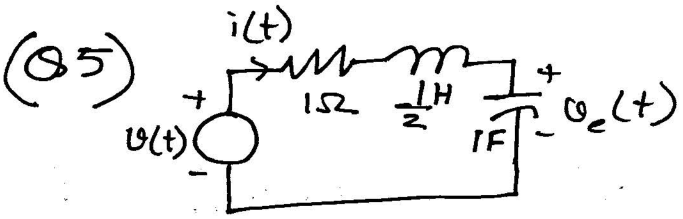
Tutorial 5 Solution

(Q5)



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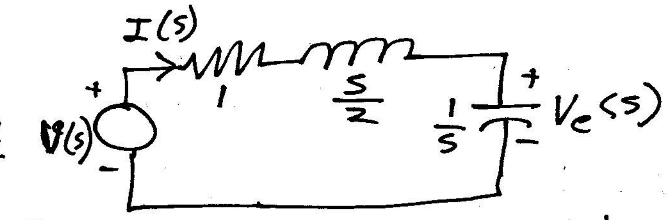


(Since all initial conditions are relaxed / zero)

$$\therefore V(s) = I(s)\left(1 + \frac{s}{2} + \frac{1}{s}\right) = I(s)\left(\frac{s^2+2s+2}{2s}\right)$$

$$\Rightarrow I(s) = \frac{2s\,V(s)}{s^2+2s+2}$$

and $V_c(s) = \frac{1}{s} I(s) = \frac{2\,V(s)}{s^2+2s+2}$

(a) $v(t) = u(t) \Rightarrow V(s) = \frac{1}{s}$

$$\therefore I(s) = \frac{2s\frac{1}{s}}{s^2+2s+2} = \frac{2}{s^2+2s+2} = \frac{2}{(s+1)^2+1^2}$$

$$\therefore i(t) = \mathcal{L}^{-1}\{I(s)\} = 2e^{-t}\sin(t)\,u(t)$$

$$V_c(s) = \frac{2}{s(s^2+2s+2)} = \frac{A}{s} + \frac{Bs+C}{s^2+2s+2}$$

(for some real values of A, B and C)

Now $As^2 + 2As + 2A + Bs^2 + Cs = 2$

$\Rightarrow A = 1$, $A + B = 0$ or $B = -1$

and $2A + C = 0$ or $C = -2$

$$\therefore V_c(s) = \frac{1}{s} - \frac{s}{s^2+2s+2} - \frac{2}{s^2+2s+2}$$

$$= \frac{1}{s} - \frac{(s+1)}{(s+1)^2+1^2} - \frac{1}{(s+1)^2+1^2}$$

$$\therefore v_c(t) = \mathcal{L}^{-1}\{V_c(s)\} = u(t)\left[1 - e^{-t}(\cos t + \sin t)\right]$$

(b) $v(t) = t\,u(t) \Rightarrow V(s) = \frac{1}{s^2}$

$$\therefore I(s) = \frac{2}{s(s^2+2s+2)}$$

$$\therefore i(t) = \mathcal{L}^{-1}\{I(s)\} = \left(1 - e^{-t}(\cos t + \sin t)\right)u(t)$$

And $V_c(s) = \dfrac{2}{s^2(s^2+2s+2)} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{Cs+D}{s^2+2s+2}$

such that

$$As^3 + 2As^2 + 2As + Bs^2 + 2Bs + 2B + Cs^3 + Ds^2 = 2$$

$$\Rightarrow (A+C)s^3 + (2A+B+D)s^2 + (2A+2B)s + 2B = 2$$

$\Rightarrow B = 1$

$2A+2B=0 \Rightarrow A = -1$

$A+C=0 \Rightarrow C = +1$

$2A+B+D=0 \Rightarrow D = 1$

$$\therefore V_c(s) = \frac{-1}{s} + \frac{1}{s^2} + \frac{s}{s^2+2s+2} + \frac{1}{s^2+2s+2}$$

$$= -\frac{1}{s} + \frac{1}{s^2} + \frac{s+1}{(s+1)^2+1^2}$$

$$\therefore v_c(t) = u(t)\left[-1 + t + e^{-t}\cos(t)\right]$$

(Q4)(a) $\mathcal{L}^{-1}\left\{ \frac{J1}{s+1+J1} - \frac{J1}{s+1-J1} - \frac{1}{(s+1+J1)^2} - \frac{1}{(s+1-J1)^2} \right\}$

