Problem Set - 8 Complete Solutions

1 (a) Find jacobian

(i) $T: x+y=u,\ y=uv \Rightarrow x=u(1-v);\ y=uv$

$$\therefore J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}1-v & v\\ -u & u\end{vmatrix}=u(1-v)+uv=u\ \text{(Ans)}$$

(ii) $T: x=2u+3v,\ y=2u-3v$

$$\therefore J=\frac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v}\end{vmatrix}=\begin{vmatrix}2 & 2\\ 3 & -3\end{vmatrix}=-6-6=-12\ \text{(Ans)}.$$

(iii) $T: x+y+z=u,\ x+y=uv;\ x=uvw$

$\Rightarrow z=u-uv,\ y=uv-uvw,\ x=uvw$

$$\therefore J=\frac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}\frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u}\\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v}\\ \frac{\partial x}{\partial w} & \frac{\partial y}{\partial w} & \frac{\partial z}{\partial w}\end{vmatrix}=\begin{vmatrix}vw & v-vw & 1-v\\ uw & u-uw & -u\\ uv & -uv & 0\end{vmatrix}$$

$$=-u^2v\ \text{(Ans)}.$$

(iv) $T: x=r\cos\phi\sin\theta,\ y=r\sin\phi\sin\theta,\ z=r\cos\theta.$

$$\therefore J=\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}=\begin{vmatrix}\cos\phi\sin\theta & \sin\phi\sin\theta & \cos\theta\\ r\cos\phi\cos\theta & r\sin\phi\cos\theta & -r\sin\theta\\ -r\sin\phi\sin\theta & r\cos\phi\sin\theta & 0\end{vmatrix}$$

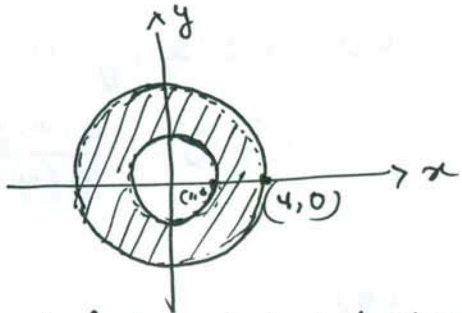
$$=r^2\sin\theta\ \text{(Ans)}.$$

Problem set - 8 Complete Solutions

1/(b). (i) The presence of $\sqrt{x^2+y^2}$ in the integrand suggests the use of polar coordinates.

Let $x = r\cos\theta$, $y = r\sin\theta$

$\therefore$ $J = \dfrac{\partial(x,y)}{\partial(r,\theta)} = r$ and the double integration changes to



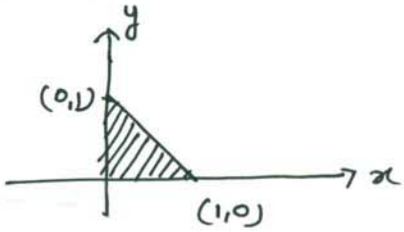
$$\iint_{R'} r.r\,dr\,d\theta,$$ where R' is bounded by $r=1$ and $r=2$

$$= 4\int_{\theta=0}^{\frac{\pi}{2}} d\theta \times \int_{r=1}^{2} r^2\,dr \quad \text{(by the help of symmetry)}$$

$$= 4\times\frac{\pi}{2}\times\left.\frac{r^3}{3}\right|_1^2 = \frac{14}{3}\pi \quad \text{(Ans.)}$$

1/(b) (ii) Let,

$$I = \iint_E e^{\frac{y}{x+y}}\,dx\,dy, \qquad E:$$



$\therefore$ E is the triangle bounded by $x=0,\ y=0,\ x+y=1$.

$\therefore$ Now use change of variable,

$$x+y=u, \quad y=uv,$$

Then. $J = \dfrac{\partial(x,y)}{\partial(u,v)} = u$ (see. 1(a)-(ii)).

