CS 60047 Autumn 2020

Advanced Graph Theory

Instructor

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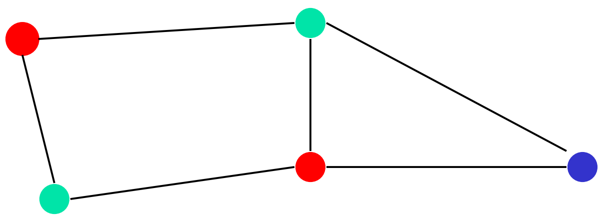
Graph Coloring Problem

- **Graph coloring** is an assignment of *"colors"*, to certain objects in a graph. Such objects can be vertices, edges, faces, or a mixture of the above

- Numerous applications: scheduling, register allocation in a microprocessor, frequency assignment in mobile radios, pattern matching, and so on …

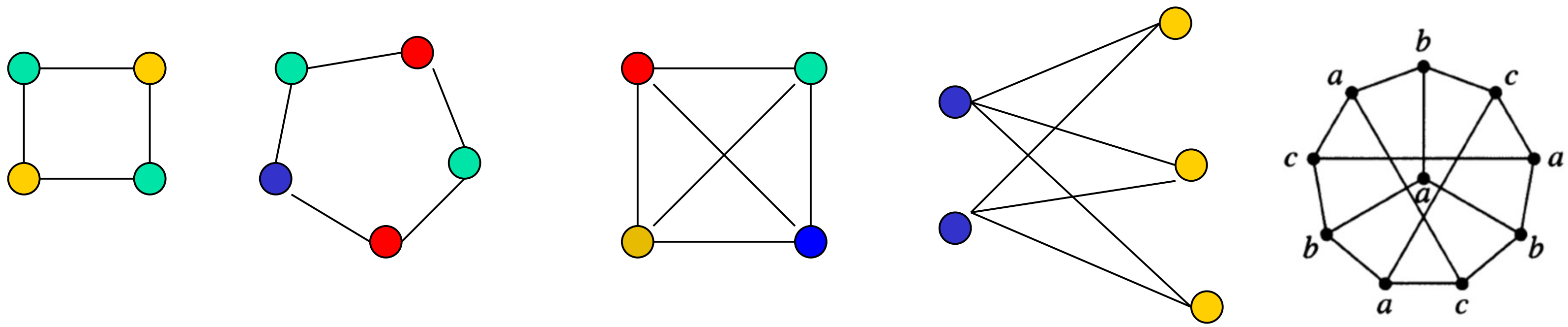
Vertex Coloring

- Assignment of colors to the vertices of the graph such that no two adjacent vertices are assigned the same color => proper coloring
- *Chromatic number* (χ): least number of colors needed to color the graph
- A graph that can be assigned (proper) *k*-coloring is *k*-colorable, and it is *k*-chromatic if its chromatic number is exactly *k*
- Equivalent to covering vertices with minimum number of independent sets, or minimum clique-cover of vertices in the complementary graph => chromatic partitioning



Vertex Coloring

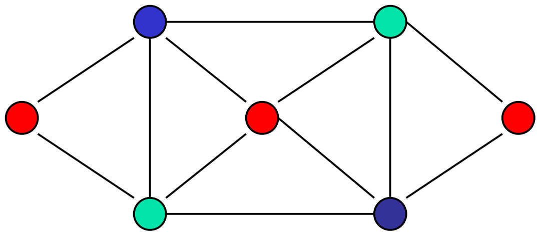
- The problem of finding a minimum coloring of a graph is NP-Hard
- The corresponding decision problem (Is there a coloring which uses at most k colors?) is NP-complete
- The chromatic number χ for C_n = 3 (n is odd) or 2 (n is even), $K_n = n$, $K_{m,n} = 2$



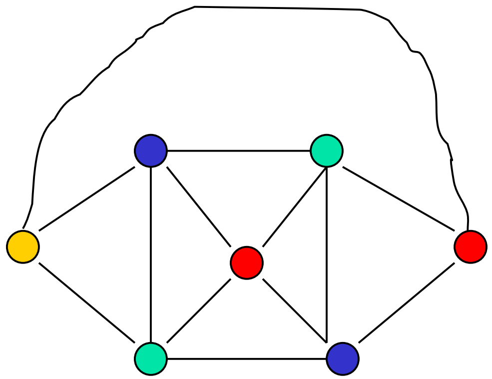
	C_4	C_5	K_4	$K_{2,3}$	PG
$\chi(G)=$	2	3	4	2	3

Planar graph coloring

- **The Four Color Theorem:** the chromatic number of a planar graph is at most 4



G1



G2

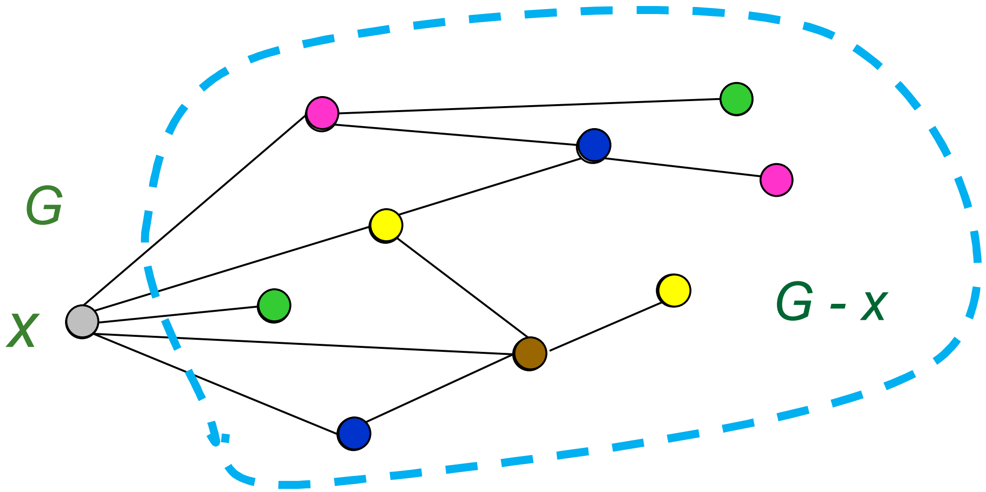
Coloring Planar Graphs

Theorem: Every planar graph G is 6-colorable

Proof: By easy induction

Basis: if $v \leq 6$, G is 6-colorable

- Find a vertex x of degree 5 or less (such a vertex always exists in a planar graph)
- Remove x; the remaining graph G – x is 6-colorable by induction hypothesis. Next, color x with a color not used by its (at most) 5 neighbors in G – x



Coloring Planar Graphs

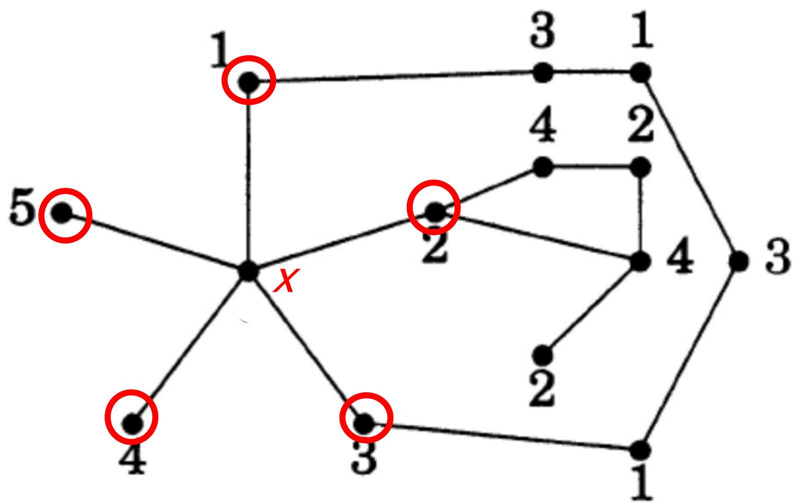
Theorem (Heawood 1890)

