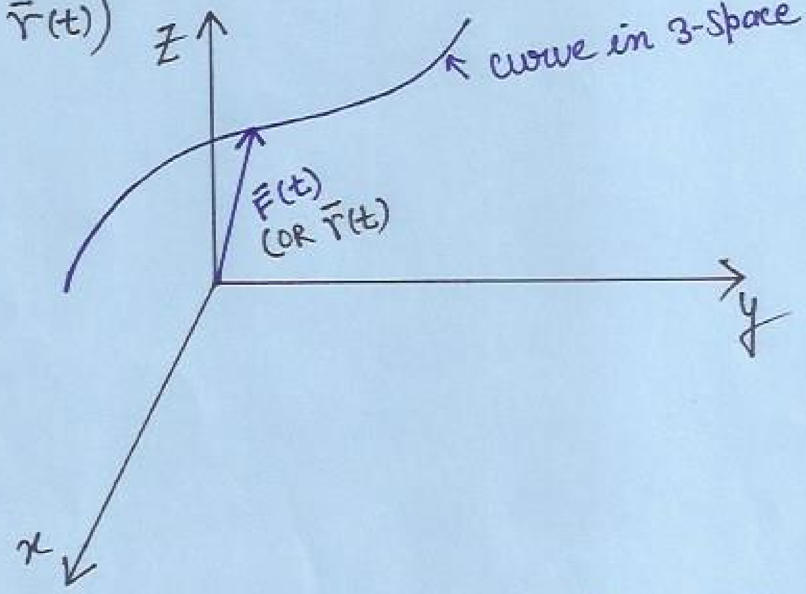
SCALAR and VECTOR Fields

Vector function of one variable: (Parametric representation of curves ~~surfaces~~)

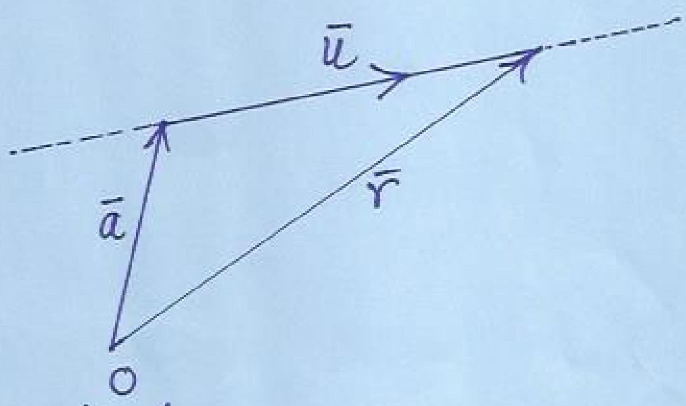
$$\vec{F}(t) = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}, \quad a \le t \le b$$

(OR $\bar{r}(t)$)



OR in 2d space $\bar{r}(t) = x(t)\,\hat{i} + y(t)\,\hat{j}, \quad a \le t \le b$

Examples! Let $\bar{a}$ be the position vector of a particular fixed point on the line and $\bar{u}$ be the vector pointing along the line.



Equation of the straight line: $\bar{r} = \bar{a} + \lambda \bar{u}$

Limit and Continuity of vector functions:

Limit: $\lim_{t \to a} |\bar{r}(t) - \bar{l}| = 0$

Continuity: The function $\bar{r}(t)$ is said to be continuous at $t = a$ if

(i) $\bar{r}(t)$ is defined in some neighbourhood of a

(ii) $\lim_{t \to a} \bar{r}(t)$ exists and

(iii) $\lim_{t \to a} \bar{r}(t) = \bar{r}(a)$.

