Assignment 6 - Problem 13.22

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1 Problem 13.22

In the SUBSET SUM problem, the input consists of a sequence of n integers $a_1, a_2, ..., a_n$ and two integers s, k, and the question is whether one can find a set $I \subseteq \{1, 2, ..., n\}$ of exactly k indices such that $\sum_{i \in I} a_i = s$. Prove that SUBSET SUM problem is W[1]-hard when parameterized by k.

1.1 Solution

We will show a reduction from EXACT UNIQUE HITTING SET problem (EUHS) to the SUBSET SUM problem. Let $(U, \mathcal{F}, k)$ be an instance of the EUHS. We will now try to construct an instance (A, s, k') of SUBSET SUM such that $(U, \mathcal{F}, k)$ is an YES-instance if and only if (A, s, k') is an YES-instance, where $A = [a_1, a_2, ..., a_n]$ is a sequence of n integers.

Let $\mathcal{F} = \{X_1, X_2, ..., X_m\}$ and $U = \{e_1, e_2, ..., e_n\}$ be the enumeration of the sets and elements in $\mathcal{F}$ and U, respectively. Define, $N_e = \{j \mid e \in X_j\}, \forall e \in U$ (the indices of the sets e belong to). Also, let $B = k + 1$. Now create,

- $A = [a_1, a_2, ..., a_n]$, where $a_i = \sum_{j \in N_{e_i}} B^{j-1}$.
- $s = \displaystyle\sum_{j=0}^{m-1} B^j$
- $k' = k$

The intuitive idea behind the construction is to take a sufficiently large base B, and map each set in $\mathcal{F}$ to a unique power of B. And each element in the universe U is transformed into an integer which is the sum of the assigned value of the sets it belong to. And we choose the value s as the sum of all the values of the assigned values of the sets in $\mathcal{F}$. This is a valid reduction, as the only way to achieve the sum is to choose elements such that they belong to each set exactly once.

1.1.1 Reduction Validity

The parameter k' is bounded by $g(k) = k$, and the reduction can be carried out in polynomial time of the instance size, as we use only at most $|\mathcal{F}| \log k$ bits at most for the computation of the integers a_i and s. Now, we will show that $(U, \mathcal{F}, k)$ is an YES-instance if and only if (A, s, k') is an YES-instance.

For the forward direction assume $Z \subseteq U$, $|Z| = k$, is a solution of the EUHS. Let $Z = \{e_{j_1}, e_{j_2}, ..., e_{j_k}\}$ be its elements. Then, $I = \{j_1, j_2, ..., j_k\}$ is a solution of the SUBSET SUM problem because, for each $1 \leq j \leq m$, it occurs in exactly one of the $N_e, e \in Z$, as for each set $X \in \mathcal{F}$, $|Z \cap X| = 1$ and since $a_i = \displaystyle\sum_{j \in N_{e_i}} B^{j-1}$, $\displaystyle\sum_{j \in I} a_j = \sum_{j=0}^{m-1} B^{j-1} = s$

For the other direction, assume $I = \{i_i, i_2, ..., i_k\}$ is a solution of the SUBSET SUM problem. Here, if we consider all the summation operations in base B, since maximum coefficient of each power of B in a_i is 1, and we sum at most k elements, there won't be any carry overs (as $B > k$) and it will be enough if we just look at the individual coefficients of all B^i separately. Since, I is a solution of SUBSET SUM problem, $\displaystyle\sum_{j \in I} a_j = s = \sum_{j=0}^{m-1} B^{j-1}$. The coefficient of each power B^i, $\forall\ 0 \leq i \leq m-1$ is 1 in s, and since there are no carryovers, each power of B comes from exactly one element a_j. As there is a unique mapping from the sets in $\mathcal{F}$ to $B^i, \forall\ 0 \leq i \leq m-1$, the corresponding elements of I in the EUHS instance, $Z = \{e_{i_1}, e_{i_2}, ..., e_{i_k}\}$, will be a solution for the EUHS.

We know that EUHS is W[1]-hard (Theorem 13.32 in book). Therefore, from the above reduction we can see that SUBSET SUM problem is also W[1]-hard.