Problem Set 2

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1. Use the bisection method with a hand calculator or computer to find the real root of $x^3 - x - 3 = 0$. Use an error tolerance of $\epsilon = 0.0001$. Graph the function $f(x) = x^3 - x - 3$ and label the root.

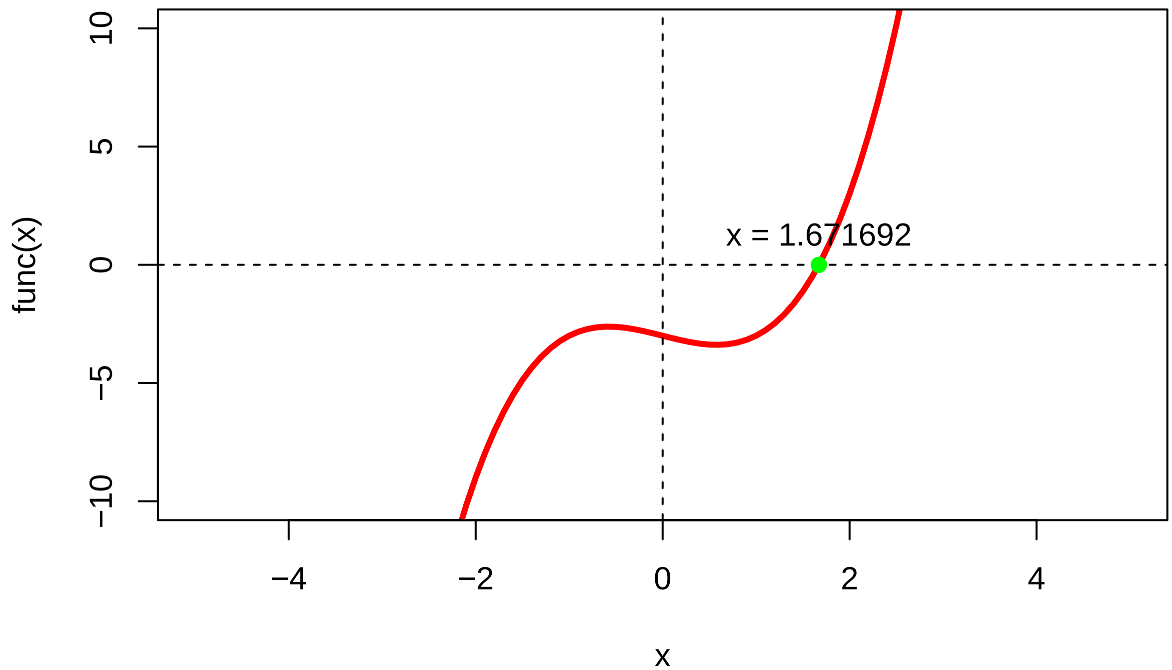
The numerical methods for finding the roots are called iterative methods.
Algorithm Bisection Method
Define $c = \frac{a+b}{2}$
If $|b - c| \leq \epsilon$ then accept root and exit
If $f(b) * f(c) \leq 0$ then a = c else b = c
Return to step 1

Solving using bisection method:

```
##    #        a        b        c         f(b)          f(c)     f(b)*f(c)
##    1 1.000000 2.000000 1.500000 3.0000000000 -1.125000e+00 -3.375000e+00
##    2 1.500000 2.000000 1.750000 3.0000000000  6.093750e-01  1.828125e+00
##    3 1.500000 1.750000 1.625000 0.6093750000 -3.339844e-01 -2.035217e-01
##    4 1.625000 1.750000 1.687500 0.6093750000  1.179199e-01  7.185745e-02
##    5 1.625000 1.687500 1.656250 0.1179199219 -1.128845e-01 -1.331133e-02
##    6 1.656250 1.687500 1.671875 0.1179199219  1.293182e-03  1.524920e-04
##    7 1.656250 1.671875 1.664062 0.0012931824 -5.610037e-02 -7.254801e-05
##    8 1.664062 1.671875 1.667969 0.0012931824 -2.747995e-02 -3.553658e-05
##    9 1.667969 1.671875 1.669922 0.0012931824 -1.311249e-02 -1.695684e-05
##   10 1.669922 1.671875 1.670898 0.0012931824 -5.914436e-03 -7.648444e-06
##   11 1.670898 1.671875 1.671387 0.0012931824 -2.311822e-03 -2.989608e-06
##   12 1.671387 1.671875 1.671631 0.0012931824 -5.096188e-04 -6.590300e-07
##   13 1.671631 1.671875 1.671753 0.0012931824  3.917071e-04  5.065487e-07
##   14 1.671631 1.671753 1.671692 0.0003917071 -5.897455e-05 -2.310075e-08
```

```
## [1] 1.671692
```

```
## [1] 1.671726
```



