SEQUENCES

References: Manual/Handbook by Dr. N. Quimpo
Adv Calculus by R.C. Buck

Definition 1. An (infinite) sequence is a function whose domain is **N** or a set equivalent to **N** (that is, in one-to-one correspondence with **N**).

It is common practice to denote a sequence f by listing the image points of f (called the terms of the sequence). For example, if

$$\begin{aligned} f(1) &= p_1 \\ f(2) &= p_2 \\ f(3) &= p_3 \\ &\vdots \\ f(n) &= p_n \\ &\vdots \end{aligned}$$

then we denote the sequence by

$$p_1, p_2, p_3, \ldots, p_n, \ldots$$

or

$$\{p_n\}.$$

The set $\{p_1, p_2, p_3, \ldots\}$ is called the trace of the sequence.

Examples:

1. If $p_n = \left((-1)^n, \frac{1}{n}\right)$, the trace of $\{p_n\}$ is the set $\left\{(-1,1), (1,\frac{1}{2}), (-1,\frac{1}{3}), \ldots\right\}$ in 2-space.
2. If $a_n = 1 + (-1)^n$, the trace of $\{a_n\}$ is the set $\{0, 1\}$ in $\Re$. Note in this example that graphing the trace of the sequence is not very helpful (unlike in Example 1) because we cannot "draw" the action of "shuttling back - and - forth" between 0 and 1. This example points out that there are important differences between a set and the terms of a sequence. (What are they?)

Our fundamental concern regarding sequences is whether they converge or diverge.

Definition 2. A sequence $\{p_n\}$ converges to a point p (or has the limit p) if, given any neighborhood N of p, there exists n_o such that

$$p_n \in N \quad \text{for all} \quad n \geq n_o.$$

In other words, all terms from index n_o onwards lie in N.

If $\{p_n\}$ converges to p, we write

$$\lim_{n\to\infty} p_n = p, \text{ or } p_n \to p,$$

and say $\{p_n\}$ is a <u>convergent</u> sequence. If $\{p_n\}$ does not converge we say it <u>diverges</u> or is a <u>divergent</u> sequence.

<u>Examples</u>:

3. $p_n = \left(\frac{1}{n}, \frac{n}{n+1}\right) \to (0,1)$.

For, given any ball of radius $\varepsilon > 0$, we can make

$$\left|\left(\frac{1}{n}, \frac{n}{n+1}\right) - (0,1)\right| = \left[\frac{1}{n^2} + \left(\frac{n}{n+1} - 1\right)^2\right]^{1/2} < \varepsilon$$

by taking

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{2}{n^2} < \varepsilon^2$$

or

$$n > \frac{\sqrt{2}}{\varepsilon}$$

4. The following series of real numbers diverge:

a. $a_n = n$ $1, 2, 3, \ldots$
b. $b_n = (-1)^n n$ $-1, 2, -3, 4, -5, \ldots$
c. $c_n = 1 + (-1)^n$ $0, 2, 0, 2, 0, 2, \ldots$
d. $d_n = (-1)^n (1 + \frac{1}{n})$

To see that $\{d_n\}$ diverges observe that alternative terms go this way - -

even terms: $1 + \frac{1}{2}, 1 + \frac{1}{4}, 1 + \frac{1}{6}, \ldots$

odd terms: $-(1 + 1), -(1 + \frac{1}{3}), -(1 + \frac{1}{5}), \ldots$

<u>Exercise</u>.

(1) Prove that if a sequence has a limit it has only one limit.

In general, a series diverges if its terms "blow up" (i.e., go to infinity), cluster about/at two or more points, or simply "go nowhere".

<u>Remark 1.</u> Convergence is a behavior not affected by a finite number of terms. For example, if $\{p_n\}$ and $\{q_n\}$are two sequences which differ only in their first 2, 000, 000 terms (all subsequent terms being equal) then $\{p_n\}$ and $\{q_n\}$ both converge or both diverge.

For completeness, we state the following familiar result about limits of sequenceness:

Theorem 1. If $\lim a_n = a$ and $\lim b_n = b$, then

a. $\lim (a_n + b_n) = a + b$
b. $\lim a_n b_n = ab$
c. $\lim \frac{a_n}{b_n} = \frac{a}{b}$ if $b \neq 0$.