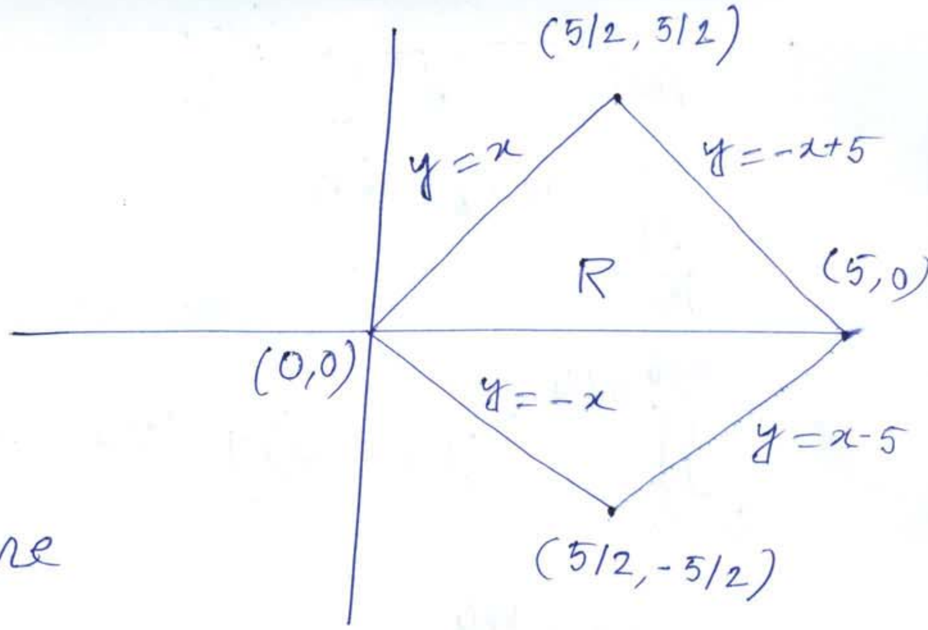
Next, E transform into E', bounded by $u=0$, $u=1$, and $v=0$ and $v=1$. Then.

$$I = \iint_{E'} e^{\frac{uv}{u}}.u.du\,dv = \iint_{E'} u.e^{v}\,du\,dv = \int_0^1 u\,du \times \int_0^1 e^v\,dv$$

$$= \left.\frac{u^2}{2}\right|_0^1 \times \left.e^v\right|_0^1$$

$$= \frac{1}{2}(e-1) \quad \text{(Ans).}$$

1.b.iii)



The region R is given on right side.

Transformations are

$x = 2u + 3v$

$y = 2u - 3v$

when $y = x \Rightarrow 2u + 3v = 2u - 3v$

$\Rightarrow v = 0$

$y = -x \Rightarrow u = 0$

$y = -x + 5 \Rightarrow 2u - 3v = -(2u + 3v) + 5$

$\Rightarrow u = 5/4$

$y = x - 5 \Rightarrow 2u - 3v = 2u + 3v - 5$

$\Rightarrow v = 5/6$

Now the new region is S is given by

$0 \le u \le 5/4 \qquad 0 \le v \le 5/6$

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 3 \\ 2 & -3 \end{vmatrix}$

$= -12$

∴ The integral is

$$\iint_R (x+y)\, dA$$

$$= \int_0^{5/6}\int_0^{5/4} \left((2u+3v) + (2u-3v) \right) |-12|\, du\, dv$$

$$= \int_0^{5/6}\int_0^{5/4} 48u\, du\, dv$$

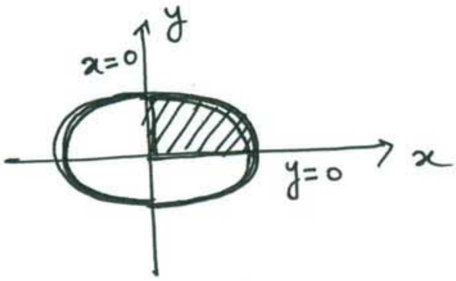
$$= \frac{5}{6} \times 24 \left[u^2\right]_0^{5/4}$$

$$= \frac{5}{\cancel{6}} \times \cancel{24}^{4} \times \frac{25}{\cancel{16}_{4}}$$

$$= \frac{125}{4}$$

1(c)

$$\iint_R \frac{\sqrt{a^2b^2 - b^2x^2 - a^2y^2}}{\sqrt{a^2b^2 + b^2x^2 + a^2y^2}}\,dx\,dy$$, R is the positive quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.



