TRIPLE INTEGRALS

Divide the region V into n sub-regions of respective volumes $\delta V_1, \delta V_2, \dots \delta V_n$. Let (x_r, y_r, z_r) be an arbitrary point in the rth sub-region.

Consider the sum

$$\sum_{j=1}^{n} f(x_j, y_j, z_j)\, \delta V_j$$

If the limit of this sum exists as $n \to \infty$ and $\delta V_j \to 0$, then

$$\iiint_V f(x,y,z)\, dV = \lim_{n\to\infty} \sum_{j=1}^{n} f(x_j, y_j, z_j)\, \delta V_j$$

Evaluation:

$$\iiint_V f(x,y,z)\, dV = \int_{z=a}^{b} \left[\int_{y=\varphi_1(z)}^{\varphi_2(z)} \left\{ \int_{x=f_1(y,z)}^{f_2(y,z)} f(x,y,z)\, dx \right\} dy \right] dz$$

Not: Similar to double integrals, the order of integration is immaterial if the limits of integration are constants.

$$\int_a^b \int_c^d \int_e^f F(x,y,z)\, dx\, dy\, dz = \int_e^f \int_c^d \int_a^b F(x,y,z)\, dz\, dy\, dx$$

$$= \int_c^d \int_e^f \int_a^b F(x,y,z)\, dz\, dx\, dy$$

Example: Evaluate $I = \int_0^a \int_0^x \int_0^{x+y} e^{x+y+z}\, dz\, dy\, dx$

$$I = \int_0^a \int_0^x e^{x+y+z}\Big|_0^{x+y} \cancel{dz}\, dy\, dx$$

$$= \int_0^a \int_0^x \left(e^{2(x+y)} - e^{x+y}\right) dy\, dx$$

$$= \int_0^a \frac{e^{2(x+y)}}{2}\Big|_0^x dx - \int_0^a e^{x+y}\Big|_0^x dx$$

$$= \frac{1}{2}\left(\int_0^a \left\{\left(e^{4x} - e^{2x}\right) - 2\left(e^{2x} - e^{x}\right)\right\} dx\right.$$

$$= \frac{1}{2} \int_0^a \left(e^{4x} - 3e^{2x} + 2e^{x}\right) dx$$

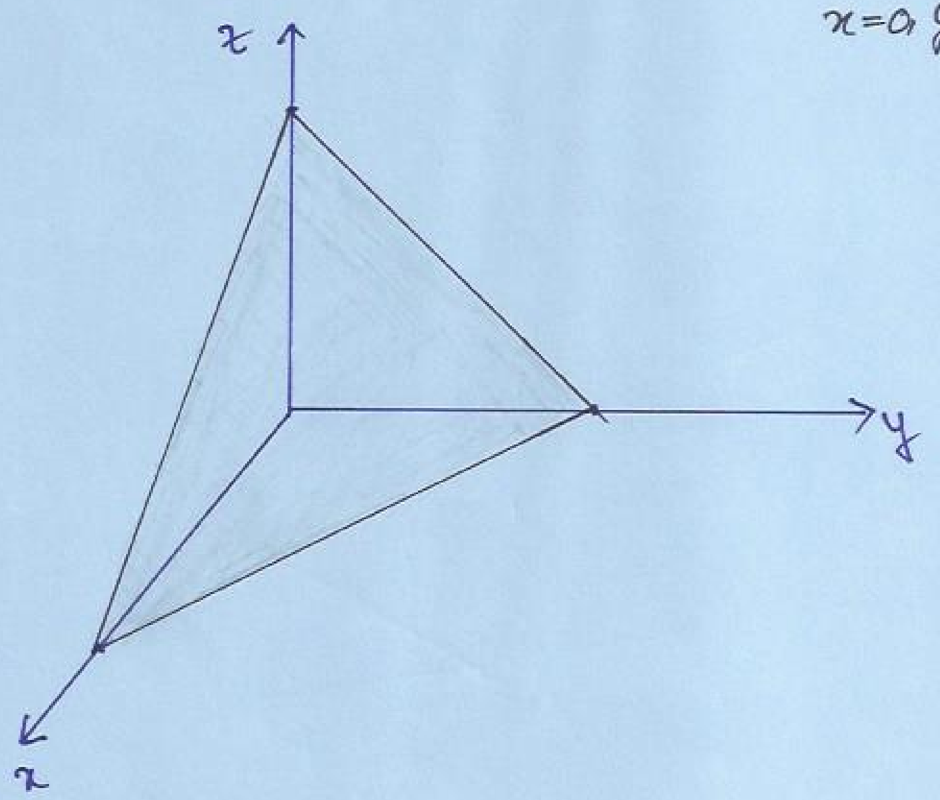
$$= \frac{1}{2}\left[\frac{e^{4x}}{4}\Big|_0^a - \frac{3}{2} e^{2x}\Big|_0^a + 2e^{x}\Big|_0^a\right]$$

