1) (a)

(i) Gauss-Jacobi

$$7x + 2y - z = 17.20$$
$$-x + 9y + 2z = 18.90$$
$$x + 5y - 11z = 28.05$$

Sol^n: This system is diagonally dominant.
Iteration formula for Gauss-Jacobi method is

$$x^{(k+1)} = \frac{1}{7}[17.20 - 2y^{(k)} + z^{(k)}]$$
$$y^{(k+1)} = \frac{1}{9}[18.90 + x^{(k)} - 2z^{(k)}]$$
$$z^{(k+1)} = \frac{1}{-11}[28.05 - x^{(k)} - 5y^{(k)}]$$

initial guess $(x^{(0)}, y^{(0)}, z^{(0)}) = (0,0,0)$

$$x^{(1)} = \frac{1}{7}[17.20 - 2\times 0 + 0] = 2.4571$$
$$y^{(1)} = \frac{1}{9}[18.90 + 0 - 2\times 0] = 2.1000$$
$$z^{(1)} = \frac{1}{-11}[28.05 - 0 - 5\times 0] = -2.5500$$

$$x^{(2)} = \frac{1}{7}[17.20 - 2\times 2.1 + (-2.55)] = 1.4929$$
$$y^{(2)} = \frac{1}{9}[18.90 + (2.4571) - 2\times(-2.55)] = 2.9397$$
$$z^{(2)} = \frac{1}{-11}[28.05 - 2.4571 - 5\times 2.1] = -1.3721$$

$$x^{(3)} = \frac{1}{7}[17.20 - 2\times 2.9397 + (-1.3721)] = 1.4212$$
$$y^{(3)} = \frac{1}{9}[18.90 + 1.4929 - 2\times(-1.3721)] = 2.5708$$
$$z^{(3)} = \frac{1}{-11}[28.05 - 1.4929 - 5\times(2.9397)] = -1.0781$$

$$\begin{cases} x^4 = \frac{1}{7}[17.20 - 2\times 2.5708 + (-1.0781)] = 1.5686 \\ y^4 = \frac{1}{9}[18.9 + 1.4212 - 2\times(-1.0781)] = 2.4975 \\ z^4 = \frac{1}{-11}[28.05 - 1.4212 - 5\times(2.5708)] = -1.2523 \end{cases}$$

$$\begin{cases} x^5 = \frac{1}{7}[17.20 - 2\times 2.4975 + (-1.2523)] = 1.5648\cancel{7} \\ y^5 = \frac{1}{9}[18.9 + 1.5686 - 2\times(-1.2523)] = 2.5526 \\ z^5 = \frac{1}{-11}[28.05 - 1.5686 - 5\times(2.4975)] = -1.2722 \end{cases}$$

$$\begin{cases} x^6 = \frac{1}{7}[17.20 - 2\times 2.5526 + (-1.2722)] = 1.5461 \\ y^{(6)} = \frac{1}{9}[18.9 + 1.548 - 2\times(-1.2722)] = 2.5566 \\ z^{(6)} = \frac{1}{-11}[28.05 - 1.548 - 5\times(2.5526)] = -1.2475 \end{cases}$$

~~Check~~

$$\begin{cases} x^{7} = \frac{1}{7}[17.20 - 2\times 2.5547 + (-1.249)] = \cancel{1.5484} \; 1.5485 \\ y^7 = \frac{1}{9}[18.9 + 1.5461 - 2\times(-1.249)] = 2.549010 \\ z^7 = \frac{1}{-11}[28.05 - 1.5461 - 5\times(2.5547)] = -1.24874 \end{cases}$$

~~Hence~~

~~$x = 1.5484$~~
~~$y = 2.549$ correct upto 3 significant Decimal~~
~~$z = -1.2473$ figure place.~~

$$\begin{cases} x^{(8)} = 1.5507 \\ y^{(8)} = 2.5492 \\ z^{(8)} = -1.2506 \end{cases} \quad \begin{cases} x^{(9)} = 1.5501 \\ y^{(9)} = 2.5502 \\ z^{(9)} = -1.2503 \end{cases} \quad \begin{cases} x^{10} = 1.5499 \\ y^{10} = 2.5501 \\ z^{10} = -1.2499 \end{cases} \quad \begin{cases} x^{11} = 1.550 \\ y^{11} = 2.550 \\ z^{11} = -1.250 \end{cases}$$

Solⁿ is $x = 1.55$, $y = 2.55$, $z = -1.25$ correct upto 3 decimal place.

