# ASSIGNMENT 9 SOLUTION



~~We know~~ Let, $\vec{F}(t) = F_1(t)\hat{i} + F_2(t)\hat{j} + F_3(t)\hat{k}$, then we have, $\lim_{t\to a} \vec{F}(t) = \hat{i}\cdot\lim_{t\to a} F_1(t) + \hat{j}\cdot\lim_{t\to a} F_2(t) + \hat{k}\cdot\lim_{t\to a} F_3(t)$.

(i) Here, $F_1(t) = \frac{\sin t}{t}$, $F_2(t) = 3\cos 3t$, $F_3(t) = e^t \tan t$

Also, $\lim_{t\to 0} F_1(t) = \lim_{t\to 0} \frac{\sin t}{t} = 1$

$\lim_{t\to 0} F_2(t) = \lim_{t\to 0} 3\cos 3t = 3$

$\lim_{t\to 0} F_3(t) = \lim_{t\to 0} e^t \tan t = e^0 \tan 0 = 0$

$\therefore \lim_{t\to 0} \vec{F}(t) = \hat{i} + 3\hat{j}$ (Ans)

(ii) Here, $F_1(t) = (t + 3t^2 \sin t)$, $F_2(t) = t^2 \cos(\frac{1}{t})$, $F_3(t) = 2$.

Clearly, $\lim_{t\to 0} F_1(t) = 0$ & $\lim_{t\to 0} F_3(t) = 2$.

Now, $|F_2(t) - 0| = |t^2 \cos(\frac{1}{t}) - 0|$

$= |t|^2 |\cos(\frac{1}{t})|$

$\leq |t|^2$ $(\because |\cos(\frac{1}{t})| \leq 1)$

$< \varepsilon$, whenever $|t| < \delta (= \sqrt{\varepsilon})$

$\Rightarrow \lim_{t\to 0} F_2(t) = 0$

$\therefore \lim_{t\to 0} \vec{F}(t) = 2\hat{k}$ (Ans)

(iii) Here, $F_2(t) = \frac{|t|}{t}$

Now, $\lim_{t\to 0+} F_2(t) = \lim_{t\to 0+} \frac{|t|}{t} = 1$

and, $\lim_{t\to 0-} F_2(t) = \lim_{t\to 0-} \frac{|t|}{t} = -1$ — which are not equal.

$\therefore \lim_{t\to 0} F_2(t)$ does not exists.

$\Rightarrow \lim_{t\to 0} \vec{F}(t)$ does not exists. (Ans)

1) (i) Let, $\phi(x, y, z) = x^2y^3 + 3xz + 2y^4 - 5$

$\therefore \vec{\nabla}\phi = (2xy^3 + 3z)\hat{i} + (3x^2y^2 + 8y^3)\hat{j} + 3x\hat{k}$

Then the normal vector to the ~~Plane~~ surface $\phi = 0$ at $(1, -2, 0)$ is, $\vec{\nabla}\phi|_{(1,-2,0)} = [2\times1\times(-2)^3 + 0]\hat{i} + [3\times(-2)^2 + 8(-2)^3]\hat{j} + 3\hat{k} = -16\hat{i} - 52\hat{j} + 3\hat{k}$

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Then the unit normal vector at $(1,-2,0)$ is,

$$\left[\frac{\nabla\phi}{|\nabla\phi|}\right]_{(1,-2,0)} = \frac{1}{\sqrt{16^2+52^2+3^2}}\left(-16\hat{i}-52\hat{j}+3\hat{k}\right)$$

And the tangent plane at $(1,-2,0)$ is,

$$(x-1)(-16) + (y+2)(-52) + (z-0)\cdot 3 = 0$$

$$\Rightarrow 16x + 52y - 3z + 88 = 0 \qquad (\underline{Ans})$$

(ii) $\phi(x,y,z) = y\log x - x^2 + xz^3 - 1$

Now, $\vec{\nabla}\phi = \left(\frac{y}{x} - 2x + z^3\right)\hat{i} + \log x\,\hat{j} + 3xz^2\hat{k}$

$\therefore$ The unit normal vector ~~at~~ to the surface $\phi = 0$ at $(1,0,1)$ is,

$$\left[\frac{\nabla\phi}{|\nabla\phi|}\right]_{(1,0,1)} = \frac{-\hat{i}+3\hat{k}}{\sqrt{1+9}} = \frac{1}{\sqrt{10}}\left(-\hat{i}+3\hat{k}\right)$$

And the tangent plane at $(1,0,-1)$ is,

$$(x-1)(-1) + (y-0)\cdot 0 + (z+1)3 = 0$$

$$\Rightarrow 3z - x + 4 = 0 \qquad (\underline{Ans})$$

(iii) $\phi(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$

$\therefore \nabla\phi = \frac{2x}{a^2}\hat{i} + \frac{2y}{b^2}\hat{j} + \frac{2z}{c^2}\hat{k}$

$\therefore$ The unit normal at $(x_0, y_0, z_0)$ is,

$$\left.\frac{\nabla\phi}{|\nabla\phi|}\right|_{(x_0,y_0,z_0)} = \frac{\frac{x_0}{a^2}\hat{i} + \frac{y_0}{b^2}\hat{j} + \frac{z_0}{c^2}\hat{k}}{\sqrt{\frac{x_0^2}{a^4} + \frac{y_0^2}{b^4} + \frac{z_0^2}{c^4}}}$$