changing ellipse to a circle by putting $x = au,\ y = bv$

$\therefore\ J = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab.$

Hence the given double integral transform into

$$\iint_{R'} \frac{\sqrt{a^2b^2 - a^2b^2u^2 - a^2b^2v^2}}{\sqrt{a^2b^2 + a^2b^2u^2 + a^2b^2v^2}}\,ab\,du\,dv = ab\iint_{R'} \frac{\sqrt{1-u^2-v^2}}{\sqrt{1+u^2+v^2}}\,du\,dv$$

where R', the new field of integration is given by the positive quadrant of the circle $u^2+v^2=1$.

Next change to polar coordinates,
put $u = r\cos\theta,\ v = r\sin\theta\ \ \therefore J = r$, then the double integration further changes to

$$ab\iint_{R''} r\frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,dr\,d\theta$$, where R'': the positive quadrant of the circle $r=1$.

$$\therefore\ \text{Int} = ab\int_0^{\frac{\pi}{2}} d\theta \times \int_0^1 r\frac{\sqrt{1-r^2}}{\sqrt{1+r^2}}\,dr = \frac{\pi}{4}ab\left(\frac{\pi}{2}-1\right)$$

(Ans.)

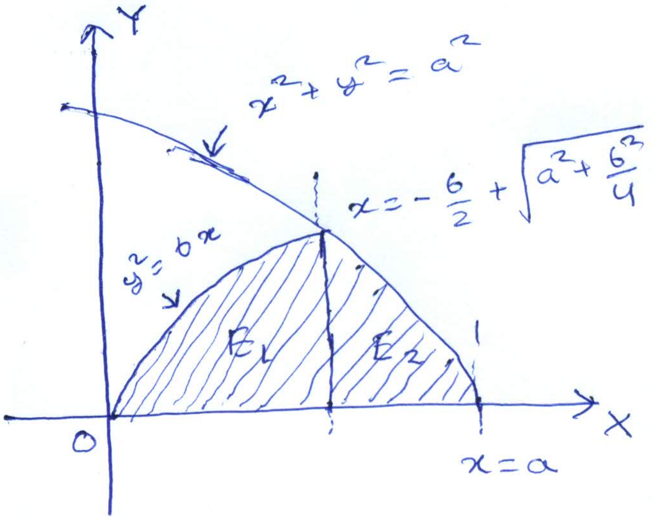
(2)

$x^2+y^2=a^2, \; y^2=bx$

$\therefore \; x^2+bx=a^2$

$\Rightarrow x^2+bx-a^2=0$

$\Rightarrow x=-\frac{b}{2} \pm \sqrt{a^2+\frac{b^2}{4}}$



$\therefore$ the curves $x^2+y^2=a^2$ and $y^2=bx$ intersects in the first quadrant at the point where $x=-\frac{b}{2}+\sqrt{a^2+\frac{b^2}{4}}=k$

Then, E is the shaded region in the above figure. We divide the domain E into two subregions E_1, E_2 both of which are quadratic with respect to y-axis.