1) (a) (ii) Gauss-Seidel

$$7x + 2y - z = 17.20$$
$$-x + 9y + 2z = 18.90$$
$$x + 5y - 11z = 28.05$$

Solⁿ: This system is diagonally dominant.

Iteration formula for Gauss-Seidel is

$$x^{(k+1)} = \frac{1}{7}[17.20 - 2y^{(k)} + z^{(k)}]$$

$$y^{(k+1)} = \frac{1}{9}[18.90 + x^{(k+1)} - 2z^{(k)}]$$

$$z^{(k+1)} = -\frac{1}{11}[28.05 - x^{(k+1)} - \cancel{5}y^{(k+1)}]$$

Initial guess $(x^{(0)}, y^{(0)}, z^{(0)}) = (0, 0, 0)$

$$\begin{cases} x^{(1)} = \frac{1}{7}[17.20 - 2\times 0 + 0] = 2.4571 \\ y^{(1)} = \frac{1}{9}[18.90 + 2.4571 - 2\times 0] = 2.3730 \\ z^{(1)} = -\frac{1}{11}[28.05 - 2.4571 - 5\times 2.373] = -1.2480 \end{cases}$$

$$\begin{cases} x^{(2)} = \frac{1}{7}[17.20 - 2\times 2.373 + (-1.248)] = 1.6009 \\ y^{(2)} = \frac{1}{9}[18.90 + 1.6001 - 2\times(-1.248)] = 2.5552 \\ z^{(2)} = -\frac{1}{11}[28.05 - 1.6001 - 5\times(2.555)] = -1.2430 \end{cases}$$

$$\begin{cases} x^{(3)} = \frac{1}{7}[17.20 - 2\times 2.555 + (-1.2432)] = 1.5495 \\ y^{(3)} = \frac{1}{9}[18.90 + 1.549 - 2\times(-1.2432)] = 2.5484 \\ z^{(3)} = -\frac{1}{11}[28.05 - 1.549 - 5\times(2.5484)] = -1.2508 \end{cases}$$

$$\begin{cases} x^{(4)} = 1.5503 \\ y^{(4)} = 2.5502 \\ z^{(4)} = -1.2499 \end{cases}$$

Check
$$\begin{cases} x^{(5)} = 1.5500 \\ y^{(5)} = 2.5500 \\ z^{(5)} = -1.2500 \end{cases}$$

Solⁿ of the system is $x = 1.550$

$y = 2.550$ correct upto

$z = -1.250$ 3 Decimals place.

Question 1

1) (b)

$$10x_1 - 2x_2 - x_3 - x_4 = 3$$

$$-2x_1 + 10x_2 - x_3 - x_4 = 15$$

$$-x_1 - x_2 + 10x_3 - 2x_4 = 27$$

$$-x_1 - x_2 - 2x_3 + 10x_4 = -9$$

(i) Gauss- Jacobi

$$x_1^{K+1} = \frac{1}{10}\left(3 + 2x_2^K + x_3^K + x_4^K\right)$$

$$x_2^{K+1} = \frac{1}{10}\left(15 + 2x_1^K + x_3^K + x_4^K\right)$$

$$x_3^{K+1} = \frac{1}{10}\left(27 + x_1^K + x_2^K + 2x_4^K\right)$$

$$x_4^{K+1} = \frac{1}{10}\left(-9 + x_1^K + x_2^K + 2x_3^K\right)$$

Initial guess: $(x_1^0, x_2^0, x_3^0, x_4^0) = (0,0,0,0)$

K	x_1^K	x_2^K	x_3^K	x_4^K
0	0	0	0	0
1	0.3000	1.5000	2.7000	-0.9000
2	0.7800	1.7400	2.7000	-0.1800
3	0.9000	1.9080	2.9160	-0.1080
4	0.9624	1.9608	2.9592	-0.0360
5.	0.9845	1.9848	2.9851	-0.0158
6.	0.9939	1.9938	2.9938	-0.0060
7.	0.9975	1.9975	2.9976	-0.0025
8.	0.9975	1.9975	2.9975	-0.0024

Thus, $x_1 = 0.997$

$x_2 = 1.997$

$x_3 = 2.997$

$x_4 = -0.002$

is the correct value upto 3 decimal places.

