Monotonic Sequence

https://www.youtube.com/watch?v=tHy3TXmZpF0

https://cpb-us-e2.wpmucdn.com/sites.uci.edu/dist/d/3128/files/2020/04/Lecture-10.pdf

Examples:

1. Two students were asked to write an nth term for the sequence 1; 16; 81; 256; ... and to write the 5th term of the sequence. One student gave the nth term as $u_n = n^4$. The other student, who did not recognize this simple law of formation, wrote $u_n = 10n^3 - 35n^2 + 50n - 24$. Which student gave the correct 5th term?

If $u_n = n^4$, then $u_1 = 1^4 = 1$, $u_2 = 2^4 = 16$, $u_3 = 3^4 = 81$, $u_4 = 4^4 = 256$, which agrees with the first four terms of the sequence. Hence the first student gave the 5th term as $u_5 = 5^4 = 625$.

If $u_n = 10n^3 - 35n^2 + 50n - 24$, then $u_1 = 1$; $u_2 = 16$; $u_3 = 81$; $u_4 = 256$, which also agrees with the first four terms given. Hence, the second student gave the 5th term as $u_5 = 601$.

Both students were correct. Merely giving a finite number of terms of a sequence does not define a unique nth term. In fact, an infinite number of nth terms is possible.

2. Explain exactly what is meant by the statement $\lim_{n\to\infty}(1-2n) = -\infty$.

If for each positive number M we can find a positive number N (depending on M) such that $a_n < -M$ for all $n > N$, then we write $\lim_{n\to\infty} = -\infty$.

In this case, $1 - 2n < -M$ when $2n - 1 > M$ or $n > \frac{1}{2}(M+1) = N$

3. Prove that a convergent sequence is bounded.

Given $\lim_{n\to\infty} a_n = a$, we must show that there exists a positive number P such that $|a_n| < P$ for all n. Now

$$|a_n| = |a_n - a + a| \le |a_n - a| + |a|$$

But by hypothesis we can find N such that $|a_n - a| < \varepsilon$ for all $n > N$, i.e.,

$$|a_n| < \varepsilon + |a| \qquad \text{for all } n > N$$

It follows that $|a_n| < P$ for all n if we choose P as the largest one of the numbers $a_1; a_2; \dots ; a_N, \varepsilon + |a|$.

4. Prove the Bolzano–Weierstrass theorem

Suppose the given bounded infinite set is contained in the finite interval [a, b]. Divide this interval into two equal intervals. Then at least one of these, denoted by $[a_1, b_1]$, contains infinitely many points. Dividing $[a_1, b_1]$ into two equal intervals, we obtain another interval, say, $[a_2, b_2]$, containing infinitely many points. Continuing this process, we obtain a set of intervals $[a_n, b_n]$, n = 1, 2, 3, ... , each interval contained in the preceding one and such that

$$b_1 - a_1 = (b - a)/2,\ b_2 - a_2 = (b_1 - a_1)/2 = (b - a)/2^2 , \dots , b_n - a_n = (b - a)/2^n$$

from which we see that $\lim_{n\to\infty}(b_n - a_n) = 0$.

This set of nested intervals corresponds to a real number which represents a limit point and so proves the theorem.

5. Prove that if $\lim_{n\to\infty} u_n$ exists, it must be unique.

We must show that if $\lim_{n\to\infty} u_n = l_1$ and $\lim_{n\to\infty} u_n = l_2$, then $l_1 = l_2$.

By hypothesis, given any $\varepsilon > 0$ we can find n_0 such that

$$|u_n - l_1| < \frac{1}{2}\varepsilon \text{ when } n > n_0, \qquad |u_n - l_2| < \frac{1}{2}\varepsilon \text{ when } n > n_0$$

Then

$$\begin{aligned} |l_1 - l_2| &= |l_1 - u_n + u_n - l_2| \\ &\le |l_1 - u_n| + |u_n - l_2| \\ &< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \end{aligned}$$

That is, $|l_1 - l_2|$ is less than any positive ε (however small) and so must be zero. Thus, $l_1 = l_2$.

