STAT5005 Qualifying Exam 2019/20

Question 1: Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables. They may not have finite expectation. Let $S_n = X_1 + \cdots + X_n$. Fix a constant $0 < p < 1$. Prove that $E(|X_1|^p) < \infty$ if and only if as $n \to \infty$,

$$\frac{S_n}{n^{1/p}} \to 0 \quad a.s.$$

Question 2:

(a) If $X_1, X_2, \ldots$ are independent random variables with $\frac{1}{2} = P(X_n = a_n) = 1 - P(X_n = -a_n)$, characterize the sequences $\{a_n, n \geqslant 1\}$ for which $\sum_{i=1}^{\infty} X_i$ converges almost surely.

(b) Suppose $\{X, Y, X_n, Y_n, n \geqslant 1\}$ are random variables defined on the same probability space. Suppose further that $X_n \xrightarrow{d} X$ and $Y_n \xrightarrow{d} Y$ as $n \to \infty$. Is it true that $X_n + Y_n \xrightarrow{d} X + Y$? What if we assume that $\{X, X_1, X_2, \ldots\}$ are independent of $\{Y, Y_1, Y_2, \ldots\}$? Justify your answer.

Question 3: Pólya's urn. A bag contains red and blue balls, with initially r red and b blue where $rb > 0$. A ball is drawn from the bag at random, its colour noted, and then returned to the bag together with a new ball of the same colour. Let R_n be the number of red balls after n such operations.

(a) Show that $\{Y_n = R_n/(n + r + b), n \geqslant 0\}$ is a martingale.

(b) Show that Y_n converges almost surely.

(c) Let T be the number of balls drawn until the first blue ball appears, and suppose that $r = b = 1$. Compute $E[T/(T + 2)]$.

(d) Suppose $r = b = 1$. Show that $P(Y_n \geqslant \frac{3}{4} \text{ for some } n) \leqslant \frac{2}{3}$.

[Note: You may use the following version of Doob's Optional Stopping Theorem: If the sequence $\{Y_n, n \geqslant 0\}$ is a bounded martingale and T is a stopping time, then the expected value of Y_T is Y_0.]