Vector Spaces

The vector Space R^n

Definition 1.

Let $(u_1, u_2, \ldots, u_n)$ be a sequence of n real numbers. The set of all such sequences is called ***n*-space (or *n*-dimensional. space)** and is denoted $\mathbf{R}^n$.

u_1 is the **first component** of $(u_1, u_2, \ldots, u_n)$.

u_2 is the **second component** and so on.

Example 1

- $\mathbf{R}^2$ is the collection of all sets of two ordered real numbers. For example, (0, 0) , (1, 2) and (-2, -3) are elements of $\mathbf{R}^2$.

- $\mathbf{R}^3$ is the collection of all sets of three ordered real numbers. For example, (0,0, 0) and (-1,3, 4) are elements of $\mathbf{R}^3$.

Definition 2.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two elements of $\mathbf{R}^n$.

We say that **u** and **v** are **equal** if $u_1 = v_1, \ldots, u_n = v_n$.

Thus two elements of $\mathbf{R}^n$ are equal if their **corresponding components** are equal.

Definition 3.

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be elements of $\mathbf{R}^n$ and let c be a scalar. **Addition** and **scalar multiplication** are performed as follows:

Addition: $\mathbf{u} + \mathbf{v} = (u_1 + v_1, \ldots, u_n + v_n)$

Scalar multiplication : $c\mathbf{u} = (cu_1, \ldots, cu_n)$

► The set $\mathbf{R}^n$ with operations of componentwise addition and scalar multiplication is an example of a **vector space,** and its elements are called **vectors.**

We shall henceforth interpret $\mathbf{R}^n$ to be a vector space.

(We say that $\mathbf{R}^n$ is **closed** under addition and scalar multiplication).

► In general, if **u** and **v** are vectors in the same vector space, then **u + v** is the diagonal of the **parallelogram** defined by **u** and **v**.

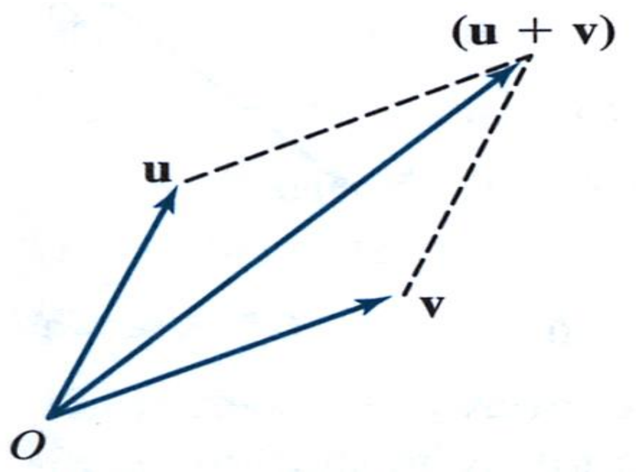


Figure 1

Example 2

Let $\mathbf{u} = (-1, 4, 3)$ and $\mathbf{v} = (-2, -3, 1)$ be elements of $\mathbf{R}^3$.

Find $\mathbf{u} + \mathbf{v}$ and $3\mathbf{u}$.

Solution: $\mathbf{u} + \mathbf{v} = (-1, 4, 3) + (-2, -3, 1) = (-3, 1, 4)$

$$3\mathbf{u} = \mathbf{3}\,(-1, 4, 3) = (-3, 12, 9)$$

Example 3

In R^2, consider the two elements (4, 1) and (2, 3).

Find their sum and give a geometrical interpretation of this sum.

we get (4, 1) + (2, 3) = (6, 4).

The vector (6, 4), the sum, is the diagonal of the parallelogram.

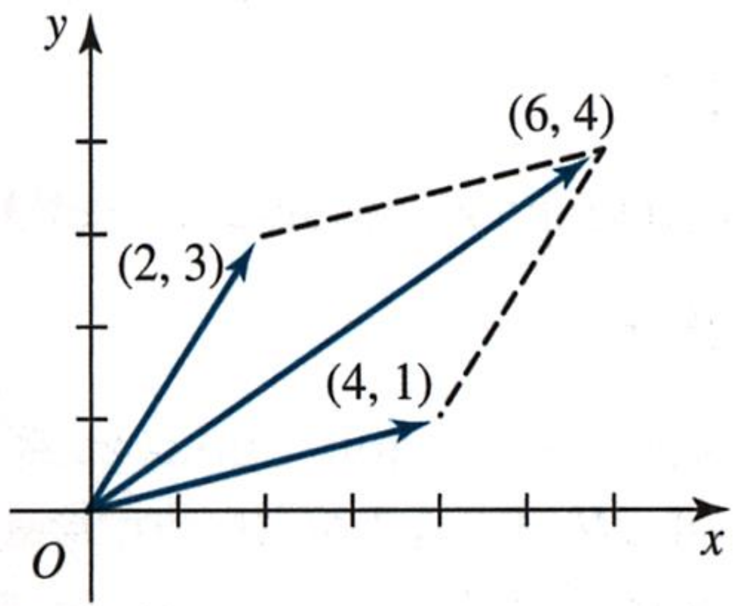


Figure 2

Example 4

Consider the scalar multiple of the vector (3, 2) by 2, we get

$$2(3, 2) = (6, 4)$$

Observe in Figure 3 that (6, 4) is a vector in the same direction as (3, 2), and 2 times it in length.

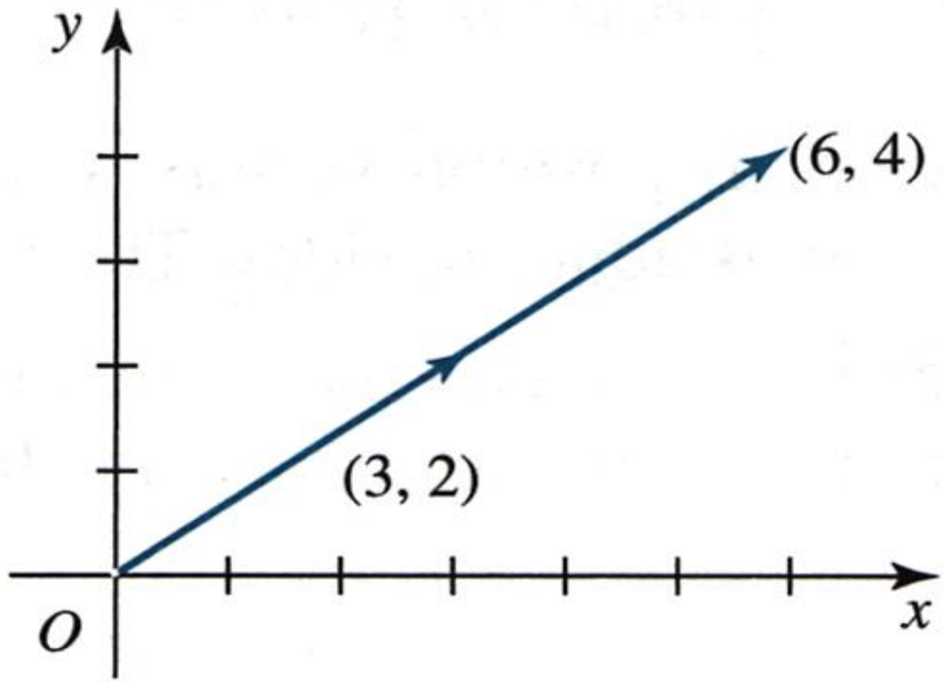


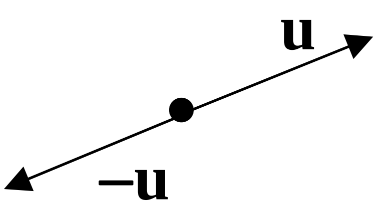
Figure 3

Zero Vector

The vector (0, 0, …, 0), having *n* zero components, is called the **zero vector** of $\mathbf{R}^n$ and is denoted $\mathbf{0}$.

Negative Vector

The vector (–1)**u** is writing –**u** and is called **the negative of u**. It is a vector having the same length (or magnitude) as **u**, but lies in the opposite direction to **u**.



Subtraction

Subtraction is performed on element of $\mathbf{R}^n$ by subtracting corresponding components.

Theorem

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let c and d be scalars.

(a) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

(b) $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$

(c) $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$

(d) $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$

(e) $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$

(f) $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$

(g) $c(d\mathbf{u}) = (cd)\mathbf{u}$

(h) $1\mathbf{u} = \mathbf{u}$

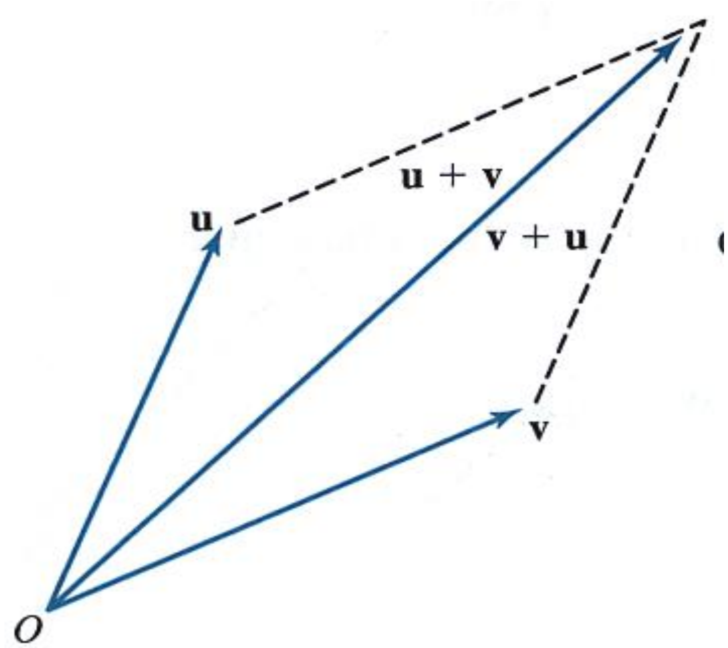


Figure 4

Commutativity of vector addition

$\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$

Example 5

Let **u** = (2, 5, –3), **v** = (–4, 1, 9), **w** = (4, 0, 2) in the vector space R^3. Determine the vector 2**u** – 3**v** + **w**.

Solution

$$\begin{aligned}\mathbf{2u}-\mathbf{3v}+\mathbf{w} &= 2(2,5,-3)-3(-4,1,9)+(4,0,2)\\ &= (4,10,-6)-(-12,3,27)+(4,0,2)\\ &= (4+12+4,10-3+0,-6-27+2)\\ &= (20,7,-31)\end{aligned}$$

Column Vectors

Row vector: $\mathbf{u} = (u_1, u_2, \ldots, u_n)$

Column vector: $\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$

We defined addition and scalar multiplication of column vectors in $\mathbf{R}^n$ in a componentwise manner:

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \text{and} \quad c\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix}$$

Dot Product, Norm, Angle, and Distance

Definition

Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ be two vectors in $\mathbf{R}^n$.
The **dot product** of **u** and **v** is denoted **u.v** and is defined
by

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + \cdots + u_n v_n$$

The dot product assigns a real number to each pair of vectors.