Then,

$$\iint_E y\,dx\,dy.$$

$$=\iint_{E_1} y\,dx\,dy+\iint_{E_2} y\,dx\,dy$$

$$=\int_{x=0}^{k} dx \int_{y=0}^{\sqrt{bx}} y\,dy+\int_{x=k}^{a}\int_{y=0}^{\sqrt{a^2-x^2}} y\,dy.$$

$$=\frac{1}{2}\int_{x=0}^{k} bx\,dx+\frac{1}{2}\int_{x=k}^{a}(a^2-x^2)\,dx.$$

$$=\frac{1}{4}bk^2+\frac{1}{2}\left[a^2x-\frac{x^3}{3}\right]_k^a$$

$$=\frac{1}{4}bk^2+\frac{1}{2}\left[a^3-\frac{a^3}{3}-a^2k+\frac{k^3}{3}\right]$$

$$=\frac{1}{3}a^3-\frac{a^2}{2}k+\frac{b}{4}k^2+\frac{1}{6}k^3 \quad \text{where } k=-\frac{b}{2}+\sqrt{a^2+\frac{b^2}{4}}$$

(3) (a) $\iiint_R (x+y+z)\,dx\,dy\,dz$ where $R: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3.$

$$= \int_0^1 dx \int_1^2 dy \int_2^3 (x+y+z)\,dz$$

$$= \int_0^1 dx \int_1^2 dy \left[\frac{(x+y+z)^2}{2}\right]_2^3$$

$$= \frac{1}{2}\int_0^1 dx \int_1^2 dy \left[(x+y+3)^2 - (x+y+2)^2\right]$$

$$= \frac{1}{2}\int_0^1 dx \int_1^2 (2x+2y+5) \cdot 1 \cdot dy.$$

$$= \frac{1}{2}\int_0^1 dx \left[\frac{(2x+2y+5)^2}{4}\right]_1^2.$$

$$= \frac{1}{8}\int_0^1 dx \left[(2x+4+5)^2 - (2x+2+5)^2\right]$$

$$= \frac{1}{8}\int_0^1 (4x+16) \cdot 2\,dx$$

$$= \int_0^1 (x+4)\,dx.$$

$$= \left[\frac{x^2}{2} + 4x\right]_0^1$$

$$= \frac{1}{2} + 4$$

$$= \frac{9}{2}$$

(6) $$\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$$

$$= \int_0^{\log 2} e^x\,dx \int_0^{x} e^y\,dy \int_0^{x+\log y} e^z\,dz$$

$$= \int_0^{\log 2} e^x\,dx \int_0^{x} e^y\,dy \left[e^z\right]_0^{x+\log y}$$

$$= \int_0^{\log 2} e^x\,dx \int_0^{x} e^y\,dy \left(e^{x+\log y} - 1\right)$$

$$= \int_0^{\log 2} e^x\,dx \int_0^{x} e^y\,(y e^x - 1)\,dy.$$

$$= \int_0^{\log 2} e^x\,dx \left[(y e^x - 1)e^y - \int e^x e^y\,dy\right]_0^{x}$$

$$= \int_0^{\log 2} e^x\,dx \left[(y e^x - 1)e^y - e^{x+y}\right]_0^{x}$$

$$= \int_0^{\log 2} e^x\,dx \left[(x e^x - 1)e^x - e^{2x} + 1 + e^x\right]$$

$$= \int_0^{\log 2} e^x\,dx \left[x e^{2x} - e^x - e^{2x} + 1 + e^x\right]$$

$$= \int_0^{\log 2} (x e^{3x} - e^{3x} + e^x)\,dx.$$

$$= \left[x\frac{e^{3x}}{3} - \int 1\cdot\frac{e^{3x}}{3}\,dx - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2}$$

$$= \left[\frac{x}{3} e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^{x} \right]_0^{\log 2}$$

$$= \frac{\log 2}{3} e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{\log 2}{3} e^{\log 2^3} - \frac{e^{\log 2^3}}{9} - \frac{e^{\log 2^3}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3} \log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3} \log 2 - \frac{19}{9}$$

(4) Let the given region be R, then R is expressed as

$0 \le z \le 1 - x - y$, $0 \le y \le 1 - x$, $0 \le x \le 1$.