Every planar graph G is 5-colorable

Proof: By induction

Basis: if $v \leq 5$, G is 5-colorable

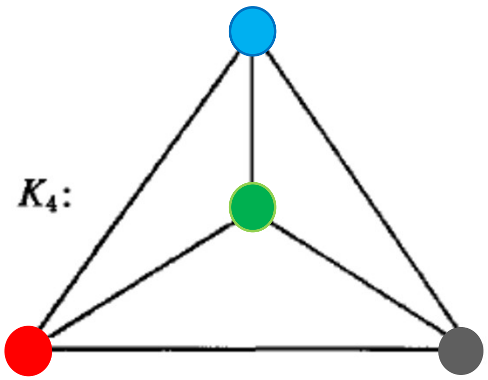
- find a vertex x of degree 5 or less (such a vertex always exists in a planar graph)
- the remaining graph G – x is 5-colorable by induction hypothesis
- use arguments based on Kempe chains and planarity

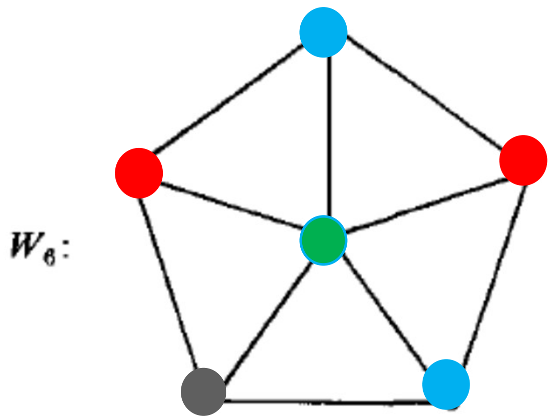


Coloring Planar Graphs

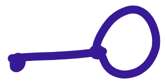
Four color theorem (Appel and Haken 1976)
Every planar graph G is 4-colorable

Result: Every planar graph with fewer than four triangles is 3-colorable





Colorability

Graphs with loops → uncolorable 

⇒ simple graphs

For any graph G, $\chi(G) \geq \omega(G)$

chromatic number

clique number

$H \subseteq G \Rightarrow \chi(H) \leq \chi(G)$

Also, $\chi(G) \geq \dfrac{|V|}{\alpha(G)}$

independence number

↳ size of the maximum independent set

$\Rightarrow$ G is 2-colorable if and only if G is bipartite (non-empty G) $\boxtimes$

Special cases

1. G is planar $\Rightarrow \chi(G) \le 4$.

2. G is K_n $\Rightarrow \chi(G) = n$

$\Rightarrow \chi(\bar{G}) = 1$

3. $\forall$ any integer $n \geq 3$,

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{if } n \text{ is odd} \end{cases}$$

4. For any G, $1 \le \chi(G) \le n$

Cartesian product of two graphs G and H

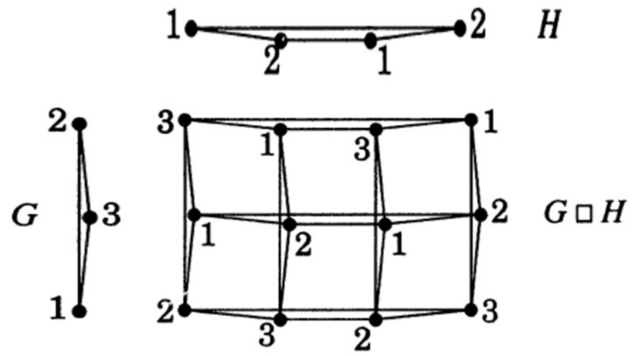
$G \square H \Rightarrow$ vertex set $V(G) \times V(H)$

