Qualify Exam. June 27, 2013

Problem from STAT5010

1. Suppose $X_1, X_2, \cdots, X_n$ is an i.i.d. sample from the distribution function

$$F_\theta(x) = \exp[-\exp(-(x-\theta))]$$

(a) Verify that $F_\theta(x)$ is a distribution function for any θ.

(b) Find the density function of $F_\theta(x)$.

(c) Find a minimal sufficient statistic for θ.

(d) Find a UMP level α test for $H_0 : \theta \geq 0$ VS. $H_1 : \theta < 0$. State clearly which theorem you have used.

(e) Invert the test in d) to obtain an $(1-\alpha)$ confidence interval.

$= -\ln U_i$

$\varepsilon_i) = e^{-e^{-\varepsilon_i}} e^{-\varepsilon_i}$

$u_i) = e^{-u_i} \cdot \left|\frac{1}{u_i}\right|$

$= e^{-u_i}$

$\sim Exp(1)$

ired Prob $P =$

2. Suppose we have four random variables $Y_i, i = 1, 2, 3, 4$, following the model $Y_i = i\theta + \varepsilon_i, i = 1, \cdots, 4$ where $\varepsilon_i, i = 1, \cdots, 4$ are i.i.d. with the distribution function

$$F(u) = \exp\{-e^{-u}\}, \ -\infty < u < \infty.$$

$+e^{-\theta})^{-1}(1+e^{-\theta}e^{-\theta}+1)^{-1}$

$(1+e^{-\theta}e^{-\theta}+1)e^{-\theta}+1)^{-1}$

(a) Calculate the probability $P_r\{Y_1 < Y_2 < Y_3 < Y_4\}$ (Hint: Let $U_i = e^{-\varepsilon_i}$, then U_i are i.i.d. following the standard exponential distribution and the required probability becomes

$\frac{\partial l}{\partial \theta} \triangleq 0 \Rightarrow \theta$

Variance $= 0$

$$P_r\{e^{-\theta}U_1 > e^{-2\theta}U_2 > e^{-3\theta}U_3 > e^{-4\theta}U_4\}$$

$\int_{e^{-\theta}u_2}^{\infty} e^{-u_1}du_1 = e^{-e^{-\theta}u_2}$ $\quad \int_{e^{-\theta}u_3}^{\infty} e^{-(e^{-\theta}+1)u_2} du_2$

$$= \int_0^\infty f_{u_4}(u_4)du_4 \int_{e^{-\theta}u_4}^\infty f_{u_3}(u_3)du_3 \int_{e^{-\theta}u_3}^\infty Pr\{U_1 > e^{-\theta}u_2\}f_{u_2}(u_2)du_2\text{)}$$

$P(\theta=0) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{24}$

$\frac{1}{(1+e^{-\theta})(e^{-\theta}+1)e^{-\theta}+1)} e^{-[(e^{-\theta}+1)e^{-\theta}+1]e^{-\theta}u_4}$ $\leftarrow \frac{1}{1+e^{-\theta}} e^{-(e^{-\theta}+1)e^{-\theta}u_3}$

Reject H_0 if $RT = \frac{P(\theta=0)}{P(\theta=\hat\theta)}$ is small

(b) Suppose that the event $\{Y_1 < Y_2 < Y_3 < Y_4\}$ is observed, what is your estimate of θ, and how do you estimate its variance?

(c) How will you test the hypothesis $H_0 : \theta = 0$ VS. $H_a : \theta \neq 0$?

(a) $\lim_{x\to-\infty} F_\theta(x) = \lim_{x\to-\infty} e^{-e^{-x+\theta}} = 0$, $\lim_{x\to\infty} F_\theta(x) = \lim_{x\to\infty} e^{-e^{-x+\theta}} = 1$, cont. func. $\frac{\partial F_\theta}{\partial x} = e^{-e^{-x+\theta}} e^{-x+\theta} \geq 0$

(b) $f_\theta(x) = e^{-e^{-x+\theta}} e^{-x+\theta}$, $x \in \mathbb{R}$

(c) $f(\vec{x}\mid\theta) = e^{-e^{\theta}\sum e^{-x_i}} \cdot e^{-\sum x_i + n\theta}$, $\sum e^{-x_i}$ is complete and sufficient stat. thus minimal.