1) b)

(ii) Gauss - Seidel

$$x_1^{k+1} = \frac{1}{10}\left(3 + 2x_2^{k} + x_3^{k} + x_4^{k}\right)$$

$$x_2^{k+1} = \frac{1}{10}\left(15 + 2x_1^{k+1} + x_3^{k} + x_4^{k}\right)$$

$$x_3^{k+1} = \frac{1}{10}\left(27 + x_1^{k+1} + x_2^{k+1} + x_3^{k}\right)$$

$$x_4^{k+1} = \frac{1}{10}\left(-9 + x_1^{k+1} + x_2^{k+1} + 2x_3^{k+1}\right)$$

Initial guess: $(x_1^0, x_2^0, x_3^0, x_4^0) = (0,0,0,0)$

K	x_1^k	x_2^k	x_3^k	x_4^k
0	0	0	0	0
1	0.3000	1.5600	2.8860	-0.1368
2.	0.8869	1.9523	2.9566	-0.0248
3.	0.9836	1.9899	2.9924	-0.0042
4.	0.9968	1.9982	2.9987	-0.0008
5.	0.9994	1.9997	2.9998	-0.0001
6.	0.9994	1.9996	2.9997	-0.0001

So,

$x_1 = 0.999$

$x_2 = 1.999$

$x_3 = 2.999$

$x_4 = -0.0$

1) c)

(i) Gauss Jacobi:

$$x_1^{(k+1)} = \frac{1}{13}\left[18 - 5x_2^{(k)} + 3x_3^{(k)} - x_4^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{12}\left[13 - 2x_1^{(k)} - x_3^{(k)} + 4x_4^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{10}\left[29 - 3x_1^{(k)} + 4x_2^{(k)} - x_4^{(k)}\right]$$

$$x_4^{(k+1)} = \frac{1}{9}\left[31 - 2x_1^{(k)} - x_2^{(k)} + 3x_3^{(k)}\right]$$

We take the initial guess:

$(x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, x_4^{(0)}) = (0, 0, 0, 0)$.

K	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_4^{(k)}$
0	0	0	0	0
1	1.3846	1.0833	2.9000	3.4444
2	1.37222	1.7590	2.5735	3.9830
3	0.9955	1.9678	2.7936	3.8018
4	0.9799	1.9519	3.0082	3.9357
5	1.0253	1.9812	2.9932	4.0125
6	1.0047	2.0005	2.9836	3.9942
7	.9965	1.9986	2.9994	3.9934
8	1.0009	1.9984	3.0012	4.0007
9	1.0008	1.9999	2.9990	4.0004
10	0.99975	2.0000	2.9997	3.9995
11	0.9999	1.9999	3.0002	3.9999

Thus, $x_1 = 1.000$, $x_2 = 2.000$, $x_3 = 3.000$, $x_4 = 4.000$

correct upto 3-decimal places

(7)

(c)(ii) Gauss seidel

$$13x_1 + 5x_2 - 3x_3 + x_4 = 18$$
$$2x_1 + 12x_2 + x_3 - 4x_4 = 13$$
$$3x_1 - 4x_2 + 10x_3 + x_4 = 29$$
$$2x_1 + x_2 - 3x_3 + 9x_4 = 31$$

solⁿ: This system is diagonally dominant. so, we apply the iteration formula as:

$$x_1^{(k+1)} = \frac{1}{13}\left[18 - 5x_2^{(k)} + 3x_3^{(k)} - x_4^{(k)}\right]$$

$$x_2^{(k+1)} = \frac{1}{12}\left[13 - 2x_1^{(k+1)} - x_3^{(k)} + 4x_4^{(k)}\right]$$

$$x_3^{(k+1)} = \frac{1}{10}\left[29 - 3x_1^{(k+1)} + 4x_2^{(k+1)} - x_4^{(k)}\right]$$

