Problem Set 3

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1.If $e^{1.3}$ is approximated by Lagrangian interpolation from the values for $e^0 = 1$, $e^1 = 2.7183$, and $e^2 = 7.3891$ what are the minimum and maximum estimates for the error? Compare to the actual error.

For n = 2

$$p_2(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} y_0 + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} y_1 + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} y_2$$
$$p_2(x) = \frac{(x-1)(x-2)}{(0-1)(0-2)} 1 + \frac{(x-0)(x-2)}{(1-0)(1-2)} 2.7183 + \frac{(x-0)(x-1)}{(2-0)(2-1)} 7.3891$$
$$p_2(x) = \frac{x^2 - 3x + 2}{(-1)(-2)} 1 + \frac{x^2 - 2x}{(1)(-1)} 2.7183 + \frac{x^2 - x}{(2)(1)} 7.3891$$
$$p_2(x) = -2.2183x^2 + 3.9366x + 1 + 3.69455x^2 - 3.69455x$$
$$p_2(x) = 1.47625x^2 + 0.24205x + 1$$

The estimated value at x = 1.3 gives:

$$p_2(1.3) = 1.47625(1.3)^2 + 0.24205(1.3) + 1 p_2(1.3) = 3.809502$$

The actual value at x = 1.3 gives:

$$f(1.3) = 3.669297$$

Finding error:

$$E_n(x, f) = |f(x) - p_2(x)| \, E_n(x, f) = |3.669297 - 3.809502|$$
$$E_n(x, f) = 0.1402057$$

The actual value is 3.669297 The estimated value is 3.809502. Therefore, the error is 0.1402057.

```
l0 = function(x) {((x-x1) * (x-x2)) / ((x0-x1) * (x0-x2))}
l1 = function(x) {(x * (x - x2)) / (x1 * (x1 - x2))}
l2 = function(x) {(x * (x - x1)) / (x2 * (x2 - x1))}
f = function(x) { exp(x) }
p = function(x) {f(x0) * l0(x) + f(x1) * l1(x) + f(x2) * l2(x)}

p(x)
```

```
## [1] 3.809502
```

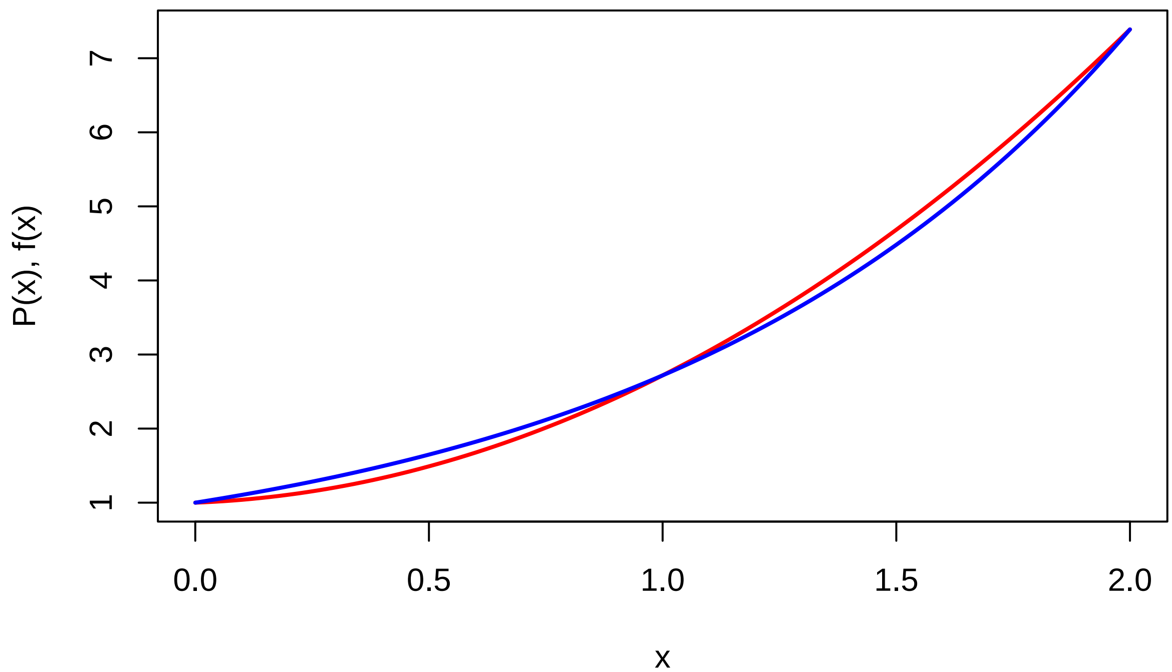
```
f(x)
```

```
## [1] 3.669297
```

```
actual_error = abs(f(x)-p(x))
actual_error
```

```
## [1] 0.1402057
```

```
curve(p, 0, 2, col = "red", lwd = 2, ylab = "P(x), f(x)")
curve(f, 0, 2, add = T, col = "blue", lwd = 2)
```




```
## function (x)
## exp(x)
```

```
## function (x)
## (2 * x - 1) * (x - 2) + x * (x - 1)
```

