Department of Statistics, The Chinese University of Hong Kong
STAT5010 Advanced Statistical Inference | Term 1, 2018-19

Final Examination on 5th December 2018

<u>Instruction to the candidates.</u> Please attempt all of the questions. Each problem carries an equal weight of 7 points. Your final score will be capped by 40, which is also the defined full mark of this exam. Good luck!

1. Determine whether or not each of the following statements is correct. If a statement is not always true, you should regard it as a "false" statement. *Note that a correct answer without a proper explanation is considered as a wrong answer. You are required to show clearly your reasoning.*

   (a) Any regular one-parameter exponential family admits a canonical form has a monotone likelihood ratio statistic.

   (b) (i.) For hypothesis $H_0 : \theta = \theta_0$, v.s. $H_1 : \theta = \theta_1$, and a continuous test statistic $T(X)$, the p-value p follows Uniform$[0, 1]$.

   (ii.) If the hypothesis is given as $H_0 : \theta = \theta_0$, v.s. $H_1 : \theta \in \Theta_1$, the p-value p follows Uniform$[0, 1]$.

   (iii.) If the hypothesis is given as $H_0 : \theta \in \Theta_0$, v.s. $H_1 : \theta \in \Theta_1$, the p-value p follows Uniform$[0, 1]$.

2. Suppose that the parameter space Θ is an open subset of $\mathbb{R}^2$, and the risk function $R(\theta, \delta)$ is continuous in θ for every estimator $\delta(X)$. Suppose further that π_n be a sequence of (possibly improper) priors on Θ such that the following conditions hold:

   (i) $r(\pi_n, \delta_0) < \infty$ for all n.

   (ii) For any open set $\Theta_0 \subset \Theta$, one has $\liminf_{n\to\infty} \int_{\Theta_0} \pi_n(\theta) d\theta > 0$, and

   (iii) $\lim_{n\to\infty} r(\pi_n, \delta_0) - r(\pi_n, \delta_{\pi_n}) = 0$,

   where δ_{π_n} is the Bayes estimate with respect to the prior π_n and δ_0 is an estimator. Conclude that δ_0 is admissible without necessarily assuming squared loss error function. *(Hint: If δ_0 is inadmissible, there exists an estimator δ' such that $R(\theta, \delta_0) \geq R(\theta, \delta')$, $R(\theta_0, \delta_0) > R(\theta_0, \delta')$ for some $\theta_0 \in \Theta$.)*

3. Find the asymptotic distribution of $\ell_n(\theta_0 + hn^{-1/2}) - \ell_n(\theta_0)$, where $h \in \mathbb{R}$ is fixed and $\ell_n(\theta)$ is the log likelihood function of a random sample $X_1, \ldots, X_n$ evaulated at θ. State also the regularity conditions needed.

4. Let $X_1, \ldots, X_n$ be i.i.d. random variables, having the exponential distribution with density $f(x; \lambda) = \lambda \exp\{-\lambda$ for $x, \lambda > 0$.

   (a) Show that $T_n = \sum_{i=1}^n X_i$ is minimal sufficient and complete for λ.

   (b) Furthermore, for given $x > 0$, it is desired to estimate the quantity $\phi = \Pr(X_1 > x; \lambda)$. Compute the Fisher information for ϕ.

(c) State the Lehmann-Scheffé theorem which ties together the ideas of completeness, sufficiency, uniqueness, and best unbiased estimation. Show that the estimator $\tilde{\phi}_n$ of ϕ defined by

$$\tilde{\phi}_n = \begin{cases} 0, & \text{if } T_n < x \\ \left(1 - \frac{x}{T_n}\right)^{n-1}, & \text{if } T_n \geq x \end{cases}$$

is the minimum variance unbiased estimator of ϕ based on $(X_1, \ldots, X_n)$. Without conducting any computations, state whether or not the variance of $\tilde{\phi}_n$ achieves the Cramér-Rao lower bound with brief justification. Show also that $E(\tilde{\phi}_k \mid T_n, \lambda) = \tilde{\phi}_n$ for $k \leq n$.

5. Let $(X_1, Y_1), \ldots (X_n, Y_n)$ be i.i.d. with $X_i \sim N(0,1)$ and $Y_i \mid X_i = x \sim N(x\theta, 1)$.

   (a) Find the maximum likelihood estimator $\hat{\theta}$ of θ.

   (b) Determine the limiting distribution of $n^{1/2}(\hat{\theta} - \theta)$.

   (c) Construct $1 - \alpha$ asymptotic confidence interval for θ based on $I(\hat{\theta})$, where $I(\theta)$ denotes the Fisher information for a single observation (X_1, Y_1).

   (d) Determine the exact distribution of $\sqrt{\sum_{i=1}^n X_i^2}(\hat{\theta} - \theta)$ and use it find the true coverage probability for the interval constructed based on observed Fisher information.

6. Let $X_1, \ldots, X_n$ be a random sample from a population on $\mathbb{R}$ with Lebesgue density f_θ. Let θ_0 and θ_1 be two constants with $\theta_0 < \theta_1$. Find a uniform most power test of size α for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$ for $f_\theta(x) = \exp\{-(x - \theta)\} I(x > \theta)$.

7. Let $X_1, \ldots, X_n \overset{i.i.d.}{\sim} N(\mu, \sigma_0^2)$ with σ_0^2 known. Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$. Show that a UMP test does *not* exist for any size $\alpha \in (0, 1)$. *(Hint: You may consider a one-sided test first. Check its power for another side of the test from which you may argue over the power of this particular one-sided UMP test.)*

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