# INTEGRAL CALCULUS

## CONVERGENCE OF IMPROPER INTEGRALS:

### PROPER INTEGRAL:

$\int_a^b f(x)\,dx$ Range of integration is finite and integrand is bounded.

### IMPROPER INTEGRAL:

Integral $\int_a^b f(x)\,dx$ is called <u>improper</u> if

(i) $a=-\infty$ and/or $b=\infty$ and $f(x)$ is bounded — first kind

(ii) $f(x)$ is unbounded at one or more points of $a \le x \le b$ — Second kind

(iii) Both (i) & (ii) type — Third kind or mixed kind.

Example: $\int_0^\infty \cos x\,dx$ — first kind

$\int_0^1 \frac{dx}{x-1}$ — Second kind

$\int_0^\infty \frac{dx}{(1-x)^2}$ — third kind

Evaluation of integrals of first kind:

(i) $\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx$

(ii) $\int_{-\infty}^b f(x)\,dx = \lim_{R\to\infty} \int_{-R}^b f(x)\,dx$

(iii) $\int_{-\infty}^\infty f(x)\,dx = \lim_{R_1\to\infty} \int_{-R_1}^c f(x)\,dx + \lim_{R_2\to\infty} \int_c^{R_2} f(x)\,dx$

OR

$$= \lim_{\substack{R_1\to\infty \\ R_2\to\infty}} \int_{-R_1}^{R_2} f(x)\,dx$$

Example -

(i) $\int_0^\infty \sin x\,dx = \lim_{R\to\infty} \int_0^R \sin x\,dx$

$$= \lim_{R\to\infty} (1-\cos R) \quad \text{does not exist !}$$

(ii) $\int_2^\infty \frac{2x^2}{x^4-1}\,dx = \lim_{R\to\infty} \int_2^R \frac{2x^2}{x^4-1}\,dx = \lim_{R\to\infty} \int_2^R \left(\frac{1}{x^2+1} + \frac{1}{x^2-1}\right) dx$

$$= \lim_{R\to\infty} \left[\int_2^R \frac{1}{x^2+1}\,dx + \frac{1}{2}\int_2^R \frac{1}{x-1}\,dx - \frac{1}{2}\int_2^R \frac{1}{x+1}\,dx\right]$$

$$= \lim_{R\to\infty} \left[\tan^{-1}R - \tan^{-1}(2) + \frac{1}{2}\ln\left(\frac{R-1}{R+1}\right) + \frac{1}{2}\ln 3\right]$$

$$= \frac{\pi}{2} - \tan^{-1}(2) + \frac{1}{2}\ln(3).$$

# Evaluation of improper integrals of the second kind

(i) If $f(x) \to \infty$ as $x \to b$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{b-\varepsilon} f(x)\,dx$$

(ii) If $f(x) \to \infty$ as $x \to a$ then

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_{a+\varepsilon}^{b} f(x)\,dx$$

(iii) If $f(x) \to \infty$ as $x \to c$ only. Here

$a < c < b$

$$\int_a^b f(x)\,dx = \lim_{\varepsilon \to 0+} \int_a^{c-\varepsilon} f(x)\,dx + \lim_{\varepsilon \to 0+} \int_{c+\varepsilon}^{b} f(x)\,dx$$

(iv) If $f(x) \to \infty$ as $x \to a$ and $x \to b$

$$\int_a^b f(x)\,dx = \lim_{\substack{\varepsilon_1 \to 0+ \\ \varepsilon_2 \to 0+}} \int_{a+\varepsilon_1}^{b-\varepsilon_2} f(x)\,dx$$

Example - 1: $\int_0^1 \frac{dx}{\sqrt{1-x}}$

$$= \lim_{\varepsilon \to 0+} \int_0^{1-\varepsilon} \frac{dx}{\sqrt{1-x}}$$

$$= \lim_{\varepsilon \to 0+} \left[-2\sqrt{1-x}\right]_0^{1-\varepsilon}$$

$$= -\lim_{\varepsilon \to 0+} 2\left(\sqrt{\varepsilon}-1\right)$$

$$= 2$$

Example - 2: $\int_0^2 \frac{dx}{2x-x^2}$

$$= \lim_{\varepsilon_1 \to 0+} \int_{\varepsilon_1}^{1} \frac{dx}{2x-x^2} + \lim_{\varepsilon_2 \to 0+} \int_1^{2-\varepsilon_2} \frac{dx}{2x-x^2}$$

