Problem from STAT5020

1. Suppose we have a collection of binary responses y_i, $i = 1, \cdots, n$, and associated k-dimensional predictor variables x_i. Define the latent variable y_i^* as

$$y_i^* = \mathrm{x}_i^T \beta + \epsilon_i, \quad i = 1, \cdots, n,$$

where the ϵ_i are independent mean-zero errors having cumulative distribution function F, and β is a k-dimensional regression parameter. Consider the model

$$Y_i = \begin{cases} 0, & \text{if } Y_i^* \geq 0 \quad \leq 0 \\ 1, & \text{if } Y_i^* < 0 \quad > 0 \end{cases}$$

(a) Show that if F is the standard normal distribution, this model is equivalent to the usual probit model for $p_i = P(Y_i = 1)$. $y_i^* \sim (X_i^T\beta, I)$ $f(Y|\beta)$

(b) Under a $N(\mu, \Sigma)$ prior for β, find the full conditional distributions for β and the Y_i^*, $i = 1, \cdots, n$.

(c) How would you modify your computational approach if, instead of the probit model, we wished to fit the logit (logistic regression) model?

$$\log\left(\frac{p_i}{1-p_i}\right) = X_i^T\beta + \varepsilon_i.$$

2. In Bayesian model comparison:

(a) Use a concrete example to illustrate how to implement the path sampling procedure for computing Bayes factor in the context of multilevel structural equation models (SEMs).

(b) Discuss other Bayesian model comparison statistics in the comparison of multilevel SEMs.

$$u_{gi} = v_g + \Lambda_{1g} \omega_{1gi} + \varepsilon_{1gi}$$

$$v_g = \mu + \Lambda_2 \omega_{2g} + \varepsilon_{2g}.$$

$$\eta_{1gi} = \cancel{\Pi_{1g}\Lambda_{1g}}\ \xi_{1gi1} + \xi_{1gi2} + \delta_{1gi}$$

STAT 5020: Topics in Multivariate Analysis

Final Examination

1. (60%) In Bayesian estimation:

   (a) (15%) Establish a SEM with mixed continuous, binary, count, and ordinal data.

   (b) (15%) Discuss model identifiability issues.

   (c) (15%) Derive the associated posterior distributions. 第五節

   (d) (15%) Describe the implementation of the posterior sampling.

2. (40%) In Bayesian model comparison:

   (a) (20%) Illustrate how to implement the path sampling procedure in computing Bayes factor in the context of multiple group SEMs.

   (b) (20%) Discuss other Bayesian model comparison statistics in the analysis of mul-grous SEMs.

2018/4 Final

STAT 5020: Topics in Multivariate Analysis

Final Examination

The Department of Medicine and Threapeatics, Community and Family Medicine, and Pharmacy at the Chinese University of Hong Kong conducted a compliance study to investigate patient nonadherence to medication. A total number of 837 ethnic Chinese patients diagnosed as suffering from hypertension were randomly selected from hospitals and clinics in Hong Kong to serve as subjects for the study. The following observed variables are associated with patients' ① nonadherence to medication, ② their knowledge of medication, and ③ their health conditions, respectively. Frequencies of (Yes '1'/No '0') are in parentheses. Dichotomous variables.

ξ_1 ①

y_1: Did you have any surplus in the previous prescribed drugs? (175/662)

y_2: Did you stop/reduce/increase the dosage? (69/768)

y_3: Did you forget to take medications? (391/446)

ξ_2 ②.

y_4: Do you feel you have hypertension? (363/474)

y_5: Do you know the reasons for taking drugs? (650/187)

y_6: Do you know the reasons for taking drugs for a long term? (605/232)

η. ③.

y_7: In the past two weeks, did you have emotional problems such as upset, hot temper, etc? (387/450)

y_8: In the past two weeks, did your health cause any difficulties in daily activities? (181/656)

y_9: In the past two weeks, did your health cause any difficulties in social activities? (177/660)

(a) Establish an SEM to analyze this dataset. Explain the model and the purpose of your study.

(b) Specify the conjugate prior distributions for the unknown parameters in the proposed model, and show the conjugacy of the specified priors.

