① Evalute the integral $\int_0^1 \frac{x^\alpha - 1}{\log x} dx$, $(\alpha > -1)$ by applying differentiating under integral sign.

$\Rightarrow$ let $\phi(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx$ ——— ①

$$\frac{d\phi}{d\alpha} = \int_0^1 \frac{x^\alpha \log x}{\log x} dx$$

$$= \int_0^1 x^\alpha dx = \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1$$

$$= \frac{1}{\alpha+1}$$

Integrate

$$\int d\phi = \int \frac{1}{\alpha+1} d\alpha$$

$\Rightarrow \phi = \log(\alpha+1) + c$ ——— ②

From ① when $\alpha = 0$, $\phi = 0$.

From ②, Putting $\alpha = 0$, we get

$$0 = \log 1 + c.$$

$$\Rightarrow c = 0$$

Hence $\phi(\alpha) = \log(\alpha+1)$

i.e., $\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(\alpha+1)$.

2 (i) $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$, where $a \geq 0,\ a \neq 1$

$\Rightarrow$ let $\phi(\alpha) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx$ ——— (3)

$$\frac{d\phi}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{\tan^{-1}(ax)}{x(1+x^2)}\right) dx$$

$$= \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{a^2-1}\int_0^\infty \left[\frac{a^2}{a^2x^2+1} - \frac{1}{1+x^2}\right] dx$$

$$= \frac{1}{a^2-1}\left[\left\{a\tan^{-1}(ax)\right\}_0^\infty - \left\{\tan^{-1}(x)\right\}_0^\infty\right]$$

$$= \frac{\pi}{2}\left[\frac{a-1}{a^2-1}\right] = \frac{\pi}{2(a+1)}.$$

Integrating w.r. to a we get

$$\int d\phi = \int \frac{\pi}{2(a+1)}\,da$$

$$\Rightarrow \phi(a) = \frac{\pi}{2}\log(a+1) + c \quad \text{——— (4)}$$

From (3) when $a = 0$, $\phi = 0$.

from (4) we get $c = 0$

Hence $\phi(a) = \frac{\pi}{2}\log(a+1)$

$$\Rightarrow \int_0^\infty \frac{\tan^{-1}(ax)}{(1+x^2)x}\,dx = \frac{\pi}{2}\log(a+1)$$

2 (ii) Let $\phi(\alpha) = \int_0^\infty e^{-x^2} \cos(2\alpha x)\, dx$ —— (5)

$$\frac{d\phi}{d\alpha} = \int_0^\infty \frac{\partial}{\partial \alpha}\left(e^{-x^2}\cos(2\alpha x)\right) dx$$

$$= \int_0^\infty (-2x) e^{-x^2} \sin(2\alpha x)\, dx$$

$$= \left[e^{-x^2}\sin(2\alpha x)\right]_0^\infty - 2\alpha \int_0^\infty e^{-x^2}\cos(2\alpha x)\, dx$$

$$= -2\alpha\phi$$

$$\Rightarrow \frac{d\phi}{d\alpha} + 2\alpha\phi = 0.$$

Integrating.

$$\phi(\alpha) = ce^{-\alpha^2} \quad \text{—— (6)}$$

In eqn (5) put $\alpha = 0$. we get

$$\phi(0) = \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2} \quad \text{(given)}$$

From (6), put $\alpha = 0$.

$$\phi(0) = c.$$

$$\Rightarrow c = \frac{\sqrt{\pi}}{2}.$$

Hence $\phi(\alpha) = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}$

$$\int_0^\infty e^{-x^2}\cos(2\alpha x)\, dx = \frac{\sqrt{\pi}}{2} e^{-\alpha^2}$$

2 (iii) $\displaystyle\int_0^t \frac{\log(1+tx)}{1+x^2}\,dx = \frac{\tan^{-1}(t)}{2}\log(1+t^2)$.

