Computational neuroscience - First assignment

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Handouts: Chapter 2, Exercise 1

I will use the Gamma distribution as given in Chapter 2.

$$I_N(t) = \frac{\lambda^N t^{N-1}}{(N-1)!} e^{-\lambda t} \tag{1}$$

We know that $t > 0$; thus, we can use the identity $x = e^{\ln x}, x > 0$

$$t^{N-1} = e^{\ln t^{N-1}}$$

Moreover, $\ln a^b = b \ln a$, thus

$$t^{N-1} = e^{\ln t^{N-1}} = e^{(N-1)\ln t} \tag{2}$$

By using (2) in equation (1) we get

$$I_N(t) = \frac{\lambda^N}{(N-1)!} e^{(N-1)\ln t} e^{-\lambda t}$$

Based on the definition of the Taylor series, we see that the Taylor expansion of exponential around N is

$$e^{(N-1)\ln t} = e^{(N-1)\ln N} + \frac{1}{1!}\left(\frac{N-1}{t}\right)^1 e^{(N-1)\ln N} + \frac{1}{2!}\left(\frac{N-1}{t}\right)^2 e^{(N-1)\ln N} + ... \Leftrightarrow$$

$$I_N(t) = \frac{\lambda^N}{(N-1)!} e^{-\lambda t} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{N-1}{t}\right)^k e^{(N-1)\ln N} \Leftrightarrow$$

$$I_N(t) = \frac{\lambda^N}{(N-1)!} e^{-\lambda t} e^{(N-1)\ln N} \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{N-1}{t}\right)^k \tag{3}$$

We can now use the Stirling's approximate factorial $k! = k^k e^{-k}\sqrt{2\pi k}$ and get

$$\sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{N-1}{t}\right)^k = \sum_{k=0}^{\infty} \frac{(N-1)^k}{k^k e^{-k}\sqrt{2\pi k}\, t^k} = \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(N-1)^k e^k}{k^k t^k \sqrt{k}} \tag{4}$$

From (3) and (4) we find

$$I_N(t) = \frac{\lambda^N}{(N-1)!} e^{-\lambda t} e^{(N-1)\ln N} \frac{1}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{(N-1)^k e^k}{k^k t^k \sqrt{k}} \tag{5}$$

Now, we should somehow convert this to a form in accordance to the Gaussian function

$$f(x) = a \cdot e^{\frac{(x-b)^2}{2c^2}}$$

or

$$f(x) = a \cdot e^{\left(\frac{x-b}{\sqrt{2}c}\right)^2}$$

We can see that we've created many constants in (5) that can be regarded simply as a. For example, we could set a as

$$a = \frac{\lambda^N}{(N-1)!} e^{(N-1)\ln N} \frac{1}{\sqrt{2\pi}}$$

or whatever is most convenient for the future transformations. However, I'm not able to continue farther from here.

Handouts: Chapter 2, Exercise 2

By definition, the Laplace transformation of $\mathcal{L}\{f(t)\}$ is $\int_0^\infty e^{-st} f(t) dt$. Similarly,

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \int_0^\infty e^{-st} \frac{1}{\sqrt{\pi t}} dt$$

$\frac{1}{\sqrt{\pi}}$ is a constant factor and we can, thus, move it outside of the integral:

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{1}{\sqrt{\pi}} \int_{t=0}^{t=\infty} e^{-st} \frac{1}{\sqrt{t}} dt \tag{1}$$

Now let us assume a new variable u, such that

$$u = st \tag{2}$$

Consequently, $t = \frac{u}{s} \Leftrightarrow \sqrt{t} = \frac{\sqrt{u}}{\sqrt{s}}$ and

$$\frac{1}{\sqrt{t}} = \frac{\sqrt{s}}{\sqrt{u}} \tag{3}$$

Moreover, by differentiating both sides of the equation (2) we find that $du = s\ dt$ and, thus,

$$dt = \frac{1}{s} du \tag{4}$$

We can now use the equations (2), (3) and (4) on the equation (1), and we get

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{1}{\sqrt{\pi}} \int_{u=0}^{u=\infty} e^{-u} \frac{\sqrt{s}}{\sqrt{u}} \frac{1}{s} du$$

We know that $\frac{\sqrt{a}}{a} = a^{\frac{1}{2}} \times a^{-1} = a^{\frac{1}{2}-1} = a^{-\frac{1}{2}} = \frac{1}{\sqrt{a}}$; hence,

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{1}{\sqrt{\pi}} \int_{u=0}^{u=\infty} e^{-u} \frac{1}{\sqrt{us}} du \tag{5}$$

Now let us assume a new variable w, such that

$$w = \sqrt{u} \tag{6}$$

Consequently,

$$u = w^2 \tag{7}$$

By differentiating both sides of the equation (6), we get $dw = \dfrac{1}{2\sqrt{u}} du$. Consequently,

$$du = 2\sqrt{u}\ dw \tag{8}$$

We can now use the equations (7) and (8) on the equation (5), and we get

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{1}{\sqrt{\pi}} \int_{w=0}^{w=\infty} e^{-w^2} \frac{1}{\sqrt{us}} 2\sqrt{u}\ dw$$

By performing simplifications and moving $\frac{2}{\sqrt{s}}$ outside the integral (we are allowed to do so because it is an integral with regard to w only), we get

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{2}{\sqrt{\pi s}} \int_{w=0}^{w=\infty} e^{-w^2}\ dw \tag{9}$$

It is known that

$$\int_0^{\infty} e^{-x^2} dx = \frac{1}{2} \int_{-\infty}^{\infty} e^{-x^2} dx \text{ (Gaussian Integral)} = \frac{1}{2}\sqrt{\pi} \tag{10}$$

From (9) and (10) we get

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{2}{\sqrt{\pi s}} \frac{1}{2} \sqrt{\pi}$$

With simplifications, we get

$$\mathcal{L}\{\frac{1}{\sqrt{\pi t}}\} = \frac{1}{\sqrt{s}}$$

Handouts: Chapter 2, Exercise 3a

Exercise 5.3