CS 60047 Autumn 2020

Advanced Graph Theory

Instructor

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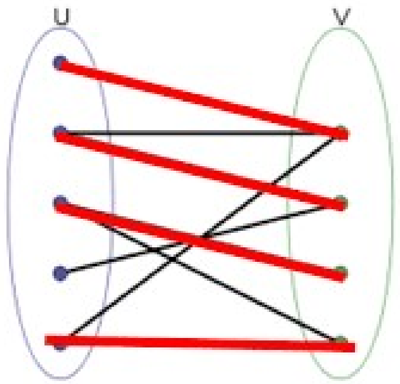
Lecture #02, #03, #04

04 September 2020: Lecture #02
09 September 2020: Lecture #03
11 September 2020: Lecture #04

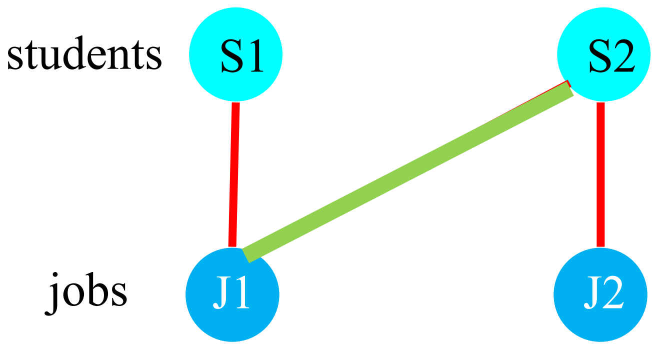
Indian Institute of Technology Kharagpur
Computer Science and Engineering

Graph Theory in Economics and Social Sciences

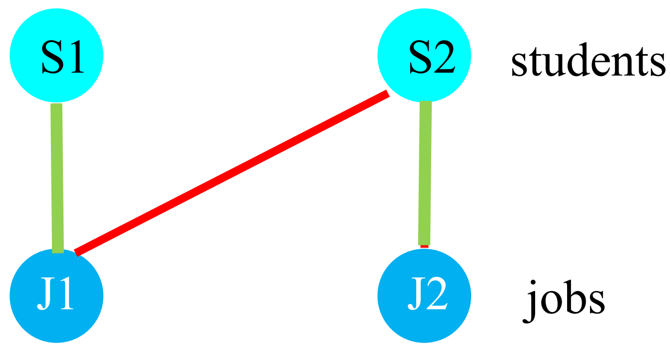


Matching theory: *Gale-Shapley* algorithm

Nobel Prize in Economics 2012



minimum matching:
only one student got placed



maximum matching:
both got placed

Solving a 500-year old problem of missing king

Courtesy: https://towardsdatascience.com/the-king-that-graph-theory-discovered-8cce31a3cd26

In August 2012, a skeleton was found while digging into a car park in Leicester. It was believed that this was Richard III — the crooked-backed English king who was killed in the Battle of Bosworth in 1485, and dumped there.

Only mitochondrial DNA (*mtDNA*) that is passed down through *maternal lines* in families, remains unchanged from generation to generation. Looking into the ancestry **graph database**, a search is made to identify a living descendent of a sister of Richard-III through a 500-year old family tree of females, and they found Michael Ibsen, who is currently alive. Michael's *mtDNA* was compared to that extracted from the skeleton, and it was a perfect match. In 2015, the exhumed body of King Richard-III was cremated in full honor.

Graph theory in forensics

Networks and Structures as Graphs

Vehicle/Distribution:

Motorways, rail, electricity grid, water, nerves, electrical circuits
Mobile network, Internet

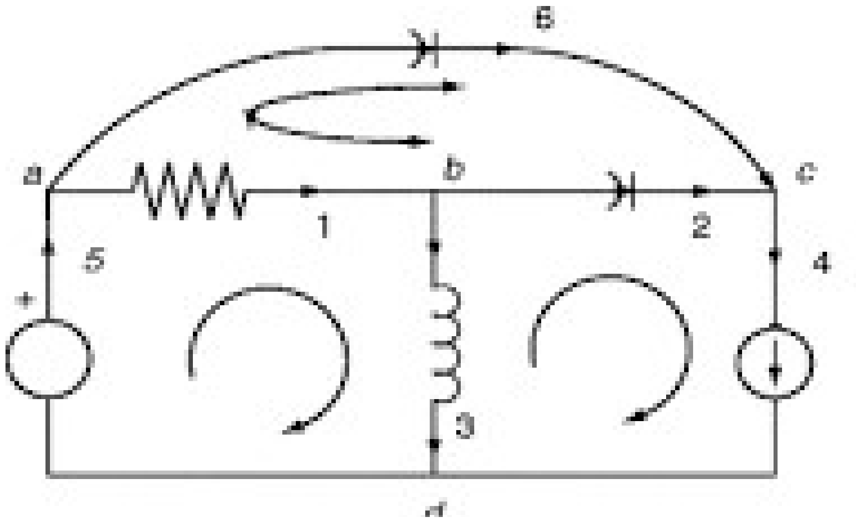
Socio-economic:

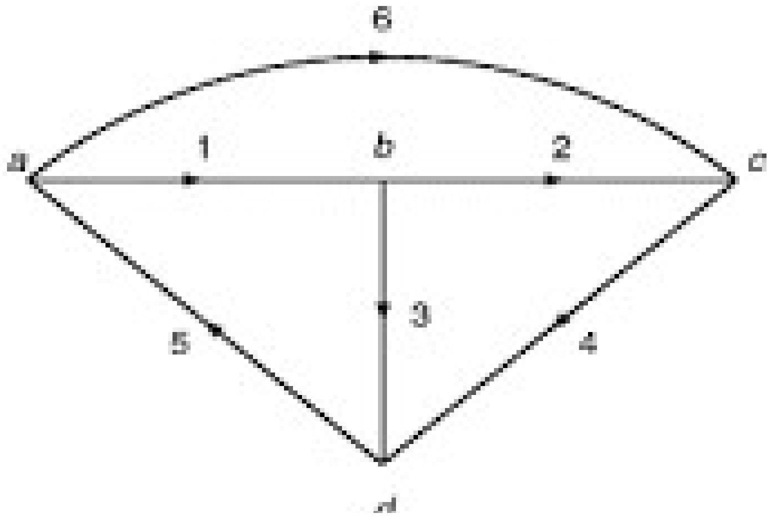
Trade, politics, friendship, Facebook, citation network, co-authorship network, Erdos distance

Spatial-temporal:

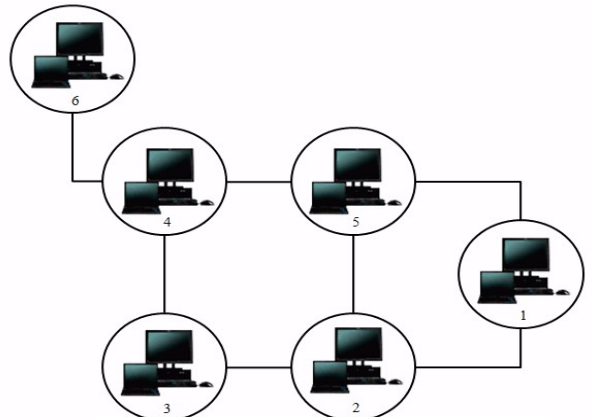
Timetables, maps, chemical structures

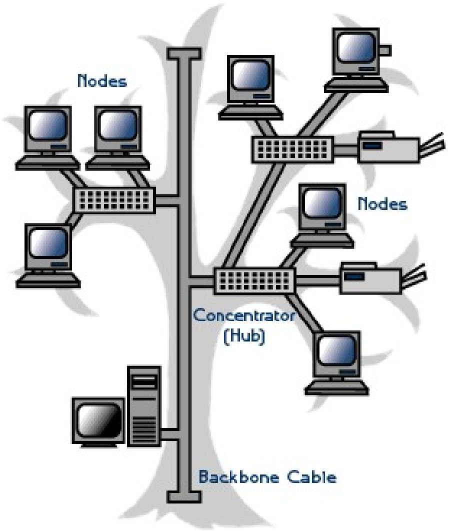
Electrical Circuits

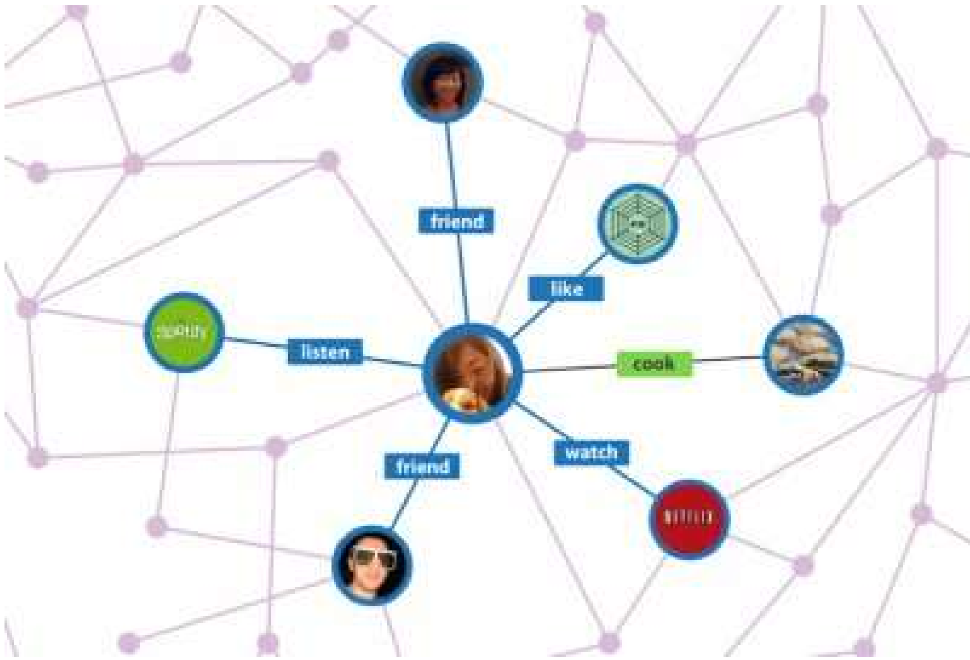




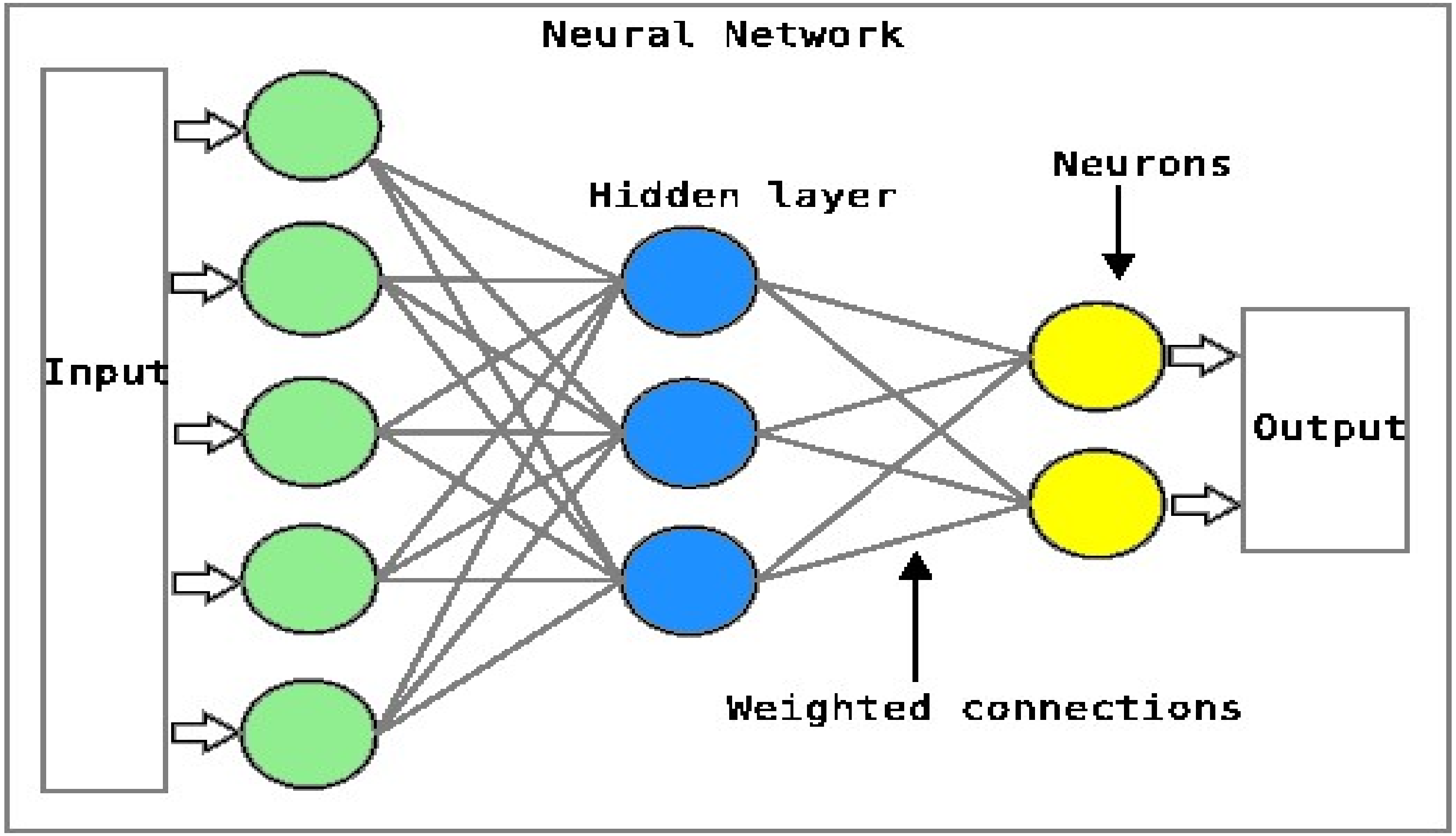
Computer Networks, IoT



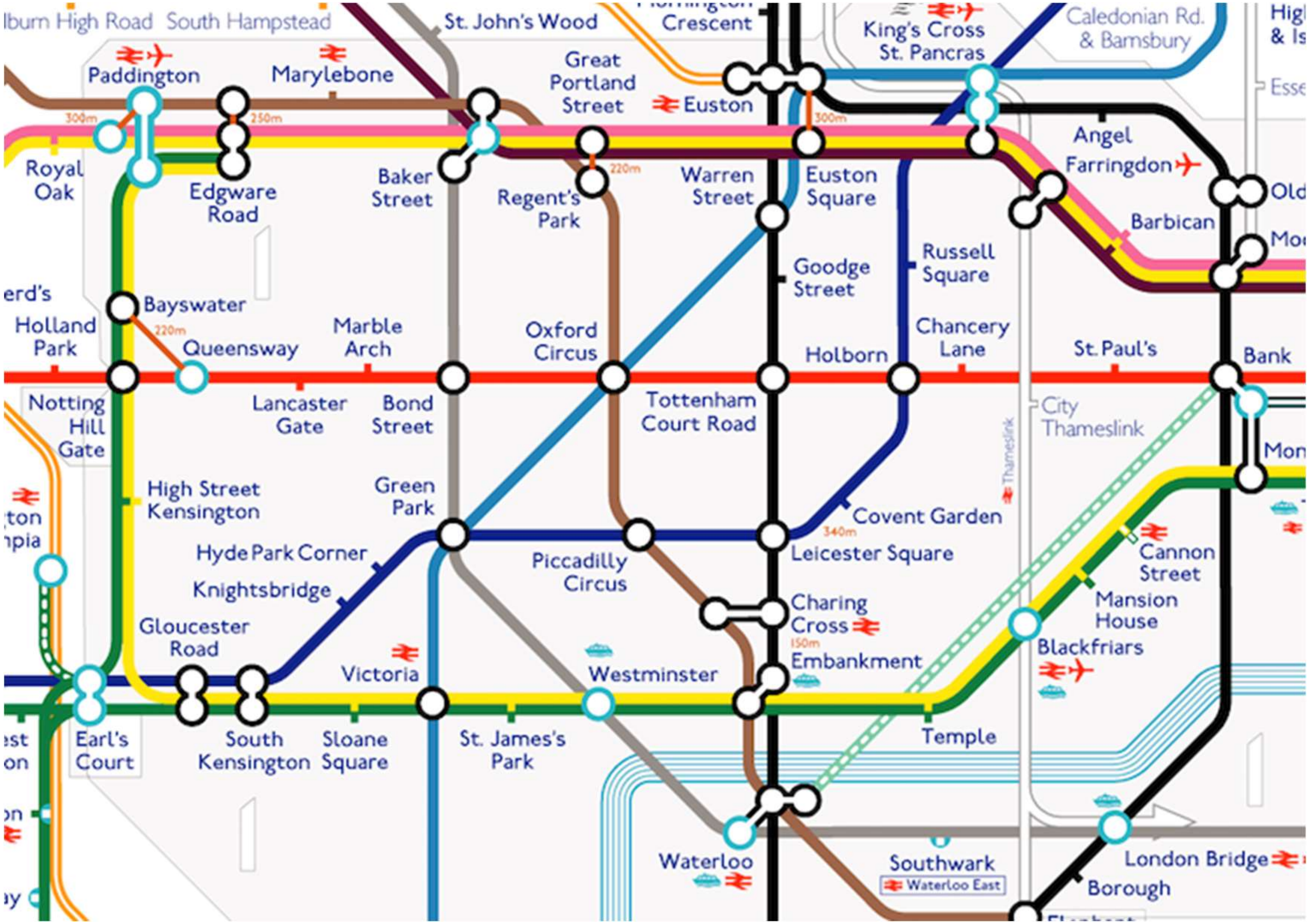




Neural Network



Tube Railway Map

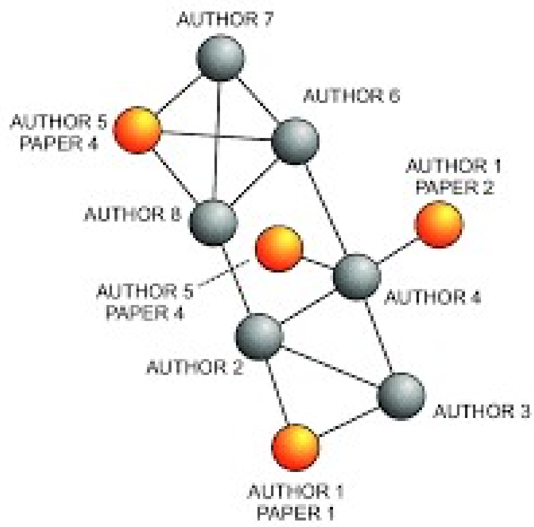


Social Networks as Graphs

Facebook, Twitter, Citation Networks, etc …

Example: Collaboration (co-authorship) networks

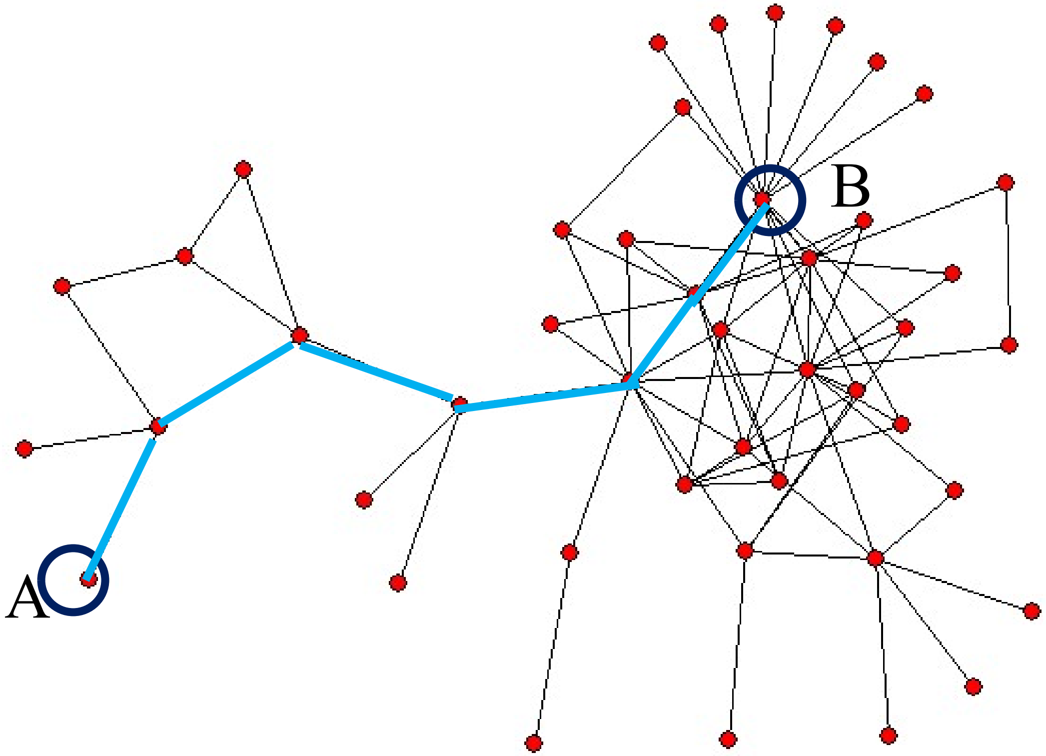
The *Author Collaboration Network* (ACN) is an undirected graph that depicts collaboration between an author and other authors in a scientific dataset. Two authors are said to collaborate when they are both listed as authors in the same *Web of Science* document.



Each author → a *node* in a graph; put an *edge* between two nodes *a* and *b*, if there is a published scientific paper with both *a* and *b* as co-authors.

Collaboration Distance between two authors *x* and *y*? The length of the Shortest Path between *x* and *y* in ACN.

Collaboration Network (ACN)



Collaboration Distance (A, B) = length of the **shortest path** between Author-A and Author-B in the ACN-graph = 6

A general question in network design: Given constraints such that degree(v) $\le \Delta$ for all $v \in V$, and diameter $\le \psi$, design $G(V, E)$ that maximizes $|V|$.

