Solution Set - 3

1. (a) Let the order of A be n.
Since A is non-singular, A^{-1} exists.
Also $\lambda \neq 0$. Therefore λ^{-1} exists.
Since λ is an eigen value of A. $\det(A-\lambda I_n)=0$.

$$\det(A^{-1}-\lambda^{-1}I_n) = (\det A)^{-1}\det(AA^{-1}-\lambda^{-1}A)$$
$$= (\det A)^{-1}(\lambda^{-1})^n \det(\lambda I_n - A)$$
$$= (\det A)^{-1}(\lambda^{-1})^n(-1)^n \det(A-\lambda I_n)$$
$$= 0, \text{ since } \det(A-\lambda I_n)=0$$

This proves that λ^{-1} is an eigen value of A^{-1}.

(b) The characteristic polynomial of $P^{-1}AP$ is $\det(P^{-1}AP - xI_n)$.

$$\det(P^{-1}AP - xI_n) = \det[P^{-1}AP - P^{-1}(xI_n)P]$$
since $P^{-1}(xI_n)P = xI_n$

$$= \det[P^{-1}(A-xI_n)P]$$
$$= \det P^{-1}\det(A-xI_n)\det P$$
$$= \det(A-xI_n)\det(P^{-1}P)$$
$$= \det(A-xI_n)\det(I_n)$$
$$= \det(A-xI_n)$$

Therefore the matrices $P^{-1}AP$ ~~have~~ & and A have the same ~~eigen values~~ characteristic polynomial and so they have the same eigen values.

(c) let $\Psi_A(x) = \det(A - xI_n) = (x-\lambda)^p \phi(x)$ where ϕ is a polynomial of degree $n-p$ and $\phi(\lambda) \neq 0$.

The characteristic polynomial of $A - \lambda I_n$ is $\det(A - \lambda I_n - xI_n)$.

$$\det(A - \lambda I_n - xI_n) = \det[A - (\lambda + x)I_n]$$
$$= (\lambda + x - \lambda)^p \phi(\lambda + x)$$
$$= x^p \mu(x), \text{ where } \mu(0) = \phi(\lambda) \neq 0.$$

This proves that 0 is a root of multiplicity p of the characteristic equation of the matrix $A - \lambda I_n$.

(2) (b) $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

~~det~~ characteristic eqn of A is

$$\det(A - \lambda I) = 0$$

or, $\begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = 0$

or, $\lambda^2 + 1 = 0$

or, $\lambda = \pm i$

A is a real matrix and the eigen values of A are not real numbers. Therefore the real matrix A has no eigen vector.

But if A be considered as a complex matrix, then the eigen vectors of A corresponding to the eigen values i and $-i$ can be obtained.

Let $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ be an eigen vector corresponding to i. ②

Then $AX = iX$ and this gives

$$-ix_1 - x_2 = 0$$
$$x_1 - ix_2 = 0$$

The equivalent system is $x_1 - ix_2 = 0$.

The solution is $k(i, 1)$, where $k \in \mathbb{C}$.

The eigen vectors are $k\begin{pmatrix} i \\ 1 \end{pmatrix}$, where k is a non-zero complex number.

<u>For $\lambda = -i$</u>

$$AX = -iX$$
$$\Rightarrow \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = -i\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

or, $-x_2 = -ix_1 \Rightarrow ix_1 + x_2 = 0$

$x_1 = -ix_2 \qquad x_1 + ix_2 = 0$

$\Rightarrow$ ~~$x_1 = +ix_2$~~

$\Rightarrow$ ~~$(x_2 \quad 2ix_1 = 0$~~

~~$ix_1 = x_2$~~ which is equivalent

i.e. to $ix_1 = x_2$

or, ~~$-x_1 = ix_2$~~

$x_1 = -ix_2$

The eigen vectors corresponding to $-i$ are $c\begin{pmatrix} 1 \\ -i \end{pmatrix}$, ~~where c~~ $c\begin{pmatrix} 1 \\ i \end{pmatrix}$, where c is a non zero complex number.