$$= \lim_{\varepsilon_1 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_{\varepsilon_1}^{1} + \lim_{\varepsilon_2 \to 0+} \frac{1}{2}\left[\ln \frac{x}{2-x}\right]_1^{2-\varepsilon_2}$$

$$= -\frac{1}{2} \lim_{\varepsilon_1 \to 0+} \ln\left(\frac{\varepsilon_1}{2-\varepsilon_1}\right) + \frac{1}{2} \lim_{\varepsilon_2 \to 0+} \ln\left(\frac{2-\varepsilon_2}{\varepsilon_2}\right)$$

$$= \infty$$

$\Rightarrow$ Integral diverges

# CONVERGENCE TEST FOR IMPROPER INTEGRALS

## – TYPE-I INTEGRALS

### COMPARISON TEST – I:

If f and g are positive or non-negative, $f \geq 0$, $g \geq 0$ and $f(x) \leq g(x)$, $\forall x$ in $[a, \infty)$, then

(i) $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges

(ii) $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges

### COMPARISON TEST – II:

Suppose $f(x) \geq 0$ & $g(x) > 0$ $\forall x > a$.

If $\lim\limits_{x\to\infty} \dfrac{f(x)}{g(x)} = K \ (\neq 0)$. Then both the integrals $\int_a^\infty f(x)\,dx$ and $\int_a^\infty g(x)\,dx$ converge or diverge together.

In case: $K = 0$ and $\int_a^\infty g(x)\,dx$ converges then

$$\int_a^\infty f(x)\,dx \text{ converges}$$

In case $K = \infty$ and $\int_a^\infty g(x)\,dx$ diverges then

$$\int_a^\infty f(x)\,dx \text{ diverges.}$$

A useful comparison test:

Consider $a > 0$ and

$$\int_a^R \frac{c}{x^n}\,dx = \begin{cases} c\ln\left(\frac{R}{a}\right), & n = 1 \\ \frac{c}{1-n}\left[\frac{1}{R^{n-1}} - \frac{1}{a^{n-1}}\right], & n \neq 1 \end{cases}$$

$$\Rightarrow \int_a^\infty \frac{c}{x^n}\,dx = \lim_{R\to\infty}\int_a^R \frac{c}{x^n}\,dx = \begin{cases} +\infty, & n \leq 1. \\ \frac{c}{(n-1)\,a^{n-1}}, & n > 1. \end{cases}$$

$\mu$-test (comparison test + above result)

Let $f(x) \geq 0$ in the interval $[a, \infty)$, $a > 0$. (OR $f(x) \leq 0$)

a) If $\exists$ $\mu > 1$ such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists then

$$\int_a^\infty f(x)\,dx \text{ is convergent.}$$

b) If $\exists$ $\mu \leq 1$ such that $\lim\limits_{x\to\infty} x^\mu f(x)$ exists and $\neq 0$, then the integral $\int_a^\infty f(x)\,dx$ is divergent and the same is true if $\lim\limits_{x\to\infty} x^\mu f(x)$ is $+\infty$ (or $-\infty$)

OR, in short:

If $\lim\limits_{x\to\infty} x f(x) = A \neq 0$ (or $= \pm\infty$), then

$$\Rightarrow \int_a^\infty f(x)\,dx \text{ diverges}$$

Test fails if $A = 0$.

Examples:



i) $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Sol: Note that $\frac{1}{x\sqrt{x^2+1}} \sim \frac{1}{x^2}$ so, let $f(x) = \frac{1}{x\sqrt{x^2+1}}$ and $g(x) = \frac{1}{x^2}$. Further $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1 \ (\neq 0)$

$\Rightarrow \int_1^\infty f(x)\,dx$ & $\int_1^\infty g(x)\,dx$ converge or diverge together.

As $\int_1^\infty \frac{dx}{x^2}$ converges $\Rightarrow \int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ converges.

OR apply $\mu$-test as $\mu = 2$.

ii) $\int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$

Let $f(x) = \frac{x^2}{\sqrt{x^5+1}} \quad \left(\sim \frac{1}{\sqrt{x}}\right)$

and $g(x) = \frac{1}{\sqrt{x}}$.

Note that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{\sqrt{x^5+1}} \cdot \sqrt{x} = 1$

As $\int_1^\infty \frac{1}{\sqrt{x}}\,dx$ diverges, by comparison test

$\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ diverges.

