Problem Set - 12

AUTUMN 2017

MATHEMATICS-I (MA10001)

1. Obtain the Laurent's series which represent the function $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in the regions $(a)\ |z|<2\ (b)\ 2<|z|<3\ (c)\ |z|>3$.

2. Find two Laurent's series expressions in power of z of the function $f(z) = \dfrac{1}{z(1+z^2)}$.

3. Find Laurent's series expansion of $f(z) = \dfrac{1}{(z-1)(z-4)}$ valid in $1<|z-2|<2$.

4. Find the principal part of the following Laurent's series

   (a) $f(z) = \dfrac{1}{5}\left[\dfrac{1}{z+2} - \dfrac{z-2}{z^2+1}\right]\ (|z|>2)$

   (b) $f(z) = \dfrac{1}{z(1+z^2)}\ (0<|z|<1 \quad \text{and} \quad |z|>1)$

   (c) $f(z) = \dfrac{1}{(z^2+1)(z^2+2)}\ (1<|z|<\sqrt{2} \quad \text{and} \quad |z|>\sqrt{2})$

   (d) $f(z) = \dfrac{\sin z}{z^4}\ (|z|>0)$

   (e) $f(z) = (z-3)\sin\dfrac{1}{z}\ (|z|>0)$.

5. Find the principal part of the Laurent expansion of the following functions at the given point:

   (a) $\dfrac{e^z}{(z-2)^2},\ z=2$

   (b) $\dfrac{\sin z}{z^3(z+1)^2},\ z=0$

6. Find the singularity of the following functions and classify them:

   i. $\dfrac{\cot z}{(z-a)^2}$ at $z=0,\ z=\infty$

   ii. $\sin\left(\dfrac{1}{1-z}\right)$ at $z=1$

   iii. $\sin z - \cos z$ at $z=\infty$

   iv. $\operatorname{cosec}\left(\dfrac{1}{z}\right)$ at $z=0$

   v. $\tan\left(\dfrac{1}{z}\right)$ at $z=0$

   vi. e^z at $z=\infty$

   vii. $\dfrac{(z-2)}{z^2}\sin\left(\dfrac{1}{z-1}\right)$ at $z=1$

viii. $\cot z$ at $z = \infty$

ix. $\sec\left(\frac{1}{z}\right)$ at $z = 0$

x. $(z-3)\sin\left(\frac{1}{z+2}\right)$ at $z = -2$

xi. $\frac{e^z}{z^2+4}$

xii. $\frac{1-e^z}{1+e^z}$ at $z = \infty$

xiii. $\frac{1}{\sin z - \cos z}$ at $z = \frac{\pi}{4}$

xiv. $z \operatorname{cosec} z$ at $z = \infty$

7. Find the poles of the following functions and determine their order:

   (a) $f(z) = \frac{1}{z^2(z-1)^3}$

   (b) $f(z) = \left(\frac{z+1}{z^2+1}\right)^2$

8. Find each pole and its order and calculate residue of each of the pole of $f(z) = \frac{z^2}{(z-1)^2(z+2)}$.

9. Find the residue of the functions:

   (a) $\frac{1}{(z^2+1)^3}$ at $z = i$

   (b) $\frac{z^2}{z^2+a^2}$ at $z = ia$

   (c) $\frac{\cot \pi z}{(z-a)^2}$ at $z = a$

   (d) $z\cos\frac{1}{z}$ at $z = 0$

10. If $0 < |z| < 1$, then express $f(z) = \frac{z}{(z-1)(z-3)}$ in a series of positive and negative powers of $(z-1)$.

11. Expand $e^{\frac{z}{z-2}}$ in a Laurent series about $z = 2$.

12. Using Cauchy Residue theorem evaluate the following integrals

   (a) $\int_C \frac{\cos 4z}{z^3} dz$, where C is the square with vertices at $2, -2, 2i, -2i$.

   (b) $\int_L \tan z dz$, where $L : |z| = 2$.

   (c) $\int_L \frac{1}{z^2(z-2)(z-3)} dz$, where $L : |z-i| = 3$.

   (d) $\int_C \frac{\cos \pi z}{z(z^2+1)} dz$, where C be the circle $|z| = 2$.