$Y \sim N(\mu_Y, \sigma_Y^2)$ $\rho(X, Y) =$

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 + \right.\right.$$

Tut - 4

1. $f_X(x) = 2(x+1)/9 \qquad -1 < x < 2$

$= 0 \qquad$ otherwise

$Y = X^2$

$\cancel{f_Y(y) = \sqrt{f_X}}$

$F_Y(y) = P(Y \le y)$

$= P(X^2 \le y)$

$= P(-\sqrt{y} \le X \le \sqrt{y})$

$$\Rightarrow F_Y(y) = \begin{cases} \frac{2}{9}\int_{-\sqrt{y}}^{\sqrt{y}} (x+1)\,dx & \sqrt{y} < 1 \\ \frac{2}{9}\int_{-1}^{\sqrt{y}} (x+1)\,dx & 1 < \sqrt{y} < 2 \\ & \sqrt{y} > 2 \\ 0 & \end{cases}$$

$$= \begin{cases} \frac{2}{9}(2\sqrt{y}) & y < 1 \\ \frac{2}{9}\left(\frac{y}{2} + \sqrt{y} + \frac{1}{2}\right) & 1 \le y < 4 \\ 0 & y \ge 4 \end{cases}$$

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$$f_Y(y) = \begin{cases} \frac{1}{9}(1+y^{-1/2}) & 1 \le y < 4 \\ 0 & \text{otherwise} \end{cases}$$

2. $$f_X(x) = \begin{cases} x/2 & 0 \le x < 1 \\ 1/2 & 1 < x \le 2 \\ \frac{3-x}{2} & 2 < x \le 3 \end{cases}$$

$$Y = (X - 3/2)^2$$

$$F_Y(y) = P(Y \le y)$$
$$= P((X - \tfrac{3}{2})^2 \le y)$$
$$= P(\tfrac{3}{2} - \sqrt{y} \le X \le \tfrac{3}{2} + \sqrt{y})$$

$$= \begin{cases} \frac{1}{2}(2\sqrt{y}) & 0 < \sqrt{y} < \frac{1}{2} \\ \frac{1}{2} + \int_{\frac{3}{2}-\sqrt{y}}^{1} \frac{x}{2}\,dx + \int_{2}^{\frac{3}{2}+\sqrt{y}} \left(\frac{3-x}{2}\right)dx & \frac{1}{2} < \sqrt{y} < \frac{3}{2} \end{cases}$$

$$f_Y(y) = F_Y'(y)$$

$$= \begin{cases} \frac{1}{2}y^{-1/2} & 0 < y < 1/4 \\ \frac{\left(\frac{3}{2}-\sqrt{y}\right)}{2}\cdot\left(\frac{1}{2\sqrt{y}}\right) + \frac{\left(\frac{3}{2}-\sqrt{y}\right)}{2}\cdot\left(\frac{1}{2\sqrt{y}}\right) & \frac{1}{4} < y < \frac{9}{4} \end{cases}$$

~~$= \begin{cases} \frac{1}{2}y^{-1/2} & 0 < y < 1/4 \\ \frac{1}{2}y^{-1/2}(\frac{5}{2} - y^{1/2}) & \frac{1}{4} < y < \frac{9}{4} \\ 0 & \text{otherwise} \end{cases}$~~

~~$\begin{cases} \frac{1}{2}y^{-1/2} & 0 < y < \frac{1}{4} \\ \frac{1}{2}y^{-1/2}(5/2 - y^{1/2}) & \frac{1}{4} < y < \frac{9}{4} \end{cases}$~~

$$= \begin{cases} \frac{1}{2} y^{-1/2} & 0 < y < \frac{1}{4} \\ \frac{1}{2} y^{-1/2}\left(\frac{3}{2} - y^{1/2}\right) & \frac{1}{4} < y < \frac{9}{4} \end{cases}$$

3. $f_X(x) = \frac{2x}{\pi^2} \quad 0 < x < \pi$

$Y = \sin X$

$$F_Y(y) = P(Y \le y)$$
$$= P(\sin X \le y)$$
$$= P(0 < X \le \sin^{-1} y) \cancel{+ P(\sin^{-1} y \le X <)}$$
$$+ P(\pi - \sin^{-1} y \le X < \pi)$$
$$= \int_0^{\sin^{-1} y} \frac{2x}{\pi^2} dx + \int_{\pi - \sin^{-1} y}^{\pi} \frac{2x}{\pi^2} dx$$

$$f_Y(y) = F_Y'(y)$$
$$= \frac{2 \sin^{-1} y}{\pi^2} \left(\frac{1}{\sqrt{1-y^2}}\right)$$
$$+ \frac{2(\pi - \sin^{-1} y)}{\pi^2} \left(\frac{1}{\sqrt{1-y^2}}\right)$$

$$f_Y(y) = \frac{2\pi}{\pi^2 \sqrt{1-y^2}}$$

$$= \begin{cases} \frac{2}{\pi\sqrt{1-y^2}} & 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