And the tangent plane through $(x_0, y_0, z_0)$ is,

$$(x-x_0)\frac{x_0}{a^2} + (y-y_0)\frac{y_0}{b^2} + (z-z_0)\frac{z_0}{c^2} = 0$$

$$\Rightarrow \frac{xx_0}{a^2} + \frac{yy_0}{b^2} + \frac{zz_0}{c^2} = 1 \qquad \left(\because \frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} + \frac{z_0^2}{c^2} = 1\right)$$

$(\underline{Ans})$

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1) $\phi(x,y,z) = \tan(x^2+y^2) - z$

$\therefore \nabla\phi = 2\sec^2(x^2+y^2)\left[x\hat{i} + y\hat{j}\right] - \hat{k}$

$= 2\sec^2(x^2+y^2)x\hat{i} + 2\sec^2(x^2+y^2)y\hat{j} - \hat{k}$

$\nabla\phi|_{(0,0,1)} = 2\sec^2(0^2+0^2)\cdot 0\cdot\hat{i} + 2\sec^2(0^2+0^2)\cdot 0\hat{j} - \hat{k} = -\hat{k}$

$\therefore \frac{\nabla\phi}{|\nabla\phi|}\Big|_{(0,0,1)} = \frac{-\hat{k}}{\sqrt{(-1)^2}} = -\hat{k}$, unit normal vector

$\therefore$

2) (i) $f(x,y) = e^x\cos y$.

$\therefore \nabla\phi = e^x\cos y\,\hat{i} - e^x\sin y\,\hat{j}$

$\therefore \nabla\phi|_{(0,\frac{\pi}{4})} = \frac{1}{\sqrt{2}}\hat{i} - \frac{1}{\sqrt{2}}\hat{j} = \frac{1}{\sqrt{2}}(\hat{i}-\hat{j})$

Let, $\vec{r} = \hat{i}+3\hat{j}$. Then the unit vector along $\vec{r}$ is,

$\hat{n} = \frac{\vec{r}}{|\vec{r}|} = \frac{1}{\sqrt{10}}(\hat{i}+3\hat{j})$

$\therefore$ Directional derivative of $f(x,y)$ along $\vec{r}$ is,

$\nabla\phi|_{(0,\frac{\pi}{4})}\cdot\hat{n} = \frac{1}{\sqrt{2}}(\hat{i}-\hat{j})\cdot\frac{1}{\sqrt{10}}(\hat{i}+3\hat{j})$

$= \frac{1}{2\sqrt{5}}(1-3) = -1/\sqrt{5}$

(ii) Here, $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$

$\therefore \vec{\nabla}f = \frac{3}{2}(x^2+y^2+z^2)^{1/2}\cdot(2x\hat{i}+2y\hat{j}+2z\hat{k})$

$= 3\sqrt{x^2+y^2+z^2}\,(x\hat{i}+y\hat{j}+z\hat{k})$

$$\therefore \nabla f\Big|_{(-1,1,2)} = 3\sqrt{1+1+4}\left(-\hat{i}+\hat{j}+2\hat{k}\right) = 3\sqrt{6}\left(-\hat{i}+\hat{j}+2\hat{k}\right)$$

$\therefore$ Directional derivative of $f(x,y,z)$ at $(-1,1,2)$ along $\hat{i}-2\hat{j}+\hat{k}$ is,

$$\nabla f\Big|_{(-1,1,2)}\cdot\hat{n} = 3\sqrt{6}\left(-\hat{i}+\hat{j}+2\hat{k}\right)\cdot\frac{(\hat{i}-2\hat{j}+\hat{k})}{\sqrt{1+4+1}}$$

$$= 3(-1-2+2) = -3$$

(iii) $f(x,y,z) = \sqrt{xy^2+2x^2z}$

$$\therefore \nabla f = \frac{1}{2\sqrt{xy^2+2x^2z}}\left[(y^2+4xz)\,\hat{i} + 2xy\,\hat{j} + 2x^2\,\hat{k}\right]$$

$\therefore$ Directional derivative of $f$ at $(2,-2,1)$ along $z$-axis is,

$$\nabla f\Big|_{(2,-2,1)}\cdot\hat{k} = \left[\frac{2x^2}{2\sqrt{xy^2+2x^2z}}\right]_{(2,-2,1)} = \frac{4}{\sqrt{2\times4+2\times4}} = 1$$

(iv) $f(x,y) = 3x^4+2y^3 \cancel{\;-\;8z\;\text{at}\;(1,1,0)} - a^2$ , $a=$ Const.

$$\therefore \nabla f = 12x^3\,\hat{i} + 6y^2\,\hat{j} \cancel{-8z}$$

$$\therefore \nabla f\Big|_{(1,1)} = 12\hat{i}+6\hat{j}.$$

Now the unit vector in the direction which makes $30^\circ$ angle to $x$-axis is,

$$\hat{n} = \cos(30^\circ)\,\hat{i} + \sin(30^\circ)\,\hat{j}$$

$$= \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}$$

$\therefore$ The required directional derivative is,

$$\nabla f\Big|_{(1,1)}\cdot\hat{n} = 6\sqrt{3}+3 = 3(1+2\sqrt{3}).$$

Here, $\phi(x,y,z) = x^2 y^2 z^4$.

$\therefore \nabla\phi = 2xy^2z^4\hat{i} + 2x^2yz^4\hat{j} + 4x^2y^2z^3\hat{k}$.