Example 1

Find the dot product of

$$\mathbf{u} = (1, -2, 4) \text{ and } \mathbf{v} = (3, 0, 2)$$

Solution

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= (1 \times 3) + (-2 \times 0) + (4 \times 2) \\ &= 3 + 0 + 8 \\ &= 11 \end{aligned}$$

Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in $\mathbf{R}^n$ and let c be a scalar. Then

1. u.v = v.u

2. (u + v).w = u.w + v.w

3. $c\mathbf{u}.\mathbf{v} = c(\mathbf{u}.\mathbf{v}) = \mathbf{u}.c\mathbf{v}$

4. u.u ≥ 0, and **u.u** $= 0$ if and only if **u = 0**

Proof

1. Let $\mathbf{u} = (u_1, u_2, \ldots, u_n)$ and $\mathbf{v} = (v_1, v_2, \ldots, v_n)$. We get

$$\begin{aligned}\mathbf{u}\cdot\mathbf{v} &= u_1 v_1 + \cdots + u_n v_n \\ &= v_1 u_1 + \cdots + v_n u_n \quad \text{by the commutative property of real numbers} \\ &= \mathbf{v}\cdot\mathbf{u}\end{aligned}$$

4. $\mathbf{u}\cdot\mathbf{u} = u_1 u_1 + \cdots + u_n u_n = (u_1)^2 + \cdots + (u_n)^2$

$(u_1)^2 + \cdots + (u_n)^2 \geq 0$, thus $\mathbf{u}\cdot\mathbf{u} \geq 0$.

$(u_1)^2 + \cdots + (u_n)^2 = 0$, if and only if $u_1 = 0, \cdots, u_n = 0$.

Thus $\mathbf{u}\cdot\mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Norm of a Vector in $\boldsymbol{R}^n$

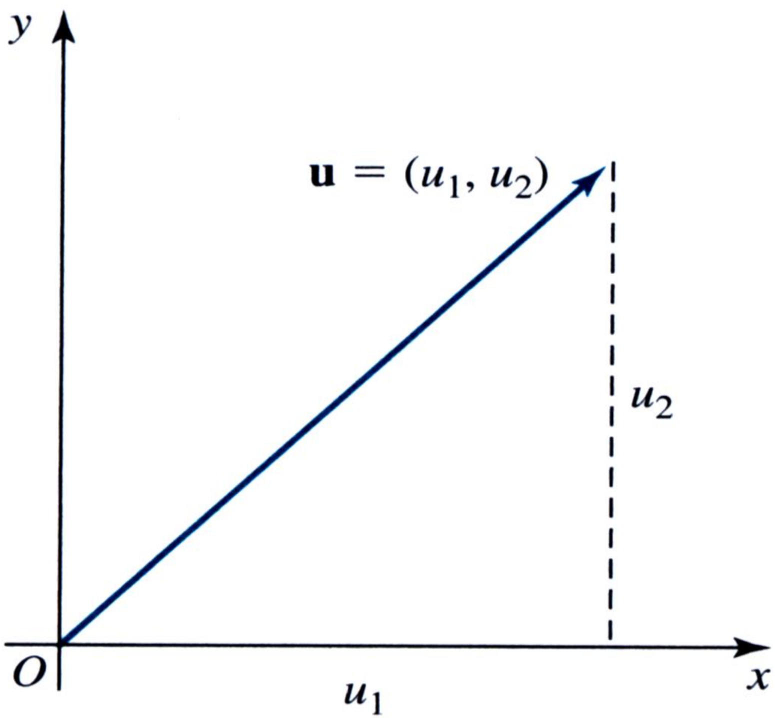


Figure 5 length of $\mathbf{u}$

Definition

The **norm** (**length** or **magnitude**) of a vector $\mathbf{u} = (u_1, \ldots, u_n)$ in $\mathbf{R}^n$ is denoted $\|\mathbf{u}\|$ and defined by

$$\|\mathbf{u}\| = \sqrt{(u_1)^2 + \cdots + (u_n)^2}$$

Note:

The norm of a vector can also be written in terms of the dot product $\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$

Example 2

Find the norm of each of the vectors $\mathbf{u} = (1, 3, 5)$ of $\mathbf{R}^3$ and $\mathbf{v} = (3, 0, 1, 4)$ of $\mathbf{R}^4$.

Solution

$$\|\mathbf{u}\| = \sqrt{(1)^2 + (3)^2 + (5)^2} = \sqrt{1 + 9 + 25} = \sqrt{35}$$

$$\|\mathbf{v}\| = \sqrt{(3)^2 + (0)^2 + (1)^2 + (4)^2} = \sqrt{9 + 0 + 1 + 16} = \sqrt{26}$$

Definition

A **unit vector** is a vector whose norm is 1.

If $\mathbf{v}$ is a nonzero vector, then the vector

$$\boxed{\mathbf{u} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}}$$

is a unit vector in the direction of $\mathbf{v}$.

This procedure of constructing a unit vector in the same direction as a given vector is called **normalizing** the vector.

Example 3

(a) Show that the vector (1, 0) is a unit vector.

(b) Find the norm of the vector (2, –1, 3). Normalize this vector.

Solution

(a) $\|(1,0)\| = \sqrt{1^2+0^2} = 1$. Thus (1, 0) is a unit vector. It can be similarly shown that (0, 1) is a unit vector in $\mathbf{R}^2$.

(b) $\|(2,-1,3)\| = \sqrt{2^2+(-1)^2+3^2} = \sqrt{14}$. The norm of (2, –1, 3) is $\sqrt{14}$.

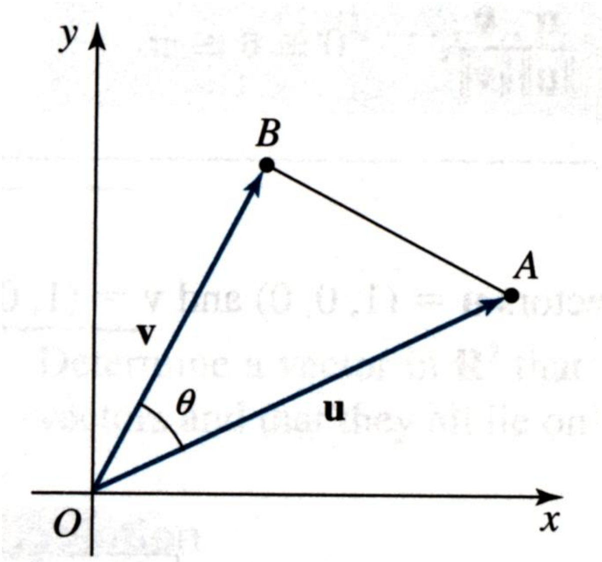
The normalized vector is

$$\frac{1}{\sqrt{14}}(2,-1,3)$$

The vector may also be written $\left(\frac{2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$.

This vector is a unit vector in the direction of (2, –1, 3).

Angle between Vectors (in R^2)



The law of cosines gives:

$$\boxed{\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|}}$$

← **Figure 6**

Angle between Vectors (in R^n)

Definition

Let $\mathbf{u}$ and $\mathbf{v}$ be two nonzero vectors in $\mathbf{R}^n$.

The **cosine of the angle** θ between these vectors is

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} \qquad 0 \le \theta \le \pi$$

Example 4

Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $\mathbf{v} = (1, 0, 1)$ in $\mathbf{R}^3$.

Solution $\mathbf{u}\cdot\mathbf{v} = (1,0,0)\cdot(1,0,1) = 1$

$$\|\mathbf{u}\| = \sqrt{1^2+0^2+0^2} = 1 \qquad \|\mathbf{v}\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

Thus $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \dfrac{1}{\sqrt{2}}$, the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°.

Orthogonal Vectors

Definition

Two nonzero vectors are **orthogonal** if the angle between them is a right angle .

Theorem 4.2

Two nonzero vectors **u** and **v** are orthogonal *if and only if* $\mathbf{u}.\mathbf{v} = \mathbf{0}$.

Proof

$$\mathbf{u}, \mathbf{v} \text{ are orthogonal} \iff \cos\theta = 0 \iff \mathbf{u}\cdot\mathbf{v} = 0$$

Example 5

Show that the following pairs of vectors are orthogonal.

(a) (1, 0) and (0, 1).

(b) (2, –3, 1) and (1, 2, 4).

Solution

(a) $(1, 0).(0, 1) = (1 \times 0) + (0 \times 1) = 0$.

The vectors are orthogonal.

(b) $(2, -3, 1).(1, 2, 4) = (2 \times 1) + (-3 \times 2) + (1 \times 4) = 2 - 6 + 4 = 0$.

The vectors are orthogonal.

Note

- (1, 0), (0,1) are orthogonal unit vectors in $\mathbf{R}^2$.
- (1, 0, 0), (0, 1, 0), (0, 0, 1) are orthogonal unit vectors in $\mathbf{R}^3$.
- (1, 0, …, 0), (0, 1, 0, …, 0), …, (0, …, 0, 1) are orthogonal unit vectors in $\mathbf{R}^n$.

Example 6

Determine a vector in $\mathbf{R}^2$ that is orthogonal to (3, –1). Show that there are many such vectors and that they all lie on a line.

Solution

Let the vector (a, b) be orthogonal to (3, –1)

We get
$$(a,b)\cdot(3,-1)=0$$
$$(a\times 3)+(b\times(-1))=0$$
$$3a-b=0$$
$$b=3a$$

Thus any vector of the form (a, 3a) is orthogonal to the vector (3, –1).

Any vector of this form can be written

$$a(1, 3)$$

The set of all such vectors lie on the line defined by the vector (1, 3).

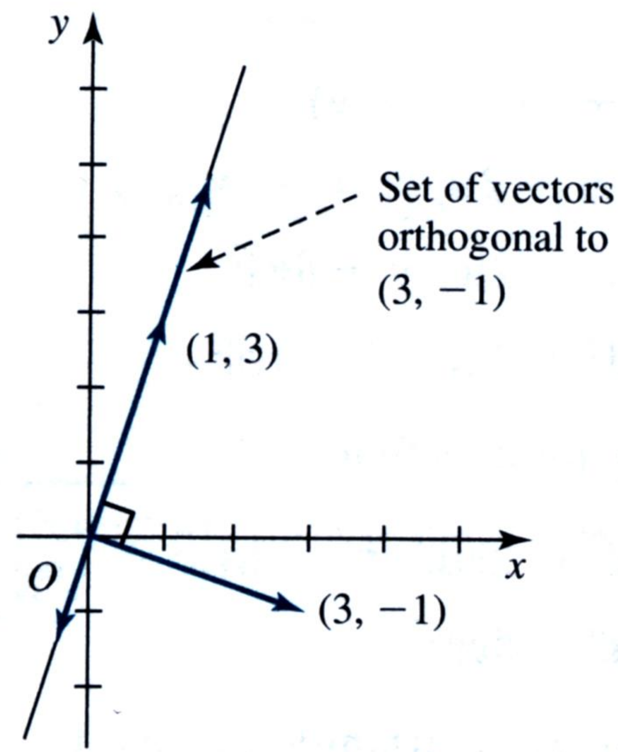


Figure 7

Theorem

Let **u** and **v** be vectors in $\mathbf{R}^n$.