4. $X \sim \text{Bin}(n, p)$

a) $Y_1 = 3X + 4$

$\cancel{F_Y(y) = P(3X + 4 \le y)}$
$\cancel{= P\left(X \le \frac{y-4}{3}\right)}$

$$P_Y(y) = P(\cancel{Y} 3X+4=y)$$

$$= P\left(X=\frac{y-4}{3}\right)$$

$$P_Y(y) = {}^{n}C_t \, p^t (1-p)^{n-t}$$ where $t = \frac{y-4}{3}$ (if y is of the form $3x+4$) 6

b) $Y_2 = X - 3$

$$P_Y(y) = P(X = y+3)$$

$$= {}^{n}C_{y+3} \, p^{y+3} (1-p)^{(n-3)-y}$$ (if $y \le n$

c) $Y_3 = X^2 + 2$

$$P_Y(y) = P(X = \sqrt{y-2})$$

d) ||| as above.

5. $X \sim Beta(a, b)$

$$f_X(x) = \begin{cases} \dfrac{x^{a-1}(1-x)^{b-1}}{B(a,b)} & 0 \le x \le 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$Y_1 = \frac{1}{1+X}$$

$$F_{Y_1}(y) = P\left(\frac{1}{1+X} \le y\right)$$

$$= P\left(X \ge \frac{1}{y} - 1\right)$$

$$= 1 - P\left(X < \frac{1}{y} - 1\right) \quad \text{if } 0.5 < y < 1$$

$$= \begin{cases} 1 - F_X\left(\frac{1}{y} - 1\right) & 0.5 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

$$f_{Y_1}(y) = f_X\left(\frac{1}{y} - 1\right) \cdot \left(\frac{-1}{y^2}\right)$$

$$f_Y(y) = \frac{\left(\frac{1}{y}-1\right)^{a-1}\left(2-\frac{1}{y}\right)^{b-1}}{\beta(a,b)} \qquad 0.5 < y < 1$$

6. $C \sim \text{Uniform}(15, 21)$.

$$f_C(x) = \begin{cases} \frac{1}{6} & 15 \le x \le 21 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_F(y) = P\left((y-32)\times\frac{5}{9} > \cancel{6}\right)$$

$$= P\left(C < \frac{5}{9}(y-32)\right)$$

$$= F_C\left(\frac{5}{9}(y-32)\right)$$

$$f_F(y) = \begin{cases} \frac{5}{54} & 59 \le y \le 69.8 \\ 0 & \text{elsewhere} \end{cases}$$

7. $f_X(x) = cx^2 \exp\{-\beta x^2\} \qquad x > 0$

$$Y = \frac{1}{2}mX^2$$

$$F_Y(y) = P(Y \le y)$$

$$= P\left(-\sqrt{\frac{2y}{m}} \le X \le \sqrt{\frac{2y}{m}}\right)$$

$$= F_X\left(\sqrt{\frac{2y}{m}}\right)$$

$$f_Y(y) = F_X'\left(\sqrt{\frac{2y}{m}}\right)\cdot\sqrt{\frac{2}{m}}\cdot\frac{1}{2\sqrt{y}}$$

$$= \begin{cases} \frac{\sqrt{2}\,c\sqrt{y}}{\sqrt{m}\,m}\exp\{-2\beta y/m\} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$$

8.

$$f_X(x) = \begin{cases} \frac{x+1}{4}, & -1 \le x \le 1 \\ \frac{3-x}{4}, & 1 \le x \le 3 \end{cases}$$

$$Y = |X| \Rightarrow Y = \begin{cases} X & x > 0 \\ -X & x < 0 \end{cases}$$

$$= P(|X| \le y)$$

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$$\cancel{f_Y(y) = \begin{cases} \frac{y+1}{4} & 0 \le y \le 1 \\ \frac{3-y}{4} & 1 \le y \le 3 \end{cases}}$$

$$F_Y(y) = P(|X| \le y) \qquad y > 0$$
$$= P(-y \le X \le y)$$

$$\Rightarrow F_Y(y) = \begin{cases} F_X(y) - F_X(-y) & 0 \le y < 1 \\ F_X(y) & 1 \le y \le 3 \end{cases}$$

$$f_Y(y) = \begin{cases} f_X(y) + f_X(-y) & 0 \le y \le 1 \\ f_X(y) & 1 \le y \le 3 \end{cases}$$

$$f_Y(y) = \begin{cases} \frac{y+1}{4} + \left(\frac{-y+1}{4}\right) = \frac{1}{2} & 0 \le y \le 1 \\ \frac{3-y}{4} & 1 \le y \le 3 \end{cases}$$

