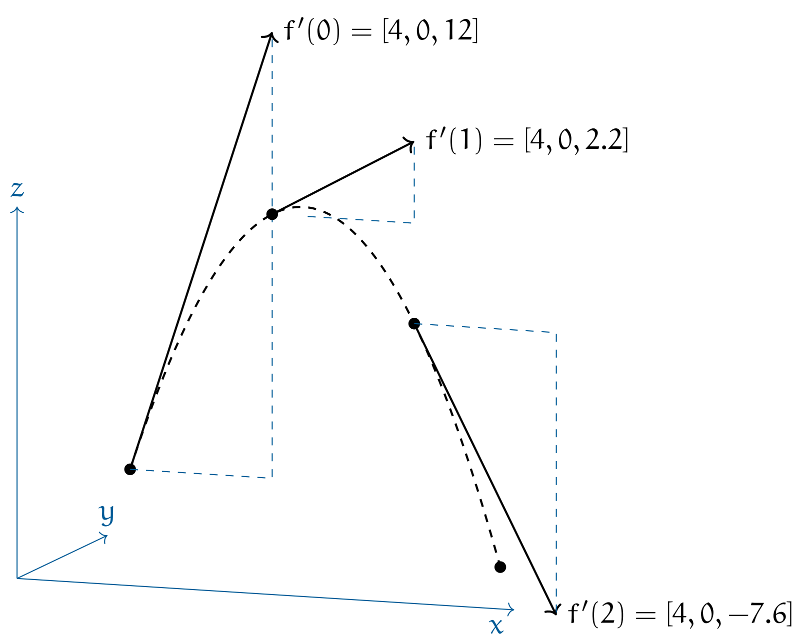


26.1 Finding the velocity vector

Now that we have its position vector, we can differentiate each component separately to get its velocity as a vector-valued function:

$$f'(t) = [4, 0, -9.8t + 12]$$

That is, the velocity is constant along the x-axis, zero along the y-axis, and decreasing with time along the z axis.

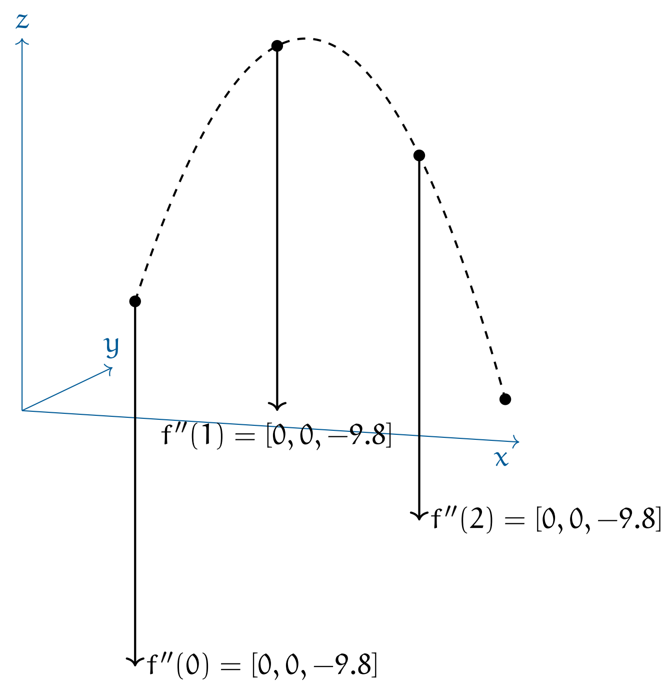


26.2 Finding the acceleration vector

Now that we have its velocity, we can get its acceleration as a vector-valued function:

$$f''(t) = [0, 0, -9.8]$$

There is no acceleration along the x or y axes. It is accelerating down at a constant $9.8m/s^2$.



Chapter 27

Fertilizer

In 1950, there were 2.5 billion people on the planet, and about 65% were malnourished. In 2019, there were 7.7 billion people on the planet, and only 15% are malnourished. How did crop yields increase so much? There were several factors: better crop varieties, reliable irrigation, increased mechanization, and affordable fertilizers.

When a plant grows, it takes molecules out of the soil and uses them to build proteins. It primarily needs the elements nitrogen (N), phosphorus (P), and potassium (K).

When you buy a bag of fertilizer at the store, it typically has three numbers on the front. For example, you might buy a bag of "24-22-4". This means that 24% of the mass of the bag is nitrogen, 22% is phosphorus, and 4% is potassium.

Potassium comes as potassium carbonate (K_2CO_3), potassium chloride (KCl), potassium sulfate (K_2SO_4), and potassium nitrate (KNO_3). Any blend of these chemicals is known as "potash". Potash is dug up out of mines.

Phosphorus is also mined, but is refined into phosphoric acid (H_3PO_4) before it is put into fertilizer.

Nitrogen is an especially interesting case for 2 reasons:

- Worldwide farmers apply more nitrogen to their soil than potassium or phosphorous combined.
- 78% of the air we breathe is nitrogen in the form of N_2, but neither plants nor animals can utilize nitrogen in that form.

27.1 The Nitrogen Cycle

Converting the N_2 in the air into a form that a plant can use is known as *nitrogen fixation*. For billions of years, there were only two ways that nitrogen fixation occurred on earth:

- The energy from lightning causes N_2 and H_2O to reconfigure as ammonia (NH_3) and nitrate (NO_3). This accounts for about 10% of all naturally occurring nitrogen fixation.
- Cyanobacteria are responsible for the rest. They convert N_2 into ammonia.

Let's say that you are eating soybeans. There is a cyanobacteria called *rhizobia* that has a symbiotic relationship with soybean plants. Rhizobia fixes nitrogen for the soybean plant. The soybean plant performs photosynthesis and gives sugars to the rhizobia.

The proteins in the soybeans contain nitrogen from the rhizobia. When you eat them, you use some of the nitrogen to build new proteins. You probably don't use all the nitrogen, so your cells release ammonia into your blood.

Ammonia likes to react with things, so your liver combines the ammonia with carbon dioxide to make urea $(CO(NH_2)_2)$. Your kidneys take the urea out of your blood and mix it with a bunch of water and salts in your bladder. When you urinate, the urea leaves your body.

If you urinate on the ground, the nearby plants can take the nitrogen out of the urea.

When you die, the nitrogen in your proteins will return to the soil as ammonia and nitrate.

For centuries, farms got their nitrogen from urine, feces, and rotting organic material. There were two challenges with this:

- Human pathogens had to be kept away from human food.
- There was simply not enough to support 7.7 billion people.

So we had to figure out how to do nitrogen fixation at an industrial level.

27.2 The Haber-Bosch Process

During World War I, two German scientists, Fritz Haber and Carl Bosch figured out how to make ammonia from N_2 and H_2 using high temperatures and pressures. This is how nearly all nitrogen fertilizer is created today.

Where do we get the H_2? From methane (CH_4) in natural gas. Today, 3-5% of the world's natural gas production is consumed in the Haber-Bosch process.

The ammonia is converted into ammonium nitrate (NH_4NO_3) or urea before it is shipped to farms.

27.3 Other nutrients

Healthy plants require several other elements that are sometimes applied as fertilizer: calcium, magnesium, and sulfur.

Finally, tiny amounts of copper, iron, manganese, molybdenum, zinc, and boron are sometimes needed.

Chapter 28

Concrete

To make concrete, you mix cement with water and an aggregate (sand or rock). The cement is usually only about 10 to 15 percent of the mixture. The cement reacts with the water, and the resulting solid binds the aggregate together. In 2019, the world consumed 4.5 billion tons of cement.

Concrete is hard and durable. The mortar between the pyramids at Giza is concrete – it is now 5000 years old. Today we use concrete to build many structures including buildings, bridges, airport runways, and dams.

There are many kinds of cement, but the most common is Portland cement. It is made by heating limestone (calcium carbonate) with clay (for silicon) in a kiln. Two things come out of the kiln: Carbon dioxide and a hard substance called "clinker". The clinker is ground up with some gypsum before it is sent to market.

The carbon dioxide is released into the atmosphere. Cement manufacture is responsible for about 8% of the world's CO_2 emissions; it is a major contributor to climate change.

Really hard concrete, like that used in a nuclear power plant, can support 3,000 kg per centimeter without being crushed. However, if you pull on two ends of a piece of concrete it comes apart pretty easily. We say that concrete can handle a lot of *compressive stress*, but not much *tensile* stress.

28.1 Steel reinforced concrete

Many places where we use concrete (like in a bridge), we need both compressive and tensile stress. Often the top of a beam is undergoing compression and the bottom of the beam is undergoing tension.

FIXME Picture here

Steel has tremendous tensile strength, but not as much compressive strength as concrete. To get both tensile *and* compressive strength, we often bury steel bars or cables inside the concrete. This is known as *steel-reinforced concrete*. The concrete generally does a very good job protecting the steel, which keeps it from rusting.

You may have heard of *rebar*. That is just short for "reinforcing bar". Typically rebar has bumps and ridges that keep the bar and the concrete from moving independently.

28.2 Recycling concrete

A lot of concrete structures only last about 100 years. When they are demolished, the concrete can be reused as aggregate in other projects. Often the concrete bits are mixed with cement and made into concrete again.

If the concrete to be reused is reinforced with steel, the steel has to be removed and recycled separately. Then the concrete is crushed into small pieces.

Chapter 29

Metals

Elements that transmit electricity well, even at low temperatures, are called *metals*. Here are some metals that you are probably familiar with: aluminum, iron, copper, tin, gold, silver, and platinum. Aluminum and iron are particularly common; together they make up about 14% of the earth's crust.

An *alloy* is a mixture of elements that includes at least one metal. Brass, for example, is an alloy of copper and zinc. Bronze is an alloy of copper and tin.

29.1 Steel

One of the most common alloys is steel, an alloy of iron and carbon. In pure iron, the molecules slip easily past each other, so pure iron is relatively soft and easily deformed. The carbon in steel prevents that slipping, thus steel is much, much harder than iron.

How much carbon? If you put less than 0.002% by weight, you end up with something very much like pure iron. As you increase the carbon, it gets harder and harder. Once it gets above about 2%, the result is very brittle.

If you add about 11% chromium to steel, you get *stainless steel* which resists rusting.

Exercise 44 Tensile Strength

The tensile strength of steel is usually between 400 MPa and 1200 MPa. A Mega Pascal (MPa) is the strength necessary to hold 1,000,000 newtons of force with a cable that has a 1 square meter cross section. Or, equivalently, to hold 1 newton of force with a cable that has a 1 square millimeter cross section.

If you have are buying a round cable that has a tensile strength of 700 Mpa and must hold a 100 kg man aloft, what the diameter of the smallest cable you can use?

Working Space

Answer on Page 402

Here are some approximate tensile strengths of ather materials:

Material	Tensile strength (MPa)
Iron	3
Concrete	4
Rubber	16
Glass	33
Wood	40
Nylon	100
Human hair	200
Aluminum	300
Steel	700
Spider webs	1000
Carbon fiber	4000

29.2 What metal for what task?

You will see copper used a lot for electrical wires in your house and appliances because it is very efficient at moving electricity (very little power is lost as heat). It is also very good a transmitting heat, so you will often see copper pots and pans.

Aluminum is less dense than copper, and is still a pretty good conductor of electricity. Thus, the overhead wires in a power system are often made of aluminum.

Aluminum is not as strong as steel, but considerably lighter. It is often used structurally where weight is a concern: skyscrapers, cars, airplanes, and ships.

Titanium is about as strong as steel, but it weights about half as much. Titanium is very difficult to work with, so it is used in places where weight and strength are very important and cost is not: airplanes and bicycles.

(Carbon fiber, which is light, strong, and very easy to work with, is replacing aluminum and titanium in many applications. 20 years ago, many expensive bicycles were made of titanium. These days the vast majority are made with carbon fiber.)

Zinc and tin are very resistant to corrosion, so they are often used as a coating to prevent steel from rusting. They are also used in many alloys for the same reason. In the United States, the penny is 97.5% zinc and only 2.5% copper.

Chapter 30

Introduction to Spreadsheets

For many real-world problems, spreadsheets are the perfect tool. In this chapter, you will be introduced to how to use a spreadsheet. There are numerous spreadsheet programs: Google Sheets, Microsoft Excel, Apple Numbers, OpenOffice Calc, etc. All of them are very similar. This instruction will use Google Sheets, but if you are using one of the others, you should be able to follow along.

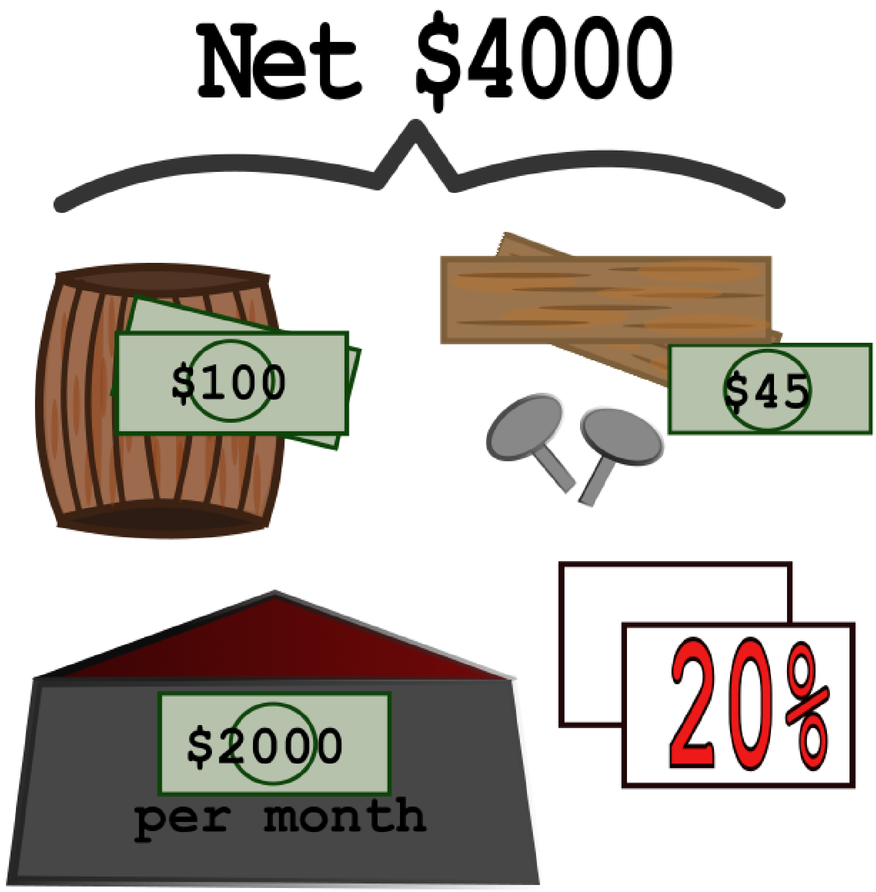
The first spreadsheet program (VisiCalc) was introduced in 1979 as a tool for finance people to play "what if" games. For example, a company might make a spreadsheet that told them how much more profit they would make if they changed from using an expensive metal to using a cheaper alloy.

In honor of its history, let's start by studying a business question: I have a friend who dreams of quitting her job to become a cooper. (A cooper makes barrels that are used for aging wine and whiskey.) She says:

- It costs $45 dollars in materials to build one barrel.
- A barrel sells for $100 dollars.

- The workshop/warehouse she wants to rent costs \$2000 per month.
- Taxes take 20% of her profits.
- She needs to make \$4000 monthly after taxes.

She has asked you, "How many barrels do I need to make each month?"



30.1 Solving It Symbolically

Many problems can be solved two ways: symbolically or numerically. To solve this problem symbolically, you would write out the facts as equations or inequalities and then do symbol manipulations until you ended up with an answer. In this case, you would let b be the number of barrels and create the following inequality:

$$(1.0 - 0.2)\,(b(100 - 45) - 2000) \geq 4000$$

You would simplify it:

$$(0.8)\,(55b - 2000) \geq 4000$$

And simplify it more:

$$44b - 1600 \geq 4000$$

If that is true, then:

$$44b \geq 5600$$

And if that is true, then:

$$b \geq \frac{1400}{11}$$

1400/11 is about 127.27, so she needs to make and sell 128 barrels each month.

That is a perfect answer, and we didn't need a spreadsheet at all. Two things:

- As problems get larger and more realistic, it gets much more difficult to solve them symbolically.
- As soon as you say "Yes, you need to make and sell 128 barrels each month." Your friend will ask "What if I make and sell 200 barrels? How much money will I make then?"

So we use a spreadsheets to solve the problem numerically.

30.2 Solving It Numerically (with a spreadsheet)

Let's get back to our example. Put labels in the A column:

- Barrels produced (per month)

- Materials cost (per barrel)
- Sale price (per barrel)
- Pre-tax earnings (per month)
- Taxes (per month)
- Take home pay (per month)

Format them any way you like. It should look something like this:

	A
1	**Barrels Produced** (per month)
2	**Materials cost** (per barrel)
3	**Sale price** (per barrel)
4	**Rent** (per month)
5	**Pretax Earnings** (per month)
6	**Taxes** (per month)
7	**Take home pay** (per month)
8	

In the B column, the first four cells are values (not formulas):

- 115 formatted as a number with no decimal point
- 45 formatted as currency
- 100 formatted as currency
- 2000 formatted as currency

It should look something like this:

	A	B
1	**Barrels Produced** (per month)	115
2	**Materials cost** (per barrel)	$45.00
3	**Sale price** (per barrel)	$100.00
4	**Rent** (per month)	$2,000.00

The next three cells in the B column will have formulas:

- B1 * (B3 - B2) - B4
- 0.2 * B5
- B5 - B6

It should look something like this:

	A	B
1	**Barrels Produced** (per month)	115
2	**Materials cost** (per barrel)	$45.00
3	**Sale price** (per barrel)	$100.00
4	**Rent** (per month)	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00
6	**Taxes** (per month)	$865.00
7	**Take home pay** (per month)	$3,460.00
8		

Now you can share this spreadsheet with your friend and she can put different values into the cells for what-if games. Like "If I can get my materials cost down to $42 per barrel, what happens to my take home pay?"

Sometimes it is nice to show a range of values for a variable or two. In this case, it might be nice to show your friend what the numbers look like if she produces 115, 120, 125, 130, 135, or 140 barrels per month.

We have one column, and now we need six. How do we duplicate cells?

1. Click B1 to select it and then shift-click on B7 to select all seven cells.
2. Copy them. (There is probably a menu item for this.)
3. Click C1 to select it
4. Paste them.

	A	B	C
1	**Barrels Produced** (per month)	115	115
2	**Materials cost** (per barrel)	$45.00	$45.00
3	**Sale price** (per barrel)	$100.00	$100.00
4	**Rent** (per month)	$2,000.00	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00	$4,325.00
6	**Taxes** (per month)	$865.00	$865.00
7	**Take home pay** (per month)	$3,460.00	$3,460.00
8			

We want the first cell in the new column to be 120. You could just type in 120, but let's do something more clever. Put a formula into that cell: $= B1 + 5$. Now the cell should show 120.

Why did we put in a formula? When we duplicate this column, this cell will always have 5 more barrels than the cell to its left.

Now let's duplicate the second column a few times. The easy way to do this is to select the cells as you did before and drag the lower-right corner to the right until column G is in the selection. When you end the drag, the copies will appear:

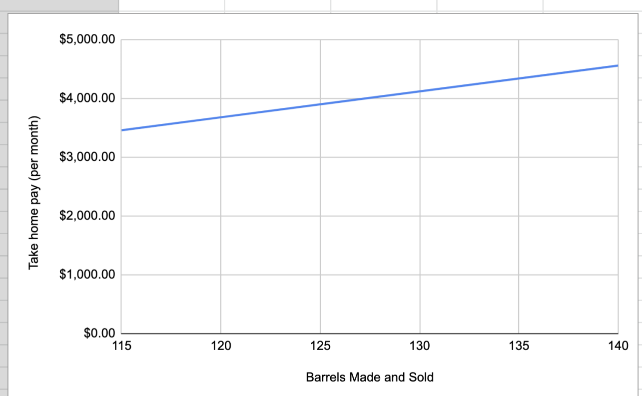
	A	B	C	D	E	F	G
1	**Barrels Produced** (per month)	115	120	125	130	135	140
2	**Materials cost** (per barrel)	$45.00	$45.00	$45.00	$45.00	$45.00	$45.00
3	**Sale price** (per barrel)	$100.00	$100.00	$100.00	$100.00	$100.00	$100.00
4	**Rent** (per month)	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00	$4,600.00	$4,875.00	$5,150.00	$5,425.00	$5,700.00
6	**Taxes** (per month)	$865.00	$920.00	$975.00	$1,030.00	$1,085.00	$1,140.00
7	**Take home pay** (per month)	$3,460.00	$3,680.00	$3,900.00	$4,120.00	$4,340.00	$4,560.00

Nice, right? Now your friend can easily see how many barrels correspond to how much take-home pay. Do you know what would be really helpful? A graph.

30.3 Graphing

Graphing is a little different on every different platform. Here is what you want the graph to look like.

	A	B	C	D	E	F	G
1	**Barrels Produced** (per month)	115	120	125	130	135	140
2	**Materials cost** (per barrel)	$45.00	$45.00	$45.00	$45.00	$45.00	$45.00
3	**Sale price** (per barrel)	$100.00	$100.00	$100.00	$100.00	$100.00	$100.00
4	**Rent** (per month)	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00	$2,000.00
5	**Pretax Earnings** (per month)	$4,325.00	$4,600.00	$4,875.00	$5,150.00	$5,425.00	$5,700.00
6	**Taxes** (per month)	$865.00	$920.00	$975.00	$1,030.00	$1,085.00	$1,140.00
7	**Take home pay** (per month)	$3,460.00	$3,680.00	$3,900.00	$4,120.00	$4,340.00	$4,560.00

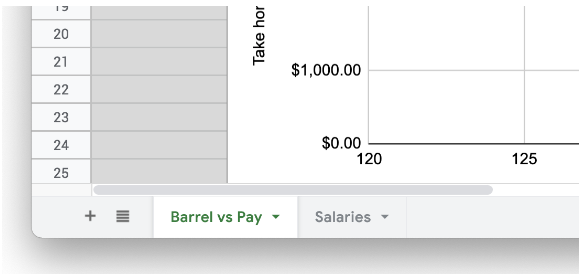


On Google Sheets:

1. Select cells B7 through G7.
2. Choose the menu item Insert -> Chart.
3. Choose the chart type (Line)
4. Add the X-axis to be B1 through G1.
5. Under the Customize tab, Set the label for the X-axis to be “Barrels Made and Sold”.
6. Delete the chart title (which is the same as the Y-axis label).

30.4 Other Things You Should Know About Spreadsheets

Your spreadsheet document can have several "Sheets". Each has its own grid of cells. The sheet has a name; usually, you call it something like "Salaries". When you need to use a value from the "Salaries" sheet in another sheet, you can specify "Salaries!A2" – that is, cell A2 on sheet "Salaries". To flip between the sheets there is usually a tab for each at the bottom of the document.



By default, the cell references are relative. That is when you write a formula in cell H5 that references the value in cell G4, the cell remembers "The cell that is one up and one to the left of me." Thus, if you copy that formula into B9, now that formula reads the value from A8.

If you want an absolute reference, you use $. If H5 references G4, G4 will be used no matter where on the sheet the formula is copied to.

You can use the $ on the row or column. In $A4, the column is absolute and the row is relative. In A$4, the row is absolute and the column is relative.

30.5 Challenge: Make a spreadsheet

You have a company that bids on painting jobs. Make a spreadsheet to help you do bids. Here are the parameters:

- The client will tell you how many square meters of wall needs to be painted.
- Paint costs $0.02 per square meter of wall
- On average, a square meter of wall takes 0.02 hours to paint.
- You can hire painters at $15 per hour.
- You add 20% to your estimated costs for a margin of error and profit.

Make a spreadsheet such that when you type in the square meters to be painted, the spreadsheet tells you how much you will spend on paint and labor. It also tells you what your bid should be.

CHAPTER 31

Compound Interest

When you loan money to someone, you typically charge them some sort of interest. The most common loan of this sort is what the bank calls a "savings account". Any money you put in the account is loaned to the bank. The bank then lends it to someone else, who pays interest to the bank. And the bank gives some of that interest to you. However, what if you leave the interest in your account? And you start making *interest on the interest*? This is known as *compound interest*.

31.1 An example with annual interest payments

Let's say that you put $1000 in a savings account that pays 6% interest every year. How much money would you have after 12 years? Let's make a spreadsheet.

	A	B	C
1	**Interest Rate**	6.00%	
2			
3	**After year:**	**Interest**	**Balance**
4	0	$0.00	1000
5	1	$60.00	$1,060.00

Create a new spreadsheet and edit the cells to look like this. All the cells in rows 1 - 4 are just values: just type in what you see.

The fifth row is all formulas:

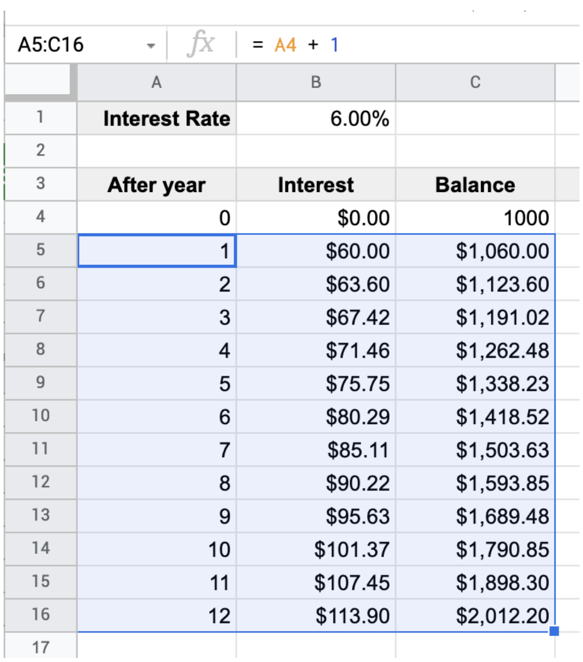
After year	Interest	Balance
= A4 + 1	= B$1 * C4	= C4 + B5

The interest rate field should be formatted as a percentage. One thing to know when dealing with percentages in the spreadsheet: if the field says "600%", its value is 6.

The cells in the Interest and Balance column should be formatted as currency.

You are about to make a bunch of copies of the cells in the fifth row, so make sure they look right.

Click on A5 and shift-click on C5 to select all three cells. Drag the lower-right corner down to fill the rows 6 - 15.



A5:C16 fx = A4 + 1

	A	B	C
1	Interest Rate	6.00%	
2			
3	After year	Interest	Balance
4	0	$0.00	1000
5	1	$60.00	$1,060.00
6	2	$63.60	$1,123.60
7	3	$67.42	$1,191.02
8	4	$71.46	$1,262.48
9	5	$75.75	$1,338.23
10	6	$80.29	$1,418.52
11	7	$85.11	$1,503.63
12	8	$90.22	$1,593.85
13	9	$95.63	$1,689.48
14	10	$101.37	$1,790.85
15	11	$107.45	$1,898.30
16	12	$113.90	$2,012.20
17			

Look at the numbers. The first interest payment is $60, but the last is $113.90. Your balance has more than doubled!

31.2 Exponential Growth

We figured this out numerically by repeatedly multiplying the balance by the interest rate. What if you wanted to know what the balance would be n years after investing P_0 dollars with an annual interest rate of r? (Note that r in our example would be 0.06, not 6.0.)

Each year, the balance is multiplied by $1+r$, so after one year, P_0 would become $P_0 \times (1+r)$. The next year you would multiply this number by $(1+r)$ again: $P_0 \times (1+r) \times (1+r)$. The next year? $P_0 \times (1+r) \times (1+r) \times (1+r)$ See the pattern? P_n is this balance after n years, then

$$P_n = P_0(1+r)^n$$

Because n is an exponent, we call this *exponential growth*. And there are few things as terrifying to a scientist as the phrase "The population is undergoing exponential growth".

31.3 Sensitivity to interest rate

For most people, the first surprising thing about compound interest is how quickly your money grows after a few years. The second thing that is surprising is how much difference a small change in the percentage rate makes.

Let's add another set of columns that shows what happens to your money if you convince the bank to pay you 8% instead of 6%.

Copy everything from columns B and C:

	A	B	C	D	E
1	Interest Rate	6.00%		6.00%	
2					
3	After year	Interest	Balance	Interest	Balance
4	0	$0.00	1000	$0.00	1000
5	1	$60.00	$1,060.00	$60.00	$1,060.00
6	2	$63.60	$1,123.60	$63.60	$1,123.60
7	3	$67.42	$1,191.02	$67.42	$1,191.02
8	4	$71.46	$1,262.48	$71.46	$1,262.48
9	5	$75.75	$1,338.23	$75.75	$1,338.23
10	6	$80.29	$1,418.52	$80.29	$1,418.52
11	7	$85.11	$1,503.63	$85.11	$1,503.63
12	8	$90.22	$1,593.85	$90.22	$1,593.85
13	9	$95.63	$1,689.48	$95.63	$1,689.48
14	10	$101.37	$1,790.85	$101.37	$1,790.85
15	11	$107.45	$1,898.30	$107.45	$1,898.30
16	12	$113.90	$2,012.20	$113.90	$2,012.20
17					

Now edit the second interest rate to be 8%:

	A	B	C	D	E
1	**Interest Rate**	6.00%		8.00%	
2					
3	**After year**	**Interest**	**Balance**	**Interest**	**Balance**
4	0	$0.00	1000	$0.00	1000
5	1	$60.00	$1,060.00	$80.00	$1,080.00
6	2	$63.60	$1,123.60	$86.40	$1,166.40
7	3	$67.42	$1,191.02	$93.31	$1,259.71
8	4	$71.46	$1,262.48	$100.78	$1,360.49
9	5	$75.75	$1,338.23	$108.84	$1,469.33
10	6	$80.29	$1,418.52	$117.55	$1,586.87
11	7	$85.11	$1,503.63	$126.95	$1,713.82
12	8	$90.22	$1,593.85	$137.11	$1,850.93
13	9	$95.63	$1,689.48	$148.07	$1,999.00
14	10	$101.37	$1,790.85	$159.92	$2,158.92
15	11	$107.45	$1,898.30	$172.71	$2,331.64
16	12	$113.90	$2,012.20	$186.53	$2,518.17

Chapter 32

Introduction to Data Visualization

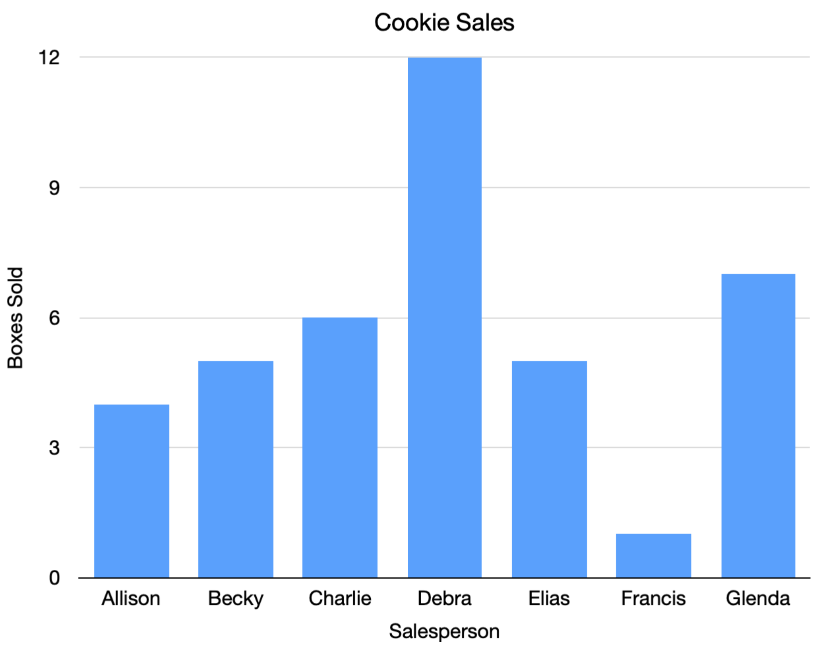
It is difficult for the human mind to look at a list of numbers and identify the patterns in them, so we often make pictures with the numbers. These pictures are called *graphs*, *charts*, or *plots*. Often the right picture can make the meaning in the data obvious. *Data visualization* is the process of making pictures from numbers.

32.1 Common Types of Data Visualizations

Depending on the type of data and what you are trying to demonstrate about it, you will use different types of data visualizations. How many types of data visualizations are there? Hundreds, but we will concentrate on just four: The bar chart, the line graph, the pie chart, and the scatter plot.

32.1.1 Bar Chart

Here is an example of a bar chart.



Each bar represents the cookie sales of one person. For example, Charlie has sold 6 boxes of cookies, so the bar goes over Charlie's name and reaches to the number 6.

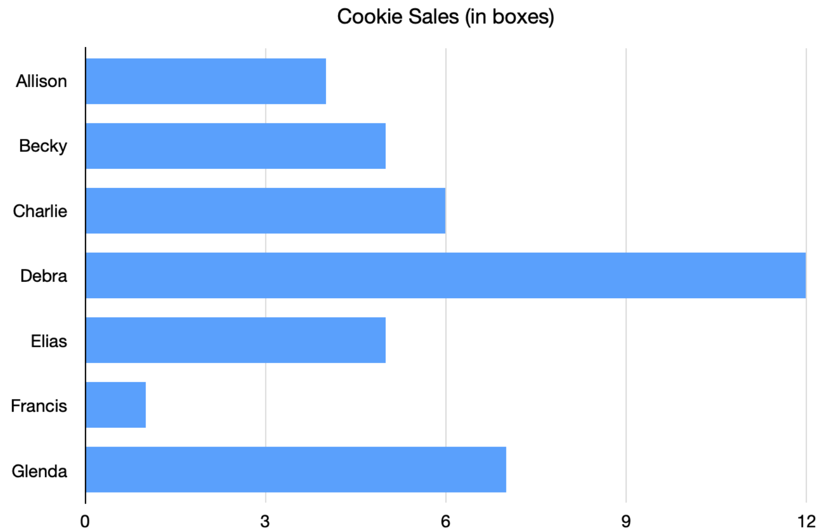
Looking at this chart, you probably think, "Wow, Debra has sold a lot more cookies than anyone else, and Francis has sold a lot fewer."

The same data could be in a table like this:

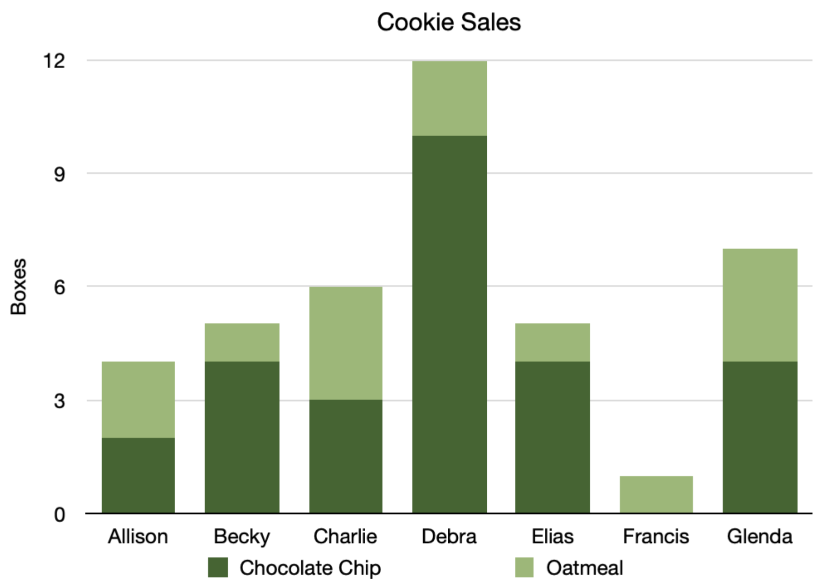
Salesperson	Boxes Sold
Allison	4
Becky	5
Charlie	6
Debra	12
Elias	5
Francis	1
Glenda	7

The table (especially a large table) is often just a bunch of numbers. A chart helps our brains understand what the numbers mean.

Bar charts can also go horizontally.



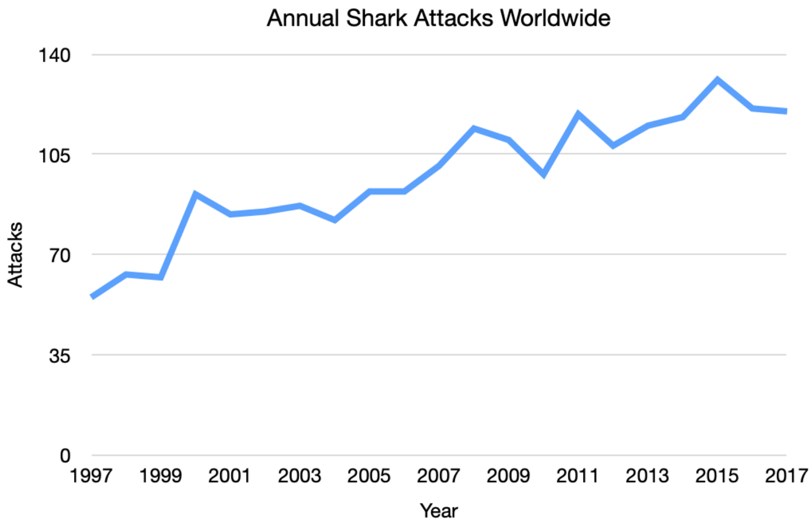
Sometimes we use colors to explain what contributed to the number.



This tells us that Becky sold more boxes of chocolate chip cookies than boxes of oatmeal cookies.

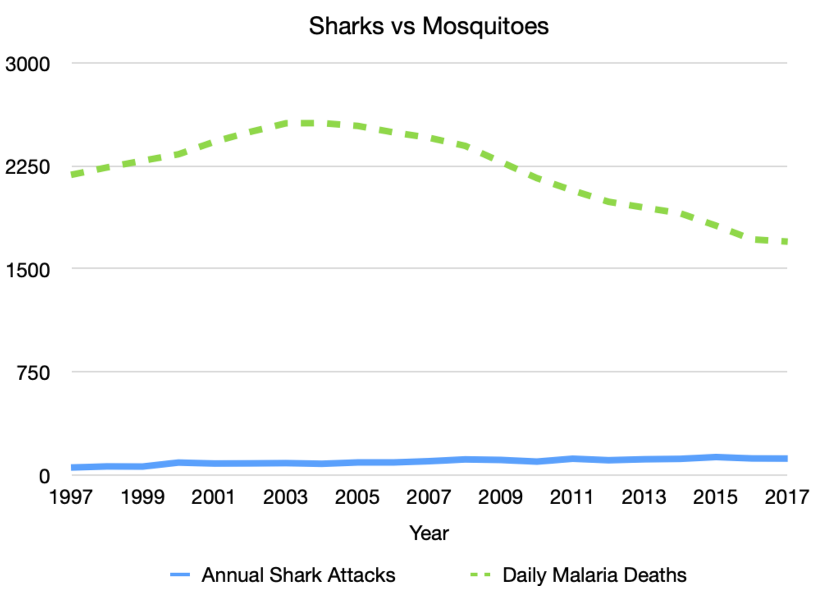
32.1.2 Line Graph

Here is a line graph.



