Signal & Networks Autnm 2018-19: Course coverage on signals
by Tapas Kumar Bhattacharya

Referred books (for signal part):

1. Signals and Systems by P. Ramesh Babu & R. Ananda Natarajan
Publisher; SCITECH PUBLICATIONS (INDIA), 3rd edition

2. Signals & Systems by A. V. Oppenheim, A. S. Willsky and S. H. Nawab,
Prentice-Hall India, 2015.

3. Networks and Systems by D. Roy Choudhury
New AGE International Publishers

Lectures 1 & 2 : on 18/07/2018 Wednesday

Given brief outline of signal portion of the course. A continuous time signal x(t) can be classified in various ways – even or odd; real or complex; periodic or non-periodic etc.

For any time t, x(t) has any (finite) value over the range of time from -∞ to +∞.
A given signal x(t) in time axis can be right shifted by an amount say τ and the description of the right shifted signal will then be x(t – τ). Similarly a left shifted signal will be x(t + τ). Asignal in general can be time shifted, time scaled and time reversed.
Introduced two special signals namely unit step signal u(t) and unit ramp signal r(t) and found out relationship between these two. Sometimes it is possible to describe a given signal x(t) in a rather neat mathematical form in terms of u(t) and r(t).

Lecture 3 : on 19/07/2018 Thursday

A brief review of the last lecture. Any signal x(t) can be shown to be sum of an even and an odd function.
Introduced the unit delta function δ(t). A thin rectangular strip of width h around the origin and height 1/h has unity area. If we allow the width h to tend to zero then height tends to +∞ then this pulse is said to be a delta function δ(t) of strength 1. The magnitude of δ(t) approaches to +∞ at t=0 and δ(t) = 0 for t ≠ 0. Although we can not specify the value of δ(t) at t = 0 but $\int_{-\infty}^{\infty} \int \delta(t)\, dt = 1$.

The multiplication of a signal with impulse function $x(t)$ means that $x(t)\delta(t) = x(0)\delta(t)$. Which means that , $\int_{-\infty}^{\infty} x(t)\delta(t)\, dt = \int_{-\infty}^{\infty} x(0)\delta(t)\, dt = x(0)$.

Similarly when $x(t)$ is multiplied with a shifted delta function $\delta(t-a)$ and integrated from t = -∞ to t = +∞ ,

$$\int_{-\infty}^{\infty} x(t)\delta(t-a)\,dt=\int_{-\infty}^{\infty} x(a)\delta(t)\,dt=x(a)$$

To continue with more interesting properties of delta functions in next class.

Lecture 4 : on 23/07/2018 Monday

Continued discussion on delta function. It was shown that $\delta(at)=\delta(t)/|a|$ for both +ve and -ve values of a . Students were asked to show $\delta(at+b)=\delta(t+\frac{b}{a})/|a|$. Then it was shown that $x(t)\frac{d\delta}{dt}=\frac{dx}{dt}(0)\delta(t)$. Importance of impulse function highlighted. If a function $x(t)$ has discontinuities (jump in the value of $x(t)$ in no time) at several times and we want to get $\frac{dx}{dt}$. We will see that where ever discontinuity exist $\frac{dx}{dt}$ can be sketched as some δ function of appropiate strength. An uncharged ideal capacitor when connected to an ideal voltage source at time, the current drawn is shown to be an impulse of strength = the amount of charge transferred.

For a given signal $x(t)$, we might like to carry out (with respect to t) shifting or scaling or reversing or all of these operations to get a new signal say $x(-4t-3)$. Remember that all these operations are done with respect to time t . It was pointed out that it is better to carry out operations from right to left i.e., first shift $x(t)$ to right by 3 units to get $x(t-3)$. Then from $x(t-3)$, get $x(4t-3)$ by time scaling. Finally do the time reversal operation on $x(4t-3)$ to get $x(-4t-3)$. This was explained with the help of a simple example.

Lectures 5 & 6 : on 25/07/2018 Wednesday

Explanation of $\delta[g(t)]$ where $g(t)$ is a general function of time. It is expected that at whatever time $g(t)=0$ there will be impulses present. So at the roots of $g(t)=0$, impulses are present. Let $b_{1,} b_{2,} .. b_i$ are the roots. However the strengths of the impulses will not be same. It was shown that for the i th root

$$\int_{-\infty}^{\infty} x(t)\delta[g(t)]dt=\int_{-\infty}^{\infty} x(t)\delta[\frac{dg}{dt}(b_i)(t-b_i)]dt=\int_{-\infty}^{\infty} x(t)\frac{\delta(t-b_i)}{|\frac{dg}{dt}(bi)|}dt$$

Therefore, for all the roots it will be:

$$\int_{-\infty}^{\infty} x(t)\delta[g(t)]dt\,dt=\sum_i \int_{-\infty}^{\infty} x(t)\frac{\delta(t-b_i)}{|\frac{dg}{dt}(bi)|}dt$$

Thus the strength of the impulses will be different : in fact it depends on the slope g(t) at that particular root.

The solution of linear differential equations.

The solution has two parts : natural response $y_n(t)$ and solution due to the forcing function $y_f(t)$. Roots of the characteristic equations decides . It was shown that $y_n(t)=C_1 e^{m_1 t}+C_2 e^{m_2 t}\ldots$ if the roots are distinct. Where as solution due to the forcing function $x(t)$ will be the linear combination of $x(t)$ and its heigher order derivatives i., $y_f(t)=k_1 x+k_2\frac{dx}{dt}+k_3\frac{d^2 x}{dt^2}+k_4\frac{d^3 x}{dt^3}+\ldots\infty$

Therefore the solution of the differential eqution

$$\frac{d^2 y}{dt^2}+a\frac{dy}{dt}+b\,y=x(t)$$

can be straight away written as :

$$y(t)=y_n(t)+y_f(t)=C_1 e^{m_1 t}+C_2 e^{m_2 t}+k_1 x+k_2\frac{dx}{dt}+k_3\frac{d^2 x}{dt^2}+k_4\frac{d^3 x}{dt^3}+\ldots\infty$$

where m_1 and m_2 are the two distinct roots of the characteristic equation $m^2+a m+b=0$

Determination of various constants:

1. first determine constants k_1, k_2 etc. by noting that $x_f(t)$ alone satiesfies the differential equation. This will give you an identity from which k_1 , k_2 etc. can be found out.