With the interval $[-3, 1.672]$, calculate how many iterations needed:
Since $\epsilon = 0.0001$, so $|a - c_n| = 0.0001$.

$$|a - c_n| \leq \left[\frac{1}{2}\right]^n (b - a)$$

$$0.0001 \leq \left[\frac{1}{2}\right]^n [1.672 - (-3)]$$

$$0.0001 \leq \left[\frac{1}{2}\right]^n (4.672)$$

$$\log_2 \frac{0.0001}{4.672} \leq \left[\frac{1}{2}\right]^n log_2$$

$$n \geq 16$$

2. The function $f(x) = -3x^3 + 2e^{x^2/2} - 1$ has values of zero near x = -0.5 and x = 0.5
   a. What is the derivative of f?

$$\frac{d}{dx}[-3x^3 + 2e^{\frac{x^2}{2}} - 1]$$

$$= -3 * \frac{d}{dx}[x^3] + 2 * \frac{d}{dx}[e^{\frac{x^2}{2}}] + \frac{d}{dx}[-1]$$

$$= -3 * 3x^2 + 2e^{\frac{x^2}{2}} * \frac{d}{dx}[\frac{x^2}{2}] + 0$$

$$= -9x^2 + 2e^{\frac{x^2}{2}} * \frac{1}{2} * \frac{d}{dx}[x^2]$$

$$= -9x^2 + 2e^{\frac{x^2}{2}}x$$
$$= -9x^2 + 2xe^{\frac{x^2}{2}}$$

b. If you begin with Newton's method at x=0, which root is reached? How many iterations to achieve an error less than $1-^{-5}$?

$$f(0) = -3(0^3) + 2e^{\frac{0^2}{2}} - 1 = 1$$
$$f'(0) = -9(0^2) + 2(0)e^{\frac{0^2}{2}} = 0$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$
$$x_2 = 0 - \frac{1}{0} = undefined$$

When it starts at $x = 0$, we get $undefined$ which prompts us to use a different starting point when using Newton's method.

c. Begin Newton's method at another starting point to get the other zero. 1st iteration: $x_1 = -0.2$

$$f(-0.2) = -3(-0.2)^3 + 2e^{\frac{-0.2^2}{2}} - 1 = 1.06440268$$
$$f'(-0.2) = -9(-0.2)^2 + 2(-0.2)e^{\frac{-0.2^2}{2}} = -0.768080536$$
$$x_2 = -0.2 - \frac{1.06440268}{-0.768080536} = 1.185795132$$
$$|x_2 - x_1| = |1.185795132 - (-0.2)| = 1.385795132$$

2nd iteration: $x_2 = 1.185795132$

$$f(1.185795132) = -3(1.185795132)^3 + 2e^{\frac{1.185795132^2}{2}} - 1 = -1.962247068$$
$$f'(1.185795132) = -9(1.185795132)^2 + 2(1.185795132)e^{\frac{1.185795132^2}{2}} = -7.864581946$$
$$x_3 = 1.185795132 - \frac{-1.962247068}{-7.864581946} = 0.9362916373$$
$$|x_3 - x_2| = |0.9362916373 - 1.185795132| = 0.2495034947$$

3rd iteration: $x_3 = 0.9362916373$

$$f(0.9362916373) = -3(0.9362916373)^3 + 2e^{\frac{0.9362916373^2}{2}} - 1 = -0.3621729375$$
$$f'(0.9362916373) = -9(0.9362916373)^2 + 2(0.9362916373)e^{\frac{0.9362916373^2}{2}} = -4.987082379$$
$$x_4 = 0.9362916373 - \frac{-0.3621729375}{-4.987082379} = 0.8636694286$$
$$|x_4 - x_3| = |0.8636694286 - 0.9362916373| = 0.07262220873$$