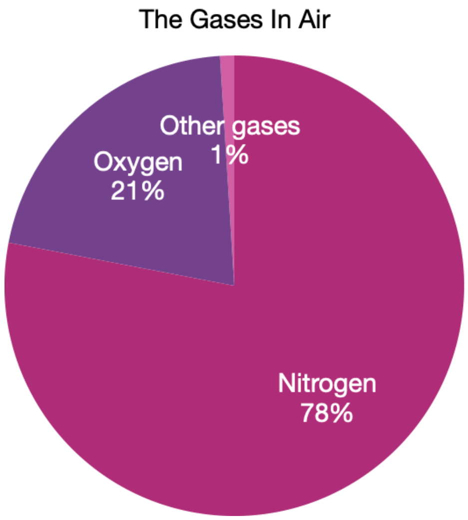
These are often used to show trends over time. Here, for example, you can see that the number of shark attacks has been increasing over time.

You can have more than one line on a graph.



32.1.3 Pie Chart

You use a pie chart when you are looking at the comparative size of numbers.



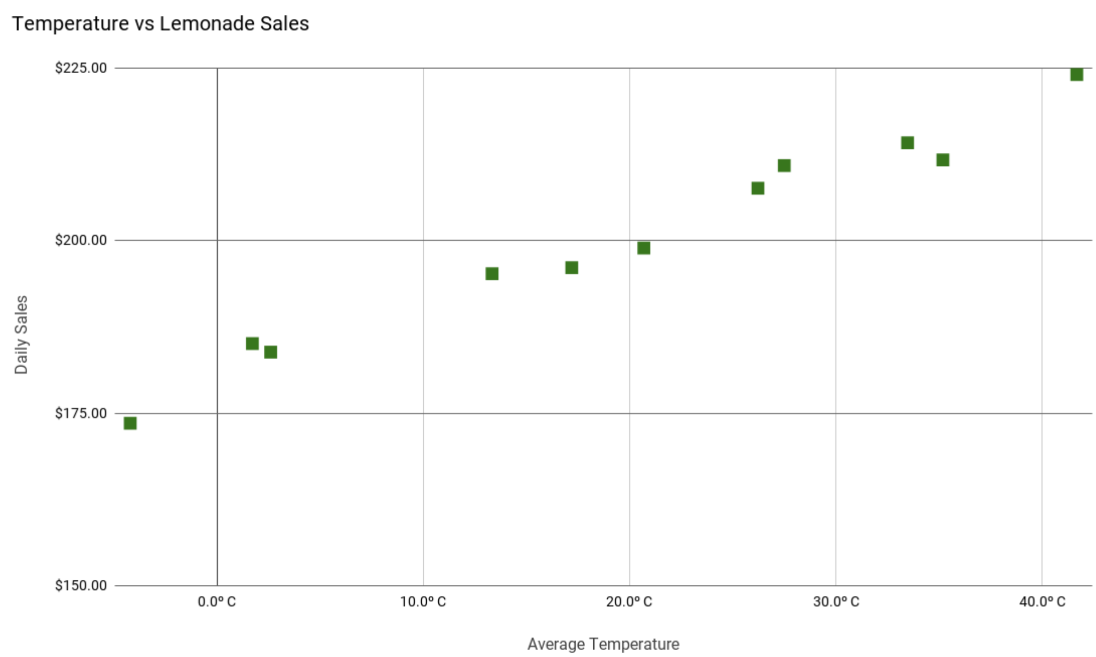
32.1.4 Scatter Plot

Sometimes you have a bunch of data points with two values and you are looking for a relationship between them. For example, maybe you write down the average temperature and the total sales for your lemonade stand on the 15th of every month:

Date	Avg. Temp.	Total Sales
15 January 2022	2.6º C	$183.85
15 February 2022	-4.2º C	$173.56
15 March 2022	13.3º C	$195.22
15 April 2022	26.2º C	$207.61
15 May 2022	27.5º C	$210.88
15 June 2022	31.3º C	$214.18
15 July 2022	33.5º C	$215.23
15 Aug 2022	41.7º C	$224.07
15 September 2022	20.7º C	$198.94
15 October 2022	17.2º C	$196.10
15 November 2022	1.7º C	$185.10
15 December 2022	0.2º C	$188.70

And you think "I wonder if I sell more lemonade on hotter days?"

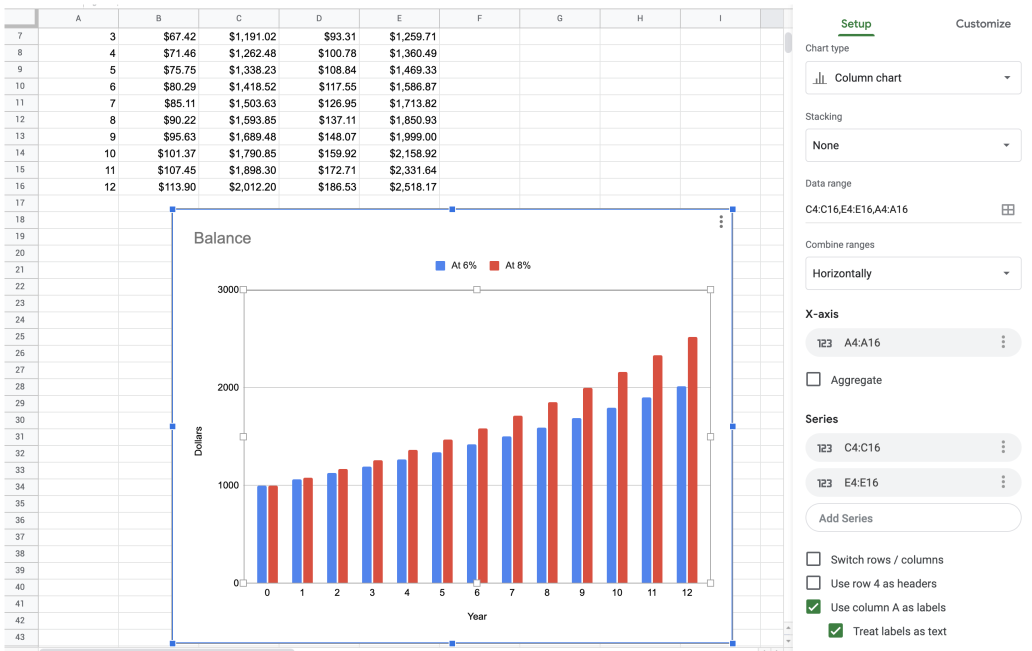
You might create a scatter plot. For each day, you put a mark that represents that temperature and the sales that day:



From this scatter plot, you can easily see that you do sell more lemonade as the temperature goes up.

32.2 Make Bar Graph

Go back to your compound interest spreadsheet and make a bar graph that shows both balances over time:



The year column should be used as the x-axis. There are two series of data that come from C4:C16 and E4:E16. Tidy up the titles and legend as much as you like.

Looking at the graph, you can see the balances start the same, but balance of the account with the larger interest rate quickly pulls away from the account with the smaller interest rate.

Chapter 33

Exponents

Let's quickly review exponents. Ancient scientists started coming up with a lot of formulas that involved multiplying the same number several times. For example, if they knew that a sphere was r centimeters in radius, its volume in milliliters was

$$V = \frac{4}{3} \times \pi \times r \times r \times r$$

They did two things to make the notation less messy. First, they decided that if two numbers were written next to each other, the reader would assume that meant "multiply them". Second, they came up with the exponent, a little number that was lifted off the baseline of the text, that meant "multiply it by itself". For example 5^3 was the same as $5 \times 5 \times 5$.

Now the formula for the volume of a sphere is written

$$V = \frac{4}{3}\pi r^3$$

Tidy, right? In an exponent expression like this, we say that r is *the base* and 3 is *the exponent*.

33.1 Identities for Exponents

What about exponents of exponents? What is $\left(5^3\right)^2$?

$$\left(5^3\right)^2 = (5 \times 5 \times 5)^2 = (5 \times 5 \times 5)(5 \times 5 \times 5) = 5^6$$

In general, for any a, b, and c:

$$\left(a^b\right)^c = a^{(bc)}$$

If you have $\left(5^3\right)\left(5^4\right)$ that is just $5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5$ or 5^7

The general rule is, for any a, b, and c

$$\left(a^b\right)\left(a^c\right) = a^{(b+c)}$$

Mathematicians *love* this rule, so we keep extending the idea of exponents to keep this rule true. For example, at some point, someone asked "What about 5^0?" According to the rule, 5^2 must equal $5^{(2+0)}$ which must equal $\left(5^2\right)\left(5^0\right)$. Thus, 5^2 must be 1. So mathematicians declared "Anything to the power of 0 is 1".

We don't typically assume that $0^0 = 1$. It is just too weird. So we say, that for any a not equal to zero,

$$a^0 = 1$$

What about $5^{(-2)}$? By our beloved rule, we know that $\left(5^{-2}\right)\left(5^5\right)$ must be equal to 5^3, right? So 5^{-2} must be equal to $\frac{1}{5^2}$.

We say, for any a not equal to zero and any b,

$$a^{-b} = \frac{1}{a^b}$$

This makes dividing one exponential expression by another (with the same base) easy:

$$\frac{a^b}{a^c} = a^{(b-c)}$$

We often say "cancel out" for this. Here I can "cancel out" x^2:

$$\frac{x^5}{x^2} = x^3$$

What about $5^{\frac{1}{3}}$? By the beloved rule, we know that $5^{\frac{1}{3}}5^{\frac{1}{3}}5^{\frac{1}{3}}$ must equal 5^1. Thus $5^{\frac{1}{3}} = \sqrt[3]{5}$.

We say, for any a and b not equal to zero and any c greater than zero,

$$a^{\frac{b}{c}} = a^b \sqrt[c]{a}$$

Before you go on to the exercises, note that the beloved rule demands a common base.

- We can combine these: $\left(5^2\right)\left(5^4\right) = 5^6$
- We cannot combine: $\left(5^2\right)\left(3^5\right)$

With that said, we note for any a,b, and c:

$$(ab)^c = (a^c)(b^c)$$

So, for example, if I were asked to simplify $\left(3^4\right)\left(6^2\right)$, I would note that $6 = 2 \times 3$, so

$$\left(3^4\right)\left(6^2\right) = \left(3^4\right)\left(3^2\right)\left(2^2\right) = \left(3^6\right)\left(2^2\right)$$

If these ideas are new to you (or maybe they have been forgotten), watch the Khan Academy's **Intro to rational exponents** video at `https://youtu.be/lZfXc4nHooo`.

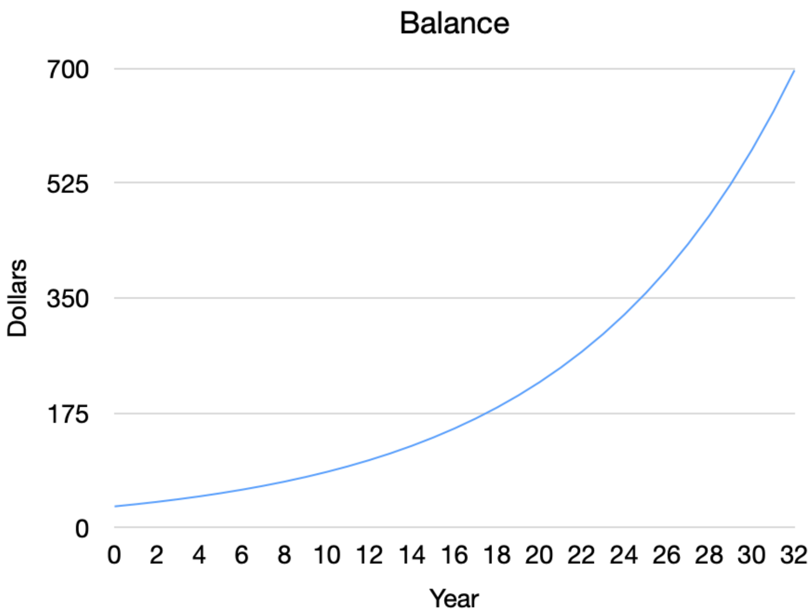
Chapter 34

Exponential Decay

In a previous chapter, we saw that an investment of P getting compound interest with an annual interest rate of r, grows exponentially. At the end of year t, your balance would be

$$P\,(1 + r)^t$$

Because r is positive, this number grows as time passes. You get a nice exponential growth curve that looks something like this:



This is \$30 invested with a 10% annual interest rate. So the formula for the balance after t years would be

$$(30)(1.1)^t$$

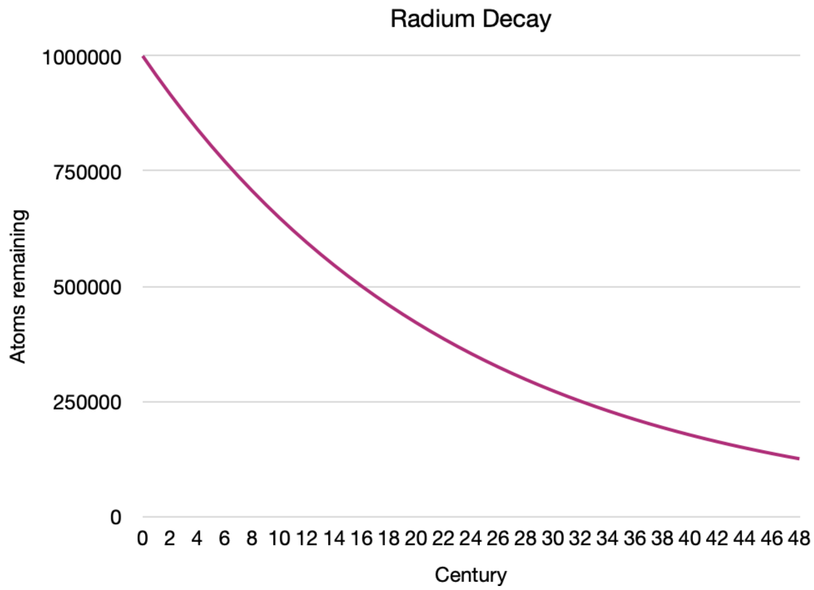
What if r were negative? This would be *exponential decay*.

34.1 Radioactive Decay

Until around 1970, there were companies making watches whose faces and hands were coated with radioactive paint. The paint usually contained radium. When a radium atom decays, it gives off some energy, loses two protons and two neutrons, and becomes becomes a different element (radon). Some of the energy given off is visible light. Thus, these watches glow in the dark.

How many of the radium atoms in the paint decay each century? About 4.24%.

Notice the quantity of atoms lost is proportional to the number of atoms you have. This is exponential decay. If we assume that we start with a million radium atoms, the number of atoms decreases over time like this:



- We start with 1,000,000 atoms.

- At 16 centuries, we have only 500,000 (half as many) left.

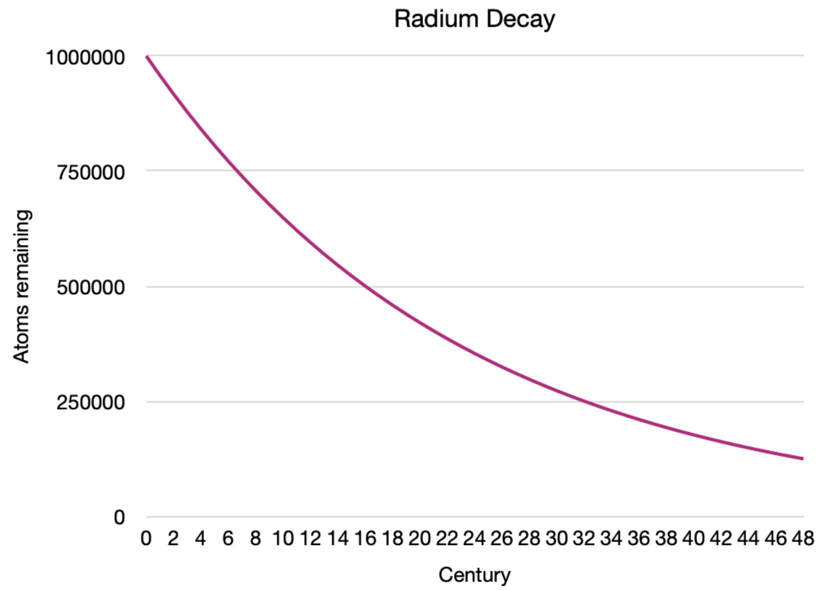
- 16 centuries after that, we have only 250,000 (half again) left.

- 16 centuries after that, we have only 125,000 (half again) left.

A nuclear chemist would say that radium has a *half-life* of 1,600 years. Note that this means that if you bought a watch with glowing hands in 1960, it will be glowing half as brightly in the year 3560.

How do we calculate the amount of radium left at the end of century t? If you start with P atoms, at the end of the t-th century you will have

$$P(1 - 0.0424)^t$$



This is exponential decay.

34.2 Model Exponential Decay

Let's say you get hired to run a company with 480,000 employees. Each year 1/8 of your employees leave the company for some reason (retirement, quitting, etc.). For some reason, you never hire any new employees.

Make a spreadsheet that indicates how many of the original 480,000 employees will still be around at the end of each year for the next 12. Then make a bar graph from that data.

Chapter 35

Logarithms

After the world had created exponents, it needed the opposite. We could talk about the quantity $? = 2^3$, that is, "What is the product of 2 multiplied by itself three times?" We needed some way to talk about $2^? = 8$, that is "2 to the what is 8?" So we developed the logarithm.

Here is an example:

$$\log_2 8 = 3$$

In English, you would say "The logarithm base 2 of 8 is 3."

The base (2, in this case) can be any positive number. The argument (8, in this case) can also be any positive number.

Try this one: What is the logarithm base 2 of 1/16?

You know that $2^{-4} = \frac{1}{16}$, so $\log_2 \frac{1}{16} = -4$.

35.1 Logarithms in Python

Most calculators have pretty limited logarithm capabilities, but python has a nice `log` function that lets you specify both the argument and the base. Start python, import the math module, and try taking a few logarithms:

```
>>> import math
>>> math.log(8,2)
3.0
>>> math.log(1/16, 2)
-4.0
```

Let's say that a friend offers you 5% interest per year on your investment for as long as you want. And you wonder, "How many years before my investment is 100 times as large?" You can solve this problem with logarithms:

```
>>> math.log(100, 1.05)
94.3872656381287
```

If you leave your investment with your friend for 94.4 years, the investment will be worth 100 times what you put in.

35.2 Logarithm Identities

The logarithm is defined this way:

$$\log_b a = c \iff b^c = a$$

Notice that the logarithm of 1 is always zero, and $\log_b b = 1$.

The logarithm of a product:

$$\log_b ac = \log_b a + \log_b c$$

This follows from the fact that $b^{a+c} = b^a b^c$. What about a quotient?

$$\log_b \frac{a}{c} = \log_b a - \log_b c$$

Exponents?

$$\log_b (a^c) = c \log_b a$$

Notice that because logs and exponents are the opposite of each other, they can cancel each other out:

$$b^{\log_b a} = a$$

and

$$\log_b (b^a) = a$$

35.3 Changing Bases

I mentioned that most calculators have pretty limited logarithm capabilities. Most calculators don't allow you to specify what base you want to work with. All scientific calculators have a button for "log base 10". So you need to know how to use that button to get logarithms for other bases. Here is the change-of-base identity:

$$\log_b a = \frac{\log_c a}{\log_c b}$$

So, for example, if you wanted to find $\log_2 8$, you would ask the calculator for $\log_{10} 8$ and then divide that by $\log_{10} 2$. You should get 3.

35.4 Natural Logarithm

When you learn about circles, you are told that the circumference of a circle is about 3.141592653589793 times its diameter. Because we use this unwieldy number a lot, we give it a name: We say "The circumference of a circle is π times its diameter."

There is a second unwieldy number that we will eventually use a lot in solving problems. It is about 2.718281828459045 (but the digits actually go on forever, just like π). We call this number e. (I'm not going to tell you why e is special now, but soon...)

Most calculators have a button labeled "ln". That is the *natural logarithm* button. It takes the log in base e.

Similarly, in python, if you don't specify a base, the logarithm is done in base e:

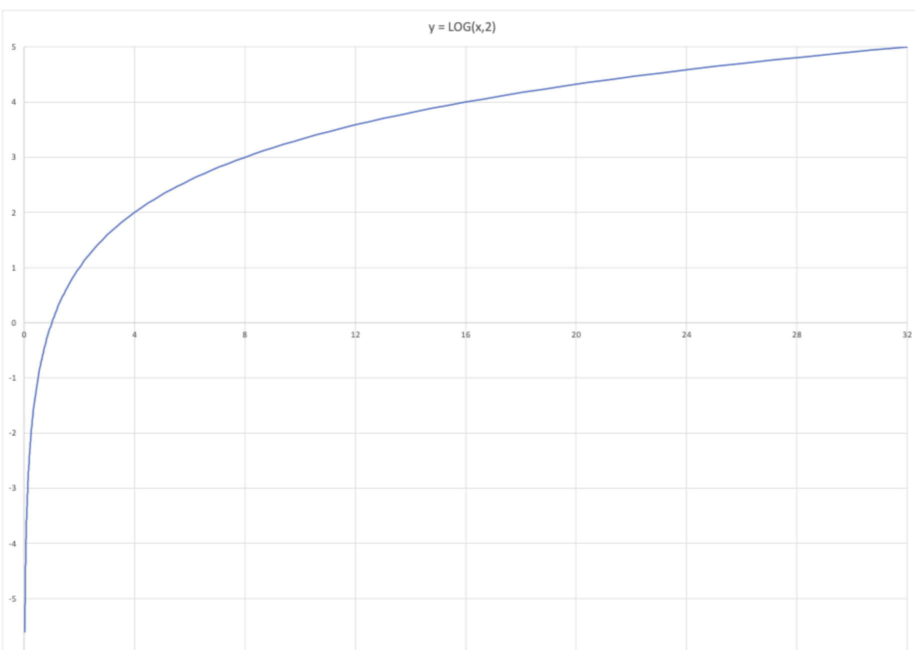
```
>>> math.log(10)
2.302585092994046
>>> math.log(math.e)
1.0
```

35.5 Logarithms in Spreadsheets

Spreadsheets have three log functions:

- `LOG` takes both the argument and the base. `LOG(8,2)` returns 3.
- `LOG10` takes just the argument and uses 10 as the base.
- `LN` takes just the argument and uses e as the base.

Here is a plot from a spreadsheet of a graph of $y = \mathrm{LOG}(x, 2)$.



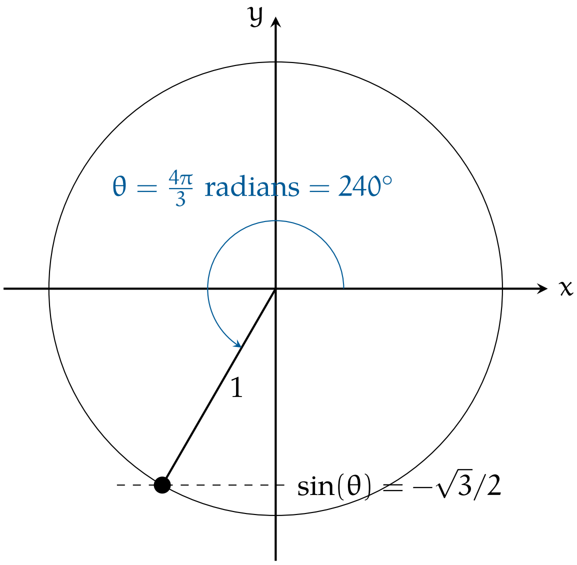
Spreadsheets also have the function `EXP(x)` which returns e^x. For example, `EXP(2)` returns 7.38905609893065.

Chapter 36

Trigometric Functions

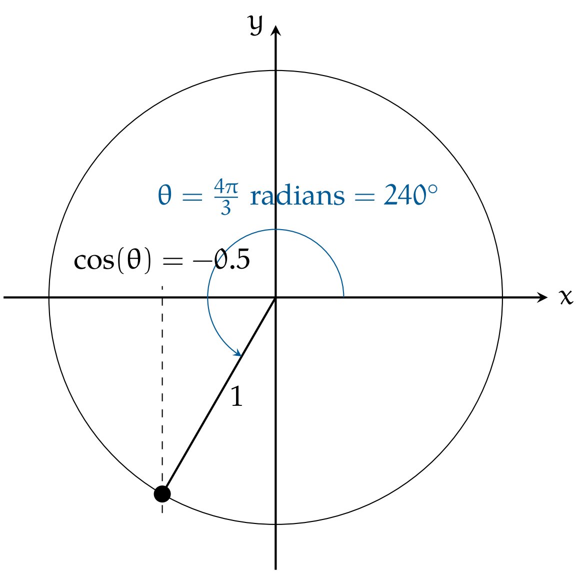
As mentioned earlier, in a right triangle where one angle is θ, the sine of θ is the length of the side opposite θ divided by the length of the hypotenuse.

The sine function is defined for any real number. We treat that real number θ as an angle, we draw a ray from the origin out to the unit circle. The y value of that point is the sine. So, for example, the $\sin(\frac{4\pi}{3})$ is $-\sqrt{3}/2$



(Note that in this section, we will be using radians instead of degrees unless otherwise noted. While degrees are more familiar to most people, engineers and mathematicians nearly always use radians when solving problems. Your calculator should have a radians mode and a degrees mode. You want to be in radians mode.)

Similarly, we define cosine using the unit circle: to find the cosine of θ, we draw a ray from the origin at the angle θ. The x component of the point where the ray intersects the unit circle is the cosine of θ.



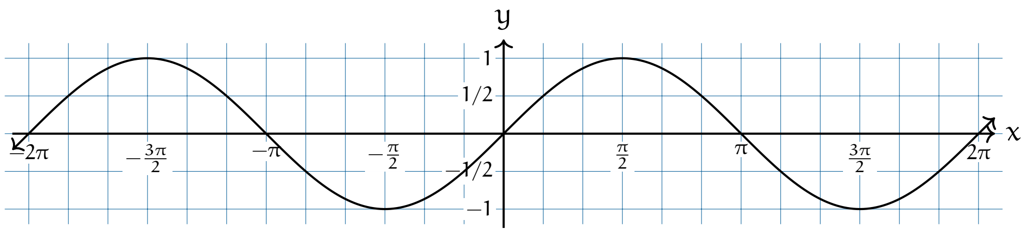
From this description, it is easy to see why $\sin(\theta)^2 + \cos(\theta)^2 = 1$. They are the legs of a right triangle with a hypotenuse of length 1.

It should also be easy to see why $\sin(\theta) = \sin(\theta + 2\pi)$: Each time you go around the circle, you come back to where you started.

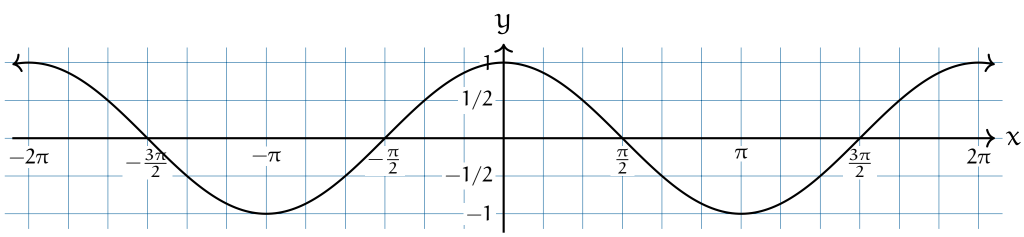
Can you see why $\cos(\theta) = \sin(\theta + \pi/2)$? Turn the picture sideways.

36.1 Graphs of sine and cosine

Here is a graph of $y = \sin(x)$:



It looks like waves, right? It goes forever to the left and right. Remembering that $\cos(\theta) = \sin(\theta + \pi/2)$, we can guess what the graph of $y = \cos(x)$ looks like:



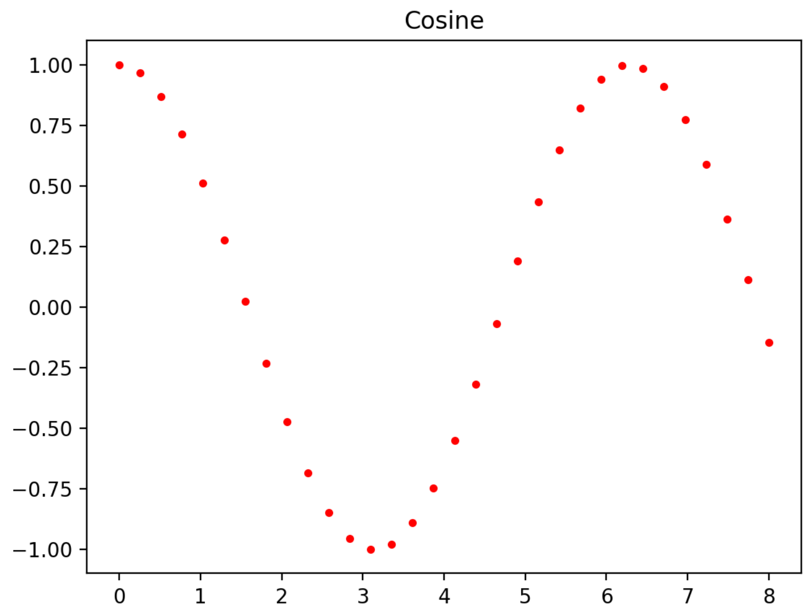
36.2 Plot cosine in Python

Create a file called `cos.py`:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
```

```
fig, ax = plt.subplots()
ax.plot(thetas, cosines, 'r.', label="Cosine")
ax.set_title("Cosine")
plt.show()
```

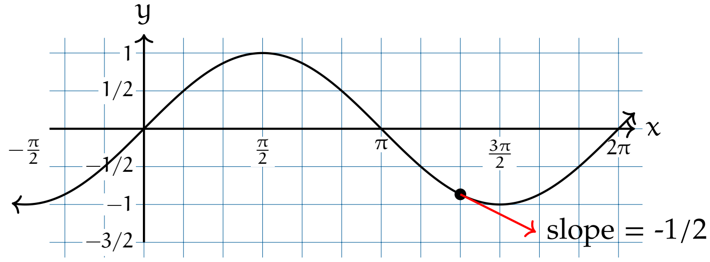
This will plot 32 points on the cosine wave between 0 and 8. When you run it, you should see something like this:



36.3 Derivatives of sine and cos

Here is a wonderful property of sine and cosine functions: At any point θ, the slope of the sine graph at θ equals $\cos(\theta)$.

For example, we know that $\sin(4\pi/3) = -(1/2)\sqrt{3}$ and $\cos(4\pi/3) = -1/2$. If we drew a line tangent to the sine curve at this point, it would have a slope of -1/2:



We say "The derivative of the sine function is the cosine function."

Can you guess the derivative of the cosine function? For any θ, the slope of the graph of the $\cos(\theta)$ is $-\sin(\theta)$.

36.4 A weight on a spring

Let's say you fill a rollerskate with heavy rocks and attach it to the wall with a stiff spring. If you push the skate toward the wall a release it, it will roll back and forth. Engineers would say "The skate will oscillate."

Intuitively, you can probably guess:

- If the spring is stronger, the skate will oscillate more times per minute.
- If the rocks are lighter, the skate will oscillate more times per minute.

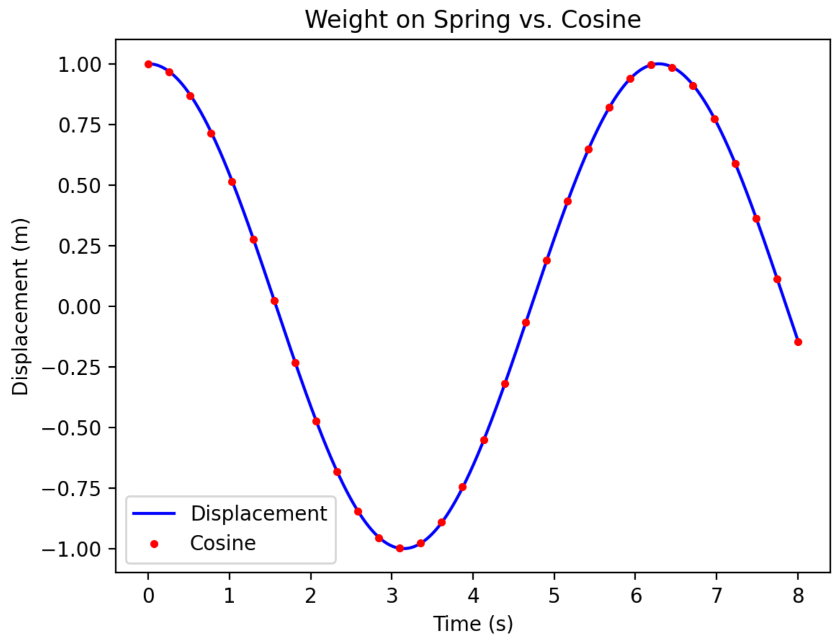
The force that the spring exerts on the skate is proportional to how far its length is from its relaxed length. When you buy a spring, the manufacturer advertises its "spring rate", which is in pounds per inch or newtons per meter. If a spring has a rate of 5 newtons per meter, which means that if stretch or compress it 10 cm, it will push back with a force of 0.5 newtons. If you stretch or compress it 20 cm, it will push back with a force of 1 newton.

Let's write a simulation of the skate-on-a-spring. Duplicate `cos.py`, and name the new copy `spring.py`. Add code to implement the simulation:

```
import numpy as np
import matplotlib.pyplot as plt

until = 8.0

# Constants
mass = 100 # kg
spring_constant = -1 # newtons per meter displacement
time_step = 0.01 # s

# Initial state
displacement = 1.0 # height above equilibrium in meters
velocity = 0.0
time = 0.0 # seconds

# Lists to gather data
```

```
displacements = []
times = []

# Run it for a little while
while time <= until:
    # Record data
    displacements.append(displacement)
    times.append(time)

    # Calculate the next state
    time += time_step
    displacement += time_step * velocity
    force = spring_constant * displacement
    acceleration = force / mass
    velocity += acceleration

# Make a plot of cosine
thetas = np.linspace(0, until, 32)
cosines = []
for theta in thetas:
    cosines.append(np.cos(theta))