2. Now in the total solution we have to find out C_1 and C_2 from the boundary conditions given.

If there are repeated roots for example in the above example if $m_1=m_2=m$ (say), then natural response will be

$$y_n(t)=(C_1+C_2 t)e^{mt}$$

if there are 3 repeated roots in a 4th order differential equation i.e., $m_1=m_2=m_3=m$ (say) then

$$y_n(t)=(C_1+C_2 t+C_3 t^2)e^{mt}+C_4 e^{m_4 t}$$

Solved some problems with students participations.

Lecture 7 : on 26/07/2018 Thursday

It is interesting to note that if the input signal $x(t)$ happens to be an exponentioal function then the solution due to the forcing function $y_f(t)$ can be very easily obtained. In fact the solution due to the exponential $x(t)$ will be once again exponential itself as any order of derivative on an exponential function retains the same exponential form.

Consider the following differential equation

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by=x(t) \quad \text{and} \quad x(t)=Ae^{j\omega t},$$

i.e., $(D^2+aD+b)y_f(t)=Ae^{j\omega t}$

It was shown that, response $y_f(t)$ will be:

For $y_f(t)=\dfrac{Ae^{j\omega t}}{(D^2+aD+b)\vdots_{D=j\omega}}$

Also note that for constant (dc) input $x(t)=A=Ae^{j0t}$ and

$$y_f(t)=\frac{Ae^{j\omega t}}{(D^2+aD+b)\vdots_{D=0}}=\frac{A}{b}$$

How this can help me? Consider $x(t)$ to be sinusoidally varying excitation, say $\sin\omega t$ or . These types of functions can be written in exponential forms as follows

$$\sin\omega t=\frac{1}{2j}[e^{(j\omega t)}-e^{(-j\omega t)}]$$

$$\cos\omega t=\frac{1}{2}[e^{(j\omega t)}+e^{(-j\omega t)}]$$

Therefore $y_f(t)$ can be easily obtained by doing a little bit of algebra on the two exponential terms.

How to solve differential equation when the excitation or input $x(t)=\delta(t)$?
Discussion on this has been started and will be continued.

Lecture 8 : on 30/07/2018 Thursday

How to solve differential equation when the excitation or input $x(t)=\delta(t)$?

1. Consider a first order differential equation with forcing function $\delta(t)$.

$$\frac{dy}{dt}+2y=\delta(t) \quad \text{with the boundary condition} \quad y(0^-)=1$$

Looking at the equation we know, to balance the equation LHS must produce an impulse to balnce the equation. This is possible if y jumps from $y(0^-)=1$ to $y(0^+)$, then $\frac{dy}{dt}$ can be an impulse and everything will be in place.

Now multiply both sides of the equation by dt and integrate from 0^- to 0^+ to get:

$$\int_{0^-}^{0^+} dy+2\int_{0^-}^{0^+} y\,dt=\int_{0^-}^{0^+} \delta(t)\,dt$$

or $y(0^+)=1+1=2$

Since $\delta(t)=0$ for $t>0^+$, the problem then boils down to solving the following differential equation with new B.C as shown below.

$$\frac{dy}{dt}+2y=0 \quad \text{With B.C} \quad y(0^+)=2$$

Characteristic equation $m+2=0$ gives $m=-2$ hence solution is :

$$y(t)=Ce^{-2t}$$

From B.C we get $C=2$. Hence the solution is $y(t)=2e^{-2t}$.

What about 2nd or higher order differential equations? Consider the following equation.

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+by=\delta(t)$$

We immediately know that $\frac{dy}{dt}$ must have a step jump so that $\frac{d^2y}{dt^2}$ gives the necessary impulse to balance the right hand side. So conclusion is y(t) will be continuous and $\frac{dy}{dt}$ will have a step jump; whch means:

$$y(0^-)=y(0^+) \quad \text{and} \quad \frac{dy}{dt}(0^+)\neq\frac{dy}{dt}(0^-) \quad .$$

For n th order differential equation $\frac{d^{n-1}y}{dt^{n-1}}$ will have a step jump and lower order differentiation terms will be continuous.

Why impulse response of a system (described by a linear differential equation) is important to know? We shall show later that if the impulse response of a system is known, we can find out the response of the system for any arbitrary input signal $x(t)$.

How to solve a differential equation when the characteristics roots are complex?

Consider the following equation:

$$\frac{d^2y}{dt^2}+a\frac{dy}{dt}+b\,y=x(t)$$

$m_{1,}m_2=\frac{1}{2}[-a\pm\sqrt{(a^2-4b)}]=\alpha\pm j\beta$ where α and β are real numbers.

If $a^2<4b$, the roots will be complex. Note that complex roots will always *appear as a pair of complex conjugate numbers.* Therefore the natural response will be :

$$y_n(t)=C_1e^{(\alpha+j\beta)t}+C_2e^{(\alpha-j\beta)t}$$

We live in a natural world, therefore $x(t)$ is a real input signal. Therefore for a real system , $y_n(t)$ too must be a real function. How this can happen? This can happen only and only when the two complex terms on the RHS are complex conjugate of each other. But the roots are already complex conjugate of each other. Therefore C_1 and C_2 must also be conjugate of each other.

Let $C_1=R\,e^{j\theta}$ then $C_2=R\,e^{-j\theta}$

Putting C_1 and C_2 in the equation $y_n(t)=C_1 e^{(\alpha+j\beta)t}+C_2 e^{(\alpha-j\beta)t}$ and doing a little bit of algebraic manipulation we get,

$$y_n(t)=2Re^{\alpha t}\cos(\beta t+\theta)$$

The value of R and θ are determined from the B.C . If α is negative, then the response is a *exponetially decaying damped sinusoid* and it finally decays down to zero with a time constant $\frac{1}{\alpha}$.
Solution due to the forcing function does not depend on characteristic roots.

You do yourself:
Try to find out the response if the roots of the characteristic equation is purely imaginary i.e., $\pm j\beta$.

