

# ETC3550/ETC5550

## Applied forecasting

Ch5. The forecasters' toolbox

[OTexts.org/fpp3/](http://OTexts.org/fpp3/)

Rob J Hyndman  
George Athanasopoulos

# FORECASTING

## PRINCIPLES AND PRACTICE

A comprehensive introduction to the latest forecasting methods using R. Learn to improve your forecast accuracy using dozens of real data examples.



3RD EDITION

**oTexts**  
MAKING OPEN ACCESS EDUCATION

# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Outline

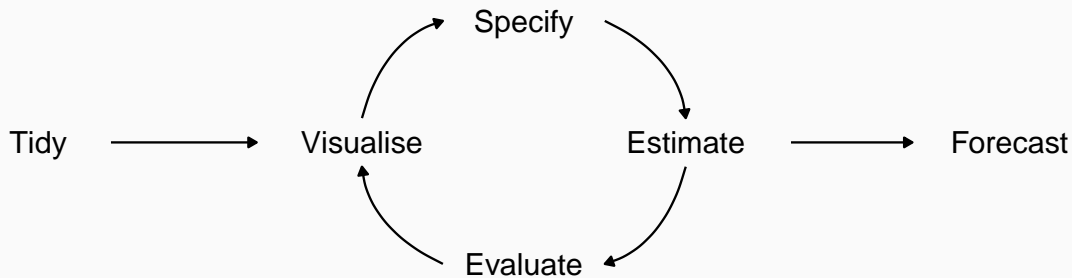
- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# A tidy forecasting workflow

The process of producing forecasts can be split up into a few fundamental steps.

- 1 Preparing data
- 2 Data visualisation
- 3 Specifying a model
- 4 Model estimation
- 5 Accuracy & performance evaluation
- 6 Producing forecasts

# A tidy forecasting workflow



# Data preparation (tidy)

```
gdppc <- global_economy |>
  mutate(GDP_per_capita = GDP / Population) |>
  select(Year, Country, GDP, Population, GDP_per_capita)
gdppc
```

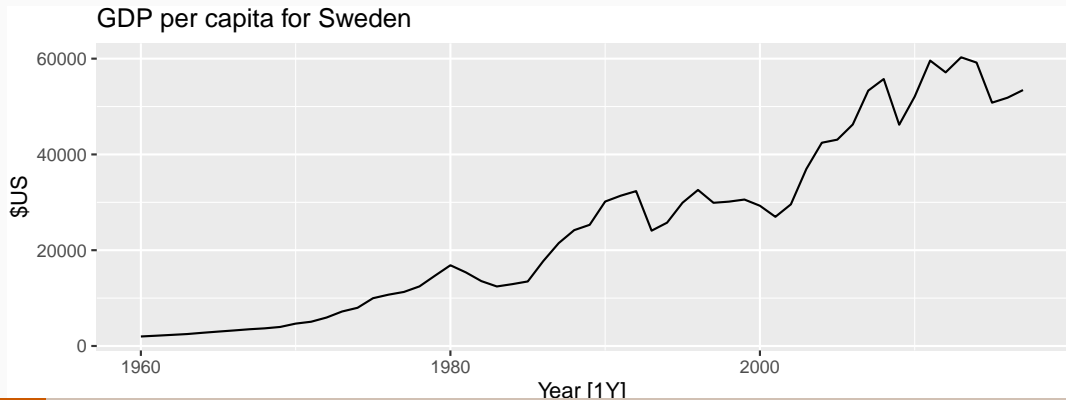
```
## # A tsibble: 15,150 x 5 [1Y]
```

```
## # Key:          Country [263]
```

##	Year	Country	GDP	Population	GDP_per_capita
##	<dbl>	<fct>	<dbl>	<dbl>	<dbl>
## 1	1960	Afghanistan	537777811.	8996351	59.8
## 2	1961	Afghanistan	548888896.	9166764	59.9
## 3	1962	Afghanistan	546666678.	9345868	58.5
## 4	1963	Afghanistan	751111191.	9533954	78.8
## 5	1964	Afghanistan	800000044.	9731361	82.2
## 6	1965	Afghanistan	1006666638.	9938414	101.

# Data visualisation

```
gdppc |>  
  filter(Country == "Sweden") |>  
  autoplot(GDP_per_capita) +  
  labs(title = "GDP per capita for Sweden", y = "$US")
```



# Model estimation

The `model()` function trains models to data.

```
fit <- gdppc |>  
  model(trend_model = TSLM(GDP_per_capita ~ trend()))  
fit
```

```
## # A mable: 263 x 2  
## # Key:      Country [263]  
##   Country      trend_model  
##   <fct>         <model>  
## 1 Afghanistan <TSLM>  
## 2 Albania      <TSLM>  
## 3 Algeria      <TSLM>  
## 4 American Samoa <TSLM>  
## 5 Andorra      <TSLM>
```



# Model estimation

The `model()` function trains models to data.

```
fit <- gdppc |>
  model(trend_model = TSLM(GDP_per_capita ~ trend()))
fit
```

```
## # A mable: 263 x 2
## # Key:      Country [263]
##   Country      trend_model
##   <fct>         <model>
## 1 Afghanistan <TSLM>
## 2 Albania     <TSLM>
## 3 Algeria     <TSLM>
## 4 American Samoa <TSLM>
## 5 Andorra     <TSLM>
```

A mable is a model table, each cell corresponds to a fitted model.

# Producing forecasts

```
fit |> forecast(h = "3 years")
```

```
## # A tibble: 789 x 5 [1Y]
```

```
## # Key:      Country, .model [263]
```

##	Country	.model	Year	GDP_per_capita	.mean
##	<fct>	<chr>	<dbl>	<dist>	<dbl>
##	1 Afghanistan	trend_model	2018	N(526, 9653)	526.
##	2 Afghanistan	trend_model	2019	N(534, 9689)	534.
##	3 Afghanistan	trend_model	2020	N(542, 9727)	542.
##	4 Albania	trend_model	2018	N(4716, 476419)	4716.
##	5 Albania	trend_model	2019	N(4867, 481086)	4867.
##	6 Albania	trend_model	2020	N(5018, 486012)	5018.
##	7 Algeria	trend_model	2018	N(4410, 643094)	4410.
##	8 Algeria	trend_model	2019	N(4489, 645311)	4489.

# Producing forecasts

```
fit |> forecast(h = "3 years")
```

A fable is a forecast table with point forecasts and distributions.

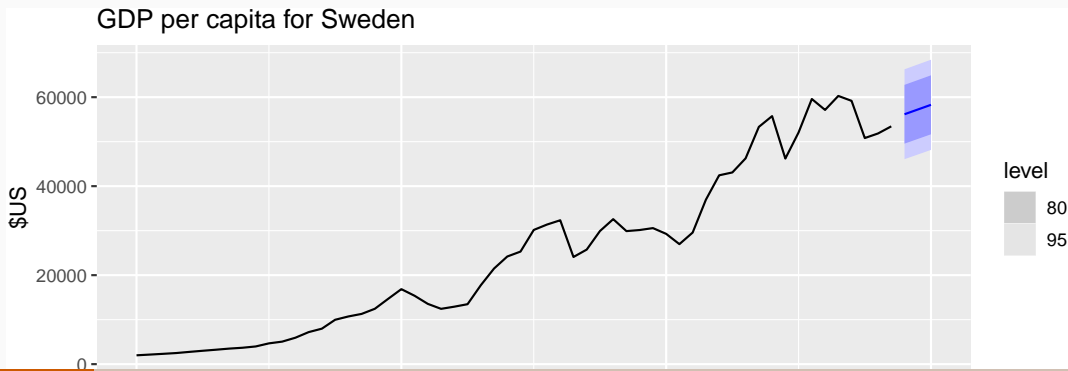
```
## # A fable: 789 x 5 [1Y]
```

```
## # Key:      Country, .model [263]
```

##	Country	.model	Year	GDP_per_capita	.mean
##	<fct>	<chr>	<dbl>	<dist>	<dbl>
##	1 Afghanistan	trend_model	2018	N(526, 9653)	526.
##	2 Afghanistan	trend_model	2019	N(534, 9689)	534.
##	3 Afghanistan	trend_model	2020	N(542, 9727)	542.
##	4 Albania	trend_model	2018	N(4716, 476419)	4716.
##	5 Albania	trend_model	2019	N(4867, 481086)	4867.
##	6 Albania	trend_model	2020	N(5018, 486012)	5018.
##	7 Algeria	trend_model	2018	N(4410, 643094)	4410.
##	8 Algeria	trend_model	2019	N(4489, 645311)	4489.