Let $\displaystyle\phi(t) = \int_0^t \frac{\log(1+tx)}{1+x^2}\,dx.$ ——— (8)

$$\frac{d\phi}{dt} = \int_0^t \frac{\partial}{\partial t}\left(\frac{\log(1+tx)}{1+x^2}\right)dx + \frac{\log(1+t^2)}{1+t^2} - 0$$

$$= \int_0^t \frac{x}{(1+tx)(1+x^2)}\,dx + \frac{\log(1+t^2)}{1+t^2}.$$

$$= -\frac{t}{1+t^2}\int_0^t \frac{dx}{1+tx} + \frac{1}{2(1+t^2)}\int_0^t \frac{2x}{1+x^2}\,dx + \frac{t}{1+t^2}\int_0^t \frac{dx}{1+x^2} + \frac{\log(1+t^2)}{1+t^2}$$

$$= -\frac{1}{1+t^2}\Big[\log(1+tx)\Big]_0^t + \frac{1}{2(1+t^2)}\Big[\log(1+x^2)\Big]_0^t + \frac{t}{1+t^2}\Big[\tan^{-1}x\Big]_0^t + \frac{\log(1+t^2)}{(1+t^2)}$$

$$= -\frac{\log(1+t^2)}{(1+t^2)} + \frac{\log(1+t^2)}{2(1+t^2)} + \frac{t.\tan^{-1}(t)}{(1+t^2)} + \frac{\log(1+t^2)}{1+t^2}$$

$$= \frac{\log(1+t^2)}{2(1+t^2)} + \frac{t}{1+t^2}\tan^{-1}(t).$$

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Integrating,

$$\phi = \frac{1}{2}\int \log(1+t^2)\cdot\frac{1}{1+t^2}\,dt + \int \frac{t\cdot\tan^{-1}(t)}{1+t^2}\,dt + c$$

$$= \frac{1}{2}\log(1+t^2)\tan^{-1}(t) - \frac{1}{2}\int\frac{2t}{1+t^2}\cdot\tan^{-1}t\,dt$$

$$+\int \frac{t\cdot\tan^{-1}t}{1+t^2}\,dt.$$

$$\Rightarrow \phi = \frac{1}{2}\log(1+t^2)\cdot\tan^{-1}(t) + c. \quad \text{—— (9)}$$

From (8), Put $t=0$. then $\phi(0)=0$.

In (9), put $t=0$, we get

$c=0$.

$$\phi(t) = \frac{1}{2}\log(1+t^2)\cdot\tan^{-1}(t).$$

$$\int_0^t \frac{\log(1+tx)}{1+x^2}\,dx = \frac{1}{2}\log(1+t^2)\cdot\tan^{-1}(t).$$

(3) Let $f(x,t) = (x+t^3)^2$

(i) Now $\int_0^1 f(x,t)\,dx$

$$= \int_0^1 (x+t^3)^2\,dx$$

$$= \int_0^1 (x^2 + 2xt^3 + t^6)\,dx$$

$$= \left[\frac{x^3}{3} + x^2t^3 + xt^6\right]_0^1$$

$$= \frac{1}{3} + t^3 + t^6.$$

(ii)

Now $\frac{d}{dt}\int_0^1 f(x,t)\,dx$

$$= \frac{d}{dt}\left(\frac{1}{3} + t^3 + t^6\right)$$

$$= 3t^2 + 6t^5$$

Now $\frac{\partial}{\partial t} f(x,t)$

$$= \frac{\partial}{\partial t}\left((x+t^3)^2\right)$$

$$= 2(x+t^3)(3t^2) = 6t^2(x+t^3)$$

Now $\int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx = \int_0^1 6t^2(x+t^3)\,dx$

$$= \left[\frac{6t^2x^2}{2} + 6t^5x\right]_0^1 = 3t^2 + 6t^5$$

(Proved)

(4)

$$f(x,t) = \begin{cases} \dfrac{xt^3}{(x^2+t^2)^2} & \text{if } x \neq 0,\ t \neq 0 \\ 0 & \text{if } x=0,\ t=0. \end{cases} \quad \text{--- (1)}$$

when $t \neq 0$.

$$F(t) = \int_0^1 f(x,t)\,dx = \int_0^1 \frac{xt^3}{(x^2+t^2)^2}\,dx$$

let $u = x^2 + t^2$ | limits.