Now that we have obtained 3rd derivative of f(x) and 1st derivative of g(x), we can now begin to obtain the roots of the 1st derivative of g(x)

Let $g(x) = x(x-1)(x-2)$, then $g\prime(x) = 3x^2 - 6x + 2$. The roots of such equation are

$$x = 1.5773502269, 0.4226497308$$

```
f = function(x) {x*(x-1)*(x-2)}
f(0.4226497308)
```

```
## [1] 0.3849002
```

```r
min_error = abs(exp(0) / 6 * f(0.4226497308))
min_error
```

```
## [1] 0.06415003
```

```r
max_error = abs(exp(2) / 6 * f(0.4226497308))
max_error
```

```
## [1] 0.4740082
```

Therefore the minimum and maximum error would be 0.06415003 and 0.4740082, respectively. The actual value is 3.669297. The estimated value is 3.809502. Therefore, the error we have obtained is 0.1402057. Comparing this value to the min and max error, it is relatively close.

2.Construct the divided difference table from these data

x	0.2	0.3	0.7	-0.3	0.1
$f(x)$	1.23	2.34	-1.05	6.51	-0.06

Use the divided-difference to interpolate for $f(0.4)$

$$f[x_0,x_1]=\frac{2.34-1.23}{0.3-(0.2)}=\frac{1.11}{0.5}=2.22$$

$$f[x_0,x_1,x_2]=\frac{\frac{-1.05-2.34}{0.7-0.3}-2.22}{0.7-(-0.2)}=\frac{-10.695}{0.9}=\frac{-713}{60}$$

$$f[x_0,x_1,x_2,x_3]=\frac{\frac{6.51-(-1.05)}{-0.3-0.7}-(\frac{-713}{60})}{0.3-(-0.2)}=\frac{1297}{150}$$

$$f[x_0,x_1,x_2,x_3,x_4]=\frac{\frac{-0.06-(6.51)}{-0.1-(-0.3)}-(\frac{1297}{150})}{0.1-(-0.2)}=\frac{-15853}{180}$$

$$\begin{aligned}P_4(x)=&1.23+(x+0.2)(2.22)\\&+(x+0.2)(x-0.3)\left(-\frac{713}{60}\right)\\&+(x+0.2)(x-0.3)(x-0.7)\left(\frac{1297}{150}\right)\\&+(x+0.2)(x-0.3)(x-0.7)(x+0.3)\left(\frac{15853}{180}\right)\end{aligned}$$

$$P_4(0.4)=1.23+2.22(0.4)+0.444+(0.4+0.2)(\frac{-713}{60}(0.4)+\frac{713}{200})+(0.4+0.2)(0.4-0.3)(\frac{1297}{50}(0.4)-\frac{9079}{1500})+$$
$$(0.4+0.2)(0.4-0.3)(0.4-0.7)(\frac{15853}{180}(0.4)+\frac{15853}{600})$$
$$P_4(0.4)=0.9986899999$$

3.You have these values for x and f(x):

x	0.2	0.3	0.7	-0.3	0.1
$f(x)$	1.23	2.34	-1.05	6.51	-0.06

Find $f(0.5)$ from cubic that starts from $x = 0.1$.

