Tutorial 2 Solution

Q.4) Time invariant or not

(a) $y(t) = x(t) + t\,x(t-1)$

NOT Time invariant

Justification : (There can be many approaches of justification. I am going to justify this with a example scenario)

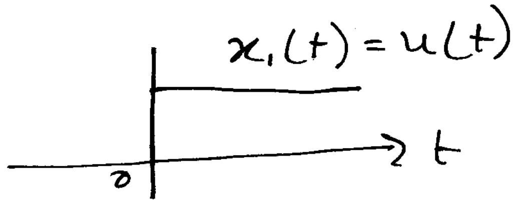
Let the input be $x_1(t) = u(t)$
Then the ouput at $t = 2$ is given by

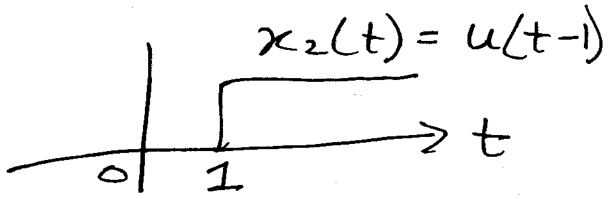
$$y_1(2) = u(2) + 2\,u(2-1)$$
$$= 1 + 2 = 3$$

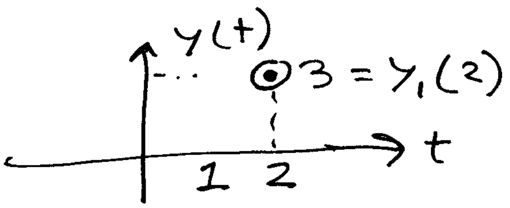
Now let us delay the input by one second then the delayed input is $x_2(t) = u(t-1)$
Now the output at $t = 3$ is given by

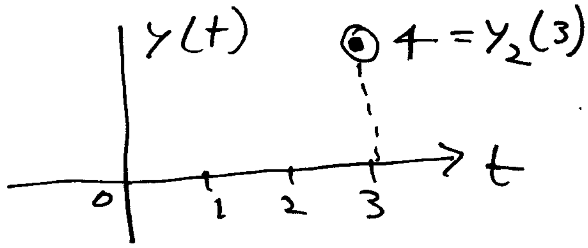
$$y_2(3) = u(3-1) + 3\,u((3-1)-1)$$
$$= 1 + 3\,u(3-1-1)$$
$$= 4$$

Observe $y_2(3) \neq y_1(2)$. $\therefore$ The output by one second is not simply delayed when the input is delayed.









Alternative Justification

When the input is $x_1(t)$ the output is

$$y_1(t) = x_1(t) + t\,x_1(t-1) \quad \text{---- (i)}$$

Now if the input is delayed by t_0

i.e. $x_2(t) = x_1(t-t_0)$ ---- (ii)

then the new output will be

$$y_2(t) = x_2(t) + t\,x_2(t-1) \quad \text{---- (iii)}$$

$$= x_1(t-1) + t\,x_1(t-t_0-1) \quad \text{---- (iv)}$$

[using (ii)]

However ~~th~~ if the system were Time invariant then the new output should be equal to the $y_1(t)$ delayed by t_0 amount

i.e. $\cancel{y_1(t)}$ $y_1^{delayed}(t) = y_1(t-t_0)$

$$\cancel{= x_1(t-t_0) + (t-t_0)}$$

$$= x_1(t-t_0) + (t-t_0)\,x_1(t-t_0-1)$$

[using (i)]

$\because$ $y_1^{delayed}(t) \neq y_2(t)$

Or the delayed version of the output is not same as the output for delayed input

$\therefore$ Not Time invariant.

(b) $y(t) = x(t)\cos(2t)$

NOT Time invariant

Justification

Input $x_1(t)$ (A)	Output $y_1(t)$ (B)	Delayed input $x_2(t)$ (C)	Output for delayed input
$x_1(t)$	$y_1(t) = x_1(t)\cos 2t$	$x_2(t) = x_1(t-t_0)$	

A	Input	$x_1(t)$
B	Output	$y_1(t) = x_1(t)\cos(2t)$
C	Delayed input	$x_2(t) = x_1(t-t_0)$
D	Output for input in (C) i.e. the output when the input is delayed	$y_2(t) = x_2(t)\cos(2t)$ $= x_1(t-t_0)\cos(2t)$
E	Output in (B) delayed	$y_1^{delayed}(t)$ $= x_1(t-t_0)\cos(2(t-t_0))$

Note that (E) and (D) are not same

i.e. $y_1^{delayed}(t) \neq y_2(t)$

$\therefore$ NOT Time invariant.

(e) $y(t) = x(-t/4)$

Not time invariant

Justification

A	INPUT	$x_1(t)$
B	OUTPUT	$y_1(t) = x_1(-t/4)$
C	Delayed input	$x_2(t) = x_1(t - t_0)$
D	Output for delayed input in (C)	$y_2(t) = x_2(-t/4)$ $= x_1(-t/4 - t_0)$ [using equation in (C)]
E	Output in B delayed	$y_1^{delayed}(t) = y_1(t - t_0)$ $= x_1\left(\frac{-(t-t_0)}{4}\right)$ [using equation (B)] $= x_1\left(-\frac{t}{4} + \frac{t_0}{4}\right)$

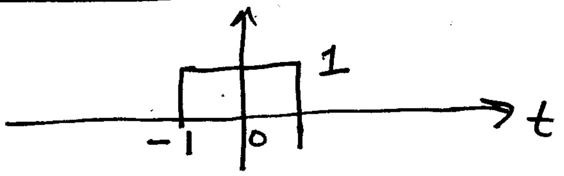
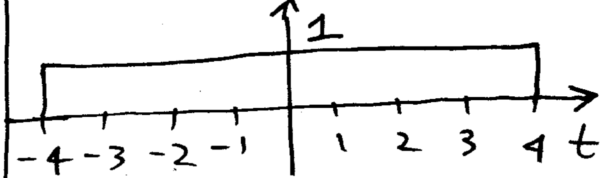
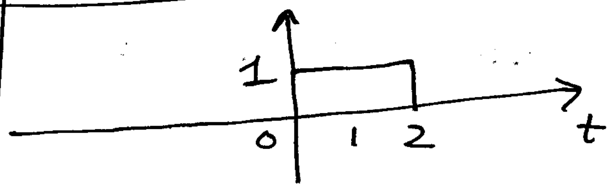
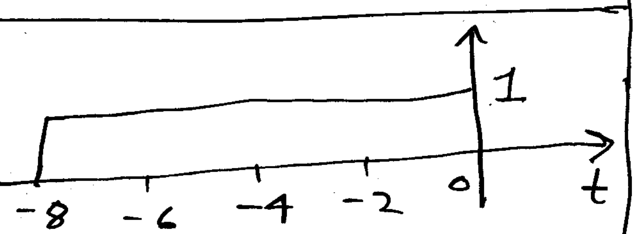
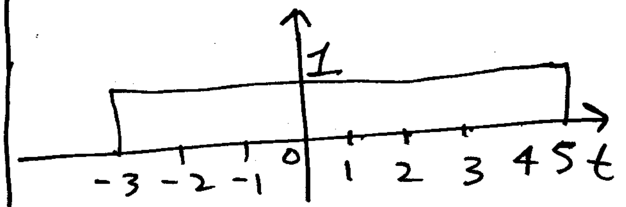
$D \neq E$

or output for delayed input $\neq$ the delayed output

$\therefore$ NOT Time invariant.

Alternative Justification

(Justification with example)

A	Input $x_1(t)$ =	
B	Output $y_1(t) = x_1(-t/4)$	
C	Input in (A) delayed by 1 second $x_2(t)$	
D	Output against the input in (C) $y_2(t)$	
E	Output in (B) delayed by 1 second	

So the output in (D) and (E) are not same.

$\therefore$ NOT TI.

Q3) Linearity and Causality

(i) $y(t) = \int_{-\alpha}^{t} x(t)\,dt$

* If $x(t) = 0$ for all t
then $y(t) = \int_{-\alpha}^{t} 0\,dt = 0$ for all t

* ~~If Let the~~ When the input is $x_1(t)$
let the output by $y_1(t)$

$\therefore \; y_1(t) = \int_{-\alpha}^{t} x_1(t)\,dt$

Now if the input is scalled by a factor 'a' that is $x_2(t) = a x_1(t)$
then the new output will be

$$y_2(t) = \int_{-\alpha}^{t} x_2(t)\,dt = \int_{-\alpha}^{t} a x_1(t)\,dt$$

$$= a\, y_1(t)$$

$\therefore$ If input is amplitude scalled, the output is also amplitude scalled by the same factor (The system is Homogeneous)

* Let the output for inputs $x_1(t)$ and $x_2(t)$ be $y_1(t)$ and $y_2(t)$ respectively

$\therefore \; y_1(t) = \int_{-\alpha}^{t} x_1(t)\,dt$

$$y_2(t) = \int_{-\infty}^{t} x_2(t)\,dt.$$

Now if the two inputs are added together, then the output will be

$$y(t) = \int_{-\infty}^{t} \big(x_1(t) + x_2(t)\big)\,dt$$

$$= \int_{-\infty}^{t} x_1(t)\,dt + \int_{-\infty}^{t} x_2(t)\,dt$$

$$= y_1(t) + y_2(t)$$

$\therefore$ The system ~~is linear.~~ obeys superposition.

Therefore system is <u>Linear</u>

Note that the value of output at any time t depends only on the input at that instant and the past input.

$\therefore$ The system is <u>Causal</u>

(ii) $y(t) = \int_{-\alpha}^{2t} x(t)\,dt$.

~~B~~	INPUT	OUTPUT	COMMENT
A	$x_1(t)$	$y_1(t) = \int_{-\alpha}^{2t} x_1(t)\,dt$	
B	$x_2(t)$	$y_2(t) = \int_{-\alpha}^{2t} x_2(t)\,dt$	
C	0	$y_0(t) = \int_{-\alpha}^{2t} 0\,dt = 0$	Homogeneous system
D	$a x_1(t)$	$y(t) = \int_{-\alpha}^{2t} a x_1(t)\,dt$ $= a y_1(t)$	Homogeneous system
E	$x_1(t) + x_2(t)$	$y(t) = \int_{-\alpha}^{2t} (x_1(t) + x_2(t))\,dt$ $= y_1(t) + y_2(t)$	The system obeys superposition

$\therefore$ The system is <u>Linear</u>

The system is <u>not causal</u>

Because ~~the~~ the output at $t = 1$

$= \int_{-\alpha}^{2} x(t)\,dt$ depends on the input ~~put~~ ~~upto~~

upto $t = 2$ (future input)

(iii) $y(t) = \int_{-\infty}^{t/2} x(t)\, dt$.

Linear (Justification is similar to problem (i) and (ii))

Non causal

Because the output at $t = (-10)$ ~~dep~~

$= \int_{-\infty}^{-5} x(t)\, dt$ which depends on input upto time $t = -5$ i.e future input.