Then,

$$\iiint_R \frac{dx\,dy\,dz}{(1 + x + y + z)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(x + y + z + 1)^3}$$

$$= \int_0^1 dx \int_0^{1-x} dy \left[\frac{1}{-2(x + y + z + 1)^2} \right]_0^{1-x-y}$$

$$= -\frac{1}{2} \int_0^1 dx \int_0^{1-x} \left[\frac{1}{4} - \frac{1}{(x + y + 1)^2} \right] dy$$

$$= -\frac{1}{2}\int_0^1 dx \left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1 dx \left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1}\right]$$

$$= -\frac{1}{2}\int_0^1 \left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right] dx$$

$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{2} - \log 2 + \frac{1}{8}\right]$$

$$= -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$

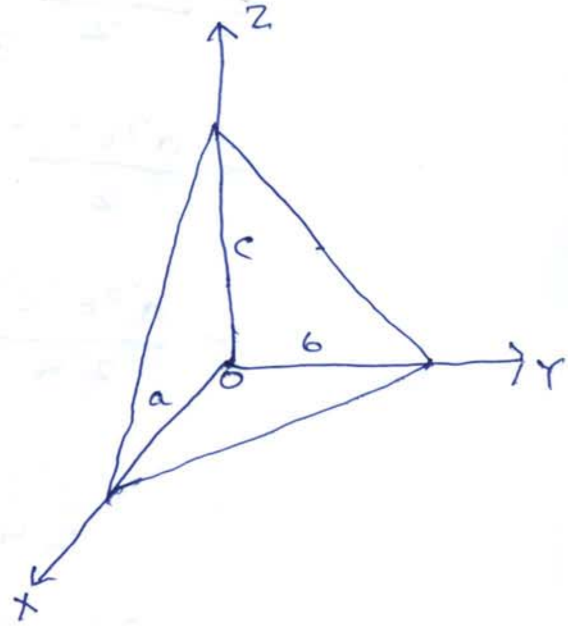
(5) Let $I = \iiint x^2 y z \, dx\, dy\, dz$ ......(1)

puting $x = au$, $y = bv$, $z = cw$

$(\Rightarrow) dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$

in (1), we get

$$I = \iiint a^2 bc\, u^2 v w\, a\,du\, b\,dv\, c\,dw$$



Then, the region of integration is bounded by

$u = 0$, $v = 0$, $w = 0$,

$u + v + w = 1$.

Then $I = \iiint_R a^3 b^2 c^2 u^2 v w \, du \, dv \, dw$

$$= \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-v} a^3 b^2 c^2 u^2 v w \, dw \, du \, dv$$

$$= a^3 b^2 c^2 \int_0^1 \int_0^{1-u} u^2 v \left[\frac{w^2}{2} \right]_0^{1-u-v} dv \, du$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 \int_0^{1-u} u^2 v (1-u-v)^2 \, dv \, du.$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 \int_0^{1-u} u^2 v \left[(1-u)^2 - 2(1-u)v + v^2 \right] dv \, du$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 \int_0^{1-u} u^2 \left[(1-u)^2 v - 2(1-u)v^2 + v^3 \right] du \, dv$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 u^2 \left[(1-u)^2 \frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4} \right]_0^{1-u} du$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 u^2 \left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4} \right] du$$

$$= \frac{a^3 b^2 c^2}{2} \int_0^1 \frac{u^2 (1-u)^4}{12} \, du.$$

$$= \frac{a^3 b^2 c^2}{24} \int_0^1 u^{3-1} (1-u)^{5-1} \, du.$$

$$= \frac{a^3 b^2 c^2}{24} B(3, 5)$$

$$= \frac{a^3 b^2 c^2}{24} \frac{\Gamma(3)\Gamma(5)}{\Gamma(8)} = \frac{a^3 b^2 c^2}{24} \cdot \frac{2!\,4!}{7!} = \frac{a^3 b^2 c^2}{2520}$$

Ans

(6) Let us convert the given integral into spherical polar co-ordinates. By putting

$x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$;

$z = r\cos\theta$.

then, $J = \dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = r^2\sin\theta$.

