Interesting Network Theorems

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1 Introduction

We already know various methods of solving a circuit problem. Mesh analysis, nodal analysis are two very basic methods often adopted to solve circuit problem. Circuit analysis by using principle of superposition is another useful way to analyze a circuit. If one wants to know the current only in a particular branch of a rather complicated network, then Thevenin or Norton's theorem may be suitably applied. We shall assume the circuit to be linear and bilateral when we want to apply superposition or Thevenin theorem. In this lecture note we shall introduce the following network theorems.

1. Substitution theorem.
2. Compensation theorem
3. Thevenin & Norton theorem
4. Tellegen Theorem.
5. Reciprocity Theorem.

1.1 Substitution theorem

The idea of this theorem can be easily understood by referring to the circuit 1(a). Suppose we have solved this circuit and current through the R_4 resistor is I_4 in the direction shown. Suppose we draw another circuit shown in 1(b) where the resistor R_4 has been replaced by a voltage source of magnitude I_4R_4 and with polarity consistent with the direction of the current. These two circuits (shown in 1(a) and 1(b)) are equivalent with current values of different branches will remain same which also means that node voltages of of these two circuits will also be same. Why? KVL equations in different loops and KCL equations at different nodes remain same in both the cases.

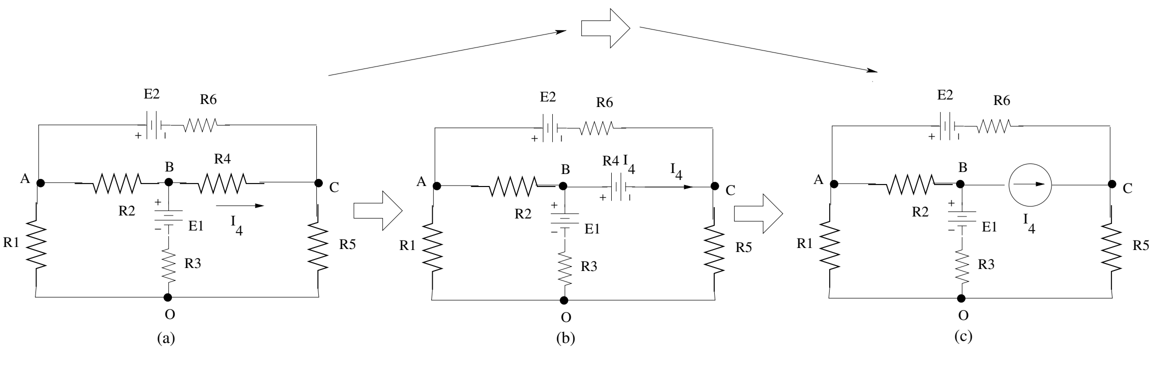

Figure 1:

The circuit of 1(a) is also equivalent to the circuit of figure 1(c) where R_4 of figure 1(a) is replaced by a current source of value I_4. The reason for this is same as explained in the last paragraph. Such a transformation is obviously applicable to any other elements as well. This theorem can be suitably applied to establish various other theorem such as Thevenin theorem etc.

1.2 Proof of Thevenin theorem

Look at the figure 2(a) where a linear network N is shown with voltage and current sources. The impedances of the networks are not shown for clarity, Z connected across AB is the load impedance. We would like to find out V_{AB} by applying superposition theorem. From substitution theorem we know the the impedance Z can be replaced by a current source of value I in the direction as shown in figure 2(b). We would like to find out V_{AB} by applying superposition theorem.