(d) $w(\theta) = -e^{\theta}$; $w'(\theta) = e^{\theta} > 0$, $w(\theta)\uparrow$, $f_\theta(x)$ is MLR

By Karlin-Rubin theorem, Reject H_0 iff $\sum e^{-x_i} < t_0$. $\alpha = P_{\theta=0}\{\sum e^{-x_i} < t_0\}$

let $y = e^{-x}$, $x = -\ln y$. $f_\theta(y) = e^{-e^{\theta}y} e^{\theta} y \cdot \left|\frac{1}{y}\right| = e^{\theta} e^{-e^{\theta}y} \sim Exp(e^{-\theta})$

$\sum_{i=1}^n y_i \sim Gamma(n, e^{-\theta})$ therefore $t_0 = Gamma(n, 1, \alpha)$

(e) $P(e^{\theta}\sum e^{-x_i} < t_0) = \alpha \Rightarrow P(\sum e^{-x_i} > t_0 e^{-\theta}) = 1-\alpha \Rightarrow \theta < \ln\frac{\sum e^{-x_i}}{t_0}$

STA 5010

Final Exam. December 10, 2003

1. Suppose that

$$Y_i = \alpha + \beta x_i + \varepsilon_i, \quad i = 1, \cdots, n$$

$\varepsilon_i's$ are i.i.d. with density function

$$f(u) = \exp\{c_2 u^2 + c_1 u + c_0\}$$

where $x_1, \cdots, x_n, c_0, c_1, c_2$ are known constants and α, β are unknown parameters.

(a) Find a minimal sufficient statistics for (α, β).

(b) Is your statistic complete?

(c) Give the UMVUE for (α, β).

2. Continue with the Last problem. Let X_{n+1} and X_{n+2} be two known numbers. What is the UMVUE of $P\{Y_{n+1} > Y_{n+2}\}$, where $Y_{n+i} = \alpha + \beta\ X_{n+i} + \varepsilon_{n+i}$.

3. Let $X_1, \cdots, X_n$ be a random Sample from $N(0, \sigma_X^2)$, and let $Y_1, \cdots, Y_m$ be a random Sample from $N(0, \sigma_Y^2)$, independent of X_s'. Define $\boxed{\lambda = \sigma_Y^2 / \sigma_X^2}$.

(a) Find the level α LRT of $H_0 : \lambda = \lambda_0$ versus $H_1 : \lambda \neq \lambda_0$.

(b) Express the rejection region of the LRT of part (a) in terms of an F random variable.

(c) Find a $1 - \alpha$ confidence interval for λ.

4. Let X_1, X_2, X_3 be a random sample of size three from a uniform $(\theta, 2\theta)$ distribution, where $\theta > 0$,

(a) Find the method of moments estimator of θ.

(b) Find the MLE, $\hat{\theta}$, and find a constant k such that $E_\theta(k\hat{\theta}) = \theta$.

(c) Which of the two estimators can be improved by using sufficiency? How?

5. Let $X_1, \cdots, X_n, X_{n+1}, \cdots, X_{n+m}$ be an i.i.d. sample from $f(x) = \frac{1}{\beta}\exp\left(-\frac{x}{\beta}\right), x > 0$. We observe the values of $X_1, \cdots, X_n$ and the event that $X_{n+1} > Y_{n+1}, \cdots, X_{n+m} > Y_{n+m}$ (The exact value of $X_{n+1}, \cdots, X_{n+m}$ are unknown, but $Y_{n+1}, \cdots, Y_{n+m}$ are observed).

(a) Find a MLE of β.

(b) Find a minimal sufficient.