$du = 2x\,dx$ | $x=0 \rightarrow u = t^2$, $x=1 \rightarrow u = 1+t^2$

$$= \int_{u=t^2}^{1+t^2} \frac{t^3}{2u^2}\,du$$

$$= \frac{t^3}{2} \left[-\frac{1}{u} \right]_{t^2}^{1+t^2}$$

$$= \frac{t^3}{2} \left[\frac{1}{t^2} - \frac{1}{1+t^2} \right]$$

$$F(t) = \frac{t}{2(1+t^2)} \quad \text{--- (2)}$$

from (1) $F(0) = \int_0^1 f(x,0)\,dx = 0.$

from (2) $F(0) = 0.$

$$\therefore \boxed{F(t) = \frac{t}{2(1+t^2)}} \quad \forall t. \quad \text{--- (3)}$$

Therefore, $F(t)$ is differentiable and.

$$F'(t) = \frac{1-t^2}{2(1+t^2)^2}, \quad \forall t. \qquad \text{—— (4)}$$

$$\underline{F'(0) = \frac{1}{2}.}$$

<u>Now</u> we compute $\frac{\partial}{\partial t} f(x,t)$ and then $\int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx$

At $x=0$, $\quad f(0,t) = 0 \quad \forall t.$

$f(0,t)$ is differentiable in t and $\frac{\partial}{\partial t} f(0,t) = 0$.

For $x \neq 0$, $\quad f(x,t)$ is differentiable in t and

$$\frac{\partial}{\partial t} f(x,t) = \frac{(x^2+t^2)^2 \cdot 3xt^2 - xt^3 \cdot 2(x^2+t^2) \cdot 2t}{(x^2+t^2)^4}$$

$$= \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}$$

So,

$$\frac{\partial}{\partial t} f(x,t) = \begin{cases} \dfrac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases} \qquad \text{—— (5)}$$

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Now,

$$\frac{\partial}{\partial t} f(x,t)\Big|_{t=0} = 0.$$

So, at $t=0$.

$$\frac{d}{dt}\int_0^1 f(x,t)\,dx\Big|_{t=0} = F'(0) = \frac{1}{2}$$

and $$\int_0^1 \left[\frac{\partial}{\partial t} f(x,t)\right]_{t=0} dx = 0.$$

So, both sides are not equal.

<u>Justification:</u>

$\frac{\partial}{\partial t} f(x,t)$ is not a continuous function of (x,t).

The denominator in equation 5 is $(x^2+t^2)^3$, has a problem near $(0,0)$.

$$\lim_{(x,t)\to(0,0)} \frac{xt^2(3x^2-t^2)}{(x^2+t^2)^3}$$

let $x = mt$

$$\Rightarrow \lim_{t\to 0} \frac{mt\,t^2(3m^2t^2-t^2)}{(m^2t^2+t^2)^3}$$

$$\Rightarrow \lim_{t\to 0} \frac{m(3m^2-1)}{t(m^2+1)^3}$$ which does not tends to 0 as $t \to 0$.

(5) Let $\phi(\alpha) = \int_0^\infty \frac{e^{-\alpha x}\sin x}{x}\,dx$, where $\alpha > 0$. —— (10)

$$\frac{d\phi}{d\alpha} = \int_0^\infty \frac{\partial}{\partial\alpha}\left(\frac{e^{-\alpha x}\sin x}{x}\right)dx$$

$$= -\int_0^\infty \frac{x e^{-\alpha x}\sin x}{x}\,dx$$

$$= -\int_0^\infty e^{-\alpha x}\sin x\,dx$$

using the result,

$$\int e^{-\alpha x}\sin x\,dx = \frac{-e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x)$$

we obtain.

$$\frac{d\phi}{d\alpha} = \left[\frac{e^{-\alpha x}}{1+\alpha^2}(\alpha\sin x + \cos x)\right]_0^\infty = \frac{-1}{1+\alpha^2}$$

Integrating w.r. to α.

$$\phi(\alpha) = -\tan^{-1}(\alpha) + c. \quad \text{—— (11)}$$

In (10) Put $\alpha = \infty$, we get

$$\phi(\infty) = \int_0^\infty \frac{0.\sin x}{x}\,dx = 0$$

From (11) we get $c = \frac{\pi}{2}$

Hence $\phi(\alpha) = \frac{\pi}{2} - \tan^{-1}(\alpha)$.