Differentiability: $\bar{r}(t)$ is said to be differentiable if

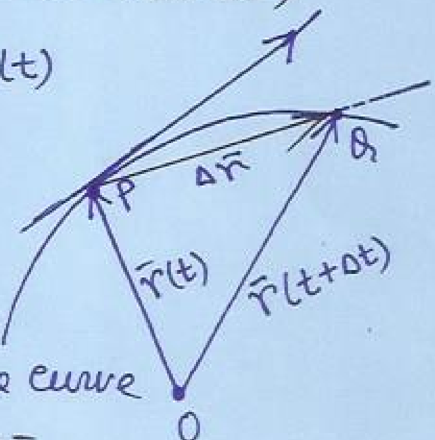
$$\lim_{\Delta t \to 0} \frac{\bar{r}(t+\Delta t) - \bar{r}(t)}{\Delta t} \text{ exists.}$$

Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the parametric representation of a curve C, then

$$\frac{d\bar{r}}{dt} = r'(t) = \frac{dx(t)}{dt}\hat{i} + \frac{dy(t)}{dt}\hat{j} + \frac{dz(t)}{dt}\hat{k}$$

Geometric representation of $r'(t)$: (tangent to a curve)

Note that the direction of $\Delta\bar{r} = \bar{r}(t+\Delta t) - \bar{r}(t)$ and $\frac{\Delta \bar{r}}{\Delta t}$ is the same.



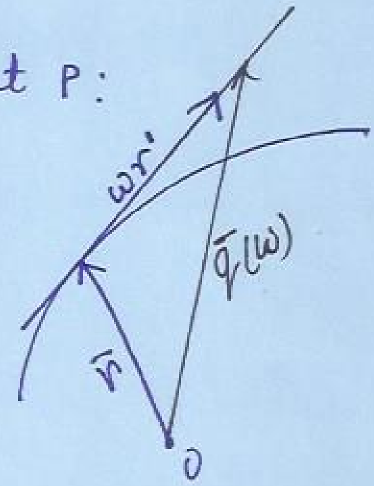
Then the limiting position of the vector $\frac{\Delta \bar{r}}{\Delta t}$, i.e., $\lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t}$ is the tangent to the curve at P.

tangent vector $= \bar{r}'(t) = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t}$.

Unit tangent vector $\bar{u} = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

Equation of the tangent to c at P:

$$\bar{q}(\omega) = \bar{r} + \omega \bar{r}'$$



Partial derivatives of a vector function:

Let $\bar{r} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ & v_1, v_2, v_3 are differentiable functions of n variables $t_1, t_2, \cdots t_n$.

Then the partial derivative of v with respect to t_j is given by

$$\frac{\partial \bar{v}}{\partial t_j} = \frac{\partial v_1}{\partial t_j}\hat{i} + \frac{\partial v_2}{\partial t_j}\hat{j} + \frac{\partial v_3}{\partial t_j}\hat{k}$$

Example: (i) Find $v'(t)$ for $v(t) = (\cos t + t^2)(t\hat{i} + \hat{j} + 2\hat{k})$

$$v'(t) = (3t^2 - t\sin t + \cos t)\hat{i} + (2t - \sin t)(\hat{j} + 2\hat{k})$$

ii) Partial derivatives:

$$\bar{r}(t_1, t_2) = a\cos t_1\, i + a\sin t_1\, j + t_2 K$$

$$\frac{\partial \bar{r}}{\partial t_1} = -a\sin t_1\, i + a\cos t_1\, j$$

$$\frac{\partial \bar{r}}{\partial t_2} = \bar{K}$$

□

VECTOR Field:

A vector field in 3d space is a 3 components vector and the compoments are function of 3 variables.

A vector field in the plane is a 2 component vector whose components are functions of two variables.

