Qualify Exam. June 27, 2013

Problem from STAT5010

1. Suppose $X_1, X_2, \cdots, X_n$ is an i.i.d. sample from the distribution function

$$F_\theta(x) = \exp[-\exp(-(x-\theta))]$$

   (a) Verify that $F_\theta(x)$ is a distribution function for any θ.

   (b) Find the density function of $F_\theta(x)$.

   (c) Find a minimal sufficient statistic for θ.

   (d) Find a UMP level α test for $H_0 : \theta \geq 0$ VS. $H_1 : \theta < 0$. State clearly which theorem you have used.

   (e) Invert the test in d) to obtain an $(1-\alpha)$ confidence interval.

$= -\ln U_i$

$\epsilon_i) = e^{-e^{-\epsilon_i}} e^{-\epsilon_i}$

$u_i) = e^{-u_i} \, y_i \left|\frac{1}{u_i}\right|$

$= e^{-u_i}$

$\sim Exp(1)$

2. Suppose we have four random variables $Y_i, i = 1, 2, 3, 4$, following the model $Y_i = i\theta + \varepsilon_i, i = 1, \cdots, 4$ where $\varepsilon_i, i = 1, \cdots, 4$ are i.i.d. with the distribution function

$$F(u) = \exp\{-e^{-u}\}, \ -\infty < u < \infty.$$

ired Prob $P =$

$+e^{-\theta})^{-1}[(1+e^{-\theta})e^{-\theta}+1]^{-1}$

$[(1+e^{-\theta})e^{-\theta}+1]e^{-\theta}+1]^{-1}$

   (a) Calculate the probability $P_r\{Y_1 < Y_2 < Y_3 < Y_4\}$ (Hint: Let $U_i = e^{-\varepsilon_i}$, then U_i are i.i.d. following the standard exponential distribution and the required probability becomes

$\frac{\partial P}{\partial \theta} \triangleq 0 \Rightarrow$

Variance $= 0$

$$P_r\{e^{-\theta}U_1 > e^{-2\theta}U_2 > e^{-3\theta}U_3 > e^{-4\theta}U_4\}$$

$$= \int_0^\infty f_{u_4}(u_4)du_4 \int_{e^{-\theta}u_4}^\infty f_{u_3}(u_3)du_3 \int_{e^{-\theta}u_3}^\infty Pr\{U_1 > e^{-\theta}u_2\} f_{u_2}(u_2)du_2\,)$$

$\int_{e^{-\theta}u_2}^\infty e^{-u_1}du_1 = e^{-e^{-\theta}u_2}$ $\quad \int_{e^{\theta}u_3}^\infty e^{-[e^{-\theta}+1]u_2} du_2$

$\frac{1}{(1+e^{-\theta})(e^{-\theta}+1)e^{-\theta}+1)} e^{-[(e^{-\theta}+1)e^{-\theta}+1]e^{-\theta}u_4}$ $\quad \leftarrow \frac{1}{1+e^{-\theta}} e^{-(e^{-\theta}+1)e^{-\theta}u_3}$

$P(\theta=0) = \frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4} = \frac{1}{24}$

Reject H_0 if $RT = \frac{P(\theta=0)}{P(\theta=\hat\theta)}$ is small

   (b) Suppose that the event $\{Y_1 < Y_2 < Y_3 < Y_4\}$ is observed, what is your estimate of θ, and how do you estimate its variance?

   (c) How will you test the hypothesis $H_0 : \theta = 0$ VS. $H_a : \theta \neq 0$?

(a) $\lim_{x\to-\infty} F_\theta(x) = \lim_{x\to-\infty} e^{-e^{-x+\theta}} = 0$, $\lim_{x\to\infty} F_\theta(x) = \lim_{x\to\infty} e^{-e^{-x+\theta}} = 1$, cont. func. $\frac{\partial F_\theta}{\partial x} = e^{-e^{-x+\theta}} e^{-x+\theta} \geq 0$

(b) $f_\theta(x) = e^{-e^{-x+\theta}} e^{-x+\theta}$, $x \in \mathbb{R}$

(c) $f(\vec{x}|\theta) = e^{-e^{\theta}\sum e^{-x_i}} \cdot e^{-\sum x_i + n\theta}$, $\sum e^{-x_i}$ is complete and sufficient stat. thus minimal.

(d) $w(\theta) = -e^{\theta}$; $w'(\theta) = e^{\theta} > 0$, $w(\theta) \uparrow$, $f_\theta(x)$ is MLR

By Karlin-Rubin theorem, Reject H_0 iff $\sum e^{-x_i} < t_0$, $\alpha = P_{\theta=0}\{\sum e^{-x_i} < t_0\}$

let $y = e^{-x}$, $x = -\ln y$, $f_\theta(y) = e^{-e^{\theta}y} e^{\theta} y \cdot |\frac{1}{y}| = e^{\theta} e^{-e^{\theta}y} \sim Exp(e^{-\theta})$

$\sum_{i=1}^n y_i \sim Gamma(n, e^{-\theta})$ therefore $t_0 = Gamma(n, 1, \alpha)$

(e) $P(e^{\theta}\sum e^{-x_i} < t_0) = \alpha \Rightarrow P(\sum e^{-x_i} > t_0 e^{-\theta}) = 1-\alpha \Rightarrow \theta < \ln \frac{\sum e^{-x_i}}{t_0}$