1) Find $f(0.05)$ using Newton's forward difference formula;

x	0	0.1	0.2	0.3	0.4
$f(x)$	1	1.2214	1.4918	1.8221	2.2255

The difference table →

x	$f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	1				
		0.2214			
0.1	1.2214		0.0490		
		0.2704		0.0109	
0.2	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.3	1.8221		0.0731		
		0.4034			
0.4	2.2255				

From Newton's forward difference formula,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{6}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{24}\Delta^4 y_0 + \cdots$$

$x = 0.05$, $x_0 = 0$, $u = \frac{x - x_0}{h} = \frac{0.05 - 0}{0.1}$

$= 0.5$.

$\therefore f(0.05)$

$$= 1 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2} \times 0.0490 + \frac{0.5(0.5-1)(0.5-2)}{6} \times 0.0109 + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24} \times 0.0023$$

$= 1.105166$

2) Find $f(1.5)$ using Newton's forward difference formula:

x	0	2	4	6	8
$f(x)$	-1	13	43	89	151

The difference table is →

x	$f(x)$	Δy	$\Delta^2 y$
0	-1		
		14	
2	13		16
		30	
4	43		16
		46	
6	89		16.
		62	
8	151		

Here $x = 1.5$, $x_0 = 0$, $h = 2$, $u = \frac{x - x_0}{h} = \frac{1.5}{2}$

$= 0.75$

From Newton's forward difference formula,

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2} \Delta^2 y_0 \cancel{+ u(u-1)} + \ldots$$

$$f(1.5) = -1 + 0.75 \times 14 + \frac{0.75(0.75-1)}{2} \times 16$$

$= 8.$

3) Find $\log_{10} 2.91$ using Newton's backward difference formula:

~~Back~~ x	2.0	2.2	2.4	2.6	2.8	3.0
$f(x)$	0.30103	0.34242	0.38021	0.41497	0.44716	0.47721

Backward difference table:

x	$y=f(x)$	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
2.0	0.30103					
		0.04139				
2.2	0.34242		−0.00360			
		0.03779		0.00057		
2.4	0.38021		$-0.00\cancel{257}$ 303		−0.00011	
		0.03476		0.00046		0.00008
2.6	0.41497		−0.00257		−0.00003	
		0.03219		0.00043		
2.8	0.44716		−0.00214			
		0.03005				
3.0	0.47721					

Here $x = 2.91$, $x_n = 3.0$, $h = 0.2$, $u = \dfrac{x - x_n}{h}$

$$= \frac{2.91 - 3.0}{0.2}$$

$$= -0.45$$

From Newton's Backward formula,

$$f(x) = y_n + u\nabla y_n + u(u+1)\frac{\nabla^2 y_n}{2!} + u(u+1)(u+2)\frac{\nabla^3 y_n}{3!}$$

$$+ u(u+1)(u+2)(u+3)\frac{\nabla^4 y_4}{4!} + u(u+1)(u+2)(u+3)(u+4)\frac{\nabla^5 y_n}{5!} + \ldots$$

$$= 0.47721 + (-0.45)\times 0.03005 + \frac{(-0.45)(-0.45+1)}{2}\times(-0.00214)$$

$$+ \frac{(-0.45)(-0.45+1)(-0.45+2)}{6}\times 0.00043 +$$

$$+ \frac{(-0.45)(-0.45+1)(-0.45+2)(-0.45+3)}{24}\times(-0.00003)$$

$$+ \frac{(-0.45)(-0.45+1)(-0.45+2)(-0.45+3)(-0.45+4)}{120}\times 0.00008$$ ~~(Neglecting 5th order difference)~~

$= \cancel{0.46388}$ $= 0.4639237$

Q4) $f(1.45)$? ? ?

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.0	0.24197				
		-0.02412			
1.1	0.21785		0.00046		
		-0.02366		0.00038	
1.2	0.19419		0.00084		-0.00004
		-0.02282		0.00034	
1.3	0.17137		0.00118		-0.00009
		-0.02164		0.00025	
1.4	0.14973		0.00143		
		-0.02021			
1.5	0.12952				

