Problem Set - 10

AUTUMN 2017

MATHEMATICS-I (MA10001)

1. Find the following limits (if exists)
   (a) $\lim_{z \to 0} \dfrac{(\text{Re}(z) - \text{Im}(z))^2}{|z|^2}$
   (b) $\lim_{z \to 0} \left[\dfrac{1}{1 - e^{\frac{1}{x}}} + iy^2\right]$
   (c) $\lim_{z \to 0} \left(\dfrac{z}{\bar{z}}\right)^2$
   (d) $\lim_{z \to 0} \dfrac{z}{|z|}$
2. Test the continuity of the following functions:
$$(a) \quad f(z) = \begin{cases} \frac{\text{Im}(z)}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \qquad (b) \quad f(z) = \begin{cases} \frac{z^2}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases} \qquad (c) \quad f(z) = \begin{cases} \frac{\text{Re}(z^2)}{|z|^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
3. Examine the continuity of $f(z)$ at $z = 0$, where $f(z)$ is
$$f(z) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
4. Show that
$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
   satisfies Cauchy Reimann equations at $z = 0$ but $f'(0)$ does not exist.
5. Show that for the function,
$$f(z) = \begin{cases} \frac{xy^2(x+iy)}{x^2+y^4} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$$
   $f'(0)$ does not exist but it satisfies Cauchy Reimann equations at $(0, 0)$.
6. Using Cauchy Reimann equations show that
   (a) $f(z) = |z|^2$ is not analytic at any point.
   (b) $f(z) = \bar{z}$ is not analytic at any point.
   (c) $f(z) = \dfrac{1}{z}$, $z \neq 0$ is analytic at all points except at the point $z = 0$.
7. Show that the function $Logz$ is analytic for all z except the point $\{z : \text{Re}\, z \leq 0, \ \text{Im}\, z = 0\}$.
8. Let $f(z) = u + iv$ be analytic in a domain D. Prove that f is constant in D if any one of the followings hold.
   (a) $f'(z)$ vanishes in D.
   (b) $\text{Re}\, f(z) = u =$ constant.
   (c) $\text{Im}\, f(z) = v =$ constant.
   (d) $|f(z)| =$ constant (non zero).
9. Show that the function $u = \cos x \cosh y$ is harmonic. Find its harmonic conjugate.
10. Show that following functions are harmonic:
   (a) $u(x, y) = 2x + y^3 - 3x^2y$

(b) $v(x, y) = e^x \sin y$

and find their harmonic conjugates and the corresponding analytic functions $f(z)$.

11. $u(r, \theta) = r^2 \cos 2\theta$ is harmonic. Find its conjugate harmonic function and the corresponding analytic function $f(z)$.

12. If $f(z)$ is analytic function of z, then prove that

    (a) $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} = 4\dfrac{\partial^2}{\partial z \partial \bar{z}}$.

    (b) $\left(\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$.

13. If $\nabla = \left(\dfrac{\partial}{\partial x} + i\dfrac{\partial}{\partial y}\right)$, then prove that (a) $\dfrac{\partial}{\partial x} = \dfrac{\partial}{\partial z} + \dfrac{\partial}{\partial \bar{z}}$ (b) $\dfrac{\partial}{\partial y} = i\left(\dfrac{\partial}{\partial z} - \dfrac{\partial}{\partial \bar{z}}\right)$ (c) $\nabla = 2\dfrac{\partial}{\partial \bar{z}}$.

14. Find the values of constants a, b, c and d such that the function $f(z) = (x^2 + axy + by^2) + i(cx^2 + dxy + y^2)$ is analytic.

15. Suppose $f(z) = u + iv$ is analytic at $z_0 \neq 0$. Show that

$$f'(z_0) = -\frac{i}{z_0}\left(\frac{\partial u}{\partial \theta} + i\frac{\partial v}{\partial \theta}\right)$$

at $z = z_0$, where (r, θ) are the polar coordinates.