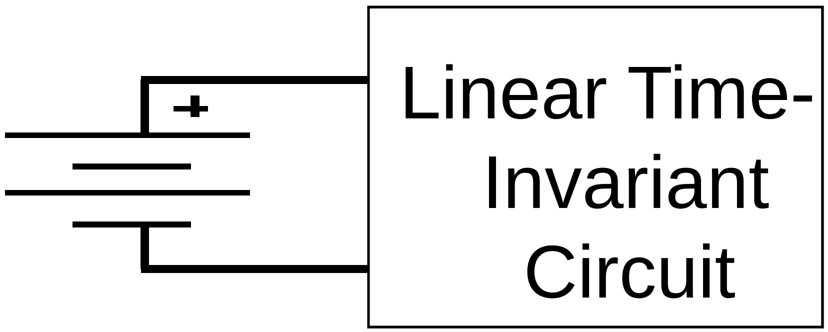
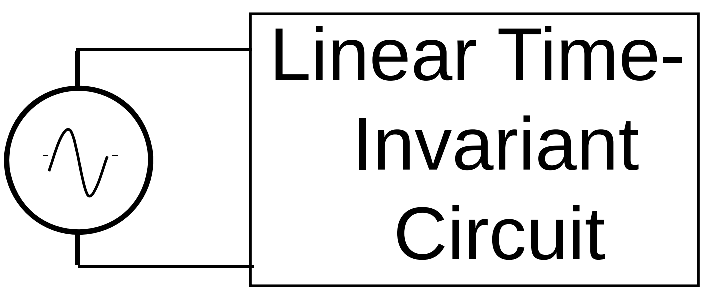
RC and RL Circuits (First-order)

Types of Circuit Excitation

Steady-state Excitation

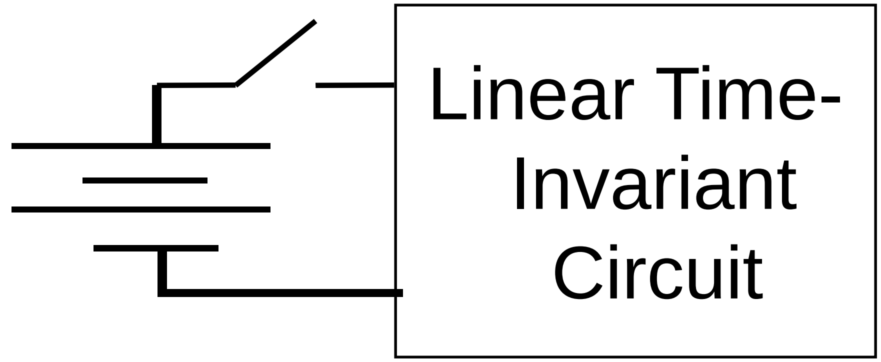


Steady-State Excitation (DC Steady-State)

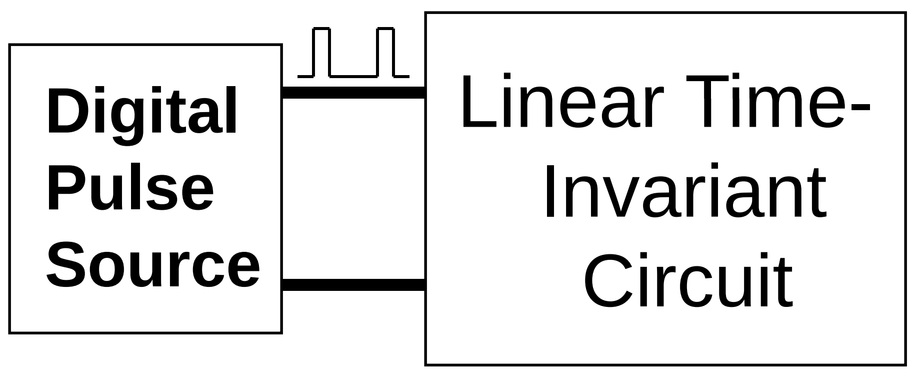


Sinusoidal (Single-Frequency) Excitation →AC Steady-State

Transient Excitation

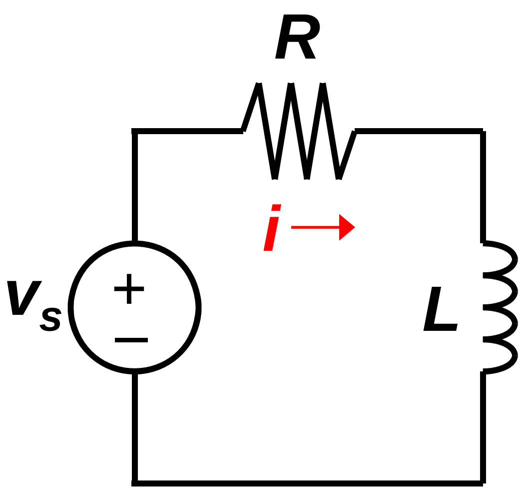


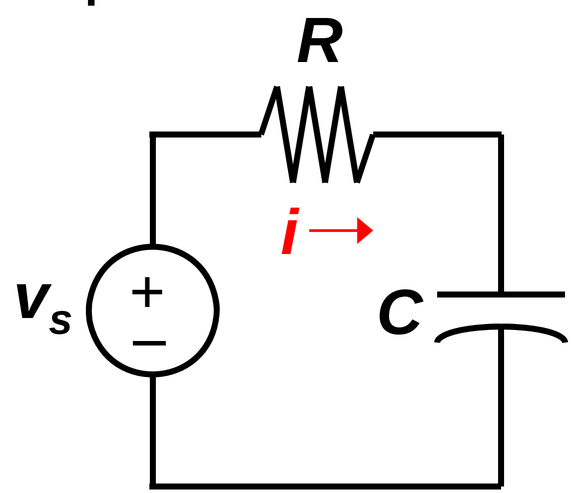
OR



First-Order Circuits

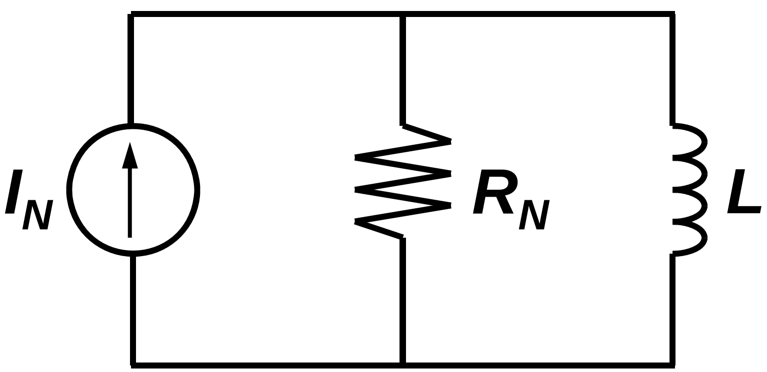
- A circuit that contains only sources, resistors and an inductor is called an ***RL circuit***.
- A circuit that contains only sources, resistors and a capacitor is called an ***RC circuit***.
- RL and RC circuits are called first-order circuits because their voltages and currents are described by first-order differential equations.