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(a) setting $\alpha = 0$, we obtain,

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}. \quad \text{—— (12)}$$

(b) substituting $x = ay$ in (12)

$$\int_0^\infty \frac{\sin x}{x}\, dx = \int_0^\infty \frac{\sin ay}{y}\, dy = \frac{\pi}{2}$$

(6) (i)

$$\text{Let } \phi(t) = \int_0^\infty \frac{(e^{-x} - e^{-tx})}{x}\, dx, \quad t > 0 \quad \text{—— (13)}$$

$$\frac{d\phi}{dt} = \int_0^\infty \frac{\partial}{\partial t}\left(\frac{e^{-x} - e^{-tx}}{x}\right) dx$$

$$= \int_0^\infty \frac{-e^{-tx}}{x}(-x)\, dx$$

$$= \int_0^\infty e^{-tx}\, dx$$

$$= \left[\frac{e^{-tx}}{-t}\right]_0^\infty = \frac{1}{t}.$$

Integrating

$$\phi = \log t + c. \quad \text{—— (14)}$$

In (13) put $t = 1$, we get $\phi(1) = 0$.

In (14) put $t = 1 \Rightarrow c = 0$.

Hence $\phi(t) = \log t$.

6 (ii) $\int_0^\infty \frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}e^{-ax}\right)dx$

Let $I = \int_0^\infty \frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}e^{-ax}\right)dx$ —— (1)

$\frac{dI}{da} = \int_0^\infty \frac{\partial}{\partial a}\left(\frac{e^{-x}}{x}\left(a-\frac{1}{x}+\frac{1}{x}e^{-ax}\right)\right)dx$

$= \int_0^\infty \frac{e^{-x}}{x}\left(1-e^{-ax}\right)dx$ —— (2)

$\frac{d^2I}{da^2} = \int_0^\infty e^{-(a+1)x}\,dx.$

$= \left[\frac{e^{-(a+1)x}}{-(a+1)}\right]_0^\infty = \frac{1}{a+1}$ —— (3)

Integrating,

$\frac{dI}{da} = \log(a+1) + C_1$ —— (4)

Put $a=0$ in (2) we get $\frac{dI}{da}=0.$

From (4) we get $C_1 = 0$

$$\frac{dI}{da} = \log(a+1)$$

$$I = \int \log(a+1)\, da$$

$$= a \cdot \log(a+1) - \int \frac{a}{a+1}\, da$$

$$= a \log(a+1) - \int \left(1 - \frac{1}{a+1}\right) da$$

$$= a \log(a+1) - a + \log(a+1) + C_2$$

$$= a \log(a+1) - a + C_2$$

$$= (a+1) \log(a+1) - a \quad \text{——— (5)}$$

Put $a = 0$ in (1), we get $I = 0$.

Put $a = 0$ in (5), we get

$$0 = C_2$$

Hence $I = (a+1)\log(a+1) - a$.

(iii) $\cancel{\int_0^{\pi} \frac{dx}{a + b\cos x}, \quad (a > 0, |b| < a)}$

6 (iii) $\int_0^1 \frac{x^a - x^b}{\log x} dx$

$$I = \int_0^1 \frac{x^a - x^b}{\log x} dx \quad \text{——— ①}$$

$$\frac{dI}{da} = \int_0^1 \frac{x^a \log x}{\log x} dx$$

$$= \int_0^1 x^a dx = \left[\frac{x^{a+1}}{a+1}\right]_0^1$$

$$\frac{dI}{da} = \frac{1}{a+1}$$

Integrating

$$I = \log(a+1) + C \quad \text{——— ②}$$

From ①, Put $a = b$. then $I = 0$.

From ② Put $a = b$.

we get $C = -\log(b+1)$

$$I = \log(a+1) - \log(b+1)$$

$$\boxed{I = \log\left(\frac{a+1}{b+1}\right)}.$$

(7) $I = \int_0^{\pi} \frac{dx}{a+b\cos x}$.