$$x_4^{(k+1)} = \frac{1}{9}\left[31 - 2x_1^{(k+1)} - x_2^{(k+1)} + 3x_3^{(k+1)}\right]$$

we take the initial guess value as

$$x_1^{(0)} = 0,\ x_2^{(0)} = 0,\ x_3^{(0)} = 0,\ x_4^{(0)} = 0.$$

$\therefore$

$$\begin{cases} x_1^{(1)} = \frac{1}{13}\left[18 - 5\times0 + 3\times0 - 0\right] = 1.38462 \\ x_2^{(1)} = \frac{1}{12}\left[13 - 2\times(1.38462) - 0 - 0\right] = 0.85256 \\ x_3^{(1)} = \frac{1}{10}\left[29 - 3\times(1.38462) + 4\times(0.85256) - 0\right] = 2.82564 \\ x_4^{(1)} = \frac{1}{9}\left[31 - 2\times1.38462 - 0.85256 + 3\times2.82564\right] = 3.9839 \end{cases}$$

$$x_1^{(2)} = \frac{1}{13}[18 - 5\times(\cdot 85256) + 3\times(2\cdot 82564) - 3\cdot 9839]$$

$$= 1\cdot 4023$$

$$x_2^{(2)} = \frac{1}{12}[13 - 2\times 1\cdot 4023 - 2\cdot 82564 + 4\times 3\cdot 9839]$$

$$= 1\cdot 9421$$

$$x_3^{(2)} = \frac{1}{10}[29 - 3\times 1\cdot 4023 + 4\times 1\cdot 9421 - 3\cdot 9839]$$

$$= 2\cdot 8578$$

$$x_4^{(2)} = \frac{1}{9}[31 - 2\times 1\cdot 4023 - 1\cdot 9421 + 3\times 2\cdot 8578]$$

$$= 3\cdot 8696$$

$$x_1^{(3)} = \frac{1}{13}[18 - 5\times 1\cdot 9421 + 3\times 2\cdot 8578 - 3\cdot 8696]$$

$$= 0\cdot 99948$$

$$x_2^{(3)} = \frac{1}{12}[13 - 2\times 0\cdot 99948 - 2\cdot 8578 + 4\times 3\cdot 8696]$$

$$= 1\cdot 9685$$

$$x_3^{(3)} = \frac{1}{10}[29 - 3\times\cdot 99948 + 4\times 1\cdot 9685 - 3\cdot 8696]$$

$$= 3\cdot 0006$$

$$x_4^{(3)} = \frac{1}{9}[31 - 2\times 0\cdot 99948 - 1\cdot 9685 + 3\times 3\cdot 0006]$$

$$= 4\cdot 00382$$

$$x_1^{(4)} = \frac{1}{13}[18 - 5\times 1\cdot 9685 + 3\times 3\cdot 0006 - 4\cdot 00382] = 1\cdot 01196$$

$$x_2^{(4)} = \frac{1}{12}[13 - 2\times 1\cdot 01196 - 3\cdot 0006 + 4\times 4\cdot 00382] = 1\cdot 99923$$

$$x_3^{(4)} = \frac{1}{10}[29 - 3\times 1\cdot 01196 + 4\times 1\cdot 99923 - 4\cdot 00382] = 2\cdot 9957$$

$$x_4^{(4)} = \frac{1}{9}[31 - 2\times 1\cdot 01196 - 1\cdot 99923 + 3\times 2\cdot 9957] = 3\cdot 996$$

$$\begin{cases} x_1^{(5)} = \frac{1}{13}[18 - 5\times 1.99923 + 3\times 2.9957 - 3.996] = 0.9996 \\ x_2^{(5)} = \frac{1}{12}[13 - 2\times 0.9996 - 2.9957 + 4\times 3.996] = 1.99909 \\ x_3^{(5)} = \frac{1}{10}[29 - 3\times 0.9996 + 4\times 1.9991 - 3.996] = 3.00016 \\ x_4^{(5)} = \frac{1}{9}[31 - 2\times 0.9996 - 1.99909 + 3\times 3.00016] = 4.00024 \end{cases}$$

