Assignment 1 - Problem 2.8

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1 Problem 2.8

In the **MIN-ONES-2-SAT** problem, we are given a 2-CNF formula ϕ and an integer k, and the objective is to decide whether there exists a satisfying assignment for ϕ with at most k variables set to true. Show that **MIN-ONES-2-SAT** admits a polynomial kernel.

1.1 Solution

Let ϕ denote the given instance of **MIN-ONES-2-SAT** problem, and $C(\phi)$ denote the clauses in it. Also, let $L(\phi)$ denote all the literals occurring in ϕ. Now, we will try to convert ϕ to an equivalent instance of **VERTEX-COVER**. Create a directed graph G_ϕ with $V(G_\phi) = L(\phi)$, and $(\overline{x}, y), (\overline{y}, x) \in E(G_\phi), \forall (x \vee y) \in C(\phi)$.

Claim 1. If there exists a directed path from x to y, then for an assignment ψ satisfying ϕ, $\psi(x) = 1 \implies \psi(y) = 1$.

Proof If G_ϕ contains the directed arc (x, y), it means $(\overline{x} \vee y) \in C(\phi)$. So, if $\psi(x) = 1$, then $\psi(\overline{x}) = 0$. Since ψ satisfies ϕ, $\psi(y) = 1$. Now by induction on path length, we can say $\psi(x) = 1 \implies \psi(y) = 1$, if there exists a directed path from x to y in G_ϕ. $\square$

Claim 2. If x and $\overline{x}$ are part of a directed cycle in G_ϕ, then there is no assignment that satisfies ϕ.

Proof If G_ϕ contains a directed cycle with both x and $\overline{x}$, then by *Claim 1*, both x and $\overline{x}$ has to be 1 in a satisfying assignment which is not possible. Hence, there is no assignment that satisfies ϕ. $\square$

So, from now on we will handle only instances that are satisfiable as otherwise, it is a NO-instance. Now let us create a closure ϕ^* from ϕ. For every pair of literal x, $y \in L(\phi)$, such that there is a directed path from $\overline{x}$ to y in G_ϕ, add the clause $(x \vee y)$ to ϕ^*. Moreover, ϕ^* contains all the clauses present in ϕ. Now ϕ^* is the required closed form of ϕ.

Claim 3. An assignment ψ satisfies ϕ if and only if ψ satisfies ϕ^*.

Proof The forward direction is quite trivial. Since ϕ^* contains all clauses in ϕ, if ψ satisfies ϕ^*, ψ satisfies ϕ as well.

Now let's prove the other direction. Assume ψ satisfies ϕ . If there exists a clause $(x \vee y) \in C(\phi^*) - C(\phi)$, then there exists a directed path from $\overline{x}$ to y in G_ϕ. So if $\psi(x) = 1$, then the clause evaluates to true. But, if $\psi(x) = 0$, then $\psi(\overline{x}) = 1$, and by *Claim 1*, $\psi(y) = 1$, and thus the clause becomes true. Hence, ψ is an assignment that satisfies ϕ^*, if it satisfies ϕ. $\square$

Now, consider ϕ^*. Let ϕ^*_+ be a collection of all the clauses in ϕ^* such that both the literals in it occur in their positive form and $L(\phi^*_+)$ be the literals that occur in ϕ^*_+. Consider the undirected graph $G_{\phi^*_+}$ with $V(G_{\phi^*_+}) = L(\phi^*_+)$ and $E(G_{\phi^*_+}) = \{(x, y) : (x \vee y) \in C(\phi^*_+)\}$.

Claim 4. There exists an assignment ψ that satisfies ϕ^* with at most k variables set to 1, if and only if there exists a vertex cover X for $G_{\phi^*_+}$ of size k.

