Stokastiska processer lab 1

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Summary

In this report we shall investigate the winning probabilities of a gamblers ruin-type markov chain. Throught the report we will be using $N = 5$, but the probability p for winning in one single isolated game, and the starting number of credits will be changed throughout the report in the effort to study different aspects of the markov chain. We will be referencing p as a "local probability" for clarity. We will be using two gamblers Kim and Robin to test out the different aspects of the Markov chain.

Problem 1

1.a

Kim plays at the casino. The local probability is $p = 0.5$. In each round, they bet one credit. If Kim wins, they profit 1 credit as a win. If Kim loses, then they loose one credit. Kim stops playing either when they've lost all their money, or when they owns 5 credits. We here examine the probability of Kim winning 5 credits. We write a function that simulates Kim's gambling:

```
en_spelomgang <- function(p, kapital)
if (runif(1) < p) {
return(kapital + 1)
} else {
return(kapital - 1)
}
kim_spelar <- function(p,kapital){
while (kapital != 0 && kapital != 5) {
kapital <- en_spelomgang(p,kapital)}
if (kapital == 0) {return (0)}
else {return(1)}
}
```

1.b

Here we will write the function sim_kim which does n simulations of Kim's gambling and counts the number of times Kim makes the desired profit i.e. 5 credits.

```
sim_kim <- function(p,kapital,n){
succ <- 0
counter <- n
```

```r
while(counter > 0){
succ <- succ + kim_spelar(p,kapital)
counter <- counter-1
}
return(succ)
}
```

We run 1000 simulations and see how many times Kim managed to win.

```r
sim_kim(0.5,1,1000)
```

```
## [1] 211
```

It seems that the probability that Kim manages to make the win is around 0.2.

1.c

We will now simulate 1000 rounds of the game with varying starting credits: $1, 2, 3$ and 4 credits and investigate how often Kim reaches 5 credits in each case. We present the results in the table below:

```r
kims_df <-data.frame("Starting credits" = 1:4,
c(sim_kim(0.5,1,1000), sim_kim(0.5,2,1000),sim_kim(0.5,3,1000),sim_kim(0.5,4,1000)))
names(kims_df)[-1] <-c("The number of times Kim wins")
knitr::kable(kims_df, digits = 2, caption = "Table 1: The number of winnings given different credits at
```

Table 1: Table 1: The number of winnings given different credits at the beginning of the game.

Starting.credits	The number of times Kim wins
1	202
2	400
3	605
4	804

We conclude that the chance of getting 5 credits is biggest when Kim has 4 credits and smallest when Kim has 1. The results align with our expectations.

problem 2

We shall now estimate the gamblers long-term probability of success using a theoretical approach. The gamblers state after n games is denoted X_n, and we shall call the state after an arbitrarily large number of games X_∞. We want to find $P(X_\infty = 5)$.

The one-step transition probability matrix P, using the function kims_matris below, can be divided into four parts. This assumes that it is written in standard form, meaning the transient states are listed first, and then the recurrent. We will denote the matrix only containing the transition probabilities of the transient states with P_t. We will denote the matrix only containing transition probabilities from a transient state to a recurrent with R. This way we can describe P as

$$P = \begin{pmatrix} P_t & R \\ 0 & I \end{pmatrix}$$

Where 0 denotes a matrix with all elements equal to zero, and I denotes the unit matrix. This is because the probability of going from a recurrent state to a transient always is zero. Similarly, the probability of going from a recurrent state to a recurrent is always 1.

The long term transition matrix P^n as $n \to \infty$ converges to

$$P = \begin{pmatrix} 0 & SR \\ 0 & I \end{pmatrix}$$

In "Introduction to probability models" (Sheldon M. Ross, 11 ed, p.232) it is stated that the matrix S can be solved for with the expression

$$S = (I - P_t)^{-1}$$

The matrix SR can therefore be constructed with little difficulty. This is done with the function SR_builder_2 below.