Exercises:

Show

a. $\lim_{n\to\infty} a_n^p = a^p$
b. $\lim_{n\to\infty} p^{a_n} = p^a$
c. $\lim_{n\to\infty} \frac{c}{n^p} = 0$
d. $\lim_{n\to\infty} \frac{1 + 2\,.\,10^n}{5 + 3\,.\,10^n} = \frac{2}{3}$
e. $\lim_{n\to\infty} 3^{2n-1} = \infty$
f. Prove that the series $\sum_1^\infty (-1)^{n-1}$ diverges.

CHAPTER 5

THE CLASSICAL RESULTS ON THE TOPOLOGY OF R^n

(based on Dr. N. Quimpo's notes)

We come now to a series of theorems which provide deep insights into the topological structure of the real line.

Before we tackle the theorems themselves, let us give them a "popular reading". In this form, the results seem commonsensical enough:

The first reads: "A line of soldiers marching up to a wall must stop before the wall or have to stop at the wall itself".

The second says: "You cannot crowd too many people in a room and still expect them to have elbow room".

The third reads: "If you open up a Russian doll, you expect to find a last doll".

In discussing these results, we shall think of the Least Upper Bound Property as our basic assumption. These theorems shall be consequences of this axiom.

Theorem 1. (*Bounded Monotone Sequence Property* or *BMSP*). In 3, a monotone sequence which is bounded converges.

Proof.

Let our sequence be $\{a_n\}$. The trace of $\{a_n\}$ is bounded so it has an LUB, say *a*. We claim that $\lim_{n\to\infty} a_n = a$.

Let $\varepsilon > 0$ be given. Since *a* is the LUB of the trace of $\{a_n\}$, then $a - \varepsilon$ is not an upper bound. Thus there exists n_o such that

$$a-\varepsilon < a_{n_0} < a < a+\varepsilon .$$

Since the sequence is increasing,

$$a-\varepsilon < a_n < a+\varepsilon \text{ for } n \geq n_o.$$

Hence

$$\lim_{n\to\infty} a_n = a . \quad \square$$

Exercise 1. Show by using *BMSP* that the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

converges.

Exercise 2. If S is a bounded set in n-space, define the diameter of S by

$$\text{diam}(S) = \sup\{|p - q| : p, q \in S\}.$$

(Note the role of the LUB Property in this definition.) Prove that if $A \subset B$, then diam (A) $\leq$ diam (B). (Hint: The problem reduces to comparing two sets of nonnegative numbers.)

Exercise 3. If A and B are any sets in n-space, define the distance between A and B by

$$d(A, B) = \inf \{p - q : p \in A, q \in B\}.$$

Prove that if $A \subset B$, then

$$d(B, C) \leq d(A, C).$$

Theorem 2. (*Nested Intervals Property* or *NIP*). Let $\{I_n\}$ be a sequence of nonempty bounded closed intervals on the line such that

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

Then

$$\bigcap_1^\infty I_n \neq \varnothing.$$

Proof.

Let $I_n = [a_n, b_n]$, $n = 1, 2, \cdots$

Exercise 4. The student is asked to show first that

$$a_1 \leq a_n \leq b_m \leq b_1 \text{ for any n, m}$$

Thus,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \text{ for each n,}$$

so $\{a_n\}$ is an increasing sequence bounded by b_1. By *BMSP*, $\{a_n\}$ converges to some a. In fact the proof of *BMSP* indicates that *a* is the least upper bound of the set $\{a_1, a_2, a_3, \cdots\}$.

Similarly, $\{b_n\}$ converges to some b, where b is the greatest lower bound of $\{b_1, b_2, b_3, \cdots\}$.

Claim: $a \leq b$.