(a) Triangle Inequality:

$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

(a) Pythagorean theorem :

$$\text{If } \mathbf{u}.\mathbf{v} = 0 \text{ then } \|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

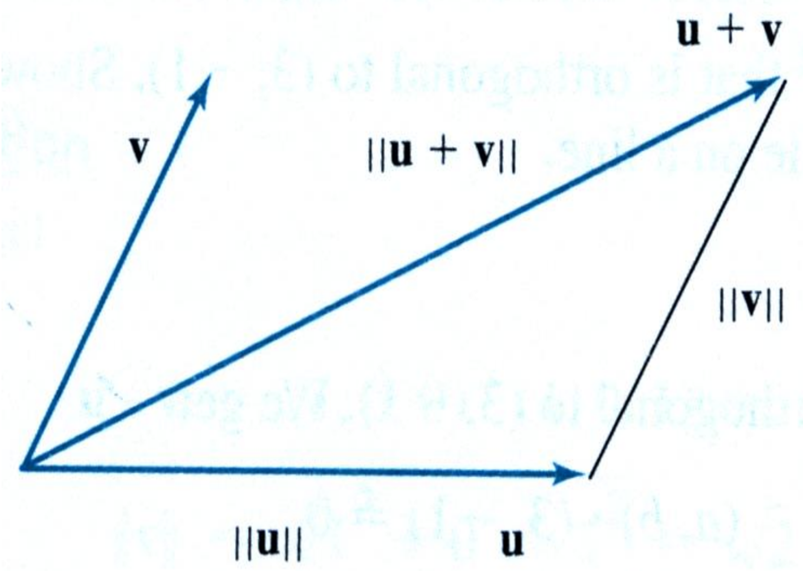


Figure 8(a)

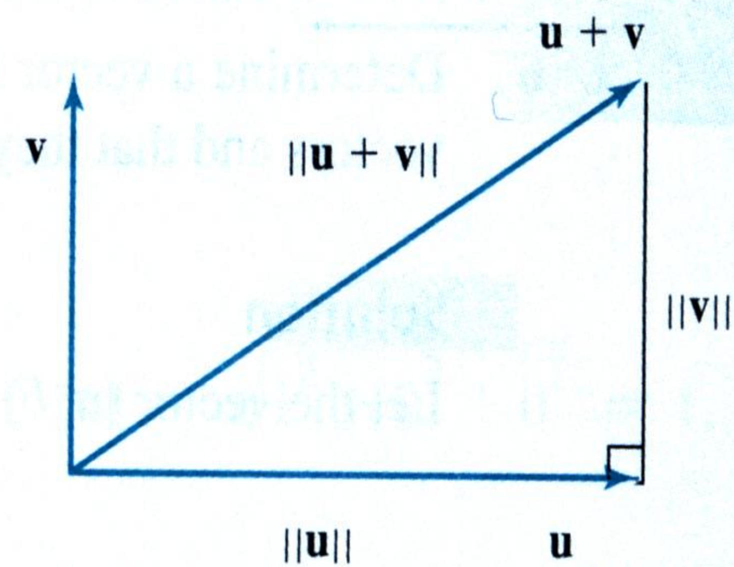


Figure 8(b)

Distance between Points

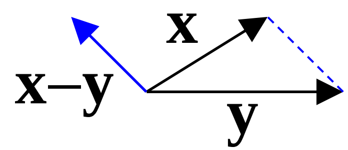
Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ and $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ be two points in $\mathbf{R}^n$.

The **distance** between **x** and **y** is denoted $d(\mathbf{x}, \mathbf{y})$ and is defined by

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

Note: We can also write this distance as follows.

$$d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$$



Note: *It is clear that* $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (*the symmetric property*)

Example 7. Determine the distance between the points $\mathbf{x} = (1, -2, 3, 0)$ and $\mathbf{y} = (4, 0, -3, 5)$ in $\mathbf{R}^4$.

Solution

$$d(\mathbf{x}, \mathbf{y}) = \sqrt{(1-4)^2 + (-2-0)^2 + (3+3)^2 + (0-5)^2}$$
$$= \sqrt{9 + 4 + 36 + 25}$$
$$= \sqrt{74}$$

General Vector Spaces

Our aim in this section will be to focus on the algebraic properties of $\mathbf{R}^n$.

Definition

A **vector space** is a set V of elements called **vectors,** having operations of ***addition*** and ***scalar multiplication*** defined on it that satisfy the following conditions.

Let u, v, and w be arbitrary elements of V, and c and d are scalars.

- **Closure Axioms**

1. The sum $\mathbf{u} + \mathbf{v}$ exists and is an element of V. (V is closed under addition.)
2. $c\mathbf{u}$ is an element of V. (V is closed under scalar multiplication.)

Example 1

(1) $V = \{ \ldots, -3, -1, 1, 3, 5, 7, \ldots \}$

V is not closed under addition because $1+3=4 \notin V$.

(2) $Z = \{ \ldots, -2, -1, 0, 1, 2, 3, 4, \ldots \}$

Z is closed under addition because

for any $a, b \in Z$, $a + b \in Z$.

Z is not closed under scalar multiplication because

½ is a scalar, for any odd $a \in Z$, $(½)a \notin Z$.

Definition of Vector Space (continued)

- **Addition Axioms**

3. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative property)
4. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ (associative property)
5. There exists an element of *V*, called the **zero vector**, denoted **0**, such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$.
6. For every element **u** of *V* there exists an element called the **negative** of **u**, denoted **–u**, such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$.

- **Scalar Multiplication Axioms**

7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
10. $1\mathbf{u} = \mathbf{u}$

A Vector Space in R^3

Let $W = \{\, a(1,0,1) \mid a \in R \,\}$. Prove that W is a vector space.

Proof

Let $\mathbf{u} = a(1,0,1)$ and $\mathbf{v} = b(1,0,1) \in W$, for some a, $b \in \mathbf{R}$.

Axiom 1: $\mathbf{u} + \mathbf{v} = a(1,0,1) + b(1,0,1) = (a+b)(1,0,1)$

$\therefore \mathbf{u} + \mathbf{v} \in W$. Thus W is closed under addition.

Axiom 2: $c\mathbf{u} = ca(1,0,1) \in W$.

Thus W is closed under scalar multiplication.

Axiom 5: Let $\mathbf{0} = (0, 0, 0) = 0(1,0,1)$,

then $\mathbf{0} \in W$ and $\mathbf{0}+\mathbf{u} = \mathbf{u}+\mathbf{0} = \mathbf{u}$ for any $\mathbf{u} \in W$.

Axiom 6: For any $\mathbf{u} = a(1,0,1) \in W$. Let $-\mathbf{u} = -a(1,0,1)$,

then $-\mathbf{u} \in W$ and $(-\mathbf{u})+\mathbf{u} = \mathbf{0}$.

Axiom 3,4 and 7~10: trivial

Vector Spaces of Matrices (M_{mn})

Let $M_{22} = \{ \begin{bmatrix} p & q \\ r & s \end{bmatrix} \mid p, q, r, s \in R \}$. Prove that M_{22} is a vector space.

Proof

$$\text{Let } \mathbf{u} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} e & f \\ g & h \end{bmatrix} \in M_{22}.$$

Axiom 1:

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

u + **v** is a 2 × 2 matrix. Thus M_{22} is closed under addition.

► ***Question***: Prove Axiom 2 and Axiom 7 .

Axiom 3 and 4:

From our previous discussions we know that 2 × 2 matrices are commutative and associative under addition (Theorem 2.2).

Axiom 5:

The 2 × 2 zero matrix is $\mathbf{0}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$, since

$$\mathbf{u}+\mathbf{0}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}=\mathbf{u}$$

Axiom 6:

If $\mathbf{u}=\begin{bmatrix}a & b\\ c & d\end{bmatrix}$, then $-\mathbf{u}=\begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}$, since

$$\mathbf{u}+(-\mathbf{u})=\begin{bmatrix}a & b\\ c & d\end{bmatrix}+\begin{bmatrix}-a & -b\\ -c & -d\end{bmatrix}=\begin{bmatrix}a-a & b-b\\ c-c & d-d\end{bmatrix}=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}=\mathbf{0}$$

In general: The set of $m \times n$ matrices, $\boldsymbol{M}_{mn}$, is a vector space.

Is the set $W=\{\begin{bmatrix}p & q\\ r & s\end{bmatrix} \mid p,q,r,s>0\}$ a vector space?

Vector Spaces of Functions

Prove that $\boldsymbol{F} = \{ f \mid f: R \to R \}$ is a vector space.

For example: $f: R \to R$, $f(x)=2x$,
$g: R \to R$, $g(x)=x^2+1$.

Let $f, g \in \boldsymbol{F}$, $c \in R$.

Axiom 1:

$f + g$ is defined by $(f + g)(x) = f(x) + g(x)$.

$\Rightarrow f + g : R \to R$

$\Rightarrow f + g \in \boldsymbol{F}$. Thus $\boldsymbol{F}$ is <u>closed under addition</u>.

Axiom 2:

cf is defined by $(cf)(x) = c \cdot f(x)$.

$\Rightarrow cf : R \to R$

$\Rightarrow cf \in \boldsymbol{F}$. Thus $\boldsymbol{F}$ is <u>closed under scalar multiplication</u>.

Vector Spaces of Functions (continued)

Axiom 5:

Let **0** be the function such that $\mathbf{0}(x) = 0$ for every $x \in R$.

0 is called the **zero function**.

We get $(f + \mathbf{0})(x) = f(x) + \mathbf{0}(x) = f(x) + 0 = f(x)$ for every $x \in R$.

Thus $f + \mathbf{0} = f$. (**0** is the **zero vector**.)

Axiom 6:

Let the function $-f$ defined by $(-f)(x) = -f(x)$.

$$\begin{aligned}[f+(-f)](x) &= f(x) + (-f)(x) \\ &= f(x) - [f(x)] \\ &= 0 \\ &= \mathbf{0}(x)\end{aligned}$$

Thus $[f + (-f)] = \mathbf{0}$, $-f$ is the negative of f.

Is the set $\mathcal{F} = \{ f \mid f(x) = ax^2 + bx + c$ for some $a, b, c \in R\}$ *a vector space*?

Theorem (useful properties)

Let V be a vector space, $\mathbf{v}$ a vector in V, $\mathbf{0}$ the **zero vector** of V, c a scalar, and 0 the zero scalar. Then

(a) $0\mathbf{v} = \mathbf{0}$

(b) $c\mathbf{0} = \mathbf{0}$

(c) $(-1)\mathbf{v} = -\mathbf{v}$

(d) If $c\mathbf{v} = \mathbf{0}$, then either $c = 0$ or $\mathbf{v} = \mathbf{0}$.

Subspaces

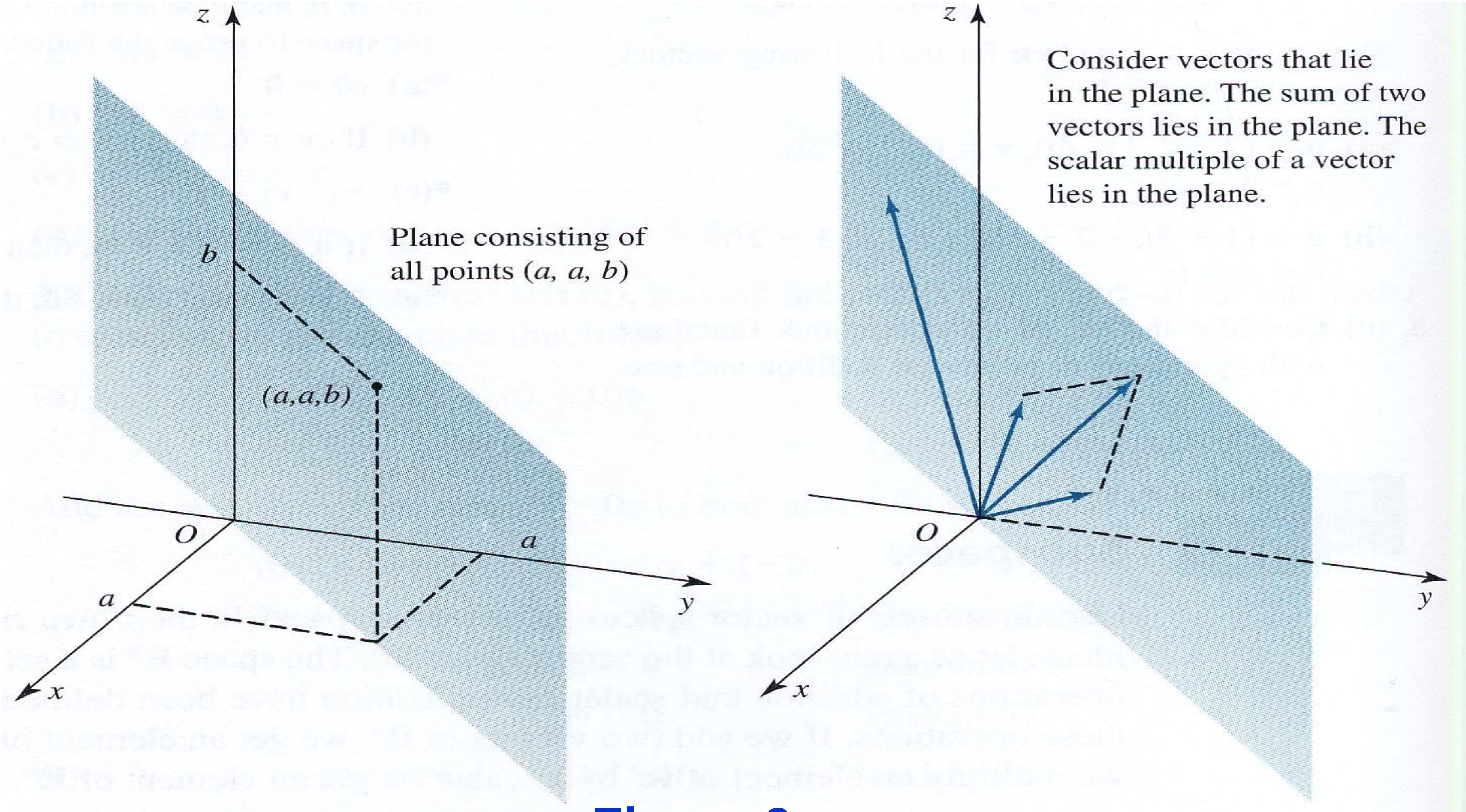


Figure 9

Note:

► In general, a subset of a vector space may or may not satisfy the closure axioms.

