Elementary Mathematics for Physics

Differential Calculus :

$$g(x) \; : \; dg = \frac{dg}{dx} dx$$

$$f(x, y, z)$$

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz$$

$\frac{\partial f}{\partial x}$: Partial derivative of f wrt x by keeping y and z fixed.

Example : $u(x, y) = x^2 + 2\text{xy}$

$$\frac{\partial u}{\partial x} = 2\text{x} + 2\text{y} \qquad \frac{\partial u}{\partial y} = 2\text{x}$$

Vector Calculus

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \qquad d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

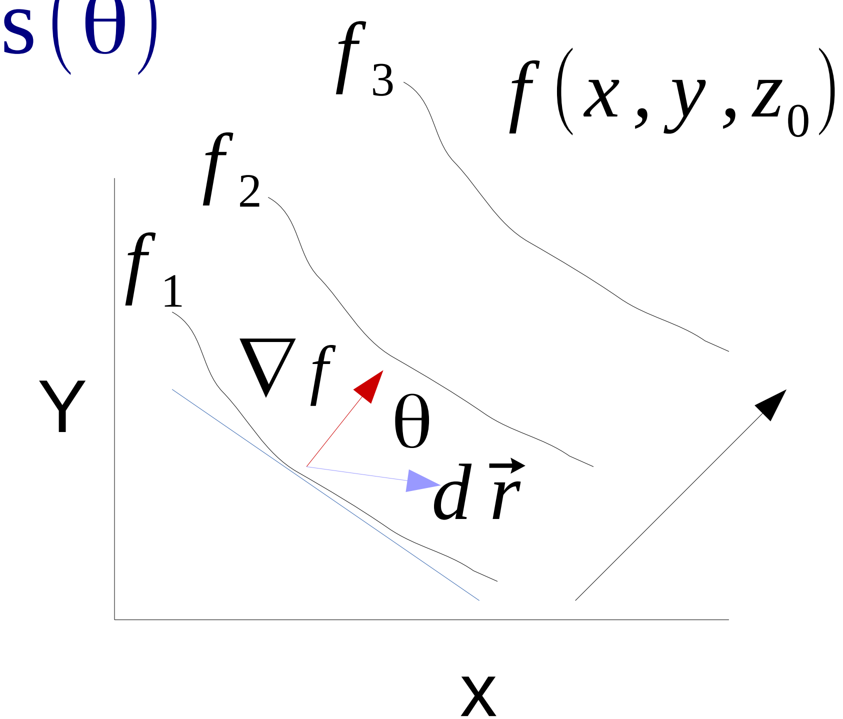
$$\text{Vector operator : } \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{Gradient : } \vec{\nabla} f(x, y, z) = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$df = \nabla f \cdot d\vec{r} = |\nabla f|\ |d\vec{r}|\ \cos(\theta)$$

df is maximum when $\theta = 0$.

It is directed towards normal at a point on a curve with constant f.



Vector Calculus

Example: $$r(x,y,z)=\sqrt{x^2+y^2+z^2}$$

what is the direction of ∇r?

Clearly, the fastest increase of r will be radially outward.

So the expectation is $\dfrac{\nabla r}{|\nabla r|}=\hat{r}$

Check:

$$\nabla r=\hat{i}\frac{\partial r}{\partial x}+\hat{j}\frac{\partial r}{\partial y}+\hat{k}\frac{\partial r}{\partial z}$$