Q2 Linear or not

(a) $\frac{dy}{dt} + 2t^2 y = t\, x(t)$

Assume The $O/P = y_1(t)$ when $I/P = x_1(t)$

$$\Rightarrow \frac{dy_1}{dt} + 2t^2 y_1 = t\, x_1(t) \quad \text{--- (i)}$$

Also assume The $O/P = y_2(t)$ when $I/P = x_2(t)$

$$\Rightarrow \frac{dy_2}{dt} + 2t^2 y_2 = t\, x_2(t) \quad \text{--- (ii)}$$

Now for the system to be linear, when the input is $ax_1(t) + bx_2(t)$ the output should by $ay_1(t) + by_2(t)$, that means the input $ax_1(t) + bx_2(t)$ and the output $ay_1(t) + by_2(t)$ ~~should~~ must satisfy the system equation

$$\frac{d}{dt}(a_1 y_1 + a_2 y_2) + 2t^2 (a_1 y_1 + a_2 y_2) = t\left(a_1 x_1(t) + a_2 x_2(t)\right) \quad \text{--- (iii)}$$

From $a_1 \times (i) + a_2 \times (ii)$ we get

$$a_1 \frac{d}{dt} y_1 + a_2 \frac{d}{dt} y_2 + 2t^2 a_1 y_1 + 2t^2 a_2 y_2$$
$$= t a_1 x_1(t) + t a_2 x_2(t)$$

$$\Rightarrow \frac{d}{dt}(a_1 y_1 + a_2 y_2) + 2t^2(a_1 y_1 + a_2 y_2)$$
$$= t(a_1 x_1 + a_2 x_2)$$

Therefor (i) and (ii) $\Rightarrow$ (iii)

So the system is linear

Side note: since (iii) is true for any value of a_1 & a_2,
putting $a_1 = 0$ & $a_2 = 0$ implies $x(t) = 0$ and $y(t) = 0$ satisfies the system equation.
Also putting only $a_2 = 0$ implies $a x_1(t)$ & $a y_1(t)$ together satisfies the system equation. So the system is homogeneous.
But I need not show the homogeneity separately, since when (iii) is satisfied homogeity is obvious

(b) $\frac{dy}{dt} + y^2 = 3x(t)$

Not linear

Justification:

Assume when i/p is $x_1(t)$, output $= y_1(t)$
and when i/p is $x_2(t)$, output $= y_2(t)$

$$\therefore \frac{dy_1}{dt} + y_1^2 = 3x_1(t) \quad \text{------ (i)}$$

$$\frac{dy_2}{dt} + y_2^2 = 3x_2(t) \quad \text{------ (ii)}$$

Lets investigate if for i/p $= a_1x_1(t) + a_2x_2(t)$
the o/p $= a_1y_1(t) + a_2y_2(t)$ or not.

$$\frac{d}{dt}(a_1y_1 + a_2y_2) + (a_1y_1 + a_2y_2)^2 \overset{?}{=} 3(a_1x_1 + a_2x_2)$$

$$\Rightarrow \left(\frac{d}{dt}(a_1y_1) - 3a_1x_1\right) + \left(\frac{d}{dt}a_2y_2 - 3a_2x_2\right) + (a_1y_1 + a_2y_2)^2 \overset{?}{=} 0$$

$$\Rightarrow -a_1y_1^2 - a_2y_2^2 + (a_1y_1 + a_2y_2)^2 \overset{?}{=} 0 \quad \text{------ (iii)}$$

[using equations (i) and (ii)]

But equation (iii) is not true. Therefore, the system is not linear.

Side note: Put $a_1 = 2$ and $a_2 = 0$ is eqn. (iii) we get $-2y_1^2 + 4y_1^2 = 0$, which can't be true. Convince yourselves/ email us.

(e) $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 3y = x\frac{dx}{dt}$

you may follow the approach of Q26. Let me follow a slightly different approach

Assume when i/p $= x_1(t)$, o/p $= y_1(t)$

$$\Rightarrow \frac{d^2y_1}{dt^2} + 5\frac{dy_1}{dt} + 3y_1 = x_1\frac{dx_1}{dt}$$

$$\Rightarrow \frac{d^2}{dt^2}(ay_1) + 5\left(\frac{d}{dt}(ay_1)\right) + 3(ay_1) = ax_1\frac{dx_1}{dt} \quad \text{---} (i)$$

Now for the system to be linear when the i/p is ax_1 the output must be $a.y_1$

Assume the system to be linear, therefore the system must satisfy

$$\frac{d^2}{dt^2}(ay_1) + 5\frac{d}{dt}(ay_1) + 3a(y_1) = (ax_1)\frac{d}{dt}(ax_1) \quad \text{---} (ii)$$

subtracting (i) from (ii) we get

$$ax_1\frac{d}{dt}(ax_1) - ax_1\frac{dx_1}{dt} = 0$$

$$\Rightarrow a^2x_1\frac{dx_1}{dt} = ax_1\frac{dx_1}{dt} \rightarrow \text{But this is not true (contradiction)}$$

∴ The system is not linear

Sidenote: I can choose $x_1(t) = \sin(\omega t)$ and $a = 2$. Then $a^2x_1\frac{dx_1}{dt} = ax_1\frac{dx_1}{dt}$ can not be true for all time. Convince yourselves or email us

An interesting Note : Suppose a system is described with a differential equation (Like the ones in Q, 2). ~~Then if the output for input $x_1(t)$~~ Suppose the system is Linear. ~~If~~ when i/p $x_1(t)$ is applied, the observed o/p is $y_1(t)$. Next when i/p $x_2(t)$ is applied the observed o/p is $y_2(t)$. Next when i/p $(x_1(t)+x_2(t))$ is applied the o/p need not be equal to $(y_1(t)+y_2(t))$ even if the system is linear. The o/p of a system described with differential equation depends on initial/ Boundary condition apart from the i/p. Therefore, for a suitable initial/ Boundary condition the output will be equal to $(y_1(t)+y_2(t))$ when the i/p is $(x_1(t)+x_2(t))$. But other initial conditions this may not hold true.

While checking for linearity, we check whether under suitable initial condition(s) (for example initially relaxed system) The output can be $(a_1 y_1 + a_2 y_2)$ when the input is $(a_1 x_1 + a_2 x_2)$.

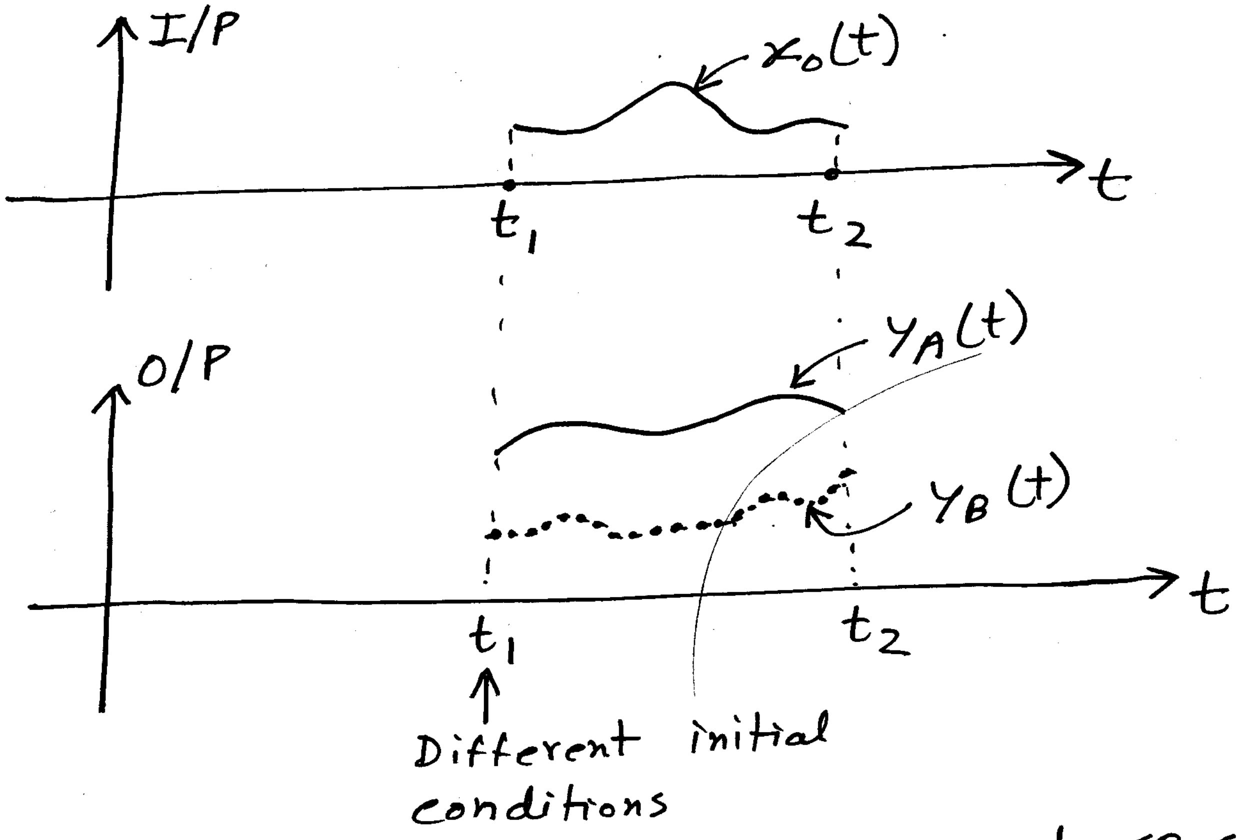
Either understand this or ignore this note safely

