STAT5005 Final Exam 2020/21
[Totally 100 marks] (3:30-6:30pm, 10 December 2020)

Instructions:

1. This is an open book examination. You may also look at course materials in a computer or pad. However, searching online or communicating with others is not allowed.
2. Please have a camera open (showing your face and hands) during the exam. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.
3. Totally 6 questions on 2 pages. If you think there is a problem with the question, please state your reason.
4. Please send a private message to the meeting (co-)host in the chatroom to use the toilet or to submit the solution early.
5. You should stop writing at 6:30pm. After finishing, please take a clear picture of your solution and send it to my email (xfang@cuhk.edu.hk). You may cc the email to yourself to keep a record.
6. Please avoid touching your keyboard except when sending the private message in 4. Use hand writing for your solutions.

Question 1: [30 marks]

(i) Suppose X_n converge in distribution to X as $n \to \infty$ and they all have finite second moments. Find a sufficient condition for $E(X_n^2) \to E(X^2)$. Justify your answer.

(ii) Suppose $\{A_n, n \geqslant 1\}$ is a sequence of independent events with $P(A_n) < 1, n \geqslant 1$ and $P(\cup_{n=1}^{\infty} A_n) = 1$. Prove that $P(A_n, i.o.) = 1$.

(iii) Suppose $X_1, X_2, \ldots$ are i.i.d. such that $E(X_1) = 0, E(X_1^2) = \sigma^2 < \infty$. Let $S_n = \sum_{i=1}^n X_i$, $n \geqslant 1$. Find a sequence of real numbers $a_1, a_2, \ldots$ such that $\{S_n^2 + a_n, n \geqslant 1\}$ is a martingale with respect to $\{\mathcal{F}_n = \sigma(X_1, \ldots, X_n), n \geqslant 1\}$. Justify your answer.

Question 2: [20 marks] Let $X_1, X_2, \ldots$ be i.i.d. with $EX_i = 0$. Let $S_n = \sum_{i=1}^n X_i$. Let $1 < p < 2$ be a constant. Prove that $E|X_i|^p < \infty$ if and only if

$$\frac{S_n}{n^{1/p}} \to 0 \quad a.s..$$

Question 3: [15 marks] Let $\{F_n, n \geqslant 1\}$ and F be distribution functions. Show that $F_n \Rightarrow F$ if and only if for any $\epsilon > 0, h > 0$ and $t \in \mathbb{R}$, there exists an N such that for $n > N$,

$$F(t-h) - \epsilon < F_n(t) < F(t+h) + \epsilon$$

Question 4: [15 marks] Prove the following properties of conditional expectation:

(i) For any random variables X and Y with finite second moment and any σ-field $\mathcal{A}$,

$$E\{X[E(Y|\mathcal{A})]\} = E\{[E(X|\mathcal{A})]Y\}.$$

(ii) Suppose X has finite expectation and Z is independent of $\{X, Y\}$. Prove that

$$E(X|Y, Z) = E(X|Y).$$

Question 5: [10 marks] Let $X_1, X_2, \ldots$ be independent with $EX_i = 0, E(X_i^2) < \infty$ for all i. Let $S_n = \sum_{i=1}^n X_i$. Prove that for any $x > 0$, we have

$$P(\max_{1\leqslant k\leqslant n} S_k \geqslant x) \leqslant \frac{E(S_n^2)}{x^2 + E(S_n^2)}.$$

Question 6: [10 marks] Let $X_1, X_2, \ldots$ be i.i.d. such that $P(X_1 = -1) = P(X_1 = 1) = \frac{1}{2}$. Let

$$S_0 = 0, \quad S_k = \sum_{i=1}^k X_i, \ k = 1, 2, \ldots.$$

Let

$$Y_0 = 0, \quad Y_k = (Y_{k-1} + X_k)^+, \ k = 1, 2, \ldots.$$

Prove that for any n, Y_n and $\max_{0\leqslant k\leqslant n} S_k$ have the same distribution.