(c) Is you Statistic in a) an UMVUE?

— END —

Qualify Exam. August 10, 2012

Problem from STAT5010

1. Let μ and Σ be the mean and covariance matrix from a bi-variate normal distribution. Let $X_1, X_2, \cdots, X_n$ be an i.i.d. sample from a truncated bi-variate normal distribution, or for $\mathrm{x} \in R^2$, the density function of X_1 is

$$f(\mathrm{x}|\mu, \Sigma, \gamma) = \begin{cases} C\frac{1}{\sqrt{\det(\Sigma)}}\exp\left\{-\frac{1}{2}(\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu)\right\}, & \text{for } (\mathrm{x}-\mu)'\Sigma^{-1}(\mathrm{x}-\mu) \le \gamma^2 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find constant C and show that C only depends on γ.

   (b) Find minimum sufficient statistic for (μ, Σ) if γ is known.

   (c) Find minimum sufficient statistic for γ if (μ, Σ) is known.

   (d) Find UMVUE for (μ, Σ) if γ is known and prove your answer.

   (e) Find UMVUE for γ if (μ, Σ) is known and prove your answer.

   (f) If all μ, Σ and γ are unknown, how do you estimate all three parameters?

2. Suppose that $X_1, \cdots, X_{n_1}$ and $Y_1, \cdots, Y_{n_2}$ are two samples from the population $N(\mu_1, \sigma^2)$ and $N(\mu_2, \sigma^2)$. Assume σ^2 is unknown.

   (a) Show that the α-level likelihood ratio test for

$$H_o : \mu_2 \le \mu_1 \text{ VS. } H_a : \mu_2 > \mu_1 \text{ is the usual two sample t-test.}$$

   (b) Can you show that the test is an uniformly most powerful test?

Qualify Exam. Decemeber 2013

Problem from STAT5010

$P(A \cup B)$. (θ_0, θ_0+1). $Y_n \le \theta_0+1$.

1. Let $X_1, \cdots, X_0$ be a random sample from Uniform $(\theta, \theta+1)$ distribution. To test $H_0: \theta = \theta_0$ Vs $H_1: \theta > \theta_0$, use the test

$$\text{reject } H_0 \text{ if } Y_n \geq 1 + \theta_0 \text{ or } Y_1 \geq k + \theta_0$$

$\beta(\theta) = P_\theta($ $)$

where k is a constant, $Y_1 = \min\{X_1, \cdots, X_n\}, Y_n = \max\{X_1, \cdots, X_n\}$

   (a) Determin k so that the test will have size α.

   (b) Find the power function of the test in a).

   (c) Prove that the test is UMP size α test.

   (d) Find values of n and k so that the UMP $\alpha = .05$ level test will have power at least .8 if $\theta > \theta_0 + 1$.

   (e) Obtain an $1 - \alpha$ level confidence interval by inverting the above test.

2. Let $X_1, X_2, \cdots, X_n$ be an sequence of i.i.d. r.v.s from $f(x|\beta) = \frac{1}{\beta}\exp(-\frac{x}{\beta}), x > 0$. We also know that $Y_1, Y_2, \cdots, Y_n$ be an sequence of i.i.d. r.v.s from a known population with density $g(y)$. Suppose that the following sample is observed $\min(X_i, Y_i), 1_{[X_i \leq Y_i]}, i = 1, \cdots, n$.

   (a) Find a MLE of β based on the observed sample.

   (b) Find a minimal suficient statistics for β.

   (c) Is the statistics in a) an UMVUE? Prove or disprove your answer.

Department of Statistics, The Chinese University of Hong Kong
STAT5010 Advanced Statistical Inference | Term 1, 2019–20

Take-home Examination

Instruction to the candidates: Please attempt all of the questions. Each problem carries an equal weight of 4 points. Your final score will be capped by 20, which is also the defined full mark of this exam. Good luck!