$\boxed{Q1}$ (i) <u>Static or dynamic</u>

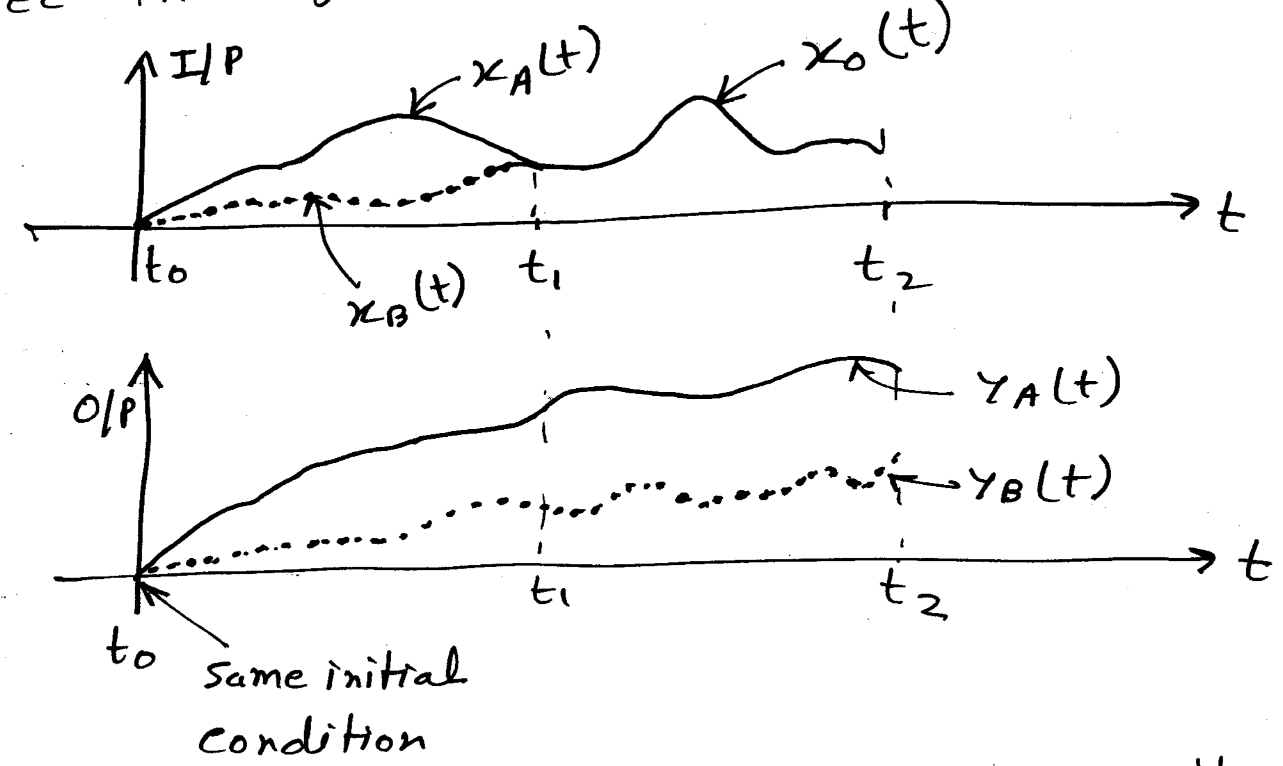
<u>Note</u>: A system is static/memoryless iff the o/p depends only on the present i/P. otherwise if the o/p depends on the past i/P in any way the the system must have memory or it must be dynamic.

Generally the systems described with (non-trivial) differential equations are dynamic. Let us understand why.

The o/p of a system (described with a differential equation) depends both on the i/p and the initial boundary ~~conti~~ condition. Suppose, I have two copies (A and B) of same the same system. I apply ~~sys~~ the same i/p $x_0(t)$ to both of them starting from time $t=t_1$ to time $t=t_2$. But the two systems have two different initial condition at $t=t_1$. Therefore, the output of system A and system B will not be same between $t=t_1$ and $t=t_2$. See the graphs below. $y_A(t)$ and $y_B(t)$ are the o/ps of two ~~syste~~ identical systems.



Now let us go back in time and ~~se~~ see what could have happened before $t=t_1$. See the graphs below.



Both systems started with same initial condition at $t=t_0$. ~~from~~ From $t=t_0$ to $t=t_1$, they received different i/ps. Therefore their states at $t=t_1$ was different

After $t=t_1$ both of them received same i/p but due to different initial conditions at $t=t_1$ their o/ps were different between $t=t_1$ and $t=t_2$.

So how could that be explained?

The systems with ~~dy~~ differential equations remembers the past i/p in the form of the initial condition

Therefore, system A and B remembered what i/p was given to them from t_0 ~~to~~ t_1 ~~So even if th~~ Between t_0 and t_1 the i/ps were different. That's why the o/ps were different between t_1 and t_2 although the i/p was same between t_1 and t_2. Clearly ~~the~~ such systems remembers past i/p & therefore they are dynamic.

<u>Conclusion</u>: Systems described with a non-trivial differential eqn. are generally dynamic

$\therefore$ (a), (b), (e) of Q.1 are dynamic systems.

(d) $y(t)$ depends on the past i/p $x(t-2)$ so this is also a dynamic system.

(ii) Linear or Non-Linear

(a) Non Linear (Particularly because of the +4 term.)

Proof : Assume ~~the~~ when i/p $= x_1(t)$ the output is $y_1(t)$

$$\Rightarrow \frac{d^2y_1}{dt^2} + 2\frac{dy_1}{dt} + y_1 + 4 = x_1(t) \quad --- (i)$$

Now ~~#~~ Lets assume that the system is linear. So if the i/p is scalled up by ~~at~~ a factor 'a', ~~the~~ ~~o/p~~ the o/p should

i.e. $x_2(t) = ax_1(t)$.

be $y_2(t) = ay_1(t)$.

$$\Rightarrow \frac{d^2}{dt^2}y_2 + 2\frac{dy_2}{dt} + y_2 + 4 = x_2(t)$$

$$\Rightarrow a\frac{d^2}{dt^2}y_1 + 2a\frac{dy_1}{dt} + ay_1 + 4 = ax_2 \quad --- (ii)$$

subtracting ($a \times$ (i)) from (ii) we get

$$4 - 4a = 0$$

$$\Rightarrow 4(1-a) = 0$$

$$\Rightarrow 4 = 0 \quad (\text{if I choose } a \neq 0)$$

— Contradiction / Impossible.

$\therefore$ The system cannot be linear

~~Non~~ Note : This system in Q 1 a) can easily be converted into a linear system by redefining the i/p $z(t) = x(t) - 4$

(b) NON LINEAR

Proof by contradiction:

Assume the system to be linear

Assume for i/p $= x_1(t)$ o/p $= y_1(t)$

$$\Rightarrow \frac{d^3}{dt^3} y_1(t) + 2\frac{d^2}{dt^2} y_1(t) + 4\frac{dy_1}{dt} + 3y_1^2 = x_1(t+1) \quad \text{---- (i)}$$

Now if we take i/p $x_2(t) = a x_1(t)$

due to assumed linearity the o/p $y_2(t) = a\, y_1(t)$

$$\Rightarrow \frac{d^3}{dt^3} y_2(t) + 2\frac{d^2}{dt^2} y_2(t) + 4\frac{dy_2}{dt} + 3y_2^2 = x_2(t+1)$$

$$\Rightarrow a\frac{d^3}{dt^3} y_1(t) + 2a\frac{d^2}{dt^2} y_1(t) + 4a\frac{dy_1(t)}{dt} + 3a^2 y_1^2 = a x_1(t+1) \quad \text{--- (ii)}$$

from (ii) $- a \times$ (i)

$$3a^2 y_1^2 - 3a y_1^2 = 0$$

$\Rightarrow a y_1^2 - y_1^2 = 0$ [$\because$ I can choose $a, \neq 0$]

$\Rightarrow (a-1)\, y_1^2 = 0$ This is not true since I can choose $a \neq 1$ and y_1 need not be zero for all time

— our assumption that the system is linear can't be true.

(c) NON LINEAR

$\boxed{\text{Brief}}$ Proof by contradiction

Assume $x_1(t) \longrightarrow y_1(t)$

$\Rightarrow D^2 y_1 + 2y_1(Dy_1) + 3t y_1 = x_1$ - - - - - (i)

Assume linear $\Rightarrow$ $ax_1 \longrightarrow ay_1$

$\Rightarrow D^2 y_2 + 2y_2(Dy_2) + 3t y_2 = x_2$

$\Rightarrow aD^2 y_1 + a^2 2 y_1 (D y_1) + a3t y_1 = a x_1$ - - - (ii)

from ~~(ii) - a x~~ (ii) $- a \times$ (i) we get

$a^2 2 y_1 (D y_1) - a 2 y_1 (D y_1) = 0$

$\Rightarrow a(a-1) y_1 (D y_1) = 0$

$\Rightarrow y_1 (D y_1) = 0$ [$\because$ I can choose $a \neq 0$, $a \neq 1$]

$\Rightarrow y_1 \left(\frac{d}{dt} y_1\right) = 0$ for all time

$\Rightarrow$ either y_1 is zero
or y_1 is a constant.

$\Rightarrow y_1$ is a constant

$\Rightarrow$ Left side of the system differential equation is constant

$\Rightarrow$ Right side $= x_1(t)$ is constant.
But I can choose $x_1(t)$ to be not constant then the above can not be true. So our assumption that the system is linear is wrong.

d) LINEAR

Proof

Assume for i/p $= x_1(t)$, o/p $= y_1(t)$
and for i/p $= x_2(t)$, o/p $= y_2(t)$

$\therefore y_1(t) = a x_1(t) + b t^2 x_1(t-2)$ ------ (i)

$y_2(t) = a x_2(t) + b t^2 x_2(t-2)$ ------ (ii)

Now if the i/p is $k_1 x_1(t) + k_2 x_2(t)$
then clearly the o/p will be

$$y(t) = a\left(k_1 x_1(t) + k_2 x_2(t)\right) + b t^2\left(k_1 x_1(t-2) + k_2 x_2(t-2)\right)$$

$$= k_1 y_1(t) + k_2 y_2(t)$$

This is true for any value of k_1 and k_2 including zeros.

$\therefore$ The system obeys superposition and Homogeneity. $\therefore$ The system is Linear

(iv) Time invariance

(a) Time invariant

~~As~~ Proof Assume when i/p $= x_1(t)$, o/p $= y_1(t)$

$$\Rightarrow \frac{d^2}{dt^2} y_1(t) + 2\frac{dy_1}{dt}(t) + y_1(t) + 4 = x_1(t) \quad \text{--- (i)}$$

(i) is true for all values of time t. so if we substitute t by $(t-t_0)$ in eqn (i), that should still remain true.

$$\therefore \frac{d^2}{dt^2} y_1(t-t_0) + 2\frac{dy_1}{dt}(t-t_0) + y_1(t-t_0) + 4 = x_1(t-t_0) \quad \text{--- (ii)}$$

[In other words, since eqn. (i) is true for all time t, that means it is also true for all ~~and~~ ~~any~~ values of t in $(t-t_0)$]

Now lets choose a new i/p which is nothing but the previous i/p delayed by time t_0 $\therefore x_2(t) = x_1(t-t_0)$

$\Rightarrow \frac{d}{dt} x_2(t) = \frac{d}{dt} x_1(t-t_0) \Rightarrow \frac{d^2}{dt^2} x_2(t) = \frac{d^2}{dt^2} x_2(t-t_0)$

Let us also delay the previous o/p by the same time t_0 $\therefore y_2(t) = y_1(t-t_0)$

$\Rightarrow \frac{d}{dt} y_2(t) = \frac{d}{dt} y_1(t-t_0) \Rightarrow \frac{d^2}{dt^2} y_2(t) = \frac{d^2}{dt^2} y_2(t-t_0)$

Now let us check whether this new (delayed) i/p together with the new (delayed) o/p satisfy the system equation

$$\frac{d^2}{dt^2} y_2(t) + 2\frac{d}{dt} y_2(t) + y_2(t) + 4 \stackrel{?}{=} x_2(t)$$

$$\Rightarrow \frac{d^2}{dt^2} y_1(t-t_0) + 2\frac{d}{dt} y_1(t-t_0) + y_1(t-t_0) + 4 \overset{?}{=} x_1(t-t_0) \quad \text{--- (iii)}$$

If equation (iii) is true then the system is time invariant.
However, from equation (ii) we know that equaing (iii) is true.
∴ The system is time invariant.

