1/10/20 Adv. Graph Theory
Take - Home - Test

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1. Since, the graph has 4 vertices and is connected, it has at least 3 edges.

3 edges

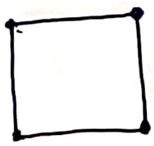
This has to be a tree. Only 2 possible trees, "path" & "star".

i)

ii)

4 edges

iii)



iv)

This is obtained by removing 2 edges from K_4. They can be distinguished on whether the removed edges share a common point or not. This exhausts all possibilities

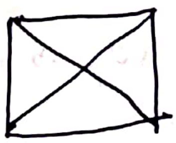
5 edges

v)

Removing 1 edge from K_4. Indistinguishable whatever edge we remove.

6 edges

vi)



Unique K_4.

$\boxed{\text{Total:- 6}}$

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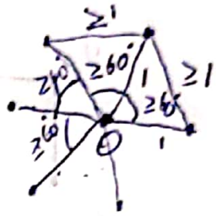
2. • Choose the n points in the plane as vertices

• There will be an edge between 2 vertices if the distance between the 2 corresponding points is 1 cm.

~~[illegible]~~

• The maximum degree for each vertex is 6.

Proof:- Let the degree of O be k.



From the figure we can see that

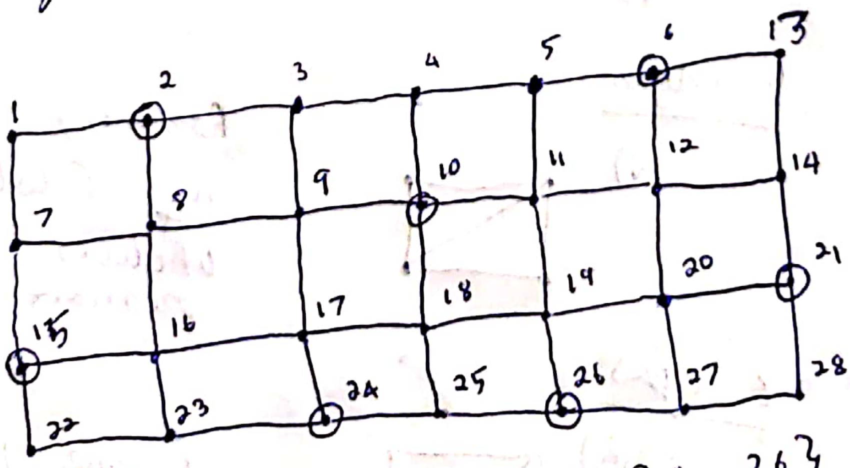
$k \cdot 60^\circ \leq 360^\circ$ {A full circle}

$K \leq 6$.

• Therefore there are $\frac{6 \times n}{2} = 3n$ edges at most.

∴ There are at most $3n$ pairs of points whose distance is 1 cm.

3.



$S = \{2, 6, 10, 15, 21, 24, 26\}$

$|S| = 7$

This is the minimum-size dominating set

Assume that there exists $|S| = 6$, where S is dominating set.

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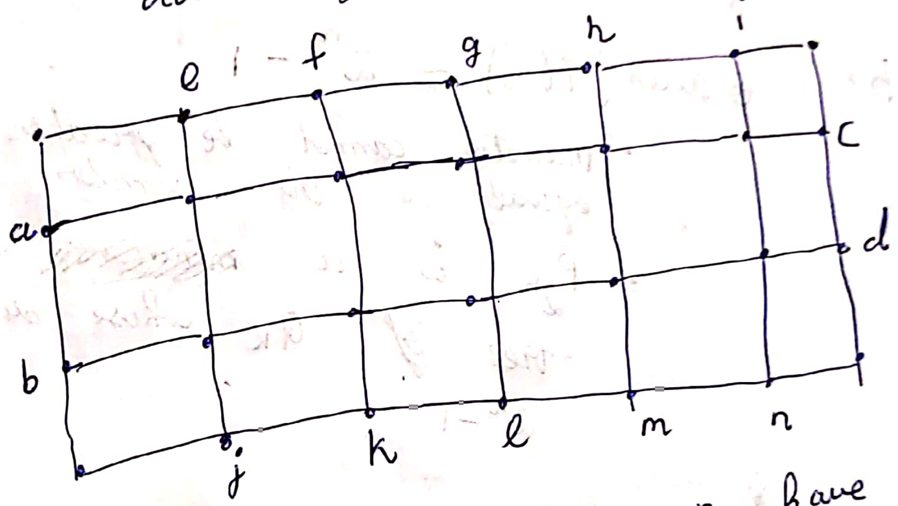
- To cover the 4 corners, there should be 4 vertices in S whose $deg(v) \le 3$.

