Fourier Series to Fourier Transform

- We have seen that for a periodic f^n. $x(t)$ satisfying Dirichlet condition :-

$$x(t) = \sum_{K=-\infty}^{\infty} c_K e^{jK\omega t} \quad \text{Fourier series}$$

where $c_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\omega t}\, dt$

T = fundamental time period and $\omega = \frac{2\pi}{T}$ = fundamental frequency.

- The question is can an aperiodic function too can be resolved into different frequency. The answer is yes!

An aperiodic signal $x(t)$ can be considered to be a periodic signal with time period $T \to \infty$ and fundamental frequency $\Delta\omega = \frac{2\pi}{T} \to 0$.

Example is a single pulse.

Under the condition $T \to \infty$ & $\Delta\omega \to 0$ let us examine the above formulas.

$$c_K = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\, e^{-jK\Delta\omega t}\, dt \quad \text{with } T \to \infty,\ \Delta\omega \to 0$$

$\therefore$ It looks like all $c_K \to 0$ & no useful information can be obtained.

However

$$\frac{c_K}{(1/T)} = \int_{-T/2}^{T/2} x(t)\, e^{-jK\Delta\omega t}\, dt$$

may converge to some value giving Fourier Co-eff density

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Replace $K\Delta\omega$ by ω

$$\frac{C_K}{1/T} = \int_{-T/2}^{T/2} x(t)\, e^{-j\omega t}\, dt \xrightarrow{\lim_{T\to\infty}} X(\omega)$$

Fourier co-eff. density.

$$\boxed{X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt}$$

Now the other formula:-

$$x(t) = \sum_{K=-\infty}^{\infty} C_K\, e^{jK\Delta\omega t} \qquad T = \frac{2\pi}{\Delta\omega}$$

$$= \sum_{K=-\infty}^{\infty} \left(\frac{C_K}{1/T}\right) \frac{1}{T}\, e^{j\underline{K\Delta\omega} t}$$

$$= \int_{\omega=-\infty}^{\infty} F(\omega)\, \frac{\Delta\omega}{2\pi}\, e^{j\omega t}$$

$$\boxed{x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega}$$

Fourier Transform pair

$$x(t) \longleftrightarrow F(\omega)$$

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Fourier Transform of some standard signals and properties :—

$$f(t) \longleftrightarrow F(\omega)$$

We know

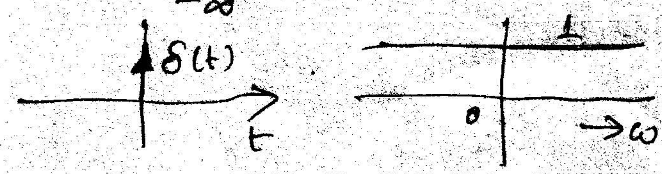
$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega \quad \& \quad F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt$$

① Fourier transform of $\delta(t)$

$x(t) = \delta(t)$

$$\therefore \mathcal{F}[\delta(t)] = \int_{-\infty}^{\infty} \delta(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$$\therefore \boxed{\mathcal{F}[\delta(t)] = 1}$$



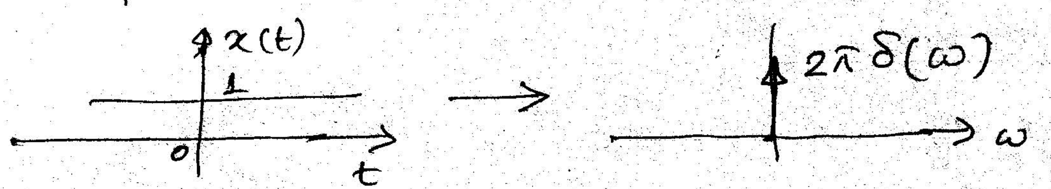
② If $x(t) = 1$ $\quad X(\omega) = ?$

$$X(1) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt = \int_{-\infty}^{\infty} 1\, e^{-j\omega t}\, dt$$

$$x(t) = 1 = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{-j\omega t}\, d\omega = 1$$

Guess $\quad = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\,\delta(\omega)\, e^{-j\omega t}\, d\omega = 1$

$$\therefore \boxed{1 \longleftrightarrow 2\pi\,\delta(\omega)}$$



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(3) Time shifting property:-

If $f(t) \Leftrightarrow F(\omega)$ then $f(t-t_0) \longleftrightarrow e^{-j\omega t_0} F(\omega)$

