Tutorial Sheet-2 **AUTUMN 2017**

MATHEMATICS-I (MA10001) *August 1, 2017*

1. Determine the following limits, if exist:

   a) $\lim_{x\to 0}\left[1+\left(\dfrac{\log\cos x}{\log\cos(x/2)}\right)^2\right]^2$

   b) $\lim_{x\to 0}\left(\sqrt{\sqrt{x}+x^3}\right)\cos\dfrac{\pi}{x}$

   c) $\lim_{x\to 1}\dfrac{x^n-1}{|x-1|},\quad n\in\mathbb{N}$

   d) $\lim_{x\to\pi/4}\dfrac{\sin x-\cos x}{\cos(2x)}$

   e) $\lim_{x\to\infty}\dfrac{x^{-1}+x^{-1/2}}{x+x^{-1/2}}$

   f) $\lim_{x\to 0}\dfrac{\sin(2x)}{\ln(x+1)}$

   g) $\lim_{x\to\pi/2}(\cos x)^{\frac{\pi}{2}-x}$

   h) $\lim_{x\to 1}\left(\tan\dfrac{\pi x}{4}\right)^{\tan\frac{\pi x}{2}}$

   i) $\lim_{x\to\infty}x^{1/x}$

   j) $\lim_{x\to\pi/2}(\tan x)^{2x-\pi}$

   k) $\lim_{x\to 0+}(\sin 2x)^{\tan 3x}$

   l) $\lim_{x\to\infty}\left(\dfrac{x+1}{x+2}\right)^x$

   m) $\lim_{x\to 0+}(\cot x)^{\frac{1}{\ln x}}$

   n) $\lim_{x\to 0}\dfrac{e^x-x-1}{\cos x-1}$

   o) $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{\cot x}{x}\right)$

   p) $\lim_{x\to 0+}\left(\dfrac{\sin^{-1}x}{x}\right)^{\frac{1}{x^2}}$

   q) $\lim_{x\to\infty}x^{1/\ln x}$

   r) $\lim_{x\to 0}\dfrac{\log_{sec(x/2)}\cos x}{\log_{\sec x}\cos(x/2)}$

   s) $\lim_{x\to 0}\left(\dfrac{1}{x}\right)^{\tan x}$

   t) $\lim_{x\to\pi/2}(\sec x)^{\cot x}$

   u) $\lim_{x\to 0}(\cos ax)^{b/x^2}$.

2. Find the Taylor polynomial of degree n about x_0 with remainder of both Cauchy and Lagranges form for the following functions:

   a) $f(x)=\log(1+x)$, degree $n=4$, centered at $x_0=0$.
   b) $f(x)=\sin x$, degree $n=3$, centered at $x_0=\pi/6$.
   c) $f(x)=\sqrt{1+x+x^2}$, degree $n=3$, centered at $x_0=0$.
   d) $f(x)=\frac{1}{x}$, degree $n=4$, centered at $x_0=1$.

3. Write the Maclaurin's formula for the function $f(x)=\sqrt[3]{1+x}$ of degree 2. Further estimate the error of the approximate equation $\sqrt[3]{1+x}\approx 1+\frac{1}{3}x-\frac{1}{9}x^2$ when $x=0.3$.

4. Suppose $P_2(x)$ be the second degree Taylor polynomial for $f(x)$ centered at $x = 10$ and $|f'''(x)| \leq 3$ for any x. Estimate the absolute value of the remainder term in both Cauchy and Lagranges form.

5. If $f(x)$, $f'(x)$, $f''(x)$ are continuous on $[a, b]$ and $f'(a) = f'(b)$. Then show that

$$f\left(\frac{a+b}{2}\right) = \frac{1}{2}[f(a) + f(b)] + \frac{(b-a)^2}{8} f''(c)$$

for some $c \in (a, b)$.

6. By Taylor series expansion, using suitable function

   a) find the value of $\sqrt{1.5}$ approximately.

   b) show that $\sin 46^\circ \approx \frac{1}{\sqrt{2}}\left(1 + \frac{\pi}{180}\right)$.

7. Expand the following polynomial with respect to the given powers.

   a) $x^4 - 4x^3 + 5x^2 + 3$ in powers of $x - 1$.

   b) $x^5 + x^3 + x$ in powers of $x + 2$.

   c) $x^5 - 5x^4 + 6x^3 - 3x^2 - 4x + 5$ in powers of $x - 3$.

8. Use Taylor's theorem to prove that

   a) $\cos x \geq 1 - \frac{x^2}{2}$ for $-\pi < x < \pi$.

   b) $x - \frac{x^3}{6} < \sin x < x$ for $0 < x < \pi$.

   c) $1 + x + \frac{x^2}{2} + \frac{x^3}{3!} < e^x < 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} e^x$ for all $x > 0$.

9. Prove that

   a) $\left|\log(1+x) - (x - \frac{x^2}{2} + \frac{x^3}{3})\right| < \frac{1}{4}$ for any $x \in [0, 1]$.

   b) $\left|\sin x - (x - \frac{x^3}{3!} + \frac{x^5}{5!})\right| < \frac{1}{7!}$ for any $x \in [-1, 1]$.

10. Find the Maclaurin's infinite series expansion for

    a) $f(x) = \cos x$ for all $x \in \mathbb{R}$.

    b) $f(x) = \log(1 + x)$ for $(-1, 1]$.

    c) $f(x) = e^x \cos x$ for all $x \in \mathbb{R}$

11. Let a function $f : [a, \infty) \to \mathbb{R}$ be twice differentiable on $[a, \infty)$ and there exists positive real numbers A and B such that $|f(x)| \leq A$, $|f''(x)| \leq B$ for all $x \in [a, \infty)$. Prove that $|f'(x)| \leq 2\sqrt{AB}$ for all $x \in [a, \infty)$.

12. Let $c \in \mathbb{R}$ and a real function f be such that f'' is continuous on some neighbourhood of c. Prove that

$$\lim_{h\to 0} \frac{f(c+h) - 2f(c) + f(c-h)}{h^2} = f''(c).$$

13. If $f(x) = \sin x = f(0) + xf'(\theta x)$, $0 < \theta < 1$,then show that

$$\lim_{x\to 0} \theta = \frac{1}{\sqrt{3}}.$$

14. Using Taylor's series formula, evaluate

a) $\displaystyle\lim_{x\to 0} \frac{\sin x}{\sqrt{1-\cos x}}$

b) $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

c) $\displaystyle\lim_{x\to 0} \frac{x - \log(1+x)}{1-\cos x}$

d) $\displaystyle\lim_{x\to 0} \frac{xe^x - \log(1+x)}{x^2}$.

15. Evaluate lthe limit $\displaystyle\lim_{t\to 0} \frac{e^{t^2} - 1 - t^2 - \frac{t^4}{2}}{\sin(t^2) - t^2}$ by using L'Hospital rule. Observe that it can be solve more easily by using taylor's series expansion of e^{t^2} and $\sin(t^2)$ about $t = 0$.

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