$$= \left(J1\, e^{-(1+J)t} - J1\, e^{-(1-J)t} - te^{-(1+J)t} - te^{-(1-J)t} \right) u(t)$$

$$= \left(e^{-t} J\left(e^{-Jt} - e^{Jt}\right) - te^{-t}\left(e^{-Jt} + e^{Jt}\right) \right) u(t)$$

$$= \left(e^{-t}\, 2\, \frac{\left(e^{Jt} - e^{-Jt}\right)}{2J} - te^{-t}\, 2\, \frac{\left(e^{Jt} + e^{-Jt}\right)}{2} \right) u(t)$$

$$= 2e^{-t}\left(\sin t - t \cos t \right) u(t)$$

(b) $\mathcal{L}^{-1}\left\{ \frac{J6}{(s-J)^2} - \frac{J6}{(s+J)^2} \right\}$

$$= J6\left(t\, u(t) \left(e^{Jt} - e^{-Jt} \right) \right)$$

$$= 6t\, u(t)\ \frac{e^{Jt} - e^{-Jt}}{2J} \times 2J^2$$

$$= -26t \sin t\ u(t)$$

(Q3)(a) $\dfrac{s^3}{(s+2)(s+3)(s+4)} = \dfrac{s^3}{(s+2)(s^2+7s+12)}$

$$= \frac{s^3}{s^3+9s^2+26s+24} = 1 - \frac{9s^2+26s+24}{s^3+9s^2+26s+24}$$

$$= 1 - \left(\frac{A}{s+2} + \frac{B}{s+3} + \frac{C}{s+4}\right)$$

such that

$$A(s+3)(s+4) + B(s+2)(s+4) + C(s+2)(s+3) = 9s^2+26s+24$$

~~for~~ putting $s=-2$

$A2 = 8 \Rightarrow A = 4$

putting $s=-3$

$B(-1) = 27 \Rightarrow B = -27$

putting $s=-4$

$C(2) = 64 \Rightarrow C = 32$

$\therefore$ Given expression

$$= 1 - \frac{4}{s+2} + \frac{27}{s+3} - \frac{32}{s+4}$$

$\therefore$ Required Laplace inverse

$$= \delta(t) - 4e^{-2t}u(t) + 27e^{-3t}u(t) - 32e^{-4t}u(t)$$

(b) $X(s) = \dfrac{3s^2+2s+2}{(s+2)^2(s+3)} = \dfrac{A}{s+3} + \dfrac{B}{s+2} + \dfrac{C}{(s+2)^2}$

such that

$A(s+2)^2 + B(s+3)(s+2) + C(s+3) = 3s^2+2s+2$

putting $s=-3$, $A = 23$

$\therefore (A+B) = 3$ (equating the coefficient of s^2)

$\Rightarrow B = -20$

$\therefore 4A + 6B + 3C = 2$ (equating the constant term)

$\Rightarrow 92 - 120 + 3C = 2 \Rightarrow C = 10$

$\therefore X(s) = \dfrac{23}{s+3} - \dfrac{20}{s+2} + \dfrac{10}{(s+2)^2}$

$\therefore \mathcal{L}^{-1}\{X(s)\} = 23e^{-3t}u(t) - 20e^{-2t}u(t) + 10te^{-2t}u(t)$

(c) $X(s) = \dfrac{4s^2-3s+5}{s(s^2+2s+5)} = \dfrac{4s^2-3s+5}{s(s-(-1+2J))(s-(-1-2J))}$

$= \dfrac{A}{s} + \dfrac{B}{s+1-2J} + \dfrac{C}{s+1+2J}$

such that

$A(s+1-2J)(s+1+2J) + Bs(s+1+2J) + Cs(s+1-2J) = 4s^2-3s+5$

putting $s=0$, $A(5) = 5 \Rightarrow A = 1$

putting $s=-1+2J$, $B(-1+2J)(4J) = 4(1-4-4J) - 3(-1+2J) + 5$

$\Rightarrow B(-8-4J) = -12-16J+3-6J+5$

$= -4-22J$

$\Rightarrow B = \dfrac{4+22J}{8+4J} = \dfrac{(4+22J)(8-4J)}{80} = \dfrac{120+160J}{80}$

$$= \frac{3+4j}{2}$$

$$\therefore C = B^{*} = \frac{3-4j}{2}$$

$$\therefore X(s) = \frac{1}{s} + \frac{3+4j}{2(s+1-2j)} + \frac{3-4j}{2(s+1+2j)}$$

$$\therefore \mathcal{L}^{-1}\{X(s)\} = u(t) + \frac{3+4j}{2} e^{-(1-2j)t} u(t) + \frac{3-4j}{2} e^{-(1+2j)t} u(t)$$

$$= \left(1 + \frac{\cancel{1}5}{2} e^{j\theta} e^{-t} e^{2jt} + \frac{5}{2} e^{-j\theta} e^{-t} e^{-2jt}\right) u(t)$$