$= \int_0^{\pi} \frac{dx}{a\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)+b\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)}$

$= \int_0^{\pi} \frac{dx}{(a+b)\cos^2\frac{x}{2} + (a-b)\sin^2\frac{x}{2}}$

$= \frac{1}{a-b}\int_0^{\pi} \frac{\sec^2\frac{x}{2}}{\frac{a+b}{a-b}+\tan^2\frac{x}{2}}\,dx$

$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}}\left[\tan^{-1}\left\{\tan\frac{x}{2}\cdot\sqrt{\frac{a-b}{a+b}}\right\}\right]_0^{\pi}$

$= \frac{2}{a-b}\sqrt{\frac{a-b}{a+b}} \times \frac{\pi}{2}$

$= \frac{\pi}{\sqrt{a^2-b^2}}$

Now diff w. r. to a

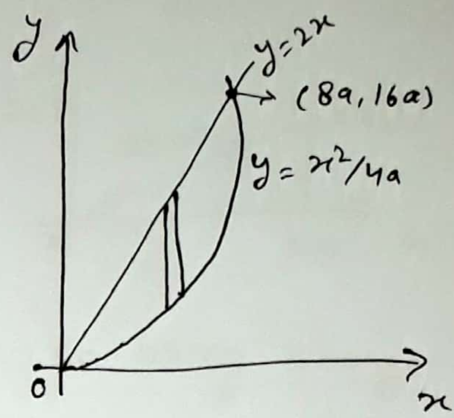
$\frac{dI}{da} = -\frac{1}{2}\frac{2\pi a}{(a^2-b^2)^{3/2}}$.

or, $\int_0^{\pi} \frac{\partial}{\partial a}\left(\frac{1}{a+b\cos x}\right)dx = -\frac{1}{2}\frac{2\pi a}{(a^2-b^2)^{3/2}}$

or, $\int_0^{\pi} \frac{-1}{(a+b\cos x)^2}\,dx = \frac{-\pi a}{(a^2-b^2)^{3/2}}$

or, $\int_0^{\pi} \frac{dx}{(a+b\cos x)^2} = \frac{\pi a}{(a^2-b^2)^{3/2}}$.

⑧ (i) $\iint_D xy\,dx\,dy$ D: x-axis, $y = 2x$, $y = \dfrac{x^2}{4a}$



Point of intersection of $y = 2x$, and $y = \dfrac{x^2}{4a}$ is $(8a, 16a)$.

$y : \dfrac{x^2}{4a} \longrightarrow 2x$

$x : 0 \longrightarrow 8a.$

$$I = \int_{x=0}^{8a} \int_{y=x^2/4a}^{2x} xy\,dy\,dx = \int_0^{8a} \left[\frac{xy^2}{2}\right]_{\frac{x^2}{4a}}^{2x} dx$$

$$= \int_0^{8a} \frac{x}{2}\left(4x^2 - \frac{x^4}{16a^2}\right) dx = \int_0^{8a} \left(2x^3 - \frac{x^5}{32a^2}\right) dx$$

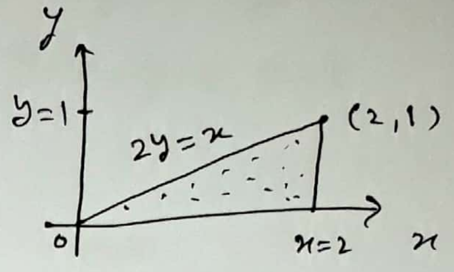
$$= \left[\frac{x^4}{2} - \frac{x^6}{32 \times 6a^2}\right]_0^{8a} = \frac{4096\,a^4}{2} - \frac{4096 \times \cancel{64}^{\,2}\, a^6}{\cancel{32} \times \cancel{6}_{\,3}\, a^2}$$

$$= 4096\,a^4 \times \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{2048}{3}a^4$$

Ans.

8 (ii) $I = \iint_D e^{x^2}\,dx\,dy \qquad D: 2y \le x \le 2 \;\&\; 0 \le y \le 1.$

$$I = \int_{y=0}^{1} \int_{x=2y}^{2} e^{x^2}\,dx\,dy$$



Here we cannot integrate w.r.t x, so using change of order, we first integrate wrt y.

