CS40032: Principles of Programming Languages
Module 03: λ-Calculus: Semantics

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Semantics of λ-Expressions

Source:

λ- Calculus Overview

Operational Semantics of Pure Functional Languages

Free and Bound Variable

- An occurrence of a variable x is said to be *bound* when it occurs in the body M of an abstraction $\lambda x.M$
- We say that λx is a *binder* whose scope is M
- An occurrence of x is *free* if it appears in a position where it is *not bound* by an enclosing abstraction on x
- For example,
  - Occurrences of x in xy and $\lambda y.xy$ are *free*
  - Occurrences of x in $\lambda x.x$ and $\lambda z.\lambda x.\lambda y.x(yz)$ are *bound*
  - In $(\lambda x.x)x$ the first occurrence of x is *bound* and the second is *free*
- In a loose parallel to C functions, consider the *bound* variables as *local* (including *parameters*) and *free* variables as *global* or *non-local*

Free and Bound Variable

- In an abstraction, the variable named is referred to as the **bound** variable and the associated λ-expression is the **body** of the abstraction
- In an expression of the form:

$$\lambda v.\ e$$

occurrences of variable v in expression e are **bound**

- All occurrences of other variables are **free**
- Example:

$$((\lambda x.\ \lambda y.\ (xy))(yw))$$

  - x, and y are **bound** in first part
  - y, and w are **free** in second part

Free and Bound Variable: Other Contexts

- $\int_0^1 x^2 dx$; $\int_0^1 A * x^2 dx$
- $\sum_{x=1}^{10} \frac{1}{x}$; $\sum_{x=1}^{10} K * \frac{1}{x}$
- $\lim_{x\to\infty} e^{-x}$; $\lim_{x\to\infty}(M + e^{-x})$
- `int succ(int x) { return x + 1; }`
- $\forall x \in \mathbb{R}, x > 1 \Rightarrow \frac{1}{x} < 1$

Free and Bound Variable

- *Definition*: An occurrence of a variable v in a λ-expression is called **bound** if it is within the scope of a λv; otherwise it is called **free**
  - A variable may occur both bound and free in the same λ-expression – for example, in $\lambda x.\ y\ \lambda y.\ y\ x$ the first occurrence of y is free and the other two are bound

Set of Free Variables

- *Definition*: The **set of free variables** in an expression E, denoted by $FV(E)$, is defined as follows:
  a. $FV(c) = \Phi$ for any constant c
  b. $FV(x) = \{x\}$ for any variable x
  c. $FV(E1\ E2) = FV(E1) \cup FV(E2)$
  d. $FV(\lambda x.\ E) = FV(E) - - \{x\}$
- A λ-expression E with no free variables ($FV(E) = \Phi$) is called **closed**

Substitution

- The notation $E[v \rightarrow E1]$ refers to the λ-expression obtained by replacing each free occurrence of the variable v in E by the λ-expression $E1$
- Naive Rules of Substitution:
  a. $v[v \rightarrow E_1] = E_1$ for any variable v
  b. $x[v \rightarrow E_1] = x$ for any variable $x \neq v$
  c. $(\lambda v.\ E)[v \rightarrow E_1] = \lambda v.\ (E[v \rightarrow E_1])$
  d. $(E_{rator}\ \ E_{rand})[v \rightarrow E_1] = ((E_{rator}[v \rightarrow E_1])(E_{rand}[v \rightarrow E_1]))$
- Does it work?

$$(\lambda y.x)[x \rightarrow (\lambda z.zw)] = \lambda y.\lambda z.zw$$

YES!

Unsafe Substitution: Example

- Consider:

$$(\lambda x.\ x)[x \to y] = \lambda x.\ (x[x \to y]) = \lambda x.\ y$$

  conflicts with a basic understanding that the names of bound variables (that is, parameters) do not matter.
- The identity function is the same whether we write it as $\lambda x.x$ or $\lambda z.z$ or $\lambda fred.fred$.
- If these do not behave the same way under substitution they would not behave the same way under evaluation and that seems wrong
- The mistake is that the substitution should only apply to **free** variables and **not bound** ones
- Here x is bound in the term so we should not substitute it
- That seems to give us what we want:

