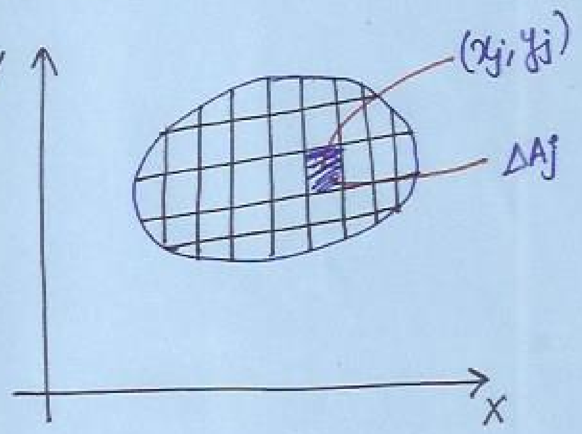
MULTIPLE INTEGRALS

Double Integrals: Let $f(x,y)$ be defined in a closed region D of the xy plane. Divide D into n subregions of area ΔA_j, $j=1,2,\dots,n$. Let (x_j,y_j) be some point of ΔA_j. Then consider



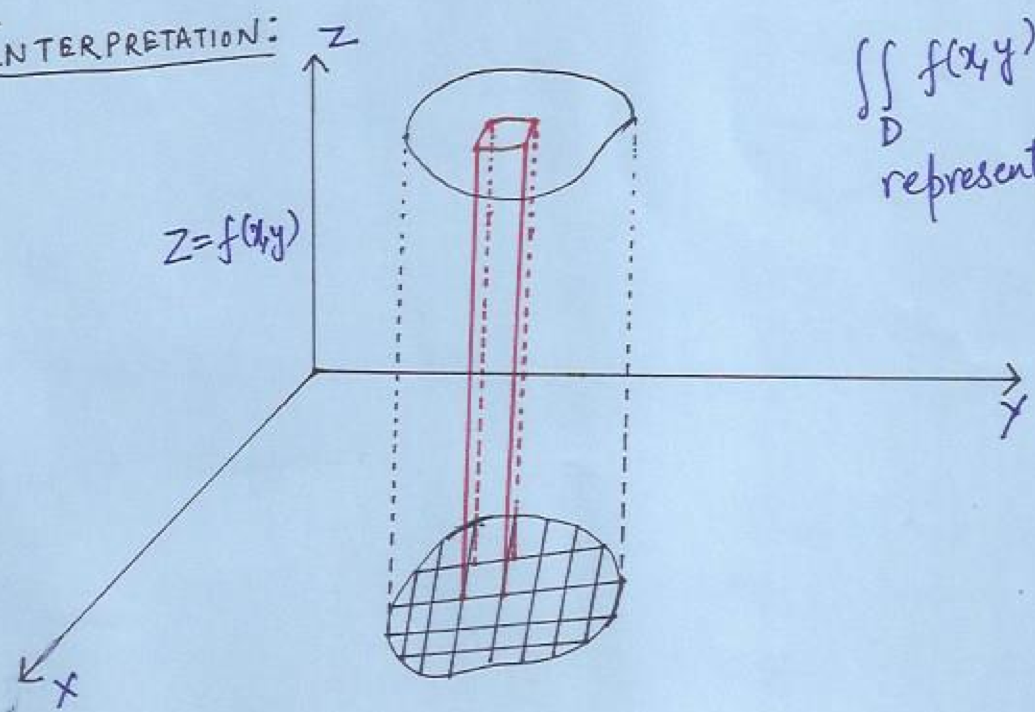
$$\lim_{n\to\infty} \sum_{j=1}^{n} f(x_j,y_j)\,\Delta A_j$$

If this limit exists, then it is denoted by

$$\iint_D f(x,y)\,dA \quad \text{or} \quad \iint_D f(x,y)\,dx\,dy$$

Note: It can be proved that the above limit exists if $f(x,y)$ is continuous or piecewise continuous in D.