# Visualising forecasts

```
fit |>  
  forecast(h = "3 years") |>  
  filter(Country == "Sweden") |>  
  autoplot(gdppc) +  
  labs(title = "GDP per capita for Sweden", y = "$US")
```



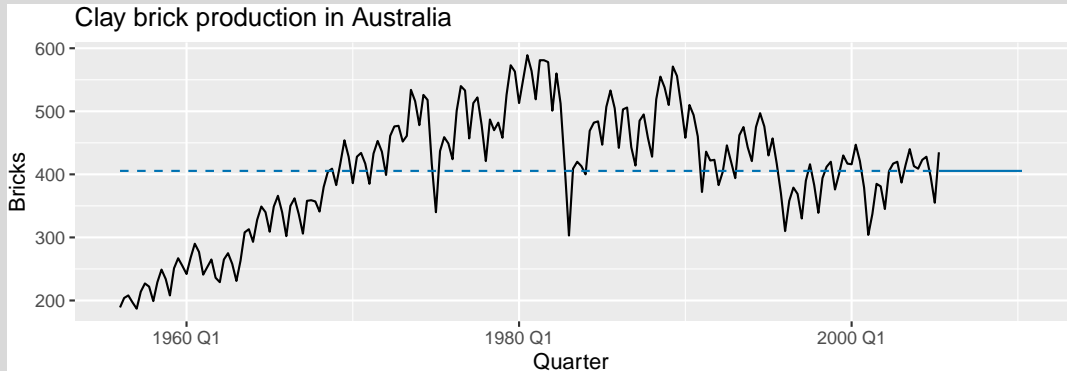
# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Some simple forecasting methods

## MEAN( $y$ ): Average method

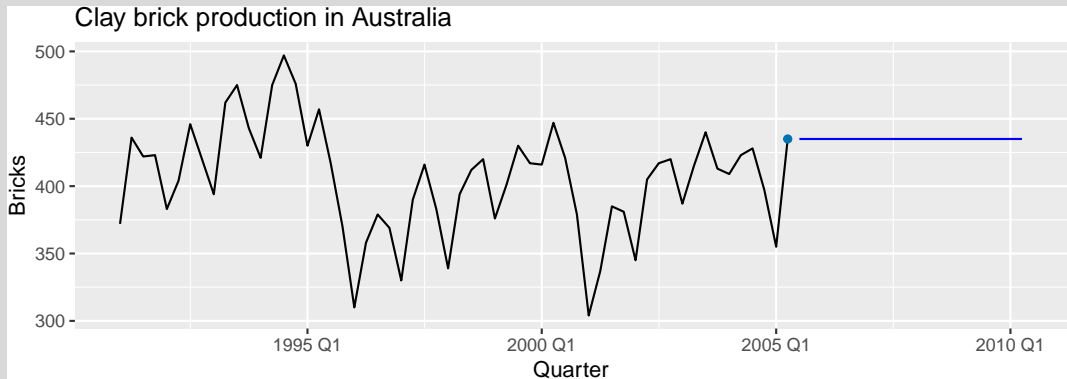
- Forecast of all future values is equal to mean of historical data  $\{y_1, \dots, y_T\}$ .
- Forecasts:  $\hat{y}_{T+h|T} = \bar{y} = (y_1 + \dots + y_T)/T$



# Some simple forecasting methods

## NAIVE( $y$ ): Naïve method

- Forecasts equal to last observed value.
- Forecasts:  $\hat{y}_{T+h|T} = y_T$ .
- Consequence of efficient market hypothesis.

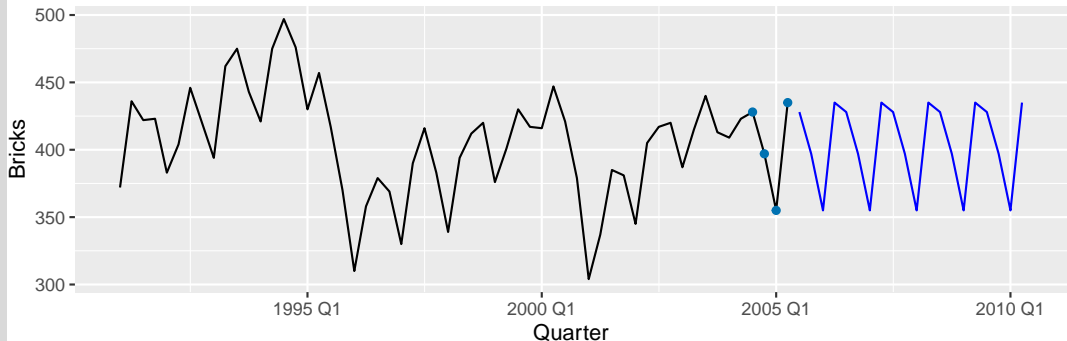


# Some simple forecasting methods

## SNAIVE( $y \sim \text{lag}(m)$ ): Seasonal naïve method

- Forecasts equal to last value from same season.
- Forecasts:  $\hat{y}_{T+h|T} = y_{T+h-m(k+1)}$ , where  $m$  = seasonal period and  $k$  is the integer part of  $(h-1)/m$ .

Clay brick production in Australia





# Some simple forecasting methods

## `RW(y ~ drift())`: Drift method

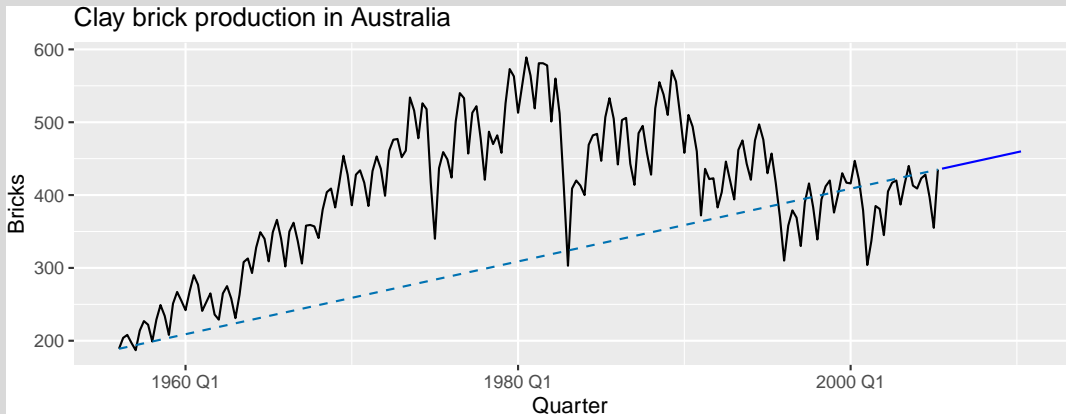
- Forecasts equal to last value plus average change.
- Forecasts:

$$\begin{aligned}\hat{y}_{T+h|T} &= y_T + \frac{h}{T-1} \sum_{t=2}^T (y_t - y_{t-1}) \\ &= y_T + \frac{h}{T-1} (y_T - y_1).\end{aligned}$$

- Equivalent to extrapolating a line drawn between first and last observations.