A vector field in 3d-Space:

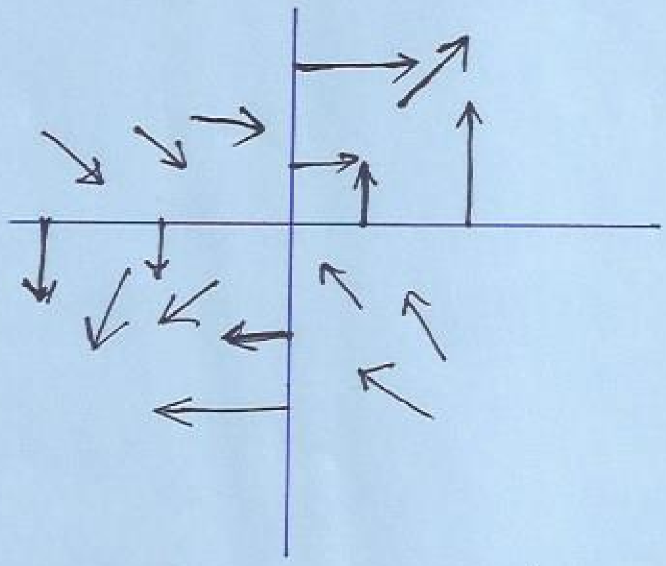
$$G(x,y,z) = f(x,y,z)\,\hat{i} + g(x,y,z)\,\hat{j} + h(x,y,z)\,\hat{k}$$

A vector field in 2d-Space:

$$K(x,y) = f(x,y)\,\hat{i} + g(x,y)\,\hat{j}$$

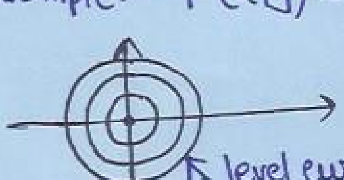
Example: Velocity of the air within a room.

Example: $U(x,y) = y i + x j$



Similarly, a scalar field is defined (temperature inside a room)

Example: $T(x,y) = x^2+y^2$; visualization through level curves $x^2+y^2 = \text{const}$.



level curves of scalar field $T(x,y) = x^2+y^2$.

(level surface in 3D case)

- Gradient of a scalar function $f(x,y,z)$ is a vector given by

$$\text{grad } f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

- Nebla or Del operator:

$$\nabla \equiv \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z}\hat{k} \quad \text{or} \quad \nabla \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

So, $\text{grad } f = \nabla f$.

- If a surface is given by $f(x,y,z) = c$, then $\nabla f(P)$ is the vector normal to the surface $f(x,y,z) = C$ at the point P.

Consider a smoth curve C on the surface passing through the point P on the surface. Let $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ be the position vector of P.

Since the curve lies on the surface, we have

$$f(x(t), y(t), z(t)) = C$$

Then $\frac{d}{dt} f(x(t), y(t), z(t)) = 0 \Rightarrow \frac{\partial f}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\cdot\frac{dz}{dt} = 0$

$$\Rightarrow \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}\right)\cdot\left(\frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}\right) = 0$$

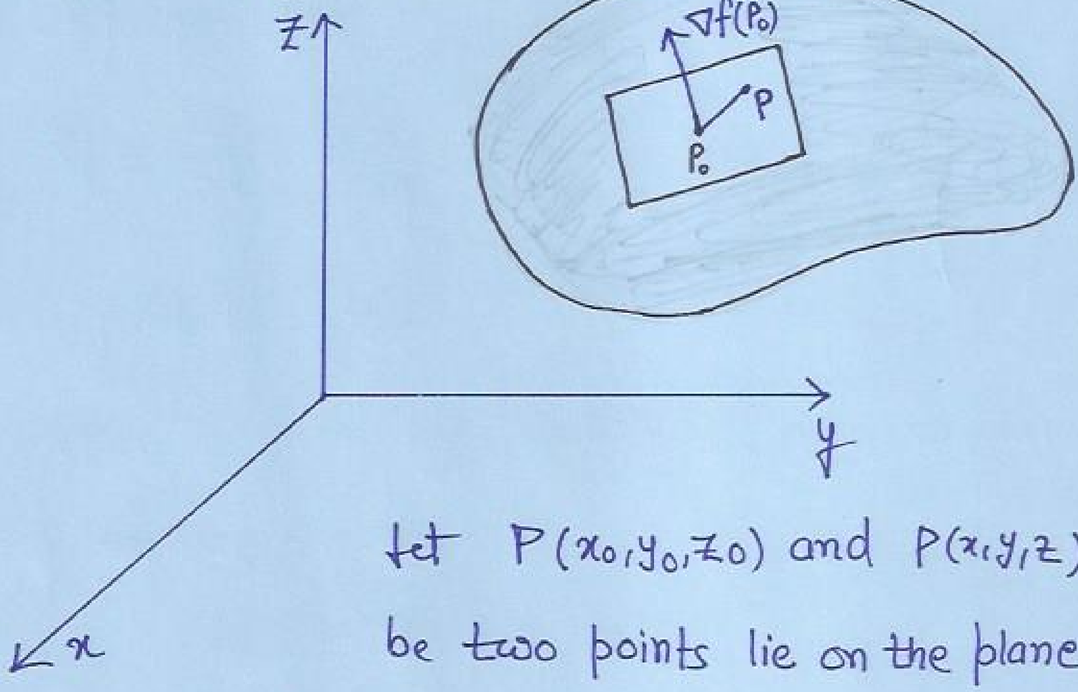
$$\Rightarrow \nabla f \cdot \bar{r}'(t) = 0$$