ALTERNATIVE SOLUTION (SIMPLER)

We will ~~use~~ back calculate o/p from i/p
If the i/P $= x_1(t)$ when o/p $= y_1(t)$
Then back calculation implies

$$x_1(t) = \frac{d^2 y_1(t)}{dt^2} + 2\frac{dy_1(t)}{dt} + y_1(t) + 4$$

$$\Rightarrow x_1(t-t_0) = \frac{d^2}{dt^2} y_1(t-t_0) + 2\frac{dy_1}{dt}(t-t_0) + y_1(t-t_0) + 4 \quad \text{---- (i)}$$

[substituting t with $(t-t_0)$]

Now if we consider a new o/p which is delayed version of prevoious o/p, i.e.

$$y_2(t) = y_1(t-t_0)$$

Then the corresponding new i/p can be back calculated as

$$x_2(t) = \frac{d^2 y_2(t)}{dt^2} + \frac{d}{dt} y_2(t) + y_2(t) + 4$$

$= \frac{d^2}{dt^2} y_1(t-t_0) + 2\frac{d}{dt} y_1(t-t_0) + y_1(t-t_0) + 4$

$= x_1(t-t_0)$ [from eqn (i)]

$\therefore x_2(t) = x_1(t-t_0)$

$\therefore$ When o/P is delayed by t_0. the corresponding i/P is also known to be delayed by same time.

$\therefore$ The system is time invariant

(b) Time invariont

Brief proof using back calculation

When o/P $= y_1(t)$ i/P $= x_1(t)$ (assume)

$\cancel{\Rightarrow \frac{d^3}{dt^3} y_1(t) + 2\frac{d^2 y_1(t)}{dt^2} + 4\frac{dy_1(t)}{dt} + 3y_1^2(t) = x_1(t+1)}$

$\cancel{\Rightarrow \frac{d^3}{dt^3} y_1(t-t_0) + 2\frac{d^2}{dt^2} y_1(t-t_0) + 4\frac{d}{dt} y_1(t-t_0) + 3y_1^2(t-t_0) = x_1(t-t_0+1)}$

~~Now the back calculated i/P ~~for the~~ corresponding to the delayed o/P by is given~~

$\cancel{y_2(t) = y_1(t-t_0)}$

$\cancel{x_2(t+1)}$

$x_1(t+1) = \frac{d^3}{dt^3} y_1(t) + 2\frac{d^2}{dt^2} y_1(t) + 4\frac{d}{dt} y_1(t) + 3y_1^2(t)$

$\Rightarrow x_1(t) = \frac{d^3}{dt^3} y_1(t-1) + 2\frac{d^2}{dt^2} y_1(t-1) + 4\frac{d}{dt} y_1(t-1) + 3y_1^2(t-1)$

[substituting t with $(t-1)$]

$$\Rightarrow x_1(t-t_0) = \frac{d^3}{dt^3} y_1(t-t_0-1) + 2\frac{d^2}{dt^2} y_1(t-t_0-1) + 4\frac{d}{dt} y_1(t-t_0-1) + 3 y_1^2(t-t_0-1) \quad \text{----- (i)}$$

[substituting t with $(t-t_0)$]

Now if we want to have a new o/p which is a delayed version of the previous o/p i.e. $y_2(t) = y_1(t-t_0)$, ~~Th~~ then we can back calculate the required i/p as

$$\boxed{\cancel{x_2(t) = \frac{d^3}{dt^3} y_2(t) + 2\frac{d^2}{dt^2} y_2(t) + 4\frac{d}{dt} y_2(t) + 3 y_2^2(t)}}$$

$$\boxed{\cancel{= \frac{d^3}{dt^3} y_1(t-t_0) + 2\frac{d^2}{dt^2} y_1(t-t_0) + 4\frac{d}{dt} y_1(t-t_0) + 3 y_1^2(t-t_0)}}$$

$$\boxed{\cancel{= x_1(t-t_0)}}$$ ~~[from eqn. (i)]~~

$$x_2(t+1) = \frac{d^3}{dt^3} y_2(t) + 2\frac{d^2}{dt^2} y_2(t) + 4\frac{d}{dt} y_2(t) + 3 y_2^2(t)$$

$$= \frac{d^3}{dt^3} y_1(t-t_0) + 2\frac{d^2}{dt^2} y_1(t-t_0) + 4\frac{d}{dt} y_1(t-t_0) + 3 y_1^2(t-t_0)$$

$$\Rightarrow x_2(t) = \frac{d^3}{dt^3} y_1(t-1-t_0) + 2\frac{d^2}{dt^2} y_1(t-1-t_0) + 4\frac{d}{dt} y_1(t-1-t_0) + 3 y_1^2(t-1-t_0)$$

[substituting t with $(t-1)$]

$= x_1(t-t_0)$ [from eqn ~~I~~ (i)]

$\therefore$ If we want to delay the o/p by some time t_0 we have to delay the i/p by the same amount of time t_0

$\therefore$ The system is time invariant.

(e) NOT Time invariant

Proof by contradiction

Assume when i/p $= x_1(t)$ o/p $= y_1(t)$

$$\Rightarrow \frac{d^2y_1(t)}{dt^2} + 2y_1(t)\frac{d}{dt}y_1(t) + 3t\,y_1(t) = x_1(t) \quad \text{---- (i)}$$

$$\Rightarrow \frac{d^2y_1}{dt^2}(t-t_0) + 2y_1(t-t_0)\frac{d}{dt}y_1(t-t_0) + 3(t-t_0)\,y_1(t-t_0) = x_1(t-t_0) \quad \text{---- (ii)}$$

[substituting t by $(t-t_0)$, since eqn (i) is true for all values of t]

Assume the system is Time Invariant

$\therefore$ If we delay the i/p $x_2(t) = x_1(t-t_0)$ the o/p should also be delayed by the same amount

$$\Rightarrow y_2(t) = y_1(t-t_0) \quad \text{---- (iii)}$$

$$\therefore \frac{d^2}{dt^2}y_2(t) + 2y_2(t)\frac{d}{dt}y_2(t) + 3t\,y_2(t) = x_2(t)$$

$$\Rightarrow \frac{d^2}{dt^2}y_1(t-t_0) + 2y_1(t-t_0)\frac{d}{dt}y_1(t-t_0) + 3t\,y_1(t-t_0) = x_1(t-t_0) \quad \text{--- (iv)}$$

[replacing $y_2(t)$ by $y_1(t-t_0)$ from equation (iii)]

Now from (iv) − (ii) we get

$$3t\, y_1(t-t_0) - 3(t-t_0)\, y_1(t-t_0) = 0$$

$$\Rightarrow (\cancel{t} - \cancel{t} + t_0)\, y_1(t-t_0) = 0$$

$\Rightarrow$ either $t_0 = 0$ [that means we cannot apply any delay]

or $y_1(t-t_0) = 0$ for all time t which need not be true.

So ~~our~~ our assumption that the system is TI was wrong

(d) NOT Time invariant (precisely because of the t^2 term in the equation)

Brief proof by contradiction

Assume when $i/p = x_1(t)$ $o/p = y_1(t)$

$$\Rightarrow y_1(t) = a x_1(t) + b t^2 x_1(t-2)$$

$$\Rightarrow y_1(t-t_0) = a x_1(t-t_0) + b (t-t_0)^2 x_1(t-t_0-2)$$

---- (i) [substituting t with $(t-t_0)$]

Now assume the system to be TI

then ~~if $i/p = x_1(t-t_0)$, o/p should be $y_1(t-t_0)$~~

~~$y_1(t-t_0)$~~ if $i/p = x_2(t) = x_1(t-t_0)$ then

o/p should be $y_2(t) = y_1(t-t_0)$

$$\Rightarrow y_2(t) = a x_2(t) + b t^2 x_2(t-2)$$

$$\Rightarrow y_1(t-t_0) = a x_1(t-t_0) + b t^2 x_1(t-2-t_0)$$

---- (ii)

from (i) − (ii)

$$6(t-t_0)^2 x_1(t-t_0-2) - 6t^2 x_1(t-t_0-2) = 0$$

$$\Rightarrow 6\left((t-t_0)^2 - t^2\right) x_1(t-t_0-2) = 0 \quad \text{for all time } t$$

$$\Rightarrow \begin{cases} \text{either } (t-t_0)^2 = t^2 \Rightarrow t_0 = 0 \Rightarrow \text{no delay} \\ x_1(t-t_0-2) = 0 \Rightarrow x_1(t) = 0 \text{ which need not be true.} \end{cases}$$

∴ Our assumption was wrong

~~(iii) CAUSAL OR NON CAUSAL~~

An Interesting Note: Suppose a system is ~~given~~ described with a differential equation. Then its output ~~at~~ depends on both the i/P & initial conditions. Therefore, it is not necessary that if we delay the i/P by some amount, the o/P will be simply delayed. That will be true only if the initial conditions are kept unchanged even when we start the i/P from a delayed time.

(iii) CAUSAL or NON CAUSAL

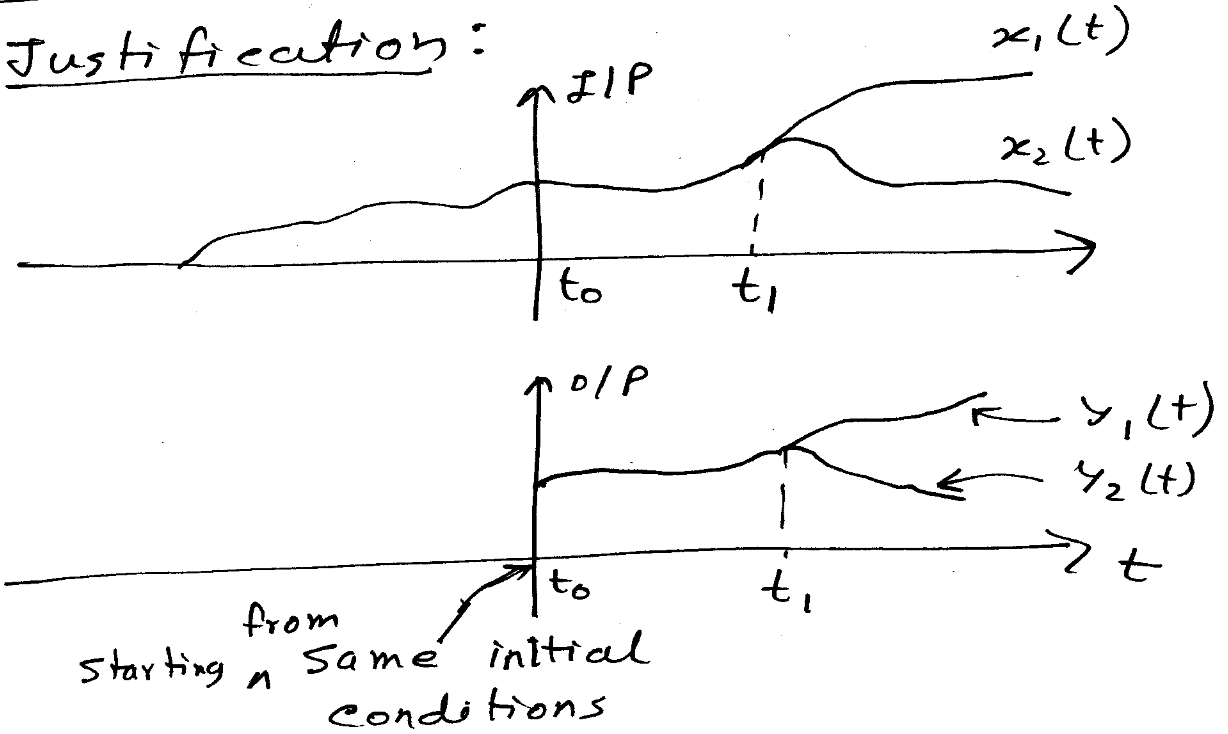
(b) Non causal

Because if I change the i/p at time $(t+1)$ the o/p ~~the~~ at time t must change.