PHYSICAL INTERPRETATION:



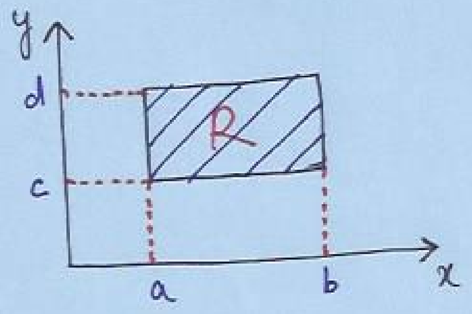
$\iint_D f(x,y)\,dA$ represents Volume

Evaluation of Double Integrals

a) If $f(x,y)$ is continuous* on rectangular region

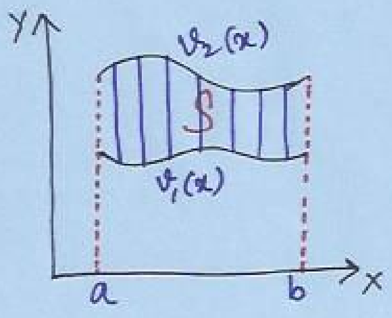
R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x,y)\, dA = \int_c^d \underbrace{\left\{\int_a^b f(x,y)\, dx\right\}}_{\psi(y)} dy = \int_a^b \underbrace{\left\{\int_c^d f(x,y)\, dy\right\}}_{\psi(x)} dx$$



* or $f(x,y)$ is defined and bounded on R.

b)



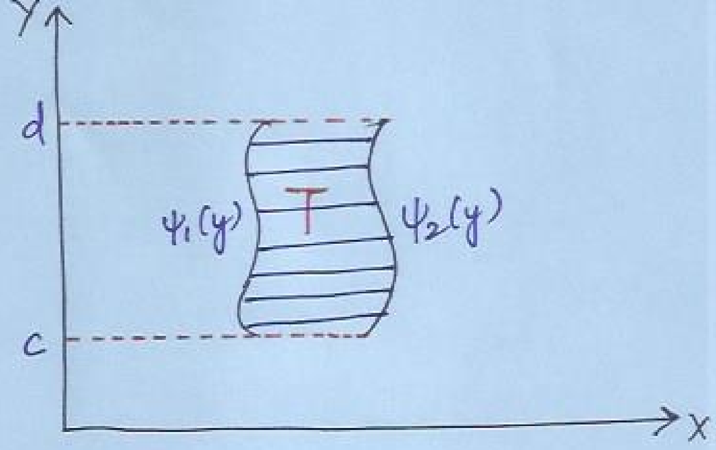
- $v_1(x)$ and $v_2(x)$ are continuous between 'a' and 'b'.
- $f(x,y)$ be defined and bounded on S.

Then,

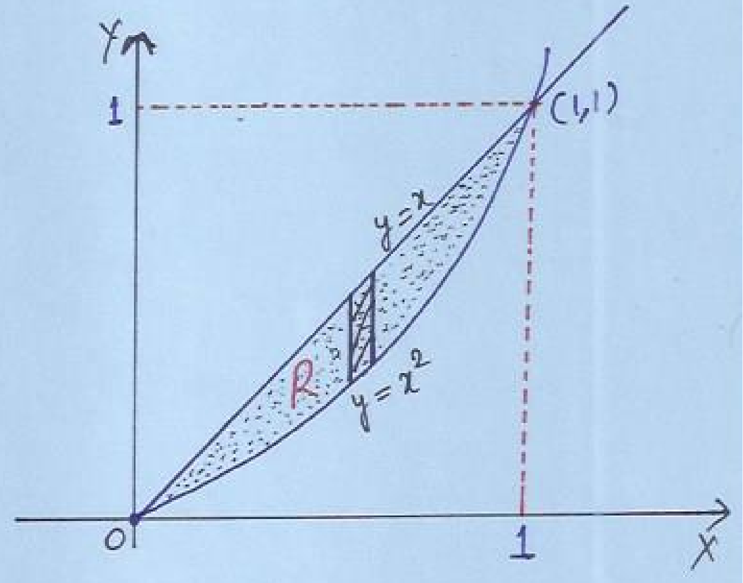
$$\iint_S f(x,y)\, dA = \int_a^b \int_{v_1(x)}^{v_2(x)} f(x,y)\, dy\, dx$$

c)

$$\iint_T f(x,y)\,dA = \int_c^d \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,dx\,dy$$



<u>Example-1:</u> Evaluate $\iint_R xy(x+y)\,dA$ where R is the region bounded by the line $y=x$ and the curve $y=x^2$.