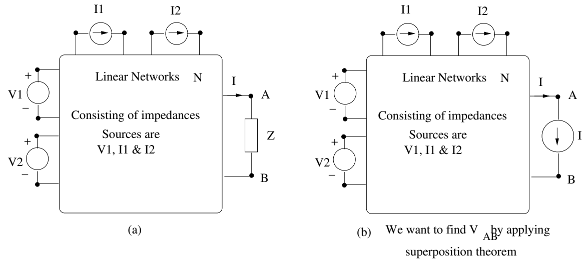

Figure 2:

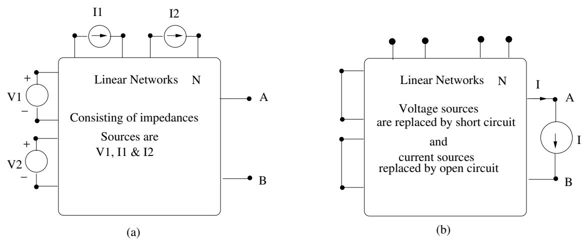

Figure 3:

Thus the network can be thought of having an extra current source of value I in addition to the sources present in the network. Now let us apply super position theorem to the equivalent network shown in figure 2(b) to calculate V_{AB}.

• First consider all the network sources to be present and deactivate the current source I showing an open circuit between AB as shown in figure 3(a). So we solve the circuit of figure 3(a) and calculate $V_{AB}|_{OC}$ and call this voltage to be V_{th}.

• Next deactivate all the sources of the network by open circuiting the current sources and short circuiting the voltage sources. Only source now present is the current source I between AB as shown in figure 3(b). We redraw the figure 3(b) and show it in figure 4(a).

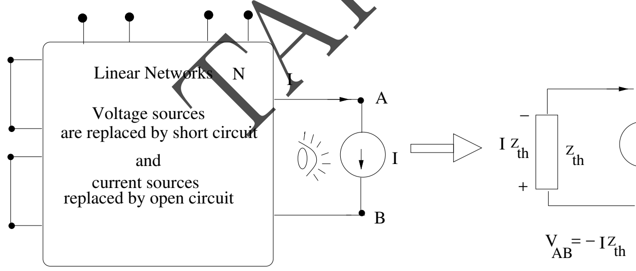

Figure 4:

With all sources of N being deactivated, we can represent its equivalent in figure 4(b) where Z_{th} is the looking in equivalent of the impedance of the network N. Thus, $V_{AB} = -IZ_{th}$. Therefore, V_{AB} of the original circuit shown in figure 2 will be:

$$V_{AB} = V_{th} - IZ_{th}$$

Now the RHS is also IZ. Therefore:

$$V_{AB} = V_{th} - IZ_{th} = IZ$$

$$\text{solving we get, } I = \frac{V_{th}}{Z + Z_{th}}$$

This proves Thevenin theorem.

1.3 Proof of Norton's theorem

Refer to figure 5(a) where current through the impedance connected across AB is I. Using substitution theorem we can replace Z by a voltage source IZ as shown in figure 5(b). Thus current through

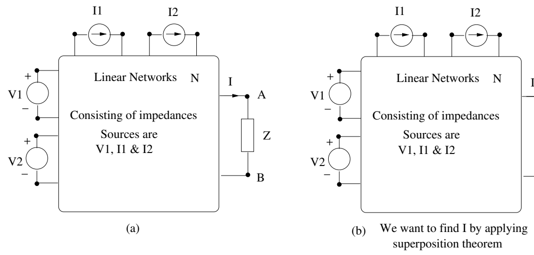

Figure 5:

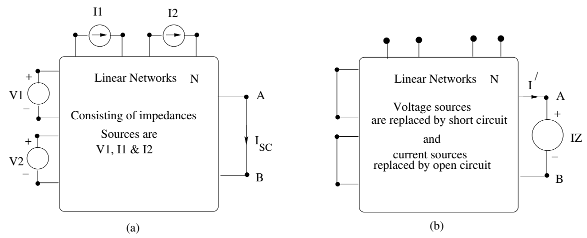

Figure 6:

the branch AB will be due all the sources. To find the contribution due to all sources present in the network, short circuit the voltage source IZ and solve for the current - let this current be called I_{sc} as shown in figure 6(a). Next step is to calculate the current I' flowing through AB considering the source IZ only with all other sources of the network deactivated as shown in figure 6(b).