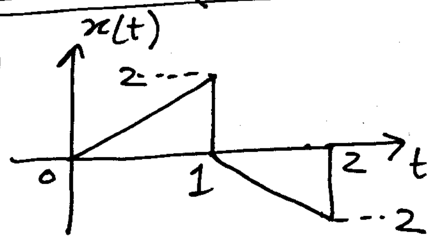
where $\theta = \tan^{-1}\left(\frac{4}{3}\right)$

$$= \left(1 + \frac{5}{2} e^{-t}\left(e^{j(2t+\theta)} + e^{-j(\theta+2t)}\right)\right) u(t)$$

$$= \left(1 + \frac{5}{2} e^{-t} \times 2\cos(2t+\theta)\right) u(t)$$

$$= \left(1 + 5e^{-t}\cos\left(2t + \tan^{-1}\left(\tfrac{4}{3}\right)\right)\right) u(t)$$

(Q2)



$$x(t) = 2r(t) - 2r(t-1) - 2u(t-1)$$
$$- 2r(t-1) + 2r(t-2) + 2u(t-2)$$
$$= 2r(t) - \left(4r(t-1) + 2u(t-1)\right)$$
$$+ \left(2r(t-2) + 2u(t-2)\right)$$

$$\therefore X(s) = \frac{2}{s^2} - \left(\frac{4}{s^2} + \frac{2}{s}\right) e^{-s} + \left(\frac{2}{s^2} + \frac{2}{s}\right) e^{-2s}$$

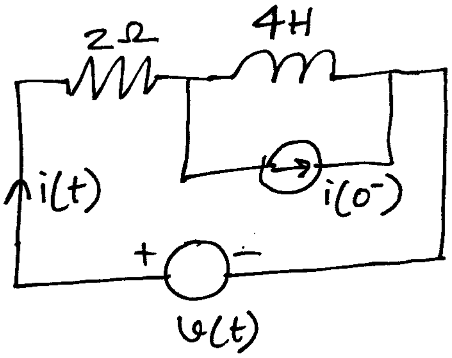
$$= X_1(s) + X_2(s) e^{-s} + X_3(s) e^{-2s} \quad \text{(as given)}$$

$$\therefore X_1(s) = \frac{2}{s^2}$$

$$X_2(s) = -\left(\frac{4}{s^2} + \frac{2}{s}\right)$$

$$X_3(s) = \left(\frac{2}{s^2} + \frac{2}{s}\right)$$

(Q6)



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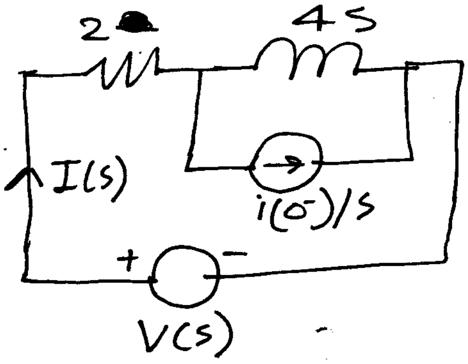

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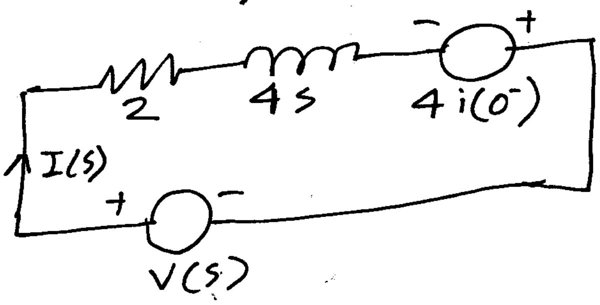
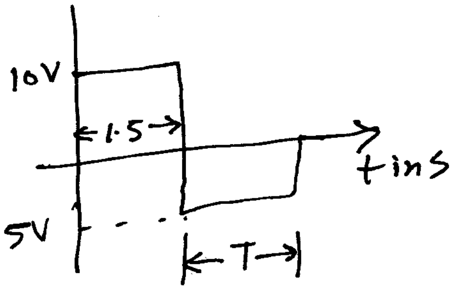