<u>Solution:</u> $I = \int_{x=0}^{1} \int_{x^2}^{x} xy(x+y)\,dy\,dx$

$$= \int_0^1 \left[\frac{y^2}{2}\cdot x^2 + x\cdot\frac{y^3}{3}\right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^4}{2} + \frac{x^4}{3} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

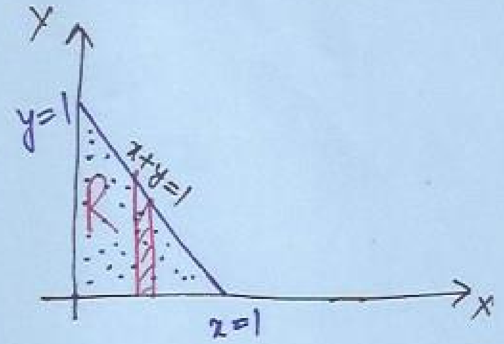
$$= \int_0^1 \left[\frac{5x^4}{6} - \frac{x^6}{2} - \frac{x^7}{3}\right] dx$$

$$= \frac{5}{6}\cdot\frac{1}{5} - \frac{1}{2}\cdot\frac{1}{7} - \frac{1}{3}\cdot\frac{1}{8} = \frac{3}{56}$$

or

$$I = \int_{y=0}^{1} \int_{x=y}^{\sqrt{y}} xy(x+y)\,dx\,dy = \dots = \frac{3}{56}$$

Example-2: Evaluate $\iint_R e^{2x+3y}\,dx\,dy$, R is a triangle bounded by $x=0$, $y=0$ and $x+y=1$.



$$I=\int_{x=0}^{1}\int_{y=0}^{1-x} e^{2x+3y}\,dy\,dx$$

$$=\int_0^1 e^{2x}\left[\frac{e^{3y}}{3}\right]_0^{1-x}dx$$

$$=\int_0^1 e^{2x}\,\frac{1}{3}\cdot\left\{e^{3-3x}-1\right\}dx$$

$$=\frac{1}{3}\int_0^1 \left(e^{3-x}-e^{2x}\right)dx = \frac{1}{3}\left[-e^{3-x}-\frac{e^{2x}}{2}\right]_0^1$$

$$=-\frac{1}{3}\left[e^2+\frac{e^2}{2}-e^3-\frac{1}{2}\right] = -\frac{1}{3}\left[\frac{3e^2}{2}-e^3-\frac{1}{2}\right]$$

Ans...

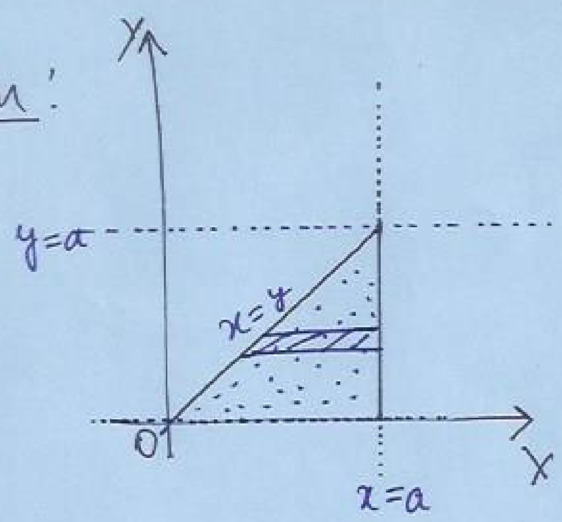
Change of Order of Integration:

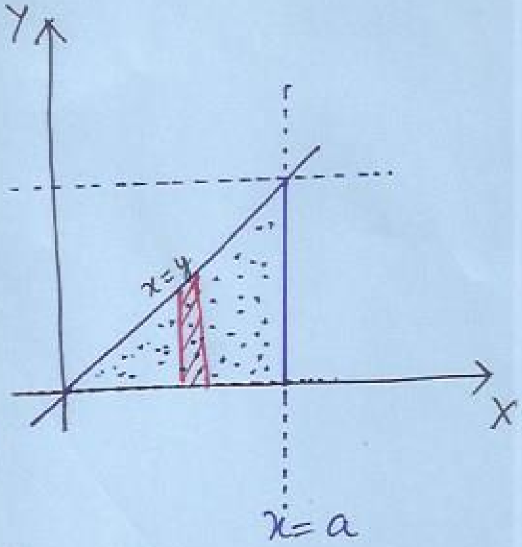
Why? To make the integration easier.

Example: Change the order of integration

$$\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy$$ and evaluate.