# Plot the data
fig, ax = plt.subplots()
ax.plot(times, displacements, 'b', label="Displacement")
ax.plot(thetas, cosines, 'r.', label="Cosine")

ax.set_title("Weight on Spring vs. Cosine")
ax.set_xlabel("Time (s)")
ax.set_ylabel("Displacement (m)")
ax.legend()
plt.show()
```

When you run it, you should get a plot of your spring and the cosine graph on the same plot.



The position of the skate is following a cosine curve. Why?

Because a sine or cosine waves happen whenever the acceleration of an object is proportional to -1 times its displacement. Or in symbols:

$$a \propto -p$$

where a is acceleration and p is the displacement from equilibrum.

Remember that if you take the derivative of the displacement, you get the velocity. And if you take the derivative of that, you get acceleration. So, the weight on the spring must follow a function f such that

$$f(t) \propto -f''(t)$$

Remember that the derivative of the $\sin(\theta)$ is $\cos(\theta)$.

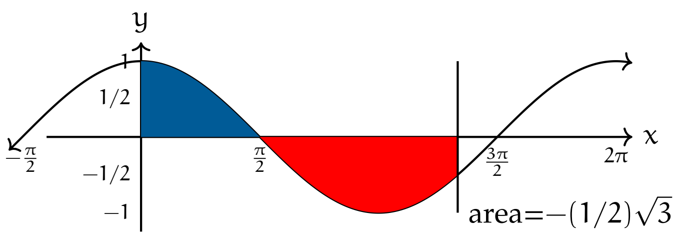
And the derivative of the $\cos(\theta)$ is $-\sin(\theta)$

Thus these sorts of waves have an almost-magical power: their acceleration is proportional to -1 times their displacement.

Thus sine waves of various magnitudes and frequencies are ubiquitous in nature and technology.

36.5 Integral of sine and cosine

If we take the area between the graph and the x axis of the cosine function (and if the function is below the x axis, it counts as negative area), from 0 to $4\pi/3$, we find that it is equal to $-(1/2)\sqrt{3}$

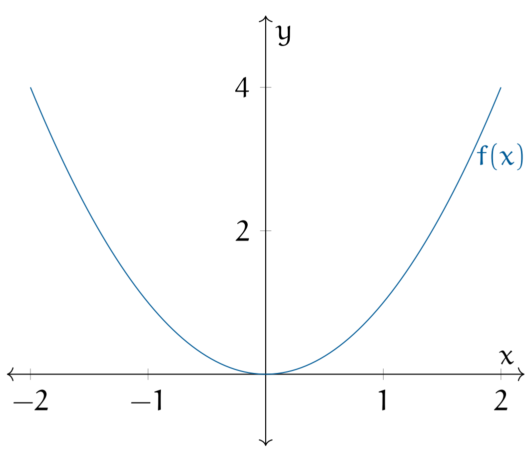


We say "The integral of the cosine function is the sine function."

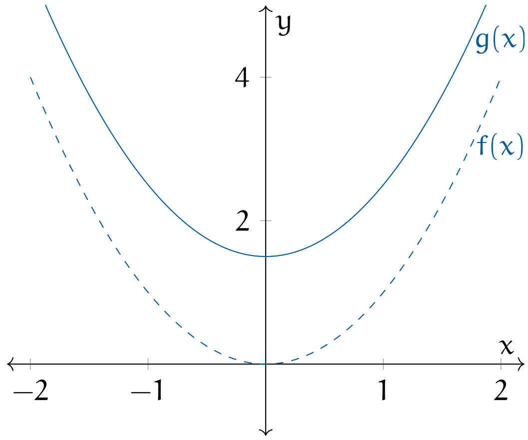
Chapter 37

Transforming Functions

Let's say I gave you the graph of a function f, like this:



And then I tell you that the function $g(x) = f(x) + 1.5$. Can you guess what the graph of g would look like? It is the same graph, just translated up 1.5:



There are four kinds of transformations that we do all the time:

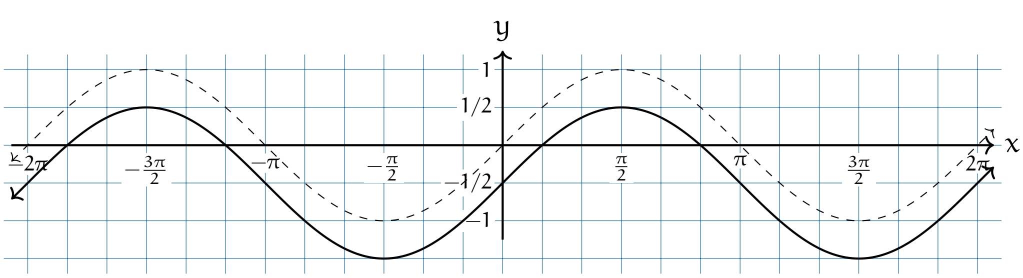
- Translation up and down in the direction of y axis (the one you just saw)
- Translation left and right in the direction of the x axis
- Scaling up and down along the y axis
- Scaling up and down along the x axis

Now I will demonstrate each of the four using the graph of $\sin(x)$.

37.1 Translation up and down

When you add a positive constant to a function, you translate the whole graph up that much. A negative constant translates it down.

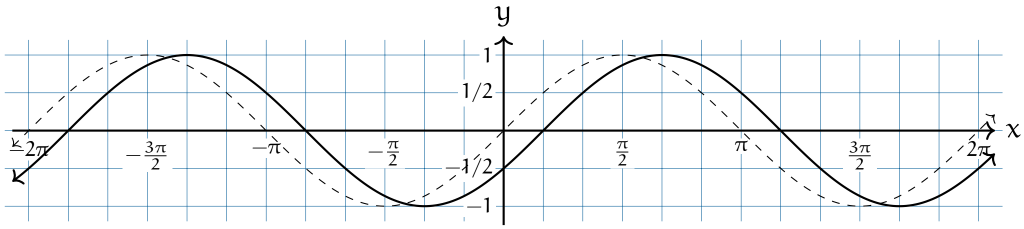
Here is the graph of $\sin(x) - 0.5$:



37.2 Translation left and right

When you add a positive number to x before running it through f, you translate the graph to the left that much. Adding a negative number translates the graph to the right.

Here is the graph of $\sin(x - \pi/6)$:



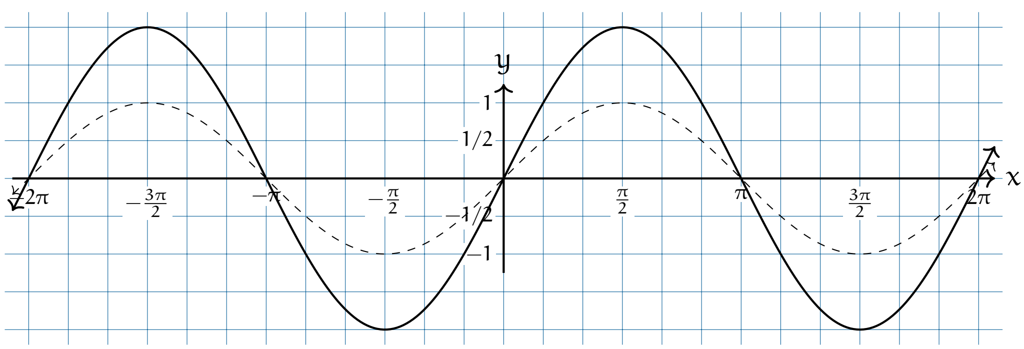
Notice the sign:

- Add to x before processing with the function translates the graph to the *left*.
- Subtract from x before processing with the function translates the graph to the *right*

37.3 Scaling up and down in the y direction

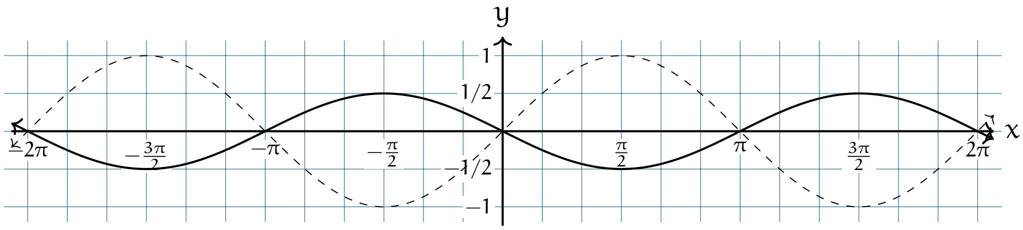
To scale the function up and down, you multiply the result of the function by a constant. If the constant is larger than 1, it stretches the function up and down.

Here is $y = 2\sin(x)$:



With a wave like this, we speak of its *Amplitude*, which you can think of as its height. The baseline that this wave oscillates around is zero. The maximum distance that it gets from that baseline is its amplitude. Thus, the amplitude here has been increased from 1 to 2.

If you multiply by a negative number, the function gets flipped. Here is $y = -0.5\sin(x)$

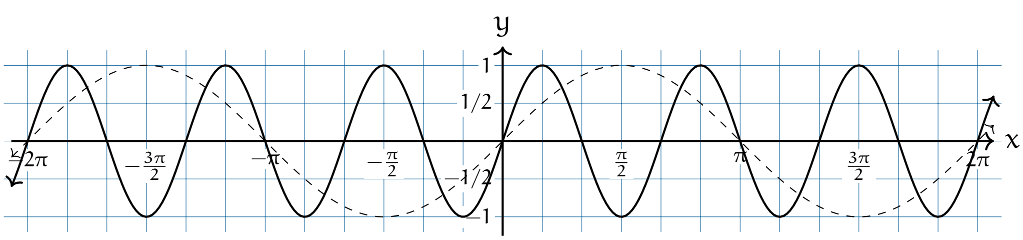


Amplitude is never negative. Thus, the amplitude of this wave is 0.5.

37.4 Scaling up and down in the x direction

If you multiply x by a number larger than 1 before running it through the function, the graph gets compressed toward zero.

Here is $y = \sin(3x)$:

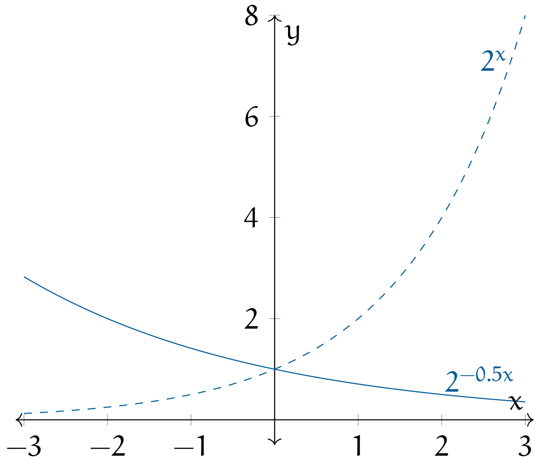


The distance between two peaks of a wave is known as its *wavelength*. The original wave had a wavelength of 2π. The compressed wave has a wavelength of $2\pi/3$.

If you multiply x by a number smaller than 1, it will stretch the function out, away from the y axis.

If you multiply x by a negative number, it will flip the function around the y axis.

Here is $y = 2^{(-0.5x)}$. Notice that it has flipped around the y axis and is stretched out along the x axis.

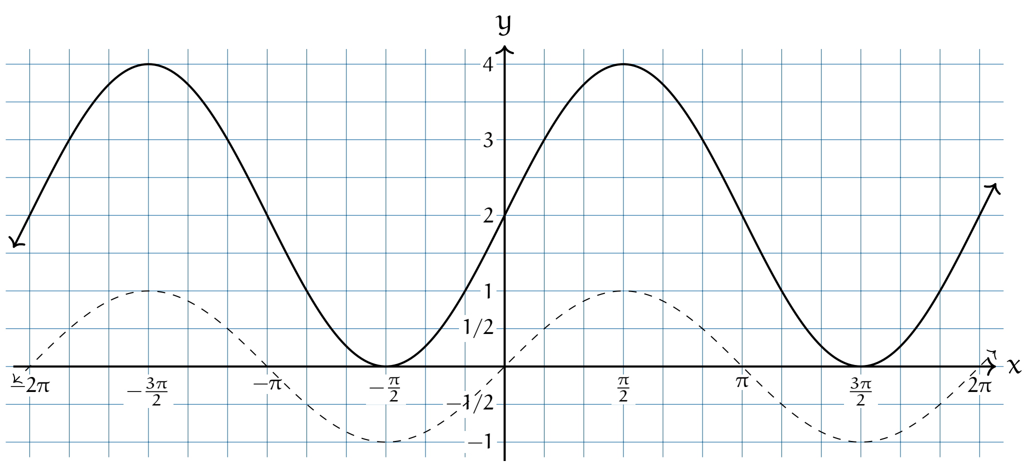


Reflection over *x*-axis	$(x, y) \to (x, -y)$
Reflection over *y*-axis	$(x, y) \to (-x, y)$
Translation	$(x, y) \to (x + a, y$
Dilation	$(x, y) \to (kx, ky)$
Rotation 90° counterclockwise	$(x, y) \to (-y, x)$
Rotation 180°	$(x, y) \to (-x, -y$

37.5 Order is important!

We can combine these transformations. This allows us, for example, to translate a function up 2 and then scale along the y axis by 3.

Here is $y = 2.0(\sin(x) + 1)$:



A function is often a series of steps. Here are the steps in $f(x) = 2(\sin(x) + 1)$:

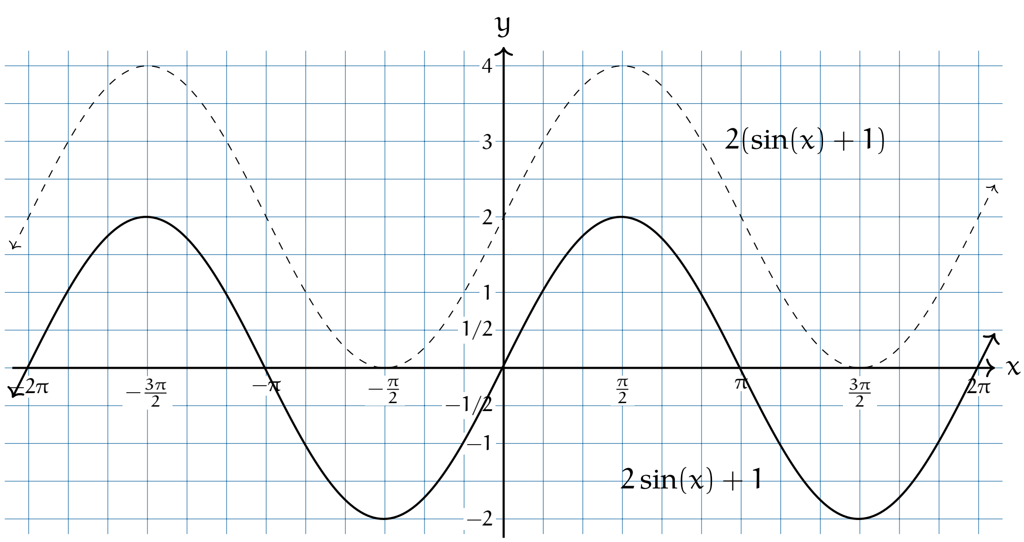
1. Take the sine of x
2. Add 1 to that
3. Multiply that by 2

What if we change the order? Here are the steps in $g(x) = 2\sin(x) + 1$:

1. Take the sine of x

2. Multiply that by 2

3. Add 1 to that

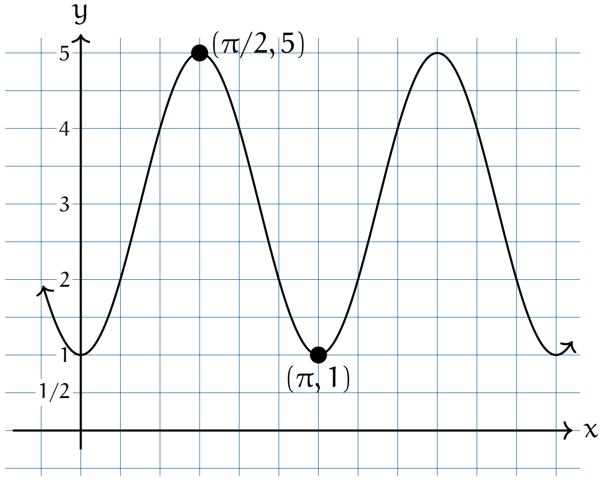


The moral: You can do multiple transformations of your function, but the order in which you do them is important.

Exercise 45 **Transforms**

Working Space

Find a function that creates a sine wave such that the top of the first crest is at the point $(\frac{\pi}{2}, 5)$ and the bottom of the trough that follows is at $(\pi, 1)$.



Answer on Page 403

Chapter 38

Sound

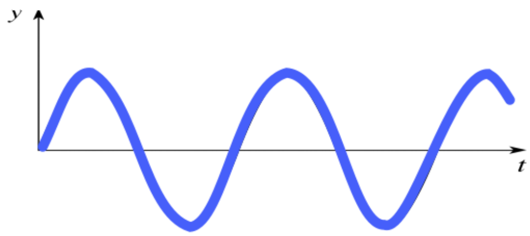
When you set off a firecracker, it makes a sound.

Let's break that down a little more: Inside the cardboard wrapper of the firecracker, there is potassium nitrate (KNO_3), sulfur (S), and carbon(C). These are all solids. When you trigger the chemical reactions with a little heat, these atoms rearrange themselves to be potassium carbonate (K_2CO_3), potassium sulfate (K_2SO_4), carbon dioxide (CO_2), and nitrogen (N_2). Note that the last two are gasses.

The molecules of a solid are much more tightly packed than the molecules of a gas. So after the chemical reaction, the molecules expand to fill a much bigger volume. The air molecules nearby get pushed away from the firecracker. They compress the molecules beyond them, and those compress the molecules beyond them.

This compression wave radiates out as a sphere; its radius growing at about 343 meters per second ("The speed of sound").

The energy of the explosion is distributed around the surface of this sphere. As the radius increases, the energy is spread more and more thinly around. This is why the firecracker seems louder when you are closer to it. (If you set off a firecracker in a sewer pipe, the sound will travel much, much farther.)



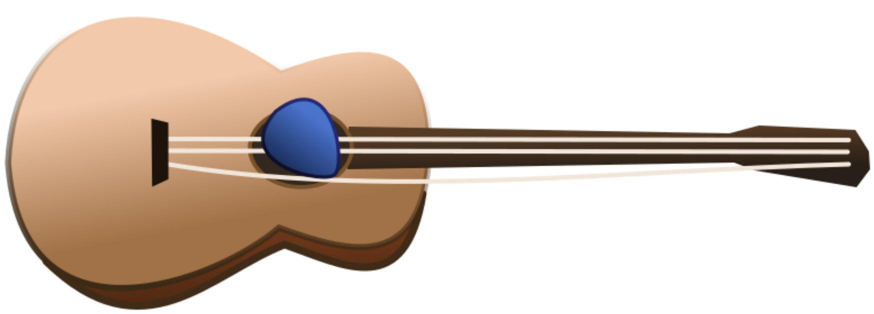
This compression wave will bounce off of hard surfaces. If you set off a firecracker 50 meters from a big wall, you will hear the explosion twice. We call the second one "an echo."

The compression wave will be absorbed by soft surfaces. If you covered that wall with pillows, there would be almost no echo.

The study of how these compression waves move and bounce is called *acoustics*. Before you build a concert hall, you hire an acoustician to look at your plans and tell you how to make it sound better.

38.1 Pitch and frequency

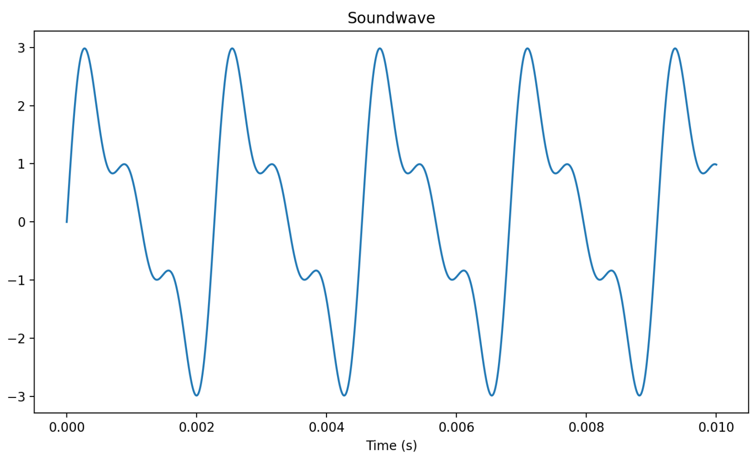
The string on a guitar is very similar to the weighted spring example. The farther the string is displaced, the more force it feels pushing it back to equilibrium. Thus, it moves back and forth in a sine wave. (OK, it isn't a pure sine wave, but we will get to that later.)



The string is connected to the center of the boxy part of the guitar, which is pushed and pulled by the string. That creates compression waves in the air around it.

If you are in the room with the guitar, those compression waves enter your ear, push and pull your eardrum, which is attached to bones that move a fluid that tickles tiny hairs, called *cilia* in your inner ear. That is how you hear.

We sometimes see plots of sound waveforms. The x-axis represents time. The y-axis represents the amount the air is compressed at the microphone that converted the air pressure into an electrical signal.



If the guitar string is made tighter (by the tuning pegs) or shorter (by the guitarist's fingers on the strings), the string vibrates more times per second. We measure the number of waves per second and we call it the *frequency* of the tone. The unit for frequency is *Hertz*: cycles per second.

Musicians have given the different frequencies names. If the guitarist plucks the lowest note on his guitar, it will vibrate at 82.4 Hertz. The guitarist will say "That pitch is low E." If the string is made half as long (by a finger on the 12th fret), the frequency will be twice as fast (164.8 Hertz), and the guitarist will say "That is E an octave up."

For any note, the note that has twice the frequency is one octave up. The note that has half the frequency is one octave down.

The octave is a very big jump in pitch, so musicians break it up into 12 smaller steps. If the guitarist shortens the E string by one fret, the frequency will be $82.4 \times 1.059463 \approx 87.3$ Hertz.

Shortening the string one fret always increases the frequency by a factor of 1.059463. Why?

Because $1.059463^1 2 = 2$. That is, if you take 12 of these hops, you end up an octave higher.

This, the smallest hop in western music, is referred to as *half step*.

Exercise 46 Notes and frequencies

The note A near the middle of the piano, is 440Hz. The note E is 7 half steps above A. What is its frequency?

Working Space

Answer on Page 403

38.2 Chords and harmonics

Of course, a guitarist seldom plays only one string at a time. Instead, he uses the frets to pick a pitch for each string and strums all six strings.

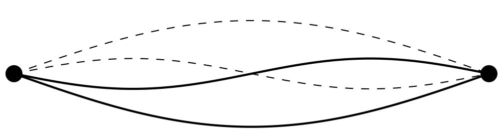
Some combinations of frequencies sound better than others. We have already talked about the octave: if one string vibrates twice for each vibration of another, they sound sweet together.

Musicians speak of "the fifth". If one string vibrates three times and the other vibrates twice in the same amount of time, they sound sweet together.

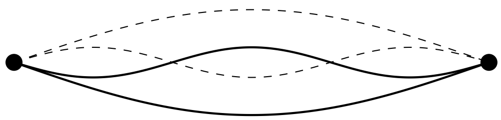
If one string vibrates 4 times while the other vibrates 3 times, they sound sweet together. Musicians call this "the third."

Each of these different frequencies tickle different cilia in the inner ear, so you can hear all six notes at the same time when the guitarist strums his guitar.

When a string vibrates, it doesn't create a single sine wave. Yes, the string vibrates from end-to-end and this generates a sine wave at what we call *the fundamental frequency*. However, there are also "standing waves" on the string. One of these standing waves is still at the centerpoint of the string, but everything to the left of the centerpoint is going up while everything to the right is going down. This creates *an overtone* that is twice the frequency of the fundamental.



The next overtone has two still points – it divides the string into three parts. The outer parts are up while the inner part is down. Its frequency is three times the fundamental frequency.



And so on: 4 times the fundamental, 5 times the fundamental, etc.

In general, tones with a lot of overtones tend to sound bright. Tones with just the fundamental sound thin.

Humans can generally hear frequencies from 20Hz to 20,000Hz (or 20kHz). Young people tend to be able to hear very high sounds better than older people.

Dogs can generally hear sounds in the 65Hz to 45kHz range.

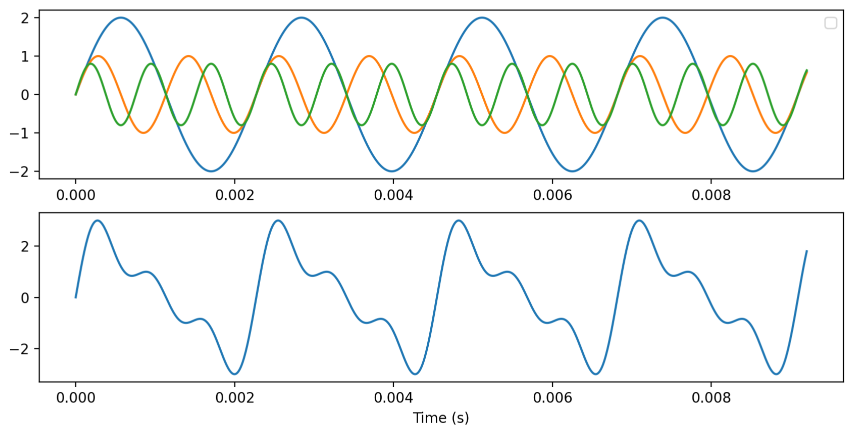
38.3 Making waves in Python

Let's make a sine wave and add some overtones to it. Create a file `harmonics.py`

```
import matplotlib.pyplot as plt
import math

# Constants: frequency and amplitude
fundamental_freq = 440.0 # A = 440 Hz
fundamental_amp = 2.0

# Up an octave
first_freq = fundamental_freq * 2.0 # Hz
first_amp = fundamental_amp * 0.5

# Up a fifth more
second_freq = fundamental_freq * 3.0 # Hz
second_amp = fundamental_amp * 0.4

# How much time to show
max_time = 0.0092 # seconds

# Calculate the values 10,000 times per second
time_step = 0.00001 # seconds

# Initialize
time = 0.0
times = []
```

```
totals = []
fundamentals = []
firsts = []
seconds = []

while time <= max_time:
    # Store the time
    times.append(time)

    # Compute value each harmonic
    fundamental = fundamental_amp * math.sin(2.0 * math.pi * fundamental_freq * time)
    first = first_amp * math.sin(2.0 * math.pi * first_freq * time)
    second = second_amp * math.sin(2.0 * math.pi * second_freq * time)

    # Sum them up
    total = fundamental + first + second

    # Store the values
    fundamentals.append(fundamental)
    firsts.append(first)
    seconds.append(second)
    totals.append(total)

    # Increment time
    time += time_step

# Plot the data
fig, ax = plt.subplots(2, 1)

# Show each component
ax[0].plot(times, fundamentals)
ax[0].plot(times, firsts)
ax[0].plot(times, seconds)
ax[0].legend()

# Show the totals
ax[1].plot(times, totals)
ax[1].set_xlabel("Time (s)")

plt.show()
```

When you run it, you should see a plot of all three sine waves and another plot of their sum:



38.3.1 Making a sound file

The graph is pretty to look at, but make a file that we can listen to.

The WAV audio file format is supported on pretty much any device, and a library for writing WAV files comes with Python. Let's write some sine waves and some noise into a WAV file.

Create a file called `soundmaker.py`

```
import wave
import math
import random

# Constants
frame_rate = 16000 # samples per second
duration_per = 0.3 # seconds per sound
frequencies = [220, 440, 880, 392] # Hz
amplitudes = [20, 125]
baseline = 127 # Values will be between 0 and 255, so 127 is the baseline
samples_per = int(frame_rate * duration_per) # number of samples per sound

# Open a file
wave_writer = wave.open('sound.wav', 'wb')

# Not stereo, just one channel
wave_writer.setnchannels(1)

# 1 byte audio means everything is in the range 0 to 255
```

```
wave_writer.setsampwidth(1)

# Set the frame rate
wave_writer.setframerate(frame_rate)

# Loop over the amplitudes and frequencies
for amplitude in amplitudes:
    for frequency in frequencies:
        time = 0.0
        # Write a sine wave
        for sample in range(samples_per):
            s = baseline + int(amplitude * math.sin(2.0 * math.pi * frequency * time))
            wave_writer.writeframes(bytes([s]))
            time += 1.0 / frame_rate

        # Write some noise after each sine wave
        for sample in range(samples_per):
            s = baseline + random.randint(0, 15)
            wave_writer.writeframes(bytes([s]))