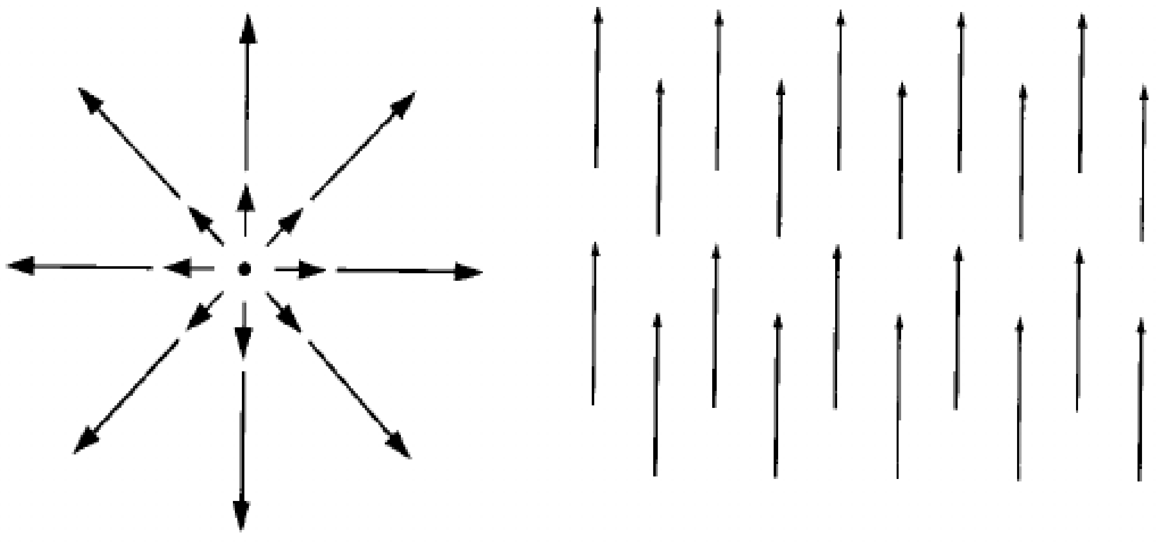
$$=\frac{1}{\sqrt{x^2+y^2+z^2}}\left[\hat{i}x+\hat{j}y+\hat{k}z\right]$$

$$=\frac{\vec{r}}{r}=\hat{r}$$

Vector Calculus

$$\text{Divergence}: \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This scalar is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in consideration.



A *source will have positive divergence*.

A *sink will have negative divergence*.

Vector Calculus

Example:

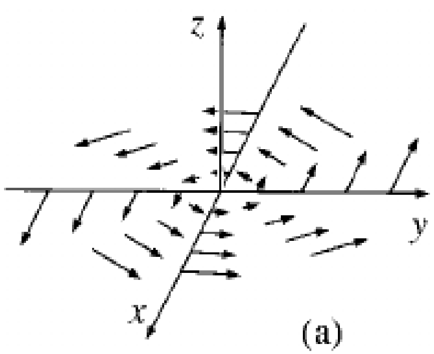
Divergence of the vector : $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$

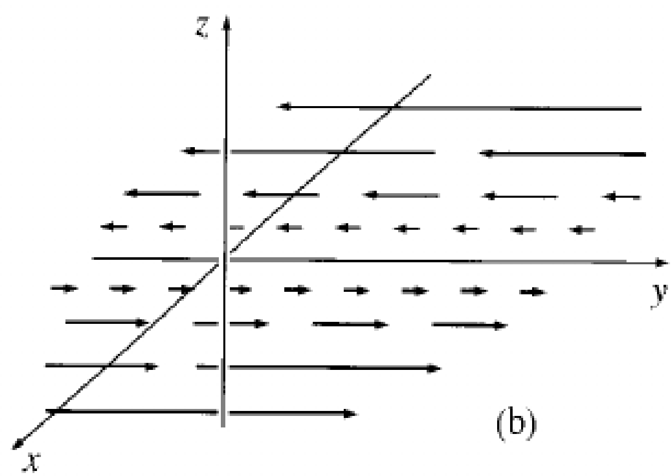
$$\vec{\nabla} \cdot \vec{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = 3$$

Vector Calculus

$$\text{Curl}: \vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$





A point with large curl is a whirlpool.

Vector Calculus

Example: Curl of the vector : $\vec{A} = \hat{i}\,y^2 - \hat{j}\,x^2 + \hat{k}\,2\mathrm{xy}$

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + \hat{j}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + \hat{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

$$= \hat{i}\left(2\mathrm{x}\right) + \hat{j}\left(-2\mathrm{y}\right) + \hat{k}\left(-2\mathrm{x} - 2\mathrm{y}\right)$$

$$\nabla\cdot\left(\nabla\times\vec{A}\right) = 2 - 2 + 0 = 0$$

Divergence of a curl is **always** zero.

$$\nabla\times\hat{r} = \nabla\times\left(\nabla r\right) = 0$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} = \frac{1}{\sqrt{3}}\left(\hat{i} + \hat{j} + \hat{k}\right)$$

Curl of a gradient is **always** zero.

