Problem Set 1

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Problem Set 1

1.Prove that $lim_{x \to -1} 2x + 1 = -1$.

Note: For any given $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$.

Draft:

$$|2x + 1 - (-1)| < \epsilon$$
$$|2x + 2| < \epsilon$$
$$|2||x - (-1)| < \epsilon$$
$$|x - (-1)| < \frac{\epsilon}{|2|}$$

End of Draft: $\delta \leq \frac{\epsilon}{2}$

Start of Formal Solution:
Given we have $\epsilon > 0$.
Let $\delta \leq \frac{\epsilon}{2}$ shows that when $|x - (-1)| < \delta$, then $|2x + 1 - (-1)| < \epsilon$

$$|x - (-1)| < \delta$$
$$|x - (-1)| < \frac{\epsilon}{2}$$
$$|2||x - (-1)| < \epsilon$$
$$|2x + 2| < \epsilon$$
$$|2x + 1 - (-1)| < \epsilon$$

Therefore, $lim_{x \to -1} 2x + 1 = -1$.

2.Determine all the numbers c which satisfy the conclusions of the Mean Value Theorem for the following function and graph using R with the point/s identified. $f(x) = x^3 - 4x^2 - 2x - 5$ on $[-10, 10]$

Let $f(x)$ be continuous for $a \leq x \leq b$, and let it be differentiable for $a < x < b$. Then there is at least one point ξ in (a, b) for which

$$f(b) - f(a) = f'(\xi)(b - a)$$
$$f\prime(\xi) = \frac{f(b) - f(a)}{b - a}$$

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## -100 - 8*x + 3*x^2
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## [1] -4.592130  7.258796
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Since f(x) is a polynomial, it is both continuous and differentiable, so $f'(x) = 3x^2 - 8x - 2.$ we then replace x with the values provided which was $[-10, 10]$

$$f(x) = x^3 - 4x^2 - 2x - 5$$
$$f(-10) = (-10)^3 - 4(-10)^2 - 2(-10) - 5$$
$$= -1000 - 400 + 20 - 5$$
$$= f(x) = -1385$$

$$f(x) = x^3 - 4x^2 - 2x - 5$$
$$f(10) = 10^3 - 4(10)^2 - 2(10) - 5$$
$$= 1000 - 400 - 20 - 5$$
$$= f(x) = 575$$

we then use this equation for the calculation of the point we need to find

$$f\prime(\xi) = \frac{f(b) - f(a)}{b - a}$$
$$a = -1385, b = 575$$
$$3\xi^2 - 8\xi - 2 = \frac{575 - (-1385)}{10 - (-10)}$$

