1.

i) The integral $\int_0^1 \frac{1}{1-x}\,dx$ is improper, since 1 is the point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ ~~satisfy~~ satisfying $0<\epsilon<1$.

$$\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{dx}{1-x} = \lim_{\epsilon\to 0^+}[-\log\epsilon] = \infty.$$

Therefore the improper integral $\int_0^1 \frac{dx}{1-x}$ is divergent.

ii) The integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is improper, since 1 is a point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0, 1-\epsilon]$ for all ϵ satisfying $0<\epsilon<1$.

$$\lim_{\epsilon\to 0^+}\int_0^{1-\epsilon}\frac{1}{\sqrt{1-x^2}}\,dx = \lim_{\epsilon\to 0^+}\left[\sin^{-1}(1-\epsilon)\right] = \frac{\pi}{2}.$$

Therefore the improper integral $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$ is convergent and $\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{2}$.

iii) The integral $\int_0^2 \frac{dx}{\sqrt{x(2-x)}}$ is improper, since 0 and 2 are points of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\epsilon, 2-\epsilon']$ for all ϵ, ϵ' satisfying $0<\epsilon<2$, $0<\epsilon'<2$.

Let us examine if $\lim_{\epsilon\to 0^+}\int_\epsilon^1 \frac{1}{\sqrt{x(2-x)}}\,dx$ and $\lim_{\epsilon'\to 0^+}\int_1^{2-\epsilon'}\frac{1}{\sqrt{x(2-x)}}\,dx$ exist.

$$\lim_{\epsilon\to 0^+}\int_{0+\epsilon}^1 \frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{\epsilon\to 0^+}\left[\sin^{-1}(x-1)\right]_\epsilon^1 = \frac{\pi}{2},$$

$$\lim_{\epsilon'\to 0^+}\int_1^{2-\epsilon'}\frac{1}{\sqrt{x(2-x)}}\,dx = \lim_{\epsilon'\to 0^+}\left[\sin^{-1}(x-1)\right]_1^{2-\epsilon'} = \frac{\pi}{2}.$$

Therefore $\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx$ is convergent and $\int_0^2 \frac{1}{\sqrt{x(2-x)}}\,dx = \pi$.

iv) Let the given integral be $\int_1^\infty \frac{1}{x\log(x)}dx$. Then 1 is a point of infinite discontinuity of the integrand. We are to examine the convergence at 1 as well as at ∞.

Let us consider the integrals $\int_1^a \frac{1}{x\log x}dx$ and $\int_a^\infty \frac{1}{x\log x}dx$ where $a \in (1,\infty)$

Let $\phi(\varepsilon) = \int_{1+\varepsilon}^a \frac{1}{x\log x}dx$ and $\Psi(X) = \int_a^X \frac{1}{x\log x}dx$; where $1<\varepsilon<a$ and $X>a$.

Now

$$\phi(\varepsilon) = \int_{1+\varepsilon}^a \frac{1}{x\log x}dx = \log(\log x)\Big|_{1+\varepsilon}^a = \log(\log a) - \log(\log(1+\varepsilon))$$

$$\Psi(X) = \int_a^X \frac{1}{x\log x}dx = \log(\log x)\Big|_a^X = \log(\log X) - \log(\log a)$$

Now

$$\int_1^\infty \frac{1}{x\log x}dx = \lim_{\varepsilon\to 0^+}\phi(\varepsilon) + \lim_{X\to\infty}\Psi(X)$$

$$= \lim_{\substack{X\to\infty \\ \varepsilon\to 0^+}} \left[\log(\log X) - \log(\log(1+\varepsilon))\right]$$

clearly limit does not exists.

Therefore the improper integral $\int_1^\infty \frac{1}{x\log x}dx$ is divergent.

v) Let the given integral be $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$. Then we are to examine the convergence at ∞.

Let us consider the integral $\phi(X) = \int_1^X \frac{1}{(1+x)\sqrt{x}}dx$, $X>1$.

Therefore, $\phi(X) = \int_1^X \frac{1}{(1+x)\sqrt{x}}dx$, $X>1$.

putting $x=t^2$, we get $dx = 2t\,dt$. Now

$$\phi(X) = \int_1^{\sqrt{X}} \frac{2t}{(1+t^2)t}dt = 2\cdot\int_1^{\sqrt{X}} \frac{dt}{(1+t^2)} = 2\left[\tan^{-1}(\sqrt{X}) - \frac{\pi}{4}\right]$$

Now $\lim_{X\to\infty}\phi(X) = 2\cdot\left[\tan^{-1}(\infty) - \frac{\pi}{4}\right] = 2\cdot\frac{\pi}{4} = \frac{\pi}{2}$.