Example: Collaboration Distance

https://www.csauthors.net/

Find the *shortest path* between Bhargab B. Bhattacharya and **Edsger W. Dijkstra**

Bhargab B. Bhattacharya
co-authored 15 papers with
Ansuman Banerjee
co-authored 1 paper with
Huibiao Zhu
co-authored 2 papers with
C. Antony R. Hoare
co-authored 1 paper with
Edsger W. Dijkstra
distance = 4

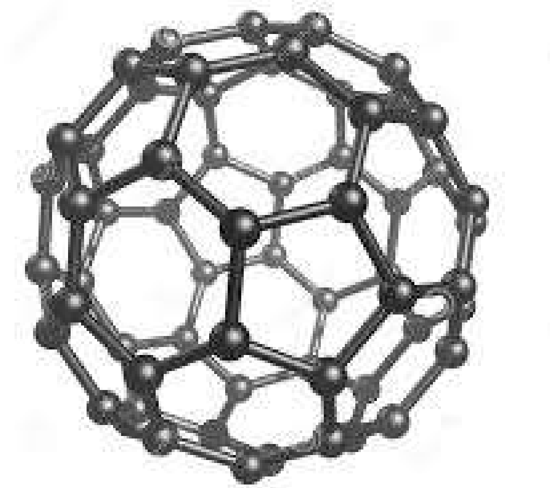
Find the *shortest path* between Bhargab B. Bhattacharya and **Paul Erdős**

Bhargab B. Bhattacharya
co-authored 2 papers with
Pradip K. Srimani
co-authored 13 papers with
Stephen T. Hedetniemi
co-authored 2 papers with
Paul Erdős
distance = 3

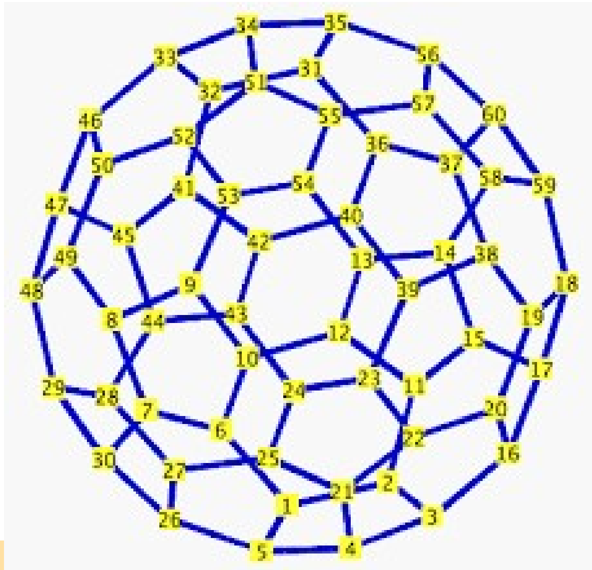
Conjecture: In the ACN graph, *Collaboration Distance* between any author listed in the Web of Science and **Paul Erdős** ≤ 6

Soccer Ball and C_{60} as Graph

Soccer Ball



Buckminsterfullerene (carbon allotrope) C_{60} diameter 0.7 *nm;* 60 single bonds, 30 double bonds; 1996 Nobel Prize

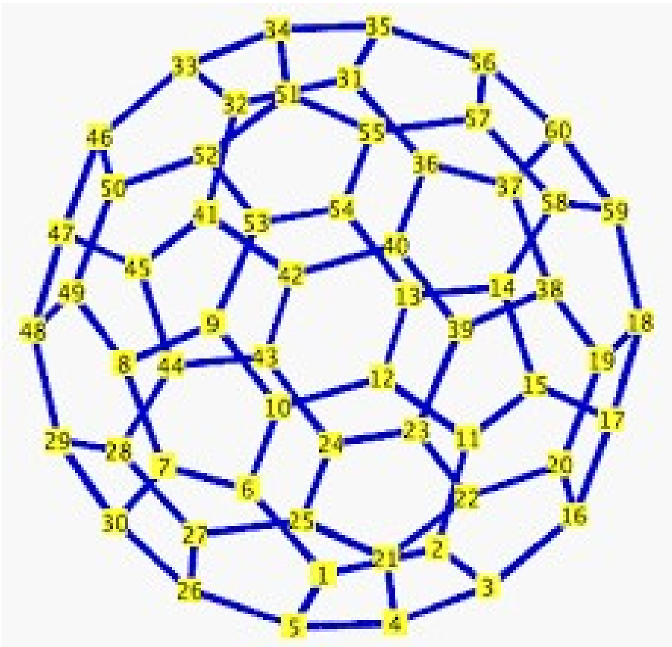


graph representation

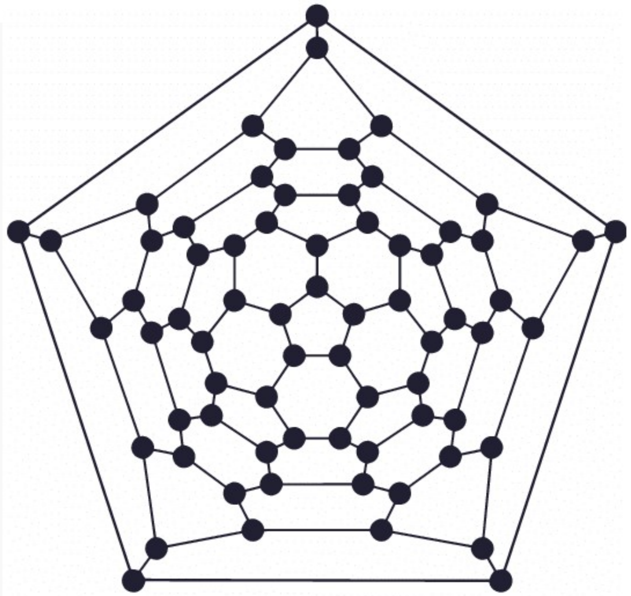
sixty nodes, *twenty* hexagons, and *twelve* pentagons; degree of each node = 3; Why these numbers?

Soccer Ball as a Planar Graph

Soccer Ball



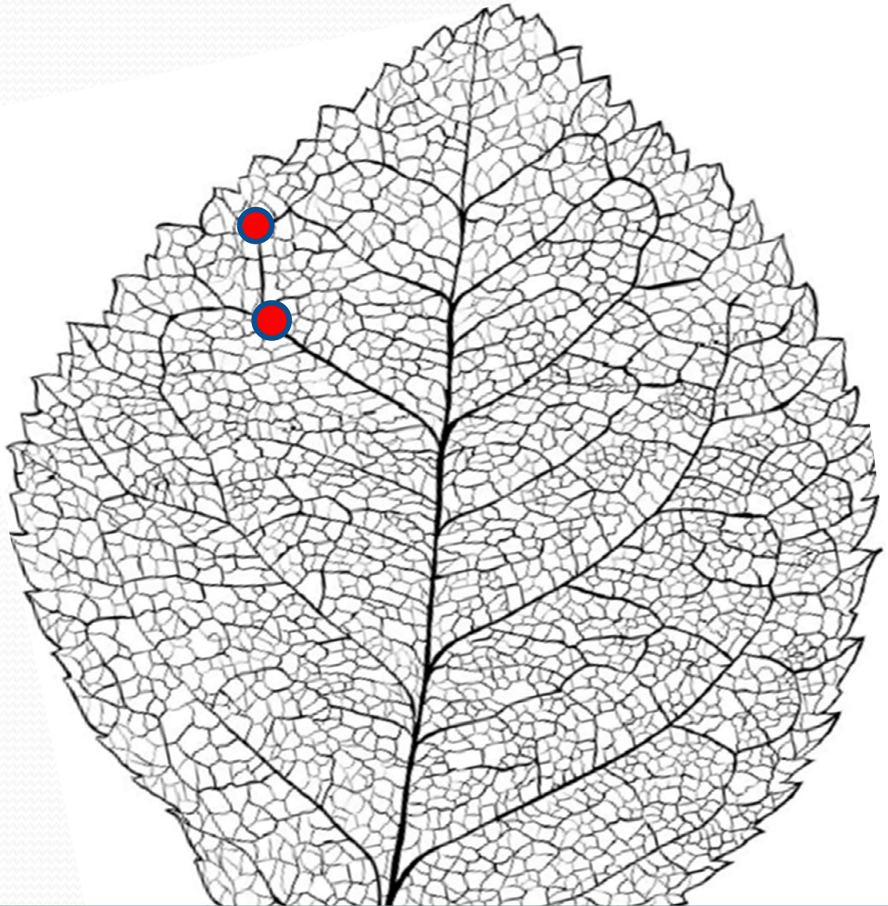
Graph representation



Drawing on a plane

Sixty nodes, twenty hexagons, and twelve pentagons; the graph can be drawn on a plane without any crossover

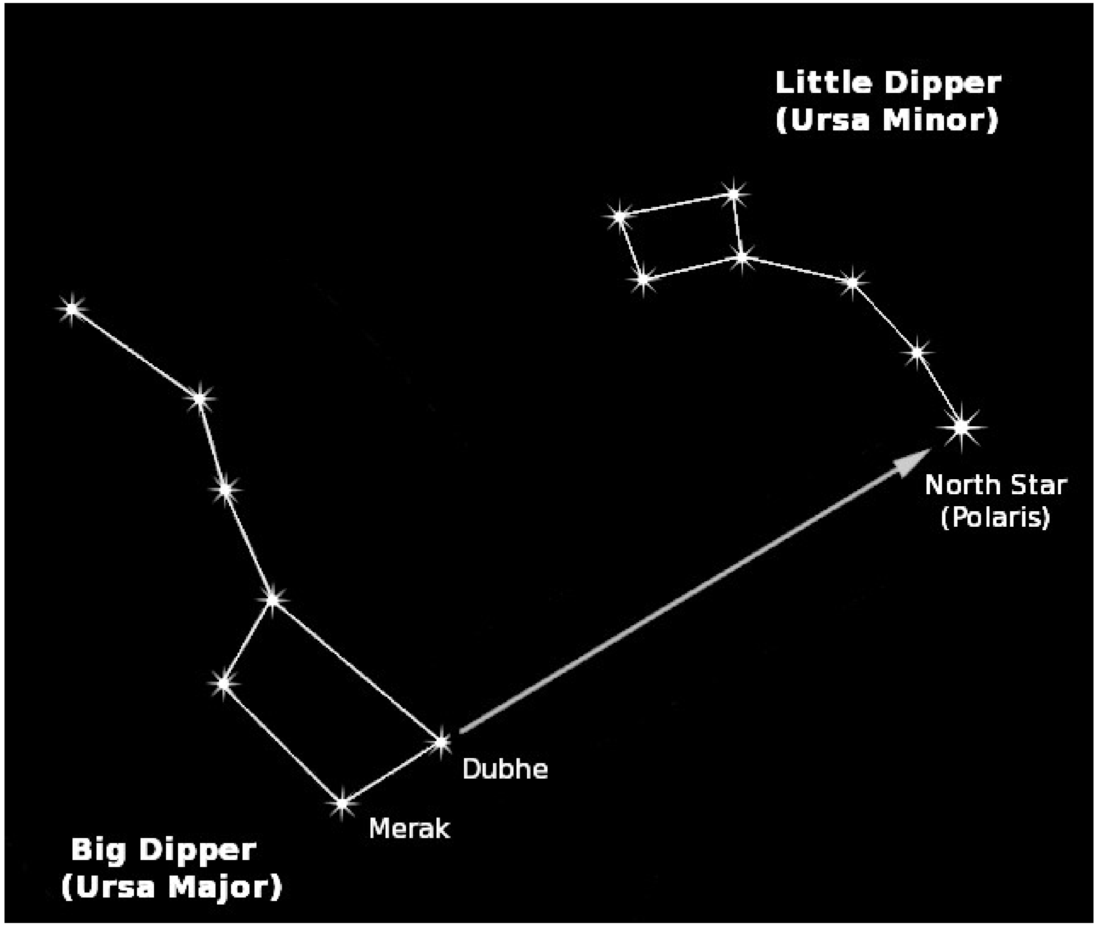
Leaf-Vein Network



What kind of graph is this? Consider junction points as nodes;
put an edge between two adjacent junctions - a planar graph

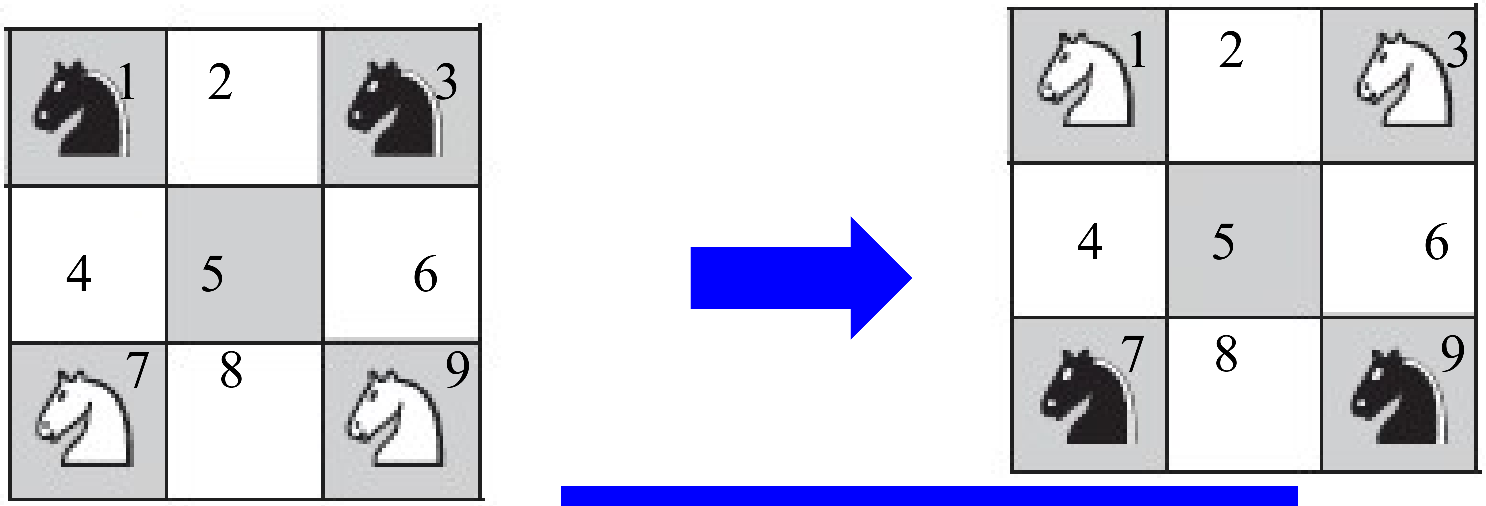
Computational botany

Sky Map

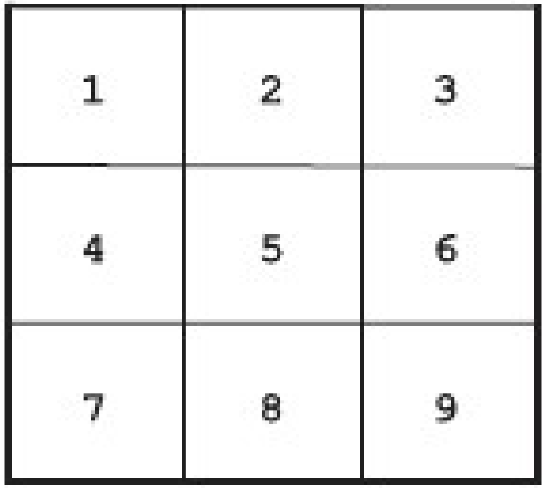


Little Dipper
(Ursa Minor)
North Star
(Polaris)
Dubhe
Merak
Big Dipper
(Ursa Major)

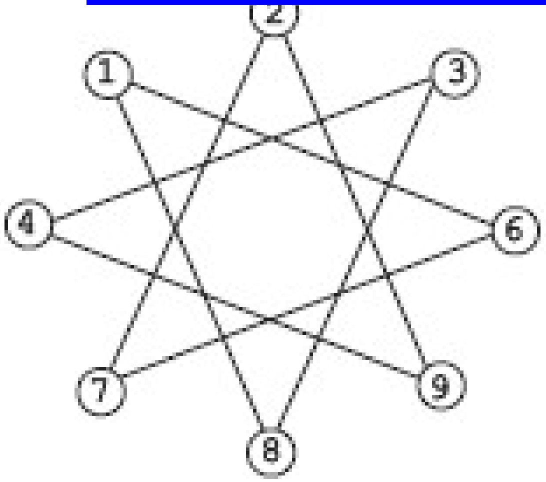
Guarini's Puzzle – Swapping of Knights



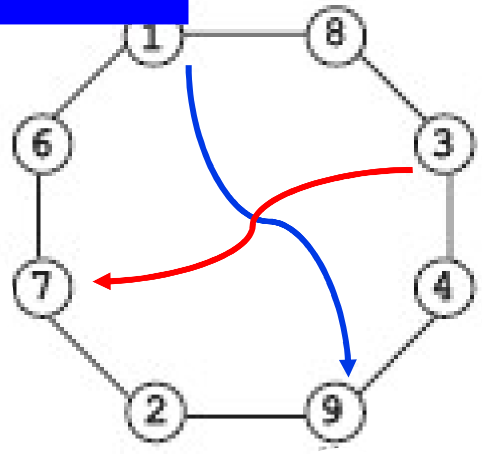
Each knight requires four moves; total 16 moves



Labeled board



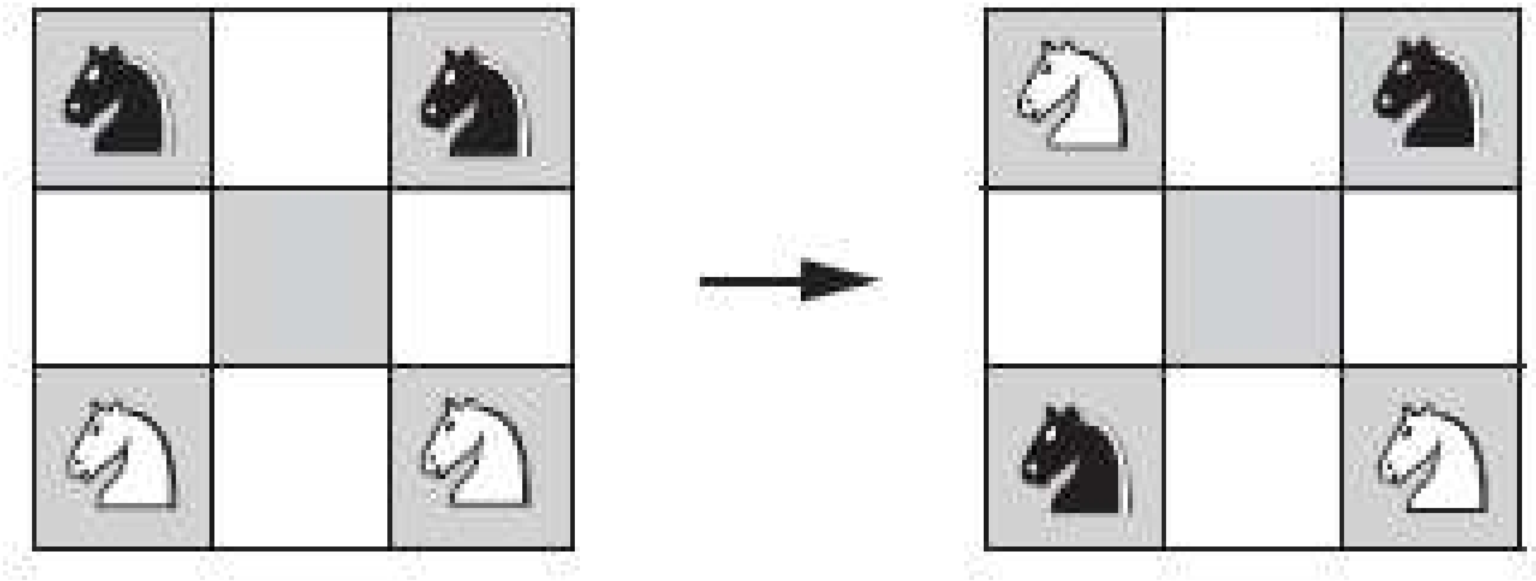
Possible moves of a knight



Redrawn graph

Guarini’s Puzzle - Four Alternating Knights

Initial Final

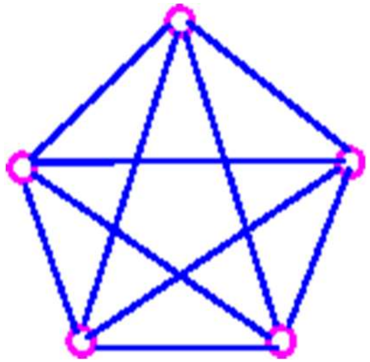


Valid chess moves ..

No cell should occupy more than one knight at a time ….

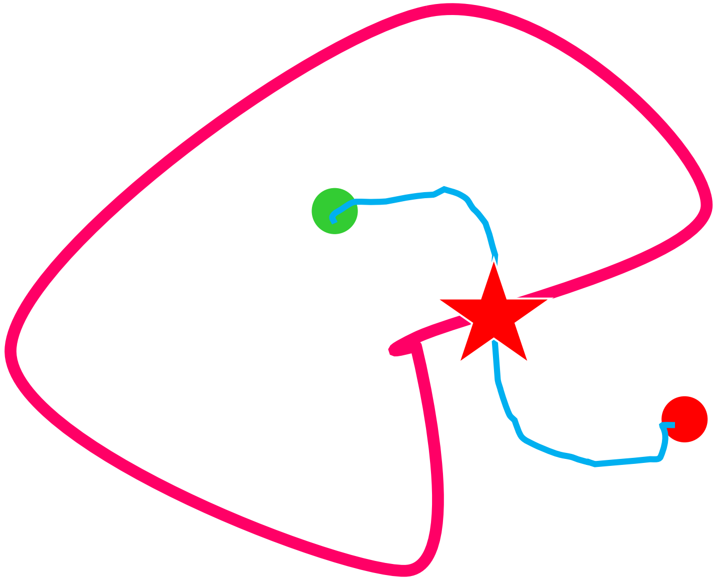
Your thoughts on it …..

The Power of Abstraction



K_5

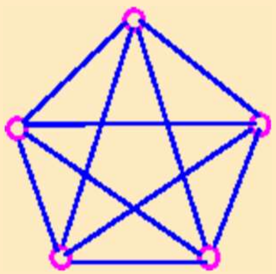
Show that K_5 is non-planar



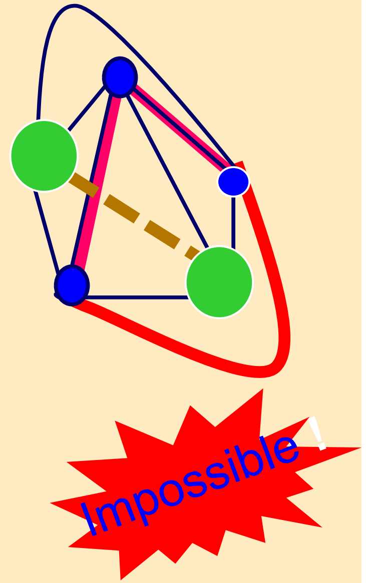
The Ramayana: Sita and Laxman-Rekha

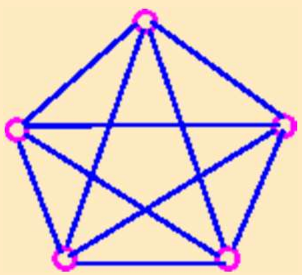
Jordan curve

Abstraction: Planarity, Genus, and Thickness of a Graph

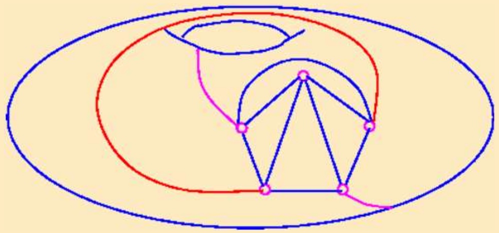


K_5: A complete graph with 5 vertices
Non-planar: cannot be drawn on a plane (genus zero) without any crossover

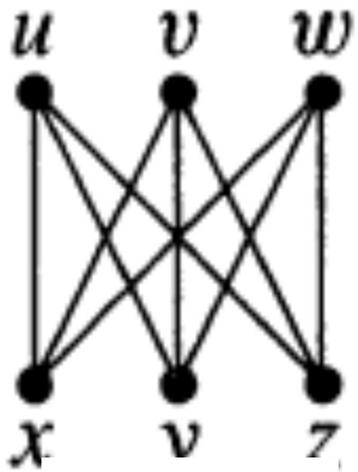




Can be embedded on a torus without any crossover



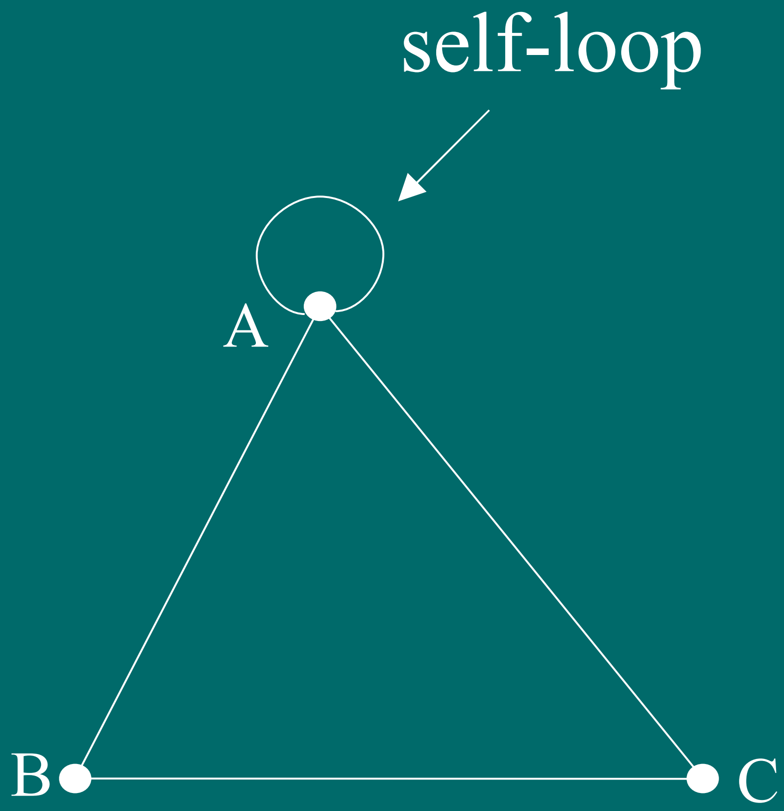
Show using Jordan curve argument that $K_{3,3}$ is non-planar