$\therefore \nabla\phi|_{(3,1,-2)} = (2\times3\times1\times16)\hat{i} + (2\times9\times1\times16)\hat{j} + 4\times9\times1\times(-8)\hat{k}$

$= 96\hat{i} + 288\hat{j} - 288\hat{k}$

$= 96(\hat{i}+3\hat{j}-3\hat{k})$

$\therefore$ The directional derivative of $\phi$ at $(3,1,-2)$ is max$^m$ along the direction of $\nabla\phi|_{(3,1,-2)}$, i.e. $\hat{i}+3\hat{j}-3\hat{k}$; and the max$^m$ directional derivative is, $|\nabla\phi|_{(3,1,-2)}| = 96\sqrt{1^2+3^2+(-3)^2} = 96\sqrt{19}$.

(Ans)

4) Here the two surfaces are,

$\phi_1(x,y,z) = x^2+y^2+z^2-6=0$

$\phi_2(x,y,z) = x^2+y^2-z = 0$

We know that the angle between two surfaces is equal to the angle between their normals.

Now, a normal to $\phi_1=0$ at $(1,-1,+2)$ is,

$\nabla\phi_1|_{(1,-1,2)} = (2x\hat{i}+2y\hat{j}+2z\hat{k})|_{(1,-1,2)} = 2\hat{i}-2\hat{j}+4\hat{k}$

Similarly, a normal to $\phi_2=0$ at $(1,-1,2)$ is,

$\nabla\phi_1|_{(1,-1,2)} = (2x\hat{i}+2y\hat{j}-\hat{k})|_{(1,-1,2)} = 2\hat{i}-2\hat{j}-\hat{k}$

Let $\theta$ be the angle between the two surfaces at $(1,-1,2)$.

Then, $\nabla\phi_1\cdot\nabla\phi_2 = |\nabla\phi_1||\nabla\phi_2|\cos\theta$ at $(1,-1,2)$.

$\Rightarrow (2\hat{i}-2\hat{j}+4\hat{k})\cdot(2\hat{i}-2\hat{j}-\hat{k}) = \sqrt{2^2+(-2)^2+4^2}\cdot\sqrt{2^2+(-2)^2+(-1)^2}\cos\theta$

$\Rightarrow \cos\theta = \frac{4}{\sqrt{24}\times3} = \frac{2}{3\sqrt{6}}$

$\Rightarrow \theta = \cos^{-1}(2/3\sqrt{6})$ (Ans)

5) Let, $\phi_1(x,y,z) = ax^2-byz-(a+2)x=0$

$\phi_2(x,y,z) = 4x^2y+z^3-4=0$.

$\therefore \nabla\phi_1 = [2ax - (a+2)]\hat{i} + (-bz)\hat{j} + (-by)\hat{k}$

$\nabla\phi_2 = 8xy\,\hat{i} + 4x^2\hat{j} + 3z^2\hat{k}$

$\therefore \nabla\phi_1\big|_{(1,-1,2)} = (a-2)\hat{i} - 2b\hat{j} + b\hat{k}$

$\nabla\phi_2\big|_{(1,-1,2)} = -8\hat{i} + 4\hat{j} + 12\hat{k}$

Now, if the two surfaces are orthogonal to each other, then,

$\nabla\phi_1 \cdot \nabla\phi_2 = 0$ at $(1,-1,2)$

$\Rightarrow -8(a-2) - 8b + 12b = 0$

$\Rightarrow 2a - b = 4$ ——— (1)

Again, as $(1,-1,2)$ is a point on the surface $\phi_1 = 0$, then

$a + 2b - (a+2) = 0$

$\Rightarrow b = 1$ ——— (2)

$\therefore$ From (1), $a = \frac{1}{2}(b+4) = \frac{5}{2}$ . ——— (3) (Ans)

(a) $r^n = (x^2+y^2+z^2)^{n/2}$

So, $\nabla r^n = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)^{n/2}$

Now, $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2}$

$= \frac{n}{2}\cdot(x^2+y^2+z^2)^{\frac{n}{2}-1}\cdot 2x = nx(x^2+y^2+z^2)^{\frac{n-2}{2}}$

$= nx\cdot r^{n-2}$

Similarly, $\frac{\partial}{\partial y}(x^2+y^2+z^2)^{n/2} = ny\cdot r^{n-2}$ and

$\frac{\partial}{\partial z}(x^2+y^2+z^2)^{n/2} = nz\, r^{n-2}$

$\therefore \nabla r^n = nr^{n-2}(x\hat{i}+y\hat{j}+z\hat{k}) = nr^{n-2}\vec{r}$ (proved)

(b) $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

L.H.S. $\nabla^2 r^n = \nabla^2(x^2+y^2+z^2)^{n/2}$

Now, $\frac{\partial}{\partial x}(x^2+y^2+z^2)^{n/2} = nx\cdot(x^2+y^2+z^2)^{\frac{n-2}{2}}$

$\frac{\partial^2}{\partial x^2}(x^2+y^2+z^2)^{n/2} = n\cdot(x^2+y^2+z^2)^{\frac{n-2}{2}} + x^2\cdot n\cdot\frac{n-2}{\cancel{2}}\cdot(x^2+y^2+z^2)^{\frac{n-4}{2}}$

$= nr^{n-2} + x^2\cdot\frac{n(n-2)}{\cancel{2}}\, r^{n-4}$

Similarly, $\frac{\partial^2}{\partial y^2} r^n = nr^{n-2} + n(n-2)y^2 r^{n-4}$ and

$\frac{\partial^2}{\partial z^2} r^n = nr^{n-2} + n(n-2)z^2 r^{n-4}$

$\therefore \nabla^2 r^n = 3nr^{n-2} + n(n-2)r^{n-2} = (n^2+n)r^{n-2}$.