Therefore the integral $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx$ is convergent and

$$\int_1^\infty \frac{1}{(1+x)\sqrt{x}}dx = \frac{\pi}{2}.$$

vi) The integral $\int_1^3 \frac{10x}{(x^2-9)^{1/3}}dx$ is improper, since 3 is the point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[1, 3-\varepsilon]$ for all ε satisfying $0<\varepsilon<2$. Now

$$\lim_{\varepsilon\to 0^+}\int_1^{3-\varepsilon}\frac{10x}{(x^2-9)^{1/3}}dx = \lim_{\varepsilon\to 0^+}\int_{-2}^{(\varepsilon^2-6\varepsilon)^{1/3}} 5.3.t\, dt \quad [\text{putting } x^2-9=t^3]$$

$$= \lim_{\varepsilon\to 0^+}\left[\frac{15}{2}\left[(\varepsilon^2-6\varepsilon)^{2/3}-4\right]\right]$$

$$= -30.$$

Therefore the improper integral $\int_1^3\frac{10x}{(x^2-9)^{1/3}}dx$ is convergent and $\int_1^3\frac{10x}{(x^2-9)^{1/3}}dx = -30.$

2.

i) Let the given integral be $\int_0^1 f(x)\,dx$, where $f(x) = \frac{1}{(x+1)(x+2)\sqrt{x(1-x)}}$. 0 and 1 are the only points of infinite discontinuity of f. $f(x)>0$ for all $x\in(0,1)$. Let us examine the convergence of the improper integrals $\int_0^{1/2} f(x)\,dx$ and $\int_{1/2}^1 f(x)\,dx$.

Convergence of $\int_0^{1/2} f(x)\,dx$ at 0.

$\lim_{x\to 0^+}\sqrt{x}\,f(x) = \frac{1}{2}$. By μ test, $\int_0^{1/2} f(x)\,dx$ is convergent ... (i)

Convergence of $\int_{1/2}^1 f(x)\,dx$ at 1.

$\lim_{x\to 1^-}\sqrt{1-x}\,f(x) = \frac{1}{6}$. By μ test $\int_{1/2}^1 f(x)\,dx$ is convergent. ... (ii)

From (i) and (ii) it follows that $\int_0^1 f(x)\,dx$ is convergent.

ii) The integral $\int_0^{\frac{\pi}{2}} \frac{1}{e^x - \cos x}dx$ is improper, since 0 is the point of infinite discontinuity of the integrand. The integrand is bounded and integrable on $[0+\epsilon, \frac{\pi}{2}]$ for all ϵ satisfying $0<\epsilon<\frac{\pi}{2}$.

Now we will check the convergence of $\int_0^{\pi/2} \frac{1}{e^x-\cos x}dx$ at 0.

$$\lim_{x\to 0^+} \frac{x.}{e^x-\cos x} = \lim_{x\to 0^+} \frac{1}{e^x+\sin x} = 1$$

By μ test $\int_0^{\pi/2} \frac{1}{e^x-\cos x}dx$ is divergent.

iii) Let us examine the convergence of the integrals $\int_0^1 \frac{x^{p-1}}{1+x}dx$ and $\int_0^1 \frac{x^{-p}}{1+x}dx$.

clearly if $p \geq 1$ then $\int_0^1 \frac{x^{p-1}}{1+x}dx$ is a proper one. If $p<1$, 0 is the point of infinite discontinuity of the integrand. Here $f(x)>0$ for all $x\in(0,1]$, where $f(x) = \frac{x^{p-1}}{1+x}$. Now

$$\lim_{x\to 0^+} \frac{x^{1-p}\cdot x^{p-1}}{1+x} = 1.$$ By μ test $\int_0^1 f(x)dx$ is convergent iff $1-p<1$ i.e. $0<p$.

For the second case $\int_0^1 \frac{x^{-p}}{1+x}dx$ is a proper one if $p\leq 0$. So if $p>0$, 0 is the point of infinite discontinuity. Here $g(x)>0$ for all $x\in(0,1]$, where $g(x) = \frac{x^{-p}}{1+x}$. Now

$\lim_{x\to 0^+} \frac{x^p \cdot x^{-p}}{1+x} = 1$. By μ test $\int_0^1 g(x)\,dx$ is convergent iff $p<1$.

Thus combining the above two cases we say that $\int_0^1 \frac{x^{p-1}+x^{-p}}{1+x}\,dx$ is convergent if $0<p<1$.

iv) Let $f(x) = \log\sin x$, $x\in(0,\frac{\pi}{2}]$. 0 is the point of infinite discontinuity of f. $f(x)>0$ for all $x\in(0,\frac{\pi}{2}]$

we have $\lim_{x\to 0^+} \sqrt{x}(\log x) = 0$ and $\lim_{x\to 0^+} \sqrt{x}\log\frac{\sin x}{x} = 0$.

i.e. $\lim_{x\to 0^+} \sqrt{x}\left[\log x + \log\frac{\sin x}{x}\right] = 0$

or $\lim_{x\to 0^+} \sqrt{x}\log(\sin x) = 0$.

