Change of Variables:

Analogous to the method of substitution in single variable. $x = g(t)$

$$\int_a^b f(x)\,dx = \int_c^d f(g(t))\,\underbrace{g'(t)}_{=\frac{dx}{dt}}\,dt$$

where $a = g(c)$ and $b = g(d)$

We can change variables in two dimensional case.

Let the variables x, y in the double integral

$$\iint_R f(x,y)\,dx\,dy$$

be changed to new-variables u, v by means of relations

$$x = \phi(u,v) \quad , \; y = \psi(u,v)$$

then double integral is transformed into

$$\iint_{R'} f\{\phi(u,v), \psi(u,v)\}\,\underbrace{|J|\,du\,dv}$$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$ Jacobian

R' is the region in uv plane which corresponds to the region R in the xy-plane.

Special Case: Cartesian to Polar Co-ordinate

$x = r\cos\theta \,, \quad y = r\sin\theta$

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

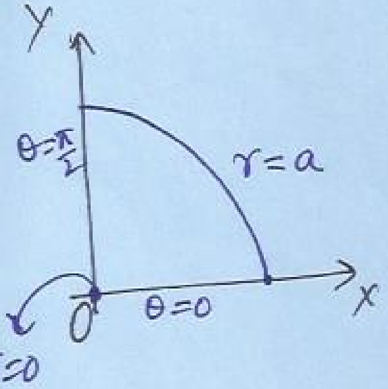
$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

Example 1: Volume of one Octant of a sphere of radius a,

$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy$$

where S is the first quadrant of the circular disk

$$x^2+y^2 \le a^2$$



Change of variables

$x = r\cos\theta \,, \quad y = r\sin\theta$

$|J| = r$

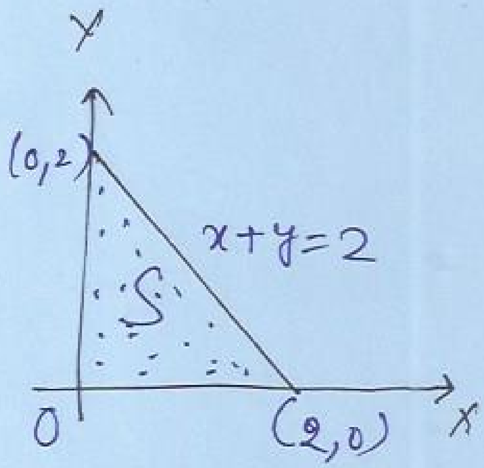
$$\iint_S \sqrt{a^2-x^2-y^2}\,dx\,dy = \iint_R \sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$\iint_R \sqrt{a^2-r^2}\;r\,dr\,d\theta = \int_{\theta=0}^{\pi/2}\int_{r=0}^{a} \sqrt{a^2-r^2}\;r\,dr\,d\theta$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right) \left. \frac{(a^2-r^2)^{3/2}}{3/2} \right|_0^a$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{2}\right) (-a^3) \cdot \frac{2}{3}$$

$$= \frac{\pi}{6} a^3 .$$



Example-2: $\iint_S e^{(y-x)/(y+x)}\, dx\,dy$

Change of variables

$$\begin{aligned} y-x&=u \\ y+x&=v \end{aligned} \quad \Rightarrow \quad x = \frac{v-u}{2}$$

$$y = \frac{v+u}{2}$$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{4} - \frac{1}{4} = -\frac{1}{2}$$

