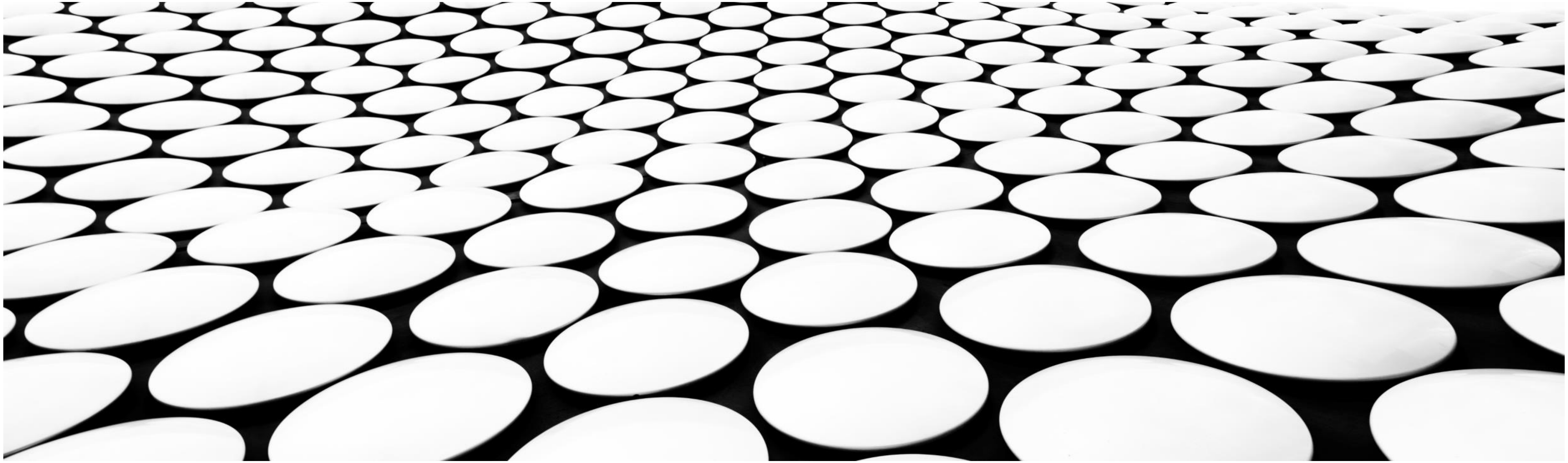
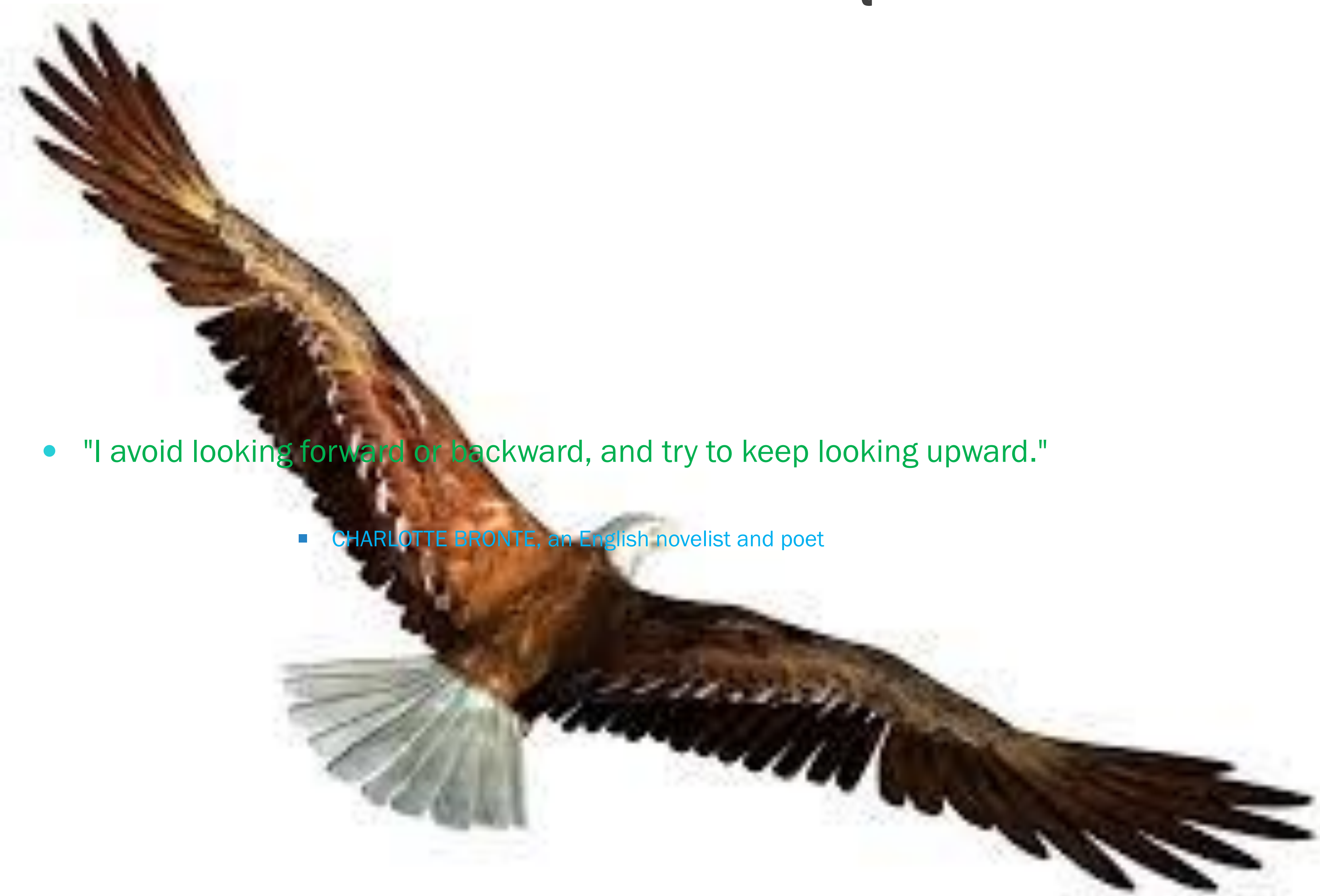
PROBABILITY