Let $g(x) = \frac{1}{\sqrt{x}}$, $x\in(0,\frac{\pi}{2}]$. Then $g(x)>0$ for all $x\in(0,\frac{\pi}{2}]$.

$\lim_{x\to 0^+} \frac{f(x)}{g(x)} = 0$ and $\int_0^{\pi/2} g(x)\,dx$ is convergent. By comparison test, $\int_0^{\pi/2} f(x)\,dx$ is convergent, i.e. $\int_0^{\pi/2} \log(\sin x)\,dx$ is convergent.

v) Let the given integral be $\int_0^\infty f(x)\,dx$, where $f(x) = \frac{1-\cos x}{x^2}$.

Then 0 is a point of infinite discontinuity of f. We are to examine the convergence at 0 as well as at ∞.

~~convergence at~~
convergence at 0

Let us consider the $\int_0^1 f(x)dx$. $f(x)>0$ for all $x\in(0,1]$.

Let $g(x)=\frac{1}{\sqrt{x}}$. Then $\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{1-\cos x}{x^{3/2}}=0$.

So by comparison test, $\int_0^1 f(x)\,dx$ is convergent as $\int_0^1 g(x)\,dx$ is convergent. —(a)

convergence at ∞.

Let us consider the integral $\int_1^\infty f(x)\,dx$. $f(x)>0$ for all $x\geq 1$. Now we can see that $f(x)<\frac{2}{x^2}$ and $\int_1^\infty \frac{2}{x^2}dx$ is convergent. So by comparison test, $\int_1^\infty f(x)\,dx$ is convergent. —(b).

Thus from (a) and (b) we have $\int_0^\infty \frac{1-\cos x}{x^2}dx$ is convergent.

vi) Let the given integral be $\int_0^\infty f(x)dx$. where $f(x)=\frac{\cos x}{\sqrt{x^3+x}}$.

Then $|f|(x)\leq\frac{1}{\sqrt{x^3+x}}$. Let $g(x)=\frac{1}{\sqrt{x^3+x}}$, $x>0$.

0 is a point of infinite discontinuity of g.

Convergence of $\int_0^1 g(x)dx$.

$g(x)>0$, for all $x\in(0,1]$. Let $u(x)=\frac{1}{\sqrt{x}}$, $x\in(0,1]$. Then $u(x)>0$ for all $x\in(0,1]$.

$\lim_{x\to 0}\frac{g(x)}{u(x)}=1$, a non-zero finite real number and $\int_0^1 u(x)dx$ is convergent. By comparison test, $\int_0^1 g(x)dx$

is convergent. — (a).

convergence of $\int_1^\infty g(x)\,dx$.

$g(x) > 0$ for all $x > 1$. Let $v(x) = \frac{1}{x^{3/2}}$, $x > 1$. Then $v(x) > 0$ for all $x > 1$.

$\lim_{x \to \infty} \frac{g(x)}{v(x)} = 1$, a non-zero finite real number and $\int_1^\infty v(x)\,dx$ is convergent. By comparison test $\int_1^\infty g(x)\,dx$ is convergent. — (b)

From (a) and (b) it follows $\int_0^\infty g(x)\,dx$ is convergent.

Since $|f|(x)$ and $g(x)$ both are positive for all $x > 0$ and $|f|(x) \le g(x)$ for all $x > 0$, $\int_0^\infty |f|(x)\,dx$ is convergent, by comparison test.

Therefore $\int_0^\infty f(x)\,dx$ is absolutely convergent and hence the given integral is convergent.

vii) Let the given integral be $\int_0^\infty f(x)\,dx$, where $f(x) = \left(\frac{1}{x^2} - \frac{1}{x \sinh x}\right)$

In the above problem 0 is not a point of infinite discontinuity. So we are to examine the convergence at ∞ only.

convergence at ∞.

Let us consider the integral $\int_1^\infty f(x)\,dx$, $f(x) > 0$ for all $x \geq 1$. Now we can see that $f(x) < \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent. So by comparison test, $\int_1^\infty f(x)\,dx$ is convergent.

So $\int_0^\infty f(x)\,dx$ is convergent.

viii) 0 is the only possible point of infinite discontinuity of the integrand. Let us examine the convergence of $\int_0^{1/2} x^{n-1}\log x\,dx$. The integrand is negetive in $(0,\frac{1}{2}]$.

Let $f(x) = -x^{n-1}\log x$, $x\in(0,\frac{1}{2}]$. Then $f(x)>0$ for all $x\in(0,\frac{1}{2}]$.