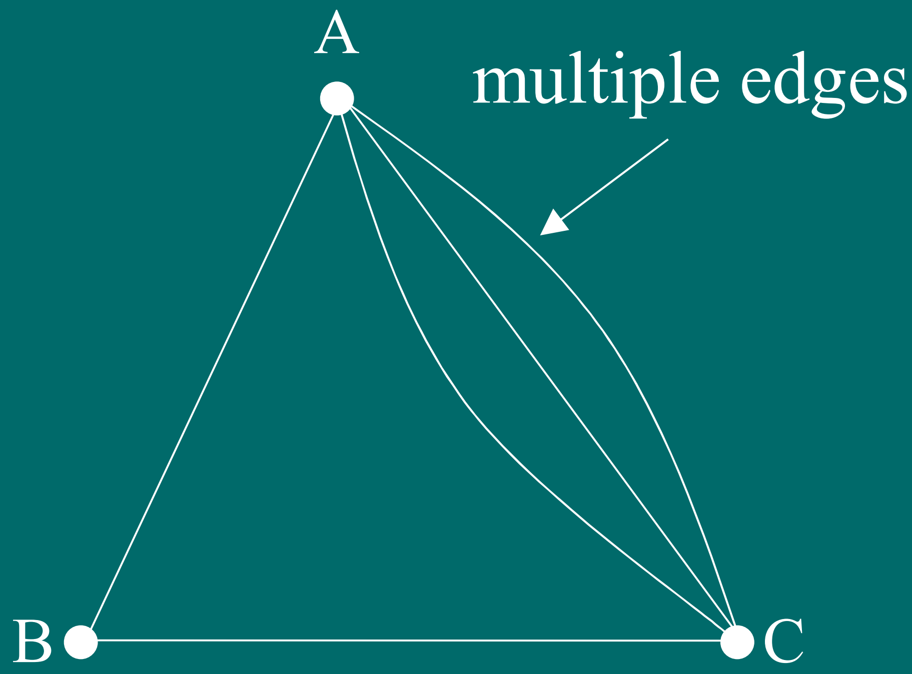


$K_{3,3}$

Graph Theory Basics

Non-Simple graphs (multi-graph)





Simple graphs: no self-loops, no multi-edges

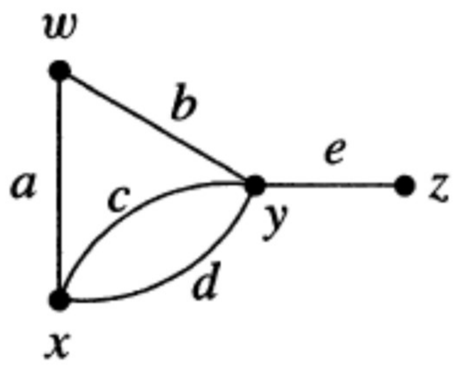
Matrix Representation of a Graph $G(V, E)$

$$\begin{array}{c|cccc} & w & x & y & z \\ \hline w & & & & \\ x & & & & \\ y & & & & \\ z & & & & \end{array}$$

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}
\begin{array}{c} \begin{array}{cccc} w & x & y & z \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 2 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

adjacency matrix

$(n \times n)$



G

$$\begin{array}{c} \\ w \\ x \\ y \\ z \end{array}
\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

incidence matrix

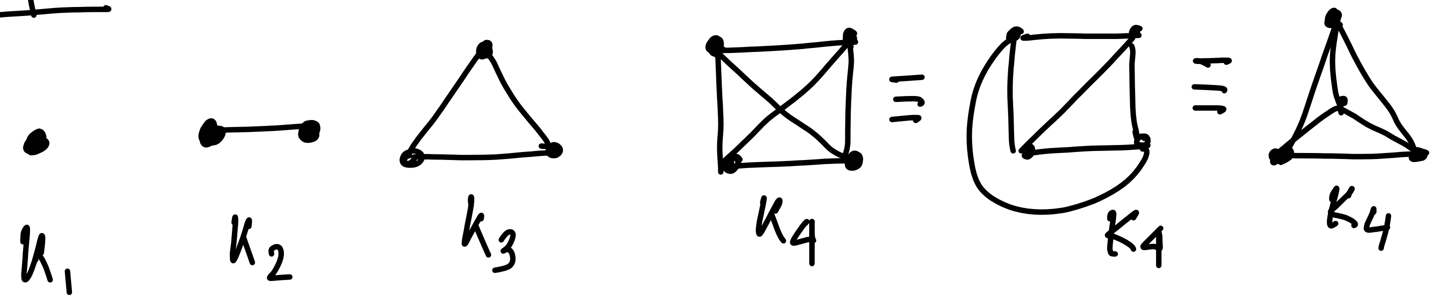
$\sim (n \times n^2)$

$|V| = n$

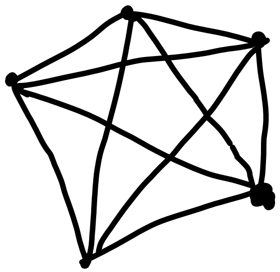
Basic concepts

1. Complete graph of n vertices (K_n)
   (every pair is adjacent)

Example

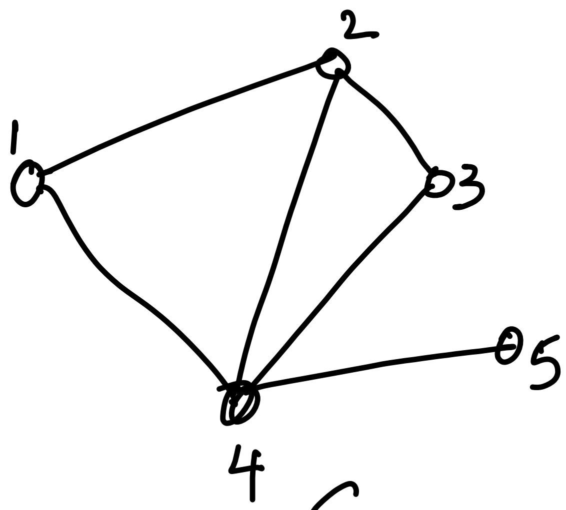


K_5

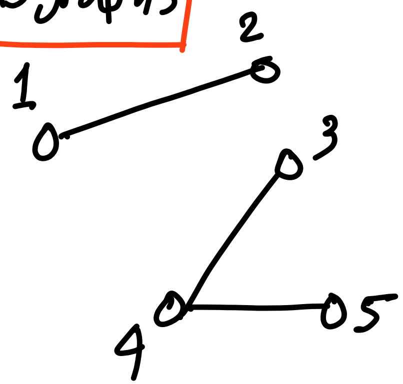


K stands for Kuratowski

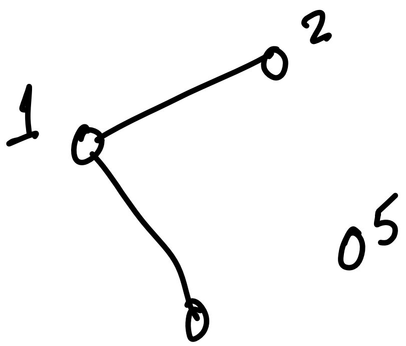
Subgraphs and induced subgraphs



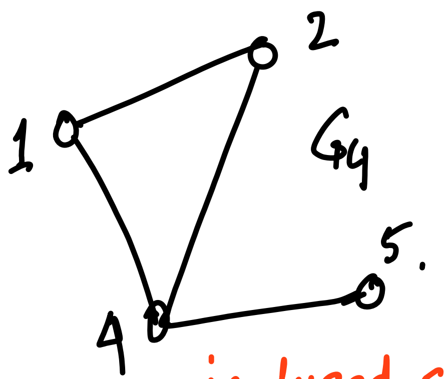
G



subgraph G_1 of G



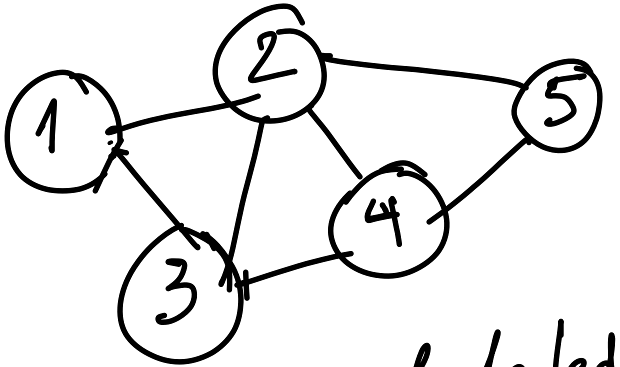
subgraph G_3



induced subgraph of G

G_1, G_2, G_3 not induced subgraph

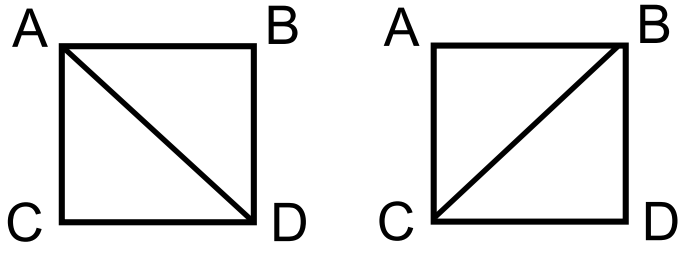
Labeled and unlabeled graph



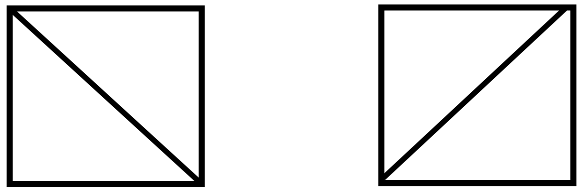
vertices are labeled
(labeled graph)

No. of distinct labeled graphs with n vertices $= 2^{\binom{n}{2}}$

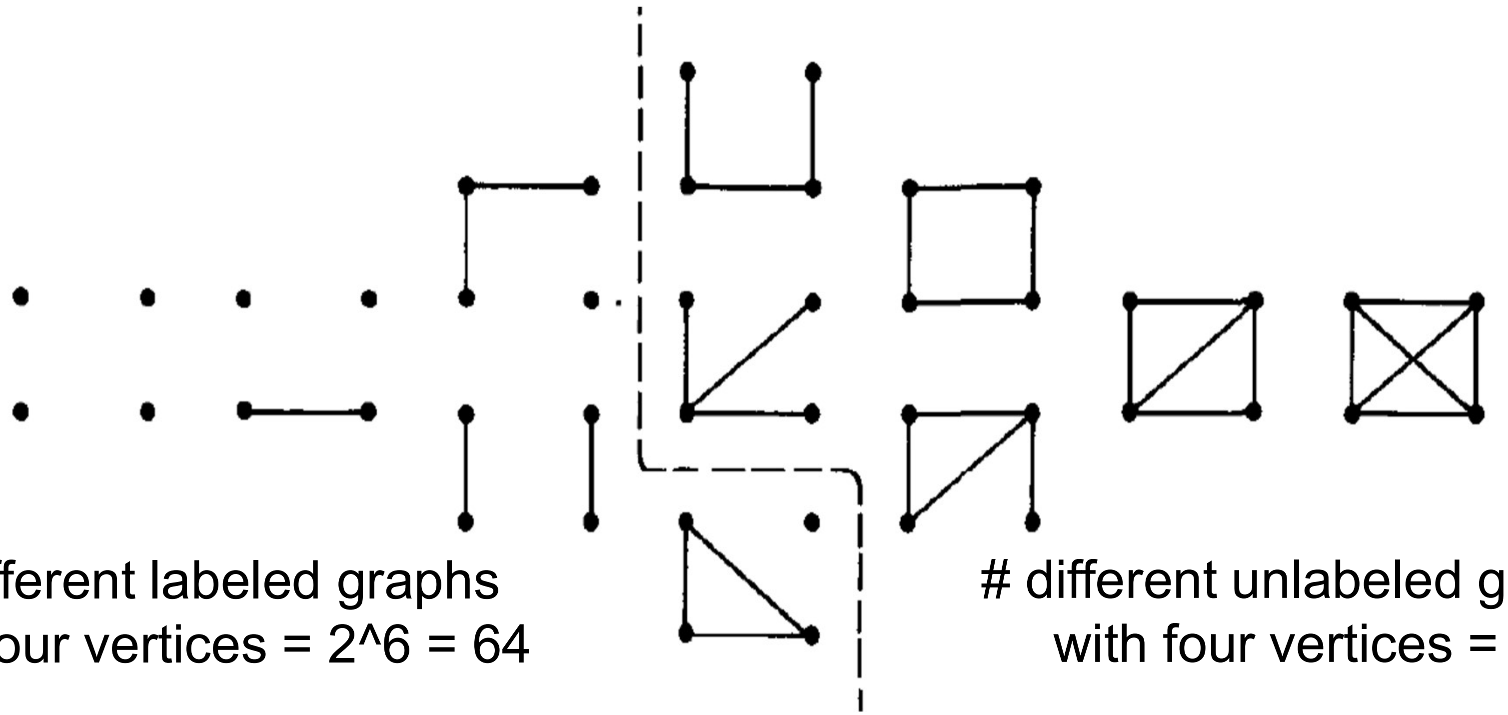
Counting labeled and unlabeled graphs



labeled graph:
counted twice



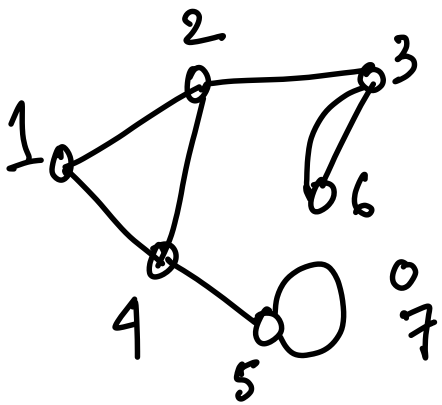
unlabeled graph:
counted once



different labeled graphs
with four vertices = 2^6 = 64

different unlabeled graphs
with four vertices = 11

Degree of a vertex (node)



Vertex	1	2	3	4	5	6	7
degree (δ)	2	3	3	3	3	2	0

Party theorem: In a gathering of n people, you will always find at least two persons who have an equal number of friends present in the same party.

In terms of graph theory, it means:

First theorem in graph theory

In any simple graph $G(V, E)$, $\exists$ at least two vertices with the same degree.

Proof: Let $|V| = n$

Case 1: No isolated vertices $\Rightarrow$ no vertex with degree 0.

What are the possible values of degrees?

δ's $\rightarrow$ $1, 2, 3, \ldots, n-1$ (because, G is simple, and a vertex can be adjacent to at most $(n-1)$ other vertices)

Number of nodes $\rightarrow$ n

$\Rightarrow$ Now the proof follows from pigeon-hole principle.

Other cases: Follows easily. $\boxtimes$

Handshaking Lemma

In a graph $G(V,E)$ with n vertices and m edges, i.e., $|V| = n$, $|E| = m$,

$$\delta(v_1) + \delta(v_2) + \cdots + \delta(v_n) = 2m$$

i.e.,

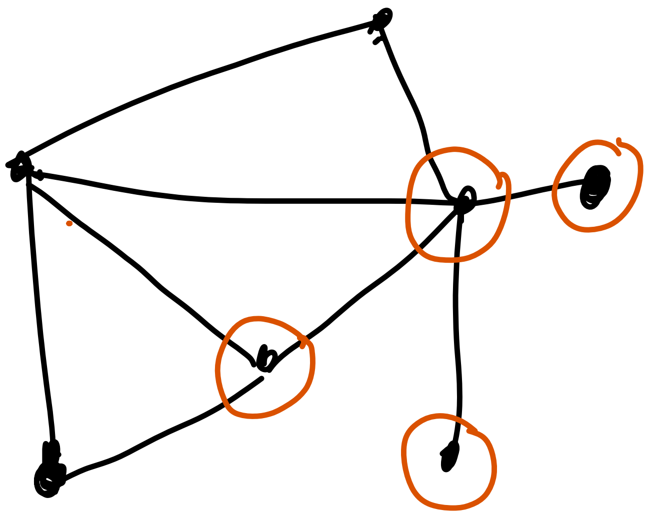
$$\boxed{\sum_{i=1}^{n} \delta(v_i) = 2m} \Rightarrow$$

degree sum is always even

$\Rightarrow$ Number of hands involved in a handshaking party is always <u>even</u>

Corollary

In any graph G, the number of odd-degree vertices must be even.

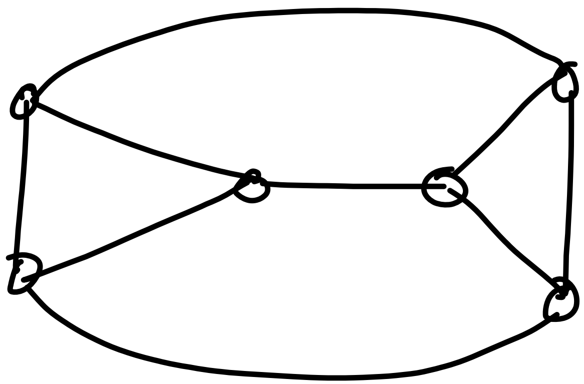


O ⇒ odd-degree vertices

4 here

Regular graphs

A graph is regular, if $\forall i$, $\delta(v_i)$ = same.

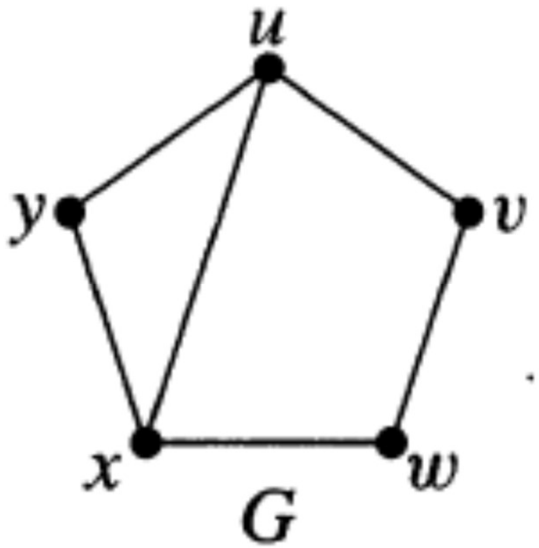


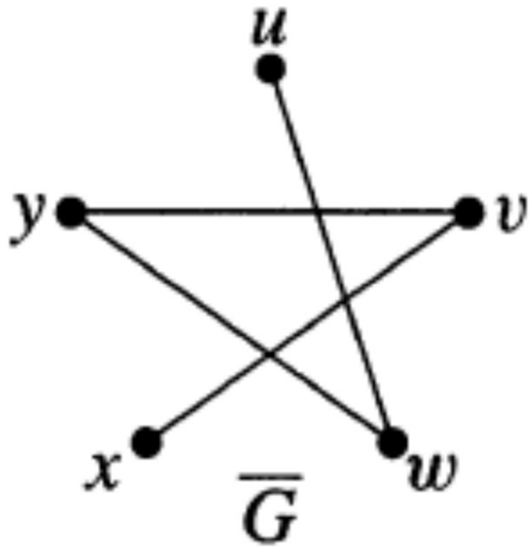
← degree of every vertex $= 3$.

cubic graph ↑

Corollary: The number of vertices in a cubic graph must always be even.

Complementary Graphs



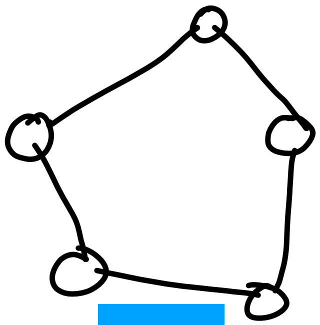


C_5 is self-complementary

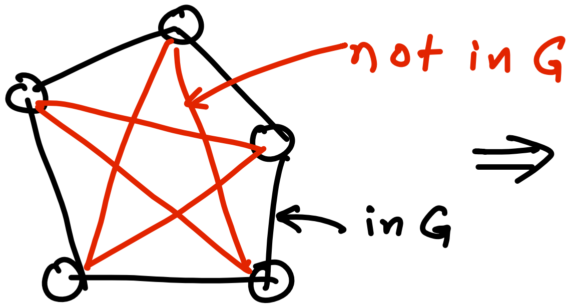
Complementary graph

In a simple graph G, the complement of G, denoted as $\bar{G}$ is defined as follows:

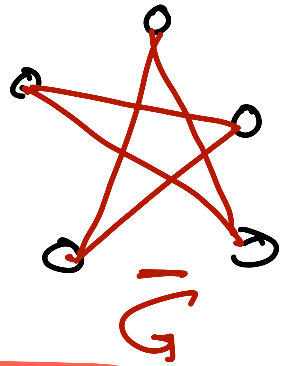
$$e_{ij} \in E(G) \iff e_{ij} \notin E(\bar{G})$$



C_5



$\Rightarrow$

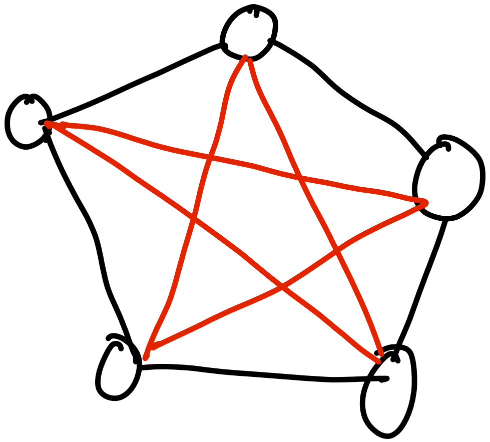


$$\boxed{\underline{\text{Claim}} \quad G \cup \bar{G} = K_n}$$

$$\boxed{|E(G)| + |E(\bar{G})| = \binom{n}{2}}$$

Claim

An n-vertex graph H is self-complementary if and only if K_n decomposes into two copies of H.

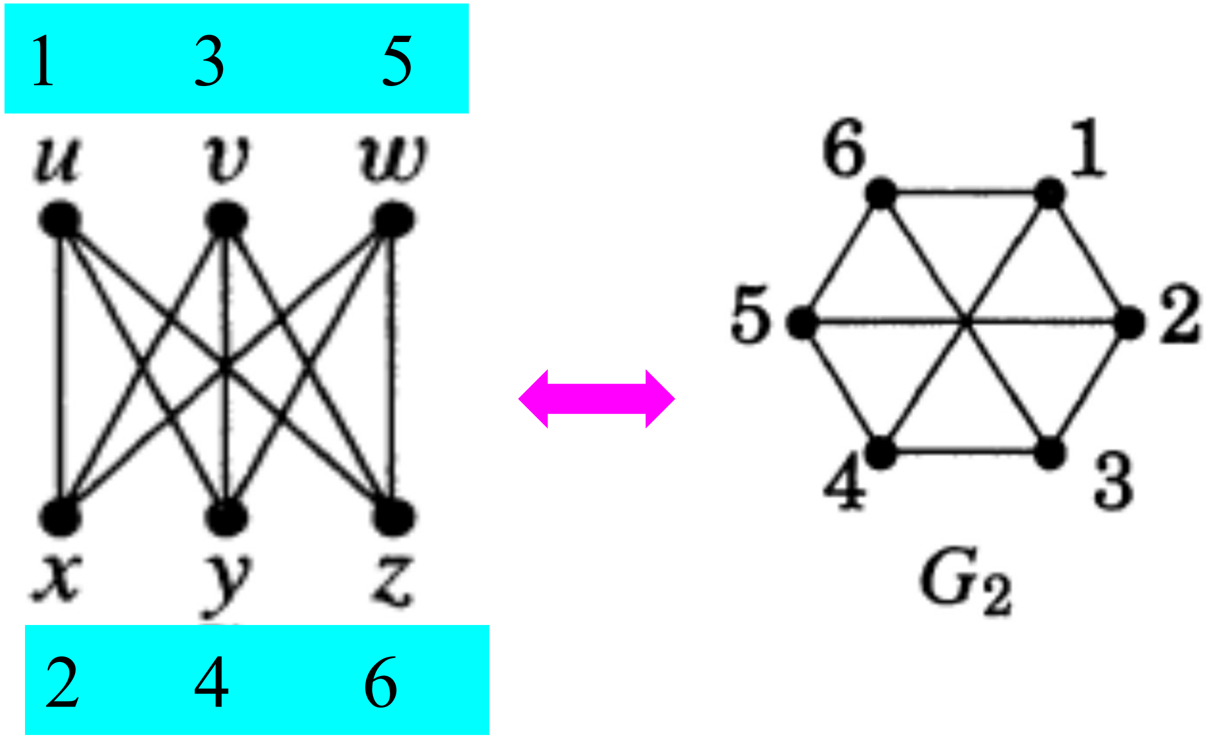


K_5

K_5 decomposes into two copies of 5-cycle (C_5).

