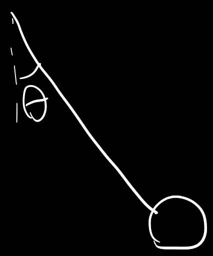

$$T=\frac{1}{2}mv^2=\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$$
$$=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)$$
$$V=mgy=mg(-r\cos\theta)$$
$$L=T-V=\frac{1}{2}m(\dot{r}^2+r^2\dot{\theta}^2)+mgr\cos\theta$$
$$=\frac{1}{2}ml^2\dot{\theta}^2+mgl\cos\theta$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)-\frac{\partial L}{\partial\theta}=0$$
$$\frac{d}{dt}(ml^2\dot{\theta})-mgl(-\sin\theta)=0$$
$$ml^2\ddot{\theta}+mgl\sin\theta=0 \longrightarrow \boxed{\ddot{\theta}=-\frac{g}{l}\sin\theta} \leftarrow \text{Equation of Motion}$$

Hamiltonian Mechanics

$H=$ Total Energy $=T+V$ (in simple cases) $\qquad p_i=\frac{\partial L}{\partial \dot{q}_i}$

$=\sum p_i\dot{q}_i-L$ (in general case)

H in Cartesian Coordinates in simple cases $=T+V=\frac{1}{2m}(p_x^2+p_y^2+p_z^2)+V$

Hamilton's Equations

$$\boxed{\frac{dq_i}{dt}=\frac{\partial H}{\partial p_i}} \quad \boxed{\frac{dp_i}{dt}=-\frac{\partial H}{\partial q_i}}$$

Spring Hamiltonian

$T=\frac{1}{2m}p_x^2 \qquad H=T+V=\frac{1}{2m}p_x^2+\frac{1}{2}kx^2$

$V=\frac{1}{2}kx^2 \qquad \boxed{\frac{dx}{dt}=\frac{\partial H}{\partial p}=\frac{p_x}{m}} \quad \boxed{\frac{dp_x}{dt}=-\frac{\partial H}{\partial x}=-kx}$

Just definition of momentum $p=mv$ $\qquad$ Hooke's Law $F=-kx$

Pendulum Hamiltonian

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$
$$= \frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta$$

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \Rightarrow \dot{\theta} = \frac{p_\theta}{ml^2}$$

$$H = p_\theta\dot{\theta} - L = \left(\frac{p_\theta^2}{ml^2}\right) - \left(\frac{1}{2}ml^2\dot{\theta}^2 + mgl\cos\theta\right)$$
$$= \left(\frac{p_\theta^2}{ml^2}\right) - \left(\frac{p_\theta^2}{2ml^2} + mgl\cos\theta\right)$$
$$\boxed{H = \frac{1}{2ml^2}p_\theta^2 - mgl\cos\theta}$$

$$\boxed{\frac{d\theta}{dt} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{ml^2}}$$

$$\frac{dp_\theta}{dt} = -\frac{\partial H}{\partial p_\theta} = -mgl\sin\theta$$
$$\downarrow$$
$$\frac{dp_\theta}{dt} = \frac{d}{dt}\left(ml^2\frac{d\theta}{dt}\right) = ml^2\ddot{\theta}$$
$$ml^2\ddot{\theta} = -mgl\sin\theta$$
$$\boxed{\ddot{\theta} = -\frac{g}{l}\sin\theta}$$

Same Eqn. We got from Lagrangian Mechanics $\longrightarrow$

Canonical Transformation

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \Longrightarrow \frac{dQ_i}{dt} = \frac{\partial H}{\partial P_i} \quad \frac{dP_i}{dt} = -\frac{\partial H}{\partial Q_i}$$

$Q_i(q_i, p_i)$ and $P_i(q_i, p_i)$ are our new coordinates. They are functions of the old coordinates. We can go the other way with $q_i(Q_i, P_i)$ and $p_i(Q_i, P_i)$.

Multivariable Chain Rule

$$\frac{d}{dt} f(a, b, c, \dots) = \frac{\partial f}{\partial a}\frac{da}{dt} + \frac{\partial f}{\partial b}\frac{db}{dt} + \frac{\partial f}{\partial c}\frac{dc}{dt} + \dots$$

$$\frac{dQ_i}{dt} = \sum \frac{\partial Q_i}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial Q_i}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial Q_i}{\partial t}\cancel{\frac{dt}{dt}}^{\,1}$$

$$\frac{dP_i}{dt} = \sum \frac{\partial P_i}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial P_i}{\partial p_i}\frac{dp_i}{dt} + \frac{\partial P_i}{\partial t}\cancel{\frac{dt}{dt}}^{\,1}$$

If we wanted Q_i and P_i to also be an explicit function of time, we would need these terms.