(e) The characteristic equation of A is $\lambda^2(1-\lambda)=0$ and therefore the eigen values of A are 0, 0, 1.

0 is an eigen value of algebric multiplicity 2; and 1 is a simple eigen value of A (i.e., of algebric multiplicity 1).

The eigen vectors corresponding to the eigen value 0 are the non-null sol^{n}s of the system

$$x+y+z=0 \Rightarrow x=-y$$
$$z=0.$$

let us take $x=c$, where c is a non zero real number

The eigen vectors are $c\begin{pmatrix}1\\-1\\0\end{pmatrix}$, where c is a non-zero real number.

The rank of the characteristic subspace is 1.

Therefore the geometric multiplicity of the eigen value 0 is 1.

The eigen vectors corresponding to the eigen value 1 are the non-null solutions of the system

$$y+z=0$$
$$x+2y+z=0.$$

The solution of the system is $c(1,-1,1)$, where $c\in\mathbb{R}$.

So the eigen vectors corresponding to $\lambda=1$ are $c\begin{pmatrix}1\\-1\\1\end{pmatrix}$, where c is a non-zero real number.

The rank of the characteristic subspace is 1

and therefore the geometric multiplicity of the eigen value 1 is 1. ③

ⓓ Let $A = \begin{pmatrix} -1 & 0 & -2i \\ 0 & 2 & 0 \\ 2i & 0 & -1 \end{pmatrix}$.

The characteristic eqn is given by

$$|A - \lambda I| = \begin{vmatrix} -1-\lambda & 0 & -2i \\ 0 & 2-\lambda & 0 \\ 2i & 0 & -1-\lambda \end{vmatrix} = 0$$

so, $(-1-\lambda)(2-\lambda)(-1-\lambda) - 2i\{-2i(2-\lambda)\} = 0$

$\Rightarrow (2-\lambda)(\lambda^2 + \lambda - 3) = 0$

which has solutions $\lambda = 1, 2, -3$.

These are the three eigen values of A.

Let $X = [x \ y \ z]^T$ be the eigen ~~value~~ vector corresponding to the eigen value $\lambda = 1$.

Then we have

$$(A - I)X = \begin{pmatrix} -2 & 0 & -2i \\ 0 & 1 & 0 \\ 2i & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

or, $-2x - 2iz = 0$
$y = 0$
$2ix - 2z = 0$

or, $z = ix$
$y = 0$.

Take $x = c$ where $c \in \mathbb{R} \setminus \{0\}$.

Hence the eigen vectors corresponding to the

eigen value $\lambda = 1$ are $c(1\ 0\ i)^T$ where c is a non-zero real number.

For $\lambda = 2$, $(A - 2I)x = 0$

i.e., $\begin{pmatrix} -3 & 0 & -2i \\ 0 & 0 & 0 \\ 2i & 0 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Which gives, $3x = -2iz$ and $2ix = 3z$

so, $x = z = 0$ and y is arbitrary.

$\therefore$ The eigen vectors corresponding to $\lambda = 2$ are $d\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, where d is a non-zero real number.

Again for $\lambda = -3$, ~~(A~~

$(A + 3I)x = 0$

i.e., $\begin{pmatrix} 2 & 0 & -2i \\ 0 & 5 & 0 \\ 2i & 0 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

From this we deduce $x = iz$ and $y = 0$,

so take $z = c$.

Then $(x, y, z) = (ic, 0, c)$

$\cancel{= i(c, 0} = c(i, 0, 1)$.

The eigen vectors corresponding to $\lambda = -3$ are $c\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}$ where $c \in \mathbb{R} \setminus \{0\}$.