Tut-5

1. $f_{XY}(x,y) = e^{-(x+y)} \quad x > 0, y > 0$

i) $$P(1 < X+Y < 2) = \int_1^2 \int_0^k e^{-(t+(k-t))} \, dt \, dk$$
$$= \int_1^2 \int_0^k e^{-k} \, dt \, dk$$
$$= \int_1^2 k e^{-k} \, dk$$
$$= 2e^{-1} - 3e^{-2}$$

ii) $$P(X<Y \mid X < 2Y) = \frac{P(X<Y, X<2Y)}{P(X<2Y)}$$
$$= \frac{P(X<Y)}{P(X<2Y)}$$

$$P(X<Y) = \cancel{\int e^{-(t+y)}dx} \int_0^\infty \int_0^w e^{-(t+w)} dt\, dw$$

$$= \int_0^\infty e^{-w} \int_0^w e^{-t} dt\, dw$$

$$= \int_0^\infty e^{-w}(1-e^{-w})\, dw$$

$$= \int_0^\infty (e^{-w} - e^{-2w})\, dw$$

$$= 1 - \frac{1}{2} = \frac{1}{2}$$

{Can be done like this, since there are 3 choices.
- $X<Y$ } symmetry
- $Y<X$
- $Y=X$ {0} continuous

$\therefore P(X<Y) = 0.5$}

$$\text{III ly } P(X<2Y) = \int_0^\infty e^{-w} \int_0^{2w} e^{-t} dt\, dw$$

$$= \int_0^\infty e^{-w}(1-e^{-2w})\, dw$$

$$= \int_0^\infty (e^{-w} - e^{-3w})\, dw$$

$$= 1 - \frac{1}{3} = \frac{2}{3}$$

$$\Rightarrow P(X<Y \mid X<2Y) = \frac{3}{4}$$

iii) $P(0<X<1 \mid Y=2)$

$$= \frac{\int_0^1 e^{-(2+t)} dt}{\int_0^\infty e^{-(2+t)} dt}$$

$$= \frac{e^{-2}(1-e^{-1})}{e^{-2}} = 1-e^{-1}$$

iv) From 1

$$P(x+y<m) = \int_0^m \int_0^k e^{-k} dt\, dk = \int_0^m k e^{-k}$$

$$\Rightarrow \frac{1}{2} = -me^{-m} + (1-e^{-m})$$

$$-\frac{1}{2} = -(1+m)e^{-m}$$

$$e^{m} = 2(m+1)$$

$$\Rightarrow m \approx 1.68$$

$k \quad e^{-k}$

$1 \quad -e^{-k}$

$-ke^{-k}\Big|_0^m + \int_0^m e^{-k}$

$-me^{-m} + (1-e^{-m})$

2.

$$f_{XY}(x,y) = x+y$$

$$f_X(x) = \int_0^1 (x+y)\,dy$$

$$f_X(x) = x + \frac{1}{2} \qquad \mu_X = \int_0^1 x\left(x+\frac{1}{2}\right)dx$$

$$f_Y(y) = y + \frac{1}{2} \qquad = 7/12 = \mu_Y$$

$$EX^2 = \frac{5}{12}, \quad EX = \frac{7}{12}$$

$$EY^2 = \frac{5}{12}, \quad EY = \frac{7}{12}$$

$$\mathrm{Var}(X) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$$

$$\mathrm{Var}(Y) = \frac{11}{144}$$

$$\mathrm{Cov}(X,Y) = E[XY] - \mu_X\mu_Y$$

$$E[XY] = \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$

$$= \int_0^1 y\int_0^1 (x^2+xy)\,dx\,dy$$

$$= \int_0^1 y\left(\frac{1}{3}+\frac{y}{2}\right)dy$$

$$= \frac{y^2}{6} + \frac{y^3}{6}\Big|_0^1 = \frac{1}{3}$$

$$\Rightarrow \mathrm{Cov}(X,Y) = \frac{1}{3} - \frac{49}{144} = \frac{-1}{144}$$

$$\Rightarrow \operatorname{var}(X+Y) = \operatorname{var}X + \operatorname{var}Y - 2\operatorname{Cov}(X,Y)$$
$$= \frac{5}{36}$$

$$\operatorname{Corr}(X,Y) = \frac{\frac{-1}{144}}{\frac{11}{144}} = -\frac{1}{11}$$