Hence, C_5 is self-complementary.

Graph Isomorphism



permutation:

u	v	w	x	y	z
1	3	5	2	4	6

CS 60047 Autumn 2020

Advanced Graph Theory

Instructor

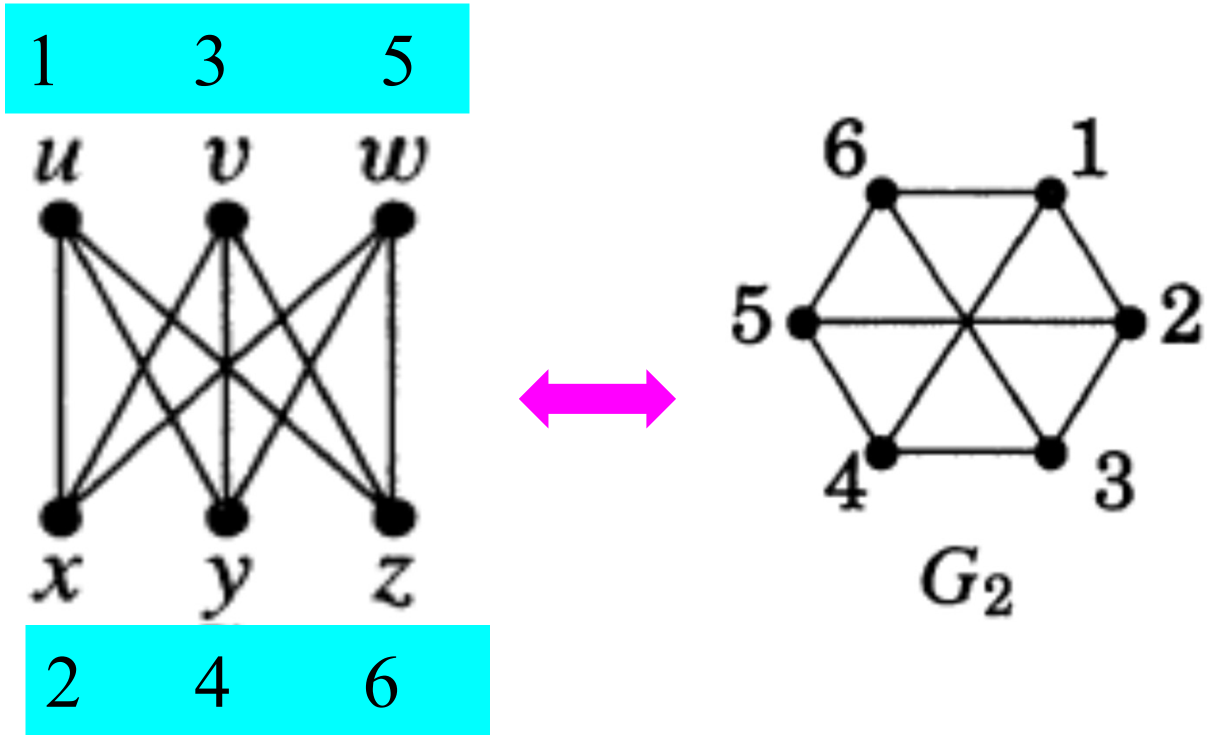
Bhargab B. Bhattacharya

Lecture #03: 09 Sept. 2020

Indian Institute of Technology Kharagpur

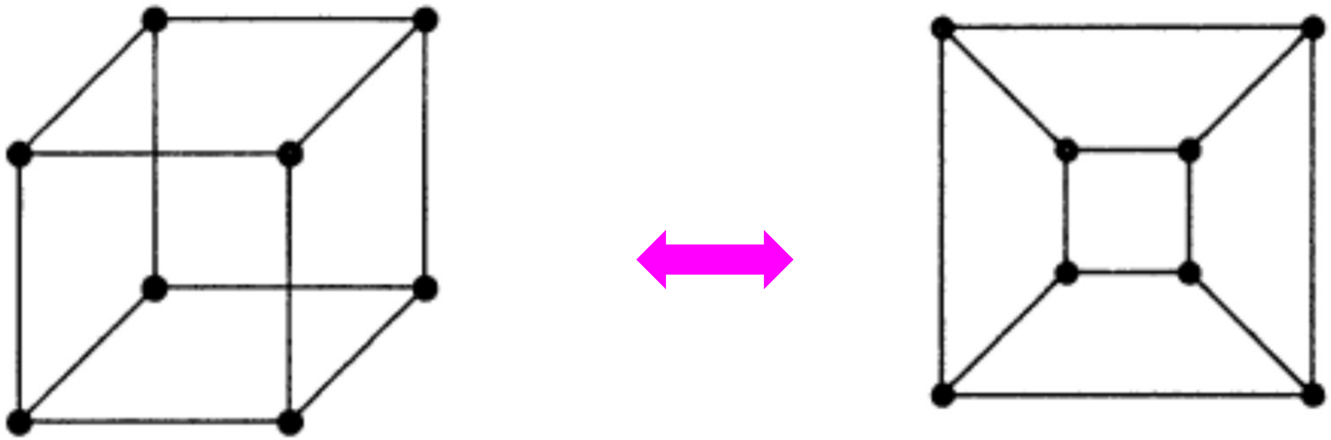
Computer Science and Engineering

Graph Isomorphism



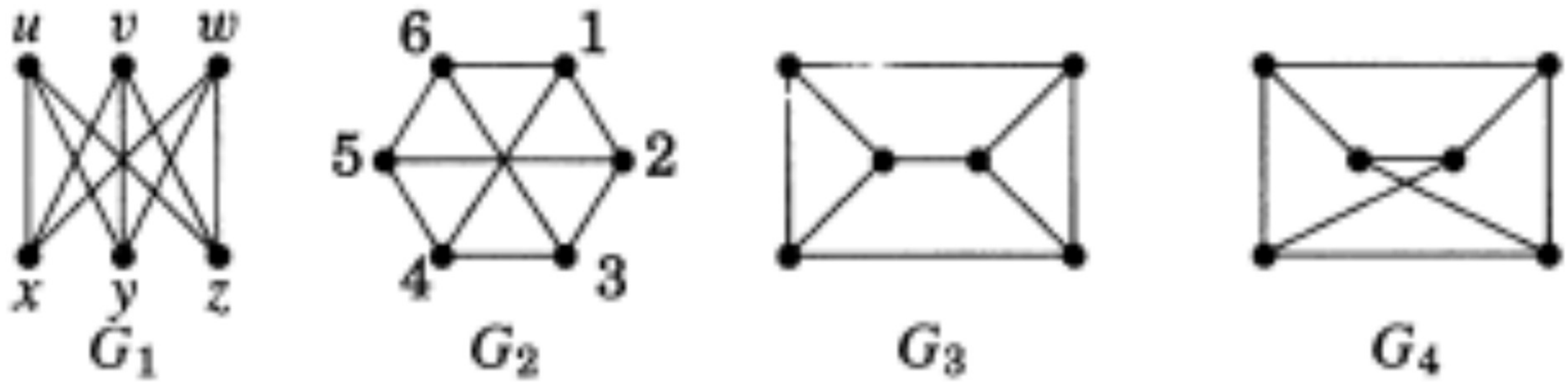
	u	v	w	x	y	z
permutation:	1	3	5	2	4	6

Graph Isomorphism



knowlegde of isomorphism is required
while counting different unlabeled graphs

Which graphs are isomorphic?



G1, G2, G4 are isomorphic; G3 is not

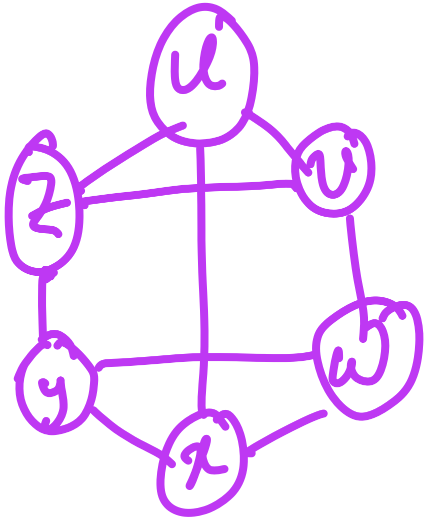
Observation

If $G(V_1, E_1)$ and $H(V_2, E_2)$ are isomorphic, then

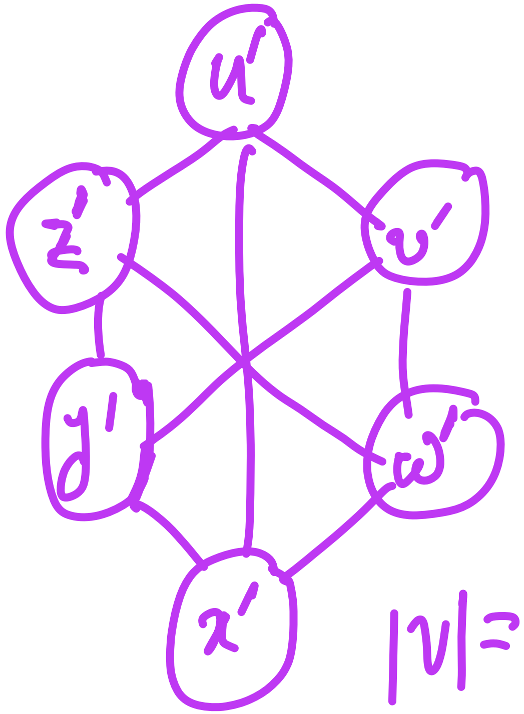
1. $|V_1| = |V_2| = n$ (say)
2. $|E_1| = |E_2|$
3. degree-vector should be the same

$$(d_1, d_2, \ldots, d_n)\big|_G \Longleftrightarrow (d_1', d_2', \ldots d_n')\big|_H$$

However, the converse is not true



$\not\Leftrightarrow$

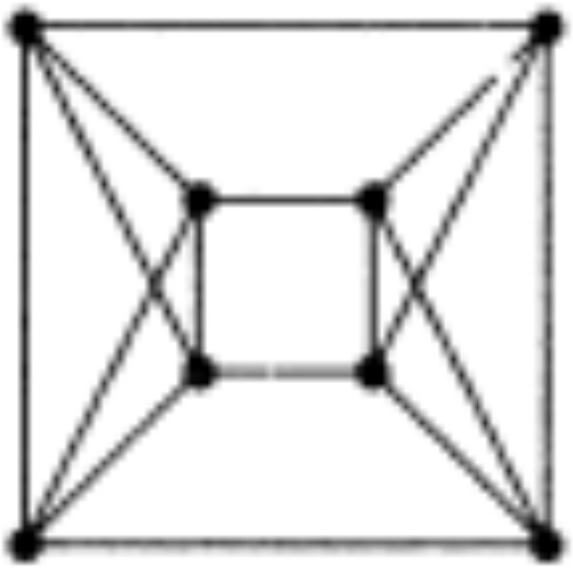


Two 3-regular non-isomorphic graphs

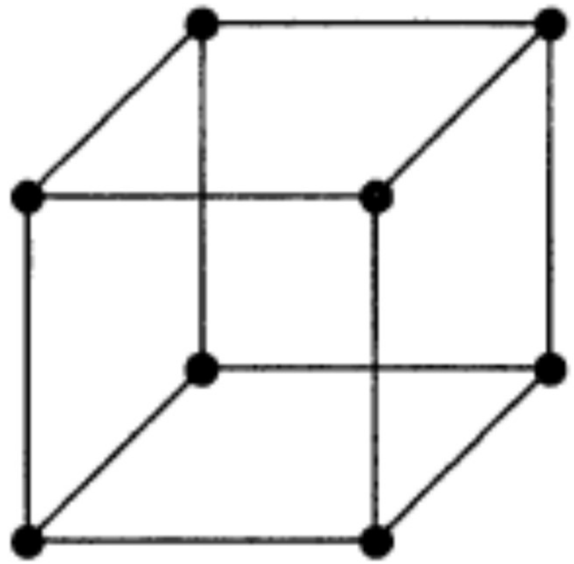
$|V|=6, \ |E|=9$ $\qquad$ $|V|=6, \ |E|=9$

$\{3,3,3,3,3,3\}$ ← degree → $\{3,3,3,3,3,3\}$

Homework



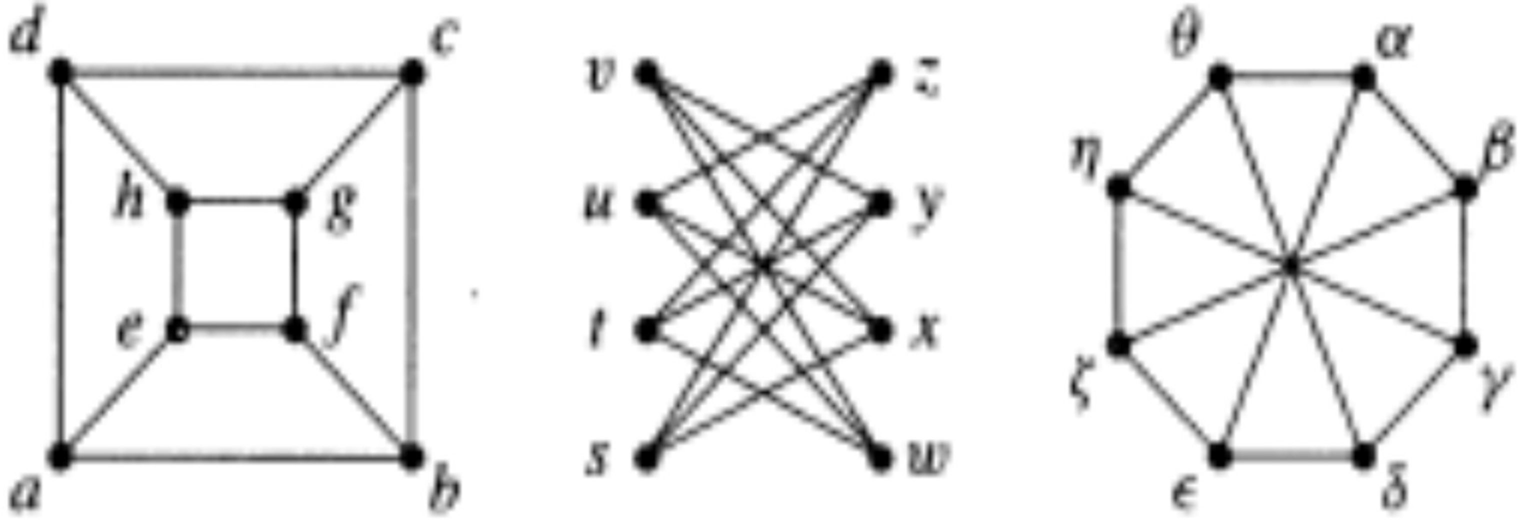
G



Q_3

Is there any connection between graph G and Hypercube Q_3?

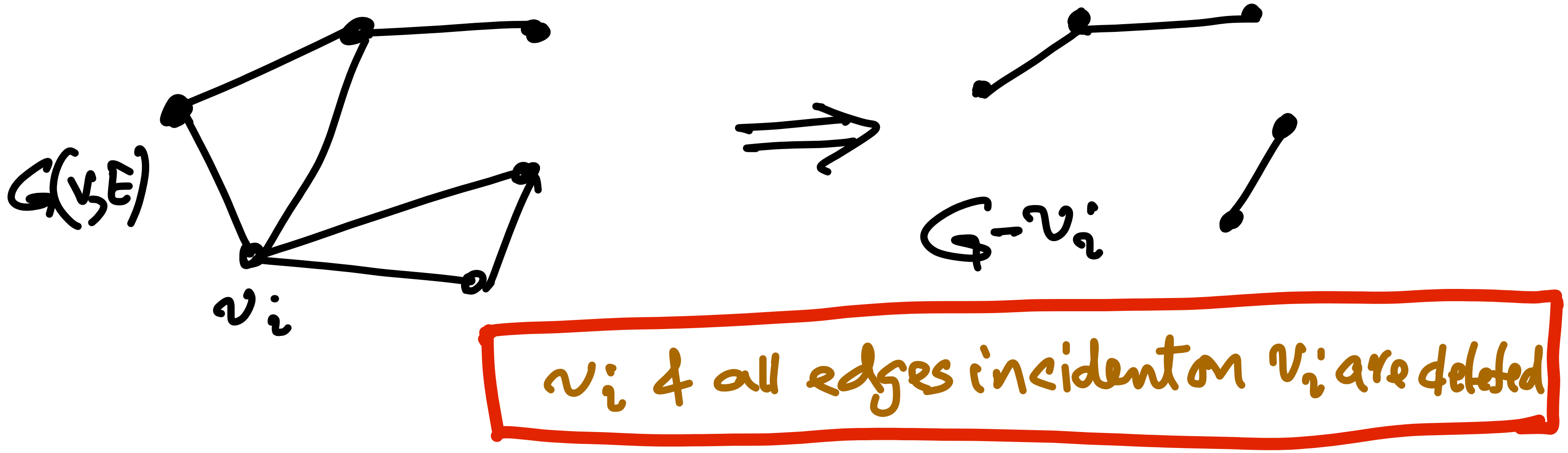
Homework
D. West (Textbook), Problem 1.1.8



Which pairs of graphs shown above are isomorphic?

Isomorphism in terms of vertex-deleted subgraphs

Subgraphs obtained by deleting a single-vertex (say v_i) from a graph $G(V, E)$, denoted as G - v_i.



Lemma (DW: Textbook, 1.3.11)

For a simple graph G with vertices $v_1, v_2, \cdots v_n$, $n \geq 3$,

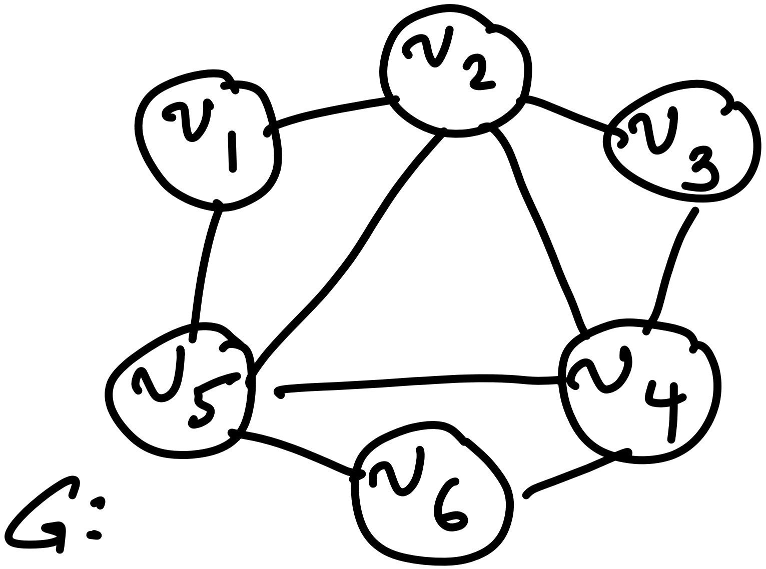
$$e(G) = \frac{\sum_i e(G - v_i)}{n-2}$$

number of edges in G

i.e., $|E|$

an expression for computing the number of edges of a graph in terms of those in vertex-deleted subgraphs

Example



$e(G) = 9, \; n = 6$

$e(G - v_1) = 7$
$e(G - v_2) = 5$
$e(G - v_3) = 7$
$e(G - v_4) = 5$
$e(G - v_5) = 5$
$e(G - v_6) = 7$

$\sum_i e(G - v_i) = 36$; $\quad n - 2 = 4 \Rightarrow \frac{36}{4} = 9$

<u>Proof</u>: Each edge has two end vertices. Hence, it is counted exactly $(n-2)$ times in $\sum e(G - v_i)$. $\boxtimes$

Ulam's Conjecture on Graph Isomorphism (1960) [see *Harary*, Graph Theory]

Let G(V1, E1) and H(V2, E2) be two graphs such that |V1| = |V2| = n, and |E1| = |E2|.

Let V1 = (v1, v2, …., vn), and V2 = (p1, p2, …, pn).

If for each i, i = 1 to n, the subgraph (G – vi) is isomorphic to (H – pj) for some j, then G and H are isomorphic.

Ulam's conjecture is still an *unsolved problem* in graph theory

Reconstruction Conjecture

Kelly and Ulam (1942)

$\boxed{\text{DW: Textbook 1.3.12}}$

Let $G(V,E)$ be a simple graph where $|V| \geq 3$.

Then G is uniquely reconstructable by the isomorphic classes of its vertex-deleted subgraphs.

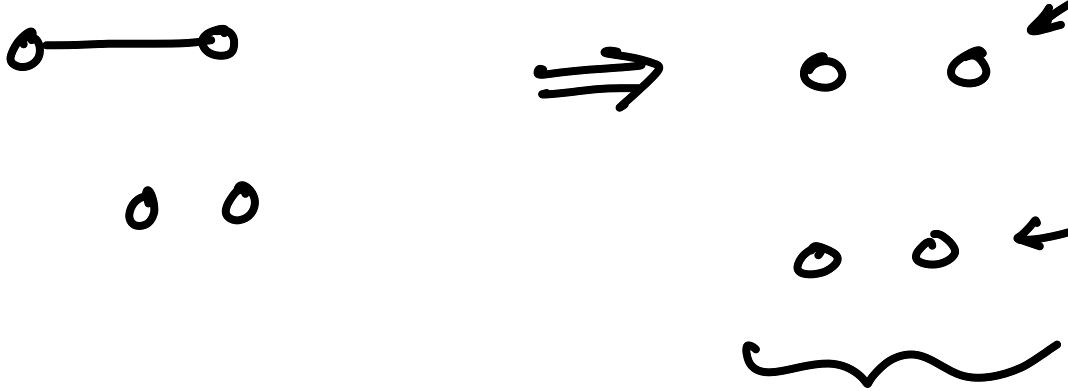
$\boxed{\text{unsolved problem in graph theory}}$

Reconstruction problem

$n \geq 3$ why?