(c) Derive the posterior distributions of the unknown parameters, and describe the posterior inference via MCMC sampling in the context of the proposed model.

(d) Explain why a model comparison is useful for model building, and illustrate how to implement the path sampling procedure in computing Bayes factor for the model comparison of the proposed model.

(e) Discuss advantages and disadvantages of the Bayesian method in the analysis of SEMs.

1. (a) $\mu \sim N(\mu_0, \Sigma_0)$. $f(\mu|y) \propto f(y|\mu) f(\mu) \propto \exp\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu) + (\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)\}$

$\propto \exp\{-\frac{1}{2}[\mu'(n\Sigma^{-1}+\Sigma_0^{-1})\mu - 2\mu'(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)]\}$

$\mu|y \sim N(\mu^* = \Sigma^*(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0), \Sigma^* = (n\Sigma^{-1}+\Sigma_0)^{-1})$

$\propto \exp\{-\frac{1}{2}[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma^{-1}\mu_0)]'(n\Sigma^{-1}+\Sigma_0^{-1})[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma^{-1}\mu_0)]\}$

(b) $f(\mu)=1$. $\mu|y \sim N(\bar{y}, \frac{1}{n}\Sigma)$

Problem from STAT5020

(c) from (a), $\mu^{(1)}|\mu^{(2)}, y \sim N(\mu_1^* + \Sigma_{12}^*\Sigma_{22}^{*-1}(\mu^{(2)}-\mu_2^*), \Sigma_{11}^* - \Sigma_{12}^*\Sigma_{22}^{*-1}\Sigma_{21}^*)$

1. Assume that y is a $p \times 1$ random vector, with the multivariate normal distribution

d) $f(y|\mu,\Sigma) = \exp\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\}|\Sigma|^{-\frac{n}{2}} = \exp\{-\frac{1}{2}tr(\Sigma^{-1}[n(\bar{x}-\mu)(\bar{x}-\mu)'+S])\}|\Sigma|^{-\frac{n}{2}}$

$$y|\mu, \Sigma \sim N(\mu, \Sigma),$$

$S=\sum(x_i-\bar{x})(x_i-\bar{x})'$

$f(\mu|\Sigma) \propto \exp\{-\frac{\kappa_0}{2}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\}|\Sigma|^{-\frac{1}{2}}$

where μ is a $p \times 1$ vector and Σ is a $p \times p$ covariance matrix, which is symmetric and positive definite.

$= \exp\{-\frac{\kappa_0}{2}tr\,\Sigma^{-1}(\mu-\mu_0)(\mu-\mu_0)'\}$

(a) When Σ is known, specify a conjugate prior distribution for μ and derive the corresponding ~~posterior distribution.~~

$f(\Sigma) \propto |\Sigma|^{-\frac{1}{2}(\rho_0+p+1)}\exp\{-\frac{1}{2}tr\,\Sigma^{-1}\Sigma_0\}$

(b) When Σ is known, specify a noninformative prior distribution for μ and derive the corresponding posterior distribution.

$\mu|\Sigma, y \sim N(\frac{n\bar{y}+\kappa_0\mu_0}{n+\kappa_0}, \frac{\Sigma}{n+\kappa_0})$

(c) If μ is partitioned into subvectors $\mu^{(1)}$ and $\mu^{(2)}$, derive the posterior conditional distribution $p(\mu^{(1)}|\mu^{(2)}, y)$ with a conjugate prior and known Σ.

$\Sigma|y \sim IW((\Sigma_0 + S + \frac{n\kappa_0}{n+\kappa_0}(\bar{x}-\mu_0)(\bar{x}-\mu_0)')^{-1}, n+\rho_0)$

(d) When Σ is unknown, a joint prior distribution for (μ, Σ) is assigned as

$f(y|\mu,\Sigma)$

$$\Sigma \sim IW(\rho_0, \Sigma_0^{-1})$$
$$\mu|\Sigma \sim N(\mu_0, \Sigma/\kappa_0),$$

$\mu | \Sigma \sim N(m_0, \Sigma)$

$\Lambda_{k_0} = m_0$

$H_0 y_k = I$

where $IW(\cdot,\cdot)$ denotes the inverse Wishart distribution, and ρ_0, Σ_0, μ_0, and κ_0 are hyperparameters. Derive the joint posterior distribution of (μ, Σ).