QUOTE OF THE DAY..



- "I avoid looking forward or backward, and try to keep looking upward."
    - CHARLOTTE BRONTE, an English novelist and poet

Probability and Statistics

Probability is the chance of an outcome in an experiment (also called event).

Event: Tossing a fair coin
Outcome: Head, Tail

Probability deals with **predicting** the likelihood of **future** events.

Statistics involves the **analysis** of the **frequency** of **past** events

Example: Consider there is a drawer containing 100 socks: 30 red, 20 blue and 50 black socks.

We can use probability to answer questions about the selection of a random sample of these socks.

- **PQ1.** What is the probability that we draw two blue socks or two red socks from the drawer?
- **PQ2.** What is the probability that we pull out three socks or have matching pair?
- **PQ3.** What is the probability that we draw five socks and they are all black?

STATISTICS

Instead, if we have no knowledge about the type of socks in the drawers, then we enter into the realm of statistics. Statistics helps us to infer properties about the population on the basis of the random sample.

Questions that would be statistical in nature are:

- **SQ1**: A random sample of 10 socks from the drawer produced one blue, four red, five black socks. What is the total population of black, blue or red socks in the drawer?
- **SQ2**: We randomly sample 10 socks, and write down the number of black socks and then return the socks to the drawer. The process is done for five times. The mean number of socks for each of these trial is 7. What is the true number of black socks in the drawer?
- etc.

PROBABILITY VS. STATISTICS

In other words:

- In probability, we are given a model and asked what kind of data we are likely to see.

- In statistics, we are given data and asked what kind of model is likely to have generated it.

Example: Measles Study

- A study on health is concerned with the incidence of childhood measles in parents of childbearing age in a city. For each couple, we would like to know how likely, it is that either the mother or father or both have had childhood measles.

- The current census data indicates that 20% adults between the ages 17 and 35 (regardless of sex) have had childhood measles.
  - This give us the probability that an individual in the city has had childhood measles.

PROBABILITY

- **Probability** can be defined as the chance of an event occurring. It can be used to quantify what the "odds" are that a specific event will occur.
  - E.g., "75% chance of snow tomorrow"

What is probability?

