Elementary Mathematics for Physics

Differential Calculus :

$$g(x)\ :\ dg = \frac{dg}{dx}dx$$

$$f(x, y, z)$$

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

$\frac{\partial f}{\partial x}$: Partial derivative of f wrt x by keeping y and z fixed.

Example : $u(x, y) = x^2 + 2xy$

$$\frac{\partial u}{\partial x} = 2x + 2y \qquad \frac{\partial u}{\partial y} = 2x$$

Vector Calculus

$$\vec{r} = x\,\hat{i} + y\,\hat{j} + z\,\hat{k} \qquad d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

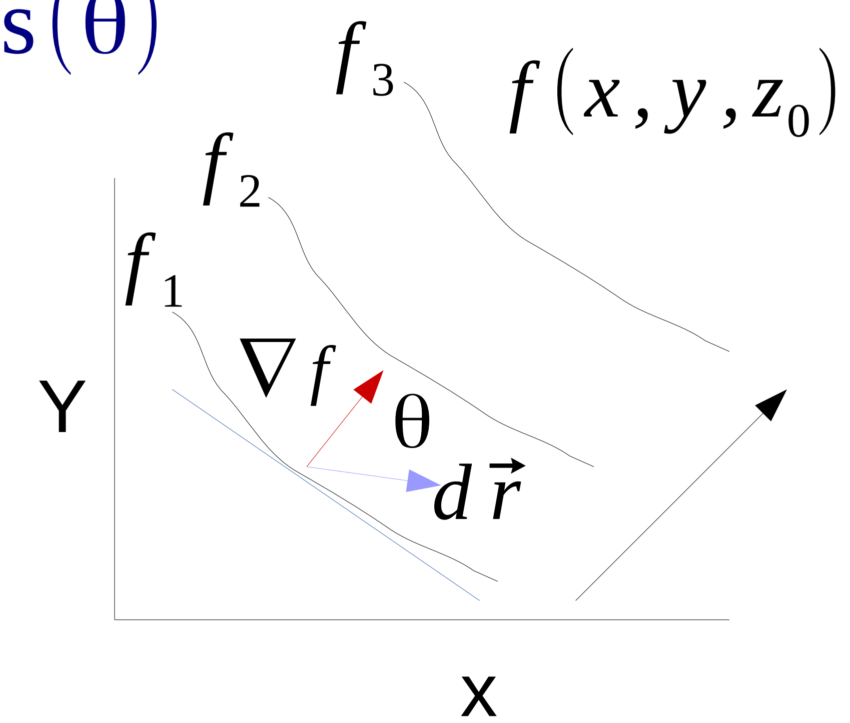
$$\text{Vector operator}: \vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\text{Gradient}: \vec{\nabla} f(x,y,z) = \hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}$$

$$df = \nabla f \cdot d\vec{r} = |\nabla f|\;|d\vec{r}|\;\cos(\theta)$$

df is maximum when $\theta = 0$.

It is directed towards normal at a point on a curve with constant f.



Vector Calculus

Example: $r(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

what is the direction of ∇r?

Clearly, the fastest increase of r will be radially outward.

So the expectation is $\dfrac{\nabla r}{|\nabla r|} = \hat{r}$

Check:

$$\nabla r = \hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}$$

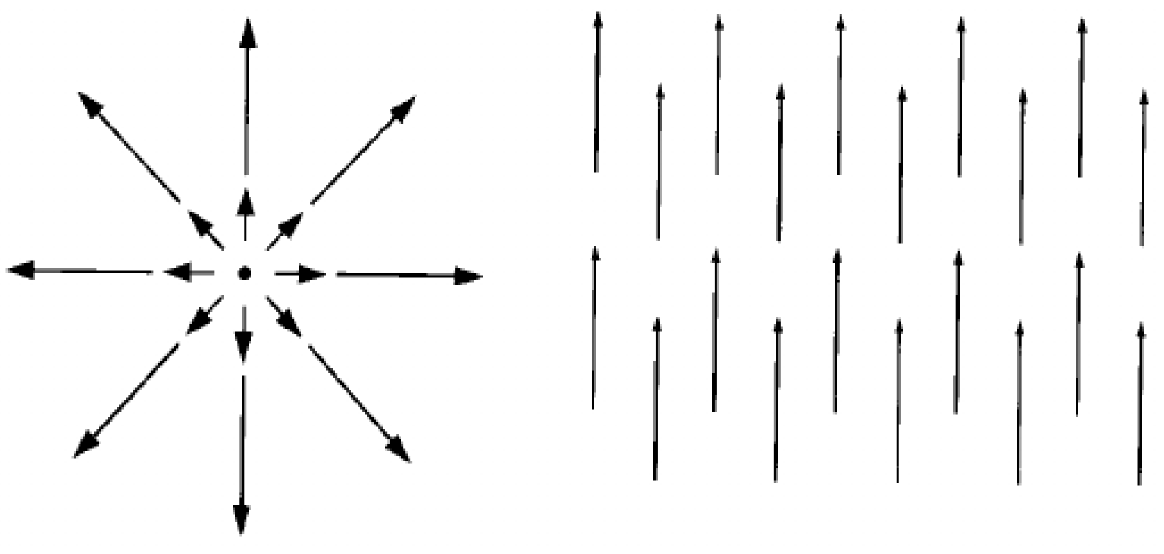
$$= \frac{1}{\sqrt{x^2 + y^2 + z^2}}\left[\hat{i}\,x + \hat{j}\,y + \hat{k}\,z\right]$$

$$= \frac{\vec{r}}{r} = \hat{r}$$

Vector Calculus

$$\text{Divergence}: \vec{\nabla}\cdot\vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

This scalar is a measure of how much the vector $\vec{A}$ spreads out (diverges) from the point in consideration.



A *source will have positive divergence*.

A *sink will have negative divergence*.