$x: 0 \to 2$

$y: 0 \to x/2$

$$I = \int_{x=0}^{2} \int_{y=0}^{x/2} e^{x^2}\,dy\,dx$$

$$= \int_0^2 \left[y e^{x^2} \right]_0^{x/2} dx$$

$$= \int_0^2 \frac{x e^{x^2}}{2}\,dx = \frac{1}{4} \int_0^2 2x e^{x^2}\,dx$$

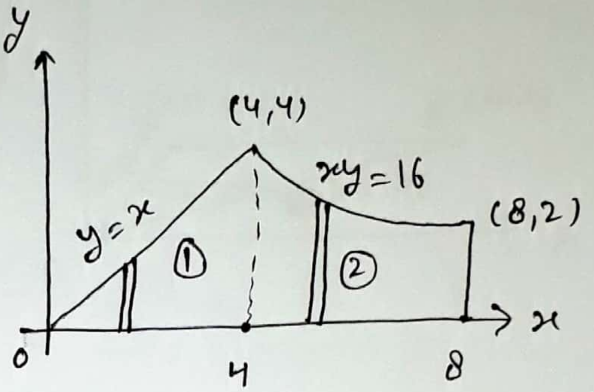
$$= \frac{1}{4} \left[e^{x^2} \right]_0^2 = \frac{1}{4}(e^4 - 1)$$

8 (iii)

$$\iint_D x^2\,dx\,dy,$$ where D is the region in the first quadrant bounded by the hyperbola $xy=16$ & lines $y=x$, $y=0$, $x=8$

Point of intersection of $xy=16$ & $y=x$ is $(4,4)$,

and $xy=16$ & $x=8$ is $(8,2)$



$$\iint_D x^2\,dx\,dy = \iint_{①} x^2\,dx\,dy + \iint_{②} x^2\,dx\,dy$$

$$= \int_0^4\int_0^x x^2\,dx\,dy + \int_4^8\int_0^{16/x} x^2\,dx\,dy$$

Limits for region ①

$x: 0 \to 4$

$y: 0 \to x$

Limits for region ②

$x: 4 \to 8$

$y: 0 \to \frac{16}{x}$

$$= \int_0^4 x^2(x-0)\,dx + \int_4^8 x^2\left(\frac{16}{x}-0\right)dx$$

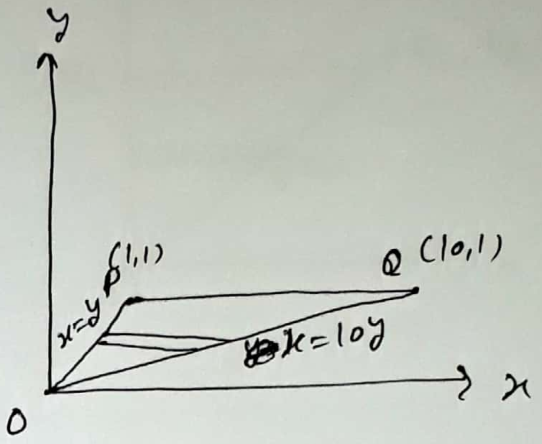
$$= \int_0^4 x^3\,dx + \int_4^8 16x\,dx$$

$$= \left[\frac{x^4}{4}\right]_0^4 + 16\left[\frac{x^2}{2}\right]_4^8 = 64 + \frac{16}{2}(64-16)$$

$$= 448$$

8 (iv)

$$\iint_D \sqrt{xy - y^2}\, dy\, dx .$$ D is a triangle with vertices (0,0), (10,1), (1,1).



Equation of OP is $x = y$

Equation of OQ is $x = 10y$

Limits

$y : 0 \to 1$

$x : y \to 10y$

$$I = \int_{y=0}^{1} \int_{x=y}^{10y} \sqrt{xy - y^2}\, dx\, dy$$

$$= \int_0^1 \left[\frac{(xy - y^2)^{3/2}}{3/2\, y} \right]_y^{10y} dy$$

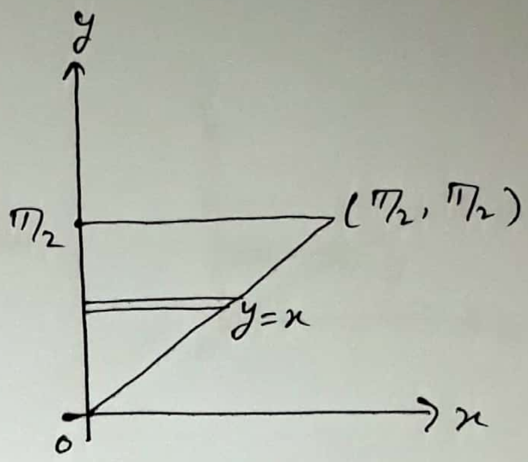
$$= \int_0^1 \frac{2}{3y} \left[(10y^2 - y^2)^{3/2} - 0 \right] dy$$

$$= \int_0^1 \frac{2}{3y} (9y^2)^{3/2}\, dy$$

$$= \int_0^1 \frac{2}{3y} \cdot (3y)^3\, dy = \int_0^1 18y^2\, dy = \frac{18}{3} = \underline{6}$$

