Between any 2 real numbers, there exists an irrational number

let $a, b \in \mathbb{R}$, where $a < b$

$$0 < \frac{1}{\sqrt{2}} < 1$$

$$0 < \frac{1}{\sqrt{2}}(b-a) < b<a$$

$$0 + a < \frac{1}{\sqrt{2}}(b-a) + a < (b-a) + a$$

$$a < \underbrace{\frac{1}{\sqrt{2}} b + \left(1 - \frac{1}{\sqrt{2}}\right) a}_{\text{irrational}} < b$$

$\therefore$ there exists an irrational number between two rational numbers

Between any 2 real numbers, there exists infinitely many rational numbers

Let $a, b \in \mathbb{R}$, where $a < b$

$b - a = c$

n - rational number ; $n > \frac{1}{c}$

$nc > 1 \rightarrow n(b-a) > 1$

$nb - na > 1$

$\therefore$ there exists r such that $na < r < nb$

$a < \frac{r}{n} < b$.

If $q_1 \in (a, b)$ where $a \neq b \in \mathbb{R}$
then a second can be formed $q_2 \in (a, q_1) \subset (a, b)$
Then a third can be found as well and it continues.