Vector Calculus

Example:

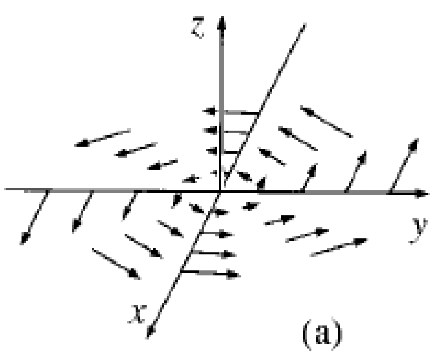
Divergence of the vector : $\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$

$$\vec{\nabla} \cdot \vec{r} = \frac{\partial r_x}{\partial x} + \frac{\partial r_y}{\partial y} + \frac{\partial r_z}{\partial z} = 3$$

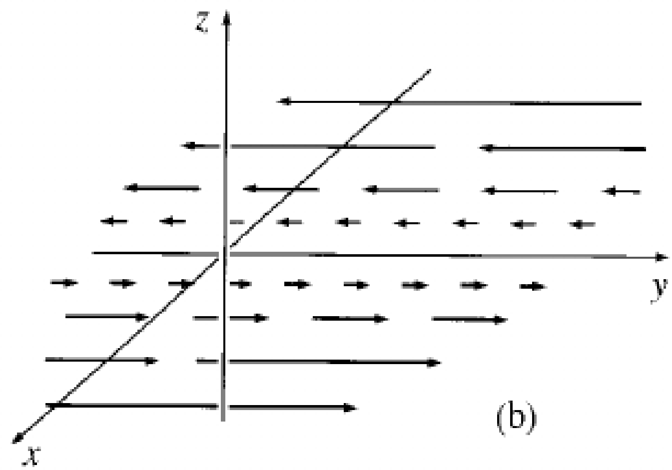
Vector Calculus

$$\text{Curl}: \vec{\nabla}\times\vec{A} = \begin{Vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_x & A_y & A_z \end{Vmatrix}$$

$$= \hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$



(a)



(b)

A point with large curl is a whirlpool.

Vector Calculus

Example: Curl of the vector : $\vec{A} = \hat{i}\, y^2 - \hat{j}\, x^2 + \hat{k}\, 2xy$

$$\vec{\nabla}\times\vec{A} = \hat{i}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\hat{j}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\hat{k}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$

$$= \hat{i}\,(2x)+\hat{j}\,(-2y)+\hat{k}\,(-2x-2y)$$

$$\nabla\cdot(\nabla\times\vec{A}) = 2-2+0 = 0$$

Divergence of a curl is **always** zero.

$$\nabla\times\hat{r} = \nabla\times(\nabla r) = 0$$

$$r=\sqrt{x^2+y^2+z^2}$$

$$\hat{r}=\frac{1}{\sqrt{3}}\left(\hat{i}+\hat{j}+\hat{k}\right)$$

Curl of a gradient is **always** zero.

Vector Calculus

(i) $\nabla \times \nabla f = 0$

(ii) $\nabla f \times \nabla g \neq 0$ (in general)

(iii) $\nabla \cdot \left(\nabla \times \vec{A} \right) = 0$

(iv) $\vec{\nabla} \times \left(\vec{\nabla} \times \vec{A} \right) = \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} \right) - \nabla^2 \vec{A}$

$$\nabla^2 = \vec{\nabla} \cdot \vec{\nabla} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

(Laplacian Operator)

Vector Calculus

Identities (Product Rules) :

(i) $\nabla(fg) = f\nabla g + g\nabla f$

(ii) $\nabla\cdot(f\vec{A}) = f(\nabla\cdot\vec{A}) + \vec{A}\cdot(\nabla f)$

(iii) $\nabla\times(f\vec{A}) = f(\nabla\times\vec{A}) - \vec{A}\times(\nabla f)$

(iv) $\nabla\cdot(\vec{A}\times\vec{B}) = \vec{B}\cdot(\nabla\times\vec{A}) - \vec{A}\cdot(\nabla\times\vec{B})$

(iv) $\nabla\times(\vec{A}\times\vec{B}) = \vec{A}(\nabla\cdot\vec{B}) - \vec{B}(\nabla\cdot\vec{A}) + (\vec{B}\cdot\nabla)\vec{A} - (\vec{A}\cdot\nabla)\vec{B}$

Vector Calculus

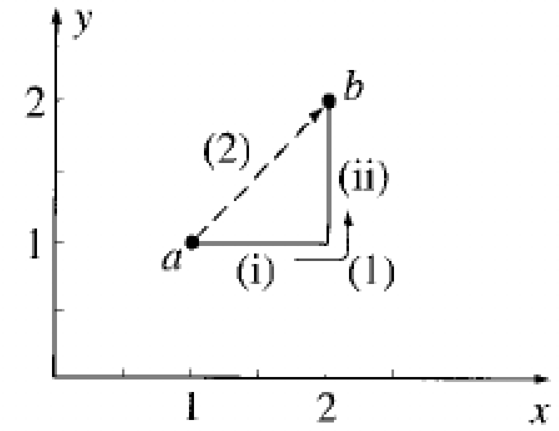
Line element : $d\vec{l} \equiv d\vec{r} \;=\; \hat{i}\,dx + \hat{j}\,dy + \hat{k}\,dz$

Surface element : $d\vec{S} \equiv d^2\vec{r} \;=\; dS\,\hat{n} \;\;;\;\; dx\;dy\,\hat{z}$

Volume element : $dV \equiv d^3\vec{r} \;=\; dx\;dy\;dz$

Integrals :

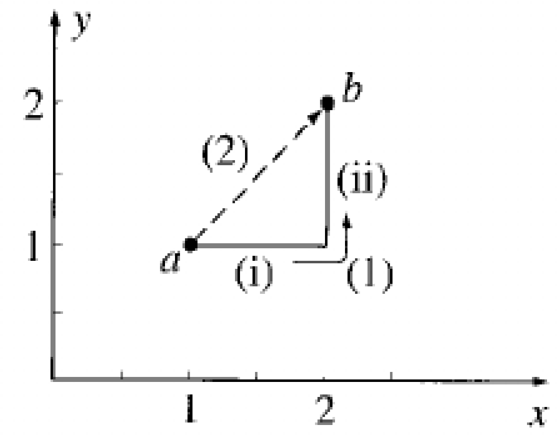
(i) Line Integral : $\int_{\vec{r},l}^{\vec{r}\,'} \vec{A}\cdot d\vec{l}$



Closed line integral : $\oint_l \vec{A}\cdot d\vec{l}$ when $\vec{r} = \vec{r}\,'$