1. Let $X_1, \dots X_n$ be a random sample from a $N(\theta, \sigma^2)$ population with σ^2 known. Consider estimating θ using the squared error loss. Let $\pi(\theta)$ be a $N(\mu, \tau^2)$ prior distribution on θ and let δ^π be the Bayes estimator of θ. Verify the following formulas for the risk function and Bayes risk.

   (a) For any constants a and b, the estimator $\delta(\boldsymbol{X}) = a\bar{\boldsymbol{X}} + b$ has risk function
   $$R(\theta, \delta) = a^2 \frac{\sigma^2}{n} + \{b - (1-a)\theta\}^2.$$

   (b) Let $\eta = \sigma^2/(n\tau^2 + \sigma^2)$. The risk function for the Bayes estimator is
   $$R(\theta, \delta^\pi) = (1-\eta)^2 \frac{\sigma^2}{n} + \eta^2(\theta - \mu)^2.$$

   (c) The Bayes risk for the Bayes estimator is
   $$B(\pi, \delta^\pi) = \tau^2 \eta.$$

2. Let X be an observation from the pdf
   $$f(x \mid \theta) = \left(\frac{\theta}{2}\right)^{|x|} (1-\theta)^{1-|x|}, \quad x \in \{-1, 0, 1\}; \theta \in [0, 1].$$

   (a) Find the MLE of θ.

   (b) Define an estimator $T(X)$ by
   $$T(X) = \begin{cases} 2 & \text{, if } x = 1 \\ 0 & \text{, otherwise} \end{cases}.$$
   Show that $T(X)$ is an unbiased estimator of θ.

   (c) Find a better estimator than $T(X)$ and prove that it is better.

3. Consider a Bayesian model in which the prior distribution for Θ is standard exponential and the density for X given Θ is
   $$f(x \mid \theta) = e^{\theta - x} I(x > \theta).$$

   (a) Find the marginal density for X and $E(X)$ in the Bayesian model.

   (b) Find the Bayes estimator for Θ under squared error loss. (Assume $X > 0$.)

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. from the uniform distribution on $(1, 2)$, and let H_n denote the harmonic average of the first n variables:
$$H_n = \frac{n}{X_1^{-1} + \ldots + X_n^{-1}}.$$
   (a) Show that $H_n \xrightarrow{p} c$ as $n \to \infty$, identifying the constant c.
   (b) Show that $\sqrt{n}(H_n - c)$ converges in distribution, and identify the limit.

5. Let $X_1, \ldots, X_n$ be i.i.d. from $N(\theta, 1)$ and let $U_1, \ldots, U_n$ be i.i.d. from a uniform distribution on $(0, 1)$, with all $2n$ variables independent. Define $Y_i = X_i U_i$, $i = 1, \ldots, n$. If the X_i and U_i are both observed, then $\bar{X}$ would be a natural estimator for θ. If only the products $Y_1, \ldots, Y_n$ are observed, then $2\bar{Y}$ may be a more responsible estimator. Determine the asymptotic relative efficiency (ARE) of $2\bar{Y}$ with respect to $\bar{X}$, where ARE of $\hat{\theta}_n$ with respect to $\tilde{\theta}_n$ is defined as the ratio $\sigma^2_{\tilde{\theta}}/\sigma^2_{\hat{\theta}}$ if $\sqrt{n}(\hat{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma^2_{\hat{\theta}})$ and $\sqrt{n}(\tilde{\theta} - \theta_0) \xrightarrow{d} N(0, \sigma^2_{\tilde{\theta}})$, respectively.

6. Suppose $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\theta, 1)$ for some $\theta \in \mathbb{R}$, and we want to estimate θ with respect to squared error loss. Show that the estimator $\delta_a(X) = \bar{X} + a$ is not a Bayes estimator for any a.

- End -

Department of Statistics, The Chinese University of Hong Kong
STAT5010 Advanced Statistical Inference | Term 1, 2018-19

Final Examination on 5th December 2018

Instruction to the candidates. Please attempt all of the questions. Each problem carries an equal weight of 7 points. Your final score will be capped by 40, which is also the defined full mark of this exam. Good luck!