Note that $\bar{r}'(t)$ is a tangent vector at P and lies in the tangent plane at P. $\Rightarrow \nabla f(P)$ is a vector normal to the surface $f(x,y,z) = C$ at P.

- Unit normal vector to a surface $f(x,y,z) = C$

$$\hat{n} = \frac{\nabla f}{|\nabla f|}$$

- Equation of the tangent plane:



Let $P(x_0, y_0, z_0)$ and $P(x, y, z)$ be two points lie on the plane.

$\Rightarrow$ $\vec{P_0P}$ lies on the tangent plane

$$\Rightarrow \vec{P_0P} \cdot \nabla f(P_0) = 0 \quad \text{(Perpendicular lines)}$$

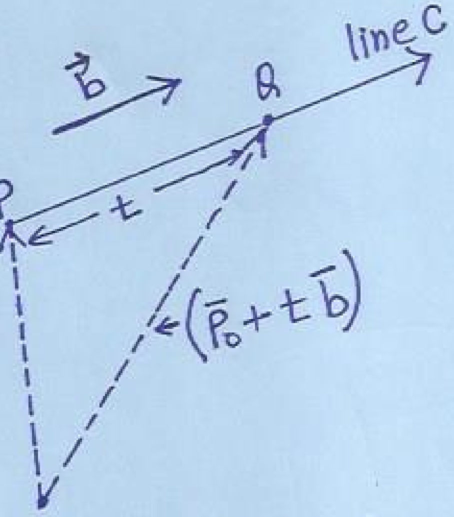
$$\Rightarrow \left((x-x_0)\hat{i} + (y-y_0)\hat{j} + (z-z_0)\hat{k}\right) \cdot \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial K}\hat{k}\right) = 0$$

$$\Rightarrow \boxed{(x-x_0)\frac{\partial f}{\partial x}(P_0) + (y-y_0)\frac{\partial f}{\partial y}(P_0) + (z-z_0)\frac{\partial f}{\partial z}(P_0) = 0}$$

- DIRECTIONAL DERIVATIVE OF $f(x,y,z)$ ALONG $\vec{b}$

  - Generalization of the notion of partial derivatives

In partial derivative: Direction is parallel to one of the coordinate axis.



Let $|\bar{b}| = 1$.