$$\therefore I(s) = \frac{V(s) + 4\, i(0^-)}{2 + 4s}$$



$$\therefore v(t) = 10u(t) - 15u(t-1.5) + 5u(t-T-1.5)$$

$$\therefore V(s) = \frac{10 - 15e^{-1.5s} + 5e^{-(T+1.5)s}}{s}$$

(a) $T = 4$ and $i(0^-) = 0$

$$\therefore I(s) = \frac{V(s)}{2+4s} = \frac{10 - 15e^{-1.5s} + 5e^{-(T+1.5)s}}{s} \times \frac{1}{2+4s}$$

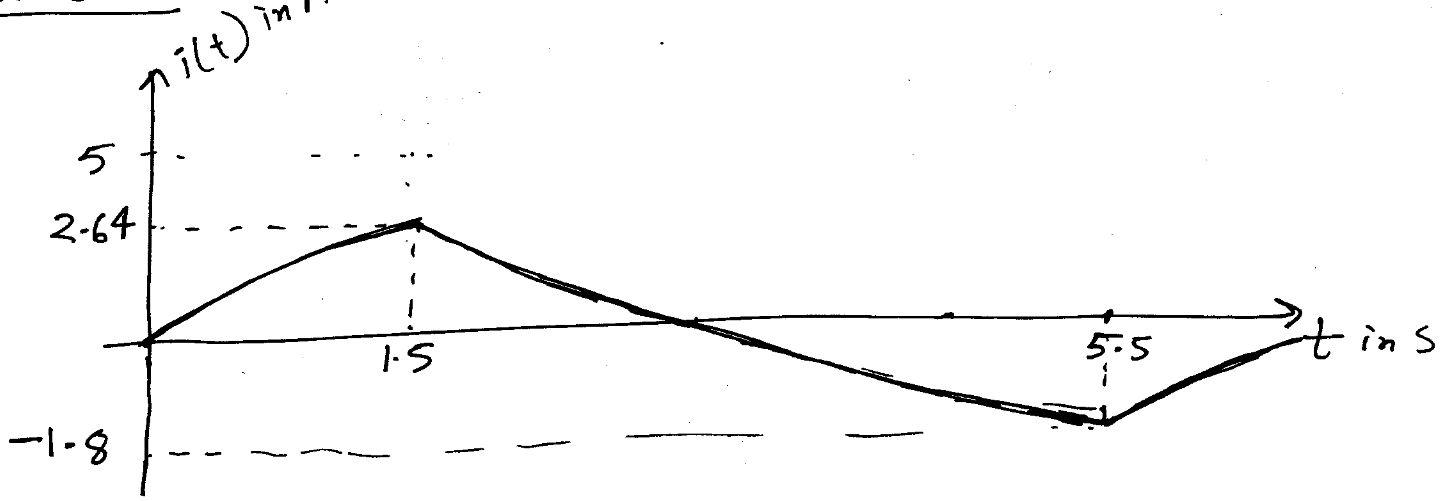
$$= \frac{\left(10 - 15e^{-1.5s} + 5e^{-5.5s}\right)}{2} \times \left(\frac{1}{s} - \frac{1}{s + 1/2}\right)$$

$$\therefore i(t) = \mathcal{L}^{-1}(I(s))$$

$$= 5u(t) - 5e^{-1/2 t} u(t) - 7.5u(t-1.5) + 7.5e^{-1/2(t-1.5)} \times u(t-1.5) + 2.5u(t-5.5) - 2.5e^{-1/2(t-5.5)} u(t-5.5)$$

$$= 5\left(1 - e^{-t/2}\right)u(t) - 7.5\left(1 - e^{-\frac{(t-1.5)}{2}}\right)u(t-1.5) + 2.5\left(1 - e^{-\frac{(t-5.5)}{2}}\right)u(t-5.5)$$

