Line Integrals:

Definitions:

Smooth curves: Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ denote the position vector of a point $P(x,y,z)$ in three dimensional space.

If $\bar{r}(t)$ posses a continuous first order derivative for all values of t under consideration then the curve is known as smooth.

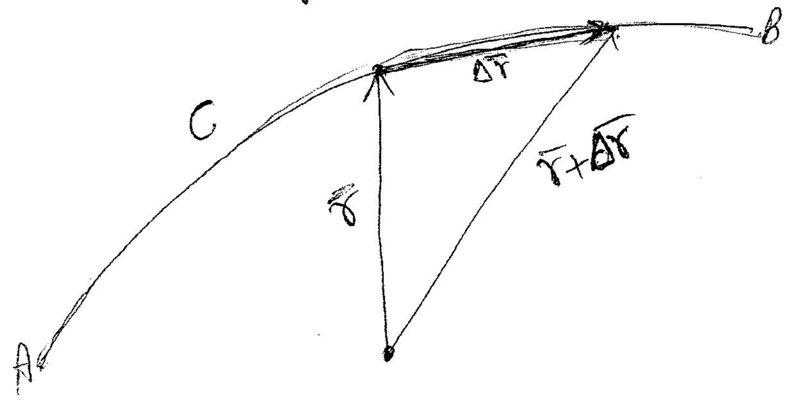
piecewise smooth if it is madeup of a finite number of smooth curves.

Simple Closed curve: A closed smooth curve which does not intersect itself anywhere is known as simple closed curve.

Smooth surfaces: A surface $\bar{r} = \bar{f}(u,v)$ is said to be smooth if $\bar{f}(u,v)$ possesses continuous first order partial derivatives.

Line integrals: (work done by a force).

Let a force $\bar{F}$ act upon a particle which is displaced along a given curve C in space from the point P whose position vector is $\bar{r}$.



first divide the curve C into a large number of small pieces.

Consider the work done when the particle moves from the position $\bar{r}$ to $\bar{r}+\Delta\bar{r}$.

On this small section of the curve C the work done is $\bar{F}\cdot\Delta\bar{r}$

Total work done $W = \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i$

The line integral is defined as:

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \lim_{N\to\infty} \sum_{i=1}^{N} \bar{F}_i \cdot \Delta\bar{r}_i}$$

<u>Evaluation of the line integral:</u>

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_a^b \bar{F}(\bar{r}(t)) \cdot \frac{d\bar{r}}{dt}\, dt}$$

In component form: $\bar{F}(\bar{r}) = \hat{i}\, F_1(x,y,z) + \hat{j}\, F_2(x,y,z) + \hat{k}\, F_3(x,y,z)$

$d\bar{r} = \hat{i}\, dx + \hat{j}\, dy + \hat{k}\, dz$. Then

$$\boxed{\int_C \bar{F}\cdot d\bar{r} = \int_C F_1\, dx + F_2\, dy + F_3\, dz}$$

Example: Find the work done by $\bar{F} = (y - x^2)\bar{i} + (z - y^2)\bar{j} + (x - z^2)\bar{k}$ over the curve $\bar{r}(t) = t\hat{i} + t^2\hat{j} + t^3\bar{k}$, $0 \le t \le 1$ from $(0,0,0)$ to $(1,1,1)$.

Solution:

$$\frac{d\bar{r}}{dt} = \hat{i} + 2t\hat{j} + 3t^2\bar{k}$$

$$\bar{F}(\bar{r}t) = (t^2 - t^2)\hat{i} + (t^3 - t^4)\hat{j} + (t - t^6)\bar{k}$$

$$= (t^3 - t^4)\hat{j} + (t - t^6)\bar{k}$$

$$\bar{F}\cdot\frac{d\bar{r}}{dt} = 2t(t^3 - t^4) + 3t^2(t - t^6)$$

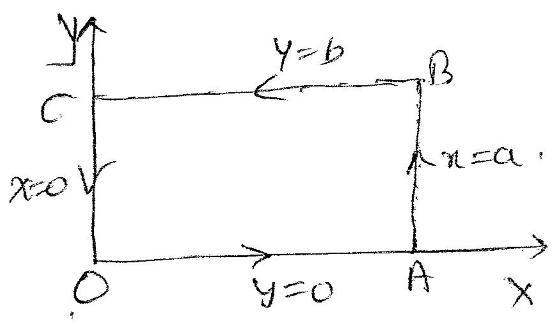
$$= 2t^4 - 2t^5 + 3t^3 - 3t^8$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{t=0}^{1} (2t^4 - 2t^5 + 3t^3 - 3t^8)\,dt$$

