Assignment 7 - Chapter 14

Kousshik Raj (17CS30022)

9-11-2020

1 Problem

Give a reduction from SUBGRAPH ISOMORPHISM parameterized by the number of edges to ODD SET, where the parameter transform is linear.

1.1 Solution

Let (G, H) be a given instance of SUBGRAPH ISOMORPHISM (SI) where we need to find a subgraph H' of G such that H' is isomorphic to H. We need to create an instance $(U, \mathcal{F}, k)$ of ODD SET, where we need to find $S \subseteq U$, $|S| \leq k$ and $|S \cap X|$ is odd, $\forall X \in \mathcal{F}$, such that (G, H) is a YES-instance if and only if $(U, \mathcal{F}, k)$ is a YES-instance. Moreover, $k = O(|E(H)|)$ should also be satisfied. Let, $|V(H)| = r$.

1.1.1 Reduction to Multicolored Instance

First, we will reduce the SI problem to the COLORED SUBGRAPH ISOMORPHISM (CSI) problem. We are given an instance (G, H) of SI, and we are required to convert it into an instance (G, H, f_1, f_2) of CSI. In CSI, we are given graphs G and H, and a coloring functions $f_1 : V(G) \to [r], f_2 : V(H) \xrightarrow{1:1} [r]$ (bijective), and we are required to find a subgraph H' of G that is isomorphic to H such that $f_1(v) = f_2(g(v))$ where $v \in V(H'), g(v) \in V(H)$, and g is the isomorphism from H' to H.

Here, we will do a Cook's reduction. Let $\mathcal{F}$ be an $(|V(G)|, r)$-perfect hash family. Let $\mathcal{L}$ be the set of bijective functions from $V(H)$ to $[r]$. Now, $(G, H) \in$ SI if and only if $\exists_{f_1 \in \mathcal{F}, f_2 \in \mathcal{L}} (G, H, f_1, f_2) \in$ CSI. We know that, $|\mathcal{F}| = O(r^{O(logr)} \log n)$ and $|\mathcal{L}| = r!$. Thus, there is a total of $O(r^r \log n)$ possibilities, and this can be computed and verified in time $O(r^r n^{O(1)})$.

Validity. The validity of the forward direction arises from the fact that for every $S \subseteq V(G)$ and $|S| = r$, there exists a coloring function $f_1 \in \mathcal{F}$, such that f_1 is injective on the subset S (it colors all the vertices of S with distinct colors). This is a property of perfect hash families we have previously seen. Now, if there exists a subgraph H' of G that is isomorphic to H, then since $V(H') \subseteq V(G)$ and $|V(H')| = r$, we will have a function $f_1 \in \mathcal{F}$, that colors all the vertices of H' distinctly. Moreover, as we try out all possible bijective coloring f_2 for H, there exists a coloring f_2 such that $f_2(g(v)) = f_1(v)$, where $v \in V(H')$, and g is the isomorphism.

For the reverse direction, it is trivial, because if $\exists_{f_1 \in \mathcal{F}, f_2 \in \mathcal{L}} (G, H, f_1, f_2) \in$ CSI, then we can just remove the coloring and we will have $(G, H) \in$ SI. Thus, this concludes the reduction.

Here, there is no change in the parameter, which is the number of edges in the graph H. Even though this is a Cook's reduction, the underlying idea behind this reduction could be used to design a better reduction from SI to CSI.

1.1.2 Reduction to ODD SET

Now, we will show a reduction from CSI to ODD SET. We have an instance (G, H, f_1, f_2) of CSI. Define $V_i = f_1^{-1}(i)\ \forall 1 \leq i \leq r$, $E_{i,j}$ as the set of edges between V_i and V_j, $E_{i,j,v}$ as a subset of $E_{i,j}$ consisting of edges incident on v. Moreover, let $v_i \in V(H)$ and $f_2(v_i) = i$. Notice that $|f_2^{-1}(i)| = 1$ as f_2 is bijective and hence v_i is unique. Now, construct $(U, \mathcal{F}, k)$ such that