$$(\lambda x.x)[x \to y] = \lambda x.x$$

Unsafe Substitution: Example

- Again, the naive substitution

  $$(\lambda x.\ (mul\ y\ x))[y \to x] \Rightarrow (\lambda x.\ (mul\ x\ x))$$

  is unsafe since the result represents a squaring operation whereas the original lambda expression does not
  - A substitution is **valid** or **safe** if no free variable in $E1$ becomes bound as a result of the substitution $E[v \to E1]$
  - An invalid substitution involves a **variable capture** or **name clash**
- Correct way would be:

  $$(\lambda x.\ (mul\ y\ x))[y \to x] \Rightarrow (\lambda z.\ (mul\ y\ z))[y \to x]$$

  $$(\lambda z.\ (mul\ y\ z))[y \to x] \Rightarrow (\lambda z.\ (mul\ x\ z))$$
- Unsafe substitutions change in semantics!

Substitution

- *Definition*: The **substitution** of an expression for a (*free*) variable in a λ-expression is denoted by $E[v \rightarrow E_1]$ and is defined as follows:
  a. $v[v \rightarrow E_1] = E_1$ for any variable v
  b. $x[v \rightarrow E_1] = x$ for any variable $x \neq v$
  c. $c[v \rightarrow E_1] = c$ for any constant c
  d. $(E_{rator}\ E_{rand})[v \rightarrow E_1] = ((E_{rator}[v \rightarrow E_1])(E_{rand}[v \rightarrow E_1]))$
  e. $(\lambda v.\ E)[v \rightarrow E_1] = (\lambda v.\ E)$ // v is not free in E
  f. $(\lambda x.\ E)[v \rightarrow E_1] = \lambda x.\ (E[v \rightarrow E_1])$ when $x \neq v$ and $x \notin FV(E_1)$
  g. $(\lambda x.\ E)[v \rightarrow E_1] = \lambda z.\ (E[x \rightarrow z][v \rightarrow E_1])$ when $x \neq v$ and $x \in FV(E_1)$, where $z \neq v$ and $z \notin FV(E\ E_1)$
- In part (g), the first substitution $E[x \rightarrow z]$ replaces the bound variable x that will capture the free x's in E_1 by an entirely new bound variable z. Then the intended substitution can be performed safely.

Substitution Example

$(\lambda y.\ (\lambda f.\ f\ x)\ y)\ [x \to f\ y]$	$\Rightarrow_\alpha$	
$\lambda z.\ ((\lambda f.\ f\ x)\ z)\ [x \to f\ y]$	$\Rightarrow$	by g) since $y \in FV(f\ y)$
$\lambda z.\ ((\lambda f.\ f\ x)\ [x \to f\ y]\ z[x \to f\ y])$	$\Rightarrow$	by d)
$\lambda z.\ ((\lambda f.\ f\ x)\ [x \to f\ y]\ z)$	$\Rightarrow$	by b)
$\lambda z.\ (\lambda g.\ (g\ x)\ [x \to f\ y])\ z$	$\Rightarrow$	by g) since $f \in FV(f\ y)$
$\lambda z.\ (\lambda g.\ g\ (f\ y))\ z$	$\Rightarrow$	by d), b), and a)

Rules

a. $v[v \to E_1] = E_1$ for any variable v
b. $x[v \to E_1] = x$ for any variable $x \neq v$
c. $c[v \to E_1] = c$ for any constant c
d. $(E_{rator}\ E_{rand})[v \to E_1] = ((E_{rator}[v \to E_1])(E_{rand}[v \to E_1]))$
e. $(\lambda v.\ E)[v \to E_1] = (\lambda v.\ E)$
f. $(\lambda x.\ E)[v \to E_1] = \lambda x.\ (E[v \to E_1])$ when $x \neq v$ and $x \notin FV(E_1)$
g. $(\lambda x.\ E)[v \to E_1] = \lambda z.\ (E[x \to z][v \to E_1])$ when $x \neq v$ and $x \in FV(E_1)$, where $z \neq v$ and $z \notin FV(E\ E_1)$

Reduction

- A λ-expression has as its meaning the λ-expression that results after all its function applications (combinations) are carried out
- Evaluating a λ-expression is called **reduction**
- Four rules of reduction
  - α-Reduction: Renaming rule
  - β-Reduction: Substitution rule
  - η-Reduction: Function Equality rule
  - δ-Reduction: Pre-defined Constants' rule