Edgeset $\Rightarrow$ (u, v) — (u', v') if

(1) $u = u'$ and $(v, v') \in E(H)$, or

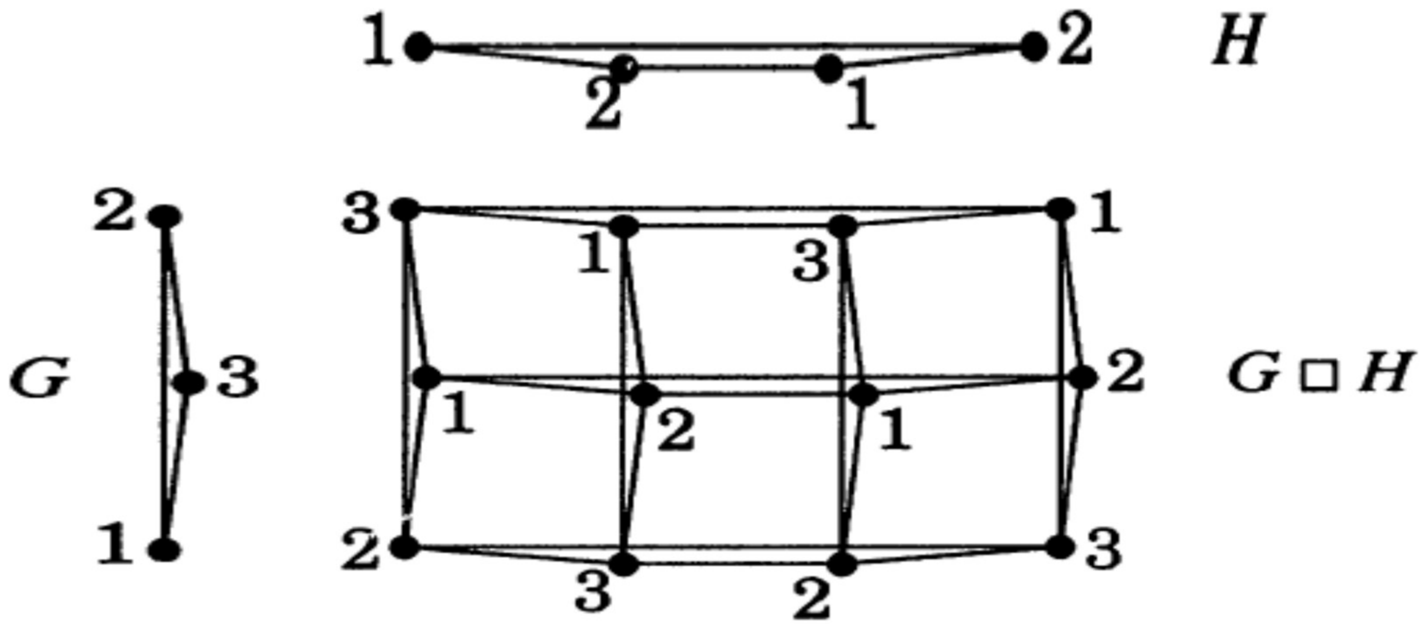
(2) $v = v'$ and $(u, u') \in E(G)$

$G \square H \cong H \square G$



Theorem: $\chi(G \square H) = \max\{\chi(G), \chi(H)\}$

Proof: Clear



Theorem: For every graph G, $\chi(G) \leq 1 + \Delta(G)$

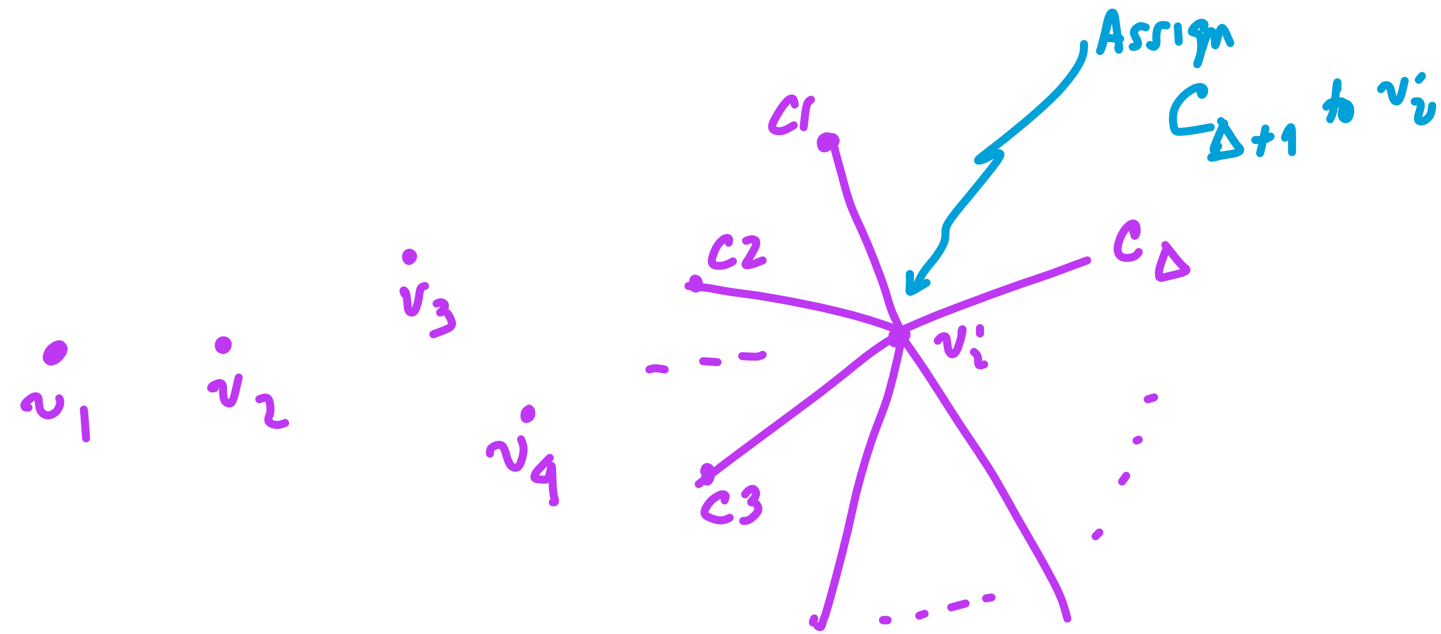
largest degree of a vertex in G

Proof by construction:

1. Label the vertices as $v_1, v_2, \ldots, v_i, \ldots, v_n$

2. Incrementally color the vertices in the same order
   (a) assign c_1 to v_1; if v_2 is adjacent to v_1, assign color c_2 to v_2, else assign color c_1 to v_2.
   (b) in general, assign to v_i, the smallest-indexed available color.

$d(v_i) = \Delta$



1) ordering of vertices
2) assign the least-indexed available color at every step.

Since $d_i \leq \Delta(G)$, some colors must be available for coloring v_i from the color set $\{c_1, c_2, \ldots, c_{\Delta+1}\}$

Hence, $\chi(G) \leq \Delta(G) + 1$.

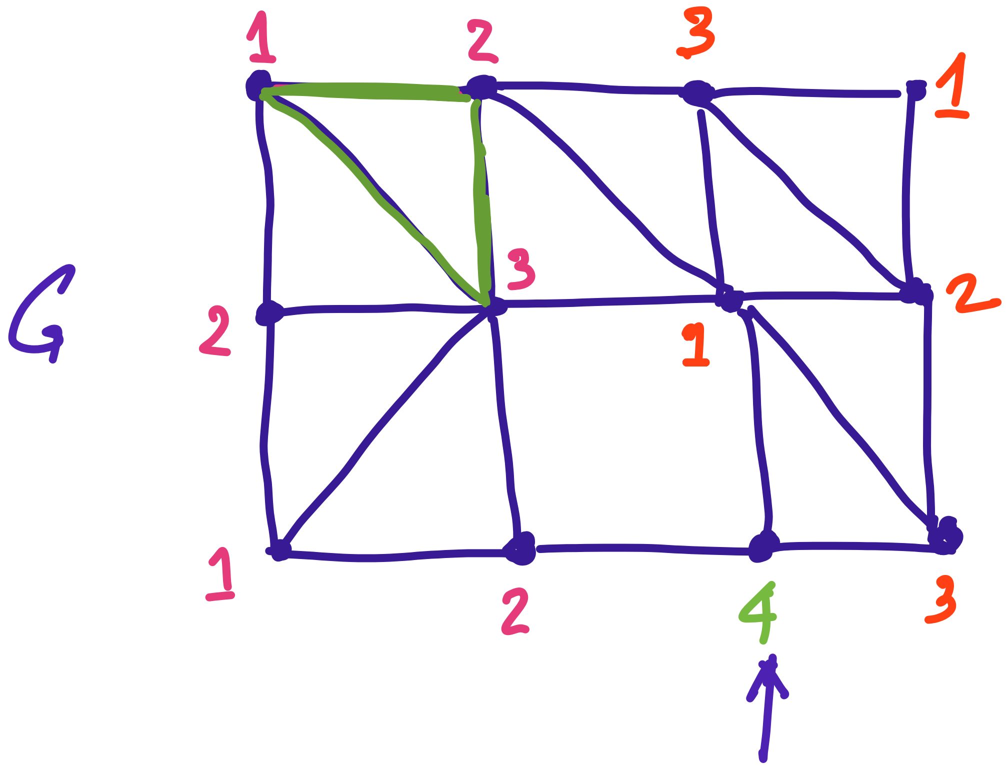
The upper bound is achieved for many graphs. For example, $\Delta(K_n) = n-1$ & $\chi(K_n) = n$.

Also, $\chi(C_n) = 3 = 1 + \Delta(C_n)$

Theorem (Brooks): If G is a connected graph of order n, then $\chi(G) \leq \Delta(G)$ unless $G = K_n$, or $n \geq 3$ is odd and $G = C_n$

Proof: See Textbook DW 5.1.22.

Example



$$\chi(G) \geq 3$$

$$\Rightarrow \chi(G) = 4$$

Graphs with large chromatic numbers

We know that $\chi(G) \geq \omega(G)$ how large?

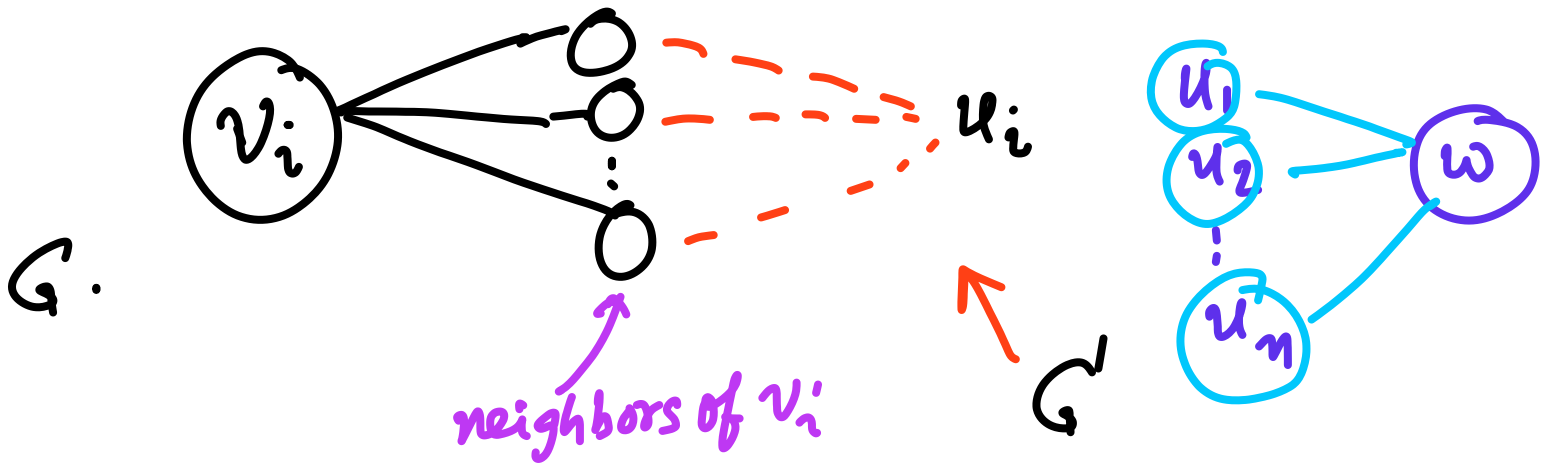
Question : can $\omega(G) = 2$, while $\chi(G)$ arbitrarily large?