For, suppose $a > b$. This implies b is not an upper bound of $\{a_1, a_2, a_3, \cdots\}$. So for some n_o,

$$a_{n_0} > b.$$

It follows that for some n,

$$b_{n_1} < a_{n_0}.$$

This contradicts the fact that $a_n \leq b_m$ for every n, m. Hence $a \leq b$. Since

$$a_n \leq a \leq b \leq b_m \text{ for each } n$$

we have

$$[a, b] \subset [a_n, b_n] \text{ for each } n$$

$$[a, b] \subset \bigcap_1^\infty [a_n, b_n]. \qquad \square$$

Note 1. In the above theorem, if the lengths of the intervals have limit

$$\lim_{n\to\infty}(b_n - a_n) = 0,$$

then a = b and the intersection is {a}.

Theorem 3. (*Bolzano-Weierstrass Theorem* for the real line). Any infinite set of real numbers which is bounded has a cluster point.

Proof.

Suppose that the set S under investigation is contained in the closed interval I_o which is of finite length, say L.

We divide I_o into two equal subintervals. One of these subintervals must contain an infinite number of points of S. Call this particular subinterval I_1.

We now divide I_1, in turn into two equal subintervals. Again one of these subintervals of I_1 must contain infinitely many points of S. Call this subinterval I_2.

Repeating the above procedure we obtain subinterval I_k in step k, which has the following description:

i. I_k contains infinitely many points of S;
ii. $I_k \subset I_{k-1}$;
iii. I_k is of length $\ell_k = \dfrac{L}{2^k}$.

Thus we obtain a sequence of closed intervals I_k which are nonempty (indeed each one contains infinitely many points of S) and satisfy

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

By the *NIP*,

$$\bigcap_1^\infty I_k \neq \varnothing.$$

Since

$$\lim_{n\to\infty} \ell_k = 0$$

then in fact the intersection is a single point c.

Claim . c is a cluster point of S.
For if $\varepsilon > 0$, the neighborhood $(c - \varepsilon, c + \varepsilon)$ contains I_k for $\frac{L}{2^k} < \varepsilon$. Now I_k contains

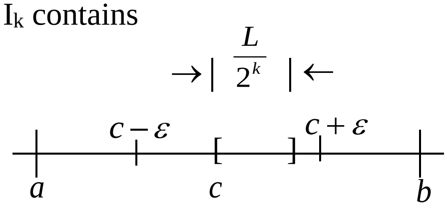

Fig. 1. "Worst case" scenario

infinitely many points of S, therefore so does $(c - \varepsilon, c + \varepsilon)$.
□

COROLLARY 4. Every bounded sequence of real numbers has a limit point.

Proof.

Exercise 5. (By the way, compare this to *BMSP*).

Exercise 6. Show that a bounded sequence of real numbers with exactly one limit point is convergent.

Exercise 7. State a 2 – space version of the *NIP* involving rectangles. Prove by applying the *NIP*.

Exercise 8. What is the general 2 – space version of the *Bolzano-Weierstrass Theorem*? Prove.

The general *Nested Interval Property* assumes the following form:

Theorem. Let $\{C_n\}$ be a sequence of nonempty closed and bounded sets in n – space such that

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

Then $\bigcap_{n=1}^{\infty} C_n$ is nonempty and closed.

Proof.

The fact that $\cap\, C_n$ is closed follows from a result seen earlier about the intersection of closed sets.

Let us assume that each C_n contains an infinite number of points. (See the exercise that follows for the other possibility.)

Choose a point in C_n and denote it by p_n. Let

$$P = \{p_1, p_2, \ldots\}.$$

Since P is an infinite set contained in the bounded set C_1, then it has a cluster point p.

Claim. $p \in \cap C_n$.

We shall show this by proving p is a cluster point of C_n for each n. Since C_n is closed for each n, it will follow that $p \in C_n$ for each n.

p is a cluster point of P so, given a neighborhood N of p, N contains infinitely many points of P. But C_n contains all points of P except, possibly, $p_1, p_2, \ldots, p_{n-1}$. Thus, N contains infinitely many points of C_n and p is a cluster point of C_n.