Lectures 9 & 10 : on 01/08/2018 Wednesday

Signals $x(t)$ in general are real. But to get analysis simplification, the real signal of the sinusoidal nature can be expressed as sum or difference of of complex exponentials as shown below

$$x(t)=\sin\omega t=\frac{1}{2j}[e^{(j\omega t)}-e^{(-j\omega t)}] \quad \text{and} \quad x(t)=\cos\omega t=\frac{1}{2}[e^{(j\omega t)}+e^{(-j\omega t)}]$$

Nothing could be more satisfying than to see the forcing function (in a differential equation) to be an exponential. Because $y_f(t)$ can be written very easily. So we shall not insist any more that $x(t)$ should be real only. Since a complex number has both magnitude and phase, so it will not be possible to show $x(t)$ in a *single graph* in order to understand how the $x(t)$ evolves with time. Therefore people use complex plane (with real and imaginary axex) to represent a complex $x(t)$ with t as a parameter. Evolution of signals $e^{j\omega t}$ and $e^{at}e^{j(\omega t+\theta)}$ where a is a real number, with time t discussed.
Energy and power of a signal defined. If energy is finite it is called an energy signal while if the power is finite it is called power signal.
Types of systems : Linear/non-linear, Time invariant/time variant, static (without memory), dynamic (with memory), causal/non-causal, invertible/non-invertible. If a system is given we shall be in a position to determine its type.

Started discussing on impulse response of a system and it’s application.

Lecture 11 : on 2/08/2018 Thursday

If the impulse response of a ***linear, time invariant system*** $h(t)$ is known , the output of the same system for any input signal $x(t)$, can be found out.

Recall for a linear system, $x(t) \rightarrow y(t)$ where x(t) is input and y(t) is output of the system, then following are true:

$a\,x(t) \rightarrow a\,y(t)$ where scaling up factor.

$a_1 x_1(t) + a_2 x_2(t) \rightarrow a_1 y_1(t) + a_2 y_2(t)$ This is scaling and superposition

Also if the system is time-invariant:

$x(t) \rightarrow y(t)$ then $x(t-a) \rightarrow y(t-a)$

Using these facts about linear, time-invariat system, it was shown that

if $\delta(t) \rightarrow h(t)$ then

$$x(t) \rightarrow \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = y(t)$$

This particular integral is called **convolution** operation on two signals x(t) and h(t) and in short hand is written as:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

It was also shown that the order of appearance of x and h inside the integral can be interchanged i.e.,

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau = h(t) * x(t)$$

Convolution is **commutative.**

In general we can **convolve** any two signals $x_1(t)$ and $x_2(t)$ if necessary and it will be written as

$$x_1(t)*x_2(t)=\int_{-\infty}^{\infty} x_1(\tau)x_2(t-\tau)\,d\tau=\int_{-\infty}^{\infty} x_2(\tau)x_1(t-\tau)\,d\tau$$

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With the help of an example, the steps involved to convolve two
signals, illustrated.
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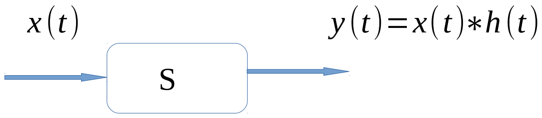
Lecture 12 : on 6/08/2018 Monday

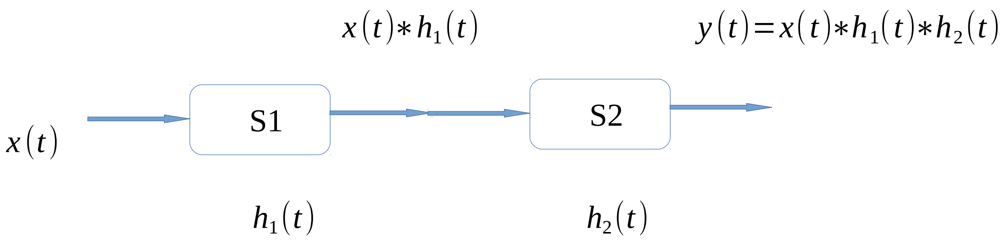
An LTI system S can be diagrammicaly represented as follows, where X(t) is iput, y(t) is output and h(t) is system’s impulse response.



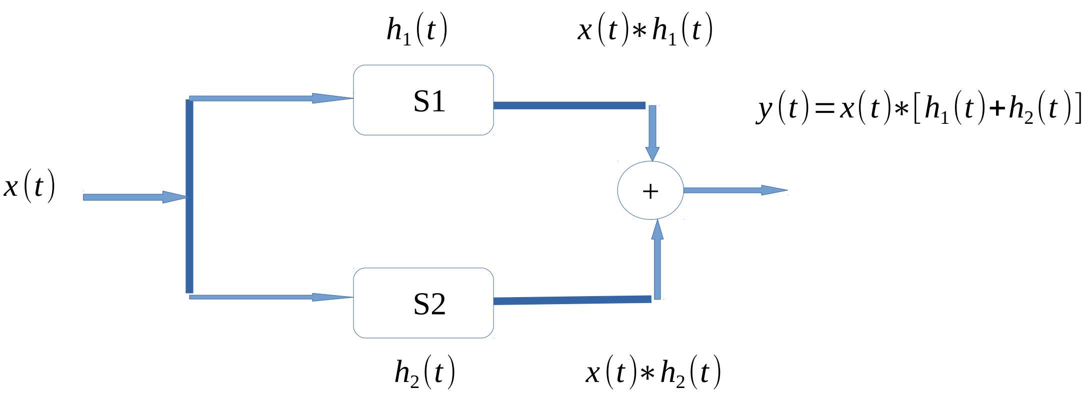
$h(t)$ = impulse response of S

If two systems S_1 and S_2 having impulse responsesare $h_1(t)$ and $h_2(t)$ are connected in series or in cascade, the equivalent system will have impulse response $h_1(t)*h_2(t)$. If the systems S_1 and S_2 are connected in parallel, the equivalent system will have impulse response $h_1(t)+h_2(t)$.