Solution to Tutorial Sheet 3

1. (a) $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \lambda \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \underset{\sim}{0}$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & \sqrt{2} \\ \sqrt{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(1-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 3\lambda = 0$$

$$\Rightarrow \lambda = 0, 3$$

When $\lambda = 0$, $\begin{pmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \sqrt{2}x_1 + x_2 = 0 \quad \text{—— (1)}$$

& $x_1^2 + x_2^2 = 1$ —— (2) [As eigen vectors are of unit length]

Let, $x_2 = a$

$\therefore x_1 = -a/\sqrt{2}$

hence, $\left(-\frac{a}{\sqrt{2}}\right)^2 + a^2 = 1$

$$\frac{3a^2}{2} = 1$$

$$a = \sqrt{\frac{2}{3}}, \quad x_1 = -\frac{1}{\sqrt{3}}$$

when $\lambda = 3$, $\begin{pmatrix} 2-3 & \sqrt{2} \\ \sqrt{2} & 1-3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \quad -x_1 + \sqrt{2}x_2 = 0 \quad \text{—(3)}$

& $\quad x_1^2 + x_2^2 = 1 \quad \text{—(4)}$

let, $x_2 = b \quad \therefore \quad x_1 = \sqrt{2}b$

$(\sqrt{2}b)^2 + b^2 = 1$

$\Rightarrow \quad 3b^2 = 1$

$\Rightarrow \quad b = \frac{1}{\sqrt{3}}$

$x_1 = \sqrt{\frac{2}{3}}$

hence, the eigen values are 0 & 3 &

eigen vectors are $\begin{pmatrix} -1/\sqrt{3} \\ \sqrt{2/3} \end{pmatrix}, \begin{pmatrix} \sqrt{2/3} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

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3. Ch. eqn. $|A - \lambda I| = 0$

$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ -1 & 3-\lambda \end{vmatrix} = 0$

$\Rightarrow (1-\lambda)(3-\lambda) + 2 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 5 = 0$

$\therefore$ The matrix A satisfies -

$A^2 - 4A + 5I = 0$

$A^2 = 4A - 5I$

$A^3 = 4A^2 - 5A$

$\vdots$

$A^6 = 4A^5 - 5A^4$

$A^4 = 4A^3 - 5A^2$

$A^4 - 4A^3 = -5A^2$

$\Rightarrow 3A^4 - 12A^3 = -15A^2$

$\therefore \quad A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$

$= -5A^4 + 8A^4 - 12A^3 + 14A^2$

$= 3A^4 - 12A^3 + 14A^2$

$= -15A^2 + 14A^2$

$= -A^2$

$= 5I - 4A.$ (Ans)

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4. $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

Ch. eqⁿ is –

$|A - \lambda I| = 0$

$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$

$\Rightarrow \lambda^3 - 2\lambda + 1 = 0$

hence, matrix A satisfies –

$A^3 - 2A + I = 0$

$A^3 = 2A - I$

$$A^9 = (A^3)^3 = (2A - I)^3$$

$$= 8A^3 - 12A^2 - I + 6A$$

$$= 16A - 8I - 12A^2 - I + 6A$$

$$= 22A - 12A^2 - 9I$$

$$= \begin{pmatrix} 22 & 0 & 44 \\ 0 & -22 & 22 \\ 0 & 22 & 0 \end{pmatrix} - \begin{pmatrix} 12 & 24 & 24 \\ 0 & 24 & -12 \\ 0 & -12 & 12 \end{pmatrix} - \begin{pmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -24 & 20 \\ 0 & -55 & 34 \\ 0 & 34 & -21 \end{pmatrix}$$

here $|A| = -1 \neq 0 \therefore A^{-1}$ exist

$$A^3 - 2A + I = 0$$

$$\Rightarrow A^{-1} = 2I - A^2$$

$$= \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$ (Ans)

5. A & P are non singular matrices.

If $A = P^{-1} D P$

Then D be a diagonal matrix having eigen values as diagonal entry.

Then we find eigen values of A.