R.H.S.

We know, $\nabla r^n = nr^{n-2}\vec{r} = n(x^2+y^2+z^2)^{\frac{n-2}{2}}(x\hat{i}+y\hat{j}+z\hat{k})$

$\nabla\cdot\nabla r^n = n\left[\frac{\partial}{\partial x}xr^{n-2} + \frac{\partial}{\partial y}yr^{n-2} + \frac{\partial}{\partial z}zr^{n-2}\right]$

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Now, $\frac{\partial}{\partial x}\left[x\cdot(x^2+y^2+z^2)^{\frac{n-2}{2}}\right]$

$= (x^2+y^2+z^2)^{\frac{n-2}{2}} + x\cdot\frac{n-2}{2}(x^2+y^2+z^2)^{\frac{n-4}{2}}\cdot 2x$

$= r^{n-2} + x^2 r^{n-4}\cdot(n-2)$

$\therefore \nabla\cdot(\nabla r^n) = n\left[3r^{n-2} + (n-2)r^{n-4}\cdot r^2\right]$

$= n r^{n-2}(3+n-2)$

$= n(n+1)r^{n-2}$, which is same as L-H-S.

$\therefore \nabla^2 r^n = \nabla\cdot(\nabla r^n)$ <proved>

(c) $\frac{\partial}{\partial x}(\ln r) = \frac{\partial}{\partial x}\left[\ln(x^2+y^2+z^2)^{1/2}\right]$

$= \frac{1}{2}\ln(x^2+y^2+z^2) = \frac{x}{x^2+y^2+z^2}$

$\frac{\partial^2}{\partial x^2}(\ln r) = \frac{(x^2+y^2+z^2) - x\cdot 2x}{(x^2+y^2+z^2)^2}$

$= \frac{y^2+z^2-x^2}{r^4}$

Simillarly, $\frac{\partial^2(\ln r)}{\partial y^2} = \frac{x^2+z^2-y^2}{r^4}$ and, $\frac{\partial^2}{\partial z^2}(\ln r) = \frac{x^2+y^2-z^2}{r^4}$

$\therefore \nabla^2 \ln r = \frac{1}{r^4}\left[\cancel{y^2}+\cancel{z^2}-\cancel{x^2}+\cancel{x^2}+z^2-\cancel{y^2}+x^2+y^2-\cancel{z^2}\right]$

$= \frac{r^2}{r^4} = \frac{1}{r^2}$ <proved>

7) (a) Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z$

$(\vec{a}\cdot\vec{r})\vec{r} = (a_1x + a_2y + a_3z)(x\hat{i} + y\hat{j} + z\hat{k})$

Now, $\frac{\partial}{\partial x}\left[x(a_1x + a_2y + a_3z)\right]$

$= a_1x + a_2y + a_3z + xa_1$

$= \vec{a}\cdot\vec{r} + a_1x$

Simillarly, $\frac{\partial}{\partial y}\left[y(a_1x + a_2y + a_3z)\right]$

$= \vec{a}\cdot\vec{r} + a_2y$ and

$\frac{\partial}{\partial z}\left[z(a_1x + a_2y + a_3z)\right] = \vec{a}\cdot\vec{r} + a_3z.$

So, $\mathrm{div}\left[(\vec{a}\cdot\vec{r})\vec{r}\right] = 3\vec{a}\cdot\vec{r} + a_1x + a_2y + a_3z$

$= 3\vec{a}\cdot\vec{r} + \vec{a}\cdot\vec{r} = 4\vec{a}\cdot\vec{r}$ (proved)

(b) Let, $\vec{v} = v_1(x,y,z)\hat{i} + v_2(x,y,z)\hat{j} + v_3(x,y,z)\hat{k}.$

$$\vec{a}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

$= (a_2v_3 - a_3v_2)\hat{i} + (a_3v_1 - a_1v_3)\hat{j} + (a_1v_2 - a_2v_1)\hat{k}.$

$$\text{L.H.S. } \vec{\nabla}\times(\vec{a}\times\vec{v}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_2v_3 - a_3v_2 & a_3v_1 - a_1v_3 & a_1v_2 - a_2v_1 \end{vmatrix}$$

~~Coefficient of $\hat{i}$ = $\frac{\partial}{\partial y}(a_1v_2 - a_2v_1) - \frac{\partial}{\partial z}(a_3v_1 - a_1v_3)$~~

$= \cancel{\frac{a_1v_2}{y} - \frac{a_2v_1}{y} + \frac{a_3v_1}{z} + \frac{a_1v_3}{z}}.$

L.H.S. $\vec{\nabla} \times (\vec{a} \times \vec{v})$

$$= \hat{i}(a_1 v_{2y} - a_2 v_{1y} - a_3 v_{1z} + a_1 v_{3z}) + \hat{j}(a_2 v_{3z} - a_3 v_{2z} - a_1 v_{2x} + a_2 v_{1x})$$
$$+ \hat{k}(a_3 v_{1x} - a_1 v_{3x} - a_2 v_{3y} + a_3 v_{2y})$$

R.H.S.