Domain in the uv-plane

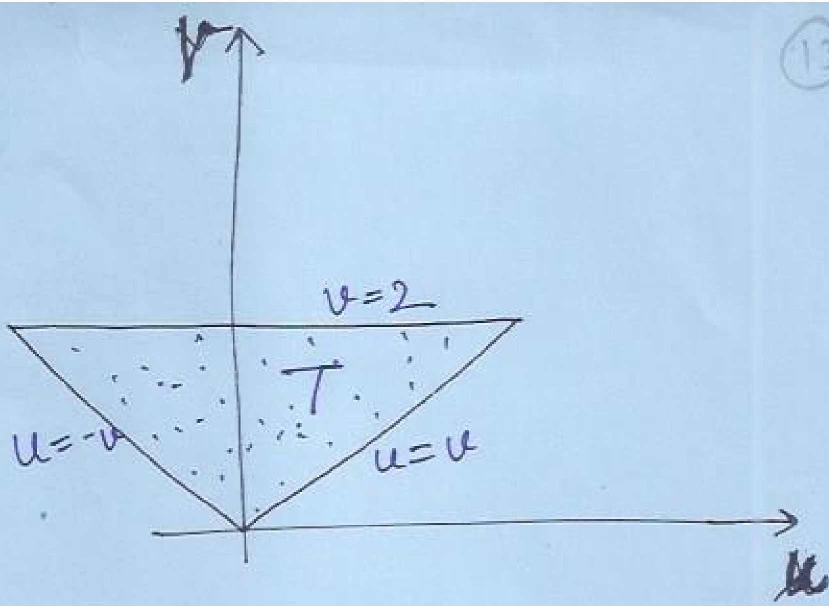
line $x=0$ maps to $v=u$

line $y=0$ maps to $v=-u$

line $x+y=2$ maps to $v=2$

$$\iint_S e^{(y-x)/(y+x)}\,dx\,dy$$

$$=\iint_T e^{u/v}\,\frac{1}{2}\,du\,dv$$



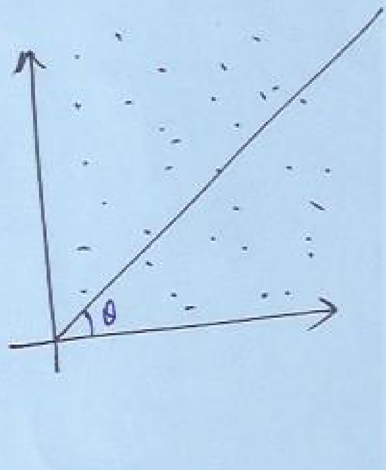
$$=\frac{1}{2}\int_{v=0}^{2}\int_{u=-v}^{v} e^{u/v}\,du\,dv$$

$$=\frac{1}{2}\int_0^2 v\left(e-\frac{1}{e}\right)dv$$

$$=e-\frac{1}{e}.$$

<u>Example</u>: Change into polar coordinates and evaluates

$$\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx$$



$$x=r\cos\theta\ ,\ y=r\sin\theta$$

$$\Rightarrow \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dy\,dx=\int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$=\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta=\int_0^{\pi/2}\frac{1}{2}\,d\theta=\frac{\pi}{4}$$

Note: Let $I = \int_0^\infty e^{-x^2} dx = \int_0^\infty e^{-y^2} dy$

$$\Rightarrow I^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$$

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = \frac{\pi}{4}$$

$$\boxed{I = \frac{\sqrt{\pi}}{2}}$$

$$\Rightarrow \boxed{\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}}$$

Example: Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing to polar coordinates.

Solution: The region of integration is bounded by

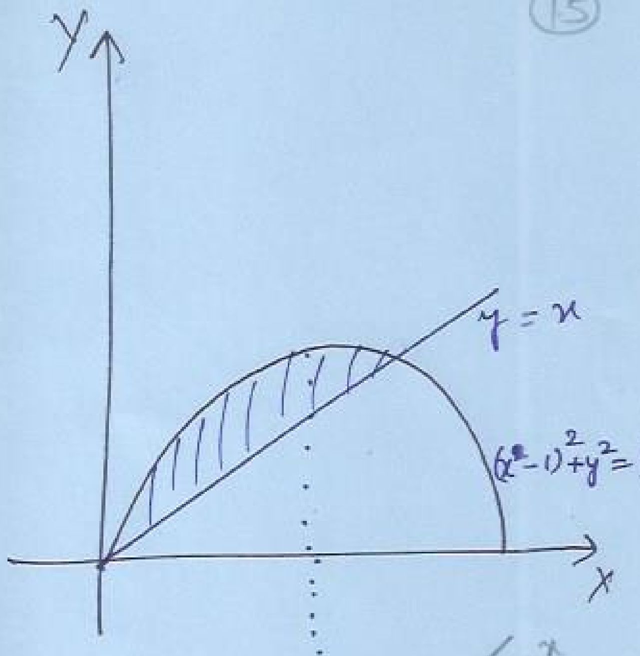
$y = x, \quad y = \sqrt{2x - x^2}, \quad x = 0 \;\&\; x = 1$