- $4\cdot(3+1) + 2\cdot(4+1) = 26$ is the total possible vertices that can be covered (as max degree is 4).

- $26 \le 28$, hence S cannot be dominating set.



- Vertices marked a, b ..., n have odd-degree and has to be matched, for a tour that comes back to the starts point

- Any vertex ~~[illegible]~~ from {e, f, g, h, i, j ... m}, has a nearest neighbour (marked) that is atleast distance 3 {as it has to cover 1 horizontal edge or atleast 3 vertical edge to reach nearest neighbour}

- |||ly for vertices {a, b, c, d} nearest marked neighbour is atleast distance 1 {it has to travel atleast 1 edge}

- So matching a-b, c-d, e-j, f-k, g-l, h-m, i-n is minimal (by weight)

Cost of Tour = Cost of Trail + Extra weight

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$= 3 \cdot |H| + 1 \cdot |V| +$ weight of matching

$= 3 \cdot 24 + 1 \cdot 21 + 2 \cdot 1 + 5 \cdot 3$

$= 72 + 21 + 2 + 15$

$= 93 + 17$

$\boxed{\text{Cost of tour} = 110}$

~~5. i) diam(T(K)) ≤ K − 1~~

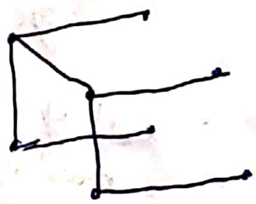
5. i) $\oplus$ $diam(T(K)) \leq 2^K - 1$

- Diameter cannot be greater than or equal to the number of vertices
- P_{2^K} is a ~~minimum~~ spanning tree of Q_K whose diameter $2^k - 1$

ii) $\oplus$ $diam(T(K)) \geq 2k - 1$

- Eccentricity of all vertices in Q_K is K. If a spanning tree of diameter less than $2K-1$ exists, then the eccentricity of the center is less than K, which is not possible. Hence, there is no spanning tree whose diameter is less than $2k-1$.

- Eg:-



Spanning tree for Q_3 with diameter $= 5$

$\Rightarrow \boxed{2K - 1 \leq diam(T(K)) \leq 2^K - 1}$

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ii) ⊕ $\text{diam}(L(Q_k)) \le k$

~~If~~

- Assume, there exists 2 edges in Q_k whose corresponding vertices in $L(Q_k)$ are at a distance $d > k$. This ~~means that~~ means that, one endpoint of one edge and ~~another~~ one endpoint of the other edge are separated by distance d in Q_k. But all ~~vert~~ pairs of vertices in Q_k have distance at most k. Hence, our assumption that the distance between 2 edges in Q_k is greater than k is wrong.

$\therefore$ $\text{diam}(L(Q_k)) \le k$

⊕ $\text{diam}(L(Q_k)) \ge k$

- Assume, $\text{diam}(L(Q_k))$ is $l < k$. This means maximum distance between any 2 vertices in Q_k is l. But by a similar argument. But this is not true in Q_k. Hence, our assumption that $\text{diam}(L(Q_k)) < k$ is wrong

$\therefore$ $\text{diam}(L(Q_k)) \ge k$

Hence, $\boxed{\text{diam}(L(Q_k)) = k}$

6.
- Lets assume each person as a vertex. And we denote a possible neighbour by an edge. Hence, this will be a complete graph.
- The number of different possible seating arrangements is the number of edge disjoint Hamiltonian cycles.

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• This is because, each edge denotes different neighbour, and since the hamiltonian cycles are edge disjoint, there is no common neighbour.

⇒ Find the number of edge disjoint Hamiltonian cycles in K_{29}.

• <u>Claim 1</u> :- There are $\frac{n}{2}$ disjoint Hamiltonian paths for even n in K_n.

