

# ETC3550/ETC5550

## Applied forecasting

Ch9. ARIMA models

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# FORECASTING

## PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

**OTexts**  
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# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

# ARIMA models

**AR:** autoregressive (lagged observations as inputs)

**I:** integrated (differencing to make series stationary)

**MA:** moving average (lagged errors as inputs)

# ARIMA models

**AR:** autoregressive (lagged observations as inputs)

**I:** integrated (differencing to make series stationary)

**MA:** moving average (lagged errors as inputs)

An ARIMA model is rarely interpretable in terms of visible data structures like trend and seasonality. But it can capture a huge range of time series patterns.

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# Stationarity

## Definition

If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

# Stationarity

## Definition

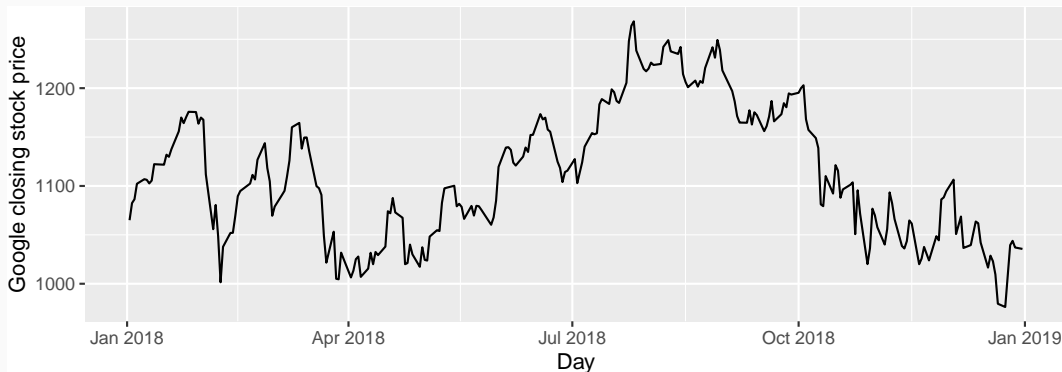
If  $\{y_t\}$  is a stationary time series, then for all  $s$ , the distribution of  $(y_t, \dots, y_{t+s})$  does not depend on  $t$ .

A **stationary series** is:

- roughly horizontal
- constant variance
- no patterns predictable in the long-term

# Stationary?

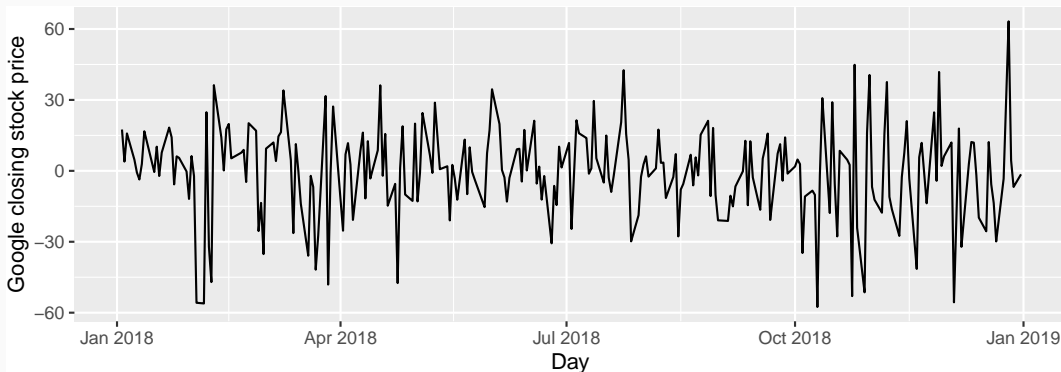
```
gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018) |>  
  autoplot(Close) +  
  labs(y = "Google closing stock price", x = "Day")
```





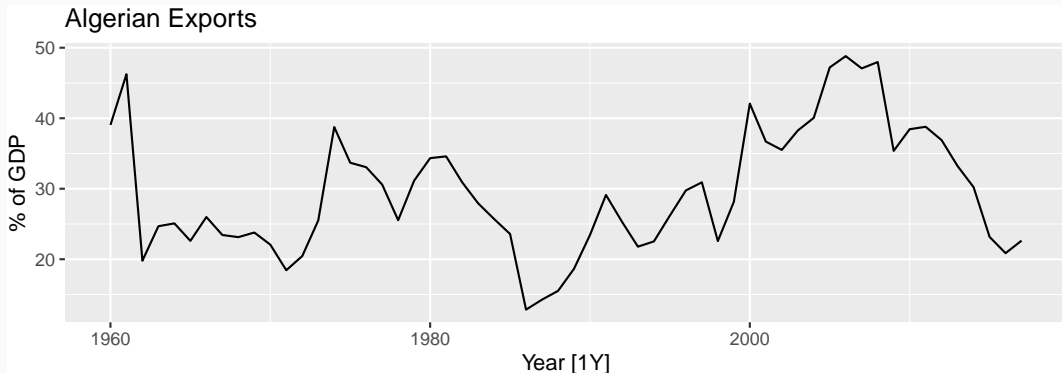
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  autoplot(difference(Close)) +  
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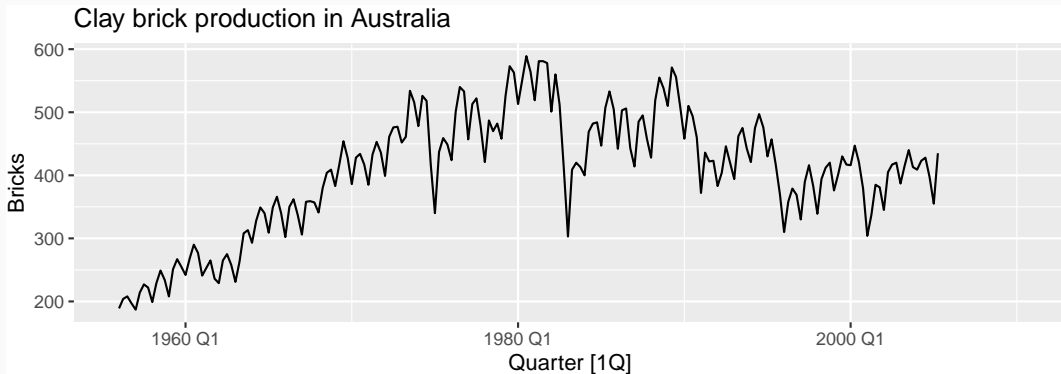
# Stationary?

```
global_economy |>  
  filter(Country == "Algeria") |>  
  autoplot(Exports) +  
  labs(y = "% of GDP", title = "Algerian Exports")
```



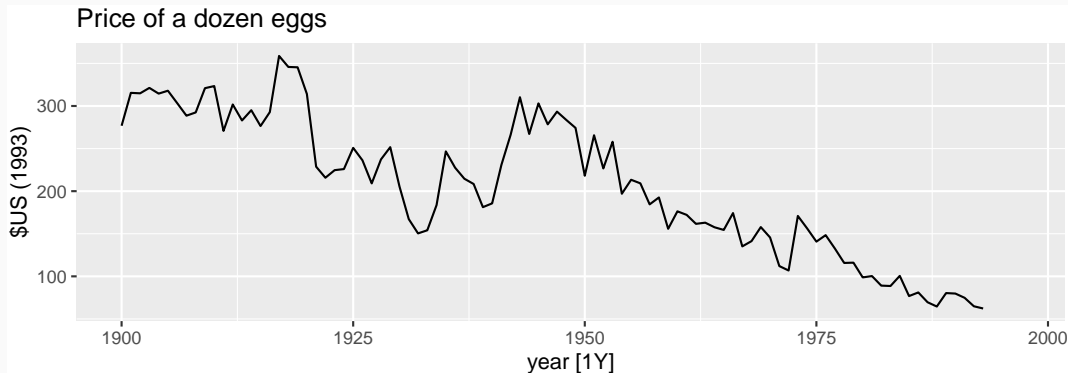
# Stationary?

```
aus_production |>  
  autoplot(Bricks) +  
  labs(title = "Clay brick production in Australia")
```



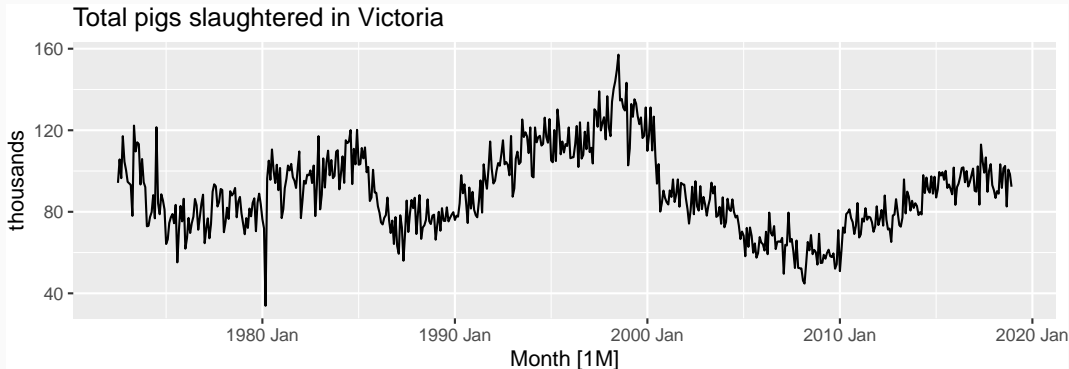
# Stationary?

```
prices |>  
  filter(year >= 1900) |>  
  autoplot(eggs) +  
  labs(y = "$US (1993)", title = "Price of a dozen eggs")
```



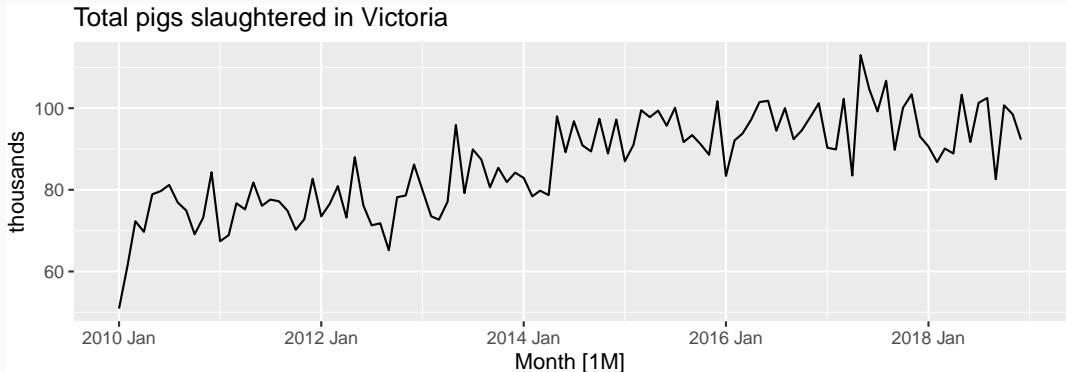
# Stationary?

```
aus_livestock |>  
  filter(Animal == "Pigs", State == "Victoria") |>  
  autoplot(Count / 1e3) +  
  labs(y = "thousands", title = "Total pigs slaughtered in Victoria")
```



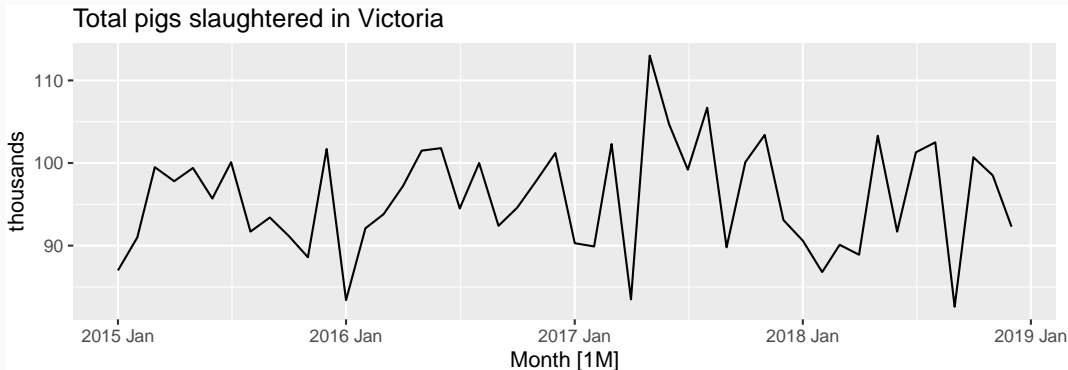
# Stationary?

```
aus_livestock |>
  filter(Animal == "Pigs", State == "Victoria", year(Month) >= 2010) |>
  autoplot(Count / 1e3) +
  labs(y = "thousands", title = "Total pigs slaughtered in Victoria")
```



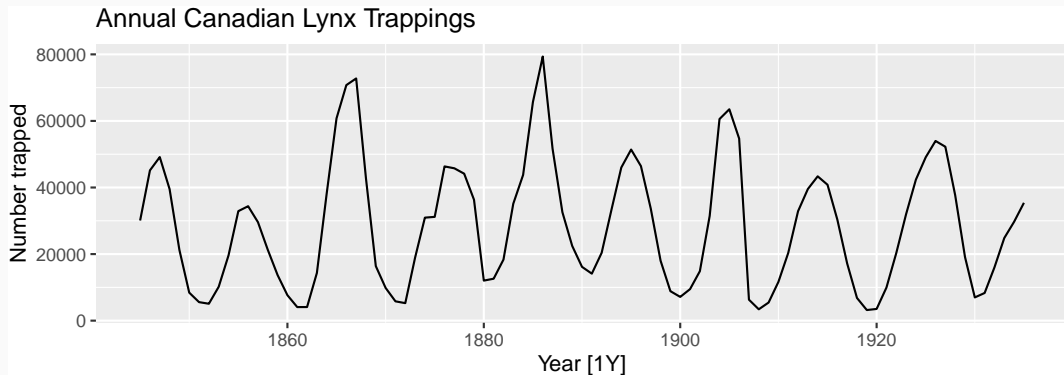
# Stationary?

```
aus_livestock |>  
  filter(Animal == "Pigs", State == "Victoria", year(Month) >= 2015) |>  
  autoplot(Count / 1e3) +  
  labs(y = "thousands", title = "Total pigs slaughtered in Victoria")
```



# Stationary?

```
pelt |>  
  autoplot(Lynx) +  
  labs(y = "Number trapped", title = "Annual Canadian Lynx Trappings")
```





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Transformations help to **stabilize the variance**.

For ARIMA modelling, we also need to **stabilize the mean**.

# Non-stationarity in the mean

## Identifying non-stationary series

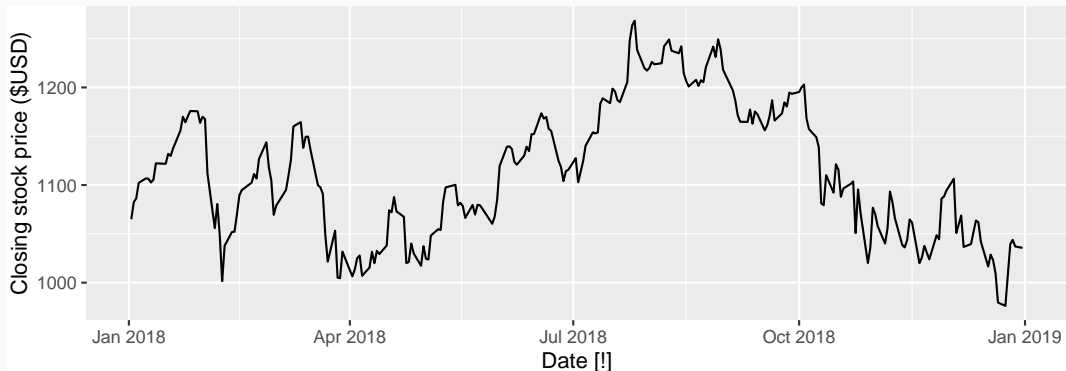
- time plot.
- The ACF of stationary data drops to zero relatively quickly
- The ACF of non-stationary data decreases slowly.
- For non-stationary data, the value of  $r_1$  is often large and positive.