x	-0.3	-0.2	0.1	0.3	0.7
$f(x)$	6.51	1.23	-0.06	2.34	-1.05

h	0.1	0.3	0.2	0.4

divided difference of f(x)

$$\frac{y_{i+1} - y_i}{x_{i+1} - x_i}$$

$$j_1 = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1.23 - 6.51}{-0.2 - (-0.3)} = \frac{-5.28}{0.1} = -52.8$$

$$j_2 = \frac{y_3 - y_2}{x_3 - x_2} = \frac{-0.06 - 1.23}{0.1 - (-0.2)} = \frac{-1.29}{0.3} = -4.3$$

$$j_3 = \frac{y_4 - y_3}{x_4 - x_3} = \frac{2.34 - (-0.06)}{0.3 - 0.1} = \frac{2.4}{0.2} = 12$$

$$j_4 = \frac{y_5 - y_4}{x_5 - x_4} = \frac{-1.05 - 2.34}{0.7 - 0.3} = \frac{-3.39}{0.4} = -8.475$$

$$A = \begin{pmatrix} 2(h_1 + h_2) & h_2 & h_0 \\ h_2 & 2(h_2 + h_3) & h_3 \\ h_0 & h_3 & 2(h_3 + h_4) \end{pmatrix} = \begin{pmatrix} 0.8 & 0.3 & 0.0 \\ 0.3 & 1.0 & 0.2 \\ 0.0 & 0.2 & 1.2 \end{pmatrix}$$

$$B = \begin{pmatrix} \frac{(j_2 - j_1)}{6} \\ \frac{(j_3 - j_2)}{6} \\ \frac{(j_4 - j_3)}{6} \end{pmatrix} = \begin{pmatrix} 291 \\ 97.8 \\ -122.85 \end{pmatrix}$$

$$S = A^{-1}B$$

$$S = \begin{pmatrix} 1.41463414634146 & -0.439024390243902 & 0.0731707317073171 \\ -0.439024390243902 & 1.17073170731707 & -0.195121951219512 \\ 0.0731707317073171 & -0.195121951219512 & 0.865853658536585 \end{pmatrix} * \begin{pmatrix} 291 \\ 97.8 \\ -122.85 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & 359.7329 & 10.7122 & -104.1604 & 0 \end{pmatrix}$$

Find $f(0.5)$ from a cubic that starts from $x = 0.1$ Based on the given, 0.7 is the only value of x that is greater than 0.5 where the ith polynomial taken is $i - 1$. 0.7 is x_5 with $i = 5 - 1 = 4$, so we'll use the $4th$ polynomial to predict on.

x	-0.3	-0.2	0.1	0.3	0.7

$$g_4(0.5) = a_4 * (0.5 - x_4)^3 + b_4 * (0.5 - x_4)^2 + c_4 * (0.5 - x_4) + d_4$$

$$a_4 = \frac{S_5 - S_4}{6 * h_4} = \frac{0 - (-104.1604)}{2.4} = 43.40016$$

$$b_4 = \frac{S_4}{2} = \frac{-104.1604}{2} = -52.0802$$

$$c_4 = \frac{y_5 - y_4}{h_4} - \frac{2 * h_4 * S_4 + h_4 * S_5}{6} = \frac{-1.05 - 2.34}{0.4} - \frac{2 * 0.4 * (-104.1604) + 0.4 * 0}{6} = 5.4131$$

$$d_4 = y_4 = 2.34$$

$$g_4(0.5) = 43.40016 * (0.5 - 0.3)^3 + -52.0802 * (0.5 - 0.3)^2 + 5.4131 * (0.5 - 0.3) + 2.34$$

$$g_4(0.5) = 1.6866$$

4.Given function $f(x) = 2xcos(2x)$. Use a central difference to compute $f'(2.0)$ and compare it using the $P'(x)$.