Position vector of the line C is: $r(t) = \bar{P}_0 + t\bar{b} = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

Using chain rule:

$$\lim_{t \to 0} \frac{f(Q) - f(P)}{t} = \frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

$\uparrow$

rate of change of f in the direction of $\vec{b}$

$$= \left(\frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} \hat{k}\right) \cdot \left(\frac{dx}{dt} \hat{i} + \frac{dy}{dt} j + \frac{dz}{dt} k\right)$$

$$= \nabla f \cdot \frac{d\bar{r}}{dt}$$

$$= \nabla f \cdot \vec{b} \qquad \left(\nabla f|_P \cdot \vec{b}\right)$$

At any point P, the directional derivative of f represents the rate of change in f along $\bar{b}$ at the point P, it is denoted by

$$D_b f = \nabla f \cdot \bar{b}$$

Remark: Directional derivative of f in the i direction

$$= \nabla f \cdot \hat{i}$$

$$= \frac{\partial f}{\partial x}.$$

Maximum rate of change of a scalar field

Note that

Rate of change of f in the direction of $\vec{b}$ is

$$D_b f = \bar{\nabla} f \circ \bar{b} = |\bar{\nabla} f|\,|\bar{b}| \cos\theta = |\bar{\nabla} f| \cos\theta$$

$$\Rightarrow \quad -|\nabla f| \leq D_b f \leq |\nabla f| \quad \text{since } -1 \leq \cos\theta \leq 1$$

$\Rightarrow$ Rate of change is maximum when θ is 0, that is, in the direction of $\bar{\nabla} f$.

$\Rightarrow$ Rate of change is minimum when θ is π, that is, in the opposite direction of $\bar{\nabla} f$.

$\Rightarrow$ Gradient vector $\bar{\nabla} f$ points in the direction in which f increases most rapidly and $-\nabla f$ points in the direction in which f decreases most rapidly.

Example: Find the unit normal to the surface $x^2+y^2-z=0$ at the point $(1,1,2)$.

Solution: Define $f = x^2+y^2-z \Rightarrow \nabla f = (2x, 2y, -1)^T$

$$\nabla f(1,1,2) = (2,2,-1)^T$$

Unit normal vector $\hat{n} = \frac{1}{\sqrt{4+4+1}}(2,2,-1)^T = (2/3, 2/3, -1/3)^T$

The other unit normal vector is $-\hat{n} = (-2/3, 2/3, -1/3)^T$.

Example: Find the directional derivative of the scalar field $f = 2x + y + z^2$ in the direction of the vector $(1,1,1)$ and evalute this at the origin.

Sol: $\nabla f = (2, 1, 2z)$

$$D_{(1,1,1)} f = \nabla f \cdot \frac{(1,1,1)}{\sqrt{3}}$$

$$= (2, 1, 2z) \cdot \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} + \frac{2}{\sqrt{3}} z$$

Value at the origin : $\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \sqrt{3}$.

Conservative vector field:

A vector field $\bar{v}$ is said to be conservative if the vector function can be written as the gradient of a scalar function f, that is, $\bar{v} = \nabla f$.

The function f is called a potential function or a potential of $\bar{v}$.

Example: Show that the vector field $\bar{F} = (2x+y, x, 2z)$ is conservative.

Sol: F is conservative if it can be written as $\bar{F} = \nabla \varphi$.

$$\Rightarrow \underbrace{\frac{\partial \varphi}{\partial x} = 2x + y}, \quad \underbrace{\frac{\partial \varphi}{\partial y} = x}, \quad \frac{\partial \varphi}{\partial z} = 2z$$

$$\varphi = x^2 + xy + h(y,z) \Rightarrow x = x + \frac{\partial h}{\partial y} \Rightarrow \frac{\partial h}{\partial y} = 0 \Rightarrow h \text{ is indep. of } y.$$

Using the last eq. $2z = 0 + \frac{dh}{dz} \Rightarrow h = z^2 + c$

$$\Rightarrow \varphi = x^2 + xy + z^2 + c$$

Ans.

