Differentiation under integral sign (Leibnitz Rule)

Let $$\Phi(\alpha) = \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$\Delta\Phi = \Phi(\alpha+\Delta\alpha) - \Phi(\alpha)$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha+\Delta\alpha)}^{u_1(\alpha)} f(x,\alpha+\Delta\alpha)\,dx + \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

$$+ \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx - \int_{u_1(\alpha)}^{u_2(\alpha)} f(x,\alpha)\,dx$$

$$= \int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx + \int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

$$- \int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx$$

Using mean value theorem:

$$\int_{u_1(\alpha)}^{u_2(\alpha)} \left[f(x,\alpha+\Delta\alpha) - f(x,\alpha)\right] dx = \Delta\alpha \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\xi_1)\,dx$$

$$\int_{u_2(\alpha)}^{u_2(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_2,\alpha+\Delta\alpha)\left[u_2(\alpha+\Delta\alpha) - u_2(\alpha)\right]$$

$$\int_{u_1(\alpha)}^{u_1(\alpha+\Delta\alpha)} f(x,\alpha+\Delta\alpha)\,dx = f(\xi_3,\alpha+\Delta\alpha)\left[u_1(\alpha+\Delta\alpha) - u_1(\alpha)\right]$$

where $\xi_1 \in (\alpha, \alpha+\Delta\alpha)$, $\xi_2 \in (u_2(\alpha), u_2(\alpha+\Delta\alpha))$, $\xi_3 \in (u_1(\alpha), u_1(\alpha+\Delta\alpha))$

Dividing by $\Delta\alpha$:

$$\frac{\Delta\phi}{\Delta\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x, \xi_1)\, dx + f(\xi_2, \alpha+\Delta\alpha)\frac{\Delta u_2}{\Delta\alpha} - f(\xi_3, \alpha+\Delta\alpha)\frac{\Delta u_1}{\Delta\alpha}$$

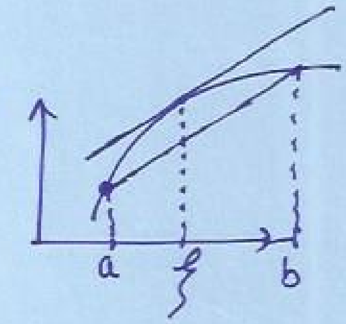
Taking the limit as $\Delta\alpha \to 0$,

$$\frac{d\phi}{d\alpha} = \int_{u_1(\alpha)}^{u_2(\alpha)} f_\alpha(x,\alpha)\, dx + f(u_2(\alpha), \alpha)\frac{du_2}{d\alpha} - f(u_1(\alpha),\alpha)\frac{du_1}{d\alpha}$$

Note: We have used the following mean value theorems in the above proof.

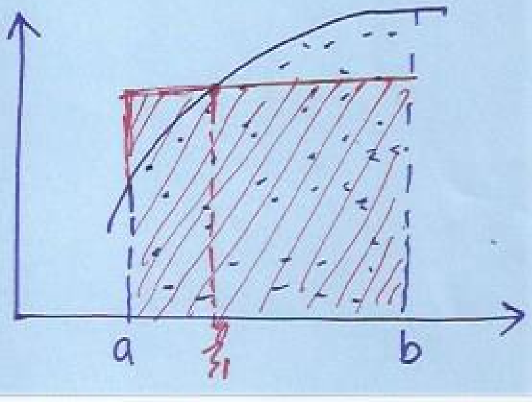
I. Lagrange mean value theorem:

$$\frac{f(b)-f(a)}{b-a} = f'(\xi) \; ; \quad \xi \in (a, b)$$



II. Mean value theorem of the integral calculus:

$$\int_a^b f(x)\, dx = (b-a) f(\xi_1) \; ; \quad \xi_1 \in (a,b)$$



A Particular Case: Assume that $u_1(\alpha)$ and $u_2(\alpha)$ are some constants. Then,

$$\frac{d\Phi(\alpha)}{d\alpha} = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx$$

OR
$$\frac{d}{d\alpha}\int_a^b f(x,\alpha)\,dx = \int_a^b \frac{\partial f}{\partial \alpha}(x,\alpha)\,dx.$$

Note: Leibnitz rule is not applicable, in general, in the case of improper integrals. In all examples given in this lecture we assume that differentiation under integral sign is valid.