$$\Rightarrow v(X|Y=y) = E[X^2|Y=y] - E[X|Y=y]^2$$

$$E[X|Y=y] = \int_0^1 x(y+x)\,dx$$
$$= \frac{1}{3} + \frac{y}{2}$$

$$E[X^2|Y=y] = \int_0^1 x^2(y+x)\,dx$$
$$= \frac{1}{4} + \frac{y}{3}$$

$$\operatorname{var}(X|Y=y) = \left(\frac{1}{4} + \frac{y}{3}\right) - \left(\frac{1}{3} + \frac{y}{2}\right)^2$$

$$f_{X|Y} = \frac{f_{XY}}{f_Y} = \frac{2(x+y)}{1+2y}$$

$$E[X|Y=y] = \int_0^1 x\left(\frac{2(x+y)}{1+2y}\right)dx$$
$$= \frac{2}{1+2y}\left(\frac{1}{3} + \frac{y}{2}\right)$$

$$\text{||ly} \quad E[X^2|Y=y] = \frac{2}{1+2y}\left(\frac{1}{4} + \frac{y}{3}\right)$$

$$\operatorname{var}(X|Y=y) = E[X^2|Y=y] - (E[X|Y=y])^2$$

3. $f_{XY}(x,y) = \frac{1}{8}(6-x-y)$ $\quad 0<x<2,\ 2<y<4$

$E(Y|X=x)$, $\operatorname{var}(Y|X=x)$, $\operatorname{Cov}(X,Y)$, $\operatorname{var}(X|Y=y)$

$$f_X(x) = \int_2^4 \frac{1}{8}(6-x-y)\,dy \qquad f_Y(y) = \int_0^{\infty} \frac{1}{8}(6-x-y)\,d$$

$$= \left(\frac{6-x}{8}\right)(2) - \frac{y^2}{16}\Big|_2^4 \qquad = \left(\frac{6-y}{8}\right)2 - \frac{1}{4}$$

$$= \frac{6-x}{4} - \frac{3}{4} \qquad = \frac{5-y}{4}$$

$$= \frac{3-x}{4}$$

$$f_{Y|X=x} = \frac{f_{XY}(y,x)}{f_X(x)} = \frac{(6-x-y)}{2(3-x)}$$

$$f_{X|Y=y} = \frac{6-x-y}{2(5-y)}$$

$$E[Y|X=x] = \int_2^4 \frac{6-x-y}{2(3-x)} \cdot y\,dy$$

$$= \frac{1}{2(3-x)}\left[\int_2^4 (6-x)y\,dy - \int_2^4 y^2\,dy\right]$$

$$= \frac{1}{2(3-x)}\left(6(6-x) - \frac{56}{3}\right)$$

$$= \frac{1}{2(3-x)}\left(\frac{52-18x}{3}\right)$$

$$= \frac{52-18x}{18-6x}$$

$$= 3 - \frac{2}{18-6x}$$

$$= 3 - \frac{1}{9-3x}$$

$$E[Y^2|X=x] = \frac{1}{2(3-x)}\left[\int_2^4 (6-x)y^2\,dy - \int_2^4 y^3\,dy\right]$$

$$= \frac{1}{2(3-x)}\left[(6-x)\frac{56}{3} - 60\right]$$

$$= \frac{1}{2(3-x)}\left[52 - \frac{56x}{3}\right]$$

$$= \frac{156 - 56x}{6(3-x)} = \frac{78 - 28x}{3(3-x)}$$

$$\text{var}(Y|X=x) = \frac{78-28x}{3(3-x)} - \left(\frac{26-9x}{3(3-x)}\right)^2$$

$$= \frac{(78-28x)\,3(3-x) - (26-9x)^2}{(3(3-x))^2}$$

$$= \frac{3(78\times 3 - 162x + 28x^2) - (26^2 - 456x + 81x^2)}{(3(3-x))^2}$$

$$= \frac{26 - 30x + 3x^2}{9(x-3)^2}$$

$$= \frac{3x^2 - 30x + 26}{3(3x^2 - 18x + 27)} \qquad \cancel{= \frac{75x^2 - 334x + 26}{9(x-3)^2}}$$

$$\text{Cov}(X,Y) = E[XY] - \mu_x \mu_y$$

$$E[XY] = \int_2^4 \int_0^2 \frac{xy}{8}(6-x-y)\,dx\,dy$$

$$= \int_2^4 \frac{y}{8} \int_0^2 ((6-y)x - x^2)\,dx\,dy$$

$$= \int_2^4 \frac{y}{8} \left(2(6-y) - \frac{8}{3}\right) dy$$

$$= \int_2^4 \frac{y}{8} \left(\frac{28 - 6y}{3}\right) dy$$

$$= \frac{1}{24} \int_2^4 (28y - 6y^2)\,dy$$

$$= \frac{1}{24} (14y^2 - 2y^3) \Big|_2^4$$

$$= \frac{1}{12}(84 - 56) = \frac{7}{3}$$

$$EX = \int_0^2 \frac{(3-x)}{4} dx = \frac{3x}{4} - \frac{x^2}{8} \Big|_0^2 = \frac{3}{2} - \frac{1}{2} = 1$$

$$EY = \int_2^4 \frac{5-y}{4} dy = \frac{5y}{4} - \frac{y^2}{8} \Big|_2^4 = \frac{5}{2} - \frac{3}{2} = 1$$