4th iteration: $x_4 = 0.8636694286$

$$f(0.8636694286) = -3(0.8636694286)^3 + 2e^{\frac{0.8636694286^2}{2}} - 1 = -0.02863794569$$
$$f'(0.8636694286) = -9(0.8636694286)^2 + 2(0.8636694286)e^{\frac{0.8636694286^2}{2}} = -4.205176438$$
$$x_5 = 0.8636694286 - \frac{-0.02863794569}{-4.205176438} = 0.8568592635$$
$$|x_5 - x_4| = |0.8568592635 - 0.8636694286| = 0.006810165069$$

5th iteration: $x_5 = 0.8568592635$

$$f(0.8568592635) = -3(0.8568592635)^3 + 2e^{\frac{0.8568592635^2}{2}} - 1 = -2.42469513 * 10^{-4}$$

$$f'(0.8568592635) = -9(0.8568592635)^2 + 2(0.8568592635)e^{\frac{0.8568592635^2}{2}} = -4.1340354068$$

$$x_6 = 0.8568592635 - \frac{-2.42469513 * 10^{-4}}{-4.1340354068} = 0.8568006115$$

$$|x_6 - x_5| = |0.8568006115 - 0.8568592635| = 0.000058652$$

6th iteration: $x_6 = 0.8568006115$

$$f(0.8568006115) = -3(0.8568006115)^3 + 2e^{\frac{0.8568006115^2}{2}} - 1 = -1.7984872 * 10^{-8}$$

$$f'(0.8568006115) = -9(0.8568006115)^2 + 2(0.8568006115)e^{\frac{0.8568592635^2}{2}} = -4.133424464$$

$$x_7 = 0.8568006115 - \frac{-1.7984872 * 10^{-8}}{-4.133424464} = 0.8568006071$$

$$|x_7 - x_6| = |0.8568006071 - 0.8568006115| = 4.35108283 * 10^{-9}$$

At the 6th iteration, we were able to reach an error less than 10^{-5} where -0.2 was the starting point.

3. Use the function from no. 2 and find the root using the secant method where $x_0 = 0$ and $x_1 = 1$. Use an error tolerance of $\epsilon = 0.001$.

$$f(x) = -3x^3 + 2e^{\frac{x^2}{2}} - 1$$

$$x_{n+1} = x_n - f(x_n) * \frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

1st iteration: $x_1 = 1, x_0 = 0$

$$f(x_1) = -3(1^3) + 2e^{\frac{1^2}{2}} - 1 = -0.7025574586$$

$$f(x_0) = -3(0^3) + 2e^{\frac{0^2}{2}} - 1 = 1$$

$$x_2 = 1 - (-0.7025574586) * \frac{1 - 0}{-0.7025574586 - 1} = 0.5873516896$$

2nd iteration: $x_2 = 0.5873516896$

$$f(x_2) = -3(0.5873516896^3) + 2e^{\frac{0.5873516896^2}{2}} - 1 = 0.7686449665$$

$$x_3 = 0.5873516896 - (-0.7686449665) * \frac{0.5873516896 - 1}{-0.7686449665 - (-0.7025574586)} = 0.8029440794$$

$$|x_3 - x_2| = |0.8029440794 - 0.5873516896| = 0.2155923898$$

3rd iteration: $x_3 = 0.8029440794$

$$f(x_3) = -3(0.8029440794^3) + 2e^{\frac{0.8029440794^2}{2}} - 1 = 0.2077417555$$

$$x_4 = 0.8029440794 - (0.2077417555) * \frac{0.8029440794 - 0.5873516896}{0.2077417555 - (0.7686449665)} = 0.8827930456$$

$$|x_4 - x_3| = |0.8827930456 - 0.8029440794| = 0.07984896617$$

4th iteration: $x_4 = 0.8827930456$

$$f(x_4) = -3(0.8827930456^3) + 2e^{\frac{0.8827930456^2}{2}} - 1 = -0.1109815908$$

$$x_5 = 0.8827930456 - (-0.1109815908) * \frac{0.8827930456 - 0.8029440794}{-0.1109815908 - (0.2077417555)} = 0.854989104$$

$$|x_5 - x_4| = |0.854989104 - 0.8827930456| = 0.02780394155$$

5th iteration: $x_5 = 0.854989104$

$$f(x_5) = -3(0.854989104^3) + 2e^{\frac{0.854989104^2}{2}} - 1 = 0.007470629366$$