α-Reduction

- *Definition*: If v and w are variables and E is a λ-expression,

$$\lambda v.\ E \Rightarrow_\alpha \lambda w.\ E[v \to w]$$

provided that w does not occur at all in E, which makes the substitution $E[v \to w]$ safe

- The equivalence of expressions under α-reduction is what makes part g) of the definition of substitution correct
- The α-reduction rule simply allows the changing of bound variables as long as there is no capture of a free variable occurrence
- The two sides of the rule can be thought of as variants of each other, both members of an equivalence class of *congruent* λ-expressions

α-Reduction

- The last example contains two α-reductions:

$$\lambda y.\ (\lambda f.\ f\ x)\ y \Rightarrow_\alpha \lambda y.\ ((\lambda f.\ f\ x)\ y)[y \to z] \Rightarrow_\alpha \lambda z.\ (\lambda f.\ f\ x)\ z$$

$$\lambda z.\ (\lambda f.\ f\ x)\ z \Rightarrow_\alpha \lambda z.\ ((\lambda f.\ f\ x)\ z)[f \to g] \Rightarrow_\alpha \lambda z.\ (\lambda g.\ g\ x)\ z$$

- Now that we have a justification of the substitution mechanism, the main simplification rule can be formally defined

β-Reduction

- *Definition*: If v is a variable and E and E_1 are λ-expressions,

$$(\lambda v.\ E)\ E_1 \Rightarrow_\beta E[v \to E_1]$$

provided that the substitution $E[v \to E_1]$ is carried out according to the rules for a safe substitution

- This β-reduction rule describes the function application rule in which the actual parameter or argument E_1 is *passed to* the function $(\lambda v.\ E)$ by substituting the argument for the formal parameter v in the function

β-Reduction

- *Definition*: If v is a variable and E and E_1 are λ-expressions,

$$(\lambda v.\ E)\ E_1 \Rightarrow_\beta E[v \rightarrow E_1]$$

  provided that the substitution $E[v \rightarrow E_1]$ is carried out according to the rules for a safe substitution
- The left side $(\lambda v.\ E)\ E_1$ of a β-reduction is called a β-**redex** – derived from *reduction expression* – and meaning an expression that can be β-reduced
- β-reduction is the main rule of evaluation in the λ-calculus
- α-reduction makes the substitutions for variables valid

β-Reduction

- The evaluation of a λ-expression consists of a series of β-reductions, possibly interspersed with α-reductions to change bound variables to avoid confusion
- Take $E \Rightarrow F$ to mean $E \Rightarrow_\beta F$ or $E \Rightarrow_\alpha F$ and let $\Rightarrow^*$ be the reflexive and transitive closure of $\Rightarrow$
- Hence:
  - For any expression E, $E \Rightarrow^* E$ and
  - For any three expressions, $(E_1 \Rightarrow^* E_2$ and $E_2 \Rightarrow^* E_3)$ implies $E_1 \Rightarrow^* E_3$
- The goal of evaluation in the λ-calculus is to reduce a λ-expression via $\Rightarrow$ until it contains no more β-redexes
- To define an *equality* relation on λ-expressions, we also allow a β-reduction rule to work backward

β-Abstraction

- *Definition*: Reversing β-reduction produces the β-**abstraction** rule,

$$E[v \to E_1] \Rightarrow_\beta (\lambda v.\ E)\ E_1$$

  and the two rules taken together give β-**conversion**, denoted by $\Leftrightarrow_\beta$
- Therefore $E \Leftrightarrow_\beta F$ if $E \Rightarrow_\beta F$ or $F \Rightarrow_\beta E$
- Take $E \Leftrightarrow F$ to mean $E \Leftrightarrow_\beta F$, $E \Rightarrow_\alpha F$ or $F \Rightarrow_\alpha E$ and let $\Leftrightarrow^*$ be the reflexive and transitive closure of $\Leftrightarrow$
- Two λ-expressions E and F are **equivalent** or **equal** if $E \Leftrightarrow^* F$