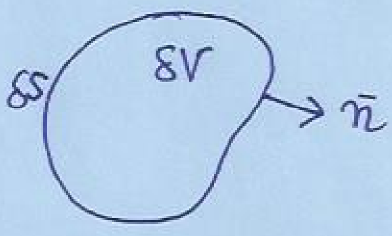
Divergence of a vector field

The divergence of a vector field $\bar{v}$ is defined as

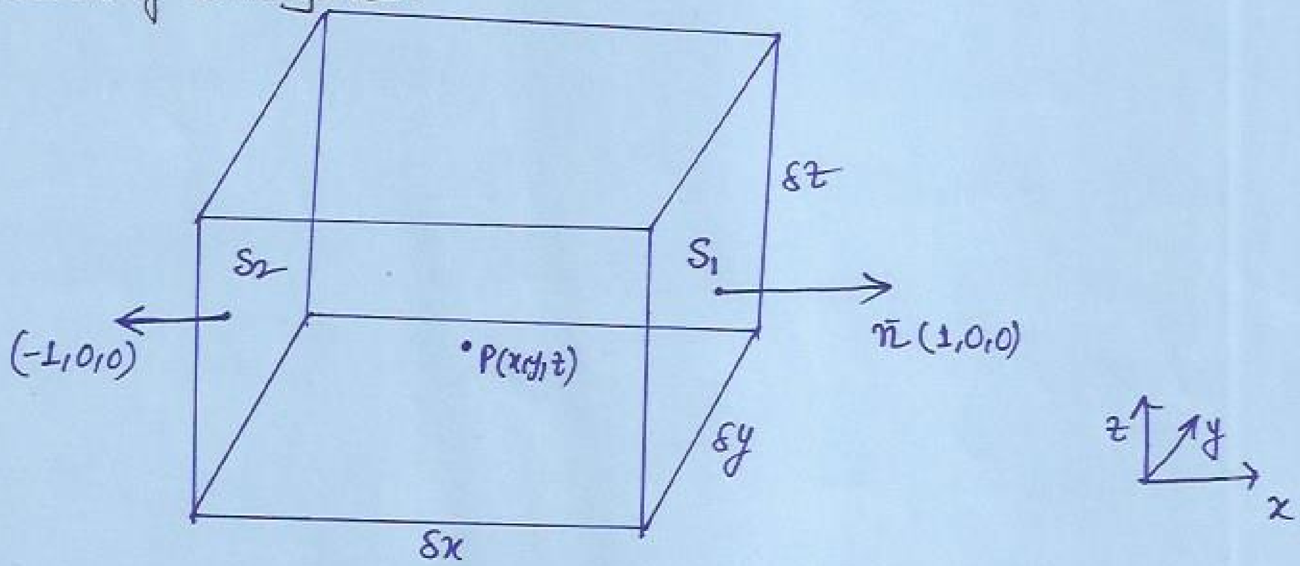
$$\operatorname{div} \bar{v} = \lim_{\delta V \to 0} \frac{1}{\delta V} \iint_{\delta S} \bar{v} \cdot \bar{n} \, ds$$

flux of the vector field v out of a small closed surface.

where δV is a small volume enclosing P with surface δS and $\bar{n}$ is the outward pointing normal to δS.



Computation of Divergence:



$$\iint_{S_1} \bar{u} \cdot \bar{n} \, ds \simeq u_1\left(x + \tfrac{\delta x}{2}, y, z\right) \delta y \, \delta z$$

$$\iint_{S_2} \bar{u} \cdot \bar{n} \, ds \simeq -u_1\left(x - \tfrac{\delta x}{2}, y, z\right) \delta y \, \delta z$$

$$\Rightarrow \iint_{S_1 + S_2} \bar{u} \cdot \bar{n} \, ds \simeq \left(u_1\left(x + \tfrac{\delta x}{2}, y, z\right) - u_1\left(x - \tfrac{\delta x}{2}, y, z\right) \right) \delta y \, \delta z$$

$$= \frac{\partial u_1}{\partial x} \delta x \, \delta y \, \delta z \simeq \frac{\partial u_1}{\partial x} \delta V$$

Similarly from other sides

$$\iint_{S_3+S_4} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_2}{\partial y}\,\delta V$$