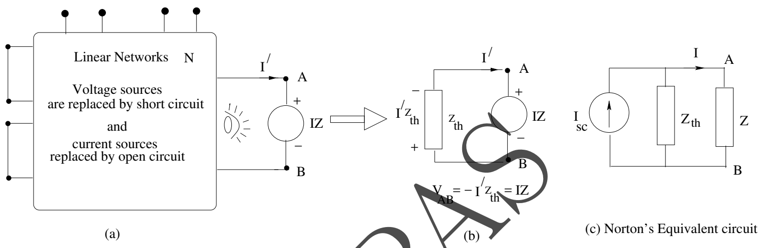

Figure 7:

The circuit shown in figure 6(b) is equivalent to 7(b) and from this we get V_{AB} to be:

$$V_{AB} = IZ = -I'Z_{th}$$

$$\text{or,}\quad I' = -\frac{IZ}{Z_{th}}$$

Finally from superposition theorem, actual current I, flowing through AB is given by:

$$I = I_{sc} - \frac{IZ}{Z_{th}}$$

Solving for I gives:

$$I = \frac{Z_{th}}{Z_{th} + Z} I_{sc}$$

This is the famous rule for current division rule between two impedances connected in parallel. Hence the Norton's equivalent circuit is obtained as shown in figure 7(c)

1.4 Compensation theorem

Suppose we have a network shown in figure 8(a) where current through the impedance Z is known to be I. Question is if the impedance is changed to $Z + \Delta Z$ what will be the new current through $Z + \Delta Z$ shown in figure 8(b)? One way of answering the question is to resolve the circuit once more with the new $Z + \Delta Z$. We shall try to suggest an alternative method to find the new current $I + \Delta I$ in 8(b) by using the substitution theorem. As a first step we can replace the $Z + \Delta Z$ by a voltage source $(I + \Delta I)(Z + \Delta Z)$ as shown in figure 8(c). Now the product $(I + \Delta I)(Z + \Delta Z)$ can be expanded and regrouped as follows

$$(I + \Delta I)(Z + \Delta Z) = IZ + I\Delta Z + \Delta I(Z + \Delta Z)$$

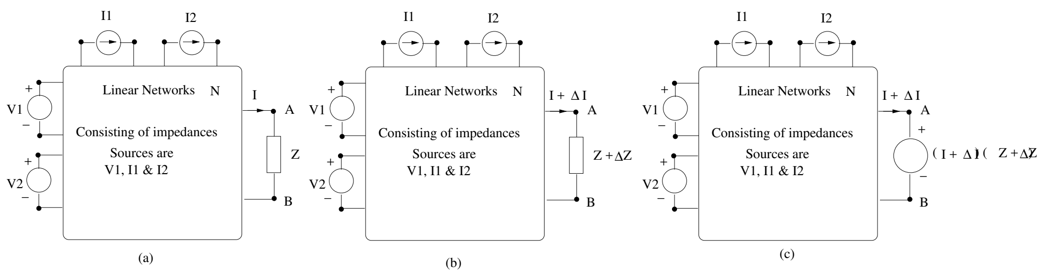

Figure 8:

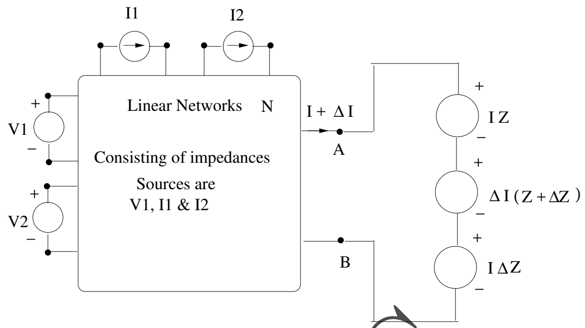

Figure 9:

So the voltage source $(I + \Delta I)(Z + \Delta Z)$ can be replaced by three voltage sources in series of values IZ, $I\Delta Z$ and $\Delta I(Z + \Delta Z)$ as shown in figure 9.