$\Delta^5 f(x)$

-0.00005

By backward interp. formula

Put $x_n = 1.5$ $\quad x = 1.45 \quad u = \frac{x - x_n}{h} = -0.5$

$$f(x) = f(x_n) + u\Delta f(x_{n-1}) + \frac{u(u+1)}{2!}\Delta f(x_{n-2}) + \cdots$$

$$f(1.45) = 0.12952 - 0.5(-0.02021) + \frac{(-0.5)\,0.5}{2}\,0.00143$$

$$+ \frac{(-0.5)(0.5)(1.5)}{6}\,0.00025$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)}{4 \times 3 \times 2}(-0.00009)$$

$$+ \frac{(-0.5)(0.5)(1.5)(2.5)(3.5)}{5!}(-0.00005)$$

$$= 0.12952 + 0.010105 - 0.00017875$$

$$- 0.000015625 + 0.00000035156$$

$$+ 0.00000109375$$

$$= 0.13949$$

Q5)

marks	0-19	20-39	40-59	60-79	80-99
no. of candidates	41	62	65	50	17

The difference table is

Number obtained	no of candidates $f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
Below 20	41			
		62		
Below 40	103		3	
		65		-18
Below 60	168		-15	
		50		-18
Below 80	218		-33	
		17		
Below 100	235			

Take newton's forward diff. formula

by $x_0 = 60$

$x = 70 \qquad x_0 = 60 \qquad h = 20$

$$f(x) = f(x_0) + \frac{(x-x_0)\Delta f(x_0)}{h} + \frac{(x-x_0)(x-x_1)\Delta^2 f(x_0)}{2!\, h^2}$$

Let $u = \frac{x-x_0}{h} \Rightarrow x = uh + x_0$

$x_r = x_0 + rh$

$x - x_r = (u - r) h$

So

$$f(x) = f(x_0) + u \Delta f(x_0) + \frac{u(u-1)}{2!} \Delta^2 f(x_0)$$

$$= 168 + 0.5 \times 50 + \frac{0.5(0.5-1)}{2} \times (-33)$$

$$= 197.125 \approx 197 \text{ candidates}$$

Q6) $f(0) = 0 \quad f\left(\frac{1}{2}\right) = -1 \quad f(1) = 0$

$x_1 = 0 \quad x_2 = \frac{1}{2} \quad x_3 = 1$

$$p(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} f(x_1)$$

$$+ \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} f(x_2)$$

$$+ \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} f(x_3)$$

$$= \cancel{\frac{x(x-1)}{0 - 1}}$$

$$\frac{(x - 0)(x - 1)}{\left(\frac{1}{2} - 0\right)\left(\frac{1}{2} - 1\right)} (-1)$$

$$= \frac{x(x-1)}{\frac{1}{2}\left(-\frac{1}{2}\right)} (-1) = 4x(x-1)$$

$$= \boxed{4x^2 - 4x}$$

$p(x) = 4x^2 - 4x$

Now

$|f(x - p(x)| =$ Remainder term

$$= |(x-0)\ (x-\tfrac{1}{2})(x-1)|\ \left|\frac{F'''(\xi)}{3!}\right|$$

$$\le \left|\frac{x(x-\frac{1}{2})(x-1)}{6}\right| = \left|\frac{x(x-1)(2x-1)}{12}\right|$$

If $0 \le x \le 1$

$$0 \le x \le 1$$

$$-1 \le x-1 \le 0$$

$$|x(x-1)| = |x^2 - x|$$

$$= \left|(x-\tfrac{1}{2})^2 - \tfrac{1}{4}\right|$$

$$= \left|\tfrac{1}{4} - (x-\tfrac{1}{2})^2\right| \le \tfrac{1}{4} \le 1$$

$$0 \le 2x \le 2$$

$$-1 \le 2x-1 \le 1$$

$$|2x-1| \le 1$$

So $|f(x) - p(x)| \le \frac{1}{12}$ Ans

(7) Prove that the sum of the Lagrangian functions or co-efficients is unity i.e

$$\sum_{r=0}^{n} w_r(x) = 1$$

Ans:

The functions

$$w_r(x) = \frac{w(x)}{(x-x_r)\,w'(x_r)} \quad (r = 0, 1, \cdots n)$$

are called the Lagrangian functions.

where $w(x) = (x-x_0)(x-x_1)\cdots(x-x_r)(x-x_{r+1})\cdots(x-x_n)$

& $w'(x_r) = (x_r - x_0)(x_r - x_1)(x_r - x_2)\cdots(x_r - x_{r-1})(x_r - x_{r+1})\cdots(x_r - x_n)$