The derivative of $f(x) = 2xcos(2x)$ is:

$$2(cos(2x) - 2xsin(2x))$$

Suppose that $2(cos(2x) - 2xsin(2x))$ at x = 2.0.

```
f=function(x){2*x*cos(2*x)}
dp = Deriv(f, "x")
dp(2.0)
```

```
## [1] 4.747133
```

So $f'(2.0) = 4.747133$
We run here from 0.05 to $0.05/2^4$:

```
n=c(1,2,2^2,2^3,2^4)
delta_x=0.05/n
dfdx=(f(2.0+delta_x)-f(2.0-delta_x))/(2*(delta_x))
error=dfdx-4.747133
round(dfdx,5);round(error,5)
```

```
## [1] 4.74358 4.74624 4.74691 4.74708 4.74712
```

```
## [1] -0.00355 -0.00089 -0.00022 -0.00006 -0.00001
```

From $0.05/2^5$ to $0.05/2^8$

```
n=c(2^5,2^6,2^7,2^8,2^9)
delta_x=0.05/n
dfdx=(f(2.0+delta_x)-f(2.0-delta_x))/(2*(delta_x))
error=dfdx-4.747133
round(dfdx,5);round(error,5)
```

```
## [1] 4.74713 4.74713 4.74713 4.74713 4.74713
```

```
## [1] 0 0 0 0 0
```

```
f=function(x){(2*x*cos(2*x))}
x=c(0.5,1,1.5,2,2.5)
d=data.frame(x, f(x))
d
```

```
##     x       f.x.
## 1 0.5  0.5403023
## 2 1.0 -0.8322937
## 3 1.5 -2.9699775
## 4 2.0 -2.6145745
## 5 2.5  1.4183109
```

```
##     x       f.x.        f1
## 1 0.5  0.5403023 -2.745192
## 2 1.0 -0.8322937 -4.275368
## 3 1.5 -2.9699775  0.710806
## 4 2.0 -2.6145745  8.065771
## 5 2.5  1.4183109        NA
```

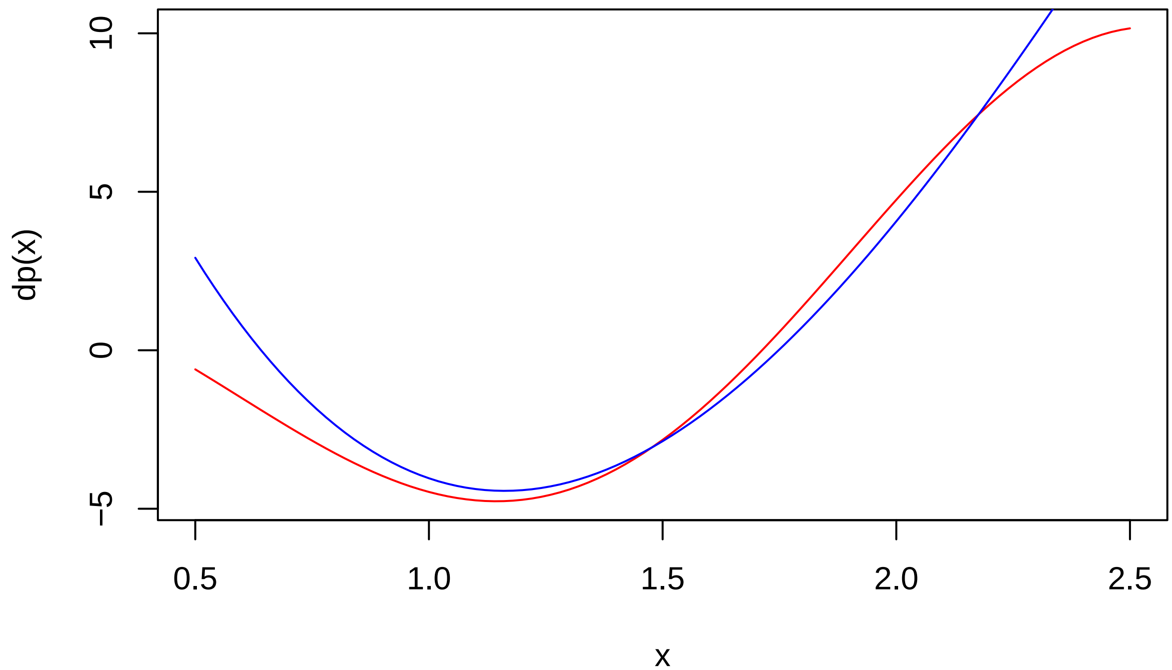
```
##     x       f.x.        f1        f2
## 1 0.5  0.5403023 -2.745192 -1.530176
## 2 1.0 -0.8322937 -4.275368  4.986174
## 3 1.5 -2.9699775  0.710806  7.354965
## 4 2.0 -2.6145745  8.065771        NA
## 5 2.5  1.4183109        NA        NA
```

```
##     x       f.x.        f1        f2       f3
## 1 0.5  0.5403023 -2.745192 -1.530176 4.344233
## 2 1.0 -0.8322937 -4.275368  4.986174 1.579194
## 3 1.5 -2.9699775  0.710806  7.354965       NA
## 4 2.0 -2.6145745  8.065771        NA       NA
## 5 2.5  1.4183109        NA        NA       NA
```

```
##     x       f.x.        f1        f2       f3        f4
## 1 0.5  0.5403023 -2.745192 -1.530176 4.344233 -1.382519
## 2 1.0 -0.8322937 -4.275368  4.986174 1.579194        NA
## 3 1.5 -2.9699775  0.710806  7.354965       NA        NA
## 4 2.0 -2.6145745  8.065771        NA       NA        NA
## 5 2.5  1.4183109        NA        NA       NA        NA
```

```
p=function(x) {-2.745192+(-1.530176)*((x - 1.0)+(x - 0.5))+
(4.344233)*((x - 1.0)*(x - 0.5)+ (x - 1.5)*(x - 0.5) +
(x - 2.0)*(x - 1.5))+ (-1.382519)*((x - 2.0)*(x - 1.5)*(x - 0.5)+
(x - 2.5)*(x - 1.5)*(x - 1.0)+
(x - 2.5)*(x - 1.5)*(x - 0.5))+
(x - 2.5)*(x-2.0)*(x-1.5)}
```