Now $\mathcal{F}\{x(t-t_0)\} = \int_{-\infty}^{\infty} x(t-t_0)\, e^{-j\omega t}\, dt$ put $t - t_0 = \tau$, $dt = d\tau$

$$= \int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega(\tau + t_0)}\, d\tau = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= e^{-j\omega t_0} X(\omega)$$

$\therefore$ $\boxed{\begin{array}{l} x(t) \longleftrightarrow X(\omega) \\ x(t-t_0) \longleftrightarrow e^{-j\omega t_0} X(\omega) \end{array}}$

(4) If $x(t) \longleftrightarrow X(\omega)$ Then $x(t)\, e^{j\omega_c t} \longleftrightarrow X(\omega - \omega_c)$

$$\mathcal{F}\{x(t)\, e^{j\omega_c t}\} = \int_{-\infty}^{\infty} x(t)\, e^{j\omega_c t}\, e^{-j\omega t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t)\, e^{-j(\omega - \omega_c)t}\, dt = X(\omega - \omega_c)$$

(5) $x(t) \longleftrightarrow X(\omega)$ then $\frac{dx}{dt} \longleftrightarrow j\omega X(\omega)$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \frac{dx}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega X(\omega))\, e^{j\omega t}\, d\omega$$

$\therefore$ $\boxed{\frac{dx}{dt} \longleftrightarrow j\omega X(\omega)}$

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(6) $x(t) \longleftrightarrow X(\omega)$ Then $t f(t) \longleftrightarrow j\dfrac{dX}{d\omega}$.

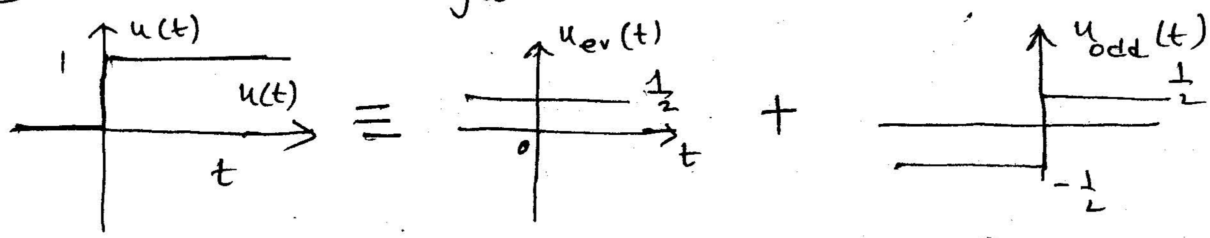
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

$$\therefore \frac{dX}{d\omega} = \int_{-\infty}^{\infty} (-jt)\, x(t)\, e^{-j\omega t}\, dt$$

$$\therefore j\frac{dX}{d\omega} = \int_{-\infty}^{\infty} t\,x(t)\, e^{-j\omega t}\, dt$$

$$\therefore t\,x(t) \longleftrightarrow j\frac{dX}{d\omega}$$

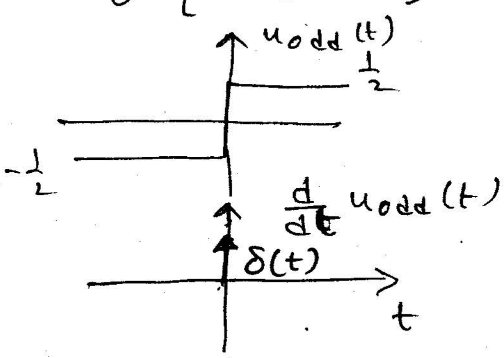
(7) $u(t) \longleftrightarrow \dfrac{1}{j\omega} + \pi\delta(\omega)$



$$\therefore \mathcal{F}\{u(t)\} = \mathcal{F}\{u_{ev}(t)\} + \mathcal{F}\{u_{odd}(t)\}.$$

Now $\mathcal{F}\{u_{ev}(t)\} = \mathcal{F}\left(\tfrac{1}{2}\right) = \tfrac{1}{2} \times 2\pi\delta(\omega) = \pi\delta(\omega)$

$\mathcal{F}\{u_{odd}(t)\} = ?$



$$\mathcal{F}\left\{\frac{d}{dt}u_{odd}(t)\right\} = \mathcal{F}\{\delta(t)\} = 1$$

$$\therefore j\omega\,\mathcal{F}\{u_{odd}(t)\} = 1$$

$$\therefore \mathcal{F}\{u_{odd}(t)\} = \frac{1}{j\omega}$$

$$\boxed{\therefore \mathcal{F}\{u(t)\} = \frac{1}{j\omega} + \pi\delta(\omega)}$$