Let

$$\frac{1}{w(x)} = \frac{A_0}{x-x_0} + \frac{A_1}{x-x_1} + \frac{A_2}{x-x_2} + \cdots + \frac{A_r}{x-x_r} + \cdots + \frac{A_n}{x-x_n}$$

$$\therefore \frac{1}{w(x)} = \frac{1}{w(x)}\Big[A_0(x-x_1)(x-x_2)\cdots(x-x_n) + A_1(x-x_0)(x-x_2)\cdots(x-x_n) + A_2(x-x_0)(x-x_1)(x-x_3)\cdots(x-x_n) + \cdots + A_r(x-x_0)\cdots(x-x_{r-1})(x-x_{r+1})\cdots(x-x_n) + \cdots + A_n(x-x_0)\cdots(x-x_{n-1})\Big] \quad \text{---} (*)$$

Since (*) is an identity in x & thus it is true for all values of x, Putting $x = x_0, x_1, \cdots x_n$ successively, we have

$$1 = A_0(x_0 - x_1)(x_0 - x_2)\cdots(x_0 - x_n)$$

$$\therefore A_0 = \frac{1}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} = \frac{1}{w'(x_0)}$$

$$1 = A_1(x_1 - x_0)(x_1 - x_2)\cdots(x_1 - x_n)$$

$$\therefore A_1 = \frac{1}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} = \frac{1}{w'(x_1)}$$

$\vdots$

$\therefore$ $A_r = \dfrac{1}{(x_r - x_0)\cdots(x_r - x_{r-1})(x_r - x_{r+1})\cdots(x_r - x_n)} = \dfrac{1}{w'(x_r)}$

$\therefore$ $A_n = \dfrac{1}{w'(x_n)}$

now substituting the values of A_i's $(i = 0, 1, \cdots n)$ in

(*) we get

$$1 = \sum_{r=0}^{n} \frac{w(x)}{(x - x_r)\,(w'(x_r))} = \sum_{r=0}^{n} w_r(x)$$

hence $\sum\limits_{r=0}^{n} w_r(x) = 1$ (proved)

(8) Use Lagrange's formula to find the value of y when $x = 102$ from the given data:

x	93	96.2	100	104.2	108.7
$y=f(x)$	11.38	12.80	14.70	17.07	19.91

Sol^n: Lagrange's Interpolation formula is

$$L(x) = W(x) \sum_{r=0}^{n} \frac{y_r}{D_r}$$

where $W(x) = (x-x_0)(x-x_1) \cdots (x-x_r) \cdots (x-x_n)$

& $D_r = (x - x_r)(x_r - x_0) \cdots (x_r - x_{r-1})(x_r - x_{r+1}) \cdots (x_r - x_n)$

To determine $L(x)$ at $x = 102$, we have the following data:

$x_0 = 93$, $x_1 = 96.2$, $x_2 = 100$, $x_3 = 104.2$, $x_4 = 108.7$

& $y_0 = 11.38$, $y_1 = 12.80$, $y_2 = 14.70$, $y_3 = 17.07$, $y_4 = 19.91$

Computational Table:

Row product →					D_r	y_r	$\frac{y_r}{D_r}$
(102−93) = 9	(93−96.2) = −3.2	(93−100) = −7	(93−104.2) = −11.2	(93−108.7) = −15.7	35449.344	11.38	0.000321
3.2	5.8	−3.8	−8	−12.5	−7052.8	12.80	−0.00181
7	3.8	2	−4.2	−8.7	1943.928	14.70	0.00756
~~11.2~~ 11.2	8	4.2	−2.2	−4.5	3725.568	17.07	0.00458
15.7	12.5	8.7	4.5	−6.7	−51477.356	19.91	−0.000387

$\sum_{r=0}^{4} \frac{y_r}{D_r} = 0.01026$

$W(102) = (102-93)(102-96.2)(102-100)(102-104.2)(102-108.7)$
$= 1538.856$

Hence $L(102) = 1538.856 \times 0.01026 = 15.789 \simeq 15.79$ Ans

(9) Find ~~the~~ by Lagrange's formula the interpolation polynomial which corresponds the following data:

x	-1	0	2	5
$y=f(x)$	9	5	3	15

Sol^n: Lagrange's interpolation formula is

$$L(x) = W(x)\left[\sum_{r=0}^{n} \frac{y_r}{D_r}\right] \quad , \quad y_r = f(x_r)$$

where
$$W(x) = (x-x_0)(x-x_1)\cdots(x-x_r)\cdots(x-x_n)$$
& $$D_r = (x-x_r)(x_r-x_0)\cdots(x_r-x_{r-1})(x_r-x_{r+1})\cdots(x_r-x_n)$$