$\Rightarrow Cov(X,Y) = \frac{4}{3}$

4. BVN (28·4, 31·6, 46·24, 54·76, 0·82)

$X \rightarrow$ age of woman

$Y \rightarrow$ age of man

$P(X \geq 30)$

~~$f_{XY}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp$~~

~~$f_{UV}(u,v) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(u^2 + v^2 - 2\rho uv\right)\right\}$~~

~~where $u = \frac{x-\mu_x}{\sigma_x}$, $v = \frac{y-\mu_y}{\sigma_y}$~~

a) $X = \sigma_x Z_1 + \mu_x$

$Y = \sigma_y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_y$

$X \sim N(\mu_x, \sigma_x^2)$

$Y \sim N(\mu_y, \sigma_y^2)$

$\Rightarrow P(X \geq 30) = 1 - \Phi\left(\frac{30-\mu_x}{\sigma_x}\right)$

$= \Phi\left(\frac{\mu_x - 30}{\sigma_x}\right)$

$= 0.4090$

b) $P(Y \geq 35 \mid X = 30)$

$z_1 = \frac{30-\mu_x}{\sigma_x}$

$Y' = \sigma_y\left(\rho\left(\frac{30-\mu_x}{\sigma_x}\right) + \sqrt{1-\rho^2} Z_2\right) + \mu_y$

$Y' = \sigma_y\sqrt{1-\rho^2} Z_2 + \rho\sigma_y \ldots$

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$\Rightarrow Y' \sim N(33.03, 15.14)$

$$P(Y' \geq 35) = 1 - \Phi\left(\frac{35 - 33.03}{\sqrt{15.14}}\right)$$

$$= 1 - \Phi\left(\frac{1.97}{3.89}\right)$$

$$= \Phi(-0.506)$$

$$= 0.3050$$

5. $(X,Y) \sim BVN(2000, 0.1, 2500, 0.01, 0.87)$

$X \sim$ Lifetime of bulb
$Y \sim$ Filament diameter

$\mu_x = 2000 \quad \sigma_x = 50$
$\mu_y = 0.1 \quad \sigma_y = 0.1$

$$P(X > 1950 \mid Y = 0.098)$$

$$Y = \sigma_y Z_1 + \mu_y$$

$$X = \sigma_x\left(\rho Z_1 + \sqrt{(1-\rho^2)}\, Z_2\right) + \mu_x$$

$\because Y = 0.098$

$$\Rightarrow Z_1 = \frac{0.098 - 0.1}{0.1} = -0.02$$

$$\cancel{X = 2000(-0.0174 + 0.493 Z_2) + 2000}$$

$$\cancel{= 0.493 Z_2 + 1965.2}$$

$$X' = 50(0.87 \times (-0.02) + \sqrt{1-0.87^2}\, Z_2) + 2000$$

$$= 50\sqrt{1-0.87^2}\, Z_2 + 1999.13$$

$$\Rightarrow X' \sim N(1999.13, 607.75)$$

$$P(X' \geq 1950) = 1 - \Phi\left(\frac{1950 - 1999.13}{24.65}\right)$$

$$= \Phi(1.99)$$

$$= 0.9767$$

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7. $$f_{XY}(x,y) = \frac{1}{y} e^{-y-\frac{x}{y}}$$

$$\cancel{f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy = \int_0^{\infty} \frac{1}{y} e^{-\left(y+\frac{x}{y}\right)} dx = \frac{1}{y}}$$