Check

$$\begin{cases} x_1^{(6)} = \frac{1}{13}[18 - 5\times 1.99909 - 3.00016 + 4\times 4.0002] = 1.0004 \\ x_2^{(6)} = \frac{1}{12}[13 - 2\times 1.0004 - 3.00016 - 4.0002] = 2.000 \\ x_3^{(6)} = \frac{1}{10}[29 - 3\times 1.0004 + 4\times 2.000 - 4.0002] = 2.999 \\ x_4^{(6)} = \frac{1}{9}[31 - 2\times 1.0004 - 2.000 + 3\times 2.999] = 3.9999 \end{cases}$$

Thus, $x_1 = 1.000$, $x_2 = 2.000$, $x_3 = 3.000$, $x_4 = 4.000$ correct upto 3-decimal places.

(2) Find root, lying between 2 & 3, of the eqn $x^3 - x - 11 = 0$ by using bisection method correct upto three decimals

Let $f(x) = x^3 - x - 11$

$f(2) = -5 < 0$, $f(3) = 13 > 0 \Rightarrow f(2)f(3) < 0$. [$\therefore$ root lying in this range]

$f'(x) = 3x^2 - 1$

$\therefore$ $f'(x) > 0 \ \forall \ x \in [2, 3] \Rightarrow$ only one root between 2 & 3.

n	a_n(-ve)	b_n(+ve)	$x_{n+1} = \frac{a_n + b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	2.125
1	2	2.5	2.25	-1.859
2	2.25	2.5	2.375	0.0215
3	2.25	2.375	2.3125	-0.946
4	2.3125	2.375	2.3438	-0.4684
5	2.3438	2.375	2.3594	-0.2252
6	2.3594	2.375	2.3672	-0.1023
7	2.3672	2.3750	2.3711	-0.0405
8	2.3711	2.3750	2.3731	-0.0087
9	2.3731	2.3750	2.3741	0.0072
10	2.3731	2.3741	2.3736	-0.00079
11	2.3736	2.3741	2.3738	0.00024
check → 12	2.3736	2.3738	2.3737	0.00008

$\therefore$ In the 11^{th} step, a_n, b_n & x_{n+1} are equal upto three significant figures. $\therefore$ root is $\boxed{2.374}$ upto three significant

Q3) (a) $2x - 3\sin x - 5 = 0$

$f(x) = 2x - 3\sin x - 5$ in $[2, 3]$

n	$a_n(-ve)$	$b_n(+ve)$	$x_{n+1} = \frac{a_n+b_n}{2}$	$f(x_{n+1})$
0	2	3	2.5	$-0.1300 < 0$
1	2.5	3	2.75	$0.3560 > 0$,
2	2.5	2.75	2.625	$0.11260 > 0$
3	2.5	2.625	2.5625	$-0.0091 < 0$
4	2.5625	2.625	2.59375	$0.05173 > 0$
5	2.5625	2.5937	2.5781	$0.02125 > 0$
6	2.5625	2.5781	2.5703	$0.00606 > 0$
7	2.5623	2.5703	2.5663	-0.00172
8	2.5663	2.5703	2.5683	$0.0021 > 0$
9	2.5663	2.5683	2.5673	$0.0002 > 0$
10	2.5663	2.5673	2.5668	$-0.0007 < 0$

Here 2.5668 is root

Upto 3 decimal places 2.567

(b)

$$x \log_{10} x = 1.2$$

$$\Rightarrow x \frac{\log x}{\log 10} = 1.2$$

$$f(x) = x \log_{10} x - 1.2$$

$[2, 3]$

n	$a_n < 0$	$b_n > 0$	x_{n+1}	$f(x_{n+1})$
0	2	3	2.5	$-0.205 < 0$
1	2.5	3	2.75	$0.0081 > 0$
2	2.5	2.75	2.625	$-0.099 < 0$
3	2.625	2.75	2.6875	$-0.046 < 0$
4	2.6875	2.75	2.71875	$-0.0190 < 0$
5	2.71875	2.75	2.7343	$-0.005 < 0$
6.	2.7343	2.75	2.7421	$0.001 > 0$
7.	2.7343	2.742	2.73815	$-0.0021 < 0$
.	2.7381	2.742	2.74006	$-0.0005 < 0$
	2.7400	2.742	2.741	$0.0003 > 0$
	2.740	2.741	2.740	$-0.0005 < 0$

correct upto 2 decimal place

answer is 2.74

4) <u>Convergence criterion of fixed pt iteration</u>

This method is based on the principle of finding a sequence $\{x_n\}$ each element of which successively approximates a real root α of the Eqn $f(x)=0$, in $[a,b]$.