# Close the file
wave_writer.close()
```

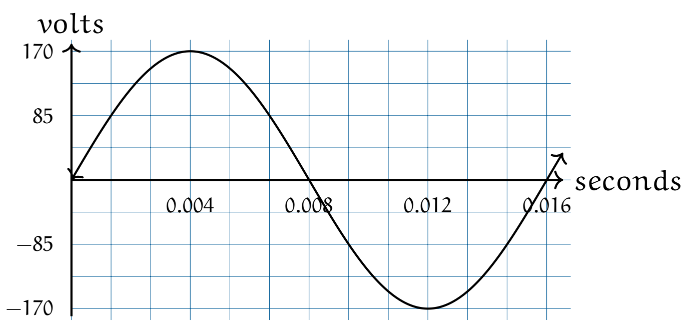
When you run it, it should create a sound file with several tones of different frequencies and volumes. Each tone should be followed by some noise.

Chapter 39

Alternating Current

We have discussed the voltage and current created by a battery. A battery pushes the electrons in one direction at a constant voltage; this is known as *Direct Current* or DC. A battery typically provides between 1.5 and 9 volts.

The electrical power that comes into your home on wires is different. If you plotted the voltage over time, it would look like this:



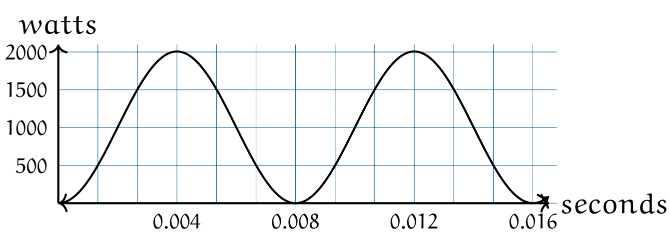
The x axis here represents ground. When you insert a two-prong plug into an outlet, one is "hot" and the other is "ground". Ground represents 0 volts and should be the same voltage as the dirt under the building.

The voltage is a sine wave at 60Hz. Your voltage fluctuates between -170v and 170v. Think for a second what that means: The power company pushes electrons at 170v and then pulls electrons at 170v. It alternates back and forth this way 60 times per second.

39.1 Power of AC

Let's say you turn on your toaster which has a resistance of 14.4 ohms. How much energy (in watts) does it change from electrical energy to heat? We know that $I = V/R$ and we know that watts of power are IV. So given a voltage of V, the toaster is consuming V^2/R watts.

However, V is fluctuating. Let's plot the power the toaster is consuming:



Another sine wave! Here is a lesser-known trig identity: $(\sin(x))^2 = \frac{1}{2} - \frac{1}{2}\cos(2x)$

So this is actually a cosine wave flipped upside down, scaled down by half the peak power, and translated up so that it is never negative. Note that it is also twice the frequency of the voltage sine wave.

If we say the peak voltage is V_p and the resistance of the toaster is R, the power is given by

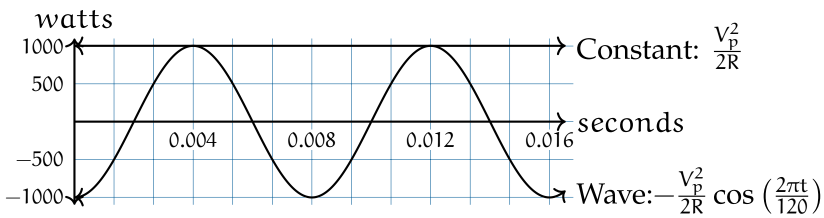
$$\frac{V_p^2}{2R} - \frac{V_p^2}{2R}\cos\left(\frac{2\pi t}{120}\right)$$

As a toaster user and as someone who pays a power bill, you are mostly interested in the average power. To get the average power, you take the area under the power graph and divide it by the amount of time.

We can think of the area under the curve as two easy-to-integrate quantities summed:

- A constant function of $y = fracV_p^2 2R$

- A wave $y = -\frac{V_p^2}{2R}\cos\left(\frac{2\pi t}{120}\right)$



When we integrate that constant function we get $\frac{tV_p^2}{2R}$

When we integrate that wave for a complete cycle we get...zero! The positive side of the wave is canceled out by the negative side.

So, the average power is $\frac{V_p^2}{2R}$ watts.

Someone at some point said "I'm used to power being V^2/R. Can we define a voltage measure for AC power such that this is always true?"

So we started using V_{rms} which is just $\frac{V_p}{\sqrt{2}}$. If you look on the back of anything that plugs into a standard US power outlet, it will say something like "For 120v". What they mean is "For 120v RMS, so we expect the voltage to fluctuate back and forth from 170v to -170v."

Notice that this is the same Root-Mean-Squared that we defined earlier, but now we know that if $y = \sin(x)$, the RMS of y is $1/\sqrt{2} \approx 0.707$.

For current, we do the same thing: If the current is AC, the power consumed by a resistor is $I_{RMS}^2 R$, where I_{RMS} is the peak current divided by $sqrt2$.

39.2 Power Line Losses

A wire has some resistance. Thinner wires tend to have more resistance than thicker ones. Aluminum wires tend to have more resistance than copper wires.

Let's say that the power that comes to your house has to travel 20 km from the generator in a cable that has about 1Ω of resistance per km. Let's say that your home is consuming 12 kilowatts of power.

If that power is 120v RMS from the generator to your home, what percentage of the power is lost heating the power line? 10 amps RMS flow through your home. When that current goes through the wire, $I^2R = (100)(20) = 2000 watts$ is lost to heat.

So the power company would need to supply 14 kilowatts of power, knowing that 2 kilowatts would be lost on the wires.

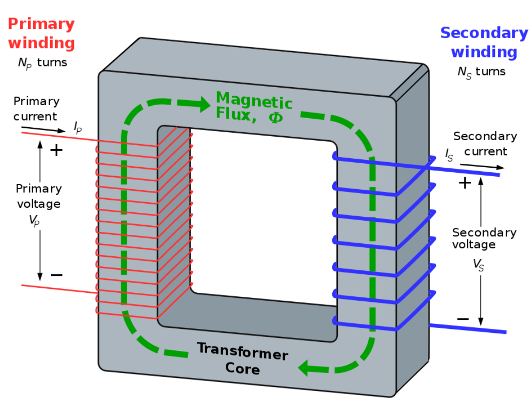
What if the power company moved the power at 120,000 volts RMS? Now only 0.01 amps RMS flow through your home. When that current goes through the wire $I^2R = (0.0001)(20) =$ .002 watts of power are lost on the power lines.

It is much, much more efficient. The only problem is that 120,000 volts would be incredibly dangerous. So the power company moves power long distances at very high voltages, like 765 kV. Before the power is brought into your home, it is converted into a lower voltage using a *transformer*.

39.3 Transformers

A transformer is a device that converts electrical power from one voltage to another. A good transformer is more than 95% efficient. The details of magnetic fields, flux, and inductance are beyond the scope of this chapter, so I am going to give a relatively simple explanation and admit that it is incomplete.

A transformer is a ring with two sets of coils wrapped around it.



(Diagram from Wikipedia)

When alternating current is run through the primary winding, it creates magnetic flux in the ring. The magnetic flux induces current in the secondary winding.

If V_P is the voltage across the primary winding and V_S is the voltage across the secondary winding, they are related by the following equation:

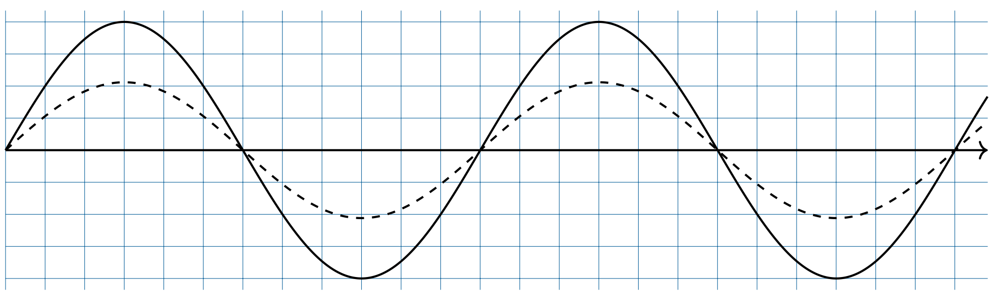
$$\frac{V_P}{V_S} = \frac{N_P}{N_S}$$

where N_P and N_S are the number of turns in the primary and secondary windings.

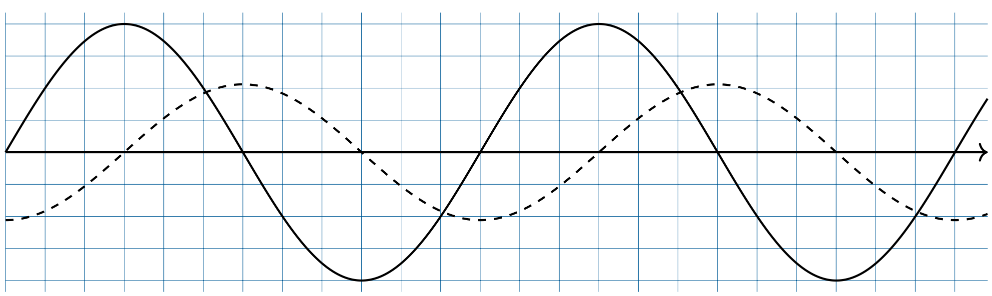
There are usually at least two transformers between you and the very high voltage lines. There are transformers at the substation that make the voltage low enough to travel on regular utility poles. On the utility poles, you will see cans that contain smaller transformers. Those step the voltage down to make the power safe to enter your home.

39.4 Phase and 3-phase power

If two waves are "in sync" we say they have the same *phase*.

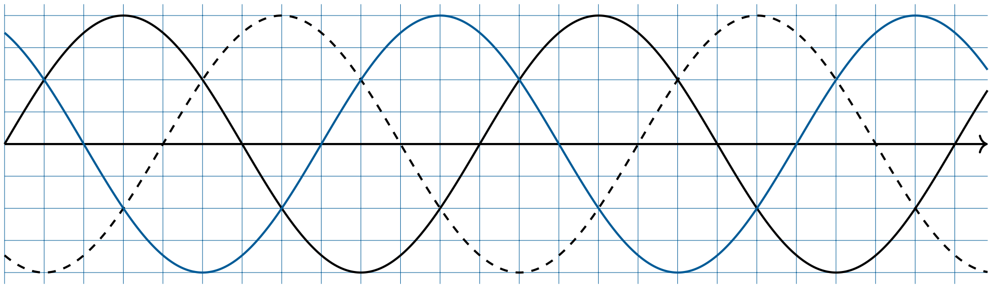


If they are the same frequency, but are not in-sync, we can talk about the difference in their phase.



Here we see that the smaller wave is lagging by $\pi/2$ or $90°$.

In most power grids, there are usually 3 wires carrying the power. The voltage on each is $2\pi/3$ out of phase with the other two:



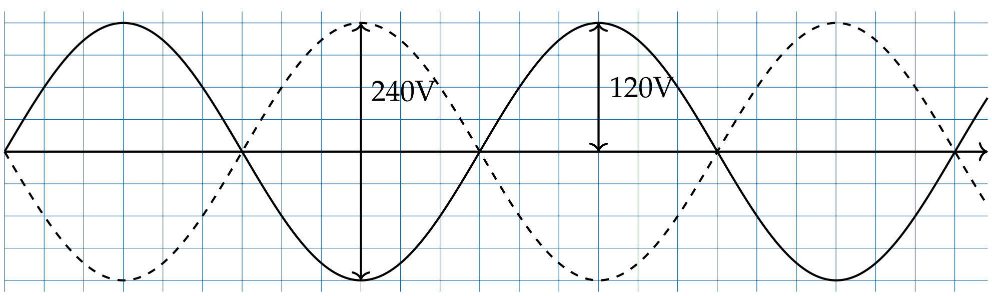
This is nice in two ways:

- While the power in each wire is fluctuating, the total power is not fluctuating at all.
- While the power plant is pushing and pulling electrons on each wire, the total number number of electrons leaving the load is zero.

(Both these assume that there each wire is attached to a load with the same constant resistance.)

In big industrial factories, you will see all three wires enter the building. Large amounts of smooth power delivery means a lot to an industrial user.

In residential settings, each home gets its power from one of the three wires. However, two wires typically carry power into the home. Each one carries 120V RMS, but they are out of phase by 180 degrees. Lights and small appliances are connected to one of the wires and ground, so they get 120V RMS. Large appliances, like air conditioners and washing machines, are connected across the two wires so they get 240V RMS.



How do you get two circuits, 180 degrees out of phase, from one circuit? Using a center-tap transformer.

FIXME: Diagram here

Chapter 40

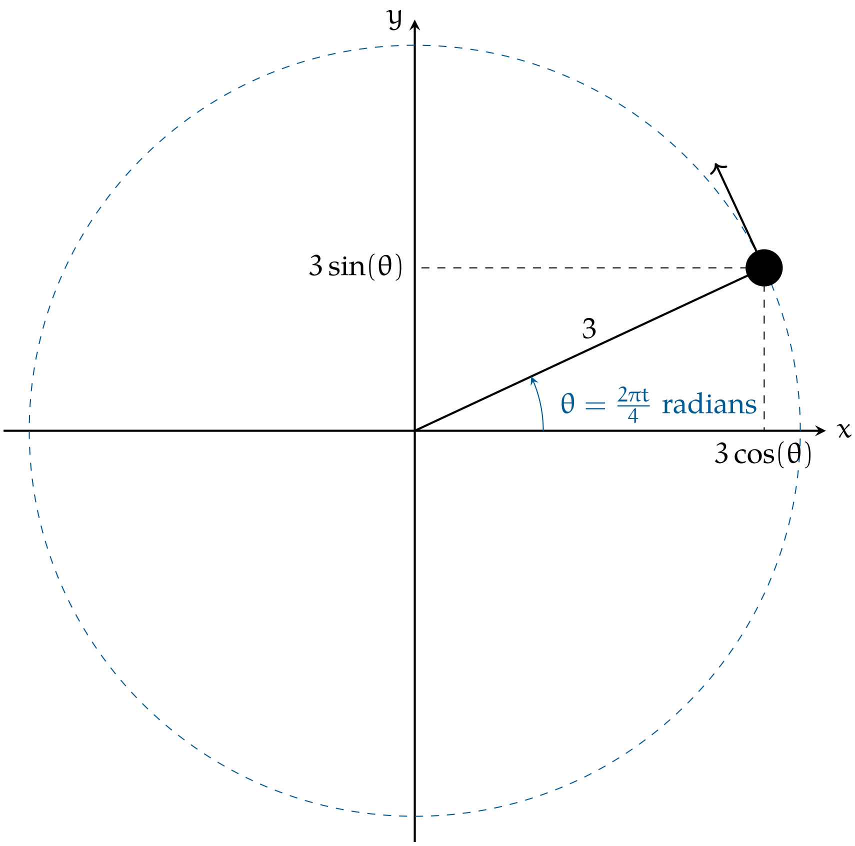
Circular Motion

Let's say you tie a 0.16 kg billard ball to a long string and begin to swing it around in a circle above your head. Let's say the string is 3 meters long, and the ball returns to where it started every 4 seconds. If you start your stopwatch as the ball crosses the x-axis, the position of the ball at any time t given by:

$$p(t) = [3\cos\left(\frac{2\pi}{4}t\right), 3\sin\left(\frac{2\pi}{4}t\right), 2]$$

(This assumes that the ball would be going counter-clockwise if viewed from above. The spot you are standing on is considered the origin $[0, 0, 0]$.)

Notice that the height is a constant – 2 meters in this case. That isn't very interesting, so we will talk just about the first two components. Here is what it would look like from above:



In this case, the radius, r, is 3 meters. The period, T is 4 seconds. In general, we say that circular motion is given by:

$$p(t) = \left[r \cos \frac{2\pi t}{T}, r \cos \frac{2\pi t}{T}\right]$$

A common question is "How fast is it turning right now?" If you divide the 2π radians of a circle by the 4 seconds it takes, you get the answer "About 1.57 radians per second." This is known as *angular velocity* and we typically represent it with the lowercase Omega: ω. (Yes, it looks a lot like a "w".) To be precise, in our example, the angular velocity is $\omega = \frac{\pi}{2}$.

Notice that this is different from the question "How fast is it going?" This ball is traveling the circumference of $6\pi \approx 18.85$ meters every 4 seconds. So the speed of the ball is about 4.71 meters per second.

40.1 Velocity

The velocity of the ball is a vector, and we can find that vector by differentiating each component of the position vector.

For any constants a and b:

Expression	Derivative
$a \sin bt$	$ab \cos bt$
$a \cos bt$	$-ab \sin bt$

Thus, in our example, the velocity of the ball at any time t is given by:

$$v(t) = \left[-\frac{3(2\pi)}{4} \sin \frac{2\pi t}{4}, \frac{3(2\pi)}{4} \cos \frac{2\pi t}{4}, 0\right]$$

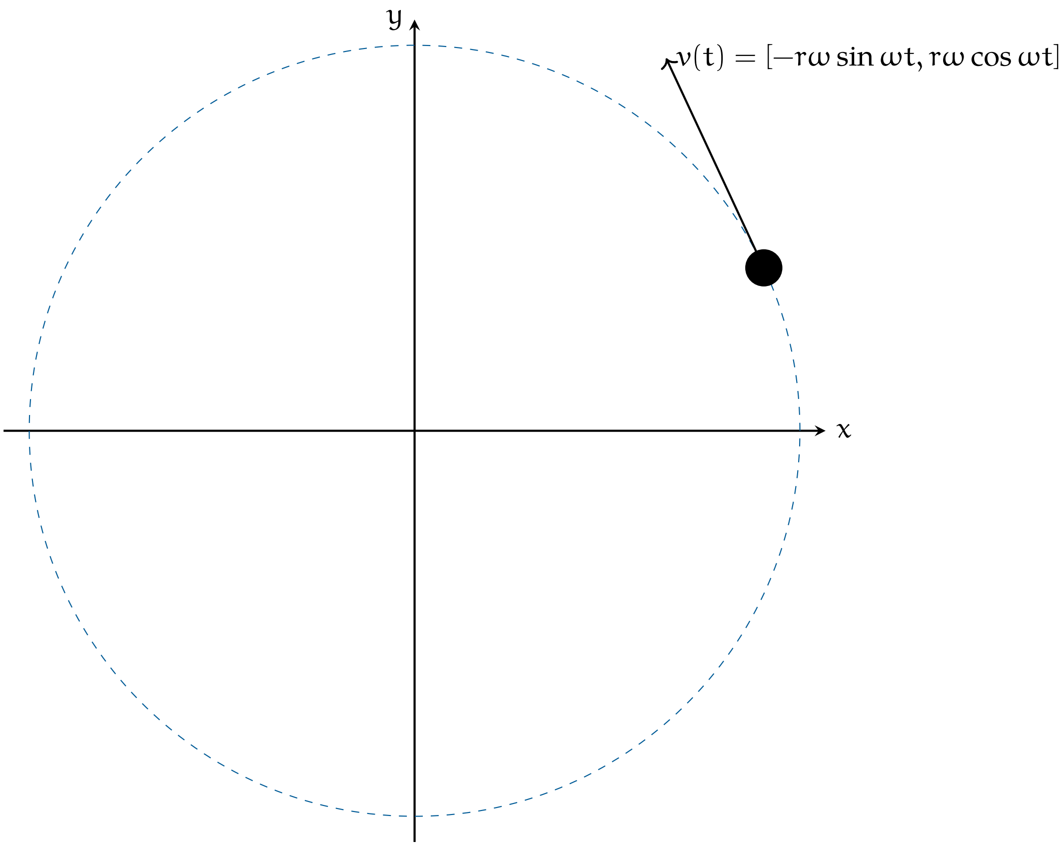
Notice that the velocity vector is perpendicular to the position vector. It has a constant magnitude.

In general, an object traveling in a circle at a constant speed has the velocity vector:

$$v(t) = [-r\omega \sin \omega t, r\omega \cos \omega t]$$

where $t = 0$ is the time that it crosses the x axis. If $\omega$is negative, that means the motion would be clockwise when viewed from above.

The magnitude of the velocity vector is $r\omega$.

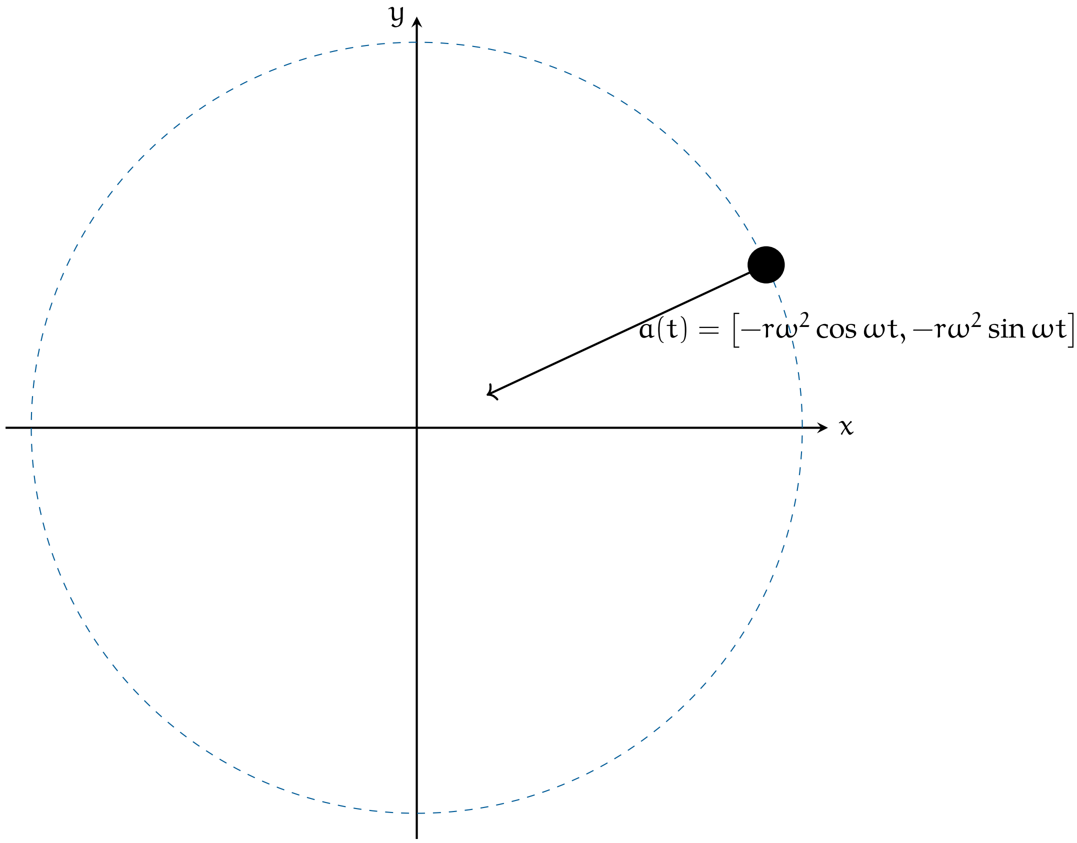


40.2 Acceleration

We can get the acceleration by differentiating the components of the velocity vector.

$$a(t) = \left[-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t\right]$$

Notice that the acceleration vector points toward the center of the circle it is traveling on. That is, when an object is traveling on a circle at a constant speed, its only acceleration is toward the center of the circle.



The magnitude of the acceleration vector is $r\omega^2$.

40.3 Centripetal force

How hard is the ball pulling against your hand? That is, if you let go, the ball would fly in a straight line. The force you are exerting on the string is what causes it to accelerate toward the center of the circle. We call this the *centripetal force*.

Recall that $F = ma$. The magnitude of the acceleration is $r\omega^2 = 3\left(\frac{2pi}{4}\right)^2 \approx 7.4$ m/s. The mass of the ball is 0.16 kg. So the force pulling against your hand is about 1.18 newtons.

The general rule is that when something is traveling in a circle at a constant speed, the centripetal force needed to keep it traveling in a circle is:

$$F = mr\omega^2$$

If you know the radius r and the speed v of the object, here is the rule:

$$F = \frac{mv^2}{r}$$

Exercise 47 Circular Motion

Just as your car rolls onto a circular track with a radius of 200 m, you realize your 0.4 kg cup of coffee is on the slippery dashboard of your car. While driving 120 km/hour, you hold the cup to keep it from sliding.
What is the maximum amount of force you would need to use (The friction of the dashboard helps you, but the max is when the friction is zero.)

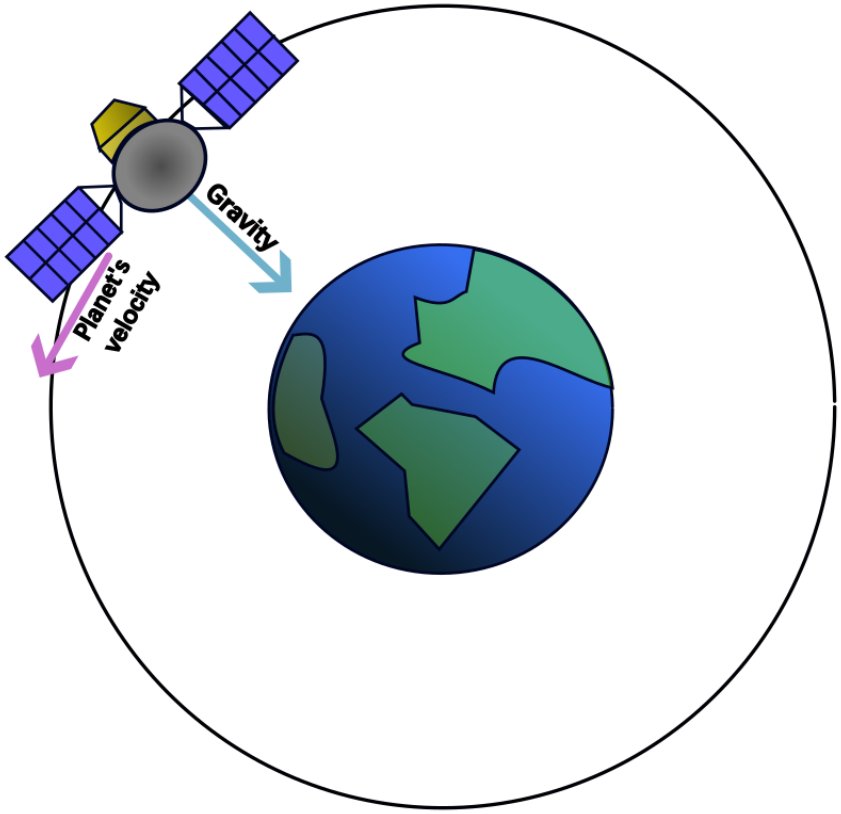
Working Space

Answer on Page 403

Chapter 41

Orbits

A satellite stays in orbit around the planet because the pull of the planet's gravity causes it to accelerate toward the center of the planet. The satellite must be moving at a very particular speed to keep a constant distance from the planet – to travel in a circular orbit. If it is moving too slowly, it will get closer to the planet. If it is going too fast, it will get farther



from the planet.

The radius of the earth is about 6.37 million meters. A satellite that is in a low orbit is typically about 2 million meters above the ground. At that distance, the acceleration due to gravity is more like $6.8\text{m}/\text{s}^2$, instead of the $9.8\text{m}/\text{s}^2$ that we experience on the surface of the planet.

How fast does the satellite need to be moving in a circle with a radius of 8.37 million meters to have an acceleration of $6.8\text{m}/\text{s}^2$? Real fast.

Recall that the acceleration vector is

$$a = \frac{v^2}{r}$$

Thus the velocity v needs to be:

$$v = \sqrt{ar} = \sqrt{6.8(8.37\text{x}10^6} = 7,544 \text{ m/s}$$

(That's 16,875 miles per hour.)

When a satellite falls out of orbit, it enters the atmosphere at that 7,544 m/s. The air rushing by generates so much friction that the satellite gets very, very hot and usually disintegrates.

41.1 Astronauts are not weightless

Some people see astronauts floating inside an orbiting spacecraft and think there is no gravity: that the astronauts are so far away that the gravity of the planet doesn't affect them. This is incorrect. The gravity might be slightly less (Maybe 6 newtons per kg instead of 9.8 newtons per kg), but the weightless they experience is because they and the spacecraft is in free fall. They are just moving so fast (in a direction perpendicular to gravity) that they don't collide with the planet.


Force of gravity continues to act
on astronuat therefore the
astronaut has weight
F_g

Exercise 48 Mars Orbit

The radius of Mars is 3.39 million meters. The atmosphere goes up another 11 km. Let's say you want to put a satellite in a circular orbit around Mars with a radius of 3.4 million meters.
The acceleration due to gravity on the surface of Mars is $3.721 \mathrm{m}/\mathrm{s}^2$. We can safely assume that it is approximately the same 11 km above the surface.
How fast does the satellite need to be traveling in its orbit? How long will each orbit take?

Working Space

Answer on Page 404

41.2 Geosynchronous Orbits

The planet earth rotates once a day. Satellites in low orbits circle the earth many times a day. Satellites in very high orbits circle less than once per day. There is a radius at which a satellite orbits exactly once per day. Satellites at this radius are known as "geosynchronous" or "geostationary" because they are always directly over a place on the planet.

The radius of a circular geosynchronous orbit is 42.164 million meters. (About 36 km above the surface of the earth.)

A geosynchronous satellite travels at a speed of 3,070 m/s.

Geosynchronous satellites are used for the Global Positioning Satellite system, weather monitoring system, and communications system.

Chapter 42

Electromagnetic Waves

Sound is a compression wave – to travel, it needs a medium to compress: air, water, etc. (Regardless of what you have seen in movies, sound does not travel through a vacuum)

Light is an electromagnetic wave – it causes fluctuations in the electric and magnetic fields that are everywhere. It can cross a vacuum, as it does to reach us from the sun.

Electromagnetic waves travel at about 300 million meters per second in a vacuum. The waves travel slower through things. For example, an electromagnetic wave travels at 225 million meters in water.

Electromagnetic waves come in different frequencies. For example, the light coming out of a red laser pointer is usually about 4.75×10^{14} Hz. The wifi data sent by your computer is carried on an electromagnetic wave too. It is usually close to 2.4×10^{6} Hz or 5×10^{6} Hz.

Because we know how fast the waves are moving, we sometimes talk about their wavelengths instead of their frequencies. The light coming out of a laser pointer is $300 \times 10^{6}/4.76 \times 10^{14} = 630 \times 10^{-9}$ m, or 630 nm.

Exercise 49 Wavelengths

A green laser pointer emits light at 5.66×10^{14} Hz. What is its wavelength in a vacuum?

Working Space

Answer on Page 404

We have given names to different ranges of the electromagnetic spectrum:

Name	Hertz	Meters
Gamma rays	$\times 10$	$\times 10$
X-rays	$\times 10$	$\times 10$
Ultraviolet	$\times 10$	$\times 10$
Blue	$\times 10$	$\times 10$
Red	$\times 10$	$\times 10$
Infrared	$\times 10$	$\times 10$
Microwaves	$\times 10$	$\times 10$
Radio waves	$\times 10$	$\times 10$

(You may have heard of "cosmic rays" and wonder why they are not listed in this table. Cosmic rays are actually the nuclei of atoms that have been stripped of their electron cloud. These particles come flying out of the sun at very high speeds. They were originally thought to be electromagnetic waves, and they were mistakenly named "rays".)

In general, the lower frequency the wave is, the better it passes through a mass. A radio wave, for example, can pass through the walls of your house, but visible light cannot. The people who designed the microwave oven, chose the frequency of 2.45 GHz because the energy from those waves tended to get absorbed in the first few inches of food that it passed through.

42.1 The greenhouse effect

Humans have dug up a bunch of long carbon-based molecules (like oil and coal) and burned them, releasing large amounts of CO_2 into the atmosphere. It is not obvious why that has made the planet warmer. The answer is electromagnetic waves.

A warm object gives off infrared electromagnetic waves. That's why, for example, motion detectors in security systems are actually infrared detectors: even in a dark room, your body gives off a lot of infrared radiation.

You may have heard of "heat-seeking missiles." These are more accurately called "Infrared homing missiles" because they follow objects giving off infrared radiation – hot things like jet engines.

The sun beams a lot of energy to our planet in the form of electromagnetic radiation: visible light, infrared, ultraviolet. (How much? At the top of the atmosphere directly facing the sun, we get 1,360 watts of radiation per square meter. That is a lot of power!)

Some of that radiation just reflects back into space. 23% is reflected by the clouds and the atmosphere, 7% makes it all the way to the surface of the planet and is reflected back into space.

The other 71% is absorbed: 48% is absorbed by the surface and 23% is absorbed by the atmosphere. All of that energy warms the planet and the atmosphere so that it gives off infrared radiation. The planet lives in equilibrium: The infrared radiation leaving our atmosphere is exactly the same amount of energy as that 71% of the radiation that it absorbs.

(If the planet absorbs more energy than it releases, the planet gets hotter. Hotter things release more infrared. When the planet is in equilibrium again, it stops getting hotter.)

So what is the problem with CO_2 and other large molecules in the atmosphere? They absorb the infrared radiation instead of letting it escape into space. Thus the planet must be hotter to maintain equilibrium.

The planet is getting hotter, and it is creating a multitude of problems:

- Weather patterns are changing, which leads to extreme floods and droughts.
- Ice and snow in places like Greenland are melting and flowing into the oceans. This is raising sea-levels.
- Biomes with biodiversity are resilient. Rapidly changing climate is destroying biodiversity everywhere, which is making these ecosystems very fragile.
- In many places, permafrost, which has trapped large amounts of methane in the ground for millenia, is melting.

That last item is particularly scary because methane is a large gas molecule – it absorbs even more infrared radiation than CO_2. As it is escapes the permafrost, the problem will get worse.

Scientists are working on four kinds of solutions:

- **Stop increasing the amount of greenhouse gases in our atmosphere.** It is hoped that non-carbon based energy systems like solar, wind, hydroelectric, and nuclear could let us stop burning carbon. Given the methane already being released, it maybe too late for this solution to work on its own.

- **Take some of greenhouse gases out of our atmosphere and sequester them somewhere.** The trunk of a tree is largely carbon molecules. When you grow a tree where there had not been one before, you are sequestering carbon inside the tree. There are also scientists that are trying to develop process that pull greenhouse gases out of the air and turn them into solids.

- **Decrease the amount of solar radiation that is absorbed by our planet and its atmosphere.** Clouds reflect a lot of radiation back into space. Could we increase the cloudiness of our atmosphere? Or maybe mirrors in orbit around our planet?

- **Adapt to the changing climate.** These scientists are assuming that global warming will continue, and are working to minimize future human suffering. How will we relocate a billion people as the oceans claim their homes? When massive heat waves occur, how will we keep people from dying? As biodiversity decreases, how can we make sure that species that are important to human existence survive?

What are the greenhouse gases and how much does each contribute to keeping the heat from exiting to space? These numbers are still being debated, but this will give you a feel:

Water vapor	H_2O	36 - 72 %
Carbon dioxide	CO_2	9 - 26 %
Methane	CH_4	4 - 9 %
Ozone	O_3	3 - 7 %

Notice that while we talk a lot about carbon dioxide, the most important greenhouse gas is actually water. Why don't we talk about it? Given the enormous surfaces of the oceans, it is difficult to imagine any way to permanently decrease the amount of water in the air. Also, a lot of water in the air is in the form of clouds that help reflect radiation before it is absorbed.

Chapter 43

How Cameras Work

Let's say it is a sunny day and you are standing in a field a few meters from a cow. You use the camera on your phone to take a picture of the cow. How does that whole process work?

43.1 The Light That Shines On the Cow

The sun is a sphere of hot gas. About 70% of the gas is hydrogen. About 28% is helium. There's also a little carbon, nitrogen, and oxygen.

Gradually, the sun is converting hydrogen into helium through a process known as "nuclear fusion". (We will talk more about nuclear fusion in a later chapter.) A lot of heat is created in this process. The heat makes the gases glow.

How does heat make things glow? The heat pushes the electrons into higher orbitals. When they back down to a lower orbital, they release a photon of energy, which travels away from the atom as an electromagnetic wave.

Heat isn't the only way to push the electrons into a higher orbital. For example, a fluorescent lightbulb is filled with gas. When we pass electricity through the gas, its electrons are moved to a higher orbital. When they fall, light is created.

What is the frequency of the wave that the photon travels on? Depending on what orbital it falls from and how far it falls, the photon created has different amounts of energy. The amount of energy determines the frequency of the electromagnetic wave.

Formula for enegy of a photon

If you want to know the amount of energy E in a photon, here is the formula:

$$E = \frac{hc}{\lambda}$$

where c is the speed of light, λ is the wavelength of the electromagnetic wave, and h Planck's constant: $6.63 \times 10^{-34} m^2 kg/s$

For example, a red laser light has a wavelength of about 630 nm. So the energy in each photon is:

$$\frac{(300 \times 10^6)(6.63 \times 10^{-34})}{630 \times 10^{-9}} = 3.1 \times 10^{-19} \text{ joules}$$

In the sun, there are several kinds of molecules and each has a few different orbitals that the electrons can live in. Thus, the light coming from the sun is made up of electromagnetic waves of many different frequencies.

We can see some of these frequencies as different colors, but some are invisible to humans, for example ultraviolet and infrared.

43.2 Light Hits the Cow

When these photons from the sun hit the cow, the hide and hairs of the cow will absorb some of the photons. These photons will become heat and make the cow feel warm. Some of the photons will not be absorbed – they will leave the cow. When you say "I see the cow," what you are really saying is "I see some photons that were not absorbed by the cow."

Different materials absorb different amounts of each wavelength. A plant, for example, absorbs a large percentage of all blue and red photons that hit it, but it absorbs only a small percentage of the green photons that hit it. Thus we say "That plant is green."

White things absorb very small percentages of photons of any visible wavelength. Black things absorb very *large* percentages of photons of any visible wavelength.

Before we go on, let's review: The sun creates photons that travel as electromagnetic waves of assorted wavelengths to the cow. Many of those photons are absorbed, but some are not. Some of those photons that are not absorbed go into the lens of our camera.

43.3 Pinhole camera

The simplest cameras have no lenses. They are just a box. The box has a tiny hole that allows photons to enter. The side of the box opposite the hole is flat and covered with film or some other photo-sensitive material.

The photons entering the box continue in the same direction they were going when they passed through the hole. Thus, the photons that entered from high, hit the back wall low. The photons that came from the left, hit the back wall on the right. Thus the image is projected onto the back wall rotated 180 degrees: What was up is down, what was on the left is on the right.

FIXME: picture here

Exercise 50 Height of the image

Working Space

Let's say that that the pinhole is exactly the same height as the shoulder of the cow and that the shoulder is directly above one hoof. Than the pinhole, the shoulder, and the hoof form a right triangle. Now, let's say that the camera is being held perpendicular to the ground. Now, the pinhole, the image of the shoulder, and the image of the hoof on the back wall of the camera also form a right triangle.
These two triangles are similar.
The shoulder is 2 meters from the hoof. The cow is standing 3 meters from the camera. The distance from the pinhole to the back wall of the camera is 3 cm. How tall is the image of the cow on the back wall of the camera?

Answer on Page 404

43.4 Lenses

Quick review: A photon leaves the sun in some random direction. It travels 150 million km from the sun and hits a cow. It is not absorbed by the cow, and heads off in a new direction. It passes through the pinhole and hits the back wall of the camera. That seems incredibly improbable, right?

It actually is kind of improbable, especially if there isn't a lot of light – like you are taking the picture at dusk. To increase the odds, we added a *lens* to the camera.

If you focus a lens on a wall, and then you draw a dot on the wall. The lens is designed such that all the photons from the dot that hit the lens get redirected to the same spot on the back wall of the camera – regardless of which path it took to get to the lens.

FIXME: illustration here

Note that the image still gets flipped. There is a *focal point* that all the photons pass through.

FIXME: illustration here

The distance from the lens to its focal point is called the lens's *focal length*. Telephoto lenses, that let you take big pictures of things that are far away, have long focal lengths. Wide-angle lenses have short focal lengths.

43.5 Sensors

The camera on your phone has a sensor on the back wall of the camera. The sensor is broken up into tiny rectangular regions called pixels. When you say a sensor is 6000 by 4000 pixels, we are saying the sensor is a grid of 24,000,000 pixels: 6000 pixels wide and 4000 pixels tall.

Each pixel has three types of cavities that take in photos. One of the cavities measures the amount of short wavelength light, like blues and violets. One of the cavities measures the long wavelength light, like reds and oranges. One of the cavities measures the intensity of wavelengths in the middle, like greens.

Thus, if your camera has a resolution of 6000×4000, the image is 72,000,000 numbers: Every one of the 24,000,000 pixels yeilds three numbers: intensity of long wavelength, mid wavelength, and long wavelength light. We call these numbers "RGB" for Red, Green, and Blue.

Chapter 44

How Eyes Work

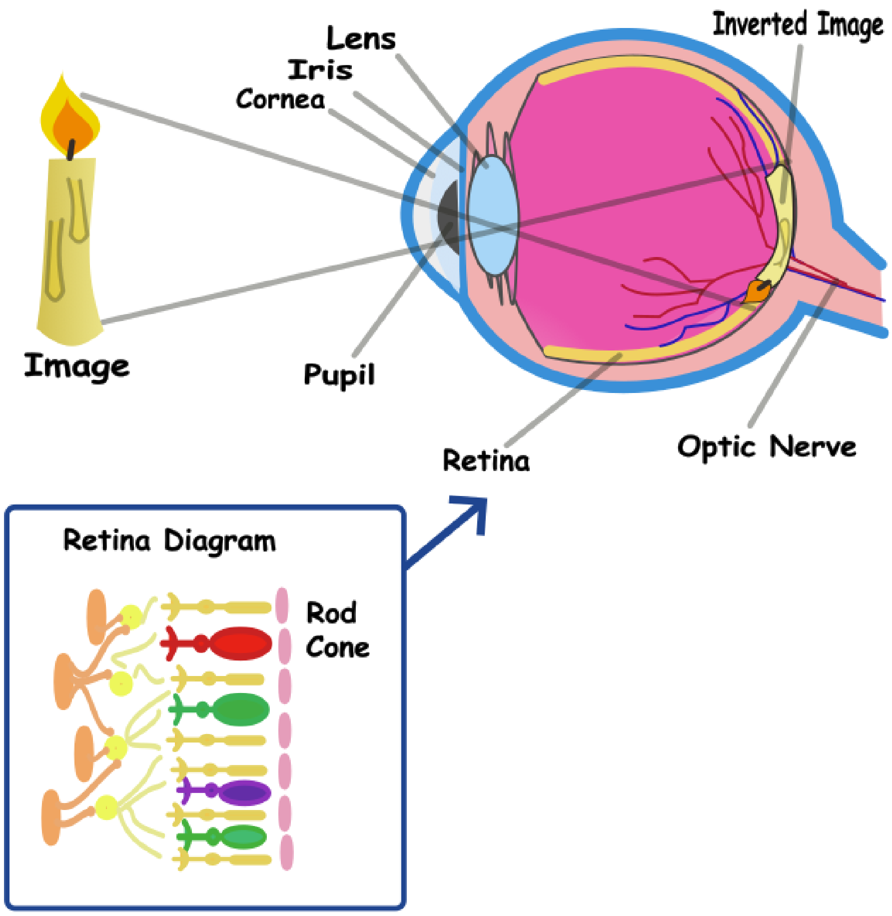
Dr. Craig Blackwell has made a great video on the mechanics of the eye. You should watch it: https://youtu.be/Z8asc2SfFHM

Mechanically, your eye works a lot like a camera. The eye is a sphere with two lenses on the front: The outer lens is called the *cornea*, and the second lens is just called "the lens."

Between the two lenses is an aperture that opens wide when there is very little light, and closes very small when there is bright light. The opening is called the *pupil* and the tissue that forms the pupil is called the *iris*. When people talk of the color of your eyes, they are talking about the color of your iris. The blackness at the center of your iris is your pupil.

There are two types of photoreceptor cells in your retina: rods and cones. The rods are more sensitive; in very dark conditions, most of our vision is provided by the rods. The cones are used when there is plenty of light, and they let us see colors.

The white part around the outside of the eyeball? That is called the *sclera*.



The walls of the eye are lined inside with the *retina*, which has sensors that pick up the light and send impulses down the optic nerve to your brain.

Just like a camera, the images are flipped when they get projected on the back of the eye.

44.1 Eye problems

Now that you know the mechanics of the eye, let's enumerate a few things that commonly go wrong with the eye.

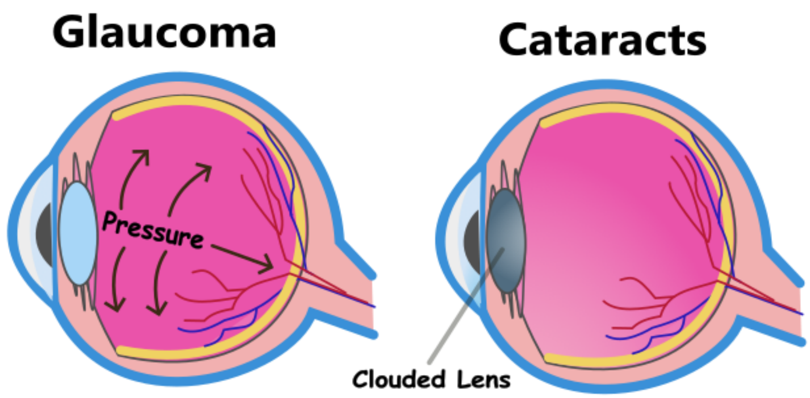
44.1.1 Glaucoma

The space between your cornea and lens is filled with a fluid called *aqueous humor*. To feed the cells of the cornea and lens, the aqueous humor carries oxygen and nutrients like blood would, but it is transparent so you can see. Aqueous humor is constantly being pumped into and out of that chamber. If aqueous humor has trouble exiting, the pressure builds up and can damage the eye. This is known as *glaucoma*.

44.1.2 Cataracts

The lens should be clear. As a person ages (and it can be accelerated by diabetes, too much exposure to sunlight, smoking, obesity, and high blood pressure), the proteins in the lens break down and clump together, becoming opaque. From the outside, the eye will look cloudy. This is called a *cataract*, and it makes it difficult for the person to see.

The problem can be corrected: The person's cloudy lens is removed and replaced with a



clear, manufactured lens.

44.1.3 Nearsightedness, farsightedness, and astigmatism

If you are in a dark room and a tiny LED is turned on, the photons from that LED can pass through your cornea in many different places. If your eye is focusing on that light correctly, all the photons should meet up at the same place on the retina.

FIXME: Diagram here

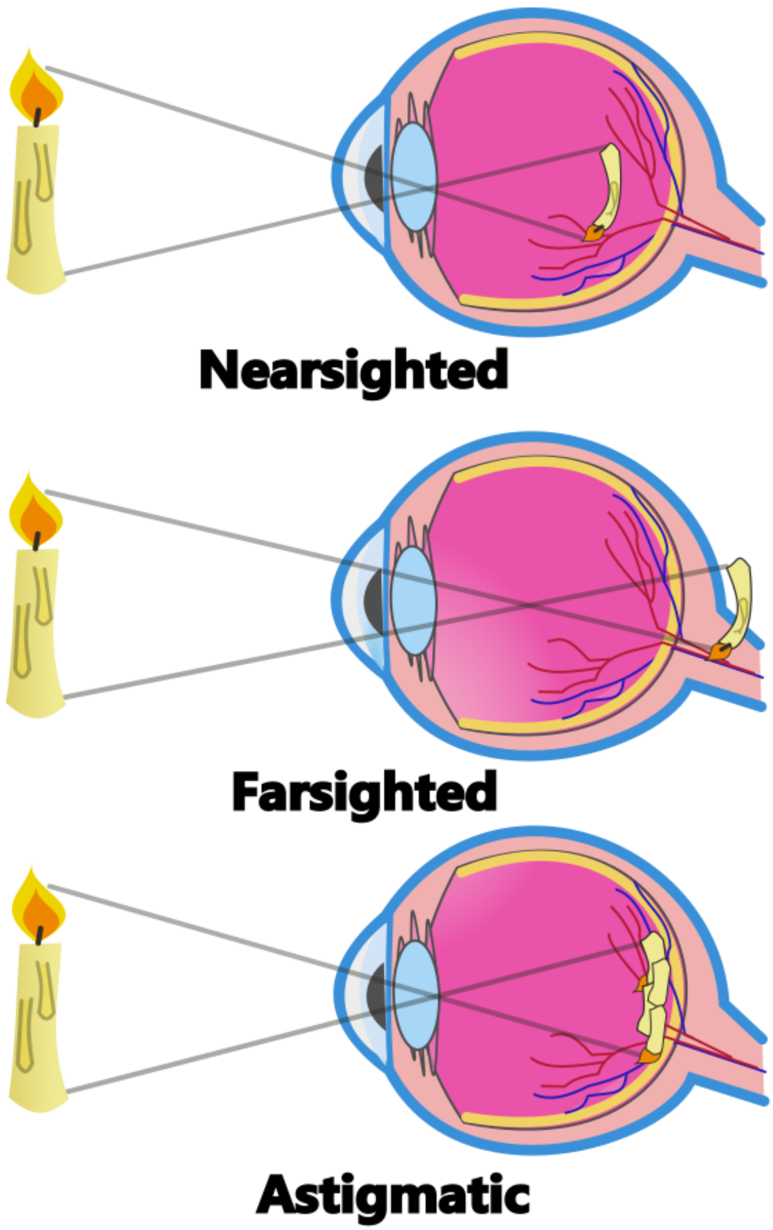
If the lenses are bending the light too much, the photons meet up before they hit the retina and get smeared a bit across the retina. To the person, the LED would appear blurry. The eye is said to be *nearsighted* or *myopic*.

If the lenses are not bending it enough, the photons would meet up behind the retina. Once again, they get smeared a bit across the retina and the LED looks blurry to the person. The eye is said to be *farsighted* or *hyperoptic*.

Your lenses are supposed to bend the photons the same amount vertically and horizontally.

If one dimension is focused, but the other is myopic or hyperoptic, the eye is said to have *astigmatism*.

Myopia, hyperoptia, and astigmatism can be corrected with glasses or contact lenses. Doctors can also do surgical corrections, usually by changing the shape of the cornea.

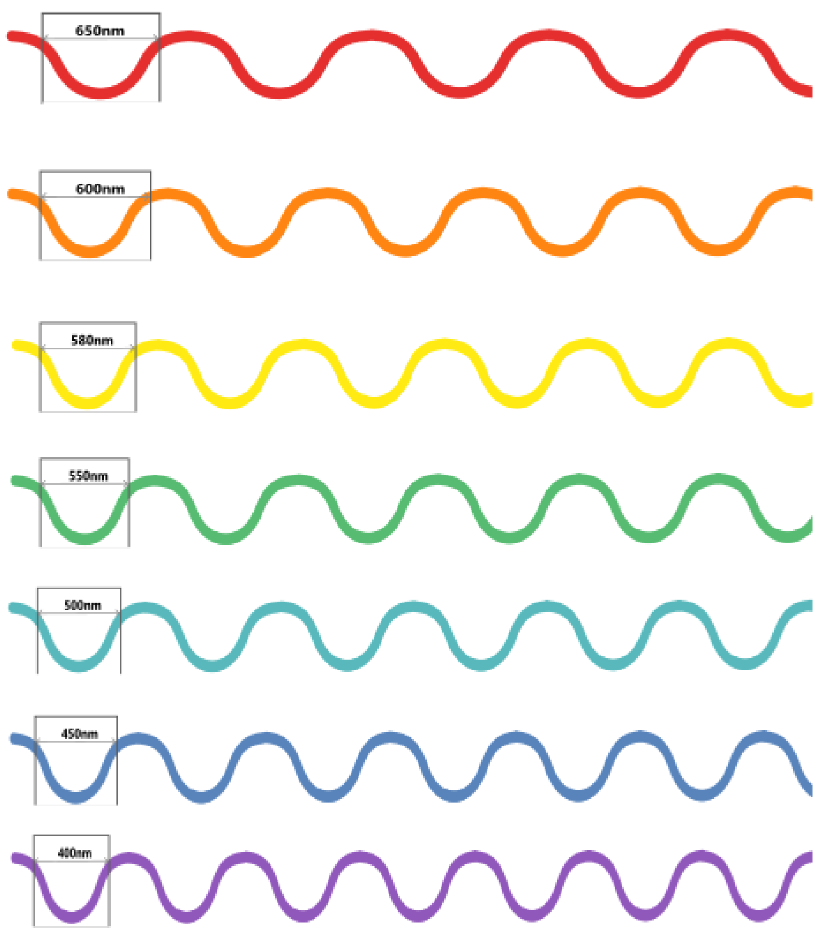


44.2 Seeing colors

TED-Ed has made a good video on how we see color. Watch it here: `https://youtu.be/l8_fZPHasdo`

When a rainbow forms, you are seeing different wavelengths separating from each other. In the rainbow:

- Red is about 650 nm.
- Orange is about 600 nm.
- Yellow is about 580 nm.
- Green is about 550 nm.
- Cyan is about 500 nm.
- Blue is about 450 nm.
- Violet is about 400 nm.



If you shine a light with a wavelength of 580 nm on a white piece of paper, you will see yellow.

However, if you shine two lights with wavelengths of 650 nm (red) and 550 nm (green), you will also see yellow.

Why? Our ears can hear two different frequencies at the same time. Why can't our eyes see two colors in the same place?

As mentioned above, the cone photoreceptors in our eyes let us see colors. There are three kinds of cones:

- Blue: Cones that are most sensitive to frequencies near 450nm.
- Green: Cones that are most sensitive to frequencies near 550nm.
- Red: Cones that let us see the frequencies up to about 700nm.

When a wavelength of 580 nm hits your retina, it excites the red and green receptors, and your brain interprets that mix as yellow.

Similarly, when light that contains both 650 nm and 550 nm waves hits your retina, it excites the red and green receptors, and your brain interprets that mix as yellow.

You can't tell the difference!

Now we know why the sensors on the camera are RGB. The camera is recording the scene as closely as necessary to fool your eye.

A TV or a color computer monitor only has three colors of pixels: red, green, and blue. By controlling the mix of them, it creates the sensation of thousands of colors to your eye.

44.3 Pigments

A color printer works oppositely: Instead of radiating colors, it puts pigments on the paper that absorb certain frequencies. A pigment that absorbs only frequencies near 650 nm (red) will appear to your eye as cyan. This makes sense because the sensation of cyan is created when your blue and green receptors are activated.

Thus, pigment colors come in:

- Cyan: absorbs frequencies around red
- Magenta: absorbs freqencies around green
- Yellow: absorbs frequencies around blue

If you buy ink for a color printer, you know there is typically a fourth ink: black. If you put cyan, magenta, and yellow pigments on paper, the mix won't absorb all the visible spectrum in a consistent manner, and our eyes are pretty sensitive to that, so we would see brown. So we add black ink to get pretty grays and blacks.

We call this approach to color CMYK (as opposed to RGB). If an artist is creating an image to be viewed on a screen, they will typically make an RGB image. If they are creating an image to be printed using pigments, they typically create a CMYK image. (Most of us don't care so much – we just let the computer do conversions between the two color spaces for us.)

Chapter 45

Images in Python

An image is usually represented as a three-dimensional array of 8-bit integers. NumPy arrays are the most commonly used library for this sort of data structure.

If you have an RGB image that is 480 pixels tall and 640 pixels wide, you will need a $480 \times 640 \times 3$ NumPy array.

There is a separate library (imageio) that:

- Reads an image file (like JPEG files) and creates a NumPy array.
- Writes a NumPy array to a file in standard image formats

Let's create a simply python program that creates a file containing an all-black image that is 640 pixels wide and 480 pixels tall. Create a file called `create_image.py`:

```
import NumPy as np
import imageio
```

```
import sys

# Check command-line arguments
if len(sys.argv) < 2:
    print(f"Usage {sys.argv[0]} <outfile>")
    sys.exit(1)

# Constants
IMAGE_WIDTH = 640
IMAGE_HEIGHT = 480

# Create an array of zeros
image = np.zeros((IMAGE_HEIGHT, IMAGE_WIDTH, 3), dtype=np.uint8)

# Write the array to the file
imageio.imwrite(sys.argv[1], image)
```

To run this, you will need to supply the name of the file you are trying to create. The extension (like .png or .jpeg) will tell imageio what format you want written. Run it now:

```
python3 create_image.py blackness.png
```

Open the image to confirm that it is 640 pixels wide, 480 pixels tall, and completely black.

45.1 Adding color

Now, let's walk through through the image, pixel-by-pixel, adding some red. We will gradually increase the red from 0 on the left to 255 on the right.