Method - I

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \frac{e^{-y}}{y} \int_0^{\infty} e^{-\frac{x}{y}} dx$$

$$= \frac{e^{-y}}{y} \times y \times 1$$

$$f_Y(y) = e^{-y}$$

$$\boxed{EY = E[\text{Gamma}(1,1)] = 1}$$

$$\boxed{\text{var}(Y) = 1}$$

$$f_{X|Y=y}(x) = \frac{\frac{1}{y} e^{-y-\frac{x}{y}}}{e^{-y}} = \frac{e^{-\frac{x}{y}}}{y}$$

$$E[E[X|Y]] = EX$$

$$E[X|Y=y] = \int_0^{\infty} \frac{x}{y} e^{-\frac{x}{y}} dx$$

$$= y$$

$$E[E[X|Y]] = \int_0^{\infty} y \cdot e^{-y} dy$$

$$= 1$$

$$\Rightarrow \boxed{EX = 1}$$

$$var(X) = var(E[X|Y]) + E[var(X|Y)]$$

$$E[X|Y=y_j] = y_j$$

$$var(X|Y=y_j) = E[X^2|Y=y_j] - E[X|Y=y_j]^2$$

$$= \int_0^\infty \frac{x^2 e^{-\frac{x}{y_j}}}{y_j} dx - y_j^2$$

$$= y_j^2$$

$$\Rightarrow E[var(X|Y=y_j)] = 2$$

$$\Rightarrow E[X|Y=y_j] = y_j = Y$$

$$var(E[X|Y]) = var(Y) = 1$$

$$\Rightarrow var(X) = 1 + 2 = 3$$

- $$cov(X,Y) = E[XY] - EX \cdot EY$$

$$= \int_0^\infty \int_0^\infty x e^{-y-\frac{x}{y}} dx\,dy - \mu_X \mu_Y$$

$$= \int_0^\infty e^{-y} \int_0^\infty x e^{-\frac{x}{y}} dx\,dy - 1$$

$$= \int_0^\infty e^{-y} \cdot y^2 dy - 1$$

$$= 1$$

$y^2 \quad e^{-y}$

$2y \quad -e^{-y}$

$2 \quad ye^{-y}$

$x \quad e^{-\frac{x}{y}}$

$1 \quad -ye^{-\frac{x}{y}}$

$-xye^{-\frac{x}{y}}\Big|_0^\infty$

$\int_0^\infty y e^{-\frac{x}{y}}$

$-y^2 (-1)$

- $$corr(X,Y) = \rho = \frac{1}{1 \cdot \sqrt{3}} = \frac{1}{\sqrt{3}}$$

Method-II

$$\frac{X}{Y} = U, \quad Y = V$$

... for the questions

8. BVN $(75, 83, 25, 16, 0.8)$ $X \sim 1^{st}$ test
$Y \sim 2^{nd}$ test

$\mu_y = 83$, $\mu_x = 75$
$\sigma_y = 4$, $\sigma_x = 5$
$\rho = 0.8$

$$X = \sigma_x Z_1 + \mu_x$$
$$Y = \sigma_y(\rho Z_1 + \sqrt{1-\rho^2} Z_2) + \mu_y$$

given $X = 80$
$\Rightarrow Z_1 = 1$

$$Y' = 4(\rho + 0.6 Z_2) + 83$$
$$= 2.4 Z_2 + (83 + 4\rho)$$

$Y' \sim N(83 + 4\rho, 2$

$$P(Y' \geq 80) = 1 - \Phi\left(\frac{-3-4\rho}{2.4}\right)$$
$$= \Phi\left(\frac{3+4\rho}{2.4}\right)$$

If $\rho = 0.8$
$$P(Y' \geq 80) = \Phi(2.58)$$
$$= 0.9951$$
$\rho = -0.8$
$$P(Y' \geq 80) = \Phi(-0.083)$$
$$= 0.4681$$

9.

BVN$(6, 4, 1, 0.5, 0.1)$

. $P(X_1 \leq 5)$

$X_1 \sim N(6, 1)$

$$\Rightarrow P(X_1 \leq 5) = \Phi\left(\frac{5-6}{1}\right)$$
$= 0.1587$

$$\frac{5-\mu_1}{\sigma_1} = z_1 \quad \Rightarrow z_1 = -1$$

$$y' = \sigma_y(\rho z_1 + \sqrt{1-\rho^2}\, z_2) + \mu_y$$

$$= 4(-0.1 + \sqrt{0.99}\, z_2) + 4$$

$$y' = 4\sqrt{0.99}\, z_2 + 3.6 \quad \Rightarrow y' \sim N(3.6, 15.84)$$

$$P(y' \le 5) = \Phi\left(\frac{5-3.6}{4\sqrt{0.99}}\right)$$

$$= 0.6368$$

- $x_2 = 6$

$$\frac{6-\mu_2}{\sigma_2} = z_2 \quad \Rightarrow z_2 = 0.5$$

$$x' = \sigma_x(\rho z_2 + \sqrt{1-\rho^2} z_1) + \mu_x$$

$$= 5(0.05 + \sqrt{0.99}\, z_1) + 6$$

$$x' = 5\sqrt{0.99} z_1 + 6.25$$

$$\Rightarrow E[x'] = 6.25$$

10.