We ~~we~~ rewrite $f(x)=0$ as $x=\varphi(x)$, here we assume φ is continuously differentiable in $[a,b]$.

Thus a root α of the given eqn satisfy $\alpha=\varphi(\alpha)$

& successive approximation are given by

$$x_{n+1}=\varphi(x_n)$$

now the convergence of $\{x_n\}$ depends upon the nature of $\varphi(x)$; because the presentation of $f(x)=0$ as $x=\varphi(x)$ is not unique.

Using Lagrange MVT, we have [Here we assuming that $\varphi(x)$ is continuously diff'ble]

$$|\alpha-x_1|=|\varphi(\alpha)-\varphi(x_0)|=|\alpha-x_0||\varphi'(\xi_1)| \text{ for } x_0<\xi_1<\alpha$$

$$|\alpha-x_2|=|\varphi(\alpha)-\varphi(x_1)|=|\alpha-x_1||\varphi'(\xi_2)| \text{ for } x_1<\xi_2<\alpha$$

$$\vdots$$

$$|\alpha-x_n|=|\varphi(\alpha)-\varphi(x_{n-1})|=|\alpha-x_{n-1}||\varphi'(\xi_n)| \text{ for } x_{n-1}<\xi_n<\alpha$$

$$|\alpha-x_{n+1}|=|\varphi(\alpha)-\varphi(x_n)|=|\alpha-x_n||\varphi'(\xi_{n+1})| \text{ for } x_n<\xi_{n+1}<\alpha$$

Thus,
$$|\alpha-x_{n+1}|=|\alpha-x_n||\varphi'(\xi_{n+1})|$$
$$=|\alpha-x_0||\varphi'(\xi_1)||\varphi'(\xi_2)|\cdots|\varphi'(\xi_{n+1})|$$

Assuming $|\varphi'(x)|\le\rho$ in $(a_0\le x\le b_0)$ we have

$$|\alpha-x_{n+1}|\le|\alpha-x_0|\rho^n$$

Thus $\lim_{n\to\infty} |\alpha - x_{n+1}| \leq |\alpha - x_0| \lim_{n\to\infty} \rho^{n+1} \to 0$ if $\rho < 1$

$\to \infty$ if $\rho > 1$

hence $\lim_{n\to\infty} x_{n+1} \to \alpha$ iff $\rho < 1$ in $[a_0, b_0]$.

i.e $|\phi'(x)| \leq \rho < 1$.

Therefore the convergence criteria is

$|\phi'(x)| < 1$ in $[a_0, b_0]$.