$$= \frac{1}{2}\left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^{a} - \frac{1}{4} + \frac{3}{2} - 2\right]$$

$$= \frac{1}{2}\left[\frac{e^{4a}}{4} - \frac{3}{2} e^{2a} + 2e^{a} - \frac{3}{4}\right]$$

$$= \frac{e^{4a}}{8} - \frac{3}{4} e^{2a} + e^{a} - \frac{3}{8}$$ An

Example: Evaluate $\iiint\limits_R \frac{dx\,dy\,dz}{(x+y+z+1)^3}$; R is the region bounded by $x=0,\ y=0,\ z=0$ & $x+y+z=1$



$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} \cdot dz\ dy\ dx$$

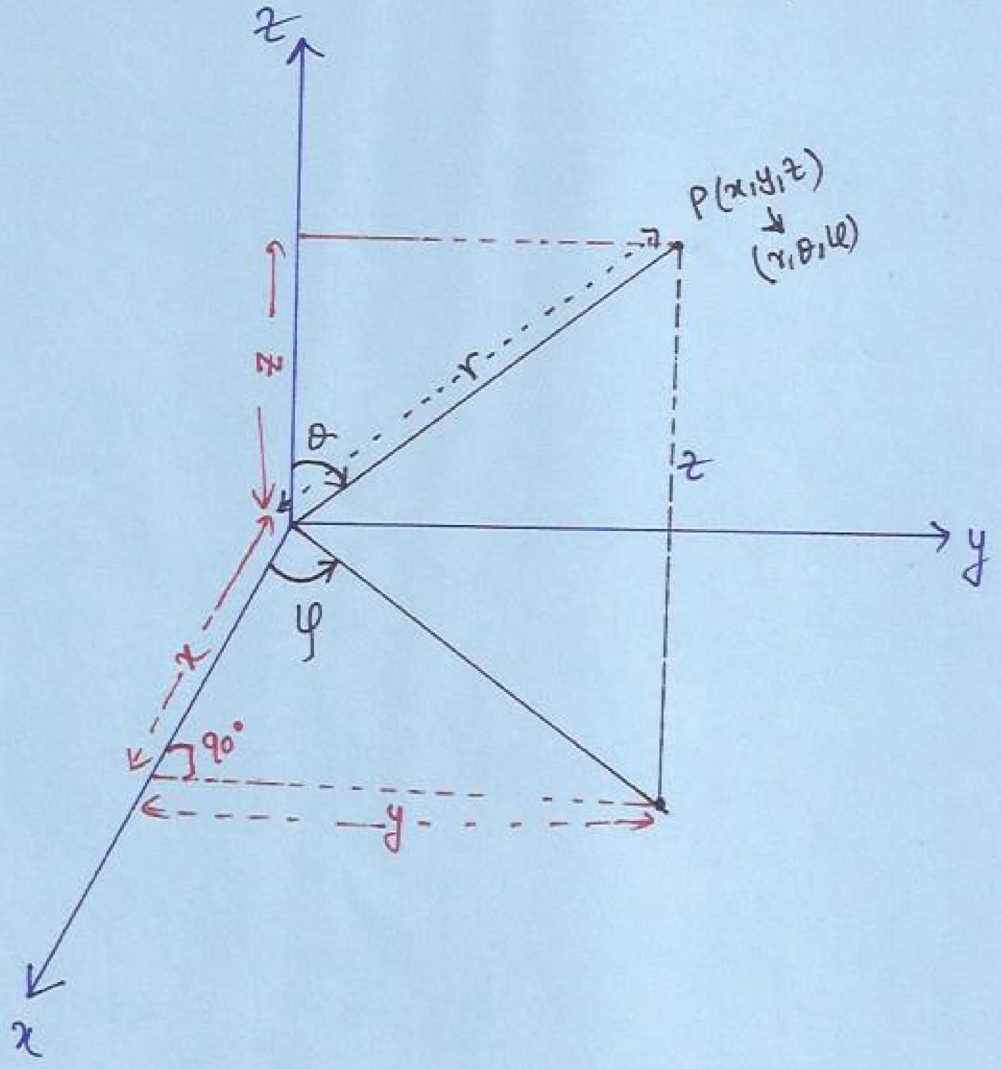
$$= \int_0^1 \int_0^{1-x} \left[-\frac{1}{2} (x+y+z+1)^{-2} \right]_0^{1-x-y} dy\ dx$$