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(8) very important property

If $z(t) = x(t) * y(t)$ Then $Z(\omega) = X(\omega) \times Y(\omega)$

Now

$$\mathcal{F}\{z(t)\} = Z(\omega) = \int_{t=-\infty}^{\infty} z(t)\, e^{-j\omega t}\, dt$$

Now $$z(t) = x(t) * y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau$$

$$\therefore Z(\omega) = \int_{t=-\infty}^{\infty} \int_{\tau=-\infty}^{\infty} x(\tau)\, y(t-\tau)\, d\tau \; e^{-j\omega t}\, dt$$

Now changing the order of integration.

$$Z(\omega) = \int_{\tau=-\infty}^{\infty} x(\tau) \int_{t=-\infty}^{\infty} y(t-\tau)\, e^{-j\omega t}\, dt \; d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, \mathcal{F}\{y(t-\tau)\}\, d\tau$$

but $y(t) \leftrightarrow Y(\omega)$

$y(t-\tau) \leftrightarrow Y(\omega)\, e^{-j\omega\tau}$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, Y(\omega)\, e^{-j\omega\tau}\, d\tau = Y(\omega) \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$= Y(\omega)\, X(\omega) = X(\omega)\, Y(\omega)$$

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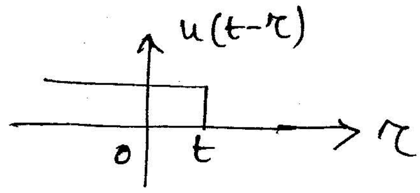
(9) Integration property.

$$x(t) \longleftrightarrow X(\omega) \text{ Then } \int_{-\infty}^{t} x(\tau)\,d\tau \longleftrightarrow \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

Now

$$\int_{-\infty}^{t} x(\tau)\,d\tau = \int_{-\infty}^{t} x(\tau)\,d\tau + \int_{t}^{\infty} 0\,d\tau$$



$$= \int_{-\infty}^{\infty} x(\tau)\,u(t-\tau)\,d\tau$$

$$\int_{-\infty}^{t} x(\tau)\,d\tau = x(t) * u(t)$$

$$\therefore \quad \mathcal{F}\left\{\int_{0}^{t} x(\tau)\,d\tau\right\} = \mathcal{F}\left\{x(t) * u(t)\right\}$$

$$= X(\omega)\,U(\omega)$$

$$= X(\omega)\left[\frac{1}{j\omega} + \pi\delta(\omega)\right]$$

$$= \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega)$$

$$\boxed{\mathcal{F}\left\{\int_{0}^{t} x(\tau)\,d\tau\right\} = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)}$$

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~~Some~~ One important observations:-

9. If $x(t)$ is a real function. and $x(t) \leftrightarrow X(\omega)$

Then $X(-\omega) = X^*(\omega)$

Proof: $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-\infty}^{\infty} x(t)\cos\omega t \, dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t \, dt$

$\therefore X(\omega) = \int_{-\infty}^{\infty} \underbrace{x(t)}_{\text{Real}} \cos\omega t \, dt - j\int_{-\infty}^{\infty} x(t)\sin\omega t \, dt$

$\therefore X(-\omega) = \int_{-\infty}^{\infty} x(t)\cos\omega t \, dt + j\int_{-\infty}^{\infty} x(t)\sin\omega t \, dt.$

$\therefore \boxed{X(-\omega) = X^*(\omega)}$ for real $x(t)$

10. If $x(t)$ is even & Real and $x(t) \leftrightarrow X(\omega)$

Then $X(\omega)$ also will be real and even.

Proof :- $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

put $t = -\tau$ $\quad dt = -d\tau$

$\therefore X(\omega) = \int_{+\infty}^{-\infty} x(\tau) e^{-j\omega(-\tau)} (-d\tau) = \int_{-\infty}^{\infty} x(\tau) e^{-j(-\omega)\tau} d\tau$ $\quad \because x(\tau) = x(-\tau)$ even.

$X(\omega) = X(-\omega)$ $\quad \therefore X(\omega)$ even.

but for real $x(t)$ $\quad X(-\omega) = X^*(\omega)$

$\therefore X(\omega) = X(-\omega) = X^*(\omega)$

$\therefore X(\omega)$ must be real (no imaginary component).

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(17) If $x(t)$ is odd and real

& $x(t) \longleftrightarrow X(\omega)$

Then $X(\omega)$ will be also odd and will be purely imaginary.