Ch. eqn is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (3-\lambda)\left[(1-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow \lambda(3-\lambda)(\lambda-2) = 0$$

$$\Rightarrow \lambda = 0, 2, 3$$

hence, $D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

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6. $A = \begin{bmatrix} 1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

$\det A = 1 - 2\times 1$
$\quad - 2\times 1$
$= 1 - 4$
$= -3$

Ch. eqn is – $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 & -2 \\ 1 & 2-\lambda & 1 \\ -1 & -1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 + (0+1-2)\lambda + 3 = 0$$

$$\Rightarrow \lambda^3 - 3\lambda^2 - \lambda + 3 = 0$$

$$\Rightarrow \lambda^2(\lambda-1) - 2\lambda(\lambda-1) - 3(\lambda-1) = 0$$

$$\Rightarrow (\lambda-1)(\lambda^2 - 2\lambda - 3) = 0$$

$$\Rightarrow (\lambda-1)(\lambda+1)(\lambda-3) = 0$$

$$\Rightarrow \lambda = 1, -1, 3$$

for $\lambda = 1$,
$$\begin{pmatrix} 0 & 2 & -2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$2x_2 - 2x_3 = 0 \quad \text{—(1)}$$
$$x_1 + x_2 + x_3 = 0 \quad \text{—(2)}$$

$$x_2 = x_3$$
$$x_1 = -x_2 - x_3$$
$$= -2x_3$$

hence, eigen vector is $= \begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}$

for $\lambda = -1$,
$$\begin{pmatrix} 2 & 2 & -2 \\ 1 & 3 & 1 \\ -1 & -1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\Rightarrow x_1 + x_2 - x_3 = 0 \quad \text{—(3)}$$
$$x_1 + 3x_2 + x_3 = 0 \quad \text{—(4)}$$

$(3) + (4) \Rightarrow$ $2x_1 + 4x_2 = 0$
$$x_1 + 2x_2 = 0$$

$x_1 = -2x_2$

$x_3 = -x_2$

hence, eigen vector is - $\begin{pmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$

for $\lambda = 3$, $\begin{pmatrix} -2 & 2 & -2 \\ 1 & -1 & 1 \\ -1 & -1 & -3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x_1 - x_2 + x_3 = 0$ —(5)

$-x_1 - x_2 - 3x_3 = 0$ —(6)

(5) + (6) $\Rightarrow$ $-2x_2 - 2x_3 = 0$

$x_3 = -x_2$

$x_1 = 2x_2$.

hence, eigen vector is - $\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}$

hence, $P = \begin{pmatrix} -2/\sqrt{6} & -2/\sqrt{6} & 2/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{6} & -1/\sqrt{6} \end{pmatrix}$

& $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

7. The cross product of $b = [0\ 1\ 0]^T$ and $x = [x_1\ x_2\ x_3]$

is $$b \times x = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{vmatrix} = x_3\hat{i} + 0\hat{j} - x_1\hat{k}$$

$$= [x_3\ \ 0\ \ -x_1]$$

Now, we have $L(x) = b \times x$

$$= M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Where M is a 3×3 matrix

let, $$M = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix}$$

Now, $$M \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = b \times x$$

$$\Rightarrow \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_3 \\ 0 \\ -x_1 \end{pmatrix}$$

So, $$M = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

hence, ch. eq^n of M is –

$$\lambda^3 - \lambda = 0$$
$$\lambda(\lambda^2 - 1) = 0$$
$$\lambda = 0, \pm 1.$$

8. given A has eigen values $-1, 1, 2, -2$ (8)

$B = A^4 - 5A^2 + 5I$

Then eigen values of B will be –

$1, 1, 1, 1$ [$\because$ if λ is an eigen value of A Then λ^m will be an eigen value of A^m]

hence, $(A+B)$ has eigen values

$0, 2, 3, -1$

hence, trace of $(A+B)$ = Sum of the eigenvalues $= 4$.

9. (a) given, $M = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$

Now, $(\bar{M})^t = \begin{bmatrix} 1 & 1+i & 2i & 9 \\ 1-i & 3 & 4 & 7-i \\ -2i & 4 & 5 & i \\ 9 & 7+i & -i & 7 \end{bmatrix}$

Now, $(\bar{M})^t = M$

hence, hermitian.

~~So~~ (b) For the sol^n see the page no. (12).

