1. Since $|a|+|b| \ge |a-b|$, and the equal sign holds only when $a \cdot b \le 0$. (1)

To maximize the likelihood function

$$L(\theta) = \prod_{i=1}^{n} \frac{1}{2} e^{-|x_i-\theta|}.$$

it suffices to ~~maximize~~ minimize

$$l(\theta) = \sum_{i=1}^{n} |x_i-\theta|.$$

Then, ~~we~~ using inequality (1), we obtain

$$\sum_{i=1}^{n} |x_i-\theta| \ge |x_{(n)}-\theta-(x_{(1)}-\theta)| + |x_{(n-1)}-\theta-(x_{(2)}-\theta)| + \cdots + |x_{(m+1)}-\theta-(x_{(m-1)}-\theta)| + |x_{(m)}-\theta|.$$

where $m=\frac{n+1}{2}$.

We know the equal sign holds when $(x_{(i)}-\theta)(x_{(n+1-i)}-\theta) \le 0$ for $i=1,\dots,m-1$.

So the range of $\theta^*$ should be $x_{(m-1)} \le \theta \le x_{(m+1)}$, and in order to achieve the minimum of last term, $\theta^* = x_{(m)}$.

When $n$ is even, $\theta^*$ can be any number between $X_{(m)}$ and $X_{(m+1)}$, where $m=\frac{n}{2}$.

$$I(\theta) = E\left[\left(\frac{\partial \log L(\theta)}{\partial\theta}\right)^2 \Big| \theta\right]$$

$= nE\left[\left(\frac{\partial \log f(x_1;\theta)}{\partial\theta}\right)^2 \Big| \theta\right]$, since $X_i$'s are iid.

$= n\,E\left[(\mathrm{sgn}(X_1-\theta))^2 | \theta\right]$ since $\frac{\partial \log f(x_1;\theta)}{\partial\theta} = \mathrm{sgn}(X_1-\theta)$ for $X_1 \ne \theta$

$= \cancel{DEf}\ n$, since $E((\mathrm{sgn}(X_1-\theta))^2|\theta) = 1$

2. Since $E(X^2)$ is finite, then $Var(X)$ is also finite and ~~set~~ denote it ~~as~~ as $\sigma^2$.

By Central Limit Theorem, we have

$$n^{\frac{1}{2}} \bar{X}_n \xrightarrow{d} N(0,\sigma^2)$$

By weak law of large number, we have

$$\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2 \xrightarrow{p} \sigma^2, \quad \bar{X} \xrightarrow{p} \mu$$

And

$$S_n^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X}) = \left(\frac{1}{n}\sum_{i=1}^{n}(X_i-\mu)^2\right) + (\bar{X}-\mu)^2$$

By continuous mapping theorem $\xrightarrow{p} \sigma^2$

By Slutsky's Theorem. we obtain $S_n \xrightarrow{p} \sigma$ since $\sqrt{x}$ is a continuous function.

$$\frac{n^{\frac{1}{2}}\bar{X}_n}{S_n} \xrightarrow{d} N(0,1)$$

If $E(X_1)=\mu \in \mathbb{R}$, then now we have ~~$n^{\frac{1}{2}}\bar{X}_n \xrightarrow{d} N(\mu,\sigma^2)$ and $S_n \xrightarrow{p} \sigma^2$~~ $\frac{n^{\frac{1}{2}}(\bar{X}_n-\mu)}{S_n} \xrightarrow{d} N(0,1)$

Then the 1-α confidence interval for μ is

$$\cancel{(n^{\frac{1}{2}}\bar{X}_n - Z_{1-\alpha/2}\cdot S_n,\ n^{\frac{1}{2}}\bar{X}_n - Z_{\alpha/2}\cdot S_n)};\ \cancel{(\bar{X}_n - Z_{1-\alpha/2}\frac{S_n}{\sqrt{n}},\ \bar{X}_n - Z_{\alpha/2}\frac{S_n}{\sqrt{n}})}$$