► However, any subset that is closed under both of these operations satisfies all the other vector space properties.

Definition

Let V be a vector space and U be a *nonempty* subset of V.

U is said to be a **subspace** of V if it is closed under addition and under scalar multiplication.

Example 1

Let U be the subset of $\mathbf{R}^3$ consisting of all vectors of the form $(a, 0, 0)$ (with zeros as second and third components and $a \in \mathbf{R}$), i.e., $U = \{(a, 0, 0) \in \mathbf{R}^3\}$.

Show that U is a subspace of $\mathbf{R}^3$.

Solution

Let $(a, 0, 0), (b, 0, 0) \in U$, and let $k \in \mathbf{R}$.

We get

$$(a, 0, 0) + (b, 0, 0) = (\mathrm{a} + b, 0, 0) \in U$$

$$k(a, 0, 0) = (k\, a, 0, 0) \in U$$

The sum and scalar product are in U.

Thus U is a subspace of $\mathbf{R}^3$. #

Geometrically, *U is the set of vectors that lie on the x-axis.*

Example 2

Let V be the set of vectors of of $\mathbf{R}^3$ of the form (a, a^2, b), namely $V = \{(a, a^2, b) \in \mathbf{R}^3 \}$.

Show that V is not a subspace of $\mathbf{R}^3$.

Solution

Let $(a, a^2, b), (c, c^2, d) \in V$.

$$(a, a^2, b) + (c, c^2, d) = (a+c,\ a^2 + c^2,\ b + d)$$
$$\neq (a + c,\ (a + c)^2,\ b + d),$$

since $a^2 + c^2 \neq (a + c)^2$.

Thus $(a, a^2, b) + (c, c^2, d) \notin V$.

V is <u>not closed</u> under addition.

V is <u>not a subspace</u>.

Example 3

Prove that the set W of 2×2 diagonal matrices is a subspace of the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

(+) Let $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} \in W.$

We get $\mathbf{u} + \mathbf{v} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} p & 0 \\ 0 & q \end{bmatrix} = \begin{bmatrix} a+p & 0 \\ 0 & b+q \end{bmatrix}$

$\Rightarrow \mathbf{u} + \mathbf{v} \in W.$

$\Rightarrow$ W is *closed under addition*.

($\cdot$) Let $c \in \mathbf{R}$. We get $c\mathbf{u} = c\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} ca & 0 \\ 0 & cb \end{bmatrix}$

$\Rightarrow c\mathbf{u} \in W.$

$\Rightarrow$ W is *closed under scalar multiplication*.

$\Rightarrow$ W is a subspace of $\boldsymbol{M}_{22}$.

The vector space of polynomials (P_n)

Example 4. ***Let P_n denoted the set of real polynomial functions of degree $\leq n$. Prove that P_n is a vector space if addition and scalar multiplication are defined on polynomials in a pointwise manner.***

Solution

Let f and $g \in \boldsymbol{P_n}$, where

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \text{ and}$$

$$g(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$$

►(+) $(f+g)(x)$

$$= f(x) + g(x)$$

$$= [a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0] + [b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0]$$

$$= (a_n + b_n)x^n + (a_{n-1} + b_{n-1})x^{n-1} + \ldots + (a_1 + b_1)x + (a_0 + b_0)$$

$(f+g)(x)$ is a polynomial of degree $\leq n$.

Thus $f + g \in \boldsymbol{P_n}$.

Then $\boldsymbol{P_n}$ is closed under addition.

▶ $(\cdot)$ Let $c \in \mathbf{R}$, $(cf)(x) = c[f(x)]$

$$= c[a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0]$$

$$= ca_n x^n + ca_{n-1}x^{n-1} + \ldots + ca_1 x + ca_0$$

$(cf)(x)$ is a polynomial of degree $\leq n$.

So $cf \in \boldsymbol{P_n}$.

Then $\boldsymbol{P_n}$ is closed under scalar multiplication.

In conclusion : By (+) and $(\cdot)$, $\boldsymbol{P_n}$ is a subspace of the vector space $\boldsymbol{F}$ of functions.

Therefore $\boldsymbol{P_n}$ is itself a vector space.

Theorem (*Very important condition*)

Let U be a subspace of a vector space V.
U contains the zero vector of V.

Note. Let $\mathbf{0}$ be the zero vector of V.

If $\mathbf{0} \notin U \Rightarrow U$ is not a subspace of V.

If $\mathbf{0} \in U \Rightarrow (+)(\cdot)$ hold $\Rightarrow U$ is a subspace of V.

$(+)(\cdot)$ failed $\Rightarrow U$ is not a subspace of V.

Caution. ***This condition is necessary but not sufficient.***

(See, for instance, *Example 2* above and *Example 5* below)

Example 5

Let W be the set of vectors of the form $(a, a, a+2)$.
Show that W is not a subspace of $\mathbf{R}^3$.

Solution

If $(a, a, a+2) = (0, 0, 0)$, then $a = 0$ and $a + 2 = 0$.

This system is inconsistent it has no solution.

Thus $(0, 0, 0) \notin W$. (The necessary condition does not hold)

$\Rightarrow$ W is not a subspace of $\mathbf{R}^3$.

Homework

Let $\boldsymbol{F} = \{ f \mid f\colon \mathbf{R} \to \mathbf{R} \}$ the vector space of functions on $\mathbf{R}$.

Which of the following are subspaces of $\boldsymbol{F}$?

(a) $\boldsymbol{W}_1 = \{ f \mid f\colon \mathbf{R} \to \mathbf{R},\ f(0)=0 \}$.

(b) $\boldsymbol{W}_2 = \{ f \mid f\colon \mathbf{R} \to \mathbf{R},\ f(0)=3 \}$.

(c) $\boldsymbol{W}_3 = \{ f \mid f\colon \mathbf{R} \to \mathbf{R},$ for some $c \in \mathbf{R},\ f(x)=c$ for every $x \}$.

Linear Combinations of Vectors

$W = \{(a, a, b) \mid a, b \in \mathbf{R}\} \subseteq \mathbf{R}^3$

$(a, a, b) = a\,(1,1,0) + b\,(0,0,1)$

$\therefore W$ is generated by (1,1,0) and (0,0,1).

e.g., $(2, 2, 3) = 2\,(1,1,0) + 3\,(0,0,1)$

$(-1, -1, 7) = -1\,(1,1,0) + 7\,(0,0,1)$.

Definition

Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in a vector space V.

We say that $\mathbf{v}$, a vector of V, is a **linear combination** of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$, if there exist scalars $c_1, c_2, \ldots, c_m$ such that $\mathbf{v}$ can be written $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_m\mathbf{v}_m$.

Example 1

The vector (5, 4, 2) is a linear combination of the vectors (1, 2, 0), (3, 1, 4), and (1, 0, 3), since it can be written

$$(5, 4, 2) = (1, 2, 0) + 2(3, 1, 4) - 2(1, 0, 3)$$

Example 2

Determine whether or not the vector (–1, 1, 5) is a linear combination of the vectors (1, 2, 3), (0, 1, 4), and (2, 3, 6).

Solution

Suppose $c_1(1,2,3)+c_2(0,1,4)+c_3(2,3,6)=(-1,1,5)$

$$(c_1, 2c_1, 3c_1)+(0, c_2, 4c_2)+(2c_3, 3c_3, 6c_3)=(-1,1,5)$$
$$(c_1+2c_3, 2c_1+c_2+3c_3, 3c_1+4c_2+6c_3)=(-1,1,5)$$

$$\Rightarrow \begin{cases} c_1 \quad\quad\;\; +2c_3 = -1 \\ 2c_1 + \;\; c_2 + 3c_3 = \;\; 1 \\ 3c_1 + 4c_2 + 6c_3 = \;\; 5 \end{cases} \Rightarrow c_1 = 1, c_2 = 2, c_3 = -1$$

Thus (–1, 1, 5) is a linear combination of (1, 2, 3), (0, 1, 4), and (2, 3, 6), where $(-1, 1, 5) = (1, 2, 3) + 2(0, 1, 4) - 1(2, 3, 6).$

Example 3

Express the vector (4, 5, 5) as a linear combination of the vectors (1, 2, 3), (–1, 1, 4), and (3, 3, 2).

Solution

Suppose $c_1(1,2,3)+c_2(-1,1,4)+c_3(3,3,2)=(4,5,5)$

$$(c_1,2c_1,3c_1)+(-c_2,c_2,4c_2)+(3c_3,3c_3,2c_3)=(4,5,5)$$
$$(c_1-c_2+3c_3,2c_1+c_2+3c_3,3c_1+4c_2+2c_3)=(4,5,5)$$
$$\Rightarrow\begin{cases} c_1-\ \ c_2+3c_3=4 \\ 2c_1+\ \ c_2+3c_3=5 \\ 3c_1+4c_2+2c_3=5 \end{cases}\Rightarrow c_1=-2r+3, c_2=r-1, c_3=r$$

Thus (4, 5, 5) can be expressed in many ways as a linear combination of (1, 2, 3), (–1, 1, 4), and (3, 3, 2):

$$(4,5,5)=(-2r+3)(1,2,3)+(r-1)(-1,1,4)+r(2,3,6)$$

Example 4

Show that the vector (3, –4, –6) cannot be expressed as a linear combination of the vectors (1, 2, 3), (–1, –1, –2), and (1, 4, 5).

Solution

Suppose

$$c_1(1,2,3)+c_2(-1,\ -1,-2)+c_3(1,4,5)=(3,-4,-6)$$

$$\Rightarrow \quad \begin{cases} c_1 - \ c_2 + \ c_3 = \ 3 \\ 2c_1 - \ c_2 + 4c_3 = -4 \\ 3c_1 - 2c_2 + 5c_3 = -6 \end{cases}$$

This system has no solution.

Thus (3, –4, –6) is not a linear combination of the vectors (1, 2, 3), (–1, –1, –2), and (1, 4, 5).

Example 5

Determine whether the matrix $\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$ is a linear combination of the matrices $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$, and $\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$ in the vector space $\boldsymbol{M}_{22}$ of 2×2 matrices.