Vector Calculus

(i) $\nabla \times \nabla f = 0$

(ii) $\nabla f \times \nabla g \neq 0$ (in general)

(iii) $\nabla \cdot (\nabla \times \vec{A}) = 0$

(iv) $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(Laplacian Operator)

Vector Calculus

Identities (Product Rules) :

(i) $\nabla\left(fg\right) = f\,\nabla g + g\nabla f$

(ii) $\nabla\cdot\left(f\vec{A}\right) = f\left(\nabla\cdot\vec{A}\right) + \vec{A}\cdot\left(\nabla f\right)$

(iii) $\nabla\times\left(f\vec{A}\right) = f\left(\nabla\times\vec{A}\right) - \vec{A}\times\left(\nabla f\right)$

(iv) $\nabla\cdot\left(\vec{A}\times\vec{B}\right) = \vec{B}\cdot\left(\nabla\times\vec{A}\right) - \vec{A}\cdot\left(\nabla\times\vec{B}\right)$

(iv) $\nabla\times\left(\vec{A}\times\vec{B}\right) = \vec{A}\left(\nabla\cdot\vec{B}\right) - \vec{B}\left(\nabla\cdot\vec{A}\right) + \left(\vec{B}\cdot\nabla\right)\vec{A} - \left(\vec{A}\cdot\nabla\right)\vec{B}$

Vector Calculus

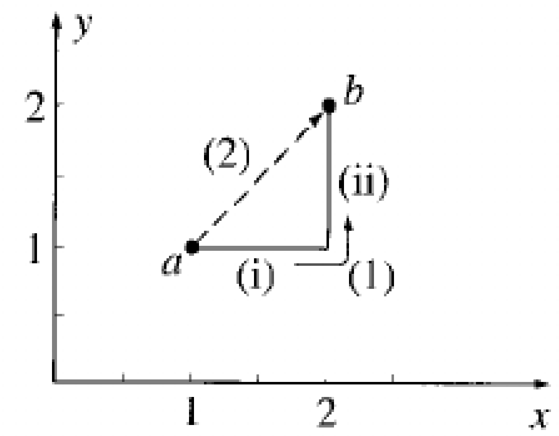
Line element : $d\vec{l} \equiv d\vec{r} \;=\; \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

Surface element : $d\vec{S} \equiv d^2\vec{r} \;=\; dS\,\hat{n} \;\;;\;\; dx\;dy\,\hat{z}$

Volume element : $d\,V \equiv d^3\vec{r} \;=\; dx\;dy\;dz$

Integrals :

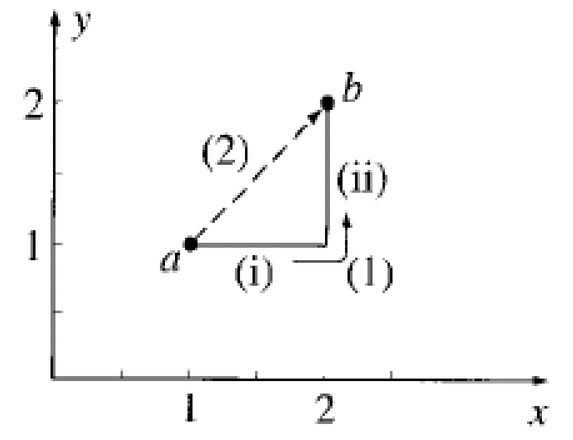
(i) Line Integral : $\int_{\vec{r},l}^{\vec{r}\,'} \vec{A}\cdot d\,\vec{l}$



Closed line integral : $\oint_l \vec{A}\cdot d\,\vec{l}$ when $\vec{r} = \vec{r}\,'$