OR apply $\mu$-test as $\mu = 1/2$ in this case.

iii) $\int_0^\infty e^{-x^2}dx = \underbrace{\int_0^1 e^{-x^2}dx}_{\text{PROPER}} + \int_1^\infty e^{-x^2}dx$

We know that: $e^{x^2} = 1 + x^2 + \frac{x^4}{\lfloor 2} + \cdots > x^2 \quad \forall x>0, x<0$

$$\Rightarrow e^{-x^2} < \frac{1}{x^2}$$

Since $\int_1^\infty \frac{1}{x^2}dx$ converges, the integral $\int_1^\infty e^{-x^2}dx$ converges.

OR $\mu = 2$ & $\lim_{x\to\infty} x^2 e^{-x^2} = 0 \Rightarrow \int_0^\infty e^{-x^2}dx$ converges

(iv) $\int_0^\infty \frac{\sin^2 x}{x^2}dx = \underbrace{\int_0^1 \frac{\sin^2 x}{x^2}dx}_{\text{PROPER}} + \int_1^\infty \frac{\sin^2 x}{x^2}dx$

Also $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{1}{x^2}$ converges

$\Rightarrow \int_0^\infty \frac{\sin^2 x}{x^2}dx$ converges.

(v) $\int_1^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}dx$

$f(x) = \frac{x\tan^{-1}x}{(1+x^4)^{1/3}} = \frac{\tan^{-1}x}{x^{1/3}(1+x^{-4})^{1/3}} \quad (\sim x^{-1/3} \text{ at } \infty)$

$g(x) = \frac{1}{x^{1/3}} \qquad \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{1/3}\cdot\tan^{-1}x}{x^{1/3}(1+x^{-4})^{1/3}} = \pi/2$

$\Rightarrow \int_1^\infty f(x)\,dx$ diverges

OR Apply $\mu$-test for $\mu = 1/3\ (<1) \to$ divergence of $\int_1^\infty f(x)dx$.

## ABSOLUTE CONVERGENCE:

**Def:** The integral $\int_a^\infty f(x)\,dx$ converges absolutely

$$\Leftrightarrow \int_a^\infty |f(x)|\,dx \text{ converges.}$$

**Def:** The integral $\int_a^\infty f(x)\,dx$ converges conditionally $\Leftrightarrow$ it converges but not absolutely.

**Example:** $\int_1^\infty \frac{\sin x}{x^2}\,dx$ converges absolutely (OR $\int_1^\infty \frac{\sin x}{x^p}\,dx, p>1$)

Note that $\frac{|\sin x|}{x^2} \leq \frac{1}{x^2}$

By comparison test $\int_1^\infty \frac{|\sin x|}{x^2}\,dx$ converges.

**Theorem:** $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty |f(x)|\,dx$ converges but the converse is not true.

**Example:** $\int_0^\infty \frac{\sin x}{x}\,dx$ converges conditionally (to be discussed later)

**DIRICHLET TEST:** Let $f, g: [a,\infty) \to \mathbb{R}$ be such that

i) $f$ is integrable on each interval $[a,b]$, $b>a$ and the integrals $\int_a^b f(x)\,dx$ are uniformly bounded, i.e., $\exists\, C>0$, s.t.

$$\left|\int_a^b f(x)\,dx\right| \leq C \quad \text{for all } b>a \quad (b<\infty)$$

ii) $g$ is monotone and bounded on $[a,\infty)$ and $\lim_{x\to\infty} g(x) = 0$

Then the improper integral $\int_a^\infty f(x)\,g(x)\,dx$ converges.

Example: $\int_1^{\infty} \frac{\sin x}{x^p}\, dx$ is convergent for $p>0$.

Let $f(x) = \sin x$ & $g(x) = \frac{1}{x^p}$

Note that $\left|\int_1^{b} \sin x\, dx\right| = |\cos(1) - \cos(b)| \leq |\cos(1)| + |\cos(b)|$

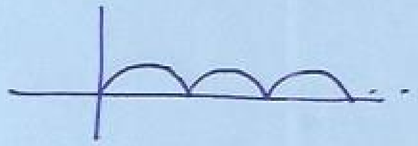
$\leq 2.$ for $1 \leq b < \infty$

Also $g(x) = \frac{1}{x^p}$ is monotone decreasing function tending to 0 as $x \to \infty$, for $p>0$.

Using Dirichlet test $\int_1^{\infty} f(x)\, g(x)\, dx$ converges for $p>0$.