(d) $y(t)$ depends on $x(t)$ (current input/present input) and $x(t-2)$ (past i/p) only. Therefore it is a causal system

(a) CAUSAL SYSTEM

Justification:



Consider I have two copies of the same / identical system described by the differential equation in Q1a.

~~In~~ To one, I apply the i/p $x_1(t)$ & to the other, I apply i/p $x_2(t)$

$x_1(t)$ and $x_2(t)$ are same upto time $t = t_1$

Now I want to find the o/p of the system between $t = t_0$ and $t = t_1$.
Also both the systems starts from the same initial conditions at $t = t_0$ (which depends on the i/p before ~~to~~ $t = t_0$)

Since the excitation and also the initial conditions are same, the o/p after solving the diff. eqn. must come out to be same for both the systems.

$\therefore\ y_1(t) = y_2(t)$ ~~between~~ upto $t = t_1$

Infact you can never find two i/ps $x_1(t)$ and $x_2(t)$ which are equal upto some time $t = t_1$ but the the o/ps are different before $t = t_1$ (given that the initial conditions are also same)

$\Rightarrow$ The o/p of the system depends on ~~past~~ present and ~~(possibly)~~ past i/p but not on future i/p.
So the system is causal.

side note: For question numbe (1 b) If I want to have same output between $t = t_0$ and $t = t_1$, then I must keep the i/P same ~~upto~~ ~~time~~ ~~t~~ between time $t = t_0 + 1$ and $t = t_1 + 1$ (and I also must start from the same initial condition at time $t = t_0$.) So the system is NOT CAUSAL.

(c) CAUSAL

Justification similar to Q 1a

Brietlly: The o/P in this case depends on (i) i/P (ii) Initial condition and (iii) time (because of the term $3ty$)

Now assume I have two copies of the same system.
Now if I start from same initial conditions at $t = t_0$, if I keep the i/P same between $t = t_0$ and $t = t_1$, (and ~~the th value at t is also~~ clearly I am considering the same time interval)

THEN all COPIES of the given system will produce same o/P.

$\boxed{Q5)}$ a) $x(t) = e^{-t}u(t)$ .... [given]

$\Rightarrow x(\tau) = e^{-\tau}u(\tau)$ ... [replacing t with τ]

$h(t) = u(t)$ .... [given]

$\Rightarrow h(t-\tau) = u(t-\tau)$ -- [replacing t with $(t-\tau)$]

$$y(t) = x(t) * h(t)$$

$$= \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau}\,\underbrace{u(\tau)}\,\underbrace{u(t-\tau)}\,d\tau$$

This term is non-zero if $\tau > 0$ and $\tau < t$. Therefore, if $t < 0$, this term is zero for all values of τ....

This term is zero for $\tau < 0$

This term is zero for $t - \tau < 0$ or $\tau > t$

$$= \begin{cases} \int_0^t e^{-\tau}u(\tau)u(t-\tau)\,d\tau \text{ ... if } t \geq 0 \\ 0 \text{ ---------- if } t < 0 \end{cases}$$

$$= \begin{cases} \int_0^t e^{-\tau} \times 1 \times 1 \,d\tau \text{ ---- if } t \geq 0 \\ 0 \text{ ---------- if } t < 0 \end{cases}$$

$$= \begin{cases} -[e^{-\tau}]_0^t & \text{if } t > 0 \\ 0 & \text{if } t < 0 \end{cases} = \begin{cases} 1 - e^{-t} & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

b) $x(t) = u(t)$ $\qquad$ $h(t) = u(t)$

$\Rightarrow x(\tau) = u(\tau)$ $\qquad$ $\Rightarrow h(t-\tau) = u(t-\tau)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = \int_{-\infty}^{\infty} u(\tau)\, u(t-\tau)\, d\tau$$

$$= \begin{cases} \int_0^t u(\tau)\, u(t-\tau)\, d\tau & \text{if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases}$$

$$= \begin{cases} \int_0^t 1\, d\tau & \text{if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases} = \begin{cases} t \text{ if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases}$$

---

c) $x(t) = e^{-t} u(t)$ $\qquad$ $h(t) = e^{-3t} u(t)$

$\Rightarrow x(\tau) = e^{-\tau} u(\tau)$ $\qquad$ $\Rightarrow h(t-\tau) = e^{-3(t-\tau)} u(t-\tau)$

$$\therefore y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} e^{-\tau} u(\tau)\, e^{-3(t-\tau)} u(t-\tau)\, d\tau$$

$$= \begin{cases} \int_0^t e^{-\tau} u(\tau)\, e^{-3t+3\tau} u(t-\tau)\, d\tau & \text{if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases}$$

$$= \begin{cases} e^{-3t} \int_0^t e^{2\tau} \times 1 \times 1\, d\tau & \text{if } t \geqslant 0 \\ 0 \text{ if } t < 0 \end{cases}$$

$$= \begin{cases} e^{-3t}\,\frac{1}{2}\left[e^{2\tau}\right]_0^t & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

$$= \begin{cases} \frac{1}{2}\,e^{-3t}\left(e^{2t} - e^{0}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases} = \begin{cases} \frac{1}{2}\left(e^{-t} - e^{-3t}\right) & \text{if } t \geq 0 \\ 0 & \text{if } t < 0 \end{cases}$$

Q6) $x(t) = e^{-3t}u(t)$

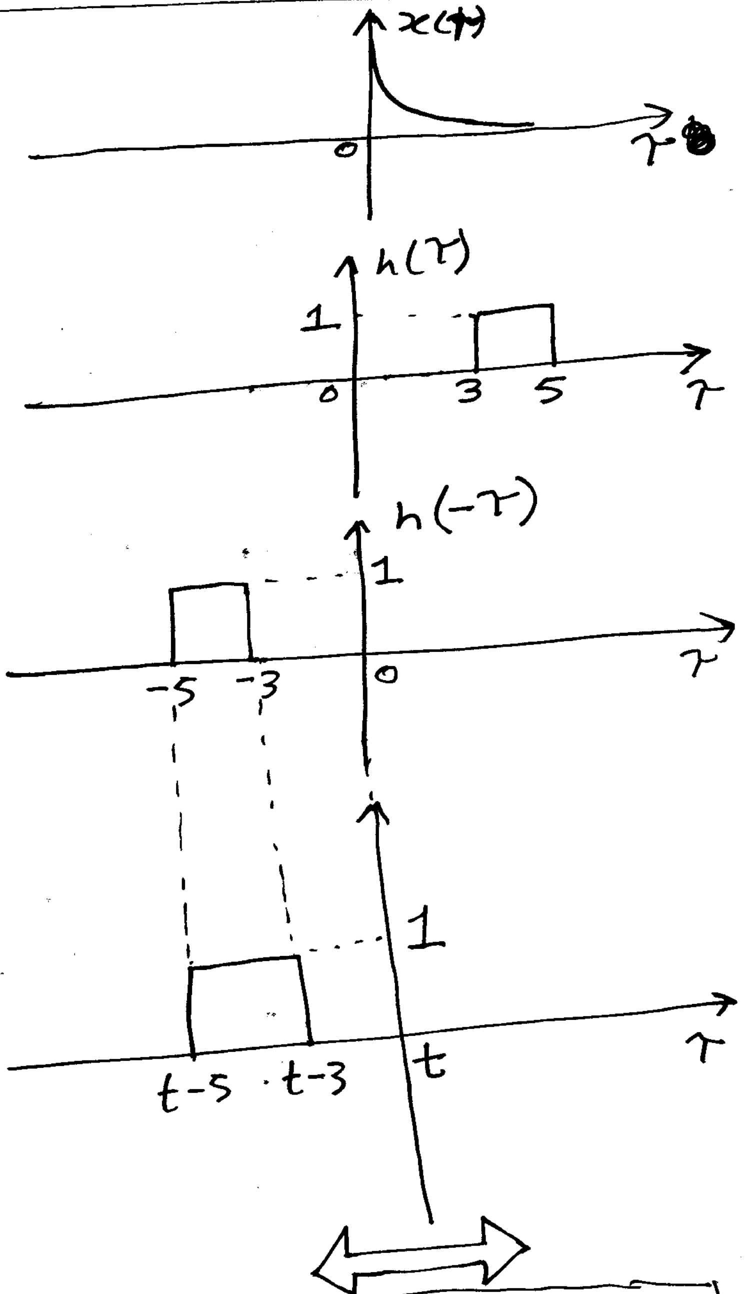
$\Rightarrow x(\tau) = e^{-3\tau}u(\tau)$

(substituting t with τ)

$h(t) = u(t-3) - u(t-5)$

$h(\tau) = u(\tau-3) - u(\tau-5)$

(substituting t with τ)

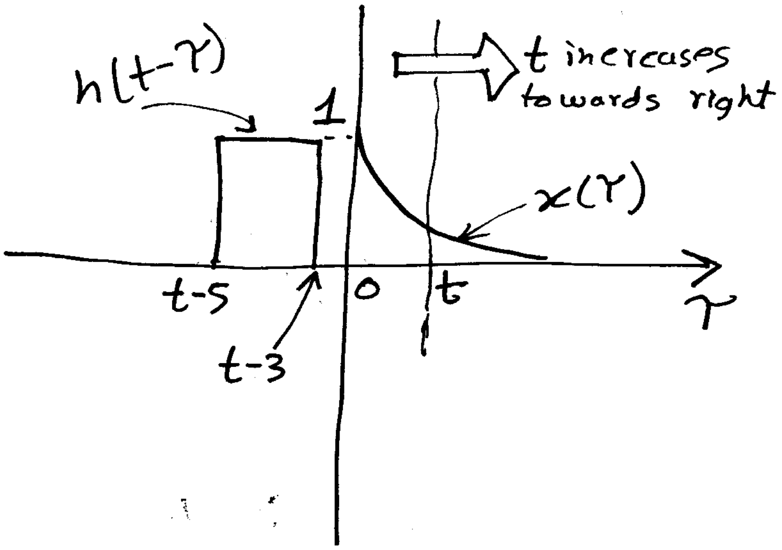


Now draw $h(-\tau)$ by flipping it horizontally about the y axis

Next, draw $h(t-\tau)$. For this simply replace ~~τ=0~~ $\tau = 0$ point with $\tau = t$, $\tau = -3$ with $\tau = t-3$, $\tau = -5$ with $\tau = t-5$ etc

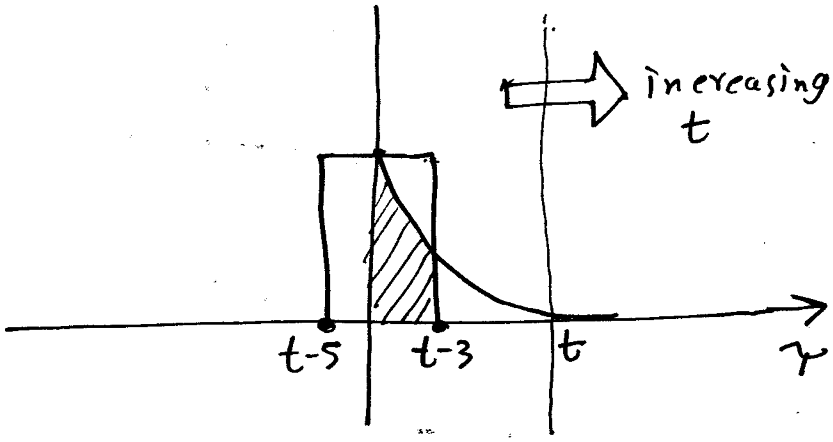
This graph moves towards right as t increases & towards left a t decreases

Next, draw $x(\tau)$ & $h(t-\tau)$ together.