If $n-1>0$, the integral $\int_0^{\frac{1}{2}} f(x)\,dx$ is a proper one, since $\lim\limits_{x\to 0^+} x^r\log x = 0$, for all $r>0$.

If $n-1\leq 0$, 0 is the only point of infinite discontinuity of f. Let m be a positive number such that $m+n-1>0$. Then $\lim\limits_{x\to 0^+} x^{m+n-1}\log x = 0$. Therefore $\lim\limits_{x\to 0^+} x^m f(x) = 0$.

Let $g(x) = \frac{1}{x^m}$, $x\in(0,\frac{1}{2}]$. Then $g(x)>0$ for all $x\in(0,\frac{1}{2}]$.

Since $\lim\limits_{x\to 0^+}\frac{f(x)}{g(x)} = 0$ and $\int_0^{\frac{1}{2}} g(x)\,dx$ is convergent if $m<1$, it follows that $\int_0^{\frac{1}{2}} f(x)\,dx$ is convergent if $m<1$.

Therefore $\int_0^1 f(x)\,dx$ is convergent if $m<1$ and $m+n-1>0$, i.e. if ~~xxxxxxxxxx~~ $1-n<m<1$ i.e. if $n>0$.

If $n=0$, the integral ~~xx~~ reduces to $\int_0^1 \frac{\log x}{x}\,dx$.

$\int_\varepsilon^1 \frac{\log x}{x}\,dx = -\frac{1}{2}(\log\varepsilon)^2 \to -\infty$ as $\varepsilon\to 0^+$ and therefore $\int_0^1 f(x)\,dx$ is divergent if $n=0$.

If $n<0$, then $x^{n-1}\geq x^{-1}$ for all $x\in(0,1]$. Since the integral $\int_0^1 \frac{\log x}{x}\,dx$ is divergent, it follows that $\int_0^1 f(x)\,dx$ is divergent. Hence the given integral is convergent if and only if $n>0$.

ix) we simply apply the comparison test. Since $e^x \geq 1$ and $\sqrt{x^2-\frac{1}{2}} \leq \sqrt{x^2} = x$, it follows that

$$\frac{e^x}{\sqrt{x^2-\frac{1}{2}}} \geq \frac{1}{\sqrt{x^2-\frac{1}{2}}} \geq \frac{1}{\sqrt{x^2}} = \frac{1}{x} > 0.$$

Since $\int_1^\infty \frac{1}{x}\,dx$ diverges, the comparison test implies $\int_1^\infty \frac{e^x}{\sqrt{x^2-\frac{1}{2}}}\,dx$ diverges as well.

x) $\int_0^1 \frac{1}{1-x^4}\,dx = \lim_{v\to 1^-} \int_0^v \frac{1}{1-x^4}\,dx$.

For $0 \leq x \leq 1$, $1-x^4 = (1-x)(1+x)(1+x^2) < 4(1-x)$. Hence $\frac{1}{(1-x^4)} > \frac{1}{4(1-x)}$.

But, $\int_0^1 \frac{1}{4(1-x)}\,dx = \lim_{v\to 1^-} -\frac{1}{4}\log|1-x|\Big|_0^v = \lim_{v\to 1^-} -\frac{1}{4}(\log|1-v|) = \infty$.

Thus $\int_0^1 \frac{1}{(1-x^4)}\,dx$ diverges.

3. Let $f(x) = \left|\frac{\sin x}{x}\right|, \quad x > 0$

$= 1, \quad x = 0.$

Then f is continuous and hence integrable on $[0, X]$, for all $X > 0$. Let us consider the integral $\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx$, where n is a positive integer.

$$\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx = \int_0^{n\pi} \frac{|\sin x|}{x} dx = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx.$$

Now $\displaystyle\int_{(r-1)\pi}^{r\pi} \frac{|\sin x|}{x} dx = \int_0^{\pi} \frac{|\sin u|}{(r-1)\pi + u} du$, $[x = (r-1)\pi + u]$

$$= \int_0^{\pi} \frac{\sin u}{(r-1)\pi + u} du$$

For all $u \in [0, \pi]$, $(r-1)\pi + u \leq r\pi$.

Therefore $\displaystyle\int_0^{\pi} \frac{\sin u}{(r-1)\pi + u} du \geq \frac{1}{r\pi} \int_0^{\pi} \sin u \, du = \frac{2}{r\pi}$.

Hence $\displaystyle\int_0^{n\pi} \left|\frac{\sin x}{x}\right| dx \geq \frac{2}{\pi}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) \quad -(a)$

As $n \to \infty$, the R.H.S of (a) gives the series $\displaystyle\sum_{n=1}^{\infty} \frac{2}{\pi n}$ which is divergent series. Hence $\displaystyle\lim_{n \to \infty} \int_0^{n\pi} f(x) dx = \infty$. This implies that the improper integral $\displaystyle\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ is divergent.