# Some simple forecasting methods

## Drift method



# Model fitting

The `model()` function trains models to data.

```
brick_fit <- aus_production |>
  filter(!is.na(Bricks)) |>
  model(
    Seasonal_naive = SNAIVE(Bricks),
    Naive = NAIVE(Bricks),
    Drift = RW(Bricks ~ drift()),
    Mean = MEAN(Bricks)
  )
```

```
## # A mable: 1 x 4
##   Seasonal_naive   Naive      Drift    Mean
##           <model> <model>    <model> <model>
## 1           <SNAIVE> <NAIVE> <RW w/ drift> <MEAN>
```

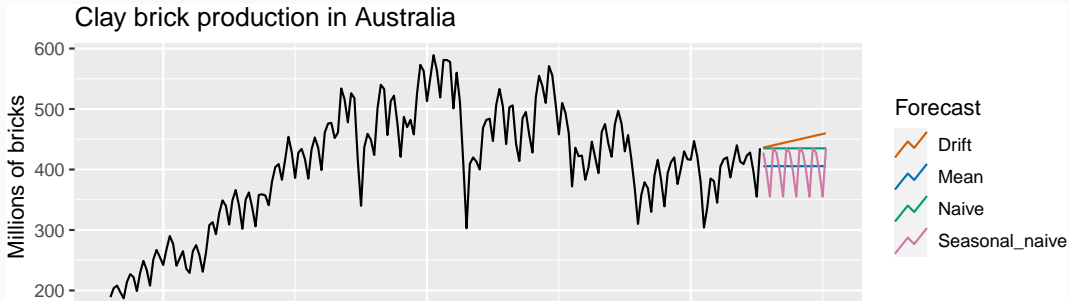
# Producing forecasts

```
brick_fc <- brick_fit |>  
  forecast(h = "5 years")
```

```
## # A fable: 80 x 4 [1Q]  
## # Key:      .model [4]  
##   .model      Quarter      Bricks .mean  
##   <chr>        <qtr>        <dist> <dbl>  
## 1 Seasonal_naive 2005 Q3 N(428, 2336) 428  
## 2 Seasonal_naive 2005 Q4 N(397, 2336) 397  
## 3 Seasonal_naive 2006 Q1 N(355, 2336) 355  
## 4 Seasonal_naive 2006 Q2 N(435, 2336) 435  
## # ... with 76 more rows
```

# Visualising forecasts

```
brick_fc |>
  autoplot(aus_production, level = NULL) +
  labs(
    title = "Clay brick production in Australia",
    y = "Millions of bricks"
  ) +
  guides(colour = guide_legend(title = "Forecast"))
```



# Facebook closing stock price

```
# Extract training data
fb_stock <- gafa_stock |>
  filter(Symbol == "FB") |>
  mutate(trading_day = row_number()) |>
  update_tsibble(index = trading_day, regular = TRUE)

# Specify, estimate and forecast
fb_stock |>
  model(
    Mean = MEAN(Close),
    Naive = NAIVE(Close),
    Drift = RW(Close ~ drift())
  ) |>
  forecast(h = 42) |>
  autoplot(fb_stock, level = NULL) +
  labs(title = "Facebook closing stock price", y = "$US") +
  guides(colour = guide_legend(title = "Forecast"))
```

# Facebook closing stock price



# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy



# Fitted values

- $\hat{y}_{t|t-1}$  is the forecast of  $y_t$  based on observations  $y_1, \dots, y_{t-1}$ .
- We call these “fitted values”.
- Sometimes drop the subscript:  $\hat{y}_t \equiv \hat{y}_{t|t-1}$ .
- Often not true forecasts since parameters are estimated on all data.

## For example:

- $\hat{y}_t = \bar{y}$  for average method.
- $\hat{y}_t = y_{t-1} + (y_T - y_1)/(T - 1)$  for drift method.

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

## Assumptions

- 1  $\{e_t\}$  uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2  $\{e_t\}$  have mean zero. If they don't, then forecasts are biased.

# Forecasting residuals

**Residuals in forecasting:** difference between observed value and its fitted value:  $e_t = y_t - \hat{y}_{t|t-1}$ .

## Assumptions

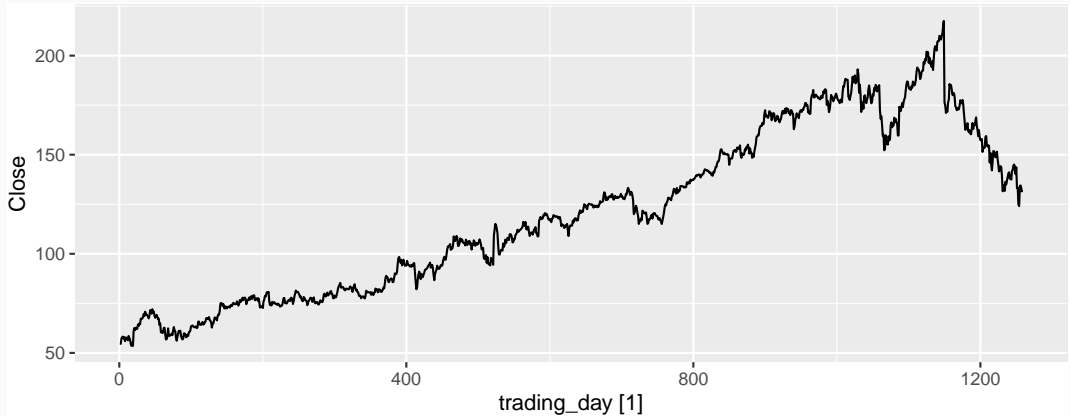
- 1  $\{e_t\}$  uncorrelated. If they aren't, then information left in residuals that should be used in computing forecasts.
- 2  $\{e_t\}$  have mean zero. If they don't, then forecasts are biased.

## Useful properties (for distributions & prediction intervals)

- 3  $\{e_t\}$  have constant variance.
- 4  $\{e_t\}$  are normally distributed.

# Facebook closing stock price

```
fb_stock |> autoplot(Close)
```



# Facebook closing stock price

```
fit <- fb_stock |> model(NAIVE(Close))
augment(fit)
```

```
## # A tsibble: 1,258 x 7 [1]
## # Key:      Symbol, .model [1]
##   Symbol .model      trading_day Close .fitted .resid .innov
##   <chr>   <chr>          <int> <dbl>   <dbl>   <dbl>   <dbl>
## 1 FB     NAIVE(Close)         1  54.7    NA     NA      NA
## 2 FB     NAIVE(Close)         2  54.6    54.7 -0.150 -0.150
## 3 FB     NAIVE(Close)         3  57.2    54.6  2.64   2.64
## 4 FB     NAIVE(Close)         4  57.9    57.2  0.720  0.720
## 5 FB     NAIVE(Close)         5  58.2    57.9  0.310  0.310
## 6 FB     NAIVE(Close)         6  57.2    58.2 -1.01  -1.01
## 7 FB     NAIVE(Close)         7  57.9    57.2  0.720  0.720
## 8 FB     NAIVE(Close)         8  55.9    57.9 -2.03  -2.03
## 9 FB     NAIVE(Close)         9  57.7    55.9  1.83   1.83
## 10 FB    NAIVE(Close)        10  57.6    57.7 -0.140 -0.140
## # with 1,248 more rows
```

# Facebook closing stock price

```
fit <- fb_stock |> model(NAIVE(Close))
augment(fit)
```

```
## # A tsibble: 1,258 x 7 [1]
## # Key:      Symbol, .model [1]
##   Symbol .model      trading_day Close .fitted .resid .innov
##   <chr>   <chr>          <int> <dbl>   <dbl>   <dbl>   <dbl>
## 1 FB     NAIVE(Close)         1  54.7    NA     NA      NA
## 2 FB     NAIVE(Close)         2  54.6    54.7  -0.150  -0.150
## 3 FB     NAIVE(Close)         3  57.2    54.6   2.64    2.64
## 4 FB     NAIVE(Close)         4  57.9    57.2   0.720   0.720
## 5 FB     NAIVE(Close)         5  58.2    57.9   0.310   0.310
## 6 FB     NAIVE(Close)         6  57.2    58.2  -1.01   -1.01
## 7 FB     NAIVE(Close)         7  57.9    57.2   0.720   0.720
## 8 FB     NAIVE(Close)         8  55.9    57.9  -2.03   -2.03
## 9 FB     NAIVE(Close)         9  57.7    55.9   1.83    1.83
## 10 FB    NAIVE(Close)        10  57.6    57.7  -0.140  -0.140
```

 $\hat{y}_{t|t-1}$  $e_t$ 

## Naïve forecasts:

$$\hat{y}_{t|t-1} = y_{t-1}$$

$$e_t = y_t - \hat{y}_{t|t-1} = y_t - y_{t-1}$$

```
## # with 1,248 more rows
```