Vector Calculus

Example: $a(1,1,0), \quad b(2,2,0)$



$$\vec{A} = y^2\hat{i} + 2x(y+1)\hat{j}$$

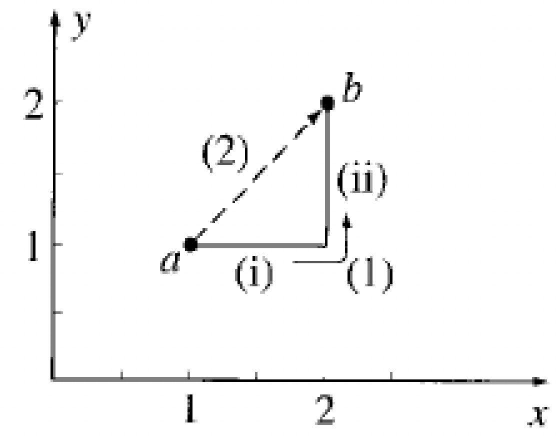
Line integral of $\vec{A}$ along paths (i) and (ii):

$$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{A}\cdot d\vec{l} = y^2 dx + 2x(y+1)dy$$

Vector Calculus

$$a(1,1,0), \quad b(2,2,0)$$



Path-1:

$$\int_{l,a}^{b} \vec{A}\cdot d\,\vec{l} = \int_{1}^{2} 1^2 dx + \int_{1}^{2} [2\cdot 2(y+1)]dy$$

Path-2:

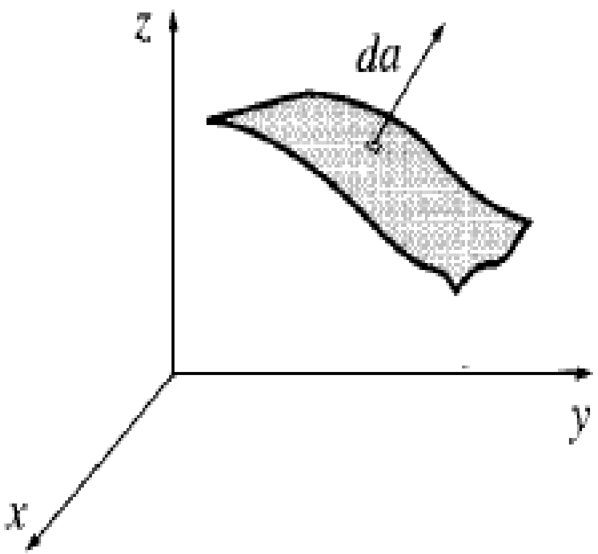
$$y = x \Rightarrow dy = dx$$

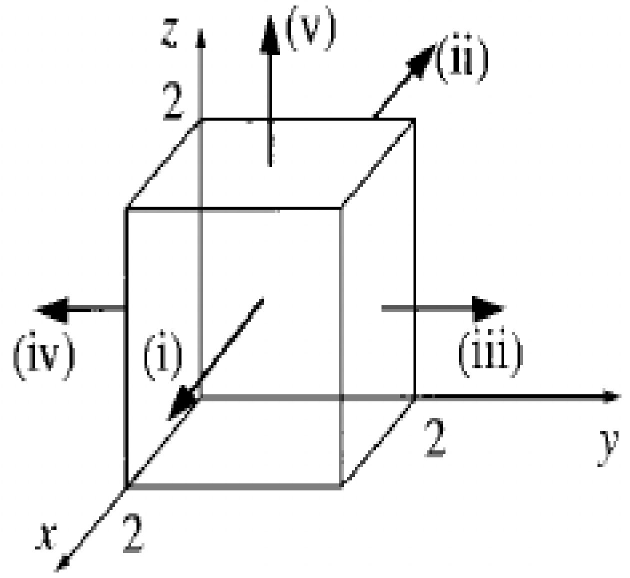
$$\int_{l,a}^{b} \vec{A}\cdot d\,\vec{l} = \int_{1}^{2} [x^2 + 2\mathrm{x}(x+1)]dx = 10$$

$$\oint_{l} \vec{A}\cdot d\,\vec{l} = 11 - 10 = 1$$

Vector Calculus

(ii) Surface Integral : $\int_S \vec{A}\cdot d\vec{S}$





$$d\vec{S}_{(i)} = dy\,dz\,\hat{i}$$
$$d\vec{S}_{(iii)} = dx\,dz\,\hat{j}$$
$$d\vec{S}_{(v)} = dx\,dy\,\hat{k}$$

Closed Surface Integral : $\oint_S \vec{A}\cdot d\vec{S}$

It is called flux of $\vec{A}$ through the surface.