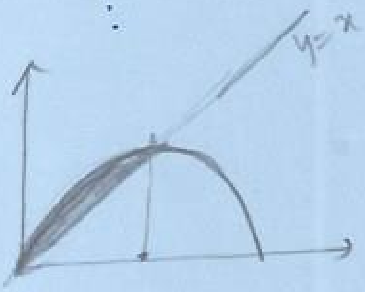
Polar Equation of the circle

$(r\cos\theta-1)^2+r^2\sin^2\theta=1$

$\Rightarrow r^2-2r\cos\theta=0$

$\Rightarrow r=2\cos\theta$





$$\int_0^1\int_x^{\sqrt{2x-x^2}}(x^2+y^2)\,dy\,dx$$

$$=\int_{\theta=\pi/4}^{\pi/2}\int_{r=0}^{2\cos\theta} r^2\cdot r\,dr\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[\frac{r^4}{4}\right]_0^{2\cos\theta}d\theta=\int_{\pi/4}^{\pi/2}4\cos^4\theta\cdot d\theta$$

$$=\int_{\pi/4}^{\pi/2}(2\cos^2\theta)^2\,d\theta=\int_{\pi/4}^{\pi/2}(1+\cos2\theta)^2\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[1+\cos^2 2\theta+2\cos2\theta\right]d\theta$$

$$=\int_{\pi/4}^{\pi/2}\left[1+\frac{1}{2}(1+\cos4\theta)+2\cos2\theta\right]d\theta$$

$$= \cdots = \frac{1}{8}(3\pi - 8)$$

Example: Evaluate the integral

$$\iint_R \sqrt{x^2+y^2}\, dx\,dy$$ by changing to polar co-ordinates, where R is the region in the x-y plane bounded by the circles $x^2+y^2=4$ and $x^2+y^2=9$.

Solution: $x = r\cos\theta$, $y = r\sin\theta$

$$|J| = r$$

$$I = \int_0^{2\pi}\int_2^3 r \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^3}{3}\right]_2^3 d\theta$$

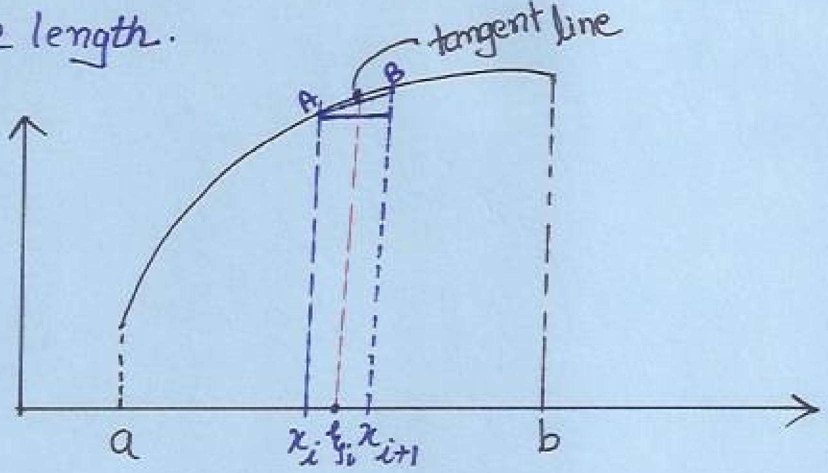
$$= \left(\frac{27-8}{3}\right) 2\pi$$

$$= \frac{19}{3} \cdot 2\pi$$

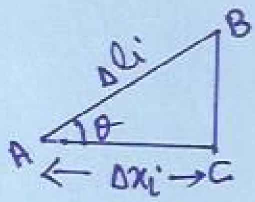
$$= \frac{38}{3} \cdot \pi$$

COMPUTING THE AREA OF A SURFACE:

Let us consider the case of 1-dimension, i.e. computation of the curve length.



length of the curve



$$L = \lim_{n\to\infty} \sum_{i=1}^{n-1} \Delta l_i$$

$$\frac{\Delta x_i}{\Delta l_i} = \cos\theta$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n-1} \sqrt{1+f'(\xi_i)^2} \cdot \Delta x_i$$

$$\Rightarrow \Delta l_i = \Delta x_i \frac{1}{\cos\theta}.$$

$$= \int_a^b \sqrt{1+(f'(x))^2} \cdot dx$$

Also $\cos\theta = \dfrac{1}{\sqrt{1+\tan^2\theta}}$

$$\frac{1}{\cos\theta} = \sqrt{1+\tan^2\theta}$$

$$= \sqrt{1+f'(\xi_i)^2}$$

In two dimension case we consider

$$\Rightarrow \Delta l_i = \Delta x_i \sqrt{1+f'(\xi_i)^2}$$

tangent plane instead of tangent line and similar to one dimensional case we get surface area.

$$\boxed{S = \iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy}$$

where D is the projection of the surface in xy-plane.

Similarly if the equation is given in the form:

$$x = \mu(y, z) \text{ or in the form } y = \psi(x, z)$$

then

$$S = \iint_{\tilde{D}} \sqrt{1 + \left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2}\, dy\, dz$$

$$S = \iint_{\tilde{\tilde{D}}} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2}\, dx\, dz.$$

where $\tilde{D}$ and $\tilde{\tilde{D}}$ are the domains in the yz and xz planes in which the given surface is projected.

<u>Example</u>: Compute the surface area of the sphere

$$x^2 + y^2 + z^2 = R^2$$

<u>Solution</u>: Equation of the surface

$$z = \sqrt{R^2 - x^2 - y^2} \quad \text{(upper half)}$$

In this case:
$$\frac{\partial z}{\partial x} = -\frac{x}{\sqrt{R^2 - x^2 - y^2}}$$

$$\frac{\partial z}{\partial y} = -\frac{y}{\sqrt{R^2 - x^2 - y^2}}$$

Domain of integration: $x^2 + y^2 \le R^2$

$$S = 2 \int_{-R}^{R} \int_{-\sqrt{R^2 - x^2}}^{+\sqrt{R^2 - x^2}} \underbrace{\sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}}_{\hookrightarrow \frac{R}{\sqrt{R^2 - x^2 - y^2}}}\, dy\, dx$$

transformation to polar coordinate gives:

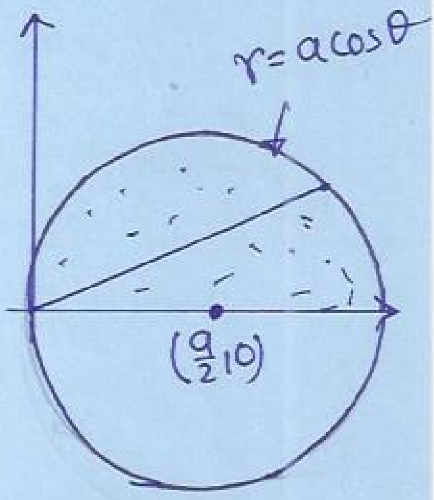
$$S = 2\int_0^{2\pi}\int_0^{R} \frac{R}{\sqrt{R^2-r^2}}\, r.\,dr\,d\theta$$

$$= 2\pi \cdot 2R\left(-\sqrt{R^2-r^2}\right)_0^R$$

$$= 4\pi R^2$$

Question: Find the area of that part of the sphere $x^2+y^2+z^2=a^2$ which is cut off by the cylinder $x^2+y^2=ax$.