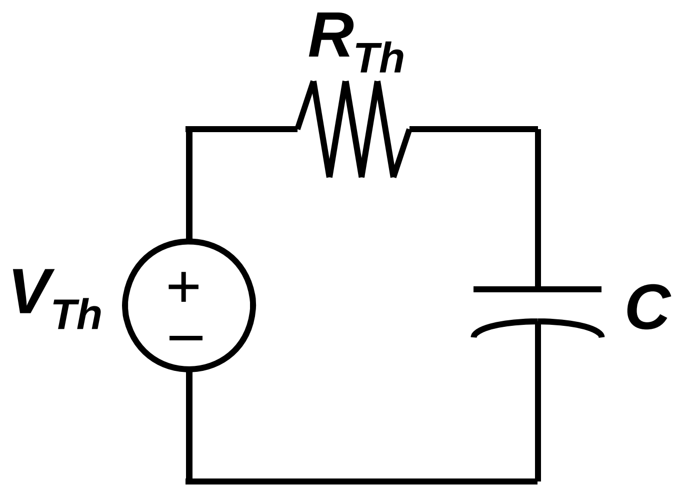




Review Concept

- Any first-order circuit can be reduced to a Thévenin (or Norton) equivalent connected to either a single equivalent impedance (inductor or capacitor).





  - In steady state, an inductor behaves like a short circuit
  - In steady state, a capacitor behaves like an open circuit

Review Concept

- The ***natural response*** of an RL or RC circuit is its behavior (*i.e.,* current and voltage) when stored energy in the inductor or capacitor is released to the resistive part of the network (containing no independent sources).

- The ***step response*** of an RL or RC circuit is its behavior when a voltage or current source **step** is applied to the circuit, or immediately after a switch state is changed.

RC Circuits

Capacitors and Stored Charge

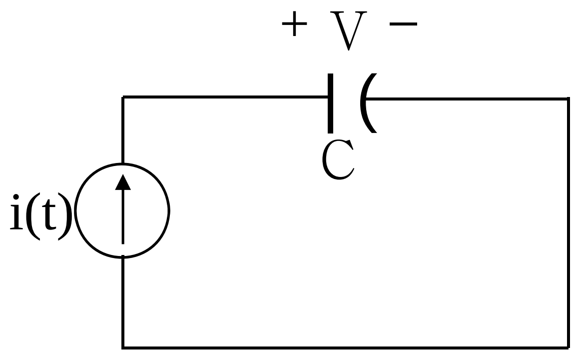
- Electrons keep on moving around and around a circuit contributing a current flow.
- Current doesn't really "flow through" a capacitor. No electrons can go through an insulator (ideal condition).
- But, we **say** that current flows through a capacitor. What we mean is that positive charge collects on one plate and leaves the other.
- A capacitor stores energy in the form of electrical charge.
- When a capacitor stores charge, it has non-zero voltage. In this case, we say the capacitor is "charged". A capacitor with zero voltage has no charge differential, and we say it is "discharged".

Capacitors in circuits

- A circuit with capacitors can be analyzed by using KVL and KCL, nodal analysis, and other similar techniques.
- The voltage across the capacitor is related to the current through it by a differential equation instead of simple Ohm’s law.