K_2:

$\bar{K}_2$:

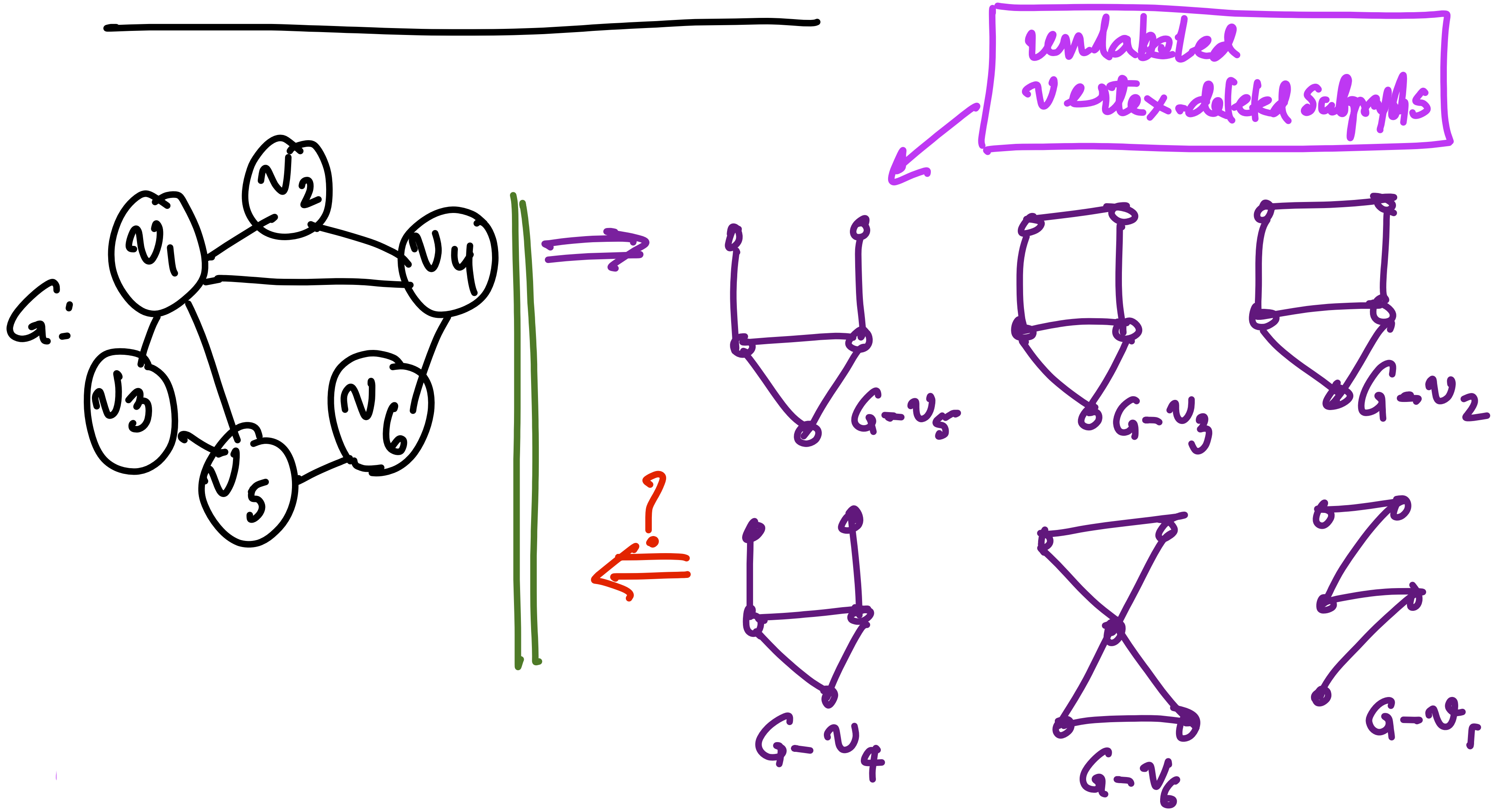


vertex-deleted subgraphs

not distinguishable,
hence reconstruction is
not unique, for $n=2$

It is not known
whether any other
such cases exist

Reconstruction Problem



Graph Isomorphism

The computational complexity of graph-isomorphism problem is not yet known (unsolved for 50 years)!

It is neither proved to be in *P* nor in the *NP*-*C*omplete Class

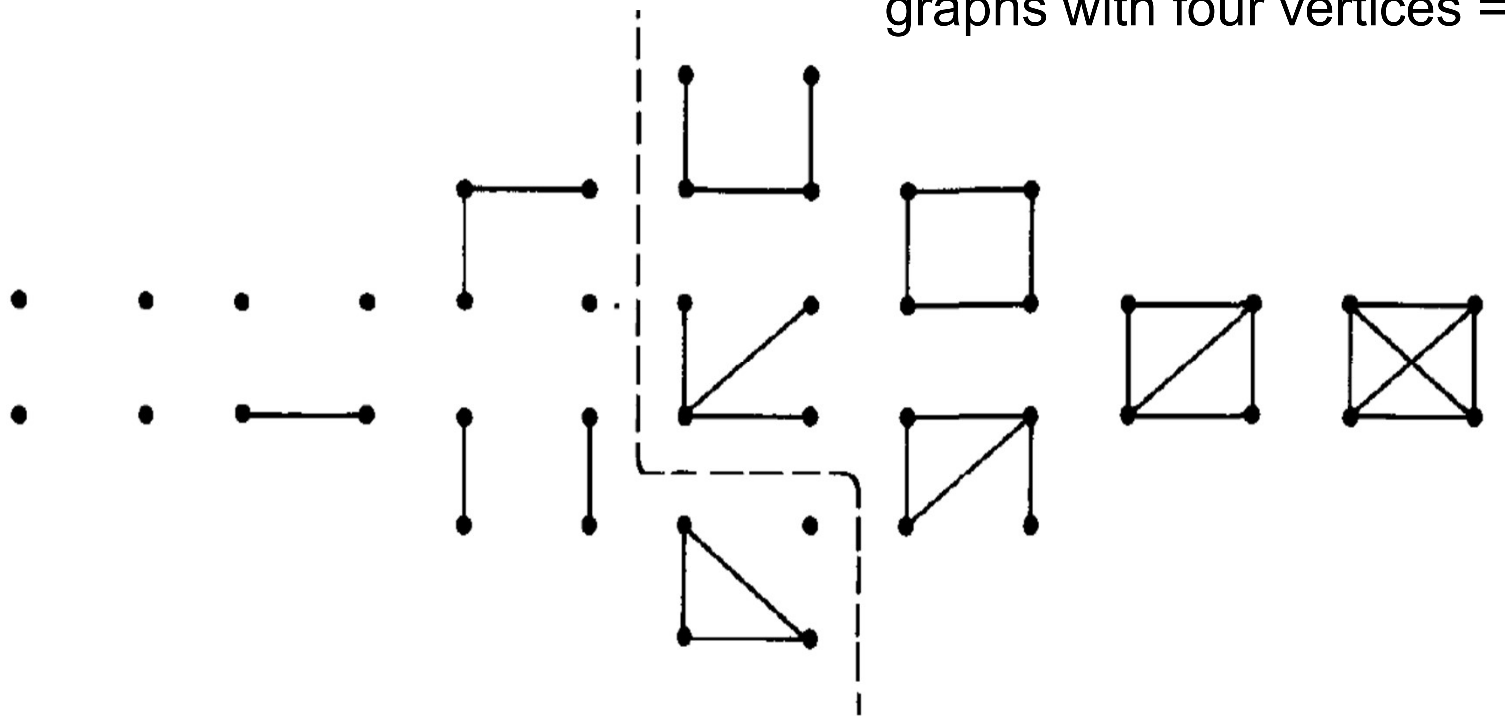
It is said to be in *NPI* (NP-Intermediate) class

Subgraph isomorphism problem is however proved to be *NP*-*C*omplete (Does *G* contain a subgraph that is isomorphic to another given graph *H*?)

GI-hard Class:
(the set of problems that are polynomially reducible from GI-problem)

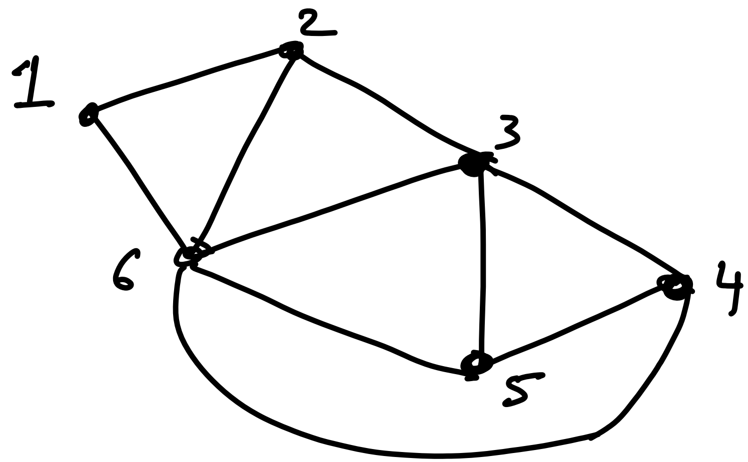
Counting unlabeled graphs

Number of different unlabeled graphs with four vertices = 11



no two pairs in the above collection of graphs are isomorphic; all subgraphs in the same isomorphic class are counted only once

Cliques and independent sets



A clique is a maximal complete sub graph.

Cliques → $\{1,2,6\}$, $\{2,3,6\}$, $\{3,4,5,6\}$

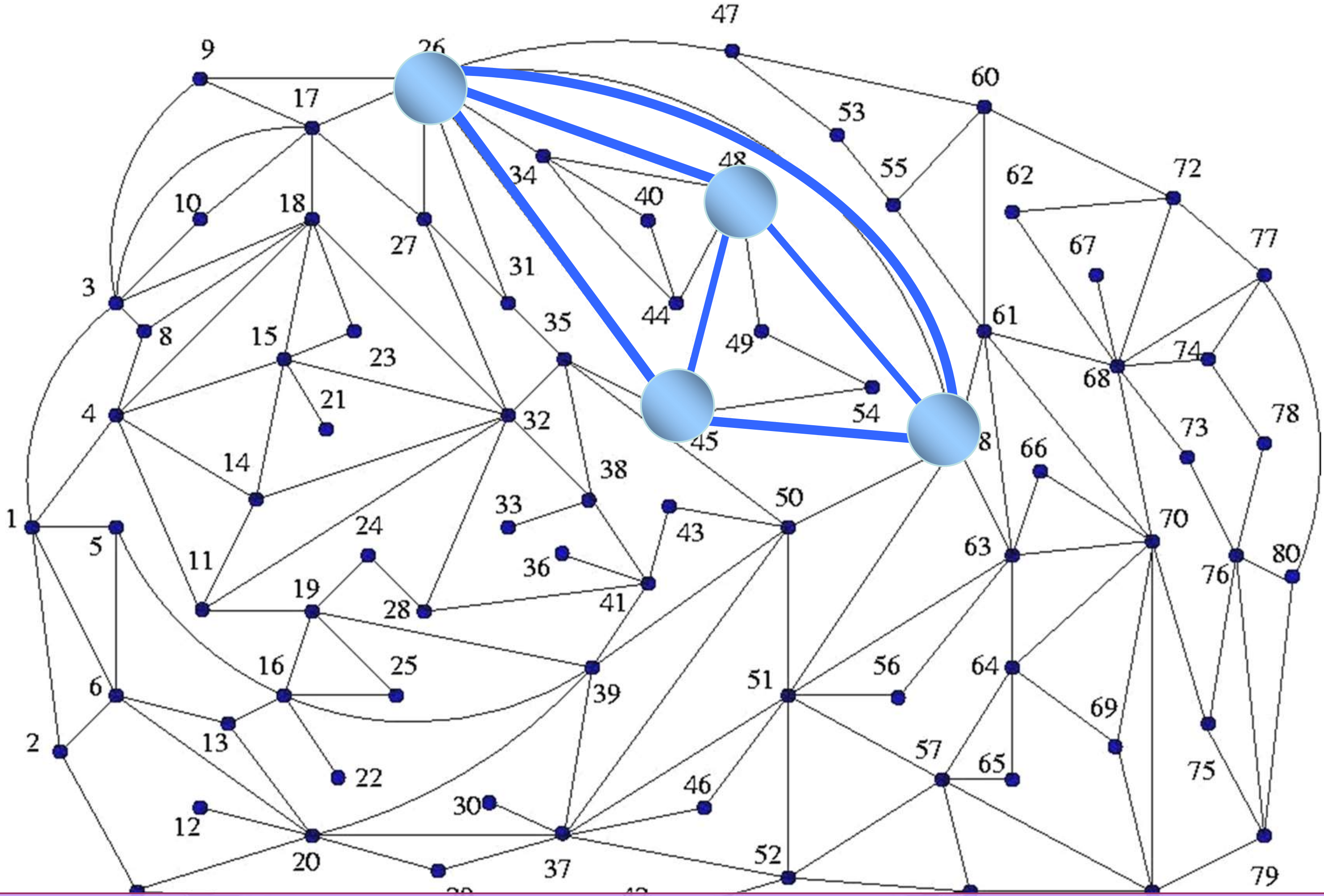
Maximum clique $= \{3,4,5,6\}$

$\{3,4,5\}$ is not a clique, because it is not maximal.

$\{2,5,6\}$ is not a clique, because it is not complete.

Does this graph contain a 4-clique?

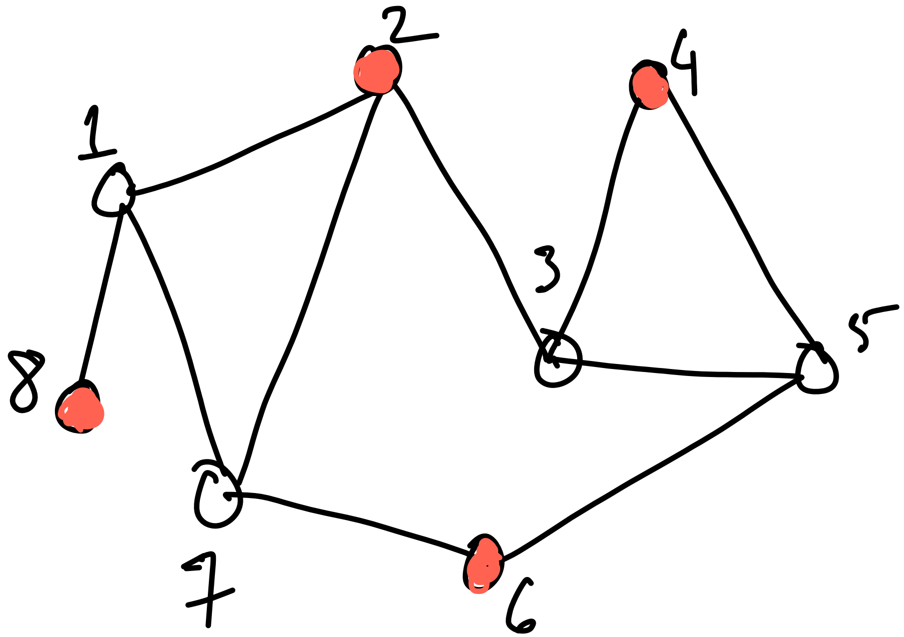
Yes, it has a 4-clique (K_4)



How to ascertain that?

Independent sets (stable set)

no pair of vertices in an IS should be adjacent



Independent set (of vertices)

$\{2, 4, 6\}$

$\{1, 4, 6\}$

imal → $\{3, 6, 8\}$

$\{2, 4, 6, 8\}$

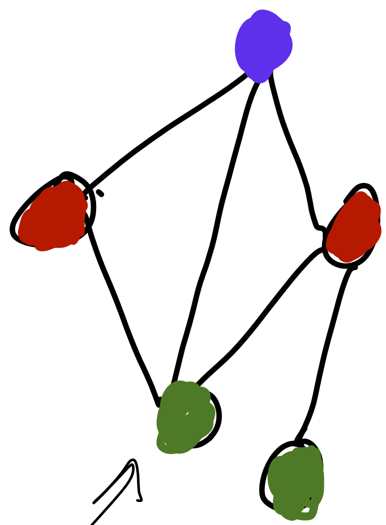
num →

Claim

A clique in G is a maximal independent set in $\overline{G}$ and vice-versa. $\boxtimes$

Coloring

A graph $G(V,E)$ is said to be properly colored if the nodes are colored in such a way that no two adjacent vertices receive the same color.

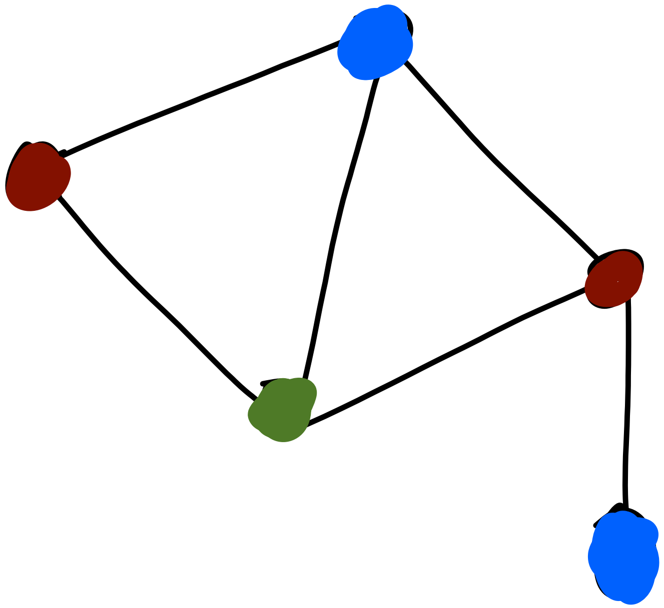


need three colors

Chromatic number $\chi(G)$

Minimum # colors needed to color G.

Clique number & chromatic number



$\chi(G) \to$ chromatic number
(min # colors needed to color G)

$\omega(G) \to$ size of the maximum clique.

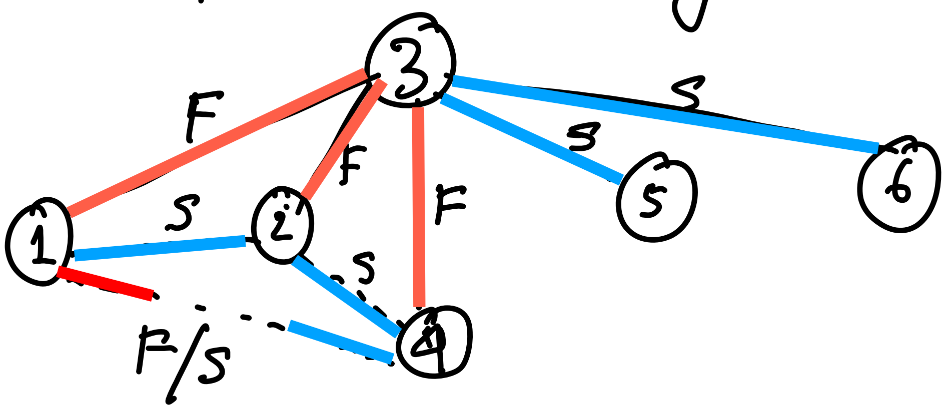
$$\boxed{\chi(G) \geq \omega(G) \text{ in any graph } G}$$

Three-friend or Three-stranger Theorem

Among any collection of six persons, there must be three mutual friends, or three mutual strangers

Proof: Persons $\rightarrow \{1, 2, 3, 4, 5, 6\}$

W.L.O.G, choose any one, say 3



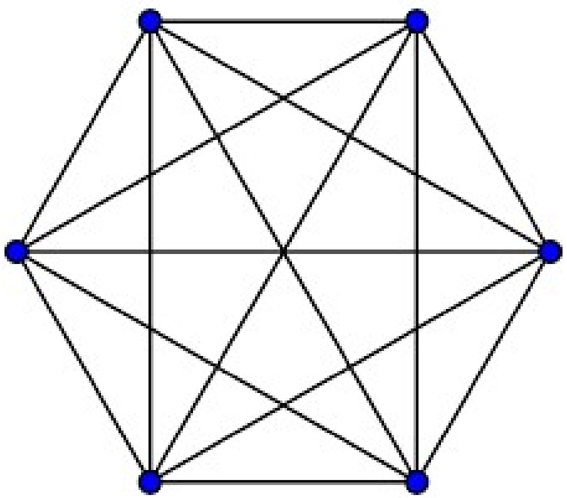
F: friend
S: stranger

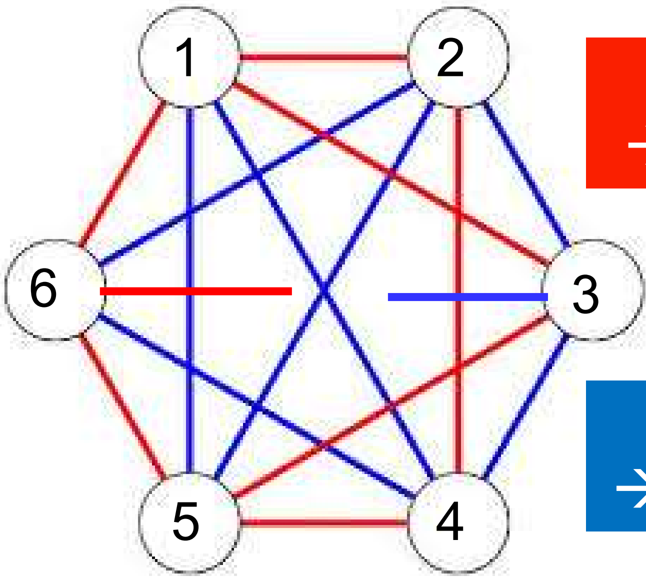
$\boxtimes$

Two equivalent statements for Three Friends/Strangers in terms of graph theory

1. In any simple graph G of six vertices, either G or $\overline{G}$ contains a K_3 (i.e., triangle). [*see Harary*]

2. If the edges of K_6 are randomly colored with two colors (say, Red or Blue), then there exists at least one monochromatic triangle in K_6



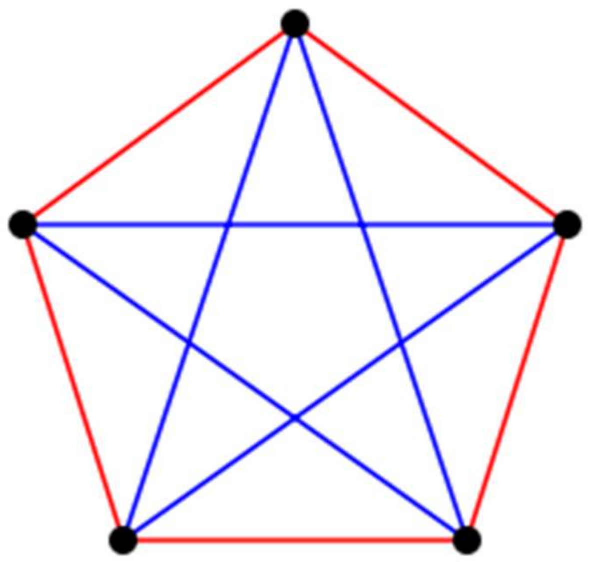


color (3, 6) with red
→ (1, 3, 6) red triangle

color (3, 6) with blue
→ (2, 3, 6) blue triangle

Is it true for K_5 as well?

Q. If the edges of K_5 are randomly colored with two colors (say, Red or Blue), does there exist at least one monochromatic triangle?



No, here is a counterexample

General question: What is the smallest positive integer $R(m, n)$ such that every red-blue edge coloring of $K_{(m+n)}$ will contain either a monochromatic K_m or a monochromatic K_n?

We have just proven that
$R(3, 3) = 6$

The values of $R(m, n)$ are called Ramsey numbers

Ramsey Numbers $R(m, n)$?

Claim: $R(m, n)$ always exists for all positive integers m and n (see Harary: *Graph Theory*)

The determination of Ramsey numbers is an unsolved problem

Some known results:
$R(3, 3) = 6$; $R(3, 4) = 9$; $R(4, 4) = 18$; $R(4, 5) = 25$;

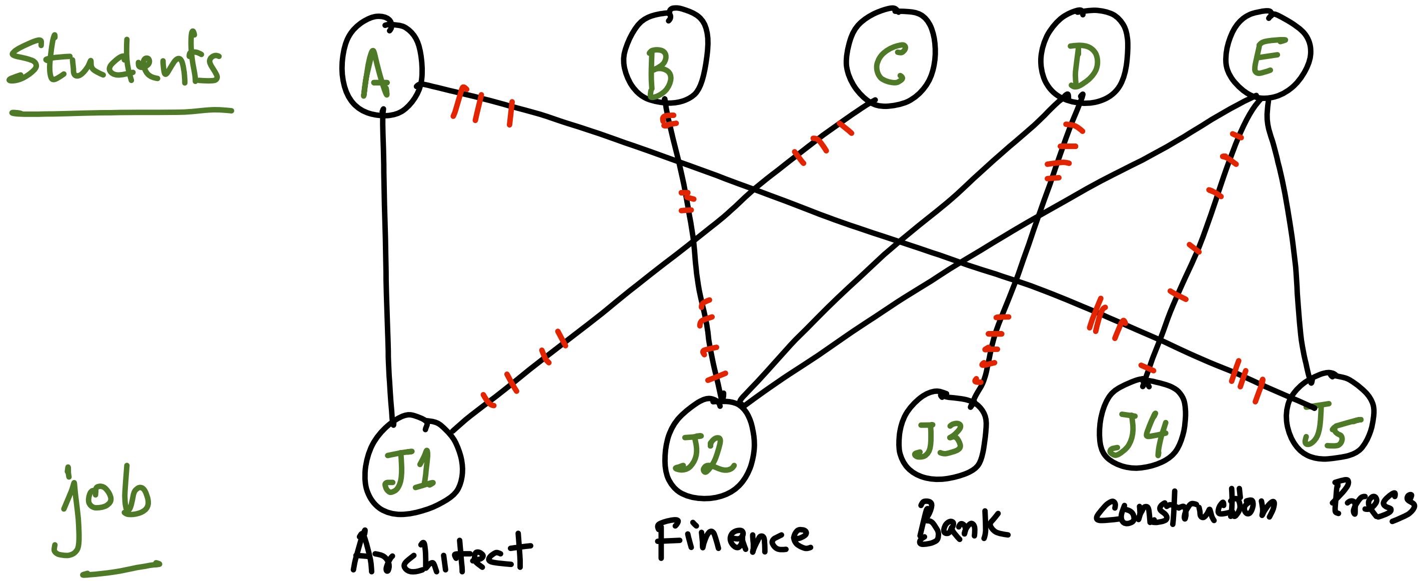
$R(5, 5)$ lies between 43 and 48, its exact value is not known (2017)!