Given
$$x_0=-1 \;,\; x_1=0 \;,\; x_2=2 \;,\; x_3=5$$
$$y_0=f(x_0)=9 \;,\; y_1=5 \;,\; y_2=3 \;,\; y_3=15$$

Computational Table:

			Row product $\longrightarrow$	D_r	y_r	$\frac{y_r}{D_r}$
$(x+1)$	-1	-3	-6	$-18(x+1)$	9	$-\frac{1}{2(x+1)}$
1	x	-2	-5	$10x$	5	$\frac{1}{2x}$
3	2	$(x-2)$	-3	$-18(x-2)$	3	$-\frac{1}{6(x-2)}$
6	5	3	$(x-5)$	$90(x-5)$	15	$\frac{1}{6(x-5)}$

$$W(x) = (x+1)\,x\,(x-2)(x-5)$$

$$\therefore L(x) = (x+1)\,x\,(x-2)(x-5)\left[-\frac{1}{2(x+1)} + \frac{1}{2x} - \frac{1}{6(x-2)} + \frac{1}{6(x-5)}\right]$$

$$= \frac{1}{6}\Big[-3x(x-2)(x-5) + 3(x+1)(x-2)(x-5) - (x+1)x(x-5) + (x+1)x(x-2)\Big]$$

$$= \frac{1}{6}\Big[(x-2)(x-5)\{-\cancel{3x}+\cancel{3x}+3\}$$

$$-(x+1)x(\cancel{x}-5-\cancel{x}+2)\Big]$$

$$= \frac{1}{6}\left[3(x^2-7x+10)+3(x^2+x)\right]$$

$$= \frac{1}{6}\left[3x^2-21x+30+3x^2+3x\right]$$

$$= \frac{1}{6}\left[6x^2-18x+30\right]$$

$$= x^2-3x+5$$

$\therefore$ Required polynomial is x^2-3x+5 Ans.

10)

(i) Here $f(x) = 4x - 3x^2$

$a = 0$; $b = 1$

$n = 10$

$\therefore h = \frac{b-a}{n} = \frac{1-0}{10} = 0.1$

i :	0	1	2	3	4	5	6	7	8	9	10
x_i :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$y_i = f(x_i)$:	0	0.37	0.68	0.93	1.12	1.25	1.32	1.33	1.28	1.17	1

The Trapezoidal rule is:

$$I_T = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1 + y_2 + \cdots + y_9)\right] = \int_0^1 (4x - 3x^2)\,dx$$

$$= \frac{0.1}{2}\left[0 + 1 + 2 \times 9.45\right]$$

$$I_T = 0.995 \text{ (Approx)}$$

(ii) The Simpson's one-third rule is

$$I_S = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{0.1}{3}\left[0 + 1 + 4 \times 5.05 + 2 \times 4.40\right]$$

$$= 1.00 \text{ (Approx)}.$$

Exact value: $\int_0^1 (4x - 3x^2)\,dx$

$= (2x^2 - x^3)\Big|_0^1$

$= 1$

Error in Trapezoidal method = Exact value − Approx. value

$= 1 - 0.995$

$= 0.005$

Error in Simpson's method $= 1 - 1$

$= 0.$

11)

$$I = \int_0^1 \frac{1}{1+x^2}\,dx = \left|\tan^{-1} x\right|_0^1 = \tan^{-1}(1) - \tan^{-1}(0)$$

$\therefore I = 0.785.$

Here, $a = 0,\ b = 1;\ n = 6;\ f(x) = \frac{1}{1+x^2}$

$\therefore h = \frac{1-0}{6} = 0.167$

i	0	1	2	3	4	5	6
x_i	0	.167	.334	.501	0.668	0.835	1
$y_i = f(x_i)$	1	0.973	0.9	0.799	0.691	0.589	0.5

i) Trapezoidal Rule:

$$I_T = \frac{h}{2}\left[y_0 + y_6 + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{h}{2}\left[1 + 0.5 + 2 \times 3.952\right]$$

$$I_T = \frac{0.167}{2}\left[1.5 + 7.904\right] = 0.785$$

$$\text{Error} = I - I_T = 0$$

ii) Simpson's one-third Rule:

$$I_S = \frac{h}{3}\left[y_0 + y_6 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.167}{3}\left[1 + 0.5 + 4 \times 2.361 + 2 \times 1.591\right]$$

$$I_S = 0.786$$

$$\text{Error} = I - I_S = -0.001$$

12)

$$I = \int_0^{\pi/2} e^{\sin x}\, dx.$$

$a = 0$; $b = \pi/2$ & $h = \frac{\pi}{12}$, $f(x) = e^{\sin x}$

$\therefore$ $n = \frac{\pi/2 - 0}{\pi/12} = 6$

i :	0	1	2	3	4	5	6
x_i :	0	$\pi/12$	$2\pi/12$	$\frac{3\pi}{12}$	$\frac{4\pi}{12}$	$\frac{5\pi}{12}$	$\pi/2$
$y_i = f(x_i)$:	1	1.2954	1.6487	2.0281	2.3774	2.6272	2.7183

i) Trapezoidal rule:

$$I_T^{Q} = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{\pi}{24}\left[3.7183 + 2 \times 9.9768\right]$$

