Beta & Gamma Function

Definition:

Beta function:

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \quad , \quad m>0,\ n>0$$

Gamma function:

$$\Gamma n = \int_0^\infty e^{-x} x^{n-1}\,dx \quad , \quad n>0.$$

Convergence of Beta function:

Case-I: $m, n \geq 1$, the integral is proper. Hence it is convergent.

Case-II: $m, n < 1$.

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \underbrace{\int_0^c x^{m-1}(1-x)^{n-1}\,dx}_{I_1} + \underbrace{\int_c^1 x^{m-1}(1-x)^{n-1}\,dx}_{I_2}$$

where $0<c<1$.

Consider

$$I_1 = \int_0^c x^{m-1}(1-x)^{n-1}\,dx$$

Then.

$$\lim_{x\to 0+} x^{\mu}\, x^{m-1}(1-x)^{n-1} = \lim_{x\to 0} x^{\mu+m-1}(1-x)^{n-1}$$

$$= 1 \quad \text{if} \quad \mu+m-1=0$$

$$\Rightarrow \mu = -m+1.$$

If $0<m<1$, then $0<\mu<1$ and hence the integral converges.

If $m\leq 0$ then $\mu \geq 1$ and hence the integral diverges.

Similarly: consider $I_2 = \int_c^1 x^{m-1} (1-x)^{n-1} dx$

$$\lim_{x \to 1} (1-x)^{\mu} \cdot x^{m-1} (1-x)^{n-1} = \lim_{x \to 1} x^{m-1} (1-x)^{\mu+n-1}$$

If $0<n<1$, the integral converges

If $n \leq 0$, the integral diverges.

Therefore

$\int_0^1 x^{m-1} (1-x)^{n-1} dx$ converges if both m & $n > 0$.

otherwise it is divergent.

Convergence of Gamma function:

$$\Gamma n = \int_0^\infty x^{n-1} e^{-x} dx$$

Case I: $n \geq 1$

The integrand is bounded in $0<x \leq a$, where a is arbitrary.

We check convergence of $\int_a^\infty x^{n-1} e^{-x} dx$

Consider $\lim_{x \to \infty} x^{\mu} f(x) = \lim_{x \to \infty} \frac{x^{\mu} \cdot x^{n-1}}{e^{x}}$

$= 0$ for all values of μ and n.

Using μ test ($\mu > 1$), the integral $\int_a^\infty x^{n-1} e^{-x} dx$ is convergent for all values of n.

Case II: Let $0<n<1$: Then

$$\int_0^{\infty} e^{-x} x^{n-1}\, dx = \int_0^{a} e^{-x}.x^{n-1}\, dx + \underbrace{\int_a^{\infty} e^{-x} x^{n-1}\, dx}_{\text{converges (see above)}}$$

Note that $\lim\limits_{x \to 0} x^{\mu} x^{n-1} e^{-x} = 1$ if $\mu+n-1=0$, i.e.,

if $\mu = 1-n$

Since n lies between 0 & 1, μ also lies between 0 & 1.

Hence $\int_0^{a} e^{-x} x^{n-1}\, dx$ is convergent.

Therefore the integral converges for $0<n<1$.

Case III Let $n \leq 0$.

$\lim\limits_{x \to 0} x^{\mu} x^{n-1} e^{-x}$ Take $\mu = 1$:

$$\lim_{x \to 0} x^{n} e^{-x} = \begin{cases} 1, & n=0 \\ \infty & n<0 \end{cases}$$

$$\Rightarrow \int_0^{a} e^{-x} x^{n-1}\, dx \text{ diverges.}$$

PROPERTIES OF BETA and GAMMA function:

a) $B(m,n) = B(n,m)$

Subst. $1-x=y$ . .

b) Evaluation of $B(m,n)$

Suppose n is a positive integer.

$$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

Integrating by parts keeping $(1-x)^{n-1}$ as first function.