4. Let $f(x) = \frac{1}{e^x - x}, \quad g(x) = \frac{1}{e^x}$.

Then $\displaystyle\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{1/(e^x - x)}{1/e^x} = \lim_{x \to \infty} \frac{e^x}{e^x - x}$

$\displaystyle= \lim_{x \to \infty} \frac{e^x}{e^x - 1} = \lim_{x \to \infty} \frac{1}{1 - e^{-x}} = 1$

Since the limit is a non-zero finite real number then by comperison test $f(x)$ and $g(x)$ converges and ~~diverg~~ diverge together. Since $\int_0^{\infty} g$ converges, $\int_0^{\infty} f$ also.

5. Let the given integral be $\int_0^\infty f(x)dx$, where $f(x)=\frac{\sin x(1-\cos x)}{x^n}$

Then 0 is a point of infinite discontinuity of f. We are to examine convergence at 0 as well as at ∞.

Convergence at 0

Let us consider the integral $\int_0^1 f(x)dx$. $f(x)>0$ for all $x\in(0,1]$. Let $g(x)=\frac{1}{x^{n-3}}$, Then

$$\lim_{x\to 0}\frac{f(x)}{g(x)}=\lim_{x\to 0}\frac{\sin x(1-\cos x)}{x^3}=\frac{1}{2},$$ a non-zero finite real number. So $\int_0^1 f(x)\,dx$ and $\int_0^1 g(x)\,dx$ converges or diverges together. Since $\int_0^1 g(x)dx$ converges when $n-3<1$ i.e. if $n<4$.

i.e. $\int_0^1 f(x)\,dx$ converges when $n<4$. — (a).

Now let us consider the integral $\int_1^\infty f(x)dx$. $f(x)>0$ for all $x>1$. Let $h(x)=\frac{1}{x^{n+1}}$, then $h(x)>0$ for all $x>1$.

$$\text{Now } \lim_{x\to\infty}\frac{f(x)}{h(x)}=\lim_{x\to\infty}\frac{\sin x(1-\cos x)}{x}=0.$$

So $\int_1^\infty f(x)\,dx$ and $\int_1^\infty h(x)dx$ converges together. Since $\int_1^\infty h(x)dx$ converges when $n+1>1$ i.e. when $n>0$. i.e. $\int_1^\infty f(x)\,dx$ converges when $n>0$ — (b).

Thus combining (a) and (b) we get $\int_0^\infty f(x)\,dx$ converges when $0<n<4$.

6. $$\int_0^\infty \frac{5\sin(4x)-4\sin(5x)}{x^2}dx=20\log\left(\frac{5}{4}\right).$$

[NOTE: $\boxed{\text{Use the theorem}}$

Let a function ϕ be continuous on $(0,\infty)$ and $\lim_{x\to 0^+}\phi(x)=\phi_0$ (finite), $\lim_{x\to\infty}\phi(x)=\phi_1$ (finite). Then

$\int_0^\infty \frac{\phi(ax)-\phi(bx)}{x}dx = (\phi_0-\phi_1)\log\left(\frac{b}{a}\right)$, where $a>0, b>0$ and $b>a$.]

In the above theorem take $\phi(x)=\frac{\sin x}{x}$. Then $\lim_{x\to 0^+}\phi(x)=1$ (finite); $\lim_{x\to\infty}\phi(x)=0$, (finite).

7. Let the given integral be $\int_0^1 f(x)\,dx$ where $f(x)=x^{m-1}(1-x)^{n-1}$. It is a proper integral if $m\geq 1$ and $n\geq 1$. 0 is the only point of infinite discontinuity of f if $m<1$ and 1 is the only point of infinite discontinuity of f if $n<1$.

Let us examine the convergence of $\int_0^{\frac{1}{2}} f(x)\,dx$ when $m<1$ and the convergence of $\int_{1/2}^1 f(x)\,dx$ when $n<1$.

convergence of $\int_0^{\frac{1}{2}} f(x)\,dx$ at 0 when $m<1$. $f(x)>0$ ~~when~~ for all $x\in(0,\frac{1}{2}]$. $\lim_{x\to 0^+} x^{1-m}\cdot f(x)=\lim_{x\to 0^+}(1-x)^{n-1}=1$ (a non-zero finite real number). By μ test, $\int_0^{\frac{1}{2}} f(x)\,dx$ is convergent iff $1-m<1$ i.e. iff $m>0$. $-(a)$

convergence of $\int_{1/2}^1 f(x)\,dx$ at 1 when $n<1$. $f(x)>0$ for all $x\in[\frac{1}{2},1)$

$\lim_{x\to 1^-}(1-x)^{1-n}f(x)=\lim_{x\to 1^-}x^{m-1}=1$, (a non zero finite real number)

By μ test, $\int_{1/2}^1 f(x)\,dx$ is convergent iff $1-n<1$ i.e. iff $n>0$. $-(b)$

Thus combining (a) and (b) we can say that $\int_0^1 f(x)\,dx$ is convergent iff $m>0, n>0$.

