STAT5030 Assignment1

Due: January 30, 2023

1. Prove that rank($\boldsymbol{x}$)=rank($\boldsymbol{x}^\top \boldsymbol{x}$).

2. Prove (or disprove) the following:

   (a) If $\boldsymbol{P}\boldsymbol{X}\boldsymbol{X}^\top \boldsymbol{P}^\top = \boldsymbol{Q}\boldsymbol{X}\boldsymbol{X}^\top \boldsymbol{Q}^\top$, then $\boldsymbol{P}\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{X}$.

   (b) If $\boldsymbol{P}\boldsymbol{X}\boldsymbol{X}^\top = \boldsymbol{Q}\boldsymbol{X}\boldsymbol{X}^\top$, then $\boldsymbol{P}\boldsymbol{X} = \boldsymbol{Q}\boldsymbol{X}$.

3. Suppose that $\boldsymbol{A}$ is $n \times p$ of rank r and that $\boldsymbol{A}$ is partitioned as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} \end{bmatrix}.$$

   where $\boldsymbol{A}_{11}$ as $r \times r$ of rank r. Prove that a generalized inverse of $\boldsymbol{A}$ is given by

$$\boldsymbol{A}^- = \begin{bmatrix} \boldsymbol{A}_{11}^- & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Let

$$\boldsymbol{A} = \begin{bmatrix} 2 & 6 \\ 1 & 3 \\ 3 & 9 \\ 5 & 15 \end{bmatrix},$$

   (a) Find the Moore-Penrose inverse.

   (b) Find a generalized inverse different from Moore-penrose inverse.

5. Let $\boldsymbol{X}_{n\times p}$ be a matrix of rank k and the first column is vector with all elements equal to 1. Further more, $\boldsymbol{J}$ is a matrix with all elements 1. Let

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{x}(\boldsymbol{x}^\top \boldsymbol{x})^- \boldsymbol{x}^\top, \\ \boldsymbol{B} &= \boldsymbol{I}_n - \boldsymbol{x}(\boldsymbol{x}^\top \boldsymbol{x})^- \boldsymbol{x}^\top, \\ \boldsymbol{C} &= \boldsymbol{x}(\boldsymbol{x}^\top \boldsymbol{x})^- \boldsymbol{x}^\top - \tfrac{1}{n}\boldsymbol{J}, \\ \boldsymbol{D} &= \boldsymbol{I}_n - \tfrac{1}{n}\boldsymbol{J}. \end{aligned}$$

   (a) Prove that $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$ are symmetric and idempotent.

   (b) Find the rank of $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ and $\boldsymbol{D}$.

6. Let

$$\boldsymbol{A} = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 2 & 0 \\ 2 & 0 & 2 \end{pmatrix},$$

(a) Find a symmetric generalized inverse for $\boldsymbol{A}$.

(b) Find a nonsymmetric generalized inverse for $\boldsymbol{A}$.

7. Prove that the system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ has at least one solution vector $\boldsymbol{x}$ if and only if rank($\boldsymbol{A}$)=rank($\boldsymbol{A}, \boldsymbol{c}$).

8. Prove that the system of equations $\boldsymbol{Ax} = \boldsymbol{c}$ has a solution if and only of for any generalized inverse $\boldsymbol{A}^-$ of $\boldsymbol{A}$

$$\boldsymbol{AA}^-\boldsymbol{c} = \boldsymbol{c}.$$

9. Prove that if $\boldsymbol{A}$ is $n \times p$ of rank $p < n$, then $\boldsymbol{A}^-$ is a left inverse of $\boldsymbol{A}$, namely, $\boldsymbol{A}^-\boldsymbol{A} = \boldsymbol{I}$

10. Let $\boldsymbol{X}$ be $m \times n$, $\boldsymbol{X}^-$is the corresponding generalized inverse, and $r(\boldsymbol{X}) = k > 0$. Prove that:

(a) $r(\boldsymbol{X}^-) \geq k$.

(b) $\boldsymbol{X}^-\boldsymbol{X}$ and $\boldsymbol{X}\boldsymbol{X}^-$ are idempotent.

(c) $r(\boldsymbol{X}^-\boldsymbol{X}) = r(\boldsymbol{X}\boldsymbol{X}^-) = k$.

(d) $\boldsymbol{X}^-\boldsymbol{X} = \boldsymbol{I}$ if and only if $r(\boldsymbol{X}) = n$.

(e) $tr(\boldsymbol{X}^-\boldsymbol{X}) = tr(\boldsymbol{X}\boldsymbol{X}^-) = k = r(\boldsymbol{X})$.

(f) If $\boldsymbol{X}^-$ is any G-inverse of $\boldsymbol{X}$, then $(\boldsymbol{X}^-)^\top$ is a G-inverse of $\boldsymbol{X}^\top$.

11. For $\boldsymbol{K} = \boldsymbol{X}(\boldsymbol{X}^\top\boldsymbol{X})^-\boldsymbol{X}^\top$, prove that:

(a) $\boldsymbol{K} = \boldsymbol{K}^\top$, $\boldsymbol{K} = \boldsymbol{K}^2$(Symmetric Idempotent).

(b) rank($\boldsymbol{K}$)=rank($\boldsymbol{X}$)=r.(rank($\boldsymbol{K}$)=tr($\boldsymbol{K}$)=rank($\boldsymbol{X}$))

(c) $\boldsymbol{KX} = \boldsymbol{X}$. $(\boldsymbol{X}^\top\boldsymbol{K} = \boldsymbol{X}^\top)$

(d) $(\boldsymbol{X}^\top\boldsymbol{X})^-\boldsymbol{X}^\top$ is a G-inverse of $\boldsymbol{X}$ for any G-inverse of $\boldsymbol{X}^\top\boldsymbol{X}$.

12. Prove the properties:

(a) The Moore-Penrose inverse is unique.

(b) $r(\boldsymbol{A}^+) = r(\boldsymbol{A})$.

(c) If $\boldsymbol{A}$ is symmetric idempotent, $\boldsymbol{A}^+ = \boldsymbol{A}$.