Systems connected in series:



Systems connected in parallel:

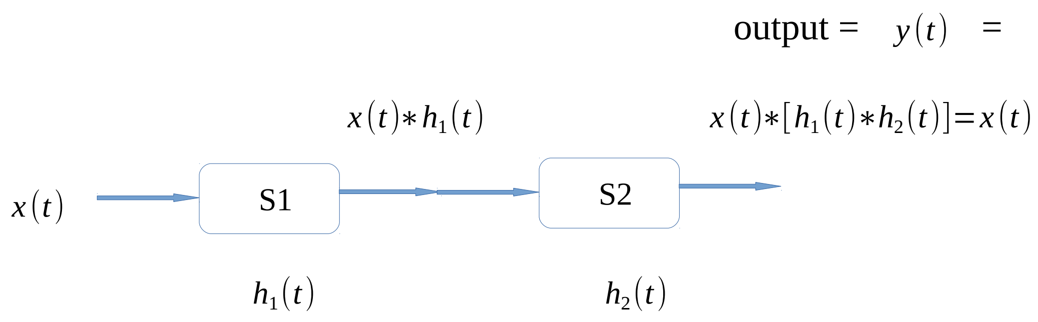


Knowing the impulse response $h(t)$ of a system, it is possible to conclude whether the system is (i) memory less or not, (ii) causal or not and (iii) invertible or not.
It was discussed and shown that:

(1) For a memory less system impulse response : $h(t)=k\delta(t)$

(2) For a causal system : $h(t)=0$ for $t<0$

(3) A system S_1 is said to be invertible, if there exists another system S_2 to be connected in series such that for the overall system output and input are same i.e., $y(t)=x(t)$.



For an invertible system $h_1(t)*h_2(t)=\delta(t)$ where $h_1(t)$ and $h_2(t)$ are the impulse responses of S_1 and S_2 respectively.

Lectures 13 & 14 : on 08/08/2018 Wednesday

Reviewed briefly the last lecture. It was pointed out that convolution operation is not only commutative but also associative and distributive, i.e.,

Commutative: $x_1(t)*x_2(t)=x_2(t)*x_1(t)$

Associative: $[x_1(t)*x_2(t)]*x_3(t)=x_1(t)*[x_2(t)*x_3(t)]$ etc.

Distributive: $x_1(t)*[x_2(t)+x_3(t)]=x_1(t)*x_2(t)+x_1(t)*x_3(t)$

It was shown that a system in BIBO (bounded input, bounded output) sense is stable if :

$\int_{-\infty}^{\infty} |h(\tau)| d\tau \leq \infty$ i.e., the integral should be finite. In other words h(t) should be absolutely integrable for a stable system.

An interesting observation:
If two systems are connected in cascade, the over all output and input relation will not be affected then position of S1 and S2 can be interchanged. Using this idea it was shown that, if the unit step response of a system is known to be y_{step}(t), then the unit impulse response of the system is $h(t)=\frac{d\, y_{step}(t)}{dt}$.

Fourier series introduced. If $x(t)$ is periodic with a fundamental time period T and if x(t) satisfies Dirichlet's conditions, then x(t) can be represented as sum of some constant value and infinite number of sinusoidal terms of different frequencies $k\omega$, where $\omega=\frac{2\pi}{T}$ is called the fundamental frequency.

$$x(t)=c_o+\sum_{k=1}^{\infty} a_k \cos k\omega t+\sum_{k=1}^{\infty} b_k \sin k\omega t$$

The values of the constants were derived to be :

$$c_o=\frac{1}{T}\int_{-T/2}^{T/2} x(t)dt=\frac{1}{T}\int_{0}^{T} x(t)$$

$$a_k=\frac{2}{T}\int_{-T/2}^{T/2} x(t)\cos k\omega t\, dt=\frac{2}{T}\int_{0}^{T} x(t)\cos k\omega t\, dt$$

$$b_k=\frac{2}{T}\int_{-T/2}^{T/2} x(t)\sin k\omega t\, dt=\frac{2}{T}\int_{0}^{T} x(t)\sin k\omega t\, dt$$

It may be noted, all the above integrations are to be carried out over a time period T . The results of the constants is independent of the lower and upper limits of the integration so long the difference of the upper and the lower limit is the time period T .

Students were asked to represent a periodic square wave x(t) in Fourier series coefficients.

Lecture 15 : on 9/08/2018 Thursday

For a periodic signal $x(t)$ if $x(t)=-x(t\pm\frac{T}{2})$ then x(t) is said to *have half wave symmetry.* The implication of this are: $C_o=0$ (i.e., no dc value) and only the odd harmonics will be present. Sine & cosine terms may be present for k = odd integers. A periodic signal $x(t)$ is said to be *quarter wave symmetric* when $x(t)$ is either *even or odd* signal and is also having *half wave symmetry*. The implication of this is that for even x(t) , $C_o=0$ with only cosine terms $(a_k\neq 0)$ k = odd
and for odd x(t) , $C_o=0$ with only sine terms $(b_k\neq 0)$ k = odd

From a mere visual look at the periodic signal , it is sometimes very easy to figure out whether the signal is half wave symmetric or quarter wave symmetric. It is needless to say that if x(t) is neither even nor odd and not has half wave symmetry, then all the terms and both odd and even harmonics may be present.

Alternative way of writing the Fourier series

$$x(t)=c_o+\sum_{k=1}^{\infty} a_k\cos k\omega t+\sum_{k=1}^{\infty} b_k\sin k\omega t=c_o+\sum_{k=1}^{\infty}\overline{C_k}\cos(k\omega t-\theta_k)$$

where $\overline{C_k}=\sqrt{a_k^2+b_k^2}$ and $\theta_k=\tan^{-1}(\frac{b_k}{a_k})$

The plot of $|\overline{C_k}|$ versus $k\omega$ is called line spectrum and tells us about the strength of each harmonic present in the signal. Similarly θ_k can be plotted versus $k\omega$ to get the phase information of the signal at a particular harmonic.

Lecture 16 : on 13/08/2018 Monday

We have seen that for periodic $x(t)$,

$$x(t)=c_o+\sum_{k=1}^{\infty} a_k \cos k\omega t+\sum_{k=1}^{\infty} b_k \sin k\omega t$$

The RHS of the above equation was manipulated and x(t) was written in a much more compact way as below.

$$x(t)=\sum_{k=-\infty}^{\infty} C_k e^{jk\omega t}$$ This is called Fourier series in complex form.

where,

$$C_k=\frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega t}dt$$

The above two formulas are worth remembering. C_k in general will be complex.

C_k is related to original a_k and b_k as per the following relations.

$$C_k=\frac{1}{2}(a_k-j\,b_k) \quad \text{and} \quad C_{-k}=\frac{1}{2}(a_k+j\,b_k)$$

With the help of student's participation found out C_k (hence a_k and b_k) for an odd periodic square wave.

For a train of periodic unit impulse signal, C_k was found to be constant (= $\frac{1}{T}$) which is independent of k i.e., all harmonics (including average value) are present with equal strengths.