Answer: yes! we can construct large graphs, which are triangle-free, but $\chi(G)$ is arbitrarily large.

Mycielski's Construction

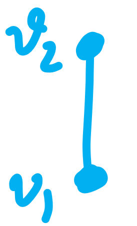
simple graph $G \implies G'$ ← simple graph

$v_1, v_2, \ldots, v_{\underline{\underline{n}}} \implies G \cup \{u_1, u_2, \ldots, u_{\underline{\underline{n}}}, w\}$



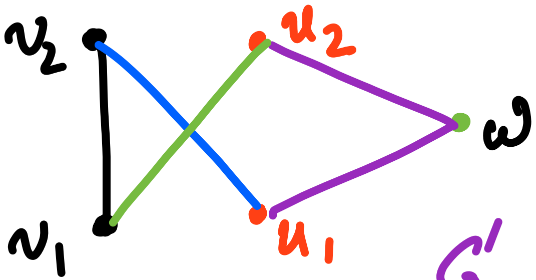
Mycielski's Construction

Example



G

$\Longrightarrow$



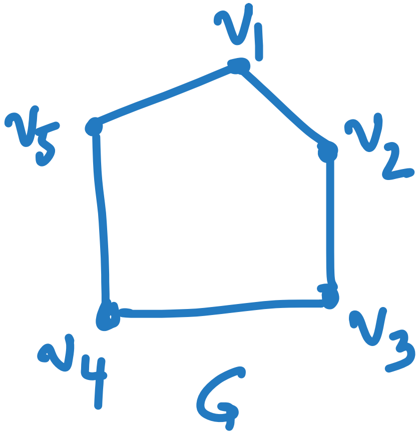
$\equiv C_5$

G'

$\omega(G) = 2, \quad \chi(G) = 2$

$\omega(G) = 2, \quad \chi(G) = 3$

Mycielski's Construction

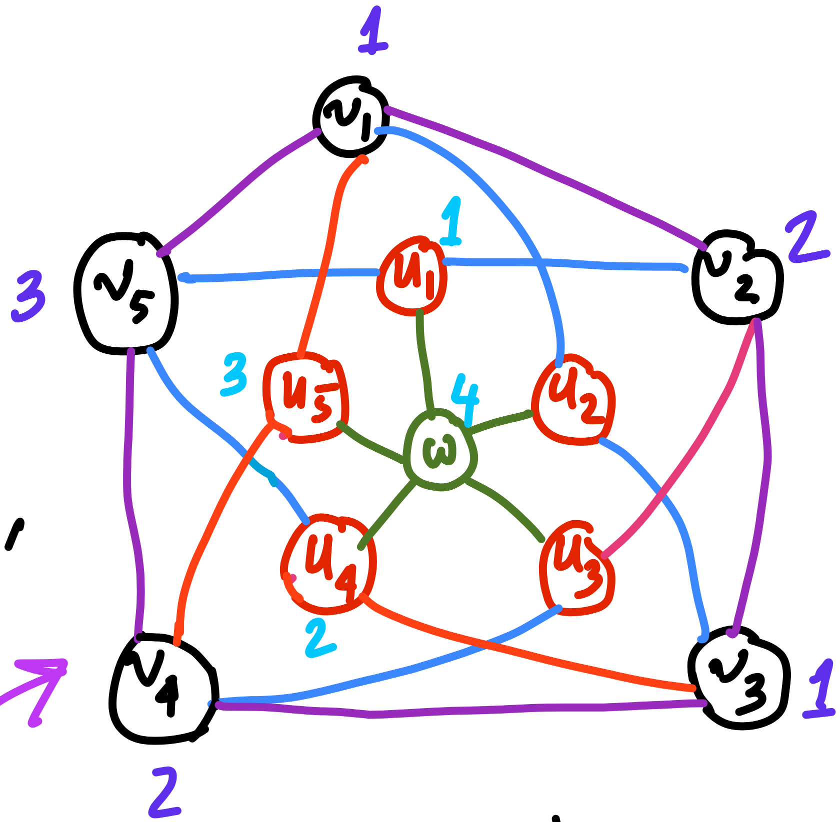


$\omega(G) = 2$

$\chi(G) = 3$

$\Rightarrow$

G'



4-chromatic Grötzsch Graph

$\omega(G') = 2$

$\chi(G') = 4$

Mycielski's Construction

$G \Longrightarrow G'$

$|V| = n$

$\omega = 2$

$\chi = k$

$|V| = 2n+1$

$\omega = 2$

$\chi = k+1$

Triangle-free large graphs

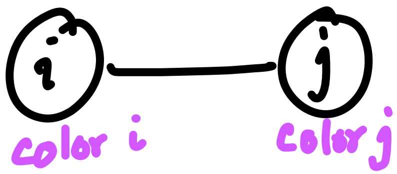
Extremal Problems

What are the smallest and largest k-chromatic graphs with n vertices?

Theorem: Every k-chromatic graph G with n vertices has at least $\binom{k}{2}$ edges.

Proof: G is optimally colored with k colors, say $1, 2, \ldots, i, j, \ldots k$.

$\exists$ an edge



in G; $\Rightarrow \binom{k}{2}$ distinct pairs of colors $\boxtimes$

The maximization problem: What is the largest r-chromatic graph with n vertices?

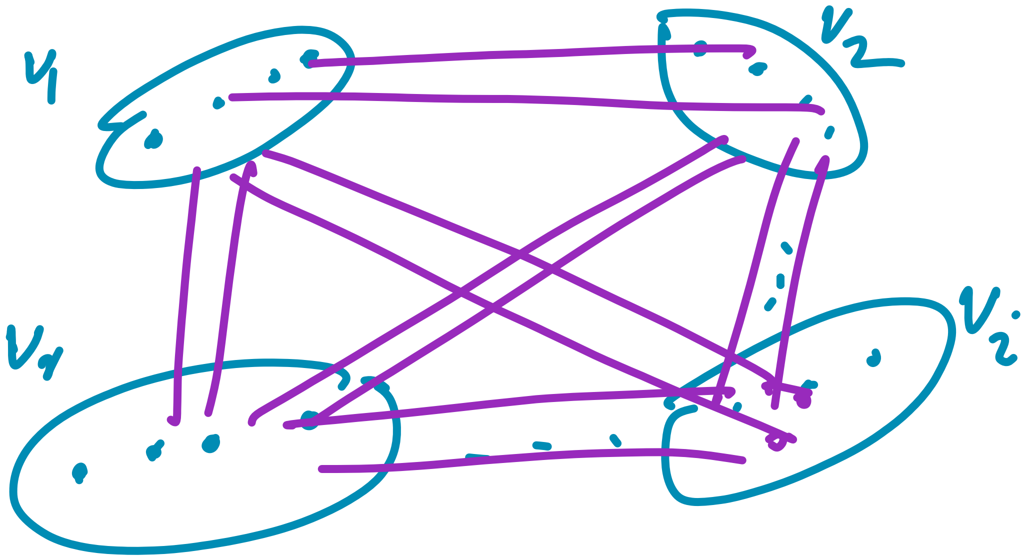
r-chromatic graph $\Longrightarrow$ G is r-partite