If $\vec{A}$ represents velocity of a fluid, flux of $\vec{A}$ will flow out of surface per unit time.

Vector Calculus

(iii) Volume Integral : $\int_V T(x,y,z)\,dV$

$$\int_V \vec{A}\,dV = \hat{i}\int_V A_x\,dV + \hat{j}\int_V A_y\,dV + \hat{k}\int_V A_z\,dV$$

Fundamental Theorem of Calculus:

$$\int_a^b \frac{df(x)}{dx}\,dx = f(b) - f(a)$$

The integral of a derivative of a function over some interval is the value of the function at the end points (or boundaries).

In vector calculus, there are three types of derivatives.

Vector Calculus

(i) Fundamental Theorem of Gradients :

$$\int_{a,l}^{b} \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

$$\text{as } dT = \nabla T \cdot d\vec{l}$$

The integral (line) of a derivative (gradient) of a function is its value at the boundary points.

The integral is independent of path.

$$\oint_{l} \vec{\nabla} T \cdot d\vec{l} = 0$$

Vector Calculus

Divergence Theorem (Gauss or Greens Theorem):

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right) dV = \oint_S \vec{A}\cdot d\vec{S}$$

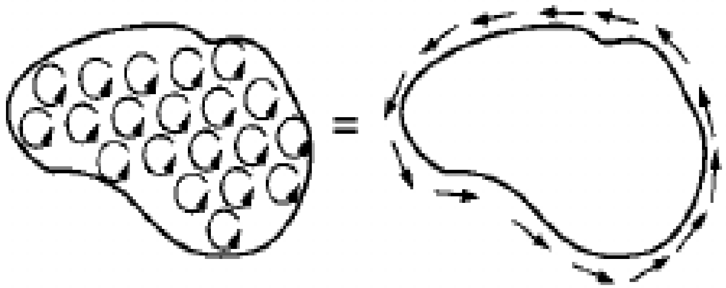
The integral (volume) of a derivative (divergence) of a vector is its value at the boundary surface enclosing the volume.

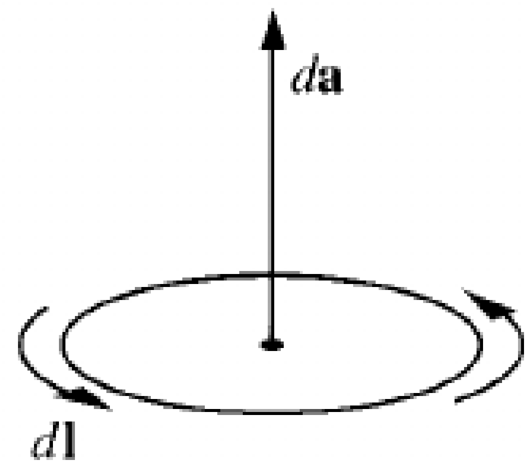
Fundamental Theorem of Curl (Stoke'sTheorem).:

$$\int_s \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_l \vec{A}\cdot d\vec{l}$$

The integral (surface) of a derivative (Curl) of a vector is its value at the boundary line enclosing the surface.

Vector Calculus





$$\int_s \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_l \vec{A}\cdot d\vec{l}$$

$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}$ depends only on the boundarly line.

$$\oint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = 0$$

Vector Calculus

Integration by Parts:

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right)$$

$$\int_a^b \frac{d}{dx}(fg)\,dx = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx$$

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b.$$

$$\vec{\nabla}\cdot\left(f\vec{A}\right) = f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f$$

$$\int_V \vec{\nabla}\cdot\left(f\vec{A}\right)dV \equiv \oint_S \left(f\vec{A}\right)\cdot d\vec{S} = \int_V \left[f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla}f\right]dV$$