β-Abstraction

- Reductions (both α and β) are allowed to sub-expressions in a λ-expression by three rules:
  1. $E_1 \Rightarrow E_2$ implies $E_1\ E \Rightarrow E_2\ E$
  2. $E_1 \Rightarrow E_2$ implies $E\ E_1 \Rightarrow E\ E_2$
  3. $E_1 \Rightarrow E_2$ implies $\lambda x.\ E_1 \Rightarrow \lambda x.\ E_2$

η-Reduction

- *Definition*: If v is a variable and E is a λ-expression (denoting a function), and v has no free occurrence in E,

$$\lambda v.\ (E\ v) \Rightarrow_\eta E$$

- Example:

$$\lambda x.\ (sqr\ x) \Rightarrow_\eta sqr$$

$$\lambda x.\ (add\ 5\ x) \Rightarrow_\eta (add\ 5)$$

  Note: $(add\ 5\ x)$ abbreviates $(add\ 5)$

- Take $E \Leftrightarrow_\eta F$ to mean $E \Rightarrow_\eta F$ or $F \Rightarrow_\eta E$

η-Reduction

- The requirement that x should have no free occurrences in E is necessary to avoid an invalid reduction such as

$$\lambda x.\ (add\ x\ x) \Rightarrow (add\ x)$$

- This rule fails when E represents some constants; for example, if 5 is a predefined constant numeral, $\lambda x.\ (5\ x)$ and 5 are not equivalent or even related
- η-reduction, justifies an extensional view of functions; that is, two functions are equal if they produce the same values when given the same arguments

$$\forall x, f(x) = g(x) \Rightarrow f = g$$

η-Reduction

- **Extensionality Theorem**: If $F_1\ x \Rightarrow^* E$ and $F_2\ x \Rightarrow^* E$ where $x \notin FV(F_1\ F_2)$, then $F_1 \Leftrightarrow^* F_2$ where $\Leftrightarrow^*$ includes η-reductions.
  Proof:

$$F_1 \Leftrightarrow_\eta \lambda x.\ (F_1\ x) \Leftrightarrow_\eta \lambda x.\ E \Leftrightarrow_\eta \lambda x.\ (F_2\ x) \Leftrightarrow_\eta F_2$$

- The rule is not strictly necessary for reducing λ-expressions and may cause problems in the presence of constants, but included for completeness

δ-Reduction

- *Definition*: If the λ-calculus has predefined constants (that is, if it is not pure), rules associated with those predefined values and functions are called *delta* rules:

- Example:

$$(add\ 3\ 5) \Rightarrow_\delta 8$$

and

$$(not\ \ true) \Rightarrow_\delta false$$

- Example:

$$\begin{array}{l}
twice = \lambda f.\ \lambda x.\ f\ (f\ x) \\
twice\ (\lambda n.\ (add\ \ n\ 1))\ 5 \Rightarrow_\beta \\
(\lambda f.\ \lambda x.\ (f\ (f\ x)))(\lambda n.\ (add\ \ n\ 1))\ 5 \Rightarrow_\beta \\
(\lambda x.\ ((\lambda n.\ (add\ \ n\ 1))((\lambda n.\ (add\ \ n\ 1))\ x)))\ 5 \Rightarrow_\beta \\
(\lambda n.\ (add\ \ n\ 1))\ ((\lambda n.\ (add\ \ n\ 1))\ 5) \Rightarrow_\beta \\
(add\ ((\lambda n.\ (add\ \ n\ 1))\ 5)\ 1) \Rightarrow_\beta \\
(add\ (add\ 5\ 1)\ 1) \Rightarrow_\delta 7
\end{array}$$

Evaluation Strategies

- Call-by-Value (CBV)
  - C / C++: the argument expression is evaluated, and the resulting value is bound to the corresponding variable in the function (frequently by copying the value into a new memory region)
- Call-by-Reference (CBR)
  - C++: a function receives an implicit reference to a variable used as argument, rather than a copy of its value
  - CBR may be simulated in languages that use CBV by making use of references, such as pointers (Call-by-Address or CBA)
- Call-by-Copy-Restore (CBCR) / Value-Result
  - Fortran (old): a special case of call by reference where the provided reference is unique to the caller (Copy-in-Copy-out)
- Call-by-Name (CBN)
  - C / C++ Macro: the arguments to a function are not evaluated before the function is called – rather, they are substituted directly into the function body
  - Lazy Evaluation
  - Call-by-Need: a memorized variant of CBN where, if the function argument is evaluated, that value is stored for subsequent uses