```
import NumPy as np
import imageio
import sys

# Check command-line arguments
if len(sys.argv) < 2:
    print(f"Usage sys.argv[0] <outfile>")
    sys.exit(1)

# Constants
IMAGE_WIDTH = 640
IMAGE_HEIGHT = 480
```

```
# Create an array of zeros
image = np.zeros((IMAGE_HEIGHT, IMAGE_WIDTH, 3), dtype=np.uint8)

for col in range(IMAGE_WIDTH):

    # Red goes from 0 to 255 (left to right)
    r = int(col * 255.0 / IMAGE_WIDTH)

    # Update all the pixels in that column
    for row in range(IMAGE_HEIGHT):
        # Set the red pixel
        image[row, col, 0] = r

# Write the array to the file
imageio.imwrite(sys.argv[1], image)
```

When you run the function to create a new image, it will be a fade from black to red as you move from left to right:

Now, inside the inner loop, update the blue channel so that it goes from zero at the top to 255 at the bottom:

```
    # Update all the pixels in that column
    for row in range(IMAGE_HEIGHT):

        # Update the red channel
        image[row,col,0] = r

        # Blue goes from 0 to 255 (top to bottom)
        b = int(row * 255.0 / IMAGE_HEIGHT)
        image[row,col,2] = b

imageio.imwrite(sys.argv[1], image)
```

When you run the program again, you will see the color fades from black to blue as you go down the left side. As you go down the right side, the color fades from red to magenta.

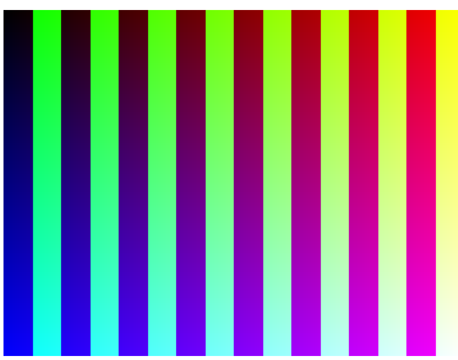
Notice that red and blue with no green looks magenta to your eye.

Now let's add some stripes of green:

```
import NumPy as np
import imageio
import sys

# Check command line arguments
if len(sys.argv) < 2:
    print(f"Usage sys.argv[0] <outfile>")
    sys.exit(1)

# Constants
IMAGE_WIDTH = 640
IMAGE_HEIGHT = 480
STRIPE_WIDTH = 40
pattern_width = STRIPE_WIDTH * 2

# Create an image of all zeros
image = np.zeros((IMAGE_HEIGHT, IMAGE_WIDTH, 3), dtype=np.uint8)

# Step from left to right
for col in range(IMAGE_WIDTH):

    # Red goes from 0 to 255 (left to right)
    r = int(col * 255.0 / IMAGE_WIDTH)

    # Should I add green to this column?
    should_green = col % pattern_width > STRIPE_WIDTH

    # Update all the pixels in that column
```

```
    for row in range(IMAGE_HEIGHT):

        # Update the red channel
        image[row,col,0] = r

        # Should I add green to this pixel?
        if should_green:
            image[row,col,1] = 255

        # Blue goes from 0 to 255 (top to bottom)
        b = int(row * 255.0 / IMAGE_HEIGHT)
        image[row,col,2] = b

imageio.imwrite(sys.argv[1], image)
```

When you run this version, you will see the previous image in half the stripes. In the other half, you will see that green fades to cyan down the left side and yellow fades to white down the right side.



45.2 Using an existing image

imageio can also be used to read in any common image file format. Let's read in an image and save each of the red, green, and blue channels out as its own image.

Create a new file called `separate_image.py`:

```
import imageio
import sys
import os

# Check command line arguments
if len(sys.argv) < 2:
```

```
    print(f"Usage {sys.argv[0]} <infile>")
    sys.exit(1)

# Read the image
path = sys.argv[1]
image = imageio.imread(path)

# What is the filename?
filename = os.path.basename(path)

# What is the shape of the array?
original_shape = image.shape

# Log it
print(f"Shape of {filename}:{original_shape}")

# Names of the colors for the filenames
colors = ['red','green','blue']

# Step through each of the colors
for i in range(3):

    # Create a new image
    newimage = np.zeros(original_shape, dtype=np.uint8)

    # Copy one channel
    newimage[:,:,i] = image[:,:,i]

    # Save to a file
    new_filename = f"{colors[i]}_{filename}"
    print(f"Writing {new_filename}")
    imageio.imwrite(new_filename, newimage)
```

Now you can run the program with any common RGB image type:

```
python3 separate_image.py dog.jpg
```

This will create three images: `red_dog.jpg`, `green_dog.jpg`, and `blue_dog.jpg`.

Chapter 46

Introduction to Polynomials

Watch Khan Academy's **Polynomials intro** video at https://youtu.be/Vm7H0VTlIco

A *monomial* is the product of a number and a variable raised to a non-negative (but possibly zero) integer power. Here are some monomials:

$3x^2$	πx^2	$7x$	$-\frac{2}{3}x^{12}$
$-2x^{15}$	$(3.33)x^{100}$	3	0

The exponent is called the *degree* of the monomial. Examples: $3x^{17}$ has degree 17, $-7x$ has degree 1, and 3.2 has degree 0 (because you can think of it as $(3.2)x^0$).

The number in the product is called the *coefficient*. Example: $3x^{17}$ has a coefficient of 3, $-2x$ has a coefficient of -2, and $(3.4)x^{1000}$ has a coefficient of 3.4.

A *polynomial* is the sum of one or more monomials. Here are some polynomials:

$4x^2 + 9x + 3.9$ $\qquad$ $\pi x^2 + \pi x + \pi$ $\qquad$ $7x + 2$

$-2x^{10} + (3.4)x - 45x^{900} - 1$ $\qquad$ 3.3 $\qquad$ $3x^{20}$

We say that each monomial is a *term* of the polynomial.

$x^{-5} + 12$ is *not* a polynomial because the first term has a negative exponent.

$x^2 - 32x^{\frac{1}{2}} + x$ is *not* a polynomial because the second term has a non-integer exponent.

$\frac{x+2}{x^2+x+5}$ is *not* a polynomial because it is not just a sum of monomials.

Exercise 51 Identifying Polynomials

Circle only the polynomials.

$-2x^3+\frac{1}{2}x+3.9(4.5)x^2 + \pi x$ $\qquad$ 7

$2x^{-10}+4x-1$ $\qquad$ $x^{\frac{2}{3}}$ $\qquad$ $3x^{20}+2x^{19}-5x^{18}$

Working Space

Answer on Page 404

We typically write a polynomial starting at the term with the highest degree and proceed in decreasing order to the term with the lowest degree:

$$2x^9 - 3x^7 + \frac{3}{4}x^3 + x^2 + \pi x - 9.3$$

This is known as *the standard form*. The first term of the standard form is called *the leading term*, and we often call the coefficient of the leading term *the leading coefficient*. We sometimes speak of the degree of the polynomial, which is just the degree of the leading term.

Exercise 52 Standard of a Polynomial

Write $21x^2 - x^3 + \pi - 1000x$ in standard form. What is the degree of this polynomial? What is its leading coefficient?

Working Space

Answer on Page 404

Exercise 53 Evaluate a Polynomial

Let $y = x^3 - 3x^2 + 10x - 12$. What is y when x is 4?

Working Space

Answer on Page 405

I would be remiss in my duties if I didn't mention one more thing about polynomials: mathematicians have defined a polynomial to be a sum of a *finite* number of monomials.

It is certainly possible to have a sum of an infinite number of monomials like this:

$$1 + \frac{1}{2}x + \frac{1}{4}x^2 + \frac{1}{8}x^3 + \frac{1}{16}x^4 + \ldots$$

This is an example of an *infinite series*; we don't consider them polynomials. Infinite series are interesting and useful, but I will not discuss them much until later in the course.

Chapter 47

Python Lists

Watch CS Dojo's **Introduction to Lists in Python** video at https://www.youtube.com/watch?v=tw7ror9x32s

To review, Python list is an indexed collection. The indices start at zero. You can create a list using square brackets.

Now you are going to write a program that makes an array of strings. Type this code into a file called `faves.py`:

```
favorites = ["Raindrops", "Whiskers", "Kettles", "Mittens"]
favorites.append("Packages")
print("Here are all my favorites:", favorites)
print("My most favorite thing is", favorites[0])
print("My second most favorite is", favorites[1])
number_of_faves = len(favorites)
print("Number of things I like:", number_of_faves)

for i in range(number_of_faves):
```

```
    print(i, ": I like", favorites[i])
```

Run it:

```
$ python3 faves.py
Here are all my favorites: ['Raindrops', 'Whiskers', 'Kettles', 'Mittens', 'Packages']
My most favorite thing is Raindrops
My second most favorite is Whiskers
Number of things I like: 5
0 : I like Raindrops
1 : I like Whiskers
2 : I like Kettles
3 : I like Mittens
4 : I like Packages
```

After you have run the code, study it until the output makes sense.

Exercise 54 Assign into list

Before you list the items, replace "Mittens" with "Gloves".

Working Space

Answer on Page 405

47.1 Evaluating Polynomials in Python

First, before you go any further, you need to know that raising a number to a power is done with ** in Python. So for example, to get 5^2, you would write `5**2`.

Back to polynomials: if you had a polynomial like $2x^3 - 9x + 12$, you could write it like this: $12x^0 + (-9)x^1 + 0x^2 + 2x^3$. We could use this representation to keep a polynomial in a Python list. We would simply store all the coefficients in order:

```
pn1 = [12,-9,0,2]
```

In the list, the index of each coefficient would correspond to the degree of that monomial. For example, in the list 2 is at index 3, so that entry represents $2x^3$.

In the last chapter, you evaluated the polynomial $x^3 - 3x^2 + 10x - 12$ at $x = 4$. Now you will write code that does that evalution. Create a file called `polynomials.py` and type in the following:

```
def evaluate_polynomial(pn, x):
    sum = 0.0
    for degree in range(len(pn)):
        coefficient = pn[degree]
        term_value = coefficient * x ** degree
        sum = sum + term_value
    return sum

pn1 = [-12.0, 10.0, -3.0, 1.0]
y = evaluate_polynomial(pn1, 4.0)
print("Polynomial 1: When x is 4.0, y is", y)
```

Run it. It should evaluate to 44.0.

47.2 Walking the list backwards

Now you are going to make a function that makes a pretty string to represent your polynomial. Here is how it will be used:

```
def polynomial_to_string(pn):
    ...Your Code Here...

pn_test = [-12.0, 10.0, 0.0, 1.0]
print(polynomial_to_string(pn1))
```

This would output:

```
1.0x**3 + 10.0x + -12.0
```

This is not as simple as you might hope. In particular:

- You should skip the terms with a coefficient of zero
- The term of degree 1 has an x, but no exponent
- The term of degree 0 has neither an x nor an exponent

- Standard form demands that you list the terms in the reverse order from that of your coefficients list. You will need to walk the list from last to first.

Add this function to your `polynomials.py` file after your `evaluate_polynomial` function:

```
def polynomial_to_string(pn):

    # Make a list of the monomial strings
    monomial_strings = []

    # Start at the term with the largest degree
    degree = len(pn) - 1

    # Go through the list backwards stop after constant term
    while degree >= 0:
        coefficient = pn[degree]

        # Skip any term with a zero coefficient
        if coefficient != 0.0:

            # Describe the monomial
            if degree == 0:
                monomial_string = "{}".format(coefficient)
            elif degree == 1:
                monomial_string = "{}x".format(coefficient)
            else:
                monomial_string = "{}x^{}".format(coefficient, degree)

            # Add it to the list
            monomial_strings.append(monomial_string)

        # Move to the previous term
        degree = degree - 1

    # Deal with the zero polynomial
    if len(monomial_strings) == 0:
        monomial_strings.append("0.0")

    # Make a string that joins the terms with a plus sign
    return " + ".join(monomial_strings)
```

Note that in a list n items, the indices go from 0 to $n-1$. So when we are walking the list backwards, we start at `len(pn) - 1` and stop at zero.

Look over the code and google the functions you aren't familar with. For example, if you want to know about the (join) function, google for "python join".

Now change your code to use the new function:

```
pn1 = [-12.0, 10.0, -3.0, 1.0]
y = evaluate_polynomial(pn1, 4.0)
print("y =", polynomial_to_string(pn1))
print("    When x is 4.0, y is", y)
```

Run the program. Does the function work?

Exercise 55 Evaluate Polynomials

Using the function that you just wrote, add a few lines of code to `polynomials.py` to evaluate the following polynomials:

- Find $4x^4 - 7x^3 - 2x^2 + 5x + 2.5$ at $x = 8.5$. It should be 16481.875
- Find $5x^5 - 9$ at $x = 2.0$. It should be 151.0

Working Space

Answer on Page 405

47.3 Plot the polynomial

We can evaluate a polynomial at many points and plot them on a graph. You are going to write the code to do this. Create a new file called `plot_polynomial.py`. Copy your `evaluate_polynomial` function into the new file.

Add a line at the beginning of the program that imports the plotting library matplotlib:

```
import matplotlib.pyplot as plt
```

After the `evaluate_polynomial` function:

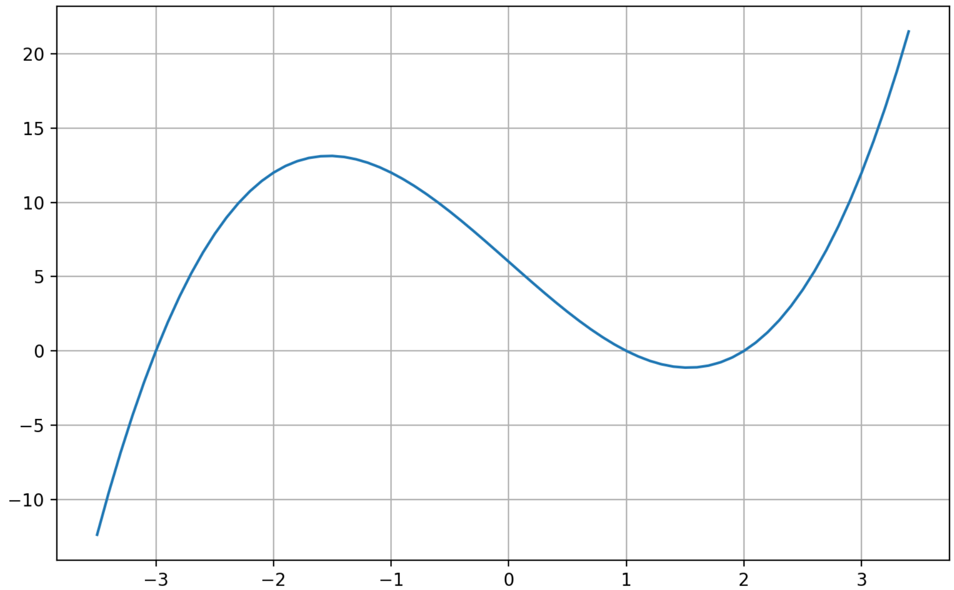
- Create a list with polynomial coefficients.

- Create two empty arrays, one for x values and one for y values.
- Fill the x array with values from -3.5 to 3.5. Evaluate the polynomial at each of these points; put those values in the y array.
- Plot them

Like this:

```
# x**3 - 7x + 6
pn = [6.0, -7.0, 0.0, 1.0]

# These lists will hold our x and y values
x_list = []
y_list = []

# Start at x=-3.5
current_x =-3.5

# End at x=3.5
while current_x <= 3.5:
    current_y = evaluate_polynomial(pn, current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.1

# Plot the curve
plt.plot(x_list, y_list)
plt.grid(True)
plt.show()
```

You should get a beautiful plot like this:



If you received an error that the matplotlib was not found, use pip to install it:

```
$ pip3 install matplotlib
```

Exercise 56 Observations

Where does your polynomial cross the y-axis? Looking at the polynomial $x^3 - 7x+6$, could you have guessed that value?

Where does your polynomial cross the x-axis? The places where a polynomial crosses the x-axis is called *its roots*. Later in the course, you will learn techniques for finding the roots of a polynomial.

Working Space

Answer on Page 405

Chapter 48

Adding and Subtracting Polynomials

Watch Khan Academy's **Adding polynomias** video at https://youtu.be/ahdKdxsTj8E

When adding two monomials of the same degree, you sum their coefficients:

$$7x^3 + 4x^3 = 11x^3$$

Using this idea, when adding two polynomials, you convert it into one long polynomial and then simplify by combining terms with the same degree. For example:

$$\begin{aligned}
&(10x^3 - 2x + 13) + (-5x^2 + 7x - 12) \\
&\qquad = 10x^3 + (-2)x + 13 + (-5)x^2 + 7x + (-12) \\
&\qquad = 10x^3 + (-5)x^2 + (-2 + 7)x + (13 - 12) \\
&\qquad = 10x^3 - 5x^2 + 5x + 1
\end{aligned}$$

Exercise 57 Adding Polynomials Practice

Working Space

Add the following polynomials:

1. $2x^3 - 5x^2 + 3x - 9$ and $x^3 - 2x^2 - 2x - 9$

2. $3x^5 - 5x^3 + 3x^2 - x - 3$ and $2x^4 - 2x^3 - 2x^2 + x - 9$

Answer on Page 405

Notice that in the second question, the degree 1 term disappears completely: $(-x) + x = 0$

One more tricky thing that can happen: Sometimes the coefficients don't add nicely. For example:

$$\pi x^2 - 3x^2 = (\pi - 3)x^2$$

That is as far as you can simplify it.

48.1 Subtraction

Now watch Khan Academy's **Subtracting polynomials** at `https://youtu.be/5ZdxnFspyP8`.

When subtracting one polynomial from the other, it is a lot like adding two polynomials. The difference: when make the two polynomials into one long polynomial, we multiply each monomial that is being subtracted by -1. For example:

$$(2x^2 - 3x + 9) - (5x^2 - 7x + 4)$$
$$= 2x^2 + (-3)x + 9 + (-5)x^2 + 7x + (-4)$$
$$= (2 - 5)x^2 + (-3 + 7)x + (9 - 4)$$
$$= -3x^2 + 4x + 5$$

Exercise 58 Subtracting Polynomials Practice

Working Space

Add the following polynomials:

1. $(2x^3 - 5x^2 + 3x - 9) - (x^3 - 2x^2 - 2x - 9)$

2. $(3x^5 - 5x^3 + 3x^2 - x - 3) - (2x^4 - 2x^3 - 2x^2 + x - 9)$

Answer on Page 405

48.2 Adding Polynomials in Python

As a reminder, in our Python code, we are representing a polynomial with a list of coefficients. The first coefficient is the constant term. The last coefficient is the leading coefficient. So, we can imagine $-5x^3 + 3x^2 - 4x + 9$ and $2x^3 + 4x^2 - 9$ would look like this: *FIXME: Diagram here*

To add the two polynomials then, we sum the coefficients for each degree. *FIXME: Diagram here*

Create a file called `add_polynomials.py`, and type in the following:

```
def add_polynomials(a, b):
    degree_of_result = len(a)
    result = []
    for i in range(degree_of_result):
        coefficient_a = a[i]
        coefficient_b = b[i]
        result.append(coefficient_a + coefficient_b)
    return result
```

```
polynomial1 = [9.0, -4.0, 3.0, -5.0]
polynomial2 = [-9.0, 0.0, 4.0, 2.0]
polynomial3 = add_polynomials(polynomial1, polynomial2)

print('Sum =', polynomial3)
```

Run the program.

Unfortunately, this code only works if the polynomails are the same length. For example, try making `polynomial1` have a larger degree than `polynomial2`:

```
# x**4 - 5x**3 + 3x**2 - 4x + 9
polynomial1 = [9.0, -4.0, 3.0, -5.0, 1.0]

# 2x**3 + 4x**2 - 9
polynomial2 = [-9.0, 0.0, 4.0, 2.0]
polynomial3 = add_polynomials(polynomial1, polynomial2)
print('Sum =', polynomial3)
```

See the problem?

Exercise 59 Dealing with polynomials of different degrees

Working Space

Can you fix the function `add_polynomials` to handle polynomials of different degrees?
Here is a hint: In Python, there is a `max` function that returns the largest of the numbers it is passed.

```
biggest = max(5,7)
```

Here `biggest` would be set to 7.
Here is another hint: If you have an array `mylist`, `i`, a non-negative integer, is only a legit index if `i < len(mylist)`.

Answer on Page 406

48.3 Scalar multiplication of polynomials

If you multiply a polynomial with a number, the distributive property applies:

$$(3.1)(2x^2 + 3x + 1) = (6.2)x^2 + (9.3)x + 3.1$$

(When we are talking about things that are more complicated than a number, we use the word *scalar* to mean "Just a number". So this is the product of a scalar and a polynomial.)

In `add_polynomials.py`, add a function to that multiplies a scalar and a polynomial:

```
def scalar_polynomial_multiply(s, pn):
    result = []
    for coefficient in pn:
        result.append(s * coefficient)
    return result
```

Somewhere near the end of the program, test this function:

```
polynomial4 = scalar_polynomial_multiply(5.0, polynomial1)
print('Scalar product =', polynomial_to_string(polynomial4))
```

Exercise 60 Subtract polynomials in Python

Working Space

Now implement a function that does subtraction using `scalar_polynomial_multiply` and `add_polynomials`.
It should look like this:

```
def subtract_polynomial(a, b):
    ...Your code here...

polynomial5 = [9.0, -4.0, 3.0, -5.0]
polynomial6 = [-9.0, 0.0, 4.0, 2.0, 1.0]
polynomial7 = subtract_polynomial(polynomial5, polynomial6)
print('Difference =', polynomial_to_string(polynomial7))
```

Answer on Page 406

Chapter 49

Multiplying Polynomials

Watch Khan Academy's **Multiplying monomials** at https://youtu.be/Vm7H0VTlIco.

To review, when you multiply two monomials, you take the product of their coefficients and the sum of their degrees:

$$(2x^6)(5x^3) = (2)(5)(x^6)(x^3) = 10x^9$$

If you have a product of more than two monomials, multiply *all* the coefficients and sum *all* the exponents:

$$(3x^2)(2x^3)(4x) = (3)(2)(4)(x^2)(x^3)(x^1) = 24x^6$$

Exercise 61 Multiplying monomials

Working Space

Multiply these monomials

1. $(3x^2)(5x^3)$

2. $(2x)(4x^9)$

3. $(-5.5x^2)(2x^3)$

4. $(\pi)(-2x^5)$

5. $(2x)(3x^2)(5x^7)$

Answer on Page 406

49.1 Multiplying a monomial and a polynomial

Watch Khan Academy's **Multiplying monomials by polynomials** at `https://youtu.be/pD2-H15ucNE`.

When multiplying a monomial and a polynomial, you use the the distributive property.

Then it is just multiplying several pairs of monomials:

$$(3x^2)(4x^3 - 2x^2 + 3x - 7)$$
$$= (3x^2)(4x^3) + (3x^2)(-2x^2) + (3x^2)(3x) + (3x^2)(-7)$$
$$= 12x^5 - 6x^4 + 9x^3 - 21x^2$$

Exercise 62 Multiplying a monomial and a polynomial

Working Space

Multiply these monomials

1. $(3x^2)(5x^3 - 2x + 3)$
2. $(2x)(4x^9 - 1)$
3. $(-5.5x^2)(2x^3 + 4x^2 + 6)$
4. $(\pi)(-2x^5 + 3x^4 + x)$
5. $(2x)(3x^2)(5x^7 + 2x)$

Answer on Page 407

49.2 Multiplying polynomials

Watch Khan Academy's **Multiplying binomials by polynomials** video at https://youtu.be/D6mivA_8L8U

When you are multiplying two polynomials, you will use the distributive property several times to make it one long polynomial. Then you will combine the terms with the same degree. For example,

$$\begin{aligned}
&(2x^2-3)(5x^2+2x-7)\\
&= (2x^2)(5x^2+2x-7) + (-3)(5x^2+2x-7)\\
&= (2x^2)(5x^2) + (2x^2)(2x) + (2x^2)(-7) + (-3)(5x^2) + (-3)(2x) + (-3)(-7)\\
&= 10^4 + 4x^3 + -14x^2 + -15x^2 + -6x + 21 = 10^4 + 4x^3 + -29x^2 + -6x + 21
\end{aligned}$$

One common form that you will see is multiplying two binomials together:

$$(2x+7)(5x+3) = (2x)(5x+3) + (7)(5x+3) = (2x)(5x) + (7)(5x) + (2x)(3) + (7)(3)$$

Notice the product has become the sum of four parts: the firsts, the inners, the outers, and the lasts. People sometimes use the mnemonic FOIL to remember this pattern, but there is a general rule that works for all product of polynomials, not just binomials. Here it is: Every term in the first will be multiplied by every term in the second, and then just add them together.

So, for example, if you have a polynomial s with three terms and you multiply it by a polynomial t with five terms, you will get a sum of 15 terms – each term is a product of two monomials, one from s and one from t. (Of course, several of those terms might have the same degree, so they will be combined together when you simplify. Thus you typically end up with a polynomial with less than 15 terms.)

Using this rule, here is how I would multiply $2x^2 - 3x + 1$ and $5x^2 + 2x - 7$:

$$\begin{aligned}
(2x^2-3x+1)(5x^2+2x-7) &= \begin{matrix} (2x^2)(5x^2) & + & (2x^2)(2x) & + & (2x^2)(-7) & + \\ (-3x)(5x^2) & + & (-3x)(2x) & + & (-3x)(-7) & + \\ (1)(5x^2) & + & (1)(2x) & + & (1)(-7) & \end{matrix}\\
&= 10x^4 + 4x^3 + (-14)x^2 + (-15)x^3 + (-6)x^2 + 21x + 5x^2 + 2x + (-7)\\
&= 10x^4 + (4-15)x^3 + (-14-6+5)x^2 + (21+2)x + (-7)\\
&= 10x^4 - 11x^3 - 15x^2 + 23x - 7
\end{aligned}$$

Note that the product (before combining terms with the same degree) has $3 \times 3 = 9$ terms – every possible combination of a term from the first polynomial and a term from the second polynomial.

One common source of error: losing track of the negative signs. You will need to be really careful. I have found that it helps to use + between all terms, and use negative coefficients to express subtraction. For example, if the problem says $4x^2 - 5x - 3$, you should work with that as $4x^2 + (-5)x + (-3)$

Exercise 63 Multiplying polynomials

Multiply the following pairs of polynomials:

1. $2x + 1$ and $3x - 2$
2. $-3x^2 + 5$ and $4x - 2$
3. $-2x - 1$ and $-3x - \pi$
4. $-2x^5 + 5x$ and $3x^5 + 2x$

Working Space

Answer on Page 407

Exercise 64 Observations

Let's say I have two polynomials, p_1 and p_2. p_1 has degree 23. p_2 has degree 12. What is the degree of their product?

Working Space

Answer on Page 407

Chapter 50

Multiplying Polynomials in Python

At this point, you have created a nice toolbox of functions for dealing with lists of coefficients as polynomials. Create a file called `poly.py` and copy the folowing functions into it:

- `evaluate_polynomial`
- `polynomial_to_string`
- `add_polynomials`
- `scalar_polynomial_multiply`
- `subtract_polynomial`

Now create another file in the same directory called `test.py`. Type this into that file:

```
import poly

polynomial_a = [9.0, -4.0, 3.0, -5.0]
print('Polynomial A =', poly.polynomial_to_string(polynomial_a))

polynomial_b = [-9.0, 0.0, 4.0, 2.0, 1.0]
print('Polynomial B =', poly.polynomial_to_string(polynomial_b))

# Evaluation
value_of_b = poly.evaluate_polynomial(polynomial_b, 3)
print('Polynomial B at 3 =', value_of_b)

# Adding
a_plus_b = poly.add_polynomials(polynomial_a, polynomial_b)
print('A + B =', poly.polynomial_to_string(a_plus_b))

# Scalar multiplication
b_scalar = poly.scalar_polynomial_multiply(-3.2, polynomial_b)
print('-3.2 * Polynomial B =', poly.polynomial_to_string(b_scalar))

# Subtraction
a_minus_b = poly.subtract_polynomial(polynomial_a, polynomial_b)
print('A - B =', poly.polynomial_to_string(a_minus_b))
```

When you run it, you should get the following:

```
Polynomial A = -5.0x^3 + 3.0x^2 + -4.0x + 9.0
Polynomial B = 1.0x^4 + 2.0x^3 + 4.0x^2 + -9.0
Polynomial B at 3 = 162.0
A + B = 1.0x^4 + -3.0x^3 + 7.0x^2 + -4.0x
-3.2 * Polynomial B = -3.2x^4 + -6.4x^3 + -12.8x^2 + 28.8
A - B = -1.0x^4 + -7.0x^3 + -1.0x^2 + -4.0x + 18.0
```

Now you are ready to implement multiplication of polynomials. The function will look like this:

```
def multiply_polynomials(a, b):
  ...Your code here...
```

It will return a list of coefficients.

In an exercise in the last chapter, you were asked " Let's say I have two polynomials, p_1 and p_2. p_1 has degree 23. p_2 has degree 12. What is the degree of their product?" The answer was $23 + 12 = 35$.

In our implementation, a polynomial of degree 23 is held in a list of length 24.

In Python we wil be trying to multiply a polynomial a and a polynomial b represented as lists. What is the degree of that product?

```
result_degree = (len(a) - 1) + (len(b) - 1)
```

Now, we need to create an array of zeros that is one longer than that. Here is a cute Python trick: if you have a list, you can replicate it using the * operator.

```
a = [5,7]
b = a * 4
print(b)
# [5, 7, 5, 7, 5, 7, 5, 7]
```

Here's how you will get a list of zeros:

```
result = [0.0] * (result_degree + 1)
```

We will step through a getting the index and value of each entry. You can do this in one line using `enumerate`:

```
for a_degree, a_coefficient in enumerate(a):
```

For each of those, we will step through the entire b polynomial. As you multiply together each term, you will add it to appropriate coefficient of the result.

Here is the whole function:

```
def multiply_polynomials(a, b): # What is the degree of the resulting
polynomial?  result_degree = (len(a) - 1) + (len(b) - 1)

    # Make a list of zeros to hold the coefficents result = [0.0] *
    (result_degree + 1)

    # Iterate over the indices and values of a for a_degree,
    a_coefficient in enumerate(a):

        # Iterate over the indices and values of b for b_degree,
        b_coefficient in enumerate(b):
```

```
            # Calculate the resulting monomial coefficient =
            a_coefficient * b_coefficient degree = a_degree + b_degree

            # Add it to the right bucket
            result[degree] = result[degree] + coefficient

    return result
```

Take a long look at that function. When you understand it, type it into `poly.py`.

In `test.py`, try out the new function:

```
# Multiplication
a_times_b = poly.multiply_polynomials(polynomial_a, polynomial_b)
print('A x B =', poly.polynomial_to_string(a_times_b))
```

This is an example of a *nested loop*. The outer loop steps through the polynomial a. For each step it takes, the inner loop steps through the entire polynomial `b`.

50.1 Something surprising about lists

You can imagine that you might want to create two very similar polynomials. Let's say polynomial c is $x^2 + 2x + 1$ and polynomial d is $x^2 - 2x + 1$. You might think you are very clever to just alter that degree 1 coefficient like this:

```
c = [1.0, 2.0, 1.0]
d = c
d[1] = -2.0
```

If you printed out `c`, you would get $[1.0, -2.0, 1.0]$. Why? You assigned two variables (`c` and `d`) to the *the same list*. So when you use one reference (`d`) to change the list, you see the change if you look at the list from either reference. *FIXME: Diagram of two references to the same list here.*

To create two separate lists, you would need to explicitly make a copy:

```
c = [1.0, 2.0, 1.0]
d = c.copy()
d[1] = -2.0
```

Chapter 51

Differentiating Polynomials

If you had a function that gave you the height of an object, it would be handy to be able to figure out a function that gave you the velocity at which it was rising or falling. The process of converting the position function into a velocity function is known as *differentiation* or *finding the derivative*.

There are a bunch of rules for finding a derivative, but differentiating polynomials only requires three:

- The derivative of a sum is equal to the sum of the derivatives.
- The derivative of a constant is zero.
- The derivative of a nonconstant monomial at^b (a and b are constant numbers, t is time) is abt^{b-1}

So, for example, if I tell you that the height in meters of quadcopter at second t is given by $2t^3 - 5t^2 + 9t + 200$. You could tell me that its vertical velocity is $6t^2 - 10t + 9$

Exercise 65 Differentation of polynomials

Differentiate the following polynomials.

Working Space

Answer on Page 407

Notice that the degree of the derivative is one less than the degree of the original polynomial. (Unless, of course, the degree of the original is already zero.)

Now, if you know that a position is given by a polynomial, you can differentate it to find the object's velocity at any time.

The same trick works for acceleration: Let's say you know a function that gives an object's velocity. To find its acceleration at any time, you take the derivative of the velocity function.

Exercise 66 Differentation of polynomials in Python

Working Space

Write a function that returns the derivative of a polynomial in `poly.py`. It should look like this:

```
def derivative_of_polynomial(pn):
  ...Your code here...
```

When you test it in `test.py`, it should look like this:

```
# 3x**3 + 2x + 5
p1 = [5.0, 2.0, 0.0, 3.0]
d1 = poly.derivative_of_polynomial(p1)
# d1 should be 9x**2 + 2
print("Derivative of", poly.polynomial_to_string(p1),"is", poly.polynomial_to_string(d1))

# Check constant polynomials
p2 = [-9.0]
d2 = poly.derivative_of_polynomial(p2)
# d2 should be 0.0
print("Derivative of", poly.polynomial_to_string(p2),"is", poly.polynomial_to_string(d2))
```

Answer on Page 407

Chapter 52

Python Classes

The built-in types, like strings have functions associated with them. So, for example, if you needed a string converted to uppercase, you would call it's `upper()` function: -

```
my_string = "houston, we have a problem!"
louder_string = my_string.upper()
```

This would set `louder_string` to "HOUSTON, WE HAVE A PROBLEM!" When a function is associated with a datatype like this, it called a *method*. A datatype with methods is known as a *class*. The data of that type is known as *instance* of that class. For example, in the example, we would say "`my_string` is an instance of the class `str`. `str` has a method called `upper`"

The function `type` will tell you the type of any data:

```
print(type(my_string))
```

This will output

```
<class 'str'>
```

A class can also define operators. `+`, for example, is redefined by `str` to concatenate strings together:

```
long_string = "I saw " + "15 people"
```

52.1 Making a Polynomial class

You have created a bunch of useful python functions for dealing with polynomials. Notice how each one has the word "polynomial" in the function name like `derivative_of_polynomial`. Wouldn't it be more elegant if you had a Polynomial class with a `derivative` method? Then you could use your polynomial like this:

```
a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])

print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)
```

And it would output:

```
2.30x^2 + 9.00 plus 2.10x^3 + 4.50x + -2.00 is 2.10x^3 + 2.30x^2 + 4.50x + 7.00
2.30x^2 + 9.00 times 2.10x^3 + 4.50x + -2.00 is 4.83x^5 + 29.25x^3 + -4.60x^2 + 40.50x + -18.0
2.30x^2 + 9.00 times 3 is 6.90x^2 + 27.00
2.30x^2 + 9.00 minus 2.10x^3 + 4.50x + -2.00 is -2.10x^3 + 2.30x^2 + -4.50x + 11.00
Derivative of 2.10x^3 + 4.50x + -2.00 is 6.30x^2 + 4.50
```

Create a file for your class definition called `Polynomial.py`. Enter the following:

```
class Polynomial:
    def __init__(self, coeffs):
        self.coefficients = coeffs.copy()

    def __repr__(self):
```

```
        # Make a list of the monomial strings
        monomial_strings = []

        # For standard form we start at the largest degree
        degree = len(self.coefficients) - 1

        # Go through the list backwards
        while degree >= 0:
            coefficient = self.coefficients[degree]

            if coefficient != 0.0:
                # Describe the monomial
                if degree == 0:
                    monomial_string = "{:.2f}".format(coefficient)
                elif degree == 1:
                    monomial_string = "{:.2f}x".format(coefficient)
                else:
                    monomial_string = "{:.2f}x^{}".format(coefficient, degree)

                # Add it to the list
                monomial_strings.append(monomial_string)

            # Move to the previous term
            degree = degree - 1

        # Deal with the zero polynomial
        if len(monomial_strings) == 0:
            monomial_strings.append("0.0")

        # Separate the terms with a plus sign
        return " + ".join(monomial_strings)

    def __call__(self, x):
        sum = 0.0
        for degree, coefficient in enumerate(self.coefficients):
            sum = sum + coefficient * x ** degree
        return sum

    def __add__(self, b):
        result_length = max(len(self.coefficients), len(b.coefficients))
        result = []
        for i in range(result_length):
            if i < len(self.coefficients):
                coefficient_a = self.coefficients[i]
            else:
                coefficient_a = 0.0
```

```
            if i < len(b.coefficients):
                coefficient_b = b.coefficients[i]
            else:
                coefficient_b = 0.0
            result.append(coefficient_a + coefficient_b)

        return Polynomial(result)

    def __mul__(self, other):

        # Not a polynomial?
        if not isinstance(other, Polynomial):
            # Try to make it a constant polynomial
            other = Polynomial([other])

        # What is the degree of the resulting polynomial?
        result_degree = (len(self.coefficients) - 1) + (len(other.coefficients) - 1)