$\varphi_{q|k} = \sigma^2 V$

$f(\mu,\Sigma|y) \propto f(y|\mu,\Sigma) f(\mu|\Sigma) f(\Sigma)$

2. A generalized linear mixed effect model (GLMM) is defined as

$$g(\mu_{it}) = \mathrm{x}_{it}^T\beta + \mathrm{z}_{it}^T\mathrm{u}_i,$$

where $g(\cdot)$ is a link function, $\mu_{it} = E(y_{it}|\mathrm{u}_i)$, y_{it} is the observation for subject i at time t, x_{it} and z_{it} are vectors of explanatory variables, u_i is a $q \times 1$ vector of subject-specific random effects, and $\mathrm{u}_i \sim N(0, \Sigma)$.

(a) Explain why the GLMM is commonly used in the analysis of longitudinal data.

(b) If $g(\cdot)$ is a log link, $q = 1$, $z_{it} = 1$, and $u_i \sim N(0, \sigma^2)$, show that

$$cov(y_{it}, y_{is}) = \exp(\mathrm{x}_{it}^T\beta + \mathrm{x}_{is}^T\beta)\{\exp(\sigma^2)(\exp(\sigma^2) - 1)\}.$$

$\log \mu_{it} = x_{it}^T\beta + u_i$

$Cov(y_{it}, y_{is}) = E[Cov(y_{it}, y_{is}|u_i)] + Cov(E(y_{it}|u_i), E(y_{is}|u_i))$

$E[Cov(y_{it}, y_{is}|u_i)] = 0$, Given u_i, $y_{it} \perp y_{is}$

$Cov(E(y_{it}|u_i), E(y_{is}|u_i)) = Cov(\mu_{it}, \mu_{is}) = e^{x_{it}^T\beta + x_{is}^T\beta} \cdot Var(e^{u_i})$

$Var(e^{u_i}) = E[e^{2u_i}] - (E[e^{u_i}])^2 = e^{2\sigma^2} - e^{\sigma^2}$

Qualify Exam. (Topics in Multivariate Analysis), December 2013

1. Suppose we have a collection of binary responses y_i, $i = 1, \cdots, n$, and associated k-dimensional predictor variables x_i. Define the latent variable y_i^* as

$$y_i^* = \mathrm{x}_i^T \beta + \epsilon_i, \quad i = 1, \cdots, n,$$

where the ϵ_i are independent mean-zero errors having cumulative distribution function F, and β is a k-dimensional regression parameter. Consider the model

$$y_i = \begin{cases} 0, & \text{if } y_i^* \geq 0 \\ 1, & \text{if } y_i^* < 0 \end{cases}.$$

   (a) Specify a conjugate prior distribution for β.

   (b) Under (a), find the full conditional distributions for β and y_i^*, $i = 1, \cdots, n$.

(a) $\beta \sim N(\gamma_0, \Sigma_0)$ $\quad p(\beta \mid y_1, y_2, \cdots, y_n)$ $\quad p(y_i = 0 \mid \beta) = p(x_i'\beta + \epsilon_i \geq 0 \mid \beta)$

$\epsilon_i = y_i^* - x_i'\beta$ $\quad \propto p(y_1, y_2, \cdots, y_n \mid \beta)\, p(\beta)$ $\quad = p(\epsilon_i \geq -x_i'\beta \mid \beta)$

$= 1 - F(-x_i'\beta)$ $\quad p(y_i = 0 \mid \beta) = \Phi(x_i'\beta)$.

$= \Phi(x_i'\beta)$

2. In Bayesian model comparison:

   (a) Use a concrete example to illustrate how to implement the path sampling procedure for computing Bayes factor in the context of mixture structural equation models (SEMs).

   (b) Discuss other Bayesian model comparison statistics in the comparison of mixture SEMs.