$$B(m,n) = \left[\frac{x^m}{m}(1-x)^{n-1}\right]_0^1 + \int_0^1 \frac{x^m}{m}(n-1)(1-x)^{n-2}\,dx$$

$$= \frac{(n-1)}{m}\int_0^1 x^m (1-x)^{n-2}\,dx$$

$$\vdots$$

$$= \frac{(n-1)(n-2)\cdots 1^{=(n-(n-1))}}{m(m+1)\cdots(m+n-2)}\int_0^1 x^{m+n-2}\,dx$$

$$= \frac{\underline{|(n-1)}}{m(m+1)\cdots(m+n-2)(m+n-1)}$$

If m is a positive integer

$$B(m,n) = \frac{\underline{|(m-1)}}{n(n+1)\cdots(n+m-1)}$$

If both m and n are integer

$$B(m,n) = \frac{\underline{|(n-1)}\;\underline{|(m-1)}}{\underline{|m+n-1}}$$

c) Evaluation of Gamma function:

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\, dx$$

integrating by parts:

$$= -x^n e^{-x}\Big|_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\, dx$$

$$\boxed{\Gamma(n+1) = n\,\Gamma(n)}$$

Note that if n is an integer

$$\Gamma(n) = (n-1)(n-2)\cdots 3\cdot 2\cdot \Gamma(1)$$

where $\Gamma(1) = \int_0^\infty e^{-x}\, dx = -e^{-x}\Big|_0^\infty = 1.$

$\Rightarrow$ $\boxed{\Gamma(n) = \lfloor(n-1), \text{ if } n \text{ is a positive int.}}$

d) $\Gamma(\tfrac{1}{2}) = \sqrt{\pi}$

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x}\, dx$$

Subst. $x = y^2 \Rightarrow dx = 2y\, dy$

$$\Rightarrow \Gamma(n) = \int_0^\infty y^{2n-1} e^{-y^2}\, 2\, dy$$

Set $n = \frac{1}{2} \Rightarrow \Gamma(\tfrac{1}{2}) = 2\int_0^\infty y^0 e^{-y^2}\, dy$

$$= 2\int_0^\infty e^{-y^2}\, dy = 2\cdot\frac{\sqrt{\pi}}{2} = \sqrt{\pi}$$

$\Rightarrow$ $\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$

Different forms of Γn :

a) $\Gamma n = \int_0^\infty e^{-x} x^{n-1}\, dx$

Subst. $x = \lambda y \Rightarrow dx = \lambda dy$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-\lambda y} . \lambda^{n-1} y^{n-1} \lambda\, dy$$

$$\Rightarrow \boxed{\int_0^\infty e^{-\lambda y} y^{n-1}\, dy = \frac{\Gamma n}{\lambda^n}}$$

b) Subst. $x^n = z \Rightarrow n x^{n-1} dx = dz$

$$\Rightarrow \Gamma n = \int_0^\infty e^{-z^{1/n}} \frac{1}{n} dz \Rightarrow \boxed{\int_0^\infty e^{-z^{1/n}} dz = n\Gamma n = \Gamma n+1}$$

c) Subst $e^{-x} = t \Rightarrow -e^{-x} dx = dt$

$$\Rightarrow \Gamma n = -\int_1^0 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt$$

$$\Rightarrow \int_0^1 \left[\ln\left(\frac{1}{t}\right)\right]^{n-1} dt = \Gamma n$$

Different forms of Beta function:

a) $B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$

Subst. $x = \frac{1}{1+y} \Rightarrow dx = -\frac{1}{(1+y)^2} dy$

$$\boxed{B(m,n) = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}} dy = \int_0^\infty \frac{y^{m-1}}{(1+y)^{m+n}} dy}$$

b) $x = \sin^2\theta \Rightarrow dx = 2\sin\theta\cos\theta\, d\theta$

$$B(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2m-1}\theta\, d\theta$$

Relation between Gamma & Beta function:

We know for m and n being integers

$$B(m,n) = \frac{\lfloor m-1 \, \lfloor n-1}{\lfloor m+n-1}$$

$$\Rightarrow \boxed{B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}}$$

This result also holds for $m, n > 0$ (not necessary only for integers)

Some other deductions:

1. $\boxed{\Gamma n\, \Gamma 1-n = \frac{\pi}{\sin n\pi}}$ $0 < n < 1$.