□

Exercise 9. Settle the case where some C_n contains only finitely many points.

Definition 1. A sequence $\{p_k\}$ in n-space is said to be a *Cauchy sequence* if, for any given $\varepsilon > 0$ there is an n_o such that

$$|p_i - p_j| < \varepsilon \quad \text{whenever} \quad i, j > n_o.$$

Exercise 10. Show that every Cauchy sequence is a bounded sequence.

Theorem 5. Any Cauchy sequence of real numbers converges to a real number.

Proof.

Suppose our Cauchy sequence does not converge. Since we are dealing with a bounded sequence, $a = \liminf a_n$ and $b = \limsup a_n$ both exist and we have $a < b$. (Why?) Suppose we take ε to be a number such that $\varepsilon < b - a$. For this ε we can find n_o such that

$$|a_n - a_m| < \varepsilon \text{ for } n, m > n_o \quad \text{or} \quad -\varepsilon < a_n - a_m < \varepsilon$$

which yields

$$a_n < \varepsilon + a_m \text{ for } n, m > n_o.$$

Fixing m, we apply the result of Exercise 4.12 to get

$$\limsup a_n = b \leq \varepsilon + a_m$$

so

$$b - \varepsilon \leq a_m \text{ for } m > n_o.$$

Again we apply Exercise 4.12 and obtain

$$b - \varepsilon \leq \liminf a_m = a.$$

But this says

$$b - a \leq \varepsilon$$

contradicting our choice of ε. Thus $a = b$ and $\lim a_n$ exists. □

Exercise 11. Show that if a sequence of real numbers is convergent, it is Cauchy.

The theorem and the exercise together say that as far as the real numbers are concerned

$$\text{“}\{a_n\} \text{ converges } \Longleftrightarrow \{a_n\} \text{ is Cauchy”.}$$

so convergence and being Cauchy are synonymous in 3. A similar result holds for n-dimensional Euclidean space. In higher mathematics, examples are given of certain other topological spaces where Cauchy sequences may not converge within the space.

Definition 2. Given a set S, a collection $\mathcal{S}$ of sets is a covering of S if S is a subset of the union of the sets in $\mathcal{S}$.

A covering $\mathcal{S}$ is called open if each set in $\mathcal{S}$ is an open set. $\mathcal{S}$ is called finite if it consists of only a finite number of sets.

Examples:

1. The collection $\{B((n, m), 1): n, m = 0, 1, 2, \ldots\}$ is an open covering of $\{(x, y): x \geq 0, y \geq 0\}$.

2. The collection $[i, i + 1] \times [j, j + 1], \; i = 0, 1; \; j = 0, 1$ is a finite covering of $[0, 2] \times [0, 2]$.

3. The collection of intervals $\left(r - \frac{r}{2}, r + \frac{r}{2}\right)$, $r \in (0, 1)$ is a covering of (0, 1) with an uncountable number of sets in the collection.

Exercise 12.

(a) Show that the unit open square $(0, 1) \times (0, 1)$ can be expressed as the union of a collection of closed disks.

(b) Show that the unit open disk
$$x^2 + y^2 < 1$$
can be expressed as the union of a collection of closed squares.

Exercise 13. Show that the balls

$$B_n = B((n, 0), n)$$

form a covering of the half-plane $x > 0$ in 2-space. (In fact, their union is the half-plane.)

Definition 3. A set C is compact if, whenever $\mathcal{S}$ is an open covering of C, there is a finite subcollection of $\mathcal{S}$ which covers C.

Theorem 6. If C is compact, then it is a closed and bounded set.

Proof. Exercise 14. Consider the following hints for the theorem in 2-space:

Boundedness: $\{B((0, 0), n): n = 1, 2, \ldots\}$ is an open covering of C.