$$x^2+y^2-ax=0 \Rightarrow \left(x-\frac{a}{2}\right)^2+y^2=\frac{a^2}{4}$$



$$S = 2.2\int_0^{\pi/2}\int_{r=0}^{a\cos\theta} \frac{a}{\sqrt{a^2-r^2}}\, r.\,dr\,d\theta$$

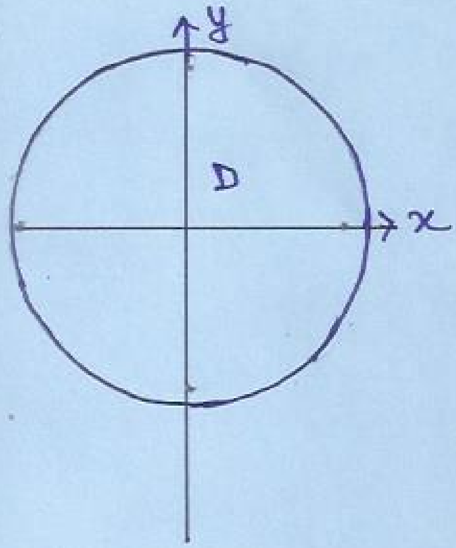
$$= 4.\, a\int_0^{\pi/2}\left(-\sqrt{a^2-r^2}\right)_0^{a\cos\theta} d\theta$$

$$= 4a.\int_0^{\pi/2}\left[-a\sin\theta + a\right] d\theta$$

$$= 4a.\left[\left[a\cos\theta\right]_0^{\pi/2} + a\{\theta\}_0^{\pi/2}\right] = 4a.\left[-a+a.\frac{\pi}{2}\right]$$

$$= 2a^2(\pi-2).$$

Ex. Determine the surface area of the part of $z = xy$ that lies in the cyl. $x^2+y^2=1$.



Solution:

$z = f(x,y) = xy$

$z_x = y$ & $z_y = x$

$$S = \iint_D \sqrt{1+x^2+y^2}\, dA$$

In polar coordinate

$$S = \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r\sqrt{1+r^2}\, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{1}{2}\,\frac{2}{3}\left[(1+r^2)^{3/2}\right]_0^1 d\theta = \frac{2\pi}{3}\left(2^{3/2}-1\right)$$ Ans.

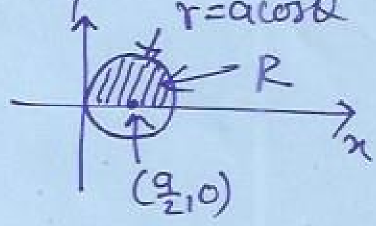
Evaluation of volume:

$$V = \iint_D \underset{f(x,y)}{\underset{\parallel}{z}}\, dx\, dy \quad \text{OR} \quad \iint_{\bar{D}} \mu(y,z)\, dy\, dz$$

OR $$\iint_{\bar{\bar{D}}} \psi(x,z)\, dx\, dz.$$

Example: Find the volume common to sphere $x^2+y^2+z^2=a^2$ and a circular cylinder $x^2+y^2=ax$.

Required volume:



$$V = 4\iint_R z \, dx\, dy$$

$$= 4\iint_R \sqrt{a^2-x^2-y^2}\, dx\, dy$$

Subst. $x = r\cos\theta \quad y = r\sin\theta$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\cos\theta} \sqrt{a^2-r^2}\; r\, dr\, d\theta$$

$$= \frac{4}{-2}\int_0^{\pi/2} \frac{2}{3}\left(a^2-r^2\right)^{3/2}\Big|_0^{a\cos\theta} d\theta$$

$$= -2\cdot\frac{2}{3}\cdot\int_0^{\pi/2}\left(a^3\sin^3\theta - a^3\right) d\theta$$

$$= \frac{-4}{3}a^3\cdot\left[\frac{2}{3}-\frac{\pi}{2}\right] = \frac{2}{9}a^3\left(3\pi-4\right)$$

Ans.