        # Make a list of zeros to hold the coefficents
        result = [0.0] * (result_degree + 1)

        # Iterate over the indices and values of a
        for a_degree, a_coefficient in enumerate(self.coefficients):

            # Iterate over the indices and values of b
            for b_degree, b_coefficient in enumerate(other.coefficients):

                # Calculate the resulting monomial
                coefficient = a_coefficient * b_coefficient
                degree = a_degree + b_degree

                # Add it to the right bucket
                result[degree] = result[degree] + coefficient

        return Polynomial(result)

    __rmul__ = __mul__

    def __sub__(self, other):
        return self + other * -1.0

    def derivative(self):

        # What is the degree of the resulting polynomial?
        original_degree = len(self.coefficients) - 1
```

```
        if original_degree > 0:
            degree_of_derivative = original_degree - 1
        else:
            degree_of_derivative = 0

        # We can ignore the constant term (skip the first coefficient)
        current_degree = 1
        result = []

        # Differentiate each monomial
        while current_degree < len(self.coefficients):
            coefficient = self.coefficients[current_degree]
            result.append(coefficient * current_degree)
            current_degree = current_degree + 1

        # No terms? Make it the zero polynomial
        if len(result) == 0:
            result.append(0.0)

        return Polynomial(result)
```

Create a second file called `test_polynomial.py` to test it:

```
from Polynomial import Polynomial

a = Polynomial([9.0, 0.0, 2.3])
b = Polynomial([-2.0, 4.5, 0.0, 2.1])

print(a, "plus", b , "is", a+b)
print(a, "times", b , "is", a*b)
print(a, "times", 3 , "is", a*3)
print(a, "minus", b , "is", a-b)

c = b.derivative()

print("Derivative of", b ,"is", c)

slope = c(3)
print("Value of the derivative at 3 is", slope)
```

Run the test code:

```
python3 test_polynomial.py
```

Chapter 53

Common Polynomial Products

In math and physics, you will run into certain kinds of polynomials over and over again. In this chapter, I am going to cover some patterns that you will want to start to recognize.

53.1 Difference of squares

Watch **Polynomial special products: difference of squares** from Khan Academy at `https://youtu.be/uNweU6I4Icw`.

If you are asked what is $(3x-7)(3x+7)$, you would use the distributive property to expand that to $(3x)(3x) + (3x)(7) + (-7)(3x) + (-7)(7)$. Two of the terms cancel each other, so this is $(3x)^2 - (7)^2$. This would simplify to $9x^2 - 49$

You will see this pattern a lot. Anytime you see $(a + b)(a - b)$, you should immediately recognize it equals $a^2 - b^2$. (Note that the order doesn't matter: $(a-b)(a+b)$ also a^2-b^2.)

Working the other way is important too: anytime you see a^2-b^2, that you should recognize that you can change that into the product $(a+b)(a-b)$. Making something into a product like this is known as *factoring*. You probably have done prime factorization of numbers like $42 = 2 \times 3 \times 7$. In the next couple of chapters you will learn to factorize polynomials.

Exercise 67 Difference of Squares

Simply the following products

1. $(2x - 3)(2x + 3)$
2. $(7 + 5x^3)(7 - 5x^3)$
3. $(x - a)(x + a)$
4. $(3 - \pi)(3 + \pi)$
5. $(-4x^3 + 10)(-4x^3 - 10)$
6. $(x + \sqrt{7})(x - \sqrt{7})$ Factor the following polynomials:
7. $x^2 - 9$
8. $49 - 16x^6$
9. $\pi^2 - 25x^8$
10. $x^2 - 5$

Working Space

Answer on Page 408

We are often interested in the roots of a polynomial, that is we want to know "For what values of x does the polynomial evaluate to zer?" For example, when you deal with falling bodies, the first question you might ask would be "How many seconds before the hammer hits the ground?" Once you have factored a polynomial into binomials, you can easily find the roots.

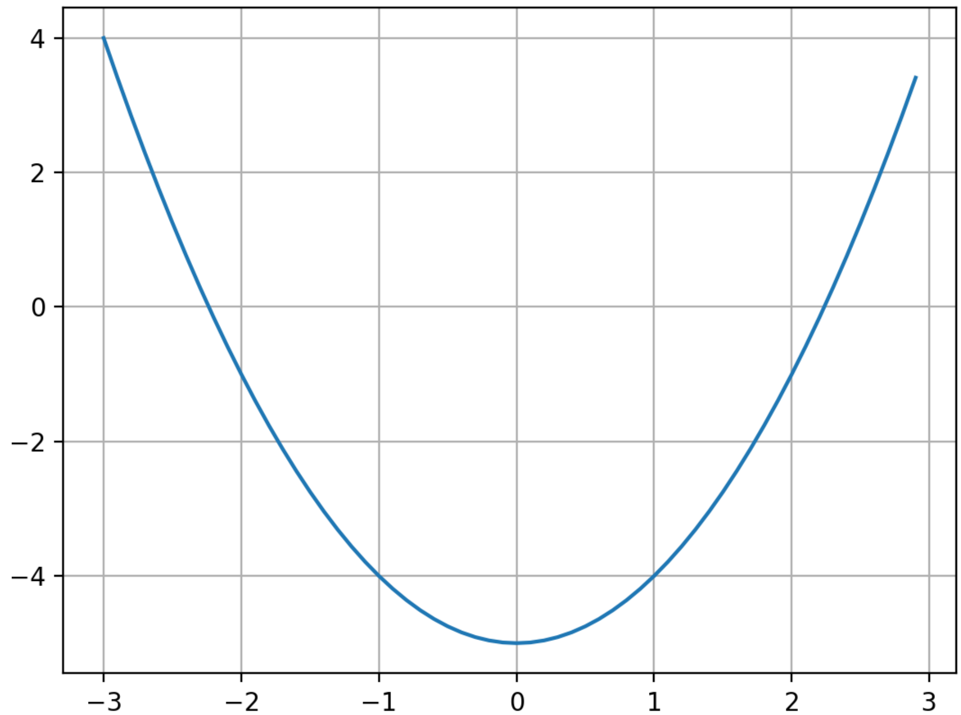
For example, what are the roots of $x^2 - 5$? You just factored it into $(x + \sqrt{5})(x - \sqrt{5})$ This product is zero if and only if one of the factors is zero. The first factor is only zero when x is $-\sqrt{5}$. The second factor is zero only when x is $\sqrt{5}$. Those are the only two roots of this polynomial.

Let's check that result. $\sqrt{5}$ is a little more than 2.2. Using your Python code, you can graph the polynomial:

```
import poly.py
import matplotlib.pyplot as plt
```

```
# x**2 - 5
pn = [-5.0, 0.0, 1.0]

# These lists will hold our x and y values
x_list = []
y_list = []

# Start at x=-3
current_x =-3.0

# End at x=3.0
while current_x < 3.0:
    current_y = poly.evaluate_polynomial(pn, current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.1

# Plot the curve
plt.plot(x_list, y_list)
plt.grid(True)
plt.show()
```

You should get a plot like this:



It does, indeed, seem to cross the x-axis near -2.2 and 2.2.

53.2 Powers of binomials

You can raise whole polynomials to exponents. For example,

$$(3x^3 + 5)^2 = (3x^3 + 5)(3x^3 + 5)$$
$$= 9x^6 + 15x^3 + 15x^3 + 25 = 9x^6 + 30x^3 + 25$$

A polynomial with two terms is called a *binomial*. $5x^9 - 2x^4$, for example, is a binomial. In this section, we are going to develop some handy techniques for raising a binomial to some power.

Looking at the previous example, you can see that for any monomials a and b, $(a+b)^2 = a^2 + 2ab + b^2$. So, for example, $(7x^3 + \pi)^2 = 49x^6 + 14\pi x^3 + \pi^2$

Exercise 68 Squaring binomials

Simply the following

1. $(x+1)^2$
2. $(3x^5+5)^2$
3. $(x^3-1)^2$
4. $(x-\sqrt{7})^2$

Working Space

Answer on Page 409

What about $(x+2)^3$? You can do it as two separate multiplications:

$$(x+2)^3 = (x+2)(x+2)(x+2)$$
$$= (x+2)(x^2+4x+4) = x^3+4x^2+4x+2x^2+8x+8$$
$$= x^3+6x^2+12x+8$$

And, in general, we can say that for any monomials a and b, $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$.

What about higher powers? $(a+b)^4$, for example? You could use the distributive property four times, but it starts to get pretty tedious.

Here is a trick. This is known as *Pascal's triangle*

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
\ldots
\end{array}$$

Each entry is the sum of the two above it.

The coefficients of each term are given by the entries in Pascal's triangle:

$$(a+b)^4 = 1a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + 1b^4$$

Exercise 69 Using Pascal's Triangle

Working Space

1. What is $(x + \pi)^5$?

Answer on Page 409

Chapter 54

Factoring Polynomials

We factor a polynomial into two or more polynomials of lower degree. For example, let's say that you wanted to factor $5x^3 - 45x$. You would note that you can factor out $5x$ from every term. Thus,

$$5x^3 - 45x = (5x)(x^2 - 9)$$

And then, you might notice that the second factor looks like the difference of squares, so

$$5x^3 - 45x = (5x)(x + 3)(x - 3)$$

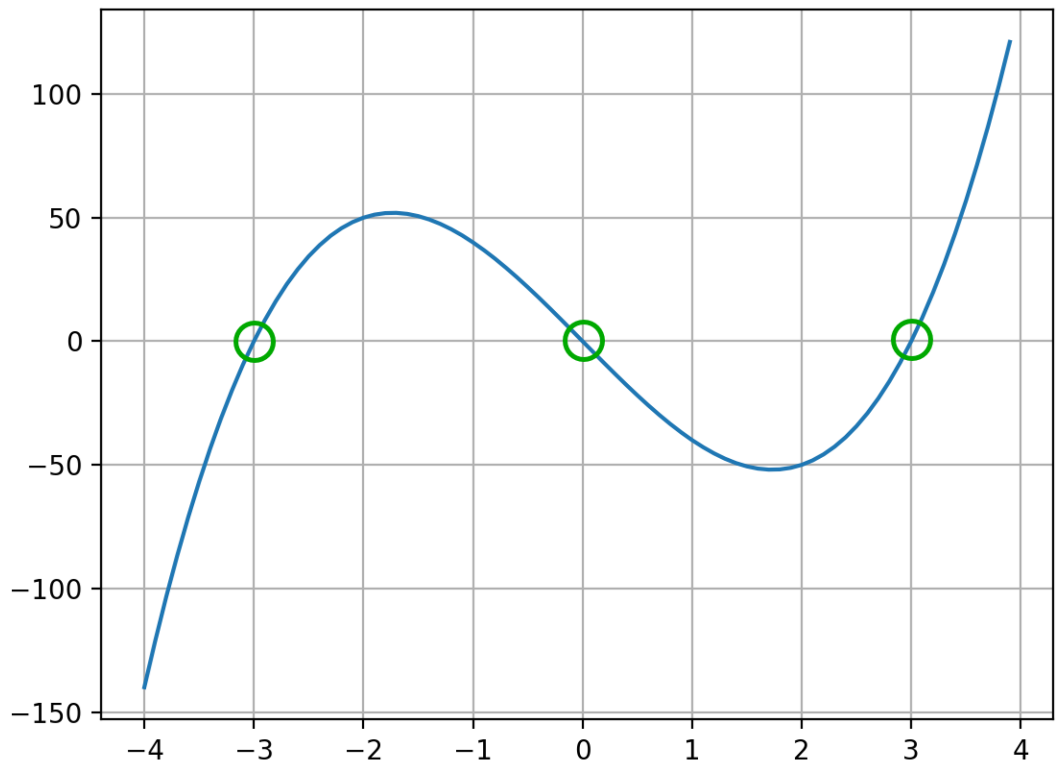
That is as far as we can factorize this polynomial.

Why do we care? The factors make it easy to find the roots of the polynomial. This polynomial evaluates to zero if and only if at least one of the factors is zero. Here we see that

- The factor $(5x)$ is zero when x is zero.
- The factor $(x + 3)$ is zero when x is -3.
- The factor$(x - 3)$ is zero when x is 3.

So looking at the factorization, you can see that $5x^3 - 45x$ is zero when x is 0, -3, or 3.

This is a graph of that polynomial with its roots circled:



54.1 How to factor polynomials

The first step when you are trying to factor a polynomial is to find the greatest common divisor for all the terms, and pull that out. In this case, the greatest common divisor will also be a monomial: its degree is the least of the degrees of the terms, its coefficient will be the greatest common divisor of the coefficients of the terms.

For example, what can you pull out of this polynomial?

$$12x^100 + 30x^31 + 42x^17$$

The greatest common divisor of the coefficients (12, 30, and 42) is 6. The least of the degrees of terms (100, 31, and 17) is 17. So you can pull out $6x^17$:

$$12x^100 + 30x^31 + 42x^17 = (6x^17)(2x^83 + 5x^14 + 7)$$

Exercise 70 Factoring out the GCD monomial

Working Space

Answer on Page 409

So, now you have the product of a monomial and a polynomial. If you are lucky, the polynomial part looks familiar, like the difference of squares or a row from Pascal's triangle.

Often you are trying factor a quadratic like $x^2 + 5x + 6$ in a pair of binomials. In this case, the result would be $(x + 3)(x + 2)$. Let's check that:

$$(x + 3)(x + 2) = (x)(x) + (3)(x) + (2)(x) + (3)(2) = x^2 + 5x + 6$$

Notice that 3 and 2 multiply to 6 and add to 5. If I were trying to factor $x^2 + 5x + 6$, I would ask myself"What are two numbers that when multiplied equal 6 and when added equal 5?" And I would might guess wrong a couple of times. For example, I might say to myself "Well, 6 times 1 is 6. Maybe those work. But 6 and 1 add 7. So those don't work."

Solving these sorts of problems are like solving a Sudoku puzzle: you try things and realize they are wrong, so you backtrack and try something else.

The numbers are sometimes negative. For example, $x^2 + 3x - 10$ factors into $(x + 5)(x - 2)$.

Exercise 71 Factoring quadratics

Working Space

Answer on Page 409

Chapter 55

Practice with Polynomials

At this point, you know all the pieces necessary to solve problems involving polynomials. In this chapter, you are going to practice using all of these ideas together.

Watch Khan Academy's **Polynomial identities introduction here**: https://youtu.be/EvNKKyhLSpQ Also watch the follow up here: https://youtu.be/-6qiO49Q180

FIXME: Lots of practice problems here

Chapter 56

Graphing Polynomials

In using polynomials to solve real-world problems, it is often handy to know what the graph of the polynomial looks like. You have many of the tools you need to start to sketch out the the graphs:

- To find where the graph crosses the y-axis, you can evaluate the polynomial at $x = 0$.
- To find where the graph crosses the x-axis, you can find the roots of the polynomial.
- To find the level spots on the graph (often the top of a hump or the bottom of a dip), you can take the derivative of the polynomial (which is a polynomial), and find the roots of that.

FIXME: Diagram of those things

For example, if you wanted to graph the polynomial $f(x) = -x^3 - x^2 + 6x$, you might plug in a few values that are easy to compute:

- $f(-2) = -8$

- $f(-1) = -6$
- $f(0) = 0$
- $f(1) = 4$
- $f(2) = 0$

So, right away we know two roots: $x = 0$ and $x = 2$. Are there others? We won't know until we factor the polynomial:

$$
\begin{aligned}
&-x^3 - x^2 + 6x \\
&= (-1x)(x^2 + x - 6) \\
&= (-1x)(x+3)(x-2)
\end{aligned}
$$

So, yes, there is a third root: $x = -3$

What about the level spots? $f'(x) = -3x^2 - 2x + 6$. Where is that zero?

$$
\begin{aligned}
&-3x^2 - 2x + 6 = 0 \\
&x^2 + \frac{2}{3}x - 2 = 0
\end{aligned}
$$

We have a formula for quadratics like this:

$$
\begin{aligned}
x &= -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} \\
&= -\frac{\frac{2}{3}}{2} \pm \frac{\sqrt{\left(\frac{2}{3}\right)^2 - 4(-2)}}{2} \\
&= -\frac{1}{3} \pm \frac{\sqrt{\frac{4}{9} + 8}}{2} \\
&= -\frac{1}{3} \pm \frac{\sqrt{\frac{85}{9}}}{2} \\
&= -\frac{1}{3} \pm \frac{\sqrt{85}}{6} \\
&\approx 1.20 \text{ and } -1.87
\end{aligned}
$$

Now you might plug those numbers in:

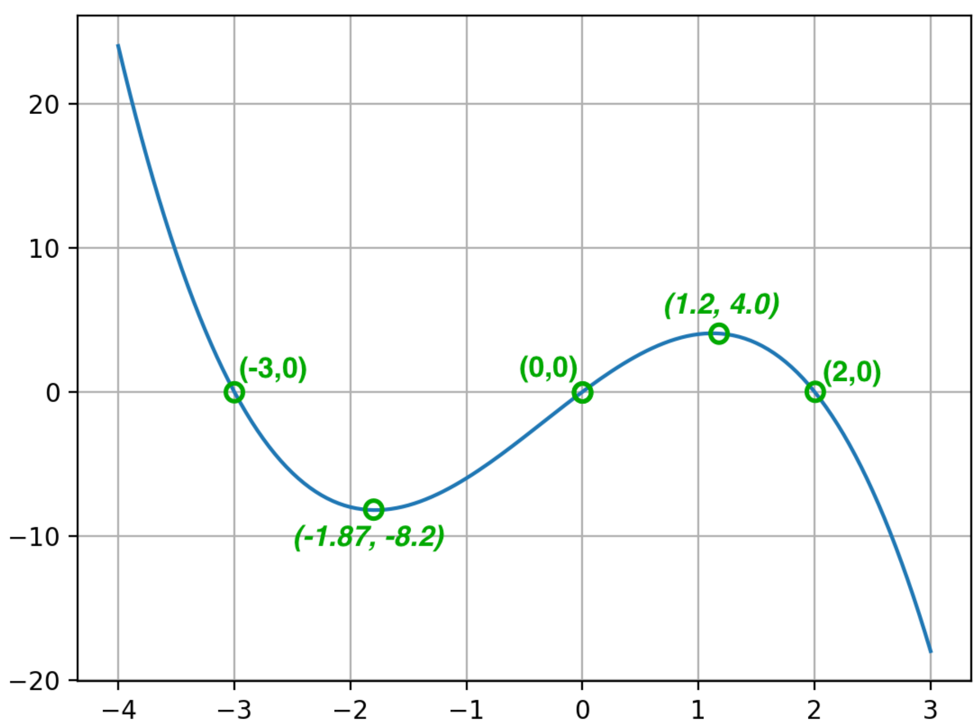
- $f(1.2) \approx 4.0$
- $f(-1.87) \approx -8.2$

56.1 Leading term in graphing

There is one more trick you need before you can draw a good graph of a polynomial. As you go father and farther to the left and right, where does the function go? That is, does the graph go up on both ends (like a smile)? Or does it go down on both ends (like a frown)? Or does the negative end go down (frowny) while the positive end go up (smiley)? Or does the negative go up (smiley) and the positive end go down (frowny)?

Assuming the polynomial is not constant, there are only those four possibilties. It is determined entirely by the leading term of the polynomial. If the degree of the leading term is even, both ends go in the same direction (both are smiley or both are frowny). If the coefficient of the leading term is positive, the positive end is smiley.

The graph we are working on has a leading term of $-1x^3$. The degree is odd, thus the ends go in different directions. The coefficient is negative, so the positive end points down. Now you can draw the graph, which should look something like this:



Chapter 57

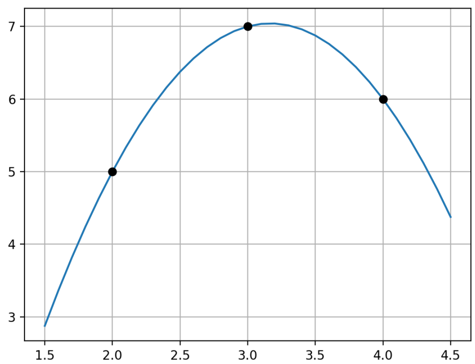
Interpolating with Polynomials

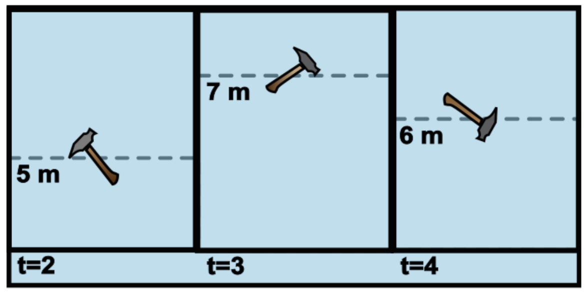
Let's say someone on a distant planet records video of a hammer being throw up into the air. They send you three random frames of the hammer in flight. Each frame has a timestamp and you can clearly see how high the hammer is in each one. Can you create a 2nd degree polynomial that explains the entire flight of the hammer?

That is, you have three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$. Can you find a, b, c such that the graph of $at^2 + bt + c = t$ passes through all three points?

The answer is yes. In fact, given any n points, there is exactly one $n-1$ degree polynomial that passes through all the points.

There are a lot of variables floating around. Let's make it concrete: The photos are taken at $t = 2$ seconds, $t = 3$ seconds, and $t = 4$ seconds. In those photos, the height of the hammer is $5m$, $7m$, and $6m$. So, we want our polynomial to pass through these points: (2, 5), (3, 7), (4,6).





How can you find that polynomial? Let's do it in small steps. Can you create a 2nd degree polynomial that is not zero at $t = 2$, but is zero at $t = 3$ and $t = 4$? Yes, you can: $(x-3)(x-4)$ has exactly two roots at $t = 3$ and $t = 4$. The value of this polynomial at $t = 2$ is $(2-3)(2-4) = 2$. We really want it to be 5m, so we can divide the whole polynomial by 2 and multiply it by 5.

Now we have the polynomial:

$$f_0(x) = \frac{5}{(2-3)(2-4)}(x-3)(x-4) = \frac{5}{2}x^2 - \frac{35}{2}x + 30$$

This is a second degree polynomial that is 5 at $t = 2$ and 0 at $t = 3$ and $t = 4$.

Now we create a polynomial that is 7 at $t = 3$ and 0 at $t = 2$ and $t = 4$:

$$f_1(x) = \frac{7}{(3-2)(3-4)}(x-2)(x-4) = -7x^2 + 42x - 56$$

Finally, we create a polynomial that is 6 at $t = 4$ and zero at $t = 2$ and $t = 3$:

$$f_2(x) = \frac{6}{(4-2)(4-3)}(x-2)(x-3) = 3x^2 - 15x + 18$$

Adding these three polynomials together gives you a new polynomial that touches all three points:

$$f(x) = \frac{5}{2}x^2 - \frac{35}{2}x + 30 - 7x^2 + 42x - 56 + 3x^2 - 15x + 18 = -\frac{3}{2}x^2 + \frac{19}{2}x - 8$$

You can test this with your `Polynomial` class. Create a file called `test_interpolation.py`. Add this code:

```
from Polynomial import Polynomial
import matplotlib.pyplot as plt
```

```
in_x = [2,3,4]
in_y = [5,7,6]

pn = Polynomial([-8, 19/2, -3/2])
print(pn)

# These lists will hold our x and y values
x_list = []
y_list = []

# Starting x
current_x = 1.5

while current_x <= 4.5:
    # Evaluate pn at current_x
    current_y = pn(current_x)

    # Add x and y to respective lists
    x_list.append(current_x)
    y_list.append(current_y)

    # Move x forward
    current_x += 0.05

# Plot the curve
plt.plot(x_list, y_list)

# Plot black circles on the given points
plt.plot(in_x, in_y, "ko")
plt.grid(True)
plt.show()
```

You should get a nice plot that shows the graph of the polynomial passing through those three points.

In general, then, if you give me any three points $(t_0, h_0), (t_1, h_1), (t_2, h_2)$, here is a second degree polynomial that pass through all three points:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2)}(x-t_1)(x-t_2)+\frac{h_1}{(t_1 - t_0)(t_1 - t_2)}(x-t_0)(x-t_2)+\frac{h_2}{(t_2 - t_0)(t_2 - t_1)}(x-t_0)(x-t_1)$$

What if you are given 9 points $((t_0, h_0), (t_1, h_1), \ldots, (t_8, h_8))$ and want to find a 8th degree polynomial that passes through all of them? Just what you would expect:

$$\frac{h_0}{(t_0 - t_1)(t_0 - t_2)\ldots(t_0 - t_8)}(x-t_1)(x-t_2)\ldots(x-t_8)+\ldots+\frac{h_8}{(t_8 - t_0)\ldots(t_8 - t_7)}(x-t_0)\ldots(x-t_7)$$

FIXME: Do I need to define summation and prod here?

The general solution is, given n points, the $n-1$ degree polynomial that goes through them is

$$y = \sum_{i=0}^{n} \left(\prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - t_j}{t_i - t_j} \right) h_i$$

That would be tedious for a person to compute, but computers love this stuff. Let's create a method that creates instances of Polynomial using interpolation.

57.1 Interpolating polynomials in python

Your method will take two lists of numbers, one contains x-values and the other contains y-values. So comment out the line that creates the polynomial in `test_interpolation.py` and create it from two lists:

```
in_x = [2,3,4]
in_y = [5,7,6]
# pn = Polynomial([-8, 19/2, -3/2])
pn = Polynomial.from_points(in_x, in_y)
print(pn)
```

Add the following method to your Polynomial class in `Polynomial.py`

```
    @classmethod
    def from_points(cls, x_values, y_values):
        coef_count = len(x_values)

        # Sums start with a zero polynomial
        sum_pn = Polynomial([0.0] * coef_count)
        for i in range(coef_count):

            # Products start with the constant 1 polynomial
            product_pn = Polynomial([1.0])
            for j in range(coef_count):

                # Must skip j=i
                if j != i:
                    # (1x - x_values[j]) has a root at x_values[j]
                    factor_pn = Polynomial([-1 * x_values[j], 1])
                    product_pn = product_pn * factor_pn
```

```
        # Scale so product_pn(x_values[i]) = y_values[i]
        scale_factor  = y_values[i] / product_pn(x_values[i])
        scaled_pn = scale_factor * product_pn

        # Add it to the sum
        sum_pn = sum_pn + scaled_pn

    return sum_pn
```

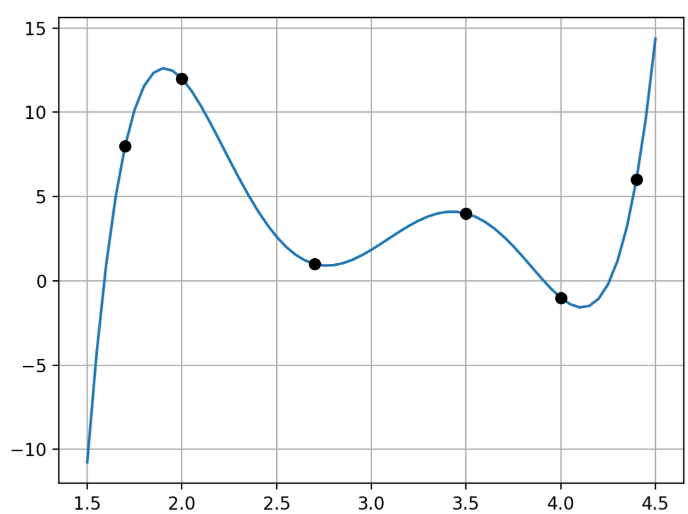
It should work exactly the same as before. You should get the same polynomial printed out as before. You shoud get the same plot of the curve passing through the three points.

How about five points? Change `in_x` and `in_y` at the start of `test_interpolation.py`:

```
in_x = [1.7, 2, 2.7, 3.5, 4, 4.4]
in_y = [8, 12, 1, 4, -1, 6]
```

You should get a polynomial that passes through all five points:

$$11.21x^5 - 171.05x^4 + 1019.44x^3 - 2957.53x^2 + 4161.78x - 2258.75$$



It should look like this:

Chapter 58

Data Tables and pandas

Much of the data that you will encounter in your career will come to you as a table. Some of these tables are spreadsheets, some are in relational databases, some will come to you as CSV files.

Typically each column will represent an attribute (like height or acreage) and each row will represent an entity (like a person or a farm). You might get a table like this:

`property_id`	`bedrooms`	`square_meters`	`estimated_value`
7927	3	921.4	$ 294,393
9329	2	829.1	$ 207,420

Typically, one of the columns is guaranteed to be unique. We call this the *primary key*. In this table, `property_id` is the primary key: every property has one, and no two properties have the same `property_id`.

58.1 Data types

Each column in a table has a type, and these usually correspond pretty nicely with types in Python.

Here are some common datatypes:

Type	Python type	Example
Integer	int	`910393`
Float	float	`-23.19`
String	string	`'Fred'`
Boolean	bool	`False`
Date	datetime.date	`2019-12-04`
Timestamps	datetime.datetime	`2022-06-10T14:05:22Z`

Sometimes it is OK to have values missing. For example, if you had a table of data about employees, maybe one of the columns would be `retirement`, a date that tells you when the person retired. People who had not yet retired would have no value in this column. We would say that they have *null* for `retirement`.

Sometimes there are constraints on what values can appear in the column. For example, if the column were `height`, it would make no sense to have a negative value.

Sometimes a column can only be one of a few values. For example, if you ran a bike rental shop, each bicycle's status would be "available", "rented", or "broken". Any other values in that column would not be allowed. We often call these columns *categorical*.

58.2 pandas

The Python community works with tables of data *a lot*, so it created the pandas library for reading, writing, and manipulating tables of data.

When working with tables, you sometimes need to go through them row-by-row. However, for large datasets, this is very slow. pandas makes it easy (and very fast) to say things like "Delete every row that doesn't have a value for height" instead of requiring you to step through the whole table.

In pandas, there are two datatypes that you use a lot:

- a `Series` is a single column of data.
- a `DataFrame` is a table of data: it has a `Series` for each column.

In the digital resources, you will fined `bikes.csv`. If you look at it in a text editor, it will start like this:

```
bike_id,brand,size,purchase_price,purchase_date,status
5636248,GT,57,277.99,1986-09-07,available
4156134,Giant,56,201.52,2005-01-09,rented
7971254,Cannondale,54,292.25,1978-02-28,available
3600023,Canyon,57,197.62,2007-02-15,broken
```

The first line is a header and tells you the name of each column. Then the values are separated by commas. (Thus the name: CSV stands for "Comma Separated Values".)

58.3 Reading a CSV with pandas

Let's make a program that reads `bikes.csv` into a pandas dataframe. Create a file called `report.py` in the same folder as `bikes.csv`.

First, we will read in the csv file. pandas has one Series that acts as the primary key; it calls this one the index. When reading in the file, we will tell it to use the `bike_id` as the index series.

If you ask a dataframe for its shape, it returns a tuple containing the number of rows and the number of columns. To confirm that we have actually read the data in, let's print those numbers. Add these lines to `report.py`:

```
import pandas as pd

# Read the CSV and create a dataframe
df = pd.read_csv('bikes.csv', index_col="bike_id")

# Show the shape of the dataframe
(row_count, col_count) = df.shape
print(f"*** Basics ***")
print(f"Bikes: {row_count:,}")
print(f"Columns: {col_count}")
```

Build it and run it. You should see something like this:

```
*** Basics ***
Bikes: 998
Columns: 5
```

Note that your table actually had 6 columns. The index series is not included in the shape.

58.4 Looking at a Series

Let's get the lowest, the highest, and the mean purchase price of the bikes. The purchase price is a series, and you can ask the dataframe for it. Add these lines to the end of your program:

```
# Purchase price stats
print("\n*** Purchase Price ***")
series = df["purchase_price"]
print(f"Lowest:{series.min()}")
print(f"Highest:{series.max()}")
print(f"Mean:{series.mean():.2f}")
```

Now when you run it, you will see a few additional lines:

```
*** Purchase Price ***
Lowest:107.37
Highest:377.7
Mean:249.01
```

What are all the brands of the bikes? Add a few more lines to your program that shows how many of each brand:

```
# Brand stats
print("\n*** Brands ***")
series = df["brand"]
series_counts = series.value_counts()
print(f"{series_counts}")
```

Now when you run it, your report will include the number of bikes for each brand from most common to least:

```
*** Brands ***
Canyon        192
BMC           173
Cannondale    170
Trek          166
GT            150
Giant         147
Name: brand, dtype: int64
```

`value_counts` returns a Series. To format this better we need to learn about accessing individual rows in a series.

58.5 Rows and the index

In an array, you ask for data using an the location (as an int) of the item you want. You can do this in pandas using `iloc`. Add this to the end of your program:

```
# First bike
print("\n*** First Bike ***")
row = df.iloc[0]
print(f"{row}")
```

When you run it, you will see the attributes of the first row of data:

```
*** First Bike ***
brand                     GT
size                      57
purchase_price        277.99
purchase_date     1986-09-07
status             available
Name: 5636248, dtype: object
```

Notice that the data coming back is actually another series.

The last line says that the name (the value for the index column) for this row is 5636248. In pandas, we usually use this to locate particular rows. For example, there is a row with `bike_id` equal to 2969341. Let's ask for one entry from the

```
print("\n*** Some Bike ***")
brand = df.loc[2969341]['brand']
print(f"brand = {brand}")
```

Now you will see the information about that bike:

```
*** Some Bike ***
brand = Cannondale
```

pandas has a few different ways of getting to that value. All of these get you the same thing:

```
brand = df.loc[2969341]['brand'] # Get row, then get value
brand = df['brand'][2969341]     # Get column, then get value
brand = df.loc[2969341, 'brand'] # One call with both row and value
```

58.6 Changing data

One of your attributes needs cleaning up. Every bike should have a status and it should be one of the following strings:"available", "rented", or "broken". Get counts for each unique value in status:

```
print("\n*** Status ***")
series = df["status"]
missing = series.isnull()
print(f"{missing.sum()} bikes have no status.")
series_counts = series.value_counts()
for value in series_counts.index:
    print(f"{series_counts.loc[value]} bikes are \"{value}\"")
```

This will show you:

```
*** Status ***
7 bikes have no status.
389 bikes are "rented"
304 bikes are "broken"
296 bikes are "available"
1 bikes are "Flat tire"
1 bikes are "Available"
```

Right away we can see two easily fixable problems: Someone typed "Available" instead of "available". Right after you read the CSV in, fix this in the data frame:

```
mask = df['status'] == 'Available'
print(f"{mask}")
df.loc[mask, 'status'] = 'available'
```

When you run this, you will see that the mask is a series with `bike_id` as the index and `False` or `True` as the value, depending on whether the row's status was equal to "Available".

When you use `loc` with this sort of mask, you are saying "Give me all the rows for which the mask is True." So, the assignment only happens in the one problematic row.

Let's get rid of the mask variable and do the same for turning `Flat tire` into `Broken`:

```
df.loc[df['status'] == 'Available', 'status'] = 'available'
df.loc[df['status'] == 'Flat tire', 'status'] = 'broken'
```

Now those problems are gone:

```
7 bikes have no status.
389 bikes are "rented"
305 bikes are "broken"
297 bikes are "available"
```

What about the rows with no values for status? We were pretty certain that the bikes were available, we could just set them to 'available':

```
missing_mask = df['status'].isnull()
df.loc[missing_mask, 'status'] = 'available'
```

Or maybe we would print out the IDs of the bikes so that we could go look for them:

```
missing_mask = df['status'].isnull()
missing_ids = list(df[missing_mask].index)
print(f"These bikes have no status:{missing_ids}")
```

But lets just keep the rows where the status is not null:

```
missing_mask = df['status'].isnull()
df = df[~missing_mask]
```

At the end of your program, write out the improved CSV:

```
df.to_csv('bikes2.csv')
```

Run the program and open `bikes2.csv` in a text editor.

58.7 Derived columns

Let's say that you want to add a column with age of the bicycle in days:

```
bike_id,brand,size,purchase_price,purchase_date,status,age_in_days
5636248,GT,57,277.99,1986-09-07,available,13061
4156134,Giant,56,201.52,2005-01-09,rented,6362
7971254,Cannondale,54,292.25,1978-02-28,available,16174
```

Your first problem is that the `purchase_date` column looks like a date, but really it is a string. So you need to convert it to a date. You can do this by applying a function to every item in the series:

```
df['purchase_date'] = df['purchase_date'].apply(lambda s: datetime.date.fromisoformat(s))
```

(With pandas, there is often more than one way to do things. pandas has a `to_datetime` function that converts every entry in a sequence to a `datetime` object. So here is another way to convert the string column in to a date column:

```
df['purchase_date'] = pd.to_datetime(df['purchase_date']).dt.date
```

You can look up `dt` and `date` if you are curious.)

Now, we can use the same trick to create a new column with the age in days:

```
today = datetime.date.today()
df['age_in_days'] = df['purchase_date'].apply(lambda d: (today - d).days)
```

When you run this, the new `bikes.csv` will have an `age_by_date` column.

Chapter 59

Data tables in SQL

Most organizations keep their data as tables inside a relational database management system. Developers talk to those systems using a language called SQL ("Structured Query Language").

Some relational database managers are pricey products you may have heard of before: Oracle, Microsoft SQL Server. Some are free: PostgreSQL or MySQL. These are server software that client programs talk to over the companies network.

There is a library, called `sqlite`, that lets us create files that hold tables. We can use SQL to create, edit, and browse those tables. sqlite is free, fast, and very easy to install. So we will use sqlite instead of a networked database management system.

If you look in your digital resources, you will find a file called `bikes.db`. I created this file using sqlite, and now you will use sqlite to access it.

In the terminal, get to the directory where `bikes.db` lives. To open the sqlite tool on that file:

```
> textbfsqlite3 bikes.db
```

(If your system complains that there is no sqlite3 tool, you need to install sqlite. See this website: https://sqlite.org/)

Please follow along: type each command shown here into the terminal and see what happens.

We mostly run SQL commands in this tool, but there are a few non-SQL commands that all start with a period. To see the tables and their columns, you can run `.schema`:

```
sqlite> .schema
CREATE TABLE bike (bike_id int PRIMARY KEY, brand text, size int,
                   purchase_price real, purchase_date date, status text);
```

That is the SQL command that I used to create the `bike` table. You can see all the columns and their types.

You want to see all the rows of data in that table?