Sketch



(b) $T = 1, \quad i(0^-) = 0$

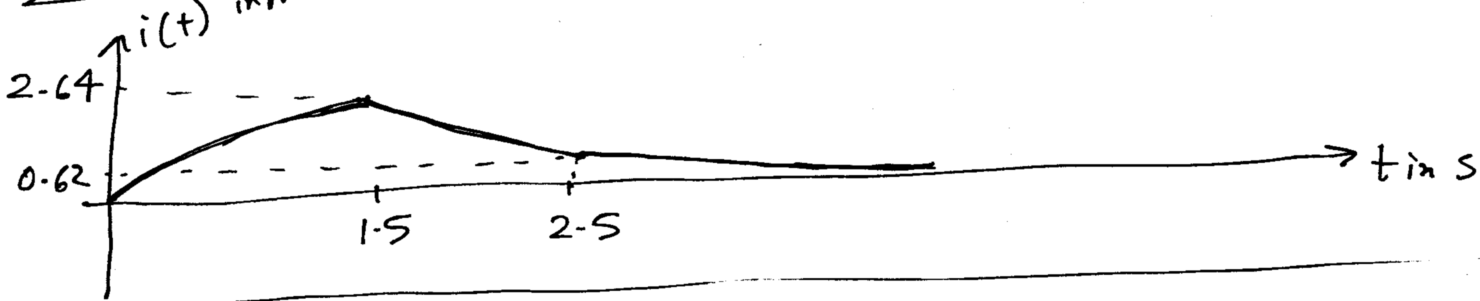
$$\therefore I(s) = \frac{V(s)}{2+4s} = \frac{10 - 15e^{-1.5s} + 5e^{-2.5s}}{s} \times \frac{1}{2+4s}$$

$$= \frac{\left(10 - 15e^{-1.5s} + 5e^{-2.5s}\right)}{2} \times \left(\frac{1}{s} - \frac{1}{s+1/2}\right)$$

$$\therefore i(t) = \mathcal{L}^{-1}\{I(s)\}$$

$$= 5\left(1 - e^{-t/2}\right)u(t) - 7.5\left(1 - e^{-(t-1.5)/2}\right)u(t-1.5) + 2.5\left(1 - e^{-(t-2.5)/2}\right)u(t-2.5)$$

Sketch



(c) $T = 2, \quad i(0^-) = 1$

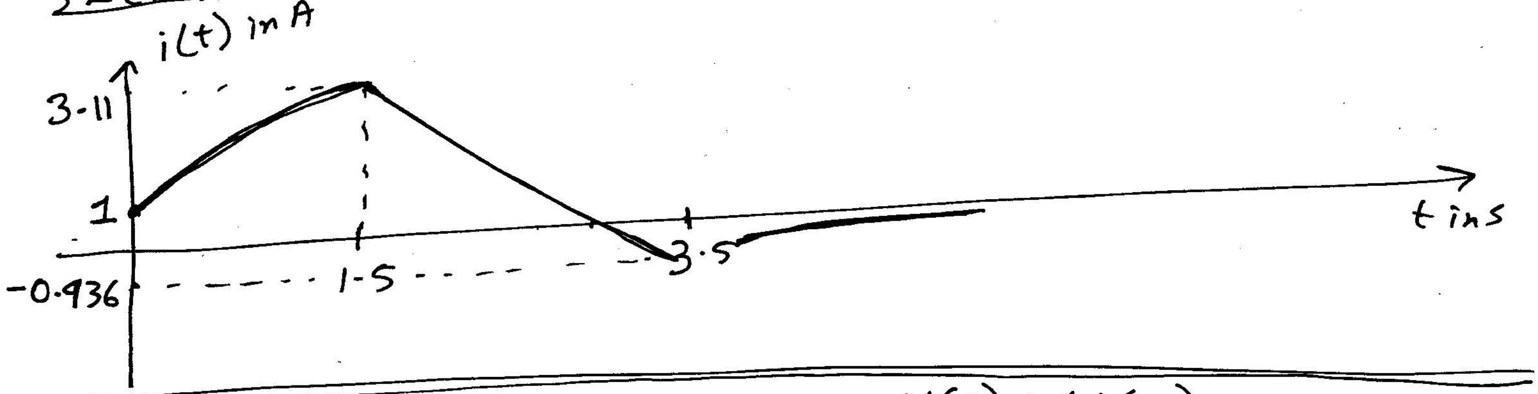
$$\therefore I(s) = \frac{V(s) + 4i(0^-)}{4s+2} = \frac{V(s)}{4s+2} + \frac{i(0^-)}{s+1/2}$$