# Facebook closing stock price

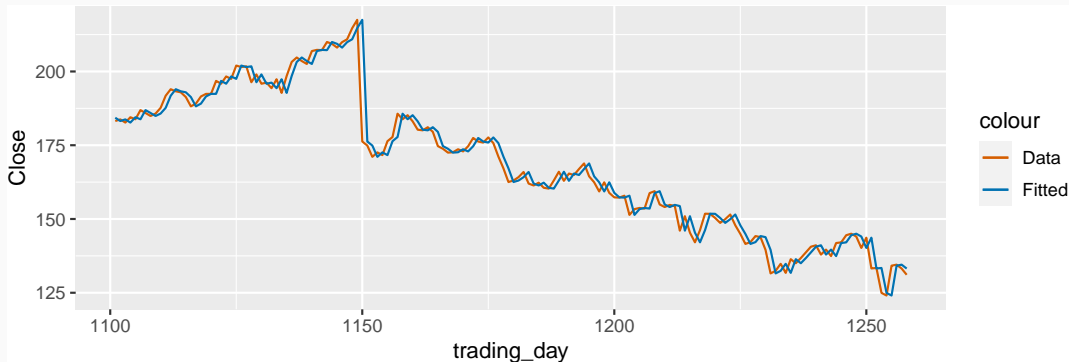
```
augment(fit) |>  
  ggplot(aes(x = trading_day)) +  
  geom_line(aes(y = Close, colour = "Data")) +  
  geom_line(aes(y = .fitted, colour = "Fitted"))
```





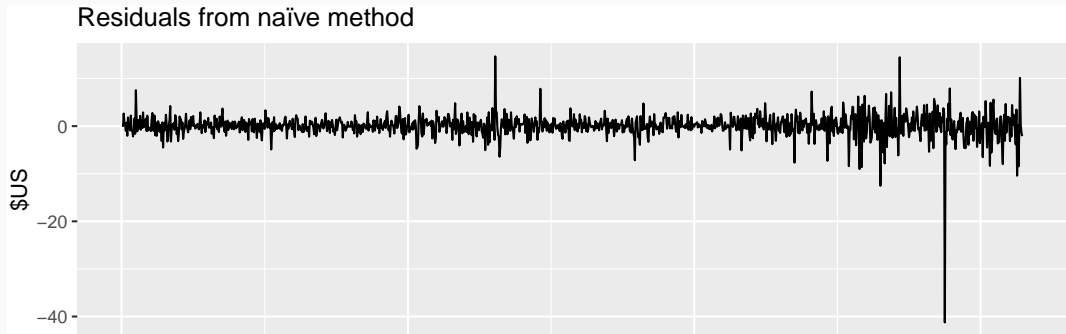
# Facebook closing stock price

```
augment(fit) |>  
  filter(trading_day > 1100) |>  
  ggplot(aes(x = trading_day)) +  
  geom_line(aes(y = Close, colour = "Data")) +  
  geom_line(aes(y = .fitted, colour = "Fitted"))
```



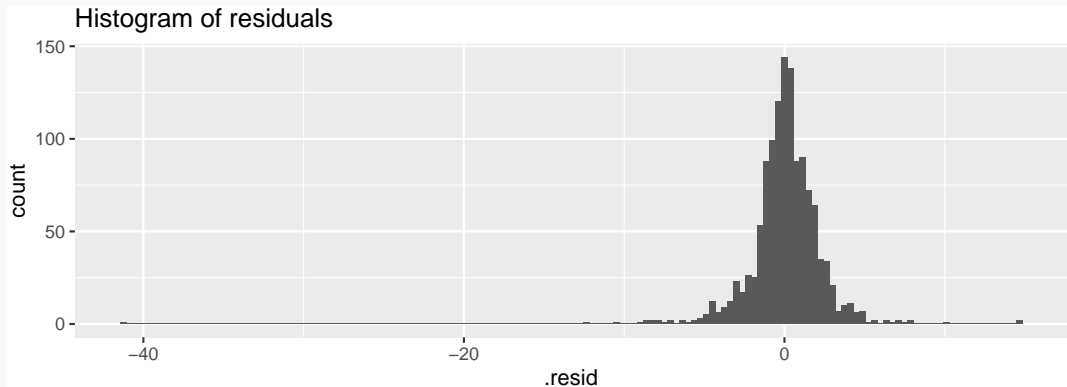
# Facebook closing stock price

```
augment(fit) |>  
  autoplot(.resid) +  
  labs(  
    y = "$US",  
    title = "Residuals from naïve method"  
  )
```



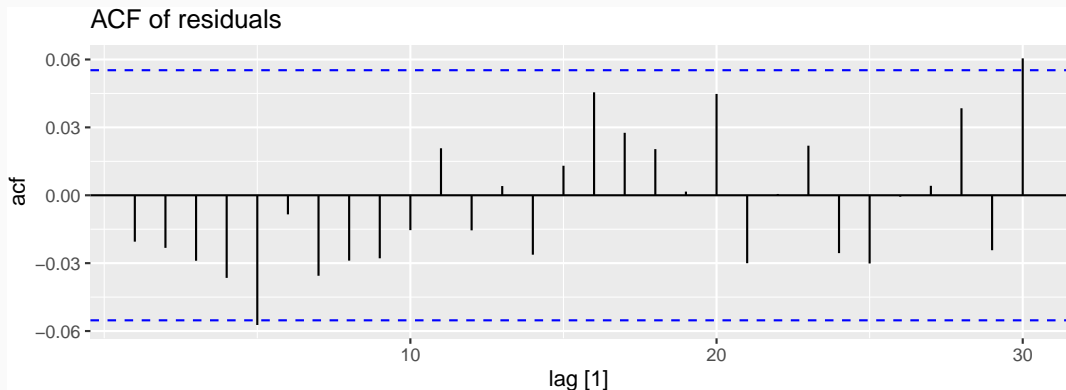
# Facebook closing stock price

```
augment(fit) |>  
  ggplot(aes(x = .resid)) +  
  geom_histogram(bins = 150) +  
  labs(title = "Histogram of residuals")
```



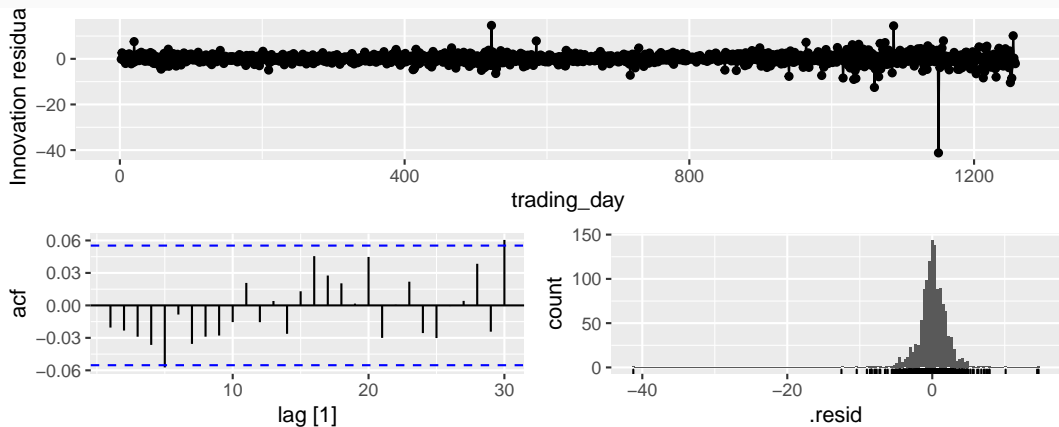
# Facebook closing stock price

```
augment(fit) |>  
  ACF(.resid) |>  
  autoplot() + labs(title = "ACF of residuals")
```



# gg\_tsresiduals() function

```
gg_tsresiduals(fit)
```



## ACF of residuals

- We assume that the residuals are white noise (uncorrelated, mean zero, constant variance). If they aren't, then there is information left in the residuals that should be used in computing forecasts.
- So a standard residual diagnostic is to check the ACF of the residuals of a forecasting method.
- We *expect* these to look like white noise.

# Portmanteau tests

$r_k$  = autocorrelation of residual at lag  $k$

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

# Portmanteau tests

$r_k$  = autocorrelation of residual at lag  $k$

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

## Box-Pierce test

$$Q = T \sum_{k=1}^{\ell} r_k^2$$

where  $\ell$  is max lag being considered and  $T$  is number of observations.

- If each  $r_k$  close to zero,  $Q$  will be **small**.
- If some  $r_k$  values large (positive or negative),  $Q$  will be **large**.



# Portmanteau tests

$r_k$  = autocorrelation of residual at lag  $k$

Consider a *whole set* of  $r_k$  values, and develop a test to see whether the set is significantly different from a zero set.

## Ljung-Box test

$$Q^* = T(T+2) \sum_{k=1}^{\ell} (T-k)^{-1} r_k^2$$

where  $\ell$  is max lag being considered and  $T$  is number of observations.