```
sqlite> select * from bike;
4997391|GT|57|269.61|2009-05-03|rented
5429447|Cannondale|50|215.91|2002-02-17|broken
5019171|Trek|58|251.17|1985-07-11|rented
3000288|Cannondale|57|211.08|1993-01-05|broken
880965|GT|52|281.75|1995-08-02|available
...
```

You will see 1000 rows of data!

The SQL language is not case-sensitive, so you can also write it like this:

```
sqlite> SELECT * FROM BIKE;
```

Often you will see SQL with just the SQL keywords in all caps:

```
sqlite> SELECT * FROM bike;
```

The semicolon is not part of SQL, but it tells sqlite that you are done writing a command and that it should be executed.

SQL lets you choose which columns you would like to see:

```
sqlite> SELECT bike_id, brand FROM bike;
4997391|GT
5429447|Cannondale
5019171|Trek
3000288|Cannondale
...
```

Using WHERE, SQL lets you choose which rows you would like to see:

```
sqlite> SELECT * FROM bike WHERE purchase_date > '2009-01-01' AND brand = 'GT';
4997391|GT|57|269.61|2009-05-03|rented
326774|GT|56|165.0|2009-06-27|available
264933|GT|52|302.43|2009-07-09|available
5931243|GT|55|173.56|2009-11-26|rented
4819848|GT|51|221.71|2009-12-11|rented
9347713|GT|52|232.32|2009-06-13|available
3019205|GT|58|262.94|2009-08-22|available
```

Using DISTINCT, SQL lets you get just one copy of each value:

```
sqlite> SELECT DISTINCT status FROM bike;
rented
broken
available

Busted
Flat tire
good
out
Rented
```

You can also edit these rows. For example, if you wanted every status that is `Busted` to be changed to `broken`. You can use an UPDATE statement:

```
sqlite> UPDATE bike SET status='broken' WHERE status='Busted';
sqlite> SELECT DISTINCT status FROM bike;
rented
broken
available

Flat tire
good
out
Rented
```

You can insert new rows:

```
sqlite> INSERT INTO bike (bike_id, brand, size, purchase_price, purchase_date, status)
   ...> VALUES (1, 'GT', 53, 123.45, '2020-11-13', 'available');
sqlite> SELECT * FROM bike WHERE bike_id = 1;
1|GT|53|123.45|2020-11-13|available
```

You can delete rows:

```
sqlite> DELETE FROM bike WHERE bike_id = 1;
sqlite> SELECT * FROM bike WHERE bike_id = 1;
```

To get out of sqlite, type `.exit`.

Exercise 72 SQL Query

Execute an SQL query that returns the `bike_id` (no other columns) of every Trek bike that cost more than $300.

Working Space

Answer on Page 409

59.1 Using SQL from Python

The people behind sqlite created a library for Python that lets you execute SQL and fetch the results from inside a python program.

Let's create a simple program that fetches and displays the bike ID and purchase date of every Trek bike that cost more than $300.

Create a file called `report.py`:

```
import sqlite3 as db

con = db.connect('bikes.db')
cur = con.cursor()
```

```
cur.execute("SELECT bike_id, purchase_date FROM bike WHERE purchase_price > 330 AND bran
rows = cur.fetchall()

today = datetime.date.today()
for row in rows:
    print(f"Bike {row[0]}, purchased {row[1]}")

con.close()
```

When you execute it, you should see:

```
> python3 report.py
Bike 4128046, purchased 2007-08-06
Bike 7117808, purchased 1995-03-12
Bike 7176903, purchased 1986-07-03
Bike 827899, purchased 2009-03-14
Bike 363983, purchased 1970-08-16
```

Chapter 60

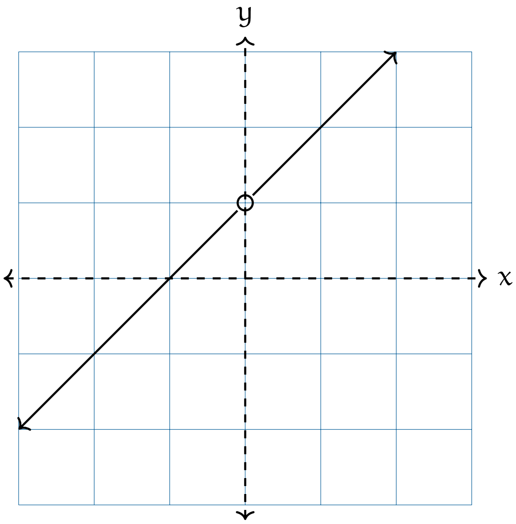
Limits

Here is a function:

$$f(x) = \frac{x^2}{x} + 1$$

This f is defined for any real number *except 0*. (You can't divide anything, including zero, by zero.)

Let's plot f:

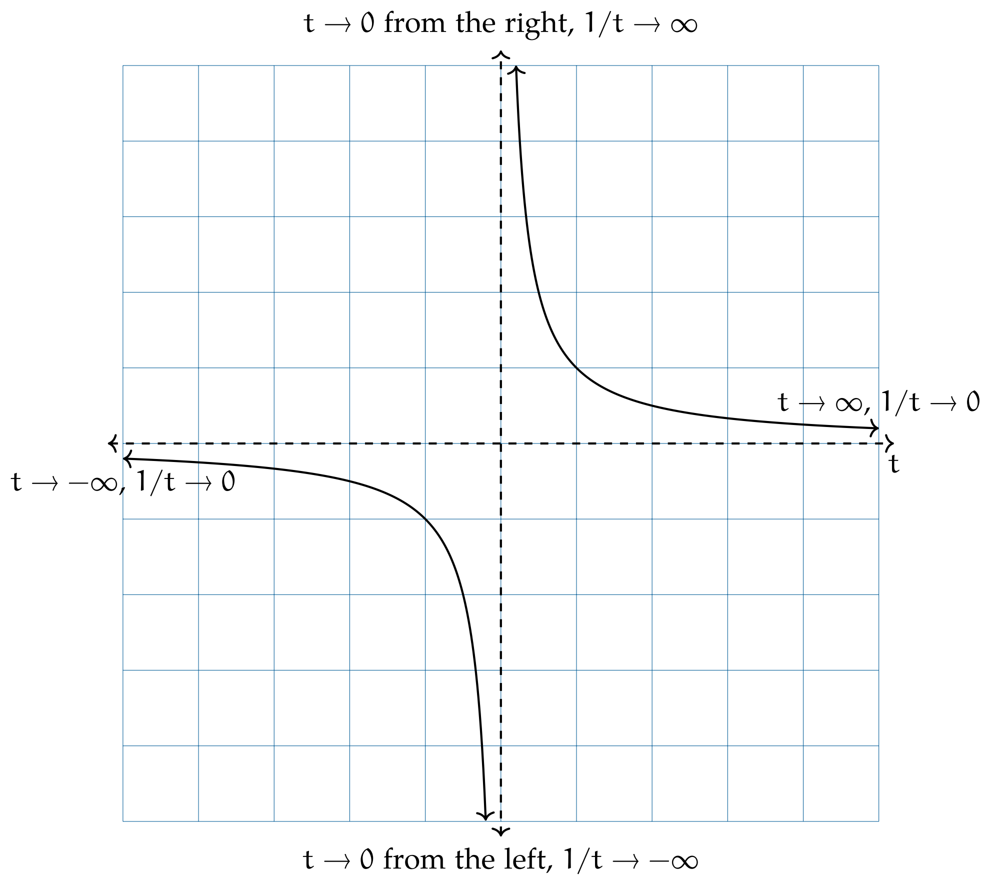


You can see that the function is the same as $x+1$ everywhere except $x=0$. You can see that as the function approaches $x=0$ from the left, the value of the function approaches 1. You can see that as the function approaches $x=1$ from the right, the value of the function approaches 1.

Mathematicians say "The *limit* of f as x approaches 0, is 1." We have a notation for this:

$$\lim_{x \to 0} f(x) = 1$$

We generally use limit whenever we mean "We are getting arbitrarily close, but we can never really get there." For example, you might say "The limit of $1/t$ as t goes to infinity is 0."



What is the limit of $1/t$ as t approaches zero? The limit isn't defined because if you approach from the right, $1/t$ goes to infinity, but if you approach from the left, $1/t$ goes to negative infinity.

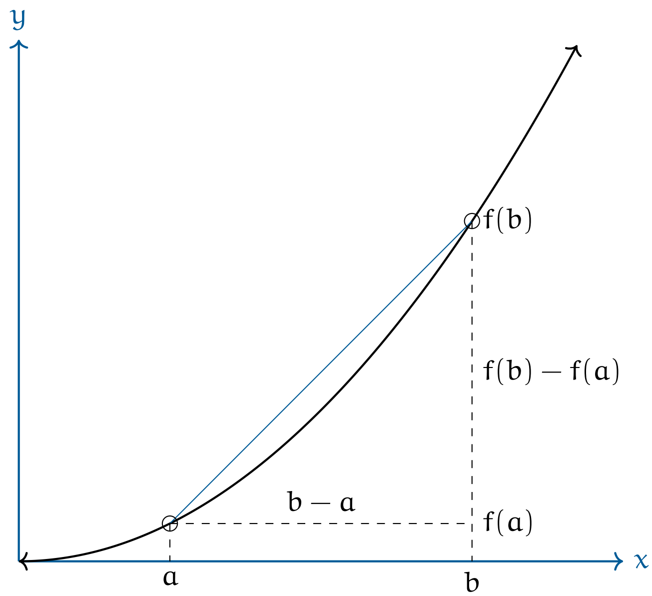
Chapter 61

Differentiation

We have done some differentiation, but you haven't been given the real definition because it is based on limits.

The idea is that we can find the slope between two points on the graph a and b like this:

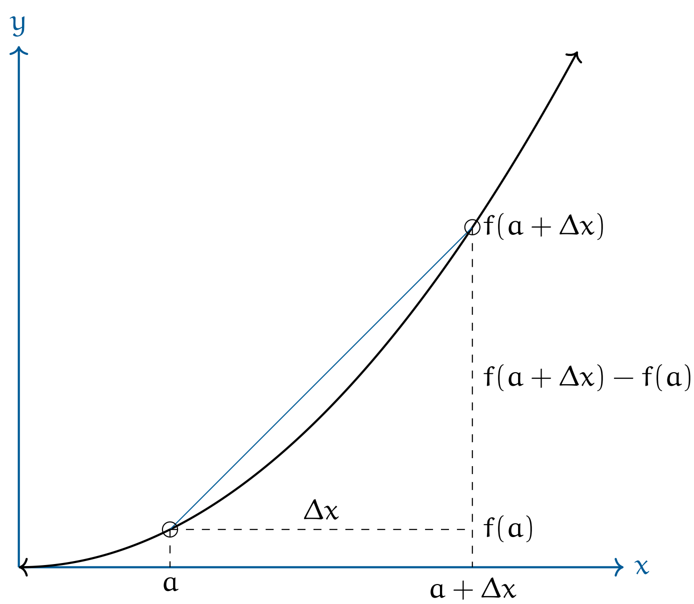
$$m = \frac{f(b) - f(a)}{b - a}$$



If we want to find the slope at a we take the limit of this as the b goes to a:

$$f'(a) = \lim_{b \to b} \frac{f(b) - f(a)}{b - a}$$

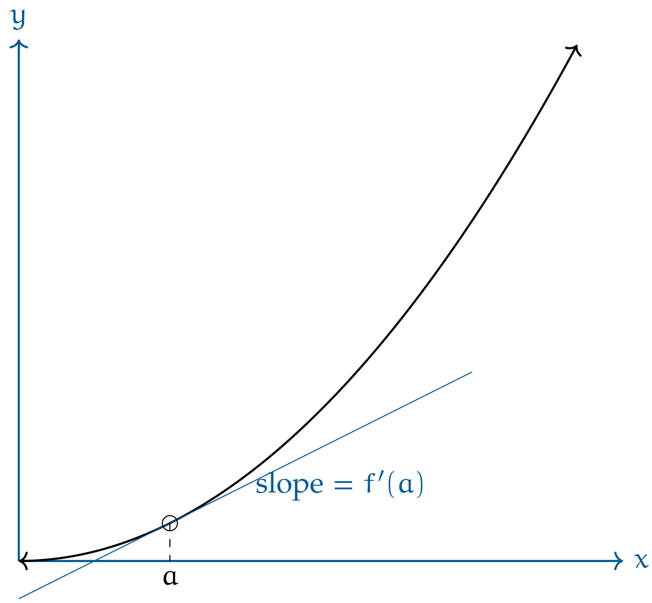
This idea is usually expressed using Δx as the difference between b and a:



Then the formula becomes:

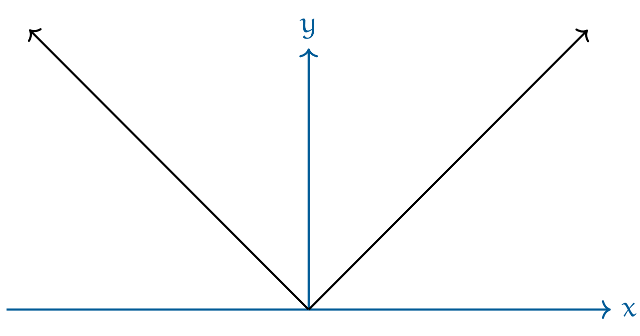
$$f'(a) = \lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x}$$

Now, at any point a we can compute the slope of the line tangent to the function at a:



61.1 Differentiability

Warning: Not every function is differentiable everywhere. For example, if $f(x) = |x|$, you get a corner at zero.



To the left of zero, the slope is -1. To the right of zero, the slope is 1. At zero? The derivative is not defined.

If a function has a derivative everywhere, it is said to be *differentiable*. Generally, you can think of differentiable functions as smooth – their graphs have no corners.

61.2 Using the definition of derivative

Let's say that you want to know the slope of $f(x) = -3x^2$ at $x = 2$. Using the definition of the derivative, that would be:

$$f'(2) = \lim_{\Delta x \to 0} \frac{f(2 + \Delta x) - f(2)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-3(2 + \Delta x)^2 - \left(-3(2)^2\right)}{\Delta x} = \lim_{\Delta x \to 0} \frac{-12 - 12\Delta x + -3(\Delta x)^2 + 12}{\Delta x} = -12$$

Chapter 62

Introduction to Discrete Probability

First, let's take care of the word *discrete* vs *discreet*. They sound exactly the same, but "discrete" means "individually separate and distinct" and "discreet" means "careful about what other people know". So you might say, "You can think of light as a continuous wave or as a blast of discrete particles." And you might say, "Please go get the box of doughnuts from the kitchen. Oh, and there are a lot of hungry people in the house, so be discreet."

When we are talking about probabilities, some problems deal with discrete quantities like "What is the probability that I will throw these three dice and the numbers that roll face up sum to 9?". There are also problems that deal with continuous properties like "What is the probability that the next bird to fly over my house will weigh between 97.2 and 98.1 grams ?" In this module, we are going to focus on the probability problems that deal with discrete quantities.

Watch Khan Academy's Introduction to Probability at https://youtu.be/uzkc-qNVoOk.

Let's say that I have a cloth sack filled with 100 marbles; 99 are red and 1 is white. If

I ask you to reach in without looking and pull out one marble, you will probably pull out a red one. We say that "There is a 1 in 100 chance that you would pull out a white marble." Or we can use percentages and say "There is a 1% chance that you will pull out a white marble." Or we can use decimals and say "There is a 0.01 probability that you will pull out a white marble." In probability, we often talk about the probability of certain events. "Pulling out a white marble" is an event, and we can give it a symbol like W. Then, in equations we use p to mean "the probability of". Thus, we can say "There is a 0.01 probability that you will pull out a white marble" which becomes the equation

$$p(W) = 0.01$$

62.1 The Probability of All Possibilities is 1.0

We know that you are either going to pull out a red marble or a white marble, so the probability of a white marble being pulled and the probability of a red marble being pulled must add up to 100%. Therefore, the odds of pulling out a red marble must be 99% or 0.99. If we let the event "Pull out a red marble" be given by the symbol R, we can say:

$$p(R) = 1.0 - P(W) = 1.0 - 0.01 = 0.99$$

Now, let's say that I make you take a marble from the bag and then toss a coin. What is the probability that you will pull a white marble and then get heads on the coin? It is the product of the two probabilities: $0.01 \times 0.5 = 0.005$, so one-half of a one percent chance. Do the probabilities still sum to 1?

- White and Heads $= 0.01 \times 0.5 = 0.005$
- White and Tails $= 0.01 \times 0.5 = 0.005$
- Red and Heads $= 0.99 \times 0.5 = 0.495$
- Red and Tails $= 0.99 \times 0.5 = 0.495$

Yes, the probabilitites of all the possibilities still add to 1.

62.2 Independence

In the last section, I told you that the probability of two events ("Pulling a red marble from the bag" and "Getting tails in a coin toss") is the product of the probability of each event: $0.99 \times 0.5 = 0.495$.

This is true if the two events are *independent*, that is the outcome of one doesn't change the probability of the other. The example I gave is independent: It doesn't matter what ball you pull from the bag, the outcome of the coin toss will always be 50-50.

What are two events that are not independent? The probability that a person is a professional basketball player and the probability that someone wears a shoe that is size 13 or larger is *not* independent. After all, height is an advantage in basketball and most tall people also have large feet. So if you know someone is a basketball player, they likely wear large shoes.

Exercise 73 Rolling Dice

If I give you three dice to roll, what is the probability that you will roll a 5 on all three dice?

Working Space

Answer on Page 409

Exercise 74 Flipping Coins

If I give you five coins to flip, what is the probability that at least one coin will come up heads?

Working Space

Answer on Page 409

62.3 Why 7 is the most likely sum of two dice

If you roll two dice, the sum will be 2 or 12 or any number in between. It is very tempting to assume that the likelihood of any of those numbers is the same. In fact, the probability of a 2 is $\frac{1}{36} \approx 3\%$ and the probability of a 7 is $\frac{1}{6} \approx 17\%$. A 7 is six times more likely than a 12! Why?

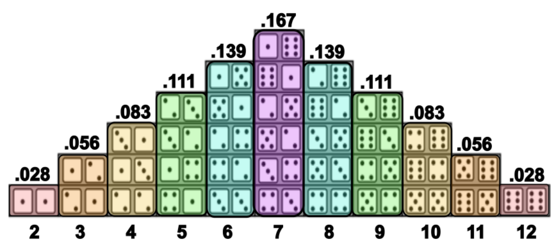
When you roll the first die, there are six possibilities with equal probability. When you roll the second die, there are six possibilities with equal probability. so there are a total of 36 possible events with equal probabilities: 1 then 1, 1 then 2, 2 then 1, 1 then 3, 3 then

1, etc. Only one of these (1 then 1) adds to 2. But six of these sum to 7: 1 then 6, 6 then 1, 2 then 5, 5 then 2, 3 then 4, 4 then 3. So a 7 is six times more likely than a 2.

Here is the complete table:

Sum							Count	Probability
2	1,1						1	1/36
3	1,2	2,1					2	1/18
4	1,3	2,2	3,1				3	1/12
5	1,4	2,3	3,2	4,1			4	1/9
6	1,5	2,4	3,3	4,2	5,1		5	5/36
7	1,6	2,5	3,4	4,3	5,2	6,1	6	1/6
8		2,6	3,5	4,4	5,3	6,2	5	5/36
9			3,6	4,5	5,4	6,3	4	1/9
10				4,6	5,5	6,4	3	1/22
11					5,6	6,5	2	1/18
12						6,6	1	1/36

When I bumped into this, I was skeptical. I decided to test it, so I rolled a pair of dice hundreds of times and made a histogram. It was a tedious and time-consuming task – just the sort of thing that we make computers do for us.



62.4 Random Numbers and Python

You are going to write a simulation of rolling dice in Python. To do this, you will need to generate a random sequence of numbers. The numbers will need to be in the range 1 to 6, and they will need to appear in the sequence with the same frequency. We say the sequence will follow *the uniform distribution*. That is, the probability is uniformly distributed among the 6 possibilities.

Start python and try a few of the different ways to generate random numbers:

```
> python3
>>> import random
>>> random.random()  # Generates a random floating point number between 0 and 1
0.6840892758539989
>>> randrange(5)      # Generates an integer in the range 0 - 4
2
```

```
>>> x = ['Rock', 'Paper', 'Scissors']
>>> random.choice(x)   # Pick a random entry from the sequence
'Paper'
>>> x
['Rock', 'Paper', 'Scissors']
>>> random.shuffle(x)   # Shuffle the order of the sequence
>>> x
['Scissors', 'Paper', 'Rock']
>>> a = list(range(30))
>>> a
[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15,
  16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 29]
>>> random.sample(a, 10) # Return 10 randomly chosen items from the sequence
[8, 7, 20, 9, 25, 13, 23, 11, 14, 16]
```

Clearly, Python has a lot of ways to do things that look random. I should be honest with you at this point: they aren't really random. The computer that you are using can't generate random data. Instead, it uses tricks to create data that looks random; we call this *pseudorandom* data. Good pseudorandom algorithms are very important for cryptography and data security.

What if you want real random data? Some companies that are using the decay of radioactive materials to generate real random data. You can pay to download it. For our purposes, Python's pseudorandom numbers are quite sufficient.

If we generate two random numbers in the range 1 through 6 and add them together, we will have simulated rolling a pair of dice. Like this:

```
>>> a = random.randrange(6) + 1
>>> b = random.randrange(6) + 1
>>> a + b
8
```

First, let's write a program that just rolls the dice 100 times and shows the result. Make a file `dice.py`:

```
import random

roll_count = 100

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
```

```
    roll = a + b
    print(f"Toss {i}: {a} + {b} = {roll}")
```

When you run it, you should see something like:

```
> python3 dice.py
Toss 0: 6 + 6 = 12
Toss 1: 4 + 4 = 8
Toss 2: 4 + 2 = 6
Toss 3: 4 + 6 = 10
Toss 4: 4 + 4 = 8
...
Toss 98: 5 + 2 = 7
Toss 99: 5 + 2 = 7
```

Now we want to count occurrences of each possible outcome. Let's use an array of integers. We will start with an array of zeros. And, for example, when we roll a 3, we'll add 1 to item 3 in the array. (We can never roll a zero or a one, so those two entries will always be zero.)

```
import random

roll_count = 100

# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b
    print(f"Toss i: a + b = roll")

    # Increment the count for roll
    counts[roll] += 1

print(f"Counts: counts")
```

When you run this, at the end you will see a count for each possible outcome :

```
...
Toss 98: 3 + 2 = 5
Toss 99: 6 + 1 = 7
Counts: [0, 0, 2, 6, 16, 11, 13, 14, 11, 11, 6, 9, 1]
```

What was the count that we expected? For example, we expected to see a 2 about once every 36 rolls, right? It might be nice to compare our count to what we expected. Add a few more lines, and we are going to increase the number of rolls. You will probably want to delete the line that prints each roll separately:

```
import random

# Can't ever be 0 or 1
p = [0.0, 0.0, 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18, 1/36]
roll_count = 1000

# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b

    # Increment the count for roll
    counts[roll] += 1

for i in range(2,13):
    print(f"i appeared counts[i] times, expected p[i] * roll_count:.1f")
```

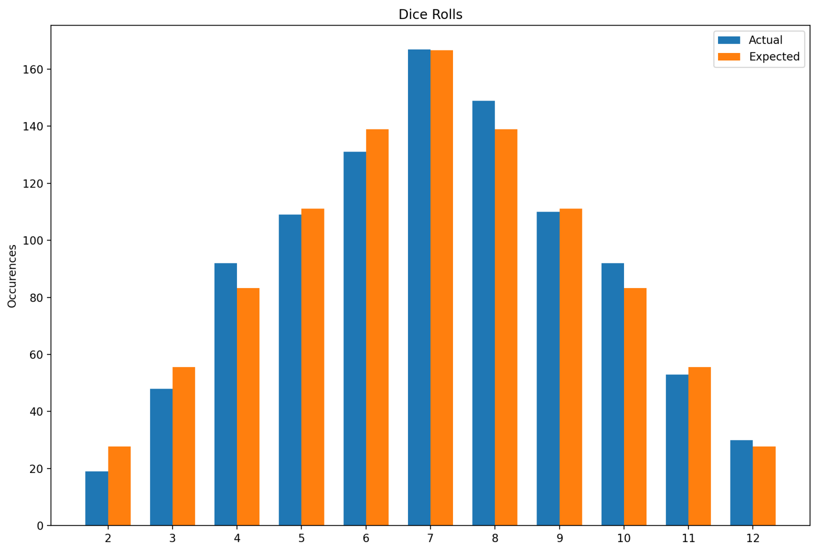
Now you should see something like:

```
2 appeared 39 times, expected 27.8
3 appeared 55 times, expected 55.6
4 appeared 84 times, expected 83.3
5 appeared 110 times, expected 111.1
6 appeared 160 times, expected 138.9
7 appeared 176 times, expected 166.7
8 appeared 124 times, expected 138.9
9 appeared 93 times, expected 111.1
10 appeared 87 times, expected 83.3
11 appeared 49 times, expected 55.6
12 appeared 23 times, expected 27.8
```

Whenever you are dealing with random numbers, the outcome will seldom be *exactly* what you expected. In this case, however, you should see that your predictions are pretty close.

62.4.1 Making a bar graph

A bar graph is a nice way to look at quantities like this. Let's make a bar graph that shows the actual count and the expected count:



We need to describe the set of rectangles, to do this we will loop through each possible roll (2 - 12) and put data in four lists for each:

```
import random
import matplotlib.pyplot as plt

# Can't ever be 0 or 1
p = [0.0, 0.0, 1/36, 1/18, 1/12, 1/9, 5/36, 1/6, 5/36, 1/9, 1/12, 1/18, 1/36]
roll_count = 1000

# Make an array containing 13 zeros
counts = [0] * 13

for i in range(roll_count):
    a = random.randrange(6) + 1
    b = random.randrange(6) + 1
    roll = a + b

    # Increment the count for roll
    counts[roll] += 1
```

```
# Gather data for bar chart
bar_width = 0.35
expected = []
actual_starts = []
expected_starts = []
labels = []
actual = []
for i in range(2,13):
    expected.append(p[i] * roll_count)
    actual.append(counts[i])
    actual_starts.append(i - bar_width/2)
    expected_starts.append(i + bar_width/2)
    labels.append(i)

fig, ax = plt.subplots()

# Create the bars
ax.bar(actual_starts, actual, bar_width, label='Actual')
ax.bar(expected_starts, expected, bar_width, label='Expected')
ax.set_xticks(labels)

# Provide labels
ax.set_ylabel('Occurences')
ax.set_title('Dice Rolls')
ax.legend()
plt.show()
```

Chapter 63

Beginning Combinatorics

Discrete probability problems often include some counting. For example, we figured out that there were 36 different ways the two dice, but all of them summed to some number 2 through 12. How many different ways could three 8-sided dice come up? We would need to count them, right? As the numbers get big we will need some tricks so we don't need to write them all down and count them one-by-one.

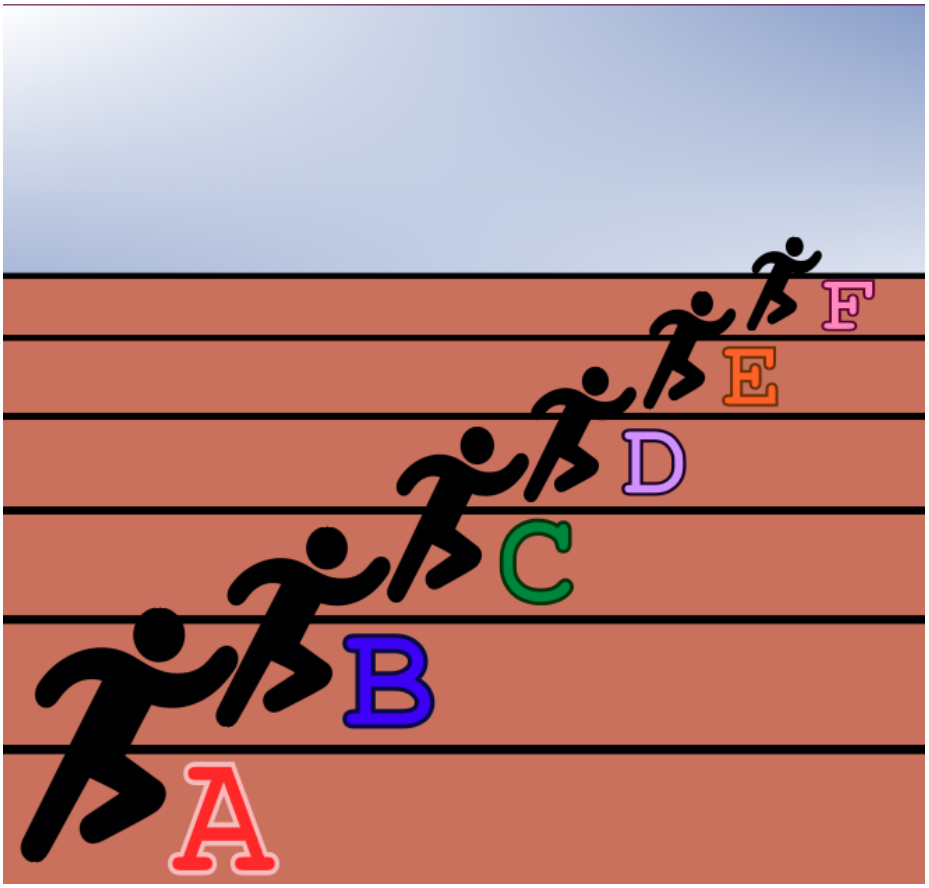
The branch of mathematics that focuses on tricks for counting is called *combinatorics*.

How can we be sure that there were 36 different configurations for the two 6-sided dice? The first die could have come up as any one of six numbers. For each of those, the second could have come up with any one of six numbers. Thus, the number of possibilities is $6 \times 6 = 36$.

How many different configurations for 3 8-sided dice? $8 \times 8 \times 8 = 8^3 = 512$.

What about seven dice, each with 20 sides? There would be $20^7 = 1,280,000,000$ configurations. See, aren't you glad we don't need to write them all down?

Now, let's say that six people (Anne, Brock, Carl, Dev, Edgar, and Fred) are going to run a race. You have to make a plaque that says who won first place, who won second place, and who won third. If you want to get all the possible plaques created beforehand, and just pull the right one out as soon as the race ends, how many plaques would you need to get engraved?



In this case, once someone has been given first place, they can't win second or third place. Thus, any of the 6 people can come in first, but once you have engraved that person's name on the plaque, there are only 5 people whose names can appear in second place. Once you have engraved that name, there are only 4 people whose names can appear in third place. Thus, you would get $6 \times 5 \times 4 = 120$ plaques engraved.

What if the plaque includes all 6 places? Then you would need $6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$ plaques engraved. We use this process often enough that we gave it a name. We say "I need 6 factorial plaques engraved." When we write a factorial, we use an exclamation point:

$$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$$

We use the word "permutation" to mean a particular ordering. This rule says n items can be ordered in $n!$ ways. Thus mathematicians actually say "If you have a list of n items then we can generate $n!$ different permutations of those items".

In Python, there is a `factorial` function in the math library:

```
> python3
>>> import math
>>> math.factorial(6)
720
```

Handy, right? Now you don't need to write a loop to calculate factorials.

Remember when we only wanted the first three names on the plaque? We can do that problem using factorials:

$$6 \times 5 \times 4 = \frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1} = \frac{6!}{3!}$$

This formulation makes it easy to figure out on any calculator with a "!" button.

The rule on this is to fill m positions from n items, it can be done this many ways:

$$\frac{n!}{(n-m)!}$$

63.0.1 Choose

Let's say that there are 12 kids in a classroom, and you need a team of 4 to wipe down the desks. How many different possible teams are there? You know that if you were giving out four different positions (Like the race gave out 1st, 2nd, and 3rd), the answer would be $12 \times 11 \times 10$ or $12!/(12-4)!$.

However, once we pick the 4 people, we don't care what order they are in, right? In this problem, the team "Anne, Brad, Carl, and Don" is the same as the team "Carl, Don, Brad, and Anne".

Thus, the quantity $12!/(12-4)!$ is many times too large because it counts each permutation separately. To get the right number, we just divide this by the number of possible permuations for a group of four people: $4!$

That gets us our answer: How many different teams of four can be chosen from 12 people?

$$\frac{12!}{(12-4)!4!} = 495$$

In combinatorics, we use this quantity a lot, so we have given it a name: *choose*

We have also given it a notation. “12 choose 4” is written like this:

$$\binom{12}{4}$$

Python has the `math.comb` function:

```
> python3
>>> import math
>>> comb(12, 4)
495
```

Chapter 64

Permutations and Sorting

In the previous chapter, we talked about permutations. If you have a list of three letters, like $[a, b, c, d]$, you can rearrange them in 4! ways:

a,b,c,d	a,b,d,c	a, d, b, c	a, d, c, b	a, c, b, d	a, c, d, b
b,a,c,d	b,a,d,c	b, d, a, c	b, d, c, a	b, c, a, d	b, c, d, a
c,b,a,d	c,b,d,a	c, d, b, a	c, d, a, b	c, a, b, d	c, a, d, b
d,b,c,a	d,b,a,c	d, a, b, c	d, a, c, b	d, c, b, a	d, c, a, b

You can make Python generate all the permutations for you:

```
from itertools import permutations
all_permutations = permutations(('a', 'b', 'c', 'd'))
for p in all_permutations:
    print(p)
```

64.1 Notation

How do we define or write down a single permutation? You could say something like "Swap the first and second items and swap the third and fourth items." However, that gets pretty difficult to read. So we usually write a permutation as two lines: the first line is before the permutation and the second line is after. Like this:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

And we can assign permutations to variables. For example, if we wanted the variable A to represent "swapping the first and second item", we would write this:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 3 & 4 \end{pmatrix}$$

And if we wanted B to represent "swapping the third and fourth item", we would write:

$$B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix}$$

Now, we can *compose* permutations together. For example, we might say:

$$B \circ A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

That is if we have the list $[a, b, c, d]$ and we apply permutation A and then permutation B, we get $[b, a, d, c]$.

Important: Note that permutations are applied from right to left. $B \circ A$ means "Applying A and then B." Why does this matter? Permutations are not necessarily commutative. That is, if you have two permutations S and T, $S \circ T$ is not always the same as $T \circ S$.

Also, note that "don't change anything" is a permutation. We call it *the identity permutation*. If you have four items, the identity permutation would be written:

$$I = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$$

(We use a capital "I" for the identity.)

64.1.1 Challenge

Find an example of two permutations S and T such that $S \circ T$ does not equal $T \circ S$.

64.2 Sorting in Python

One of the common forms of permutation in software is sorting. Sorting is putting data in a particular order. For example, in Python, if you had a list of numbers, you can sort it in ascending order like this:

```
my_grades = [92, 87, 76, 99, 91, 93]
grades_worst_to_best = sorted(my_grades)
```

Do you want to sort backwards?