Construct Turán graph $T_{n,r}$ $\Rightarrow$ Complete r-partite graph with n vertices s.t. partite sets differ in size by at most 1

$T_{n,r}$:

$$\sum_{i=1}^{r} |V_i| = n$$

$$\forall_{i,j} \; |V_i| - |V_j| \leq 1.$$



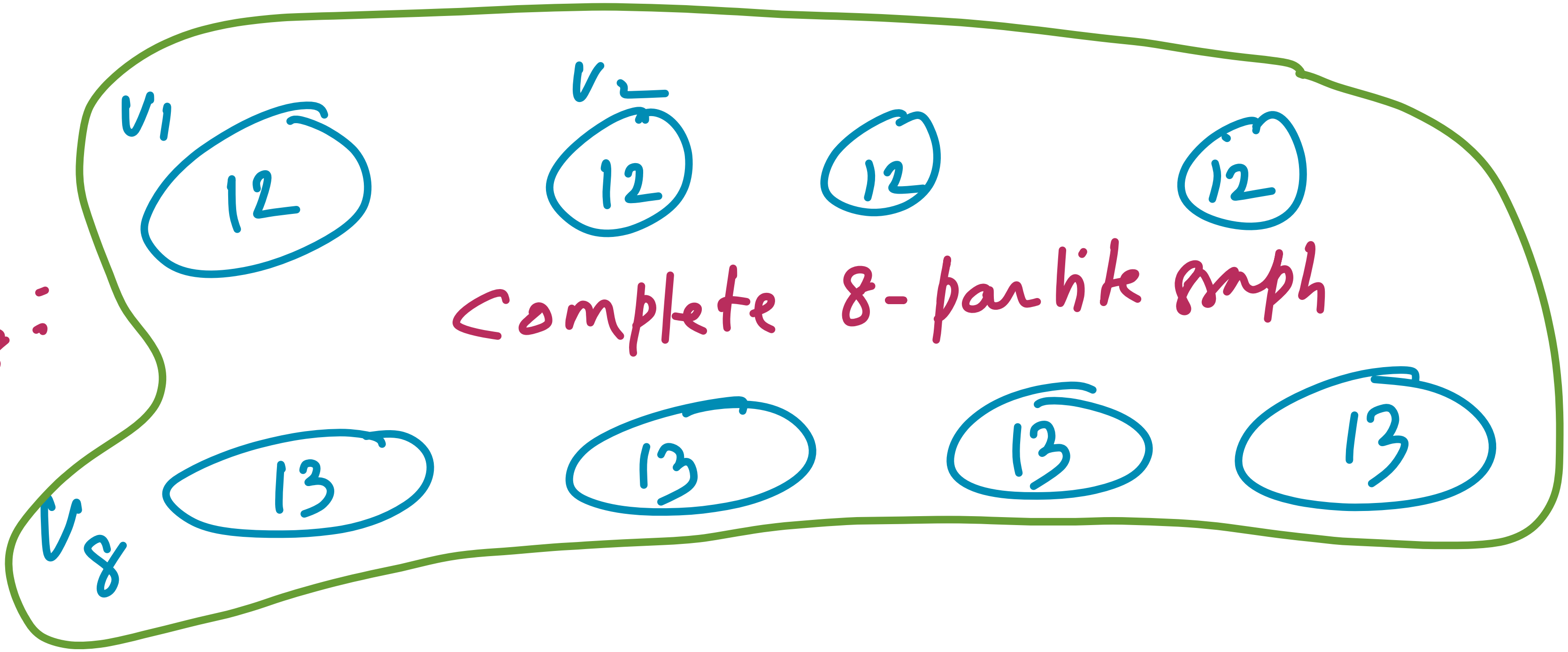
r-colorable : $\chi(G) = r$

Example:

$n = 100, \quad r = 8$

$$\left\lfloor \frac{n}{r} \right\rfloor = 12$$

$T_{100,8}$:



$$|V_i| = \left\lfloor \frac{n}{r} \right\rfloor \text{ or } \left\lceil \frac{n}{r} \right\rceil$$

Lemma (DW: 5.2.8)

Among simple r-colorable graphs with n vertices, $T_{n,r}$ is the unique graph with maximum number of edges.

Proof:

X $\qquad$ Y

$|X| \qquad |Y|$

when $|X| \cdot |Y|$ is maximized?

$|X| - |Y|$ minimized

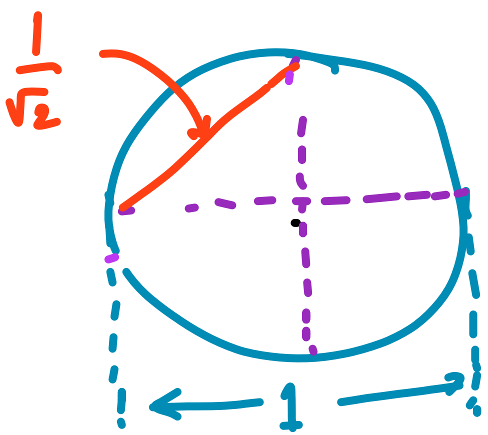
Theorem (Turan 1941)

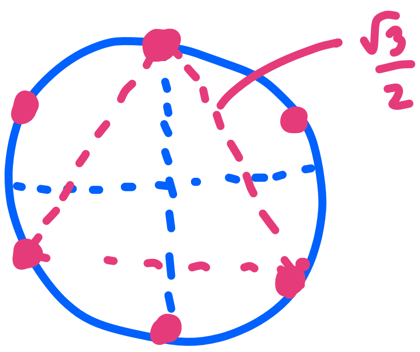
Among all n-vertex simple graphs with no $(r+1)$-clique, $T_{n,r}$ has the maximum number of edges.

Proof: Reading Assignment (DW: 5.2.9)

Example

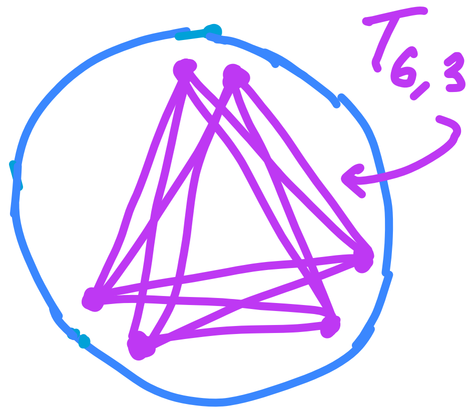
In a circular city of diameter 1, we want to position 6 police cars such that the pairs of cars separated by distance more than $\frac{1}{\sqrt{2}}$, are maximized.





6 cars equispaced around circumference

$\#$ good pairs $= \binom{6}{2} - 6 = 9$

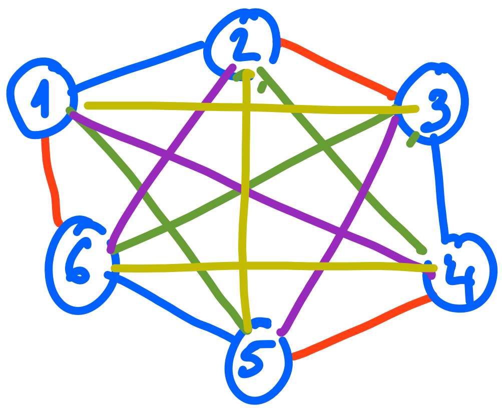


$\#$ good pairs

$= \underline{\underline{12}}$ ✓

Edge coloring

In a league with 6 teams, each team has to play with every other. A team can play one match everyday. Schedule the games in fewest possible days.



