

STAT5030 Assignment 2

Due: Feb 16, 2023

1. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, \dots, Y_n)^\top \sim N(\alpha \mathbf{1}, \sigma^2 \mathbf{I})$. Define $U = \sum_{i=1}^n (Y_i - \bar{Y})^2 / \sigma^2$ and $V = n(\bar{Y} - \alpha)^2 / \sigma^2$. Find the distributions of U and V , and show that these two random variables are independent.
2. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, \dots, Y_n)^\top \sim N(\mu \mathbf{1}, \mathbf{I})$. Let

$$\begin{aligned}\bar{Y} &= \frac{\sum_{i=1}^n Y_i}{n}, \\ Q_1 &= n\bar{Y}^2, \\ Q_2 &= \sum_{i=1}^n (Y_i - \bar{Y})^2.\end{aligned}$$

- (a) Prove that \bar{Y} and Q_2 are independent.
 - (b) Prove that Q_1 and Q_2 are independent.
 - (c) Find the distributions of Q_1 and Q_2 .
3. Let the $n \times 1$ vector $\mathbf{Y} = (Y_1, \dots, Y_n)^\top \sim N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. Let $q_1 = \mathbf{Y}^\top \mathbf{A}_1 \mathbf{Y}$, $q_2 = \mathbf{Y}^\top \mathbf{A}_2 \mathbf{Y}$ and $\mathbf{T} = \mathbf{B} \mathbf{Y}$, where \mathbf{B} is $r \times n$ and $\mathbf{A}_1, \mathbf{A}_2$ are symmetric. Prove the followings:
 - (a) $E(q_1) = \text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma}) + \boldsymbol{\mu}^\top \mathbf{A}_1 \boldsymbol{\mu}$.
 - (b) $\text{Var}(q_1) = 2\text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_1 \boldsymbol{\Sigma}) + 4\boldsymbol{\mu}^\top \mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_1 \boldsymbol{\mu}$.
 - (c) $\text{Cov}(q_1, q_2) = 2\text{tr}(\mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\Sigma}) + 4\boldsymbol{\mu}^\top \mathbf{A}_1 \boldsymbol{\Sigma} \mathbf{A}_2 \boldsymbol{\mu}$.
 - (d) $\text{Cov}(\mathbf{Y}, q_1) = 2\boldsymbol{\Sigma} \mathbf{A}_1 \boldsymbol{\mu}$.
 - (e) $\text{Cov}(\mathbf{T}, q_1) = 2\mathbf{B} \boldsymbol{\Sigma} \mathbf{A}_1 \boldsymbol{\mu}$.
4. Suppose \mathbf{y} is $N_3(\boldsymbol{\mu}, \sigma^2 \mathbf{I})$ and let $\boldsymbol{\mu}^\top = [3, -2, 1]$ and

$$\begin{aligned}\mathbf{A} &= \frac{1}{3} \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}, \\ \mathbf{B} &= \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix},\end{aligned}$$

- (a) Find the distribution of $\mathbf{y}^\top \mathbf{A} \mathbf{y} / \sigma^2$.
 - (b) Are $\mathbf{y}^\top \mathbf{A} \mathbf{y}$ and $\mathbf{B} \mathbf{y}$ independent?
 - (c) Are $\mathbf{y}^\top \mathbf{A} \mathbf{y}$ and $y_1 + y_2 + y_3$ independent?

5. X_1, X_2, X_3, X_4 be a random sample of size 4 from a distribution which is $N(0, \sigma^2)$. Let $Q = X_1X_2 - X_3X_4$. Does Q/σ^2 has a χ^2 distribution? Justify your answer.
6. Suppose \mathbf{y} is $N_n(\mu\mathbf{1}, \Sigma)$ where

$$\Sigma = \sigma^2 \begin{pmatrix} 1 & \rho & \cdots & \rho \\ \rho & 1 & \cdots & \rho \\ \vdots & \vdots & & \vdots \\ \rho & \rho & \cdots & 1 \end{pmatrix}.$$

Derive the distribution of

$$\frac{\sum_{i=1}^n (y_i - \bar{y})^2}{\sigma^2(1 - \rho)}.$$

7. Let $\mathbf{Y} = (Y_1, Y_2, Y_3)^\top$. $E(\mathbf{Y}) = (2, 3, 4)^\top$ and the covariance matrix of \mathbf{Y} is

$$\Sigma = \begin{pmatrix} 2 & 0 & -1 \\ 0 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix}.$$

Let $U = \sum_{i=1}^3 (Y_i - \bar{Y})^2$. Find the expected value of U .

8. Let $\mathbf{Y} = (Y_1, \dots, Y_n)^\top$ with $E(\mathbf{Y}) = \mu\mathbf{1}$ and the covariance matrix of \mathbf{Y} be $\sigma^2\mathbf{I}$. Let

$$U = \sum_{i < j} (Y_i - Y_j)^2.$$

- (a) Find the expected value of U .
- (b) Find a constant k such that kU is an unbiased estimator of σ^2 .
9. Let Y_1, \dots, Y_n be random variables with equal means μ , variances σ^2 and covariances $\rho\sigma^2$. Assume ρ known. Let

$$U = \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

Find a constant k such that kU is an unbiased estimator of σ^2 .