9(i)

$$I = \int_{x=0}^{\pi/2} \int_{y=x}^{\pi/2} \frac{\sin y}{y} \, dy \, dx$$



changing the order of integration

limits.

$y : 0 \rightarrow \pi/2$

$x : 0 \rightarrow y$

$$I = \int_{y=0}^{\pi/2} \int_{x=0}^{y} \frac{\sin y}{y} \, dx \, dy$$

$$I = \int_{0}^{\pi/2} \frac{\sin y}{y} [x]_0^y \, dy$$

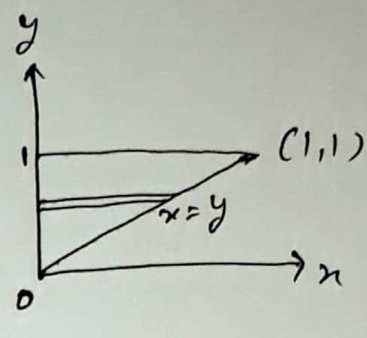
$$I = \int_{0}^{\pi/2} \sin y \, dy$$

$$I = [-\cos y]_0^{\pi/2}$$

$$I = -[0 - 1]$$

$$I = 1$$

9(ii) $I = \int_{x=0}^{1}\int_{y=x}^{1} e^{y^2}\,dx\,dy$



Changing the order of integration.

Limits.

$y : 0 \to 1$

$x : 0 \to y$

$$I = \int_{y=0}^{1}\int_{x=0}^{y} e^{y^2}\,dx\,dy$$

$$= \int_{y=0}^{1} y e^{y^2}\,dy$$

$$= \frac{1}{2}\int_{0}^{1} 2y e^{y^2}\,dy$$

$$= \frac{1}{2}\left[e^{y^2}\right]_0^1$$

$$\boxed{I = \frac{1}{2}(e-1)}$$

9(iii)

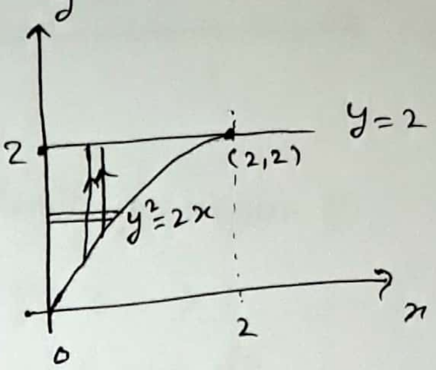
$$I = \int_{y=0}^{2} \int_{x=0}^{y^2/2} \frac{y}{\sqrt{x^2+y^2+1}} \, dx \, dy$$

changing the order of integration.

Limits

$x : 0 \to 2$

$y : \sqrt{2x} \to 2$



$$I = \int_{x=0}^{2} \int_{y=\sqrt{2x}}^{2} \frac{2 \times y}{2\sqrt{x^2+y^2+1}} \, dy \, dx$$

$$I = \frac{1}{2} \int_{x=0}^{2} \left[\frac{(x^2+y^2+1)^{1/2}}{1/2} \right]_{y=\sqrt{2x}}^{2} dx$$

$$I = \int_0^2 \left(\sqrt{x^2+5} - \sqrt{x^2+2x+1} \right) dx$$

$$I = \int_0^2 \left[\sqrt{x^2+5} - (x+1) \right] dx$$

Using. $\left[\int \sqrt{x^2+a^2} \, dx = \frac{x\sqrt{x^2+a^2}}{2} + \frac{a^2}{2} \ln\left(x+\sqrt{x^2+a^2}\right). \right]$