Solution:





$$\int_{y=0}^{a}\int_{x=y}^{a}\frac{x}{x^2+y^2}\,dx\,dy=\int_{x=0}^{a}\int_{y=0}^{x}\frac{x}{x^2+y^2}\,dy\,dx$$

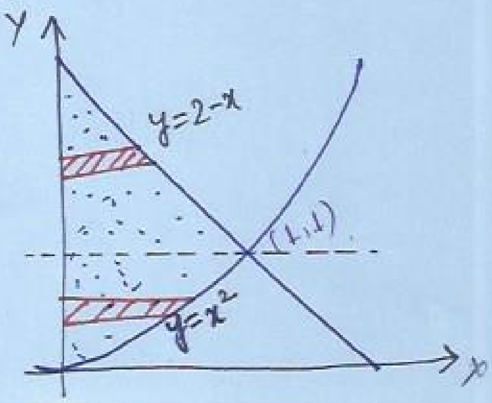
$$=\int_{x=0}^{a}\left[x\cdot\frac{1}{x}\cdot\tan^{-1}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$=\int_0^a\left[\tan^{-1}(1)-\tan^{-1}(0)\right]dx$$

$$=\frac{\pi}{4}(x)_0^a=\frac{\pi a}{4}$$

Example: Change the order of integration

$$\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx \text{ and evaluate.}$$



Solution: $\int_0^1\int_{y=x^2}^{2-x} xy\,dy\,dx$

$$= \int_{y=0}^{1} \int_{x=0}^{\sqrt{y}} xy \, dx \, dy + \int_{y=1}^{2} \int_{x=0}^{2-y} xy \, dx \, dy$$

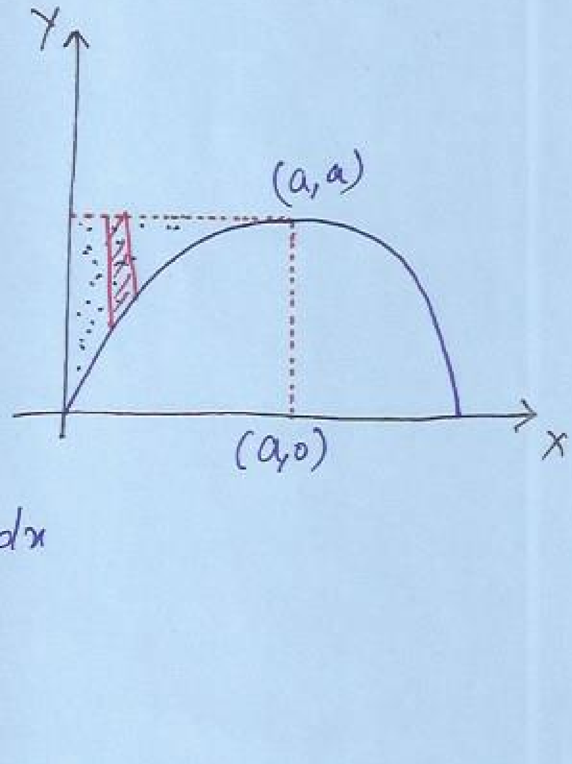
$$= \ldots$$

$$= \frac{3}{8}.$$ Ans...

Example: change the order of integration

$$\int_{y=0}^{a} \int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy \cdot \log(x+a)}{(x-a)^2} \, dx \, dy$$ and evaluate.

Solution:



$$\int_{y=0}^{a} \int_{x=0}^{a-\sqrt{a^2-y^2}} \ldots \, dx \, dy = \int_{x=0}^{a} \int_{y=\sqrt{a^2-(x-a)^2}}^{a} \frac{xy \log(x+a)}{(x-a)^2} \, dy \, dx$$

$$= \int_0^a \frac{x \log(x+a)}{(x-a)^2} \cdot \frac{1}{2} \cdot \left[a^2 - \{a^2 - (x-a)^2\} \right] dx$$

$$= \frac{1}{2} \int_0^a x \log(x+a) \, dx$$

$$= \frac{1}{2} \left[\left\{ \frac{x^2}{2} \log(x+a) \right\}_0^a - \int_0^a \frac{x^2}{2} \cdot \frac{1}{(x+a)} \, dx \right]$$