# Example: Google stock price

```
google_2018 <- gafa_stock |>  
  filter(Symbol == "GOOG", year(Date) == 2018)
```

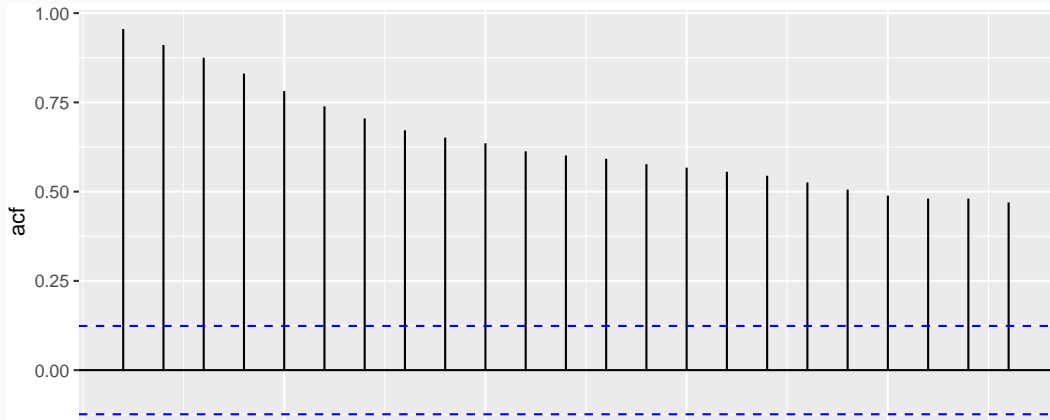
# Example: Google stock price

```
google_2018 |>  
  autoplot(Close) +  
  labs(y = "Closing stock price ($USD)")
```



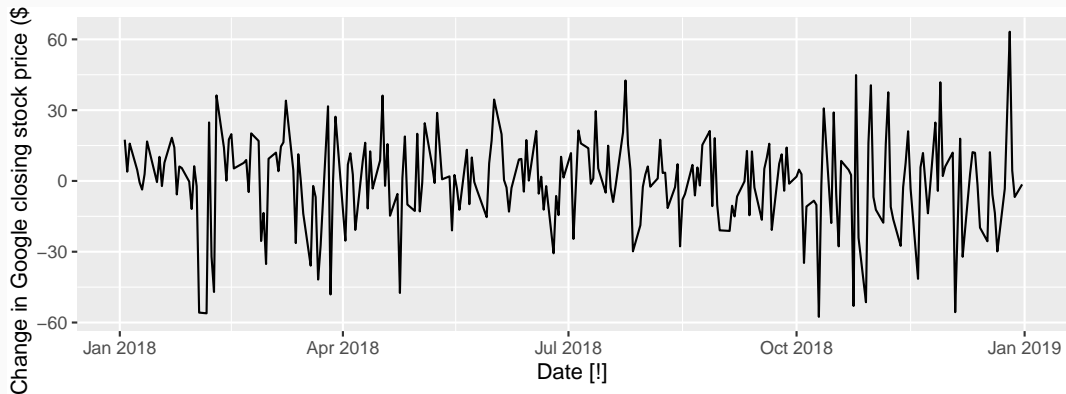
# Example: Google stock price

```
google_2018 |>  
  ACF(Close) |>  
  autoplot()
```



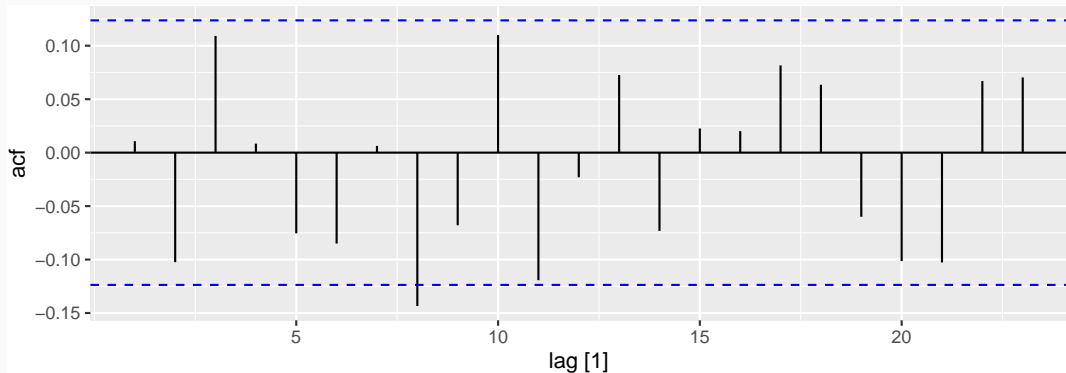
# Example: Google stock price

```
google_2018 |>  
  autoplot(difference(Close)) +  
  labs(y = "Change in Google closing stock price ($USD)")
```



# Example: Google stock price

```
google_2018 |>  
  ACF(difference(Close)) |>  
  autoplot()
```





# Differencing

- Differencing helps to **stabilize the mean**.
- The differenced series is the *change* between each observation in the original series:  $y'_t = y_t - y_{t-1}$ .
- The differenced series will have only  $T - 1$  values since it is not possible to calculate a difference  $y'_1$  for the first observation.

# Random walk model

If differenced series is white noise with zero mean:

$$y_t - y_{t-1} = \varepsilon_t \quad \text{or} \quad y_t = y_{t-1} + \varepsilon_t$$

where  $\varepsilon_t \sim NID(0, \sigma^2)$ .

- Very widely used for non-stationary data.
- This is the model behind the **naïve method**.
- Random walks typically have:
  - ▶ long periods of apparent trends up or down
  - ▶ Sudden/unpredictable changes in direction
- Forecast are equal to the last observation
  - ▶ future movements up or down are equally likely.

# Random walk with drift model

If differenced series is white noise with non-zero mean:

$$y_t - y_{t-1} = c + \varepsilon_t \quad \text{or} \quad y_t = c + y_{t-1} + \varepsilon_t$$

where  $\varepsilon_t \sim NID(0, \sigma^2)$ .

- $c$  is the **average change** between consecutive observations.
- If  $c > 0$ ,  $y_t$  will tend to drift upwards and vice versa.
- This is the model behind the **drift method**.

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Occasionally the differenced data will not appear stationary and it may be necessary to difference the data a second time:

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$$\begin{aligned}y_t'' &= y_t' - y_{t-1}' \\&= (y_t - y_{t-1}) - (y_{t-1} - y_{t-2}) \\&= y_t - 2y_{t-1} + y_{t-2}.\end{aligned}$$

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- $y_t''$  will have  $T - 2$  values.
- In practice, it is almost never necessary to go beyond second-order differences.

## Seasonal differencing

A seasonal difference is the difference between an observation and the corresponding observation from the previous year.

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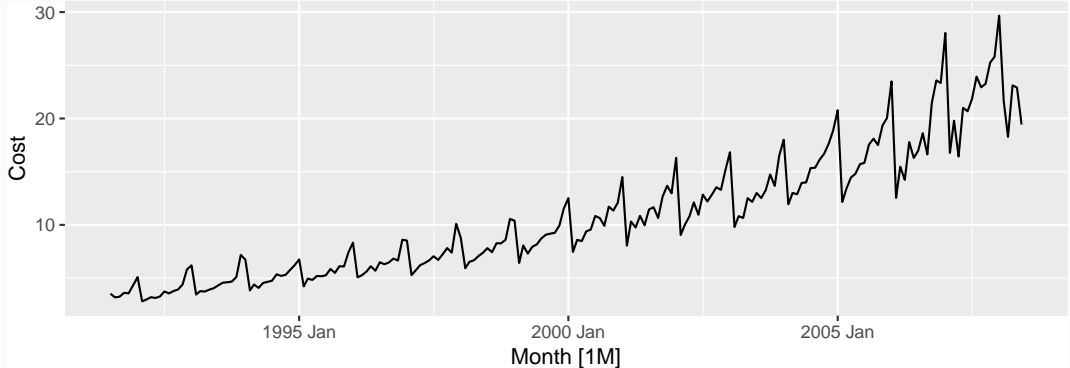
- For monthly data  $m = 12$ .
- For quarterly data  $m = 4$ .

# Antidiabetic drug sales

```
a10 <- PBS |>  
  filter(ATC2 == "A10") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

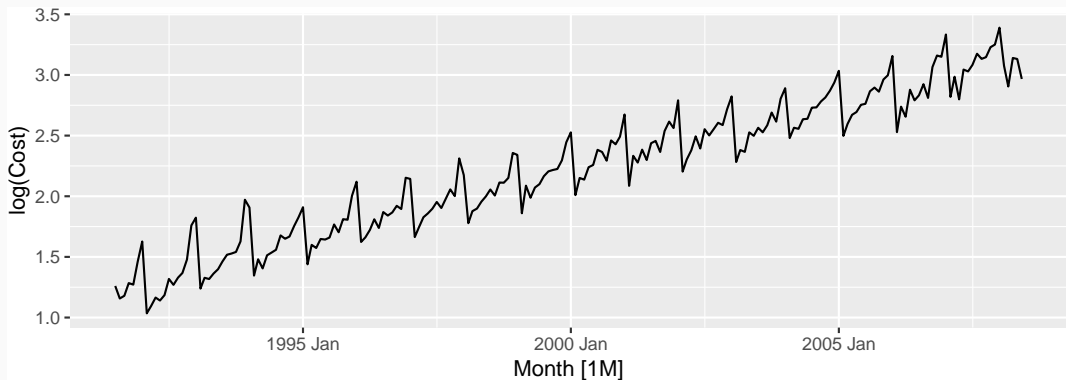
# Antidiabetic drug sales

```
a10 |> autoplot(  
  Cost  
)
```



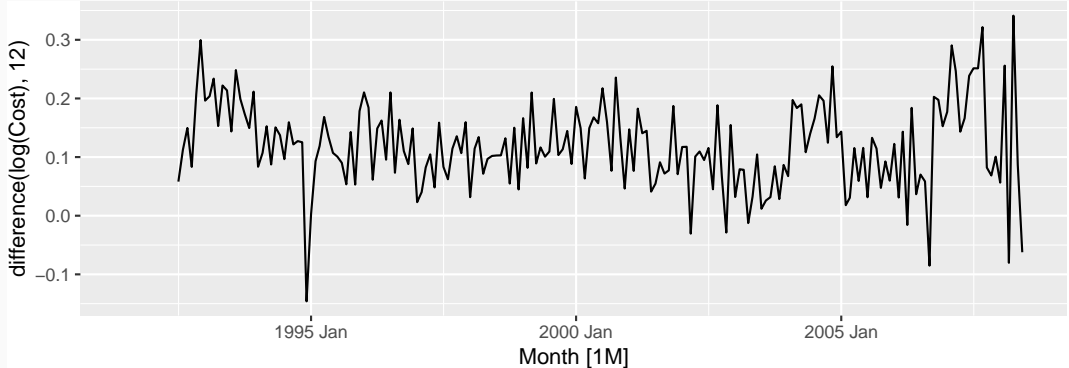
# Antidiabetic drug sales

```
a10 |> autoplot(  
  log(Cost)  
)
```



# Antidiabetic drug sales

```
a10 |> autoplot(  
  log(Cost) |> difference(12)  
)
```

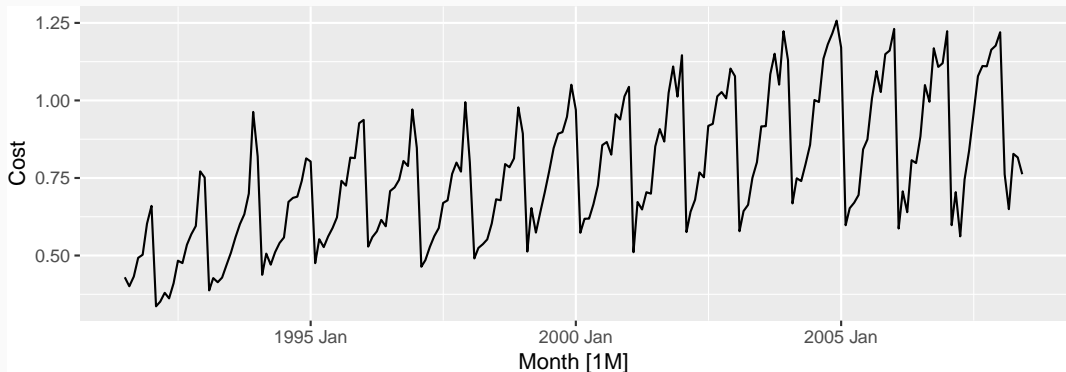


# Corticosteroid drug sales

```
h02 <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

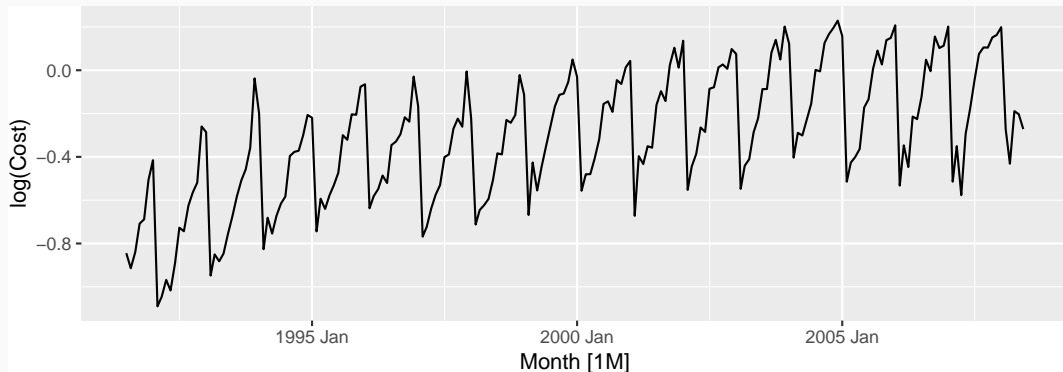
# Corticosteroid drug sales

```
h02 |> autoplot(  
  Cost  
)
```



# Corticosteroid drug sales

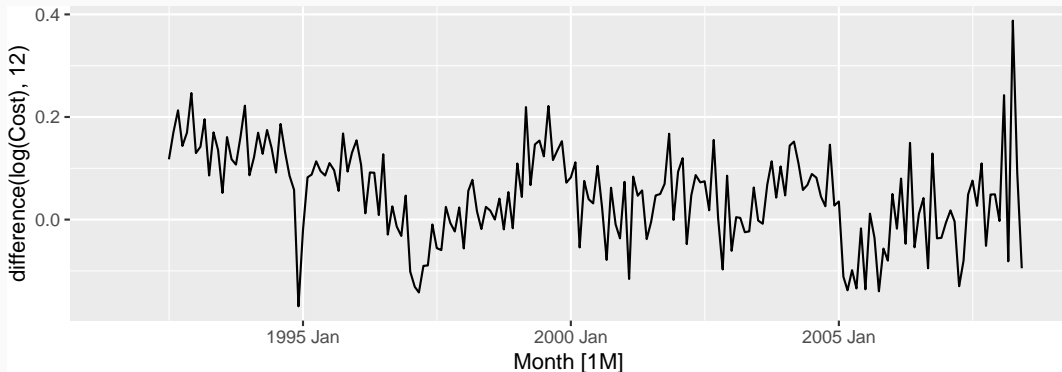
```
h02 |> autoplot(  
  log(Cost)  
)
```





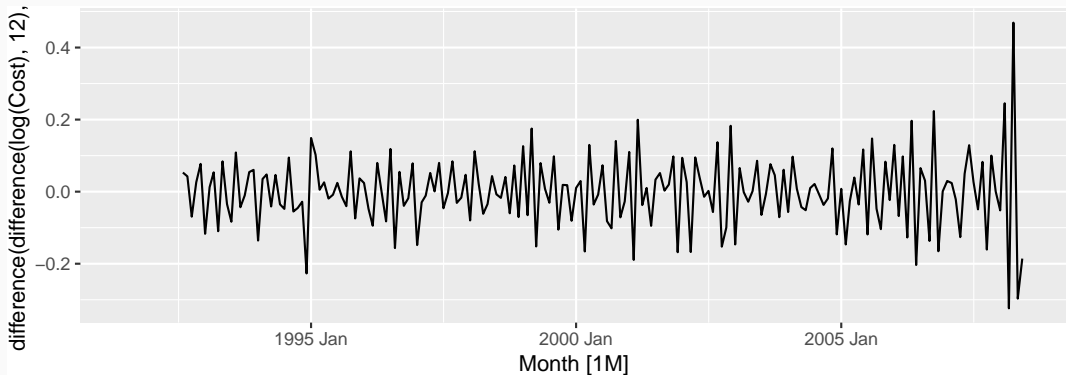
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12)  
)
```



# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12) |> difference(1)  
)
```



# Corticosteroid drug sales

- Seasonally differenced series is closer to being stationary.
- Remaining non-stationarity can be removed with further first difference.