$$x_6 = 0.854989104 - (0.007470629366) * \frac{0.854989104 - 0.8827930456}{0.007470629366 - (-0.1109815908)} = 0.8567426629$$

$$|x_6 - x_5| = |0.8567426629 - 0.854989104| = 0.0017535589$$

6th iteration: $x_6 = 0.8567426629$

$$f(x_6) = -3(0.8567426629^3) + 2e^{\frac{0.8567426629^2}{2}} - 1 = 2.39490687 * 10^{-4}$$

$$x_7 = 0.8567426629 - (2.39490687 * 10^{-4}) * \frac{0.8567426629 - 0.854989104}{2.39490687 * 10^{-4} - (0.007470629366)} = 0.8571548525$$

$$|x_7 - x_6| = |0.8571548525 - 0.8567426629| = 0.000412189648$$

4. Consider the system

$$10.2x + 2.4y - 4.5z = 14.067,$$
$$-2.3x - 7.7y + 11.1z = -0.996,$$
$$-5.5x - 3.2y + 0.9z = -12.645$$

a. Present the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 10.2 & 2.4 & -4.5 & 14.067 \\ -2.3 & -7.7 & 11.1 & -0.996 \\ -5.5 & -3.2 & 0.9 & -12.645 \end{array}\right)$$

In fraction form

$$\left(\begin{array}{ccc|c} \frac{51}{5} & \frac{12}{5} & -\frac{9}{2} & \frac{14067}{1000} \\ -\frac{23}{10} & -\frac{77}{10} & \frac{111}{10} & -\frac{249}{250} \\ -\frac{11}{2} & -\frac{16}{5} & \frac{9}{10} & -\frac{2529}{200} \end{array}\right)$$

b. Solve the system using Ax= LUx= Ly = b and round the final answer to 4 decimal digits.

Note

$$A = \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{bmatrix}$$

Using Gaussian Elimination method

Here

$$A = \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ -2.3 & -7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{bmatrix}$$

$$R_2 \leftarrow R_2 - (-0.2255)R1[L_{2,1} = -0.2255]$$

$$= \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ 0 & -7.1588 & 10.0853 \\ -5.5 & -3.2 & 0.9 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (-0.5392)R_1[L_{3,1} = -0.5392]$$

$$= \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ 0 & -7.1588 & 10.0853 \\ 0 & -1.9059 & -1.5265 \end{bmatrix}$$

$$R_3 \leftarrow R_3 - (0.2662)R_2[L_{3,2} = 0.2662]$$

$$= \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ 0 & -7.1588 & 10.0853 \\ 0 & 0 & -4.2115 \end{bmatrix}$$

Given LU Decomposition

$$U = \begin{bmatrix} 10.2 & 2.4 & -4.5 \\ 0 & -7.1588 & 10.0853 \\ 0 & 0 & -4.2115 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -0.2255 & 1 & 0 \\ -0.5392 & 0.2662 & 1 \end{bmatrix}$$

Using LU First solving Ly = b for y by forward substitution.

```
##            [,1]
## [1,] 14.067000
## [2,]  2.176109
## [3,] -5.639354
```

Using LU: Then solve Ux = y for x by back substitution.

```
##          [,1]
## [1,] 1.597527
## [2,] 1.582455
## [3,] 1.339037
```

c. Find the residual vector if the correct solution is x=1.4531001, y=-1.5891949, z=-0.2748947

```
A=matrix(c(10.2,2.4,-4.5,-2.3,-7.7,11.1,-5.5,-3.2,0.9), nrow = 3, byrow = T)
b=matrix(c(14.067,-0.996,-12.645), nrow = 3, byrow = T)
x_bar=matrix(c(1.4531001, -1.5891949, -0.2748947))
r=b-A%*%x_bar
r
```

```
##           [,1]
## [1,]  1.822421
## [2,] -6.839339
## [3,] -9.490968
```

5. Compute the Frobenius norm, maximum column sum, and maximum row sum of the matrix:

$$\begin{pmatrix} 10.2 & 2.4 & 4.5 \\ -2.3 & 7.7 & 11.1 \\ -5.5 & -3.2 & 0.9 \end{pmatrix}$$