Closedness: Suppose $p \in \partial C$ but $p \notin C$. Take a collection of concentric closed balls centered at p. Show the complement of these balls form an open covering of C.

Exercise 15. If S is a bounded set in 2-space then, given $\varepsilon > 0$, we can find a finite set of points

$$p_1, p_2, \ldots, p_n$$

in S such that each point of S lies in at least one of the disks

$$B(p_i, \varepsilon)\ , \quad i = 1, 2, \ldots, n.$$

Lemma 7. If S is an open covering of A in n-space, then there is a countable subcovering of S which also covers A.

Proof. (For 2-space)

We shall denote by R the collection of all open disks

B(u, v, r) where u, v, r are rational

Then R is made up of a countable number of sets and R forms a covering of 2-space. We can assume the sets in R are listed as

$$B_1, B_2, B_3, \ldots$$

If p is a point of A which lies in set $O \in \mathsf{S}$, (note that there may be more than one such O), then we can find B_k such that

$$p \in B_k \subseteq O.$$

(This is **Exercise** (16). Hint: Let p = (x, y) and take various cases for x, y.)

In fact there are infinitely many such B_k. (Why?) Let us choose the one with the smallest index k and denote its index by k(p). Note that $k(p_1)$ can be equal to $k(p_2)$ even for $p_1 \neq p_2$.

Now, choose one O to correspond to $B_{k(p)}$ and call it k(p). Thus we have the correspondence

$$p \leftrightarrow O_{k(p)}\ .$$

The collection of such $O_{k(p)}$ is a countable subcovering of S which covers A. □

Theorem 8. (*Heine-Borel*). A closed and bounded set in n-space is compact.

Proof.

Because of Lemma 7, we need only show: Given a countable open covering

$$O_1, O_2, O_3, \ldots$$

of C, we can find a finite subset of the O_i's covering C. (This is called a "finite subcovering" of $\{O_1, O_2, \ldots\}$.)

Define new open sets from the O_i's as follows: Let

$$\begin{aligned} V_1 &= O_1 \\ V_2 &= O_1 \cup O_2 \\ &\vdots \\ V_{n+1} &= V_n \cup O_{i+1} \end{aligned}$$

The V_i form an open covering of C also. Moreover,

$$V_1 \subset V_2 \subset V_3 \subset \cdots$$

Now define

$$\begin{aligned} C_1 &= C \setminus V_1 \\ C_2 &= C \setminus V_2 \\ &\vdots \\ C_k &= C \setminus V_k \\ &\vdots \end{aligned}$$

The C_k are bounded closed sets satisfying

$$C \supset C_1 \supset C_2 \supset \cdots$$

There are two possibilities: Either all the C_k are nonempty or $C_\ell = \varnothing$ for some ℓ.

If $C_k \neq \varnothing$ for each k, the generalized version of the *NIP* applies so there is q such that

$$q \in \bigcap_{k=1}^{\infty} C_k.$$

So $q \in C_k = C \setminus V_k$ meaning $q \in C$ but $q \notin V_k$ for each k, contradicting the fact that $\{V_k\}$ is a covering for C.

Thus $C_\ell = \varnothing$ which implies $C_{\ell+1}, C_{\ell+2}, \ldots$ are all empty. Since $C_\ell = C \setminus V_\ell$ then V_ℓ must contain C. Hence

$$O_1, O_2, \ldots O_\ell$$

cover C. □

SEQUENCES

References: Manual/Handbook by Dr. N. Quimpo
Adv Calculus by R.C. Buck

Definition 1. An (infinite) sequence is a function whose domain is **N** or a set equivalent to **N** (that is, in one-to-one correspondence with **N**).