SR represents the probabilities of ending up in either state 5 or 0, depending on which state we started in. Because state 5 is written before state 0, the probability that the gambler will end up in state 5 if they started in state 1 is on row 1, column 1 in SR. Note that if we´re considering the entire transition matrix P, this element is found in row 1, column 5. The transition matrix P is constructed below.

```
##one-step transiton matrix for a Gamblers Ruin-type Markov chain with p=p, N=5.

kims_matris <- function(p) {
matrix(c( 0   , p   , 0   , 0 , 0 , 1-p ,
          1-p , 0   , p   , 0 , 0 , 0   ,
          0   , 1-p , 0   , p , 0 , 0   ,
          0   , 0   , 1-p , 0 , p , 0   ,
          0   , 0   , 0   , 0 , 1 , 0   ,
          0   , 0   , 0   , 0 , 0 , 1  ),
nrow = 6, byrow = TRUE)
}
```

```
##Constructs the long-term probability matrix SR taking in the probability for positive
##outcome (p), and the one-step transiton matrix (matrix).

SR_builder_2 <- function(p, matrix){
  unit <- diag(1, nrow=4, ncol=4)
  P_t_matrix <- matrix(c( matrix[1,1],matrix[1,2],matrix[1,3],matrix[1,4],
                          matrix[2,1],matrix[2,2],matrix[2,3],matrix[2,4],
                          matrix[3,1],matrix[3,2],matrix[3,3],matrix[3,4],
                          matrix[4,1],matrix[4,2],matrix[4,3],matrix[4,4]), nrow = 4, byrow = TRUE)

  R_matrix <- matrix(c( matrix[1,5],matrix[1,6],
                        matrix[2,5],matrix[2,6],
                        matrix[3,5],matrix[3,6],
                        matrix[4,5],matrix[4,6]), nrow = 4, byrow = TRUE)

  SR <- (solve(unit-P_t_matrix))%*%R_matrix
```

```r
  return(SR)
}

SR_builder_2(0.5, kims_matris(0.5))
```

```
##      [,1] [,2]
## [1,]  0.2  0.8
## [2,]  0.4  0.6
## [3,]  0.6  0.4
## [4,]  0.8  0.2
```

As discussed above, the first column of SR describes the probability that $X_\infty = 5$, for the respective starting states $1, 2, 3, 4$. This result implies that in the simulation in problem 1, when simulating 1000 games (from start to ruin/fortune), we could expect that ($X_\infty = 5$) 200 times if we start with 1 SEK, 400 times if we start with 2 SEK, 600 times if we start with 3 SEK, and 800 times if we start with 4 SEK.

Comparing this with the results in table 1 of our simulation in problem 1, we can say that the simulated number of games that were won coincide well with the expected number of won games for the respective starting states.

Problem 3

In most, or all (rounded to nearest casino) casinos the goal is to maximize profits. It is therefore unlikely that a gamblers ruin-type game would be set to 0.5. We shall now investigate the probability for $X_\infty = 5$ when $p = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9$. We shall do this both theoretically and empirically, using problem 2 and problem 1 respectively. We shall only consider the case of starting with 1 credit.

```r
##Empirical approach: Simulating the long-term probability of ending up in state 5 if we start in state

simu_prob = rep(0, times = 9)
for (i in 1:length(simu_prob)) {
  simu_prob[i] = sim_kim(i/10, 1, 3000)/3000
}

#simu_prob will be used in the table below.
```

```r
##Theoretical approach: Constructing SR and extracting the probability of ending up
##in state 5 if we start in state 1.

kim_theoretical_prob = rep(0, times = 9)
for (i in 1:length(kim_theoretical_prob)) {
  kim_theoretical_prob[i] = SR_builder_2(i/10, kims_matris(i/10))[1,1]
}

#theo_prob will be used in the table below.
```

```r
kims_df <- data.frame("Local probability" = 1:9 / 10,
sim_P = simu_prob, SR_P = kim_theoretical_prob)

names(kims_df)[-1] <- c("$\\mathbb{P}(X_\\infty = 5)$ (simulation)",
"$\\mathbb{P}(X_\\infty = 5)$ (theoretical from $\\mathbf{SR}$)")
knitr::kable(kims_df, digits = 2, caption = "Tabell 2: Simulated and theoretical probabilities of winnin
```

Table 2: Tabell 2: Simulated and theoretical probabilities of winning the gamblers ruin for different local probabilites, starting with 1 SEK, N=5.