Clearly if $t-3<0$ then $x(\tau)$ & $h(t-\tau)$ do not overlap & therefore o/p $y(t)$ will be zero.

$\therefore\ y(t)=0$ for $t-3<0$ or $t<3$



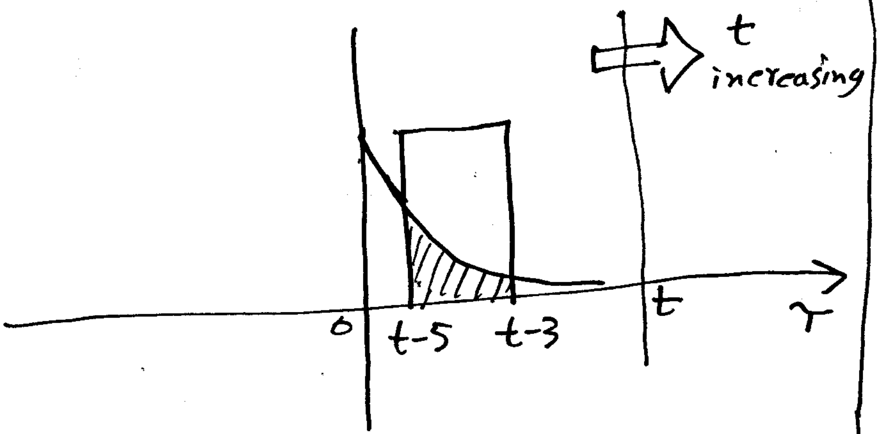
when $t-3\geq 0$ and $t-5\leq 0$

i.e. $3\leq t\leq 5$

partial overlap occurs.

$$y(t)=\int_{0}^{t-3} e^{-3\tau}\,d\tau$$

$$=\frac{1}{3}\left(1-e^{-3(t-3)}\right)$$



when $t-5\geq 0$

i.e $t\geq 5$

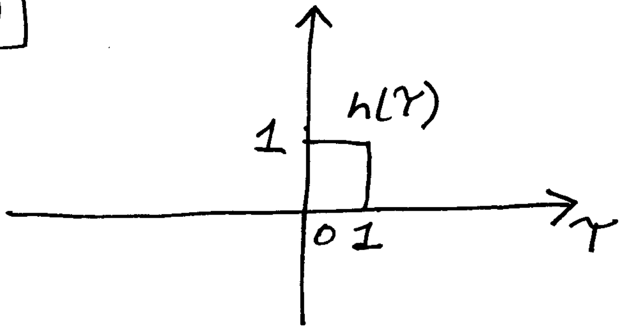
complete overlap

$$y(t)=\int_{t-5}^{t-3} e^{-3\tau}\,d\tau$$

$$=\frac{1}{3}\left(e^{-3(t-5)}-e^{-3(t-3)}\right)$$

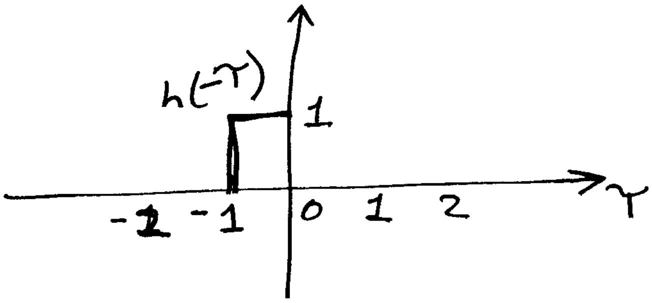
$$\therefore\ y(t)=\begin{cases} 0 & \text{if } t\leq 3 \\ \frac{1}{3}\left(1-e^{-3(t-3)}\right) & \text{if } 3\leq t\leq 5 \\ \frac{1}{3}\left(e^{-3(t-5)}-e^{-3(t-3)}\right) & \text{if } t\geq 5 \end{cases}$$

$\boxed{Q7)}$



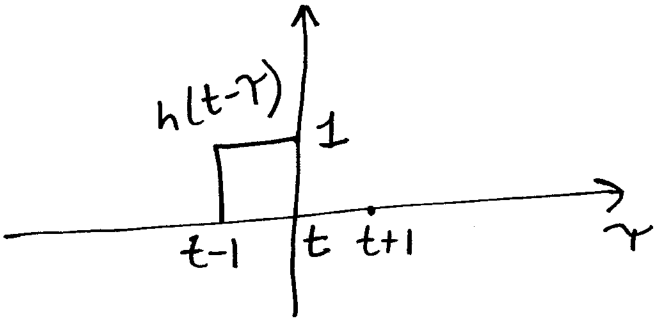
Simply draw the graph $h(t)$, but replace the variable t with τ.

⇩



Flip arround y axis. (horizontally)

⇩



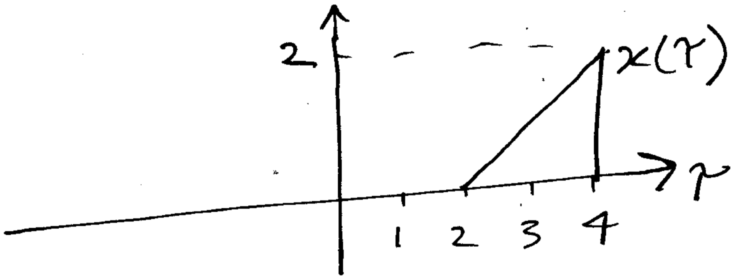
Just replace

$\tau = 0$ with $\tau = t$

$\tau = -1$ with $\tau = t-1$

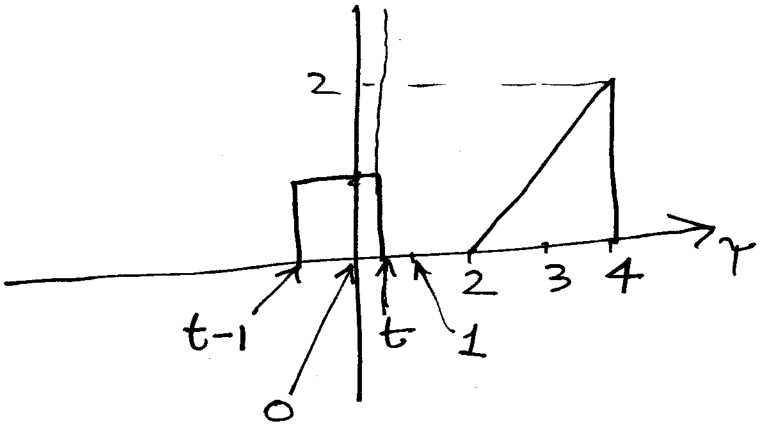
$\tau = 1$ with $\tau = t+1$

⋮ etc



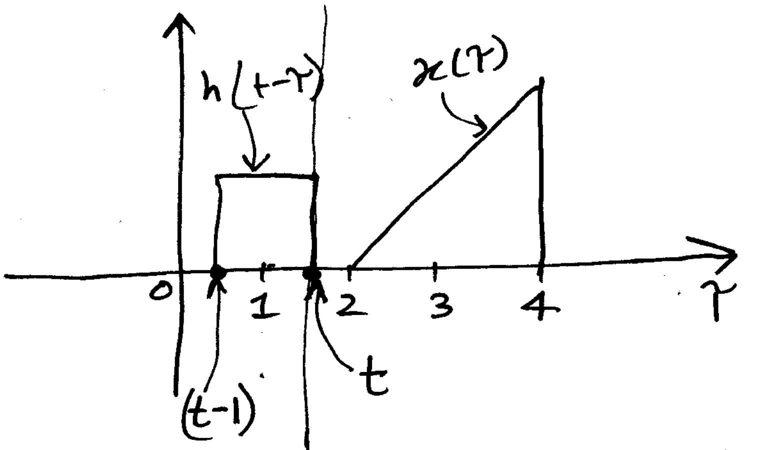
~~Draw $x(\tau)$ vs. τ~~

Draw τ vs. $x(\tau)$



Draw $x(\tau)$ and $h(t-\tau)$ together.

P.T.O

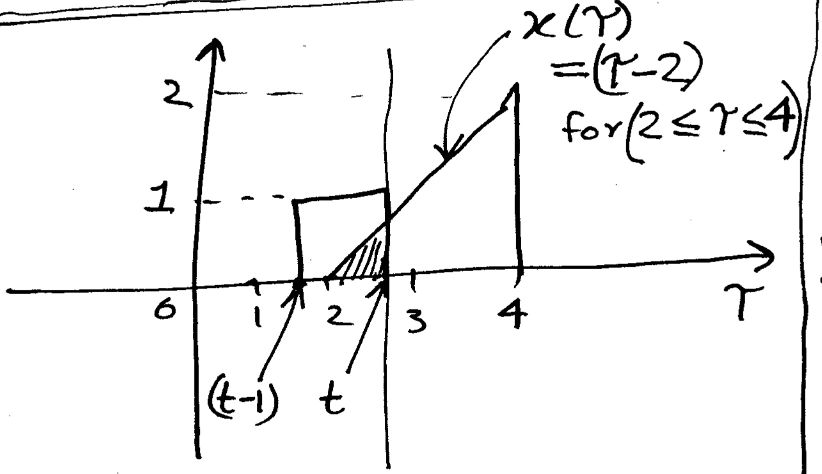


This line is the indicator for time t. As t increases this line moves right. The graph of $h(t-\tau)$ also moves right with this indicator