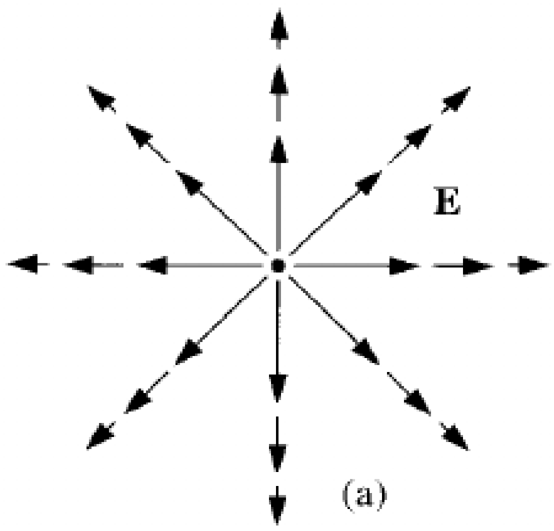
$$\int_V f\left(\vec{\nabla}\cdot\vec{A}\right)dV = -\int_V \left[\vec{A}\cdot\vec{\nabla}f\right]dV + \oint_S \left(f\vec{A}\right)\cdot d\vec{S}$$

Maxwell's Equations for Electromagnetism

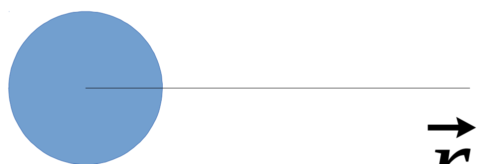
- A set of four (two scalar and two vector) first order partial differential equations involving electric and magnetic fields, and their sources.
- These are outcome of different laws of electromagnetism, namely, (i) Gauss's Law, (ii) Ampere's Law and (iii) Faraday's Law, and the equation of continuity for electric charge and current densities.
- They lead to second order partial differential equations for electric and magnetic fields, describing electromagnetic waves.

Electrostatics

Gauss's Law:



$$\text{Electric Flux}: \oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\text{encl}}}{\epsilon_0}$$



$\vec{r}$

$$E(r)\times(4\pi r^2)=\frac{Q}{\epsilon_0}$$

$$\vec{E}(\vec{r})=\frac{Q}{4\pi\epsilon_0 r^2}\hat{r}$$

$$\oint_S \vec{E}\cdot d\vec{S} = \frac{Q_{\mathrm{encl}}}{\epsilon_0}$$

Divergence Theorem

$$\int \left[\vec{\nabla}\cdot\vec{E} - \frac{1}{\epsilon_0}\int \rho(\vec{r})\right] dV = 0$$

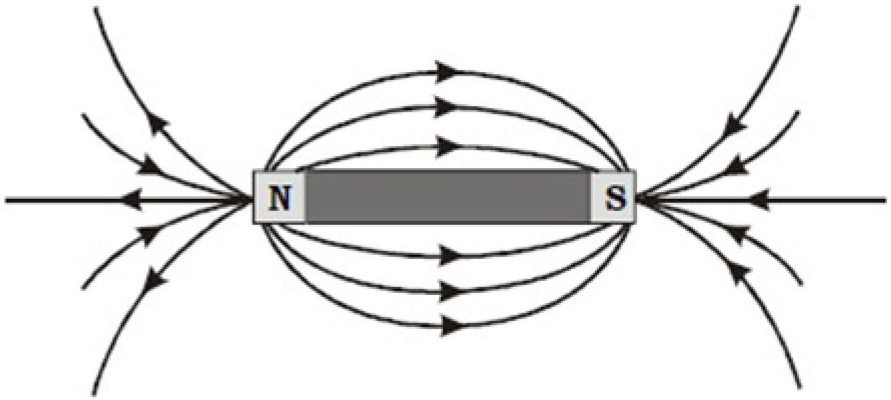
$$\vec{\nabla}\cdot\vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$$ (Differential form of Gauss's Law)

Electric Field diverges/converges from/towards positive/negative charge.

The divergence is nonzero due to the possibility of the presence/dominance of one kind of charge.

Magnetostatics

Only one kind of magnetic
pole(charge) is not possible.