$$f(x,y) = 1 - \alpha(1-2x)(1-2y)$$

$$f_X(x) = \int_0^1 (1-\alpha(1-2x)(1-2y))\,dy$$

$$= 1 - \alpha(1-2x)\int_0^1 (1-2y)\,dy$$

$$= 1$$

$$f_Y(y) = 1$$

$$\Rightarrow EX = EY = \frac{1}{2}$$

$$EY^2 = EX^2 = \int_0^1 x^2\,dx = \frac{1}{3}$$

$$\Rightarrow \text{var}\,Y = \text{var}\,X = \frac{1}{12}$$

~~cov(X, Y) =~~

$$EXY = \int_0^1\int_0^1 xy\,\{1-\alpha(1-2x)(1-2y)\}\,dx\,dy$$

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$$= \int_0^1 y\left(\int_0^1 x - \alpha(1-2y)\int_0^1 x(1-2x)\,dx\right) dy$$

$$= \int_0^1 y\left(\frac{1}{2} - \alpha(1-2y)\left(-\frac{1}{6}\right)\right) dy$$

$$= \frac{1}{2}\int_0^1 \left(y + \frac{\alpha y(1-2y)}{3}\right) dy$$

$$= \frac{1}{2}\left(\frac{1}{2} + \frac{\alpha}{3}\left(-\frac{1}{6}\right)\right)$$

$$= \frac{1}{4} - \frac{\alpha}{36}$$

$$\mathrm{cov}(X,Y) = EXY - \mu_x \mu_y$$

$$= -\frac{\alpha}{36}$$

$$\mathrm{cor}(X,Y) = \frac{-\frac{\alpha}{36}}{\frac{1}{12}} = -\frac{\alpha}{3}$$

$$\Rightarrow -3 \leq \alpha \leq 3$$

correlation has to be 0 for independence $\Rightarrow \alpha = 0$

11. a) $P(X \leq 1, Y \geq 2) = P(\{(0,2), (0,3), (0,4), (1,2), (1,3), (1,4)\})$

$$= 0.356$$

b) $$P_X(x) = \begin{cases} 0.21 & x=0 \\ 0.298 & x=1 \\ 0.277 & x=2 \\ 0.215 & x=3 \end{cases}$$

$$P_Y(y) = \begin{cases} 0.267 & y=1 \\ 0.397 & y=2 \\ 0.302 & y=3 \\ 0.034 & y=4 \end{cases}$$

$EX = 0.298 + 2 \times 0.277 + 3 \times 0.215$
$= 1.497$

$EY = 2.103$

$EX^2 = 3.34$ $\Rightarrow \mathrm{Var} X = 1.099$

$EY^2 = 5.117$ $\mathrm{Var} Y = 0.694$

~~$EXY = \sum\sum xy P_X(x, y)$~~

$$EXY = \sum_{y=1}^{4} \sum_{x=0}^{3} xy \, P_{XY}(x, y)$$
$$= 3.279$$

$$\mathrm{Cov}(X,Y) = 3.279 - 1.497 \times 2.103$$
$$= 0.13$$

$$\rho(X, Y) = \frac{0.13}{\sigma_X \sigma_Y} = 0.149$$

c) $$P(Y \geq 2 \mid X = 1) = \frac{P(Y \geq 2, X = 1)}{P(X = 1)}$$

$$= 0.688$$

12. a) $P(X < 0.5, \; X < Y \leq X + 0.25)$

$$= \int_0^{0.5} \int_x^{x+0.25} f_{XY} \, dy \, dx$$

$$= \int_0^{0.5} \int_x^{x+0.25} 8xy \, dy \, dx$$

$$= \int_0^{0.5} 4x \left((x+0.25)^2 - x^2 \right) dx$$

$$= \int_0^{0.5} 4x \left(0.5x + \frac{1}{16} \right) dx$$

$$= \int_0^{0.5} (2x^2 + 0.25x) \, dx = 0.114$$

b) $$\frac{\int_{1/6}^{3/4} f_{XY}(\tfrac{1}{6}, y)\, dy}{\int_{1/6}^{1} f_{XY}(\tfrac{1}{6}, y)\, dy}$$

$$= \frac{\int_{1/6}^{3/4} 8(\tfrac{1}{6})\, y\, dy}{\int_{1/6}^{1} 8(\tfrac{1}{6})\, y\, dy}$$