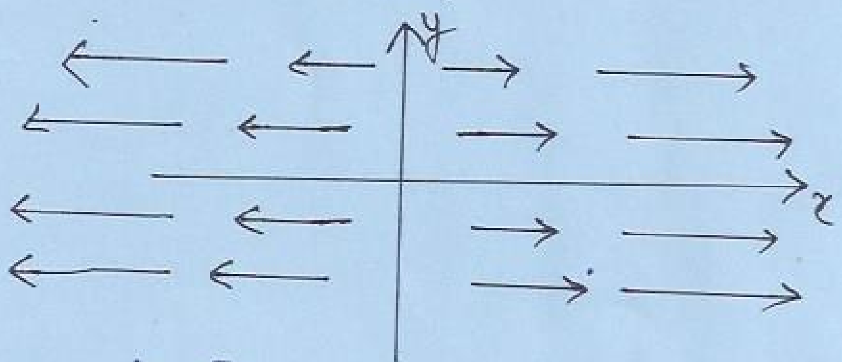
& $$\iint_{S_5+S_6} \bar{u}\cdot\bar{n}\, ds \simeq \frac{\partial u_3}{\partial z}\,\delta V$$

Therefore, $\operatorname{div} \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$

OR $\boxed{\operatorname{div} \bar{u} = \bar{\nabla}\cdot\bar{u}}$

Physical Interpretation: Divergence can be interpreted as the rate of expansion or compression of the vector field.

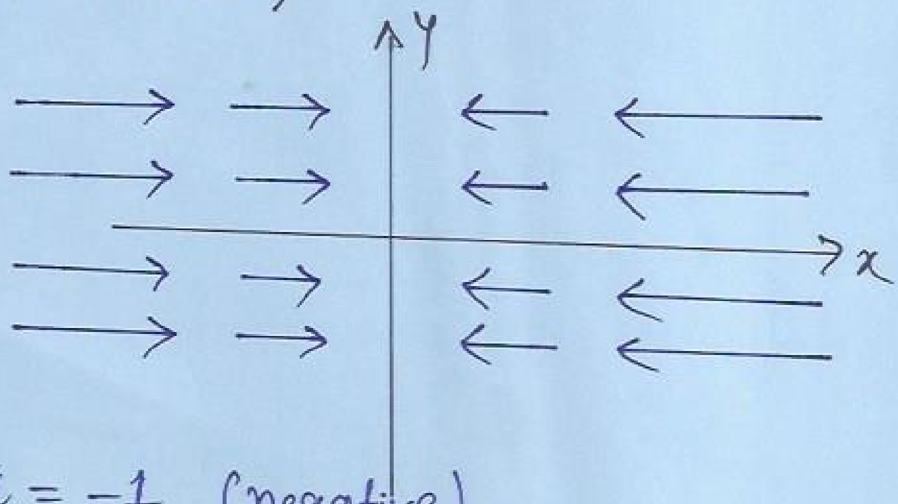
Example-1: $\bar{u} = (x, 0, 0)$



Tendency of fluid is EXPANSION.

$\operatorname{div} \bar{u} = 1$ (positive)