- $U = \{\bigcup_{1 \leq i \leq r} V_i\} \cup \{\bigcup_{1 \leq i < j \leq r} E_{i,j}\}$
- $k = r + |E(H)|$
- $\forall 1 \leq i \leq r$, $\mathcal{F}$ contains V_i, and $\forall (v_i, v_j) \in E(H)$, $\mathcal{F}$ contains $E_{i,j}$
- $\forall (v_i, v_j) \in E(H)$, $\mathcal{F}$ contains $E'_{i,j,v} = E_{i,j,v} \cup (V_i - \{v\}), \forall v \in V_i$ and $E'_{i,j,w} = E_{i,j,w} \cup (V_j - \{w\}), \forall w \in V_w$

Validity. It is obvious that $r \leq 2|E(H)|$, which means $k \leq 3|E(H)|$ and thus $k = O(|E(H)|)$. We will now prove that (G, H, f_1, f_2) is a YES-instance if and only if $(U, \mathcal{F}, k)$ is a YES-instance.

For the forward direction, let H' be a colorful subgraph of G that is isomorphic to H with color of the vertices mapping each other being the same, where $V(H') = \{u_1, u_2, ..., u_r\}$ and $u_i \in V_i$. Let g be the isomorphism from H' to H such that $g(u_i) = v_i, u_i \in V(H'), v_i \in V(H)$ (by definition of CSI, $f_1(u_i) = f_2(v_i)$). Then we claim that the set $S = \{\bigcup_{1 \leq i \leq r}\{u_i\}\} \cup \{\bigcup_{(u_i,u_j) \in E(H')} e_{i,j}\}$ where $e_{i,j}$ is the edge between vertices u_i and u_j, has an odd intersection with every set in $\mathcal{F}$. This is true for the sets V_i and $E_{i,j}$ as they have exactly one element from them in S. Now consider the sets $E'_{i,j,v}$. WLOG, we can assume $v \in V_i$. If the endpoint of the edge $e_{i,j}$ is v, then it can be seen that $v = u_i$ and $E'_{i,j,v} \cap S = \{e_{i,j}\}$ which means the intersection is of size one. If v is not the endpoint, then $v \neq u_i$ and $E'_{i,j,v} \cap S = \{v\}$, and thus again the intersection is of size one, which is odd. Since, $E'_{i,j,v}$ is present if and only if $(v_i, v_j) \in E(H)$, there always exists a corresponding edge $e_{i,j}$ in H'.

For the other direction, let $S \subseteq U$ be a set of size at most k such that the size of its intersection with every set in $\mathcal{F}$ is odd. Since there are $r + |E(H')| = k$ disjoint sets in $\mathcal{F}$, corresponding to V_i and $E_{i,j}$, we have $|S| = k$, and each of those disjoint sets contain exactly one element from S. Let, $V_i \cap S = \{u_i\}$ (vertices) and $E_{i,j} \cap S = \{e_{i,j}\}$ (edges). Now, we claim that these vertices and edges form a colorful subgraph H' of G that is isomorphic to H, with $u_i \in V(H')$ mapping to $v_i \in V(H)$, where $f_1(u_i) = f_2(v_i) = i$. Let's take an edge $(v_i, v_j) \in E(H)$. Then the corresponding edge in the graph H' should be $e_{i,j}$. We have to show that u_i and u_j are indeed the endpoints of the edge $e_{i,j}$ as they map to the vertices v_i and v_j in the isomorphic graph. Assume for the sake of contradiction, u_i is not an endpoint of $e_{i,j}$. Then, $u_i \notin E'_{i,j,u_i}$ and $e_{i,j} \notin E'_{i,j,u_i}$. This means that the intersection of S with E_{i,j,u_i} is empty, which contradicts the assumption that S is a solution of the ODD SET problem. Therefore, u_i and u_j are indeed the endpoint of the edge $e_{i,j}$. And since all edges corresponding to the edges in H are present in H' and there are only $|E(H)|$ elements corresponding to the edges present in S, there are no other edges in H'. Thus, H' is isomorphic to H, with the mapped vertices having the same color. Thus, this concludes the reduction.

We first reduced the SI problem to the CSI problem (under Cook's reduction), where there is no change in parameter. We then reduced the CSI to ODD SET, where the parameter transform is linear. Thus, we have shown a reduction of SUBGRAPH ISOMORPHISM parameterized by the number of edges to ODD SET where the parameter transform is linear.