Solutions to CNS 2013 PT4 - Kuramoto

4.a

-Expanding $r\exp(i\Theta)=\frac{1}{N}\sum_{j=1}^{N}\exp(i\theta_j)$, we have $r\cos(\Theta)$ equals $\frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j)$ and $r\sin(\Theta)$ equals $\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j)$. Then,

$$\dot{\theta}_i=\omega_i-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i-\theta_j)$$
$$\dot{\theta}_i=\omega_i-\frac{K}{N}\sin(\theta_i)\sum_{j=1}^{N}\cos(\theta_j)+\frac{K}{N}\cos(\theta_i)\sum_{j=1}^{N}\sin(\theta_j)$$
$$\dot{\theta}_i=\omega_i-K\sin(\theta_i)r\cos(\Theta)+K\cos(\theta_i)r\sin(\Theta)$$

$$\dot{\theta}_i=\omega_i-Kr\sin(\theta_i-\Theta).$$

4.b

-When θ_j equals Θ, then r equals 1: therefore, r is a measure of a number of population that have the same phase.

-Θ represents an average phase of the population.

4.c

-The phase fixed point of each phase oscillator is $\sin^{-1}(\omega_i/(Kr))+\Theta$. Jacobian matrix of each oscillator at the phase fixed point is $-Kr\cos(\sin^{-1}(\omega_i/(Kr)))$. Then, the phase fixed point is linearly stable if and only if $-Kr\cos(\sin^{-1}(\omega_i/(Kr)))<0$,i.e. $\omega_i\in(-Kr,Kr)$: it is unstable if and only if $-Kr\cos(\sin^{-1}(\omega_i/(Kr)))>0$, i.e. $\omega_i<-Kr$ or $\omega_i>Kr$.