$I_T = 3.09864$, correct upto 5-decimal places.

ii) Simpson one-third rule:

$$I_S = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{\pi}{36}\left[3.7183 + 4 \times 5.9507 \times 2 \times 4.0261\right]$$

$I_S = 3.10436$, correct upto five decimal places.

$$I = \int_0^1 \cos x \, dx.$$

$a = 1$, $b = 0$; $n = 5$; $f(x) = \cos x$

$$h = \frac{1-0}{5} = 0.2$$

Here, $n = 5$ (odd number), so we choose Trapezoidal method to evaluate the given integration over Simpson's one-third rule, as in Simpson's one-third rule the number of sub-interval (n) should be 'even'.

i	0	1	2	3	4	5
x_i	0	0.2	0.4	0.6	0.8	1
$y_i = f(x_i)$	1	0.9801	0.9211	0.8253	0.6967	0.5403

The Trapezoidal rule is:

$$I_T = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}\left[1 + 0.5403 + 2 \times 3.4232\right]$$

$I_T = 0.8387$, correct upto four decimal places.

14) $I = \int_0^1 e^{-x^2}\,dx$

$a = 0$; $b = 1$; $n = 10$; $f(x) = e^{-x^2}$

$h = \frac{b-a}{n} = 0.1$

i :	0	1	2	3	4	5	6	7	8	9	10
x_i :	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
y_i :	1	0.99	0.96	0.91	0.85	0.78	0.70	0.61	0.53	0.44	0.37

$\therefore I_S = \frac{h}{3}\left[y_0 + y_{10} + 4(y_1+y_3+y_5+y_7+y_9) + 2(y_2+y_4+y_6+y_8)\right]$

$I_S = \frac{0.1}{3}\left[1 + 0.37 + 4 \times 3.73 + 2 \times 3.04\right]$

$I_S = 0.746.$

Now, ϵ_S is ~~given by~~ bounded by,

$$cM_4 \le \epsilon_S \le cM_4^*$$

$$c = -\frac{(b-a)^5}{180 \times n^4}$$

; M_4 & M_4^* are the largest & smallest value of $f^4(x)$

$$\Rightarrow c = -\frac{(b-a)}{180} h^4$$

$$\Rightarrow c = -\frac{1}{1800000}$$

By considering the derivative $f^5(x)$, we find that largest values of: ~~$f^4(x) = 12e^{-x^2} - 48x^2e^{-x^2} + 16x^4e^{-x^2}$~~

$f^4(x) = e^{-x^2}(12 - 48x^2 + 16x^4)$ in $[0,1]$, occurs at $x = 0$ & smallest value ~~occur~~ occurs at $x = (2.5 - 0.5\sqrt{10})^{1/2}$

$\therefore M_4^* = 12$ & $M_4 = -7.419$

$\therefore \frac{-7.419}{1800000} \le \epsilon_S \le \frac{12}{1800000} \Rightarrow -0.0000052 \le \epsilon_S \le 0.000007$

15) Find $y = \ln x$.

Let $f(x) = \ln x$.

x	$y = \ln x$
2	0.69315
2.5	0.91629
3.0	1.09861

Lagrange's Interpolation formula:

$$L(x) = \sum_{r=0}^{n} f(x_r) \times \left(\prod_{\substack{k=0 \\ k \neq r}}^{n} \frac{(x - x_k)}{(x_r - x_k)} \right)$$

$$\therefore L(x) = \frac{(x-2.5)(x-3.0)}{(2-2.5)\times(2-3.0)} f(x_0) + \frac{(x-2)(x-3)}{(2.5-2)(2.5-3)} f(x_1) + \frac{(x-2)(x-2.5)}{(3-2)(3-2.5)} f(x_2)$$

$$= \frac{(x-2.5)(x-3.0)}{(-0.5)\times(-1.0)} \times 0.69315 + \frac{(x-2)(x-3)}{0.5\times(-0.5)} \times 0.91629 + \frac{(x-2)(x-2.5)}{1 \times 0.5} \times 1.09861$$

$$= (2x^2 - 11x + 15) \times 0.69315 - (4x^2 - 20x + 24) \times 0.91629 + (2x^2 - 9x + 10) \times 1.09861$$

$$= -0.08164x^2 + 0.81366x - 0.60761,$$

(Required quadratic poly.)