If $t < 2$

no overlap

$\therefore y(t) = 0$



If $t \geq 2$ and $t \leq 3$

i.e $2 \leq t \leq 3$

partial overlap

$$y(t) = \int_2^t x(\tau)\, h(t-\tau)\, d\tau$$

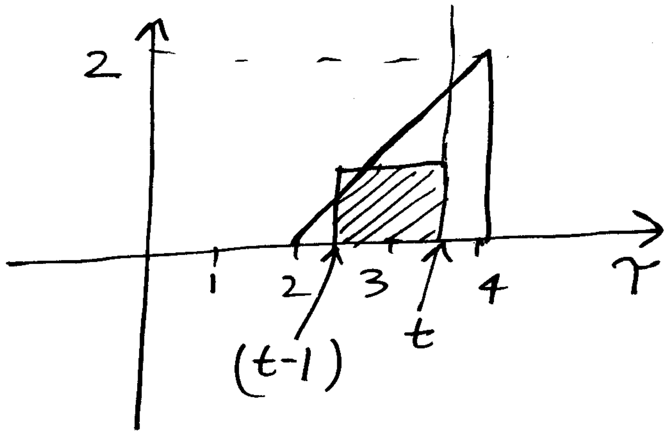
$$= \int_2^t (\tau - 2) \times 1\, d\tau$$

$$= \left[\frac{\tau^2}{2} - 2\tau\right]_2^t$$

$$= \frac{t^2}{2} - 2t + 2$$

$$= \frac{1}{2}(t-2)^2$$

P.T.O



If $t \geq 3$ but $t \leq 4$

i.e $3 \leq t \leq 4$

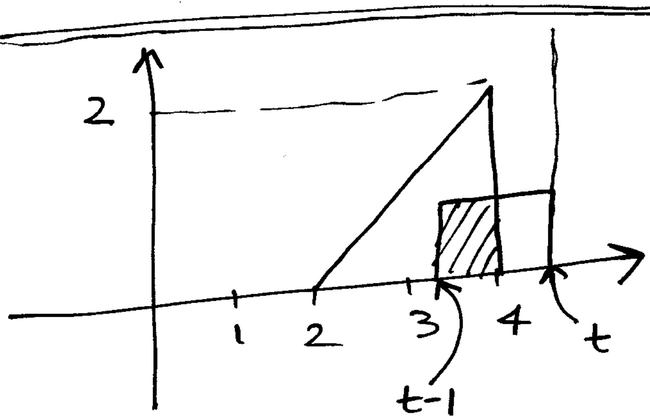
~~partial~~ full overlap

$$y(t) = \int_{t-1}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-1}^{t} (\tau - 2)\, d\tau$$

$$= \frac{t^2}{2} - 2t - \frac{(t-1)^2}{2} + 2(t-1)$$

$$= t - 5/2$$



for $t \geq 4$ but $t \leq 5$

or $4 \leq t \leq 5$

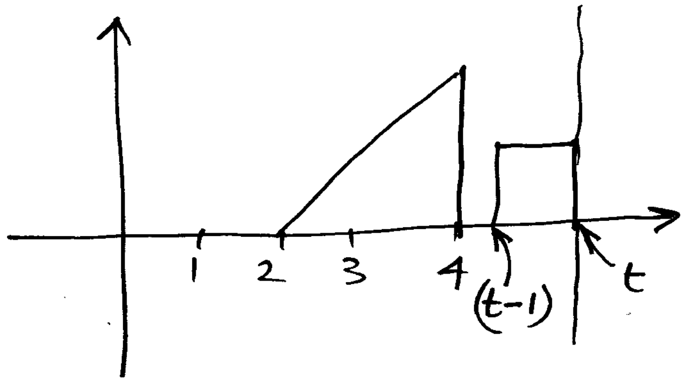
partial overlap

$$y(t) = \int_{t-1}^{4} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-1}^{4} (\tau - 2)\, d\tau$$

$$= 8 - 8 - \frac{(t-1)^2}{2} + 2(t-1)$$

$$= (t-1)\left(\frac{5-t}{2}\right)$$



If $(t-1) > 4$ or $t > 5$

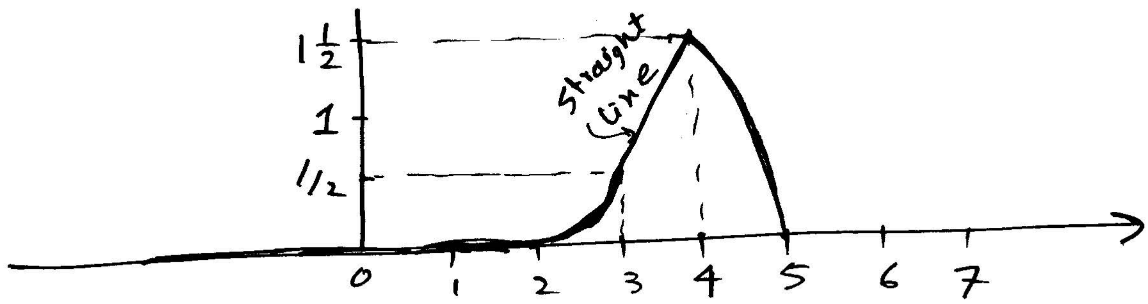
no overlap

$\Rightarrow y(t) = 0$

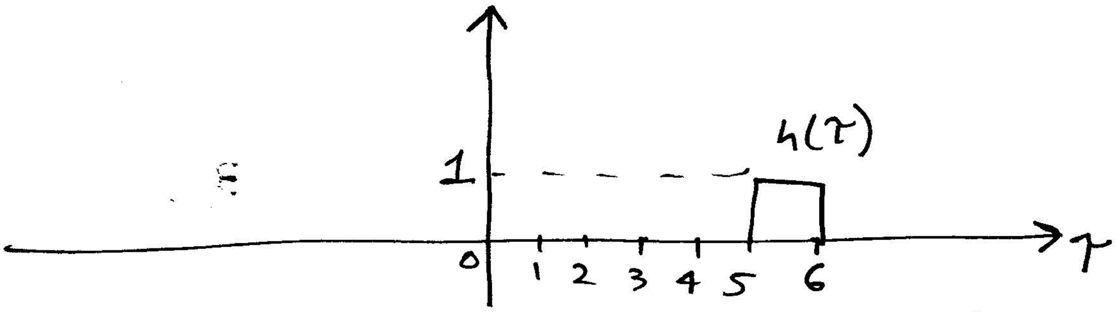
P.T.O

$$\therefore y(t) = \begin{cases} 0 & \text{if } t < 2 \\ \frac{1}{2}(t-2)^2 & \text{if } 2 \le t \le 3 \\ t - 3/2 & \text{if } 3 \le t \le 4 \\ (t-1)(5-t)/2 & \text{if } 4 \le t \le 5 \\ 0 & \text{if } t > 5 \end{cases}$$

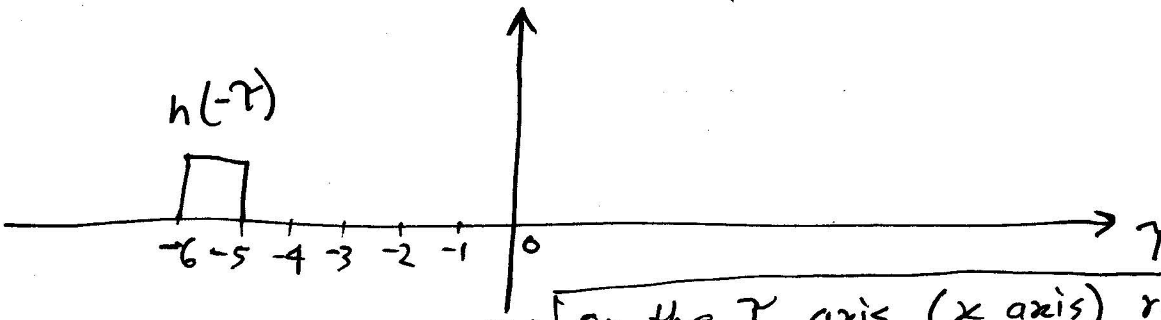
sketch of $y(t)$



$\boxed{Q8)}$

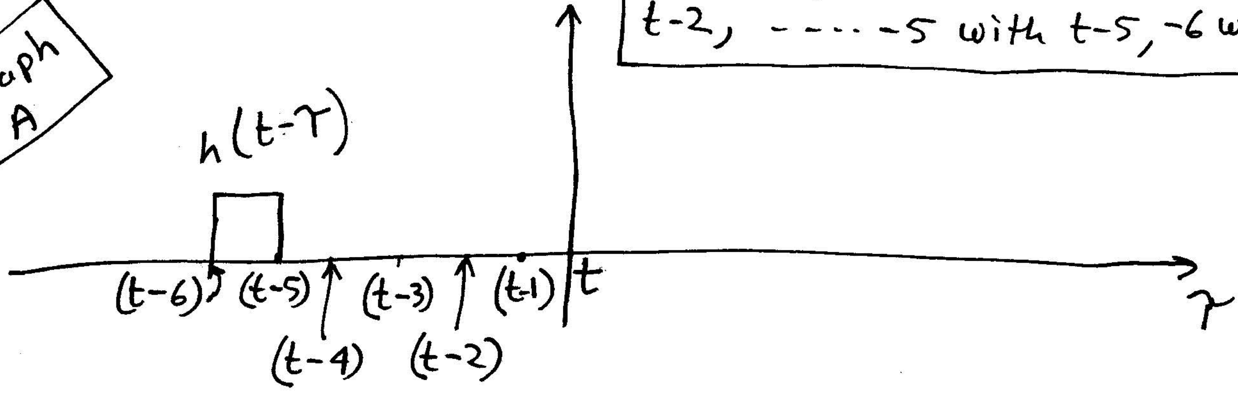


$\Downarrow$ flip around Y axis

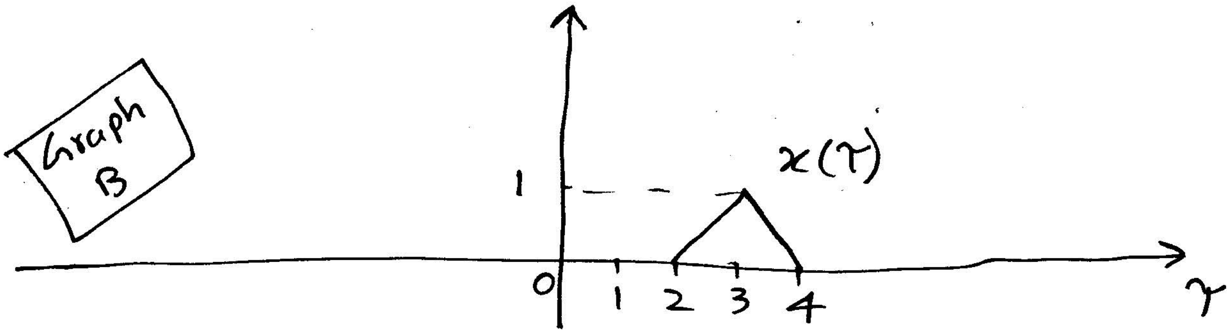


$\Downarrow$ On the τ axis (x axis) replace 0 with t, -1 with t-1, -2 with t-2, -----5 with t-5, -6 with t-6