$$= \frac{10 - 15e^{-1.5s} + 5e^{-3.5s}}{s} \times \frac{1}{4s+2} + \frac{1}{s+1/2}$$

$$= \frac{10 - 15e^{-1.5s} + 5e^{-3.5s}}{2} \times \left(\frac{1}{s} - \frac{1}{s+1/2}\right) + \frac{1}{s+1/2}$$

$$\therefore\ i(t) = \mathcal{L}^{-1}\{I(s)\}$$
$$= 5(1-e^{-t/2})u(t) - 7.5(1-e^{-(t-1.5)/2})u(t-1.5) + 2.5(1-e^{-(t-3.5)/2})u(t-3.5) + e^{-t/2}u(t)$$

Sketch



(Q7) If $R=0$ then $I(s) = \dfrac{V(s) + 4i(0^-)}{4s}$

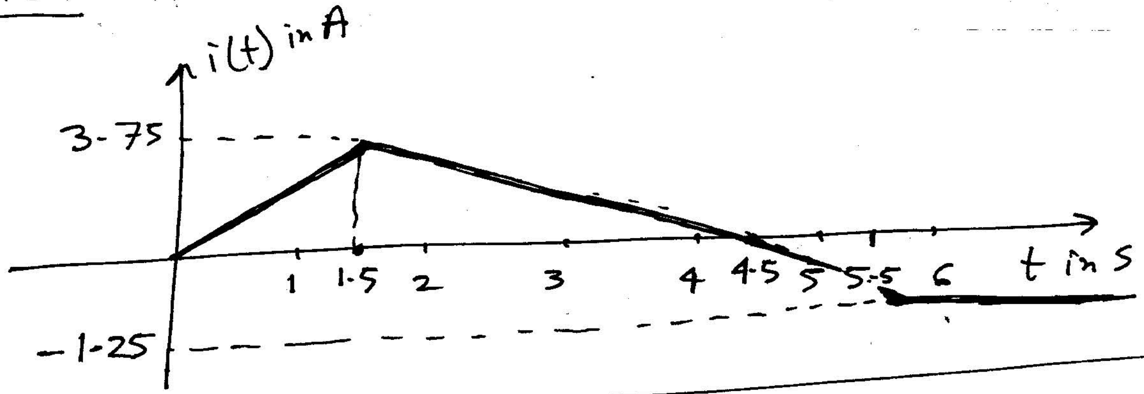
The circuit diagram, $v(t)$, $V(s)$ are all same as Q6

(a) $T = 4$ and $i(0^-) = 0$

$$\therefore\ I(s) = \frac{V(s)}{4s} = \frac{10 - 15e^{-1.5s} + 5e^{-5.5s}}{4s^2}$$

$$\therefore\ i(t) = \mathcal{L}^{-1}\{I(s)\} = 2.5t\,u(t) - 3.75(t-1.5)u(t-1.5) + 1.25(t-5.5)u(t-5.5)$$

Sketch

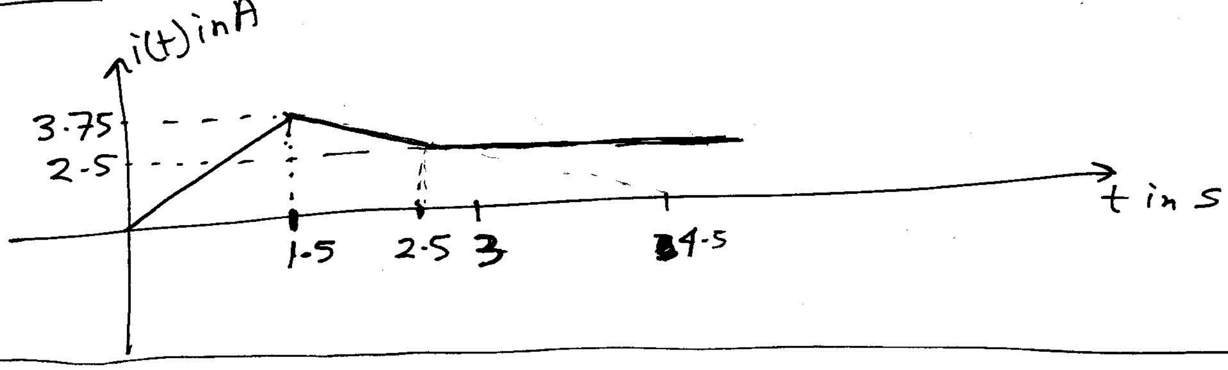


(b) $T = 1 \qquad i(0^-) = 0$

$$\therefore I(s) = \frac{V(s)}{4s} = \frac{10 - 15e^{-1.5s} + 5e^{-2.5s}}{4s^2}$$

$$\therefore i(t) = 2.5\,t\,u(t) - 3.75(t-1.5)\,u(t-1.5) + 1.25(t-2.5) \times u(t-2.5)$$