Proof.

See the proofs in any standard reference (e.g., R.C. Buck, *Advanced Calculus*, pp. 43 - 44).

We can use sequences to throw more light on ideas previously described. For instance:

Theorem 2. The closure of a set S in n – space is the set of all limits of converging sequences of points from S.

Proof.

We shall show each set described is a subset of the other.

(1) If $\{p_n\}$ is a sequence of points in S and $\lim p_n = p$, then $p \in \bar{S}$. For it is an easy exercise to show that p cannot be an exterior point of S.
(2) If $q \in \bar{S}$, then we can find a sequence $\{q_n\}$ of points of S which converges to q as follows:

If $q \in S$, then we can simply take $q_n = q$ for each n.

If $q \notin S$, then q must be a cluster point for S. (Why?) Now take a suitable decreasing sequence of neighborhood of q, for example,

$$B_n = B(q, \frac{1}{n}), \quad n = 1, 2, \ldots$$

There is always a point of S in each such neighborhood so we can choose a point of S in B_n and call it q_n. Clearly,

$$q_n \to q. \qquad \blacklozenge$$

Corollary 3. A set S is closed if and only if it contains the limit of every converging sequence $\{p_n\}$ of points lying in S.

Proof.

($\Rightarrow$) Since S is closed, $\bar{S} = S$. Now cite the theorem.

($\Leftarrow$) If p is a boundary point of S, then we can construct a sequence of points of S converging to p. By the theorem, p lies in S. $\blacklozenge$

Two notions related to the concept of "limit of a sequence", are used a lot in higher analysis. These are the "limit superior" and "limit inferior" of a sequence. To discuss them requires introducing the analog for sequences of the concept of "cluster point of a set".

Definition 3. A point p is a <u>limit point</u> for the sequence $\{p_n\}$ if, for any neighborhood $\mathbf{N}$ of p, there are infinitely many subscripts n for which $p_n \in \mathbf{N}$.

Example:

5. The sequence $\{p_n\}$ where

$$p_n = \left((-1)^n \frac{1}{n}, 1 + (-1)^n \right)$$

has the limit points (0, 0) and (0, 2).

Remarks.

2. We shift the emphasis to <u>subscripts</u> instead of the points themselves because a sequence differs from a set in this aspect, namely, terms of a sequence may involve repetition of points. (See Example 2.)

3. Mark the distinction between "limit" and "limit point":

limit of a sequence	*limit point of a sequence*
"for all subscripts $n > n_o$, $p_n \in \mathbf{N}$"	"for infinitely many subscripts n, $p_n \in \mathbf{N}$"

To understand the two concepts better, it will be necessary to work out the following exercises:

Exercises.

(2) Show that the real sequence

$$a_n = \{1 + (-1)^n\}$$

has two limit points but no limit.

(3) Show that the limit of a sequence is a limit point of the sequence. (Clearly, the converse is not always true.)

(4) Give an example of a sequence with one limit point but which diverges.

(5) Construct a sequence with more than two limit points.

(6) Can a sequence have infinitely many limit points? no limit point?

(7) Given a sequence

$$p_1, p_2, \ldots, p_i, \ldots$$

the sequence

$$q_1, q_2, \ldots, q_j, \ldots$$

is called a <u>subsequence</u> of $\{p_i\}$ if

i.) To each j, there corresponds an i such that $q_j = p_i$.

ii.) Whenever $j_1 < j_2$ then $i_1 < i_2$.

More informally, a subsequence is obtained from a sequence $\{p_i\}$when we form a new sequence by selecting only some of the terms of $\{p_i\}$.

We denote a subsequence of $\{p_i\}$ by

$$p_{i_1}, p_{i_2}, p_{i_3}, \ \ldots$$

or $\{ p_{i_k} \}$. Note here that it is k which runs through 1, 2, 3, …

Now for the exercise: Prove that if p is a limit point of $\{p_n\}$, there is a subsequence $\{ p_{n_k} \}$ of $\{p_n\}$ which converges to p.