Vector Calculus

Example: $a\left(1,1,0\right), \quad b\left(2,2,0\right)$



$$\vec{A} = y^2\hat{i} + 2\mathrm{x}\left(y+1\right)\hat{j}$$

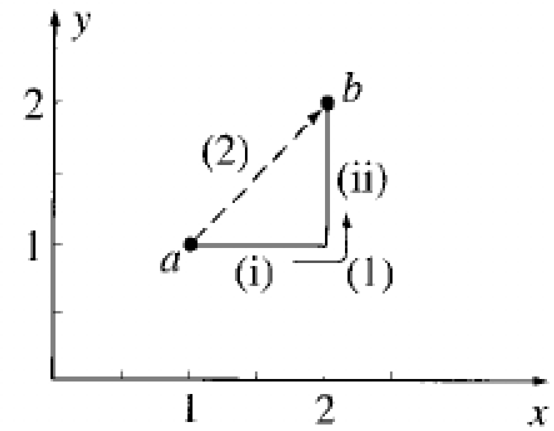
Line integral of $\vec{A}$ along paths (i) and (ii):

$$d\vec{l} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

$$\vec{A}\cdot d\vec{l} = y^2 dx + 2\mathrm{x}\left(y+1\right)dy$$

Vector Calculus

$$a(1,1,0),\quad b(2,2,0)$$



Path-1:

$$\int_{l,a}^{b}\vec{A}\cdot d\vec{l} = \int_{1}^{2}1^2 dx+\int_{1}^{2}[2\cdot 2(y+1)]dy$$

Path-2:

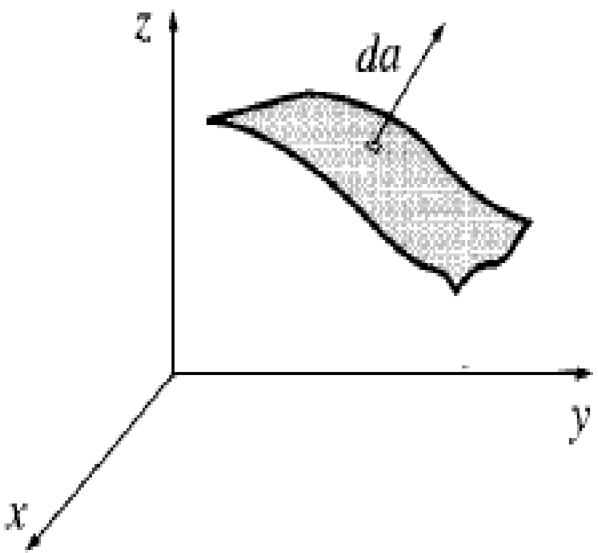
$$y = x \Rightarrow dy = dx$$

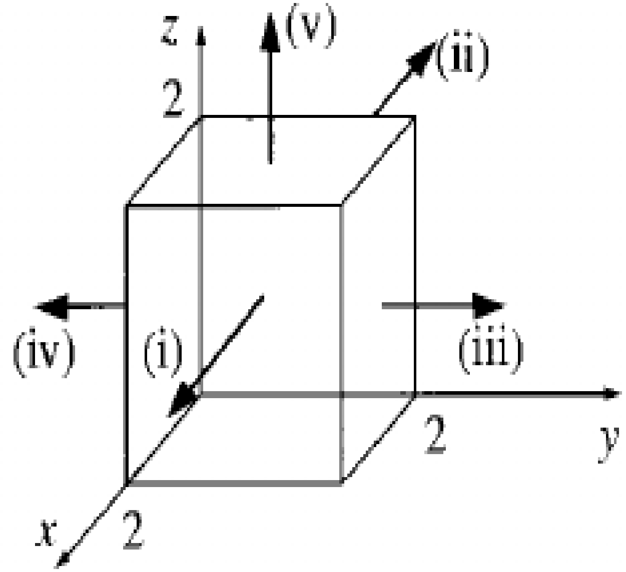
$$\int_{l,a}^{b}\vec{A}\cdot d\vec{l} = \int_{1}^{2}[x^2+2\mathrm{x}(x+1)]dx = 10$$

$$\oint_{l}\vec{A}\cdot d\vec{l} = 11-10 = 1$$

Vector Calculus

(ii) Surface Integral : $\int_S \vec{A} \cdot d\vec{S}$





$$d\vec{S}_{(i)} = dy\,dz\,\hat{i}$$
$$d\vec{S}_{(iii)} = dx\,dz\,\hat{j}$$
$$d\vec{S}_{(v)} = dx\,dy\,\hat{k}$$

Closed Surface Integral : $\oint_S \vec{A} \cdot d\vec{S}$

It is called flux of $\vec{A}$ through the surface.

If $\vec{A}$ represents velocity of a fluid, flux of $\vec{A}$ will flow out of surface per unit time.

Vector Calculus

(iii) Volume Integral : $\int_V T(x, y, z)\, dV$

$$\int_V \vec{A}\, dV = \hat{i} \int_V A_x\, dV + \hat{j} \int_V A_y\, dV + \hat{k} \int_V A_z\, dV$$

Fundamental Theorem of Calculus:

$$\int_a^b \frac{df(x)}{dx}\, dx = f(b) - f(a)$$

The integral of a derivative of a function over some interval is the value of the function at the end points (or boundaries).

In vector calculus, there are three types of derivatives.

Vector Calculus

(i) Fundamental Theorem of Gradients :

$$\int_{a,l}^{b} \vec{\nabla} T \cdot d\vec{l} = T(b) - T(a)$$

$$\text{as } dT = \nabla T \cdot d\vec{l}$$

The integral (line) of a derivative (gradient) of a function is its value at the boundary points.

The integral is independent of path.

$$\oint_l \vec{\nabla} T \cdot d\vec{l} = 0$$

Vector Calculus

Divergence Theorem (Gauss or Greens Theorem):

$$\int_V \left(\vec{\nabla}\cdot\vec{A}\right) dV = \oint_S \vec{A}\cdot d\vec{S}$$

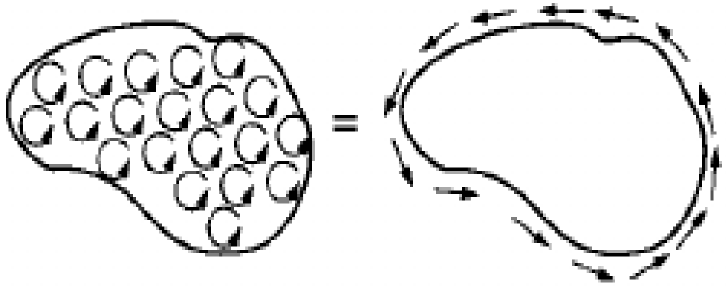
The integral (volume) of a derivative (divergence) of a vector is its value at the boundary surface enclosing the volume.