$$i = C\frac{dV}{dt}$$

CAPACITORS

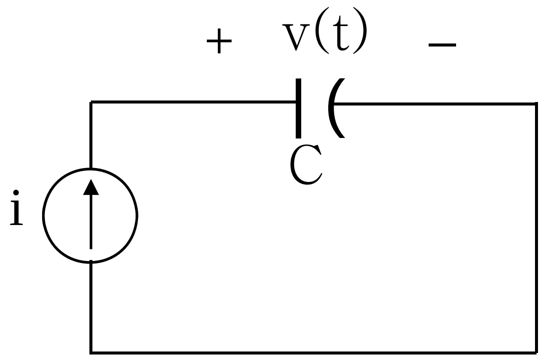


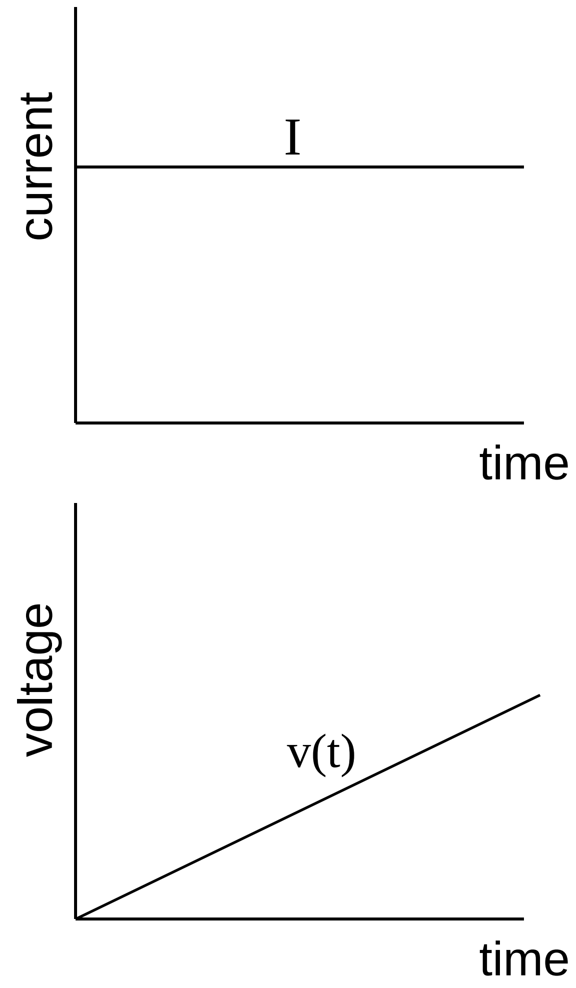
capacitance is analyzed by

$$i = C\frac{dV}{dt}$$

So $$\frac{dV}{dt} = \frac{i}{C}$$

Charging a Capacitor with a constant current





$$\frac{dv(t)}{dt} = \frac{I}{C}$$

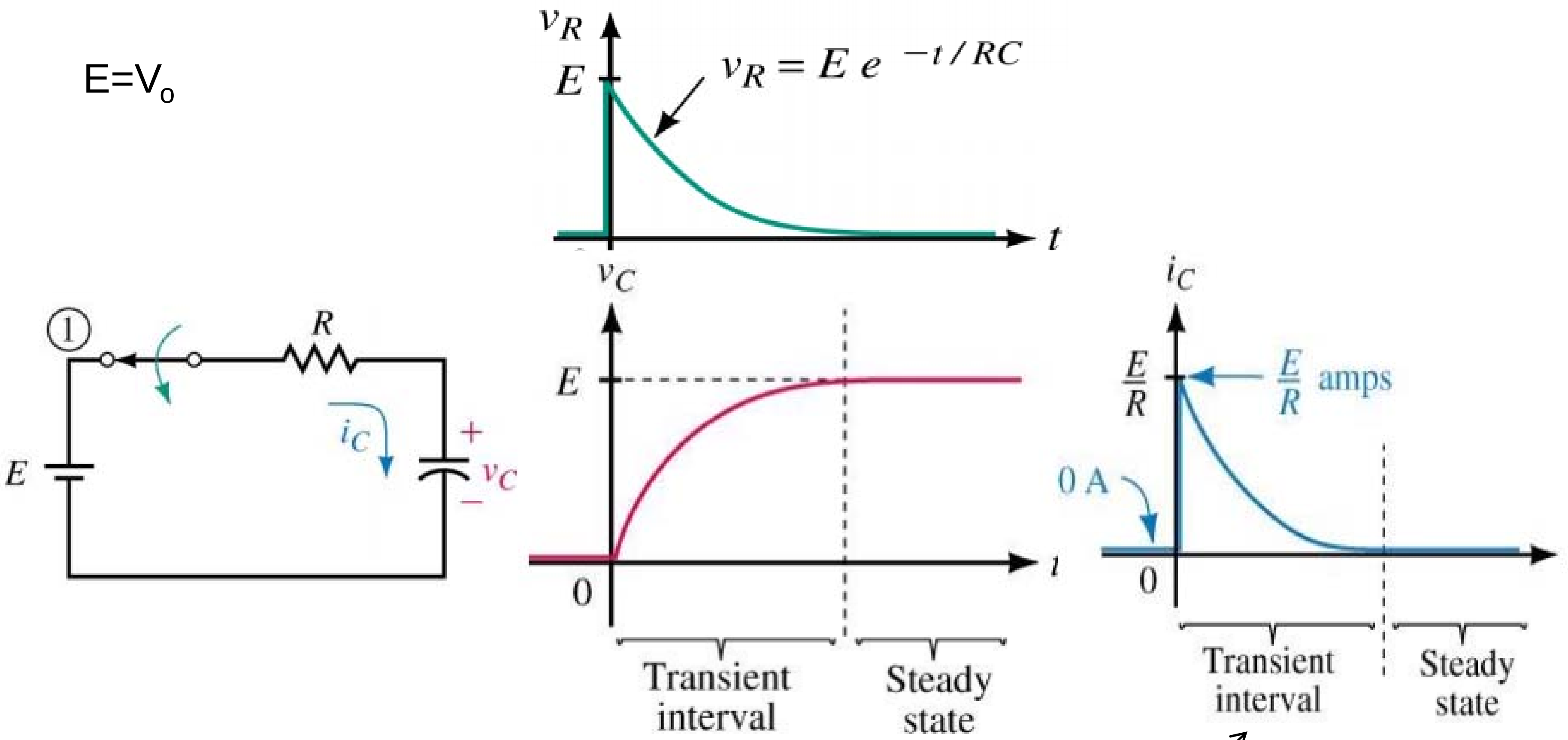
Integrating both sides,

$$\int_0^t \frac{dv(t)}{dt}\,dt = \int_0^t \frac{I}{C}\,dt$$

$$v(t) = \int_0^t \frac{I}{C}\,dt = \frac{I \times t}{C}$$

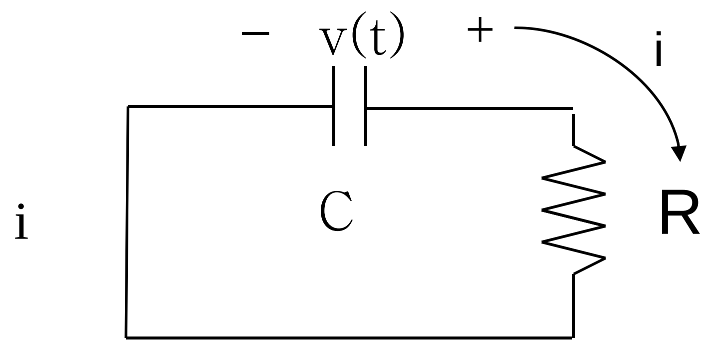
Charging a Capacitor with a constant voltage

$E = V_o$



$$v_C(t) = V_o(1 - e^{-t/RC}) \quad \text{and} \quad i(t) = \frac{V_o}{R} e^{-t/RC}$$

Discharging a Capacitor through a resistor



Time constant
τ = RC

$$\frac{dv(t)}{dt} = -\frac{i(t)}{C} = -\frac{v(t)}{RC}$$

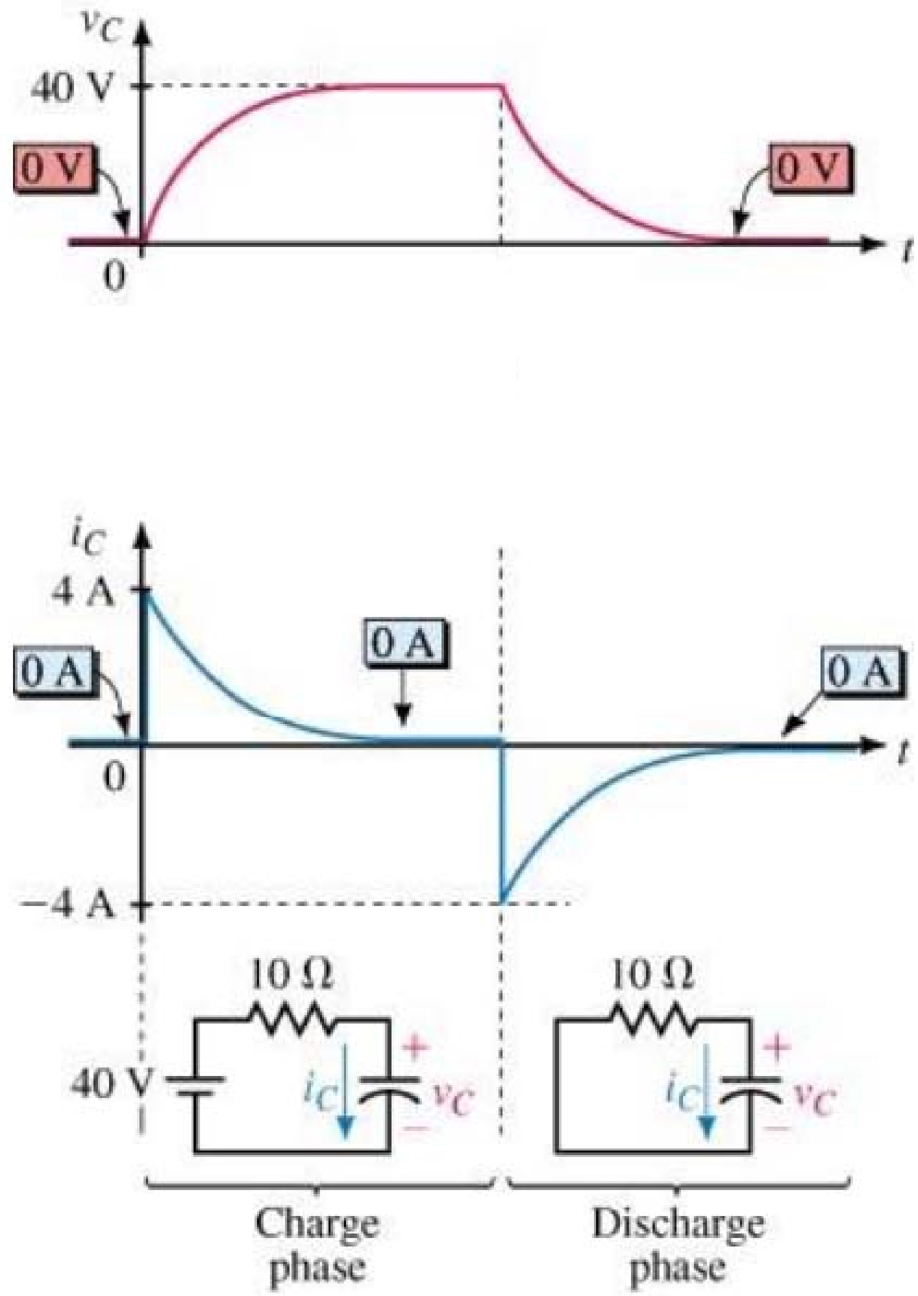
This is an elementary differential equation, whose solution is the exponential:

$$\boxed{v(t) = V_0 e^{-t/\tau}}$$

Since: $$\frac{d}{dt} e^{-t/\tau} = -\frac{1}{\tau} e^{-t/\tau}$$

and $$\boxed{i(t) = \frac{V_o}{R} e^{-t/RC}}$$

Capacitor charging and discharging



Time Constant τ

Analogy of ***time constant (discharging)***

-At $t = \tau$, the voltage has reduced to $1/e$ (~0.37) of its initial value.