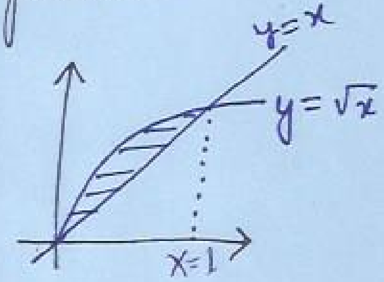
$$= \frac{1}{2} \left[\left\{ \frac{a^2}{2} \log(2a) \right\} - \frac{1}{2} \int_0^a \left[(x-a) + \frac{a^2}{x+a} \right] dx \right]$$

$$= \frac{1}{2}\left[\frac{a^2\log(2a)}{2} - \frac{1}{2}\left\{\frac{a^2}{2} - a^2 + a^2\log(2a) - a^2\log a\right\}\right]$$

$$= \frac{a^2}{8}\left[1 + 2\log a\right]$$

Example → Change the order of integration

$$\int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dy\,dx$$



Ans: $\int_0^1 \int_{y^2}^{y} f(x,y)\,dx\,dy$

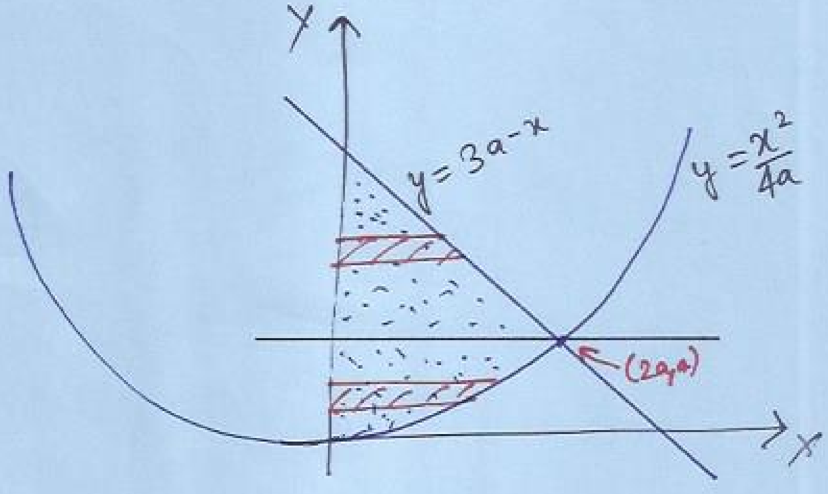
Q: Change the order of integration in

$$I = \int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-x} F(x,y)\,dy\,dx$$

Solution:

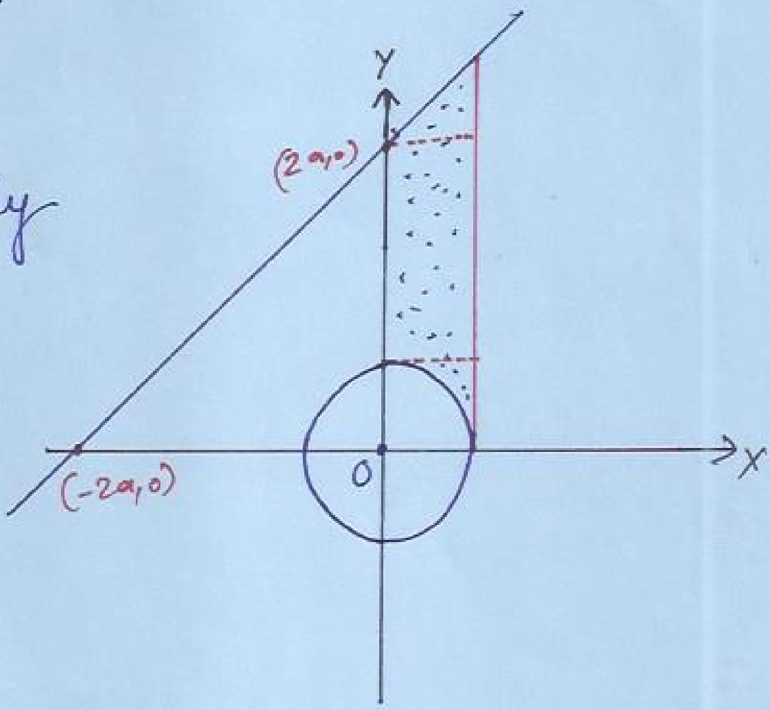
$$I = \int_0^a \int_0^{2\sqrt{ay}} F(x,y)\,dx\,dy$$