Solution: In one day 3 teams can play $\Rightarrow$ "matching"

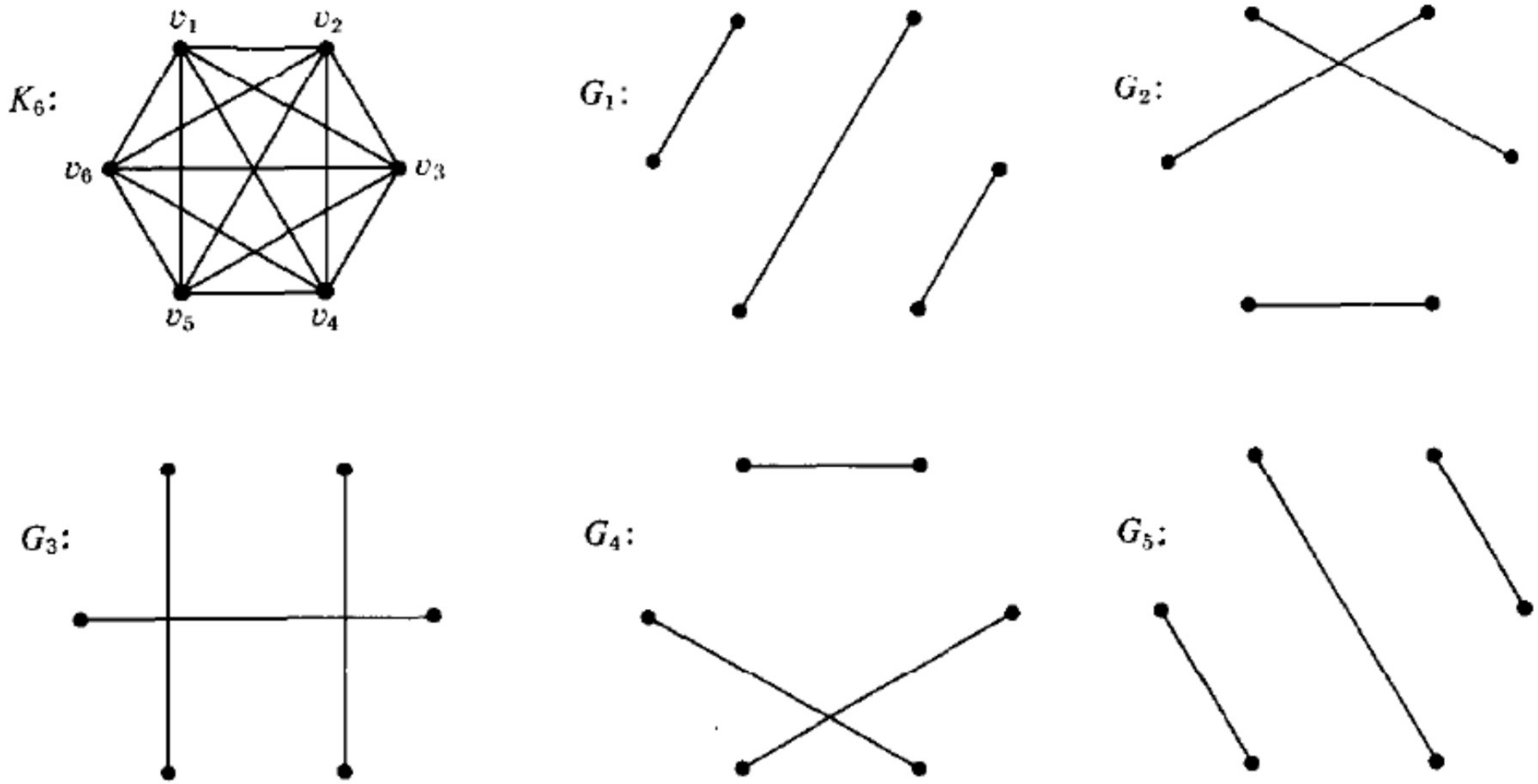
Total number of matches $= \binom{6}{2} = 15$

Needs at least 5 days

$\Rightarrow$ Edge coloring with 5 colors

optimal

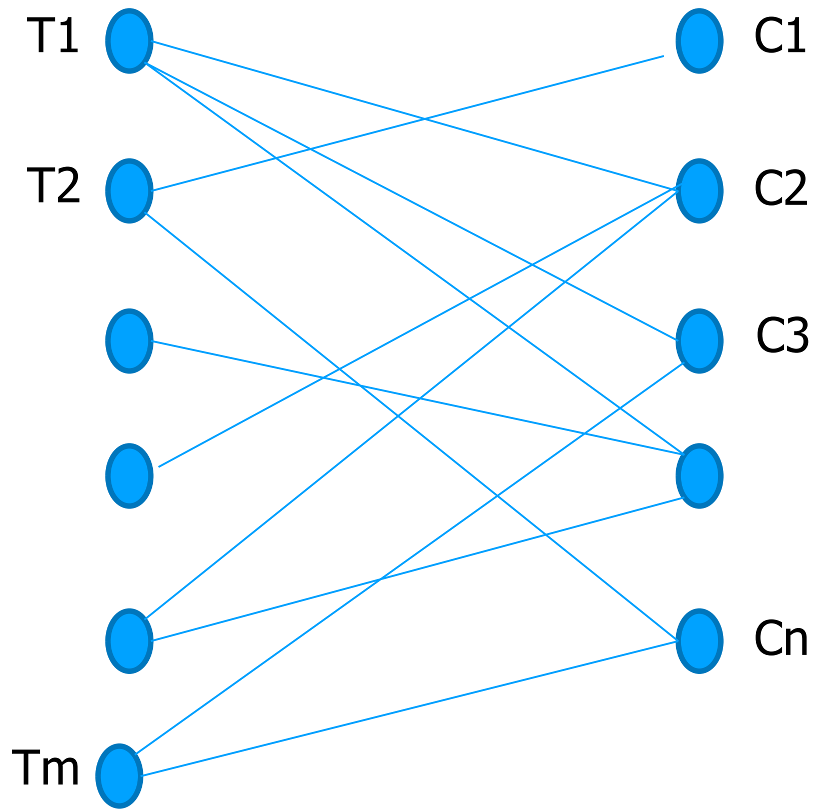
Decomposing edges of K_6 into edge-disjoint 1-regular graphs



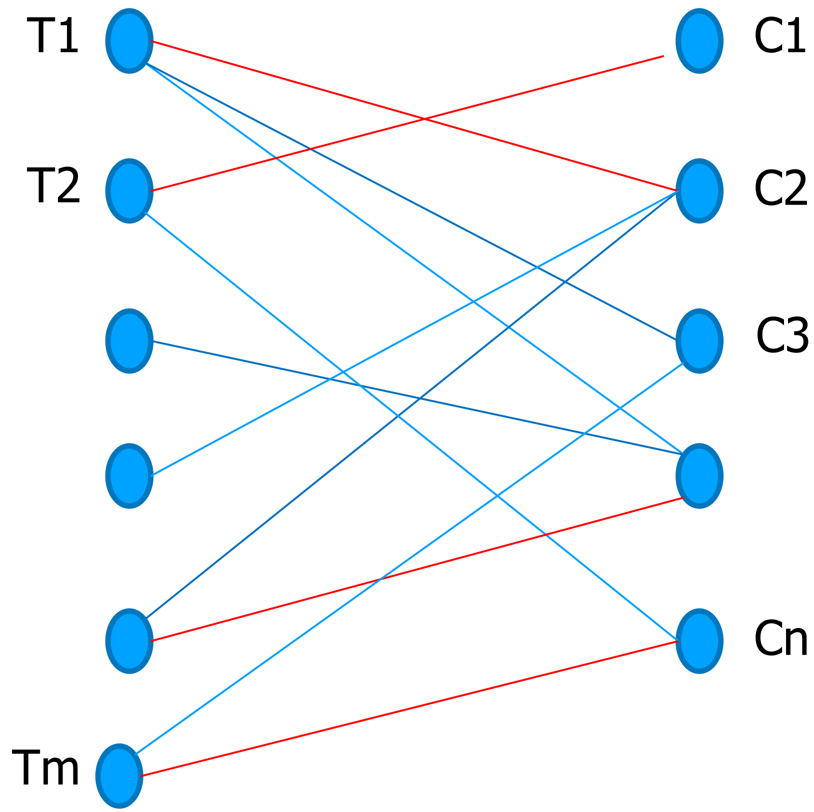
Also known as 1-factorization (a factor is a subgraph that spans all vertices)

Time-tabling problems

- *m* teachers, *n* classes
- Teacher *i* is required to teach class *j*
- In a given period, a teacher can be in at most one class, and a *class* can have at most one teacher.
- Design a timetable with minimum # of periods
- *Properly* color the edges of G with as few colors as possible



Time-tabling problems



Definition: A k-edge coloring of G is a labeling $f: E(G) \to S$, $|S| = k$.

