CHAPTER 5

THE CLASSICAL RESULTS ON THE TOPOLOGY OF R^n

(based on Dr. N. Quimpo's notes)

We come now to a series of theorems which provide deep insights into the topological structure of the real line.

Before we tackle the theorems themselves, let us give them a "popular reading". In this form, the results seem commonsensical enough:

The first reads: "A line of soldiers marching up to a wall must stop before the wall or have to stop at the wall itself".

The second says: "You cannot crowd too many people in a room and still expect them to have elbow room".

The third reads: "If you open up a Russian doll, you expect to find a last doll".

In discussing these results, we shall think of the Least Upper Bound Property as our basic assumption. These theorems shall be consequences of this axiom.

Theorem 1. (*Bounded Monotone Sequence Property* or *BMSP*). In 3, a monotone sequence which is bounded converges.

Proof.

Let our sequence be $\{a_n\}$. The trace of $\{a_n\}$ is bounded so it has an LUB, say *a*. We claim that $\lim_{n\to\infty} a_n = a$.

Let $\varepsilon > 0$ be given. Since *a* is the LUB of the trace of $\{a_n\}$, then $a - \varepsilon$ is not an upper bound. Thus there exists n_o such that

$$a-\varepsilon < a_{n_0} < a < a+\varepsilon .$$

Since the sequence is increasing,

$$a-\varepsilon < a_n < a+\varepsilon \text{ for } n \geq n_o.$$

Hence

$$\lim_{n\to\infty} a_n = a . \quad \square$$

Exercise 1. Show by using *BMSP* that the sequence

$$a_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}$$

converges.

Exercise 2. If S is a bounded set in n-space, define the diameter of S by

$$\text{diam (S)} = \sup\{|p - q| : p, q \in S\}.$$

(Note the role of the LUB Property in this definition.) Prove that if $A \subset B$, then diam (A) $\leq$ diam (B). (Hint: The problem reduces to comparing two sets of nonnegative numbers.)

Exercise 3. If A and B are any sets in n-space, define the distance between A and B by

$$d(A, B) = \inf \{|p - q| : p \in A, q \in B\}.$$

Prove that if $A \subset B$, then

$$d(B, C) \leq d(A, C).$$

Theorem 2. (*Nested Intervals Property* or *NIP*). Let $\{I_n\}$ be a sequence of nonempty bounded closed intervals on the line such that

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

Then

$$\bigcap_1^\infty I_n \neq \varnothing.$$

Proof.

Let $I_n = [a_n, b_n]$, $n = 1, 2, \cdots$

Exercise 4. The student is asked to show first that

$$a_1 \leq a_n \leq b_m \leq b_1 \text{ for any n, m}$$

Thus,

$$a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \text{ for each n,}$$

so $\{a_n\}$ is an increasing sequence bounded by b_1. By *BMSP*, $\{a_n\}$ converges to some a. In fact the proof of *BMSP* indicates that *a* is the least upper bound of the set $\{a_1, a_2, a_3, \cdots\}$.

Similarly, $\{b_n\}$ converges to some b, where b is the greatest lower bound of $\{b_1, b_2, b_3, \cdots\}$.

Claim: $a \leq b$.

For, suppose $a > b$. This implies b is not an upper bound of $\{a_1, a_2, a_3, \cdots\}$. So for some n_o,

$$a_{n_0} > b.$$

It follows that for some n,

$$b_{n_1} < a_{n_0}.$$

This contradicts the fact that $a_n \leq b_m$ for every n, m. Hence $a \leq b$. Since

$$a_n \leq a \leq b \leq b_m \text{ for each } n$$

we have

$$[a, b] \subset [a_n, b_n] \text{ for each } n$$

$$[a, b] \subset \bigcap_{1}^{\infty} [a_n, b_n]. \qquad \square$$

Note 1. In the above theorem, if the lengths of the intervals have limit

$$\lim_{n\to\infty}(b_n - a_n) = 0,$$

then a = b and the intersection is {a}.

Theorem 3. (*Bolzano-Weierstrass Theorem* for the real line). Any infinite set of real numbers which is bounded has a cluster point.

Proof.

Suppose that the set S under investigation is contained in the closed interval I_o which is of finite length, say L.

We divide I_o into two equal subintervals. One of these subintervals must contain an infinite number of points of S. Call this particular subinterval I_1.

We now divide I_1, in turn into two equal subintervals. Again one of these subintervals of I_1 must contain infinitely many points of S. Call this subinterval I_2.

Repeating the above procedure we obtain subinterval I_k in step k, which has the following description:

i. I_k contains infinitely many points of S;
ii. $I_k \subset I_{k-1}$;
iii. I_k is of length $\ell_k = \dfrac{L}{2^k}$.