Note.

The above integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$, $m>0, n>0$ is called the Beta function and it is denoted by $B(m,n)$.

Thus we have to find $B(\frac{5}{2},\frac{7}{2})$.

Now $B(\frac{5}{2},\frac{7}{2}) = \frac{\Gamma(\frac{5}{2}).\Gamma(\frac{7}{2})}{\Gamma(6)} = \frac{3\pi}{2^8}$ [using $\Gamma(n+1)=n\Gamma(n)$ for $n>0$.]

[using $B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m>0, n>0$.]

8. Let the given integral be $\int_0^\infty f(x)\,dx$, where $f(x) = e^{-x}.x^{m-1}$.

If $m \geq 1$, 0 is not a point of infinite discontinuity of f. f has an infinite discontinuity at 0 if $m<1$.

convergence at 0. $(m<1)$

$f(x)>0$ for all $x \in (0,1]$. Let $g(x) = x^{m-1}$, $x\in(0,1]$. Then $g(x)>0$ for all $x\in(0,1]$ and $\lim_{x\to 0} \frac{f(x)}{g(x)} = 1$, a non-zero finite number.

$\int_0^1 g(x)\,dx$ is convergent iff $1-m<1$ i.e. iff $m>0$.

convergence at ∞.

$f(x)>0$ for all $x\geq 1$. Let $g(x) = \frac{1}{x^2}$, $x\geq 1$. Then $g(x)>0$ for all $x\geq 1$ and $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim \frac{x^{m+1}}{e^x} = 0$ for all m.

As the integral $\int_1^\infty g(x)\,dx$ is convergent, i.e. the integral $\int_1^\infty f(x)\,dx$ is convergent for all m.

Hence the given ~~is~~ integral is convergent iff $m>0$.

<u>Note</u>

The integral $\int_0^\infty x^{m-1} e^{-x}\,dx$, $m>0$ is called the Gamma function and it is denoted by $\Gamma(m)$.

i.e. we have to find $\Gamma(2019)$. Using $\Gamma(n+1) = n\Gamma(n)$, where $n\in\mathbb{N}$.

we can say that $\Gamma(2019) = 2018!$.

9. Let $\sqrt{x} = t$ or $x = t^2$ or $dx = 2t\,dt$

$$\int_0^1 x^4 (1-\sqrt{x})^5 dx = \int_0^1 (t^2)^4 (1-t)^5 (2t\,dt)$$

$$= 2\int_0^1 t^9 (1-t)^5 dt$$

$$= 2\beta(10,6)$$

$$= 2 . \frac{\Gamma(10)\Gamma(6)}{\Gamma(16)}$$

$$= \cancel{2 . \frac{\Gamma(9) . \Gamma(5)}{\Gamma(15)}}$$

$$= 2 . \frac{9!\,5!}{15!}$$

$$= 2 . \frac{5!}{10\times 11\times 12\times 13\times 14\times 15}$$

$$= \frac{1}{11\times 13\times 7\times 15} = \frac{1}{15015} .$$

10.
$$I = \int_0^\infty \frac{x^a}{a^x}\,dx \quad , \quad a>1.$$

putting $a^x = e^t \Rightarrow x\log a = t \Rightarrow x = \frac{t}{\log a}$.

$\Rightarrow dx = \frac{dt}{\log a}$.

i.e.
$$I = \int_0^\infty \frac{t^a}{(\log a)^a} \cdot e^{-t} \cdot \frac{dt}{\log a}$$
$$= \frac{1}{(\log a)^{a+1}} \int_0^\infty e^{-t} \cdot t^{(a+1)-1}\,dt$$
$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