Now we decide to apply superposition theorem to circuit of figure 9. Let only the voltage source IZ along with all the network sources are present as shown in figure 10(a). The current must be I because this circuit is nothing but the original circuit shown in figure 8(a). Now consider the other

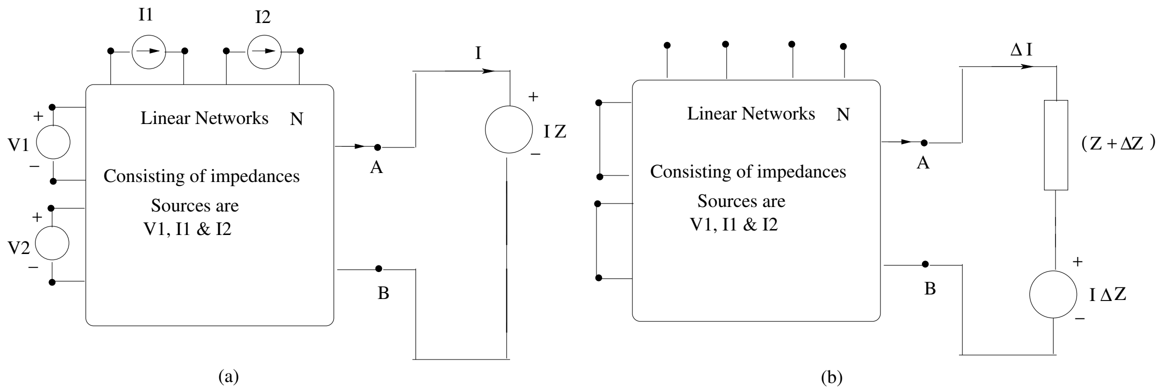

Figure 10:

two voltage sources namely $I\Delta Z$ and $\Delta I(Z + \Delta Z)$ to be present along with all network sources deactivated in 10(b). In presence of these two sources only, current must be ΔI. Now we apply the substitution theorem in the reverse way and replace the voltage source $\Delta I(Z + \Delta Z)$ by the impedance $(Z + \Delta Z)$ to get the circuit as shown in figure 10(b). In this circuit only one voltage source of value $I\Delta Z$ is present. All the sources present in the network N are deactivated. Idea is to solve this rather simple circuit to solve for ΔI. Finally the current through $Z + \Delta Z$ in the original network will be $I + \Delta I$.

2 Tellegen's Theorem

Before we start, let us recall that any network is a collection of different elements (branches) which are interconnected. Before solving the network it is our prerogative to arbitrarily assign the direction of currents in different elements and assign the polarity of the voltages of different elements. We shall assume that in an element current enters through the +ve terminal of the voltage drop as pictorially depicted in figure 11 for three elements. The way we have assumed the direction of

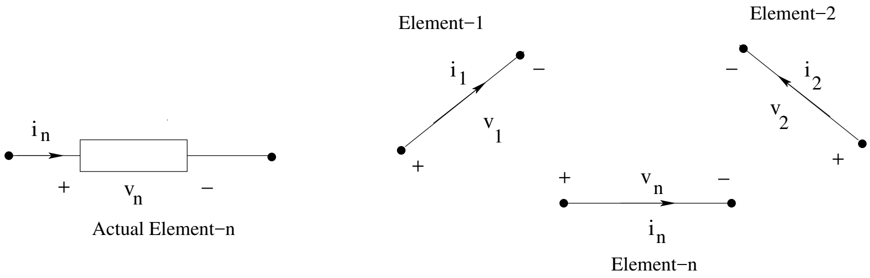

Figure 11:

current and polarity of the voltage across a particular element, the product of this two gives the power absorbed by that element. If the product happens to be +ve then the element is really absorbing power and if the product comes out to be negative then the element is delivering power. From the law of conservation of power in a circuit, the sum of all products calculated for all the elements will be zero.

$$\sum_{k=1}^{n} v_k \times i_k = 0$$

Where, n is the total number of elements in the network.

For two (or more) networks, A and B having same topology or same graph

$$\sum_{k=1}^{n} v_k \times i'_k = \sum_{k=1}^{n} v'_k \times i_k = 0$$

where,

$$\begin{aligned} v_k &= \text{Voltage across the k th element of the network-A} \\ i_k &= \text{current through the k th element of the network-A} \\ v'_k &= \text{Voltage across the k th element of the network-B} \\ i'_k &= \text{current through the k th element of the network-B} \\ n &= \text{number of elements of network-A} = \text{number of elements of network-B} \end{aligned}$$

Topology or graph of a network is drawn by replacing the branch elements with straight lines and nodes are indicated by bullets. In figure 12(a) and (b) the graphs of two networks A and B are shown which are same in appearance. The arrows in the elements show the assumed direction of the currents in various elements. Similarly voltage across each elements are also shown with assumed direction of polarities.