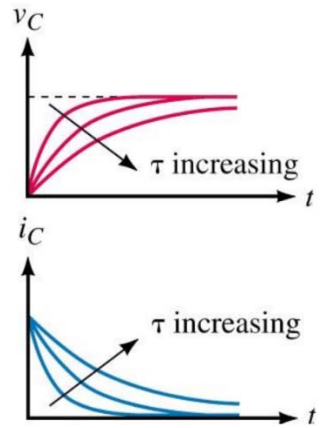
Time constant corresponds to the frequency at which the output signal power drops to half the value it has at low frequencies (determines bandwidth)

-At $t = 5\tau$, the voltage has reduced to less than 1% of its initial value.

$$\boxed{\tau = RC} \quad \text{(sec)}$$

Analogy of ***time constant (charging)***

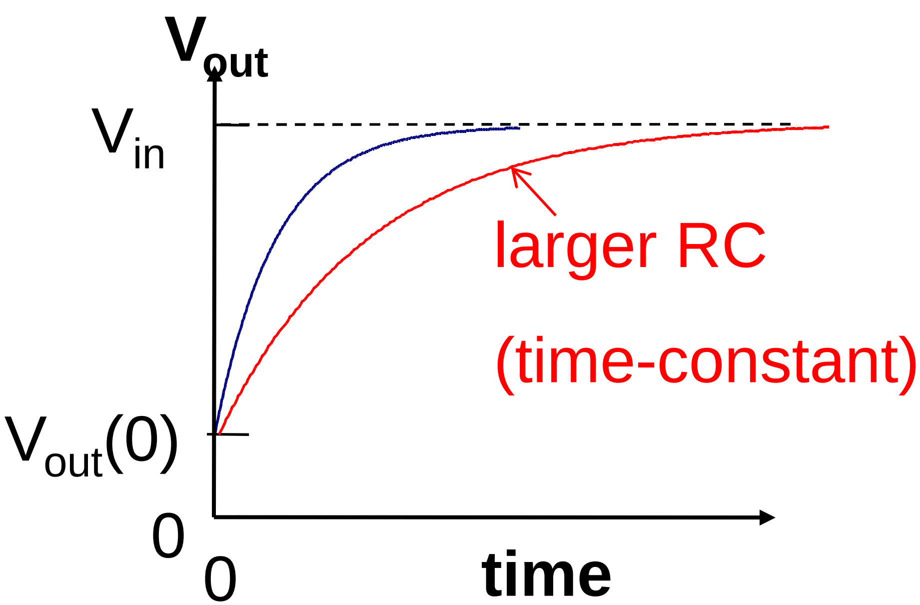
- At $t = \tau$, the voltage has increased to $1-(1/e)$ (~0.63) of its final value.
- At $t = 5\tau$, the voltage has reduced to more than 99% of its final value.

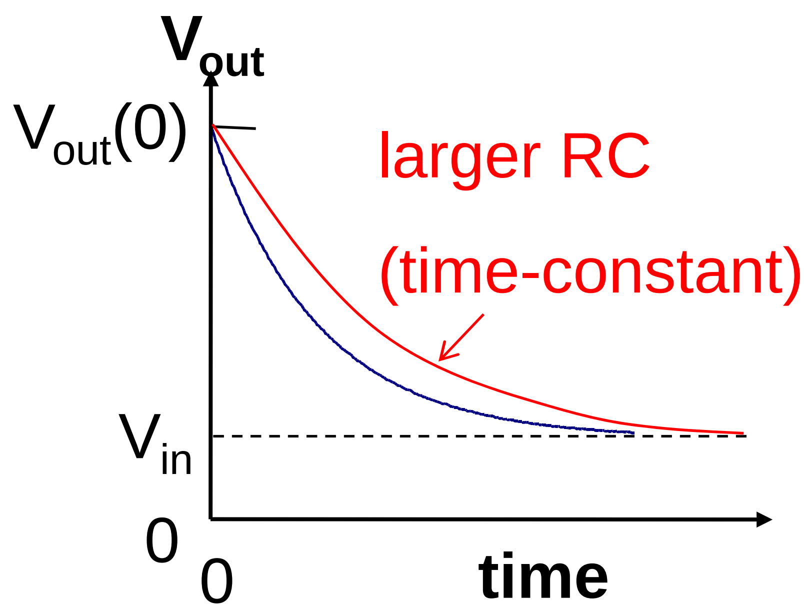


Practical Insight

$$V_{out}(t) = V_{in} + \left(V_{out}(0) - V_{in}\right)e^{-t/(RC)}$$

- $V_{out}(t)$ starts at $V_{out}(0)$ and goes to V_{in} asymptotically.
- The difference between the two values decays exponentially.
- The rate of convergence depends on RC. The bigger RC is, the slower the convergence.





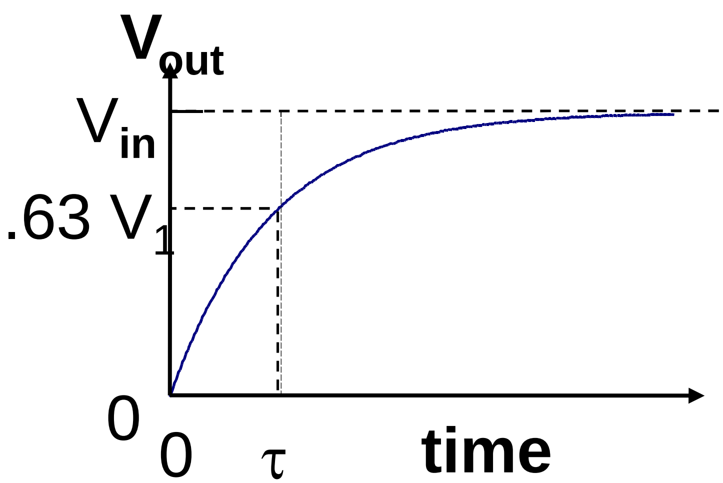
Time Constant (again)

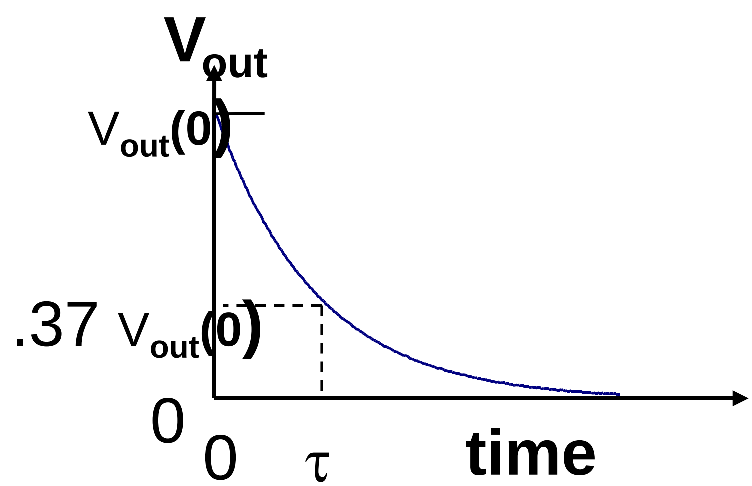
$$V_{out}(t) = V_{in} + \left(V_{out}(0) - V_{in}\right)e^{-t/(RC)}$$

- The value RC is called the **time constant**.
- After 1 time constant has passed (t = RC), the above works out to:

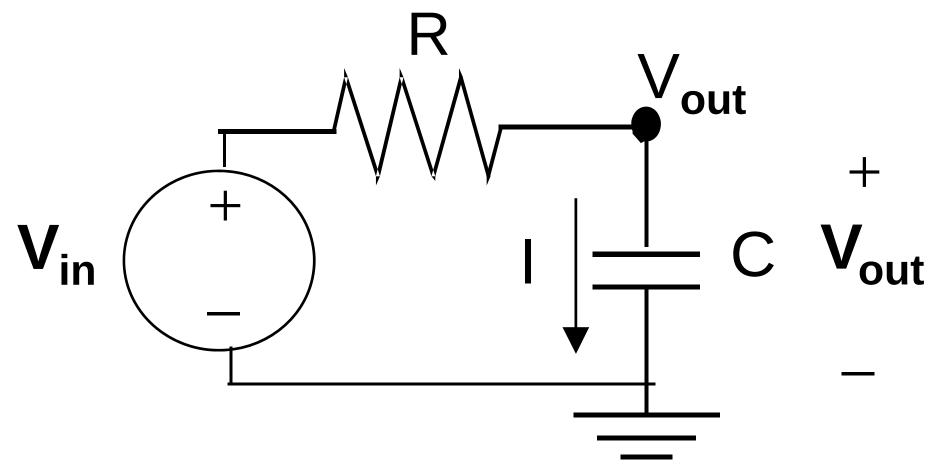
$$V_{out}(t) = 0.63\ V_{in} + 0.37\ V_{out}(0)$$

- So after 1 time constant, $V_{out}(t)$ has completed 63% of its transition, with 37% left to go.
- After 2 time constants, only 0.37^2 left to go.



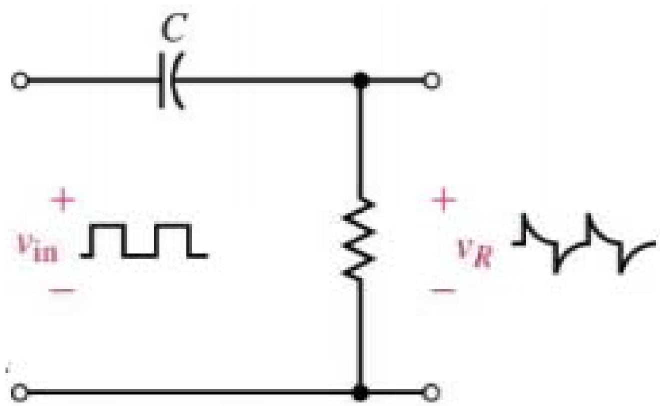


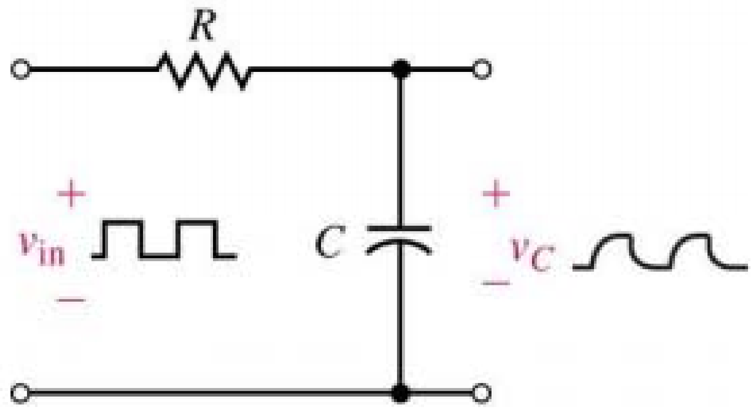
Transient vs. Steady-State



- When V_{in} does not match up with V_{out} , due to an abrupt change in V_{in} for example, V_{out} will begin its **transient period** where it exponentially decays to the value of V_{in}.
- After a while, V_{out} will be close to V_{in} and be nearly constant. We call this **steady-state**.
- In steady state, the current through the capacitor is (approx) zero. **The capacitor behaves like an open circuit in steady-state**.
- $I = C\, dV_{out}/dt$, and V_{out} is constant in steady-state.

Wave shaping circuits



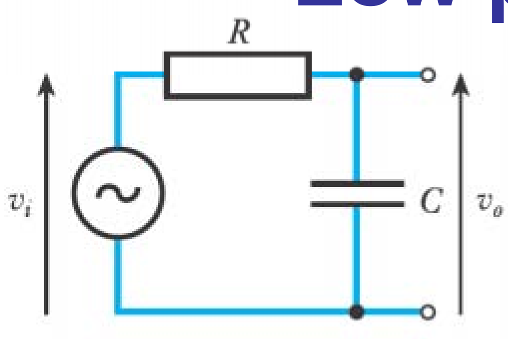


Filters

Frequency selective circuits

- Low pass
- High pass
- Band pass
- Band stop and notch
- All pass

Low pass filter (RC first order)



$$\frac{v_o}{v_i} = \frac{Z_C}{Z_R + Z_C} = \frac{-j\frac{1}{\omega C}}{R - j\frac{1}{\omega C}} = \frac{1}{1 + j\omega CR} = \frac{1}{1 + j\frac{\omega}{\omega_c}} = \frac{1}{1 + j\frac{f}{f_c}}$$

At low frequencies, ω is small and the voltage gain is approximately 1.

At high frequencies, the magnitude of ωCR becomes more significant and the gain of the network decreases.

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1 + (\omega CR)^2}}$$

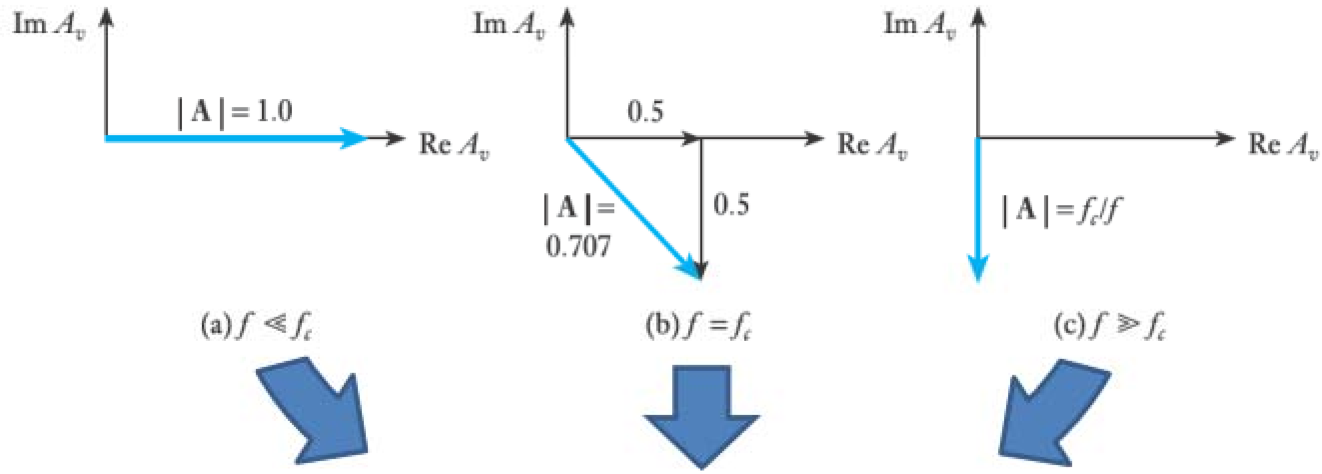
When the value of **ωCR is equal to 1**, this gives:

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

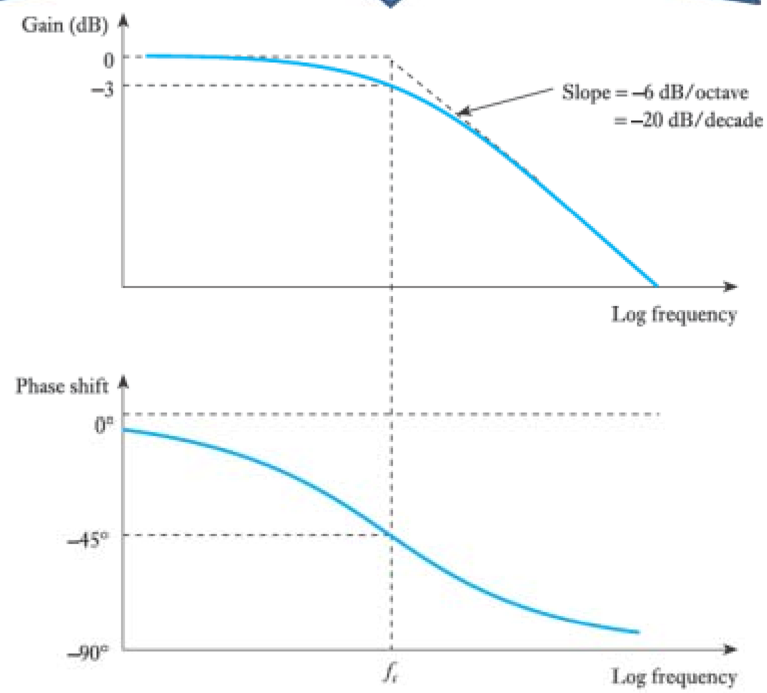
Since power gain is proportional to the square of the voltage gain, this is half of power gain (or a fall of 3 dB) compared with the gain at high frequencies.

The frequency, in which the power gain half of the maximum value, is called **cut-off frequency** of the circuit.

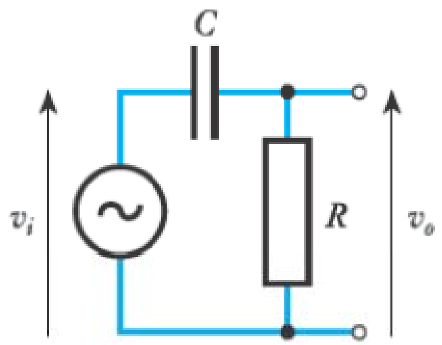
Phasor diagrams of the gain at different frequencies.



Gain and phase responses (or Bode diagram) for the low-pass *RC* network.



High pass filter (RC first order)



$$\frac{v_o}{v_i} = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R - j\dfrac{1}{\omega C}} = \frac{1}{1 - j\dfrac{1}{\omega CR}} = \frac{1}{1 - j\dfrac{1}{(2\pi f)(1/2\pi f_c)}} = \frac{1}{1 - j\dfrac{f_c}{f}}$$

At high frequencies, ω is large and the voltage gain is approximately 1.

At lower frequencies $1/\omega CR$ becomes more significant and the gain of the network decreases.

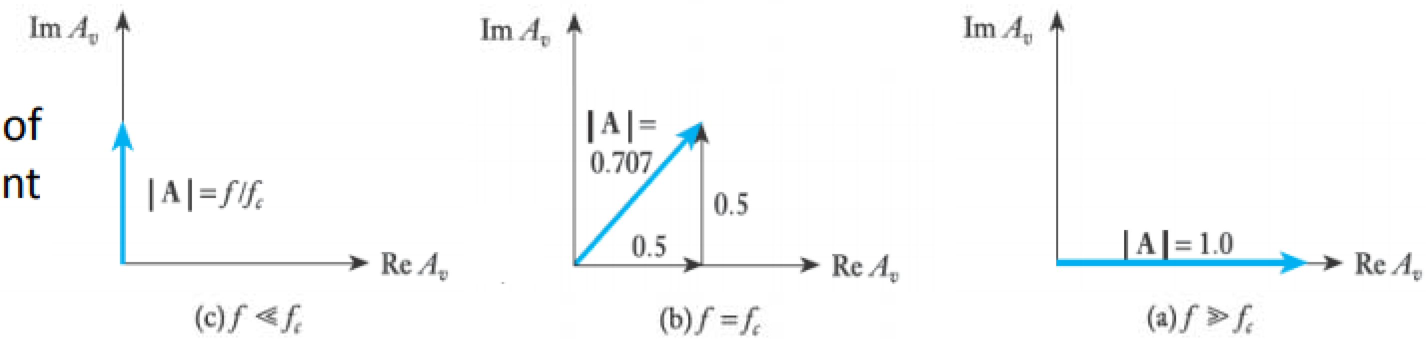
$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1^2 + \left(\dfrac{1}{\omega CR}\right)^2}}$$

The frequency where the value of $1/\omega CR$ is equal to 1, the voltage gain amplitude is:

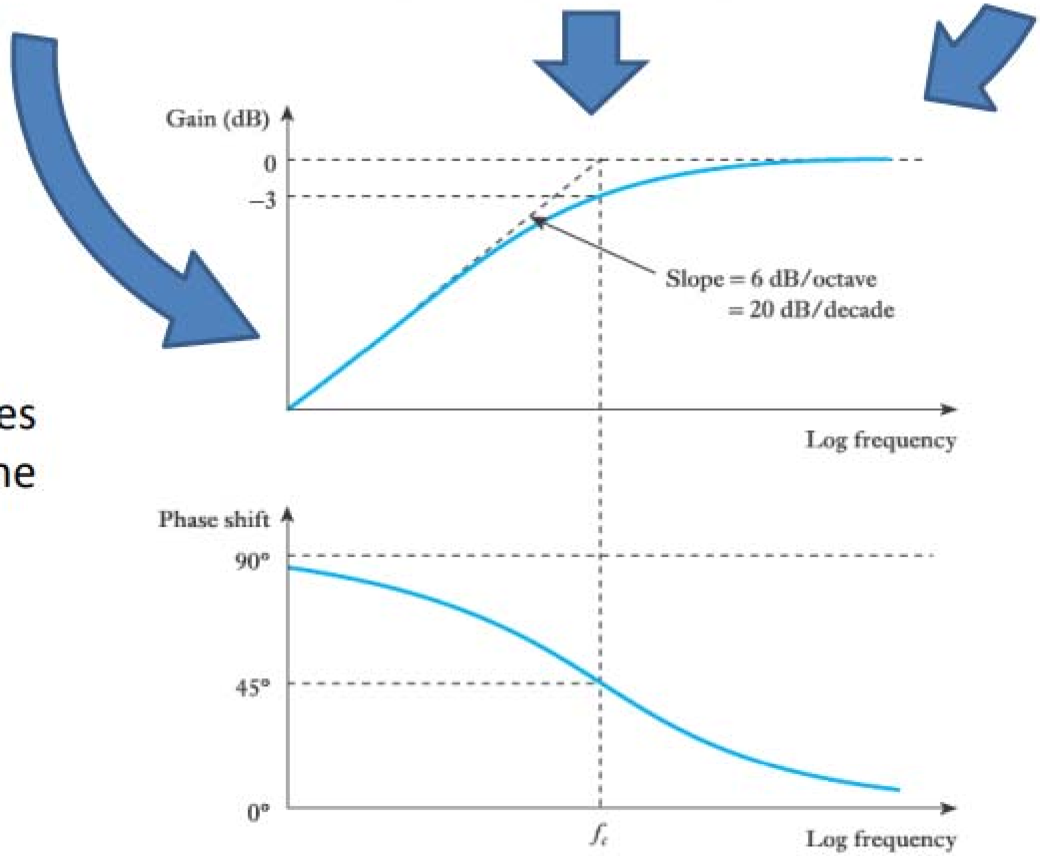
$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

Since power gain is proportional to the square of the voltage gain, this is half of power gain (or a fall of 3 dB) compared with the gain at high frequencies.
The frequency, in which the power gain half of the maximum value, is called **cut-off frequency** of the circuit.