For a 1-D system, we have

$$\begin{bmatrix} \dot Q \\ \dot P \end{bmatrix} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} \begin{bmatrix} \dot q \\ \dot p \end{bmatrix}$$

By Hamilton's Equations, we have

$$\begin{bmatrix} \dot q \\ \dot p \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \dot Q \\ \dot P \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{bmatrix}$$

$$\begin{bmatrix} \dot Q \\ \dot P \end{bmatrix} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix}$$

Last Step

$$\frac{\partial H}{\partial q} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial q} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial q}$$

$$\frac{\partial H}{\partial p} = \frac{\partial H}{\partial Q}\frac{\partial Q}{\partial p} + \frac{\partial H}{\partial P}\frac{\partial P}{\partial p}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{bmatrix} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{bmatrix}$$

$$\begin{bmatrix} \dot Q \\ \dot P \end{bmatrix} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial P}{\partial q} \\ \frac{\partial Q}{\partial p} & \frac{\partial P}{\partial p} \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{bmatrix}$$

$$M = \text{Jacobian} = \begin{bmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{bmatrix}$$

If you're doing Physics. If you're doing math.

$$J = \Omega = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

By Hamilton's Equations,

$$\begin{bmatrix} \dot Q \\ \dot P \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial Q} \\ \frac{\partial H}{\partial P} \end{bmatrix}$$

$$\boxed{MJM^T = J \text{ for a canonical transform.}}$$

Testing if Euler Methods are Canonical Using Springs

$\frac{dp}{dt} = -kx$ and $\frac{dx}{dt} = \frac{p}{m}$ are the equations of motion

Euler

$$x_{n+1} = x_n + \Delta t \frac{dx}{dt} = x_n + \Delta t \frac{p_n}{m}$$
$$p_{n+1} = p_n + \Delta t \frac{dp}{dt} = p_n - \Delta t k x_n$$

$$\Rightarrow \text{Jacobian} = \begin{bmatrix} \frac{dx_{n+1}}{dx_n} & \frac{dx_{n+1}}{dp_n} \\ \frac{dp_{n+1}}{dx_n} & \frac{dp_{n+1}}{dp_n} \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{m}\Delta t \\ -\Delta t k & 1 \end{bmatrix} = M$$

$$MJM^T = \begin{bmatrix} 1 & \frac{1}{m}\Delta t \\ -\Delta t k & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -\Delta t k \\ \frac{1}{m}\Delta t & 1 \end{bmatrix} = \left(1 + \frac{k}{m}\Delta t^2\right) \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Euler is not canonical.

Semi-Implicit Euler v1

$$x_{n+1} = x_n + \Delta t \frac{dx}{dt} = x_n + \Delta t \frac{p_{n+1}}{m} = x_n + \Delta t \frac{1}{m}\left(p_n - \Delta t k x_n\right) = x_n + \Delta t \frac{p_n}{m} - \frac{1}{m}\Delta t^2 k x_n$$
$$= \left(1 - \frac{1}{m}\Delta t^2 k\right) x_n + \Delta t \frac{p_n}{m}$$
$$p_{n+1} = p_n + \Delta t \frac{dp}{dt} = p_n - \Delta t k x_n$$

Jacobian is
$$\begin{bmatrix} \frac{dx_{n+1}}{dx_n} & \frac{dx_{n+1}}{dp_n} \\ \frac{dp_{n+1}}{dx_n} & \frac{dp_{n+1}}{dp_n} \end{bmatrix} = \begin{bmatrix} 1 - \frac{k}{m}\Delta t^2 & \frac{\Delta t}{m} \\ -k\Delta t & 1 \end{bmatrix} = M$$

$$MJM^T = \left(1 - \cancel{\frac{k}{m}\Delta t^2} - \left(\cancel{-\frac{k}{m}\Delta t^2}\right)\right) J = J$$

Semi-Implicit Euler v1 is canonical

Semi-Implicit Euler v2

$$x_{n+1} = x_n + \Delta t \frac{dx}{dt} = x_n + \frac{p_n}{m}\Delta t$$
$$p_{n+1} = p_n + \Delta t \frac{dp}{dt} = p_n + \Delta t\left(-k x_{n+1}\right) = p_n - k\Delta t\left(x_n + \frac{p_n}{m}\Delta t\right) = \left(1 - \frac{k}{m}\Delta t^2\right) p_n - k\Delta t x_n$$

$$\text{Jacobian} = \begin{bmatrix} 1 & \frac{1}{m}\Delta t \\ -k\Delta t & \left(1 - \frac{k}{m}\Delta t^2\right) \end{bmatrix} = M$$

$$MJM^T = \left(1 - \cancel{\frac{k}{m}\Delta t^2} - \left(\cancel{-\frac{k}{m}\Delta t^2}\right)\right) J = J$$

Semi-Implicit Euler v2 is canonical.