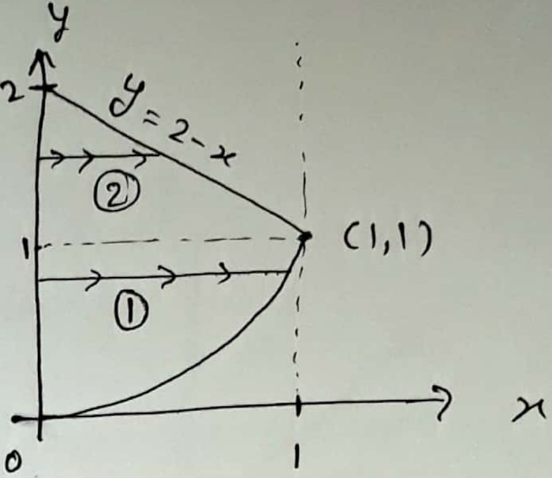
$$I = \left[\frac{x\sqrt{x^2+5}}{2} + \frac{5}{2} \ln\left(x+\sqrt{x^2+5}\right) - \frac{x^2}{2} - x \right]_0^2$$

$$I = 3 + \frac{5}{2} \left(\ln 5 - \ln \sqrt{5} \right) - 4$$

$$\boxed{I = \frac{5}{4} \ln 5 - 1}$$

9(iv) $I=\int_{x=0}^{1}\int_{y=x^2}^{2-x} xy\,dx\,dy$

Changing the order of integration



$$I=\iint_{①} xy\,dx\,dy+\iint_{②} xy\,dx\,dy$$

Limits for region ①.

$y = 0 \longrightarrow 1$

$x = 0 \longrightarrow \sqrt{y}$

Limits for region ②

$y = 1 \longrightarrow 2$

$x = 0 \longrightarrow 2-y.$

$$I=\int_{y=0}^{1}\int_{x=0}^{\sqrt{y}} xy\,dx\,dy+\int_{y=1}^{2}\int_{x=0}^{2-y} xy\,dx\,dy$$

$$I=\int_0^1 y\left[\frac{x^2}{2}\right]_0^{\sqrt{y}}dy+\int_1^2 y\left[\frac{x^2}{2}\right]_0^{2-y}dy$$

$$I=\int_0^1 \frac{y^2}{2}\,dy+\int_1^2 \frac{y(2-y)^2}{2}\,dy$$

$$I=\frac{1}{6}+\frac{1}{2}\int_1^2(4y+y^3-4y^2)\,dy$$

$$I=\frac{1}{6}+\frac{1}{2}\left[2y^2+\frac{y^4}{4}-\frac{4}{3}y^3\right]_1^2$$

$$I=\frac{1}{6}+\frac{1}{2}\left[\left(8+4-\frac{32}{3}\right)-\left(2+\frac{1}{4}-\frac{4}{3}\right)\right]$$

$$I=\frac{1}{6}+\frac{1}{2}\left[10-\frac{1}{4}-\frac{28}{3}\right]=I=\frac{1}{6}+\frac{5}{24}=\frac{3}{8}$$

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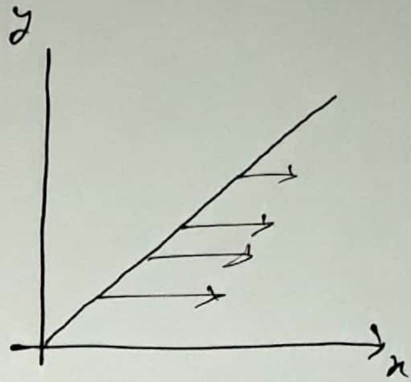
9(v) $I = \int_{x=0}^{\infty} \int_{y=0}^{x} e^{-xy} y \, dy \, dx$



Changing the order of integration

Limits

$y : 0 \to \infty$

$x : y \to \infty.$

$$I = \int_{y=0}^{\infty} \int_{x=y}^{\infty} e^{-xy} y \, dy \, dx$$

$$I = \int_{y=0}^{\infty} y \left[\frac{e^{-xy}}{-y} \right]_{y}^{\infty} dy$$

$$I = \int_{0}^{\infty} - \left[0 - e^{-y^2} \right] dy$$

$$I = \int_{0}^{\infty} e^{-y^2} dy = \frac{\sqrt{\pi}}{2}$$