We show that the graph of$f'(x)$ and P_4'



Comparison of answers

```r
answers = data.frame(answer1,answer2,dp(2))
colnames(answers) = c("Central diff ", "Finite diff ", "Actual ")
print(answers,row.names = FALSE)
```

```
##  Central diff  Finite diff   Actual
##       4.747133     4.747133 4.747133
```

5.Use Trapezoid rule to estimate $\int_1^2 \frac{1}{x^2}dx$. Accurate within 0.001.

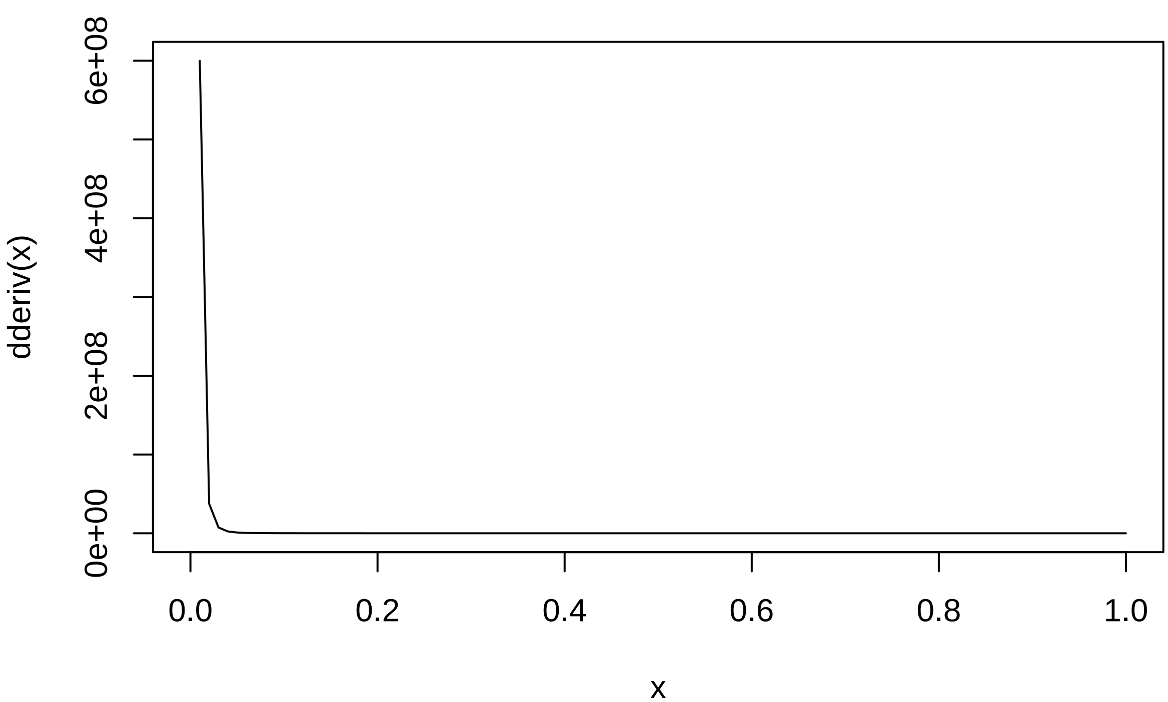
Let $f(x)$ be a continuous function over $[a, b]$, having a second derivative $f''(x)$ over this interval. If M is the maximum value of $|f''(x)|$ over $[a, b]$, then the upper bounds for the error T_n to estimate $\int_a^b f(x)dx$ is