Q5) Find root of $5x^3 - 20x + 3 = 0$ by fixed point iteration

$$f(x) = 5x^3 - 20x + 3$$

$$f(0) = 3$$

$$f(1) = 5 - 20 + 3 = -12$$

so $\exists$ root btween $[0,1]$

$$5x^3 - 20x + 3 = 0$$

$$x(5x^2 - 20) = -3$$

$$\Rightarrow x = \frac{3}{20 - 5x^2}$$

$$\phi(x) = \frac{3}{20 - 5x^2}$$

$$\phi'(x) = \frac{-3(-10x)}{(20 - 5x^2)^2} = \frac{30x}{(20 - 5x^2)^2}$$

$|\phi'(x)|$ in $[0,1]$

$$\max \left| |\phi'(0)|, |\phi'(1)| \right|$$

$$= \max \left| 0, \frac{30}{(15)^2} \right| < 1$$

so take $\phi(x) = \dfrac{3}{20-5x^2}$

so $x = \dfrac{3}{20-5x^3} = \phi(x)$

$\phi(x_n) = \dfrac{3}{20-5x_n^2}$ $\quad x_{n+1} = \phi(x_n)$

n	x_n	$\phi(x_n)$
0	0	0.15
1	0.15	0.15084
2	0.15084	0.15085
3	0.15085	0.150858

Ans 0.151 correct upto 3 decimal places

(6) Fixed pt iteration correct upto 3. decimal places

$$\sin x = 10(x-1)$$

Let $f(x) = \sin x - 10(x-1)$

now $f(1) = 0.0175 > 0$ & $f(1.5) = -4.97 < 0$

we write the eq^n as

$$x = \frac{1}{10}[\sin x + 10] = \phi(x).$$

here $|\phi'(x)| = \left|\frac{\cos x}{10}\right| < \frac{1}{10} < 1$

Thus $x = \frac{\sin x + 10}{10} = \phi(x)$ gives us a convergent seq^n of iteration.

$\boxed{\text{Iteration formula } x_{n+1} = \phi(x_n)}$

we take $x_0 = 1$.

n	x_n	$\phi(x_n)$
0	1	1.0017
1	1.0017	1.0017
2	1.0017	1.0017

Thus, 1.002 is the root of the given Eq^n, ~~that is~~ correct upto three decimals places.

(7) Let $f(x) = x^2 + \ln x - 2$

$\therefore f(1) = -1 < 0$
$f(2) = 2.69315 > 0$ $\Big\}$ $f(x)$ has one root between 1 and 2.

i) <u>Fixed-Point iteration:</u>

$f(x) = 0$

$\Rightarrow x^2 + \ln x - 2 = 0$

$\Rightarrow x = \sqrt{2 - \ln x}$

We write, $x = \sqrt{2 - \ln x} = \varphi(x)$.

$$\therefore \varphi'(x) = -\frac{1}{2x\sqrt{2 - \ln x}}$$

Now $\text{Max}\left[|\varphi'(1)|, |\varphi'(2)|\right] = \text{Max}\left[0.35, 0.21\right]$

$= 0.35 < 1$.

Since $|\varphi'(x)|$ is a decreasing fn. &

$\max\left[|\varphi'(1)|, |\varphi'(2)|\right] < 1$,

$|\varphi'(x)| < 1$ between 1 and 2.

$\therefore x = \sqrt{2 - \ln x} = \varphi(x)$ gives us a convergent sequence of iteration.

Fixed-Point iteration formula:

$x_{n+1} = \varphi(x_n)$, where x_n is the nth approximation of the root of $f(x) = 0$

Take $x_0 = 1$.

n	x_n	$\varphi(x_n)$
0	1	1.4142
1	1.4142	1.2859
2	1.2859	1.3223
3	1.3223	1.3117
4	1.3117	1.3148
5	1.3139	1.3142
6	1.3142	1.3141
7	1.3141	1.3141

(ii) Newton Raphson iteration formula:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$f(x) = x^2 + \ln x - 2$ and $f'(x) = 2x + 1/x$.

n	x_n	$f(x_n)$	$f'(x_n)$	$-\frac{f(x_n)}{f'(x_n)}$
0	1	-1	3	0.3333
1	~~0.~~ 1.3333	0.0653	3.4166	-0.0191
2	1.3142	0.0003	3.3893	-0.0001
3	1.3141	0.0000	3.3892	0
4	1.3141			

No. of iterations (fixed-point) = 7

No. of iterations (Newton-Raphson) = 4.

Q) (8) $f(x) = x \sin x + \cos x$

$f'(x) = x \cos x$

$$\frac{f(x)}{f'(x)} = \tan x + \frac{1}{x}$$

Iterative formula is $x_{n+1} = x_n + h_n$

n	x_n	$h_n = -f(x_n) \mid f'(x_n)$
0	3.142	−0.3183
1	2.8233	−0.0247
2	2.7986	−0.0002
3.	2.7984	$\longrightarrow$ iteration stop

So root is 2.7984

9) Newton-Raphson iteration formula:

$$x_{n+1} = x_n + h_n$$

where $h_n = -\dfrac{f(x_n)}{f'(x_n)}$ $(n = 0, 1, 2, 3, \ldots)$

$$f'(x_n) \neq 0 \quad (n = 0, 1, 2, 3)$$

Let $f(x) = 10^x + x - 4$

$\therefore f'(x) = 10^x \ln 10 + 1$

$f(0) = 10^0 + 0 - 4 = -3$

$f(0.5) = 10^{0.5} + 0.5 - 4 = -0.3377223 < 0$

$f(0.8) = 10^{0.8} + 0.8 - 4 = 3.1095734 > 0$

$\therefore f(x) = 0$ has a root which lies between 0.5 & 0.8.