Example: Show that $\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx$ does not converge.

$$\int_0^{\infty} \left|\frac{\sin x}{x}\right| dx = \sum_{n=0}^{\infty} \int_{n\pi}^{(n+1)\pi} \frac{|\sin x|}{x}\, dx$$



Subst. $x = n\pi + y$, then

$$\sum_{n=0}^{\infty} \int_0^{\pi} \frac{|\sin(n\pi + y)|}{n\pi + y}\, dy = \sum_{n=0}^{\infty} \int_0^{\pi} \frac{|(-1)^n \sin y|}{n\pi + y}$$

$$= \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + y}\, dy \geq \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + \pi}\, dy$$

$$= \sum_{n=0}^{\infty} \frac{1}{(n+1)} \cdot \frac{2}{\pi} \rightarrow \text{divergent series.}$$

$$\Rightarrow \sum_{n=0}^{\infty} \int_0^{\pi} \frac{\sin y}{n\pi + y}\, dy \text{ diverges}$$

and hence the improper integral

$$\int_0^{\infty} \frac{|\sin x|}{x}\, dx \text{ diverges.}$$

Example: Test the convergence of $\int_0^\infty \frac{\sin x}{x} \cdot e^{-x}\, dx$

$$\int_0^\infty \frac{\sin x}{x} e^{-x}\, dx = \underbrace{\int_0^1 \frac{\sin x}{x} \cdot e^{-x}\, dx}_{\text{PROPER}} + \int_1^\infty \frac{\sin x}{x} e^{-x}\, dx$$

Note that $\int_1^b \frac{\sin x}{x}\, dx \leq \int_1^b \sin x\, dx \leq 2$

Further, $e^{-x}$ is monotone and bounded, and $\lim_{x\to\infty} e^{-x} = 0$

Hence by Dirichlet's test $\int_0^\infty \frac{\sin x}{x} e^{-x}\, dx$ converges.

Example: $\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx \qquad a > 0$

$$\int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx = \underbrace{-\int_a^\infty e^{-x} \frac{\cos x}{x^2}}_{\substack{\text{Converges} \\ \text{(Similar as above)}}} + \underbrace{\int_a^\infty \frac{\cos x}{x^2}\, dx}_{\substack{\text{converges} \\ \text{(conv. absolulety)}}}$$

$\Rightarrow \int_a^\infty (1-e^{-x}) \frac{\cos x}{x^2}\, dx$ converges.

## INTEGRAL OF THE TYPE:

$$\int_{-\infty}^b f(x)\, dx$$

Subst. $x = -t$ : $\int_{-b}^\infty f(-t)\, dt$

## Review of convergence test for $\int_a^\infty f(x)dx$

**Comparison Tests:** Let $0 \le f(x) \le g(x)$ .

**(I)**

$(a)$ $\int_a^\infty g(x)dx$ conveges $\Rightarrow$ $\int_a^\infty f(x)dx$ conveges

$(b)$ $\int_a^\infty f(x)dx$ diverges $\Rightarrow$ $\int_a^\infty g(x)dx$ diverges

**(II)**

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \mathrm{k}$$

$(a)$ if $k \neq 0$ then $\int_a^\infty f(x)dx$ and $\int_a^\infty g(x)dx$ behave the same

$(b)$ if $k = 0$ and $\int_a^\infty g(x)dx$ conveges then $\int_a^\infty f(x)dx$ conveges

$(c)$ if $k = \infty$ and $\int_a^\infty g(x)dx$ diverges then $\int_a^\infty f(x)dx$ diverges

**Test Integral:**

$$\int_a^\infty \frac{1}{x^p}\, dx \quad \text{converges for } p > 1 \text{ \& diverges if } p \le 1$$

$\boldsymbol{\mu}$ **− test**: Comparison test (II) with $g(x) = \dfrac{1}{x^\mu}$

**Dirichlet's Test:**

If (1) $\left|\int_a^b f(x)dx\right| < C$ for all $b > a$, (2) $g$ is monotone, bounded and $\lim_{x\to\infty} g(x) = 0$, then

$$\int_a^b f(x)g(x)\, dx \text{ conveges}$$

## CONVERGENCE OF IMPROPER INTEGRALS OF SECOND TYPE:

TEST INTEGRAL:

$$\int_a^b \frac{dx}{(x-a)^n}$$

$$\int_a^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{(x-a)^n} = \lim_{\varepsilon\to 0+} \frac{1}{1-n}\left[\frac{1}{(b-a)^{n-1}} - \frac{1}{\varepsilon^{n-1}}\right]$$

if $n \neq 1$.