Proof It can be seen that if there is an assignment ψ that satisfies ϕ^* and $T = \psi^{-1}(1)$ such that $|T| \leq k$, then $X = T \cap L(\phi^*_+)$ is a vertex cover for $G_{\phi^*_+}$ and moreover $|X| \leq k$. Indeed, since making each clause in ϕ^* true, is same as making each clause in ϕ^*_+ true, which is further equivalent to covering all the edges in $G_{\phi^*_+}$, we can say that X is a vertex cover of $G_{\phi^*_+}$.

For the other direction, let X be an inclusion-wise minimal vertex cover for $G_{\phi^*_+}$ such that $|X| \leq k$ and let ψ be an assignment such that $\psi^{-1}(1) = X$. It is obvious that ψ satisfies ϕ^*_+. We further extend this to show that ψ satisfies ϕ^*. Lets assume that ψ doesn't satisfy ϕ^*. Then there exists a clause which evaluates to false in ϕ^*. Since the clause cannot have both positive literals as ψ satisfies ϕ^*_+, it either has to have one negative literal or two negative literals.

If it has one negative literal, then let's assume the clause is $(\overline{x}, y)$. Since this clause evaluates to false, $\psi(x) = 1$ and $\psi(y) = 0$. Since $\psi(x) = 1$, then $x \in X$. Now, there exists a clause $(x \vee z) \in C(\phi^*_+)$ such that $\psi(z) = 0$, as otherwise X will not be an inclusion-wise minimal vertex cover. Since ψ^* is a closure, $(z \vee y) \in C(\phi^*)$. Since, $\psi(y) = 0$ and $\psi(z) = 0$, this clause evaluates to false. But this cannot be the case as $(z \vee y) \in C(\phi^*_+)$ and ψ satisfies ϕ^*_+.

If it has both negative literals, then let's assume the clause is $(\overline{x}, \overline{y})$. Since this clause evaluates to false, $\psi(x) = \psi(y) = 1$. Since $\psi(x) = 1$ and $\psi(y) = 1$, then $x, y \in X$. Now, there exists clauses $(x \vee a), (y \vee b) \in C(\phi^*_+)$ such that $\psi(a) = \psi(b) = 0$, as otherwise X will not be an inclusion-wise minimal vertex cover. Since ψ^* is a closure, $(a \vee b) \in C(\phi^*)$. Since, $\psi(a) = \psi(b) = 0$, this clause evaluates to false. But this cannot be the case as $(a \vee b) \in C(\phi^*_+)$ and ψ satisfies ϕ^*_+. Therefore, our assumption that ψ doesn't satisfy ϕ^* is wrong. $\square$

Hence, finding a satisfying assignment for ϕ^* with at most k variables set to 1 is equivalent to finding a minimal vertex cover of size k for $G_{\phi^*_+}$. And by *Claim 3*, this is equivalent to finding a satisfying assignment for ϕ with at most k variables set to 1. Hence, if we find a polynomial kernel for **VERTEX-COVER**, we can say that **MIN-ONES-2-SAT** also admits a polynomial kernel.

Claim 5. **VERTEX-COVER** admits a $O(k^2)$ kernel.

Proof Let the given problem instance be (I, k). First let's reduce the problem instance.

- **RED 1:** Remove all isolated vertices.
- **RED 2:** Remove vertices with degree greater than k, and add it to the vertex cover X, and decrease k by 1.

Let $(I^{'}, k')$ be an instance where none of the above reduction rules are applicable. If the number of vertices in $I^{'}$ is greater than $2k^2$ or number of edges in $I^{'}$ is greater than k^2, then it is a NO-instance. This is because, each vertex has at least one edge incident in it, and at most k. So, choosing k vertices can cover at most k^2 edges. And there can only be at most $2k^2$ vertices. Otherwise, output $(I^{'}, k^{'})$.

This means that **VERTEX-COVER** admits an $O(k^2)$ kernel. $\square$

Therefore, by *Claim 5*, *Claim 4*, *Claim 3* we can say that **MIN-ONES-2-SAT** admits a kernel of $O(k^2)$, which is a polynomial kernel.