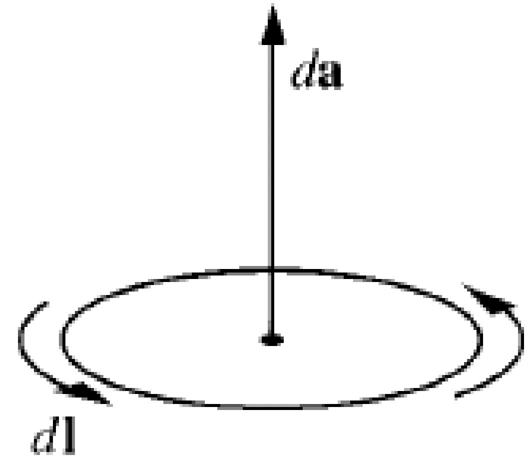
Fundamental Theorem of Curl (Stoke'sTheorem).:

$$\int_s \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_l \vec{A}\cdot d\,\vec{l}$$

The integral (surface) of a derivative (Curl) of a vector is its value at the boundary line enclosing the surface.

Vector Calculus





$$\int_s \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = \oint_l \vec{A}\cdot d\vec{l}$$

$\int_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S}$ depends only on the boundarly line.

$$\oint_S \left(\vec{\nabla}\times\vec{A}\right)\cdot d\vec{S} = 0$$

Vector Calculus

Integration by Parts:

$$\frac{d}{dx}(fg) = f\left(\frac{dg}{dx}\right) + g\left(\frac{df}{dx}\right)$$

$$\int_a^b \frac{d}{dx}(fg)\,dx = fg\Big|_a^b = \int_a^b f\left(\frac{dg}{dx}\right)dx + \int_a^b g\left(\frac{df}{dx}\right)dx$$

$$\int_a^b f\left(\frac{dg}{dx}\right)dx = -\int_a^b g\left(\frac{df}{dx}\right)dx + fg\Big|_a^b.$$

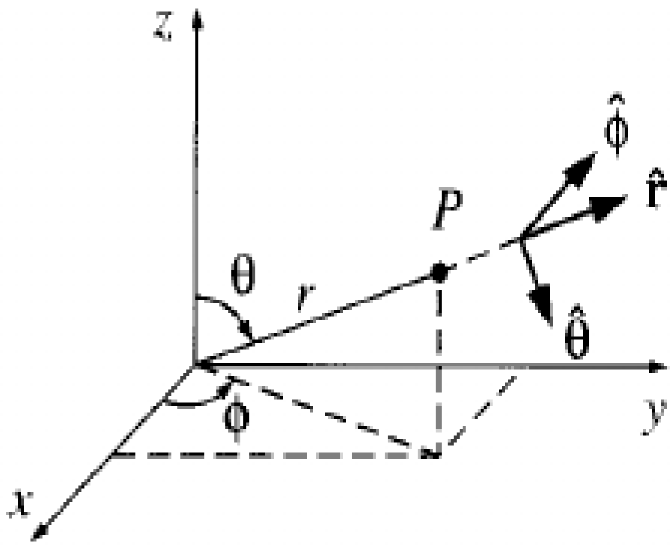
$$\nabla\cdot\left(f\vec{A}\right) = f\,\nabla\cdot\vec{A} + \vec{A}\cdot\nabla f$$

$$\int_V \vec{\nabla}\cdot\left(f\vec{A}\right)dV \equiv \oint_S \left(f\vec{A}\right)\cdot d\vec{S} = \int_V \left[f\,\vec{\nabla}\cdot\vec{A} + \vec{A}\cdot\vec{\nabla} f\right]dV$$

$$\int_V f\left(\vec{\nabla}\cdot\vec{A}\right)dV = -\int_V \left[\vec{A}\cdot\vec{\nabla} f\right]dV + \oint_S \left(f\vec{A}\right)\cdot d\vec{S}$$

Curvilinear Coordinate Systems

I) Spherical Polar Coordinates :



$$\vec{r} = \hat{i}\,x + \hat{j}\,y + \hat{k}\,z$$

$$\vec{r} = \hat{r}\,r$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{r} \times \hat{\theta} = \hat{\varphi}$$

$$\hat{\theta} \times \hat{\varphi} = \hat{r}$$

$$\hat{\varphi} \times \hat{r} = \hat{\theta}$$

$$x = r\sin(\theta)\cos(\varphi)\;;\; y = r\sin(\theta)\sin(\varphi)\;;\; z = r\cos(\theta)$$

$$\hat{r} \cdot \hat{\theta} = 0$$

$$\hat{r} = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - \hat{k}\sin\theta$$

$$\hat{\varphi} = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

Spherical Polar Coordinates :

$$\hat{r} = \hat{i}\sin\theta\cos\varphi + \hat{j}\sin\theta\sin\varphi + \hat{k}\cos\theta$$

$$\hat{\theta} = \hat{i}\cos\theta\cos\varphi + \hat{j}\cos\theta\sin\varphi - \hat{k}\sin\theta$$

$$\hat{\varphi} = -\hat{i}\sin\varphi + \hat{j}\cos\varphi$$

$$\Downarrow$$

$$\hat{i} = \hat{r}\sin\theta\cos\varphi + \hat{\theta}\cos\theta\cos\varphi - \hat{\varphi}\sin\varphi$$

$$\hat{j} = \hat{r}\sin\theta\sin\varphi + \hat{\theta}\cos\theta\sin\varphi + \hat{\varphi}\cos\varphi$$

$$\hat{k} = \hat{r}\cos\theta - \hat{\theta}\sin\theta$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\frac{\partial}{\partial x} \rightarrow \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \varphi}{\partial x}\right)\frac{\partial}{\partial \varphi} \qquad \text{(Chain rule)}$$

Spherical Polar Coordinates :