If  $y'_t = y_t - y_{t-12}$  denotes seasonally differenced series, then twice-differenced series is

$$\begin{aligned}y_t^* &= y'_t - y'_{t-1} \\&= (y_t - y_{t-12}) - (y_{t-1} - y_{t-13}) \\&= y_t - y_{t-1} - y_{t-12} + y_{t-13} .\end{aligned}$$

# Seasonal differencing

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- it makes no difference which is done first—the result will be the same.
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It is important that if differencing is used, the differences are interpretable.

# Interpretation of differencing

- first differences are the change between **one observation and the next**;
- seasonal differences are the change between **one year to the next**.

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- seasonal differences are the change between **one year to the next**.

But taking lag 3 differences for yearly data, for example, results in a model which cannot be sensibly interpreted.



# Unit root tests

## Statistical tests to determine the required order of differencing.

- 1 Augmented Dickey Fuller test: null hypothesis is that the data are non-stationary and non-seasonal.
- 2 Kwiatkowski-Phillips-Schmidt-Shin (KPSS) test: null hypothesis is that the data are stationary and non-seasonal.
- 3 Other tests available for seasonal data.

# KPSS test

```
google_2018 |>  
  features(Close, unitroot_kpss)
```

```
## # A tibble: 1 x 3  
##   Symbol kpss_stat kpss_pvalue  
##   <chr>      <dbl>      <dbl>  
## 1 GOOG      0.573      0.0252
```

# KPSS test

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##   <chr>      <dbl>      <dbl>  
## 1 GOOG      0.573      0.0252
```

```
google_2018 |>  
  features(Close, unitroot_ndiffs)
```

```
## # A tibble: 1 x 2  
##   Symbol ndiffs  
##   <chr>   <int>  
## 1 GOOG     1
```

# Automatically selecting differences

STL decomposition:  $y_t = T_t + S_t + R_t$

Seasonal strength  $F_s = \max\left(0, 1 - \frac{\text{Var}(R_t)}{\text{Var}(S_t + R_t)}\right)$

If  $F_s > 0.64$ , do one seasonal difference.

```
h02 |>
  mutate(log_sales = log(Cost)) |>
  features(log_sales, list(unitroot_nsdiffs, feat_stl))

## # A tibble: 1 x 10
##   nsdiffs trend_~1 seaso~2 seaso~3 seaso~4 spikin~5 linea~6 curva~7 stl_e~8
##   <int>     <dbl>   <dbl>   <dbl>   <dbl>     <dbl>   <dbl>   <dbl>   <dbl>
## 1       1     0.957   0.955       6       8 3.78e-10     2.92  -0.831  -0.237
## # ... with 1 more variable: stl_e_acf10 <dbl>, and abbreviated variable
## #   names 1: trend_strength, 2: seasonal_strength_year,
## #   3: seasonal_peak_year, 4: seasonal_trough_year, 5: spikiness
```

# Automatically selecting differences

```
h02 |>
  mutate(log_sales = log(Cost)) |>
  features(log_sales, unitroot_nsdiffs)
```

```
## # A tibble: 1 x 1
##   nsdiffs
##   <int>
## 1      1
```

```
h02 |>
  mutate(d_log_sales = difference(log(Cost), 12)) |>
  features(d_log_sales, unitroot_ndiffs)
```

```
## # A tibble: 1 x 1
##   ndiffs
##   <int>
```

# Backshift notation

A very useful notational device is the backward shift operator,  $B$ , which is used as follows:

$$By_t = y_{t-1}$$

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Two applications of  $B$  to  $y_t$  **shifts the data back two periods**:

$$B(By_t) = B^2y_t = y_{t-2}$$



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Two applications of  $B$  to  $y_t$  **shifts the data back two periods**:

$$B(By_t) = B^2y_t = y_{t-2}$$

For monthly data, if we wish to shift attention to “the same month last year”, then  $B^{12}$  is used, and the notation is  $B^{12}y_t = y_{t-12}$ .

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The backward shift operator is convenient for describing the process of *differencing*.

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A first-order difference can be written as

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t$$

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A first-order difference can be written as

$$y'_t = y_t - y_{t-1} = y_t - By_t = (1 - B)y_t$$

Similarly, if second-order differences (i.e., first differences of first differences) have to be computed, then:

$$y''_t = y_t - 2y_{t-1} + y_{t-2} = (1 - B)^2 y_t$$

# Backshift notation

- Second-order difference is denoted  $(1 - B)^2$ .
- *Second-order difference* is not the same as a *second difference*, which would be denoted  $1 - B^2$ ;
- In general, a  $d$ th-order difference can be written as

$$(1 - B)^d y_t$$

- A seasonal difference followed by a first difference can be written as

$$(1 - B)(1 - B^m)y_t$$

# Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

# Backshift notation

The “backshift” notation is convenient because the terms can be multiplied together to see the combined effect.

$$\begin{aligned}(1 - B)(1 - B^m)y_t &= (1 - B - B^m + B^{m+1})y_t \\ &= y_t - y_{t-1} - y_{t-m} + y_{t-m-1}.\end{aligned}$$

For monthly data,  $m = 12$  and we obtain the same result as earlier.

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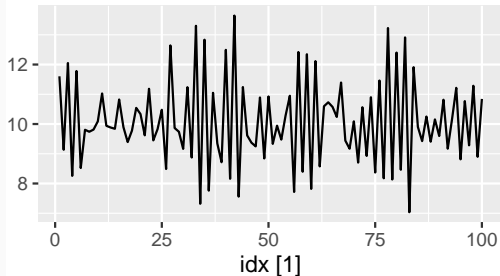
# Autoregressive models

## Autoregressive (AR) models:

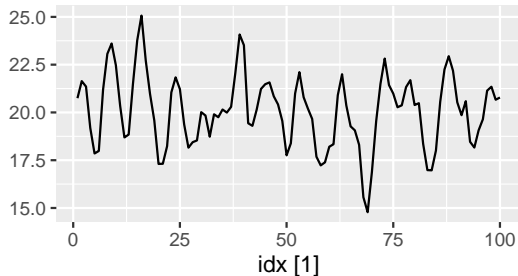
$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t,$$

where  $\varepsilon_t$  is white noise. This is a multiple regression with **lagged values** of  $y_t$  as predictors.

AR(1)



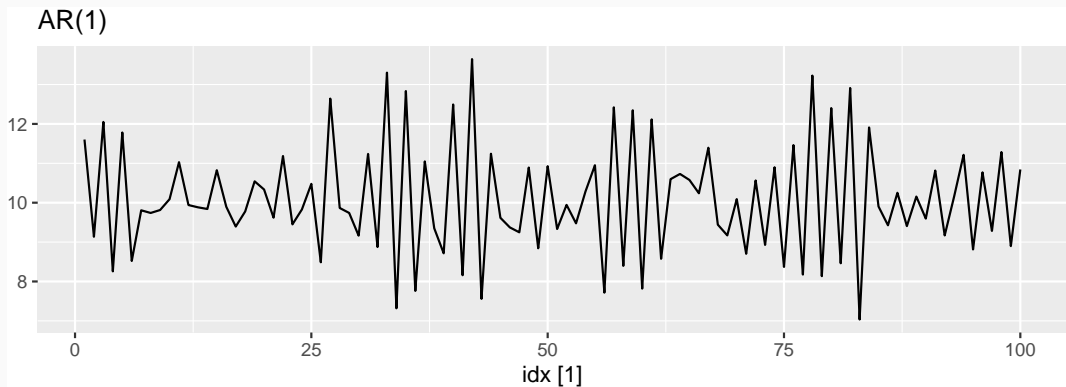
AR(2)



# AR(1) model

$$y_t = 18 - 0.8y_{t-1} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# AR(1) model

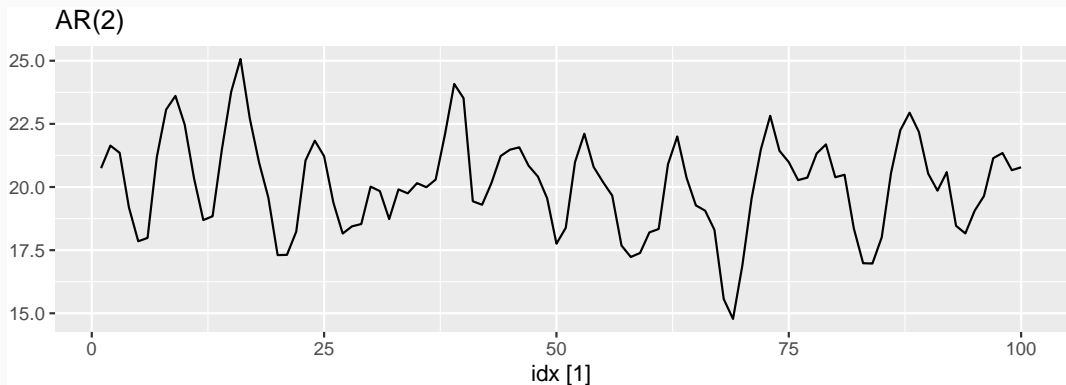
$$y_t = c + \phi_1 y_{t-1} + \varepsilon_t$$

- When  $\phi_1 = 0$ ,  $y_t$  is **equivalent to WN**
- When  $\phi_1 = 1$  and  $c = 0$ ,  $y_t$  is **equivalent to a RW**
- When  $\phi_1 = 1$  and  $c \neq 0$ ,  $y_t$  is **equivalent to a RW with drift**
- When  $\phi_1 < 0$ ,  $y_t$  tends to **oscillate between positive and negative values.**

# AR(2) model

$$y_t = 8 + 1.3y_{t-1} - 0.7y_{t-2} + \varepsilon_t$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# Stationarity conditions

We normally restrict autoregressive models to stationary data, and then some constraints on the values of the parameters are required.

## General condition for stationarity

Complex roots of  $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$  lie outside the unit circle on the complex plane.

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## General condition for stationarity

Complex roots of  $1 - \phi_1 z - \phi_2 z^2 - \dots - \phi_p z^p$  lie outside the unit circle on the complex plane.

- For  $p = 1$ :  $-1 < \phi_1 < 1$ .
- For  $p = 2$ :  
 $-1 < \phi_2 < 1$        $\phi_2 + \phi_1 < 1$        $\phi_2 - \phi_1 < 1$ .
- More complicated conditions hold for  $p \geq 3$ .
- Estimation software takes care of this.

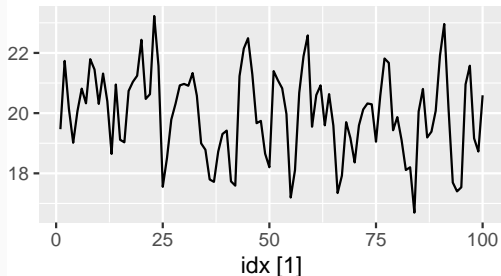
# Moving Average (MA) models

## Moving Average (MA) models:

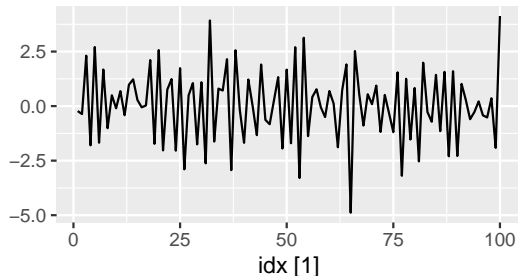
$$y_t = c + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} + \cdots + \theta_q \varepsilon_{t-q},$$

where  $\varepsilon_t$  is white noise. This is a multiple regression with **past errors** as predictors. *Don't confuse this with moving average smoothing!*

MA(1)



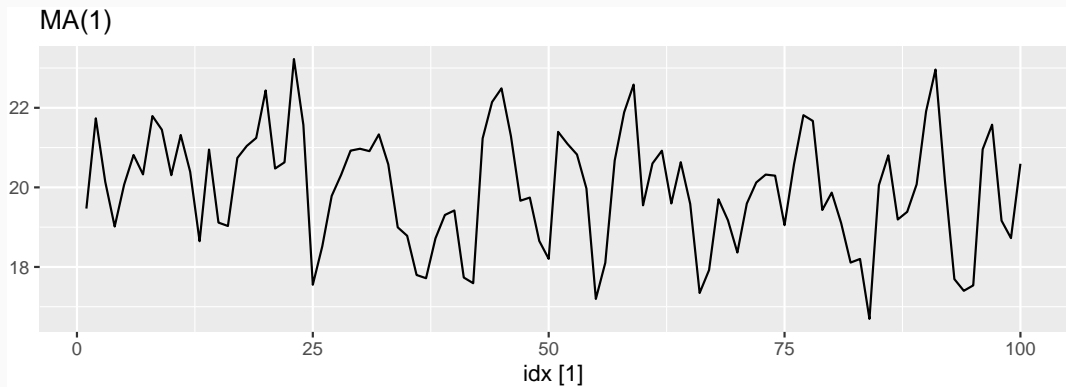
MA(2)



# MA(1) model

$$y_t = 20 + \varepsilon_t + 0.8\varepsilon_{t-1}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$

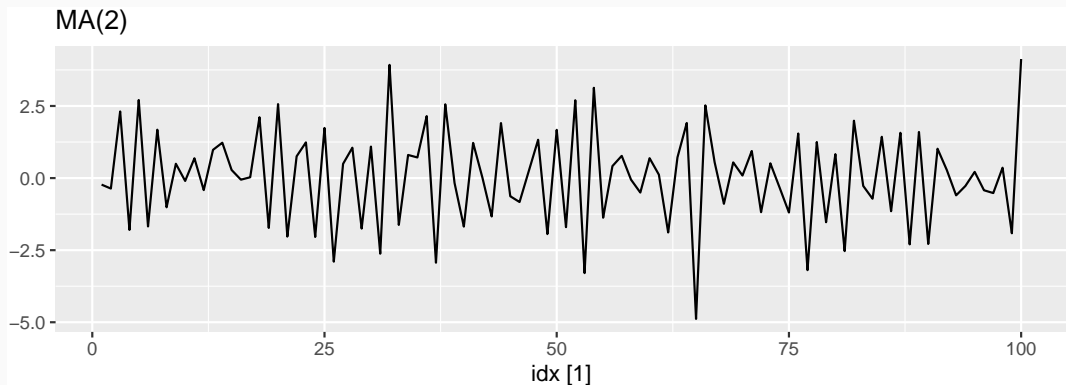




# MA(2) model

$$y_t = \varepsilon_t - \varepsilon_{t-1} + 0.8\varepsilon_{t-2}$$

$$\varepsilon_t \sim N(0, 1), \quad T = 100.$$



# MA( $\infty$ ) models

It is possible to write any stationary AR( $p$ ) process as an MA( $\infty$ ) process.

## Example: AR(1)

$$\begin{aligned}y_t &= \phi_1 y_{t-1} + \varepsilon_t \\&= \phi_1(\phi_1 y_{t-2} + \varepsilon_{t-1}) + \varepsilon_t \\&= \phi_1^2 y_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&= \phi_1^3 y_{t-3} + \phi_1^2 \varepsilon_{t-2} + \phi_1 \varepsilon_{t-1} + \varepsilon_t \\&\dots\end{aligned}$$

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Provided  $-1 < \phi_1 < 1$ :

$$y_t = \varepsilon_t + \phi_1 \varepsilon_{t-1} + \phi_1^2 \varepsilon_{t-2} + \phi_1^3 \varepsilon_{t-3} + \dots$$

# Invertibility

- Any  $MA(q)$  process can be written as an  $AR(\infty)$  process if we impose some constraints on the MA parameters.
- Then the MA model is called “invertible”.
- Invertible models have some mathematical properties that make them easier to use in practice.
- Invertibility of an ARIMA model is equivalent to forecastability of an ETS model.

# Invertibility

## General condition for invertibility

Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

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Complex roots of  $1 + \theta_1 z + \theta_2 z^2 + \dots + \theta_q z^q$  lie outside the unit circle on the complex plane.

- For  $q = 1$ :  $-1 < \theta_1 < 1$ .
- For  $q = 2$ :  
 $-1 < \theta_2 < 1 \quad \theta_2 + \theta_1 > -1 \quad \theta_1 - \theta_2 < 1$ .
- More complicated conditions hold for  $q \geq 3$ .
- Estimation software takes care of this.

# ARIMA models

## Autoregressive Moving Average models:

$$y_t = c + \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} \\ + \theta_1 \varepsilon_{t-1} + \cdots + \theta_q \varepsilon_{t-q} + \varepsilon_t.$$

# ARIMA models

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- Predictors include both **lagged values of  $y_t$  and lagged errors**.
- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.



# ARIMA models

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- Predictors include both **lagged values of  $y_t$  and lagged errors.**
- Conditions on AR coefficients ensure stationarity.
- Conditions on MA coefficients ensure invertibility.

## Autoregressive Integrated Moving Average models

- Combine ARMA model with **differencing.**
- $(1 - B)^d y_t$  follows an ARMA model.

# ARIMA models

## Autoregressive Integrated Moving Average models

### ARIMA( $p, d, q$ ) model

AR:  $p$  = order of the autoregressive part

I:  $d$  = degree of first differencing involved

MA:  $q$  = order of the moving average part.

- White noise model: ARIMA(0,0,0)
- Random walk: ARIMA(0,1,0) with no constant
- Random walk with drift: ARIMA(0,1,0) with const.
- AR( $p$ ): ARIMA( $p,0,0$ )
- MA( $q$ ): ARIMA(0,0, $q$ )

# Backshift notation for ARIMA

## ■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or  $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

## ■ ARIMA(1,1,1) model:

$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR}(1) & \text{First} & & \text{MA}(1) \\ & \text{difference} & & \end{array}$$

# Backshift notation for ARIMA

## ■ ARMA model:

$$y_t = c + \phi_1 B y_t + \dots + \phi_p B^p y_t + \varepsilon_t + \theta_1 B \varepsilon_t + \dots + \theta_q B^q \varepsilon_t$$

or  $(1 - \phi_1 B - \dots - \phi_p B^p) y_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$

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$$\begin{array}{ccccc} (1 - \phi_1 B) & (1 - B) y_t & = & c + (1 + \theta_1 B) \varepsilon_t \\ \uparrow & \uparrow & & \uparrow \\ \text{AR(1)} & \text{First} & & \text{MA(1)} \\ & \text{difference} & & \end{array}$$

Expand:  $y_t = c + y_{t-1} + \phi_1 y_{t-1} - \phi_1 y_{t-2} + \theta_1 \varepsilon_{t-1} + \varepsilon_t$

# R model

## Intercept form

$$(1 - \phi_1 B - \dots - \phi_p B^p) y'_t = c + (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

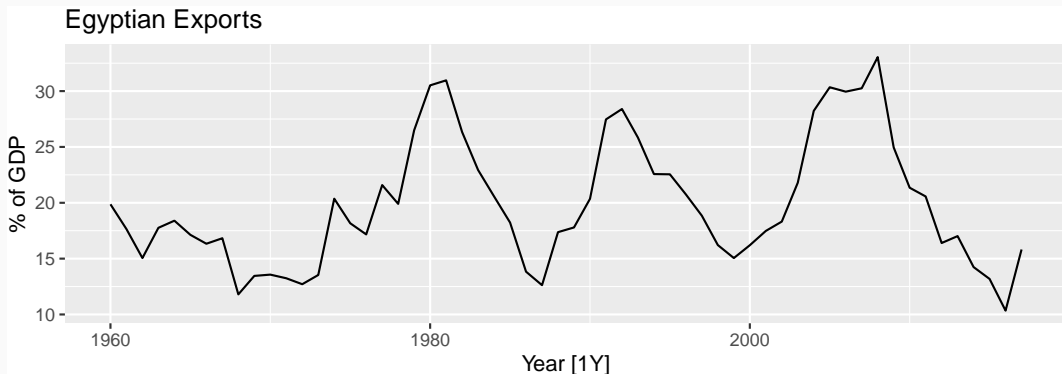
## Mean form

$$(1 - \phi_1 B - \dots - \phi_p B^p)(y'_t - \mu) = (1 + \theta_1 B + \dots + \theta_q B^q) \varepsilon_t$$

- $y'_t = (1 - B)^d y_t$
- $\mu$  is the mean of  $y'_t$ .
- $c = \mu(1 - \phi_1 - \dots - \phi_p)$ .
- fable uses intercept form

# Egyptian exports