Sketch



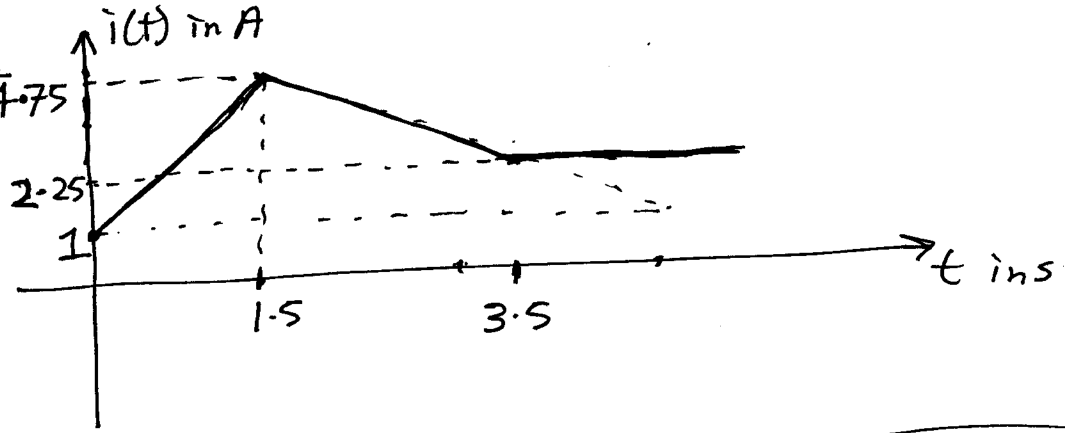
(c) $T = 2, \qquad i(0^-) = 1$

$$\therefore I(s) = \frac{V(s) + 4\,i(0^-)}{4s} = \frac{V(s)}{4s} + \frac{i(0^-)}{s}$$

$$= \frac{10 - 15e^{-1.5s} + 5e^{-3.5s}}{4s^2} + \frac{1}{s}$$

$$\therefore i(t) = 2.5\,t\,u(t) - 3.75(t-1.5)\,u(t-1.5) + 1.25(t-3.5)\,u(t-3.5) + u(t)$$

Sketch



(Q1) (a) $t\cos(t)\,u(t)$

$$\mathcal{L}\{\cos(t)\,u(t)\} = \frac{s}{s^2+1}$$

$$\therefore \mathcal{L}\{t\cos(t)\,u(t)\} = \frac{d}{ds}\left(\frac{s}{s^2+1}\right) \qquad \left[\because \mathcal{L}\{t\,x(t)\} = \frac{d}{ds}X(s)\right]$$

$$= \frac{1(s^2+1) - 2s\times s}{(s^2+1)^2} = \frac{1-s^2}{(s^2+1)^2}$$

(b) $t\sin(t)\,u(t)$

$$\mathcal{L}\{\sin(t)\,u(t)\} = \frac{1}{s^2+1}$$

$$\therefore \mathcal{L}\{t\sin(t)\,u(t)\} = \frac{d}{ds}\left(\frac{1}{s^2+1}\right) \qquad \left[\because \mathcal{L}\{t\,x(t)\} = \frac{d}{ds}X(s)\right]$$

$$= \frac{-2s}{(s^2+1)^2}$$

(c) $$\mathcal{L}\{e^{-t}\,t\cos(t)\,u(t)\} = \frac{1-(s+1)^2}{((s+1)^2+1)^2} \qquad \left[\because \mathcal{L}\{e^{-at}x(t)\} = X(s+a)\right]$$

$$= \frac{-s^2-2s}{(s^2+2s+2)^2}$$

(d) $$\mathcal{L}\{e^{t}\,t\sin(t)\,u(t)\} = \frac{-2(s+1)}{((s+1)^2+1)^2} \qquad \left[\because \mathcal{L}\{e^{-at}x(t)\} = X(s+a)\right]$$

$$= \frac{-2s-2}{(s^2+2s+2)^2}$$