Solution

Suppose
$$c_1\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} + c_2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + c_3\begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

Then

$$\begin{bmatrix} c_1 + 2c_2 & -3c_2 + c_3 \\ 2c_1 + 2c_3 & c_1 + 2c_2 \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix}$$

$$\begin{cases} c_1 + 2c_2 = -1 \\ -3c_2 + c_3 = 7 \\ 2c_1 + 2c_3 = 8 \\ c_1 + 2c_2 = -1 \end{cases}$$

This system has the unique solution $c_1 = 3$, $c_2 = -2$, $c_3 = 1$. Therefore

$$\begin{bmatrix} -1 & 7 \\ 8 & -1 \end{bmatrix} = 3\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} - 2\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Example 6

Determine whether the function $f(x) = x^2 + 10x - 7$ is a linear combination of the functions $g(x) = x^2 + 3x - 1$ and $h(x) = 2x^2 - x + 4.$

Solution

Suppose $c_1 g + c_2 h = f.$

Then

$$c_1(x^2 + 3x - 1) + c_2(2x^2 - x + 4) = x^2 + 10x - 7$$

$$(c_1 + 2c_2)x^2 + (3c_1 - c_2)x - c_1 + 4c_2 = x^2 + 10x - 7$$

$$\Rightarrow \begin{cases} c_1 + 2c_2 = 1 \\ 3c_1 - c_2 = 10 \\ -c_1 + 4c_2 = -7 \end{cases} \quad \Rightarrow c_1 = 3,\ c_2 = -1 \quad \Rightarrow f = 3g - h.$$

Spanning Sets

Definition

The vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are said to **span** a vector space if every vector in the space can be expressed as a *linear combination* of these vectors.

In this case $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m\}$ is called a **spanning set**.

Example 7

Show that the vectors (1, 2, 0), (0, 1, –1), and (1, 1, 2) span $\mathbf{R}^3$.

Solution

Let (x, y, z) be an arbitrary element of $\mathbf{R}^3$.

Suppose $$(x, y, z) = c_1(1, 2, 0) + c_2(0, 1, -1) + c_3(1, 1, 2)$$

$$\Rightarrow \quad (x, y, z) = (c_1 + c_3, 2c_1 + c_2 + c_3, -c_2 + 2c_3)$$

$$\Rightarrow \begin{cases} c_1 \phantom{{}+c_2} + c_3 = x \\ 2c_1 + c_2 + c_3 = y \\ -c_2 + 2c_3 = z \end{cases} \Rightarrow \begin{cases} c_1 = 3x - y - z \\ c_2 = -4x + 2y + z \\ c_3 = -2x + y + z \end{cases}$$

$$\Rightarrow (x, y, z) = (3x - y - z)(1,2,0) + (-4x + 2y + z)(0, 1, -1) + (-2x + y + z)(1, 1, 2)$$

$\Rightarrow$The vectors (1, 2, 0), (0, 1, –1), and (1, 1, 2) span $\mathbf{R}^3$.

Example 8

Show that the following matrices span the vector space M_{22} of 2×2 matrices.

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution

Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_{22}$ (an arbitrary element).

We can express this matrix as follows:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

proving the result.

Theorem

Let $\mathbf{v}_1, \ldots, \mathbf{v}_m$ be vectors in a vector space V. Let U be the set consisting of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.
Then U is a subspace of V spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_m$.
U is said to be the vector space **generated** by $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Proof

(+) Let $\mathbf{u}_1 = a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m$ and $\mathbf{u}_2 = b_1\mathbf{v}_1 + \ldots + b_m\mathbf{v}_m \in U$.

Then $\mathbf{u}_1 + \mathbf{u}_2 = (a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m) + (b_1\mathbf{v}_1 + \ldots + b_m\mathbf{v}_m)$
$= (a_1 + b_1)\,\mathbf{v}_1 + \ldots + (a_m + b_m)\,\mathbf{v}_m$

$\Rightarrow \mathbf{u}_1 + \mathbf{u}_2$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

$\Rightarrow \mathbf{u}_1 + \mathbf{u}_2 \in U$.

$\Rightarrow U$ is closed under vector addition.

(·)Let $c \in \mathbf{R}$. Then

$$c\mathbf{u}_1 = c(a_1\mathbf{v}_1 + \ldots + a_m\mathbf{v}_m)$$
$$= ca_1\mathbf{v}_1 + \ldots + ca_m\mathbf{v}_m$$

$\Rightarrow c\mathbf{u}_1$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

$\Rightarrow c\mathbf{u}_1 \in U$.

$\Rightarrow U$ is closed under scalar multiplication.

Thus U is a subspace of V.

By the definition of U, every vector in U can be written as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_m$.

Thus $\mathbf{v}_1, \ldots, \mathbf{v}_m$ span U.

Example 9

Consider the vector space $\mathbf{R}^3$.

The vectors (–1, 5, 3) and (2, –3, 4) are in $\mathbf{R}^3$.

Let U be the subset of $\mathbf{R}^3$ consisting of all vectors of the form

$$c_1(-1, 5, 3) + c_2(2, -3, 4)$$

Then U is a subspace of $\mathbf{R}^3$ spanned by (–1, 5, 3) and (2, –3, 4).

The following are examples of some of the vectors in U, obtained by given c_1 and c_2 various values.

$$c_1 = 1, c_2 = 0; \text{vector } (-1, 5, 3)$$
$$c_1 = 0, c_2 = 1; \text{vector } (2, -3, 4)$$
$$c_1 = 0, c_2 = 0; \text{vector } (0, 0, 0)$$
$$c_1 = 2, c_2 = 3; \text{vector } (4, 1, 18)$$

We can visualize U. U is made up of all vectors in the plane defined by the vectors (–1, 5, 3) and (2, –3, 4).

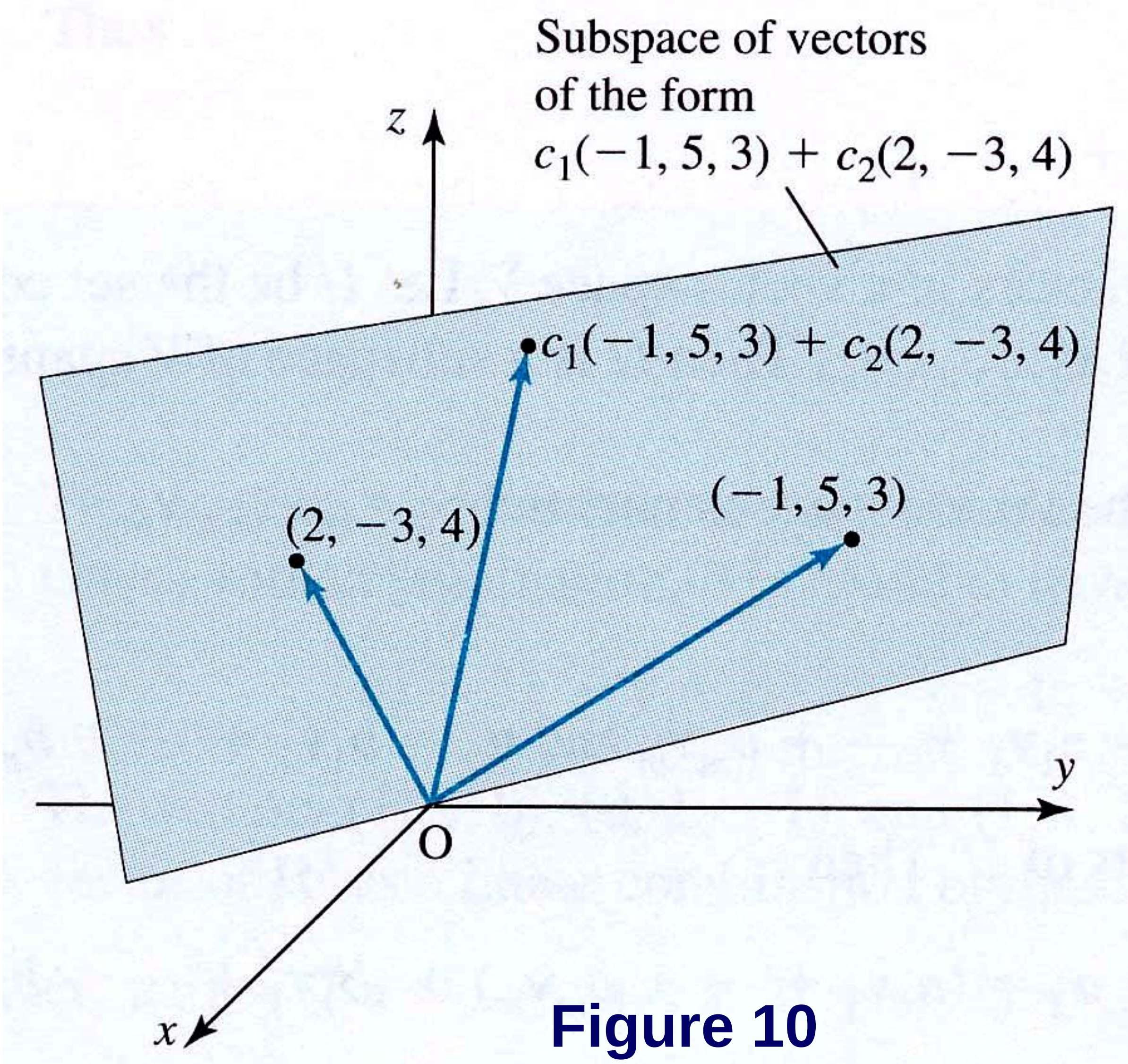

Subspace of vectors
of the form
$c_1(-1, 5, 3) + c_2(2, -3, 4)$
z
$c_1(-1, 5, 3) + c_2(2, -3, 4)$
(−1, 5, 3)
(2, −3, 4)
y
O
x

Figure 10

We can generalize this result. Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be vectors in the space $\mathbf{R}^3$.

The subspace U generated by $\mathbf{v}_1$ and $\mathbf{v}_2$ is the set of all vectors of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$.

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are not colinear, then U is the plane defined by $\mathbf{v}_1$ and $\mathbf{v}_2$.

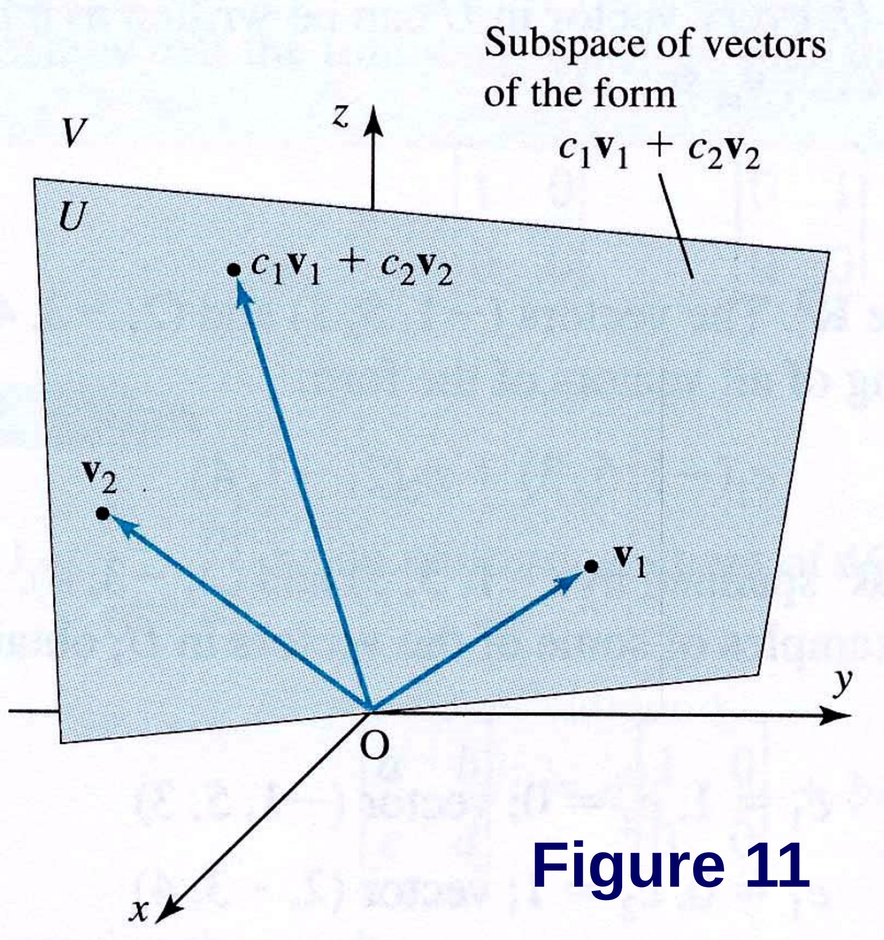


Figure 11

If $\mathbf{v}_1$ and $\mathbf{v}_2$ are vectors in $\mathbf{R}^3$ that are not colinear, then we can visualize U as a plane in three dimensions.