```
global_economy |>  
  filter(Code == "EGY") |>  
  autoplot(Exports) +  
  labs(y = "% of GDP", title = "Egyptian Exports")
```



# Egyptian exports

```
fit <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports))
report(fit)
```

```
## Series: Exports
## Model: ARIMA(2,0,1) w/ mean
##
## Coefficients:
##          ar1          ar2          ma1    constant
##          1.676    -0.8034    -0.690         2.562
## s.e.    0.111     0.0928     0.149         0.116
##
## sigma^2 estimated as 8.046:  log likelihood=-142
## AIC=293   AICc=294   BIC=303
```

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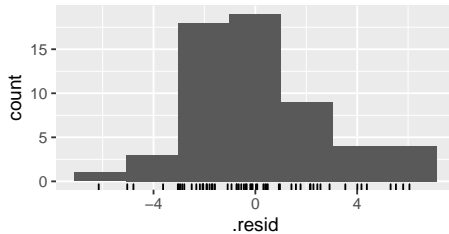
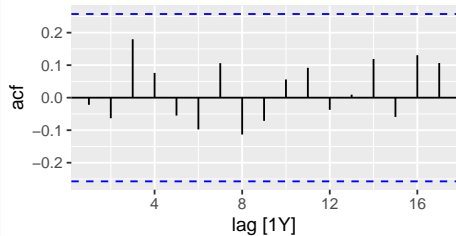
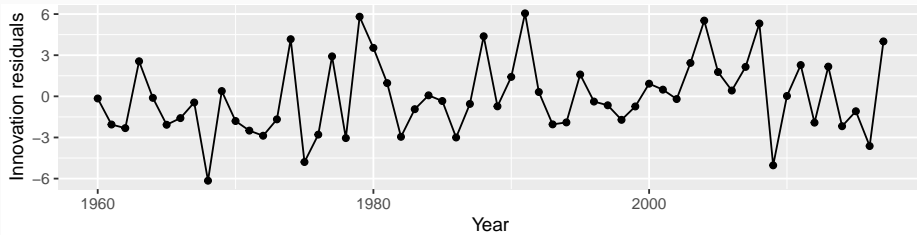
## ARIMA(2,0,1) model:

$$y_t = 2.56 + 1.68y_{t-1} - 0.80y_{t-2} - \underbrace{0.69\varepsilon_{t-1}} + \varepsilon_t,$$



# Egyptian exports

```
gg_tsresiduals(fit)
```



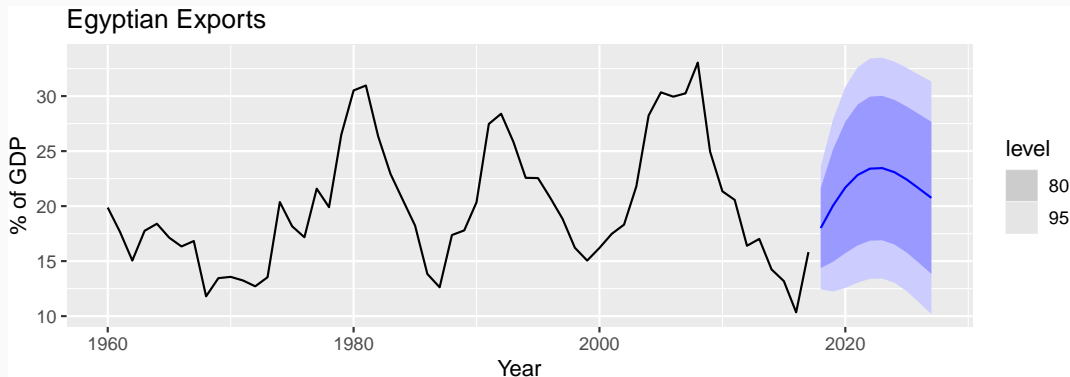
# Egyptian exports

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 10, dof = 4)
```

```
## # A tibble: 1 x 4  
##   Country      .model      lb_stat lb_pvalue  
##   <fct>        <chr>        <dbl>    <dbl>  
## 1 Egypt, Arab Rep. ARIMA(Exports)    5.78    0.448
```

# Egyptian exports

```
fit |>  
  forecast(h = 10) |>  
  autoplot(global_economy) +  
  labs(y = "% of GDP", title = "Egyptian Exports")
```



# Understanding ARIMA models

- If  $c = 0$  and  $d = 0$ , the long-term forecasts will go to zero.
- If  $c = 0$  and  $d = 1$ , the long-term forecasts will go to a non-zero constant.
- If  $c = 0$  and  $d = 2$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 0$ , the long-term forecasts will go to the mean of the data.
- If  $c \neq 0$  and  $d = 1$ , the long-term forecasts will follow a straight line.
- If  $c \neq 0$  and  $d = 2$ , the long-term forecasts will follow a quadratic trend.

# Understanding ARIMA models

## Forecast variance and $d$

- The higher the value of  $d$ , the more rapidly the prediction intervals increase in size.
- For  $d = 0$ , the long-term forecast standard deviation will go to the standard deviation of the historical data.

## Cyclic behaviour

- For cyclic forecasts,  $p \geq 2$  and some restrictions on coefficients are required.
- If  $p = 2$ , we need  $\phi_1^2 + 4\phi_2 < 0$ . Then average cycle of length  $(2\pi) / \left[ \arccos(-\phi_1(1 - \phi_2)/(4\phi_2)) \right]$ .

# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
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# Maximum likelihood estimation

Having identified the model order, we need to estimate the parameters  $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ .

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Having identified the model order, we need to estimate the parameters  $c, \phi_1, \dots, \phi_p, \theta_1, \dots, \theta_q$ .

- MLE is very similar to least squares estimation obtained by minimizing

$$\sum_{t=1}^T e_t^2$$

- The `ARIMA()` function allows CLS or MLE estimation.
- Non-linear optimization must be used in either case.
- Different software will give different estimates.



# Partial autocorrelations

**Partial autocorrelations** measure relationship between  $y_t$  and  $y_{t-k}$ , when the effects of other time lags —  $1, 2, 3, \dots, k-1$  — are removed.

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$\alpha_k$  =  $k$ th partial autocorrelation coefficient

= equal to the estimate of  $\phi_k$  in regression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k} + \varepsilon_t.$$

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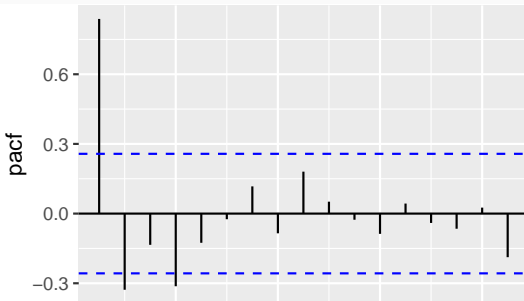
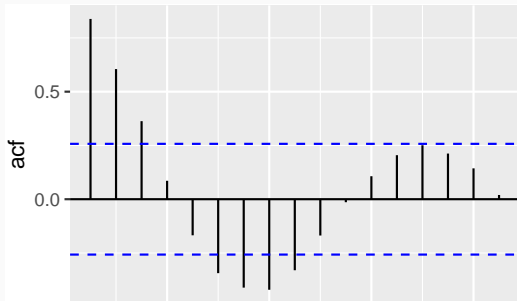
= equal to the estimate of  $\phi_k$  in regression:

$$y_t = c + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \dots + \phi_k y_{t-k} + \varepsilon_t.$$

- Varying number of terms on RHS gives  $\alpha_k$  for different values of  $k$ .
- $\alpha_1 = \rho_1$
- same critical values of  $\pm 1.96/\sqrt{T}$  as for ACF.
- Last significant  $\alpha_k$  indicates the order of an AR model.

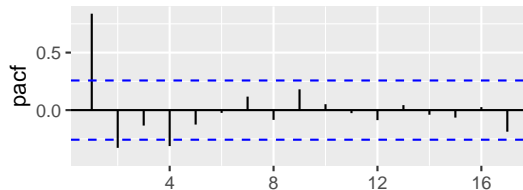
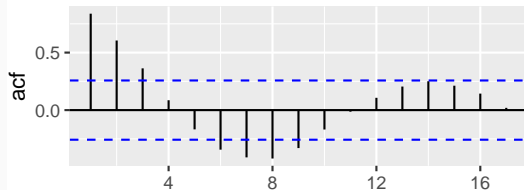
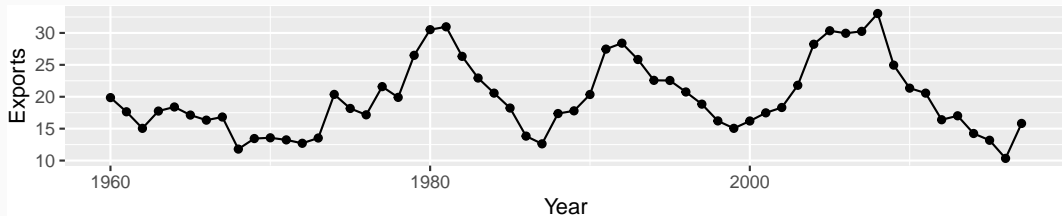
# Egyptian exports

```
egypt <- global_economy |> filter(Code == "EGY")  
egypt |>  
  ACF(Exports) |>  
  autoplot()  
egypt |>  
  PACF(Exports) |>  
  autoplot()
```



# Egyptian exports

```
global_economy |>  
  filter(Code == "EGY") |>  
  gg_tsdisplay(Exports, plot_type = "partial")
```



# ACF and PACF interpretation

## AR(1)

$$\begin{aligned}\rho_k &= \phi_1^k && \text{for } k = 1, 2, \dots; \\ \alpha_1 &= \phi_1 && \alpha_k = 0 \quad \text{for } k = 2, 3, \dots\end{aligned}$$

So we have an AR(1) model when

- autocorrelations exponentially decay
- there is a single significant partial autocorrelation.