10. $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 \\ 0 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 2$$

for $\lambda = 1$, $\begin{bmatrix} 0 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ a \end{bmatrix} = 0 \Rightarrow a = 0$

for $\lambda = 2 \Rightarrow \begin{bmatrix} -1 & 2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ b \end{bmatrix} = 0$

$$\Rightarrow -1 + 2b = 0 \Rightarrow b = 1/2$$

$$\therefore \boxed{a + b = 1/2}.$$

11. (a) Given, $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

A will have real eigen values

iff $A^T = A$

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

$$\sin\theta = -\sin\theta$$

$$2\sin\theta = 0$$

$$\boxed{\theta = n\pi}$$ n is an integer.

(*) For the sol^n of question no 12. see the pages 12 & 13.

11.(b) Given, $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

~~Ch. eqn of A is —~~

$$\cancel{\lambda^2 - 2\cos\theta\,\lambda + 1 = 0}$$

$$\cancel{\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 1}}{2}}$$

$$P^{-1}AP = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}^{-1} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}^{T} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \cos\theta + i\sin\theta & \sin\theta - i\cos\theta \\ -i\cos\theta - \sin\theta & -i\sin\theta + \cos\theta \end{pmatrix} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} \cos\theta + i\sin\theta + i\sin\theta + \cos\theta & i\cos\theta - \sin\theta + \sin\theta - i\cos\theta \\ -i\cos\theta - \sin\theta + \sin\theta + i\cos\theta & \cos\theta - i\sin\theta - i\sin\theta + \cos\theta \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} 2(\cos\theta + i\sin\theta) & 0 \\ 0 & 2(\cos\theta - i\sin\theta) \end{pmatrix}$$

$$= \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$$

Also We can find the eigen values of A, then D will be the diagonal matrix having eigen values as its diagonal entries.

13. Given, A is an $n \times n$ matrix which is both Hermitian and Unitary.

hence, $(\bar{A})^t = A$ —(1)

& $(\bar{A})^t A = I$ —(2)

using, (1) in (2) we get:

$$\boxed{A^2 = I}$$

14. let, $A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$

$= P + Q$

Now, $A = \begin{pmatrix} 1 & 2+i & 1-i \\ 2-i & 1+2i & 3 \\ 2+i & 2 & 1+i \end{pmatrix}$

$A^\theta = \begin{pmatrix} 1 & 2+i & 2-i \\ 2-i & 1-2i & 2 \\ 1+i & 3 & 1-i \end{pmatrix}$

hence, $P = \frac{1}{2}(A + A^\theta)$

$= \frac{1}{2}\begin{pmatrix} 2 & 4+2i & 3-2i \\ 4-2i & 2 & 5 \\ 3+2i & 5 & 2 \end{pmatrix}$

Now, $P^\theta = \frac{1}{2}\begin{pmatrix} 2 & 4+2i & 3-2i \\ 4-2i & 2 & 5 \\ 3+2i & 5 & 2 \end{pmatrix}$

$= P$

$$Q = \frac{1}{2}\left(A - A^{\theta}\right)$$

$$= \frac{1}{2}\begin{pmatrix} 0 & 0 & -1 \\ 0 & 4i & 1 \\ 1 & -1 & 2i \end{pmatrix}$$

$$Q^{\theta} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & -4i & -1 \\ -1 & 1 & -2i \end{pmatrix} = -Q$$

hence, Q is Skew Hermitian.

hence, $A = P + Q$

Where $P = \frac{1}{2}(A + A^{\theta})$ = Hermitian

$Q = \frac{1}{2}(A - A^{\theta})$ = Skew "

15. Let $A = \begin{pmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{pmatrix}$

$$A^* = (\bar{A})^T = \begin{pmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{pmatrix}$$

~~Then~~ It, A is unitary ~~max~~ ~~if~~ then it satisfies the relation $AA^* = I$ — (1)

$$AA^* = \begin{pmatrix} \alpha+i\gamma & -\beta+i\delta \\ \beta+i\delta & \alpha-i\gamma \end{pmatrix} \begin{pmatrix} \alpha-i\gamma & \beta-i\delta \\ -\beta-i\delta & \alpha+i\gamma \end{pmatrix}$$

$$= \begin{pmatrix} \alpha^2+\beta^2+\gamma^2+\delta^2 & 0 \\ 0 & \alpha^2+\beta^2+\gamma^2+\delta^2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad [\text{from } (1)]$$