~~where $Z_\alpha$ =~~ $\left(\bar{X}_n - Z_{\alpha/2}\cdot\frac{S_n}{\sqrt{n}},\ \bar{X}_n + Z_{\alpha/2}\frac{S_n}{\sqrt{n}}\right)$

3. (i) Consider a prior for $p$ is Beta$(\alpha,\beta)$, then the Bayes estimator under square loss function is

$$\delta_{\alpha,\beta}(x) = \frac{x+\alpha}{n+\alpha+\beta}$$

Then the risk for $\delta_{\alpha,\beta}(x)$ is

$$R(\theta, \delta_{\alpha,\beta}) = E_\theta\left(\left(\theta - \frac{x+\alpha}{n+\alpha+\beta}\right)^2\right)$$

$$\cancel{= \frac{\quad}{(n+\alpha+\beta)^2}p^2 + \frac{1}{(n+\alpha+\beta)^2}E_p(x+\alpha)^2 - 2p\frac{1}{n+\alpha+\beta}E_p(x+\alpha)}$$

$$= \frac{1}{(n+\alpha+\beta)^2}E_\theta\left(x+\alpha-\theta n-\theta\alpha-\theta\beta\right)^2$$

$$= \frac{1}{(n+\alpha+\beta)^2}E_\theta\left(x-\theta n+(1-\theta)\alpha-\theta\beta\right)^2$$

$$= \frac{1}{(n+\alpha+\beta)^2}\left[n\theta(1-\theta)+\left[(1-\theta)\alpha-\theta\beta\right]^2\right]$$

$$= \frac{1}{(n+\alpha+\beta)^2}\left(\alpha^2+\theta^2((\alpha+\beta)^2-n)+\theta(n-2\alpha(\alpha+\beta))\right)$$

When $\alpha=\frac{\sqrt{n}}{2}$, $\beta=\frac{\sqrt{n}}{2}$, $R(\theta,\delta_{\alpha,\beta})$ doesn't depend on $\theta$. By Theorem, $\delta_{\alpha,\beta}$ is the unique Bayes estimator and thus unique minimax estimator.

$$\cancel{r_{\alpha,\beta}(\theta,\Lambda) =} \frac{1}{4(n+1+2\sqrt{n})} \quad r(\theta,\tilde{\theta}) = \sup_{\theta\in[0,1]} R(\theta,\tilde{\theta}) = \frac{1}{4(n+1+2\sqrt{n})}$$

$$\tilde{\theta} = \frac{x+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$$

The likelihood function is

$$L(\theta) = \prod_{i=1}^{n}\binom{n}{x_i}\theta^{x_i}(1-\theta)^{n-x_i}$$

To maximize $L(\theta)$, we obtain

$$\hat{\theta} = \frac{X}{n}$$

$$R(\theta,\hat{\theta}) = E_\theta\left[\left(\theta-\frac{X}{n}\right)^2\right]$$

$$= \frac{\theta(1-\theta)}{n}$$

$$r(\theta,\hat{\theta}) = \sup_{\theta\in[0,1]} R(\theta,\hat{\theta}) = \frac{1}{4n} > r(\theta,\tilde{\theta})$$

Therefore, $\tilde{\theta} = \dfrac{X+\frac{\sqrt{n}}{2}}{n+\sqrt{n}}$ and $\hat{\theta}$ is not minimax.

ii) $\lim_{n\to\infty} \frac{\sup_\theta R(\hat{\theta}_n,\theta)}{\sup_\theta R(\tilde{\theta}_n,\theta)} = \lim_{n\to\infty} \frac{n+1+2\sqrt{n}}{n} = \lim_{n\to\infty} \frac{1+\frac{1}{n}+\frac{2}{\sqrt{n}}}{1} = 1$, by L'Hopital's rule.