$k_1\mathbf{v}_1$ and $k_2\mathbf{v}_2$ will be vectors on the same lines as $\mathbf{v}_1$ and $\mathbf{v}_2$.

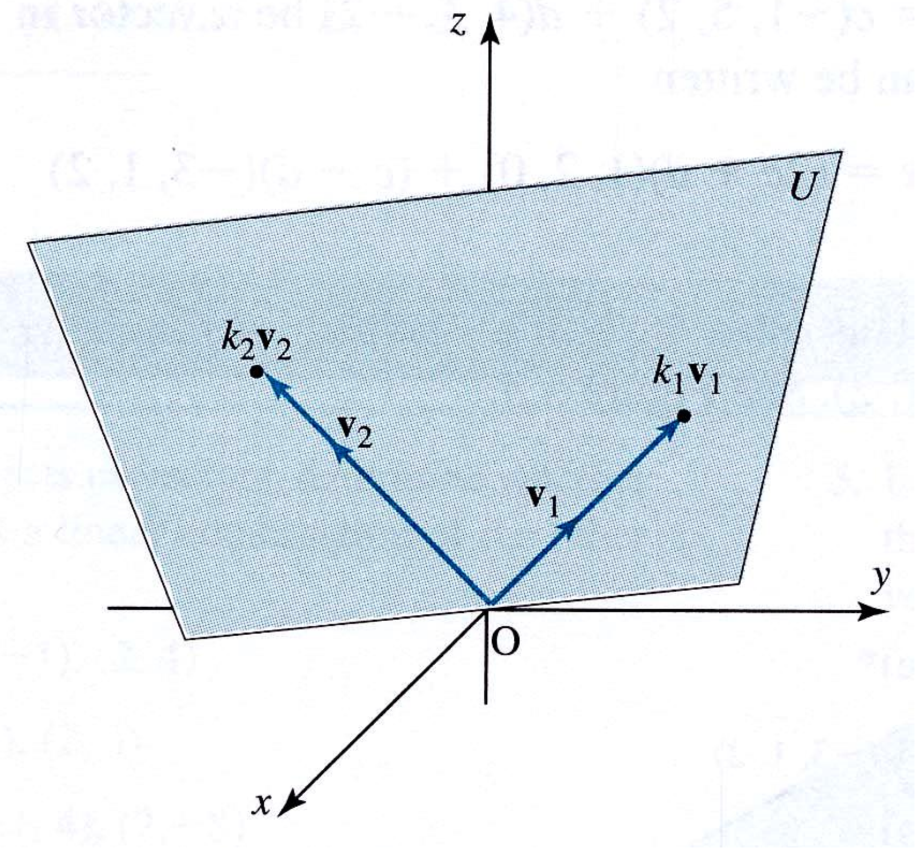


Figure 12

Example 10

Let U be the subspace of $\mathbf{R}^3$ generated by the vectors (1, 2, 0) and (–3, 1, 2). Let V be the subspace of $\mathbf{R}^3$ generated by the vectors (–1, 5, 2) and (4, 1, –2). Show that $U = V$.

Solution

($U \subseteq V$) Let $\mathbf{u}$ be a vector in U. Let us show that $\mathbf{u}$ is in V.

Since $\mathbf{u}$ is in U, there exist scalars a and b such that

$$\mathbf{u} = a(1, 2, 0) + b(-3, 1, 2) = (a - 3b, 2a + b, 2b)$$

Let us see if we can write $\mathbf{u}$ as a linear combination of (–1, 5, 2) and (4, 1, –2).

$$\mathbf{u} = p(-1, 5, 2) + q(4, 1, -2) = (-p + 4q, 5p + q, 2p - 2q)$$

Such p and q would have to satisfy

$$\begin{aligned} -p + 4q &= a - 3b \\ 5p + q &= 2a + b \\ 2p - 2q &= 2b \end{aligned}$$

This system of equations has unique solution $p = \dfrac{a+b}{3}, q = \dfrac{a-2b}{3}$.
Thus **u** can be written

$$\mathbf{u} = \frac{a+b}{3}(-1, 5, 2) + \frac{a-2b}{3}(4, 1, -2)$$

Therefore, **u** is a vector in *V*.

$(V \subseteq U)$ Let **v** be a vector in *V*. Let us show that **v** is in *U*.
Since **v** is in *V*, there exist scalars *c* and *d* such that

$$\mathbf{v} = c(-1, 5, 2) + d(4, 1, -2)$$

It can be shown that

$$\mathbf{v} = (2c+d)(1, 2, 0) + (c-d)(-3, 1, 2)$$

Therefore, **v** is a vector in *U* and hence *U*=*V*.

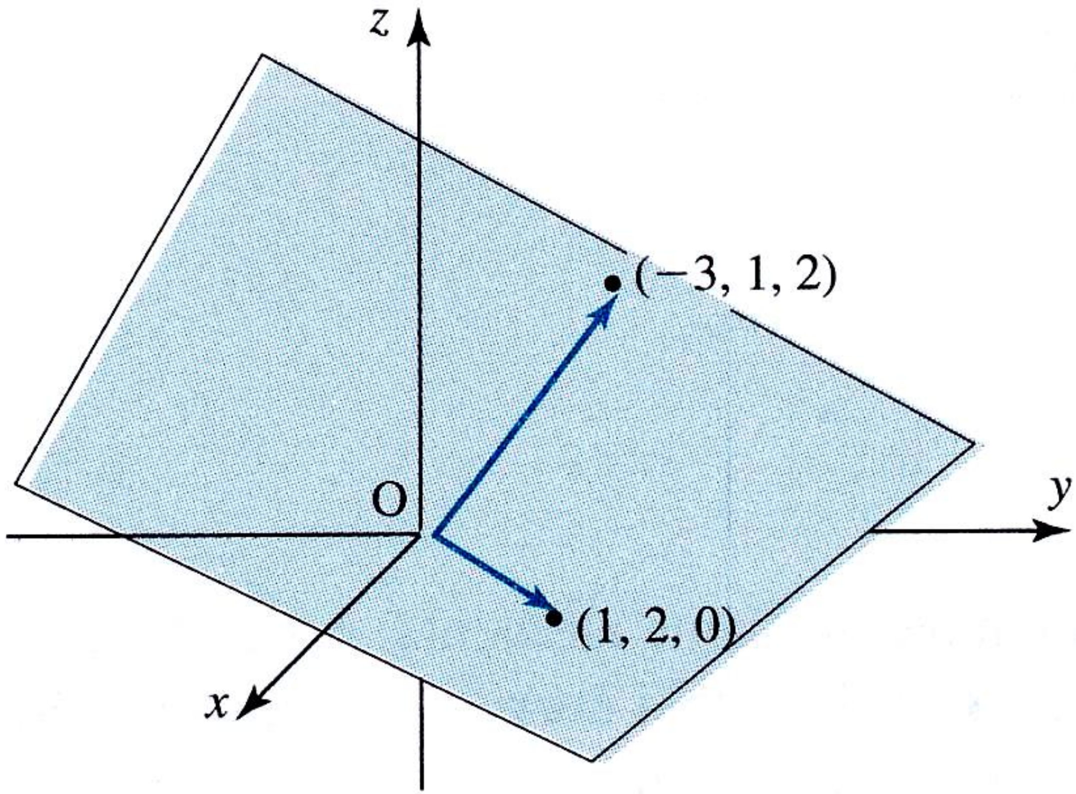


Figure 13

Example 11

Let U be the vector space generated by the functions $f(x) = x + 1$ and $g(x) = 2x^2 - 2x + 3$. Show that the function $h(x) = 6x^2 - 10x + 5$ lies in U.

Solution

h will be in the space generated by f and g if there exist scalars a and b such that

$$a(x + 1) + b(2x^2 - 2x + 3) = 6x^2 - 10x + 5$$

This given

$$2bx^2 + (a - 2b)x + a + 3b = 6x^2 - 10x + 5$$

$$\Rightarrow \quad \begin{aligned} 2b &= 6 \\ a - 2b &= -10 \\ a + 3b &= 5 \end{aligned}$$

This system has the unique solution $a = -4$, $b = 3$.

Thus $\quad -4(x + 1) + 3(2x^2 - 2x + 3) = 6x^2 - 10x + 5$

Linear Dependence and Independence

The concepts of dependence and independence of vectors are useful tools in constructing "efficient" spanning sets for vector spaces – sets in which there are no redundant vectors.

Definition

(a) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ in a vector space V is said to be **linearly dependent** if there exist scalars $c_1, \ldots, c_m$, not all zero, such that $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$

(b) The set of vectors $\{ \mathbf{v}_1, \ldots, \mathbf{v}_m \}$ is **linearly independent** if $c_1\mathbf{v}_1 + \ldots + c_m\mathbf{v}_m = 0$ can only be satisfied when $c_1 = 0, \ldots, c_m = 0$.

Example 1

Show that the set {(1, 2, 3), (–2, 1, 1), (8, 6, 10)} is linearly dependent in $\mathbf{R}^3$.

Solution

Suppose $c_1(1,2,3)+c_2(-2,1,1)+c_3(8,6,10)=\mathbf{0}$

$$\Rightarrow \quad (c_1, 2c_1, 3c_1)+(-2c_2, c_2, c_2)+(8c_3, 6c_3, 10c_3)=\mathbf{0}$$

$$(c_1-2c_2+8c_3,\ 2c_1+c_2+6c_3,\ 3c_1+c_2+10c_3)=\mathbf{0}$$

$$\Rightarrow \begin{cases} c_1-2c_2+8c_3=0 \\ 2c_1+c_2+6c_3=0 \\ 3c_1+c_2+10c_3=0 \end{cases} \Rightarrow \quad \begin{aligned} c_1 &= 4 \\ c_2 &= -2 \\ c_3 &= -1 \end{aligned}$$

Thus $\quad 4(1,2,3)-2(-2,1,1)-(8,6,10)=\mathbf{0}$

The set of vectors is linearly dependent.

Example 2

Show that the set {(3, –2, 2), (3, –1, 4), (1, 0, 5)} is linearly independent in $\mathbf{R}^3$.

Solution

Suppose $c_1(3,-2,2)+c_2(3,-1,4)+c_3(1,0,5)=\mathbf{0}$

$$\Rightarrow \ (3c_1,-2c_1,2c_1)+(3c_2,-c_2,4c_2)+(c_3,0,5c_3)=\mathbf{0}$$

$$(3c_1+3c_2+c_3,-2c_1-c_2,2c_1+4c_2+5c_3)=\mathbf{0}$$

$$\Rightarrow \begin{cases} 3c_1+3c_2+c_3=0 \\ -2c_1-c_2=0 \\ 2c_1+4c_2+5c_3=0 \end{cases}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, and $c_3 = 0$. Thus the set is linearly independent.

Example 3

Consider the functions $f(x) = x^2 + 1$, $g(x) = 3x - 1$, $h(x) = -4x + 1$ of the vector space P_2 of polynomials of degree ≤ 2.
Show that the set of functions $\{ f, g, h \}$ is linearly independent.