Erdös and Szekeres (1935): $R(m, n) \leq \binom{m+n-2}{m-1}$

RAMSEY NUMBERS

$n \backslash m$	2	3	4	5	6	7
2	2	3	4	5	6	7
3	3	6	9	14	18	23
4	4	9	18			

Job hunting problem



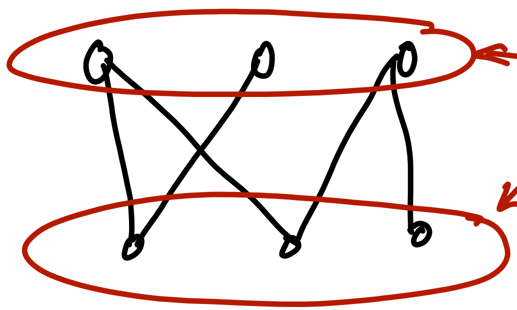
Maximum matching

Matrimonial alliance problem

⟹ ① Bipartite graphs
② Matching

Bipartite graphs

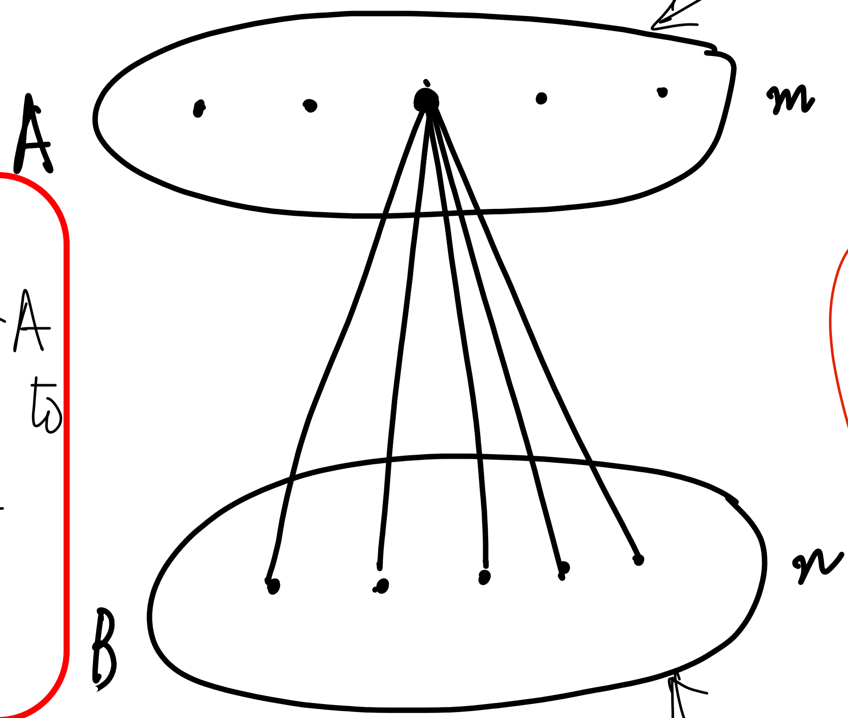
A graph $G(V,E)$ is bipartite if $V(G)$ can be partitioned into two independent sets.



each one is an independent set

Complete Bipartite Graph $K_{m,n}$

independent set



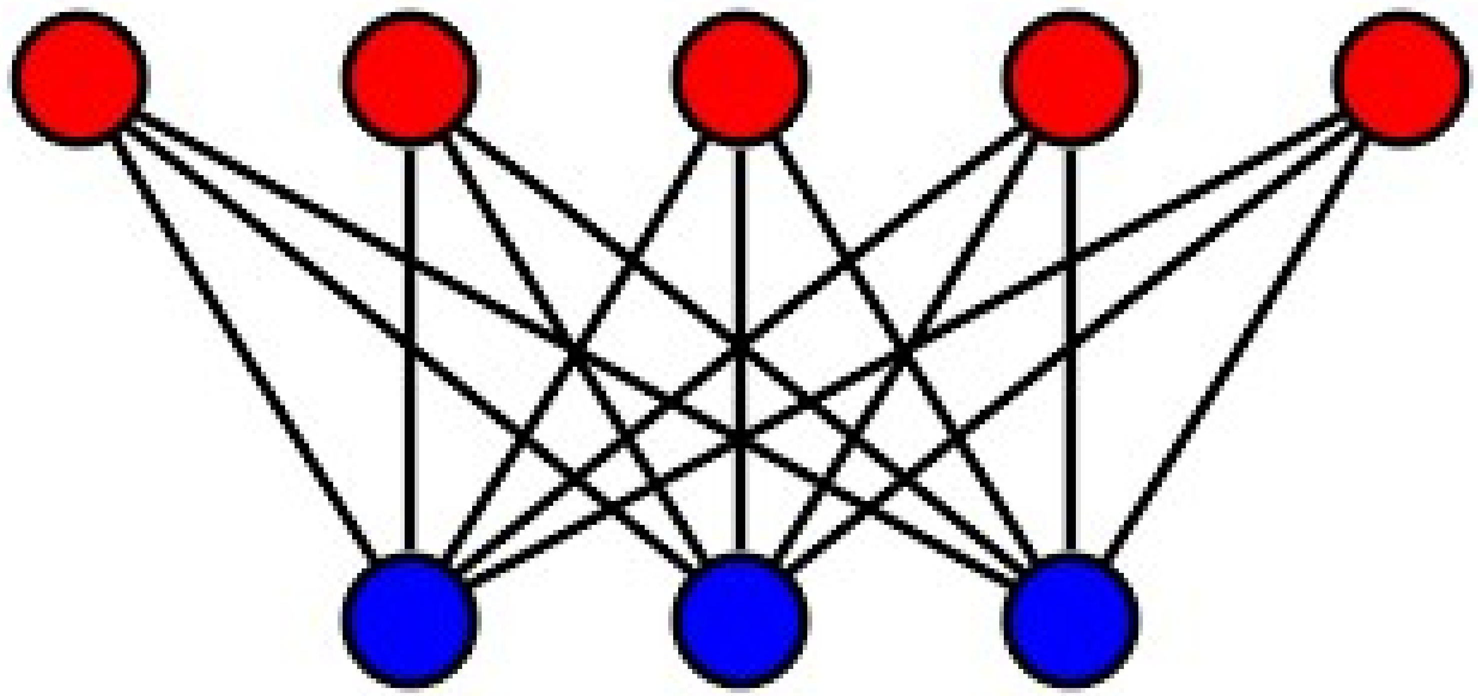
independent set

every vertex in set A is adjacent to every vertex in set B

$K_{m,n}$ Complete bipartite graph

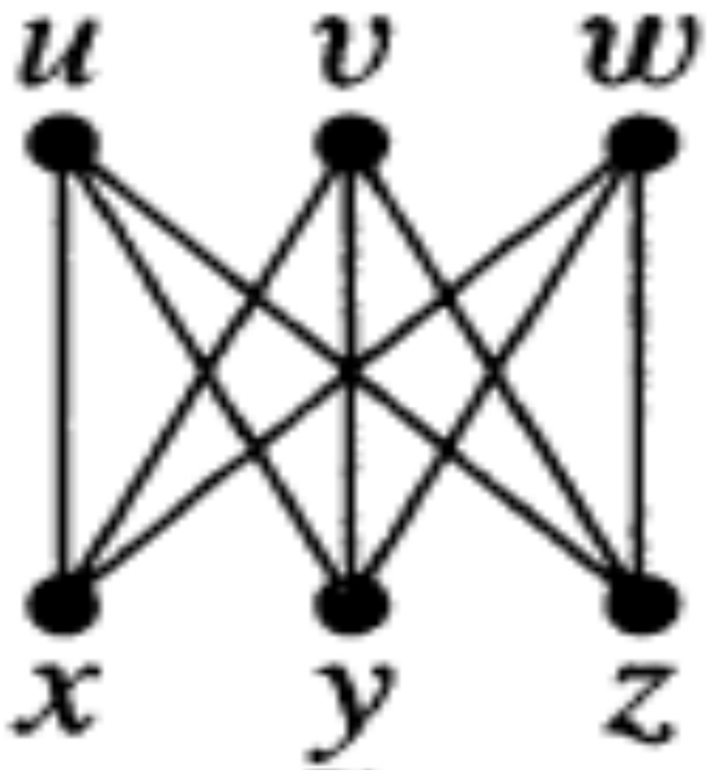
$$\boxed{|E| = m * n}$$

Complete Bipartite Graph $K_{5,3}$



Same as $K_{3,5}$

Complete Bipartite Graph $K_{3,3}$



DW: Textbook 1.8.7

Hyperactubes $Q_k(V,E)$

$v \in V \implies k$- bit binary tuples

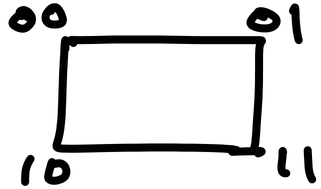
$|V| = 2^k$

if $HD(v_i, v_j) = 1$

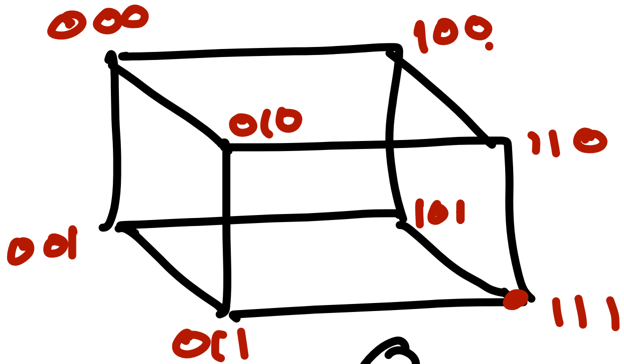
Hamming distance

Example

Q_1



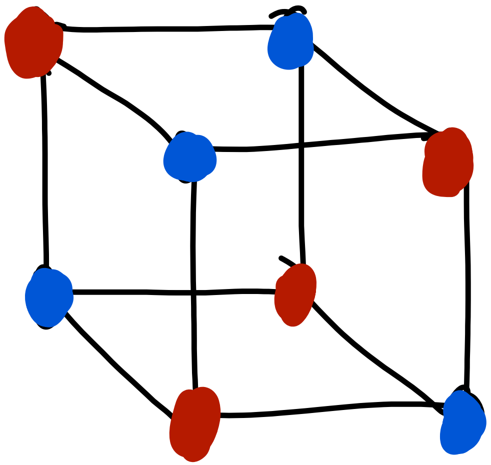
Q_2



Q_3

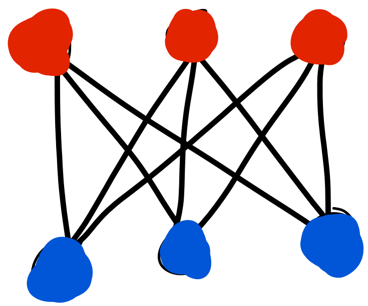
Q_k can be recursively constructed by joining two copies of Q_{k-1}

A graph G is bipartite if and only if G is 2-colorable.

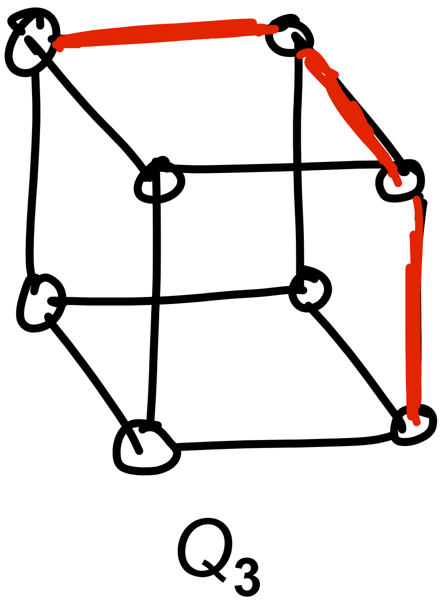


← Hypercube (Q_3)

Q_3 is bipartite.



$K_{3,3}$ is bipartite, i.e., 2-colorable



Claim

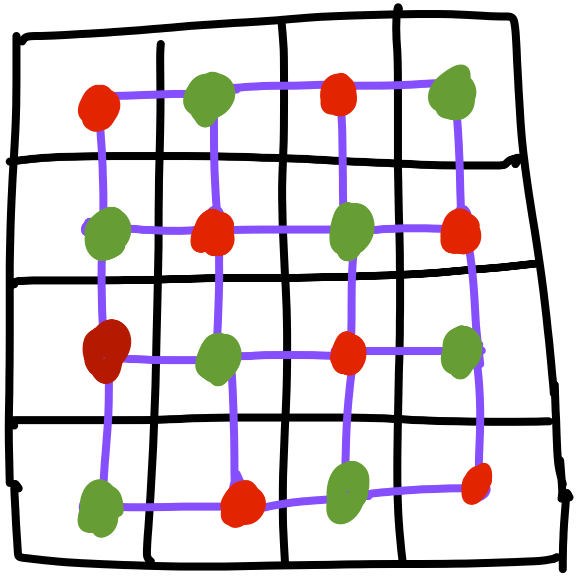
① For any $k > 0$, Q_k is bipartite.

Proof: Trivial ☒

② Q_k is k-regular

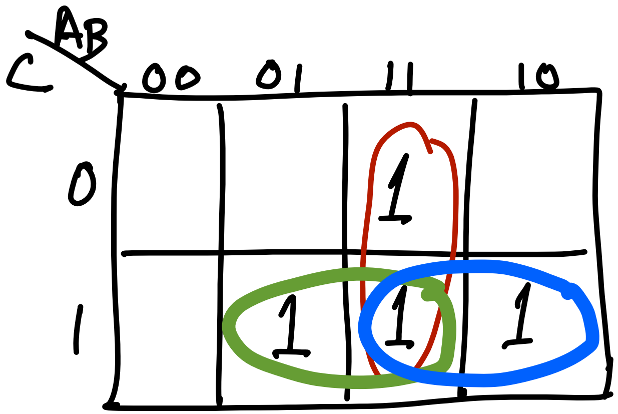
③ $\text{diameter}(Q_k) = k$

④ $e(Q_k) = \dfrac{k \cdot 2^k}{2} = k \cdot 2^{k-1}$



$\Rightarrow$ Dual graph of a Chessboard is bipartite

$\Rightarrow$ Rectangular grid-graph is bipartite.



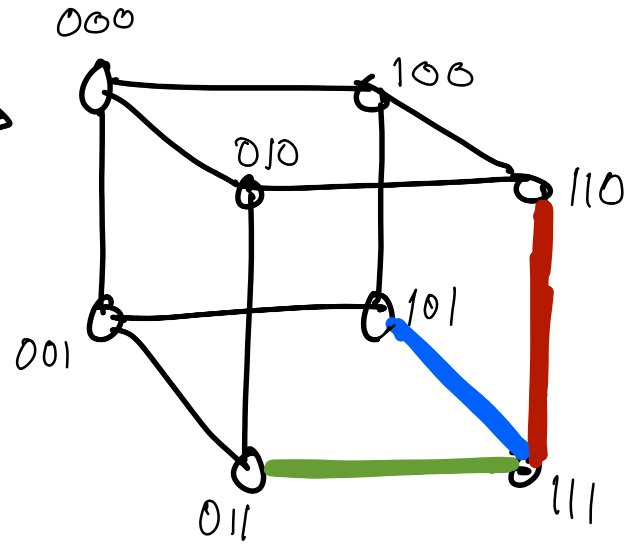
Karnaugh-map

$$f = AB + BC + CA \Rightarrow$$

$\uparrow$

carry-function

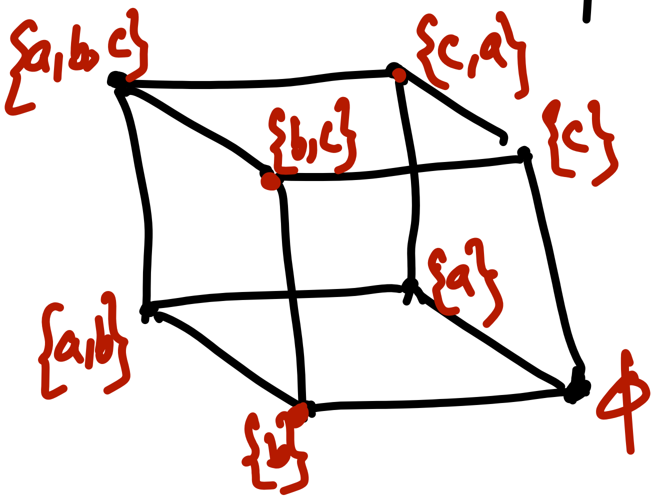
Boolean K-map is equivalent to hypercube



Adjacency Graph of Karnaugh-Map is Bipartite

Subset Containment Relationships

Let S be a set of k elements:
$S = \{s_1, s_2, \dots, s_k\}$. Then the containment relationships among the subsets of S can be represented by Q_k.



__Example:__ $k = 3$

$S = \{a, b, c\}$

$\#$ subsets of $S = 2^3 = 8$

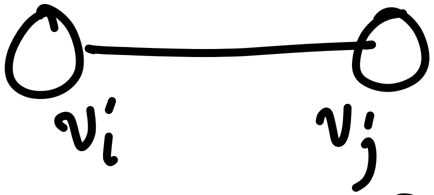
Homework

Construct a graph $G(V,E)$ as follows:

i) each node v_i is a permutation of $(1\ 2\ 3\ 4\ \ldots\ n)$

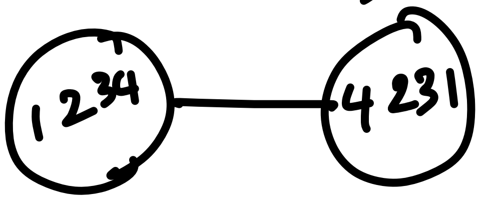
Thus, $|V| = n!$

(ii)



$\Rightarrow$ if v_j can be obtained from v_i by interchanging the numbers in two positions

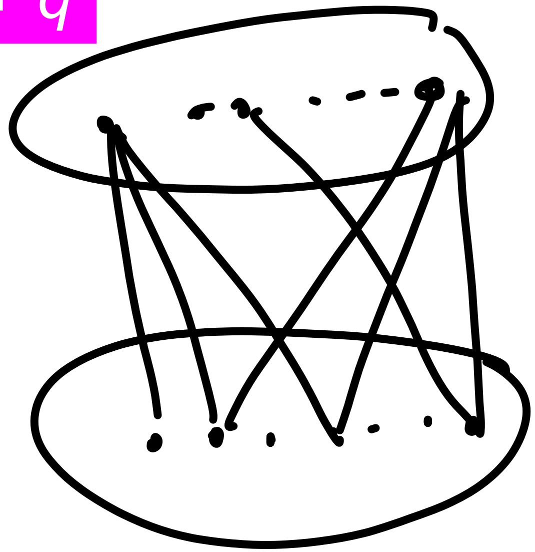
Ex:



Maximum number of edges in a bipartite graph $G(V, E)$ having n vertices

Let $G(V,E)$, $|V| = n$, be a bipartite graph. Then $|E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$.

$n = p + q$



$\left\lfloor \frac{n}{2} \right\rfloor$

$\left\lceil \frac{n}{2} \right\rceil$

$\Longrightarrow |E| \leq \left\lfloor \frac{n^2}{4} \right\rfloor$ ⊠

$e(K_{p,q}) \leq pq \leq [(p + q)/2]^2 \leq n^2/4$, *since GM ≤ AM*

Mantel's Theorem (1907)

DW: Textbook 1.3.23

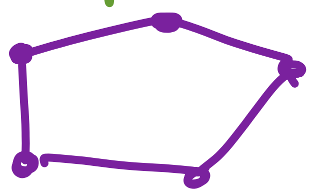
The maximum number of edges in an n-vertex triangle-free simple graph is

$$\left\lfloor \frac{n^2}{4} \right\rfloor .$$

Proof: Left as reading assignment.

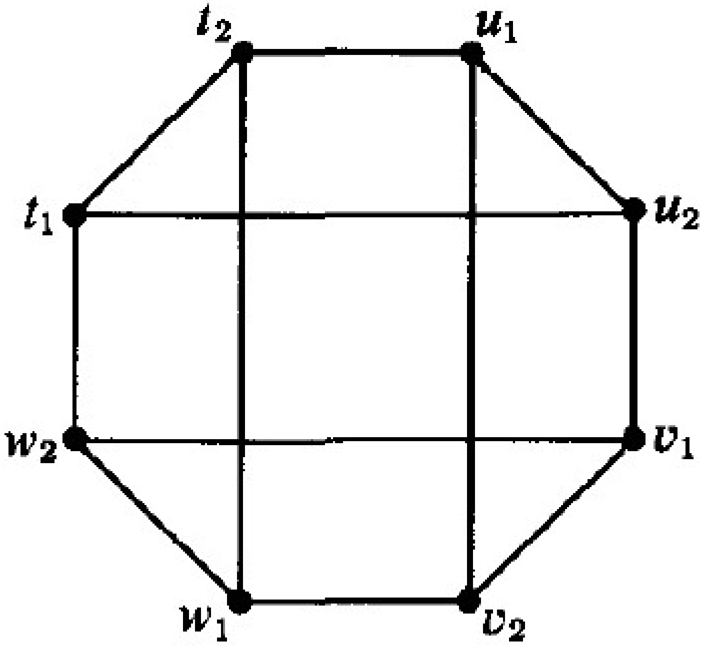
Note: We have proved a similar result for bipartite graphs.

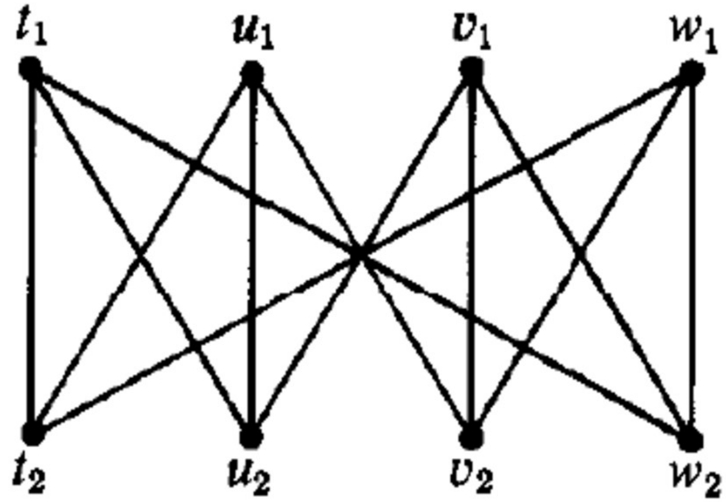
G:



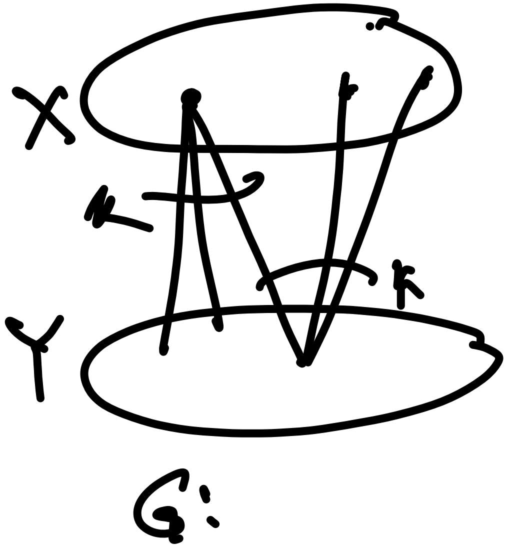
G is tringle-free, but not bipartite.