$\vec{a}(\nabla \cdot \vec{v}) - (\vec{a} \cdot \nabla)\vec{v}$

$$= (a_1 \hat{i} + a_2 \hat{j} + a_3 \hat{k})(v_{1x} + v_{2y} + v_{3z}) - \left(a_1 \frac{\partial}{\partial x} + a_2 \frac{\partial}{\partial y} + a_3 \frac{\partial}{\partial z}\right)(v_1 \hat{i} + v_2 \hat{j} + v_3 \hat{k})$$

$$= \hat{i}\left[a_1(\cancel{v_{1x}} + v_{2y} + v_{3z}) - (\cancel{a_1 v_{1x}} + a_2 v_{1y} + a_3 v_{1z})\right]$$
$$+ \hat{j}\left[a_2(v_{1x} + \cancel{v_{2y}} + v_{3z}) - (a_1 v_{2x} + \cancel{a_2 v_{2y}} + a_3 v_{2z})\right]$$
$$+ \hat{k}\left[a_3(v_{1x} + v_{2y} + \cancel{v_{3z}}) - (a_1 v_{3x} + a_2 v_{3y} + \cancel{a_3 v_{3z}})\right]$$

$$= \hat{i}(a_1 v_{2y} - a_2 v_{1y} - a_3 v_{1z} + a_1 v_{3z}) + \hat{j}(a_2 v_{3z} - a_3 v_{2z} - a_1 v_{2x} + a_2 v_{1x})$$
$$+ \hat{k}(a_3 v_{1x} - a_1 v_{3x} - a_2 v_{3y} + a_3 v_{2y})$$

$= L.H.S.$

Hence proved.

2(a) L.H.S. $\nabla\cdot(f\nabla g)$

$= \nabla\cdot\left[f(\hat{i}g_x+\hat{j}g_y+\hat{k}g_z)\right]$

$= \frac{\partial}{\partial x}(fg_x)+\frac{\partial}{\partial y}(fg_y)+\frac{\partial}{\partial z}(fg_z)$

$= fg_{xx}+f_xg_x+fg_{yy}+f_yg_y+fg_{zz}+f_zg_z$

$= f(g_{xx}+g_{yy}+g_{zz})+f_xg_x+f_yg_y+f_zg_z$

R.H.S. $f\nabla^2 g+\nabla f\cdot\nabla g$

$= f(g_{xx}+g_{yy}+g_{zz})+(f_x\hat{i}+f_y\hat{j}+f_z\hat{k})\cdot(g_x\hat{i}+g_y\hat{j}+g_z\hat{k})$

$= f(g_{xx}+g_{yy}+g_{zz})+f_xg_x+f_yg_y+f_zg_z$

which is same as L.H.S.

$\therefore \nabla\cdot(f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g$ <proved>

(b) $\nabla f\times\nabla g=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ f_x & f_y & f_z\\ g_x & g_y & g_z\end{vmatrix}$

$=(f_yg_z-f_zg_y)\hat{i}+(f_zg_x-f_xg_z)\hat{j}+(f_xg_y-f_yg_x)\hat{k}$

L.H.S. $\nabla\cdot(\nabla f\times\nabla g)$

$= \frac{\partial}{\partial x}(f_yg_z-f_zg_y)+\frac{\partial}{\partial y}(f_zg_x-f_xg_z)+\frac{\partial}{\partial z}(f_xg_y-f_yg_x)$

$= f_yg_{xz}+f_{xy}g_z-f_zg_{xy}-f_{xz}g_y+f_{yz}g_x+f_zg_{xy}-f_xg_{yz}-f_{xy}g_z$

$+f_xg_{yz}+f_{xz}g_y-f_yg_{xz}-f_{yz}g_x \quad =0.$ <proved>

9) Let, $\vec{\omega} = \omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}$

Then, $\vec{v} = \vec{\omega} \times \vec{r}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} = (\omega_2 z - \omega_3 y)\hat{i} + (\omega_3 x - \omega_1 z)\hat{j} + (\omega_1 y - \omega_2 x)\hat{k}$$

Now, $$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \omega_2 z - \omega_3 y & \omega_3 x - \omega_1 z & \omega_1 y - \omega_2 x \end{vmatrix}$$

$$= (\omega_1 + \omega_1)\hat{i} - (-\omega_2 - \omega_2)\hat{j} + (\omega_3 + \omega_3)\hat{k}$$

$$= 2(\omega_1\hat{i} + \omega_2\hat{j} + \omega_3\hat{k}) = 2\vec{\omega}$$

$$\Rightarrow \vec{\omega} = \frac{1}{2}(\vec{\nabla} \times \vec{v})$$ (Proved)

10) We have, $\vec{F} = \frac{\vec{r}}{r^2} = \frac{1}{x^2+y^2+z^2}(x\hat{i} + y\hat{j} + z\hat{k})$

Now, $$\vec{\nabla} \cdot \vec{F} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$$

$$= \sum \left[\frac{1}{x^2+y^2+z^2} - \frac{2x^2}{(x^2+y^2+z^2)^2}\right]$$

$$= \frac{3}{x^2+y^2+z^2} - \frac{2(x^2+y^2+z^2)}{(x^2+y^2+z^2)^2} = \frac{1}{x^2+y^2+z^2} = \frac{1}{r^2}$$

$\Rightarrow \vec{\nabla} \cdot \vec{F} \neq 0 \Rightarrow \vec{F}$ is not Solenoidal.

