Chapter 1

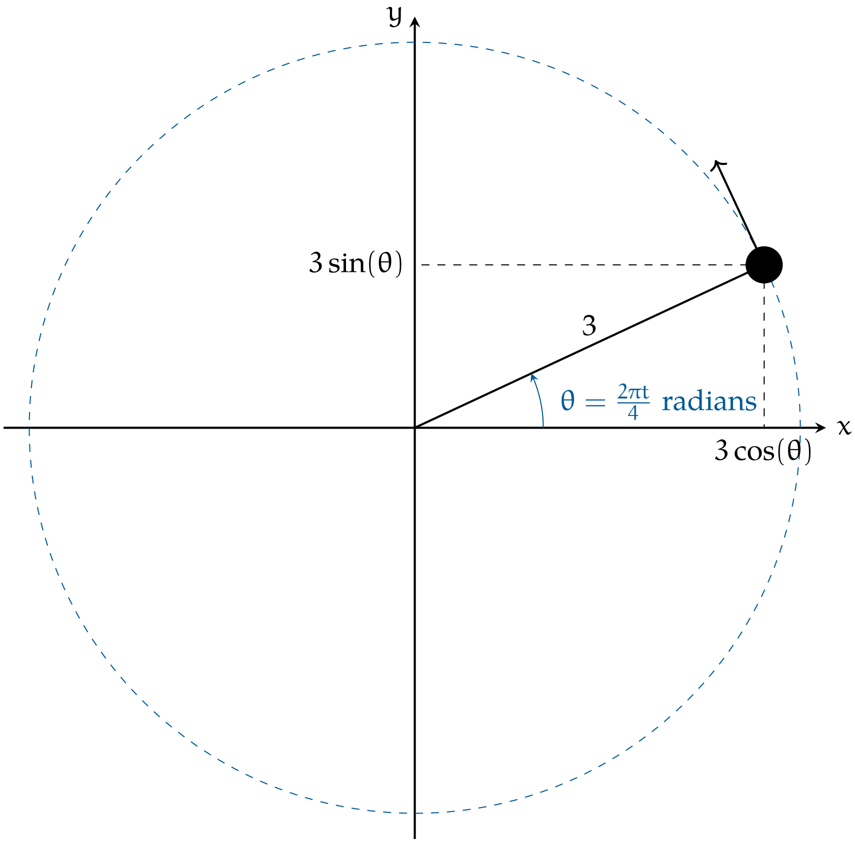
Circular Motion

Let's say you tie a 0.16 kg billard ball to a long string and begin to swing it around in a circle above your head. Let's say the string is 3 meters long, and the ball returns to where it started every 4 seconds. If you start your stopwatch as the ball crosses the x-axis, the position of the ball at any time t given by:

$$p(t) = [3\cos\left(\frac{2\pi}{4}t\right), 3\sin\left(\frac{2\pi}{4}t\right), 2]$$

(This assumes that the ball would be going counter-clockwise if viewed from above. The spot you are standing on is considered the origin $[0, 0, 0]$.)

Notice that the height is a constant – 2 meters in this case. That isn't very interesting, so we will talk just about the first two components. Here is what it would look like from above:



In this case, the radius, r, is 3 meters. The period, T is 4 seconds. In general, we say that circular motion is given by:

$$p(t) = \left[r\cos\frac{2\pi t}{T}, r\cos\frac{2\pi t}{T}\right]$$

A common question is "How fast is it turning right now?" If you divide the 2π radians of a circle by the 4 seconds it takes, you get the answer "About 1.57 radians per second." This is known as *angular velocity* and we typically represent it with the lowercase Omega: ω. (Yes, it looks a lot like a "w".) To be precise, in our example, the angular velocity is $\omega = \frac{\pi}{2}$.

Notice that this is different from the question "How fast is it going?" This ball is traveling the circumference of $6\pi \approx 18.85$ meters every 4 seconds. So the speed of the ball is about 4.71 meters per second.

1.1 Velocity

The velocity of the ball is a vector, and we can find that vector by differentiating each component of the position vector.