- What do we mean by
  - Prob (• on a die) = 1/6
  - Prob (living creature on Mars) = 1/2

SAMPLE SPACES AND PROBABILITY

- An **outcome** is the result of a single trial of a probability (random) experiment.
- A **sample space** is the set of all possible outcomes of a probability experiment.
- An **event** consists of outcomes.
- Probability(Event), P(E)

Experiment	Sample Space
Toss a coin	Head, Tail
Roll a die	1, 2, 3, 4, 5, 6
Answer a true/false question	True, False
Toss two coins	HH, HT, TH, TT

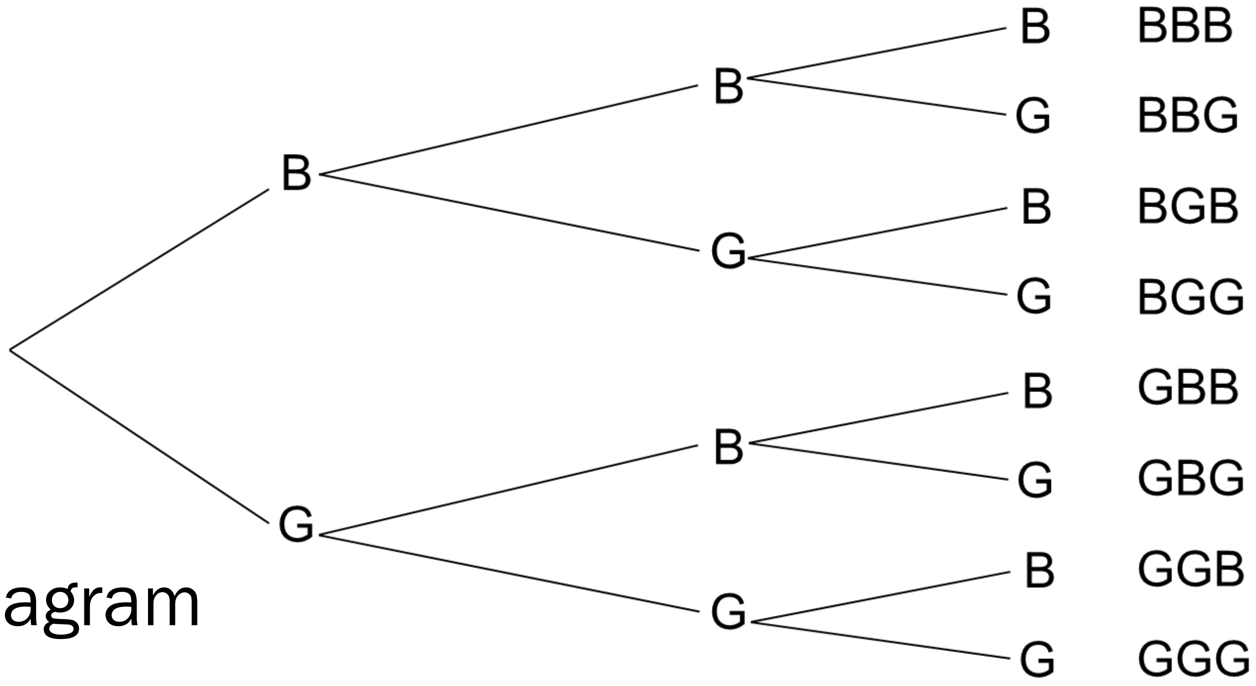
EXAMPLE: ROLLING DICE

Find the sample space for rolling two dice.

Die 1	Die 2					
	1	2	3	4	5	6
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
5	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
6	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

EXAMPLE: GENDER OF CHILDREN

Find the sample space for the gender of the children if a family has three children. Use B for boy and G for girl.



Tree diagram

SAMPLE SPACES AND PROBABILITY

There are three basic interpretations of probability:

- Classical probability
- Empirical probability
- Subjective probability

Frequentist approach

Bayesian approach

SAMPLE SPACES AND PROBABILITY

Classical probability uses sample spaces to determine the probability that an event will happen and assumes that all outcomes in the sample space are equally likely to occur.

$$\boxed{P(E)=\frac{n(E)}{n(S)}=\frac{\text{\# of desired outcomes}}{\text{Total \# of possible outcomes}}}$$

EXAMPLE: GENDER OF CHILDREN

If a family has three children, find the probability that two of the three children are girls.

Sample Space:

BBB BBG BGB BGG GBB GBG GGB GGG

Three outcomes (BGG, GBG, GGB) have two girls.

The probability of having two of three children being girls is 3/8, or 0.375.