Also, $$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x/r^2 & y/r^2 & z/r^2 \end{vmatrix}$$

$$= \sum \left[\frac{\partial}{\partial y}\left(\frac{z}{r^2}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^2}\right)\right]\hat{i}$$

$$= \sum \left[-\frac{2z}{r^3} \cdot \frac{\partial}{\partial y}(r) + \frac{2y}{r^3} \cdot \frac{\partial}{\partial z}(r)\right]\hat{i} \quad \text{--- (1)}$$

Now, $$\frac{\partial}{\partial y}(r) = \frac{\partial}{\partial y}\left(\sqrt{x^2+y^2+z^2}\right) = \frac{\cancel{2}y}{\cancel{2}\sqrt{x^2+y^2+z^2}} = \frac{y}{r}$$

$$\frac{\partial}{\partial z}(r) = \frac{z}{r}$$

$\therefore$ From (1), $$\vec{\nabla} \times \vec{F} = \sum \left[-\frac{2yz}{r^4} + \frac{2yz}{r^4}\right]\hat{i} = 0$$

$\Rightarrow \vec{F}$ is not irrotational.

11) Given that, $\vec{A}$ & $\vec{B}$ are irrotational.

Then, $\nabla \times \vec{A} = 0$ & $\nabla \times \vec{B} = 0$. —— (1)

Now, $\nabla \cdot (\vec{A} \times \vec{B}) = \sum \frac{\partial}{\partial x}(\vec{A} \times \vec{B}) \cdot \hat{i}$ $\quad \left[\because \nabla \cdot \vec{A} = \sum \hat{i} \cdot \frac{\partial \vec{A}}{\partial x}\right]$

$$= \sum \hat{i} \cdot \left[ \frac{\partial \vec{A}}{\partial x} \times \vec{B} + \vec{A} \times \frac{\partial \vec{B}}{\partial x} \right]$$

$$= \sum \hat{i} \cdot \left(\frac{\partial \vec{A}}{\partial x} \times \vec{B}\right) - \sum \hat{i} \cdot \left(\frac{\partial \vec{B}}{\partial x} \times \vec{A}\right)$$

$$= \left(\sum \hat{i} \times \frac{\partial \vec{A}}{\partial x}\right) \cdot \vec{B} - \left(\sum \hat{i} \times \frac{\partial \vec{B}}{\partial x}\right) \cdot \vec{A}$$

$$= \vec{B} \cdot (\nabla \times \vec{A}) - \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \quad \left[\because \vec{P} \cdot (\vec{Q} \times \vec{R}) = (\vec{P} \times \vec{Q}) \cdot \vec{R}\right]$$

$$= 0 \quad [\text{using } (1)]$$

$\Rightarrow$ $\vec{A} \times \vec{B}$ is solenoidal.

12) $\nabla \cdot (r^n \vec{r}) = \frac{\partial}{\partial x}[r^n x] + \frac{\partial}{\partial y}[r^n y] + \frac{\partial}{\partial z}[r^n z]$

$$= \left[r^n + x \frac{\partial r^n}{\partial x}\right] + \left[r^n + y \frac{\partial r^n}{\partial y}\right] + \left[r^n + z \frac{\partial r^n}{\partial z}\right]$$

$$= 3r^n + \left(x \frac{\partial r^n}{\partial x} + y \frac{\partial r^n}{\partial y} + z \frac{\partial r^n}{\partial z}\right)$$

$$\cancel{= 3r^n + [\; n r^{n-1} \partial}$$

$$= 3r^n + n r^{n-1} \left(x \frac{\partial r}{\partial x} + y \frac{\partial r}{\partial y} + z \frac{\partial r}{\partial z}\right)$$

$$= 3r^n + n r^{n-1} \left(x \cdot \frac{x}{r} + y \cdot \frac{y}{r} + z \cdot \frac{z}{r}\right) \quad \left(\because \frac{\partial r}{\partial x} = \frac{x}{r}\right)$$

$$= 3r^n + n r^{n-2} (x^2 + y^2 + z^2)$$

$$= (n+3) r^n \quad (\because r^2 = x^2 + y^2 + z^2)$$

So, if $\nabla \cdot (r^n \vec{r}) = 0$

$\Rightarrow (n+3) r^n = 0$

$\Rightarrow n = -3 \quad [\because r \neq 0]$

$\therefore$ $r^n \vec{r}$ is solenoidal if $n = -3$.

Ex 13) $\vec{v} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$

$$\text{Curl } \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$$= \hat{i}(2yx^2z - 2yx^2z) + \hat{k}(2xy^2z - 2xy^2z) + \hat{k}(2xyz^2 - 2xyz^2)$$
$$= \vec{0}$$

Hence, the vector field is conservative.
So, there exists a scalar potential $f(x, y, z)$
such that $\vec{v} = \nabla f$

i.e. $xyz(yz\hat{i} + xz\hat{j} + xy\hat{k}) = f_x\hat{i} + f_y\hat{j} + f_z\hat{k}$.

$\therefore f_x = xy^2z^2$

$\Rightarrow f = \dfrac{y^2z^2x^2}{2} + g_1(y, z) \xrightarrow{\text{①}} \Rightarrow f_y = yx^2z^2 + g_{1y}$

But, $f_y = yx^2z^2 \quad \therefore g_{1y} = 0 \Rightarrow g_1 = g_1(z)$.

From ①, $f_z = x^2y^2z + g_{1z}$.

But, $f_z = x^2y^2z \quad \therefore g_{1z} = 0 \Rightarrow g_1 = \text{constant}$
$= c$ (say)

$\therefore f(x, y, z) = \dfrac{x^2y^2z^2}{2} + c$.