$$\lim_{n\to\infty} \frac{R(\hat{\theta}_n,\theta)}{R(\tilde{\theta}_n,\theta)} = \lim_{n\to\infty} \frac{(n+1+2\sqrt{n})(1-\theta)\theta}{4n} \quad (1-\theta)\theta$$

$$= \lim_{n\to\infty} \frac{\left(\frac{\theta(1-\theta)}{n}\right)}{\left(\frac{1}{4(n+1+2\sqrt{n})}\right)} = \lim_{n\to\infty} \frac{4\theta(1-\theta)\cdot(n+1+2\sqrt{n})}{n} = 4\theta(1-\theta) < 1, \text{ when } \theta \neq \tfrac{1}{2}.$$

b) Consider the prior for $\theta$ is $N(0,m^2)$, then the Bayes estimator

$$\delta_m(x) = \frac{\left(\frac{n\bar{X}_n}{1}\right)}{\left(\frac{n}{1}+\frac{1}{m^2}\right)}$$

$$= \frac{(n\bar{X}_n)m^2}{nm^2+1}$$

$$= \frac{n\bar{X}_n}{n+\frac{1}{m^2}}$$

$$r_m = \frac{1}{\frac{n}{1}+\frac{1}{m^2}}$$

As $m^2\to\infty$, $r_m \uparrow \frac{1}{n}$

We know $\sup_\theta R(\theta,\bar{X}_n) = \frac{\sigma^2}{n} = \frac{1}{n} = \lim_{m^2\to\infty} r_m$

By Theorem. $\bar{X}_n$ is minimax.

Consider again the prior for $\theta$ is $N(0,m^2)$, $m\neq 0$, then the

$$R_m(\theta,\delta_m) = \frac{1}{n+\frac{1}{m^2}} < R(\theta,\bar{X}_n) = \frac{1}{n}, \quad \forall\theta\in[0,\infty)$$

Therefore, $\bar{X}_n$ is inadmissible.

4. (a) For any $\varepsilon>0$, we have

$$P(\|X_n-X\|>\varepsilon) \leq \frac{E\|X_n-X\|}{\varepsilon}$$

$$\to 0, \text{ as } n\to\infty$$

Therefore, we have $X_n \xrightarrow{P} X$

(b) Consider the case of $k=1$. $U$ is a r.v. from uniform(0,1) distribution.

And $X_n = \begin{cases} n & 0\leq U\leq \frac{1}{n} \\ 0, & \text{otherwise} \end{cases}$

Then we have $X_n \xrightarrow{P} 0$ as $n\to\infty$. and

$E|X_n-0| = 1$, doesn't goes to 0, as $n\to\infty$.

5. $P_0(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{x_i^2}{2\sigma^2}\right\}$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n \exp\left\{-\frac{\sum x_i^2}{2\sigma^2}\right\}$$

$$P_1(x) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{\sum(x_i-\theta_{i0})^2}{2\sigma^2}\right\}$$

$$\frac{P_1(x)}{P_0(x)} = \exp\left\{-\frac{\sum(x_i-\theta_{i0})^2}{2\sigma^2} + \frac{\sum x_i^2}{2\sigma^2}\right\}$$

$$= \exp\left\{\frac{-\sum\theta_{i0}^2 + 2\sum\theta_{i0}x_i}{2\sigma^2}\right\}.$$

which is a non-decreasing function of $\sum\theta_{i0}x_i$.

We know under $H_0$,

$$\sum\theta_{i0}x_i \sim N(0, \sigma^2\sum\theta_{i0}^2)$$

Therefore,

$$\frac{\sum\theta_{i0}x_i}{\sqrt{\sigma^2\sum\theta_{i0}^2}} \sim N(0,1)$$

Then the test is proposed as follows.

$$\phi(x) = \begin{cases} 1, & T(x) > k \\ 0, & T(x) \leq k \end{cases}$$

where $k = Z_{\alpha/2}$, $T(x) = \frac{\sum\theta_{i0}x_i}{\sqrt{\sigma^2\sum\theta_{i0}^2}}$.

By Neyman-Pearson lemma, we know the test $\phi(x)$ is the most powerful.