# ACF and PACF interpretation

## $AR(p)$

- ACF dies out in an exponential or damped sine-wave manner
- PACF has all zero spikes beyond the  $p$ th spike

So we have an  $AR(p)$  model when

- the ACF is exponentially decaying or sinusoidal
- there is a significant spike at lag  $p$  in PACF, but none beyond  $p$

# ACF and PACF interpretation

## MA(1)

$$\begin{aligned}\rho_1 &= \theta_1 / (1 + \theta_1^2) & \rho_k &= 0 & \text{for } k = 2, 3, \dots; \\ \alpha_k &= -(-\theta_1)^k / (1 + \theta_1^2 + \dots + \theta_1^{2k})\end{aligned}$$

So we have an MA(1) model when

- the PACF is exponentially decaying and
- there is a single significant spike in ACF



# ACF and PACF interpretation

## MA( $q$ )

- PACF dies out in an exponential or damped sine-wave manner
- ACF has all zero spikes beyond the  $q$ th spike

So we have an MA( $q$ ) model when

- the PACF is exponentially decaying or sinusoidal
- there is a significant spike at lag  $q$  in ACF, but none beyond  $q$

# Information criteria

## Akaike's Information Criterion (AIC):

$$\text{AIC} = -2 \log(L) + 2(p + q + k + 1),$$

where  $L$  is the likelihood of the data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  if  $c = 0$ .

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## Corrected AIC:

$$\text{AICc} = \text{AIC} + \frac{2(p + q + k + 1)(p + q + k + 2)}{T - p - q - k - 2}.$$

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Good models are obtained by minimizing either the AIC, AICc or BIC. Our preference is to use the AICc.

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# How does ARIMA() work?

## A non-seasonal ARIMA process

$$\phi(B)(1 - B)^d y_t = c + \theta(B)\varepsilon_t$$

Need to select appropriate orders:  $p, q, d$

## Hyndman and Khandakar (JSS, 2008) algorithm:

- Select no. differences  $d$  and  $D$  via KPSS test and seasonal strength measure.
- Select  $p, q$  by minimising AICc.
- Use stepwise search to traverse model space.

## How does ARIMA() work?

$$\text{AICc} = -2 \log(L) + 2(p + q + k + 1) \left[ 1 + \frac{(p+q+k+2)}{T-p-q-k-2} \right].$$

where  $L$  is the maximised likelihood fitted to the *differenced* data,  $k = 1$  if  $c \neq 0$  and  $k = 0$  otherwise.



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**Step1:** Select current model (with smallest AICc) from:

ARIMA(2,  $d$ , 2),   ARIMA(0,  $d$ , 0),   ARIMA(1,  $d$ , 0),   ARIMA(0,  $d$ , 1)

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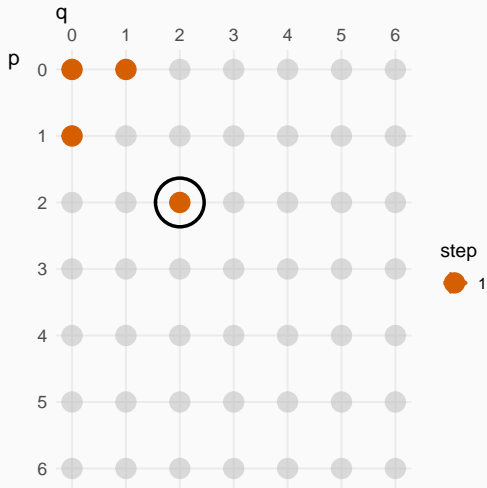
**Step 2:** Consider variations of current model:

- vary one of  $p, q$ , from current model by  $\pm 1$ ;
- $p, q$  both vary from current model by  $\pm 1$ ;
- Include/exclude  $c$  from current model.

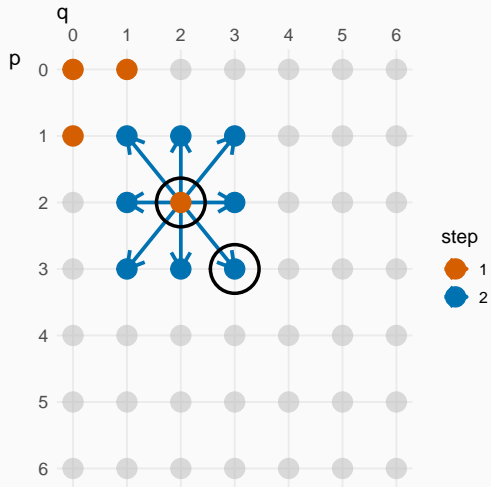
Model with lowest AICc becomes current model.

**Repeat Step 2 until no lower AICc can be found.**

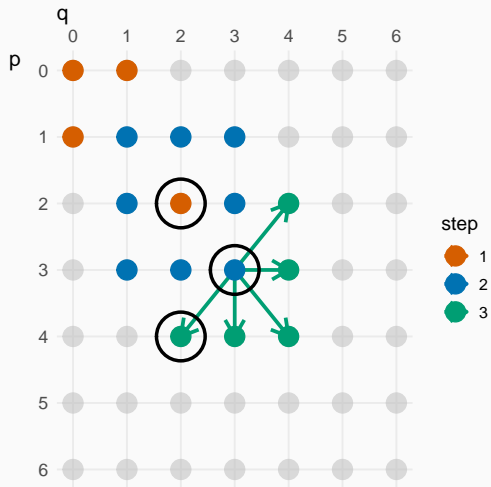
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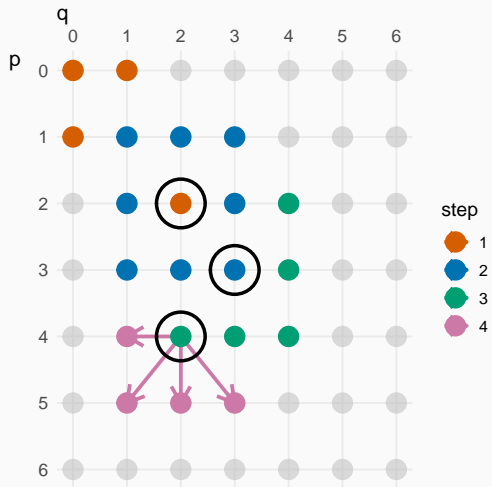
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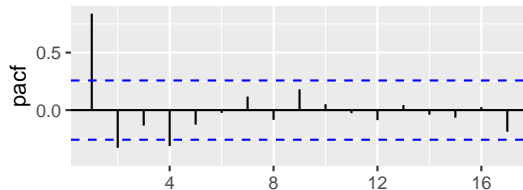
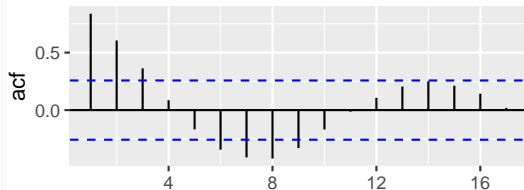
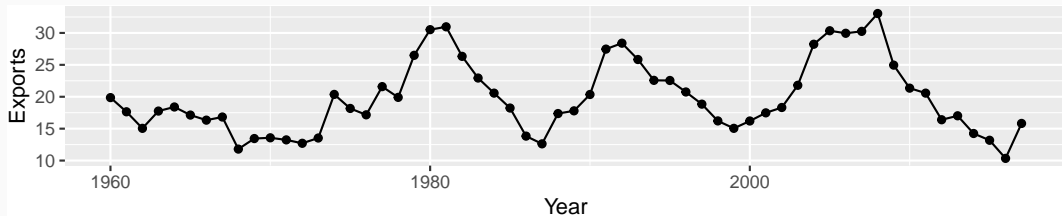


# How does ARIMA() work?



# Egyptian exports

```
global_economy |>  
  filter(Code == "EGY") |>  
  gg_tsdisplay(Exports, plot_type = "partial")
```



# Egyptian exports

```
fit1 <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports ~ pdq(4, 0, 0)))
report(fit1)
```

```
## Series: Exports
## Model: ARIMA(4,0,0) w/ mean
##
## Coefficients:
##          ar1      ar2      ar3      ar4  constant
##          0.986  -0.172   0.181  -0.328      6.692
## s.e.    0.125    0.186   0.186   0.127      0.356
##
## sigma^2 estimated as 7.885:  log likelihood=-141
## AIC=293   AICc=295   BIC=305
```



# Egyptian exports

```
fit2 <- global_economy |>
  filter(Code == "EGY") |>
  model(ARIMA(Exports))
report(fit2)
```

```
## Series: Exports
```

```
## Model: ARIMA(2,0,1) w/ mean
```

```
##
```

```
## Coefficients:
```

```
##          ar1          ar2          ma1    constant
```

```
##          1.676   -0.8034   -0.690         2.562
```

```
## s.e.    0.111    0.0928    0.149         0.116
```

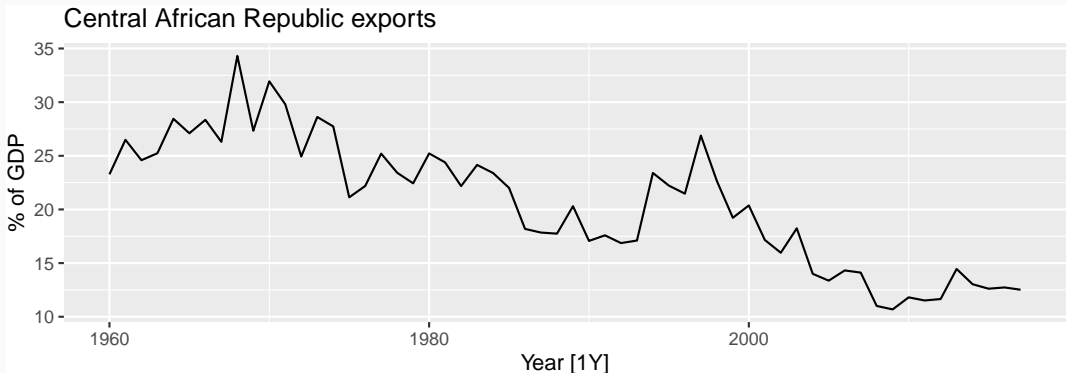
```
##
```

```
## sigma^2 estimated as 8.046:  log likelihood=-142
```

```
## AIC=293   AICc=294   BIC=303
```

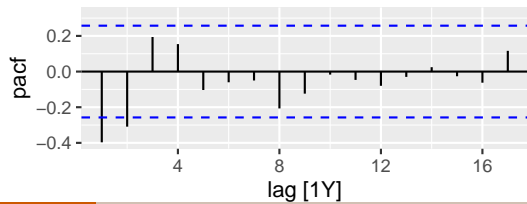
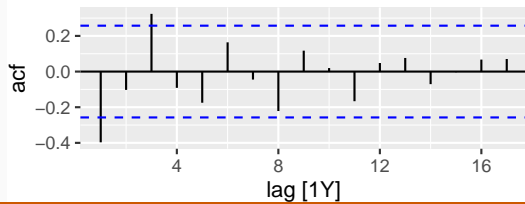
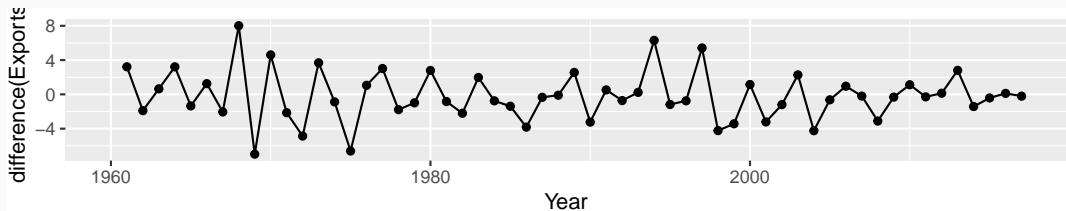
# Central African Republic exports

```
global_economy |>  
  filter(Code == "CAF") |>  
  autoplot(Exports) +  
  labs(title = "Central African Republic exports", y = "% of GDP")
```



# Central African Republic exports

```
global_economy |>  
  filter(Code == "CAF") |>  
  gg_tsdisplay(difference(Exports), plot_type = "partial")
```



# Central African Republic exports

```
caf_fit <- global_economy |>
  filter(Code == "CAF") |>
  model(
    arima210 = ARIMA(Exports ~ pdq(2, 1, 0)),
    arima013 = ARIMA(Exports ~ pdq(0, 1, 3)),
    stepwise = ARIMA(Exports),
    search = ARIMA(Exports, stepwise = FALSE)
  )
```

# Central African Republic exports

```
caf_fit |> pivot_longer(!Country,  
  names_to = "Model name",  
  values_to = "Orders"  
)
```

```
## # A mable: 4 x 3
```

```
## # Key:      Country, Model name [4]
```

##	Country	`Model name`	Orders
##	<fct>	<chr>	<model>
## 1	Central African Republic	arima210	<ARIMA(2,1,0)>
## 2	Central African Republic	arima013	<ARIMA(0,1,3)>
## 3	Central African Republic	stepwise	<ARIMA(2,1,2)>
## 4	Central African Republic	search	<ARIMA(3,1,0)>

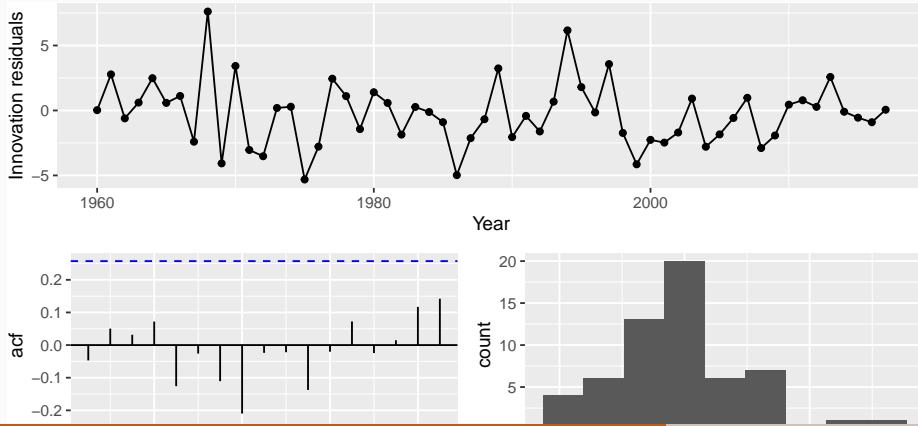
# Central African Republic exports

```
glance(caf_fit) |>  
  arrange(AICc) |>  
  select(.model:BIC)
```

```
## # A tibble: 4 x 6  
##   .model    sigma2 log_lik   AIC  AICc   BIC  
##   <chr>      <dbl>   <dbl> <dbl> <dbl> <dbl>  
## 1 search      6.52   -133.  274.  275.  282.  
## 2 arima210     6.71   -134.  275.  275.  281.  
## 3 arima013     6.54   -133.  274.  275.  282.  
## 4 stepwise    6.42   -132.  274.  275.  284.
```