Phasor diagrams of the gain at different frequencies.



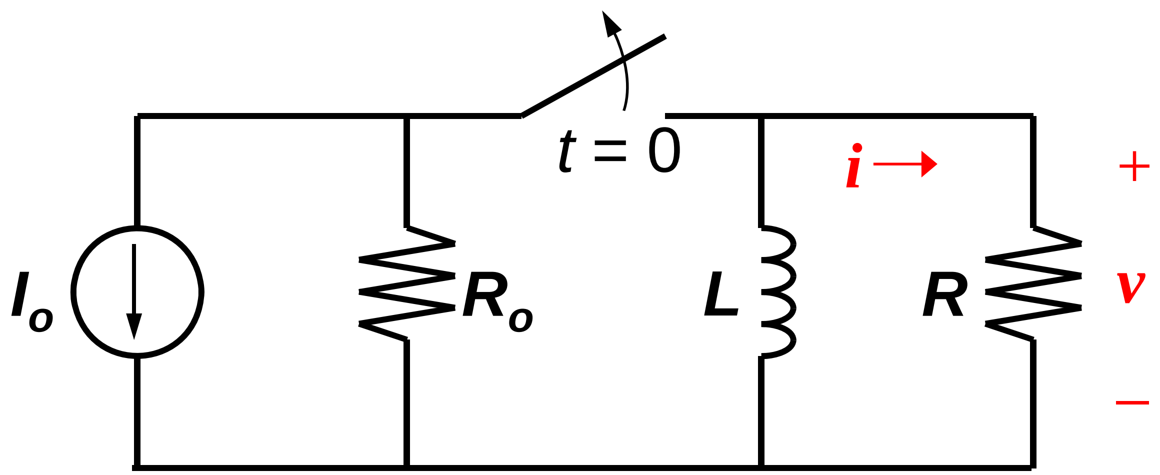
Gain and phase responses (or Bode diagram) for the high-pass *RC* network.



RL Circuits

Natural Response of an RL Circuit

- Consider the following circuit, for which the switch is closed for $t < 0$, and then opened at $t = 0$:



Notation:

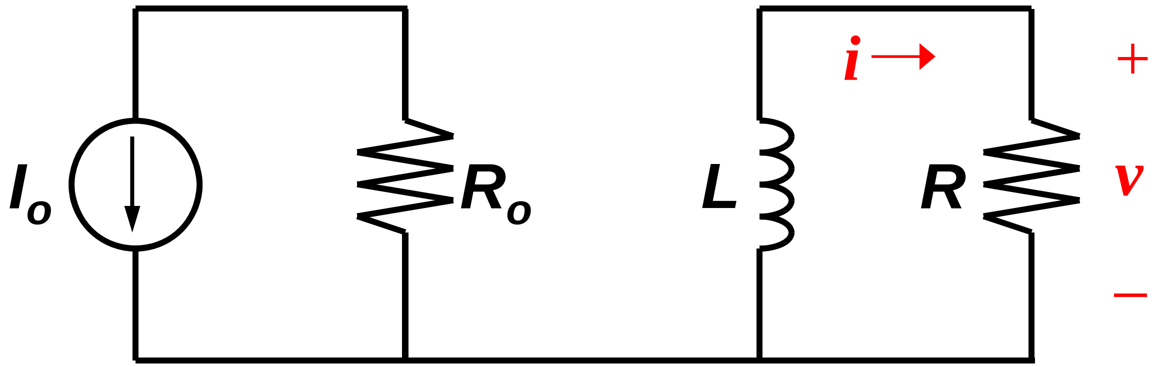
0^- is used to denote the time just prior to switching

0^+ is used to denote the time immediately after switching

- The current flowing in the inductor at $t = 0^-$ is I_o

Solving for the Current ($t \geq 0$)

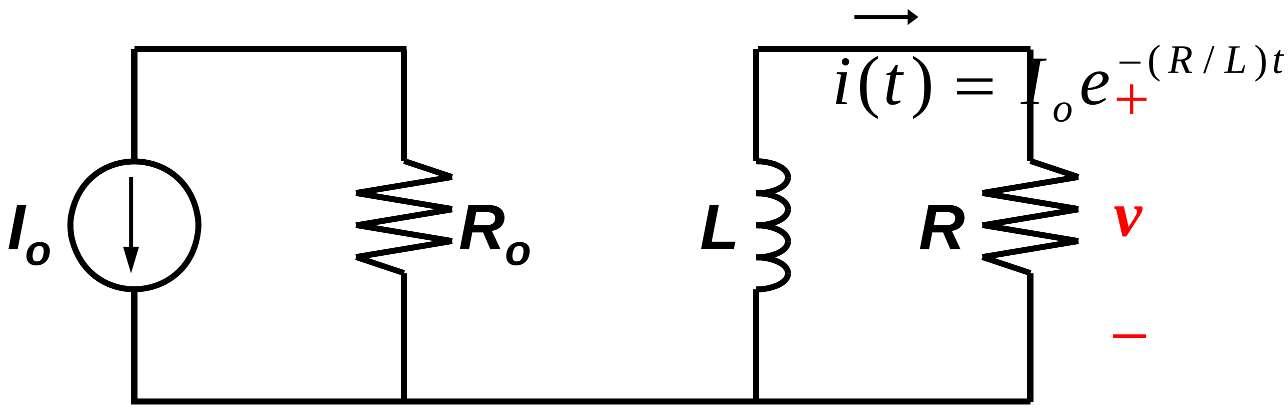
- For t > 0, the circuit reduces to



- Applying KVL to the LR circuit yields first-order D.E.:

- Solution: $i(t) = i(0)e^{-(R/L)t}$ = $I_0 e^{-(R/L)t}$

Solving for the Voltage (t > 0)



- Note that the voltage changes abruptly (step response):

$$v(0^-) = 0$$

$$\text{for } t > 0,\ \ v(t) = iR = I_o R e^{-(R/L)t}$$

$$\Rightarrow\ \ v(0^+) = I_0 R$$

Time Constant τ

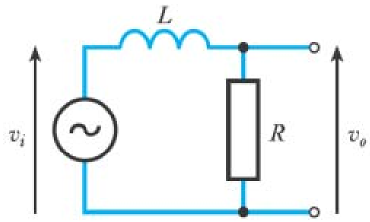
- In the example, we found that

$$i(t) = I_o e^{-(R/L)t} \quad \text{and} \quad v(t) = I_o R e^{-(R/L)t}$$

- ***Time constant*** $\boxed{\tau = \frac{L}{R}}$ (sec)

  - At $t = \tau$, the current has reduced to $1/e$ (~0.37) of its initial value.
  - At $t = 5\tau$, the current has reduced to less than 1% of its initial value.

Low pass Filter (LR)



$$\frac{v_o}{v_i} = \frac{Z_R}{Z_R + Z_L} = \frac{R}{R + j\omega L} = \frac{1}{1 + j\omega\dfrac{L}{R}} = \frac{1}{1 + j\dfrac{\omega}{\omega_c}} = \frac{1}{1 + j\dfrac{f}{f_c}}$$

At low frequencies, ω is small and the voltage gain is approximately 1.

At high frequencies, the magnitude of $\omega L/R$ becomes more significant and the gain of the network decreases.