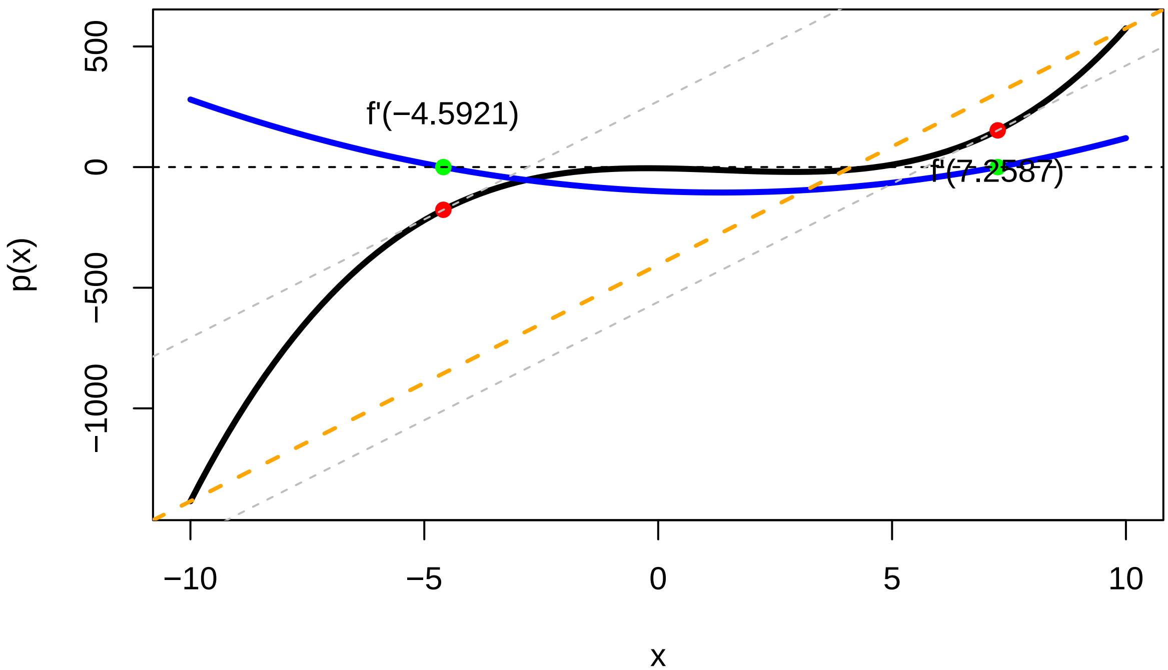
$$3\xi^2 - 8\xi - 2 = 98$$

$$3\xi^2 - 8\xi - 100 = 0$$

we can now solve for the roots of the equation using the following

$$x = \frac{-b \pm \sqrt{b^2 - 4(a)(c)}}{2a}$$
$$x = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(3)(-100)}}{2(3)}$$
$$x = \frac{8 \pm \sqrt{64 - 12(-100)}}{6}$$
$$x = \frac{8 \pm \sqrt{64 + 1200}}{6}$$
$$x = \frac{8 \pm \sqrt{1264}}{6}$$
$$x_0 \approx 7.258796, x_1 \approx -4.592130$$

the solutions on $[-10, 10]$ are $x_0 \approx 7.258796, x_1 \approx -4.592130$



3.Find the point c that satisfies the mean value theorem for integrals on the interval [-1,1]. The function is $f(x) = 3e^2$

Using the Mean Value Theorem

$$MeanValueTheorem = f'(c) = \frac{f(b) - f(a)}{b - a}$$

We then derive:

$$f'(x) = \frac{d}{dx}(2e^x)$$
$$= 2e^x$$
$$f(1) = 2e^1 = 2e$$
$$f(-1) = 2e^{-1}$$

Average rate of change:

$$\frac{f(1) - f(-1)}{1 - (-1)}$$
$$= \frac{2e^1 - 2e^{-1}}{2}$$
$$= e - \frac{1}{e}$$

Finding c:

$$f'(c) = e - \frac{1}{e}$$
$$2e^c = e - \frac{1}{e}$$

$$e^c = \frac{e - \frac{1}{e}}{2}$$

$$c = ln(\frac{e - \frac{1}{e}}{2})$$

$$\therefore c \approx ln(\frac{e - \frac{1}{e}}{2})$$

4.Consider the function $f(x) = cos(\frac{x}{2})$
a. Find the fourth Taylor polynomial for f at $x = \pi$

$$f(x) = \cos(\frac{x}{2}), f(\pi) = 0$$

$$f^1(x) = -\frac{\sin(\frac{x}{2})}{2}, f^1(\pi) = -\frac{1}{2}$$

, which is -0.5

$$f^2(x) = -\frac{\cos(\frac{x}{2})}{4}, f^2(\pi) = 0$$

$$f^3(x) = \frac{\sin(\frac{x}{2})}{8}, f^3(\pi) = -\frac{1}{8}$$

, which is 0.0125

$$f^4(x) = \frac{\cos(\frac{x}{2})}{16}, f^4(\pi) = 0$$

$$f^5(x) = -\frac{\sin(\frac{x}{2})}{32}, f^5(\pi) = -\frac{1}{32}$$

, which is -0.03125 to find the fourth Taylor Polynomial, we use the following equation:

$$p_4 = f(x_0) + \frac{f^1(x_0)}{1!}(x - x_0) + \frac{f^2(x_0)}{2!}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \frac{f^4(x_0)}{4!}(x - x_0)^4$$

$$p_4 = 0 + \frac{-\frac{1}{2}}{1!}(x - \pi) + \frac{0}{2!}(x - \pi)^2 + \frac{\frac{1}{8}}{3!}(x - \pi)^3 + \frac{0}{4!}(x - \pi)^4$$

$$p_4 = \frac{-\frac{1}{2}}{1!}(x - \pi) + \frac{\frac{1}{8}}{3!}(x - \pi)^3$$

b. Use the fourth Taylor polynomial to approximate $cos(\frac{\pi}{2})$.
To find the fourth Taylor Polynomial, we use the following equation:

$$p_4 = f(x_0) + \frac{f^1(x_0)}{1!}(x - x_0) + \frac{f^2(x_0)}{2!}(x - x_0)^2 + \frac{f^3(x_0)}{3!}(x - x_0)^3 + \frac{f^4(x_0)}{4!}(x - x_0)^4$$