Note: $||A||_1 = 18.0 \; ||A||_\infty$

$$||A||_f = \sqrt{(10.2)^2 + (2.4)^2 + (4.5)^2 + (-2.3)^2 + (7.7)^2 + (11.1)^2 + (-5.5)^2 + (-3.2)^2 + (0.9)^2}$$
$$= \sqrt{104.04 + 5.76 + 20.25 + 5.29 + 59.29 + 123.21 + 30.25 + 10.24 + 0.81}$$
$$= \sqrt{359.14}$$
$$= 18.9509894201$$
$$\approx 18.951$$

Maximum column sum

$$||A||_1 = max_{1 \leq j \leq n} \sum_i^n = 1|a_{ij}$$

Maximum row sum

$$||A||_\infty = max_{1 \leq j \leq n} \sum_j^n = 1|a_{ij}$$

Frobenius Norm:

$$||A||_f = \sqrt{\sum_{i=1}^m \sum_{j=1}^n a_{ij}^2}$$

The Frobenius norm is 18.95, the maximum column sum is 18.0 while the maximum row sum is 21.1

6. Solve the system of equations given, starting with the initial vector of [0,0,0]:

$$A = \begin{pmatrix} 2 & 5 & 1 \\ 1 & 7 & 2 \\ 1 & 2 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 13 \\ 11 \\ 1 \end{pmatrix}$$

a. Solve using the Jacobi method with 2 digit precision.

$$Dx = -(L+U)x + b$$

Jacobi Formula

```
A=matrix(c(2,5,1,1,7,2,1,2,-5), nrow = 3, byrow = T)
b=matrix(c(13,11,1), nrow = 3, byrow = T)
solve(A,b)
```

```
##         [,1]
## [1,] 4.3125
## [2,] 0.6875
## [3,] 0.9375
```

```r
L=matrix(c(0,0,0,1,0,0,1,2,0), nrow = 3, byrow = T)
D=matrix(c(2,0,0,0,7,0,0,0,-5), nrow = 3, byrow = T)
U=matrix(c(0,5,1,0,0,2,0,0,0), nrow = 3, byrow = T)
b=matrix(c(13,11,1),nrow = 3, byrow = T)
A==L+D+U
```

```
##      [,1] [,2] [,3]
## [1,] TRUE TRUE TRUE
## [2,] TRUE TRUE TRUE
## [3,] TRUE TRUE TRUE
```

Note: Let $x^0 = (0{,}0{,}0)$

$$x^1 = -D^{-1}(L+U)(0,0,0)^T + D^{-1}b$$

Iterative formula

```
## Solution after 15 iterations:
```

```
##           [,1]
## [1,] 4.3113678
## [2,] 0.6877812
## [3,] 0.9374450
```

b. Solve using the Gauss-Seidel method with 2 digit precision.

$$(L+D)x = -Ux + b$$

$$x^{n+1} = -(L+D)^{-1}Ux^n + (L+D)^{-1}b$$

```
## Solution after 10 iterations:
```

```
##           [,1]
## [1,] 4.3129180
## [2,] 0.6874139
## [3,] 0.9375492
```

c. Solve for $\bar{e}$ if the true solution is $x = (1.5, 0.33, 0.45)^T$

```r
A=matrix(c(2,5,1,1,7,2,1,2,-5), nrow = 3, byrow = T)
b=matrix(c(13,11,1), nrow = 3, byrow = T)
x_bar=matrix(c(1.5, 0.33, 0.45))
r=b-A%*%x_bar
e_bar=solve(A)%*%r
abs(e_bar)
```

```
##        [,1]
## [1,] 2.8125
## [2,] 0.3575
## [3,] 0.4875
```

```r
p=function(x){1+1/2*x+3/2*x^2}
p=function(x){{(x-1961)(x-1971)}/{(1951-1961)(1951-1871)}(2.8)}
curve(p,1950,1955,1960,1965,1970,1975)
points(c(0,-1,1),c(1,2,3), pch =19)
text(c(0,-1,1),c(1,2,3),c("(0,1)","(-1,2)","(1,3)"),
     pos = c(3,4,4))
```

$$f\left(x\right)=\frac{\left(x-1961\right)\left(x-1971\right)}{\left(1951-1961\right)\left(1951-1871\right)}\left(2.8\right)+\frac{\left(x-1951\right)\left(x-1971\right)}{\left(1961-1951\right)\left(1961-1971\right)}\left(3.2\right)+\frac{\left(x-1951\right)\left(x-1961\right)}{\left(1971-1951\right)\left(1971-1961\right)}\left(4.5\right)$$