Proof of Tellegen theorem

Let us prove that

$$\sum_{k=1}^{n} v_k \times i'_k = 0$$

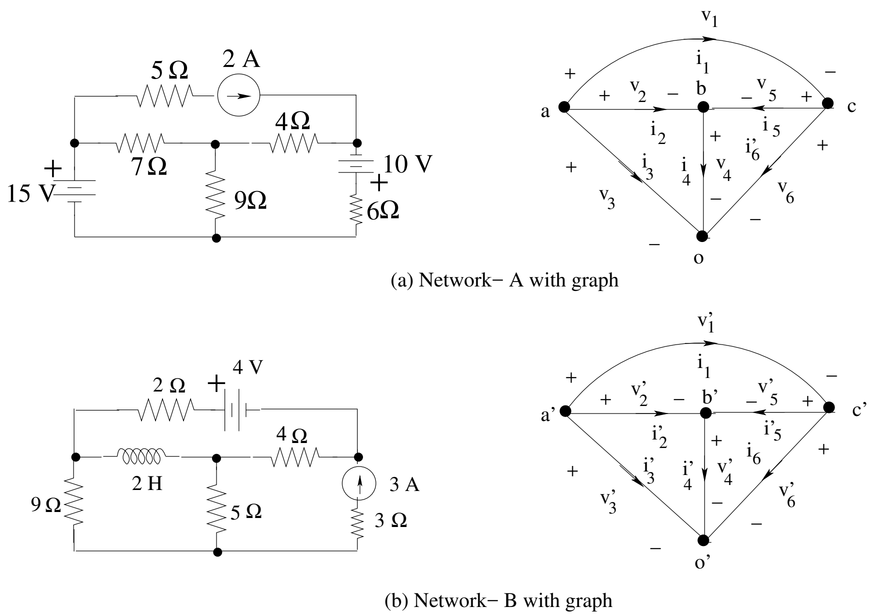

Figure 12:

Now,

$$
\begin{aligned}
\sum_{k=1}^{n} v_k \times i'_k &= v_1 \times i'_1 + v_2 \times i'_2 + v_3 \times i'_3 + v_4 \times i'_4 + v_5 \times i'_5 + v_6 \times i'_6 \\
\text{Now,}\ \ v_1 \times i'_1 &= (v_a - v_c) \times i'_1 \\
v_2 \times i'_2 &= (v_a - v_b) \times i'_2 \\
v_3 \times i'_3 &= (v_a - v_o) \times i'_3 \\
v_4 \times i'_4 &= (v_b - v_o) \times i'_4 \\
v_5 \times i'_5 &= (v_c - v_b) \times i'_5 \\
v_6 \times i'_6 &= (v_c - v_0) \times i'_6 \\
\text{Adding these,}\ \sum_{k=1}^{n} v_{kA} \times i'_k &= v_a(i'_1 + i'_2 + i'_3) + v_b(i'_4 - i'_2 - i'_5) \\
&= +v_c(i'_6 + i'_5 - i'_1) + v_o(-i'_3 - i'_4 - i'_6) \\
\text{Applying KCL at nodes} &\quad a',\ b',\ c' \text{ and } o' \text{ of Network-B} \\
i'_1 + i'_2 + i'_3 &= 0 \\
i'_4 - i'_2 - i'_5 &= 0 \\
i'_6 + i'_5 - i'_1 &= 0 \\
-i'_3 - i'_4 - i'_6 &= 0 \\
\therefore \sum_{k=1}^{6} v_k \times i'_k &= 0
\end{aligned}
$$

Thus we have proved the Tellegen theorem.

$$v_1 \times i'_1 + v_2 \times i'_2 + v_3 \times i'_3 + v_4 \times i'_4 + v_5 \times i'_5 + v_6 \times i'_6 = 0$$

