1. Determine the limits as $(x, y) \to (0, 0)$ of the following functons, if they exist:

(a) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(b) $f(x,y) = \begin{cases} \log\left(\dfrac{y}{x}\right), & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{|x|}{y^2}\exp\left(-\dfrac{|x|}{y^2}\right), & y \neq 0; \\ 0, & y = 0. \end{cases}$

(d) $f(x,y) = \begin{cases} \dfrac{x^2+y^2}{\tan(xy)}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(e) $f(x,y) = \begin{cases} \dfrac{x^2y}{x^4+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(f) $f(x,y) = \begin{cases} \log\left(\dfrac{\sqrt{x^2+y^2}+x}{\sqrt{x^2+y^2}-x}\right), & y \neq 0; \\ 0, & y = 0. \end{cases}$

(g) $f(x,y) = \begin{cases} \sin\left(\dfrac{x}{y}\right) + \sin\left(\dfrac{y}{x}\right), & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(h) $f(x,y) = \cos^3(\sqrt{x^2+y^2})$.

(i) $f(x,y) = \begin{cases} \dfrac{\sin(x^2y+xy^2)}{xy}, & xy \neq 0; \\ 0, & xy = 0. \end{cases}$

(j) $f(x,y) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(k) $f(x,y,z) = \begin{cases} \dfrac{xy^2z^2}{x^4+y^4+z^8}, & (x,y,z) \neq (0,0,0); \\ 0, & (x,y,z) = (0,0,0). \end{cases}$

2. Using $\epsilon - \delta$ method, prove the followings:

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{y^2+x^2} = 0,$

(b) $\displaystyle\lim_{(x,y)\to(-1,-1)} (xy - 2x^2) = -1,$

(c) $\displaystyle\lim_{(x,y)\to(1,0)} \frac{(x-1)^2\ln x}{y^2+(x-1)^2} = 0,$

(d) $\displaystyle\lim_{(x,y)\to(-2,2)} \frac{x^2-y^2}{y+x} = -4,$

(e) $\displaystyle\lim_{(x,y)\to(0,0)} xy\frac{x^2-y^2}{y^2+x^2} = 0,$

(f) $\displaystyle\lim_{(x,y)\to(0,0)} x\sin x\cos y = 0,$

(g) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2}{\sqrt{y^2+x^2}} = 0,$

(h) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2y^2}{y^2+x^2} = 0,$

(i) $\displaystyle\lim_{(x,y)\to(1,1)} (x^2+y^2-1) = 1,$

(j) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^4y-3x^2y^3+y^5}{(x^2+y^2)^2} = 0,$

(k) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0.$

(l) $\displaystyle\lim_{(x,y)\to(0,0)} \left[y\sin\left(\frac{x}{y}\right) + x\sin\left(\frac{y}{x}\right)\right] = 0.$

3. Using $\epsilon - \delta$ method, show that the following functions are continuous:

(a) $f(x,y) = \begin{cases} xy, & (x,y) \neq (2,3); \\ 6, & (x,y) = (2,3). \end{cases}$

(b) $f(x,y) = \begin{cases} \dfrac{5x^2y^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(c) $f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

(d) $f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & (x,y) \neq (0,0); \\ 0, & (x,y) = (0,0). \end{cases}$

4. Discuss the continuity of the following functions at $(0, 0)$:

(a) $f(x,y)=\begin{cases}\dfrac{1}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(b) $f(x,y)=\begin{cases}\dfrac{x^3y^3}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(c) $f(x,y)=\begin{cases}\dfrac{|xy|}{\sqrt{x^2+y^2}}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(d) $f(x,y)=\begin{cases}\dfrac{|xy|}{xy}, & xy\neq 0;\\ 0, & xy=0.\end{cases}$

(e) $f(x,y)=\begin{cases}\dfrac{e^{xy}}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(f) $f(x,y)=\begin{cases}\dfrac{3x^2y-y^3}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(g) $f(x,y)=\begin{cases}\dfrac{x^3}{x^2+y^2}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$

(h) $f(x,y)=\begin{cases}\dfrac{\sin xy}{xy}, & xy\neq 0;\\ 1, & xy=0.\end{cases}$

(i) $f(x,y)=\begin{cases}x\sin\dfrac{1}{y}+y\sin\dfrac{1}{x}, & xy\neq 0;\\ 0, & xy=0.\end{cases}$

5. For what values of $n$, the following function $f$ is continuous at $(0,0)$:

$$f(x,y)=\begin{cases}\dfrac{2xy}{(x^2+y^2)^n}, & (x,y)\neq(0,0);\\ 0, & (x,y)=(0,0).\end{cases}$$

6. Find the values of $c$ for which the following functions $f$ are continuous at $(0,0)$:

(a) $f(x,y)=\begin{cases}\dfrac{x^4-y^4}{x^2+y^2}, & (x,y)\neq(0,0);\\ c, & (x,y)=(0,0).\end{cases}$

(b) $f(x,y)=\begin{cases}x^2\log(x^2+y^2), & (x,y)\neq(0,0);\\ c, & (x,y)=(0,0).\end{cases}$

(c) $f(x,y)=\begin{cases}\dfrac{\sin(x^2+y^2)}{x^2+y^2}, & (x,y)\neq(0,0);\\ c, & (x,y)=(0,0).\end{cases}$

(d) $f(x,y)=\begin{cases}\dfrac{x^3+y^3}{x^2+y^2}, & (x,y)\neq(0,0);\\ c, & (x,y)=(0,0).\end{cases}$

(e) $f(x,y)=\begin{cases}\dfrac{(x-1)^2\log x}{(x-1)^2+y^2}, & (x,y)\neq(1,0);\\ c, & (x,y)=(1,0).\end{cases}$

(f) $f(x,y)=\begin{cases}\dfrac{e^{-(x^2+y^2)}-1}{x^2+y^2}, & (x,y)\neq(0,0);\\ c, & (x,y)=(0,0).\end{cases}$

(g) $f(x,y)=\begin{cases}\exp\left(-\dfrac{1}{|x-y|}\right), & x\neq y;\\ c, & x=y.\end{cases}$

7. Do the following functions have any point of discontinuities? Explain.

(a) $f(x,y)=\dfrac{x-y}{1+x+y}$,

(b) $f(x,y)=\dfrac{x-y}{1+x^2+y^2}$.

8. Find the point of discontinuities of the following functions.

(a) $f(x,y)=\dfrac{1}{\sin^2\pi x+\sin^2\pi y}$,

(b) $f(x,y)=\dfrac{1}{\sin\pi x}+\dfrac{1}{\sin\pi y}$.