$\therefore$ ~~dx dy dz~~

$$I = \iiint (x^2+y^2+z^2)\, dx\, dy\, dz$$

$$= \iiint r^2\, J\, dr\, d\theta\, d\phi$$

$$= \iiint r^2\, r^2\sin\theta\, dr\, d\theta\, d\phi$$

$$= \iiint r^4\sin\theta\, dr\, d\theta\, d\phi.$$

Then, the integral over the volume enclosed by the sphere $x^2+y^2+z^2=1$, i.e, $r=1$ becomes.

$$I = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\int_{r=0}^{1} r^4\, dr\, \sin\theta\, d\theta\, d\phi$$

$$= \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi}\sin\theta\, d\theta \int_{r=0}^{1} r^4\, dr.$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\, d\theta \left(\frac{r^5}{5}\right)_0^1$$

$$= \frac{1}{5}\int_0^{2\pi} d\phi\, [-\cos\theta]_0^{\pi}$$

$$= \frac{2}{5}\int_0^{2\pi} d\phi = \frac{2}{5}[\phi]_0^{2\pi} = \frac{4\pi}{5} \quad \underline{\text{Ans}}$$

(7) $\iiint_R y\,dV$ where R is the region lies below the plane $z = x+1$ above the xy plane and between the cylinders $x^2+y^2=1$ and $x^2+y^2=4$.

Let us convert the given integral into cylindirical coordinates.

Put $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

then, $dx\,dy\,dz = J\,dr\,d\theta\,dz$

where $J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & \sin\theta & 0 \\ -r\sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$

$\therefore$ then, the limits of integration will be r from 1 to 4, θ from 0 to 2π and z from 0 to $r\cos\theta + 1$.

$\therefore$ the given integral will be

$$\iiint_R y\,dV$$

$$= \int_{r=1}^{4}\int_{\theta=0}^{2\pi}\int_{z=0}^{r\cos\theta+1} r\sin\theta \cdot r\,dr\,d\theta\,d\phi z$$

$$= \int_{r=1}^{4} r^2\,dr \int_{\theta=0}^{2\pi}\sin\theta\,d\theta \int_{z=0}^{r\cos\theta+1} d\phi$$

$$= \int_{r=1}^{4} r^2\,dr \int_{0}^{2\pi}(r\cos\theta+1)\sin\theta\,d\theta.$$

$$= \int_{1}^{4} r^2\,dr \int_{0}^{2\pi}(r\sin\theta\cos\theta + \sin\theta)\,d\theta.$$

$$= \int_1^4 r^2 dr \int_0^{2\pi} \left(\frac{r}{2} \sin 2\theta + \sin\theta\right) d\theta.$$

$$= \int_1^4 r^2 dr \left[-\frac{r}{4} \cos 2\theta - \cos\theta\right]_0^{2\pi}$$

$$= \int_1^4 r^2 dr \left[-\frac{r}{4} \cdot 1 - \cos 2\pi + \frac{r}{4} \cos 0 + \cos 0\right]$$

$$= \int_1^4 r^2 dr \cdot [0]$$

$$= 0$$

8

$$x^2 + z^2 = 4$$

$$\Rightarrow 2x + 2z\frac{\partial z}{\partial x} = 0$$

$$\Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = 0$$

$$\therefore 1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = 1 + \frac{x^2}{z^2}$$

$$= \frac{x^2+z^2}{z^2}$$

$$= \frac{4}{4-x^2}$$

Hence, the required surface area is

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$$

$$= 8\int_0^2 \int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\, dx\, dy$$

$$= 16\int_0^2 \frac{1}{\sqrt{4-x^2}} \cdot \left[\sqrt{4-x^2}\right] dx$$

$$= 16\int_0^2 dx$$

$$= 16 \times 2 = 32$$

9) $x^2+y^2+z^2=9$

$\Rightarrow \frac{\partial z}{\partial x} = -\frac{x}{z}, \quad \frac{\partial z}{\partial y} = -\frac{y}{z}$