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

put $t = -\tau$

$$\therefore X(\omega) = \int_{+\infty}^{-\infty} x(-\tau)\, e^{-j\omega(-\tau)}\, (-d\tau)$$

$\because$ $x(t)$ is odd $\therefore x(-\tau) = -x(\tau)$

$$X(\omega) = \int_{+\infty}^{-\infty} -x(\tau)\, e^{-j\omega(-\tau)}\, (-d\tau)$$

$$= -\int_{-\infty}^{\infty} x(\tau)\, e^{-j(-\omega)\tau}\, d\tau = -X(-\omega)$$

$$\therefore X(\omega) = -X(-\omega) \quad \therefore \underline{X(\omega) \text{ is odd}}$$

or $\boxed{X(\omega) = -X^*(\omega)}$ $\quad \because X(-\omega) = X^*(\omega)$ for real $x(t)$

This is possible if $X(\omega)$ is purely imaginary.

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(12) $x(t) \longleftrightarrow X(\omega)$ Then $x(at) \longleftrightarrow \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$

Case (i) $a>0$ i,e $a = +ve$

Now

$$\mathcal{F}\{x(at)\} = \int_{t=-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

Let $\tau = at$ $\therefore d\tau = a\,dt$

$$\therefore \mathcal{F}\{x(at)\} = \int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\omega\frac{\tau}{a}}\, \frac{d\tau}{a} = \frac{1}{a}\int_{-\infty}^{\infty} x(\tau)\, e^{-j\left(\frac{\omega}{a}\right)\tau}\, d\tau$$

$$\therefore \boxed{\mathcal{F}\{x(at)\} = \frac{1}{a} X\left(\frac{\omega}{a}\right)}$$

Case (ii) $a<0$ a $-ve$ let $b = -a$ $b>0$

$$\mathcal{F}\{x(at)\} = \int_{t=-\infty}^{\infty} x(at)\, e^{-j\omega t}\, dt$$

$$= \int_{t=-\infty}^{\infty} x(-bt)\, e^{-j\omega t}\, dt$$

put $\tau = -bt$ $d\tau = -b\,dt$

$$\mathcal{F}\{x(at)\} = \int_{\tau=+\infty}^{-\infty} x(\tau)\, e^{-j\omega\frac{\tau}{(-b)}}\, \frac{d\tau}{(-b)}$$

$$= \frac{1}{b}\int_{\tau=-\infty}^{\infty} x(\tau)\, e^{-j\frac{\omega}{a}\tau}\, d\tau$$

$$= \frac{1}{b} X\left(\frac{\omega}{a}\right)$$

$$\mathcal{F}\{at\} = \frac{1}{|a|} X\left(\frac{\omega}{a}\right)$$

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(13) Principle of Duality:

If $x(t) \longleftrightarrow X(\omega)$

Then $X(t) \longleftrightarrow 2\pi\, x(-\omega)$.

Now $$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

Replacing t by y.

$$X(\omega) = \int_{-\infty}^{\infty} x(y)\, e^{-j\omega y}\, dy$$

* Now replace ω by t on both sides.

$$X(t) = \int_{-\infty}^{\infty} x(y)\, e^{-jty}\, dy$$

Now replace y by $-\omega$ $\quad y = -\omega$

$$X(t) = \int_{\infty}^{-\infty} x(-\omega)\, e^{-j(-\omega)t}\, (-d\omega) = \int_{-\infty}^{\infty} x(-\omega)\, e^{j\omega t}\, d\omega$$

or $$X(t) = 2\pi \times \frac{1}{2\pi}\int_{-\infty}^{\infty} x(-\omega)\, e^{j\omega t}\, d\omega.$$

$$X(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\, x(-\omega)\, e^{j\omega t}\, d\omega$$

$$\therefore \boxed{X(t) \longleftrightarrow 2\pi\, x(-\omega)}$$

Pictorially

time	frequency
$x(t)$ ⟷	$X(\omega)$
$X(t)$ ⟷	$2\pi x(-\omega)$

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Alternative proof o Duality

$$x(t) \longleftrightarrow X(\omega)$$

then $X(t) \longleftrightarrow 2\pi x(-\omega)$

Proof:
$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

and
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Replace t by $-t$ in both the sides

$$x(-t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

$$\therefore \quad 2\pi x(-t) = \cancel{\frac{1}{2\pi}}\int_{-\infty}^{\infty} X(\omega) e^{-j\omega t} d\omega$$

Now replace t by ω
and ω by t

$$2\pi x(-\omega) = \int_{-\infty}^{\infty} X(t) e^{-j\omega t} dt = \mathcal{F}\{X(t)\}.$$

$$\therefore \quad \boxed{X(t) \longleftrightarrow 2\pi x(-\omega)} \quad \text{proved.}$$

Application

time	frequency
$\delta(t) \longleftrightarrow$	1
$1 \longleftrightarrow$	$2\pi\delta(-\omega) = 2\pi\delta(\omega)$

$\because \delta(-\omega) = \delta(\omega)$.