1. Determine whether or not each of the following statements is correct. If a statement is not always true, you should regard it as a "false" statement. *Note that a correct answer without a proper explanation is considered as a wrong answer. You are required to show clearly your reasoning.*

   (a) Any regular one-parameter exponential family admits a canonical form has a monotone likelihood ratio statistic.

   (b) (i.) For hypothesis $H_0: \theta = \theta_0$, v.s. $H_1: \theta = \theta_1$, and a continuous test statistic $T(X)$, the p-value p follows Uniform$[0, 1]$.

   (ii.) If the hypothesis is given as $H_0: \theta = \theta_0$, v.s. $H_1: \theta \in \Theta_1$, the p-value p follows Uniform$[0, 1]$.

   (iii.) If the hypothesis is given as $H_0: \theta \in \Theta_0$, v.s. $H_1: \theta \in \Theta_1$, the p-value p follows Uniform$[0, 1]$.

2. Suppose that the parameter space Θ is an open subset of $\mathbb{R}^2$, and the risk function $R(\theta, \delta)$ is continuous in θ for every estimator $\delta(X)$. Suppose further that π_n be a sequence of (possibly improper) priors on Θ such that the following conditions hold:

   (i) $r(\pi_n, \delta_0) < \infty$ for all n.

   (ii) For any open set $\Theta_0 \subset \Theta$, one has $\liminf_{n\to\infty} \int_{\Theta_0} \pi_n(\theta) d\theta > 0$, and

   (iii) $\lim_{n\to\infty} r(\pi_n, \delta_0) - r(\pi_n, \delta_{\pi_n}) = 0$,

   where δ_{π_n} is the Bayes estimate with respect to the prior π_n and δ_0 is an estimator. Conclude that δ_0 is admissible without necessarily assuming squared loss error function. *(Hint: If δ_0 is inadmissible, there exists an estimator δ' such that $R(\theta, \delta_0) \geq R(\theta, \delta')$, $R(\theta_0, \delta_0) > R(\theta_0, \delta')$ for some $\theta_0 \in \Theta$.)*

3. Find the asymptotic distribution of $\ell_n(\theta_0 + hn^{-1/2}) - \ell_n(\theta_0)$, where $h \in \mathbb{R}$ is fixed and $\ell_n(\theta)$ is the log likelihood function of a random sample $X_1, \ldots, X_n$ evaulated at θ. State also the regularity conditions needed.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables, having the exponential distribution with density $f(x; \lambda) = \lambda \exp\{-\lambda$ for $x, \lambda > 0$.

   (a) Show that $T_n = \sum_{i=1}^n X_i$ is minimal sufficient and complete for λ.

   (b) Furthermore, for given $x > 0$, it is desired to estimate the quantity $\phi = \Pr(X_1 > x; \lambda)$. Compute the Fisher information for ϕ.

(c) State the Lehmann-Scheffé theorem which ties together the ideas of completeness, sufficiency, uniqueness, and best unbiased estimation. Show that the estimator $\tilde{\phi}_n$ of ϕ defined by

$$\tilde{\phi}_n = \begin{cases} 0, & \text{if } T_n < x \\ \left(1 - \frac{x}{T_n}\right)^{n-1}, & \text{if } T_n \geq x \end{cases}$$

is the minimum variance unbiased estimator of ϕ based on $(X_1, \ldots, X_n)$. Without conducting any computations, state whether or not the variance of $\tilde{\phi}_n$ achieves the Cramér-Rao lower bound with brief justification. Show also that $E(\tilde{\phi}_k \mid T_n, \lambda) = \tilde{\phi}_n$ for $k \leq n$.

5. Let $(X_1, Y_1), \ldots (X_n, Y_n)$ be i.i.d. with $X_i \sim N(0, 1)$ and $Y_i \mid X_i = x \sim N(x\theta, 1)$.

   (a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

   (b) Determine the limiting distribution of $n^{1/2}(\hat{\theta} - \theta)$.