Evaluation Strategies


```
#include <iostream>
using namespace std;

void f(int a, int b) { a++; b--; return; }             // CBV
void g(int& a, int& b) { a++; b--; return; }           // CBR
void h(int* pa, int* pb) { (*pa)++; (*pb)--; return; } // CBA
#define m_f(a, b) ( a * b )                            // CBN

int main() {
    int x = 3, y = 4, z = 5;
    f(x, y);
    cout << x << " " << y << endl;        // CBV = 3 4
    g(x, y);
    cout << x << " " << y << endl;        // CBR = 4 3
    h(&x, &y);
    cout << x << " " << y << endl;        // CBA = 5 2
    g(z, z);
    cout << z << endl;                    // CBR = 5, CBCR = 6 or 4
    cout << m_f(x + 1, y + 1) << endl;    // CBN = x + y + 1 = 8

    return 0;
}
```

Reduction Strategies

Definition: A λ-expression is in **normal form** if it contains no β-redexes (and no δ-rules in an applied λ calculus), so that it cannot be further reduced using the β-rule or the δ-rule.

An expression in normal form has no more function applications to evaluate.

Reduction Strategies

Questions:

1. Can every λ-expression be reduced to a normal form?
2. Is there more than one way to reduce a particular λ-expression?
3. If there is more than one reduction strategy, does each one lead to the same normal form expression?
4. Is there a reduction strategy that will guarantee that a normal form expression will be produced?

Reduction Strategies

1. **Can every λ-expression be reduced to a normal form?**

No. Consider:

$$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$$
$$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$$
$$(\lambda x.\ x\ x)(\lambda x.\ x\ x) \Rightarrow$$
$$\ldots$$

Reduction Strategies

2. **Is there more than one way to reduce a particular λ-expression?**

Yes. Consider:

$$(\lambda x.\ \lambda y.\ (add\ y\ ((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5$$

Path 1: OUTERMOST
$(\lambda x.\ \lambda y.\ (add\ y\ ((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda y.\ (add\ y\ ((\lambda z.\ (mul\ 7\ z))\ 3)))\ 5 \Rightarrow_\beta$
$(add\ 5\ ((\lambda z.\ (mul\ 7\ z))\ 3)) \Rightarrow_\beta (add\ 5\ (mul\ 7\ 3)) \Rightarrow_\delta (add\ 5\ 21) \Rightarrow_\delta 26$

Path 2: INNERMOST
$(\lambda x.\ \lambda y.\ (add\ y((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda x.\ \lambda y.\ (add\ y\ (mul\ x\ 3)))\ 7\ 5 \Rightarrow_\beta (\lambda x.\ (add\ 5\ (mul\ x\ 3)))\ 7 \Rightarrow_\beta$
$(add\ 5\ (mul\ 7\ 3)) \Rightarrow_\delta (add\ 5\ 21) \Rightarrow_\delta 26$

Path 3: MIXED
$(\lambda x.\ \lambda y.\ (add\ y((\lambda z.\ (mul\ x\ z))\ 3)))\ 7\ 5 \Rightarrow_\beta$
$(\lambda x.\ \lambda y.\ (add\ y\ (mul\ x\ 3)))\ 7\ 5 \Rightarrow_\beta (\lambda y.\ (add\ y\ (mul\ 7\ 3)))\ 5 \Rightarrow_\delta$
$(\lambda y.\ (add\ y\ 21))\ 5 \Rightarrow_\beta (add\ 5\ 21) \Rightarrow_\delta 26$

Reduction Strategies

3. **If there is more than one reduction strategy, does each one lead to the same normal form expression?**

No. Consider:

$$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x))$$

Path 1:
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow 5$
Path 2:
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
$(\lambda y.\ 5)((\lambda x.\ x\ x)(\lambda x.\ x\ x)) \Rightarrow$
...

Reduction Strategies

4. **Is there a reduction strategy that will guarantee that a normal form expression will be produced?**

Mathematician Curry proved that if an expression has a normal form, then it can be found by leftmost reduction.