We know $B(m,n) = \frac{\Gamma m\, \Gamma n}{\Gamma m+n}$, putting $m = 1-n$

$$\Rightarrow B(1-n, n) = \Gamma 1-n\, \Gamma n = \int_0^\infty \frac{y^{n-1}}{(1+y)}\, dy = \frac{\pi}{\sin n\pi} \quad \text{if } 0<n<1$$

(Complicated, Residue th. comp. ana.)

2. $\boxed{\Gamma n+1\, \Gamma 1-n = \frac{n\pi}{\sin n\pi}}$

3. Put $n = 1/2$ in 1. $\Rightarrow \Gamma\frac{1}{2}\, \Gamma\frac{1}{2} = \pi$

$$\Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

4. We know

$$B(m,n) = 2\int_0^{\pi/2} \cos^{2m-1}\theta \, \sin^{2n-1}\theta \, d\theta = \frac{\Gamma m \, \Gamma n}{\Gamma m+n}$$

let $2m-1 = p$ & $2n-1 = q$

$$\Rightarrow \boxed{\int_0^{\pi/2} \cos^p\theta \, \sin^q\theta \, d\theta = \frac{\Gamma\frac{(p+1)}{2} \, \Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}}$$

let $p = 0$ then

$$\boxed{\int_0^{\pi/2} \sin^q\theta \, d\theta = \frac{\Gamma\frac{q+1}{2}}{\Gamma\frac{q+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

let $q = 0$ then

$$\boxed{\int_0^{\pi/2} \cos^p\theta \, d\theta = \frac{\Gamma\frac{p+1}{2}}{\Gamma\frac{p+2}{2}} \cdot \frac{\sqrt{\pi}}{2}}$$

let $p = 0$, $q = 0$ then

$$\frac{\pi}{2} = \frac{1}{2}\left(\Gamma\frac{1}{2}\right)^2 \Rightarrow \boxed{\Gamma\frac{1}{2} = \sqrt{\pi}}$$

5. $B(m,n) = B(m+1,n) + B(m,n+1)$

$$R.H.S = \frac{\Gamma m+1 \, \Gamma n}{\Gamma m+n+1} + \frac{\Gamma m \, \Gamma n+1}{\Gamma m+n+1}$$

$$= \frac{\Gamma m \, \Gamma n}{\Gamma m+n+1}(m+n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} = B(m,n).$$

Example - 1: Evaluate

$$\int_0^1 x^4 (1-\sqrt{x})^5 \, dx$$

Let $\sqrt{x} = t$ or $x = t^2 \Rightarrow dx = 2t \, dt$

$$\int_0^1 t^8 (1-t)^5 \, 2t \, dt$$

$$= 2\int_0^1 t^9 (1-t)^5 \, dt$$

$$= 2 \cdot B(10,6) = 2 \cdot \frac{\Gamma 10 \, \Gamma 6}{\Gamma 16} = 2 \cdot \frac{\lfloor 9 \, \lfloor 5}{\lfloor 15} = \frac{1}{15015}$$

Example 2. Show that $\int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \frac{\pi}{\sqrt{2}}$

$$I = \int_0^{\pi/2} (\cot\theta)^{1/2} d\theta = \int_0^{\pi/2} \cos^{1/2}\theta \, \sin^{-1/2}\theta \, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right) \Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)}{2}$$

$$= \frac{1}{2} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1-\frac{1}{4}\right)$$

$$= \frac{1}{2} \cdot \frac{\pi}{\sin(\pi/4)}$$

$$= \frac{\pi}{\sqrt{2}}.$$