   (c) Construct $1 - \alpha$ asymptotic confidence interval for θ based on $I(\hat{\theta})$, where $I(\theta)$ denotes the Fisher information for a single observation (X_1, Y_1).

   (d) Determine the exact distribution of $\sqrt{\sum_{i=1}^n X_i^2}(\hat{\theta} - \theta)$ and use it find the true coverage probability for the interval constructed based on observed Fisher information.

6. Let $X_1, \ldots, X_n$ be a random sample from a population on $\mathbb{R}$ with Lebesgue density f_θ. Let θ_0 and θ_1 be two constants with $\theta_0 < \theta_1$. Find a uniform most power test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for $f_\theta(x) = \exp\{-(x - \theta)\}I(x > \theta)$.

7. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma_0^2)$ with σ_0^2 known. Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Show that a UMP test does *not* exist for any size $\alpha \in (0, 1)$. *(Hint: You may consider a one-sided test first. Check its power for another side of the test from which you may argue over the power of this particular one-sided UMP test.)*

- End -

Final Examination of 5010 STAT TONY SHI JIASHENG.

1. True / False, give explanation.

① Th6.12 of Lehmann (Ch 1).

② $\hat{F}_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1\{X_i \le x\}$, $X_1, \cdots, X_n$ sample from P (all pdf).
Then $\hat{F}_n(x)$ is UMVUE of $F(x)$.

③ p is exponential family admits one-parameter canonical form.
then it has monotone likelihood function.

2. $X_1, \cdots, X_n$ iid $\sim U(0,\theta_1)$, $Y_1 \cdots Y_n$ iid $\sim U(0,\theta_2)$

① Find complete sufficient statistic for (θ_1, θ_2) $(X_{(n)}, Y_{(n)})$

② Find UMVUE of $\frac{\theta_1}{\theta_2}$. ~~C(n)~~ $c(n) \cdot \frac{X_{(n)}}{Y_{(n)}}$.

3. $X = YZ$, $Y \sim N(\theta, 1)$, $P(Z=1) = P(Z=-1) = -\frac{1}{2}$.

① $T(X) = X$ is sufficient but not complete.

② If there exists a complete sufficient statistic for θ.

4. $f_\theta(x) = e^{-(x-\theta)} \cdot 1\{x > \theta\}$. (Refer to Problem 4.7.1 of Keener).

① Find complete sufficient statistic for θ. $(T_n = \sum_{i=1}^{n} X_i)$

② $\phi = P(X_1 > x)$ ~~Nor $P(X_1 \le x)$~~. Find Fisher info $I(\phi)$.

③ State Lehmann-Scheffé Theorem ties together with sufficiency, completeness, uniqueness, ~~bias~~ unbiased estimation.
Show that $\tilde{\phi}_n = 1\{T_n > x\} \cdot \left(1 - \frac{x}{T_n}\right)^{n-1}$ is UMVUE of ϕ.

5. $(X_1, Y_1), \cdots, (X_n, Y_n)$ are iid, $X_i \sim N(0,1)$, $Y_i | X_i = x \sim N(\theta x, 1)$.

① Find MLE of θ

② Find limiting dist of $\sqrt{n}(\hat{\theta} - \theta)$ (MLE 极限分布) $\Rightarrow N(0, \frac{1}{I(\theta)})$

③ Construct $1-\alpha$ Asymptotic confidence interval for θ based on $I(\hat{\theta})$, $I(\theta)$ is Fisher info based on one observation (X_1, Y_1).

④ Give the exact dist of $\sqrt{\sum_{i=1}^{n} X_i^2}\,(\hat{\theta}-\theta)$.

6. $H_0: \theta=\theta_0$ v.s. $H_1: \theta=\theta_1$, Find the UMPT (uniform most powerful) of size α ~~for~~ R $f_X(x) = e^{-(x-\theta)} \cdot 1\{x>\theta\}$.