Example: Show that

$$\int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx = \frac{\pi}{2}\ln(1+a) \text{ if } a \geq 0.$$

Let
$$\varphi(a) = \int_0^\infty \frac{\tan^{-1} ax}{x(1+x^2)}\,dx$$

$$\Rightarrow \varphi'(a) = \int_0^\infty \frac{1}{(1+x^2)(1+a^2x^2)}\,dx$$

$$= \int_0^\infty \frac{1}{(1-a^2)}\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right]dx$$

$$= \frac{1}{(1-a^2)}\left[\tan^{-1}x - a\tan^{-1}ax\right]_0^\infty = \frac{1}{(1-a^2)}\frac{\pi}{2}(1-a)$$

$$\Rightarrow \varphi'(a) = \frac{\pi}{2(1+a)}$$

Integrading

$$\varphi(a) = \frac{\pi}{2} \ln(1+a) + c$$

Not that $\varphi(0) = 0$

$$\Rightarrow 0 = \frac{\pi}{2} \ln(1) + c \Rightarrow c = 0$$

$$\Rightarrow \varphi(a) = \frac{\pi}{2} \ln(1+a)$$

Example: Prove $\int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\frac{\alpha^2}{4}}$

$$\varphi(\alpha) = \int_0^\infty e^{-x^2} \cos \alpha x \, dx$$

$$\varphi'(\alpha) = -\int_0^\infty e^{-x^2} \sin \alpha x \cdot x \, dx$$

Integrating right hand side by parts

$$\varphi'(\alpha) = \frac{e^{-x^2}}{2} \sin \alpha x \Big|_0^\infty + \int_0^\infty \left(-\frac{e^{-x^2}}{2}\right) \cos \alpha x \cdot \alpha \cdot dx$$

$$= -\frac{\alpha}{2} \varphi(\alpha)$$

$$\Rightarrow \frac{\varphi'(\alpha)}{\varphi(\alpha)} = -\frac{\alpha}{2} \Rightarrow \ln \varphi(\alpha) = -\frac{\alpha^2}{4} + c$$

$$\Rightarrow \varphi(\alpha) = c_1 e^{-\alpha^2/4}$$

Note that $\varphi(0) = \int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$

$$\Rightarrow \sqrt{\pi}/2 = c_1$$

$$\Rightarrow \int_0^\infty e^{-x^2} \cos \alpha x \, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2/4}.$$

Example: Starting with a suitable integral, show that

$$\int_0^x \frac{dx}{(x^2+a^2)^2} = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2(x^2+a^2)}$$

Solution: Consider $\varphi(a,x) = \int_0^x \frac{dx}{(x^2+a^2)} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\Big|_0^x$

$$= \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

Diff. w.r.t a:

$$\frac{\partial \varphi}{\partial a} = \int_0^x -\frac{1}{(x^2+a^2)^2}\cdot 2a\, dx = \frac{1}{a}\frac{1}{\left(1+\frac{x^2}{a^2}\right)}\left(-\frac{x}{a^2}\right) - \frac{1}{a^2}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\Rightarrow \int_0^x \frac{1}{(x^2+a^2)^2}\,dx = \frac{1}{2a^3}\tan^{-1}\left(\frac{x}{a}\right) + \frac{x}{2a^2}\frac{1}{(x^2+a^2)}$$

Example: Let $\varphi(\alpha) = \int_\alpha^{\alpha^2} \frac{\sin \alpha x}{x}\,dx$. Find $\varphi'(\alpha)$ where $\alpha \neq 0$.

$$\varphi'(\alpha) = \int_\alpha^{\alpha^2} \frac{\cos \alpha x}{x}\cdot x\,dx + 2\alpha\cdot\frac{\sin\alpha^3}{\alpha^2} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{\sin\alpha x}{\alpha}\Big|_\alpha^{\alpha^2} + \frac{2\sin\alpha^3}{\alpha} - \frac{\sin\alpha^2}{\alpha}$$

$$= \frac{3\sin\alpha^3 - 2\sin\alpha^2}{\alpha}.$$