A normal order reduction can have either of the following outcomes:

1. It reaches a unique (up to α-conversion) normal form λ-expression
2. It never terminates

Unfortunately, there is no algorithmic way to determine for an arbitrary λ-expression which of these two outcomes will occur

Reduction Strategies: Normal and Applicative Order

Two important orders of rewriting:

- **Normal Order** – rewrite the outermost (leftmost) occurrence of a function application.
  - *This is equivalent to call by name*.
- **Applicative Order** – rewrite the innermost (leftmost) occurrence of a function application first.
  - *This is equivalent to call by value*.

Normal order evaluation always gives the same results as lazy evaluation, but may end up evaluating an expression more times.

Order of Evaluation

- Example:

$$\begin{aligned} \mathit{double}\ x &= x + x \\ \mathit{average}\ x\ y &= (x + y)/2 \end{aligned}$$

- Using prefix notation:

$$\begin{aligned} \mathit{double}\ x &= \mathit{plus}\ x\ x \\ \mathit{average}\ x\ y &= \mathit{divide}\ (\mathit{plus}\ x\ y)\ 2 \end{aligned}$$

- Evaluate:

$$\mathit{double}\ (\mathit{average}\ 2\ 4)$$

Order of Evaluation

- Evaluate:

$$\textit{double}\ (\textit{average}\ 2\ 4)$$

- Using normal order of evaluation:
  $\textit{double}\ (\textit{average}\ 2\ 4) \Rightarrow \textit{plus}\ (\textit{average}\ 2\ 4)\ (\textit{average}\ 2\ 4) \Rightarrow$
  $\textit{plus}\ (\textit{divide}\ (\textit{plus}\ 2\ 4)\ 2)\ (\textit{average}\ 2\ 4) \Rightarrow$
  $\textit{plus}\ (\textit{divide}\ 6\ 2)\ (\textit{average}\ 2\ 4) \Rightarrow \textit{plus}\ 3\ (\textit{average}\ 2\ 4) \Rightarrow$
  $\textit{plus}\ 3\ (\textit{divide}\ (\textit{plus}\ 2\ 4)\ 2) \Rightarrow \textit{plus}\ 3\ (\textit{divide}\ 6\ 2) \Rightarrow$
  $\textit{plus}\ 3\ 3 \Rightarrow 6$
  - Notice that $(\textit{average}\ 2\ 4)$ was evaluated twice ... lazy evaluation would cache the results of the first evaluation
- Using applicative order of evaluation:
  $\textit{double}\ (\textit{average}\ 2\ 4) \Rightarrow \textit{double}\ (\textit{divide}\ (\textit{plus}\ 2\ 4)\ 2) \Rightarrow$
  $\textit{double}\ (\textit{divide}\ 6\ 2) \Rightarrow \textit{double}\ 3 \Rightarrow \textit{plus}\ 3\ 3 \Rightarrow 6$

Order of Evaluation

- Consider:
  $my\_if\ \ True\ x\ y = x$
  $my\_if\ \ False\ x\ y = y$
- Evaluate:
  $my\_if\ \ (less\ 3\ 4)\ (plus\ 5\ 5)\ (divide\ 1\ 0)$
- Using normal order of evaluation:
  $my\_if\ \ (less\ 3\ 4)\ (plus\ 5\ 5)\ (divide\ 1\ 0) \Rightarrow$
  $my\_if\ \ True\ (plus\ 5\ 5)\ (divide\ 1\ 0) \Rightarrow (plus\ 5\ 5) \Rightarrow$
  10
- Using applicative order of evaluation:
  $my\_if\ \ (less\ 3\ 4)\ (plus\ 5\ 5)\ (divide\ 1\ 0) \Rightarrow$
  $my\_if\ \ True\ (plus\ 5\ 5)\ (divide\ 1\ 0) \Rightarrow$
  $my\_if\ \ True\ 10\ (divide\ 1\ 0) \Rightarrow$
  $DIVIDE\ \ BY\ \ ZERO\ \ ERROR$

Properties of Order of Evaluation: Strictness

Two important properties of evaluation order:

- If there is any evaluation order that will terminate and that will not generate an error, normal order evaluation will terminate and will not generate an error.
- ANY evaluation order that terminates without error will give the same result as any other evaluation order that terminates without error.