$$= \frac{0.542}{0.648}$$

$$= \frac{0.356}{0.648}$$

$$= 0.55$$

c) $$f_{Y|X=1/6}(x,y) = \frac{\frac{4}{3}y}{\int_{1/6}^{1} \frac{4}{3} y\, dy}$$

$$= \frac{72y}{35} \qquad \frac{1}{6} < y < 1$$

$$E[Y \mid X = \tfrac{1}{6}] = 40.95 \text{ min}$$

d) $$f_X = \int_x^1 f_{XY}\, dy = \int_x^1 8xy\, dy$$

$$f_X = 4x(1 - x^2)$$

$$f_Y = \int_0^y 8xy\, dx$$

$$\Rightarrow EX = \frac{8}{15} \; , \; EY = \frac{4}{5}$$

$$EX^2 = \frac{1}{3} \; , \; EY^2 = \frac{2}{3}$$

$$\text{Var}X = \frac{11}{225} \; , \; \text{Var}Y = \frac{2}{75}$$

$$\text{Cov}(X, Y) = \int_0^1 \int_x^1 xy f_{XY} \, dx \, dy - \mu_X \mu_Y$$

$$= 8 \int_0^1 x^2 \int_x^1 y^2 dy \, dx - \frac{32}{75}$$

$$= \frac{8}{3} \int_0^1 x^2 (1 - x^3) \, dx - \frac{32}{75}$$

$$= \frac{4}{9} - \frac{32}{75}$$

$$= \frac{4}{225}$$

$$\rho(X, Y) = 0.492$$

13. $f_{XY}(x, y) = \frac{1}{4}(1 + xy) \; , \; |x| < 1 \; , \; |y| < 1$

$$P(2X < Y) = P(X < 0, Y > 0) + P(X < 0, Y > 2X, Y < 0) + P(X > 0, Y > 0, Y > 2X)$$

$$= \frac{1}{4} \int_{-1}^{0} \int_0^1 (1 + xy) \, dy \, dx + \frac{1}{4} \int_{-0.5}^{0} \int_{2x}^{0} (1 + xy) \, dy \, dx + \frac{1}{4} \int_{-1}^{-0.5} \int_{-1}^{0} (1 + xy) \, dy \, dx$$

$$+ \frac{1}{4} \int_0^{0.5} \int_{2x}^{1} (1 + xy) \, dy \, dx$$

$$= \frac{1}{4} \left[\int_{-1}^{0} \left(1 + \frac{x}{2}\right) dx + \int_{-0.5}^{0} (-2x - 2x^3) \, dx + \int_{-1}^{-0.5} \left(1 - \frac{x}{2}\right) dx + \int_0^{0.5} \left(1 - 2x - 2x^3 + \frac{x}{2}\right) dx \right.$$

$$=\frac{1}{4}\left[\frac{3}{4}+\frac{9}{32}+\frac{7}{16}+\frac{9}{32}\right]$$

$$=\frac{1}{2}$$

$$P(|X+Y|<1) = P(-1 \le X+Y \le 1)$$

$$\iint (1+(\mu)(k-\mu))\,d\mu\,dk$$

$$= 2P(X<0, Y>0)$$
$$+ 2P(X \ge 0, Y \ge 0, X+Y<1)$$
$$+ P(X<0, Y<0, X+Y>-1)$$

$$= 2\int_{-1}^{0}\int_{0}^{1}\frac{1}{4}(1+xy)\,dy\,dx$$
$$+ 2\int_{0}^{1}\int_{0}^{k}\frac{1}{4}(1+\mu k-\mu^2)\,d\mu\,dk$$
$$+ \int_{-1}^{0}\int_{k}^{0}\frac{1}{4}(1+\mu k-\mu^2)\,d\mu\,dk$$

$$= \frac{3}{8}+\frac{1}{4}\int_{0}^{1}\left(k+\frac{k^3}{6}\right)dk$$
$$+ \frac{1}{4}\int_{-1}^{0}\left(-k-\frac{k^3}{6}\right)dk$$

$$= \frac{31}{48}$$

$$= 2\int_{-1}^{0}\int_{0}^{1}\frac{1}{4}(1+xy)\,dy\,dx$$
$$+ 2\int_{0}^{1}\int_{0}^{k}\frac{1}{4}(1+xy)\,dy$$
$$+ 2\int_{0}^{1}\int_{0}^{k}\frac{1}{4}(1+\mu(k-\mu))\,d\mu\,dk$$

$k-\mu$

$k-\mu \ge 0$

$k<\mu$

$k-\mu>-1$

$$= 0.5\int_{-1}^{0}\left(1+\frac{x}{2}\right)dx$$
$$+ 0.5\int_{0}^{1}\left(k+\frac{k^3}{2}-\frac{k^3}{3}\right)dk$$

$k>\mu-1$

$k+1$

$$= \frac{3}{8}+\frac{13}{48}$$

$X, Y \sim U(-1,1)$

$\Rightarrow EX = EY = 0$

$$\mathrm{var}(X) = \mathrm{var}(Y) = \frac{(1+1)^2}{12} = \frac{1}{3}$$

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