Now, $1+\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$

$= 1+\frac{x^2}{z^2}+\frac{y^2}{z^2}$

$= \frac{x^2+y^2+z^2}{z^2} = \frac{9}{9-x^2-y^2}$ — (1)

put $x = r\cos\theta$, $y = r\sin\theta$. Then (1) becomes $\frac{9}{9-r^2}$

$x^2+y^2 = 3y$

$\Rightarrow r^2 = 3r\sin\theta$

$\Rightarrow r = 3\sin\theta$

Hence the required surface area is

$= \iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$

$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{9-r^2}}\, r\, d\theta\, dr$

$= 12\int_0^{\pi/2}\left(\int_0^{3\sin\theta} \frac{r\,dr}{\sqrt{9-r^2}}\right) d\theta$

$$= 12 \int_0^{\pi/2} \left[-\sqrt{9-r^2} \right]_0^{3\sin\theta} d\theta$$

$$= 12 \int_0^{\pi/2} \left[-\sqrt{9-9\sin^2\theta} + 3 \right] d\theta$$

$$= 36 \int_0^{\pi/2} \left[1 - \cos\theta \right] d\theta$$

$$= 36 \left[\theta - \sin\theta \right]_0^{\pi/2}$$

$$= 18(\pi - 2)$$

10) $x^2 + y^2 = a^2$ - - - - - - - - (1)

$x + y + z = a$ . . . . (2)

The projection of the surface area on xy plane is a circle $x^2+y^2=a^2$

$$x + y + z = a$$

$$\Rightarrow \frac{\partial z}{\partial x} = -1, \quad \frac{\partial z}{\partial y} = -1$$

$$\therefore \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{3}$$

Hence the required surface area is

$$= 4\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{3}\,.\, dx\,dy$$

$$= 4\sqrt{3}\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4\sqrt{3}\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4\sqrt{3}\,.\,\frac{a^2}{2}\cdot\frac{\pi}{2} \quad = \sqrt{3}\,\pi a^2$$

11) $y^2+z^2=4x$, $x=5$

The required volume is

$$V = \int_0^5 dx \int_{-2\sqrt{x}}^{2\sqrt{x}} dy \int_{-\sqrt{4x-y^2}}^{\sqrt{4x-y^2}} dz$$

$$= 4\int_0^5 \int_0^{2\sqrt{x}} \int_0^{\sqrt{4x-y^2}} dx\,dy\,dz$$

$$= 4\int_0^5 \int_0^{2\sqrt{x}} \sqrt{4x-y^2}\,dy$$

$$= 4\int_0^5 \left[\frac{y}{2}\cdot\sqrt{4x-y^2}+\frac{4x}{2}\sin^{-1}\frac{y}{2\sqrt{x}}\right]_0^{2\sqrt{x}}$$

$$= 4\pi\int_0^5 x\,dx = 50\pi$$

12) $x^2+y^2=1$ —— ①

$x+y+z=3$ —— ②

$z=0$ —— ③

Required volume is

$$= \iiint dx\,dy\,dz$$

$$= \iint dx\,dy\,[z]_0^{3-x-y}$$

$$= \iint (3-x-y)\,dx\,dy$$

putting $x = r\cos\theta$, $y = r\sin\theta$ we get,

$$= \iint (3 - r\cos\theta - r\sin\theta)\, r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (3r - r^2\cos\theta - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{3r^2}{2} - \frac{r^3\cos\theta}{3} - \frac{r^3}{3}\sin\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(3/2 - \frac{1}{3}\cos\theta - \frac{1}{3}\sin\theta\right) d\theta$$

$$= \left[\frac{3}{2}\theta - \frac{1}{3}\sin\theta + \frac{1}{3}\cos\theta\right]_0^{2\pi} = 3\pi$$

13) $x^2+y^2=4 \Rightarrow y = \pm\sqrt{4-x^2}$

$y+z=4 \Rightarrow z$ varies from 0 to $4-y$

x ,, ,, -2 to 2

∴ The required volume

$$V = \iiint dx\,dy\,dz$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{4-y} dx\,dy\,dz$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (4-y)\,dy$$

$$= \int_{x=-2}^{2} \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= 8\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 8\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$$

$$= 16\pi$$