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\cos\theta = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$$

$$\frac{\partial}{\partial x} \rightarrow \left(\frac{\partial r}{\partial x}\right)\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)\frac{\partial}{\partial \theta} + \left(\frac{\partial \varphi}{\partial x}\right)\frac{\partial}{\partial \varphi}$$

$$\tan\varphi = \frac{y}{x}$$

$$\frac{\partial}{\partial x} = \sin\theta\cos\varphi\left(\frac{\partial}{\partial r}\right) - \frac{\sin\varphi}{r\sin\theta}\left(\frac{\partial}{\partial \varphi}\right) + \frac{\cos\theta\cos\varphi}{r}\left(\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\varphi\left(\frac{\partial}{\partial r}\right) + \frac{\cos\varphi}{r\sin\theta}\left(\frac{\partial}{\partial \varphi}\right) + \frac{\cos\theta\sin\varphi}{r}\left(\frac{\partial}{\partial \theta}\right)$$

$$\frac{\partial}{\partial z} = \cos\theta\left(\frac{\partial}{\partial r}\right) - \frac{\sin\theta}{r}\left(\frac{\partial}{\partial \theta}\right)$$

Spherical Polar Coordinates :

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\left(\frac{\partial}{\partial \theta}\right) + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}$$

$$\vec{\nabla} T = \hat{r}\frac{\partial T}{\partial r} + \hat{\theta}\frac{1}{r}\left(\frac{\partial T}{\partial \theta}\right) + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}$$

$$dT = \vec{\nabla} T \cdot d\vec{r} \equiv \frac{\partial T}{\partial r}dr + \frac{\partial T}{\partial \theta}d\theta + \frac{\partial T}{\partial \varphi}d\varphi$$

$$\Rightarrow d\vec{r} = \hat{r}\,dr + \hat{\theta}\,r\,d\theta + \hat{\varphi}\,r\sin\theta\,d\varphi$$

Spherical Polar Coordinates :

Volume Elements:

$$dV = (dr)(r\,d\theta)(r\sin\theta\,d\varphi) = r^2\sin\theta\,dr\,d\theta\,d\varphi$$

Surface Elements:

$$dS_r = (r\,d\theta)(r\sin\theta\,d\varphi) = r^2\sin\theta\,d\theta\,d\varphi$$

$$dS_\theta = dr(r\sin\theta\,d\varphi) = r\sin\theta\,dr\,d\varphi$$

$$dS_\varphi = dr(r\,d\theta) = r\,dr\,d\theta$$

Spherical Polar Coordinates :

$$\nabla^2 T = \vec{\nabla}\cdot\vec{\nabla} T$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial T}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial}{\partial \varphi}\left(\frac{\partial T}{\partial \varphi}\right)$$

$$\vec{A} = \hat{r}\, A_r + \hat{\theta}\, A_\theta + \hat{\varphi}\, A_\varphi$$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\left(A_\varphi\right)$$

$$\vec{\nabla}\times\vec{A} = \hat{r}\,\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial \theta}\left(\sin\theta\, A_\varphi\right) - \frac{\partial A_\theta}{\partial \varphi}\right]$$

$$+ \hat{\theta}\,\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \varphi} - \frac{\partial}{\partial r}\left(r\, A_\varphi\right)\right] \quad + \hat{\varphi}\,\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, A_\theta\right) - \frac{\partial A_r}{\partial \theta}\right]$$

Spherical Polar Coordinates :

$$\vec{A} = \hat{r}\,A_r + \hat{\theta}\,A_\theta + \hat{\varphi}\,A_\varphi = \hat{i}\,A_x + \hat{j}\,A_y + \hat{k}\,A_z$$

$$A_x = \vec{A}\cdot\hat{i} = \left(\vec{A}\cdot\hat{r}\right)\sin\theta\cos\varphi + \left(\vec{A}\cdot\hat{\theta}\right)\cos\theta\cos\varphi - \left(\vec{A}\cdot\hat{\varphi}\right)\sin\varphi$$

$$= A_r\sin\theta\cos\varphi + A_\theta\cos\theta\cos\varphi - A_\varphi\sin\varphi$$

$$A_y = A_r\sin\theta\sin\varphi + A_\theta\cos\theta\sin\varphi + A_\varphi\cos\varphi$$

$$A_z = A_r\cos\theta - A_\theta\sin\theta$$

Divergence : $\vec{\nabla}\cdot\vec{A} = \dfrac{\partial A_x}{\partial x} + \dfrac{\partial A_y}{\partial y} + \dfrac{\partial A_z}{\partial z}$

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\,A_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\left(A_\varphi\right)$$

$$\vec{\nabla} \times \vec{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\sin\theta\, A_{\varphi}\right) - \frac{\partial A_{\theta}}{\partial\varphi}\right]\hat{r}$$

$$+\ \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{\partial}{\partial r}\left(rA_{\varphi}\right)\right]\hat{\theta}$$

$$+\ \frac{1}{r}\left[\frac{\partial A_r}{\partial\theta} + \frac{\partial}{\partial r}\left(rA_{\theta}\right)\right]\hat{\varphi}$$

Spherical Wave:

$$\nabla^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial}{\partial\varphi}\right)$$

Wave equation in three dimensions:

$$\nabla^2\Psi(\vec{r},t) = \frac{1}{v^2}\frac{\partial^2\Psi(\vec{r},t)}{\partial t^2}$$

Spherically symmetric wave equation:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Psi(r,t)}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}$$

$$\left[\frac{\partial^2\Psi}{\partial r^2} + \frac{2}{r}\frac{\partial\Psi}{\partial r}\right] = \frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}$$

Spherical Wave:

$$\left[\frac{\partial^2\Psi}{\partial r^2}+\frac{2}{r}\frac{\partial\Psi}{\partial r}\right]=\frac{1}{v^2}\frac{\partial^2\Psi(r,t)}{\partial t^2}$$

$$\text{Ansatz}:\Psi(r,t)=\frac{1}{r}f(r,t)$$

$$\frac{\partial^2 f(r,t)}{\partial r^2}=\frac{1}{v^2}\frac{\partial^2 f(r,t)}{\partial t^2}$$