For any constants a and b:

Expression	Derivative
$a \sin bt$	$ab \cos bt$
$a \cos bt$	$-ab \sin bt$

Thus, in our example, the velocity of the ball at any time t is given by:

$$v(t) = \left[-\frac{3(2\pi)}{4} \sin \frac{2\pi t}{4}, \frac{3(2\pi)}{4} \cos \frac{2\pi t}{4}, 0 \right]$$

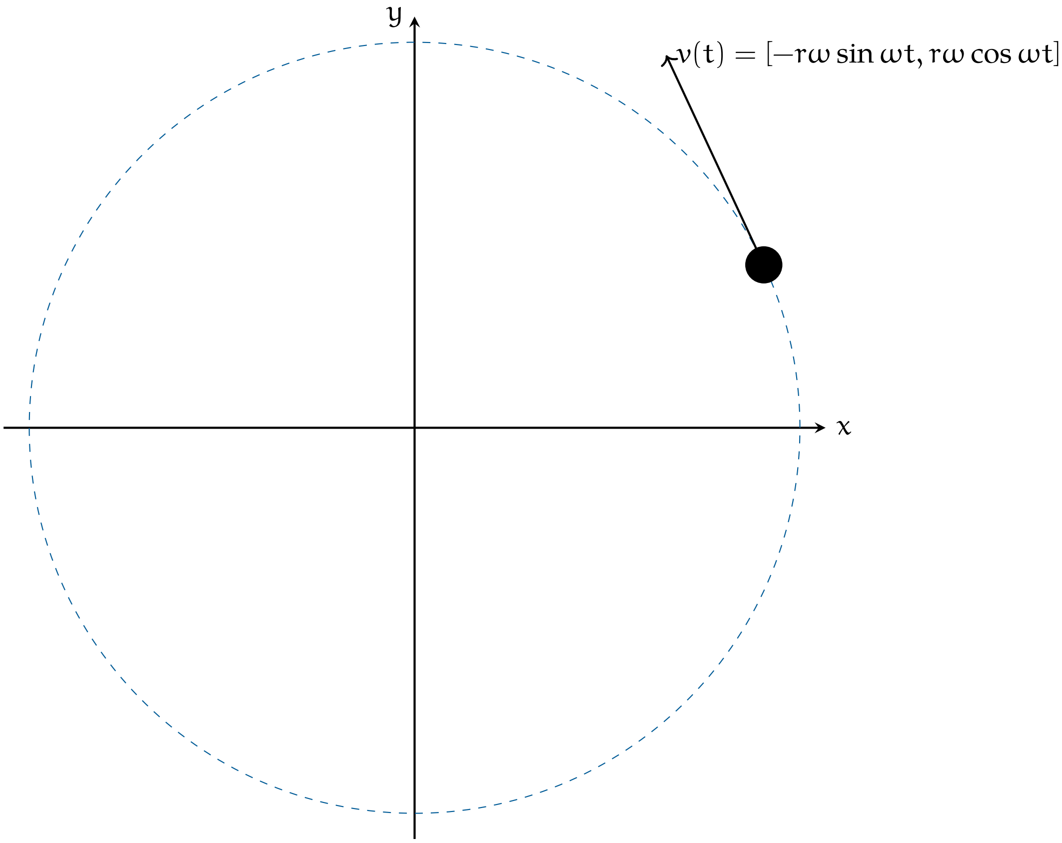
Notice that the velocity vector is perpendicular to the position vector. It has a constant magnitude.

In general, an object traveling in a circle at a constant speed has the velocity vector:

$$v(t) = [-r\omega \sin \omega t, r\omega \cos \omega t]$$

where $t = 0$ is the time that it crosses the x axis. If $\omega$is negative, that means the motion would be clockwise when viewed from above.

The magnitude of the velocity vector is $r\omega$.