In the same sum of products (element wise) of voltage of network-B and current of network-A will be zero as shown below.

$$v'_1 \times i_1 + v'_2 \times i_2 + v'_3 \times i_3 + v'_4 \times i_4 + v_5 \times i_5 + v'_6 \times i_6 = 0$$

In general therefore, for two networks having n elements each and whose topologies are identical, the following expression is true which is in fact the Tellegen,s theorem.

$$\sum_{k=1}^{n} v_k \times i'_k = \sum_{k=1}^{n} v'_k \times i_k = 0$$

2.1 Theory of reciprocity

A network is said to be reciprocal if the ratio between response to excitation remains unchanged if we interchange the position of the response and excitation. Suppose you have several impedances connected to form a network without any internal source.

Now you identify two terminals where you decide to apply input excitation (either voltage or current source). Similarly you identify another pair of terminals (may be called output terminals) where output response (either voltage or current) is obtained. Such a network may be loosely called a TWO PORT network. A two port network is shown in figure 13 where port-1 has two terminals

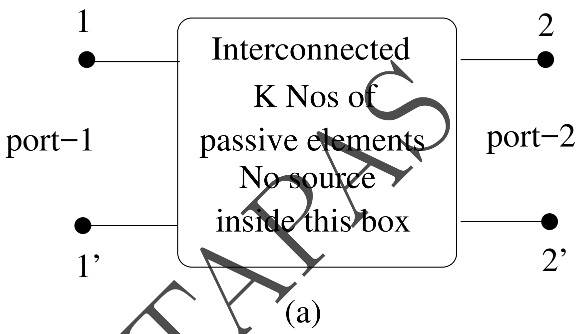

Figure 13:

1 & $1'$ and port-2 has two terminals 2 and $2'$. Either of these two ports can be used as input and output ports. The network is within the box having no sources but having resistances or in general impedances.

2.1.1 General view

In figure 14(a), network is shown to be excited with voltages v_1 and v_2 from port-1 and port-2 respectively. Current drawn by the network from these two sources are respectively i_1 and i_2 .

Now refer to figure 14(b), where the same network is shown but this time excited voltages are v'_1 and v'_2 and the currents drawn are i'_1 and i'_2. Tellegen's theorem is now applied to the circuits of

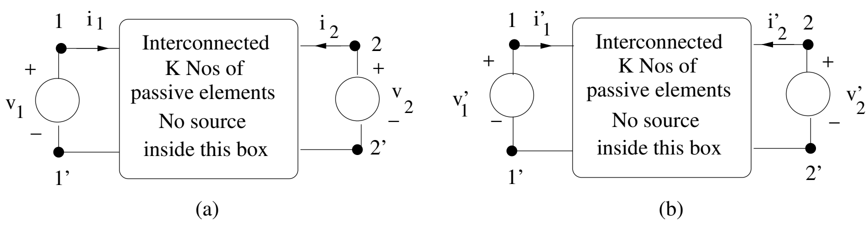

Figure 14:

figure 14(a) and (b).

$$v_1 \times (-i'_1) + v_2 \times (-i'_2) + \sum_k v_k \times i'_k \quad = \quad v'_2 \times (-i_2) + v'_1 \times (-i_1) + \sum_k v'_k \times i_k$$

$$\begin{aligned}
\text{but, } v_k &= Z_k \times i_k \\
\text{and } v'_k &= Z_k \times i'_k \\
v_1 \times (-i'_1) + v_2 \times (-i'_2) + \sum_k Z_k \times i_k \times i'_k &= v'_2 \times (-i_2) + v'_1 \times (-i_1) + \sum_k Z_k \times i'_k \times i_k \\
v_1 \times (-i'_1) + v_2 \times (-i'_2) &= v'_2 \times (-i_2) + v'_1 \times (-i_1) \\
\text{Finally, } v_1 \times i'_1 + v_2 \times i'_2 &= v'_2 \times i_2 + v'_1 \times i_1
\end{aligned}$$