<u>Definition 4</u>. The largest limit point of a bounded real sequence $\{a_n\}$ is called the <u>limit superior</u> of the sequence, and is denoted by

$$\lim_{n\to\infty} \sup a_n.$$

The smallest limit point of a bounded real sequence $\{a_n\}$ is called the <u>limit inferior</u> of the sequence, and is denoted by

$$\lim_{n\to\infty} \inf a_n.$$

<u>Note 4.</u> If the sequence $\{a_n\}$ is not bounded, we extend the above definition as follows: If $\{a_n\}$ is not bounded above, we define

$$\lim_{n\to\infty} \sup a_n = +\infty$$

If $\{a_n\}$ is bounded above but not below and if $\{a_n\}$ has no finite limit superior, then we define

$$\lim_{n\to\infty} \sup a_n = -\infty$$

The limit inferior of $\{a_n\}$ can now be defined in a general way as follows:

$$\lim_{n\to\infty} \inf a_n = -\lim_{n\to\infty} \sup b_n \text{ , where } b_n = - a_n \text{ for } n = 1, 2, \ldots$$

<u>Lemma 4.</u> Let $\{a_n\}$ be a bounded real sequence with

$$\lim_{n\to\infty} \sup a_n = a.$$

Then for any $\varepsilon > 0$,

(i) $a_n \le a + \varepsilon$ for all but a finite number of n, and
(ii) $a_n \ge a - \varepsilon$ for infinitely many n.

<u>Proof.</u>

(ii) is easier to show so its proof shall be given first. For the proof of (i), we shall need a result which comes later (Corollary 5.4).

(ii): If $a_n \geq a - \varepsilon$ for finitely many n only, this contradicts the fact that a is a limit point: Every neighborhood of a must contain a_n for infinitely many n and $[a - \varepsilon, +\infty)$ is a neighborhood of a.

(i): Suppose there exist infinitely many n for which

$$a_n > a + \varepsilon ,$$

say, $n_1, n_2, n_3, \ldots n_k, \ldots$ We shall show that, contrary to hypothesis, $\{a_n\}$ has a cluster point larger than a. If infinitely many of these a_n are equal, say, to a number α, then α is a limit point of $\{a_n\}$. Since $\alpha > a$, then we have a contradiction.

If only finitely many of these numbers are equal, then the trace of $\{a_{n_k}\}$ is an infinite set and

$$a + \varepsilon < a_{n_k} < b$$

where b is an upper bound for the bounded sequence $\{a_n\}$. By Theorem 3 of the next chapter, the trace of $\{a_{n_k}\}$ has a cluster point β (**Exercise** (8): Show $\beta > a$.) Accordingly, $\{a_{n_k}\}$ has a limit point β with $\beta > a$. Again we have a contradiction.

Remark 5. The converse of the lemma is also true. (**Exercise** (9).) Hence the condition given in the lemma can be taken as the definition of lim sup a_n.

Exercise 10. State and prove a result for lim inf a_n analogous to the Lemma.

Exercises.

(11) Find lim sup a_n and lim inf a_n if

(a) $a_n = (-1)^n(1 + \frac{1}{n})$

(b) $a_n = n + 1 + (-1)^n(n + \frac{1}{n})$

(c) $a_n = \sin(n\frac{\pi}{3})$

(d) $a_n = n^2\sin^2(n\frac{\pi}{2})$

(12) If $a \leq a_n \leq b$ for all but a finite number of n, show that

$$\liminf_{n\to\infty} a_n \geq a$$

$$\limsup_{n\to\infty} a_n \leq b.$$

(13) (a) Prove in general that

$$\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(b) Construct an example of bounded sequences $\{a_n\}$ and $\{b_n\}$ where

$$\limsup_{n\to\infty} (a_n + b_n) < \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$$

(14) Prove that a bounded sequence $\{a_n\}$ converges if and only if $\lim_{n\to\infty} \sup a_n$ and $\lim_{n\to\infty} \inf a_n$ are both finite and equal. (What is $\lim_{n\to\infty} a_n$?)

(15) Prove that for any sequence $\{a_n\}$ (bounded or unbounded)

$$\lim \inf a_n \leq \lim \sup a_n.$$

(Here you can treat $-\infty$ and $+\infty$ as "numbers" satisfying $-\infty < c < +\infty$ for any real number c.)