Arbitrary Function of Position and Momentum → Arbitrary Function of Time

$F(p_i, q_i) \rightarrow F(t)$ For example, $F = \vec{L} = \vec{q} \times \vec{p}$, where $\vec{L}$ is angular momentum.

$$F(p_i, q_i) = \begin{bmatrix} p_i \\ \vdots \\ q_i \\ \vdots \end{bmatrix}$$ eventually.

$$\frac{dF}{dt} = \sum \frac{\partial F}{\partial q_i}\frac{dq_i}{dt} + \frac{\partial F}{\partial p_i}\frac{dp_i}{dt} \qquad + \frac{\partial F}{\partial t}\frac{dt}{dt}$$

If F were also an explicit function of time.

$$\frac{dF}{dt} = \sum \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i}$$

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \rightarrow$$

Poisson Bracket

$$\{F, G\} = \sum \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \Rightarrow \frac{dF}{dt} = \{F, H\}$$

Rules of Poisson Algebra

1. $\{X, Y\} = -\{Y, X\}$ — Anticommutivity
2. $\{aX + bY, Z\} = a\{X, Z\} + b\{Y, Z\}$, $\{X, aY + bZ\} = a\{X, Y\} + b\{X, Z\}$ — Bilinearity
3. $\{XY, Z\} = \{X, Z\}Y + X\{Y, Z\}$ — Leibniz Rule
4. $\{X, \{Y, Z\}\} + \{Y, \{Z, X\}\} + \{Z, \{X, Y\}\} = 0$ — Jacobi Identity

If in canonical coordinates

5. $\{p_i, p_j\} = 0$

   $\{q_i, q_j\} = 0$

   $\{q_i, p_j\} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{else} \end{cases}$

Important Consequences

$$\{F(q_i, p_i), p_i\} = \frac{\partial F}{\partial q_i} = -\{p_i, F(q_i, p_i)\}$$

and

$$\{q_i, F(q_i, p_i)\} = \frac{\partial F}{\partial p_i} = -\{F(q_i, p_i), q_i\}$$

Lie Series

Start with $f(t) \approx \sum_{k=0}^{n} \frac{f^{(k)}(t_0)}{k!}(t-t_0)^k = f(t_0) + \Delta t \frac{df}{dt} + \frac{\Delta t^2}{2!}\frac{d^2 f}{dt^2} + \frac{\Delta t^3}{3!}\frac{d^3 f}{dt^3} + \cdots$

Replace time derivatives w/ Poisson Brackets

$$f(t) \approx f(t_0) + \Delta t \{f, H\} + \frac{\Delta t^2}{2!}\{\{f, H\}, H\} + \frac{\Delta t^3}{3!}\{\{\{f, H\}, H\}, H\} + \cdots$$

First-order Approximation

$$\begin{bmatrix} q_{n+1} \\ p_{n+1} \end{bmatrix} = \underset{\substack{\uparrow \\ \text{Bad Euler}}}{\begin{bmatrix} q_n \\ p_n \end{bmatrix}} + \Delta t \left\{ \begin{bmatrix} q_n \\ p_n \end{bmatrix}, H \right\} = \begin{bmatrix} q_n \\ p_n \end{bmatrix} + \Delta t \begin{bmatrix} \frac{\partial H}{\partial p_n} \\ -\frac{\partial H}{\partial q_n} \end{bmatrix}$$

Shift Operator

$$f(t) \approx \sum_{k=0}^{n} \frac{(t-t_0)^k}{k!} \frac{d^k f}{dt^k} \rightarrow \underbrace{\left(\sum_{k=0}^{n} \frac{\Delta t^k}{k!} \frac{d^k}{dt^k} \right)}_{\text{Shift Operator}} f(t_0) = f(t_0 + \Delta t)$$

$$\hat{S} = \sum_{k=0}^{n} \frac{\Delta t^k}{k!} \frac{d^k}{dt^k} = \sum_{k=0}^{n} \frac{\Delta t^k}{k!} \left(\frac{d}{dt} \right)^k = \sum_{k=0}^{n} \frac{1}{k!} \left(\Delta t \frac{d}{dt} \right)^k \qquad \hat{H} f = \{f, H\}$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \Rightarrow \boxed{\hat{S} = e^{\Delta t \frac{d}{dt}}} \xrightarrow{\text{Hamiltonian Mechanics}} e^{\Delta t \{\cdot, H\}} \rightarrow e^{\Delta t \hat{H}}$$