Assumption: equally likely

Rounding Rule for Probabilities

Probabilities should be expressed as reduced fractions or rounded to two or three decimal places. When the probability of an event is an extremely small decimal, it is permissible to round the decimal to the first nonzero digit after the decimal point.

E.g., 0.25, 0.127, 0.0004

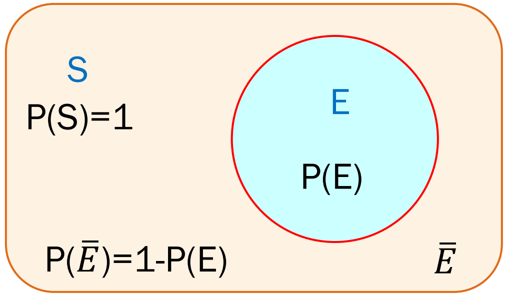
Probability Rules

- $0 \leq P(E) \leq 1$.
- If an event E cannot occur (i.e., the event contains no members in the sample space), its probability is 0.
- If an event E is certain, then the probability of E is 1.
- The sum of the probabilities of all the outcomes in the sample space is 1.

SAMPLE SPACES AND PROBABILITY

The ***complement of an event E***, denoted by $\bar{E}$, is the set of outcomes in the sample space that are not included in the outcomes of event E.

$$P(E) = 1 - P(\bar{E})$$



Find the complement of each event.

Event	Complement of the Event
Rolling a die and getting *a 4*	Getting a 1, 2, 3, 5, or 6
Selecting a letter of the alphabet and getting a vowel	Getting a consonant (assume y is a consonant)
Selecting a month and getting a month that begins with a J	Getting February, March, April, May, August, September, October, November, or December
Selecting a day of the week and getting a weekday	Getting Saturday or Sunday

EXAMPLE: RESIDENCE OF PEOPLE

If the probability that a person lives in an industrialized country of the world is 1/5 , find the probability that a person does not live in an industrialized country.

$$P(\text{Not living in industrialized country})$$

$$= 1 - P(\text{living in industrialized country})$$

$$= 1 - \frac{1}{5} = \boxed{\frac{4}{5}}$$

Empirical probability relies on actual experience to determine the likelihood of outcomes.

$$\boxed{P(E) = \frac{f}{n} = \frac{\text{frequency of desired class}}{\text{Sum of all frequencies}}}$$

* While the classical probability works with theoretical number of possible outcomes, the empirical probability works with a sample.

EXAMPLE : BLOOD TYPES

In a sample of 50 people, 21 had type O blood, 22 had type A blood, 5 had type B blood, and 2 had type AB blood. Set up a frequency distribution and find the following probabilities.

a. A person has type O blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total 50	

$$P(\mathrm{O}) = \frac{f}{n} = \frac{21}{50}$$

b. A person has type A or type B blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total 50	

$$P(\text{A or B}) = \frac{22}{50} + \frac{5}{50}$$
$$= \frac{27}{50}$$

c. A person has neither type A nor type O blood.

Type	Frequency
A	$\cancel{22}$
B	5
AB	2
O	$\cancel{21}$
Total 50	

$$P(\text{neither A nor O})$$
$$= \frac{5}{50} + \frac{2}{50}$$
$$= \frac{7}{50}$$

d. A person does not have type AB blood.

Type	Frequency
A	22
B	5
AB	2
O	21
Total	50

$$P(\text{not AB})$$
$$= 1 - P(\text{AB})$$
$$= 1 - \frac{2}{50} = \frac{48}{50} = \frac{24}{25}$$

SAMPLE SPACES AND PROBABILITY

Subjective probability uses a probability value based on an educated guess or estimate, employing opinions and inexact information.