11. See problem no. 11's solution after problem no. 12's solution.

12) i) Substitute, $y = e^{-x} \Rightarrow dy = -e^{-x}\,dx$

$$\int_0^1 \left(\log \frac{1}{y}\right)^{n-1} dy = \int_0^1 \left(\log \frac{1}{y}\right)^{n-1} y \cdot \frac{dy}{y}$$

$$= \int_\infty^0 \left(\log e^{x}\right)^{n-1} e^{-x} \; \frac{-e^{-x}\,dx}{e^{-x}}$$

$$= \int_0^\infty e^{-x}\, x^{n-1}\, dx = \Gamma(n) \quad \text{(by definition)}$$

ii) $$\int_0^{\pi/2} \tan^p \theta \; d\theta$$

$$= \int_0^{\pi/2} \sin^p \theta \, \cos^{-p} \theta \; d\theta$$

$$= \frac{1}{2} \, 2 \int_0^{\pi/2} \sin^p \theta \, \cos^{-p} \theta \; d\theta$$

$$= \frac{1}{2} \, B\left(\frac{p+1}{2}, \frac{-p+1}{2}\right)$$ [it exists if $p > -1$ & $-p > -1$ $\Rightarrow p > -1$ & $p < 1$ $\Rightarrow -1 < p < 1$]

$$= \frac{1}{2} \, \frac{\Gamma\left(\frac{p+1}{2}\right) \cdot \Gamma\left(\frac{-p+1}{2}\right)}{\Gamma(1)}$$

$$= \frac{1}{2} \, \Gamma\left(\frac{-p+1}{2}\right) \Gamma\left(1 - \frac{-p+1}{2}\right)$$

$$= \frac{1}{2} \, \frac{\pi}{\sin\left(\frac{-p+1}{2} \cdot \pi\right)} = \frac{1}{2} \, \frac{\pi}{\sin\left(\frac{\pi}{2} - \frac{p\pi}{2}\right)}$$ [By Problem 11]

$$= \frac{\pi}{2} \sec \frac{p\pi}{2}$$

The values of p for which the integral exists are $\{p : -1 < p < 1\}$.

iii) We have $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m>0$, $n>0$

$(x-a)+(b-x) = b-a \Rightarrow \frac{x-a}{b-a} + \frac{b-x}{b-a} = 1$

Let $\frac{x-a}{b-a} = y$. Then $\frac{b-x}{b-a} = 1-y$,

$dx = (b-a)\,dy$

As $x \to a$, $y \to 0$; as $x \to b$, $y \to 1$.

$$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx = \int_0^1 (b-a)^{m+n-1} y^{m-1}(1-y)^{n-1}\,dy$$

$$= (b-a)^{m+n-1}\int_0^1 y^{m-1}(1-y)^{n-1}\,dy$$

$$= (b-a)^{m+n-1} B(m,n), \quad m>0,\ n>0$$

iv) Let $x^6 = t$. Then $dx = \frac{dt}{6\,t^{5/6}}$

$$\int_0^1 \frac{1}{(1-x^6)^{1/6}}\,dx = \int_0^1 (1-t)^{-\frac{1}{6}} \cdot \frac{1}{6t^{5/6}}\,dt$$

$$= \frac{1}{6}\int_0^1 t^{\frac{1}{6}-1}(1-t)^{(1-\frac{1}{6})-1}\,dt$$

$$= \frac{1}{6} B\left(\frac{1}{6}, 1-\frac{1}{6}\right), \text{ since } 0 < \frac{1}{n} < 1$$

$$= \frac{1}{6}\,\frac{\Gamma(\frac{1}{6})\,\Gamma(1-\frac{1}{6})}{\Gamma(1)}$$

$$= \frac{1}{6}\,\frac{\pi}{\sin\frac{\pi}{6}} \quad \text{[By Problem II.]}$$

$$= \frac{1}{6} \cdot \frac{\pi}{\frac{1}{2}} = \frac{\pi}{3} \quad \text{[Proved]}$$

V. $$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} = \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta} \cdot \frac{\cos\theta}{2\sqrt{\sin\theta}}\,d\theta \quad \left[\begin{array}{l}\text{Putting} \\ x^2 = \sin\theta \\ dx = \frac{\cos\theta\,d\theta}{2\sqrt{\sin\theta}}\end{array}\right]$$

$$= \frac{1}{2}\int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta$$

$$= \frac{1}{4}\, 2\int_0^{\pi/2} \cancel{\sin}\ \sin^{1/2}\theta\,d\theta$$

$$= \frac{1}{4} B\left(\frac{3}{4}, \frac{1}{2}\right)$$

$$\left[\begin{array}{l}\text{Using} \\ \int_0^{\pi/2} \sin^m\theta \cos^n\theta\,d\theta \\ = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right), \\ m, n > -1\end{array}\right.$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{4}\right)}$$

$$= \frac{1}{4} \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\frac{1}{4}\Gamma\left(\frac{1}{4}\right)} = \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{4}\right)} \quad \text{--- (1)}$$

$$\int_0^1 \frac{dx}{\sqrt{1+x^4}} \cancel{=} \cancel{\int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta\ \sec\theta}}} \quad \left[\begin{array}{l}\text{Putting} \\ x^2 = \tan\theta \\ 2x\,dx = \sec^2\theta\,d\theta \\ dx = \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}}\end{array}\right.$$

$$= \int_0^{\pi/4} \frac{\sec^2\theta\,d\theta}{2\sqrt{\tan\theta}\ \sec\theta}$$

$$= \int_0^{\pi/4} \frac{d\theta}{2\cos\theta\sqrt{\frac{\sin\theta}{\cos\theta}}} = \int_0^{\pi/4} \frac{2\,d\theta}{2\sqrt{2}\sqrt{\sin 2\theta}}$$

$$-\frac{1}{\sqrt{2}}\int_0 \frac{d\theta}{\sqrt{\sin 2\theta}} = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-\frac{1}{2}}\phi\, d\phi$$