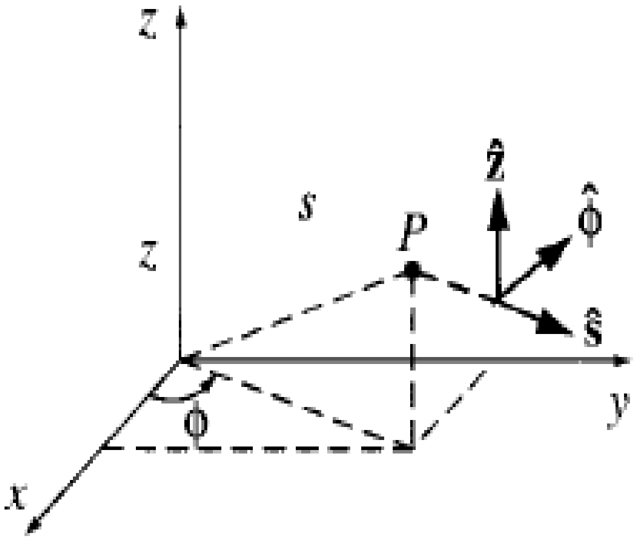
(One-dimensional wave equation)

$$\Rightarrow\ f(r,t)=a_1 f_1(r+vt)+a_2 f_2(r-vt)$$

$$\Rightarrow\ \Psi(r,t)=\frac{a_1}{r}f_1(r+vt)+\frac{a_2}{r}f_2(r-vt)$$

(Spherically symmetric wave)

Cylindrical Coordinates :



$$x = s\cos\varphi \; ; \; y = s\sin\varphi \; ; \; z = z$$

$$\hat{s} = \hat{i}\cos\varphi + \hat{j}\sin\varphi$$

$$\hat{\varphi} = \hat{i}(-\sin\varphi) + \hat{j}\cos\varphi$$

$$\hat{z} = \hat{k}$$

$$\hat{s}\times\hat{\varphi} = \hat{z} \; ; \quad \hat{\varphi}\times\hat{z} = \hat{s} \; ; \quad \hat{z}\times\hat{s} = \hat{\varphi}$$

$$\hat{i} = \hat{s}\cos\varphi - \hat{\varphi}\sin\varphi \; ; \; \hat{j} = \hat{s}\sin\varphi + \hat{\varphi}\cos\varphi \; ; \; \hat{k} = \hat{z}$$

$$s = \sqrt{x^2 + y^2} \; ; \; \tan\varphi = y/x \; ; \; z$$

Cylindrical Coordinates :

$$\frac{\partial}{\partial x} = \left(\frac{\partial s}{\partial x}\right)\frac{\partial}{\partial s} + \left(\frac{\partial \varphi}{\partial x}\right)\frac{\partial}{\partial \varphi}$$

$$= \cos\varphi\left(\frac{\partial}{\partial s}\right) - \sin\varphi\left(\frac{\partial}{\partial \varphi}\right)$$

$$\frac{\partial}{\partial y} = \sin\varphi\left(\frac{\partial}{\partial s}\right) + \cos\varphi\left(\frac{\partial}{\partial \varphi}\right)$$

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

$$\vec{\nabla} T = \hat{s}\frac{\partial T}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial T}{\partial \varphi} + \hat{k}\frac{\partial T}{\partial z}$$

Cylindrical Coordinates :

$$dT = \vec{\nabla} T \cdot d\vec{r} \equiv \frac{\partial T}{\partial s} + \frac{\partial T}{\partial \varphi} + \frac{\partial T}{\partial z}$$

$$\vec{\nabla} T = \hat{s}\frac{\partial T}{\partial s} + \hat{\varphi}\frac{1}{s}\frac{\partial T}{\partial \varphi} + \hat{k}\frac{\partial T}{\partial z}$$

$$d\vec{r} = \hat{s}\, ds + \hat{\varphi}\, s\, d\varphi + \hat{z}\, dz$$

$$dV = (ds)(s\, d\varphi)(dz) = s\, ds\, d\varphi\, dz$$

$$dS_s = (s\, d\varphi)(dz) = s\, d\varphi\, dz$$

$$dS_\varphi = (ds)(dz) = ds\, dz$$

$$dS_z = (ds)(s\, d\varphi) = s\, ds\, d\varphi$$

Cylindrical Coordinates :

$$\nabla^2 \equiv \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}\right)+\frac{1}{s^2}\frac{\partial^2}{\partial \varphi^2}+\frac{\partial^2}{\partial z^2}$$

$$\vec{A} = \hat{s}\,A_s+\hat{\varphi}\,A_\varphi+\hat{z}\,A_z$$

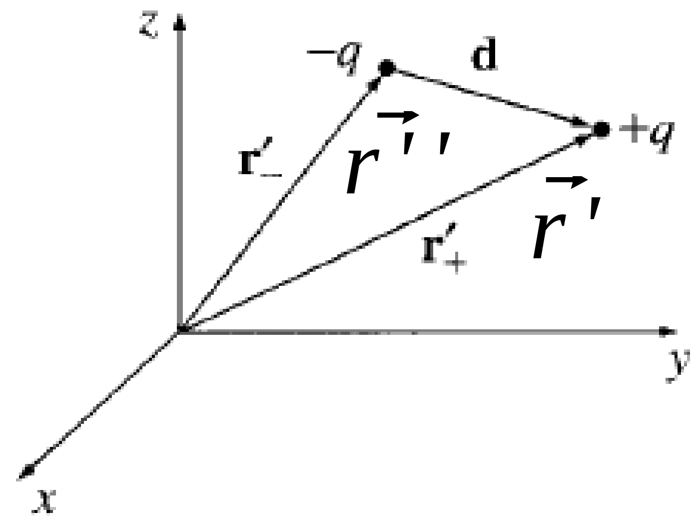
$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(s\,A_s\right)+\frac{1}{s}\frac{\partial A_\varphi}{\partial \varphi}+\frac{\partial A_z}{\partial z}$$

$$\vec{\nabla}\times\vec{A} = \hat{s}\left(\frac{1}{s}\frac{\partial A_z}{\partial \varphi}-\frac{\partial A_\varphi}{\partial z}\right)+\hat{\varphi}\left(\frac{\partial A_s}{\partial z}-\frac{\partial A_z}{\partial s}\right) + \hat{z}\,\frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\,A_\varphi\right)-\frac{\partial A_s}{\partial \varphi}\right]$$

Electric Dipole Moment

Physical Dipole:

(Equal and opposite charges)



$$\text{Dipole Moment}: \vec{p} = q\vec{r}' - q\vec{r}'' = q\vec{d}$$

Dipole Moment of Collection of Point Charges:

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}'_i \qquad \text{Total charge}: Q = \sum_{i=1}^{n} q_i$$

Dipole Moment of a Distribution of Charges:

Total charge : $Q = \int_V \rho(\vec{r}\,')\, d^3\vec{r}\,'$; $Q = \int_S \sigma\, dS_n$

Total Dipole Moment : $\vec{p} = \int_V \rho(\vec{r}\,')\, \vec{r}\,'\, d^3\vec{r}\,'$

Electric Potential due to $\vec{p}$: $V_{dip} = \dfrac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$

Dipole moment may be nonzero for any Q.

