CNS 2013

Solutions to Paul Tiesinga Problem Set 1 - Exercise 3

Marije ter Wal

3.a Analytical solution of $y'' + by' + y = 0$

The analytical solution to $y'' + by' + y = 0$ can be found by trying the function $y(x) = C\exp(\lambda x)$. This will give the characteristic equation:

$$\lambda^2 + b\lambda + 1 = 0 \tag{1}$$

and hence the solution:

$$y(x) = C_1\exp(\lambda_1 x) + C_2\exp(\lambda_2 x) \tag{2}$$

λ_1 and λ_2 are given by:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4}}{2} \tag{3}$$

This gives the following characterization of the fixed points:

- if $-\infty < b \leq -2$ the solution blows up (unstable node)
- if $-2 < b < 0$ the solution is blow-up oscillatory (unstable spiral)
- if $b = 0$ the solution is oscillatory (limit cycle)
- if $0 < b < 2$ the solution is damped oscillatory (stable spiral)
- if $2 \leq b < \infty$ the solution is damped (stable node)

3.b Rewrite and analyze as a system of first order ODEs

In order to rewrite the second order ODE to a system of two first order ODE, we define a new variable, let's call it y_1, as the derivative of the original variable y:

$$y_1 = y' \tag{4}$$

Then, by definition:

$$\begin{aligned} y_1' &= y'' && (5) \\ Y_1 &= y && (6) \end{aligned}$$

with Y_1 being the primitive of y_1. For the derivative of y_1 we thus find:

$$y_1' = -by_1 - Y_1 \tag{7}$$

Redefining Y_1 as a new variable y_2 gives us the second differential equation:

$$y_2' = Y_1' = y_1 \tag{8}$$

leaving us with a system of two coupled first order ODEs:

$$\begin{aligned} y_1' &= -by_1 - y_2 & (9)\\ y_2' &= y_1 & (10) \end{aligned}$$

We can linearize this system around the origin, by computing the Jacobian:

$$J_{(0,0)} = \begin{pmatrix} -b & -1 \\ 1 & 0 \end{pmatrix} \tag{11}$$

The stability of the fixed point can be determined based on the eigenvalues, which are given by:

$$\lambda_{1,2} = \frac{-\mathrm{tr}J \pm \sqrt{\mathrm{tr}J^2 - 4\mathrm{det}J}}{2} \tag{12}$$

with the trace $\mathrm{tr}J$ and the determinant $\mathrm{det}J$ of the Jacobian matrix given by:

$$\begin{aligned} \mathrm{tr}J &= -b + 0 = -b & (13)\\ \mathrm{det}J &= -b * 0 - -1 * 1 = 1 & (14) \end{aligned}$$

As you can see, and could have expected, this gives the same form as we found in a, and of course the characterization is also the same.

3.c Characterize stability of origin for $y'' + by'^3 + y = 0$

We transform the system to a set of first order ODEs:

$$\begin{aligned} y_1' &= y_2 & (15)\\ y_2' &= -by_2^3 - y_1 & (16) \end{aligned}$$

Linearizing this system around the origin will yield $\mathrm{tr}J = 0$ and hence is inconclusive. To benifit from the oscillatory dynamics that we can deduce from a zero trace, let's transform the system to polar coordinates:

$$\begin{aligned} y_1 &\equiv r\cos\theta & (17)\\ y_2 &\equiv r\sin\theta & (18) \end{aligned}$$

This gives:

$$\begin{aligned} y_1' &= r'\cos\theta - r\theta'\sin\theta & (19)\\ y_2' &= r'\sin\theta + r\theta'\cos\theta & (20) \end{aligned}$$

In order to find r', we make use of the fact that $\cos^2\theta + \sin^2\theta = 1$, so we multiply the above equations by $\cos\theta$ and $\sin\theta$, respectively, and add them up:

$$\begin{aligned} y_1'\cos\theta + y_2'\sin\theta &= r'\cos^2\theta - r\theta'\sin\theta\cos\theta + r'\sin^2\theta + r\theta'\cos\theta\sin\theta \quad (21) \\ &= r' \quad (22) \end{aligned}$$

Furthermore, we know:

$$\begin{aligned} y_1' &= y_2 = r\sin\theta \quad (23) \\ y_2' &= -by_2^3 - y_1 = -br^3\sin^3\theta - r\cos\theta \quad (24) \end{aligned}$$

which leaves us with:

$$\begin{aligned} r' &= r\sin\theta\cos\theta + (-br^3\sin^3\theta - r\cos\theta)\sin\theta \quad (25) \\ &= r\sin\theta\cos\theta + -br^3\sin^4\theta - r\cos\theta\sin\theta \quad (26) \\ &= -br^3\sin^4\theta \quad (27) \end{aligned}$$

From this equation we can readily see that the radius and hence the amplitude of the oscillations will decrease for all $b > 0$, while it will increase for $b < 0$. The fixed point in the origin therefore is characterized by:

- if $b < 0$ the solution blows up (unstable spiral)
- if $b = 0$ the solution is oscillatory (limit cycle)
- if $b > 0$ the solution is damped oscillatory (stable spiral)

As you might have noticed when plotting trajectories, the behaviour for $b > 0$ grows increasingly slow when the radius decreases and seems to never reach the origin. This is because of slow θ dynamics, not due to instability (write down the ODE for θ to see why).