$$\text{Error in } T_n \leq \frac{M(b-a)^2}{12n^2}.$$

We can use the idea of error bound to know the number of subintervals to get the accurate value of the integral given a certain error tolerance. In this case, the error tolerance is 0.001. We begin by determining the value of M, the maximum value of $|f''(x)|$ over $[1, 2]$ for $f(x) = \frac{1}{x^2}$.

so $f''(x) = 6/x^4$. The second derivative is at its maximum when x approaches either positive of negative infinity, as long as $x \neq 0$. Which can be seen here

```
curve(dderiv)
```



In this case, we will be using $x = 1$

$$M = |f''(1)| = \frac{6}{(1)^4} = 6$$

From the error-bound Equation of Error in $T_n \leq \frac{M(b-a)^2}{12n^2}$, we have

$$\frac{6(2-1)^2}{12n^2} \leq 0.001$$
$$\frac{6}{12n^2} \leq 0.001$$

$$\frac{1}{2n^2} \leq 0.001$$

$$2n^2 \geq \frac{1}{0.001}$$

$$\frac{2n^2}{2} \geq \frac{1000}{2}$$

$$n^2 \geq 500$$

$$n \geq \sqrt{500}$$

$$n \geq 10\sqrt{5}$$

Since n must be an integer, we take $n = 22$ to satisfy the equation as $10\sqrt{5} = 22.360679775$.

```r
f = function(x) {1/x^2}

# Define the limits of integration
a = 1
b = 2

# Number of intervals
n = 22

# Width of each interval
h = (b - a) / n

# Trapezoid rule formula
int = (h / 2) * (f(a) + 2 * sum(sapply(1:(n - 1), function(i) f(a + i * h))) + f(b))

actual = integrate(f, lower = 1, upper = 2)
```

```
## [1] "Estimated value:  0.500301170899931"
```

```
## [1] "Actual value:  0.5"
```

```
## [1] "Error:  0.000301170899930925"
```

for $n = 13$, we are guaranteed

$$\left| \int_1^2 \frac{1}{x^2} - M_n \right| \leq 0.001$$

6.Use Simpson's rule to estimate $\int_1^2 \frac{1}{x^2}dx$. Accurate within 0.001.

Let $f(x)$ be a continuous function over $[a,b]$, having a fourth derivative $f^{(4)}(x)$ over this interval. If M is the maximum value of $\left|f^{(4)}(x)\right|$ over $[a,b]$, then the upper bounds for the error S_n to estimate $\int_a^b f(x)dx$ is

$$\text{Error in } S_n \leq \frac{M(b-a)^5}{180n^4}$$

We can use the idea of error bound to know the number of subintervals to get the accurate value of the integral given a certain error tolerance. In this case, the error tolerance is 0.001. We begin by determining the value of M, the maximum value of $|f^{(4)}(x)|$ over $[1,2]$ for $f(x) = \frac{1}{x^2}$.

```
## function (x)
## 120/x^6
```

so $f^{(4)}(x) = 120/x^6$. The fourth derivative is at its maximum when x approaches either positive of negative infinity, as long as $x \neq 0$.

