Monotonic Sequence

https://www.youtube.com/watch?v=tHy3TXmZpF0

https://cpb-us-e2.wpmucdn.com/sites.uci.edu/dist/d/3128/files/2020/04/Lecture-10.pdf

Examples:

1. Two students were asked to write an nth term for the sequence 1; 16; 81; 256; ... and to write the 5th term of the sequence. One student gave the nth term as $u_n = n^4$. The other student, who did not recognize this simple law of formation, wrote $u_n = 10n^3 - 35n^2 + 50n - 24$. Which student gave the correct 5th term?

If $u_n = n^4$, then $u_1 = 1^4 = 1$, $u_2 = 2^4 = 16$, $u_3 = 3^4 = 81$, $u_4 = 4^4 = 256$, which agrees with the first four terms of the sequence. Hence the first student gave the 5th term as $u_5 = 5^4 = 625$.

If $u_n = 10n^3 - 35n^2 + 50n - 24$, then $u_1 = 1$; $u_2 = 16$; $u_3 = 81$; $u_4 = 256$, which also agrees with the first four terms given. Hence, the second student gave the 5th term as $u_5 = 601$.

Both students were correct. Merely giving a finite number of terms of a sequence does not define a unique nth term. In fact, an infinite number of nth terms is possible.

2. Explain exactly what is meant by the statement $\lim_{n\to\infty}(1-2n) = -\infty$.

If for each positive number M we can find a positive number N (depending on M) such that $a_n < -M$ for all $n > N$, then we write $\lim_{n\to\infty} = -\infty$.

In this case, $1 - 2n < -M$ when $2n - 1 > M$ or $n > \frac{1}{2}(M+1) = N$

3. Prove that a convergent sequence is bounded.

Given $\lim_{n\to\infty} a_n = a$, we must show that there exists a positive number P such that $|a_n| < P$ for all n. Now

$$|a_n| = |a_n - a + a| \le |a_n - a| + |a|$$

But by hypothesis we can find N such that $|a_n - a| < \varepsilon$ for all $n > N$, i.e.,

$$|a_n| < \varepsilon + |a| \qquad \text{for all } n > N$$

It follows that $|a_n| < P$ for all n if we choose P as the largest one of the numbers $a_1; a_2; \dots; a_N, \varepsilon + |a|$.

4. Prove the Bolzano–Weierstrass theorem

Suppose the given bounded infinite set is contained in the finite interval [a, b]. Divide this interval into two equal intervals. Then at least one of these, denoted by $[a_1, b_1]$, contains infinitely many points. Dividing $[a_1, b_1]$ into two equal intervals, we obtain another interval, say, $[a_2, b_2]$, containing infinitely many points. Continuing this process, we obtain a set of intervals $[a_n, b_n]$, n = 1, 2, 3, ... , each interval contained in the preceding one and such that

$$b_1 - a_1 = (b - a)/2,\ b_2 - a_2 = (b_1 - a_1)/2 = (b - a)/2^2,\ \dots,\ b_n - a_n = (b - a)/2^n$$

from which we see that $\lim_{n \to \infty} (b_n - a_n) = 0$.

This set of nested intervals corresponds to a real number which represents a limit point and so proves the theorem.

5. Prove that if $\lim_{n \to \infty} u_n$ exists, it must be unique.

We must show that if $\lim_{n \to \infty} u_n = l_1$ and $\lim_{n \to \infty} u_n = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\varepsilon > 0$ we can find n_0 such that

$$|u_n - l_1| < \frac{1}{2}\varepsilon \text{ when } n > n_0, \qquad |u_n - l_2| < \frac{1}{2}\varepsilon \text{ when } n > n_0$$

Then

$$\begin{aligned} |l_1 - l_2| &= |l_1 - u_n + u_n - l_2| \\ &\leq |l_1 - u_n| + |u_n - l_2| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

That is, $|l_1 - l_2|$ is less than any positive ε (however small) and so must be zero. Thus, $l_1 = l_2$.

Exercises:

Show

a. $\lim_{n \to \infty} a_n^p = a^p$

b. $\lim_{n \to \infty} p^{a_n} = p^a$

c. $\lim_{n \to \infty} \frac{c}{n^p} = 0$

d. $\lim_{n \to \infty} \frac{1 + 2 \,.\, 10^n}{5 + 3 \,.\, 10^n} = \frac{2}{3}$

e. $\lim_{n \to \infty} 3^{2n-1} = \infty$

f. Prove that the series $\sum_1^{\infty} (-1)^{n-1}$ diverges.