$$\vdots$$

$$= \frac{1}{2}\left[\ln 2 - \frac{5}{8}\right]$$ Ans

Change of Variables in TRIPLE integrals:

- Cartesian co-ordinate (x,y,z) to Spherical polar coordinates



$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$

Note that $\quad x^2+y^2+z^2 = r^2$

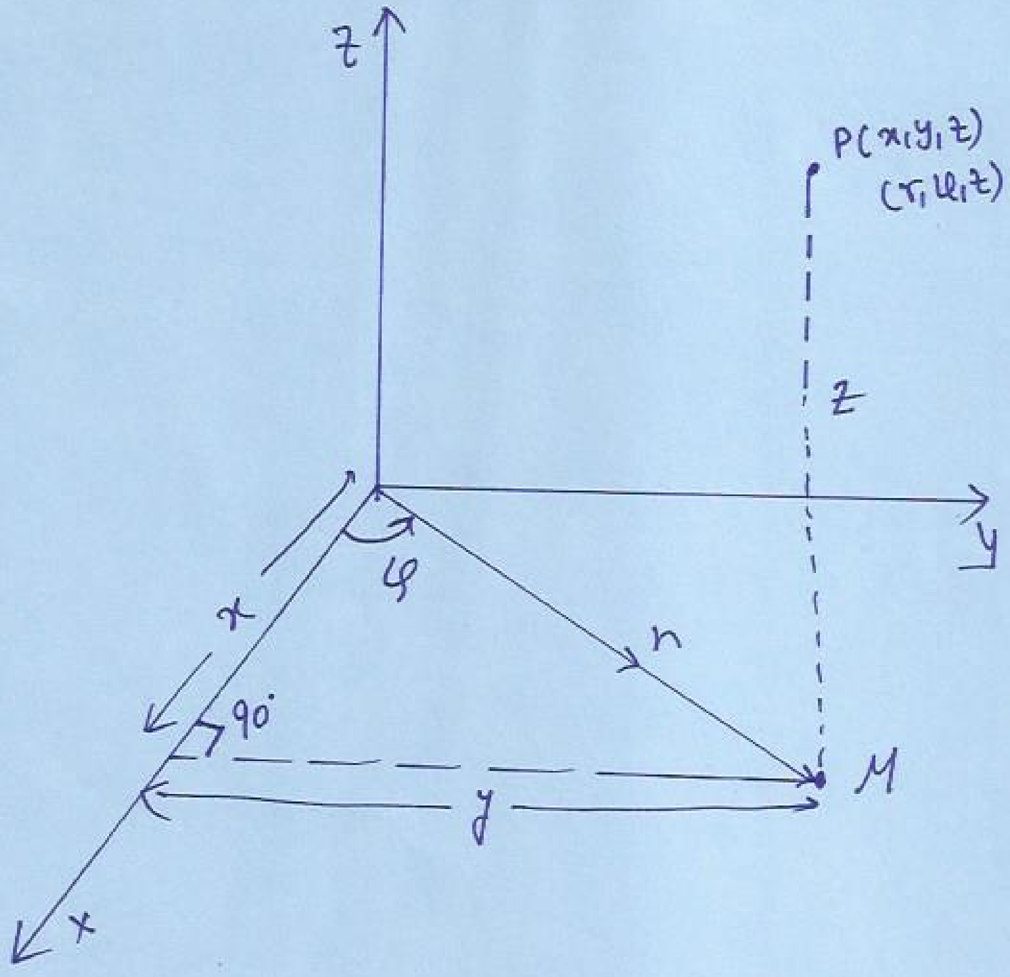
$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\theta\sin\varphi & r\cos\theta\sin\varphi & r\sin\theta\cos\varphi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{D}} f(r\sin\theta\cos\varphi,\, r\sin\theta\sin\varphi,\, r\cos\theta)\, r^2\sin\theta\, dr\,d\theta\,d\varphi.$$