- My preferences:  $\ell = 10$  for non-seasonal data,  $h = 2m$  for seasonal data (where  $m$  is seasonal period).
- Better performance, especially in small samples.

# Portmanteau tests

- If data are WN,  $Q^*$  has  $\chi^2$  distribution with  $(\ell - K)$  degrees of freedom where  $K$  = no. parameters in model.
- When applied to raw data, set  $K = 0$ .
- $\text{lag} = \ell$ ,  $\text{dof} = K$

```
augment(fit) |>  
  features(.resid, ljung_box, lag = 10, dof = 0)
```

```
## # A tibble: 1 x 4  
##   Symbol .model      lb_stat lb_pvalue  
##   <chr>   <chr>      <dbl>    <dbl>  
## 1 FB     NAIVE(Close)    12.1     0.276
```

# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Forecast distributions

- A forecast  $\hat{y}_{T+h|T}$  is (usually) the mean of the conditional distribution  $y_{T+h} \mid y_1, \dots, y_T$ .
- Most time series models produce normally distributed forecasts.
- The forecast distribution describes the probability of observing any future value.

# Forecast distributions

Assuming residuals are normal, uncorrelated,  $\text{sd} = \hat{\sigma}$ :

**Mean:**  $y_{T+h|T} \sim N(\bar{y}, (1 + 1/T)\hat{\sigma}^2)$

**Naïve:**  $y_{T+h|T} \sim N(y_T, h\hat{\sigma}^2)$

**Seasonal naïve:**  $y_{T+h|T} \sim N(y_{T+h-m(k+1)}, (k+1)\hat{\sigma}^2)$

**Drift:**  $y_{T+h|T} \sim N(y_T + \frac{h}{T-1}(y_T - y_1), h\frac{T+h}{T}\hat{\sigma}^2)$

where  $k$  is the integer part of  $(h - 1)/m$ .

Note that when  $h = 1$  and  $T$  is large, these all give the same approximate forecast variance:  $\hat{\sigma}^2$ .

# Prediction intervals

- A prediction interval gives a region within which we expect  $y_{T+h}$  to lie with a specified probability.
- Assuming forecast errors are normally distributed, then a 95% PI is

$$\hat{y}_{T+h|T} \pm 1.96\hat{\sigma}_h$$

where  $\hat{\sigma}_h$  is the st dev of the  $h$ -step distribution.

- When  $h = 1$ ,  $\hat{\sigma}_h$  can be estimated from the residuals.

# Prediction intervals

```
brick_fc |> hilo(level = 95)
```

```
## # A tsibble: 80 x 5 [1Q]
```

```
## # Key:           .model [4]
```

##	.model	Quarter	Bricks	.mean	`95%`
##	<chr>	<qtr>	<dist>	<dbl>	<hilo>
##	1 Seasonal_naive	2005 Q3	N(428, 2336)	428	[333, 523]95
##	2 Seasonal_naive	2005 Q4	N(397, 2336)	397	[302, 492]95
##	3 Seasonal_naive	2006 Q1	N(355, 2336)	355	[260, 450]95
##	4 Seasonal_naive	2006 Q2	N(435, 2336)	435	[340, 530]95
##	5 Seasonal_naive	2006 Q3	N(428, 4672)	428	[294, 562]95
##	6 Seasonal_naive	2006 Q4	N(397, 4672)	397	[263, 531]95
##	7 Seasonal_naive	2007 Q1	N(355, 4672)	355	[221, 489]95
##	8 Seasonal_naive	2007 Q2	N(435, 4672)	435	[301, 569]95
##	9 Seasonal_naive	2007 Q3	N(428, 7008)	428	[264, 592]95

# Prediction intervals

- Point forecasts often useless without a measure of uncertainty (such as prediction intervals).
- Prediction intervals require a stochastic model (with random errors, etc).
- For most models, prediction intervals get wider as the forecast horizon increases.
- Use `level` argument to control coverage.
- Check residual assumptions before believing them.
- Usually too narrow due to unaccounted uncertainty.

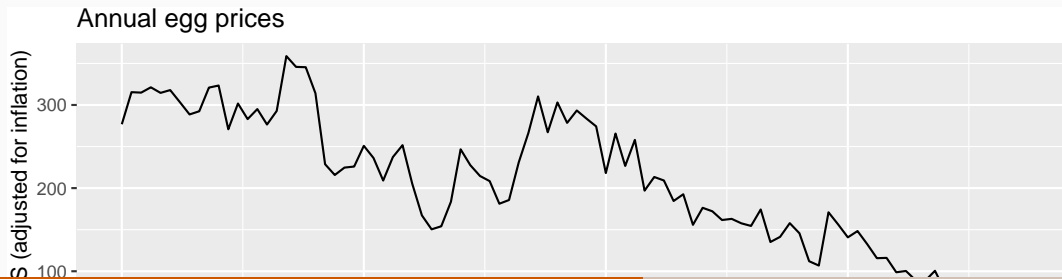


# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Modelling with transformations

```
eggs <- prices |>
  filter(!is.na(eggs)) |>
  select(eggs)
eggs |> autoplot() +
  labs(
    title = "Annual egg prices",
    y = "$US (adjusted for inflation)"
  )
```



# Modelling with transformations

Transformations used in the left of the formula will be automatically back-transformed. To model log-transformed egg prices, you could use:

```
fit <- eggs |>
  model(RW(log(eggs) ~ drift()))
fit
```

```
## # A mable: 1 x 1
##   `RW(log(eggs) ~ drift())`
##                                     <model>
## 1                                 <RW w/ drift>
```

# Forecasting with transformations

```
fc <- fit |>
  forecast(h = 50)
fc
```

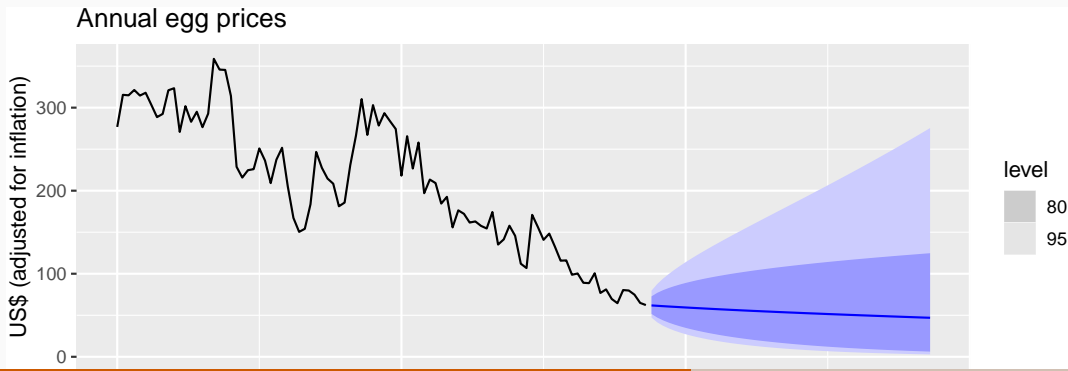
```
## # A tibble: 50 x 4 [1Y]
```

```
## # Key:   .model [1]
```

##	.model	year	eggs	.mean
##	<chr>	<dbl>	<dist>	<dbl>
##	1 RW(log(eggs) ~ drift())	1994	t(N(4.1, 0.018))	61.8
##	2 RW(log(eggs) ~ drift())	1995	t(N(4.1, 0.036))	61.4
##	3 RW(log(eggs) ~ drift())	1996	t(N(4.1, 0.055))	61.0
##	4 RW(log(eggs) ~ drift())	1997	t(N(4.1, 0.074))	60.6
##	5 RW(log(eggs) ~ drift())	1998	t(N(4.1, 0.093))	60.2
##	6 RW(log(eggs) ~ drift())	1999	t(N(4, 0.11))	59.8
##	7 RW(log(eggs) ~ drift())	2000	t(N(4, 0.13))	59.4
##	8 RW(log(eggs) ~ drift())	2001	t(N(4, 0.15))	59.0

# Forecasting with transformations

```
fc |> autoplot(eggs) +  
  labs(  
    title = "Annual egg prices",  
    y = "US$ (adjusted for inflation)"  
  )
```



# Bias adjustment

- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

# Bias adjustment

- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

## Back-transformed means

Let  $X$  be have mean  $\mu$  and variance  $\sigma^2$ .

Let  $f(x)$  be back-transformation function, and  $Y = f(X)$ .