Seperable Hamiltonian

$H = F(p_i) + G(q_i)$ For video games, $H = T(p_i) + V(q_i)$

$$\hat{T} f = \{f, T\} \qquad \hat{V} f = \{f, V\} \Rightarrow \hat{H} = \hat{T} + \hat{V} \Rightarrow f(t_0 + \Delta t) = e^{\Delta t \hat{H}} f(t_0) = e^{\Delta t (\hat{T} + \hat{V})} f(t_0)$$

$$f(t_0 + \Delta t) \approx e^{\Delta t \hat{T}} e^{\Delta t \hat{V}} f(t_0) = e^{\Delta t \hat{T}} e^{\Delta t \hat{V}} \begin{bmatrix} p_n \\ q_n \end{bmatrix}$$

Both $e^{\Delta t \hat{T}}$ and $e^{\Delta t \hat{V}}$ are symplectic/canonical

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} \approx e^{\Delta t \hat{T}} e^{\Delta t \hat{V}} \begin{bmatrix} p_n \\ q_n \end{bmatrix}$$

Force $= -\frac{\partial V}{\partial q}$

$$\approx e^{\Delta t \hat{T}} \left(\begin{bmatrix} p_n \\ q_n \end{bmatrix} + \Delta t \left\{ \begin{bmatrix} p_n \\ q_n \end{bmatrix}, V \right\} + \frac{\Delta t^2}{2!} \left\{ \left\{ \begin{bmatrix} p_n \\ q_n \end{bmatrix}, V \right\}, V \right\} + \cdots \right)$$

$$\approx e^{\Delta t \hat{T}} \left(\begin{bmatrix} p_n \\ q_n \end{bmatrix} + \Delta t \begin{bmatrix} \{p_n, V\} \\ \{q_n, V\} \end{bmatrix} \right) = e^{\Delta t \hat{T}} \left(\begin{bmatrix} p_n \\ q_n \end{bmatrix} + \Delta t \begin{bmatrix} -\frac{\partial V}{\partial q_n} \\ \underset{\substack{\uparrow \\ 0}}{\frac{\partial V}{\partial p_n}} \end{bmatrix} \right) = e^{\Delta t \hat{T}} \begin{bmatrix} p_n + \Delta t F_n \\ q_n \end{bmatrix} = e^{\Delta t \hat{T}} \begin{bmatrix} p_{n+1} \\ q_n \end{bmatrix}$$

$$\approx \begin{bmatrix} p_{n+1} \\ q_n \end{bmatrix} + \Delta t \left\{ \begin{bmatrix} p_{n+1} \\ q_n \end{bmatrix}, T \right\} + \cdots \approx \begin{bmatrix} p_{n+1} \\ q_n \end{bmatrix} + \Delta t \begin{bmatrix} \{p_{n+1}, T\} \\ \{q_n, T\} \end{bmatrix} = \begin{bmatrix} p_{n+1} \\ q_n + \Delta t \frac{\partial T}{\partial p_{n+1}} \end{bmatrix} = \begin{bmatrix} p_{n+1} \\ q_n + \Delta t \frac{p_{n+1}}{m} \end{bmatrix}$$

All that implies

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n + \Delta t F_n \\ q_n + \Delta t \frac{p_{n+1}}{m} \end{bmatrix} \leftarrow \text{Semi-Implicit Euler v1}$$

$$\begin{bmatrix} p_{n+1} \\ q_{n+1} \end{bmatrix} = \begin{bmatrix} p_n + \Delta t F_{n+1} \\ q_n + \Delta t \frac{p_n}{m} \end{bmatrix} \leftarrow \text{Semi-Implicit Euler v2}$$

where $F_n = F(q_n)$ and $F_{n+1} = F(q_{n+1})$

Difference Between $e^{\Delta t\hat{H}}$ and $e^{\Delta t\hat{T}}e^{\Delta t\hat{V}}$

$$e^{\Delta t\hat H}f = f + \Delta t\hat H f + \frac{\Delta t^2}{2!}\hat H^2 f + \cdots = f + \Delta t \hat H f + \frac{\Delta t^2}{2!}(\hat T+\hat V)^2 f + \cdots = f + \Delta t\hat H f + \frac{\Delta t^2}{2}(\hat T^2 + \hat T\hat V + \hat V\hat T + \hat V^2) f + \cdots$$