It is common practice to denote a sequence f by listing the image points of f (called the terms of the sequence). For example, if

$$\begin{aligned} f(1) &= p_1 \\ f(2) &= p_2 \\ f(3) &= p_3 \\ &\vdots \\ f(n) &= p_n \\ &\vdots \end{aligned}$$

then we denote the sequence by

$$p_1, p_2, p_3, \ldots, p_n, \ldots$$

or

$$\{p_n\}.$$

The set $\{p_1, p_2, p_3, \ldots\}$ is called the trace of the sequence.

Examples:

1. If $p_n = \left((-1)^n, \frac{1}{n}\right)$, the trace of $\{p_n\}$ is the set $\left\{(-1,1), (1,\frac{1}{2}), (-1,\frac{1}{3}), \ldots\right\}$ in 2-space.
2. If $a_n = 1 + (-1)^n$, the trace of $\{a_n\}$ is the set $\{0, 1\}$ in $\Re$. Note in this example that graphing the trace of the sequence is not very helpful (unlike in Example 1) because we cannot "draw" the action of "shuttling back - and - forth" between 0 and 1. This example points out that there are important differences between a set and the terms of a sequence. (What are they?)

Our fundamental concern regarding sequences is whether they converge or diverge.

Definition 2. A sequence $\{p_n\}$ converges to a point p (or has the limit p) if, given any neighborhood N of p, there exists n_o such that

$$p_n \in N \quad \text{for all} \quad n \geq n_o.$$

In other words, all terms from index n_o onwards lie in N.

If $\{p_n\}$ converges to p, we write

$$\lim_{n\to\infty} p_n = p, \text{ or } p_n \to p,$$

and say $\{p_n\}$ is a <u>convergent</u> sequence. If $\{p_n\}$ does not converge we say it <u>diverges</u> or is a <u>divergent</u> sequence.

<u>Examples</u>:

3. $p_n = \left(\frac{1}{n}, \frac{n}{n+1}\right) \to (0,1)$.

For, given any ball of radius $\varepsilon > 0$, we can make

$$\left|\left(\frac{1}{n}, \frac{n}{n+1}\right) - (0,1)\right| = \left[\frac{1}{n^2} + \left(\frac{n}{n+1} - 1\right)^2\right]^{1/2} < \varepsilon$$

by taking

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{2}{n^2} < \varepsilon^2$$

or

$$n > \frac{\sqrt{2}}{\varepsilon}$$

4. The following series of real numbers diverge:

a. $a_n = n$ $\quad$ 1, 2, 3, …
b. $b_n = (-1)^n n$ $\quad$ -1, 2, -3, 4, -5, …
c. $c_n = 1 + (-1)^n$ $\quad$ 0, 2, 0, 2, 0, 2, …
d. $d_n = (-1)^n (1 + \frac{1}{n})$

To see that $\{d_n\}$ diverges observe that alternative terms go this way - -

even terms: $1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, \ldots$

odd terms: $-(1 + 1), -(1 + \frac{1}{3}), -(1 + \frac{1}{5}), \ldots$

<u>Exercise</u>.

(1) Prove that if a sequence has a limit it has only one limit.

In general, a series diverges if its terms “blow up” (i.e., go to infinity), cluster about/at two or more points, or simply “go nowhere”.

<u>Remark 1.</u> Convergence is a behavior not affected by a finite number of terms. For example, if $\{p_n\}$ and $\{q_n\}$are two sequences which differ only in their first 2, 000, 000 terms (all subsequent terms being equal) then $\{p_n\}$ and $\{q_n\}$ both converge or both diverge.

For completeness, we state the following familiar result about limits of sequenceness:

Theorem 1. If $\lim a_n = a$ and $\lim b_n = b$, then

a. $\lim (a_n + b_n) = a + b$
b. $\lim a_n b_n = ab$
c. $\lim \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$.

Proof.

See the proofs in any standard reference (e.g., R.C. Buck, *Advanced Calculus*, pp. 43 - 44).

We can use sequences to throw more light on ideas previously described. For instance:

Theorem 2. The closure of a set S in n – space is the set of all limits of converging sequences of points from S.