Magnetic lines of force bends (curls);
but unlike electric lines of force, they
do not diverge/converge from/to a point.

$$\oint_S \vec{B} \cdot \overrightarrow{dS} = 0$$

$$\nabla \cdot \vec{B} = 0$$ (Differential form of no one's law)

Faraday's Law

Induced emf: $$\varepsilon = -\frac{d\Phi}{dt}$$

$$\oint_l \vec{E} \cdot \vec{dl} = -\frac{d\Phi}{dt}$$

$$\int_S \left(\vec{\nabla} \times \vec{E} \right) \cdot \vec{dS} = -\int_S \frac{d\vec{B}}{dt} \cdot \vec{dS}$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$ (Differential form of Faraday's Law)

Ampere's Law

$$\oint \vec{B} \cdot \vec{dl} = \mu_0 I_{encl}$$

$$\int_S \left(\vec{\nabla} \times \vec{B} \right) \cdot \vec{dS} = \mu_0 \int_S \vec{J} \cdot \vec{dS}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

(Differential form of Ampere's Law)

(i) $\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$

(ii) $\nabla \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

(iv) $\nabla \times \vec{B} = \mu_0 \vec{J}$

$$\vec{\nabla} \cdot (\vec{\nabla} \times \vec{B}) = \mu_0 \vec{\nabla} \cdot \vec{J}$$

lhs = 0 ; rhs $\neq 0$ (in general)

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t} \neq 0$$

Maxwell's Equations

$$\vec{\nabla}\cdot\vec{E}=\frac{\rho\left(\vec{r}\right)}{\epsilon_0} \quad ; \quad \nabla\cdot\vec{B}=0 \quad ; \quad \vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\times\vec{B} \;=\; \mu_0\left(\vec{J}+\epsilon_0\frac{\partial\vec{E}}{\partial t}\right) \;=\; \mu_0\left(\vec{J}+\vec{J}_d\right)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B}\right) \;=\; \mu_0\vec{\nabla}\cdot\vec{J} \;+\; \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\vec{\nabla}\cdot\vec{E}\right)$$

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot\vec{J} \;=\; 0$$

Electromagnetic Waves

$$\rho(\vec{r}) = 0 \quad ; \quad \vec{J}(\vec{r}) = 0 \text{ in free space.}$$

(i) $\nabla \cdot \vec{E} = 0$ (ii) $\nabla \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

These coupled first-order partial differential equations can be decoupled:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t} \right)$$

Electromagnetic Waves

$$\rho(\vec{r}) = 0 \quad ; \quad \vec{J}(\vec{r}) = 0 \;\; \text{in free space}.$$

(i) $\nabla \cdot \vec{E} = 0$ (ii) $\nabla \cdot \vec{B} = 0$

(iii) $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$ (iv) $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \dfrac{\partial \vec{E}}{\partial t}$

These coupled first-order partial differential equations can be decoupled:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right)$$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \vec{\nabla} \times \left(\mu_0 \epsilon_0 \frac{\partial \vec{B}}{\partial t} \right)$$

Electromagnetic Waves

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{E}) = \vec{\nabla}\times\left(-\frac{\partial\vec{B}}{\partial t}\right)$$

$$\vec{\nabla}\times(\vec{\nabla}\times\vec{B}) = \vec{\nabla}\times\left(\mu_0\epsilon_0\frac{\partial\vec{E}}{\partial t}\right)$$

$$\text{(i)}\quad \vec{\nabla}(\vec{\nabla}\cdot\vec{E})-\nabla^2\vec{E} = -\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{B}) = -\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2}$$

$$\text{(ii)}\quad \vec{\nabla}(\vec{\nabla}\cdot\vec{B})-\nabla^2\vec{B} = \mu_0\epsilon_0\frac{\partial}{\partial t}(\vec{\nabla}\times\vec{E}) = -\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

$$\text{(i)}\quad \nabla^2\vec{E}=\mu_0\epsilon_0\frac{\partial^2\vec{E}}{\partial t^2} \qquad \text{(ii)}\quad \nabla^2\vec{B}=\mu_0\epsilon_0\frac{\partial^2\vec{B}}{\partial t^2}$$

Electromagnetic Waves

(i) $\nabla^2 \vec{E} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{E}}{\partial t^2}$ (ii) $\nabla^2 \vec{B} = \mu_0 \epsilon_0 \frac{\partial^2 \vec{B}}{\partial t^2}$

Each component of electric and magnetic fields satisfies wave-like equations:

$$\nabla^2 y = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$

Electromagnetic waves (for electric and magnetic fields) propagate with speed:

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} = 3 \times 10^8 \text{ m/s}$$

Implication: *Light is an electromagnetic wave*

Monochromatic Electromagnetic Wave

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \qquad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

(Empty Space)

Suppose wave is monochromatic and propagating along z direction

Plane waves as the fields are uniform in planes that are perpendicular to x-direction.

$$\tilde{E}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)} \; ; \; \tilde{B}(z,t) = \tilde{B}_0 e^{i(kz-\omega t)}$$

Physical fields are real parts of $\tilde{E}$ and $\tilde{B}$.