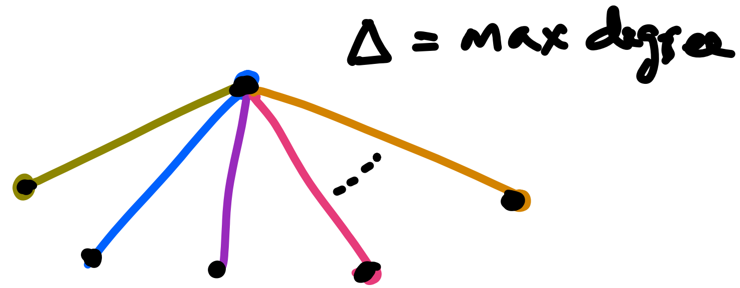
An edge-coloring is proper if all edges incident on a vertex have different colors.

If G can be properly edge-colored with k-colors $\Rightarrow$ k edge colorable.

Edge chromatic number (chromatic index) $\chi'(G)$ for a loopless graph G is the least k for which G is k-edge colorable

Observation (DW 7.1.3 – 7.1.10)

1. $\chi'(G) \geq \Delta(G) \iff$

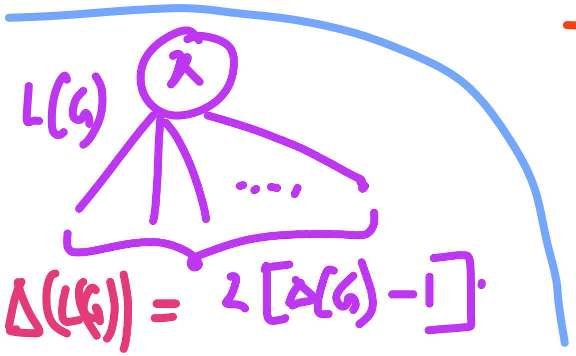


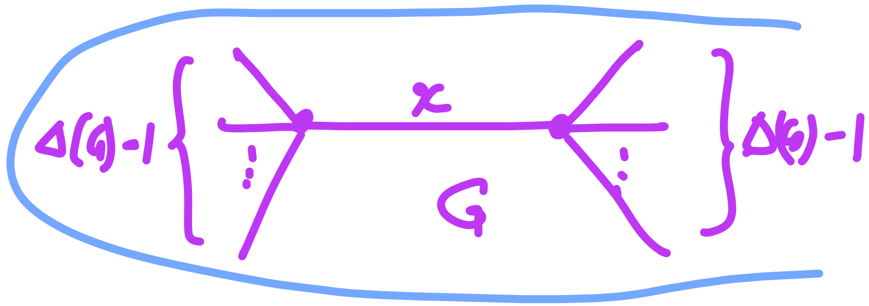
2. $\chi'(G) \leq 2\Delta(G) - 1$

$$\boxed{\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1}$$

Proof: $\chi'(G) = \chi(L(G)) \leq \Delta(L(G)) + 1$

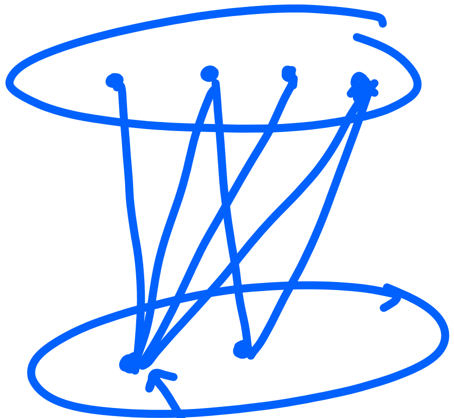
$$\leq 2\Delta(G) - 1$$





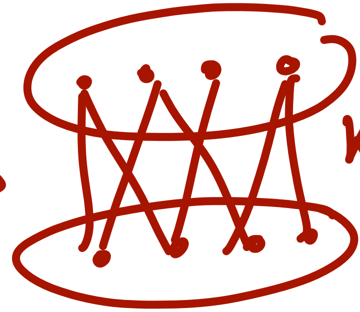
König's Theorem (DW: 7.1.7)

If G is bipartite, then $\chi'(G) = \Delta(G)$

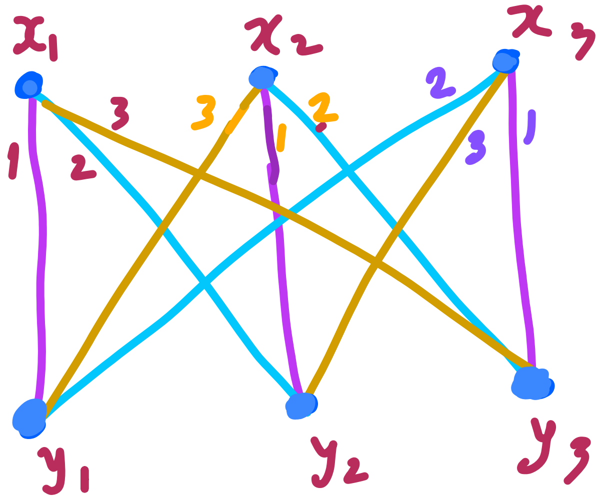


$\Delta(G) = 4$

$\Rightarrow$ Make it balanced + complete



$K_{4,4}$

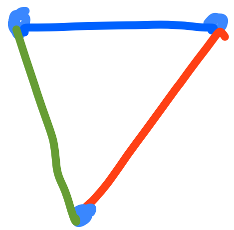


$x_1 \rightarrow$ color $1, 2, 3$

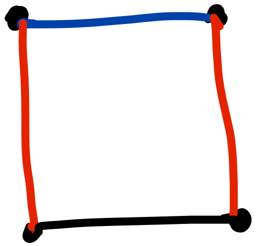
$x_2 \rightarrow$ " $3, 1, 2$

$x_3 \rightarrow$ " $2, 3, 1$

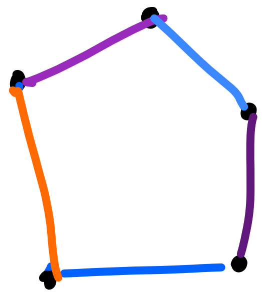
Examples



$\chi'(C_3) = 3$



$\chi'(C_4) = 2$



$\chi'(C_5) = 3$

odd cycle $\Rightarrow \chi' = 3$

even cycle $\Rightarrow \chi' = 2$

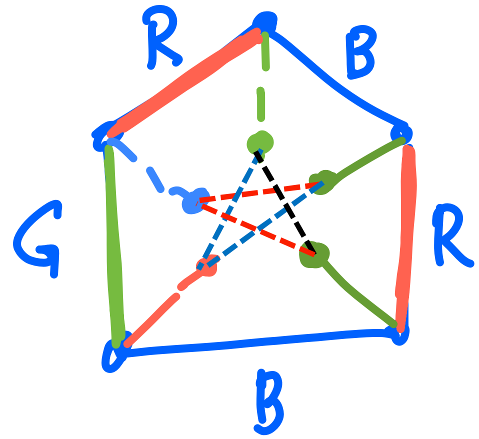
$\chi'(PG) = 4$ [shows $\chi'(G) = \Delta(G) + 1$]

$$\chi'(K_{m,n}) = \max\{m, n\}$$

$\chi'(PG) = 4$ [DW: 7.1.9]

Proof: PG comprises two vertex-disjoint 5-cycles; $\Delta(PG) = 3$

$\Rightarrow \chi'(PG) \geq 3$



R	B	G
2	2	1

R	B	G
3	1	1

Not possible

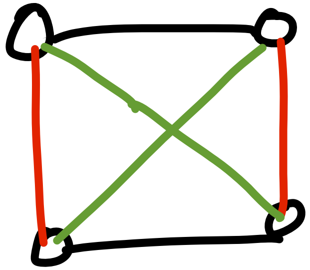
The inner 5-star ($\approx$ 5-cycle) cannot be colored with G, B and R.