Solution

Suppose

$$c_1 f + c_2 g + c_3 h = \mathbf{0}$$

Since for any real number x,

$$c_1(x^2+1)+c_2(3x-1)+c_3(-4x+1)=\mathbf{0}$$

Consider three convenient values of x. We get

$$\begin{aligned}
x &= \ \ 0: c_1 - c_2 + c_3 = 0 \\
x &= \ \ 1: 2c_1 + 2c_2 - 3c_3 = 0 \\
x &= -1: 2c_1 - 4c_2 + 5c_3 = 0
\end{aligned}$$

It can be shown that this system of three equations has the unique solution

$$c_1 = 0, c_2 = 0, c_3 = 0$$

Thus $c_1 f + c_2 g + c_3 h = \mathbf{0}$ implies that $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.
The set $\{ f, g, h \}$ is linearly independent.

Theorem

A set consisting of two or more vectors in a vector space is linearly dependent *if and only if* it is possible to express one of the vectors as a linearly combination of the other vectors.

Example 4

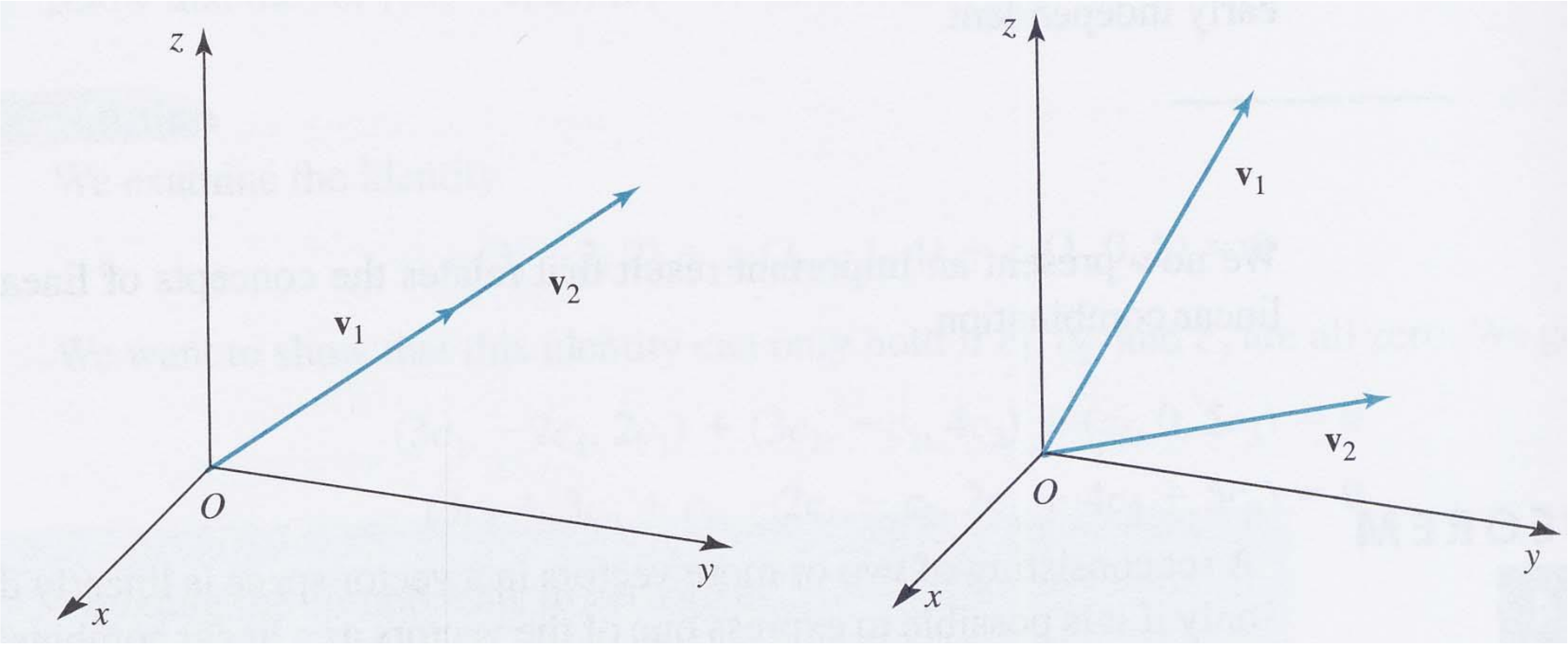
The set of vectors {$\mathbf{v}_1$=(1, 2, 1) , $\mathbf{v}_2$ =(-1, -1, 0) , $\mathbf{v}_3$ = (0, 1,1)}

is linearly dependent, since $\mathbf{v}_3 = \mathbf{v}_1 + \mathbf{v}_2$.

Thus, $\mathbf{v}_3$ is a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

m

Linear Dependence of $\{v_1, v_2\}$

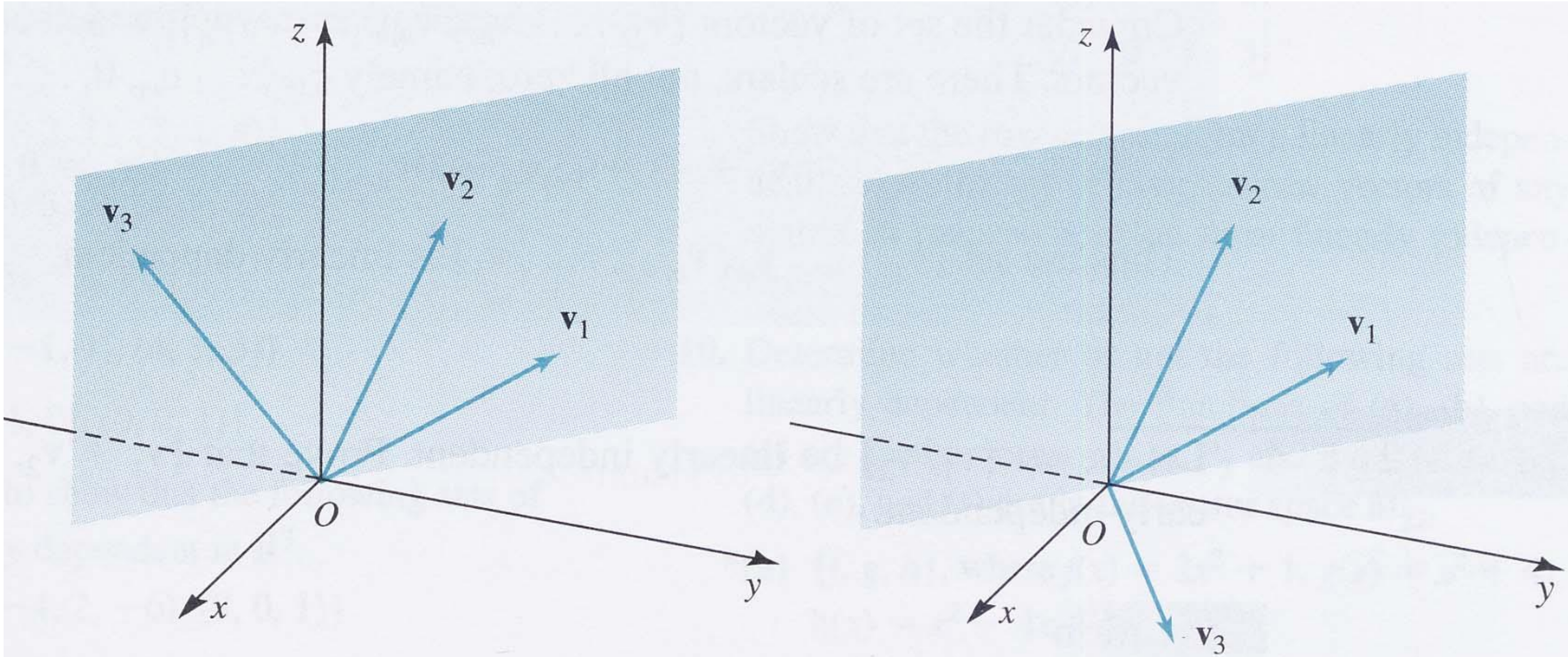


$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly dependent; vectors lie on a line

$\{\mathbf{v}_1, \mathbf{v}_2\}$ linearly independent; vectors do not lie on a line

Figure 14 Linear dependence and independence of $\{\mathbf{v}_1, \mathbf{v}_2\}$ in $\mathbf{R}^3$.

Linear Dependence of $\{v_1, v_2, v_3\}$



$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly dependent; vectors lie in a plane

$\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ linearly independent; vectors do not lie in a plane

Figure 15 Linear dependence and independence of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ in $\mathbf{R}^3$.

Theorem

Let V be a vector space. Any set of vectors in V that contains the zero is linearly dependent.

Proof

Consider the set { $\mathbf{0}$, $\mathbf{v}_2$, …, $\mathbf{v}_m$ }, which contains the zero vectors. Let us examine the identity

$$c_1\mathbf{0}+c_2\mathbf{v}_2+\cdots+c_m\mathbf{v}_m=\mathbf{0}$$

We see that the identity is true for $c_1 = 1$, $c_2 = 0$, …, $c_m = 0$ (not all zero).

Thus the set of vectors is linearly dependent, proving the theorem.

Theorem

Let the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be linearly dependent in a vector space V. Any set of vectors in V that contains these vectors will also be linearly dependent.

Example 5

The set of vectors

$\{\mathbf{v}_1=(1, 2, 1)\ , \mathbf{v}_2 =(-1, -1, 0)\ , \mathbf{v}_3 = (0, 1,1)\ , \mathbf{v}_4 =(1,\ 1, 1)\ \}$

is linearly dependent, since it contains the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ which are linearly dependent.

Definition

A finite set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is called a **basis** for a vector space V if the set spans V and is linearly independent.

Standard Basis

The set of n vectors

$$\{(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, \ldots, 1)\}$$

is a basis for $\mathbf{R}^n$. This basis is called the **standard basis** for $\mathbf{R}^n$.

How to prove it?

Example 1

Show that the set {(1, 0, –1), (1, 1, 1), (1, 2, 4)} is a basis for $\mathbf{R}^3$.

Solution

(span)

Let (x_1, x_2, x_3) be an arbitrary element of $\mathbf{R}^3$.

Suppose

$$(x_1, x_2, x_3) = a_1(1, 0, -1) + a_2(1, 1, 1) + a_3(1, 2, 4)$$

$$\Rightarrow \begin{cases} a_1 + a_2 + a_3 = x_1 \\ a_2 + 2a_3 = x_2 \\ -a_1 + a_2 + 4a_3 = x_3 \end{cases} \Rightarrow \begin{cases} a_1 = 2x_1 - 3x_2 + x_3 \\ a_2 = -2x_1 + 5x_2 - 2x_3 \\ a_3 = x_1 - 2x_2 + x_3 \end{cases}$$

Thus the set spans the space.

(linearly independent)

Consider the identity

$$b_1(1, 0, -1) + b_2(1, 1, 1) + b_3(1, 2, 4) = (0, 0, 0)$$

The identity leads to the system of equations

$$\begin{aligned} b_1 + b_2 + b_3 &= 0 \\ b_2 + 2b_3 &= 0 \\ -b_1 + b_2 + 4b_3 &= 0 \end{aligned}$$

$\Rightarrow$ $b_1 = 0$, $b_2 = 0$, and $b_3 = 0$ is the unique solution.
Thus the set is linearly independent.

$\Rightarrow$ $\{(1, 0, -1), (1, 1, 1), (1, 2, 4)\}$ spans $\mathbf{R}^3$ and is linearly independent.

$\Rightarrow$ It forms a basis for $\mathbf{R}^3$.