Taylor series expansion about  $\mu$ :

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2f''(\mu).$$

# Bias adjustment

- Back-transformed point forecasts are medians.
- Back-transformed PI have the correct coverage.

## Back-transformed means

Let  $X$  be have mean  $\mu$  and variance  $\sigma^2$ .

Let  $f(x)$  be back-transformation function, and  $Y = f(X)$ .

Taylor series expansion about  $\mu$ :

$$f(X) = f(\mu) + (X - \mu)f'(\mu) + \frac{1}{2}(X - \mu)^2f''(\mu).$$

$$E[Y] = E[f(X)] = f(\mu) + \frac{1}{2}\sigma^2f''(\mu)$$



# Bias adjustment

**Box-Cox back-transformation:**

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f(x) = \begin{cases} e^x & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} e^x & \lambda = 0; \\ (1 - \lambda)(\lambda x + 1)^{1/\lambda - 2} & \lambda \neq 0. \end{cases}$$

# Bias adjustment

**Box-Cox back-transformation:**

$$y_t = \begin{cases} \exp(w_t) & \lambda = 0; \\ (\lambda W_t + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

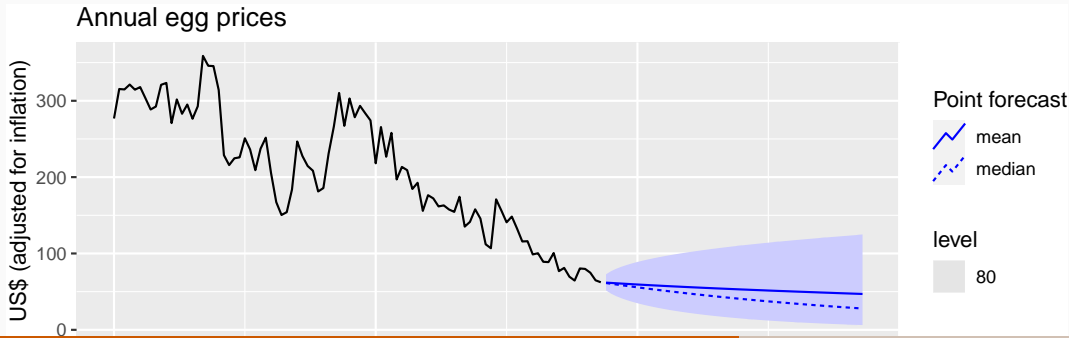
$$f(x) = \begin{cases} e^x & \lambda = 0; \\ (\lambda x + 1)^{1/\lambda} & \lambda \neq 0. \end{cases}$$

$$f''(x) = \begin{cases} e^x & \lambda = 0; \\ (1 - \lambda)(\lambda x + 1)^{1/\lambda - 2} & \lambda \neq 0. \end{cases}$$

$$E[Y] = \begin{cases} e^{\mu} \left[ 1 + \frac{\sigma^2}{2} \right] & \lambda = 0; \\ (\lambda \mu + 1)^{1/\lambda} \left[ 1 + \frac{\sigma^2(1-\lambda)}{2(\lambda \mu + 1)^2} \right] & \lambda \neq 0. \end{cases}$$

# Bias adjustment

```
fc |>  
  autoplot(eggs, level = 80, point_forecast = lst(mean, median)) +  
  labs(  
    title = "Annual egg prices",  
    y = "US$ (adjusted for inflation)"  
  )
```



# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Forecasting and decomposition

$$y_t = \hat{S}_t + \hat{A}_t$$

- $\hat{A}_t$  is seasonally adjusted component
  - $\hat{S}_t$  is seasonal component.
- 
- Forecast  $\hat{S}_t$  using SNAIVE.
  - Forecast  $\hat{A}_t$  using non-seasonal time series method.
  - Combine forecasts of  $\hat{S}_t$  and  $\hat{A}_t$  to get forecasts of original data.

# US Retail Employment

```
us_retail_employment <- us_employment |>
  filter(year(Month) >= 1990, Title == "Retail Trade") |>
  select(-Series_ID)
us_retail_employment
```

```
## # A tsibble: 357 x 3 [1M]
##       Month Title      Employed
##       <mth> <chr>      <dbl>
## 1 1990 Jan Retail Trade 13256.
## 2 1990 Feb Retail Trade 12966.
## 3 1990 Mar Retail Trade 12938.
## 4 1990 Apr Retail Trade 13012.
## 5 1990 May Retail Trade 13108.
## 6 1990 Jun Retail Trade 13183.
## 7 1990 Jul Retail Trade 13170.
## 8 1990 Aug Retail Trade 13160.
## 9 1990 Sep Retail Trade 13112.
```

# US Retail Employment

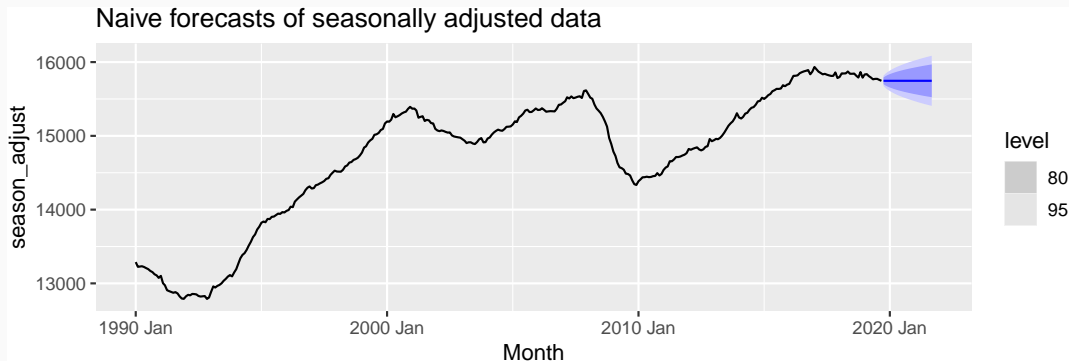
```
dcmp <- us_retail_employment |>
  model(STL(Employed)) |>
  components() |>
  select(-.model)
dcmp
```

```
## # A tsibble: 357 x 6 [1M]
```

##		Month	Employed	trend	season_year	remainder	season_adjust
##		<mth>	<dbl>	<dbl>	<dbl>	<dbl>	<dbl>
##	1	1990 Jan	13256.	13288.	-33.0	0.836	13289.
##	2	1990 Feb	12966.	13269.	-258.	-44.6	13224.
##	3	1990 Mar	12938.	13250.	-290.	-22.1	13228.
##	4	1990 Apr	13012.	13231.	-220.	1.05	13232.
##	5	1990 May	13108.	13211.	-114.	11.3	13223.
##	6	1990 Jun	13183.	13192.	-24.3	15.5	13207.
##	7	1990 Jul	13170.	13172.	-23.2	21.6	13193.
##	8	1990 Aug	13160.	13151.	-9.52	17.8	13169.

# US Retail Employment

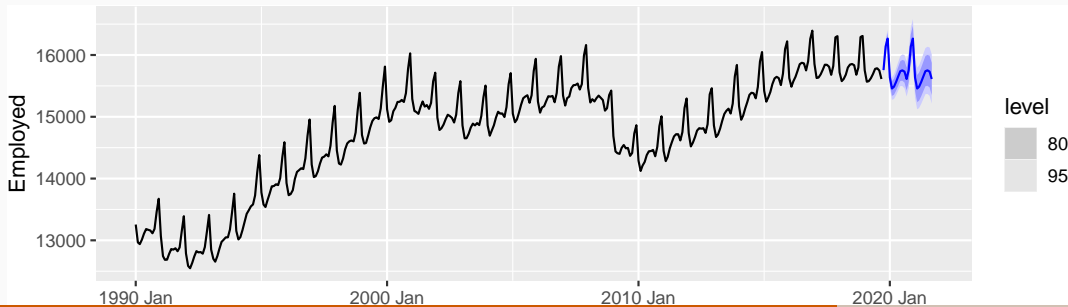
```
dcmp |>  
  model(NAIVE(season_adjust)) |>  
  forecast() |>  
  autoplot(dcmp) +  
  labs(title = "Naive forecasts of seasonally adjusted data")
```





# US Retail Employment

```
us_retail_employment |>  
  model(stlf = decomposition_model(  
    STL(Employed ~ trend(window = 7), robust = TRUE),  
    NAIVE(season_adjust)  
  )) |>  
  forecast() |>  
  autoplot(us_retail_employment)
```



# Decomposition models

`decomposition_model()` creates a decomposition model

- You must provide a method for forecasting the `season_adjust` series.
- A seasonal naive method is used by default for the `seasonal` components.
- The variances from both the seasonally adjusted and seasonal forecasts are combined.