# Central African Republic exports

```
caf_fit |>  
  select(search) |>  
  gg_tsresiduals()
```



# Central African Republic exports

```
augment(caf_fit) |>  
  filter(.model == "search") |>  
  features(.innov, ljung_box, lag = 10, dof = 3)
```

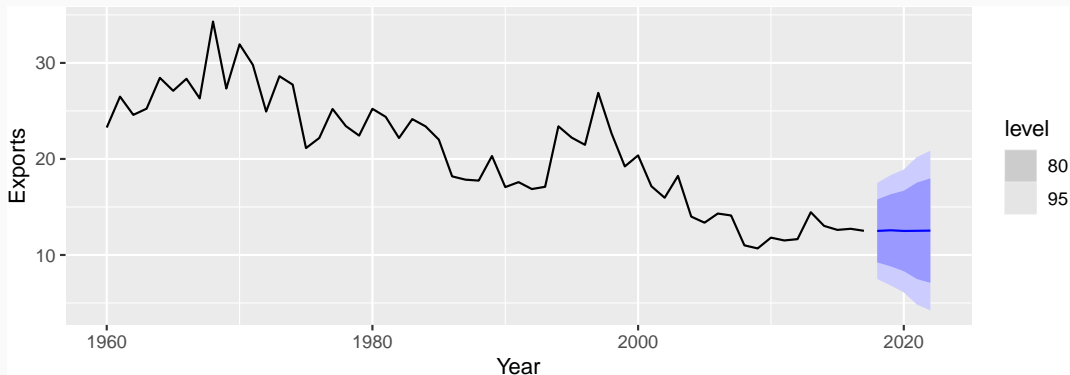
```
## # A tibble: 1 x 4
```

##	Country	.model	lb_stat	lb_pvalue
##	<fct>	<chr>	<dbl>	<dbl>
## 1	Central African Republic	search	5.75	0.569



# Central African Republic exports

```
caf_fit |>  
  forecast(h = 5) |>  
  filter(.model == "search") |>  
  autoplot(global_economy)
```



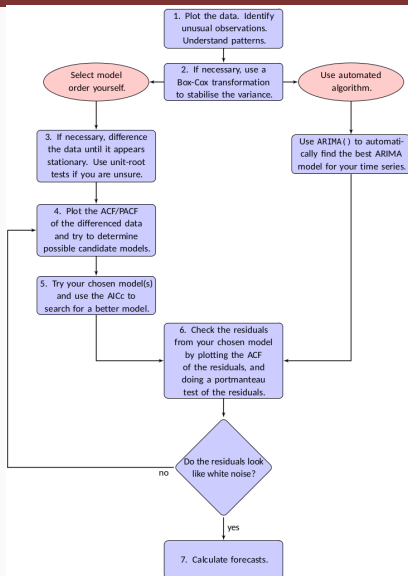
# Modelling procedure with ARIMA ( )

- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 If the data are non-stationary: take first differences of the data until the data are stationary.
- 4 Examine the ACF/PACF: Is an  $AR(p)$  or  $MA(q)$  model appropriate?
- 5 Try your chosen model(s), and use the AICc to search for a better model.
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

# Automatic modelling procedure with `ARIMA()`

- 1 Plot the data. Identify any unusual observations.
- 2 If necessary, transform the data (using a Box-Cox transformation) to stabilize the variance.
- 3 Use ARIMA to automatically select a model.
- 4
- 5
- 6 Check the residuals from your chosen model by plotting the ACF of the residuals, and doing a portmanteau test of the residuals. If they do not look like white noise, try a modified model.
- 7 Once the residuals look like white noise, calculate forecasts.

# Modelling procedure



# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting**
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

# Point forecasts

- 1 Rearrange ARIMA equation so  $y_t$  is on LHS.
- 2 Rewrite equation by replacing  $t$  by  $T + h$ .
- 3 On RHS, replace future observations by their forecasts, future errors by zero, and past errors by corresponding residuals.

Start with  $h = 1$ . Repeat for  $h = 2, 3, \dots$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} \left[ 1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4 \right] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$



# Point forecasts

## ARIMA(3,1,1) forecasts: Step 1

$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} [1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

# Point forecasts

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$$(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3)(1 - B)y_t = (1 + \theta_1 B)\varepsilon_t,$$

$$\begin{aligned} [1 - (1 + \phi_1)B + (\phi_1 - \phi_2)B^2 + (\phi_2 - \phi_3)B^3 + \phi_3 B^4] y_t \\ = (1 + \theta_1 B)\varepsilon_t, \end{aligned}$$

$$\begin{aligned} y_t - (1 + \phi_1)y_{t-1} + (\phi_1 - \phi_2)y_{t-2} + (\phi_2 - \phi_3)y_{t-3} \\ + \phi_3 y_{t-4} = \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

$$\begin{aligned} y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3 y_{t-4} + \varepsilon_t + \theta_1 \varepsilon_{t-1}. \end{aligned}$$

## Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

### ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

# Point forecasts (h=1)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## ARIMA(3,1,1) forecasts: Step 2

$$y_{T+1} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \varepsilon_{T+1} + \theta_1\varepsilon_T.$$

## ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+1|T} = (1 + \phi_1)y_T - (\phi_1 - \phi_2)y_{T-1} - (\phi_2 - \phi_3)y_{T-2} \\ - \phi_3y_{T-3} + \theta_1e_T.$$

## Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

## Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

### ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

## Point forecasts (h=2)

$$y_t = (1 + \phi_1)y_{t-1} - (\phi_1 - \phi_2)y_{t-2} - (\phi_2 - \phi_3)y_{t-3} \\ - \phi_3y_{t-4} + \varepsilon_t + \theta_1\varepsilon_{t-1}.$$

### ARIMA(3,1,1) forecasts: Step 2

$$y_{T+2} = (1 + \phi_1)y_{T+1} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2} + \varepsilon_{T+2} + \theta_1\varepsilon_{T+1}.$$

### ARIMA(3,1,1) forecasts: Step 3

$$\hat{y}_{T+2|T} = (1 + \phi_1)\hat{y}_{T+1|T} - (\phi_1 - \phi_2)y_T - (\phi_2 - \phi_3)y_{T-1} \\ - \phi_3y_{T-2}.$$



# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

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$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

- $v_{T+1|T} = \hat{\sigma}^2$  for all ARIMA models regardless of parameters and orders.
- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

# Prediction intervals

## 95% prediction interval

$$\hat{y}_{T+h|T} \pm 1.96\sqrt{v_{T+h|T}}$$

where  $v_{T+h|T}$  is estimated forecast variance.

- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

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# Prediction intervals

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- Multi-step prediction intervals for ARIMA(0,0,q):

$$y_t = \varepsilon_t + \sum_{i=1}^q \theta_i \varepsilon_{t-i}.$$

$$v_{T|T+h} = \hat{\sigma}^2 \left[ 1 + \sum_{i=1}^{h-1} \theta_i^2 \right], \quad \text{for } h = 2, 3, \dots$$

- AR(1): Rewrite as MA( $\infty$ ) and use above result.
- Other models beyond scope of this subject.

# Prediction intervals

- Prediction intervals **increase in size with forecast horizon.**
- Prediction intervals can be difficult to calculate by hand
- Calculations assume residuals are **uncorrelated** and **normally distributed.**
- Prediction intervals tend to be too narrow.
  - ▶ the uncertainty in the parameter estimates has not been accounted for.
  - ▶ the ARIMA model assumes historical patterns will not change during the forecast period.
  - ▶ the ARIMA model assumes uncorrelated future errors

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# Seasonal ARIMA models

ARIMA	$\underbrace{(p, d, q)}$	$\underbrace{(P, D, Q)_m}$
	↑	↑
	Non-seasonal part of the model	Seasonal part of of the model

where  $m$  = number of observations per year.

# Seasonal ARIMA models

E.g.,  $\text{ARIMA}(1, 1, 1)(1, 1, 1)_4$  model (without constant)



# Seasonal ARIMA models

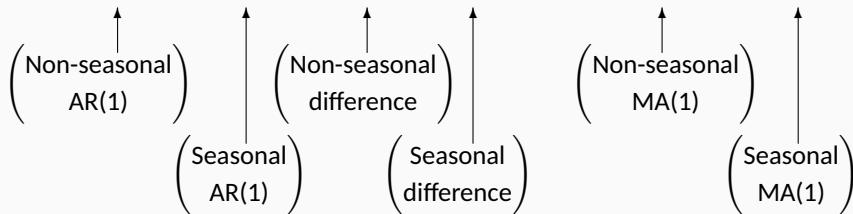
E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

# Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$



# Seasonal ARIMA models

E.g., ARIMA(1, 1, 1)(1, 1, 1)<sub>4</sub> model (without constant)

$$(1 - \phi_1 B)(1 - \Phi_1 B^4)(1 - B)(1 - B^4)y_t = (1 + \theta_1 B)(1 + \Theta_1 B^4)\varepsilon_t.$$

All the factors can be multiplied out and the general model written as follows:

$$\begin{aligned} y_t = & (1 + \phi_1)y_{t-1} - \phi_1 y_{t-2} + (1 + \Phi_1)y_{t-4} \\ & - (1 + \phi_1 + \Phi_1 + \phi_1 \Phi_1)y_{t-5} + (\phi_1 + \phi_1 \Phi_1)y_{t-6} \\ & - \Phi_1 y_{t-8} + (\Phi_1 + \phi_1 \Phi_1)y_{t-9} - \phi_1 \Phi_1 y_{t-10} \\ & + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \Theta_1 \varepsilon_{t-4} + \theta_1 \Theta_1 \varepsilon_{t-5}. \end{aligned}$$

# Common ARIMA models

The US Census Bureau uses the following models most often:

ARIMA(0,1,1)(0,1,1)<sub>m</sub> with log transformation

ARIMA(0,1,2)(0,1,1)<sub>m</sub> with log transformation

ARIMA(2,1,0)(0,1,1)<sub>m</sub> with log transformation

ARIMA(0,2,2)(0,1,1)<sub>m</sub> with log transformation

ARIMA(2,1,2)(0,1,1)<sub>m</sub> with no transformation

# Seasonal ARIMA models

The seasonal part of an AR or MA model will be seen in the seasonal lags of the PACF and ACF.

**ARIMA(0,0,0)(0,0,1)<sub>12</sub> will show:**

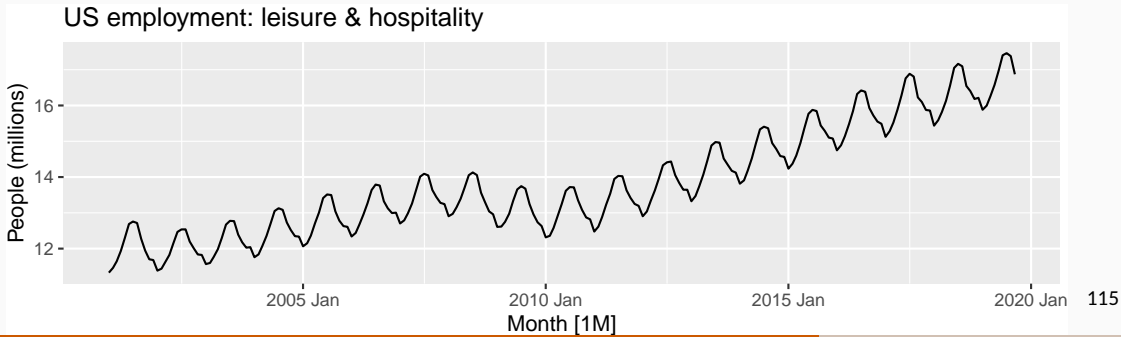
- a spike at lag 12 in the ACF but no other significant spikes.
- The PACF will show exponential decay in the seasonal lags; that is, at lags 12, 24, 36, ....

**ARIMA(0,0,0)(1,0,0)<sub>12</sub> will show:**

- exponential decay in the seasonal lags of the ACF
- a single significant spike at lag 12 in the PACF.

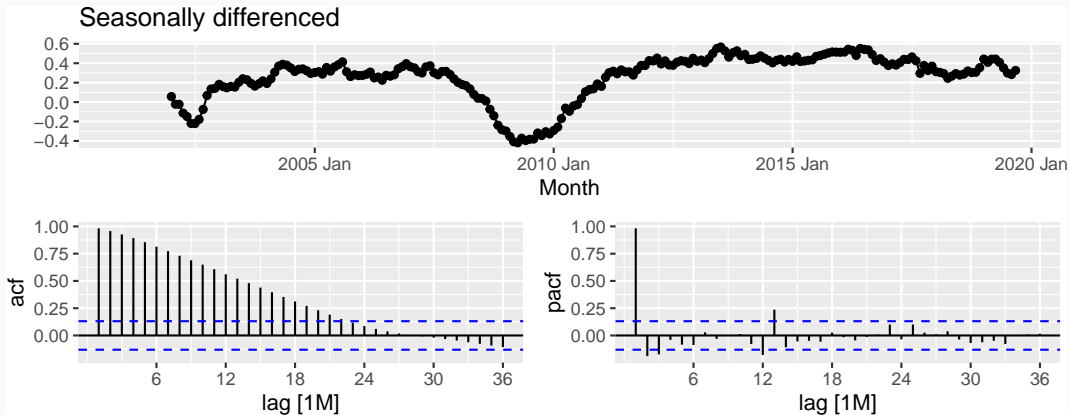
# US leisure employment

```
leisure <- us_employment |>
  filter(Title == "Leisure and Hospitality", year(Month) > 2000) |>
  mutate(Employed = Employed / 1000) |>
  select(Month, Employed)
autoplot(leisure, Employed) +
  labs(title = "US employment: leisure & hospitality", y = "People (millions)")
```



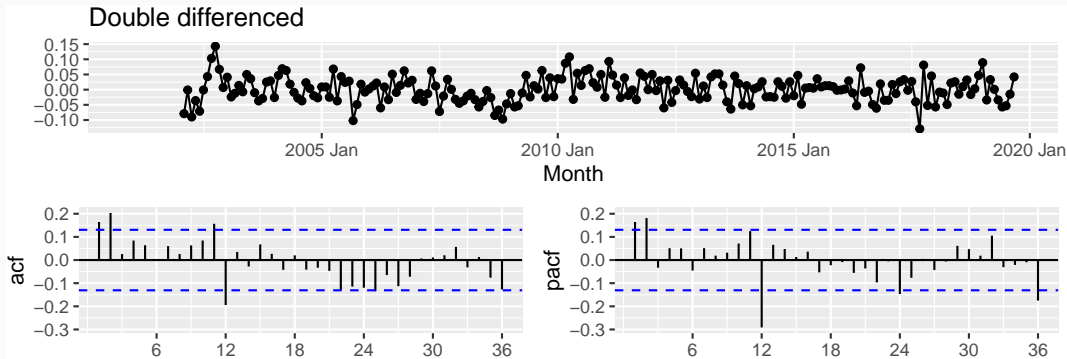
# US leisure employment

```
leisure |>  
  gg_tsdisplay(difference(Employed, 12), plot_type = "partial", lag = 36) +  
  labs(title = "Seasonally differenced", y = "")
```



# US leisure employment

```
leisure |>  
  gg_tsdisplay(difference(Employed, 12) |> difference(),  
    plot_type = "partial", lag = 36  
  ) +  
  labs(title = "Double differenced", y = "")
```