$\therefore \alpha^2+\beta^2+\gamma^2+\delta^2 = 1.$

16. $M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix}$ where $a = e^{\frac{2i\pi}{3}}$

We have to show $M\bar{M} = 3I$

$$\bar{M} = \begin{pmatrix} 1 & \frac{1}{\bar{a}^2} & \frac{1}{\bar{a}} \\ 1 & \bar{a} & \bar{a}^2 \end{pmatrix}$$

$$M\bar{M} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a^2 & a \\ 1 & a & a^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \bar{a}^2 & \bar{a} \\ 1 & \bar{a} & \bar{a}^2 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 1+\bar{a}^2+\bar{a} & 1+\bar{a}+\bar{a}^2 \\ 1+a+a^2 & 1+a^2\bar{a}^2+a\bar{a} & 1+a^2\bar{a}+a\bar{a}^2 \\ 1+a+a^2 & 1+a\bar{a}^2+a^2\bar{a} & 1+a\bar{a}+a^2\bar{a}^2 \end{pmatrix}.$$

$$\bar{a} = e^{-\frac{2i\pi}{3}} \qquad \bar{a}^2 = e^{-\frac{4i\pi}{3}}$$

$a^3 = 1$ Also $\bar{a}^3 = 1$

~~so a is the~~ $\Rightarrow 1+\bar{a}+\bar{a}^2 = 0$

$\Rightarrow 1+a+a^2 = 0$

$$M\bar{M} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} = 3I$$

Hence, $\frac{\bar{M}}{3} = M^{-1}$ (proved).

9. (b)

$$A = \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix}$$

for A to be Hermitian —

$$A = \bar{A}^t$$

$$\Rightarrow \begin{bmatrix} 3 & x+2i & yi \\ 3-2i & 0 & 1+zi \\ yi & 1-xi & -1 \end{bmatrix} = \begin{bmatrix} 3 & 3+2i & -yi \\ x-2i & 0 & 1+xi \\ -yi & 1-zi & -1 \end{bmatrix}$$

equating the elements from both the matrix we get —

$$3+2i = x+2i, \quad -yi = yi,$$

$$1+xi = 1+zi$$

$$\text{or,} \quad x = 3, \quad y = 0, \quad z = x.$$

hence, Solⁿ is $x = 3, y = 0, z = 3$

12. (a) given, $A = \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix}$

The given matrices are similar if $\exists$ an invertible matrix P such that

$$A = P^{-1}BP \quad \text{or} \quad PA = BP$$

Let, $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ we shall determine a, b, c & d such that $PA = BP$ and then check whether P is non-singular.

$$PA = BP$$

$$\Rightarrow \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 5 & 5 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

or, $$\begin{bmatrix} 5a-2b & 5a \\ 5c-2d & 5c \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ -3a+4c & -3b+4d \end{bmatrix}$$

Equating the corresponding elements, we obtain the system of equations-

$$5a - 2b = a + 2c$$
$$5a = b + 2d$$
$$5c - 2d = -3a + 4c$$
$$5c = -3b + 4d$$

or, $$4a - 2b - 2c = 0$$
$$5a - b - 2d = 0$$
$$3a + c - 2d = 0$$
$$3b + 5c - 4d = 0$$

a Solution to this System of eq^{n}s is -

$a = 1,\ b = 1,\ c = 1,\ d = 2.$

$\therefore\ P = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$ which is a non Singular matrix.

hence, matrices A & B are similar.

b) $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$ & $B = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

Now for any non-singular 2×2 matrix P,

$P^{-1}AP = P^{-1}I_2P = I_2 = A.$

i.e. we can say that A being the matrix I_2, there is no matrix other than itself which is similar to it. Therefore B is not similar to A.