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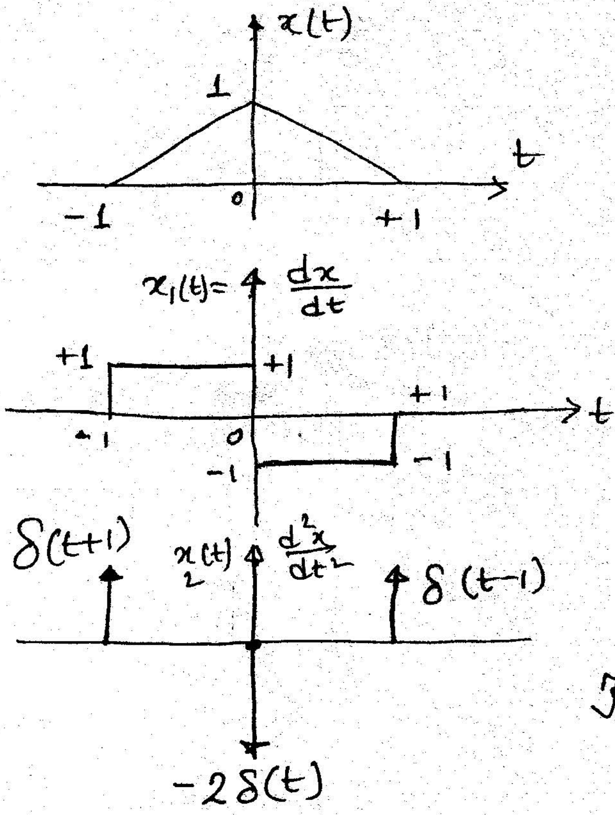
Fourier Transform of a Δ-pulse.



$\mathcal{F}\{x(t)\} = ?$

To get this we first calculate

$\mathcal{F}\{x_2(t)\} = \mathcal{F}\left\{\frac{d^2x}{dt^2}\right\}$.

$= \mathcal{F}\{\delta(t+1)\} - 2\mathcal{F}\{\delta(t)\}$.
$+ \mathcal{F}\,\delta(t-1)$

$\therefore \mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = e^{j\omega} - 2 + e^{-j\omega}$

$\mathcal{F}\left\{\frac{d^2x}{dt^2}\right\} = \left(e^{j\omega} + e^{-j\omega}\right) - 2$

$= 2\cos\omega - 2$

$= 2\left(1 - 2\sin^2\frac{\omega}{2} - 1\right)$.

$(j\omega)^2\, \mathcal{F}\{x(t)\} = -4\sin^2\frac{\omega}{2}$

$\therefore -\omega^2\, \mathcal{F}\{x(t)\} = -4\sin^2\frac{\omega}{2}$

$\therefore \mathcal{F}\{x(t)\} = \frac{4}{\omega^2} \times \sin^2\frac{\omega}{2} = \left(\frac{\sin\frac{\omega}{2}}{\omega/2}\right)^2$

Also note

$j\omega\, \mathcal{F}\{x_1(t)\} = j\omega\mathcal{F}\left(\frac{dx}{dt}\right) = \left(\frac{\sin \omega/2}{\omega/2}\right)^2$

$\therefore \mathcal{F}\left\{\frac{dx}{dt}\right\} = \frac{1}{j\omega}\left(\frac{\sin \omega/2}{\omega/2}\right)^2 = \mathcal{F}\{x_1(t)\}$.

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Area under $x(t)$

and Area under $X(\omega)$

Now $X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$

put $\omega = 0$ on both sides.

$\therefore X(0) = \int_{-\infty}^{\infty} x(t) dt$ = Area under $x(t)$.

Also $x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$

put $t = 0$ on both sides:-

$x(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) d\omega$

$\therefore \int_{-\infty}^{\infty} X(\omega) d\omega = 2\pi x(0)$ = area under the f^n $X(\omega)$.

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