Definition: A function f is *strict* in an argument if that argument is always evaluated whenever an application of f is evaluated.

If a function is strict in an argument, we can safely evaluate the argument first if we need the value of applying the function.

Lazy Evaluation and Strictness Analysis

We can use lazy evaluation on an ad-hoc basis (e.g. for *if*), for all arguments.

For all arguments, for some implementations of functional languages we can improve efficiency using strictness analysis.

$plus\ a\ b$ is strict in both arguments
$if\ x\ y\ z$ is strict in x, but not in y and z

We can do some analysis and sometimes decide if a user-defined function is strict in some of its arguments:

Lazy Evaluation and Strictness Analysis

Examples:

- $double\ x$ is strict in x
- $squid\ n\ x\ =\ if\ \ n = 0\ then\ x + 1\ else\ x - n$ is strict in n and x
- $crab\ n\ x\ =\ if\ \ n = 0\ then\ x + 1\ else\ n$ is strict in n but not x

If a function is strict in an argument x, it is correct to pass x by value, even with normal order evaluation semantics.

It is not always decidable whether a function is strict in an argument – if we do not know, pass using lazy evaluation.

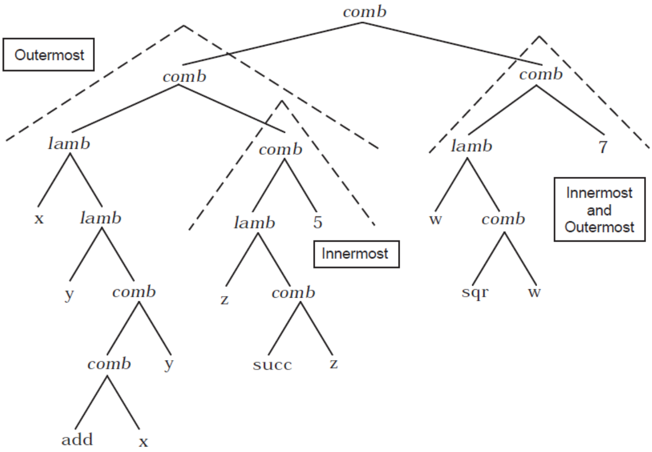
Reduction Strategies: Normal and Applicative Order

- *Definition*:
  A **normal order reduction** always reduces the *leftmost outermost* β-redex (or δ-redex) first.
  An **applicative order reduction** always reduces the *leftmost innermost* β-redex (or δ-redex) first.

- *Definition*:
  For any λ-expression of the form $E = ((\lambda x.\ B)\ A)$, we say that β-redex E is outside any β-redex that occurs in B or A and that these are inside E.
  A β-redex in a λ-expression is outermost if there is no β-redex outside of it, and it is innermost if there is no β-redex inside of it.

- Use AST for detection

AST of λ-expression



β-redexes in $(((\lambda x.\ \lambda y.\ (add\ x\ y))\ ((\lambda z.\ (succ\ z))\ 5))\ ((\lambda w.\ (sqr\ w))\ 7))$

Order of Evaluation

- Applicative Order (*leftmost innermost*)
  $((\lambda n.\ (add\ 5\ n))\ 8) \Rightarrow$
  $((\lambda n.\ (add5\ n))\ 8) \Rightarrow$ – $add5 : N \rightarrow N$ is curried
  $(add5\ 8) \Rightarrow 13$
  - Eager Evaluation
  - Call-by-Value (CBV)
  - Curried functions $(f\ x\ y\ z)$ use eager reduction
- Normal Order (*leftmost outermost*)
  $((\lambda n.\ (add\ 5\ n))\ 8) \Rightarrow (add\ 5\ 8) \Rightarrow 13$
  - Lazy Evaluation
  - Call-by-Name (CBN)
  - Function $f(x, y, z)$ use lazy reduction

Order of Evaluation


```
          ((lambda (x) (+ x x)) (* 2 3))
                      /         \
                 lazy/           \eager
                    /             \
                   /               \
      (+ (* 2 3) (* 2 3))    ((lambda (x) (+ x x)) 6)
                  \                /
                   \              /
                    \            /
                     \          /
                        (+ 6 6)
```