2.1.2 One port excited with voltage & other port kept shorted

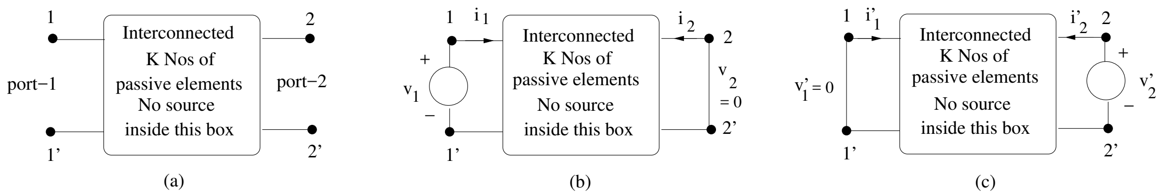

Figure 15:

Consider the same network as shown in figure 15(a) where port-1 has been used as input port with input voltage signal v_1. Our response is the current i_2 with the output port-2 shorted. We would like to know the the ratio between:

$$\text{in figure 15(b): } \frac{\text{response}}{\text{excitation}} = \frac{i_2}{v_1}$$

Consider once again the same network as shown in figure 15(b) where port-2 has been used as input port with input voltage signal v'_2. Our response is the current i'_1 with the output port-1 shorted. In the same way, we would like to know the ratio between:

$$\text{in figure 15(c) : } \frac{\text{response}}{\text{excitation}} = \frac{i'_1}{v'_2}$$

We will now find out the relationship between these two ratios, $\frac{i_2}{v_1}$ and $\frac{i'_1}{v'_2}$ by applying Tellegen's theorem. Let us calculate the sum of product of voltage & currents in the networks shown in figures 15(b) and 15(c) and equate them.

$$\begin{aligned}
v_1 \times (-i'_1) + 0 \times (-i'_2) + \sum_k v_k \times i'_k &= v'_2 \times (-i_2) + 0 \times (-i_1) + \sum_k v'_k \times i_k \\
v_1 \times (-i'_1) + \sum_k v_k \times i'_k &= v'_2 \times (-i_2) + \sum_k v'_k \times i_k \\
\text{but, } v_k &= Z_k \times i_k \\
\text{and } v'_k &= Z_k \times i'_k \\
v_1 \times (-i'_1) + \sum_k Z_k \times i_k \times i'_k &= v'_2 \times (-i_2) + \sum_k Z_k \times i'_k \times i_k \\
\text{or, } v_1 \times (-i'_1) &= v'_2 \times (-i_2) \\
\text{Finally, } \frac{i_2}{v_1} &= \frac{i'_1}{v'_2}
\end{aligned}$$

2.1.3 One port excited with current & other port kept opened

The situation is depicted in figure 16. Port-1 is excited with a current of i_1 and as a response we get a voltage v_2 at port-2 as shown in figure 16(b). In the second case, port-2 is excited with a current of i'_2 and as a response we get a voltage v'_1 at port-1. Now applying Tellegen's theorem we get,

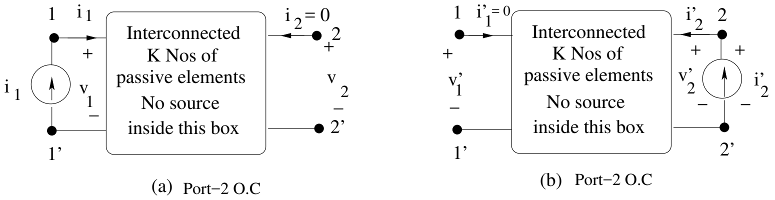

Figure 16:

$$
\begin{aligned}
v_1 \times (-i'_1) + v_2 \times (-i'_2) &= v'_1 \times (-i_1) + v'_2 \times (-i_2) \\
\text{or, } v_1 \times 0 + v_2 \times (-i'_2) &= v'_1 \times (-i_1) + v'_2 \times 0 \\
v_2 \times (-i'_2) &= v'_1 \times (-i_1) \\
\text{Finally, } \frac{v_2}{i_1} &= \frac{v'_1}{i'_2}
\end{aligned}
$$

In this case also, ratio of response to excitation remains same in both the circuits.