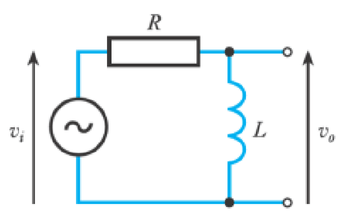
$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1 + \left(\omega\dfrac{L}{R}\right)^2}}$$

When the value of $\omega L/R$ is equal to 1, this gives

$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

This situation corresponds to a cut-off frequency.

High pass Filter (LR)



$$\frac{v_o}{v_i} = \frac{Z_L}{Z_R + Z_L} = \frac{j\omega L}{R + j\omega L} = \frac{1}{1 + \dfrac{R}{j\omega L}} = \frac{1}{1 - j\dfrac{R}{\omega L}} = \frac{1}{1 - j\dfrac{\omega_c}{\omega}} = \frac{1}{1 - j\dfrac{f_c}{f}}$$

At high frequencies, ω is large and the voltage gain is approximately 1.

At low frequencies, the magnitude of $R/\omega L$ becomes more significant and the gain of the network decreases.

$$|\text{voltage gain}| = \frac{1}{\sqrt{1+\left(\frac{R}{\omega L}\right)^2}}$$

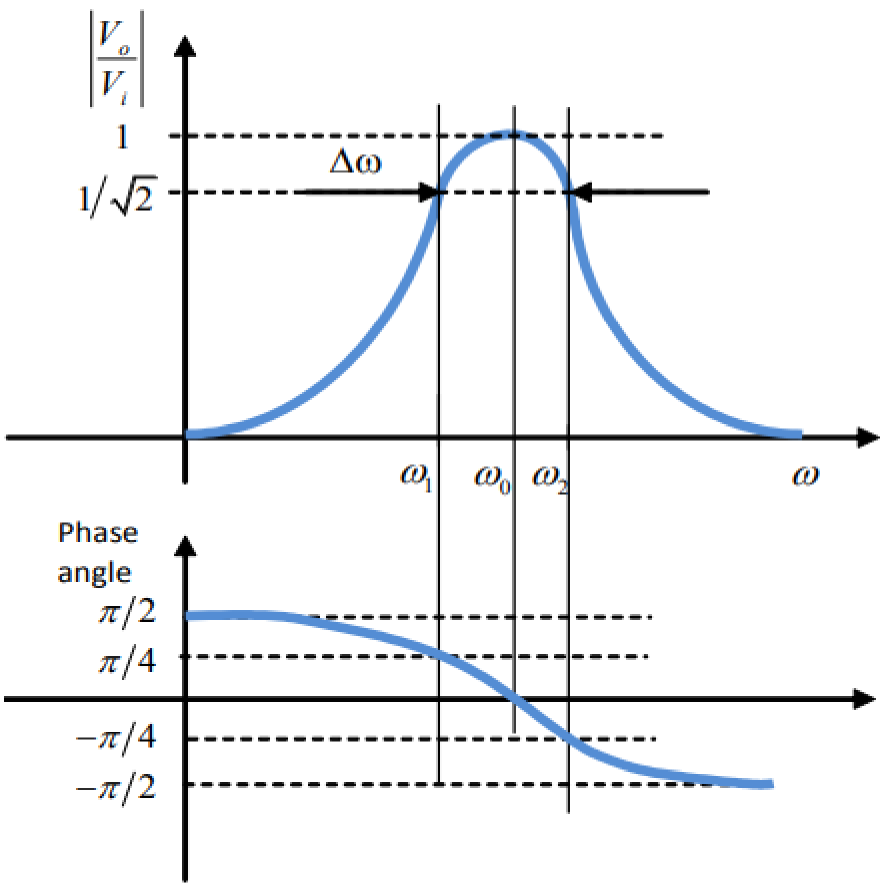
When the value of $R/\omega L$ is equal to 1, this gives:

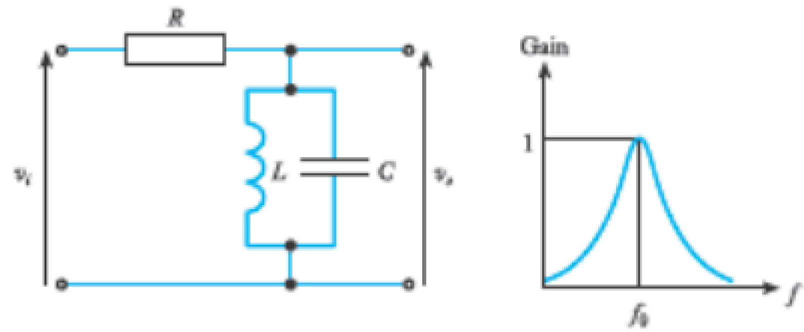
$$|\,\text{voltage gain}\,| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}} = 0.707$$

This situation corresponds to a cut-off frequency.

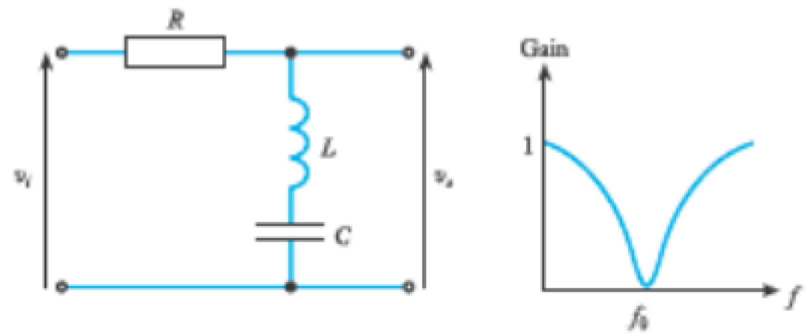
RLC first order filter (band-pass & band stop)

The combination of inductors and capacitors allows the production of filters with a very sharp cut-off. Simple *LC* filters can be produced using the series and parallel resonant circuits.





(a) A parallel *LC* network

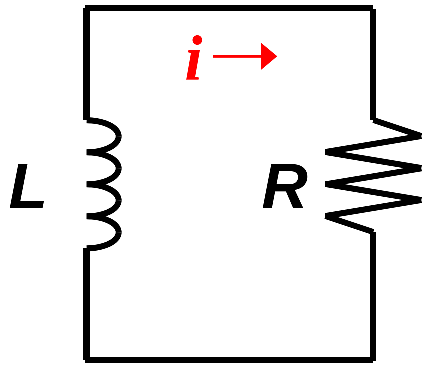


(b) A series *LC* network

$$f_0 = \frac{1}{2\pi\sqrt{LC}}$$

Natural Response Summary

RL Circuit



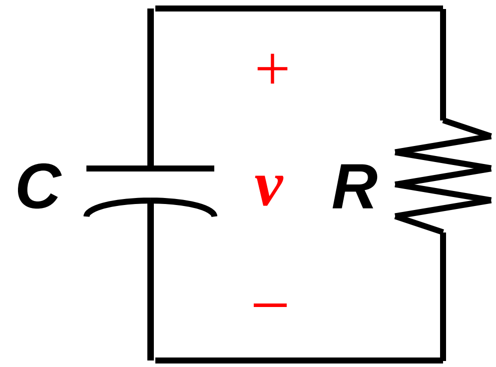
- Inductor current cannot change instantaneously

$$i(0^-) = i(0^+)$$

$$i(t) = i(0)e^{-t/\tau}$$

- time constant $\tau = \dfrac{L}{R}$

RC Circuit



- Capacitor voltage cannot change instantaneously

$$v(0^-) = v(0^+)$$

$$v(t) = v(0)e^{-t/\tau}$$

- time constant $\tau = RC$