Don't commute, so $\hat T\hat V \neq \hat V\hat T$

$$e^{\Delta t\hat T}e^{\Delta t\hat V}f = e^{\Delta t\hat T}\left(f + \Delta t\hat V f + \frac{\Delta t^2}{2!}\hat V^2 f + \cdots\right) = e^{\Delta t\hat T}f + \Delta t e^{\Delta t\hat T}\hat V f + \frac{\Delta t^2}{2!}e^{\Delta t\hat T}\hat V^2 f + \cdots$$

$$= \left(f + \Delta t\hat T f + \frac{\Delta t^2}{2!}\hat T^2 f + \cdots\right) + \Delta t\left(\hat V f + \Delta t\hat T\hat V f + \cdots\right) + \frac{\Delta t^2}{2!}\left(\hat V^2 f + \cdots\right)$$

$$= f + \Delta t(\hat T + \hat V)f + \Delta t^2\left(\frac{\hat T^2}{2} + \hat T\hat V + \frac{\hat V^2}{2}\right)f + \cdots$$

$$\left(e^{\Delta t\hat T}e^{\Delta t\hat V} - e^{\Delta t(\hat T+\hat V)}\right)f = \left(\left[\cancel{1+\Delta t(\hat T+\hat V)} + \Delta t^2\left(\cancel{\frac{\hat T^2}{2}} + \hat T\hat V + \cancel{\frac{\hat V^2}{2}}\right) + \cdots\right] - \left[\cancel{1+\Delta t(\hat T+\hat V)} + \Delta t^2\left(\cancel{\frac{\hat T^2}{2}} + \frac{\hat T\hat V}{2} + \frac{\hat V\hat T}{2} + \cancel{\frac{\hat V^2}{2}}\right) + \cdots\right]\right)f = \left(\Delta t^2\left(\hat T\hat V - \left(\frac{\hat T\hat V}{2} + \frac{\hat V\hat T}{2}\right)\right) + \cdots\right)f$$

$$= \left(\frac{\Delta t^2}{2}(\hat T\hat V - \hat V\hat T) + \cdots\right)f = \boxed{\left(\frac{\Delta t^2}{2}[\hat T,\hat V] + \cdots\right)f}$$

Where $[\hat A,\hat B] = \hat A\hat B - \hat B\hat A$ is the commutator.

$$\Rightarrow e^{\Delta t\hat T}e^{\Delta t\hat V} \approx e^{\Delta t(\hat T+\hat V+\frac{\Delta t}{2}[\hat T,\hat V]+\cdots)} \Rightarrow \boxed{\hat H_s \approx \hat T + \hat V + \frac{\Delta t}{2}[\hat T,\hat V]+\cdots}$$

← Shadow Hamiltonian for Semi-Implicit Euler v1

By a similarly tedious argument or variable substitution,

$$e^{\Delta t\hat V}e^{\Delta t\hat T} \approx e^{\Delta t(\hat T+\hat V+\frac{\Delta t}{2}[\hat V,\hat T]+\cdots)} \Rightarrow \boxed{\hat H_s\approx \hat T+\hat V+\frac{\Delta t}{2}[\hat V,\hat T]+\cdots}$$

← Shadow Hamiltonian for Semi-Implicit Euler v2

Higher-Order Symplectic Integrators

Start w/ Shadow Hamiltonian: $\hat H_{s1} = \hat T+\hat V+\underline{\frac{\Delta t}{2}[\hat T,\hat V]}+\cdots$ $\qquad \hat H_{s2} = \hat T+\hat V+\underline{\frac{\Delta t}{2}[\hat V,\hat T]}+\cdots$

Get rid of these terms.

$$[\hat V,\hat T] = -[\hat T,\hat V] \Rightarrow \hat H_{s1}+\hat H_{s2} = 2(\hat T+\hat V+\cdots)$$

Higher-order terms

Δt^3 and higher

$\Downarrow$

Do $$\underbrace{\underbrace{\left(e^{\Delta t\hat T/2}e^{\Delta t\hat V/2}\right)}_{\text{Semi-Implicit Euler v1}}\underbrace{\left(e^{\Delta t\hat V/2}e^{\Delta t\hat T/2}\right)}_{\text{Semi-Implicit Euler v2}}}_{\text{Verlet Method v1}} \approx e^{\Delta t\hat H+\cdots}$$

$$\boxed{\text{Verlet Method v1 is better because you only need to calculate the force once.}}$$

$$\underbrace{\left(e^{\Delta t\hat V/2}e^{\Delta t\hat T/2}\right)\left(e^{\Delta t\hat T/2}e^{\Delta t\hat V/2}\right)}_{\text{Verlet Method v2}} \approx e^{\Delta t\hat H+\cdots}$$