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14) $F = y\hat{i} + (x - 2xz)\hat{j} - xy\hat{k}$

$\nabla \times F = x\hat{i} + y\hat{j} - 2z\hat{k}$

A normal to $x^2+y^2+z^2 = a^2$ is

$\nabla(x^2+y^2+z^2) = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\therefore$ The unit normal is,

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}$$

Now, $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds = \iint_R (\nabla\times\vec{F})\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|}$

$$= \iint_R (x\hat{i}+y\hat{j}-2z\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}+z\hat{k}}{a}\right)\cdot\frac{dx\,dy}{z/a}$$

$$= \int_{x=-a}^{a}\int_{y=-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dy\,dx$$

$\boxed{\text{Using, } z=\sqrt{a^2-x^2-y^2}}$

~~Using the fact that $z=\sqrt{a^2-x^2-y^2}$~~

To evaluate the double integral, transform to polar coordinate $(\rho,\phi)$, where $x=\rho\cos\phi$, $y=\rho\sin\phi$, and $dy\,dx$ is replaced by $\rho\,d\rho\,d\phi$. The double integral becomes

$$\int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3\rho^2-2a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\frac{3(\rho^2-a^2)+a^2}{\sqrt{a^2-\rho^2}}\,\rho\,d\rho\,d\phi = \int_{\phi=0}^{2\pi}\int_{\rho=0}^{a}\left[-3\rho\sqrt{a^2-\rho^2}+\frac{a^2\rho}{\sqrt{a^2-\rho^2}}\right]d\rho\,d\phi$$

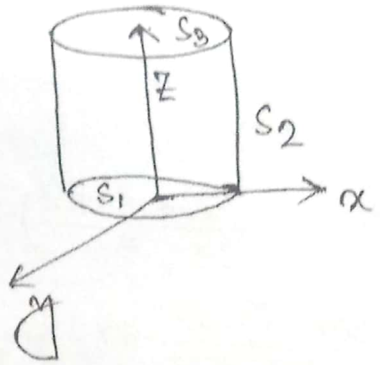
$$= \int_{\phi=0}^{2\pi}\left[(a^2-\rho^2)^{3/2} - a^2\sqrt{a^2-\rho^2}\Big|_{\rho=0}^{a}\right]d\phi$$

$$= \int_{\phi=0}^{2\pi}(a^3-a^3)\,d\phi = 0. \quad \text{(Ans)}$$

(15)

Verify Gauss theorem for $\vec{F} = x\hat{i} - y^2\hat{j} + z^2\hat{k}$ over the region bounded by $x^2+y^2=4, z=0, z=4$.

Solⁿ:-




$n_1 = -\hat{k}, \quad n_2 = \hat{k}$

$\therefore n_3 = \left(\frac{x}{2}, \frac{y}{2}\right)$.

$\therefore \iint \vec{F}\cdot\hat{n}\, ds$

$= \iint_{S_1} \vec{F}\cdot\hat{n}_1\, ds + \iint_{S_2} \vec{F}\cdot\hat{n}_2\, ds + \iint_{S_3} \vec{F}\cdot\hat{n}_3\, ds$

$= -\iint_{S_1} z^2\, ds + \iint_{S_2} \left(\frac{x^2}{2} - \frac{y^3}{2}\right) ds + \iint_{S_2} z^2 ds$

$= 0 + \iint_{S_2} \left(\frac{x^2}{2} - \frac{y^3}{2}\right) ds + \iint_{S_2} 4^2\, ds$

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$$= \iint \left(\frac{x^2}{2} - \frac{y^3}{2}\right) ds + 16 \iint_{S_2} ds$$

$$= \iint_{S_2} \left(\frac{x^2}{2} - \frac{y^3}{2}\right) ds + \pi \times 4 \times 16$$

Now, let $x = 2\cos\theta$

$y = 2\sin\theta$

$\therefore ds = r\,d\theta\,dz = 2\,d\theta\,dz.$

$$\therefore \iint_{S_2} \left(\frac{x^2}{2} - \frac{y^3}{2}\right) ds$$

$$= \int_{\theta=0}^{2\pi} \int_{z=0}^{4} \left(4\cos^2\theta - 8\sin^3\theta\right) d\theta$$

$= 16\pi.$

$$\therefore \iint_S \vec{F} \cdot \hat{n}\, ds = 64\pi + 16\pi = 80\pi.$$

Now, $\iiint_V \operatorname{div} \vec{F}\, dv$

$$= \iiint_V (1 - 2y + 2z)\, dv$$

$$= \int_{z=0}^{4} \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (1-2y+2z)\, dx\, dy\, dz.$$

$$= \int_{z=0}^{4} \int_{x=-2}^{2} \left(2\sqrt{4-x^2} + 4z\sqrt{4-x^2}\right) dx$$

$$= \int_{z=0}^{4} \int_{x=-2}^{2} (4z+2)\sqrt{4-x^2}\, dx\, dz.$$

$$= 2\pi \int_0^4 (4z+2)\, dz.$$

$$= 2\pi \cdot (32+8) = 80\pi.$$

Hence the theorem is verified.