ii) Cartesian Coordinates (x,y,z) to Cylindrical coordinates (r,φ,z)

$$(x,y,z) \to (r,\varphi,z)$$



$$x = r\cos\varphi$$
$$y = r\sin\varphi$$
$$z = z$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \varphi} & \frac{\partial z}{\partial z} \end{vmatrix} = r$$

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_{\tilde{D}} f(r\cos\varphi, r\sin\varphi, z)\, r\,dr\,d\varphi\,dz$$

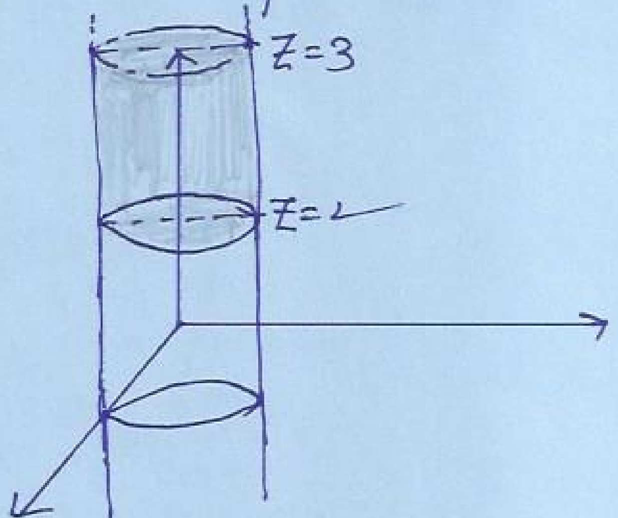
Ex: Changing to cylindrical coordinate, evaluate

$$\iiint z(x^2+y^2)\,dx\,dy\,dz\ ; \qquad \begin{array}{l} x^2+y^2 \le 1 \\ 2 \le z \le 3 \end{array}$$

Solution: $x = r\cos\theta \quad y = r\sin\varphi \quad z = z$

Note that $x^2+y^2 = r^2$

& $J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \vdots & & \\ \vdots & & \end{vmatrix} = r$



$$\iiint z(x^2+y^2)\,dx\,dy\,dz = \int_{z=2}^{3}\int_{\varphi=0}^{2\pi}\int_{r=0}^{1} z\cdot r^2\, r\, dr\, d\varphi\, dz$$

$$= \int_2^3\int_0^{2\pi} \frac{1}{4} z\, d\varphi\, dz$$

$$= \frac{1}{4}\, 2\pi\, \frac{1}{2}(9-4) = \frac{5\pi}{4}.$$

Example: Evaluate $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \frac{1}{\sqrt{x^2+y^2+z^2}}\,dz\,dy\,dx$ by changing into spherical polar coordinate.