Isomorphic?





Property



Let $G(V,E)$ be a bipartite graph, with partitions X, Y; $V = X \cup Y$

$|V| = |X| + |Y|$, $X \cap Y = \phi$

If G is k-regular, then

$|X| = |Y|$

degree of each node $= k$

DW: 1.3.9

Proof: $k|X| = |E| = k|Y| \implies |X| = |Y|$ (for $k > 0$) $\boxtimes$

$\implies$ Regular bipartite graphs admit a balanced bipartition on the set of vertices

CS 60047 Autumn 2020

Advanced Graph Theory

Instructor

Bhargab B. Bhattacharya

Lecture #04: 11 Sept. 2020

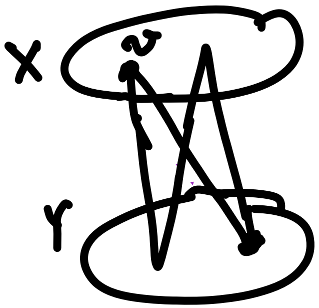
Indian Institute of Technology Kharagpur
Computer Science and Engineering

Theorem [König, 1936]:

DW: 1.2.18

A graph is bipartite if and only if it has no odd cycle.

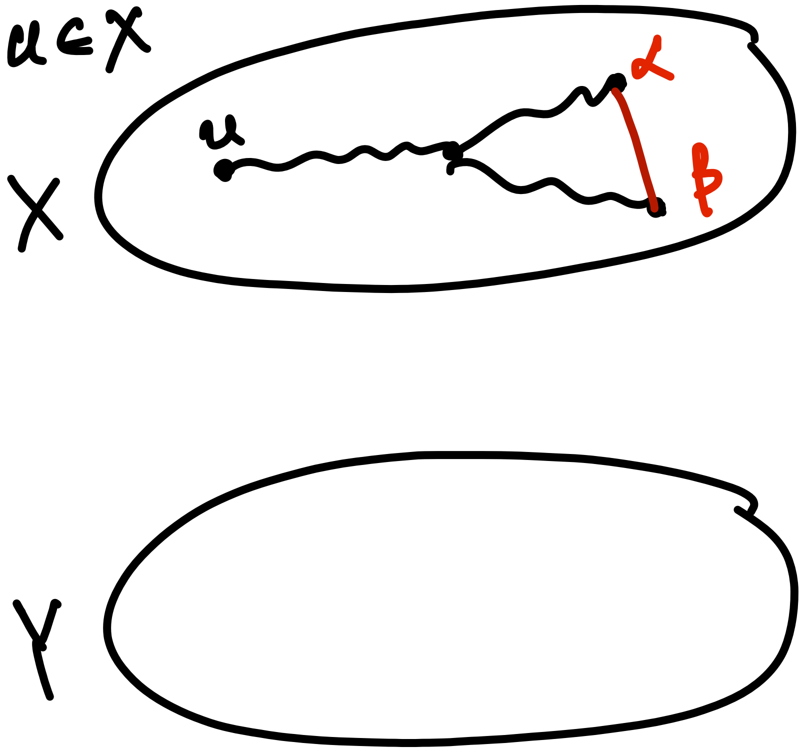
Proof: Necessity (only if): Let G be a bipartite graph. Starting from an arbitrary node say $v \in X$, one can return to v only using even number of steps.



Since both X and Y are independent sets, G cannot have any odd cycle.

Sufficiency (If):

Let $G(V,E)$ be a graph with no odd cycle.
Assume G is connected.



$X = \{v \in V \mid dist(u,v) = \text{even}\}$

$Y = \{v \in V \mid dist(u,v) = \text{odd}\}$

$\Rightarrow$ edge (α, β) cannot exist

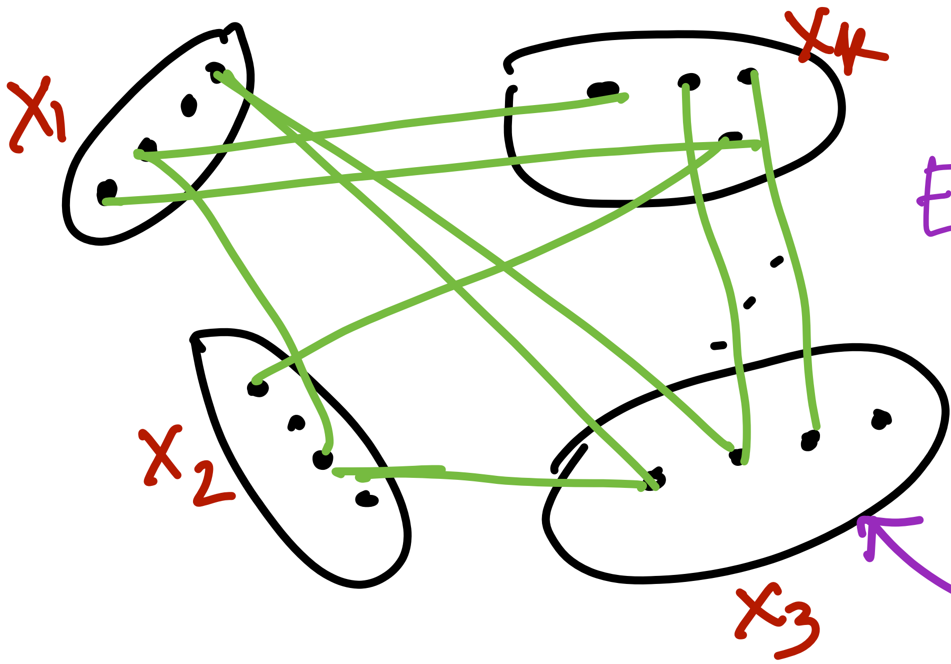
$\Rightarrow$ X is an independent set.

Similarly, Y is an indep. set.

$\Rightarrow$ G is bipartite.

Generalizations

k-partite graphs $G(V,E)$



$\forall\, i, j,\ X_i \cap X_j = \phi$

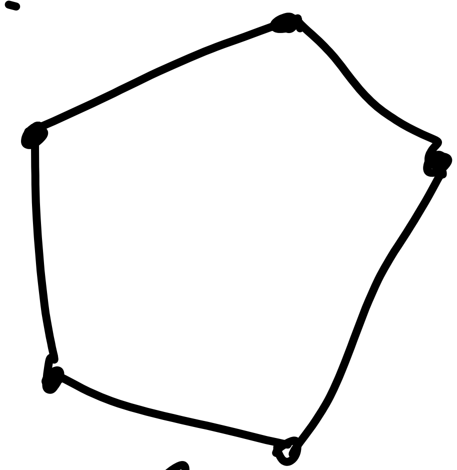
$V = \bigcup_i X_i$

$E\{e_{ij} \mid v_i, v_j \in X_p\} = \phi$

$p = 1, 2, \dots, k$

independent set

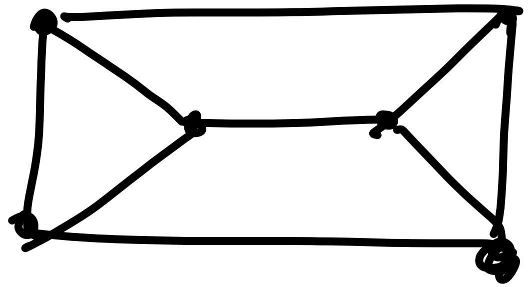
Example 1:



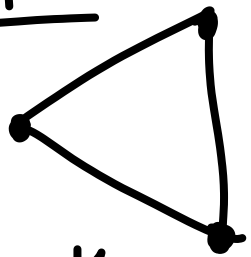
C_5

$\Rightarrow$ cannot be bipartite

$\Uparrow$

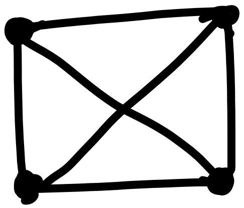


Example 2:



K_3

$+$

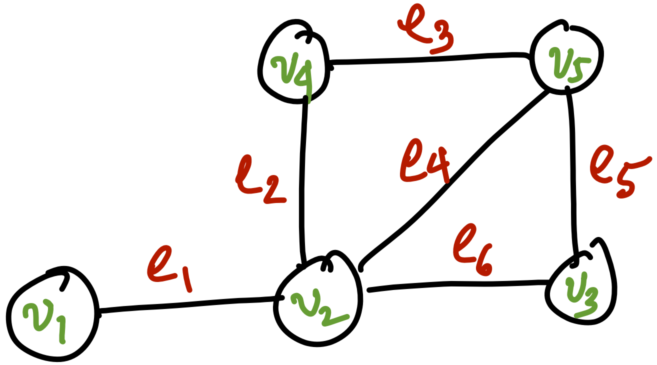


K_4

$= \overline{K}_{3,4}$

$K_m + K_n = \overline{K}_{m,n}$

Connectedness



1. **Walk:** an alternating sequence of vertices and edges, beginning & ending with vertices

λ_1: $v_1 e_1 v_2 e_4 v_5 e_4 v_2 e_6 v_3$

$\Rightarrow v_1, v_2, v_5, v_2, v_3$ (shorter notation for simple graphs)

closed walk if $v_1 = v_n$

$\Leftarrow$ should ensure connectedness

2. **Trail:** a walk where all edges are distinct

λ_2: $v_1 \xrightarrow{e_1} v_2 \xrightarrow{e_4} v_5 \xrightarrow{e_3} v_4 \xrightarrow{e_2} v_2 \xrightarrow{e_6} v_3$ $\leftarrow$ all distinct

3. **Path:** a trail in which all vertices are distinct

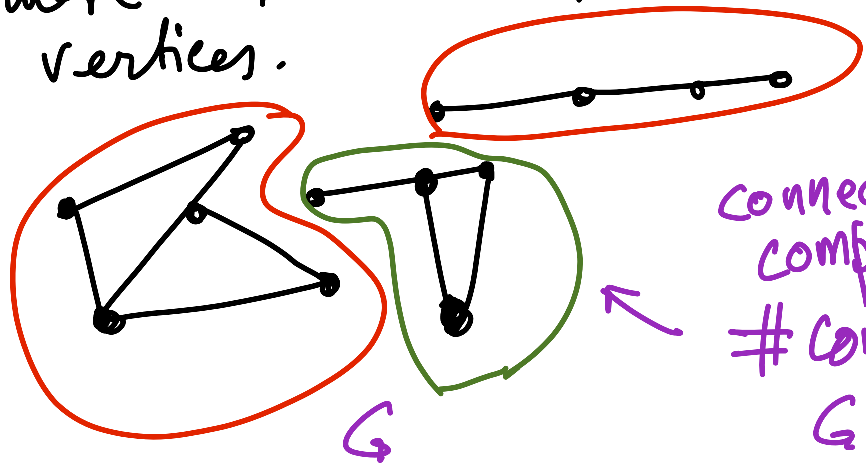
λ_3: $v_1\ v_2\ v_4\ v_5\ v_3$

	W	T	P
λ_1	✓		
λ_2	✓	✓	
λ_3	✓	✓	✓

Path comprising n vertices  $\Rightarrow P_n$

Connectedness

A graph $G(V,E)$ is said to be connected if there exists a path between every pair of vertices.



connected components (i.e. maximal)

\# components = 3

G is not connected; it has three CC's.

A cycle is path where the start and end vertices are the same. $\Rightarrow C_n$ (cycle through n vertices)

Girth of a graph G

$\Rightarrow$ length of the shortest cycle in G.

Circumference of a graph G

$\Rightarrow$ length of the longest cycle in G.

No cycles $\Rightarrow$ girth and circumference may be considered ∞

Distance between two vertices in G:

$d(u,v) \rightarrow$ length of the shortest path between u and v in G: this path is called geodesic (u,v)

\# edges

if u and v are disconnected, then $d(u,v) = \infty$

Diameter of a graph G $\Rightarrow$

length of the longest geodesic

$\Rightarrow \max_{\forall u,v} \min d(u,v)$

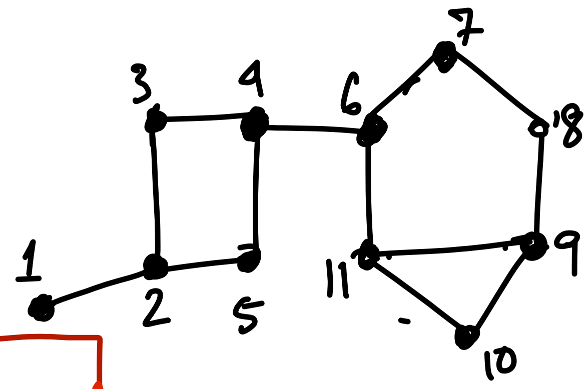
Example

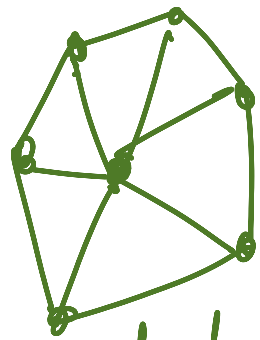
Girth = 3 (9, 10, 11)

Circumference = 6 (6, 7, 8, 9, 10, 11)

distance (3, 9) = 4 (3, 4, 6, 9)

Diameter = 6 {1 to 8/9/10}





wheel (W_n)

Diameter (K_n) = 1

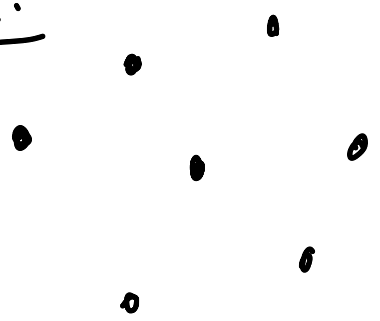
Diameter (W_n) = 2 for $n > 4$

Diameter of $Q_k = k$

Lemma (DW: 1.2.11)

Every graph with n vertices and k edges has at least $(n-k)$ components

Proof:



$|V| = n$, $|E| = 0$

$\Rightarrow$ # components $= n$

$\Rightarrow$ addition of each edge may reduce the # components by at most one

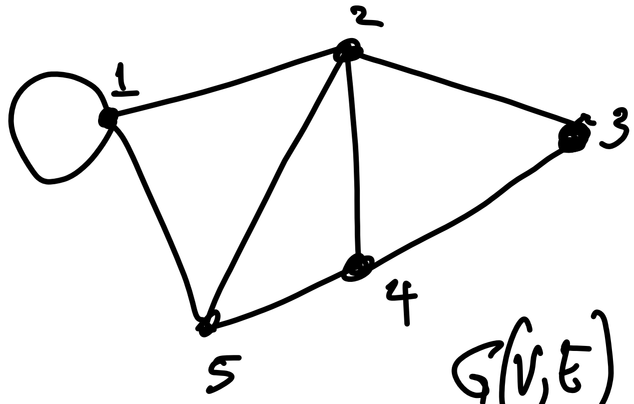
$\Rightarrow$ addition of k edges

$\Rightarrow$ # CC's $\geq n-k$ $\boxtimes$

Few notation

$d(v) \to$ degree of vertex v

$d(1) = 4$, $d(5) = 3$



$G(V, E)$

$\delta(v) \to$ minimum degree in graph G.

$\delta(G) = 2$

order $n(G) = |V|$

size $e(G) = |E|$

$\Delta(v) \to$ maximum degree in Graph G

$\Delta(G) = 4$

If $\delta(G) = \Delta(G) = k \Rightarrow G$ is k-regular

Regular graphs

A graph G is k-regular if

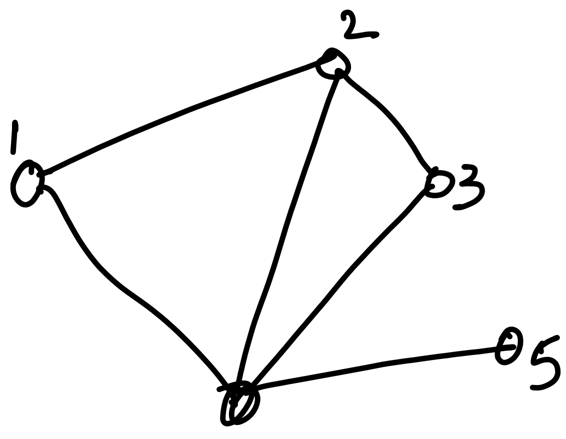
$$\delta(G) = \Delta(G) = k$$

$$|E| = \frac{nk}{2}$$

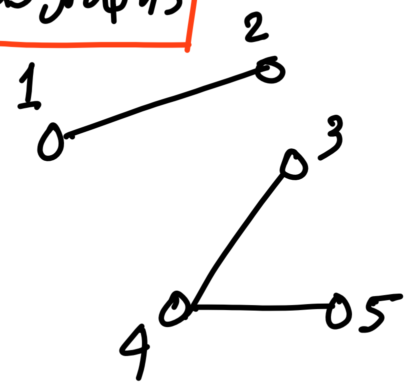
A 51-vertex graph cannot be 11-regular.

why?

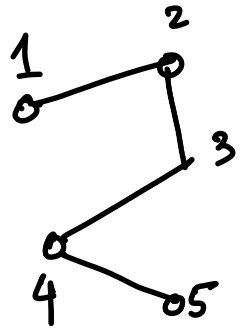
Subgraphs and induced subgraphs



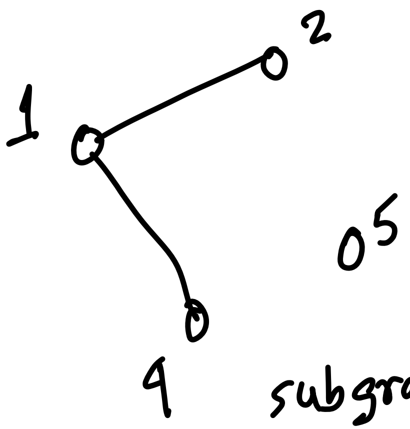
G



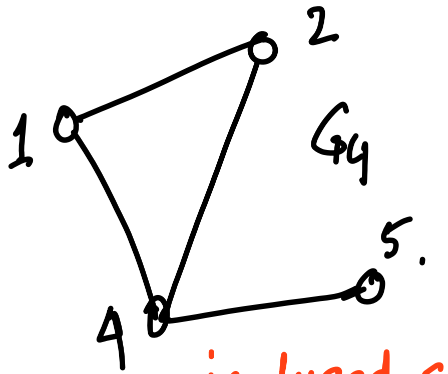
subgraph G_1 of G



Connected subgraph G_2 of G



subgraph G_3



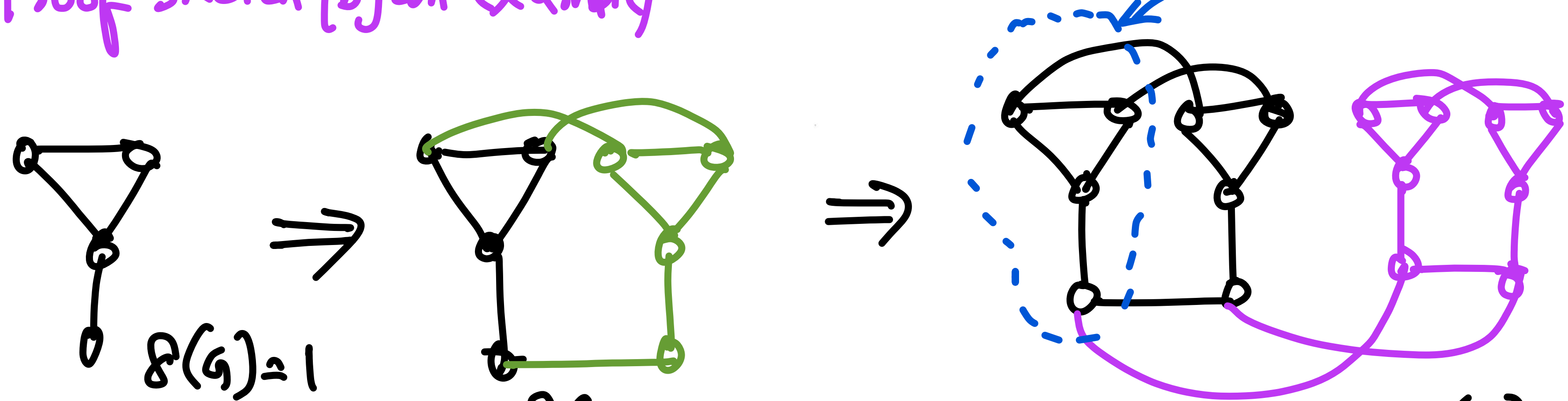
G_4

induced subgraph of G

G_1, G_2, G_3 not induced subgraph

Claim: For any graph G, there exists a regular graph F that contains G as an induced subgraph

Proof sketch (by an example)



$\delta(G) = 1$
$\Delta(G) = 3$

$\delta(G) = 2$
$\Delta(G) = 3$

$\delta(G) = 3 = \Delta(G)$

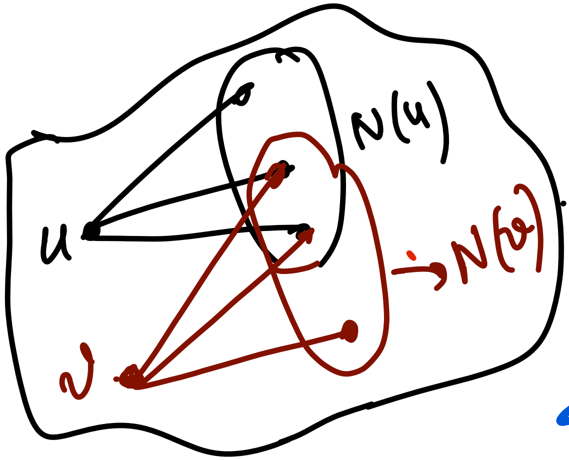
Claim (DW: 1.3.15)

Let G be a simple n-vertex graph s.t. $\delta(G) \geq \frac{1}{2}(n-1)$. Then G is connected.

Proof: pick up u, v, assume not adjacent.

$$|N(u) \cup N(v)| \leq n-2$$

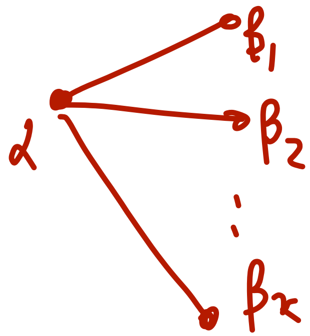
(u, v are not in the union)



$$|N(u) \cap N(v)|$$

$$= |N(u)| + |N(v)| - |N(u) \cup N(v)|$$

$$\geq \frac{n-1}{2} + \frac{n-1}{2} - (n-2) \geq 1 \qquad \boxtimes$$



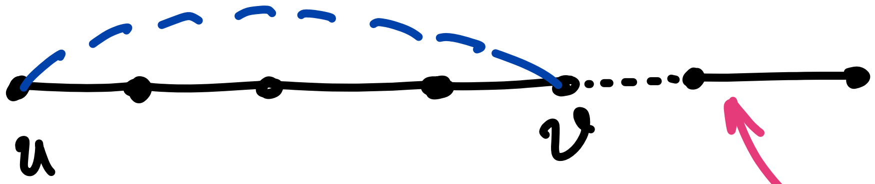
$N(\alpha) = (\beta_1, \beta_2, \ldots, \beta_k)$

neighborset of vertex α

Claim (DW: 1.2.25)

If in a graph G, $\delta(G) \geq 2$, then G contains a cycle

Proof:



a maximal path P in G;
let u be an end vertex in P.

Now, $d(u) \geq 2$, and since P is maximal, $|N(u) \cap P| \geq 2$.

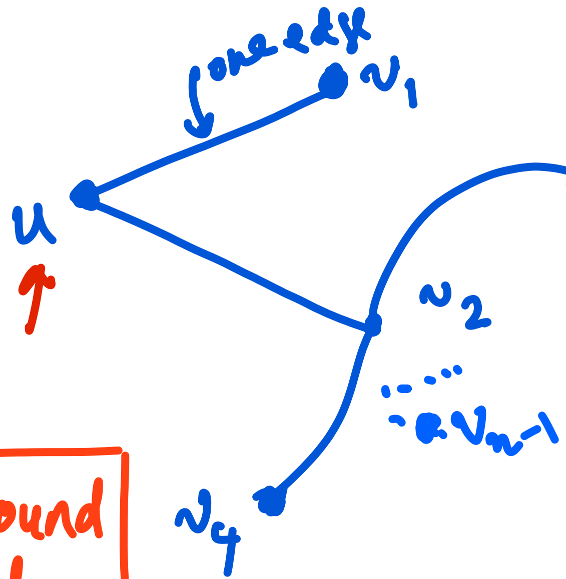
Hence G contains a cycle $\boxtimes$

Connectedness

Claim: The minimum number of edges in a connected graph with n vertices is $(n-1)$.