Proof.

We shall show each set described is a subset of the other.

(1) If $\{p_n\}$ is a sequence of points in S and $\lim p_n = p$, then $p \in \bar{S}$. For it is an easy exercise to show that p cannot be an exterior point of S.
(2) If $q \in \bar{S}$, then we can find a sequence $\{q_n\}$ of points of S which converges to q as follows:

If $q \in S$, then we can simply take $q_n = q$ for each n.

If $q \notin S$, then q must be a cluster point for S. (Why?) Now take a suitable decreasing sequence of neighborhood of q, for example,

$$B_n = B(q, \frac{1}{n}), \quad n = 1, 2, \ldots$$

There is always a point of S in each such neighborhood so we can choose a point of S in B_n and call it q_n. Clearly,

$$q_n \to q. \qquad \blacklozenge$$

Corollary 3. A set S is closed if and only if it contains the limit of every converging sequence $\{p_n\}$ of points lying in S.

Proof.

($\Rightarrow$) Since S is closed, $\bar{S} = S$. Now cite the theorem.

($\Leftarrow$) If p is a boundary point of S, then we can construct a sequence of points of S converging to p. By the theorem, p lies in S. $\blacklozenge$

Two notions related to the concept of "limit of a sequence", are used a lot in higher analysis. These are the "limit superior" and "limit inferior" of a sequence. To discuss them requires introducing the analog for sequences of the concept of "cluster point of a set".

Definition 3. A point p is a limit point for the sequence $\{p_n\}$ if, for any neighborhood **N** of p, there are infinitely many subscripts n for which $p_n \in \mathbf{N}$.

Example:

5. The sequence $\{p_n\}$ where

$$p_n = \left((-1)^n \frac{1}{n}, 1 + (-1)^n \right)$$

has the limit points (0, 0) and (0, 2).

Remarks.

2. We shift the emphasis to subscripts instead of the points themselves because a sequence differs from a set in this aspect, namely, terms of a sequence may involve repetition of points. (See Example 2.)

3. Mark the distinction between "limit" and "limit point":

limit of a sequence	*limit point of a sequence*
"for all subscripts $n > n_o$, $p_n \in \mathbf{N}$"	"for infinitely many subscripts n, $p_n \in \mathbf{N}$"

To understand the two concepts better, it will be necessary to work out the following exercises:

Exercises.

(2) Show that the real sequence

$$a_n = \{1 + (-1)^n\}$$

has two limit points but no limit.

(3) Show that the limit of a sequence is a limit point of the sequence. (Clearly, the converse is not always true.)

(4) Give an example of a sequence with one limit point but which diverges.

(5) Construct a sequence with more than two limit points.

(6) Can a sequence have infinitely many limit points? no limit point?

(7) Given a sequence

$$p_1, p_2, \ldots, p_i, \ldots$$

the sequence

$$q_1, q_2, \ldots, q_j, \ldots$$

is called a subsequence of $\{p_i\}$ if

i.) To each j, there corresponds an i such that $q_j = p_i$.

ii.) Whenever $j_1 < j_2$ then $i_1 < i_2$.

More informally, a subsequence is obtained from a sequence $\{p_i\}$when we form a new sequence by selecting only some of the terms of $\{p_i\}$.

We denote a subsequence of $\{p_i\}$ by

$$p_{i_1}, p_{i_2}, p_{i_3}, \ \ldots$$

or $\{ p_{i_k} \}$. Note here that it is k which runs through 1, 2, 3, …

Now for the exercise: Prove that if p is a limit point of $\{p_n\}$, there is a subsequence $\{ p_{n_k} \}$ of $\{p_n\}$ which converges to p.