# Outline

- 1 A tidy forecasting workflow
- 2 Some simple forecasting methods
- 3 Residual diagnostics
- 4 Distributional forecasts and prediction intervals
- 5 Forecasting with transformations
- 6 Forecasting and decomposition
- 7 Evaluating forecast accuracy

# Training and test sets



- A model which fits the training data well will not necessarily forecast well.
- A perfect fit can always be obtained by using a model with enough parameters.
- Over-fitting a model to data is just as bad as failing to identify a systematic pattern in the data.
- The test set must not be used for *any* aspect of model development or calculation of forecasts.
- Forecast accuracy is based only on the test set.

# Forecast errors

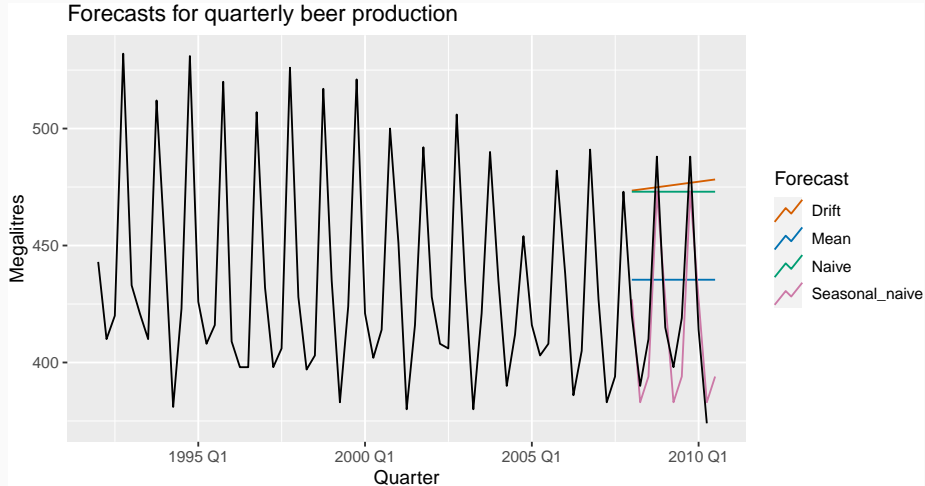
Forecast “error”: the difference between an observed value and its forecast.

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T},$$

where the training data is given by  $\{y_1, \dots, y_T\}$

- Unlike residuals, forecast errors on the test set involve multi-step forecasts.
- These are *true* forecast errors as the test data is not used in computing  $\hat{y}_{T+h|T}$ .

# Measures of forecast accuracy



# Measures of forecast accuracy

$y_{T+h}$  =  $(T + h)$ th observation,  $h = 1, \dots, H$

$\hat{y}_{T+h|T}$  = its forecast based on data up to time  $T$ .

$$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T}$$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e_{T+h}^2)$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

$$\text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)}$$

# Measures of forecast accuracy

$y_{T+h}$  =  $(T + h)$ th observation,  $h = 1, \dots, H$

$\hat{y}_{T+h|T}$  = its forecast based on data up to time  $T$ .

$e_{T+h} = y_{T+h} - \hat{y}_{T+h|T}$

$$\text{MAE} = \text{mean}(|e_{T+h}|)$$

$$\text{MSE} = \text{mean}(e_{T+h}^2)$$

$$\text{MAPE} = 100\text{mean}(|e_{T+h}|/|y_{T+h}|)$$

$$\text{RMSE} = \sqrt{\text{mean}(e_{T+h}^2)}$$

- MAE, MSE, RMSE are all scale dependent.
- MAPE is scale independent but is only sensible if  $y_t \gg 0$  for all  $t$ , and  $y$  has a natural zero.



# Measures of forecast accuracy

## Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

where  $Q$  is a stable measure of the scale of the time series  $\{y_t\}$ .

Proposed by Hyndman and Koehler (IJF, 2006).

For non-seasonal time series,

$$Q = (T - 1)^{-1} \sum_{t=2}^T |y_t - y_{t-1}|$$

works well. Then MASE is equivalent to MAE relative to a naïve method.

# Measures of forecast accuracy

## Mean Absolute Scaled Error

$$\text{MASE} = \text{mean}(|e_{T+h}|/Q)$$

where  $Q$  is a stable measure of the scale of the time series  $\{y_t\}$ .

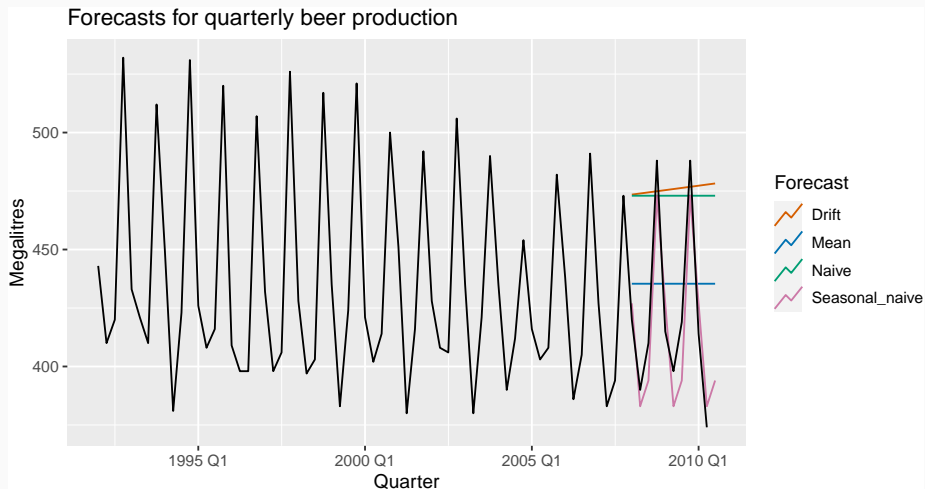
Proposed by Hyndman and Koehler (IJF, 2006).

For seasonal time series,

$$Q = (T - m)^{-1} \sum_{t=m+1}^T |y_t - y_{t-m}|$$

works well. Then MASE is equivalent to MAE relative to a seasonal naïve method.

# Measures of forecast accuracy



# Measures of forecast accuracy

```
recent_production <- aus_production |>
  filter(year(Quarter) >= 1992)
train <- recent_production |>
  filter(year(Quarter) <= 2007)
beer_fit <- train |>
  model(
    Mean = MEAN(Beer),
    Naive = NAIVE(Beer),
    Seasonal_naive = SNAIVE(Beer),
    Drift = RW(Beer ~ drift())
  )
beer_fc <- beer_fit |>
  forecast(h = 10)
```

# Measures of forecast accuracy

```
accuracy(beer_fit)
```

```
## # A tibble: 4 x 6
##   .model      .type    RMSE    MAE  MAPE  MASE
##   <chr>      <chr>    <dbl> <dbl> <dbl> <dbl>
## 1 Drift      Training  65.3   54.8  12.2   3.83
## 2 Mean      Training  43.6   35.2   7.89   2.46
## 3 Naive     Training  65.3   54.7  12.2   3.83
## 4 Seasonal_naive Training  16.8   14.3   3.31   1
```

```
accuracy(beer_fc, recent_production)
```

```
## # A tibble: 4 x 6
##   .model      .type    RMSE    MAE  MAPE  MASE
##   <chr>      <chr>    <dbl> <dbl> <dbl> <dbl>
## 1 Drift      Test     64.9   58.9  14.6   4.12
## 2 Mean      Test     38.4   34.8   8.28   2.44
## 3 Naive     Test     62.7   57.4  14.2   4.01
## 4 Seasonal_naive Test     14.3   13.4   3.17   0.937
```