Proof:

start with a vertex



For each of $(n-1)$ remaining vertices, one needs at least one edge to connect it to the rest of the graph $\boxtimes$

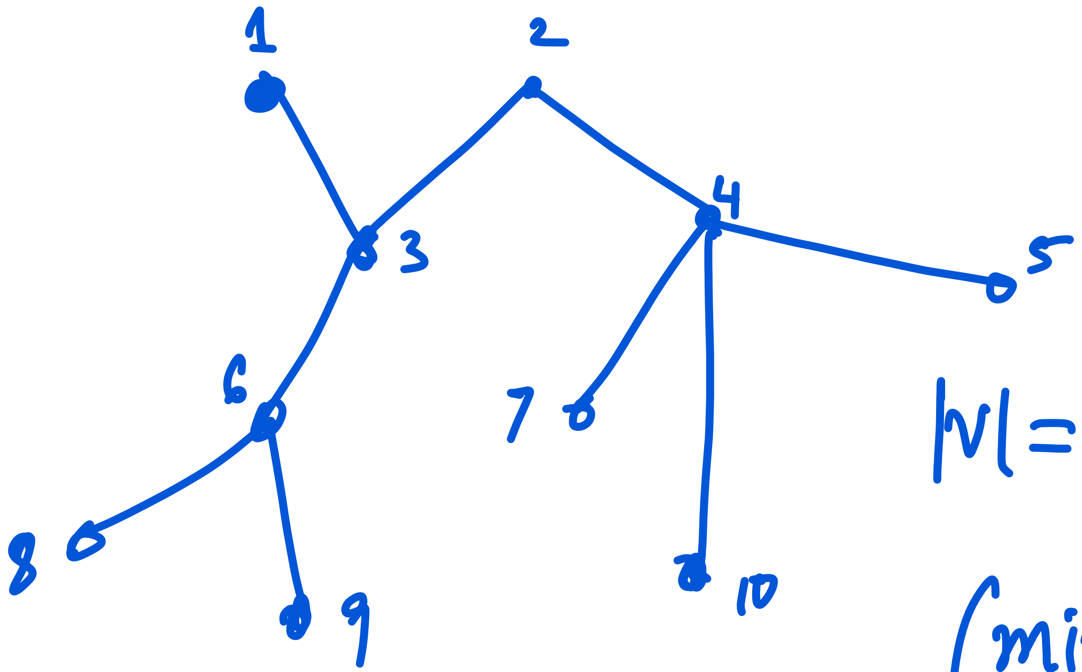
The lower bound is achieved by a tree.

You can add more edges if we want

Lower bound achieved by trees

Path P_5 # edges = 4

vertices = 5



$|V| = 10$; $|E| = 9$

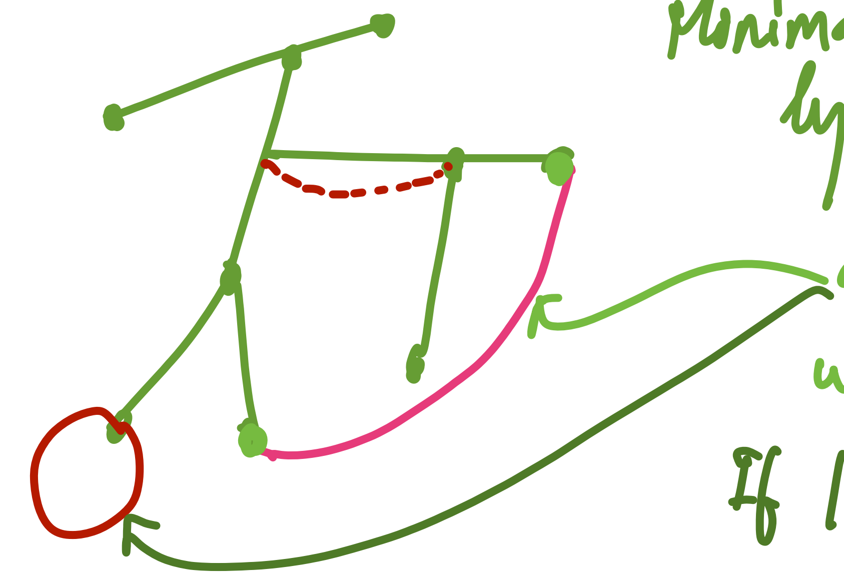
(minimally connected)

when $|E| = n-1$

Tree

Claim: If an n-vertex graph G has n or more edges, then G must have a cycle.

Case 1: Assume G is connected, and $|E| = n$.



Minimal connectivity is achieved by putting $(n-1)$ edges.

addition of one edge (red) will create a cycle.

If $|E| > n \Rightarrow$ more cycles.

Case-2 : G is disconnected,

$|V| = n, \quad |E| \geq n$

Assume two-components

$(k-1)$ edges

k-vertex component

$(n-k-1)$ edges

$(n-k)$-vertex component

For minimal connectivity (without any cycle)

edges $= (k-1) + (n-k-1)$

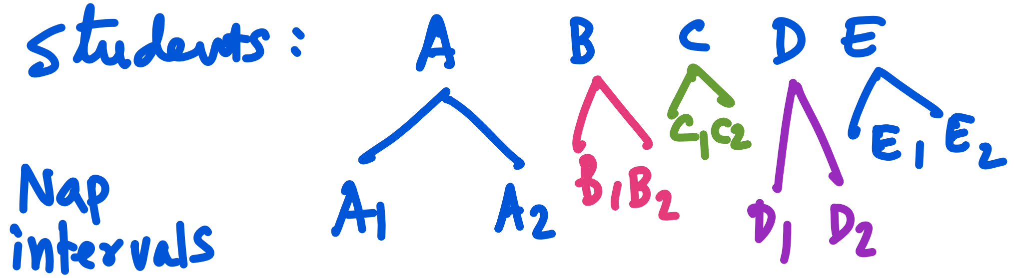
$= n-2$

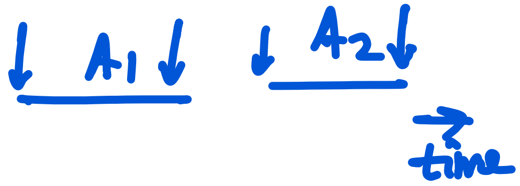
need to add at least 2 edges

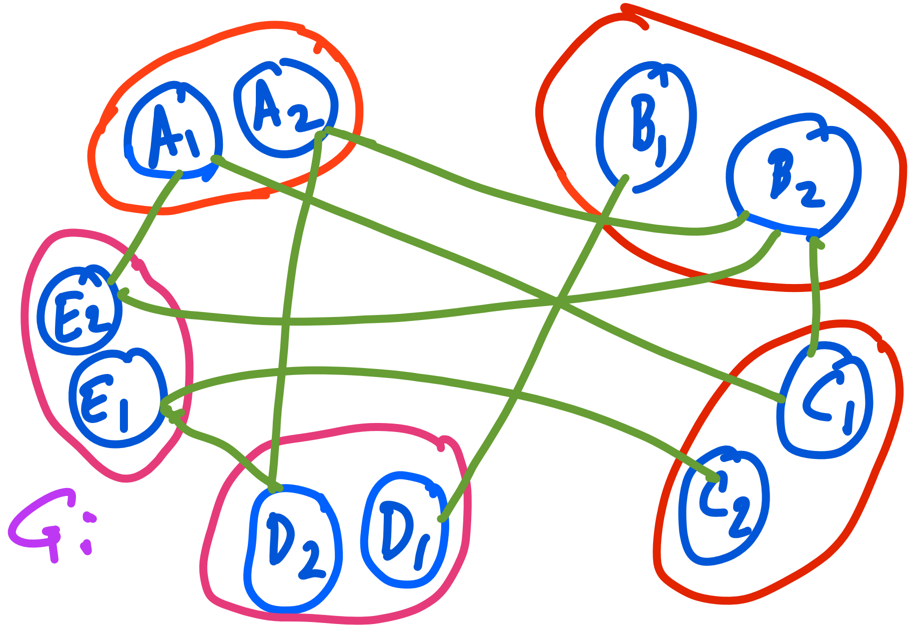
$\Downarrow$ leads to two cycles.

During a lecture each of five students fell asleep for some duration exactly twice. For each pair of students, there was some moment when both of them were asleep. Show that there exist a moment when at least three students were sleeping simultaneously.

Classroom Nap Problem







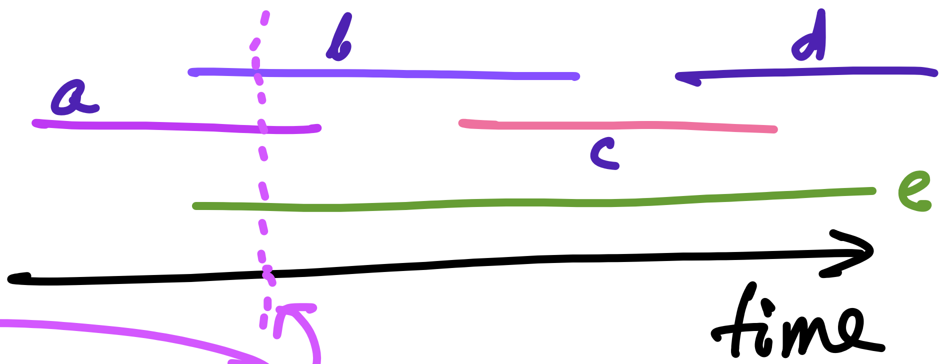
$\binom{5}{2} \Rightarrow 10$

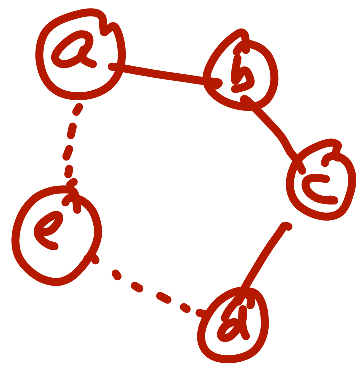
G has 10 vertices and 10 edges

$\Rightarrow$ G has a cycle.

G: $|V| = 10$, $|E| = 10 \Rightarrow$ has a cycle.

nodes in G represent time-intervals



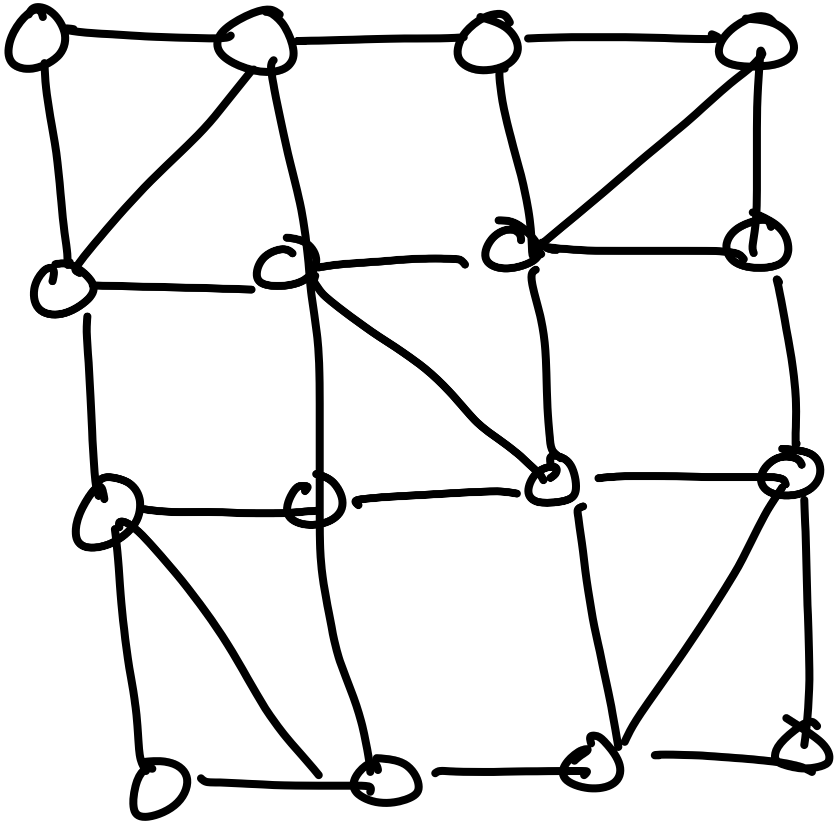


at least three intervals must overlap in time.

Helly property

two nodes in G are adjacent if their corresponding time intervals overlap.

Homework

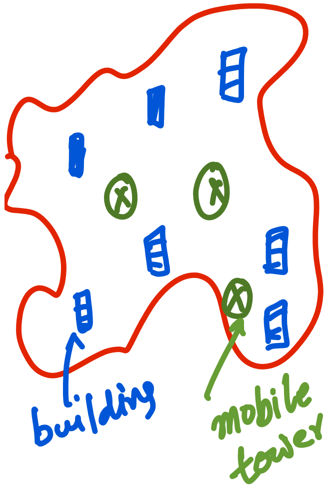


G:

Find girth, circumference, diameter of G

Dominating set

Five-queens Problem



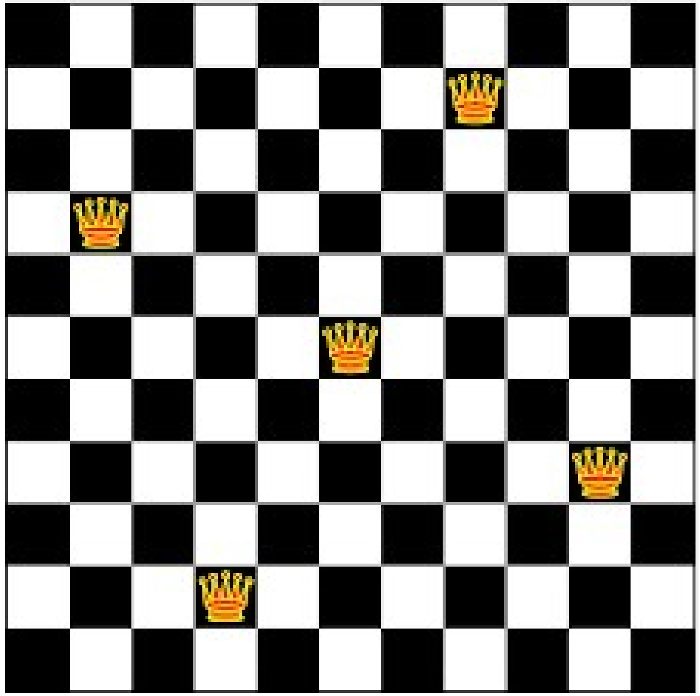
Definition: Given a graph $G(V,E)$

A vertex $u \in V$, is said to dominate a vertex v, if

(i) $v = u$, or

(ii) v is adjacent to u,
i.e., $\text{distance}(u,v) \leq 1$

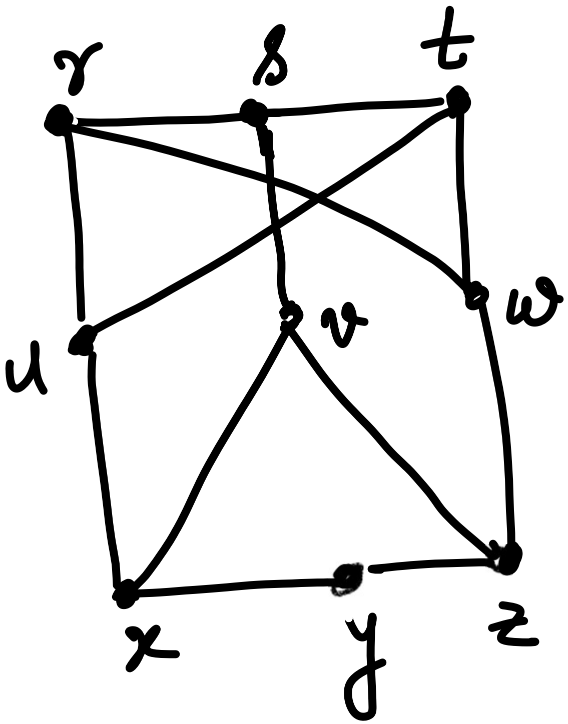
$\Rightarrow$ a vertex in a graph G dominates each of its neighbors and itself.



A subset of vertices $S \subseteq V$ is called a dominating set of G, if each $v \in V \setminus S$, has a neighbor in S.

A dominating set is minimal if no vertex can be removed.

Dominating Set : Examples



$S_1 = \{r, t, x, z\}$

$S_2 = \{s, u, w, y\}$

$S_3 \ \{s, t, y\}$

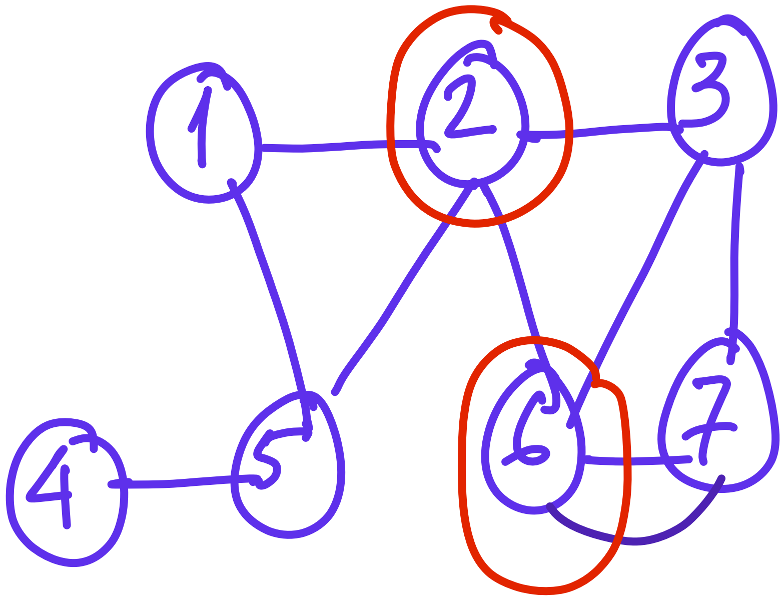
minimal

minimum

$S_i \Rightarrow$ independent dominating set

$S_3 \Rightarrow$ not independent, because $s - t$

Dominance



Two bests: 2, 6

Best two: 3, 5

"Two bests" are not the "best two"
(k best-selections may not be the best-k selection!)

Finding minimum dominating set ⇒ NP-hard
⇒ greedy may not lead to optimality

Dominance

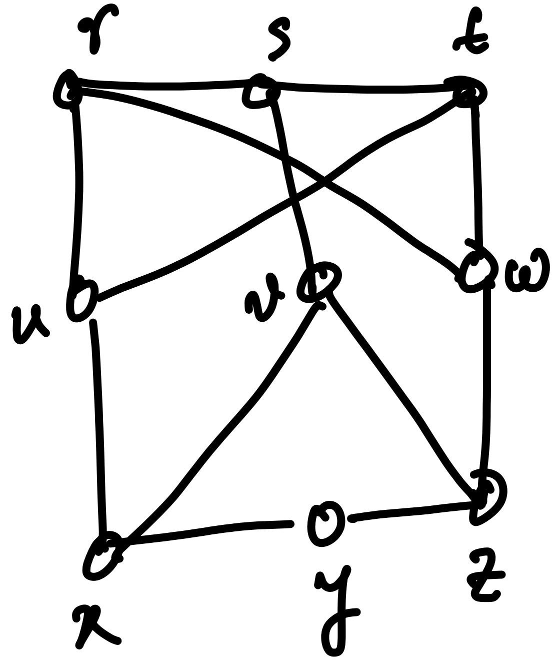
Lemma (DW: 3.1.33)
A set S of vertices in a graph is an independent dominating set if and only if it is a maximal independent set.

Proof: S is an independent. S will be maximal, if every vertex in $G \setminus S$ is adjacent to at least one vertex in S.

$\boxtimes$

Domination number $\gamma(G)$

$\rightarrow$ Cardinality of the minimum dominating set.



Minimum dominating set $= \{s, t, y\}$

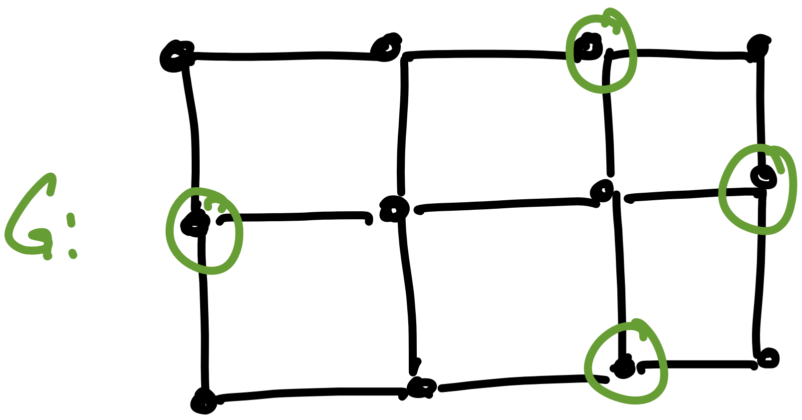
$\gamma(G) \leq 3$

Since $\Delta(G) = 3$, a dominating set with two vertices can cover at most $(2 \times 4) = 8$ vertices.

Since $|V| = 9$, $\gamma(G) = 3$ $\boxtimes$

Example: City-road watching problem

$|V| = 12$



$\gamma(G) \le 4$

$\boxed{\text{We claim that } \gamma(G) = 4}$

Proof: Assume false, i.e., $\gamma(G) = 3$

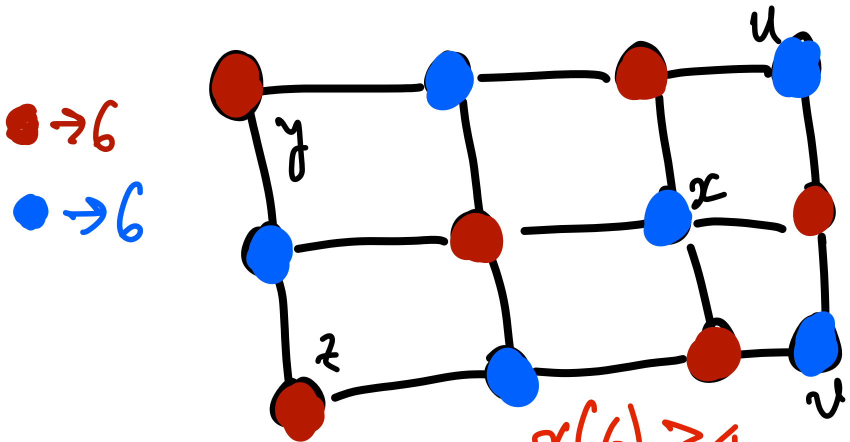
$3R \to$ dominates only $3R \Rightarrow$ Contrad.

$3B \to$ dominates only $3B \Rightarrow$ Contradic.

$2R, 1B \to d(B) = 3 \Rightarrow 5R \Rightarrow$ Contrdc.

$\hookrightarrow d(B) = 4 \Rightarrow$ choose "x"

$\Rightarrow S = \{x, y, z\} \Rightarrow u, v$ not covered.



$\gamma(G) \ge 4 \Leftarrow 2B, 1R \Rightarrow$ similar contradiction. $\boxtimes$

Homework

1. Construct a regular, simple cubic graph, with ten vertices.

2. Count the number of 6-cyles in $K_{5,\,5}$.

3. Prove that a self-complementary graph with n vertices exists if and only if n or n-1 is divisible by 4.

4. Prove that a bipartite graph G admits a unique bipartition of the vertex set if and only if G is connected.

5. Let G be a connected simple graph that does not have P_4 or C_4 as an induced subgraph. Prove that $\gamma(G) = 1$, where $\gamma(G)$ denotes the domination number of G.

In the next class, students will be asked to discuss the solutions.