Solution: First passage times

1. We write

$$I_N(t) = \frac{\lambda^N}{(N-1)!}\exp(-\lambda t + (N-1)\log t)$$

We write the exponent as a Taylor series expansion around its maximum. This maximum is obtained for $t_0 = (N-1)/\lambda$, We obtain:

$$\begin{aligned} -\lambda t + (N-1)\log t &\approx -\lambda t_0 + (N-1)\log t_0 - \frac{1}{2}(t-t_0)^2\frac{N-1}{t_0^2} \\ &\approx -N + N\log\frac{N}{\lambda} - \frac{1}{2}(t-t_0)^2\frac{\lambda^2}{N} \end{aligned}$$

where in the last step we have ignored terms of order 1. We use the Stirling formula for the approximation of the factorial:

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n (1 + O(1/n))$$

Combining these two approximations gives:

$$I_N(t) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(t-t_0)^2}{2\sigma^2}\right)$$

with $\sigma^2 = N/\lambda^2$.