$$+ \int_a^{3a} \int_0^{3a-y} F(x,y)\,dx\,dy$$



Example: Change the order of integration in

$$I=\int_0^a \int_{\sqrt{a^2-x^2}}^{x+2a} F(x,y)\,dy\,dx$$

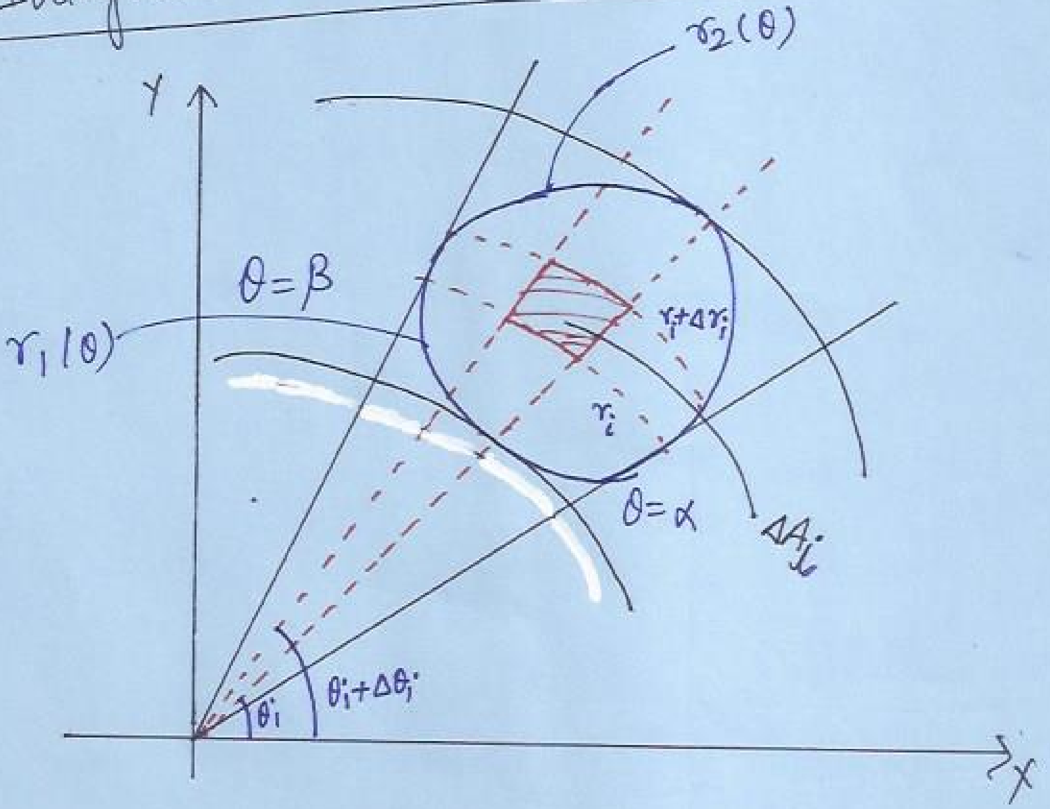


Solution: $I=\int_0^a \int_{\sqrt{a^2-y^2}}^{a} f(x,y)\,dx\,dy$

$$+\int_a^{2a}\int_0^a F(x,y)\,dx\,dy$$

$$+\int_{2a}^{3a}\int_{y=2a}^{a} F(x,y)\,dx\,dy$$

Double Integral in Polar Co-ordinate



$$\Delta A_i = (r_i + \Delta r_i)^2 \frac{\Delta\theta_i}{2} - r_i^2 \frac{\Delta\theta_i}{2}$$

$$= \cancel{2r_i\Delta\Delta}\cdot(2r_i\Delta r_i + \Delta r_i^2)\frac{\Delta\theta_i}{2}$$

$$= \left(\frac{2r_i + \Delta r_i}{2}\right)\cdot \Delta r_i \Delta\theta_i$$

$$= \left(r_i + \frac{\Delta r_i}{2}\right)\Delta r_i \Delta\theta_i$$

$$= r_i^* \Delta r_i \Delta\theta_i$$

$$I = \lim_{n\to\infty} \sum_{j=1}^{n} f(r_j, \theta_j)\, \Delta A_j$$

$$I = \int_{\theta=\alpha}^{\beta} \int_{r=r_1(\theta)}^{r=r_2(\theta)} f(r,\theta)\, r\, dr\, d\theta$$

Example → Compute area of first quadrant of a circle.

$$A = \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r\, dr\, d\theta = \frac{a^2}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4}$$