$$= \frac{29}{60}. \quad \text{Ans.}$$

Example: 2: Evaluate $\int_C \bar{F}\cdot d\bar{r}$ $\quad \bar{F} = (x^2+y^2)\hat{i} - 2xy\hat{j}$

C: rectangle in xy plane bounded by ~~y=0, y=b, x=0, x=a~~
$y=0, x=a, y=b, x=0$



$$\int_C \vec{F}\cdot d\bar{r} = \int_C \left((x^2+y^2)\hat{i} - 2xy\hat{j}\right)\cdot\left(dx\,\hat{i} + dy\,\hat{j}\right).$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C \left[(x^2+y^2)\,dx - 2xy\,dy\right].$$

Along OA: $y=0$, $dy=0$ & x varies from 0 to a.

$$\int_{OA} \bar{F}\cdot d\bar{r} = \int_0^a x^2\,dx = \frac{a^3}{3}.$$

Along AB: $$\int_{AB} \bar{F}\cdot d\bar{r} = \int_0^b -2a\cdot y\,dy = -ab^2.$$

Along BC: $$\int_{BC} \bar{F}\cdot d\bar{r} = \int_a^0 (x^2+b^2)\,dx$$
$$= -\left[\frac{a^3}{3} + ab^2\right]$$

Along DO: $$\int_{CO} \bar{F}\cdot d\bar{r} = \int_b^0 0\cdot dy = 0$$

$$\Rightarrow \int_C \bar{F}\cdot d\bar{r} = -2ab^2$$

Ans.

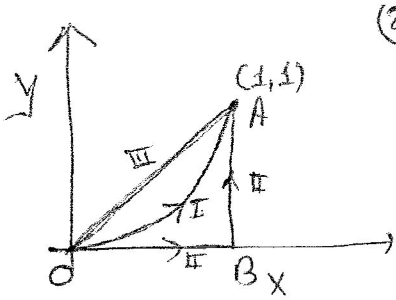
Example: If $\bar{F} = y\hat{i} - x\hat{j}$ Evaluate $\int_C \bar{F}\cdot d\bar{r}$ from $(0,0)$ to $(1,1)$ along the following ~~that~~ path:

i) the parabola $y = x^2$

ii) the straight line $(0,0)$ to $(1,0)$ and then to $(1,1)$

iii) the straight line joining $(0,0)$ to $(1,1)$.

Solution:



$$\bar{r} = x\bar{i} + y\bar{j}$$

$$d\bar{r} = dx\,i + dy\,j$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_C (yi - xj)\cdot(dx\,i + dy\,j)$$

$$= \int_C y\,dx - x\,dy.$$

i) parabola $y = x^2 \Rightarrow dy = 2x\,dx$. x varies from 0 to 1.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 x^2\,dx - x\cdot 2x\,dx$$

$$= \int_0^1 -x^2\,dx = -\frac{1}{3}.$$

ii) along OB & then BA:

$$\int_C \bar{F}\cdot d\bar{r} = \int_{OB}(y\,dx - x\,dy) + \int_{BA}(y\,dx - x\,dy)$$

along OB, $y = 0$ $dy = 0$

along BA $x = 1$ $dx = 0$,

$$\int_C \bar{F}\cdot d\bar{r} = \int_{y=0}^{1} -dy = -1.$$

iii) Along straight line $\bar{O}A$. along $\bar{O}A$ $y = x$.
$dy = dx$.

$$\int_C \bar{F}\cdot d\bar{r} = \int_0^1 (x\,dx - x\,dx) = 0.$$

Ans.