Solution:

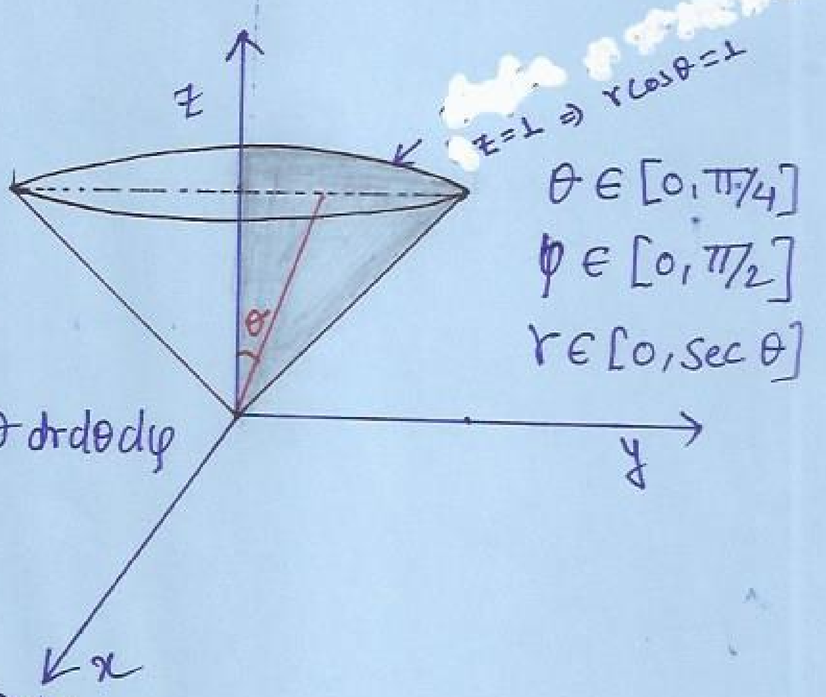
$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta \qquad x^2+y^2+z^2 = r^2$$

x varies from 0 to 1

y varies from 0 to $y=\sqrt{1-x^2}$ i.e., $y^2+x^2=1$

z varies from $\sqrt{x^2+y^2}$, i.e. $z^2 = x^2+y^2$ (cone) to 1



$\theta \in [0, \pi/4]$

$\phi \in [0, \pi/2]$

$r \in [0, \sec\theta]$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_{r=0}^{\sec\theta} \frac{1}{r}\cdot r^2\sin\theta\,dr\,d\theta\,d\varphi$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \frac{1}{2}\sec^2\theta\sin\theta\,d\theta\,d\varphi$$

$$= \frac{\pi}{4}\int_0^{\pi/4} \sec\theta\tan\theta\,d\theta$$

$$= \frac{\pi}{4}\sec\theta\Big|_0^{\pi/4} = \frac{(\sqrt{2}-1)\pi}{4}$$

Ex. Evaluate $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{1}{\sqrt{1-x^2-y^2-z^2}}\, dz\, dy\, dx$$

by changing to spherical polar coordinates.

Sol:

$$x = r\sin\theta\cos\varphi \qquad y = r\sin\theta\sin\varphi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta.$$

$$I = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{\pi/2} \int_{r=0}^{1} \frac{r^2\sin\theta}{\sqrt{1-r^2}}\, dr\, d\varphi\, d\theta$$

First evaluate: $\displaystyle\int_0^1 \frac{r^2}{\sqrt{1-r^2}}\, dr$ subst $r = \sin t$, $dr = \cos t\, dt$

$$= \int_0^{\pi/2} \frac{\sin^2 t}{\cos t} \cos t\, dt = \frac{\pi}{4}$$

$$I = \frac{\pi}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta\, d\varphi\, d\theta$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2} \cdot \left[-\cos\theta\right]_0^{\pi/2}$$

