Problem set 5 Computational neuroscience
(Hand in exercise 1&2 on Thursday October 15th, 2015;
exercise 3 on Thursday October 22nd.)

Problem 1: *Entropy rate of a Poisson process.*
Consider a Poisson process with rate r. The entropy rate is the number of bits per second.
(a) Assume that $r\Delta t$ is much smaller than 1. Calculate the entropy of the binary process which gives 1 (a spike) with probability $r\Delta t$, and 0 (no spike) with probability $1-r\Delta t$. Determine from this the entropy rate.
(b) The interspike interval distribution is exponential: $P(t)=r\exp(-rt)$. Determine the entropy per interspike interval and convert it to an entropy rate.

Problem 2. Estimating the bias in entropy estimates and methods to correct for this bias. We consider the stimulus response relation $p(r|s)$ with 4 possible responses and 3 possible stimuli which occur with equal probability, $p(s)=1/3$.

$$p(r|s) = \begin{pmatrix} \frac{2}{3} & \frac{5}{18} & \frac{1}{18} \\ \frac{5}{18} & \frac{2}{3} & \frac{1}{18} \\ \frac{1}{36} & \frac{1}{36} & \frac{4}{9} \\ \frac{1}{36} & \frac{1}{36} & \frac{4}{9} \end{pmatrix}$$

(a) Determine what the exact mutual information between r and s is.
(b) To analyze numerically what the mutual information is we need to generate N pairs (r_i,s_i), $i=1,..,N$. Take $N=3N_s$ so that each stimulus is presented N_s times. To generate the resulting r_i given the stimulus, take the corresponding column in $p(r|s)=[a\ b\ 1-a-b]$, define the cumulative probability $[a\ a+b\ 1]$, draw a random number r uniformly distributed between 0 and 1 and set $r_i=1$ when $r<a$, $r_i=2$ when $r>=a$ & $r<b$, and $r_i=3$ otherwise. Write a matlab program that generates N of these samples given $p(r|s)$ as input.
(c) Write a matlab program to naively estimate $p(r|s)$ based on N samples, where $p(r_1|s)$ for given s is (#of times r1 is obtained)/N_s etc. Test that your program produces the correct $p(r|s)$ matrix.
(d) Estimate the response and noise entropy (H(R) and H(R|S) in the notation of the lectures) as a function of N. According to the work of Strong et al (Physical Review Letters, 1998, vol 80, p 197), the entropy estimate can be written as a power of N:

$H(N) = H_\infty + \frac{a}{N} + \frac{b}{N^2}$, where H_∞ the entropy is that you are interested in. Extrapolate your numerical result to (1/N)->0 and check that it matches the analytical estimate in part a. How big is the bias in the entropy estimate when N=30? Note: you need to think about how to best use the samples you generate so that you do not keep on regenerating the same samples.

Problem 3. Estimating the bias in entropy estimates and methods to correct for this bias. In the paper by Ince et al (Neural Networks, 2010, vol 23, p 710) a different bias correction procedure is proposed based on shuffling and the marginal distributions. This only is useful if you have multiple, correlated responses. So we apply this method to a different distribution, where there are two neurons (A, B) who can each give 3 responses, hence in total there are 9 different responses and two stimuli (occurring with equal probability).

$$p(r_A, r_B \mid s_1) = \begin{pmatrix} \frac{4}{18} & \frac{3}{18} & \frac{1}{18} \\ \frac{3}{18} & \frac{4}{18} & \frac{1}{18} \\ \frac{1}{18} & 0 & \frac{1}{18} \end{pmatrix} \quad \text{and} \quad p(r_A, r_B \mid s_2) = \begin{pmatrix} \frac{1}{18} & \frac{7}{144} & \frac{1}{144} \\ \frac{1}{18} & \frac{4}{18} & \frac{3}{18} \\ \frac{1}{18} & \frac{3}{18} & \frac{4}{18} \end{pmatrix}$$

a) Determine the exact entropy (you can give a numerical value, but use the exact probabilities).
b) Determine the bias in the mutual entropy estimate for N=100 and 1000 samples
c) Implement the shuffling procedure of the Ince et al paper. (They mention a toolbox, you are free to use that to compare the accuracy of your answers). For this method each of the entropies (response and noise) is written as
$H(N) = H_{naive} - (H_{shuffle} - H_{ind})$.
The mean of the term in the parenthesis is zero, but the bias in 'shuffle' term is as large in the naïve estimate, hence in effect this corrections adds nothing in the mean but subtracts the bias. In the following we define the different terms (we give the formulas for H(R|S), H(R) is obtained in a similar way).
The samples from $p(r_A, r_B \mid s)$ are denoted by (r_{Ai},r_{Bi},s_i), from which the estimate $\hat{p}(r_A, r_B \mid s)$ is obtained by counting. In the shuffled version, for each s separately, all r_{Ai} are shuffled using matlab's randperm function, r_{Bi} can stay the same. And then $\hat{p}_{shuffled}(r_A, r_B \mid s)$ is determined from these new samples from which $H_{shuffle}$ is calculated. For the independent version, the marginal distributions are determined for each s, $\hat{p}(r_A \mid s)$ and $\hat{p}(r_B \mid s)$ from which you get $\hat{p}_{ind}(r_A, r_B \mid s) = \hat{p}(r_A \mid s)\hat{p}(r_B \mid s)$ and thus H_{ind}.
d) Show that the procedure in c) reduces the bias in the entropy and thus the mutual information estimates.