Thus we obtain a sequence of closed intervals I_k which are nonempty (indeed each one contains infinitely many points of S) and satisfy

$$I_1 \supset I_2 \supset I_3 \supset \cdots$$

By the *NIP*,

$$\bigcap_{1}^{\infty} I_k \neq \varnothing.$$

Since

$$\lim_{n\to\infty} \ell_k = 0$$

then in fact the intersection is a single point c.

Claim . c is a cluster point of S.
For if $\varepsilon > 0$, the neighborhood $(c - \varepsilon, c + \varepsilon)$ contains I_k for $\frac{L}{2^k} < \varepsilon$. Now I_k contains

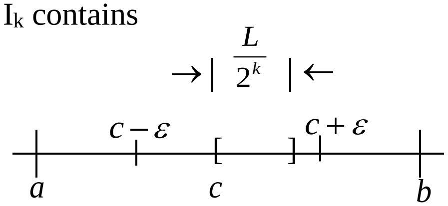

Fig. 1. “Worst case” scenario

infinitely many points of S, therefore so does $(c - \varepsilon, c + \varepsilon)$.
□

COROLLARY 4. Every bounded sequence of real numbers has a limit point.

Proof.

Exercise 5. (By the way, compare this to *BMSP*).

Exercise 6. Show that a bounded sequence of real numbers with exactly one limit point is convergent.

Exercise 7. State a 2 – space version of the *NIP* involving rectangles. Prove by applying the *NIP*.

Exercise 8. What is the general 2 – space version of the *Bolzano-Weierstrass Theorem*? Prove.

The general *Nested Interval Property* assumes the following form:

Theorem. Let $\{C_n\}$ be a sequence of nonempty closed and bounded sets in n – space such that

$$C_1 \supset C_2 \supset C_3 \supset \cdots$$

Then $\bigcap_{n=1}^{\infty} C_n$ is nonempty and closed.

Proof.

The fact that $\cap\, C_n$ is closed follows from a result seen earlier about the intersection of closed sets.

Let us assume that each C_n contains an infinite number of points. (See the exercise that follows for the other possibility.)

Choose a point in C_n and denote it by p_n. Let

$$P = \{p_1, p_2, \ldots\}.$$

Since P is an infinite set contained in the bounded set C_1, then it has a cluster point p.

Claim. $p \in \cap C_n$.

We shall show this by proving p is a cluster point of C_n for each n. Since C_n is closed for each n, it will follow that $p \in C_n$ for each n.

p is a cluster point of P so, given a neighborhood N of p, N contains infinitely many points of P. But C_n contains all points of P except, possibly, $p_1, p_2, \ldots, p_{n-1}$. Thus, N contains infinitely many points of C_n and p is a cluster point of C_n.

□

Exercise 9. Settle the case where some C_n contains only finitely many points.

Definition 1. A sequence $\{p_k\}$ in n-space is said to be a *Cauchy sequence* if, for any given $\varepsilon > 0$ there is an n_o such that

$$|p_i - p_j| < \varepsilon \quad \text{whenever} \quad i, j > n_o.$$

Exercise 10. Show that every Cauchy sequence is a bounded sequence.

Theorem 5. Any Cauchy sequence of real numbers converges to a real number.

Proof.

Suppose our Cauchy sequence does not converge. Since we are dealing with a bounded sequence, $a = \liminf a_n$ and $b = \limsup a_n$ both exist and we have $a < b$. (Why?) Suppose we take ε to be a number such that $\varepsilon < b - a$. For this ε we can find n_o such that

$$|a_n - a_m| < \varepsilon \text{ for } n, m > n_o \quad \text{or} \quad -\varepsilon < a_n - a_m < \varepsilon$$

which yields

$$a_n < \varepsilon + a_m \text{ for } n, m > n_o.$$

Fixing m, we apply the result of Exercise 4.12 to get

$$\limsup a_n = b \leq \varepsilon + a_m$$

so

$$b - \varepsilon \leq a_m \text{ for } m > n_o.$$

Again we apply Exercise 4.12 and obtain

$$b - \varepsilon \leq \liminf a_m = a.$$

But this says

$$b - a \leq \varepsilon$$

contradicting our choice of ε. Thus $a = b$ and $\lim a_n$ exists. □

Exercise 11. Show that if a sequence of real numbers is convergent, it is Cauchy.

The theorem and the exercise together say that as far as the real numbers are concerned

$$\text{“}\{a_n\} \text{ converges} \iff \{a_n\} \text{ is Cauchy”.}$$

so convergence and being Cauchy are synonymous in 3. A similar result holds for n-dimensional Euclidean space. In higher mathematics, examples are given of certain other topological spaces where Cauchy sequences may not converge within the space.

Definition 2. Given a set S, a collection $\mathcal{S}$ of sets is a covering of S if S is a subset of the union of the sets in $\mathcal{S}$.

A covering $\mathcal{S}$ is called open if each set in $\mathcal{S}$ is an open set. $\mathcal{S}$ is called finite if it consists of only a finite number of sets.