Examples: weather forecasting, predicting outcomes of sporting events

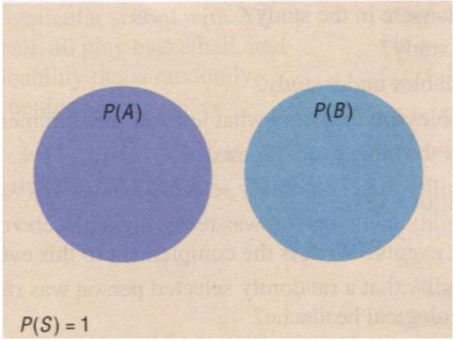
ADDITION RULES FOR PROBABILITY

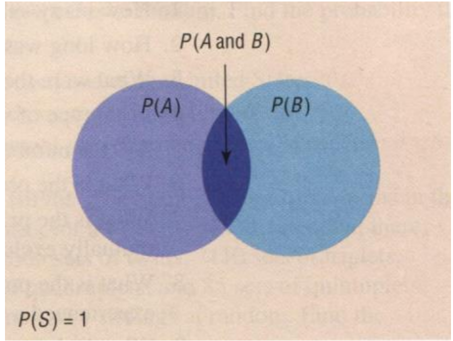
- Two events are **mutually exclusive events** if they cannot occur at the same time (i.e., they have no outcomes in common)

Addition Rules

$$P(A \text{ or } B) = P(A) + P(B) \quad \text{Mutually Exclusive}$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \quad \text{Not M. E.}$$





Determine which events are mutually exclusive and which are not, when a single die is rolled.

a. Getting an odd number and getting an even number

Getting an odd number: 1, 3, or 5

Getting an even number: 2, 4, or 6

Mutually Exclusive

b. Getting a 3 and getting an odd number

Getting a 3: 3

Getting an odd number: 1, 3, or 5

Not M. E.

EXAMPLE: R&D EMPLOYEES

The corporate research and development centers for three local companies have the following number of employees:

U.S. Steel 110

Alcoa 750

Bayer Material Science 250

If a research employee is selected at random, find the probability that the employee is employed by U.S. Steel or Alcoa.

$$P(\text{U.S. Steel or Alcoa}) = P(\text{U.S. Steel}) + P(\text{Alcoa})$$

$$= \frac{110}{1110} + \frac{750}{1110} = \frac{860}{1110} = \frac{86}{111}$$

EXAMPLE: MEDICAL STAFF

In a hospital unit there are 8 nurses and 5 physicians; 7 nurses and 3 physicians are females.

If a staff person is selected, find the probability that the subject is a nurse or a male.

Staff	Females	Males	Total
Nurses	7	1	8
Physicians	3	2	5
Total	10	3	13

$$P(\text{Nurse or Male}) = P(\text{Nurse}) + P(\text{Male}) - P(\text{Male Nurse})$$

$$= \frac{8}{13} + \frac{3}{13} - \frac{1}{13} = \boxed{\frac{10}{13}}$$

MULTIPLICATION RULES

- Two events *A* and *B* are **independent events** if the fact that *A* occurs does not affect the probability of *B* occurring.

Multiplication Rules

$$P(A \text{ and } B) = P(A) \cdot P(B) \quad \text{Independent}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A) \quad \text{Dependent}$$

EXAMPLE: TOSSING A COIN

A coin is flipped and a die is rolled. Find the probability of getting a head on the coin and a 4 on the die.

Independent Events

$$P(\text{Head and } 4) = P(\text{Head}) \cdot P(4)$$
$$= \frac{1}{2} \cdot \frac{1}{6} = \boxed{\frac{1}{12}}$$

This problem could be solved using sample space.

H1, H2, H3,H4, H5, H6, T1, T2, T3, T4, T5, T6

EXAMPLE: SURVEY ON STRESS

A Harris poll found that 46% of Americans say they suffer great stress at least once a week. If three people are selected at random, find the probability that all three will say that they suffer great stress at least once a week.

Independent Events

$$P(\text{S and S and S}) = P(\text{S}) \cdot P(\text{S}) \cdot P(\text{S})$$

$$= (0.46)(0.46)(0.46)$$

$$= \boxed{0.097}$$

EXAMPLE: UNIVERSITY CRIME

At a university in western Pennsylvania, there were 5 burglaries reported in 2003, 16 in 2004, and 32 in 2005. If a researcher wishes to select at random two burglaries to further investigate, find the probability that both will have occurred in 2004.

Dependent Events

$$P(\mathrm{C}_1 \text{ and } \mathrm{C}_2) = P(\mathrm{C}_1) \cdot P(\mathrm{C}_2 | \mathrm{C}_1)$$

$$= \frac{16}{53} \cdot \frac{15}{52} = \frac{60}{689}$$

$P(C_1 \ \ and \ C_2)$ = P(*getting both from 2004*)

$$= \frac{\#\ desired\ cases}{Total\ \#\ possible\ cases} = \frac{{}_{16}C_2}{{}_{53}C_2} = \frac{60}{689}$$

Year	No.
2003	5
2004	16
2005	32
Total	**53**

CONDITIONAL PROBABILITY

- **Conditional probability** is the probability that the second event *B* occurs given that the first event *A* has occurred.