Maxwell's equations impose special constrants :

$$\nabla\cdot\vec{E} = 0 \; ; \; \nabla\cdot\vec{B} = 0 \Rightarrow$$

$$\left(\tilde{E}_0\right)_z = 0 \text{ and } \left(\tilde{B}_0\right)_z = 0$$

Electromagnetic waves are transverse.

Monochromatic Electromagnetic Wave

- The electric and magnetic fields are perpendicular to the direction of propagation.

$$\text{Faraday's Law}: \ \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

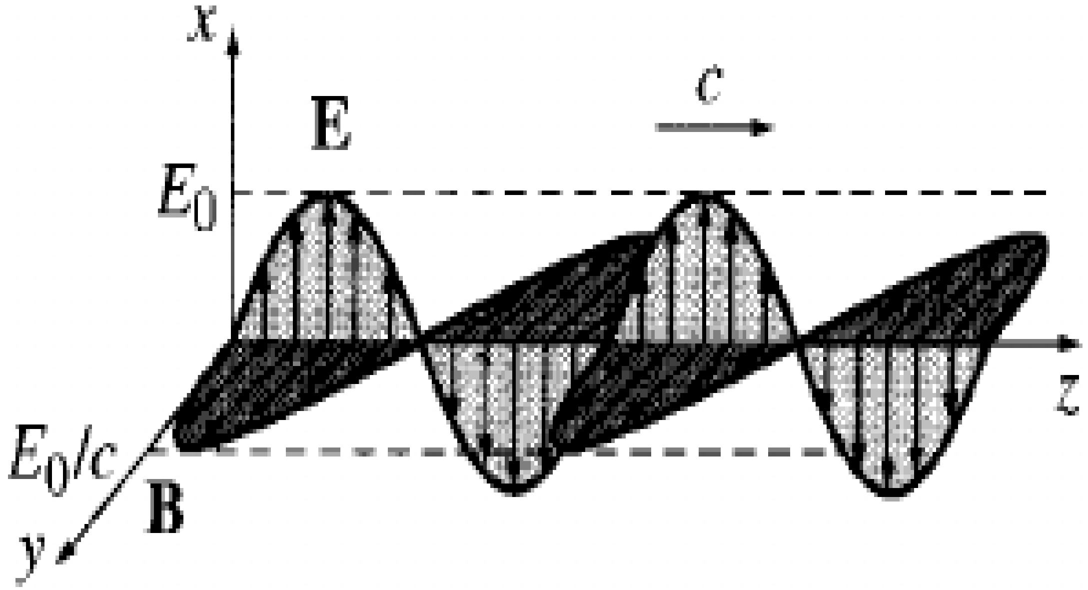
$$\Rightarrow -k\left(\tilde{E}_0\right)_y = \omega\left(\tilde{B}_0\right)_x \ ; \ k\left(\tilde{E}_0\right)_x = \omega\left(\tilde{B}_0\right)_y$$

$$\Rightarrow \tilde{B}_0 = \frac{k}{\omega}\left(\hat{z}\times\tilde{E}_0\right)$$

$\vec{E}$ and $\vec{B}$ are in phase and mutually perpendicular.

$$\text{Real amplitudes}: \ B_0 = \frac{k}{\omega}E_0 = \frac{1}{c}E_0.$$

Monochromatic Electromagnetic Wave



$$\vec{E}(\vec{r},t) = E_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \delta)\hat{n} \quad (\hat{k} \text{ is direction of propagation})$$

$$\vec{B}(\vec{r},t) = \frac{E_0}{c}\cos(\vec{k}\cdot\vec{r} - \omega t + \delta)(\hat{k}\times\hat{n})$$

$$\hat{k}\cdot\hat{n} = 0 \;;\; \hat{k}\cdot(\hat{k}\times\hat{n}) = 0.$$