Example 2

Show that { f, g, h }, where $f(x) = x^2 + 1$, $g(x) = 3x - 1$, and $h(x) = -4x + 1$ is a basis for P_2.

Solution

(linearly independent) see Example 3 of the previous section.

(span). Let p be an arbitrary function in P_2.

p is thus a polynomial of the form

$$p(x) = bx^2 + cx + d$$

Suppose $p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$

for some scalars a_1, a_2, a_3.

This gives

$$\begin{aligned} bx^2 + cx + d &= a_1(x^2 + 1) + a_2(3x - 1) + a_3(-4x + 1) \\ &= a_1 x^2 + (3a_2 - 4a_3)x + (a_1 - a_2 + a_3) \end{aligned}$$

$$\Rightarrow \begin{cases} a_1 = b \\ 3a_2 - 4a_3 = c \\ a_1 - a_2 + a_3 = d \end{cases} \Rightarrow \begin{cases} a_1 = b \\ a_2 = 4b - 4d - c \\ a_3 = 3b - 3d - c \end{cases}$$

Thus the polynomial p can be expressed

$$p(x) = a_1 f(x) + a_2 g(x) + a_3 h(x)$$

The functions f, g, and h span P_2.

They form a basis for P_2.

Theorem

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V.
If $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is a set of more than n vectors in V, then this set is linearly dependent.

Proof

Suppose

$$c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m = 0 \qquad (1)$$

We will show that values of $c_1, \ldots, c_m$ are not all zero.

The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis for V. Thus each of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_m$ can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.
Let

$$\begin{aligned} \mathbf{w}_1 &= a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n \\ &\vdots \\ \mathbf{w}_m &= a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n \end{aligned}$$

Substituting for $\mathbf{w}_1, \ldots, \mathbf{w}_m$ into Equation (1) we get

$$c_1(a_{11}\mathbf{v}_1 + a_{12}\mathbf{v}_2 + \cdots + a_{1n}\mathbf{v}_n) + \cdots + c_m(a_{m1}\mathbf{v}_1 + a_{m2}\mathbf{v}_2 + \cdots + a_{mn}\mathbf{v}_n) = \mathbf{0}$$

Rearranging, we get

$$(c_1a_{11} + c_2a_{21} + \cdots + c_ma_{m1})\mathbf{v}_1 + \cdots + (c_1a_{1n} + c_2a_{2n} + \cdots + c_ma_{mn})\mathbf{v}_n = \mathbf{0}$$

Since $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linear independent,

$$\begin{gathered} a_{11}c_1 + a_{21}c_2 + \cdots + a_{m1}c_m = 0 \\ \vdots \\ a_{1n}c_1 + a_{2n}c_2 + \cdots + a_{mn}c_m = 0 \end{gathered}$$

Since $m > n$, there are many solutions in this system.

Thus the set $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ is linearly dependent.

Theorem

Any two bases for a vector space V consist of the same number of vectors.

Proof

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ be two bases for V.

By Theorem 4.10,

$$m \le n \text{ and } n \le m$$

Thus $n = m$.

Definition

If a vector space V has a basis consisting of n vectors, then the **dimension** of V is said to be n. We write **dim(V)** for the dimension of V.

- V is **finite dimensional** if such a finite basis exists.
- V is **infinite dimensional** otherwise.

Example 3

Consider the set {{1, 2, 3), (-2, 4, 1)} of vectors in $\mathbf{R}^3$.
These vectors generate a subspace V of $\mathbf{R}^3$ consisting of all vectors of the form

$$\mathbf{v} = c_1(1, 2, 3) + c_2(-2, 4, 1)$$

The vectors (1, 2, 3) and (-2, 4, 1) span this subspace.

Furthermore, since the second vector is not a scalar multiple of the first vector, the vectors are linearly independent.
Therefore {{1, 2, 3), (-2, 4, 1)} is a basis for V.
Thus dim(V) = 2.
We know that V is, in fact, a plane through the origin.

Theorem

(a) The origin is a subspace of $\mathbf{R}^3$. The dimension of this subspace is defined to be zero.

(b) The one-dimensional subspaces of $\mathbf{R}^3$ are lines through the origin.

(c) The two-dimensional subspaces of $\mathbf{R}^3$ are planes through the origin.

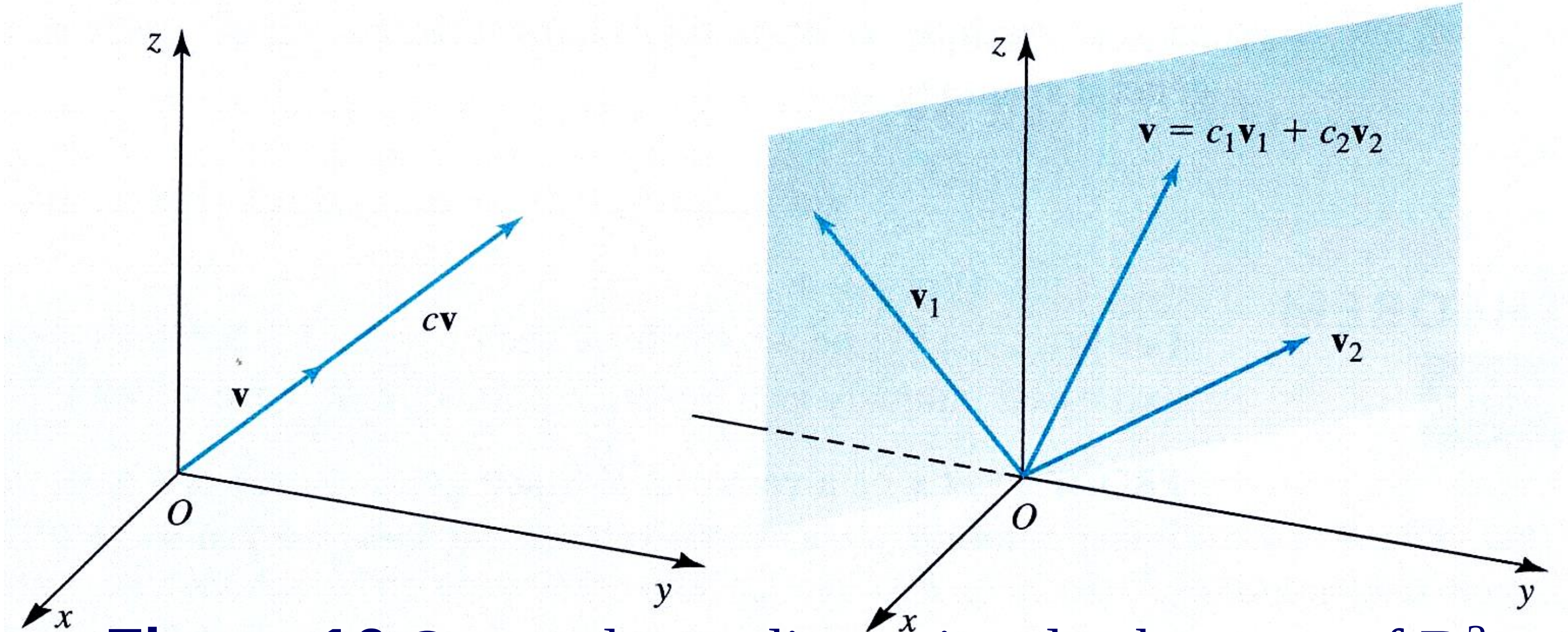


Figure 16 One and two-dimensional subspaces of $\mathbf{R}^3$

Proof

(a) Let V be the set $\{(0, 0, 0)\}$, consisting of a single elements, the zero vector of $\mathbf{R}^3$. Let c be the arbitrary scalar. Since

$$(0, 0, 0) + (0, 0, 0) = (0, 0, 0) \text{ and } c(0, 0, 0) = (0, 0, 0)$$

V is closed under addition and scalar multiplication. It is thus a subspace of $\mathbf{R}^3$. The dimension of this subspaces is defined to be zero.

(b) Let $\mathbf{v}$ be a basis for a one-dimensional subspace V of $\mathbf{R}^3$. Every vector in V is thus of the form $c\mathbf{v}$, for some scalar c. We know that these vectors form a line through the origin.

(c) Let $\{\mathbf{v}_1, \mathbf{v}_2\}$be a basis for a two-dimensional subspace V of $\mathbf{R}^3$. Every vector in V is of the form $c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. V is thus a plane through the origin.

Theorem

Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis for a vector space V. Then each vector in V can be expressed uniquely as a linear combination of these vectors.

Proof

Let $\mathbf{v}$ be a vector in V. Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, we can express $\mathbf{v}$ as a linear combination of these vectors. Suppose we can write

$$\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n \text{ and } \mathbf{v} = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

Then

$$a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n = b_1\mathbf{v}_1 + \cdots + b_n\mathbf{v}_n$$

giving

$$(a_1 - b_1)\mathbf{v}_1 + \cdots + (a_n - b_n)\mathbf{v}_n = \mathbf{0}$$

Since $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Thus $(a_1 - b_1) = 0, \ldots, (a_n - b_n) = 0$, implying that $a_1 = b_1, \ldots, a_n = b_n$. There is thus only one way of expressing $\mathbf{v}$ as a linear combination of the basis.

Theorem

Let V be a vector space of dimension n.

(a) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of n <u>linearly independent</u> vectors in V, then S is a basis for V.

(b) If $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of n vectors V that <u>spans</u> V, then S is a basis for V.

Let V be a vector space, $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a set of vectors in V.

(a) $\dim(V) = |S|$.

(b) S is a linearly independent set.

(c) S spans V.

$\left.\right\}$ S is a basis of V.

Example 4

Prove that the set B={(1, 3, -1), (2, 1, 0), (4, 2, 1)} is a basis for $\mathbf{R}^3$.

Solution

Since dim($\mathbf{R}^3$)=|B|=3. It suffices to show that this set is linearly independent or it spans $\mathbf{R}^3$.

Let us check for linear independence. Suppose

$$c_1(1,3,-1)+c_2(2,1,0)+c_3(4,2,1)=(0,0,0)$$

This identity leads to the system of equations

$$\begin{aligned} c_1+2c_2+4c_3&=0\\ 3c_1+c_2+2c_3&=0\\ -c_1+c_3&=0 \end{aligned}$$

This system has the unique solution $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Thus the vectors are linearly independent.

The set {(1, 3, -1), (2, 1, 0), (4, 2, 1)} is therefore a basis for $\mathbf{R}^3$.

Theorem

Let V be a vector space of dimension n. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ be a set of m linearly independent vectors in V, where $m < n$.

Then there exist vectors $\mathbf{v}_{m+1}, \ldots, \mathbf{v}_n$ such that

$\{\mathbf{v}_1, \ldots, \mathbf{v}_m, \mathbf{v}_{m+1}, \ldots, \mathbf{v}_n\}$ is a basis of V.

Example 5

State (with a brief explanation) whether the following statements are true or false.

(a) The vectors (1, 2), (–1, 3), (5, 2) are linearly dependent in $\mathbf{R}^2$.

(b) The vectors (1, 0, 0), (0, 2, 0), (1, 2, 0) span $\mathbf{R}^3$.

(c) {(1, 0, 2), (0, 1, -3)} is a basis for the subspace of $\mathbf{R}^3$ consisting of vectors of the form $(a, b, 2a - 3b)$.

(d) Any set of two vectors can be used to generate a two-dimensional subspace of $\mathbf{R}^3$.

Solution

(a) True: The dimension of $\mathbf{R}^2$ is two. Thus any three vectors are linearly dependent.

(b) False: The three vectors are linearly dependent. Thus they cannot span a three-dimensional space.

(c) True: The vectors span the subspace since

$$(a, b, 2a - 3b) = a(1, 0, 2) + b(0, 1, -3)$$

The vectors are also linearly independent since they are not colinear.

(d) False: The two vectors must be linearly independent.