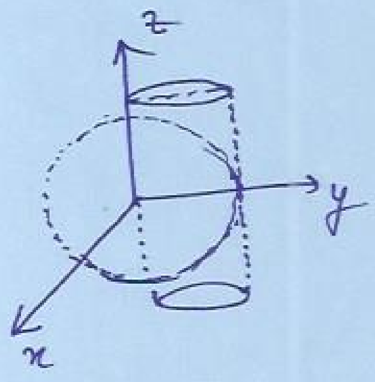
$$= \frac{\pi^2}{8} \cdot 1.$$

$$= \frac{\pi^2}{8}$$ Ans

Q. Using triple integral find the volume common to a sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

$$V = \iiint_V dx\,dy\,dz = \iiint_V dz\,dy\,dx$$

$$= 4\int_0^a \int_{y=0}^{\sqrt{ax-x^2}} \int_{z=0}^{\sqrt{a^2-x^2-y^2}} dz\,dy\,dx$$



$$= 4\int_0^a \int_0^{\sqrt{ax-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx$$

⋮

proceed as in double integral

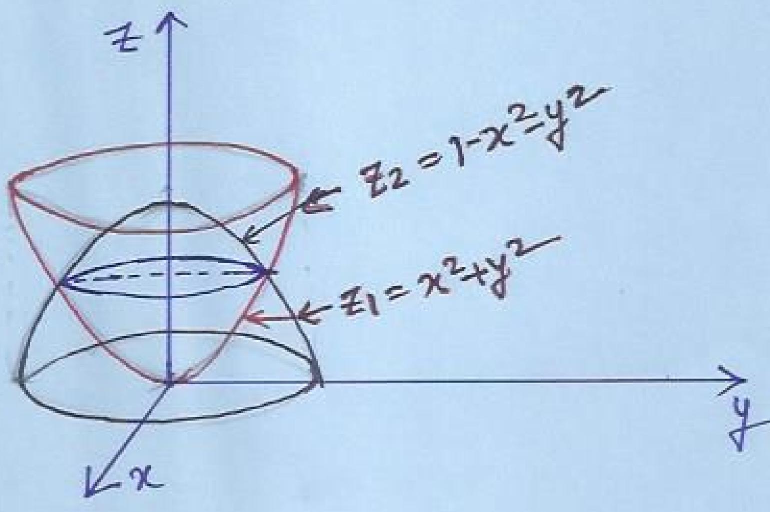
$$= \frac{2}{9}a^3\left(\pi - \frac{4}{3}\right).$$

Q. Find the volume of the solid formed by two paraboloids: $z_1 = x^2+y^2$ & $z_2 = 1-x^2-y^2$

Intersecting curve:

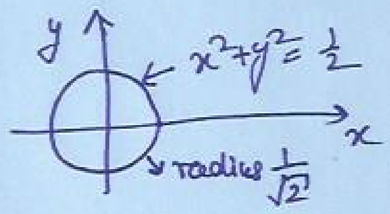
$x^2+y^2 = 1-x^2-y^2$

$\Rightarrow x^2+y^2 = \frac{1}{2}$



$$V = \iiint_V dx\,dy\,dz = \int_{x=-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{y=-\sqrt{\frac{1}{2}-x^2}}^{+\sqrt{\frac{1}{2}-x^2}} \int_{x^2+y^2}^{1-x^2-y^2} dz\,dy\,dx$$

Projection on xy plane:



Changing to cylindrical coordinates

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$V = 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} \int_{z=r^2}^{1-r^2} r\,dz\,dr\,d\theta$$

$$= 4\int_0^{\pi/2} \int_{r=0}^{\frac{1}{\sqrt{2}}} r\cdot(1-r^2-r^2)\,dr\,d\theta$$

$$= 2\pi \int_0^{\frac{1}{\sqrt{2}}} r(1-2r^2)\,dr$$

$$= 2\pi\left[\frac{1}{2}\left(\frac{1}{2}-0\right) - \frac{2}{4}\cdot\left(\frac{1}{4}-0\right)\right]$$

$$= 2\pi\cdot\left[\frac{1}{4}-\frac{1}{8}\right]$$

$$= 2\pi\cdot\frac{1}{8}$$

$$= \frac{\pi}{4} \quad \text{Ans.}$$