Tutorial Sheet-1 **AUTUMN 2017**

MATHEMATICS-I (MA10001) *August 1, 2017*

1. Determine the values of c which satisfy the Rolle's theorem for the function $f(x) = x^3 - 3x^2 - x + 3$ on $[-1, 3]$.

2. Without finding the derivative of the function $f(x)$, prove that all roots of the given function $f(x) = (x+1)(x-1)(x-2)(x-3)$ are real. can we generalize this result in case of a polynomial of degree n with n distinct real roots.

3. Prove that $f(x) = x^3 - 7x^2 + 25x + 8$ has exactly one real root.

4. If $f(x) = (x - a)^m (x - b)^n$, where $m, n \in \mathbb{N}$. Use Rolle's theorem to show that the point where $f'(x)$ vanishes divides the line segment $a \leq x \leq b$ in the ratio $m : n$.

5. Prove that if $f(x)$ is any polynomial over $\mathbb{R}$ and $f'(x)$ is the derivative of $f(x)$ then between any two consecutive roots of f' there lies atmost one root of f.

6. Use Rolle's theorem to prove the following:

   i. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function on $[0, 1]$ satisfying the condition $\int_0^1 f(x)dx = 0$. Then there exists $c \in (0, 1)$ such that

   $$f(c) = \int_0^c f(x)dx.$$

   ii. Let $f : [a, b] \to \mathbb{R}$ be a continuous function on $[a, b]$ and $f''(x)$ exists for all $x \in (a, b)$. Let $a < c < b$, then there exists a point ξ in (a, b) such that

   $$f(c) = \frac{b-c}{b-a} f(a) + \frac{c-a}{b-a} f(b) + \frac{1}{2}(c-a)(c-b)f''(\xi).$$

7. Let α, β, γ be three real numbers such that $\alpha + \beta + \gamma = 0$. If f_1, f_2 and f_3 are three continuous functions on $[a, b]$ and differentiable on (a, b) such that $f_i(a) \neq f_i(b)$ for $i = 1, 2, 3$, then there exists a point c in (a, b) such that

   $$\frac{\gamma}{f_1(b) - f_1(a)} f_1'(c) + \frac{\alpha}{f_2(b) - f_2(a)} f_2'(c) + \frac{\beta}{f_3(b) - f_3(a)} f_3'(c) = 0.$$

8. If $p(x)$ is a polynomial and $\alpha \in \mathbb{R}$, prove that between any two real roots of $p(x) = 0$ there is a root of $p'(x) + \alpha p(x) = 0$.

9. Prove that the equation $x^4 - 4x - 1 = 0$ has exactly two real roots.

10. i. Suppose that $f(0) = -3$ and $f'(x) \leq 5$ for all x. Use Lagrange's mean value theorem to find the largest possible value of $f(2)$.

    ii. Use Lagrange's mean value theorem to estimate $\sqrt[3]{28}$.

11. Let $a \in \mathbb{R}$. Prove that if f and g are differentiable functions with $f'(x) \leq g'(x)$ for every x in some interval containing a and if $f(a) = g(a)$, then $f(x) \leq g(x)$ for every $x \geq a$.

12. Apply Lagrange's mean value theorem to show that

    i. $\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$.

    ii. $3\cos^{-1}(x) - \cos^{-1}(3x - 4x^3) = \pi$ for all $x \in (-\frac{1}{2}, \frac{1}{2})$.

13. Suppose f be continuous on [0,2] and differentiable on (0,2) and satisfies $f(0) = 0$, $f(2) = 2$. Prove that there exist $c \in$ (0,2) such that $f'(c) = \frac{1}{f(c)}$.

14. If $f(x)$ and $\phi(x)$ are continuous on $[a,b]$ and differentiable on (a,b), then show that

$$\begin{vmatrix} f(a) & f(b) \\ \phi(a) & \phi(b) \end{vmatrix} = (b-a) \begin{vmatrix} f(b) & f'(c) \\ \phi(b) & \phi'(c) \end{vmatrix}, a < c < b.$$

15. Use Lagrange's mean value theorem to prove Bernoulli's inequality: for all $x > 0$ and for all $n \in \mathbb{N}$, $(1+x)^n > 1 + nx$.

16. i. $f : [0,1] \to \mathbb{R}$ be continuous function on [0,1], differentiable on (0,1) and such that $f'(x) - f(x) \geq 0$ for all $x \in [0,1]$ and $f(0) = 0$. Prove that $f(x) \geq 0$ for all $x \in [0,1]$.

    ii. Let f be continuous on $[a,b]$ and differentiable on (a,b) and $f(x) \neq 0$ on (a,b). Prove that there exist $c \in$ (a,b) such that

$$\frac{f'(c)}{f(c)} = \frac{1}{a-c} + \frac{1}{b-c}.$$

17. Prove that

    i. $\frac{2x}{\pi} < \sin\ x < x$ for $0 < x < \frac{\pi}{2}$.

    ii. $x < \tan\ x < \frac{4x}{\pi}$ for $0 < x < \frac{\pi}{4}$.

    iii. $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$ for $0 < x < 1$.

    iv. $\frac{x}{1+x} < \log(1+x) < x$ for all $x > 0$. Hence deduce that

$$\log\ \frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log\ 2,$$

    n being a positive integer.

18. Show that the function $f(x) = x^n + px + q$ with $p > 0$, cannot have more than two real roots for n is even and more than three for n is odd.

19. i. Let f be continuous on $[a,b]$, $a > 0$ and differentiable on (a,b). Prove that there exists $c \in (a,b)$ such that

$$\frac{bf(a) - af(b)}{b-a} = f(c) - cf'(c).$$

    ii. If f is differentiable on [0,1], show by Cauchy's mean value theorem that the equation $f(1) - f(0) = \frac{f'(x)}{2x}$ has at least one solution in $(0,1)$.

iii. Let f be continuous on $[a,b]$ and differentiable on (a,b). Using Cauchy's Mean value theorem show that if $a \geq 0$ then there exist $x_1, x_2, x_3 \in (a,b)$ such that

$$f'(x_1) = (b+a)\frac{f'(x_2)}{2x_2} = (b^2+ba+a^2)\frac{f'(x_3)}{3x_3^2}\,.$$

20. Using Cauchy's Mean value theorem show that $1 - \frac{x^2}{2} < \cos x$ for $x \neq 0$.