Example:

A spherical shell of radius R

Surface charge density : $\sigma = \alpha\cos\theta$

Example:

$$Q = \int_s \sigma \, dS_R = R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \left(\alpha \cos\theta\right)$$

$$= -\alpha R^2 \int_0^{\pi} \cos\theta \; d\left(\cos\theta\right) \int_0^{2\pi} d\varphi = 0$$

$$\vec{p} = \int_s \sigma \left(R \hat{r}\right) dS_R$$

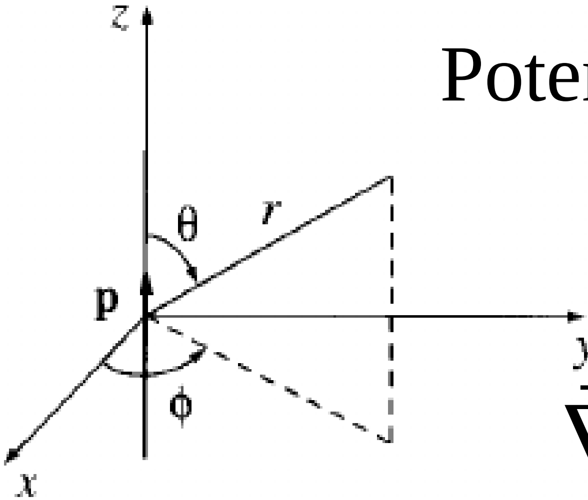
$$= R^2 \int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi \left(\alpha \cos\theta\right) \times R\left(\hat{i} \sin\theta \cos\varphi + \hat{j} \sin\theta \sin\varphi + \hat{k} \cos\theta\right)$$

$$= -\hat{k} \alpha R^3 \int_0^{\pi} \cos^2\theta \; d\left(\cos\theta\right) \int_0^{2\pi} d\varphi = \frac{4\pi}{3} \alpha R^3 \hat{k}$$

$$\text{Potential : } V\left(R, \theta\right) = \frac{\vec{p} \cdot \hat{r}}{4\pi \epsilon_0 R^2} = \frac{\alpha}{3\epsilon_0} R \cos\theta$$

It is an example of pure dipole.

Electric Field of a Pure Dipole

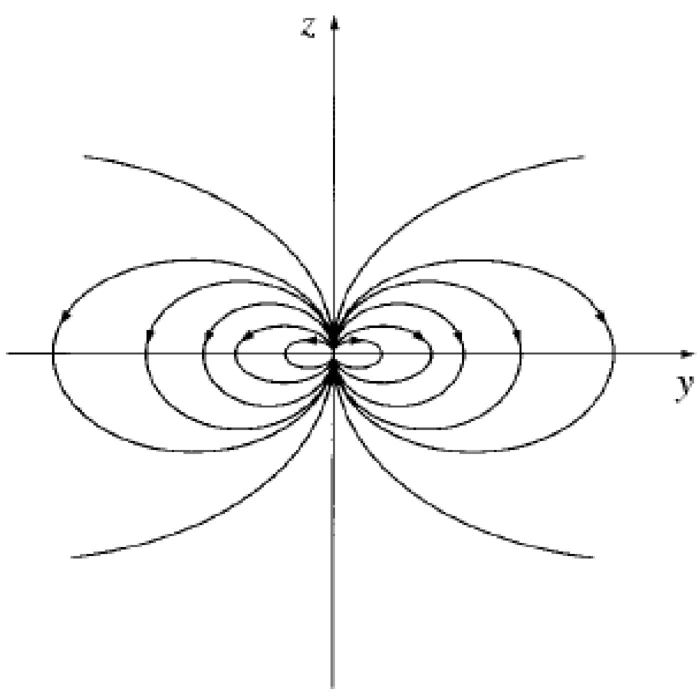


$$\text{Potential}: V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2} = \frac{p\cos\theta}{4\pi\epsilon_0 r^2}$$

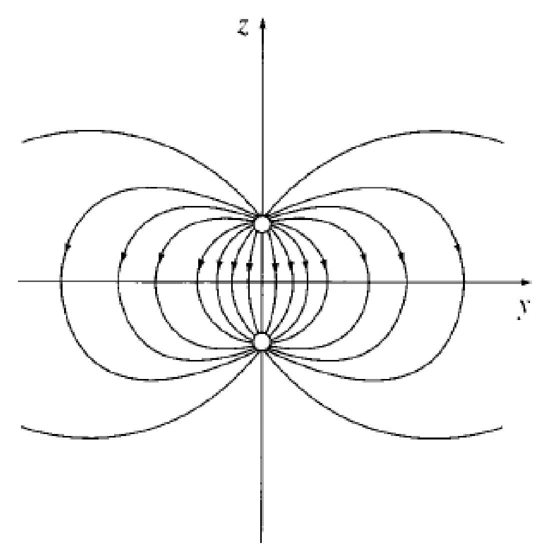
$$\vec{E}_{dip} = -\nabla V_{dip}$$

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\left(\frac{\partial}{\partial\theta}\right) + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}$$

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r} + \sin\,\theta\,\hat{\theta}\right)$$



(a) Field of a "pure" dipole



(a) Field of a "physical" dipole

Electrically Polarized Medium

Neutral atoms get polarized if electric field is applied.