Differentiation of periodic function:

$$x(t)=\sum_{k=-\infty}^{\infty} C_k e^{jk\omega t}$$

or $$\frac{dx(t)}{dt}=\sum_{k=-\infty}^{\infty} jkw\,C_k e^{jk\omega t}=\sum_{k=-\infty}^{\infty} C'_k e^{jk\omega t}$$

$\frac{dx(t)}{dt}$ too will be periodic with complex Fourier coefficient $C'_k = jkw\,C_k$.

Sometimes it is advantageous to calculate first C'_k and then get $C_k = \frac{C'_k}{j\,k\,w}$.

Lecture 17 : on 16/08/2018 Thursday

Last lecture reviewed. In many cases, integration by parts is necessary to get C_k for a given $x(t)$. One can simplify calculation, if C'_k for $\frac{dx(t)}{dt}$ (or heigher order derivatives till you get some sort of impulse function) is calculated first and then get C_k by using the relation $C_k = \frac{C'_k}{j\,k\,w}$.

Magnitude spectra of a real periodic signal was shown to be a even function where as the phase spectra was shown to be an odd function. As an example, a periodic even square pulse of width d and amplitude A with a time period T was analysed . Fourier coefficient C_k was found to be real and equal to :

$$C_k = \frac{A\,d}{T}\frac{\sin\frac{k\,\omega\,d}{2}}{\frac{k\,\omega\,d}{2}}$$

$|C_k|$ Vs $k\,\omega$ was sketched. The envelop of the plot is called a *sinc* function ($\frac{\sin\theta}{\theta}$)

Gave examples of some signals which are periodic but can not be expanded into a Fourier series as they don't satisfy Dirichlet's conditions.

Lecture 18 : on 20/08/2018 Monday

While discussing about power and energy of a general signal $x(t)$, we defined

$$P_{av} = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt$$

Average power of a periodic signal $x(t)$ then will be

$$P_{av} = \frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt$$ where T is the fundamental time period of $x(t)$.

In terms of Fourier coefficients , the power was shown to be

$$P_{av}=\frac{1}{T}\int_{-T/2}^{T/2}|x(t)|^2\,dt=\sum_{k=-\infty}^{\infty}|C_k|^2$$

When a Fourier series is truncated to a finite number of terms say N ,

$$x(t)=\sum_{k=-\infty}^{\infty}C_k e^{jk\omega t}\simeq\sum_{k=-N}^{N}C_k e^{jk\omega t}$$

Let

$$x_N(t)=\sum_{k=-N}^{N}D_k e^{jk\omega t}$$

and then the error is:

$$e(t)=\sum_{k=-\infty}^{\infty}C_k e^{jk\omega t}-\sum_{k=-N}^{N}D_k e^{jk\omega t}$$

We want to find D_k so that $e(t)$ in the sense of least square is minimum.

Now $e(t)$ can be writtens as

$$e(t)=\sum_{k>|N|}C_k e^{jk\omega t}+\sum_{k=-N}^{N}C_k e^{jk\omega t}-\sum_{k=-N}^{N}D_k e^{jk\omega t}$$

or, $$e(t)=\sum_{k>|N|}C_k e^{jk\omega t}+\sum_{k=-N}^{N}(C_k-D_k)e^{jk\omega t}$$

Now sum of the sqared error over a period,

$$\frac{1}{T}\int_{-T/2}^{T/2}|e(t)|^2\,dt=\sum_{k>|N|}^{\infty}|C_k|^2+\sum_{k=-N}^{N}|C_k-D_k|^2$$

The first term on RHS is constant for a chosen value of N . So $D_k=C_k$ will ensure minimum error.

The conclusion is : The same Fourier coefficients C_k calculated, can be used to the truncated series and this will ensure minimum error in the least squared sense.

Lecture 19 : on 23/08/2018 Thursday

If the periodic signal $x(t)$ has discontinuities, the value of the signal calculated from the RHS will always converge to the average value at these points of discontinuties. Not only that at the point of discontinuties there will be overshoot and undershoot present. This is known as Gibb's phenomen. The overshoot or undershoot is restricted to about 9% of the value of the function at the points of discontinuities.

Fourier transform is introduced. The idea is whether it is possible to know the frequency content of an **aperiodic signal**. Iin such cases, it was shown that Fourier coefficient Density per Hz, $X(\omega)$ will be meaningful. Starting from Fourier series concepts, following two important results were derived.

Fourier Transform:

$$X(\omega)=\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

$$x(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$$

$x(t)$ and $X(\omega)$ form a Fourier transform pair.. If $x(t)$ is known we can get $X(\omega)$ from the first equation and $X(\omega)$ is known, second equation can be used to get .

Fourier transform of x(t) is: $F\{x(t)\}=X(\omega)=\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$

Fourier inverse of $X(\omega)$ is : $F^{-1}\{X(\omega)\}=x(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega$

Tansform pair is indicated by: $x(t)\Leftrightarrow X(\omega)$

Lecture 20 : on 27/08/2018 Monday

The following two equations will occupy the center stage so far as Fourier transform and its inverse is concerned for fourier transform pair $x(t) \Leftrightarrow X(\omega)$.

$$F\{x(t)\} = X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ And

$$F^{-1}\{X(\omega)\} = x(t) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

Two interesting observations:

Area under the curve $x(t)$ is $A_{time\ domain} = \int_{t=-\infty}^{\infty} x(t) dt = X(0)$ (put $\omega=0$) in first equation). In other words, area under the time domain curve is nothing but the value of $X(\omega)$ at $\omega=0$.

Similarly by putting $t=0$ on both sides of the second equation, we get,

$$x(0) = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} X(\omega) d\omega = \frac{1}{2\pi} \times A_{frequency\ domain}$$

or Area under the frequency domain curve (Fourier transform curve) will be

$$A_{frequency\ domain} = 2\pi x(0) = \int_{\omega=-\infty}^{\infty} X(\omega) d\omega$$

It was shown that if $x(t)$ is a real even function, $X(\omega)$ too will also be even having real part only. Similarly, if x(t) is a real odd function, $X(\omega)$ too will be also odd having imaginary part part only.

Fourier transform of single rectangular pulse of amplitude A , width d and centered around $t=0$ is obtained as:

$X(\omega) = A d \frac{\sin(\omega d/2)}{(\omega d/2)} = A d\, sinc(\omega d/2)$, As expected this is an even function of ω having no imaginary part.