Examples:

1. The collection $\{B((n, m), 1): n, m = 0, 1, 2, \ldots\}$ is an open covering of $\{(x, y): x \geq 0, y \geq 0\}$.

2. The collection $[i, i + 1] \times [j, j + 1]$, $i = 0, 1$
   $j = 0, 1$
   is a finite covering of $[0, 2] \times [0, 2]$.

3. The collection of intervals $\left(r - \frac{r}{2}, r + \frac{r}{2}\right)$, $r \in (0, 1)$ is a covering of (0, 1) with an uncountable number of sets in the collection.

Exercise 12.

(a) Show that the unit open square $(0, 1) \times (0, 1)$ can be expressed as the union of a collection of closed disks.

(b) Show that the unit open disk

$$x^2 + y^2 < 1$$

can be expressed as the union of a collection of closed squares.

Exercise 13. Show that the balls

$$B_n = B((n, 0), n)$$

form a covering of the half-plane $x > 0$ in 2-space. (In fact, their union is the half-plane.)

Definition 3. A set C is compact if, whenever $\mathcal{S}$ is an open covering of C, there is a finite subcollection of $\mathcal{S}$ which covers C.

Theorem 6. If C is compact, then it is a closed and bounded set.

Proof. Exercise 14. Consider the following hints for the theorem in 2-space:

Boundedness: {B((0, 0), n): n = 1, 2, . . .} is an open covering of C.

Closedness: Suppose $p \in \partial C$ but $p \notin C$. Take a collection of concentric closed balls centered at p. Show the complement of these balls form an open covering of C.

Exercise 15. If S is a bounded set in 2-space then, given $\varepsilon > 0$, we can find a finite set of points

$$p_1, p_2, \ldots, p_n$$

in S such that each point of S lies in at least one of the disks

$$B(p_i, \varepsilon)\ , \quad i = 1, 2, \ldots, n.$$

Lemma 7. If S is an open covering of A in n-space, then there is a countable subcovering of S which also covers A.

Proof. (For 2-space)

We shall denote by R the collection of all open disks

B(u, v, r) where u, v, r are rational

Then R is made up of a countable number of sets and R forms a covering of 2-space. We can assume the sets in R are listed as

$$B_1, B_2, B_3, \ldots$$

If p is a point of A which lies in set $O \in \mathsf{S}$, (note that there may be more than one such O), then we can find B_k such that

$$p \in B_k \subseteq O.$$

(This is **Exercise** (16). Hint: Let p = (x, y) and take various cases for x, y.)

In fact there are infinitely many such B_k. (Why?) Let us choose the one with the smallest index k and denote its index by k(p). Note that $k(p_1)$ can be equal to $k(p_2)$ even for $p_1 \neq p_2$.

Now, choose one O to correspond to $B_{k(p)}$ and call it k(p). Thus we have the correspondence

$$p \leftrightarrow O_{k(p)}\ .$$

The collection of such $O_{k(p)}$ is a countable subcovering of S which covers A. □

Theorem 8. (*Heine-Borel*). A closed and bounded set in n-space is compact.

Proof.

Because of Lemma 7, we need only show: Given a countable open covering

$$O_1, O_2, O_3, \ldots$$

of C, we can find a finite subset of the O_i's covering C. (This is called a "finite subcovering" of $\{O_1, O_2, \ldots\}$.)

Define new open sets from the O_i's as follows: Let

$$\begin{aligned} V_1 &= O_1 \\ V_2 &= O_1 \cup O_2 \\ &\vdots \\ V_{n+1} &= V_n \cup O_{i+1} \end{aligned}$$

The V_i form an open covering of C also. Moreover,

$$V_1 \subset V_2 \subset V_3 \subset \cdots$$

Now define

$$\begin{aligned} C_1 &= C \setminus V_1 \\ C_2 &= C \setminus V_2 \\ &\vdots \\ C_k &= C \setminus V_k \\ &\vdots \end{aligned}$$

The C_k are bounded closed sets satisfying

$$C \supset C_1 \supset C_2 \supset \cdots$$

There are two possibilities: Either all the C_k are nonempty or $C_\ell = \varnothing$ for some ℓ.

If $C_k \neq \varnothing$ for each k, the generalized version of the *NIP* applies so there is q such that

$$q \in \bigcap_{k=1}^{\infty} C_k.$$

So $q \in C_k = C \setminus V_k$ meaning $q \in C$ but $q \notin V_k$ for each k, contradicting the fact that $\{V_k\}$ is a covering for C.

Thus $C_\ell = \varnothing$ which implies $C_{\ell+1}, C_{\ell+2}, \ldots$ are all empty. Since $C_\ell = C \setminus V_\ell$ then V_ℓ must contain C. Hence

$$O_1, O_2, \ldots O_\ell$$

cover C. $\square$