(Positive and negative charges are separated in equilibrium.)

$$\vec{p} = \alpha \vec{E}$$

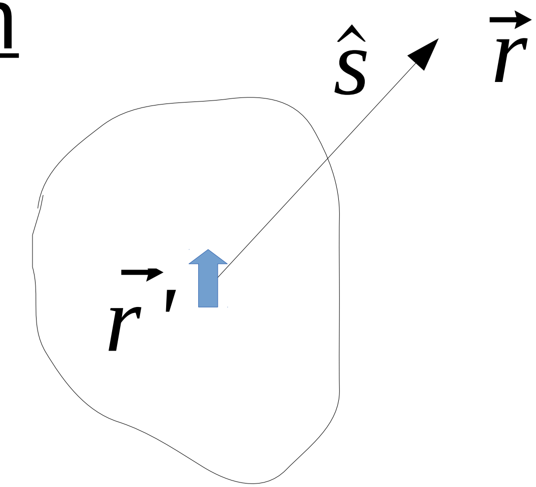
Substance consisting of neutral atoms will have huge number of dipoles pointing along $\vec{E}$.

$$\text{Polarization}: \vec{P} = \frac{\vec{p}}{V}$$

What will be the field produced by the substance?

Electrically Polarized Medium

$$d\vec{p} = \vec{P}\,dV$$



$$\vec{s} = \vec{r} - \vec{r}\,'$$

$$V(r) = \frac{1}{4\pi\epsilon_0}\int_V \frac{\vec{P}(\vec{r}\,')\cdot\hat{s}}{s^2}\,d^3\vec{r}\,'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \vec{P}\cdot\vec{\nabla}'\left(\frac{1}{s}\right)d^3\vec{r}\,'$$

$$= \frac{1}{4\pi\epsilon_0}\int_V \left[\vec{\nabla}\cdot\left(\frac{1}{s}\vec{P}\right) - \frac{1}{s}\left(\vec{\nabla}'\cdot\vec{P}\right)\right]d^3\vec{r}\,'$$

$$= \frac{1}{4\pi\epsilon_0}\left[\oint_S \left(\frac{1}{s}\right)\vec{P}\cdot\hat{n}\,dS' - \int_V \frac{1}{s}\left(\vec{\nabla}'\cdot\vec{P}\right)d^3\vec{r}\,'\right]$$

Electrically Polarized Medium

$$\text{Surface charge density : } \sigma_b = \vec{P} \cdot \hat{n}$$

$$\text{Volume charge density : } \rho_b = -\nabla \cdot \vec{P}$$

$$V = \frac{1}{4\pi\epsilon_0}\left[\oint_S \frac{1}{s}\sigma_b(\vec{r}\,')\,dS' + \int_V \frac{1}{s}\rho_b(\vec{r}\,')\,d^3\vec{r}\,'\right]$$

The knowledge of *bound* surface and volume charge distributions will determine the electric potential.

For uniform polarization, only surface charge will be present.

Dielectric medium

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \quad \text{(linear dielectrics)}$$

χ_e : electric susceptibility

Displacement Vector : $\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon \vec{E}$

Permitivity : $\epsilon = \epsilon_0 (1 + \chi_e)$

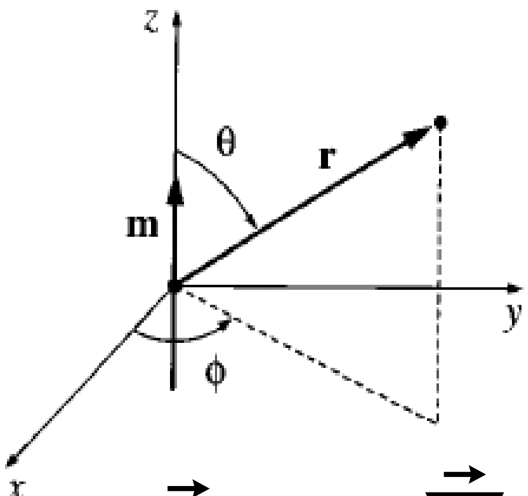
Dielectric constant : $K = \frac{\epsilon}{\epsilon_0} = 1 + \chi_e$

Magnetic Vector Potential

What is the dipole potential for producing magnetic field (analogous to field produced by electric dipole)?

$$V_{dip} = \frac{\vec{p}\cdot\hat{r}}{4\pi\epsilon_0 r^2}$$

$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{r}}{r^2}$$



$$\vec{A}_{dip} = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\varphi}$$

$$\vec{B}_{dip} = \vec{\nabla}\times\vec{A}_{dip} = \mu_0\frac{m}{4\pi r^3}\left[2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right]$$

$$\vec{E}_{dip} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\,\hat{r}+\sin\theta\,\hat{\theta}\right)$$

Magnetic medium

In presence of magnetic field, magnetic medium will have many tiny magnetic dipoles.

$$\text{Magnetization}: \vec{M} = \frac{\vec{m}}{V} \text{ (analogous to } \vec{P}\text{)}$$

$$\text{Volume bound current density}: J_b = \nabla \times \vec{M}$$

$$\text{Surface bound current density}: K_b = \vec{M} \times \hat{n}$$

Magnetic Medium

$$\vec{B} = \mu_0 \vec{H} \text{ (in free space)}$$

$\vec{B}$ induces magnetization in magnetic medium.

$$\vec{M} = \chi_m \vec{H} \quad \text{(linear medium)}$$

χ_m : magnetic susceptibility

Magnetic Field : $\vec{B} = \mu_0 \left(\vec{H} + \vec{M} \right) = \mu \vec{H}$

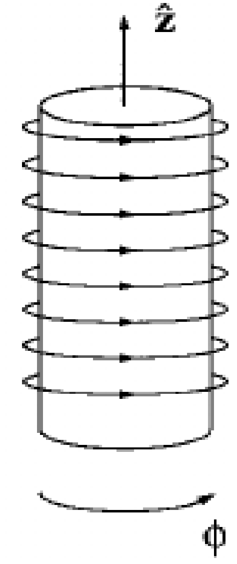
Permeability : $\mu = \mu_0 \left(1 + \chi_m \right)$

Solenoid filled with magnetic material:

$$\vec{H} = N I \hat{k}$$

$$\vec{B} = \mu_0 (1+\chi_m) \vec{H}$$

$$\vec{M} = \chi_m \vec{H}$$



$$\vec{K}_b = \vec{M} \times \hat{n} = \chi_m (\vec{H} \times \vec{n}) = \chi_m N I \hat{\varphi}$$

$$\vec{J}_b = \nabla \times \vec{M} = \nabla \times (\chi_m \vec{H}) = \chi_m \vec{J}_f$$

Unless free current flows through the material, bound current will be at the surface only.