Using the ideas of calculating area under $x(t)$ or $X(\omega)$, it was shown that the following integral has a value equal to π

$sinc(\theta)=\int_{-\infty}^{+\infty}\frac{\sin\theta}{\theta}d\theta=\pi$ This result to be used later.

Fourier transform of $x(t)=\delta(t)$:

$$F\{\delta(t)\}=\int_{t=-\infty}^{\infty}\delta(t)e^{-j\omega t}dt=\int_{t=-\infty}^{\infty}\delta(t)dt=1$$

Fourier transform of $x(t)=1$:

To find this out quickly we use the inverse formula, i.e.,

$$x(t)=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$ Now, $x(t)=1$

$$1=\frac{1}{2\pi}\int_{\omega=-\infty}^{\infty}X(\omega)e^{j\omega t}d\omega$$

Can we guess $X(\omega)$ so that integration of RHS will return us 1? The answer is yes and gussing $X(\omega)=2\pi\delta(\omega)$ will indeed make RHS to be 1.

Therefore, $1\Leftrightarrow 2\pi\delta(\omega)$.

Why not get $X(\omega)$ by putting $x(t)=1$ in the formula

$$F\{1\}=X(\omega)=\int_{t=-\infty}^{\infty}x(t)e^{-j\omega t}dt$$. Please try.

Lectures 21 & 22 : on 29/08/2018 Wednesday

Consider the following integral where a is a positive real number

$\int_{-\infty}^{+\infty}\frac{\sin a\theta}{\theta}d\theta$, it can be easily seen that this integral too has a value eqal to π and its height at $\theta=0$ is a .

The first zero crossings of the function $\frac{\sin a\theta}{\theta}$ on the left and right of the origin are respectively at $-\pi/a$ and $+\pi/a$. Now if we allow $a\rightarrow\infty$, the function $\frac{\sin a\theta}{\theta}$ will approach an impulse function $\pi\delta(\theta)$.

In short, $\lim_{a\to\infty}\frac{\sin a\theta}{\theta}=\pi\delta(\theta)$. This result can be utilised to find out Fourier transform from of $x(t)=1$ from the first principle as follows.

$$F\{1\}=X(\omega)=\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt=\int_{t=-\infty}^{\infty}(\cos\omega t-j\sin\omega t)dt$$

or, $F\{1\}=\int_{t=-\infty}^{\infty}\cos\omega t\,dt=2\int_{t=0}^{\infty}\cos\omega t\,dt=\lim_{t\to\infty}\frac{2\sin\omega t}{\omega}=2\pi\delta(\omega)$

So we have,

$F\{\delta(t)\}=1$ or $\delta(t)\Leftrightarrow 1$

$F\{1\}=2\pi\delta(\omega)$ or $1\Leftrightarrow 2\pi\delta(\omega)$

Fourier transform of unit step function was shown to be

$F\{u(t)\}=\frac{1}{j\omega}+\pi\delta(\omega)$ the result was obtained by decomposing u(t) in its even and odd parts.

Some properties of Fourier Tranasform were explained.

If $x(t)\Leftrightarrow X(\omega)$ then

1. Time shifting property

$x(t-a)\Leftrightarrow e^{(-j\omega a)}X(\omega)$ shifting in time causes a modulation in frequency domain.

2. $x(t)e^{j\omega_c t}\Leftrightarrow X(\omega-\omega_c)$ modulation in time domain causes shifting in frequency domain.

3. $t\,x(t)\Leftrightarrow j\frac{dX}{d\omega}$

4. $x(at)\Leftrightarrow\frac{1}{|a|}X\left(\frac{\omega}{a}\right)$ Scaling in time domain also causes scaling (in opposite way) in frequency and also in magnitude.

5. $x(-t) \Leftrightarrow X(-\omega)$ reversal in time domain causes reversal in frequency domain as well.

6. Duality principle:

if $x(t) \Leftrightarrow X(\omega)$ then,

$$X(t) \Leftrightarrow 2\pi x(-\omega)$$

Students found out

$$F\{e^{-at}u(t)\} = \frac{1}{(j\omega + a)}$$

It was pointed out how to sketch $|X(\omega)|$ and corresponding phase angles.

Please bring out errors if any in the above equations to me.

Lectures 23 : on 30/08/2018 Thursday

Last lectures findings were emphasised. Two more important properties explained and proved.

7. Fourier transform of convolved signals in time time domain:

if $x_1(t) \Leftrightarrow X_1(\omega)$ and $x_2(t) \Leftrightarrow X_2(\omega)$

then $F\{x_1(t) * x_2(t)\} = X_1(\omega) X_2(\omega)$ convolution in time domain means multiplication in frequency domain. An imortant result.

8. Fourier transform of an Integrated signal

if $x(t) \Leftrightarrow X(\omega)$ then

$$F\{\int_{\tau=-\infty}^{t} x(\tau) d\tau\} = \frac{X(\omega)}{j\omega} + \pi X(\omega)\delta(\omega) = \frac{X(\omega)}{j\omega} + \pi X(0)\delta(\omega)$$

Energy of an aperiodic signal:

Recall that energy of a **periodic signal** was infinite (power was finite). For aperiodic real signal energy of the signal in time domain is:

$$W = \int_{t=-\infty}^{\infty} x^2(t) dt .$$

In general if energy W of an aperiodic signal is finite, then its Fourier Transform will exist. This is called Parseval's condition for existence of Fourier series.
If x(t) satisfies Perseval,s condition the integral

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt$$ will converge giving a Fourier transform of $x(t)$.

If $x(t) \Leftrightarrow X(\omega)$, then it was shown that energy can be calculated either in time domain or in frequency domain as follows:

$$W = \int_{t=-\infty}^{\infty} x^2(t) dt = \frac{1}{2\pi} \int_{\omega=-\infty}^{\infty} |X(\omega)|^2 dt$$

Application of Fourier Transform to find the response of an LTI system

An R , L series circuit (with initial condition relaxed) is taken to be a system and we want to find out the response $i(t)$ when the system is excited by a unit step voltage. We can get $i(t)$, by solving the differwential equation which describes the system.

The goal set here is to get i(t) by applying FT.

The steps involved will be as follows:

1. Write down the governing differential equation of the system.
2. Take Fourier transform of both the sides. Note that the differential equation in time domain gets transformed to algebraic equation in frequency domain.