# US leisure employment

```
fit <- leisure |>
  model(
    arima012011 = ARIMA(Employed ~ pdq(0, 1, 2) + PDQ(0, 1, 1)),
    arima210011 = ARIMA(Employed ~ pdq(2, 1, 0) + PDQ(0, 1, 1)),
    auto = ARIMA(Employed, stepwise = FALSE, approx = FALSE)
  )
fit |> pivot_longer(everything(),
  names_to = "Model name",
  values_to = "Orders"
)
```

```
## # A mable: 3 x 2
## # Key:      Model name [3]
##   `Model name`      Orders
##   <chr>             <model>
## 1 arima012011    <ARIMA(0,1,2)(0,1,1)[12]>
## 2 arima210011    <ARIMA(2,1,0)(0,1,1)[12]>
```

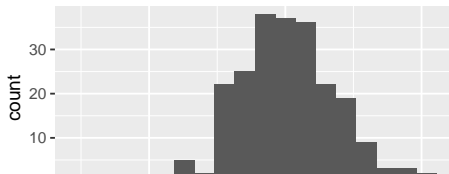
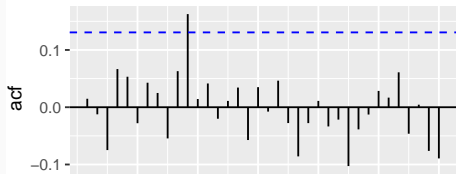
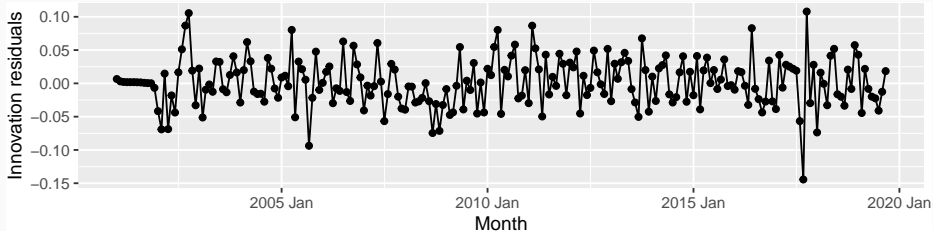
# US leisure employment

```
glance(fit) |>  
  arrange(AICc) |>  
  select(.model:BIC)
```

```
## # A tibble: 3 x 6  
##   .model      sigma2 log_lik   AIC   AICc   BIC  
##   <chr>      <dbl>   <dbl> <dbl> <dbl> <dbl>  
## 1 auto        0.00142    395. -780. -780. -763.  
## 2 arima210011 0.00145    392. -776. -776. -763.  
## 3 arima012011 0.00146    391. -775. -775. -761.
```

# US leisure employment

```
fit |>  
  select(auto) |>  
  gg_tsresiduals(lag = 36)
```



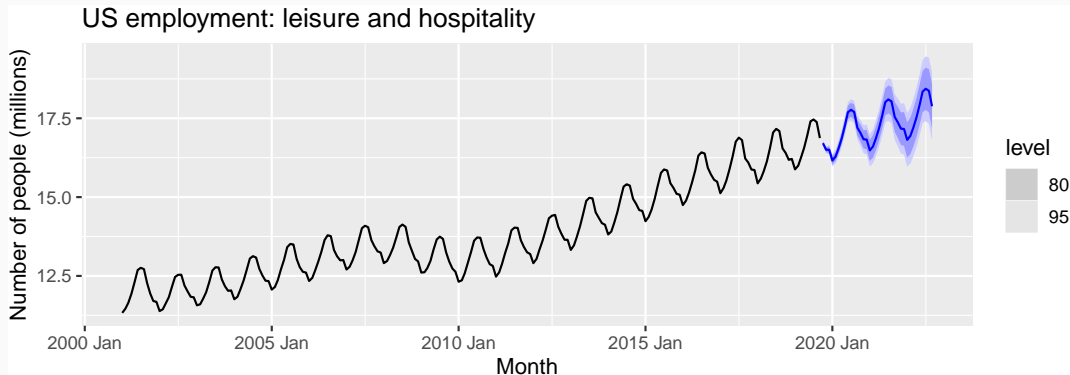
# US leisure employment

```
augment(fit) |> features(.innov, ljung_box, lag = 24, dof = 4)
```

```
## # A tibble: 3 x 3  
##   .model      lb_stat lb_pvalue  
##   <chr>      <dbl>    <dbl>  
## 1 arima012011  22.4      0.320  
## 2 arima210011  18.9      0.527  
## 3 auto        16.6      0.680
```

# US leisure employment

```
forecast(fit, h = 36) |>  
  filter(.model == "auto") |>  
  autoplot(leisure) +  
  labs(title = "US employment: leisure and hospitality", y = "Number of people (millions)")
```

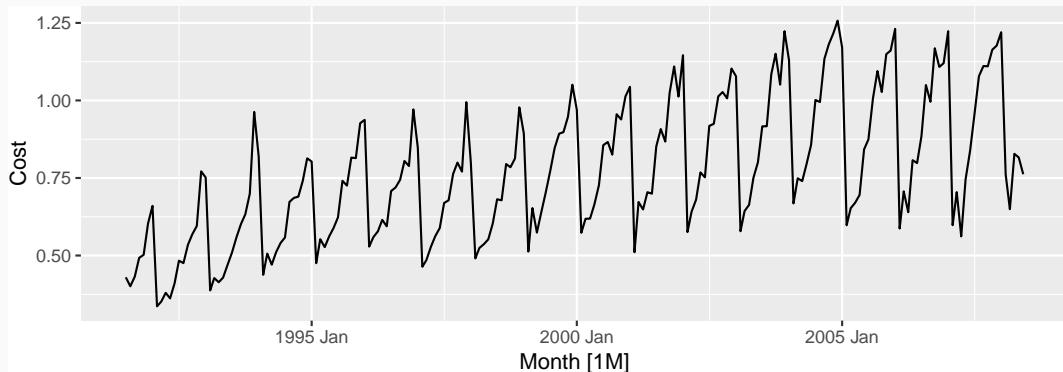


# Corticosteroid drug sales

```
h02 <- PBS |>  
  filter(ATC2 == "H02") |>  
  summarise(Cost = sum(Cost) / 1e6)
```

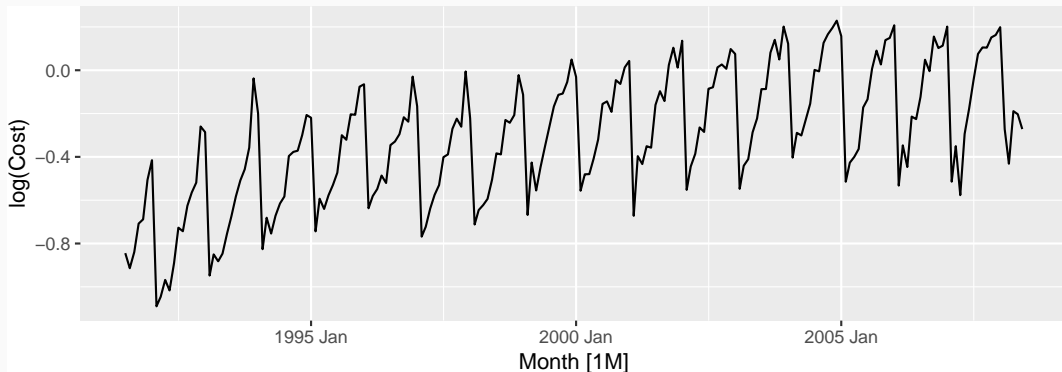
# Corticosteroid drug sales

```
h02 |> autoplot(  
  Cost  
)
```



# Corticosteroid drug sales

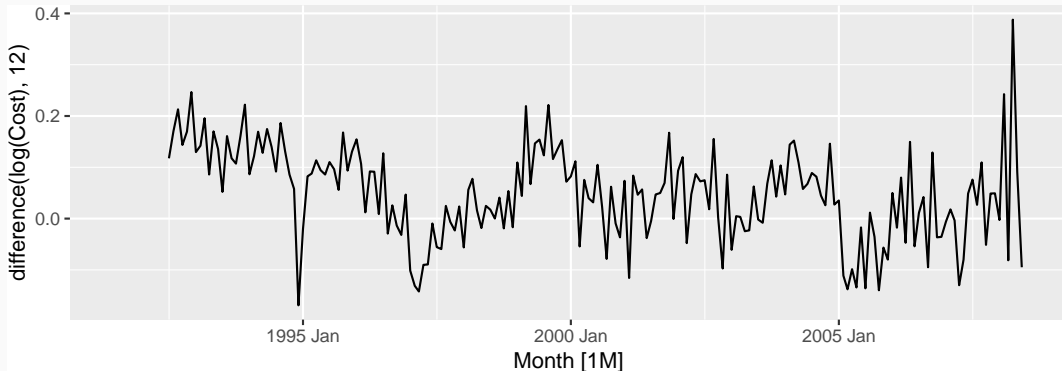
```
h02 |> autoplot(  
  log(Cost)  
)
```





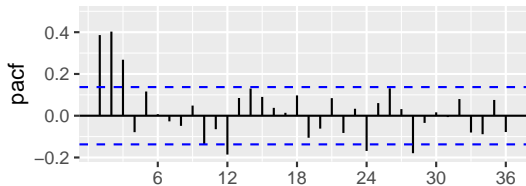
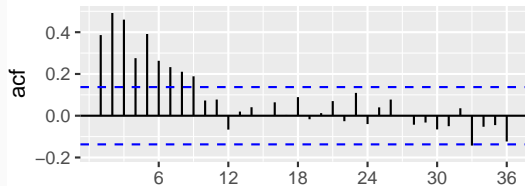
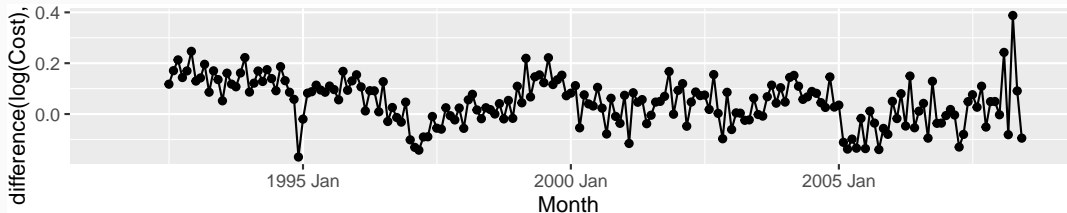
# Corticosteroid drug sales

```
h02 |> autoplot(  
  log(Cost) |> difference(12)  
)
```



# Corticosteroid drug sales

```
h02 |> gg_tsdisplay(difference(log(Cost), 12),  
  lag_max = 36, plot_type = "partial"  
)
```



# Corticosteroid drug sales

- Choose  $D = 1$  and  $d = 0$ .
- Spikes in PACF at lags 12 and 24 suggest seasonal AR(2) term.
- Spikes in PACF suggests possible non-seasonal AR(3) term.
- Initial candidate model:  $\text{ARIMA}(3,0,0)(2,1,0)_{12}$ .

## Corticosteroid drug sales

.model	AICc
ARIMA(3,0,1)(0,1,2)[12]	-485
ARIMA(3,0,1)(1,1,1)[12]	-484
ARIMA(3,0,1)(0,1,1)[12]	-484
ARIMA(3,0,1)(2,1,0)[12]	-476
ARIMA(3,0,0)(2,1,0)[12]	-475
ARIMA(3,0,2)(2,1,0)[12]	-475
ARIMA(3,0,1)(1,1,0)[12]	-463

# Corticosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))
report(fit)
```

```
## Series: Cost
```

```
## Model: ARIMA(3,0,1)(0,1,2)[12]
```

```
## Transformation: log(Cost)
```

```
##
```

```
## Coefficients:
```

	ar1	ar2	ar3	ma1	sma1	sma2
	-0.160	0.5481	0.5678	0.383	-0.5222	-0.1768
s.e.	0.164	0.0878	0.0942	0.190	0.0861	0.0872

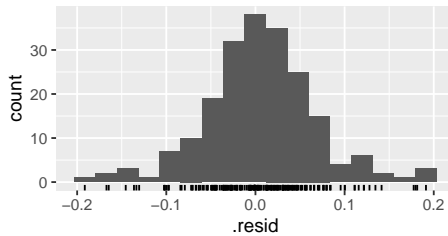
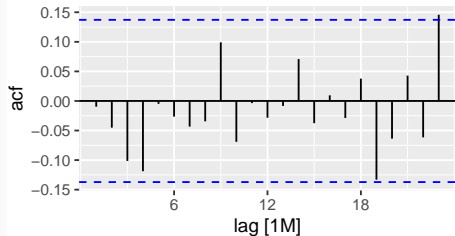
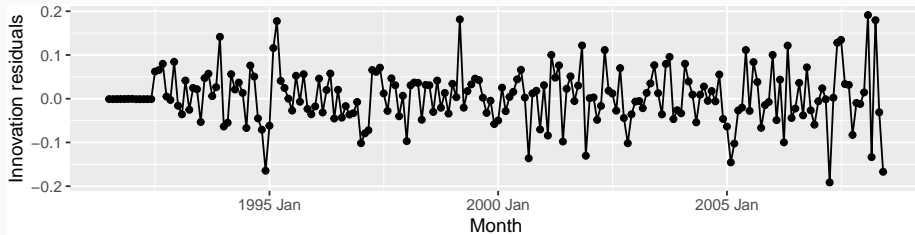
```
##
```

```
## sigma^2 estimated as 0.004278: log likelihood=250
```

```
## AIC=-486 AICc=-485 BIC=-463
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 6)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      50.7      0.0104
```

# Corticosteroid drug sales

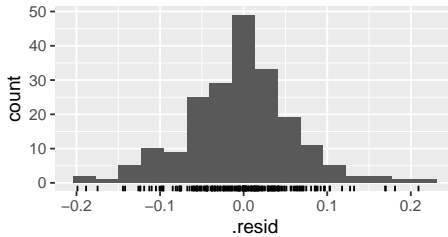
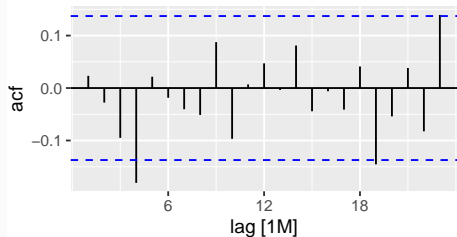
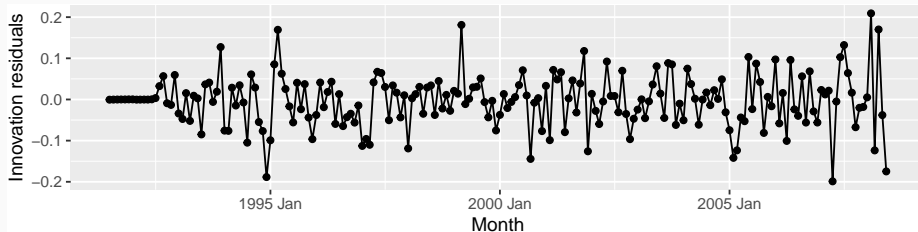
```
fit <- h02 |> model(auto = ARIMA(log(Cost)))  
report(fit)
```

```
## Series: Cost  
## Model: ARIMA(2,1,0)(0,1,1)[12]  
## Transformation: log(Cost)  
##  
## Coefficients:  
##          ar1      ar2      sma1  
##      -0.8491  -0.4207  -0.6401  
## s.e.   0.0712   0.0714   0.0694  
##  
## sigma^2 estimated as 0.004387:  log likelihood=245  
## AIC=-483   AICc=-483   BIC=-470
```



# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 3)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 auto      59.3    0.00332
```