Example: If $\bar{F} = (3x^2+6y)\bar{i} - 14yz\,\hat{j} + 20xz^2\,\bar{k}$ evaluate $\int_c \bar{F}\cdot d\bar{r}$ where c is the straight line joining $(0,0,0)$ to $(1,1,1)$.

Solution: Equation of the line:

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (parameter)}$$

$$x=t \qquad y=t \qquad z=t$$

$$\bar{r} = t\bar{i} + t\bar{j} + t\bar{k}$$

$$\frac{d\bar{r}}{dt} = (\bar{i}+\bar{j}+\bar{k})$$

$$F = (3t^2+6t)\bar{i} - 14t^2\,\bar{j} + 20t^3\,\bar{k}$$

$$\int_c \bar{F}\cdot d\bar{r} = \int_0^1 [(3t^2+6t)\hat{i} - 14t^2 j + 20t^3\,\bar{k}]\cdot[\bar{i}+\bar{j}+\bar{k}]\,dt$$

$$= \int_0^1 [3t^2+6t - 14t^2 + 20t^3]\,dt$$

$$= \int_0^1 [-11t^2 + 6t + 20t^3]\,dt$$

$$= \frac{-11}{3} + \frac{6}{2}^3 + \frac{20}{4}^5$$

$$= \frac{13}{3}. \quad \text{Ans}.$$

Example: Find the total work done in moving a particle in a force field $\bar{F} = 3xy\,\hat{i} - 5z\,\hat{j} + 10x\,\hat{k}$ along the curve $x = t^2+1$, $y = 2t^2$, $z = t^3$ from $t=1$ to 2.

Solution:

$$\int_c \bar{F}\cdot d\bar{r} = \int_c \cancel{3xy}\; 3(t^2+1)(2t^2)(2t) - 5t^3(4t) + 10(t^2+1)3t^2\,dt$$

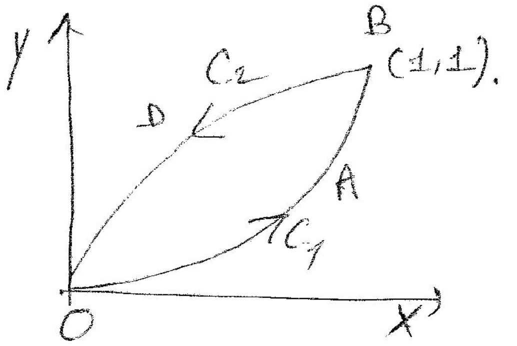
$$= 303 \qquad \text{Ans}.$$

Note: The integral around a closed curve, $\oint$, is called Circulation integral.

Example: Find the circulation of F around the curve C where $\bar{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C in the curve $y=x^2$ for (0,0) to (1,1) and the curve $y^2=x$ from (1,1) to (0,0).

Solution: $\bar{r} = x\hat{i} + y\hat{j}$

$d\bar{r} = \hat{i}\,dx + \hat{j}\,dy$



$$F\cdot d\bar{r} = (2x+y^2)dx + (3y-4x)dy$$

$$\int_C \bar{F}\cdot d\bar{r} = \int_{C_1} \bar{F}\cdot d\bar{r} + \int_{C_2} \bar{F}\cdot d\bar{r}$$

Along OAB: $y=x^2$ $dy = 2x\,dx$.

$$\int_{C_1} \bar{F}\cdot d\bar{r} = \int_{C_1} (2x+x^4)dx + (3\cdot x^2 - 4x)\cdot 2x\,dx.$$

$$= \int_0^1 (x^4+6x^3-8x^2+2x)\,dx$$

$$= \left[\frac{x^5}{5} + 6\frac{x^4}{4} - 8\cdot\frac{x^3}{3} + 2\frac{x^2}{2}\right]_0^1$$

$$= \frac{1}{30}.$$

Along C_2: $x=y^2$ $dx = 2y\,dy$. y varies from 0 to 1.

$$\int_{C_2} \bar{F}\cdot d\bar{r} = \int_{y=1}^{0} (2y^2+y^2)\,2y\,dy + (3y-4\cdot y^2)\cdot \cancel{2y\,dx}\,dy.$$

$$= -\int_0^1 6y^3-4y^2+3y$$

$$= -5/3.$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}$$ Ans.