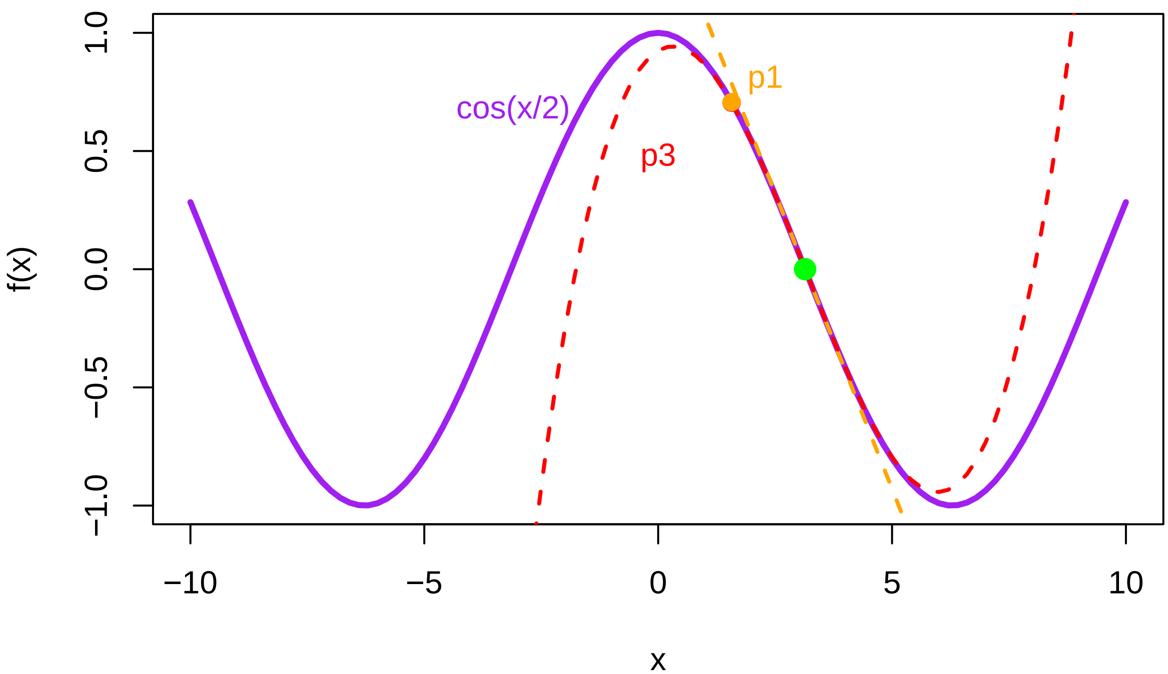
$$p_4 = 0 + \frac{-\frac{1}{2}}{1!}(\frac{\pi}{2} - \pi) + \frac{0}{2!}(\frac{\pi}{2} - \pi)^2 + \frac{\frac{1}{8}}{3!}(\frac{\pi}{2} - \pi)^3 + \frac{0}{4!}(\frac{\pi}{2} - \pi)^4$$

$$p_4 = -\frac{1}{2}(-\frac{\pi}{2}) + \frac{1}{48}(-\frac{\pi}{2})^3$$

$$p_4 = \frac{\pi}{4} - \frac{\pi^3}{324}$$

$$ActualValue = 0.7071068$$

$$EstimatedValue \approx 0.7046527$$



c. Use the fourth Taylor polynomial to bound the error By Taylor's Theorem with remainder, there exists a ε in the interval $(\frac{\pi}{2}, \pi)$ such that the remainder when approximating $cos(\pi/4)$ by the fourth Taylor polynomial satisfies

$$R_4\left(\frac{\pi}{2}\right) = \frac{f^5(\frac{\pi}{2})}{5!}(x-\pi)^5$$

We do not know the exact value of c, we find an upper bound on $R_4\left(\frac{\pi}{2}\right)$ by determining the maximum value of $|f^5(x)|$ on the interval $(\frac{\pi}{2}, \pi)$.

