Problem set 3, Computational Neuroscience, September 24th 2015, please hand in before October 1st 2015.
(Problem 3 might carry over to the next session; please check Blackboard.)

1. **Playing with networks.**
The following is a simplified model for a regular spiking pyramidal cell

$$100\dot{v} = 0.7(v+60)(v+40) - u + I$$
$$\dot{u} = 0.03(-2(v+60) - u)$$

with the reset conditions: if v>35 (i.e. action potential) then v=-50 and u=u+100.

a) Implement this model in Matlab (do not use the built-in integrator, the Euler method would be OK, but modified Euler is better).
b) Study the properties of the model

- for what value of I does the neuron spike (you need to first find typical values, use the phase plane for that)?
- Describe the bifurcation at the onset of spiking.
- how does the firing rate vary with I?
- For Cognitive neuroscience master students: what type of spiking behavior does this neuron show?

c) Implement a network of 'N' of these neurons and couple them according a general connection matrix (this means that at the reset condition you send currents to other neurons to which the neuron that just spiked is connected).
d) Pick a connection matrix and coupling strength. Explore! This could include plotting the rastergram (dots) or determining how the firing rate depends on connection strength. Use your imagination as budding scientist.

2. **Phase resetting curves**
The phase resetting curve (PRC) is defined as the change in phase $\Delta(\phi) = \phi' - \phi$ due to a pulse at phase ϕ. It can be determined by applying a set of pulses at different times but there are also analytical methods (see for instance http://www.scholarpedia.org/article/Isochron).

The method we will explore here is to associate to each point in phase space (1D in part **a**, 2D in part **b**) an asymptotic phase, so that when a pulse moves from the system from one point in phase to another point, the change in phase can be read off directly from this change of position in phase space.

a) Calculate the PRC of the leaky integrate and fire model neuron, $\tau\dot{V} = I - V$, with I>1 and when V>=1 then reset to V=0. The membrane time constant tau and I are parameters.
Step 1: associate a phase with each value of V;
Step 2: determine the change in V after a current pulse $\Delta I \delta(t - t_{pulse})$;
Step 3: find the change in phase corresponding to that ΔV.

b) Consider the following model

$$\dot{R} = (1 - R^2)R$$
$$\dot{\phi} = R$$

Step 1: determine the limit cycle and show that it is stable. The phase after a long time should look like $\phi(t) = \phi_0 + t$, with $\theta = \phi_0$ being the desired phase;
Step 2: calculate the (asymptotic) phase θ for each initial condition (R, ϕ);
Step 3: apply a delta pulse in either R or x (which refers to the original cartesian coordinates, and would in the neuron case correspond to the current, but it is the harder calculation), determine the change in asymptotic phase and thereby the PRC.

You can choose whether you want to calculate the PRC analytically or numerically, but it is probably easier to find the exact solution from an arbitrary initial condition.

3. **The Kuramoto model.** Under the assumption that the state of each neuron can be represented by the phase and that the effect of synaptic input can be captured in terms of sinusoidal PRC the dynamics of a network can be written as

$$\dot{\theta}_i = \omega_i - \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_i - \theta_j),$$

here K is the coupling strength, N is the number of oscillators, θ_i is the phase of ith oscillator and ω_i is its frequency.
a) Derive the following mean field equation:

$$\dot{\theta}_i = \omega_i - Kr\sin(\theta_i - \Theta) \text{ with } r\exp(i\Theta) = \frac{1}{N}\sum_{j=1}^{N}\exp(i\theta_j)$$

b) How should you interpret r and Θ?
c) When can you find stable solutions of the mean field equation, assuming that r and Θ are fixed?
d) Determine using simulation the solution as a function of K. Use a network of N=1000 neurons, with a oscillation frequency ω drawn from normal distribution with zero mean and a standard deviation of 0.1.

Step 1. Plot r and Θ as function of time for different values of K.
Step 2. Plot the stationary value of r, using step 1 to determine how long to simulate, as a function of K.
Step 3. There are locked oscillators whose phase is constant (note the mean frequency has been made zero without loss of generality), and those that are not locked, whose phase will vary with time (relate this to your answer to part c). Plot the fraction of locked oscillators as a function of K. Explain your findings.