Local.probability	$\mathbb{P}(X_\infty = 5)$ (simulation)	$\mathbb{P}(X_\infty = 5)$ (theoretical from **SR**)
0.1	0.00	0.00
0.2	0.00	0.00
0.3	0.02	0.02
0.4	0.07	0.08
0.5	0.20	0.20
0.6	0.38	0.38
0.7	0.59	0.58
0.8	0.74	0.75
0.9	0.88	0.89

Problem 4

4.a

Robin plays at a different casino where you can bet as much as you want at the beginning of each game round. In the case of a win, Robin receives twice what they bet. Robin always starts with 1 credit, and stops playing when they loses all their money or when they reach 5 credits.

Robin follows a slightly different strategy to Kims. If Robin has 1 credit, then they bet it. If Robin has 2 credits then they bet both of them. If Robin has 3 credits they bet 2 credits, and if Robin has 4 credits they bet 1 credit. All in effort to reach 5 credits, but not go above.

The function robins_matrix returns the transition matrix for a markov chain that describes Robin's chances of winning one full game, where p is the local probability.

```
robins_matris <- function(p) {
matrix(c( 0  , p , 0   , 0, 0, 1-p,
          0  , 0 , 0   , p, 0, 1-p,
        1-p  , 0 , 0   , 0, p, 0  ,
          0  , 0 , 1-p, 0, p, 0  ,
          0  , 0 , 0   , 0, 1, 0  ,
          0  , 0 , 0   , 0, 0, 1 ),
nrow = 6, byrow = TRUE)
}
```

4.b

At this point it could be of interest to investigate which strategy is the best in terms of maximazing winnings (or minimizing losses…), Kims or Robins. We can do this by calculating the probability of winning the game for different local probabilities by constructing a table and comparing the respective probabilities for Kim and Robin. We have already done this for Kim in problem 3, and we will reuse the theoretical result produced there. Robins probability of winning can be calculated similarly:

```
##Calculating Robins probability of winning the game for different
##local probabilities.
```

```r
rob_theoretical_prob = rep(0, times = 9)
for (i in 1:length(rob_theoretical_prob)) {
  rob_theoretical_prob[i] = SR_builder_2(i/10, robins_matris(i/10))[1,1]
}

##Constructing the table

kim_rob_df <- data.frame("Local probability" = 1:9 / 10,
rob_prob = rob_theoretical_prob, kim_prob = kim_theoretical_prob)

names(kim_rob_df)[-1] <- c("Robins probability",
"Kims probability")
knitr::kable(kim_rob_df, digits = 2, caption = "Table 3: Winning probabilities for Robin and Kim, when s
```

Table 3: Table 3: Winning probabilities for Robin and Kim, when starting with 1 SEK, N=5.

Local.probability	Robins probability	Kims probability
0.1	0.00	0.00
0.2	0.01	0.00
0.3	0.05	0.02
0.4	0.11	0.08
0.5	0.20	0.20
0.6	0.32	0.38
0.7	0.47	0.58
0.8	0.63	0.75
0.9	0.81	0.89

In table 3 we see that for $p < 0.5$, Robins probability for winning a game is larger than Kims, but only slightly. When $p = 0.5$, the probabilities are equal(rounded to two decimals). When $p > 0.5$, Kims probability is larger than Robins, but only slightly. Given our previous results in table 1, it is quite likely that these winning probabilities will increase relative to the starting number of credits. This is however only a conjecture, which could be checked rather easily.

With this in mind, it would be wise to gamble boldly (Robins strategy) when $p < 0.5$, and cautiosly (Kims strategy) when $p > 0.5$. At least when the starting with 1 credit.