Example 2: $\bar{u} = (-x, 0, 0)$

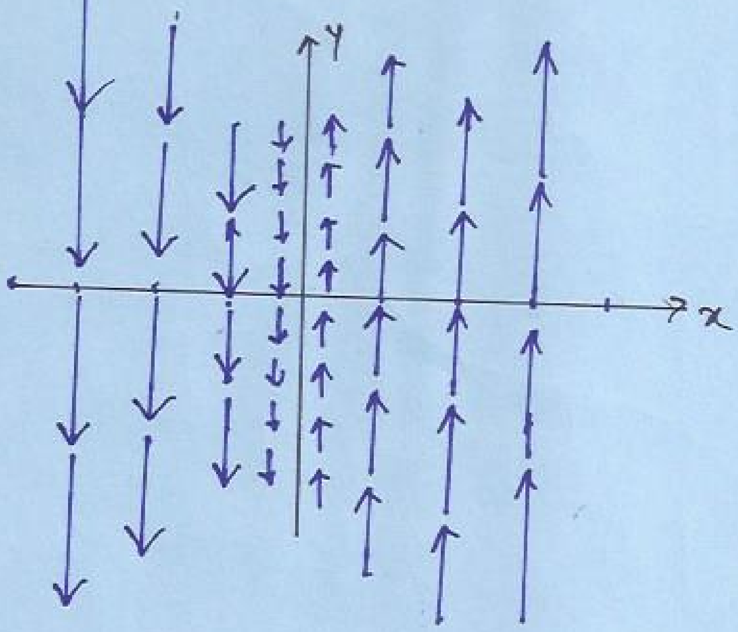


Tendency of the fluid is compression

$\operatorname{div} \bar{u} = -1$ (negative)

Example: (3):

$\bar{u} = (0, x, 0)$



Neither expanding nor contracting

$$\mathrm{div}(\bar{u}) = 0$$

Note: A vector field $\bar{v}$ for which $\bar{\nabla}\cdot\bar{v} = 0$ everywhere is said to be solenoidal. The relation $\mathrm{div}\,\bar{v} = 0$ is also known as the condition of incompressibility.

CURL OF A VECTOR FIELD:

Curl of a vector field is given by

$$\mathrm{Curl}\,\bar{F} = \bar{\nabla}\times\bar{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_1 & v_2 & v_3 \end{vmatrix}$$

where $\bar{v} = (v_1\hat{i} + v_2\hat{j} + v_3\hat{k})$

$$= \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}.$$

Physical Interpretation: Its signifies the tendency of ROTATION.

The vector $\mathrm{Curl}\,\bar{v}$ is directed along the axis of rotation with magnitude twice the angular speed.

Example: (Same as discussed in divergence section)

i) $\bar{u} = (x, 0, 0)$

$$\nabla \times u = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & 0 & 0 \end{vmatrix} = i \cdot 0 - j \cdot 0 + k \cdot 0 = 0$$

No sense of rotation

ii) $\bar{u} = (-x, 0, 0)$

Again $\nabla \times u = 0 \Rightarrow$ No sense of rotation

iii) $\bar{u} = (0, x, 0)$

$$\nabla \times u = \hat{k}$$

→ Rotation is about an axis in the z-direction.

NOTE: A vector field $\bar{u}$ for which $\nabla \times \bar{u} = 0$ everywhere is said to be irrotational.

Curl and Conservative vector field: Suppose $\bar{u}$ is conservative, i.e.,

$$\bar{u} = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$

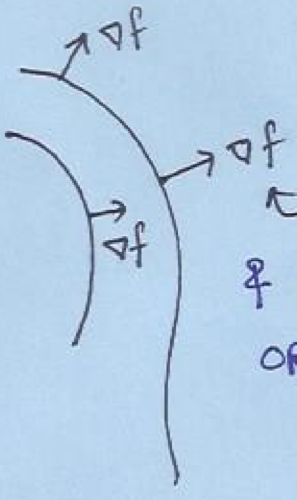
$$\nabla \times \bar{u} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} & \frac{\partial \varphi}{\partial z} \end{vmatrix} = i\left(\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial y}\right) \right) + j\left(\frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial x}\right) - \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial z}\right) \right)$$

$$+ k\left(\frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial y}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial x}\right) \right) = 0$$

Any vector field that can be written as the gradient of a scalar field is IRROTATIONAL.

Summary

a) GRADIENT



← Normal vector

& max rate of change of a scalar field

OR greatest rate of change (increase) of a function

b) DIVERGENCE:

Tendency of compression or expansion

c) CURL:

Tendency to rotate

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot \bar{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z}$$

If $\nabla \cdot \bar{u} = 0$, $\bar{u}$ is said to be SOLENOIDEL

If $\nabla \times \bar{u} = 0$, $\bar{u}$ is said to be IRROTATIONAL

If $\bar{u} = \nabla f$, $\bar{u}$ is said to be CONSERVATIVE.