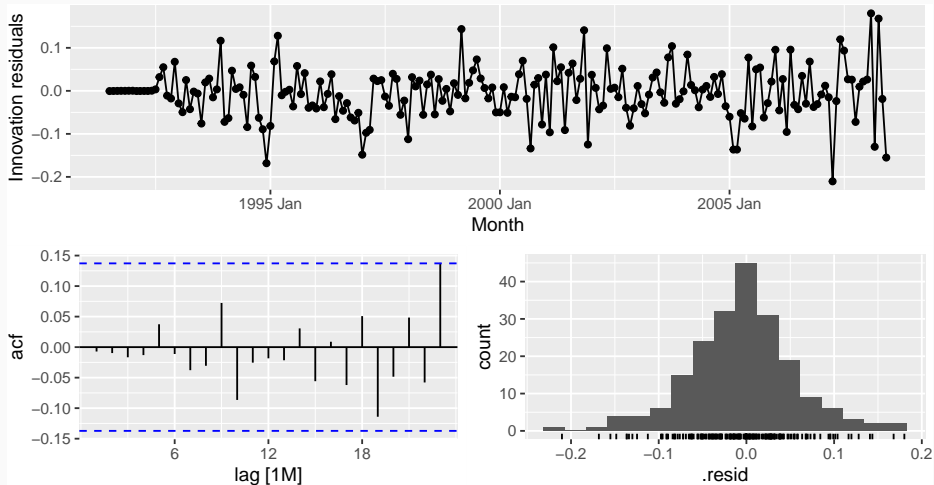
# Corticosteroid drug sales

```
fit <- h02 |>
  model(best = ARIMA(log(Cost),
    stepwise = FALSE,
    approximation = FALSE,
    order_constraint = p + q + P + Q <= 9
  ))
report(fit)
```

```
## Series: Cost
## Model: ARIMA(4,1,1)(2,1,2)[12]
## Transformation: log(Cost)
##
## Coefficients:
##          ar1      ar2      ar3      ar4      ma1      sar1      sar2      sma1      sma2
##      -0.0425  0.210  0.202  -0.227  -0.742  0.621  -0.383  -1.202  0.496
## s.e.   0.2167  0.181  0.114   0.081   0.207  0.242   0.118   0.249  0.213
##
## sigma^2 estimated as 0.004049:  log likelihood=254
```

# Corticosteroid drug sales

```
gg_tsresiduals(fit)
```



# Corticosteroid drug sales

```
augment(fit) |>  
  features(.innov, ljung_box, lag = 36, dof = 9)
```

```
## # A tibble: 1 x 3  
##   .model lb_stat lb_pvalue  
##   <chr>    <dbl>    <dbl>  
## 1 best      36.5      0.106
```

# Corticosteroid drug sales

Training data: July 1991 to June 2006

Test data: July 2006–June 2008

```
fit <- h02 |>
  filter_index(~ "2006 Jun") |>
  model(
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 0) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 2) + PDQ(2, 1, 0)),
    ARIMA(log(Cost) ~ 0 + pdq(3, 0, 1) + PDQ(1, 1, 0))
    # ... #
  )

fit |>
  forecast(h = "2 years") |>
  accuracy(h02)
```

## Corticosteroid drug sales

.model	RMSE
ARIMA(3,0,1)(1,1,1)[12]	0.0619
ARIMA(3,0,1)(0,1,2)[12]	0.0621
ARIMA(3,0,1)(0,1,1)[12]	0.0630
ARIMA(2,1,0)(0,1,1)[12]	0.0630
ARIMA(4,1,1)(2,1,2)[12]	0.0631
ARIMA(3,0,2)(2,1,0)[12]	0.0651
ARIMA(3,0,1)(2,1,0)[12]	0.0653
ARIMA(3,0,1)(1,1,0)[12]	0.0666
ARIMA(3,0,0)(2,1,0)[12]	0.0668

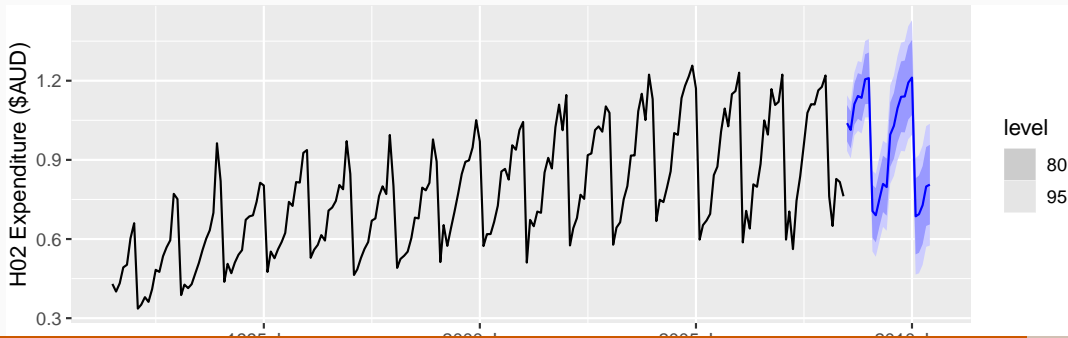
# Corticosteroid drug sales

- Models with lowest AICc values tend to give slightly better results than the other models.
- AICc comparisons must have the same orders of differencing.  
But RMSE test set comparisons can involve any models.
- Use the best model available, even if it does not pass all tests.



# Corticosteroid drug sales

```
fit <- h02 |>  
  model(ARIMA(Cost ~ 0 + pdq(3, 0, 1) + PDQ(0, 1, 2)))  
fit |>  
  forecast() |>  
  autoplot(h02) +  
  labs(y = "H02 Expenditure ($AUD)")
```



# Outline

- 1 Stationarity and differencing
- 2 Non-seasonal ARIMA models
- 3 Estimation and order selection
- 4 ARIMA modelling in R
- 5 Forecasting
- 6 Seasonal ARIMA models
- 7 ARIMA vs ETS

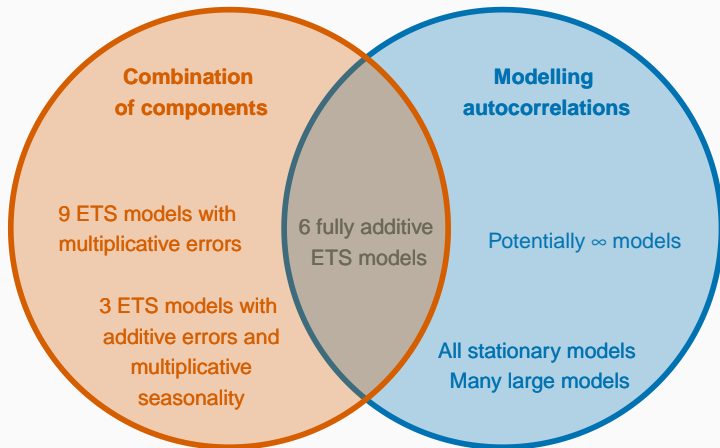
# ARIMA vs ETS

- Myth that ARIMA models are more general than exponential smoothing.
- Linear exponential smoothing models all special cases of ARIMA models.
- Non-linear exponential smoothing models have no equivalent ARIMA counterparts.
- Many ARIMA models have no exponential smoothing counterparts.
- ETS models all non-stationary. Models with seasonality or non-damped trend (or both) have two unit roots; all other models have one unit root.

# ARIMA vs ETS

## ETS models

## ARIMA models



# Equivalences

ETS model	ARIMA model	Parameters
ETS(A,N,N)	ARIMA(0,1,1)	$\theta_1 = \alpha - 1$
ETS(A,A,N)	ARIMA(0,2,2)	$\theta_1 = \alpha + \beta - 2$ $\theta_2 = 1 - \alpha$
ETS(A,A <sub>d</sub> ,N)	ARIMA(1,1,2)	$\phi_1 = \phi$ $\theta_1 = \alpha + \phi\beta - 1 - \phi$ $\theta_2 = (1 - \alpha)\phi$
ETS(A,N,A)	ARIMA(0,0,m)(0,1,0) <sub>m</sub>	
ETS(A,A,A)	ARIMA(0,1,m + 1)(0,1,0) <sub>m</sub>	
ETS(A,A <sub>d</sub> ,A)	ARIMA(1,0,m + 1)(0,1,0) <sub>m</sub>	

# Example: Australian population

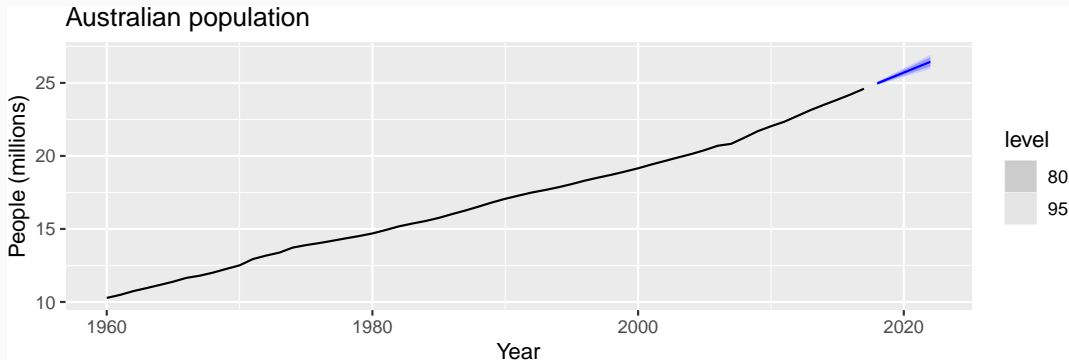
```
aus_economy <- global_economy |>
  filter(Code == "AUS") |>
  mutate(Population = Population / 1e6)
aus_economy |>
  slice(-n()) |>
  stretch_tsibble(.init = 10) |>
  model(
    ets = ETS(Population),
    arima = ARIMA(Population)
  ) |>
  forecast(h = 1) |>
  accuracy(aus_economy) |>
  select(.model, ME:RMSSE)
```

```
## # A tibble: 2 x 8
```

##	.model	ME	RMSE	MAE	MPE	MAPE	MASE	RMSSE
##	<chr>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>

# Example: Australian population

```
aus_economy |>  
  model(ETS(Population)) |>  
  forecast(h = "5 years") |>  
  autoplot(aus_economy) +  
  labs(title = "Australian population", y = "People (millions)")
```



# Example: Cement production

```
cement <- aus_production |>
  select(Cement) |>
  filter_index("1988 Q1" ~ .)
train <- cement |> filter_index(. ~ "2007 Q4")
fit <- train |>
  model(
    arima = ARIMA(Cement),
    ets = ETS(Cement)
  )
```



# Example: Cement production

```
fit |>  
  select(arima) |>  
  report()
```

```
## Series: Cement  
## Model: ARIMA(1,0,1)(2,1,1)[4] w/ drift  
##  
## Coefficients:  
##          ar1      ma1      sar1      sar2      sma1      constant  
##          0.8886  -0.237   0.081   -0.234  -0.898         5.39  
## s.e.    0.0842   0.133   0.157    0.139   0.178         1.48  
##  
## sigma^2 estimated as 11456:  log likelihood=-464  
## AIC=941   AICc=943   BIC=957
```

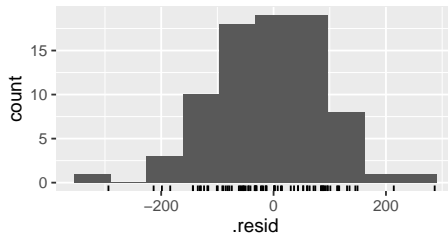
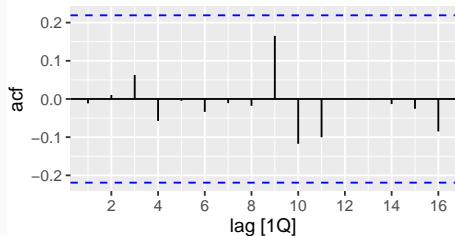
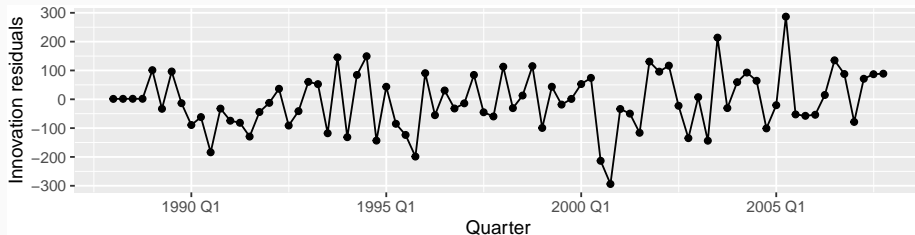
# Example: Cement production

```
fit |>  
  select(ets) |>  
  report()
```

```
## Series: Cement  
## Model: ETS(M,N,M)  
## Smoothing parameters:  
##   alpha = 0.753  
##   gamma = 1e-04  
##  
## Initial states:  
## l[0] s[0] s[-1] s[-2] s[-3]  
## 1695 1.03  1.05  1.01 0.912  
##  
## sigma^2:  0.0034  
##  
## AIC AICc BIC  
## 1104 1106 1121
```

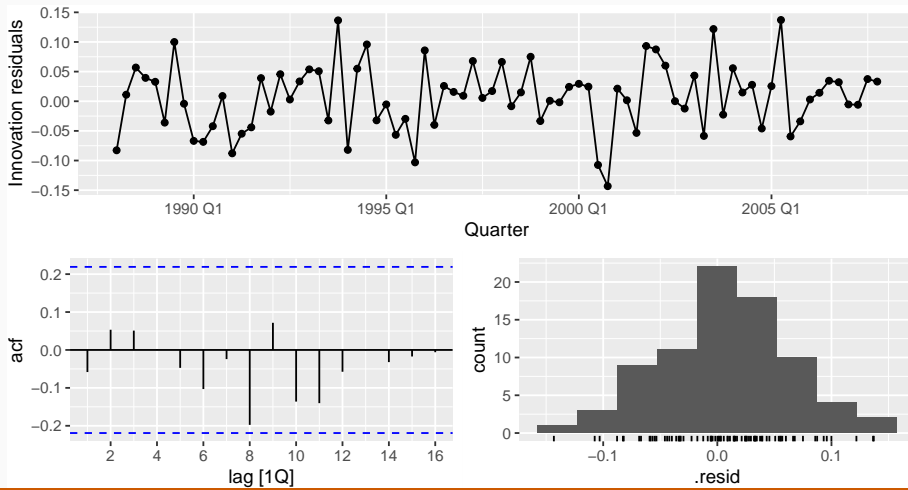
# Example: Cement production

```
gg_tsresiduals(fit |> select(arima), lag_max = 16)
```



# Example: Cement production

```
gg_tsresiduals(fit |> select(ets), lag_max = 16)
```



# Example: Cement production

```
fit |>
  select(arima) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>    <dbl>
## 1 arima      6.37      0.783
```

# Example: Cement production

```
fit |>
  select(ets) |>
  augment() |>
  features(.innov, ljung_box, lag = 16, dof = 6)
```

```
## # A tibble: 1 x 3
##   .model lb_stat lb_pvalue
##   <chr>    <dbl>    <dbl>
## 1 ets      10.0      0.438
```

# Example: Cement production

```
fit |>
  forecast(h = "2 years 6 months") |>
  accuracy(cement) |>
  select(-ME, -MPE, -ACF1)
```

```
## # A tibble: 2 x 7
##   .model .type  RMSE    MAE  MAPE  MASE RMSSE
##   <chr>  <chr> <dbl> <dbl> <dbl> <dbl> <dbl>
## 1 arima  Test   216.  186.  8.68  1.27  1.26
## 2 ets    Test   222.  191.  8.85  1.30  1.29
```

# Example: Cement production

```
fit |>  
  select(arima) |>  
  forecast(h = "3 years") |>  
  autoplot(cement) +  
  labs(title = "Cement production in Australia", y = "Tonnes ('000)")
```