Proof :- let the vertices be $v_0, v_1 \dots v_{n-1}$

• For path $i = 1, \dots \frac{n}{2}$, construct

$v_{0+i}\ v_{n-1-i}\ v_{1+i}\ v_{n-2-i} \dots v_{\frac{n}{2}-1+i}\ v_{\frac{n}{2}+i}$

• Its easy to see that the above construction leads to $\frac{n}{2}$ disjoint Hamiltonian paths when the subscripts are taken modulo n

• <u>Claim 2</u> :- There are $\frac{n-1}{2}$ disjoint Hamiltonian cycles of odd n in K_n.

<u>Proof</u>:- • Consider the graph K_{n-1} & a new vertex v_0. Construct $\frac{n-1}{2}$ disjoint Ham. paths by Claim 1. Note that they always have different start & end points for different paths.

Now join the start & end vertex through the vertex V_0.

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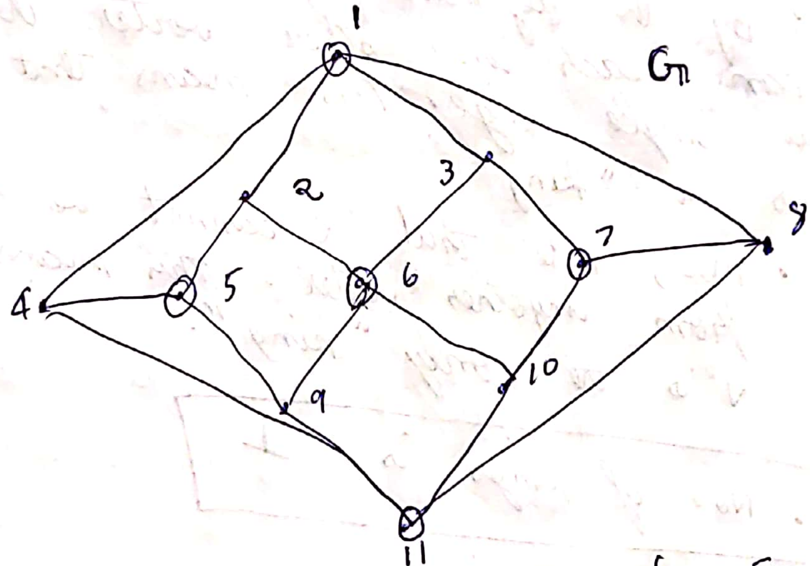
Thus we have $\frac{n-1}{2}$ disjoint Hamiltonian cycles for odd n in K_n.

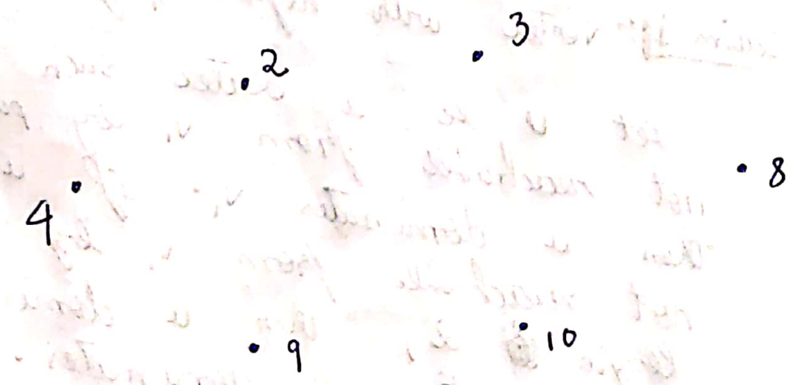
$\Rightarrow \frac{29-1}{2} = 14$ disjoint Hamiltonian cycles

Thus, the seating arrangement carries on for 14 days

7.



Remove the vertices $S=\{1, 5, 6, 7, 11\}$ we get the graph $G-S$,



which has 6 components. But $|S|=5$

$\Rightarrow w(G-S) > |S| \Rightarrow$ $\boxed{\text{No Hamiltonian cycle}}$

$w(G-S)$ is no. of connected components in $G-S$

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9. Consider a tournament of n vertices T_n.

Case - 1

- There is a vertex with 0-in degree.

Let the vertex with 0-indegree be v.

- By the definition of tournament, v can reach any other vertex through a single edge. This means that v is a "king"
- Also, note that v cannot be reached from anyother vertex. This means that v is the only "king".

$\boxed{\therefore \text{No. of kings is } 1}$

Case - 2

- There is no vertex with 0-in degree

Claim I :- vertex with highest out-degree is king (v_1).

Let u be a vertex such that u is not reachable from v_1 by path of 1, then u dominates v_1. If u is not reachable from v_1 by path of length 2, then u dominates all those vertices v_1 dominates. Thus, u has a outdegree greater than v_1, which is a contradiction.

