2.a

-Solution of ODE is

$$v(t;v_0)=v_0e^{-\frac{t}{\tau}}+I-Ie^{-\frac{t}{\tau}}$$

where $v(t)=v_0$. Then, the natural period of the neuron T is

$$T=-\tau\ln\left(1-\frac{1}{I}\right).$$

We associate phase of the neuron $\phi\in[0,1]$ and time $0\le t$ by

$$\phi=t \bmod T.$$

When $0\le t\le T$, ϕ equals $\frac{t}{T}$. We can parameterize the voltage by using phase by

$$v(\phi;v_0)=v_0e^{-\frac{\phi T}{\tau}}+I-Ie^{-\frac{\phi T}{\tau}}.$$

-To determine PRC of the leaky integrate and fire neuron, we assume that, at time $0\le t^{'}\le T$, a pulse input $I(t-t^{'})=\varepsilon\delta(t-t^{'})$ is injected into the neuron, phase of which is $\phi^{'}=\frac{t^{'}}{T}$: the voltage of the neuron will jump from $v(\phi^{'};0)$ to $v(\phi^{'};0)+\varepsilon$. Subsequently, the phase jumps from $\phi^{'}$ to $\phi^{''}$: $\phi^{''}$ is determined by

$$v(\phi^{''};0)=v(\phi^{'};0)+\varepsilon.$$

As a function of phase of the neuron $\phi^{'}$ and magnitude of the pulse input ε ,PRC is given by

$$PRC(t^{'},\varepsilon)=\phi^{''}-\phi^{'}$$

where $\phi^{''}=-\frac{\tau}{T}\ln\left(e^{-\frac{\phi^{'}T}{\tau}}-\frac{\varepsilon}{I}\right)$.

2.b

-Due to symmetry of the problem, we assume that the asymptotic phase θ equals $\phi + f(R)$: we can determine $f(R)$ by

$$1 = \frac{d\phi}{dt} + \frac{df}{dR}\frac{dR}{dt}.$$

Then $f(R)$ equals $\ln[R/(1+R)]+c$ where c is a constant. Because $f(1)$ equals 0, c equals $\ln(2)$. Therefore, the asymptotic phase $\theta(\phi, R) = \phi + \ln[2R/(1+R)]$

-We assume that an input arrives when phase of the system is ϕ and causes change only in the first component of the system, i.e. Δx. Then, the new $\tilde{R}$ is $\sqrt{1+2\Delta x\cos(2\pi\phi)+\Delta x^2}$ and new $\tilde{\phi}$ is $\tan^{-1}[\sin(2\pi\phi)/(\cos(2\pi\phi)+\Delta x)]$. With $\tilde{R}$ and $\tilde{\phi}$ we can determine a new phase of the system that is equal $\theta(\tilde{\phi}, \tilde{R})$. Then, PRC is $\theta(\tilde{\phi}, \tilde{R}) - \phi$.