Conditional Probability

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

$$P(A \text{ and } B) = P(A) \cdot P(B|A)$$

EXAMPLE: PARKING TICKETS

The probability that Sam parks in a no-parking zone *and gets a parking ticket is 0.06,* and the probability that Sam cannot find a legal parking space and has to park in the no-parking zone is 0.20. On Tuesday, Sam arrives at school and has to park in a no-parking zone. Find the probability that he will get a parking ticket.

Conditional Probability

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

N = parking in a no-parking zone

T = getting a ticket

$$P(T \mid N) = \frac{P(N \text{ and } T)}{P(N)} = \frac{0.06}{0.20} = \boxed{0.30}$$

EXAMPLE: WOMEN IN THE MILITARY

A recent survey asked 100 people if they thought women in the armed forces should be permitted to participate in combat. The results of the survey are shown.

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

a. Find the probability that the respondent answered yes (*Y*), given that the respondent was a female (*F*).

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

$$P(Y|F) = \frac{P(F \text{ and } Y)}{P(F)} = \frac{\frac{8}{100}}{\frac{50}{100}} = \frac{8}{50} = \boxed{\frac{4}{25}}$$

b. Find the probability that the respondent was a male (*M*), given that the respondent answered no (*N*).

Gender	Yes	No	Total
Male	32	18	50
Female	8	42	50
Total	40	60	100

$$P(M|N)=\frac{P(N \text{ and } M)}{P(N)}=\frac{\frac{18}{100}}{\frac{60}{100}}=\frac{18}{60}=\boxed{\frac{3}{10}}$$

EXAMPLE: BOW TIES

The Neckware Association of America reported that 3% of ties sold in the United States are bow ties (*B*). If 4 customers who purchased a tie are randomly selected, find the probability that at least 1 purchased a bow tie.

$$P(B) = 0.03,\ P(\bar{B}) = 1 - 0.03 = 0.97$$

$$P(\text{no bow ties}) = P(\bar{B}) \cdot P(\bar{B}) \cdot P(\bar{B}) \cdot P(\bar{B})$$

$$= (0.97)(0.97)(0.97)(0.97) = 0.885$$

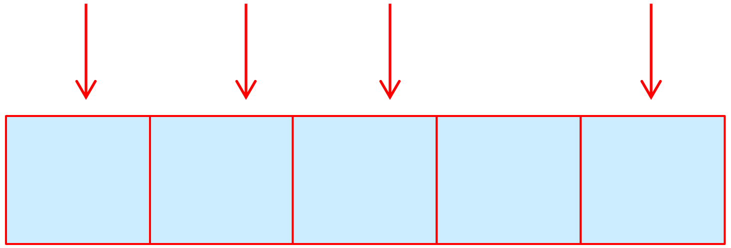
$$P(\text{at least 1 bow tie}) = 1 - P(\text{no bow ties})$$

$$= 1 - 0.885 = \boxed{0.115}$$

COUNTING RULES

- In a sequence of n events in which the first one has k_1 possibilities and the second event has k_2 and the third has k_3, and so forth, the total number of possibilities of the sequence will be

$$k_1 \times k_2 \times k_3 \times \cdot \times k_n$$



EXAMPLE: PAINT COLORS

A paint manufacturer wishes to manufacture several different paints. The categories include

Color: red, blue, white, black, green, brown, yellow

Type: latex, oil

Texture: flat, semigloss, high gloss

Use: outdoor, indoor

How many different kinds of paint can be made if you can select one color, one type, one texture, and one use?

$$\left(\begin{array}{c}\text{\# of}\\ \text{colors}\end{array}\right)\left(\begin{array}{c}\text{\# of}\\ \text{types}\end{array}\right)\left(\begin{array}{c}\text{\# of}\\ \text{textures}\end{array}\right)\left(\begin{array}{c}\text{\# of}\\ \text{uses}\end{array}\right)$$

$$7 \cdot 2 \cdot 3 \cdot 2$$

$$\boxed{\text{84 different kinds of paint}}$$

COUNTING RULES

- **Factorial** is the product of all the positive numbers from 1 to a number.

$$n! = n(n-1)(n-2)\cdots 3\cdot 2\cdot 1$$

$$0! = 1$$

- We have five students. How many line ups are possible?
  - 5x4x3x2x1=5!
- We have 45 different cars. How many line ups are possible?
  - 45!

