Computational Neuroscience 2015, Problem set 1
(Thursday – September 10th ; Hand-in on September 17th)

1. *In order to find fixed points and thereby construct bifurcation diagrams, roots of nonlinear equations will need to be found. In this problem you will be introduced to Newton's method and the matlab rootfinding routine fzero. The generic equation to be solved is f(x)=0.*

a. Consider the case where you are at x_n and you want to find a next point $x_{n+1}=x_n+h$ that is closer to the root, i.e. $f(x_n+h)=f(x_n)+hf'(x_n)+h.o.t.=0$, hence a step in the right direction is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. This is Newton's method. Solve by hand using Newton's method the equation $f(x) = x^2 - 2 = 0$ (i.e. find the square root of 2).

b. Implement Newton's method in matlab for f(x)=x exp(-x)+1 (just to make sure this is not a nonsense function, use 'fplot' to determine it does have a root)

c. To construct a bifurcation diagram, roots have to found as a function of a parameter 'a'. For instance, f(x,a)=x exp(-x)+a. Find out for what a-values you can expect solutions and plot them as a function of 'a'.

d. It is often better to use existing Matlab functions because they are written such that problems with division by zero do not crash your calculation. Read the help file for 'fzero'. Specifically, find out how to find a zero of a function with a parameter a. Construct using 'fzero' the bifurcation diagram as in c.

2. *To plot the voltage trace of a neuron or the spike trains of a large-scale network, you will have to solve coupled systems of ordinary differential equations. The easiest algorithm is the Euler method, but it is very bad, as it represents the worst trade off between accuracy and rounding errors and it should therefore be avoided at all cost. Better methods are second-order and fourth order Runge-Kutta (the name indicates their global accuracy) and the routines 'ode45' , 'ode23' etc provided with Matlab.*

The differential equation is written as y'=f(x,y), with x the independent and y the dependent variable. The goal is to find y as a function of x, starting from an initial point x_0 where $y(x_0)=y_0$. We denote the integration step by h and the resulting points are $x_n=x_0+nh$ and $y_n=y(x_0+nh)$

The Euler method is $y_{n+1}=y_n+hf(x_n,y_n)$

A second order Runge-Kutta ('modified Euler') is
$$\tilde{y}_{n+1} = y_n + hf(x_n, y_n)$$
$$y_{n+1} = y_n + \tfrac{1}{2}h(f(x_n, y_n) + f(x_n + h, \tilde{y}_{n+1}))$$
Here the wiggle denotes an intermediate (predictor) variable that is used to estimate the derivative at the end point of the integration step.

The fourth order Runge-Kutta (RK4) is:

$$y_{n+1} = y_n + \tfrac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$k_1 = hf(x_n, y_n)$$
$$k_2 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_1)$$
$$k_3 = hf(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}k_2)$$
$$k_4 = hf(x_n + h, y_n + k_3)$$

a. Solve the differential equation y'=f(x,y)=-2x-y with y(0)=-1 on the interval [0 1], using Euler and either the second or fourth order RK depending on the level of your ambition. Use different h values (0.5, 0.25, etc). Comment on the relative accuracy of the numerical solutions (use that the analytical answer is y(x)=-3exp(-x)-2x+2, this helps in your comparison). It may help to estimate the power of h with which the error term grows.

b. Read the documentation for the matlab function 'ode45' and calculate & plot the solution to the aforementioned differential equation and compare it with the analytical answer.

The above methods deal with scalar y values. Most problems deal with coupled differential equations, where y has multiple components. All the above formulas can be used as long as you replace each occurrence of y, f and k by the appropriate vector. A key issue here is the order of the computation, you need to calculate all components of k_1 in RK4 before you can move on and calculate k_2 etc. To practice this we will consider the equation for a pendulum (I is a constant).

$$\frac{d^2\theta}{dt^2} = -\sin(\theta) + I$$

c. The equation for a pendulum is a second order equation, whereas numerically we can deal only with coupled first order equations. Show that the above equation is equivalent with the following two coupled first order differential equations (specify what y_1 and y_2 is).

$$y_1' = y_2$$
$$y_2' = I - \sin(y_1)$$

d. Implement and test the modified Euler method for these equations. Make a plot of the curves in the phase plane (y_2 versus y_1) for different values of I, use values below and above 1.

e. Implement the same differential equation using ode45. How can you incorporate parameters of the differential equation in ode45? (Hint: anonymous functions)

3. Consider the second-order differential equation $y'' + by' + y = 0$ for different values of b.

a. Determine the analytical solution of this equation and characterize the nature of these solutions (remember the course 'Trillingen & golven'?).

b. Write this system as two coupled first order equations and determine the stability of the fixed point at the origin. What is the nature of the solutions found in a. for different b values when they are plotted in the phase plane?

c. Find the stability of the equilibrium point at the origin for the nonlinear equation

$y'' + b(y')^3 + y = 0$ (You can do this qualitatively by considering a few trajectories)