# Time series cross-validation

## Traditional evaluation

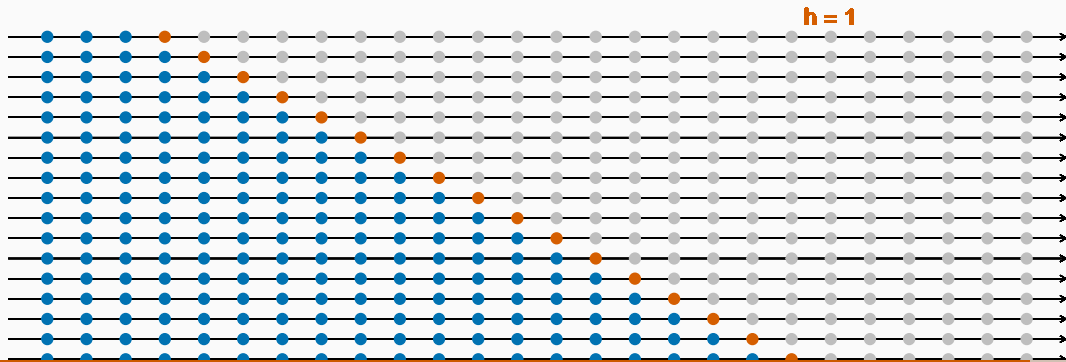


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

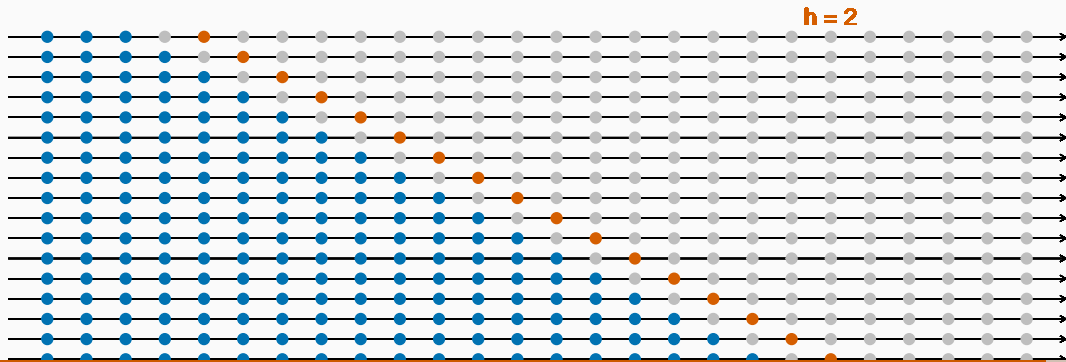


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation



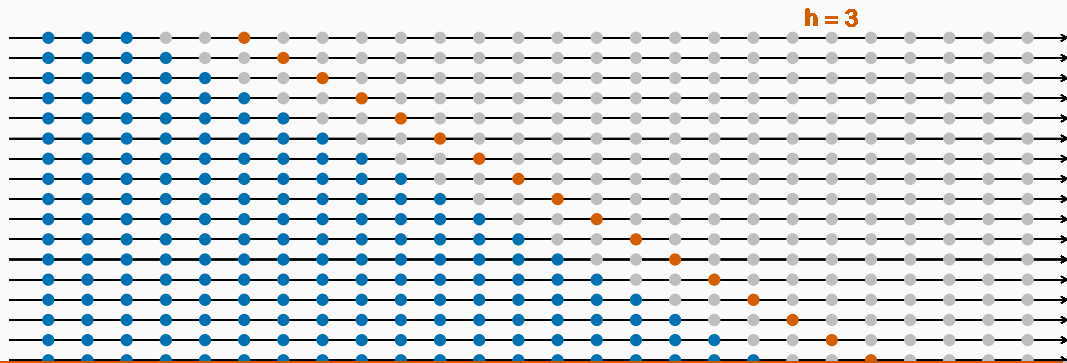


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

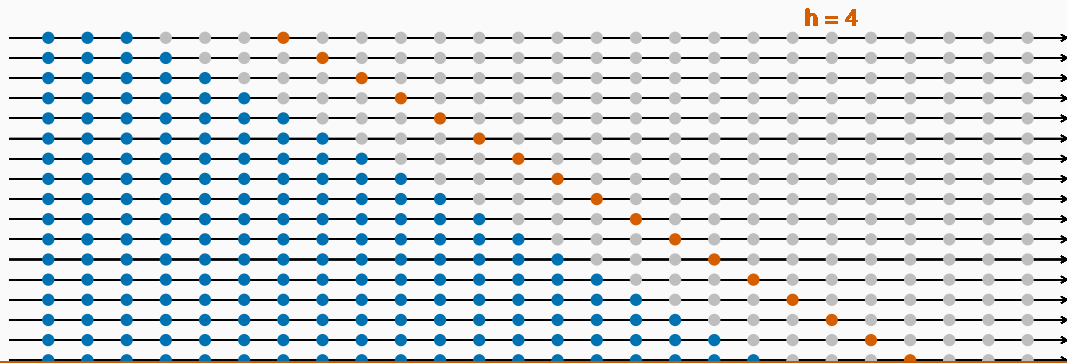


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation

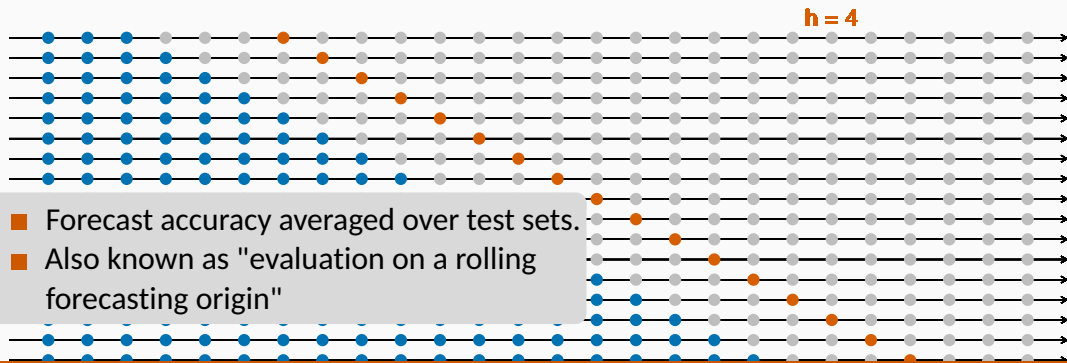


# Time series cross-validation

## Traditional evaluation



## Time series cross-validation



# Time series cross-validation

Stretch with a minimum length of 3, growing by 1 each step.

```
fb_stretch <- fb_stock |>
  stretch_tsibble(.init = 3, .step = 1) |>
  filter(.id != max(.id))
```

```
## # A tsibble: 790,650 x 4 [1]
## # Key:           .id [1,255]
##   Date           Close trading_day   .id
##   <date>         <dbl>         <int> <int>
## 1 2014-01-02     54.7             1     1
## 2 2014-01-03     54.6             2     1
## 3 2014-01-06     57.2             3     1
## 4 2014-01-02     54.7             1     2
## 5 2014-01-03     54.6             2     2
## 6 2014-01-06     57.2             3     2
## 7 2014-01-07     57.9             4     2
```

# Time series cross-validation

Estimate RW w/ drift models for each window.

```
fit_cv <- fb_stretch |>
  model(RW(Close ~ drift()))
```

```
## # A mable: 1,255 x 3
## # Key:      .id, Symbol [1,255]
##      .id Symbol `RW(Close ~ drift())`
##    <int> <chr>          <model>
## 1      1 FB           <RW w/ drift>
## 2      2 FB           <RW w/ drift>
## 3      3 FB           <RW w/ drift>
## 4      4 FB           <RW w/ drift>
## # ... with 1,251 more rows
```

# Time series cross-validation

Produce one step ahead forecasts from all models.

```
fc_cv <- fit_cv |>  
  forecast(h = 1)
```

```
## # A fable: 1,255 x 5 [1]  
## # Key:      .id, Symbol [1,255]  
##      .id Symbol trading_day      Close .mean  
##    <int> <chr>      <dbl>      <dist> <dbl>  
## 1      1 FB          4 N(58, 5.8)  58.4  
## 2      2 FB          5 N(59, 2.7)  59.0  
## 3      3 FB          6 N(59, 1.9)  59.1  
## 4      4 FB          7 N(58, 2.2)  57.7  
## # ... with 1,251 more rows
```

# Time series cross-validation

```
# Cross-validated  
fc_cv |> accuracy(fb_stock)  
# Training set  
fb_stock |>  
  model(RW(Close ~ drift())) |>  
  accuracy()
```

	RMSE	MAE	MAPE
Cross-validation	2.418	1.469	1.266
Training	2.414	1.465	1.261

A good way to choose the best forecasting model is to find the model with the smallest RMSE computed using time series cross-validation.