Thus, v_1 can reach any other vertex in length at max 2.

$\Rightarrow \boxed{v_1 \text{ is king}}$

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Since v_1 has non-zero indegree, among the vertices that dominates v_1 find the one with highest out-degree and let's call this v_2.

Claim 2:- v_2 is a king

Let ~~Assume that~~ u be a vertex that is not reachable from v_2 by a path of length 1, that is u dominates v_2 (such a vertex exists because v_2 has non-zero indegree). Assume that u is not reachable from v_2 by a path of length 2, then u also dominates v_1. Since there is no 2 length path from v_2 to u, all vertices dominated by v_2 is dominated by u. This contraddicts our assumption that v_2 has highest outdegree among those that dominate v_1.

Thus, $\boxed{v_2 \text{ is a king}}$

~~claim~~ Illy, find v_3 as the highest outdegree among those that dominate v_2.

Claim 3:- $v_3 \neq v_1$, v_3 is a king

v_3 cannot be v_1, as v_3 dominates v_2 and v_2 dominates v_1. Illy, by the above argument v_3 is king.

Thus, T_n has atleast 3 kings v_1, v_2, v_3

$\boxed{\therefore T_n \text{ cannot have exactly 2 kings}}$

10. Assume that K_5 is graceful. Then it is possible to label the vertices from 0, 1, ... 10 such that the edges get a distinct number $|f_a - f_b|$ when f_a & f_b are the values assigned to their endpoints

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- Since, 10 must be present in the edge, we ~~choose~~ should choose 0 & 10 for the labels.

- Since, 9 must be present in the edge, we should choose 1 or 9 for the label. We will choose 1 (we can always replace the numbers as $10-x$, and the answer won't change).

- Now, to get 8 in the edge label, the choices we have are
  i) choose 9. But this means $|10-9|=1$ & $|1-0|=1$, is repeated
  ii) choose 2. Again we have repeated 1 ($|1-0|$, $|2-1|$).
  iii) Choose 8. Possible

- Note that K_5 is symmetric w.r.t all vertices, so the order of numbers doesn't matter.

- Now, the labels in the edges are 10, 9, 8, 7, 2, 1. Whatever number we add for the last node will get repeated edge number

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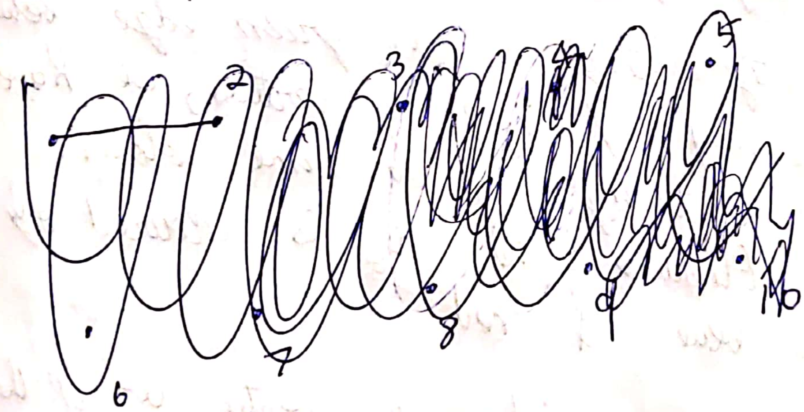
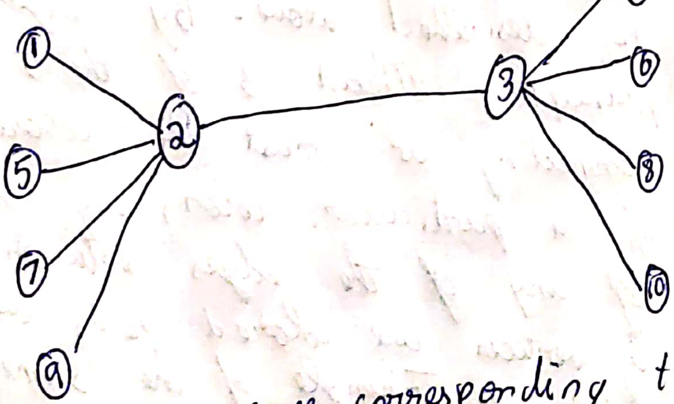
Node	Repeated number
10	Already present
9	1 (from 10 & 9)
8	Already present
7	7 (from 0 & 7)
6	2 (from 8 & 6)
5	5 (from 0 & 5, 10 & 5)
4	4 (from 8 & 4, 0 & 4)
3	2 (from & 3)
2	2 (from 0 & 2)
1	Already prese
0	Already present

$\boxed{\text{Thus, } K_5 \text{ is not graceful}}$

11. Prufer code (P) :- 2, 3, 2, 3, 2, 3, 2 3

$|P| = 8 \Rightarrow n = 10$ (Number of vertices)





Tree corresponding to the Prufer code.