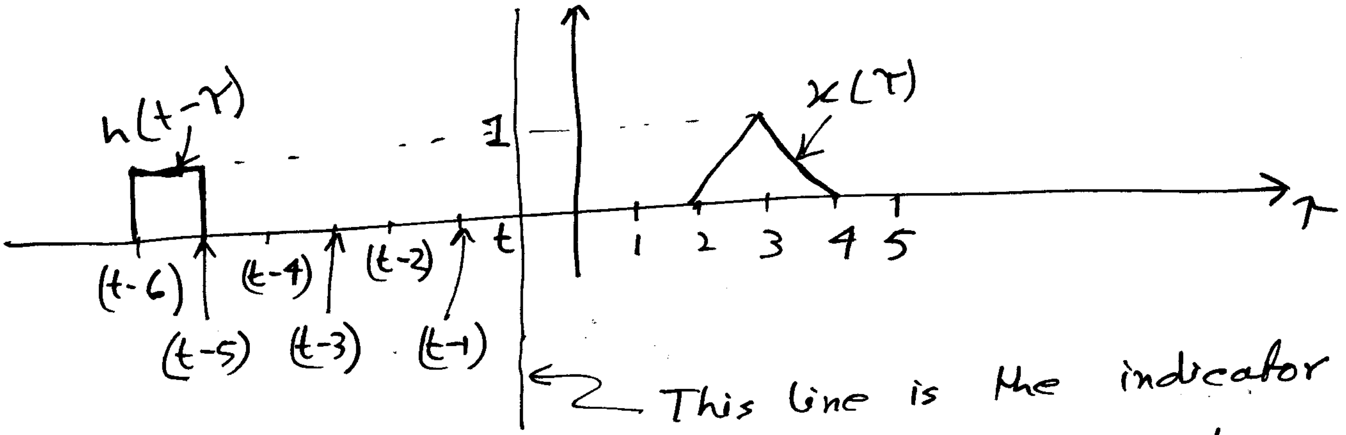
Graph A



Graph B



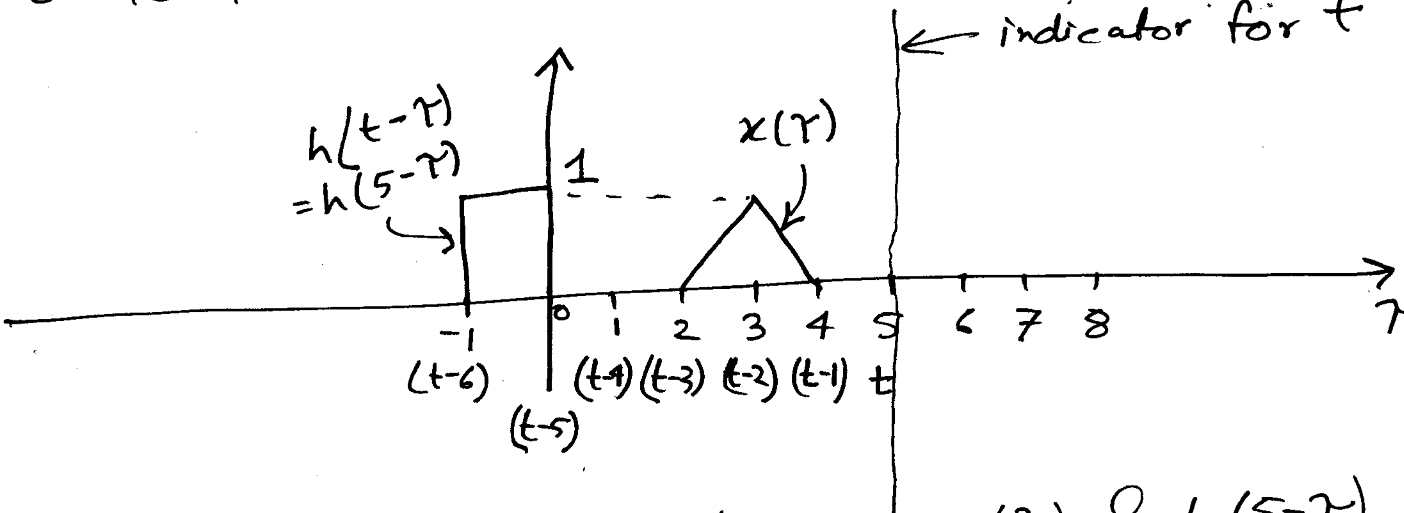
Now draw $\boxed{\text{Graph A}}$ and $\boxed{\text{Graph B}}$ together



This line is the indicator for time t. As t increases this line together with the graph of $h(t-\tau)$ moves right.

~~Observe~~

Since we want to compute output at $t=5$, i.e. $y(5)$, we will move the indicator for t to the point $\tau=5$ as shown below.



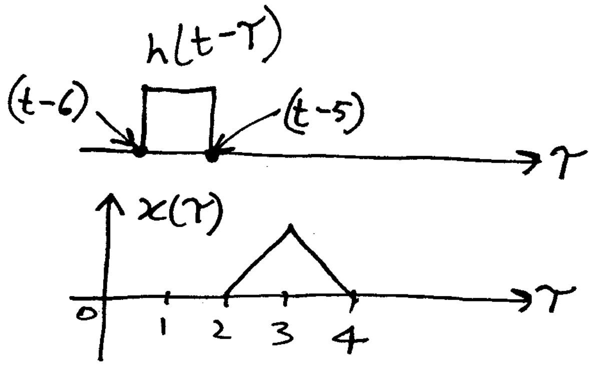
There is no overlap between $x(\tau)$ & $h(5-\tau)$

$\therefore\ y(5) = 0$

Let us now ~~to~~ compute ~~the~~ $y(t)$ briefly

~~for $t \le 7 \Rightarrow$ no overlap $\Rightarrow y(t) = 0$~~

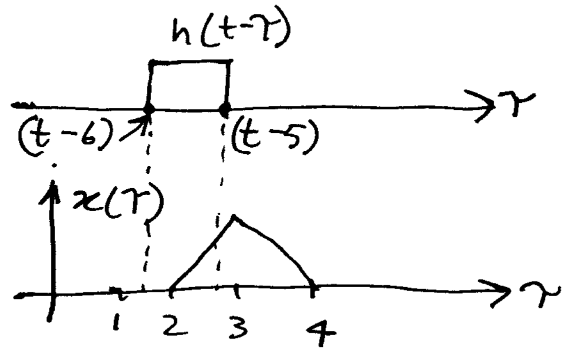
~~for $7 \le t \le 8$ $\quad y(t) = \int_{2}$~~



if $t-5 < 2$ or $t < 7$

$\Rightarrow$ no overlap

$\therefore y(t) = 0$



Note: $x(\tau) = \tau - 2$ for $2 \le \tau \le 3$

and $x(\tau) = 4 - \tau$ for $3 \le \tau \le 2$

If $t-5 \geq 2$ and $(t-6) \leq 2$

i.e $t \geq 7$ and $t \leq 8$

partial overlap

$$y(t) = \int_{2}^{t-5} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{2}^{t-5} (\tau - 2) \times 1 \, d\tau$$

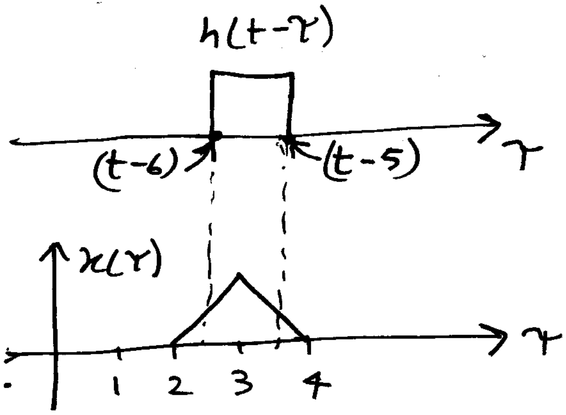
$$= \left[\frac{\tau^2}{2} - 2\tau\right]_{2}^{t-5}$$

$$= \frac{(t-5)^2}{2} - 2(t-5) - 2 + 4$$

$$= \frac{1}{2}\left(t^2 - 14t + 49\right)$$

$$= \frac{1}{2}(t-7)^2$$

P.T.O



$$x(\tau) = \begin{cases} \tau - 2 & \text{for } 2 \le \tau \le 3 \\ 4 - \tau & \text{for } 3 \le \tau \le 4 \end{cases}$$

If $t-5 \geq 3$ and $t-5 \leq 4$

or $t \geq 8$ and $t \leq 9$

full overlap

$$y(t) = \int_{t-6}^{t-5} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-6}^{t-5} x(\tau) \times 1 \; d\tau$$

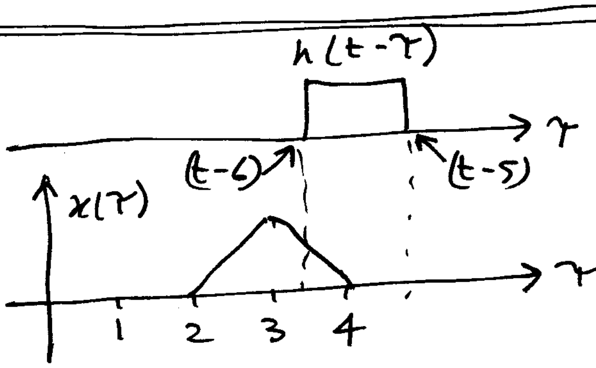
$$= \int_{t-6}^{3} x(\tau)\, d\tau + \int_{3}^{t-5} x(\tau)\, d\tau$$

$$= \int_{t-6}^{3} (\tau - 2)\, d\tau + \int_{3}^{t-5} (4-\tau)\, d\tau$$

$$= \left[\frac{\tau^2}{2} - 2\tau\right]_{t-6}^{3} + \left[4\tau - \frac{\tau^2}{2}\right]_{3}^{t-5}$$

$$= 17t - t^2 - 71{\cdot}5$$

(simplify yourselves and check my answer)



If $t-6 \geq 3$ and $t-6 \leq 4$

or $t \geq 9$ and $t \leq 10$

Partial overlap

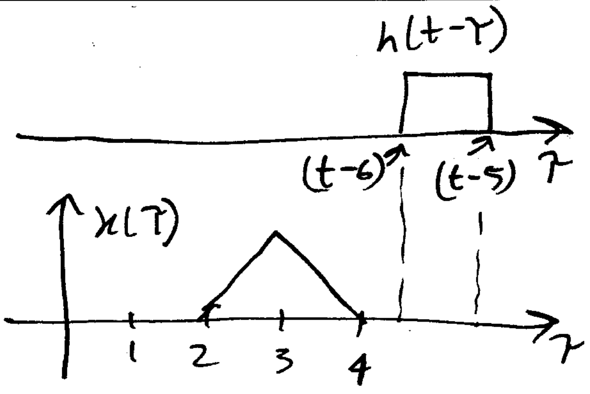
$$y(t) = \int_{t-6}^{4} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{t-6}^{4} (4-\tau)\, d\tau$$

$$= \left[4\tau - \frac{\tau^2}{2}\right]_{t-6}^{4}$$

$$= \frac{1}{2}(t-10)^2$$

(Please check)



If $t-6>4$ or $t>10$

no overlap

$y(t)=0$

$$\therefore y(t)=\begin{cases} 0 & \text{if } t\leq 7 \\ (t-7)^2/2 & \text{if } 7\leq t\leq 8 \\ 17t-t^2-71.5 & \text{if } 8\leq t\leq 9 \\ (t-10)^2/2 & \text{if } 9\leq t\leq 10 \\ 0 & \text{if } t\geq 10 \end{cases}$$