COUNTING RULES

- **Permutation** is an arrangement of objects in a specific order. Order matters.

$$ {}_nP_r = \frac{n!}{(n-r)!} \qquad = \underbrace{n(n-1)(n-2)\cdots(n-r+1)}_{r \text{ items}} $$

- We have 10 students. We select 4 and line them up. How many line ups are possible?
  - ${}_{10}P_4 = \frac{10!}{(10-4)!} = \frac{10!}{6!}$, or 10x9x8x7
- We have 45 marbles. We select 9 and line them up. How many possible line ups?
  - ${}_{45}P_9$

COUNTING RULES

- **Combination** is a grouping of objects. Order does not matter.

$$ {}_{n}C_{r} = \frac{n!}{(n-r)!\,r!} $$

- We have 10 students. We select 4. How many selections are possible? (no ordering)
  - ${}_{10}C_{4} = \frac{10!}{(10-4)!4!}$
- We have 45 marbles. We select 9. How many possible line ups are possible?
  - ${}_{45}C_{9}$

EXAMPLE: BUSINESS LOCATION

Suppose a business owner has a choice of 5 locations in which to establish her business. She decides to rank each location according to certain criteria, such as price of the store and parking facilities. How many different ways can she rank the 5 locations?

$$\underset{5}{\begin{pmatrix}\text{first}\\\text{choice}\end{pmatrix}} \cdot \underset{4}{\begin{pmatrix}\text{second}\\\text{choice}\end{pmatrix}} \cdot \underset{3}{\begin{pmatrix}\text{third}\\\text{choice}\end{pmatrix}} \cdot \underset{2}{\begin{pmatrix}\text{fourth}\\\text{choice}\end{pmatrix}} \cdot \underset{1}{\begin{pmatrix}\text{fifth}\\\text{choice}\end{pmatrix}}$$

$$\boxed{\text{120 different ways to rank the locations}}$$

Using factorials, $5! = 120$.

Using permutations, ${}_5P_5 = 120$.

Suppose the business owner in Example 4–42 wishes to rank only the top 3 of the 5 locations. How many different ways can she rank them?

$$\begin{pmatrix}\text{first}\\ \text{choice}\end{pmatrix}\begin{pmatrix}\text{second}\\ \text{choice}\end{pmatrix}\begin{pmatrix}\text{third}\\ \text{choice}\end{pmatrix}$$

$$5 \quad \cdot \quad 4 \quad \cdot \quad 3$$

$$\boxed{\text{60 different ways to rank the locations}}$$

Using permutations, ${}_5P_3 = 60$.

EXAMPLE: TELEVISION ADS

The advertising director for a television show has 7 ads to use on the program.

If she selects 1 of them for the opening of the show, 1 for the middle of the show, and 1 for the ending of the show, how many possible ways can this be accomplished?

Since order is important, the solution is

Or, ${}_7P_3 = 7 * 6 * 5 = 210$

Hence, there would be 210 ways to show 3 ads.

$$ {}_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = 210 $$

EXAMPLE: BOOK REVIEWS

A newspaper editor has received 8 books to review. He decides that he can use 3 reviews in his newspaper. How many different ways can these 3 reviews be selected?

The placement in the newspaper is not mentioned, so order does not matter. We will use combinations

$$ {}_{8}C_{3} = \frac{8!}{5!3!} = 8!/\left(5!3!\right) = 56 $$

EXAMPLE: COMMITTEE SELECTION

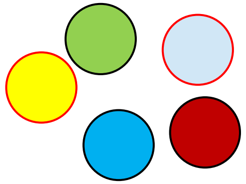
In a club there are 7 women and 5 men. A committee of 3 women and 2 men is to be chosen. How many different possibilities are there?

There are not separate roles listed for each committee member, so order does not matter. We will use combinations.

There are $35 \cdot 10 = 350$ different possibilities.

$$\text{Women: } {}_7C_3 = \frac{7!}{4!3!} = 35, \quad \text{Men: } {}_5C_2 = \frac{5!}{3!2!} = 10$$

COUNTING EXERCISE



1. (Factorial) We have five marbles with all different colors. We line them up. How many different results can we have?
5	4	3	2	1

     5x4x3x2x1 = 5!