12. Choose an arbitrary vertex v of K_{66}

~~Proof~~

- v has 65 edges connecting to it. By pigeonhole principle, at least 17 of these edges must be of a particular colour, lets say yellow.

- If any of the edges between these vertices are yellow, we have a triangle. Otherwise, the K_{17} is colored by 3 colors only.

- Now consider a vertex u of the K_{17} described above. By pigeonhole principle, atleast 6 of other 16 edges must have a same colour, lets say green.

- If there is a green edge between any of these 6 vertices, we have a triangle of the same colour. Otherwise, the K_6 is coloured only blue & red.

- Now consider a vertex w of the K_6 described above. By pigeonhole principle, atleast 3 of the 5 edges connecting w must be colored of a particular color, say red. If any of the edges between the 3 vertices are colored red, we have a triangle of the same colour.

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- Otherwise K_3 is completely colored by blue, which means we have a monochromatic triangle.

Thus, K_{66} colored by 4 colors has a monochromatic triangle.

8. We can try to recursively build a tree that can differentiate between dissection that are separable by a sequence of cuts and one that is not.

Algorithm:-

- First initalise the input as a root node.
- Check if there are no more cuts to be made. If there are none, return the node.
- See, if there is a horizontal or vertical cut, that goes through entire length or width of the input. If yes, separate the input into 2 pieces along the cut, and call the algorithm recursively on the 2 pieces. Attach the 2 outputs (which is a tree) as the left and right child of the current node

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- If there is no such cut, just return the node.

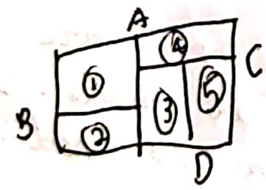
Now, we have built a tree corresponding to the geometry of dissection. To differentiate between the 2 types of dissection, just count the number of leaves.

i) If there exists a sequence of cuts that separates the into all the small rectangles, the number of leaves will be the same as the total number of rectangles we have to separate out

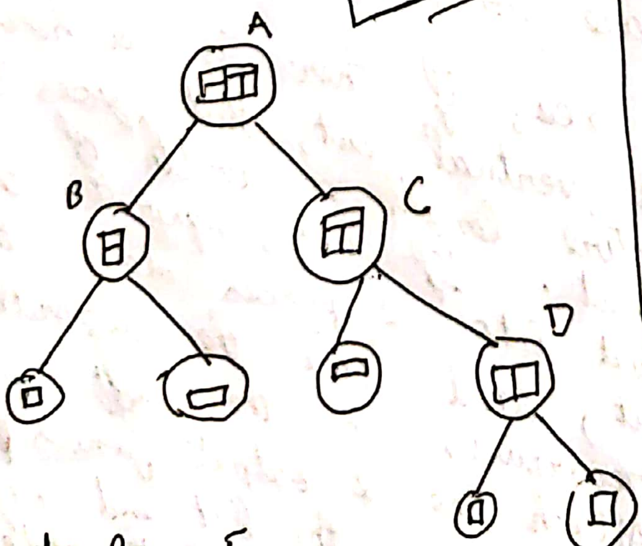
ii) If there doesn't exist a sequence, the number of leaves will be less than the rectangles.

Example

i)



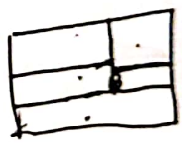
Possible

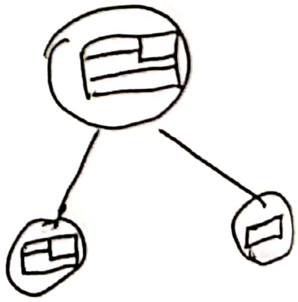


Rectangles = 5
Leaves = 5

Example

ii)





Rectangles = 4
Leaves = 2

Not possible