In this case, we will be using $x = 1$

$$M = |f^{(4)}(1)| = \frac{120}{(1)^6} = 120$$

From the error-bound for Simpson's Rule of Error in $S_n \leq \frac{M(b-a)^5}{180n^4}$, we have

$$\frac{120(2-1)^2}{180n^4} \leq 0.001$$

$$\frac{120}{180n^4} \leq 0.001$$

$$\frac{2}{3n^4} \leq 0.001$$

$$3n^4 \geq \frac{2}{0.001}$$

$$\frac{3n^4}{3} \geq \frac{2000}{3}$$

$$n^4 \geq \frac{2000}{3}$$

$$n \geq \sqrt[4]{\frac{2000}{3}}$$

Since n must be an integer and subintervals must be even, we take $n = 6$ to satisfy the equation as $\sqrt[4]{\frac{2000}{3}} = 5.081327482$.

```r
f = function(x) {1/x^2}

# Define the limits of integration
a = 1
b = 2

# Number of intervals
n = 6
```

```r
simpsons_rule <- function(f, a, b, n) {
  if (n %% 2 != 0) {
    stop("Number of subintervals must be even.")
  }

  h <- (b - a) / n
  integral <- f(a) + f(b)  # endpoints

  # Odd indexed points
  odd_sum <- sum(sapply(seq(1, n, by = 2), function(i) f(a + i * h)))
  # Even indexed points
  even_sum <- sum(sapply(seq(2, n-1, by = 2), function(i) f(a + i * h)))

  integral <- integral + 4 * odd_sum + 2 * even_sum
  integral <- integral * h / 3

  return(integral)
}

actual = integrate(f, lower = 1, upper = 2)
```

```
## [1] "Estimated value:  0.500090885144998"
```

```
## [1] "Actual value:  0.5"
```

```
## [1] "Error:  9.088514499761e-05"
```

for $n = 6$, we are guaranteed

$$\left| \int_1^2 \frac{1}{x^2} - M_n \right| \leq 0.001$$

7.Use four iterations of Romberg integration to estimate $\pi = \int_0^1 \frac{4}{1+x^2}dx$. Comment on the accuracy of the result.

```
romberg_integration = function(num_of_iter, func, lower_bound, upper_bound) {
    deltax_array = array(data = NA, dim = num_of_iter+1)
    T_array = array(data = NA, dim = num_of_iter+1)
    Tx_array = array(data = NA, dim = num_of_iter+1)

    ptr = 1
    prev = 1
    after = 2
    deltax = (upper_bound - lower_bound) / 2

    while(ptr < num_of_iter+2) {
        deltax_array[ptr] = deltax
        num_of_subintervals = 2^num_of_iter
        temp = 0
        x = 1

        while(x < 2^ptr) {
            temp = temp + 2 * func(lower_bound + x * deltax)
            x = x + 1
        }

        T_d = (deltax/2) * (func(lower_bound) + temp + func(upper_bound))
        T_array[ptr] = T_d

        if(ptr > 1) {
            Tx = ((2^(2*prev)) * T_array[after] - T_array[prev]) / (2^(2*prev) - 1)
            Tx_array[prev] = Tx
            prev = prev + 1
            after = after + 1
        }

        ptr = ptr + 1
        deltax = deltax / 2
    }

    result_df <- data.frame(deltax = deltax_array, T = T_array, Tx = Tx_array + 1)
    result_df$deltax <- deltax_array
    result_df$T <- T_array
    result_df$Tx <- Tx_array

    return(list(Tx_array, result_df))
}

func_to_integrate = function(x) {
    return(4 / (1 + x^2))
}

result = romberg_integration(4, func_to_integrate, 0, 1)
answer = result[1]
result_dataframe = result[2]
```

```r
result_dataframe
```

```
## [[1]]
##    deltax        T       Tx
## 1 0.50000 3.100000 3.141569
## 2 0.25000 3.131176 3.139509
## 3 0.12500 3.138988 3.140973
## 4 0.06250 3.140942 3.141432
## 5 0.03125 3.141430       NA
```

```r
answer
```

```
## [[1]]
## [1] 3.141569 3.139509 3.140973 3.141432       NA
```

We can see the estimated answer at the 4th iteration = 3.141432
If we use the pracma library, it gives 3.141593 with 7 iterations.

```r
f <- function(x) {
  4 / (1 + x^2)
}

rom <- romberg(f, 0.2, 1.5)
rom
```

```
## $value
## [1] 3.141593
##
## $iter
## [1] 7
##
## $rel.error
## [1] 0
```

Error = |Expected - Actual|

```r
3.141593 - 3.141432
```

```
## [1] 0.000161
```