2. (Permutation) We have five marbles with all different colors. We select three and line them up. How many different results can we have?
5	4	3

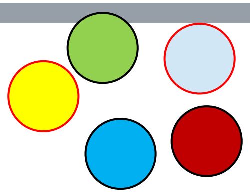
     5x4x3 = ${}_{5}P_{3}$

3. We have numbers 1 through 7. How many different even numbers of five digits can we make?

?	?	?	?	3

6x5x4x3x 3

COUNTING EXERCISE



4. (Combination) We have five marbles with all different colors. We select two out of the five. How many different results can we have?

   - E.g., {A,B}, {C,E},…

     Out of 5, we select 2, so we have ${}_5C_2$ possible cases.

   - ${}_nC_r=\binom{n}{r} = \frac{n!}{r!(n-r)!}$. ${}_5C_2=\binom{5}{2} = \frac{5!}{3!2!}$.

5. Gamma Function

   - $\Gamma(n)=(n\text{-}1)\,\Gamma(n\text{-}1)$, $\Gamma(1)=1$.
   - $\Gamma(4)=$?? $\Gamma(4)=3\text{x}\Gamma(3) = 3\text{x}2\text{x}\Gamma(2) = 3\text{x}2\text{x}1 = 3! = 6$
   - $\Gamma(n)=(n\text{-}1)!$
   - $\Gamma(3.5)=$?? $\Gamma(3.5)=2.5! = (2.5)(1.5)(0.5)(-0.5)(-1.5)\ldots$??
   - $\Gamma(0.5)=$?? $\Gamma(0.5)=\sqrt{\pi}$

COUNTING

6. Counting Exercise

- We line up eight dogs. How many possible cases do we have? : 8!
- We have 13 children. We line 7 of them up. How many possible cases do we have? : $_{13}P_7 = \frac{13!}{(13-7)!}$
- We have 50 students, and select 5 representatives. How many possible cases do we have?: $_{50}C_5 = \frac{50!}{45! \times 5!}$
- We line up 2 red balls and 3 blue balls. How many possible cases do we have? $5!/(3! \times 2!)$

PROBABILITY AND COUNTING RULES

The counting rules can be combined with the probability rules to solve probability problems.

For example, you can compute the probability of outcomes of many experiments, such as getting a full house when 5 cards are dealt or selecting a committee of 3 women and 2 men from a club consisting of 10 women and 10 men.

$$\boxed{P(E) = \frac{n(E)}{n(S)} = \frac{\text{\# of desired outcomes}}{\text{Total \# of possible outcomes}}}$$

EXAMPLE

A store has 6 *TV Graphic magazines and 8 Newstime* magazines on the counter. If two customers purchased a magazine, find the probability that one of each magazine was purchased.

TV Graphic: One magazine of the 6 magazines

Newstime: One magazine of the 8 magazines

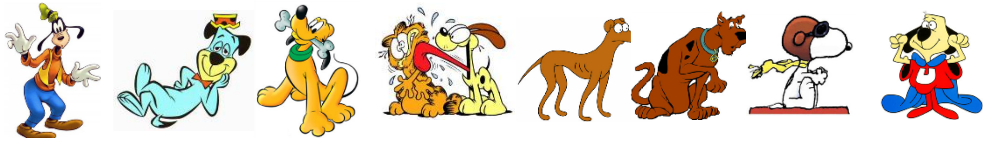
Total: Two magazines of the 14 magazines

$$\text{Probability} = \frac{{}_6C_1 \cdot {}_8C_1}{{}_{14}C_2} = \frac{6 \cdot 8}{91} = \boxed{\frac{48}{91}}$$

We select four dogs out of eight and line them up. What is the probability that Goofy is on the third position?

$$\text{Probability} = \frac{\#desired\ cases}{\#total\ possible\ cases}$$

- # total possible cases = ${}_8P_4$
- # desired cases = ${}_7P_3$
- Probability = ${}_7P_3 / {}_8P_4 = 1/8$



- We select five dogs. What is the probability that Pluto is selected?
  - # total possible cases = ${}_8C_5$
  - # desired cases = ${}_7C_4$
  - Probability = ${}_7C_4 / {}_8C_5$ = 5/8
  - Or, from symmetry, the chance is 5/8.