$$= \begin{cases} \frac{1}{(1-n)(b-a)^{n-1}} & \text{if } n<1 \\ \infty & \text{if } n>1. \end{cases}$$

For $n=1$:

$$\int_a^b \frac{dx}{(x-a)} = \lim_{\varepsilon\to 0+} \int_{a+\varepsilon}^b \frac{dx}{x-a} = \lim_{\varepsilon\to 0+} \Big[\ln|x-a|\Big]\Big|_{a+\varepsilon}^{b}$$

$$= \lim_{\varepsilon\to 0+} \left[\ln(b-a) - \ln\varepsilon\right] = \infty$$

Hence: $\int_a^b \frac{dx}{(x-a)^n}$ converges if $n<1$ and diverges if $n \geq 1$.

Note: Notation: $\int_{a+}^b f(x)\,dx$ ($f(x)$ becomes unbounded at $x=a$)

For the case $\int_a^{b-} f(x)\,dx$ we can set $x = b-t$ and get $\int_{0+}^{b-a} f(b-t)\,dt$.

Example: Test the convergence of $\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$.

Note that the integrand is unbounded at upper end.

Set $3-x = t \Rightarrow dx = -dt$

$$\int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}} = \int_0^3 \frac{dt}{t\sqrt{(3-t)^2+1}}$$

Let $g(t) = \frac{1}{t}$ $\quad \left( \underbrace{\frac{1}{t\sqrt{(3-t)^2+1}}}_{=:f(t)} \times t = \frac{1}{\sqrt{(3-t)^2+1}} \right)$

Note that $\lim\limits_{t\to 0} \frac{f(t)}{g(t)} = \lim\limits_{t\to 0} \frac{1}{\sqrt{(3-t)^2+1}} = \frac{1}{\sqrt{10}}$

$\Rightarrow \int_0^3 \frac{dx}{(3-x)\sqrt{x^2+1}}$ diverges since $\int_0^3 g(t)\,dt$ diverges.

Example: $\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$

Notice: $\left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| \le \frac{1}{\sqrt[3]{x-\pi}}$ and

$\int_{\pi}^{4\pi} \frac{1}{\sqrt[3]{x-\pi}}\,dx$ converges

$\Rightarrow \int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}$ converges absolutely.

Note: Improper integrals of the third kind can be expressed in terms of improper integrals of the first and second kind.

## Review of convergence test for $\int_{a+}^{b} f(x)dx$

**Comparison Tests:** Let $0 \le f(x) \le g(x)$ .

**(I)**

$(a)$ $\quad \int_{a+}^{b} g(x)dx$ conveges $\Rightarrow$ $\int_{a+}^{b} f(x)dx$ conveges

$(b)$ $\quad \int_{a+}^{b} f(x)dx$ diverges $\Rightarrow$ $\int_{a+}^{b} g(x)dx$ diverges

**(II)**

$$\lim_{x \to a+} \frac{f(x)}{g(x)} = \mathrm{k}$$

$(a)$ if $k \neq 0$ then $\int_{a+}^{b} f(x)dx$ and $\int_{a+}^{b} g(x)dx$ behave the same

$(b)$ if $k = 0$ and $\int_{a+}^{b} g(x)dx$ conveges then $\int_{a+}^{b} f(x)dx$ conveges

$(c)$ if $k = \infty$ and $\int_{a+}^{b} g(x)dx$ diverges then $\int_{a+}^{b} f(x)dx$ diverges

**Test Integral:**

$$\int_{a}^{b} \frac{1}{(x-a)^p}\, dx \quad \text{converges for } p < 1 \text{ \& diverges if } p \ge 1$$

**$\mu -$ test:**

if $\exists\ 0 < \mu < 1$ such that $\lim_{x \to a+} (x-a)^{\mu} f(x)$ exsits then $\int_{a+}^{b} f(x)\, dx$ conveges absolutely

if $\exists\ \mu \ge 1$ such that $\lim_{x \to a+} (x-a)^{\mu} f(x)$ exsits $(\neq 0, \text{it may be} \pm \infty)$ then $\int_{a+}^{b} f(x)\, dx$ diverges

**Dirichlet's Test:**

If $\;(1)\; \left| \int_{a+\epsilon}^{b} f(x)dx \right| < C, \;\forall\; b > a, \;(2)\; g$ is monotone, bounded and $\lim_{x \to a} g(x) = 0$, then

$$\int_{a+}^{b} f(x)g(x)\, dx \text{ conveges}$$