Definition 4. The largest limit point of a bounded real sequence $\{a_n\}$ is called the limit superior of the sequence, and is denoted by

$$\lim_{n\to\infty} \sup a_n.$$

The smallest limit point of a bounded real sequence $\{a_n\}$ is called the limit inferior of the sequence, and is denoted by

$$\lim_{n\to\infty} \inf a_n.$$

Note 4. If the sequence $\{a_n\}$ is not bounded, we extend the above definition as follows: If $\{a_n\}$ is not bounded above, we define

$$\lim_{n\to\infty} \sup a_n = +\infty$$

If $\{a_n\}$ is bounded above but not below and if $\{a_n\}$ has no finite limit superior, then we define

$$\lim_{n\to\infty} \sup a_n = -\infty$$

The limit inferior of $\{a_n\}$ can now be defined in a general way as follows:

$$\lim_{n\to\infty} \inf a_n = -\lim_{n\to\infty} \sup b_n \text{, where } b_n = - a_n \text{ for } n = 1, 2, \ldots$$

Lemma 4. Let $\{a_n\}$ be a bounded real sequence with

$$\lim_{n\to\infty} \sup a_n = a.$$

Then for any $\varepsilon > 0$,

(i) $a_n \le a + \varepsilon$ for all but a finite number of n, and
(ii) $a_n \ge a - \varepsilon$ for infinitely many n.

Proof.

(ii) is easier to show so its proof shall be given first. For the proof of (i), we shall need a result which comes later (Corollary 5.4).

(ii): If $a_n \geq a - \varepsilon$ for finitely many n only, this contradicts the fact that a is a limit point: Every neighborhood of a must contain a_n for infinitely many n and $[a - \varepsilon, +\infty)$ is a neighborhood of a.

(i): Suppose there exist infinitely many n for which

$$a_n > a + \varepsilon ,$$

say, $n_1, n_2, n_3, \ldots n_k, \ldots$ We shall show that, contrary to hypothesis, $\{a_n\}$ has a cluster point larger than a. If infinitely many of these a_n are equal, say, to a number α, then α is a limit point of $\{a_n\}$. Since $\alpha > a$, then we have a contradiction.

If only finitely many of these numbers are equal, then the trace of $\{a_{n_k}\}$ is an infinite set and

$$a + \varepsilon < a_{n_k} < b$$

where b is an upper bound for the bounded sequence $\{a_n\}$. By Theorem 3 of the next chapter, the trace of $\{a_{n_k}\}$ has a cluster point β (**Exercise** (8): Show $\beta > a$.) Accordingly, $\{a_{n_k}\}$ has a limit point β with $\beta > a$. Again we have a contradiction.

Remark 5. The converse of the lemma is also true. (**Exercise** (9).) Hence the condition given in the lemma can be taken as the definition of lim sup a_n.

Exercise 10. State and prove a result for lim inf a_n analogous to the Lemma.

Exercises.

(11) Find lim sup a_n and lim inf a_n if

(a) $a_n = (-1)^n(1 + \frac{1}{n})$

(b) $a_n = n + 1 + (-1)^n(n + \frac{1}{n})$

(c) $a_n = \sin(n\frac{\pi}{3})$

(d) $a_n = n^2\sin^2(n\frac{\pi}{2})$

(12) If $a \leq a_n \leq b$ for all but a finite number of n, show that

$$\liminf_{n\to\infty} a_n \geq a$$

$$\limsup_{n\to\infty} a_n \leq b.$$

(13) (a) Prove in general that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(b) Construct an example of bounded sequences $\{a_n\}$ and $\{b_n\}$ where

$$\limsup_{n\to\infty} (a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(14) Prove that a bounded sequence $\{a_n\}$ converges if and only if $\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \inf a_n$ are both finite and equal. (What is $\lim_{n\to\infty} a_n$?)

(15) Prove that for any sequence $\{a_n\}$ (bounded or unbounded)

$$\lim \inf a_n \le \lim \sup a_n.$$

(Here you can treat $-\infty$ and $+\infty$ as "numbers" satisfying $-\infty < c < +\infty$ for any real number c.)