$$\therefore \ln(2.7) = L(2.7) = 0.9941164$$

Here $y = f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -1/x^2$

$f'''(x) = +\frac{2}{x^3}$

Hence the error in approximating,

$R_n(x) = \frac{(x-2)(x-2.5)(x-3)}{3!} f'''(\xi)$, where $2 < \xi < 3$

$= \frac{(x-2)(x-2.5)(x-3)}{6} \frac{2}{\xi^3}$

Now $\left|\frac{1}{\xi^3}\right| < \frac{1}{2^3} = \frac{1}{8}$.

when $x = 2.7$,

$|R_n(x)| \leq \left|\frac{(2.7-2)(2.7-2.5)(2.7-3)}{6} \cdot \frac{2}{8}\right| = 0.00175$.

Hence the error estimation is,

$|R_n(x)| \leq 0.00175$

16) We have $f(x) = \ln(1+x)$,

$f(1) = 0.693147$

$f(1.1) = 0.741937$

The Lagrange interpolating polynomial is obtained as

$$P_1(x) = \frac{x-1.1}{1-1.1} * (0.693147) + \frac{(x-1)}{1.1-1} * (0.741937)$$

$\Rightarrow P(1.04) = 0.712663$.

Now, the error in linear interpolation is given by,

$$TE = \frac{1}{2!}(x-x_0)(x-x_1) f''(\xi), \quad x_0 < \xi < x_1$$

Hence, $|TE| \le \frac{1}{2} \max_{1 \le x \le 1.1} |(x-x_0)(x-x_1)| \max_{1 \le x \le 1.1} |f''(x)|$

Since maximum of $(x-x_0)(x-x_1)$ is obtained at

$$x = \frac{x_0+x_1}{2}$$

& $f''(x) = \frac{1}{(1+x^2)}$, we get,

$$|TE| \le \frac{1}{2} \frac{(x_1-x_0)^2}{4} \max_{1 \le x \le 1.1} \left|\frac{1}{(1+x)^2}\right|$$

$$|TE| \le \frac{(0.1)^2}{8} * \frac{1}{4}$$

$\therefore |TE| \le 0.0003125$

17) The maximum error in linear interpolation is given by:

$$|TE| \leq \frac{h^2}{8} M_2 \text{, where } h = \text{step-size}.$$

$$\& \quad M_2 = \max_{0 \leq x \leq 1} |f''(x)|$$

$$= \max_{0 \leq x \leq 1} |30(1+x)^4|$$

$$M_2 = 480$$

$\therefore$ so, we choose h so that,

$$60h^2 \leq 0.00005$$

$$\Rightarrow \quad h \leq 0.00091.$$

18) Error in quadratic interpolation based on the points x_{i-1}, x_i & x_{i+1} is given by,

$$TE = \frac{(x-x_{i-1})\,(x-x_i)\,(x-x_{i+1})}{3!} f'''(\xi), \quad \cancel{x_{i-1} \le \xi \le x_{i+1}}$$

$$x_{i-1} < \xi < x_{i+1}$$

Take, $t = \frac{x-x_i}{h}$,

so, $$TE = \frac{(t-1)\,t\,(t+1)}{6} h^3 f'''(\xi)$$

$$-1 < \xi < 1$$

The extreme value of $g(t) = (t-1)\,t\,(t+1)$
$= t^3 - t$ occurs at $t = \pm \frac{1}{\sqrt{3}}$

Now, $\max\{|g(t)|\} = \frac{2}{3\sqrt{3}}$

Hence, $$|TE| \le \frac{h^3}{9\sqrt{3}} \max|f''(\xi)|$$

We have $f(x) = x^2 \ln(x)$, which gives

$$f''(x) = \frac{2}{x}$$

$$\max_{5 \le x \le 10} |f'''(x)| = \frac{2}{5}$$

Hence we ~~or~~ choose h such that,

$$\frac{h^3}{9\sqrt{3}}\left(\frac{2}{5}\right) \le 0.000005$$

$$\Rightarrow h \le 0.0580.$$