Hence, you need an additional color $\boxtimes$

Thus, $\chi'(PG) = 4$

Chromatic index of K_n : $\chi'(K_n) = ?$

Case 1: $n \to$ even, Claim: $\chi'(K_n) = n-1$

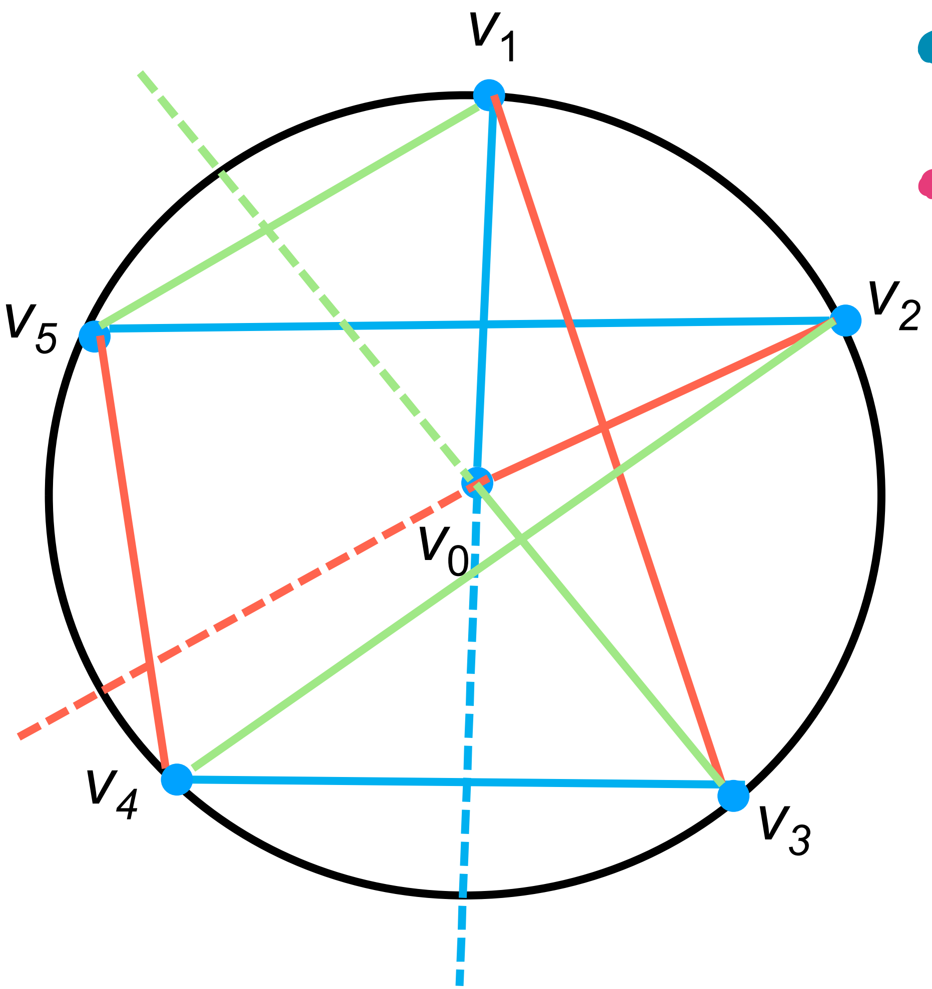


$n = 4$, $\chi'(K_4) = 3$

Idea: let $n = 2k$

Can we partition the edges of K_n into $(n-1)$ disjoint matchings?

$\chi'(K_6) = ?$



Construction procedure

- $\{(v_0, v_1), (v_2, v_5), (v_3, v_4)\}$
- $\{(v_0, v_2), (v_1, v_3), (v_4, v_5)\}$

5 disjoint matching

$$\boxed{\chi'(K_6) = 5}$$

This argument can be generalized to show $\chi'(K_n) = n-1$, when n = even

Let the vertices be $v_0, v_1, v_2, v_3, v_4, v_5$

$\chi'(K_5) = ?$

$\boxed{\chi'(K_n) = n \text{ when } n \text{ is odd.}}$

$\chi'(K_n) = ?$ when $n =$ odd.

Proof: # edges in $K_n = \dfrac{n(n-1)}{2}$

Since n is odd, perfect matching does not exist, we can match $\left(\dfrac{n-1}{2}\right)$ edges in one pass. Hence, at least n passes are needed to cover all edges $\Rightarrow \chi'(K_n) \geq n$

Now: $\chi'(K_{n+1}) = n \Rightarrow$ delete the extra vertex $\Rightarrow \chi'(K_n) = n$.

We have seen that

$$\Delta(G) \leq \chi'(G) \leq 2\Delta(G) - 1$$

Vizing theorem (DW: 7.1.10) Vadim Vizing

If G is a simple graph, then

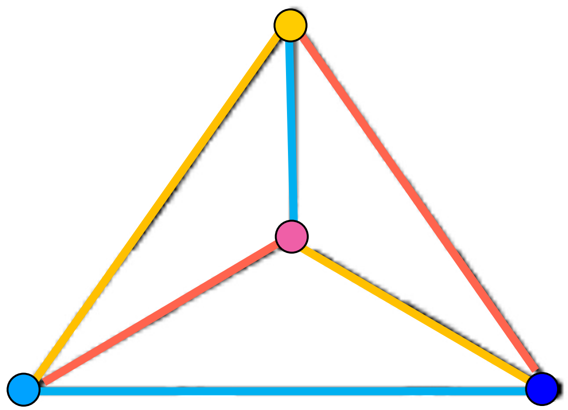
$$\chi'(G) \leq \Delta(G) + 1$$

i.e.,

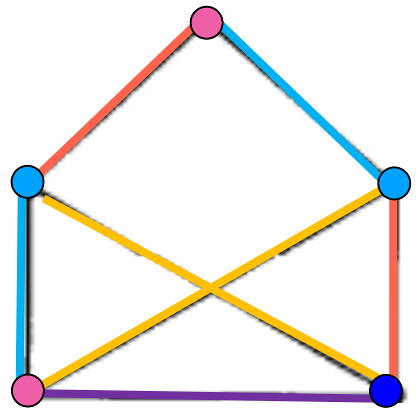
$$\boxed{\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1}$$

Vizing theorem

$$\boxed{\Delta(G) \leq \chi'(G) \leq \Delta(G)+1}$$



$\Delta = 3,\ \chi = 4,\ \chi' = 3$



$\Delta = 3,\ \chi = 3,\ \chi' = 4$

Summary: Edge Coloring of Graphs

- Definition: A proper k- edge coloring of a graph $G=(V,E)$ is a mapping $f: E \rightarrow \{1,2,\ldots,k\}$ such that adjacent edges receive distinct colors
- If the maximum degree is Δ then clearly $k \geq \Delta$
- Theorem (Vizing): Any graph has either Δ-coloring or $(\Delta+1)$-coloring (for edges)
- Designing a time-table is a proper edge coloring of a graph
- Theorem: A bipartite graph admits Δ-coloring for edges
- Edges of the complete graph K_n admit Δ-coloring when n is even, and $(\Delta+1)$-coloring when n is odd