```
my_grades = [92, 87, 76, 99, 91, 93]
grades_best_to_worst = sorted(my_grades, reverse=True)
```

Note that `sorted` makes a new list with the correct order. If you want to sort the array in place, you can use the `sort` method:

```
my_grades = [92, 87, 76, 99, 91, 93]
my_grades.sort(reverse=True)
```

64.3 Inverses

Think for a second about this permutation:

$$S = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}$$

You could say this permutation shuffles a list a bit. What is its inverse? That is, what is the permutation that unshuffles the items back to where they were originally?

$$S^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

That is, the original moved an item in the first spot to the third spot. The inverse must move whatever was in the third spot back to the first spot.

(Notation note: Because in multiplication, $b \times b^{-1} = 1$, we use "to the negative one" to indicate inverses in lots of places.)

Mechanically, how do you find the inverse? Flip the rows, and then sort the columns using the top number:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix} \text{ flip } \rightarrow \begin{pmatrix} 3 & 4 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix} \text{ sort } \rightarrow \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 1 & 2 \end{pmatrix}$$

Let's say you have two permutations A and B. Permuting by B and then A would look like this:

$$C = A \circ B$$

If you know A^{-1} and B^{-1}, what is C^{-1}? You would undo-A and then undo-B, so

$$C^{-1} = B^{-1} \circ A^{-1}$$

64.4 Cycles

Here is a permutation:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix}$$

When this is applied, whatever is at 1 gets moved to 2, 2 gets moved to 4, and 4 gets moved to 1. That is a *cycle*: $1 \rightarrow 2 \rightarrow 4$ and then it goes back to 1. It involves three locations, so we say it is a *3-cycle*.

There is another cycle in this permutation: $3 \rightarrow 5$ and then it goes back to 3.

Because these cycles share no members, we say the cycles are *disjoint*.

Every permutation can be broken down into a collection of disjoint cycles.

$$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{pmatrix} = (1 \rightarrow 2 \rightarrow 4)(3 \rightarrow 5)$$

The first handy thing about this notation is that it makes it easy for us to describe the inverse: we just run the cycles backward:

$$T^{-1} = (4 \to 2 \to 1)(5 \to 3)$$

Starting with the list $[a, b, c, d, e]$, lets repeatedly apply the permutation T

Initial	a, b, c, d, e
T applied	d, a, e, b, c
$T \circ T$ applied	b, d, c, a, e
$T \circ T \circ T$ applied	a, b, e, d, c
$T \circ T \circ T \circ T$ applied	d, a, c, b, e
$T \circ T \circ T \circ T \circ T$ applied	b, d, e, a, c
$T \circ T \circ T \circ T \circ T \circ T$ applied	a, b, c, d, e

This permutation, results in six combinations, and then it loops back on itself. The number of combinations is the least common multiple of all the cycles. In this case, there is a 3-cycle and a 2-cycle. The least common multiple of 2 and 3 is 6.

Chapter 65

Conditional Probability

Let's say there is a virus going around, and there is a vaccine for it that requires 2 shots. You are working at a school, and you are wondering how effective the vaccines are. Some students are unvaccinated, some have had one shot, and some have had two shots. One day you test all 644 students to see who has the virus. You end up with the following table:

	V_0	V_1	V_2
T_+	88 students	36 students	96 students
T_-	92 students	76 students	256 students

Here are what the symbols mean:

- V_0: student has had zero vaccination shots
- V_1: student has had one vaccination shot
- V_2: student has had both vaccination shots
- T_+: student tested positive for the virus
- T_-: student tested negative for the virus

So, for example, your data indicates that 76 students who had only one of the two shots and tested negative for the virus.

Your principal has a few questions. The first is "If I put five randomly chosen students in a study group together, what is the probabiltiy that one of them has the virus?"

The first thing you might do is make a new table that shows what is the probability of a randomly chosen student being in any particular group. You just divide each entry by 644 (the total number of students).

	V_0	V_1	V_2
T_+	$p(V_0 \text{ AND } T_+) = 13.7\%$	$p(V_1 \text{ AND } T_+) = 5.6\%$	$p(V_2 \text{ AND } T_+) = 14.9\%$
T_-	$p(V_0 \text{ AND } T_-) = 14.3\%$	$p(V_1 \text{ AND } T_-) = 11.8\%$	$p(V_2 \text{ AND } T_-) = 39.8\%$

(In this table, I expressed the number as a percentage with a decimal point – you had to round off the numbers. If you wanted exact answers, you would have to keep each as a fraction: 36 students represents $\frac{9}{161}$ of the student body.)

65.1 Marginalization

Now we can sum across the columns and rows.

	V_0	V_1	V_2	sum
T_+	0.137	0.056	0.149	$p(T_+) = 0.342$
T_-	0.143	0.118	0.398	$p(T_+) = 0.547$
sum	$p(V_0) = 0.280$	$p(V_1) = 0.174$	$p(V_2) = 0.547$	

If a child is chosen randomly from the entire student body, there is a 34.2% that the student has tested positive for the virus. And there is 17.4% chance that the student has one shot of the vaccine.

This summing of the probabilities across one dimension is known as *marginalizing*. Marginalization is just summing across all the variables that you don't care about. You don't care who has the virus, just the probability that a student has not received even one shot of the vaccine? You marginalize all the vaccine statuses.

To answer the principal's question, the easy thing to do is find the answer of the opposite "if I put five randomly chosen students in a study group together, what is the probabiltiy that *none* of them has tested positive for the virus?"

The chance that a randomly chosen student doesn't have the virus ($p(T_-)$ is 54.7%. Thus the chance that 5 randomly chosen students don't have the virus is $0.547 \times 0.547 \times 0.547 \times 0.547 \times 0.547 = 0.0489$ Thus the probability of the opposite is $1.0 - 0.0489 = 0.951$

The answer, then, is "If you put 5 kids in a study group together, there is a 95.1 % probability that at least one of them has the virus."

65.2 Conditional Probability

Now the principal asks you, "What if I make a group of 5 kids who have had both shots of the vaccine? What are the odds that one of them has tested positive for the virus?"

This involves the idea of *Conditional probability*. You want to know the odds that a student doesn't have the virus given that the student has had both shots of the vaccine.

There is a mathematical notation for this:

$$p(T_{-}|V_2)$$

That is the probability that a student who has had both vaccination shots will test negative for the virus.

How would you calculate this? You would count all the students who had a positive test *and* both vaccination shots, which you would divide by the total number of students who had both vaccination shots.

$$p(T_{-}|V_2) = \frac{256}{96 + 256} = \frac{8}{11} \approx 72.7\%$$

If we are working from the probabilities, you can get the same result this way: Divide the probability that a randomly chosen student had a positive test *and* both vaccination shots by the probabiltiy that a student had both vaccination shots:

$$p(T_{-}|V_2) = \frac{p(T_{-} \text{ AND } V_2)}{p(T_{-})} = \frac{0.398}{0.547} \approx 72.7\%$$

Notice that this is different from $p(V_2|T_{-})$, which is the probability that a student has had both vaccinations, given they tested negative for the virus.

Back to the principal's question: "If you have 5 students who have had both vaccinations, what is the probability that all of them tested negative for the virus?" The probability that one student is virus-free is $\frac{8}{11}$, so the probability that 5 students are virus-free is $\frac{8}{11}^5 \approx 0.203$. So, there is a 79.6% chance that at least one of the five has the virus.

65.3 Chain Rule for Probability

You just used this equality: For any events A and B

$$p(A|B) = \frac{p(A \text{ AND } B)}{p(B)}$$

This is more commonly written like this:

$$p(A \text{ AND } B) = \frac{p(A|B)}{p(B)}$$

This is an abstract way of writing the idea, but the idea itself is pretty intuitive: The probability that I'm going to buy and ticket and win the lottery is equal to the probability that I buy a ticket times the probability that I win, given that I have bought a ticket. (Here A is "win the lottery" and B is "buy a ticket".)

This is known as *The Chain Rule of Probability*. And we can chain together as many events as we want: The probability that you are going to die in the car that you bought with your winnings from the lottery ticket you bought is:

$$p(W \text{ AND } X \text{ AND } Y \text{ AND } Z) = p(W|X \text{ AND } Y \text{ AND } Z)p(X|Y \text{ AND } Z)p(Y|Z)p(Z)$$

where

- W = Dying in car accident
- X = Buying a car with lottery winnings
- Y = Winning the lottery
- Z = Buying a lottery ticket

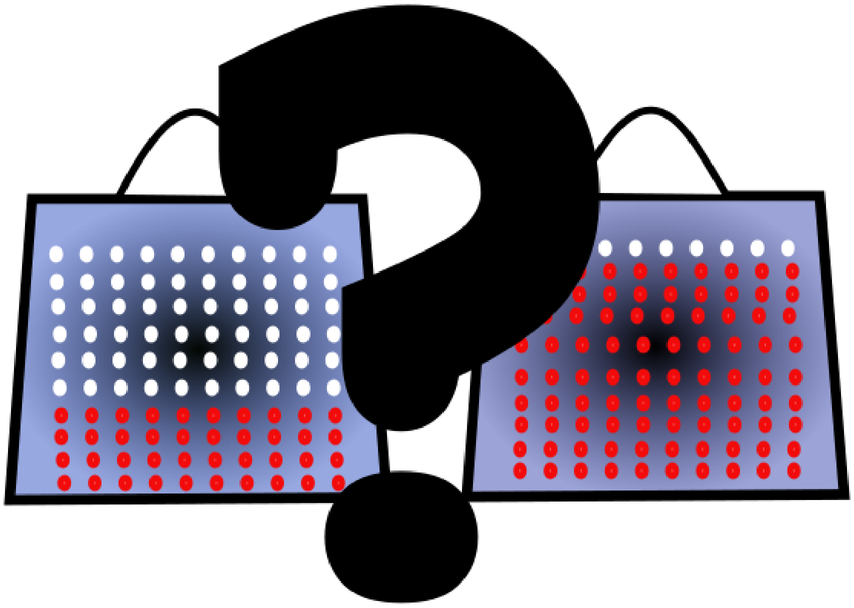
In English, then, the equation says:

"The probability that you will die in a car accident, buy a car with lottery winnings, win the lottery, and buy a lottery ticket is equal to the probability that you buy a lottery ticket times the probability that you win the lottery (given that you have bought a ticket) times the probability that buy a car with those lottery winnings (given that bought a ticket and won) times the probability that you crash that car (given that you have bought the car, won the lottery, and bought a ticket)."

Chapter 66

Bayes' Theorem

Let's say that you are holding two bags of marbles. You know that one bag contains 60 white marbles and 40 red marbles. And you know that the other holds 10 white marbles and 90 red marbles. You don't know which is which – and you can't see the marbles.



I say "Guess which bag is mostly red marbles." You pick one.

"What is the probability that this is the bag that is mostly red marbles?" You think "50 percent and there is also a 50 percent probabilitiy that it is the mostly-white-marbles bag."

Then you pick one marble from the bag. It is red. Now you must update your beliefs. It is more likely that this is the mostly-red-marbles bag. What is the probability now?

Bayes Theorem gives you the rule for updating your beliefs based on new data.

66.1 Bayes Theorem

Let's say you have two events or conditions C and D. C is "The person has a cough" and D is "The person is waiting to see a doctor."

Using the chain rule of probability, we now have two ways to calculate $p(C \text{ AND } D)$:

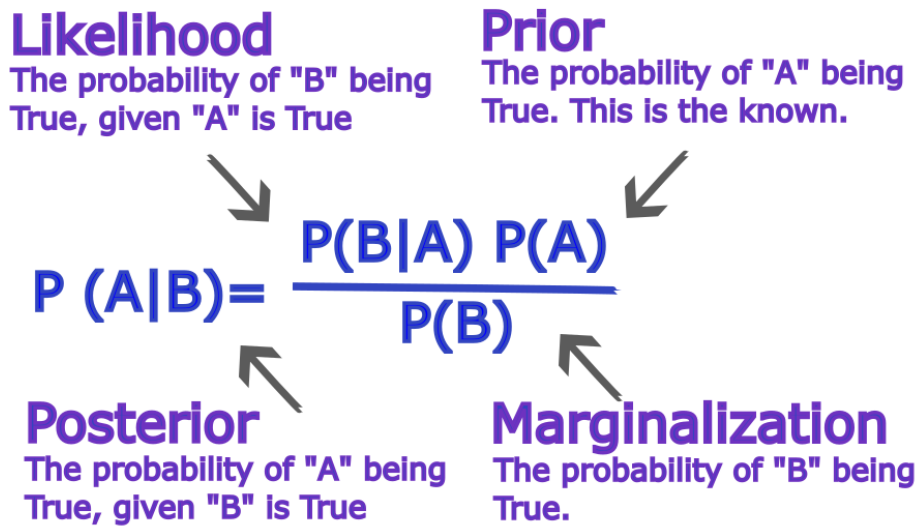
$$p(C \text{ AND } D) = p(C|D)p(D)$$

(The probability the person is at the doctor multiplied by the probability they have a cough if they are at the doctor.)

or

$$p(C \text{ AND } D) = p(C|D)p(D)$$

(The probabilitiy the person has a cough multiplied by the probabilitiy they are at the doc-



tor if they have a cough.)

Thus:

$$p(D|C) = \frac{p(C|D)p(D)}{P(C)}$$

Now you can calculate $p(D|C)$ (in this case, the probability that you are waiting to see a doctor given that you have a cough.) if you know:

- $p(C|D)$ (The probability that you have a cough given that you are waiting to see a doctor)
- $p(D)$ (The probability that you are waiting for a doctor for any reason.)
- $p(C)$ (The probability that you have a cough anywhere)

Pretty much all modern statistical methods (including most artificial intelligence) are based on this formula, which is known as Bayes' Theorem. It was written down by Thomas Bayes before he died in 1761. It was then found and published after his death.

66.2 Using Bayes' Theorem

Back to the example at the beginning. To review:

- There are two bags that look exactly the same.
- Bag W has 60 white marbles and 40 red marbles.
- Bag R has 10 white marbles and 90 red marbles.
- You pull one marble from the selected bag – it is red.

What is the probability that the selected bag is Bag R? Intuitively, you know that the probability is now more than 0.5. What is the exact number?

In terms of conditional probability, we say we are looking for "the probability that the selected bag is Bag R, given that you drew a red marble?" or $p(B_R|D_R)$, where B_R is "The selected bag is Bag R" and D_R is "You drew a red marble from the selected bag".

From Bayes' Theorem, we can write:

$$p(B_R|D_R) = \frac{P(D_R|B_R)P(B_R)}{P(D_R)}$$

$P(D_R|B_R)$ is just the probability of drawing a red marble given that the selected bag is Bag R. That is easy to calculate: There are 100 marbles in the bag, and 90 are red. Thus $P(D_R|B_R) = 0.9$.

$P(B_R)$ is just the probability that you chose Bag R before you drew out a marble. Both bags look the same, so $P(B_R) = 0.5$. This is called *the prior* because it represents what you thought the probability was before you got more information.

$P(D_R)$ is the probability of drawing a red marble. There was 0.5 probability that you put your hand into Bag W (in which 40 of the 100 marbles are red) and a 0.5 probability that you put your hand into Bag R (in which 90 of the 100 marbles are red). So

$$P(D_R) = 0.5\frac{40}{100} + 0.5\frac{90}{100} = 0.65$$

Putting it together:

$$p(B_R|D_R) = \frac{P(D_R|B_R)P(B_R)}{P(D_R)} = \frac{(0.9)(0.5)}{0.65} = \frac{9}{13} \approx 0.69$$

Thus, given that you have pulled a red marble, there is about a 69% chance that you have selected the bag with 90 red marbles.

66.3 Confidence

Bayes' Theorem, then, is about updating your beliefs based on evidence. Before you drew out the red marble, you selected one bag thinking it might contain 90 red marbles. How certain were you? 0.0 being complete disbelief and 1.0 entirely confidence, you were 0.5. After pulling out the red marble, you were about 0.69 confident that you had chosen the bag with 90 red marbles.

The question "How confident are you in your guess?" is very important in some situations. For example in medicine, diagnoses often lead to risky interventions. Few diagnoses come with 100% confidence. All doctors should know how to use Bayes' Theorem.

And in a trial, a jury is asked to determine if the accused person is guilty of a crime. Few jurors are ever 100% certain. In some trials, Bayes' Theorem is a really important tool.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 8)

To get the train to 20 meters per second in 120 seconds, you must accelerate it with a constant rate of $\frac{1}{6} m/s^2$. You remember that $F = ma$, so $F = 2400 \times \frac{1}{6}$. Thus, you will push the train with a force of 400 newtons for the 120 seconds before the bomb goes off.

Answer to Exercise 2 (on page 9)

$$F = G\frac{m_1 m_2}{r^2} = (6.674 \times 10^{-11})\frac{(6.8^3)(6 \times 10^{24})}{(10^5)^2} = 6.1 \times 10^6$$

About 6 million newtons.

Answer to Exercise 3 (on page 14)

The average hydrogen atom has a mass of 1.00794 atomic mass units.

The average oxygen atom has a mass of 15.9994.

$2 \times 1.00794 + 15.9994 = 18.01528$ atomic mass units.

Answer to Exercise 4 (on page 15)

From the last exercise, you know that 1 mole of water weighs 18.01528 grams. So 200 grams of water is about 11.1 moles. So you need to burn 11.1 moles of methane.

What does one mole of methane weigh? Using the periodic table: $12.0107 + 4 \times 1.00794 = 16.04246$ grams.

$16.0424 \times 11.10 = 178.1$ grams of methane.

Answer to Exercise 5 (on page 20)

At the top of the ladder, the cannonball has $(9.8)(5)(3) = 147$ joules of potential energy.

At the bottom, the kinetic energy $\frac{1}{2}(5)v^2$ must be equal to 147 joules. So $v^2 = \frac{294}{5}$. Thus it is going about 7.7 meters per second.

(Yes, a tiny amount of energy is lost to air resistance. For a dense object moving at these relatively slow speeds, this energy is neglible.)

Answer to Exercise 6 (on page 26)

$$4.5\ \cancel{\text{kWh}} \left(\frac{3.6 \times 10^6\ \cancel{\text{joules}}}{1\ \cancel{\text{kWh}}}\right)\left(\frac{1\ \text{calories}}{4.184\ \cancel{\text{joules}}}\right) = \frac{(4.5)(3.6 \times 10^6)}{4.184} = 1.08 \times 10^6 \text{calories}$$

Answer to Exercise ?? (on page 26)

$$\frac{0.1 \cancel{\text{gallons}}}{2 \cancel{\text{minutes}}} \left(\frac{3.7854 \cancel{\text{liters}}}{1 \cancel{\text{gallons}}} \right) \left(\frac{1000 \text{ milliliters}}{1 \cancel{\text{liters}}} \right) \left(\frac{1 \cancel{\text{minutes}}}{60 \text{ seconds}} \right) =$$
$$\frac{(0.1)(3.7854)(1000)}{(2)(60)} \text{ ml/second} = 3.1545 \text{ ml/second}$$

Answer to Exercise 8 (on page 31)

Paul is exerting $(70)(9.8)$ newtons of force at 4 meters from the fulcrum, so he is creating a torque of 2,744 newton-meters of torque on the see-saw. Jan is creating $(50)(9.5) = 490$ newtons of force.

If r is the distance from the fulcrum to Jan's seat, to balance $490r = 2744$, so $r = 5.6$ meters.

Answer to Exercise 9 (on page 33)

To lift the barrel would require $136 \times 9.8 = 1,332.8$ newtons of force.

Letting L be the length of the ramp:

$$300 = \frac{2}{L} 1332.8$$

So $L = 8.885$ meters.

Answer to Exercise 9 (on page 34)

$$583 = (70)(2.2)\frac{53}{n}$$

Thus $n = 14$ teeth.

Answer to Exercise 11 (on page 35)

We are looking for r, the radius of the piston head in meters. The area of the piston head is πr^2.

The pressure in pascals of the brake fluid is given by $12/(\pi r^2)$.

$$2,500,000 = \frac{12}{\pi r^2}$$

So $r = \sqrt{\frac{12}{\pi \times 2.5 \times 10^6}} = 0.001236077446474$ meters.

Answer to Exercise 12 (on page 48)

Equilibrium will be achieved when the box has displaced 10 kg of water. That is, when it has displaced 0.01 cubic meters.

The area of the base of the box is 0.12 square meters. So if the box sinks x meters into the water it will displace $0.12x$ cubic meters.

Thus at equilibrium $x = \frac{0.01}{0.12} \approx 0.083$ m. So, the box will sink 8.3 cm into the water before reaching equilibrium.

Answer to Exercise 13 (on page 50)

$$p = dgh = (900)(3.721)(5) = 16,744.5 \text{ Pa}$$

Answer to Exercise 14 (on page 53)

$$E_C = (1200)(0.4)(T - 10) = 480T - 4800$$

Total energy stays constant:

$$0 = (12600T - 252000) + (900T - 72000) + (480T - 4800)$$

Solving for T gets you $T = 23.52° C$.

Answer to Exercise 15 (on page 54)

During the 3 minutes, you want the coffee to give off as much of its heat as possible, so you want to maximize the difference between the temperature of the coffee and the temperature of the room around it.

You wait until the last moment to put the milk in.

Answer to Exercise 16 (on page 58)

$$\mu = \frac{1}{6}(87 + 91 + 98 + 65 + 87 + 100) = 88$$

Answer to Exercise 17 (on page 60)

The mean of your grades is 88.

The variance, then is

$$\sigma^2 = \frac{1}{6}\left((87-88)^2 + (91-88)^2 + (98-88)^2 + (61-88)^2 + (87-88)^2 + (100-88)^2\right) = \frac{784}{6} = 65\frac{1}{3}$$

The standard deviation is the square root of that: $\sigma = 8.083$ points.

Answer to Exercise 18 (on page 61)

In order the grades are 65, 87, 87, 91, 98, 100. The middle two are 87 and 91. The mean of those is 89. (Speed trick: The mean of two numbers is the number that is half-way between.)

Answer to Exercise 19 (on page 71)

The formula for the RMS is "=SQRT(SUMSQ(A2:A1001)/COUNT(A2:A1001))".

Answer to Exercise 20 (on page 82)

$V = IR$ so $I = \frac{V}{R} = \frac{24V}{6\Omega} = 4A$.

Answer to Exercise ?? (on page 84)

There is a total resistance of 8Ω, so your 16V will push 2A of current around the circuit.

2A going through a 5Ω resistor represents a 10V drop.

2A going through a 3Ω resitor represents a 6V drop.

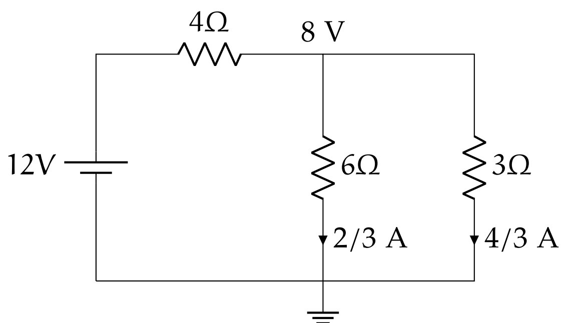
Answer to Exercise 22 (on page 86)

The effective resistance of the 6Ω and the 3Ω is 2Ω because

$$\frac{1}{R_T} = \frac{1}{6} + \frac{1}{3} == \frac{1}{2}$$

So the battery experiences a resistance of $4\Omega + 2\Omega = 6\Omega$. A 12V will push 2A through a resistance of 6Ω.

The voltage drop across the 4Ω resistor is $2A \times 4\Omega = 8V$. Thus there will be a 4V drop across the two resistors in parallel. So 2/3 A will flow through the 6Ω resistor. 4/3 A will flow through the 3Ω resistor.



Answer to Exercise 23 (on page 88)

$$F = K\frac{|q_1 q_2|}{r^2} = (8.988 \times 10^9)\frac{(-5 \times 10^{-9})(-5 \times 10^{-9})}{0.12^2} = \frac{224.7 \times 10^{-9}}{0.0144} = 15.6 \times 10^{-6}$$

15.6 micronewtons.

Answer to Exercise 24 (on page 94)

The volume of the sphere (in cubic meters) is

$$\frac{4}{3}\pi(1.5)^3 = 4.5\pi \approx 14.14$$

The mass (in kg) is $14.14 \times 7800 \approx 110,269$

The kinetic energy (in joules) is

$$k = \frac{110269 \times 5^2}{2} = 1,378,373$$

About 1.4 million joules.

Answer to Exercise 25 (on page 95)

In your mind, you can dissemble the tablet into a sphere (made up of the two ends) and a cylinder (between the two ends)

The volume of the sphere (in cubic millimeters) is

$$\frac{4}{3}\pi(2)^3 = \frac{32}{3}\pi \approx 33.5$$

Thus the cylinder part has to be $90 - 33.5 = 56.5$ cubic mm. The cylinder part has a radius of 2 mm. If the length of the cylinder part is x, then

$$\pi 2^2 x = 56.5$$

Thus $x = \frac{56.5}{4\pi} \approx 4.5$ mm.

The cylinder part of the table needs to be 4.5mm. Thus the entire tablet is 8.5mm long.

Answer to Exercise 26 (on page 104)

$180° - (92° + 42°) = 46°$

Answer to Exercise 27 (on page 105)

$360°$

Answer to Exercise 28 (on page 108)

10 because $6^2 + 8^2 = 10^2$

12 because $5^2 + 12^2 = 13^2$

8 because $8^2 + 15^2 = 17^2$

$3\sqrt{2} \approx 4.24$ because $3^2 + 3^2 = \left(3\sqrt{2}\right)^2$

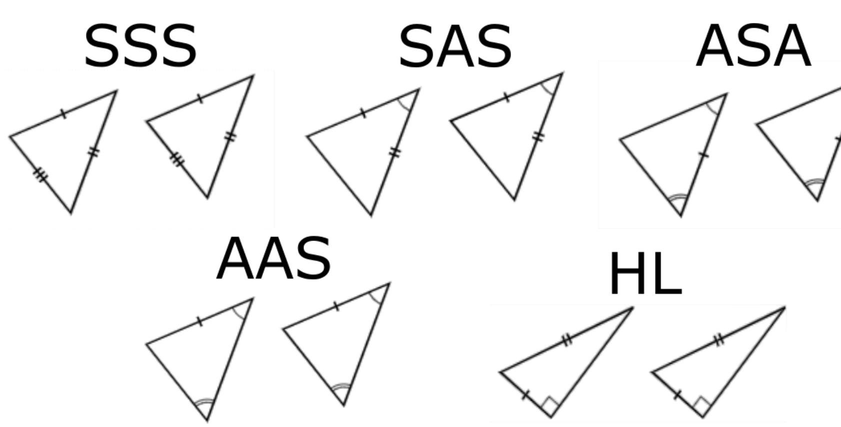
Answer to Exercise 29 (on page 115)

Congruent by the Side-Side-Right Congruency Test.

Congruent by the Side-Side-Side Congruency Test.

Congruent by the Side-Angle-Angle Congruency Test.

We don't know if they are congruent. The measured angle is not between the measured sides.



Answer to Exercise 30 (on page 119)

- $[1,2,3]+[4,5,6]=[5,7,9]$
- $[-1,-2,-3,-4]+[4,5,6,7]=[3,3,3,3]$
- $[\pi,0,0]+[0,\pi,0]+[0,0,\pi]=[\pi,\pi,\pi]$

Answer to Exercise 31 (on page 119)

To get the net force, you add the two forces:

$$F=[4.2,5.6,9.0]+[-100.2,30.2,-9.0]=[-96,35.8,0.0] \text{ newtons}$$

Answer to Exercise 32 (on page 121)

- $2\times[1,2,3]=[2,4,6]$
- $[-1,-2,-3,-4]\times-3=[3,6,9,12]$
- $\pi[\pi,2\pi,3\pi]=\pi^2,2\pi^2,3\pi^2]$

Answer to Exercise 33 (on page 123)

- $|[1,1,1]|=\sqrt{3}\approx 1.73$
- $|[-5,-5,-5]|=|-5\times[1,1,1]|=5\sqrt{3}\approx 8.66$
- $|[3,4,5]+[-2,-3,-4]|=|[1,1,1]|=\sqrt{3}\approx 1.73$

Answer to Exercise 34 (on page 126)

The momentum of the first car is 12,000 kg m/s in the north direction.

The momentum of the second car is 24,000 kg m/s in the east direction.

The new object will be moving northeast. What angle is the angle compared with the east?

$$\theta = \arctan \frac{12,000}{24,000} \approx 0.4636 \text{ radians } \approx 26.565 \text{ degrees north of east}$$

The magnitude of the momentum of the new object is $\sqrt{12,000^2 + 24,000^2} \approx 26,833$ kg m/s

Its new mass is 2,5000 kg. So the speed will be $26,833/2,500 = 10.73$ m/s.

Answer to Exercise 35 (on page 128)

The original forward momentum was 1.2 kg m/s. The original kinetic energy is $(1/2)(0.4)(3^2)$ $= 1.8$ joules.

Let s be the post-collision speed of the ball that had been at rest. Let x and y be the forward and sideways speeds (post-collision) of the other ball. Conservation of kinetic energy says

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(x^2 + y^2) = 1.8$$

Forward momentum is conserved:

$$0.4\frac{s}{\sqrt{2}} + 0.4x = 1.2$$

Which can be rewritten:

$$x = 3 - \frac{s}{\sqrt{2}}$$

Sideways momentum stays zero:

$$(0.4)\frac{s}{\sqrt{2}} - 0.4y = 0.0$$

Which can be rewritten:

$$y = \frac{s}{\sqrt{2}}$$

Substituting into to the conservation of kinetic energy equation above:

$$(1/2)(0.4)(s^2) + (1/2)(0.4)(\left(3 - \frac{s}{\sqrt{2}}\right)^2 + \left(\frac{s}{\sqrt{2}}\right)^2 = 1.8$$

Which can be rewritten:

$$s^2 - \frac{3}{\sqrt{2}}s + 0 = 0$$

There are two solutions to this quadratic: $s = 0$ (before collision) and $s = \frac{3}{\sqrt{2}}$. Thus,

$$y = \frac{3}{2}$$

and

$$x = 3 - \frac{3}{2} = \frac{3}{2}$$

So both balls careen off at 45° angles at the exact same speed.

Answer to Exercise 36 (on page 130)

- $[1, 2, 3] \cdot [4, 5, -6] = 4 + 10 - 18 = -4$
- $[\pi, 2\pi] \cdot [2, -1] = 2\pi - 2\pi = 0$
- $[0, 0, 0, 0] \cdot [10, 10, 10, 10] = 0 + 0 + 0 + 0 = 0$

Answer to Exercise 37 (on page 131)

- $[1,0] \cdot [0,1] = 0$. The angle must be $\pi/2$.

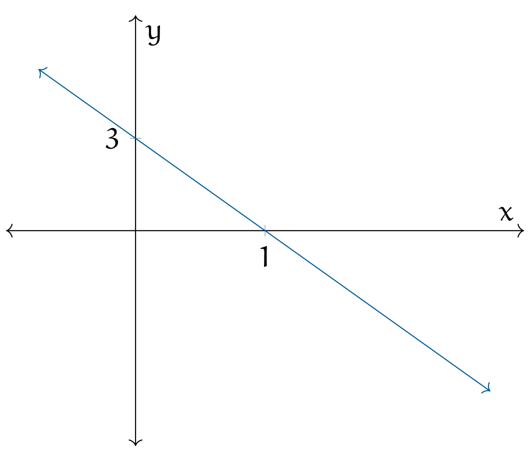
- $[3,4] \cdot [4,3] = 24$. $|[3,4]||[4,3]|\cos(\theta) = 24$. $\cos(\theta) = \frac{24}{(5)(5)}$. $\theta = \arccos(\frac{24}{25}) \approx 0.284$ radians.

Answer to Exercise 38 (on page 136)

You can only take the square root of nonnegative numbers, so the function is only defined when $x - 3 \geq 0$. Thus the domain is all real numbers greater than or equal to 3.

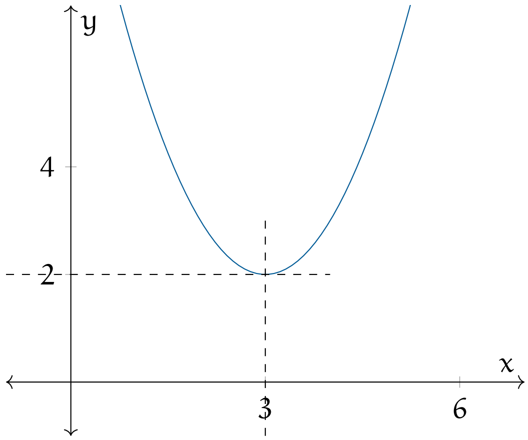
Answer to Exercise 39 (on page 137)

The graph of this function is a line. Its slope is -3. It intersects the y axis at $(0,3)$



Answer to Exercise 40 (on page 140)

This graph is the graph of $y = x^2$ that has been moved to the right by three units and up two units:



To prevent any horizontal line from containing more than one point of the graph, you would need to use the left or the right side: Either $\{x \in \mathbb{R} | x \leq 3\}$ or $\{x\mathbb{R} | x \geq 3\}$. Most people will choose the right side; the rest of the solution will assume that you did too.

To find the inverse we swap x and y: $x = (y-3)^2 + 2$

The we solve for y to get the inverse: $y = \sqrt{x-2} + 3$

You can take the square root of nonnegative numbers. So the function $f^{-1}(x) = \sqrt{x-2}+3$ is defined whenever x is greater than or equal to 2.

Answer to Exercise 41 (on page 145)

Solve for when the velocity is zero.

$t = \frac{12}{9.8} = 1.22$ seconds after release.

Answer to Exercise 42 (on page 158)

For what t is $-4.9t^2 + 12t + 2 = 0$? Start by dividing both sides of the equation by -4.9.

$$t^2 - 2.45t - 0.408 = 0$$

The roots of this are at

$$x = -\frac{b}{2} \pm \frac{\sqrt{b^2 - 4c}}{2} = -\frac{-2.45}{2} \pm \frac{\sqrt{(-2.45)^2 - 4(-0.408)}}{2} = 1.22 \pm 1.36$$

We only care about the root after we release the hammer ($t > 0$).

$1.22 + 1.36 = 2.58$ seconds after releasing the hammer, it will hit the ground.

Answer to Exercise 43 (on page 163)

The force of gravity is $9.8 \times 45 = 441$ newtons.

At any speed s, the force of wind resistance is $0.05 \times s^2 = 0.05s^2$ newtons.

At terminal velocity, $0.05s^2 = 441$.

Solving for s, we get $s = \sqrt{\frac{441}{0.05}}$

Thus, terminal velocity should be about 94 m/s.

Answer to Exercise 44 (on page 176)

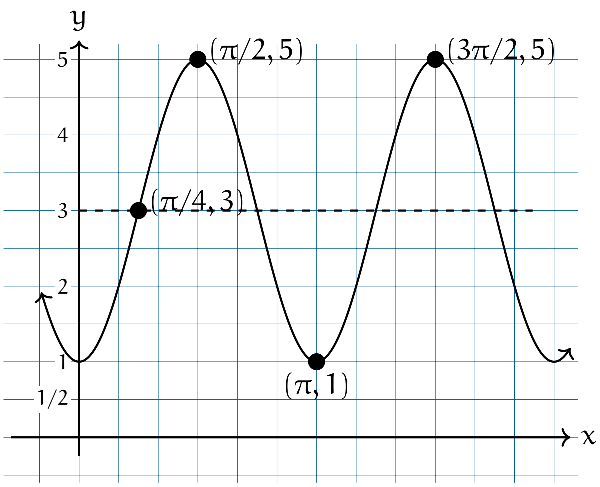
On earth, holding a 100 kg man aloft requires 980 Newtons of force.

$980/700 = 1.4$, so you need a cable with a cross-section area of 1.4 square millimeters.

$$\pi r^2 = 1.4$$

So $r = \sqrt{1.4/\pi} \approx .67$ millimeters. So the cable would have to have a diameter of at least 1.34 millimeters.

Answer to Exercise 45 (on page 225)



This wave has an amplitude of 2. Its baseline has been translated up to 3.

This wave has wavelength of π. A sine wave usually has a wavelength of 2π, so we need to compress the x axis by a factor of 2.

The wave first crosses its baseline at $pi/4$. The sine wave starts by crossing its baseline, so we need to translate the curve right by $\pi/4$.

$$f(x) = 2\sin(2x - \frac{\pi}{4}) + 3$$

Answer to Exercise 46 (on page 230)

A is 440 Hz. Each half-step is a multiplication by $\sqrt[12]{2} = 1.059463094359295$ So the frequency of E is $(440)(2^{7/12}) = 659.255113825739859$

Answer to Exercise 47 (on page 246)

$$\frac{120 \text{ km}}{1 \text{hour}} = \frac{1000 \text{ m}}{1 \text{ km}} \frac{120 \text{ km}}{1 \text{hour}} \frac{1 \text{ hour}}{3600 \text{ seconds}} = 33.3 \text{ m/s}$$

$$F = \frac{mv^2}{r} = \frac{0.4(33.3)^2}{200} = 2.2 \text{ newtons}$$

Answer to Exercise 47 (on page 251)

$$v = \sqrt{3.721(3.4 \times 10^6)} = 3,557 \text{ m/s}$$

The circular orbit is $2\pi(3.4 \times 10^6) = 21.4 \times 10^6$ meters in circumference.

The period of the orbit is $(21.4 \times 10^6)/3,557 \approx 6,000$ seconds.

Answer to Exercise 49 (on page 254)

$$\frac{300 \times 10^6}{5.66 \times 10^{14}} = 530 \times 10^{-9} = 530 \text{ nm}$$

Answer to Exercise 50 (on page 260)

The two triangles are similar, one is 2 m and 3m. The other is x cm and 3 cm.

The image of the cow is 2 cm tall.

Answer to Exercise 51 (on page 278)

$\boxed{-2x^3 + \frac{1}{2}x + 3.9}$ $\boxed{(4.5)x^2 + \pi x}$ $\boxed{7}$

$2x^{-10} + 4x - 1$ $x^{\frac{2}{3}}$ $\boxed{3x^{20} + 2x^{19} - 5x^{18}}$

Answer to Exercise 52 (on page 279)

Standard form would be $-x^3 + 21x^2 - 1000x + \pi$. The degree is 3. The leading coefficient is -1

Answer to Exercise 53 (on page 279)

$4^3 - (3)(4^2) + (10)(4) - 12 = 64 - 48 + 40 - 12$. So $y = 44$

Answer to Exercise 54 (on page 282)

```
favorites[3] = "Gloves"
```

Answer to Exercise 55 (on page 285)

```
pn2 = [2.5, 5.0, -2.0, -7.0, 4.0]
y = evaluate_polynomial(pn2, 8.5)
print("Polynomia 2: When x is 8.5, y is", y)

pn3 = [-9.0, 0.0, 0.0, 0.0, 0.0, 5.0]
y = evaluate_polynomial(pn3, 2.0)
print("Polynomial 3: When x is 2.0, y is", y)
```

Answer to Exercise 56 (on page 287)

The polynomial crosses the y-axis at 6. When x is zero, all the terms are zero except the last one. Thus you can easily tell that $x^3 - 7x + 6$ will cross the y-axis at $y = 6$.

Looking at the graph, you tell that the curve crosses the y-axes near -3, 1 and 2. If you plug those numbers into the polynomial, you would find that it evalutes to zero at each one. Thus, $x = -3$, $x = 1$, and $x = 2$ are roots.

Answer to Exercise 57 (on page 290)

$3x^3 - 7x^2 + x - 18$ and $3x^5 - 7x^3 + x^2 - 12$

Answer to Exercise 58 (on page 291)

$x^3 - 3x^2 + 5x$ and $x^5 - 3x^3 + 5x^2 - 2x + 6$

Answer to Exercise 59 (on page 292)

```
def add_polynomials(a, b):
    degree_of_result = max(len(a), len(b))
    result = []
    for i in range(degree_of_result):
        if i < len(a):
            coefficient_a = a[i]
        else:
            coefficient_a = 0.0

        if i < len(b):
            coefficient_b = b[i]
        else:
            coefficient_b = 0.0

        result.append(coefficient_a + coefficient_b)
    return result
```

Answer to Exercise 60 (on page 293)

```
def subtract_polynomial(a, b):
    neg_b = scalar_polynomial_multiply(-1.0, b)
    return add_polynomials(a, neg_b)
```

Answer to Exercise 61 (on page 296)

$(3x^2)(5x^3) = 15x^5$

$(2x)(4x^9) = 8x^{10}$

$(-5.5x^2)(2x^3) = -11x^5$

$(\pi)(-2x^5) = -2\pi x^5$

$(2x)(3x^2)(5x^7) = 30x^{10}$

Answer to Exercise 62 (on page 297)

$(3x^2)(5x^3 - 2x + 3) = 15x^6 - 6x^3 + 6x^2$

$(2x)(4x^9 - 1) = 8x^{10} - 2x$

$(-5.5x^2)(2x^3 + 4x^2 + 6) = 11x^5 - 22x^4 + 33x^2$

$(\pi)(-2x^5 + 3x^4 + x) = -2\pi x^5 + 3\pi x^4 + \pi x$

$(2x)(3x^2)(5x^7 + 2x) = 30x^{10} + 12x^4$

Answer to Exercise 63 (on page 299)

$(2x + 1)(3x - 2) = 6x^2 - x - 2$

$(-3x^2 + 5)(4x - 2) = -12x^3 + 6x^2 + 20x - 10$

$(-2x - 1)(-3x - \pi) = 6x^2 + (4 + 2\pi)x + \pi$

$(-2x^5 + 5x)(3x^5 + 2x) = -6x^{10} + 12x^6 + 10x^2$

Answer to Exercise 64 (on page 299)

The degree of the product is determined by the term that is the product of the highest degree term in p_1 and the highest degree term in p_2. Thus, the product of a degree 23 polynomial and a degree 12 polynomial has degree 35.

Answer to Exercise 65 (on page 306)

Answer to Exercise 66 (on page 307)

```
def derivative_of_polynomial(pn):

    # What is the degree of the resulting polynomial?
    original_degree = len(pn) - 1
    if original_degree > 0:
```

```
        degree_of_derivative = original_degree - 1
    else:
        degree_of_derivative = 0

    # We can ignore the constant term (skip the first coefficient)
    current_degree = 1
    result = []

    # Differentiate each monomial
    while current_degree < len(pn):
        coefficient = pn[current_degree]
        result.append(coefficient * current_degree)
        current_degree = current_degree + 1

    # No terms? Make it the zero polynomial
    if len(result) == 0:
        result.append(0.0)

    return result
```

Answer to Exercise 67 (on page 316)

$(2x - 3)(2x + 3) = 4x^2 - 9$

$(7 + 5x^3)(7 - 5x^3) = 49 - 25x^6$

$(x - a)(x + a) = x^2 - a^2$

$(3 - \pi)(3 + \pi) = 9 - \pi^2$

$(-4x^3 + 10)(-4x^3 - 10) = 16x^6 - 100$

$(x + \sqrt{7})(x - \sqrt{7}) = x^2 - 7$

$x^2 - 9 = (x + 3)(x - 3)$

$49 - 16x^6 = (7 + 4x^3)(7 + 4^3)$

$\pi^2 - 25x^8 = (\pi + 5x^4)(\pi - 5x^4)$

$x^2 - 5 = (x + \sqrt{5})(x - \sqrt{5})$

Answer to Exercise 68 (on page 319)

$(x+1)^2 = x^2 + 2x + 1$

$(3x^5 + 5)^2 = 9x^{1}0 + 30x^5 + 25$

$(x^3 - 1)^2 = x^6 - 2x^3 + 1$

$(x - \sqrt{7})^2 = x^2 - 2x\sqrt{7} + 7$

Answer to Exercise 69 (on page 320)

$(x+\pi)^5 = x^5 + 5\pi x^4 + 10\pi^2 x^3 + 10\pi^3 + x^2 + 5\pi^2 x + \pi^5$

Answer to Exercise 70 (on page 323)

Answer to Exercise 71 (on page 323)

Answer to Exercise 72 (on page 348)

```
SELECT bike_id FROM bike WHERE purchase_price > 330 AND brand='Trek'
```

Answer to Exercise 73 (on page 361)

probability of all 5's $= \frac{1}{6} \times \frac{1}{6} \times \frac{1}{6} = \left(\frac{1}{6}\right)^3 = \frac{1}{216} \approx 0.0046$

Answer to Exercise 73 (on page 361)

probability of at least one heads = 1.0 - probability of all tails $= 1.0 - \left(\frac{1}{2}\right)^5 = 1.0 - \frac{1}{32} = \frac{31}{32} =\approx 0.97$