Putting $2\theta = \phi$, $d\theta = \frac{d\phi}{2}$

$$= \frac{1}{2\sqrt{2}} \cdot \frac{1}{2} B\left(\frac{1}{4}, \frac{1}{2}\right)$$

$$= \frac{1}{4\sqrt{2}} \frac{\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3}{4}\right)} \quad \text{—— (2)}$$

Using $\int_0^{\pi/2} \sin^m\theta \cos^n\theta\, d\theta = \frac{1}{2} B\left(\frac{m+1}{2}, \frac{n+1}{2}\right)$, $m, n > -1$

Multiplying (1) & (2) we have

$$\int_0^1 \frac{x^2\,dx}{\sqrt{1-x^4}} \times \int_0^1 \frac{dx}{\sqrt{1+x^4}}$$

$$= \frac{1}{4\sqrt{2}} \cdot \left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \frac{\pi}{4\sqrt{2}}$$

b) We have $\frac{\Gamma(n)\,\Gamma(1-n)}{\Gamma(1)} = B(n, 1-n)$

Now $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$, $m>0, n>0$

let $x = \frac{t}{1+t}$. Then $dx = \frac{dt}{(1+t)^2}$

As $x \to 0+$, $t \to 0+$; as $x \to 1-$, $t \to \infty$

Therefore $B(m,n) = \int_0^\infty \left(\frac{t}{1+t}\right)^{m-1} \left(\frac{1}{1+t}\right)^{n-1} \frac{1}{(1+t)^2}\,dt$

$$= \int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

So, $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \;;\; m>0, n>0$

$$B(n, 1-n) = \int_0^\infty \frac{x^{n-1}}{(1+x)}\,dx, \quad \cancel{0<m<1} \; 0<n<1$$

$$= \frac{\pi}{\sin n\pi}$$

Hence $\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$ [proved].

13. We have $I = \int_{-1}^{1} (1+x)^n (1-x)^n \, dx$.

Let $1+x = 2t$. Then, $dx = 2dt$ and $1-x = 2(1-t)$. We obtain

$$I = 2^{2n+1} \int_0^1 t^n (1-t)^n dt = 2^{2n+1} \beta(n+1, n+1)$$

i.e. $$I = 2^{2n+1} \frac{\Gamma(n+1) . \Gamma(n+1)}{\Gamma(2n+2)} = \frac{2^{2n+1} . (n!)^2}{(2n+1)!}$$

14. We know that

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = \beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta \quad -(a)$$

Setting $m = n$, we get

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \beta(n,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1}(2\theta) \, d\theta .$$

Substituting, $2\theta = \frac{\pi}{2} - \phi$, we get $d\theta = -\frac{1}{2} d\phi$. Hence we get

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{-1}{2^{2n-1}} \int_{+\pi/2}^{-\pi/2} \cos^{2n-1}(\phi) \, d\phi$$

$$= \frac{1}{2^{2n-1}} \int_{-\pi/2}^{\pi/2} \cos^{2n-1}\phi \, d\phi = \frac{2}{2^{2n-1}} \int_0^{\pi/2} \cos^{2n-1}(\theta) \, d\theta \quad -(b)$$

Since $\cos\theta$ is even function.

Setting $m = \frac{1}{2}$ in (a), we obtain

$$\frac{\Gamma(n)\Gamma(1/2)}{\Gamma(n+\frac{1}{2})} = 2\int_0^{\pi/2} \cos^{2n-1}(\theta) \, d\theta . \quad -(c)$$

Comparing Eqs (b) and (c), we have

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)} = \frac{1}{2^{2n-1}}\left[\frac{\Gamma(n)\cdot\Gamma(\frac{1}{2})}{\Gamma(n+\frac{1}{2})}\right]$$

or $\Gamma(2n) = \frac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\cdot\Gamma(n+\frac{1}{2})$, (Since $\Gamma(\frac{1}{2}) = \sqrt{\pi}$). — (d)

which is the required result.

Setting $n = \frac{1}{4}$ in eq (d) we obtain

$$\Gamma\left(\frac{1}{2}\right) = \frac{2^{-1/2}}{\sqrt{\pi}}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right)$$

or $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \pi\sqrt{2}$.