$$-0.02209709, -0.03125000$$

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## [1] -0.02209709 -0.03125000
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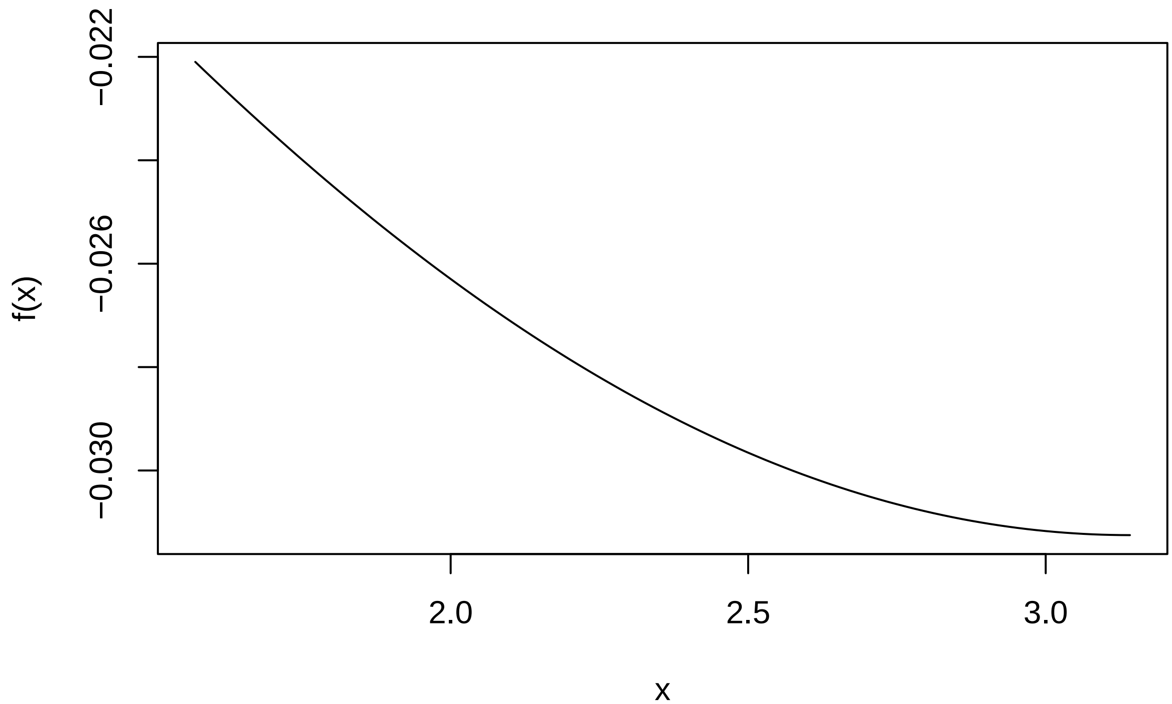

$$R_5\left(\frac{\pi}{2}\right) \leq \frac{-0.03125000}{5!}(\frac{\pi}{2}-\pi)^5 \approx 0.002490395$$

$$cos\left(\frac{\pi}{4}\right) \approx p_4\left(\frac{\pi}{2}\right) + R_4\left(\frac{\pi}{2}\right) = 0.7046527 + 0.002490395$$

$$cos\left(\frac{\pi}{4}\right) \approx 0.7071431$$

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## [1] 0.7071431
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## [1] 3.631381e-05
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5.If fl(x) is the machine approximated number of a real number x and ϵ is the corresponding relative error, then show that $fl(x) = (1-\epsilon)x$

Let:
$fl(x) = ApproximateValue$
$x = RealValue$
$\epsilon = RelativeError$

Relative Error:

$$\epsilon = |\frac{x - fl(x)}{x}|$$

Then we solve for fl(x):

$$\epsilon|x| = |x - fl(x)|$$

$$fl(x) = x - \epsilon|x|$$
$$fl(x) = (1-\epsilon)x$$

6.For the following numbers x and their corresponding approximations x_A, find the number of significant digits in x_A with respect to x and find the relative error.

$$Error = TrueValue - ApproximateValue$$
$$RelativeError = Error/TrueValue$$

a. $x = 451.01, x_A = 451.023$

the real number $x = 451.01$ can be represented as $451.01 = (-1)^0$ x 0.45101 x 10^3

we have $s = 0, \beta = 10^3, e = 4, d_1 = 4, d_2 = 5, d_3 = 1, d_4 = 0, d_5 = 1$

Suppose that the true value $x = 451.01$ and the approximate value $x_A = 451.023$. Then the Error(x_A) and the Relative Error (x_A):

Error $=$ 451.01 – 451.023 $=$ –0.013

Relative Error $=$ $\frac{451.01-451.023}{451.01}$ $=$ –0.00002882419

If x_A is an approximation to x, then we say that x_A approximates x to r significant β-digits if

$$|x - x_A| \leq \frac{1}{2}\beta^{s-r+1}$$

where

$$\beta^s \leq |x|$$

Since $10^2 < 451.01 = x$, $s = 2$ and to determine r, we get

$$|x - x_A| = 0.013 \leq \frac{1}{2} \times 10^{2-r+1} = \frac{1}{2} \times 10^{-1} = 0.05$$
$$2 - r + 1 = -1$$
$$r = 4$$
$$|451.01 - 451.023| \leq \frac{1}{2}10^{2-4+1}$$
$$0.013 \leq \frac{1}{2}0.1$$
$$0.013 \leq 0.05$$

which is true, so the approximate number has 4 significant digits and the relative error is -0.00002882419

b. $x = -0.04518, x_A = -0.045113$

Suppose that the true value $x = -0.04518$ and the approximate value $x_A = -0.045113$. Then the Error(x_A) and the Relative Error (x_A):