STAT 5010 Syllabus

Chaojie Wang

May 27, 2015

1 Probability Theory

	Without Replacement	With Replacement
Ordered	$n!/(n-r)!$	n^r
Unordered	$\binom{n}{r}$	$\binom{n+r-1}{r}$

2 Transformation and Expectation

Theorem 2.1.10: $Y = F_X(X) \Rightarrow P(Y \leq y) = y$, $0 < y < 1$.

Hint: $F_X^{-1}(y) = \inf\{x : F_X(x) \geq y\}$

Theorem 2.4.1: Leibnitz's Rule

$$\frac{d}{d\theta}\int_{a(\theta)}^{b(\theta)} f(x,\theta)dx = f(b(\theta),\theta)\frac{d}{d\theta}b(\theta) - f(a(\theta),\theta)\frac{d}{d\theta}a(\theta) + \int_{a(\theta)}^{b(\theta)} \frac{\partial}{\partial\theta} f(x,\theta)dx$$

3 Common Families of Distributions

Theorem 3.4.2: X follows exponential family

$$E(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)) = -\frac{\partial}{\partial \theta_j} \log c(\boldsymbol{\theta})$$

$$Var(\sum_{i=1}^{k} \frac{\partial w_i(\boldsymbol{\theta})}{\partial \theta_j} t_i(X)) = -\frac{\partial^2}{\partial \theta_j^2} \log c(\boldsymbol{\theta}) - E(\sum_{i=1}^{k} \frac{\partial^2 w_i(\boldsymbol{\theta})}{\partial \theta_j^2} t_i(X))$$

Lemma 3.6.5 Stein's Lemma: $X \sim n(\theta, \sigma^2)$

$$E[g(X)(X-\theta)] = \sigma^2 E g'(X)$$

4 Multiple Random Variable

5 Properties of a Random Sample

Theorem 5.4.4: The pdf of $X_{(j)}$

$$f_{X_{(j)}}(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)[F_X(x)]^{j-1}[1-F_X(x)]^{n-j}$$

Metropolis Algorithm: $Y \sim f_Y(y)$ and $V \sim f_V(v)$, to generate $Y \sim f_Y$

0. Generate $V \sim f_V$. Set $Z_0 = V$. For $i = 1, 2, \cdots$

1. Generate $U_i \sim$ uniform(0,1), $V \sim f_V$, and calculate

$$\rho_i = \min\{\frac{f_Y(V_i)}{f_V(V_i)} \frac{f_V(Z_i)}{}\}$$

6 Principles of Data Reduction

Check $T(\mathbf{X})$ is sufficient statistic:

(a) Theorem 6.2.2: If $p(\mathbf{x}|\theta)/q(T(\mathbf{x}|\theta)$ is constant of θ

(b) Theorem 6.2.6: Factorization Theorem $f(\mathbf{x}|\theta) = g(T(\mathbf{x})|\theta)h(\mathbf{x})$

(c) Theorem 6.2.10: $f(x|\boldsymbol{\theta})$ is exponential family

(d) Theorem 6.2.13: $f(\mathbf{x}|\theta)/f(\mathbf{y}|\theta)$ is constant iff $T(\mathbf{x}) = T(\mathbf{y})$

Check $S(\mathbf{X})$ is ancillary statistic:

Example 6.2.18: Location family. Example 6.2.19: Scale family

Check $T(\mathbf{X})$ is complete statistic:

Theorem 6.2.21: If $E_\theta g(T) = 0$ for all θ implies $P_\theta(g(T) = 0) = 1$ for all θ

Theorem 6.2.25: $f(x|\boldsymbol{\theta})$ is exponential family

Theorem 6.2.24: $T(\mathbf{X})$ is complete and minimal sufficient statistic $\Rightarrow$ $T(\mathbf{X})$ independent of ancillary statistic. Remark: Used under the exponential family