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we have $\phi(x,y,z) = x^2+y^2+z^2-1=0$.

outward normal to the surface

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2(\hat{i}x+\hat{j}y+\hat{k}z)}{2\sqrt{x^2+y^2+z^2}} = \hat{i}x+\hat{j}y+\hat{k}z$$

$(\because x^2+y^2+z^2=1)$

$$\iint(\nabla\times\vec{A})\,\vec{ds} = \iint_S \vec{B}\cdot\vec{ds} = \iint_R \vec{B}\cdot\hat{n}\,\frac{dx\,dy}{|\hat{n}\cdot\hat{k}|} \quad [\text{where } \vec{B} = \nabla\times\vec{A}]$$

$$\vec{B} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -yz^2 & -y^2z \end{vmatrix} = -i(-2yz+2yz)-j\cdot 0+\hat{k}(0+1)$$

$$= \hat{k}$$

$$\therefore \iint_S(\nabla\times\vec{A})\,ds = \iint_R \frac{z\,dx\,dy}{z} = \iint_R dx\,dy$$

In polar coordinates, $dx\,dy = r\,dr\,d\theta$

$$\iint_S(\nabla\times\vec{A})\cdot\vec{ds} = \int_0^1 r\,dr\int_0^{2\pi}d\theta = \pi.$$

Also, $\vec{A}|_{z=0} = (2x-y)\,i$ and $\vec{dr} = \hat{i}\,dx+\hat{j}\,dy$

$$\oint_c \vec{A}\cdot\vec{dr} = \int(2x-y)\,dx, \quad \text{But } x=\cos\theta,\ y=\sin\theta$$

$$dx = -\sin\theta\,d\theta$$

$$\oint_c \vec{A}\cdot\vec{dr} = \int_0^{2\pi}(2\cos\theta-\sin\theta)(-\sin\theta)\,d\theta$$

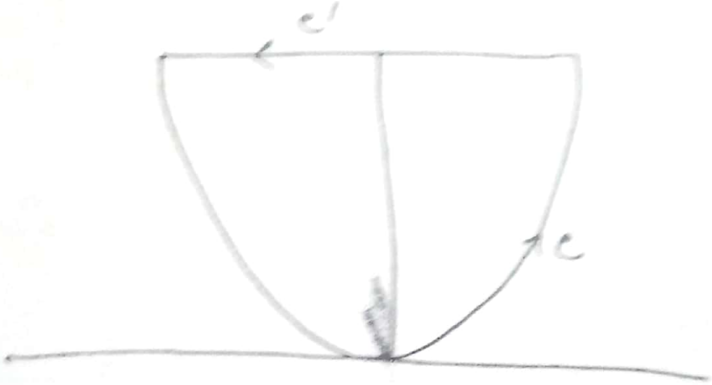
$$= 2\int_0^{2\pi}\sin\theta\,d(\sin\theta)+\int_0^{2\pi}\sin^2\theta\,d\theta$$

$$= 2\left[\frac{1}{2}\sin^2\theta\right]_0^{2\pi}+\int_0^{2\pi}\frac{1-\cos 2\theta}{2}\,d\theta = \pi.$$

$$\therefore \oint_c \vec{A}\cdot\vec{dr} = \iint_S(\nabla\times\vec{A})\cdot\vec{ds}$$

Hence Stokes theorem is verified.

17)




The closed curve $c \cup c'$ bounds the region D.

$P = 1 + xy^2$, $Q = -x^2 y$

$\frac{\partial Q}{\partial x} = -2xy$, $\frac{\partial P}{\partial y} = 2xy$

Applying Greens theorem to the region D, we have

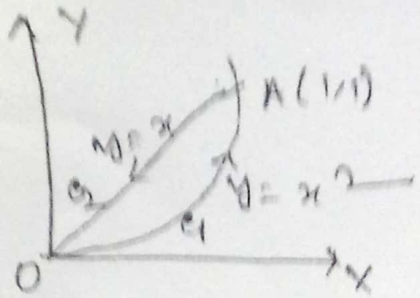
$$\int_{c \cup c'} (1 + xy^2)dx - x^2 y\,dy = \iint_D (-2xy - 2xy)\,dA$$

$$= \iint_D -4xy\,dA = \int_{-1}^{1}\int_{x^2}^{1} -4xy\,dy\,dx = 0.$$

Now, $\int_{c'} (1 + xy^2)dx - x^2 y\,dy = \int_{1}^{-1} (1 + t \cdot 1^2)\,dt$

$$= \left(t + \frac{t^2}{2}\right)\Big|_{1}^{-1} = -2$$

So, $\int_c xy^2\,dx - x^2\,dy = -(-2) = 2$.




$$\int_{c_1} (xy+y^2)\,dx + x^2\,dy = \int_0^1 \left[(x\cdot x^2 + (x^2)^2)\,dx + (x^2)(2x)\,dx\right]$$
$(\because y = x^2)$

$$= \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}.$$

$$\int_{c_2} (xy+y^2)\,dx + x^2\,dy = \int_1^0 (x\cdot x + x^2)\,dx + x^2\,dx \quad (\because y = x$$

$$= \int_1^0 3x^2\,dx = -1$$

$$\therefore \int_c (xy+y^2)\,dx + x^2\,dy = \frac{19}{20} - 1 = -\frac{1}{20}.$$

To verify Green's thm, we have.

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy+y^2)\right] dx\,dy$$

$$= \iint_R (x-2y)\,dx\,dy = \int_{x=0}^1 \int_{y=x^2}^{x} (x-2y)\,dy\,dx.$$

$$= \int_0^1 \left[\int_{x^2}^{x} (x-2y)\,dy\right] dx = \int_0^1 (xy-y^2)_{x^2}^{x}\,dx = -\frac{1}{20}.$$

.O. ⇒ stokes thm is verified.