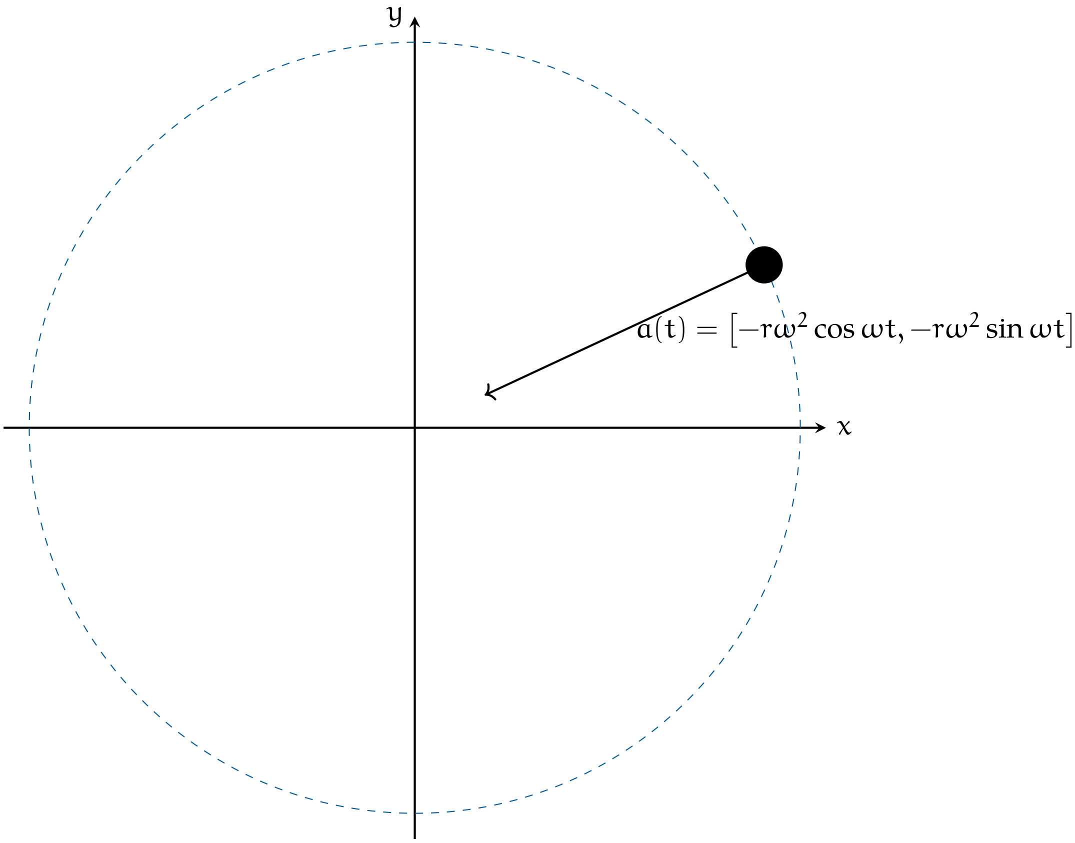


1.2 Acceleration

We can get the acceleration by differentiating the components of the velocity vector.

$$a(t) = \left[-r\omega^2 \cos \omega t, -r\omega^2 \sin \omega t\right]$$

Notice that the acceleration vector points toward the center of the circle it is traveling on. That is, when an object is traveling on a circle at a constant speed, its only acceleration is toward the center of the circle.



The magnitude of the acceleration vector is $r\omega^2$.

1.3 Centripetal force

How hard is the ball pulling against your hand? That is, if you let go, the ball would fly in a straight line. The force you are exerting on the string is what causes it to accelerate toward the center of the circle. We call this the *centripetal force*.

Recall that $F = ma$. The magnitude of the acceleration is $r\omega^2 = 3\left(\frac{2pi}{4}\right)^2 \approx 7.4$ m/s. The mass of the ball is 0.16 kg. So the force pulling against your hand is about 1.18 newtons.

The general rule is that when something is traveling in a circle at a constant speed, the centripetal force needed to keep it traveling in a circle is:

$$F = mr\omega^2$$

If you know the radius r and the speed v of the object, here is the rule:

$$F = \frac{mv^2}{r}$$

Exercise 1 **Circular Motion**

Just as your car rolls onto a circular track with a radius of 200 m, you realize your 0.4 kg cup of coffee is on the slippery dashboard of your car. While driving 120 km/hour, you hold the cup to keep it from sliding.
What is the maximum amount of force you would need to use (The friction of the dashboard helps you, but the max is when the friction is zero.)

Working Space

Answer on Page 7

This is a draft chapter from the Kontinua Project. Please see our website (`https://kontinua.org/`) for more details.

Appendix A

Answers to Exercises

Answer to Exercise 1 (on page 6)

$$\frac{120\text{ km}}{1\text{hour}} = \frac{1000\text{ m}}{1\text{ km}}\frac{120\text{ km}}{1\text{hour}}\frac{1\text{ hour}}{3600\text{ seconds}} = 33.3\text{ m/s}$$

$$F = \frac{mv^2}{r} = \frac{0.4(33.3)^2}{200} = 2.2\text{ newtons}$$