STAT5005 Final Exam 2019/20
[Totally 100 marks] (2:30-6:30pm, 5 December 2019)

Instructions:

1. This is an open book examination.

2. You are required to work independently and should not discuss with others. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.

3. After finishing, please take a clear picture of your solution and send it to my email (xfang@cuhk.edu.hk). The deadline is 6:30pm on December 5.

4. Totally 8 questions on 2 pages. If you think there is a problem with the question, please state your reason.

Question 1: [10 marks]
Prove that if $\{X_n, n \geqslant 1\}$ are i.i.d. random variables with $P(X_1 = 0) < 1$ and $S_n = \sum_{i=1}^n X_i, n \geqslant 1$, then for every $c > 0$ there exists an integer $n = n_c$ such that $P(|S_n| > c) > 0$.

Question 2: [10 marks]
State the conditional Minkowski inequality and prove it using Hölder's inequality for the conditional expectation.

Question 3: [10 marks]
For each $n \geqslant 1$, let $\{X_{n,j}, j \geqslant 1\}$ be a sequence of independent random variables. Then $\sup_{j \geqslant 1} |X_{n,j}| \to 0$ in probability as $n \to \infty$ if and only if $\forall \varepsilon > 0$, $\sum_{j=1}^{\infty} P(|X_{n,j}| > \varepsilon) \to 0$ as $n \to \infty$.

Question 4: [20 marks]
(i) Let μ be a probability measure on $\mathbb{R}$ and φ be its characteristic function. Show that

$$\mu(\{a\}) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita} \varphi(t) dt, \quad \forall\, a \in \mathbb{R}.$$

(ii) Let X be a random variable with $P(X \in h\mathbb{Z}) = 1$ for some $h > 0$, where $\mathbb{Z}$ is the integer set. Let φ be the characteristic function of X. Prove that

$$P(X = x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx} \varphi(t) dt \quad \text{for } x \in h\mathbb{Z}.$$

Question 5: [15 marks]
Let $X_1, X_2, \ldots$ be i.i.d. random variables. In statistical problems, likelihood ratios $U_n = \Pi_{i=1}^n g(X_i)/\Pi_{i=1}^n f(X_i)$ are encountered, where f, g are density functions, each being a candidate for the actual density of X_i. If g vanishes whenever f does, show that $\{U_n, n \geqslant 1\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ when f is the true density.

Question 6: [10 marks]
Suppose $\{S_n, \mathcal{F}_n, n \geqslant 1\}$ is a martingale. Prove that for any finite stopping time T,

$$E|S_T| \leqslant \lim_{n\to\infty} E|S_n|.$$

Question 7: [15 marks]
Let $(X_i, Y_i), i \geqslant 1$ be i.i.d. L_2 random vectors with $EX_1 = EY_1 = 0$, and $\mathcal{F}_n = \sigma(X_1, Y_1, \ldots, X_n, Y_n)$, $S_n = \sum_{i=1}^n X_i$, $U_n = \sum_{i=1}^n Y_i$. Prove that for any integrable stopping time T w.r.t. $\{\mathcal{F}_n\}$, the identity $E(S_T U_T) = (ET)(EX_1Y_1)$ holds.

Question 8: [10 marks]
Let $X_n = b^n Y_n, n \geqslant 1, b > 1$, where $\{Y_n\}$ are bounded i.i.d. random variables. Prove that

$$\frac{1}{b_n}\sum_{i=1}^n X_i \to 0 \quad \text{a.s.}$$

provided $b_n/b^n \to \infty$.