3. Do necessary algebraic manipulation to get expression for $I(\omega)$.

4. Finally to get i(t), take **inverse Fourier transform** of $I(\omega)$, i.e.,

$$i(t)=F^{-1}\{I(\omega)\}$$

Please point out errors if any in the above equations to me.

Lecture 24 : on 3/09/2018 Monday

Application of FT in system (circuits):

The idea of applying FT to solve for current i(t) is explained for any input excitation $x(t)$ to an R-L series circuit.

The differential equation in time domain for the system considered is:

$$R i+L\frac{di}{dt}=x(t)$$

Taking FT of both sides gives:

$RI(\omega)+Lj\omega I(\omega)=X(\omega)$ Note that it is now algebraic equation.

Or, $I(\omega)=\frac{X(\omega)}{(R+j\omega L)}$ where $R+j\omega L$ may be called system function.

Case(i): Let $x(t)=\delta(t)$, i.e, we want to find out the impulse response.

Then, $I(\omega)=\frac{F\{\delta(t)\}}{(R+j\omega L)}=\frac{1}{(R+j\omega L)}$

or, $I(\omega)=\frac{1}{L}\frac{1}{(j\omega+\frac{R}{L})}$ Now recall, $F^{-1}\{\frac{1}{(j\omega+a)}\}=e^{-at}u(t)$

Therfore, $i(t)=\frac{1}{L}e^{-\frac{R}{L}t}u(t)$, Same result we got by solving differential equation in time domain.

Case(ii): Let $x(t)=1$
In this case,

$$I(\omega)=\frac{F\{1\}}{(R+j\omega L)}=\frac{2\pi\delta(\omega)}{(R+j\omega L)}=\frac{2\pi\delta(\omega)}{R}$$

Taking Fourier inverse of both sides:

$$i(t)=\frac{1}{R}$$

Point to be noted here is that $x(t)=1$, means input is 1 from $t=-\infty$ to $t=+\infty$. In other words any transient term present long back must have died down; giving you the steady state solution.

Case(iii): Let $x(t)=u(t)$ i.e.,, we want to get step response.

Since $x(t)=u(t)$, therefore

$$F\{u(t)\}=\frac{1}{j\omega}+\pi\delta(\omega)$$

So, $I(\omega)=\frac{\frac{1}{j\omega}+\pi\delta(\omega)}{(R+j\omega L)}=\frac{1}{j\omega(R+j\omega L)}+\frac{\pi\delta(\omega)}{(R+j\omega L)}$

or, $I(\omega)=\frac{1}{j\omega(R+j\omega L)}+\frac{\pi\delta(\omega)}{R}$

The first term on RHS can be written as sum of two terms (by partial fraction method), and we get:

$$I(\omega)=\frac{\frac{1}{R}}{j\omega}+\frac{-(\frac{1}{R})}{(j\omega+\frac{R}{L})}+\frac{\pi\delta(\omega)}{R}$$

Now we are in a position to take inverse transform and get,

$$i(t)=\frac{1}{R}u(t)-(\frac{1}{R})e^{-(\frac{R}{L})t}u(t)=\frac{1}{R}\{1-e^{-(\frac{R}{L})t}\}u(t)$$ Familiar result.

Case(iii): When excitation is sinusoidal:

Let $x(t)=\cos\omega_c t$. It is expected that Fourier transform of this signal can not be a continuous function of ω .
Since $x(t)$ has a single frequency ω_c , therefore we expect some imulse function at $+\omega_c$ and at $-\omega_c$

and $I(\omega)=\frac{X(\omega)}{(R+j\omega L)}$

How to get $X(\omega)$?

Now $x(t)=\cos\omega_c t=\frac{1}{2}e^{j\omega_c t}+\frac{1}{2}e^{-j\omega_c t}$

Therefore $X(\omega)=\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)$ As expected.

Now

$$Ri+L\frac{di}{dt}=x(t)=\cos\omega_c t=\frac{1}{2}e^{j\omega_c t}+\frac{1}{2}e^{-j\omega_c t}$$

Thus, $I(\omega)=\frac{\pi\delta(\omega-\omega_c)+\pi\delta(\omega+\omega_c)}{(R+j\omega L)}=\frac{\pi\delta(\omega-\omega_c)}{(R+j\omega L)}+\frac{\pi\delta(\omega+\omega_c)}{(R+j\omega L)}$

or, $I(\omega)=\frac{\pi\delta(\omega-\omega_c)}{(R+j\omega_c L)}+\frac{\pi\delta(\omega+\omega_c)}{(R-j\omega_c L)}$

Taking inverse transformation:

$$i(t)=\frac{1}{2(R+j\omega_c L)}e^{j\omega_c t}+\frac{1}{2(R-j\omega_c L)}e^{-j\omega_c t}$$

After simplification:

$i(t)=\frac{1}{(R^2+\omega_c^2L^2)}[R\cos\omega_c t+\omega_c L\sin\omega_c t]=\frac{1}{\sqrt{(R^2+\omega_c^2L^2)}}\cos(\omega_c t-\theta)$, the familiar result.
where,

$\theta=\tan^{-1}\frac{\omega_c L}{R}$. It may be noted that the solution obtained is steady state solution because input signal $\cos\omega_c t$, exists for $-\infty<t<+\infty$.

Students were asked to solve the same problem when $x(t)=\cos\omega_c t\,u(t)$. This time you will see that apart from the steady state term another transient term will appear in the solution.

Please point out errors if any in the above equations to me.

Lecture 25 & 26 : on 5/09/2018 Wednesday

Fourier Transform to Laplace Transform
We know that

$$F\{x(t)\}=X(\omega)=\int_{t=-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

For many function may not have FT when it is not a energy signal (or fails to satisfy Perseval'condition) as mentioned earlier. Now such a function $x(t)$ if multiplied by an exponentially decaying function $e^{-\sigma t}$ then the FT of $x(t)e^{-\sigma t}$ will exist. Where %sigma is a chosen suitable to make $x(t)e^{-\sigma t}$ exponentially decaying.

So,

$$F\{x(t)e^{-\sigma t}\}=\int_{t=-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt=\int_{t=-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

Defining $s=\sigma+j\omega$

$$F\{x(t)e^{-\sigma t}\}=\int_{t=-\infty}^{\infty} x(t)e^{-\sigma t}e^{-j\omega t}dt=\int_{t=-\infty}^{\infty} x(t)e^{-st}dt$$

Obviously, the integral on the RHS will be a function of $s=\sigma+j\omega t$. and we say,

$$\int_{t=-\infty}^{\infty} x(t)e^{-st}dt = X(s)$$ to be the Laplace transform of $x(t)$.