Computations:-

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$
0	0.5	-0.3377223	8.2814134	0.0407808
1	0.5407808	0.0143887	8.9982779	-0.0015990
2	0.5391818	0.0000240	8.9688837	-0.0000027
3	0.5391791	-0.0000001	8.9688342	0.0000000
4	0.5391791	-0.0000001	8.9688342	0.0000000

$\therefore x = 0.539179$ is the root of $f(x) = 0$, correct to 6 decimal places.

10) Let $f(x) = x^3 N$

$f'(x) = 3x^2$

Newton - Raphson iterative formula:

$$x_{n+1} = x_n + h_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$= \frac{2x_n^3 + N}{3x_n^2}$$

Put $N = 10$ $\Rightarrow$ $x_{n+1} = (2x_n^3 + 10)/3x_n^2$

$f(x) = x^3 - 10$, $f'(x) = 3x^2$

$\cancel{n}$	$\cancel{x_n}$	$\cancel{(2x_n+10)/3x_n^2}$
$\cancel{0}$		

$f(2.1) = (2.1)^3 - 10 = -0.739 < 0$

$f(2.2) = (2.2)^3 - 10 = 0.648 > 0$.

$f(x)$ has a root between 2.1 and 2.2

n	x_n	$(2x_n^3+10)/3x_n^2$
0	2.1	~~1.07332~~ 2.15586
1	2.15586	2.15444
2	2.15444	2.15443
3	2.15443	2.15443.

$x = 2.1544$ is the root of $f(x) = 0$ correct to 4 decimal places.

11) Let $f(x) = x^5 - a$

$f'(x) = 5x^4$

~~$f(1.2) = -0.51168 < 0$~~

~~ft~~

Newton- Raphson iterative procedure:

$$x_{n+1} = x_n + h_n = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^5 - a}{5x_n^4}$$

$$= \frac{4x_n^5 + a}{5x_n^4} = \frac{1}{5}\left(4x_n + \frac{a}{x_n^4}\right)$$

~~Here $f(1.2) = -$~~

To evaluate $\sqrt[5]{3}$, Let $f(x) = x^5 - 3$

$f'(x) = 5x^4$.

$$\therefore x_{n+1} = \frac{1}{5}\left(4x_n + \frac{3}{x_n^4}\right)$$

$f(1.2) = -0.51168 < 0$

$f(1.5) = 4.59375 > 0$

n	x_n	$x_{n+1} = \frac{1}{5}\left(4x_n + \frac{3}{x_n^4}\right)$
0	1.2	~~1.01787~~ 1.24935
1	1.24935	1.24575
2	1.24575	1.24573
3	1.24573	1.24573

$\therefore x = 1.2457$ i.e $\sqrt[5]{3} = 1.2457$ upto 4 decimal places.

12) Formula (i) : $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)}$

$$= x_n - \frac{2f(x_n)}{f'(x_n)}$$

$f(x) = \cancel{x^3 + 15.87x + 24.34}$

$x^3 - x^2 - x + 1$ and $f'(x) = 3x^2 - 2x - 1$

$f(0) = 1$, Take $x_0 = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
0	0	1	-1
1	2	3	7
2	1.1429	0.0438	0.6329
3	1.0046	0.00004	0.0185
4	1	0	0

Now $f'(1) = 3 - 2 - 1 = 0$.

$\therefore$ $x = 1$ is a double root of $f(x) = 0$.

formula (ii): $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.

Take $x_0 = 0$.

n	x_n	$f(x_n)$	$f'(x_n)$
0	0	1	-1
1	1	0	0

$\therefore$ $x = 1$ is a double root of $f(x) = 0$

by Newton-Raphson method.