Error $=$ –0.04518 – (–0.045113) $=$ 0.000067

Relative Error $=$ $\frac{-0.04518-(-0.045113)}{-0.04518}$ $=$ 0.001482957

If x_A is an approximation to x, then we say that x_A approximates x to r significant β-digits if

$$|x - x_A| \leq \frac{1}{2}\beta^{s-r+1}$$

where

$$\beta^s \le |x|$$

Since $10^{-2} < 0.04518 = x$, $s = -2$ and to determine r, we get

$$|x - x_A| = 0.000067 \le \frac{1}{2} \times 10^{-2-r+1} = \frac{1}{2} \times 10^{-3} = 0.0005$$

$$-2 - r + 1 = -3$$

$$r = 2$$

$$|-0.04518 - (-0.045113)| \le \frac{1}{2} 10^{-2-2+1}$$

$$0.000067 \le \frac{1}{2} 10$$

$$0.000067 \le 5$$

which is true, so the approximate number has 2 significant digits and the relative error is 0.001482957

c. $x = 23.4604, x_A = 23.4213$

Suppose that the true value $x = 23.4604$ and the approximate value $x_A = 23.4213$. Then the Error(x_A) and the Relative Error (x_A):

Error $=$ 23.4604 – 23.4213 $=$ 0.0391

Relative Error $= \frac{23.4604 - 23.4213}{23.4604} = 0.001666638$

If x_A is an approximation to x, then we say that x_A approximates x to r significant β-digits if

$$|x - x_A| \le \frac{1}{2} \beta^{s-r+1}$$

where

$$\beta^s \le |x|$$

Since $10^1 < 23.4604 = x$, $s = 1$ and to determine r, we get

$$|x - x_A| = 0.0391 \le \frac{1}{2} \times 10^{1-r+1} = \frac{1}{2} \times 10^{-1} = 0.05$$

$$1 - r + 1 = -1$$

$$r = 2$$

$$|23.4604 - 23.4213| \le \frac{1}{2} 10^{1-2+1}$$

$$0.0391 \le \frac{1}{2} 1$$

$$0.0391 \le 0.5$$

which is true, so the approximate number has 2 significant digits and the relative error is 0.001666638

7.Find the condition number for the following functions
a.$f(x) = 2x^2$
The derivative of the equation is $f'(x) = 4x$

$$CN = \left| \frac{f'(x)}{f(x)} x \right|$$

$$= \left| \frac{4x}{2x^2} x \right|$$

$$= \frac{4}{2}$$
$$CN = 2$$

b.$f(x) = 2\pi^x$
The derivative of the equation is $f'(x) = 2\pi^x * ln\pi$

$$CN = \left| \frac{f'(x)}{f(x)} x \right|$$

$$= \left| \frac{2\pi^x ln\pi}{2\pi^x} x \right|$$
$$CN = |x| ln\pi$$

c.$f(x) = 2b^x$
The derivative of the equation is $f'(x) = 2b^x lnb$

$$CN = \left| \frac{f'(x)}{f(x)} x \right|$$

$$= \left| \frac{2b^x lnb}{2b^x} x \right|$$
$$CN = |x| lnb$$

8.Determine if the following series converges or diverges. If it converges determine its sum.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Listing the sequence gives us

$$n = 1, 2, 3, ...$$
$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, ..., \frac{1}{2^n}$$

We can see that the common ratio is $r = \frac{1}{2}$ Therefore:

$$S_n = \frac{a_1(1 - r^n)}{1 - r}$$

$$= \frac{\frac{1}{2}(1 - \frac{1}{2}^n)}{1 - \frac{1}{2}}$$

$$S_n = 1 - \frac{1^n}{2}$$

Now, the summation of the series 1 starting at 1 as n $\rightarrow \infty$ is equal to the limit of S as n $\rightarrow \infty$.

$$S_n = \frac{\frac{1}{2}(1)}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = \frac{\frac{1}{2}}{\frac{1}{2}} = 1$$

$$lim_{n \rightarrow \infty} S_n = 1$$

So, the series is convergent and its sum is 1.