Note with $\sigma=0$, $X(s)=X(w)$

We shall here discuss about **one sided Laplace transform**. What is it?
We shall always put a restriction in $x(t)$ as follows: $x(t)=0$ for $t<0$ and $x(t)$ exist for $t>0$ or in compact form the signal is written as $x(t)u(t)$. This type of signal is called **causal signal**. So for a causal signal x(t), its Laplace transform will be:

$$X(s)=\int_{t=0}^{\infty} x(t)e^{-st}dt$$ and this is called one sided Laplace transform.

To accommodate delta (impulse function) $\delta(t)$, the lower limit should be changed to $0^{-¿}$ with out any loss of generality. So,

$$X(s)=\int_{t=0}^{\infty} x(t)e^{-st}dt = \int_{t=0^{-1}}^{\infty} x(t)e^{-st}dt$$

It may be noted that the value of σ (for one sided L.T) must a positive number and greater than some critical value of σ_c so that $x(t)e^{-\sigma t}$ becomes a decaying function and LT integral converges. Fortunately σ will be behind the seen, we need not bother about σ while finding LT of a given function. The condition $\sigma>\sigma_c$ is called the region of convergence (ROC).

The inverse formula : Knowing $X(s)$, how to get $x(t)$?

Recall, Fourier transform of $x(t)e^{-\sigma t}\}$ is given by:

$$F\{x(t)e^{-\sigma t}\} = \int_{t=-\infty}^{\infty} x(t)e^{-(\sigma+j\omega)t}dt$$

Therefore,

$$x(t)e^{-\sigma t}=\frac{1}{2\pi}\int_{t=-\infty}^{\infty} X(\sigma+j\omega t)e^{j\omega t}d\omega$$

Multiplying both sides by e^{%sigma t}

or, $$x(t)=\frac{1}{2\pi}\int_{t=-\infty}^{\infty} X(\sigma+j\omega t)e^{(\sigma+j\omega)t}d\omega$$

Now put, $\sigma+j\omega=s$

Thus, $j d\omega=ds$ and limits of integration will be $\sigma-j\infty$ to $\sigma+j\infty$

$$x(t)=\frac{1}{2\pi j}\int_{t=\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st}ds=L^{-1}\{X(s)\}$$ is Laplace inverse of X(s).

Although x(t) can be obtained by evaluating the above integral, general practise is to use Laplace transform table where

$x(t)\Leftrightarrow X(s)$ are available for most of the useful $x(t)$ used in linear time invariant system analysis. It was shown that:

1. $L\{\delta(t)\}=1$ which also means $L^{-1}\{1\}=\delta(t)$

2. $L\{u(t)\}=\frac{1}{s}$ and $L^{-1}\{\frac{1}{s}\}=u(t)$

3. $L\{e^{at}u(t)\}=\frac{1}{s-a}$ and $L^{-1}\frac{1}{s-a}=e^{at}u(t)$

4. $L\{\cos\omega t\, u(t)\}=\frac{s}{s^2+\omega^2}$ $L^{-1}\{\frac{s}{s^2+\omega^2}\}=\cos\omega t\, u(t)$

5. $L\{\sin\omega t\, u(t)\}=\frac{\omega}{s^2+\omega^2}$ $L^{-1}\{\frac{\omega}{s^2+\omega^2}\}=\sin\omega t\, u(t)$

There are many more.
Students are advised to consult some standard text book where where LT and the corresponding inverse are tabulated for more function. The above 5 functions are most commonly used in circuit analysis and may be memorised.

Check for any mistake in mathematical equations.

Lecture 27 : on 6/09/2018 Thursday

After reviewing last lectures, some of the important properties of LT explained.

1. If $x(t) \Leftrightarrow X(s)$

then $L\{x(t-a)\} = e^{-as} X(s)$

2. If $x(t) \Leftrightarrow X(s)$

then $L\{e^{-at} x(t)\} = X(s+a)$

3. If $x(t) \Leftrightarrow X(s)$

then $L\{t\,x(t)\} = \frac{-dX(s)}{ds}$

Derivative property:

If $x(t) \Leftrightarrow X(s)$

then $L\{\frac{dx}{dt}] = s\,X(s) - x(0)$

Also $L\{\frac{d^2x}{dt^2}] = s^2 X(s) - s\,x(0) - \dot{x}(0)$ where $\dot{x}$ means $\frac{dx}{dt}$

Also $L\{\frac{d^3x}{dt^3}] = s^3 X(s) - s^2 x(0) - s\,\dot{x}(0) - \ddot{x}(0)$ and so on.

If $x(0)$ and all its derivatives are 0, then

$L\{\frac{dx}{dt}] = s\,X(s)$, $L\{\frac{d^2x}{dt^2}] = s^2 X(s)$, $L\{\frac{d^3x}{dt^3}] = s^3 X(s)$ and so

Integration property:

If $x(t) \Leftrightarrow X(s)$

then $L\{\int_0^t x(\tau)d\tau\}=\frac{X(s)}{s}$

Application of LT in circuit (system) analysis.

1. A pure inductance L is connected across a voltage source $v(t)u(t)$. Let us assume $i(0)=0$ i.e., no initial current. We wish to find out current i(t). KVL in time domain is

$$L\frac{di}{dt}=v(t)u(t)$$

Going to s-domain by taking LT of both the sides:

$LsI(s)=V(s)$ This algebraic equation is KVL in s-domain.

Hence $I(s)=\frac{V(s)}{sL}$, therefore $i(t)=L^{-1}\{\frac{V(s)}{sL}\}$

In the s-domain the circuit can be redrawn with a $V(s)$ voltage and an imedance sL connected across the source. The conclusion is that an inductance in time domain can be replaced by sL in s-domain. But remember this can be done if and only if the initial current through the inductor is zero.

2. If the inductor had some initial current $i(0)$, then the inductor can be replaced by a parallel combination of sL and $i(0)/s$ in s-domain.

Therefore it suggests that we don’t have to even write the differential equation and we can straightway draw the circuit in s-domain and solve the circuit in s-domain (which will yield only algebraic equations (KVL/KCL etc.). Finally taking Laplace inverse of the required quantity, time domain solution will be obtained.

Check for any mistake in mathematical equations.