STAT5005 Final Exam 2019/20
[Totally 100 marks] (2:30-6:30pm, 5 December 2019)

Instructions:

1. This is an open book examination.
2. You are required to work independently and should not discuss with others. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.
3. After finishing, please take a clear picture of your solution and send it to my email (xfang@cuhk.edu.hk). The deadline is 6:30pm on December 5.
4. Totally 8 questions on 2 pages. If you think there is a problem with the question, please state your reason.

Question 1: [10 marks]
Prove that if $\{X_n, n \geqslant 1\}$ are i.i.d. random variables with $P(X_1 = 0) < 1$ and $S_n = \sum_{i=1}^n X_i, n \geqslant 1$, then for every $c > 0$ there exists an integer $n = n_c$ such that $P(|S_n| > c) > 0$.

Question 2: [10 marks]
State the conditional Minkowski inequality and prove it using Hölder's inequality for the conditional expectation.

Question 3: [10 marks]
For each $n \geqslant 1$, let $\{X_{n,j}, j \geqslant 1\}$ be a sequence of independent random variables. Then $\sup_{j\geqslant 1} |X_{n,j}| \to 0$ in probability as $n \to \infty$ if and only if $\forall \varepsilon > 0$, $\sum_{j=1}^{\infty} P(|X_{n,j}| > \varepsilon) \to 0$ as $n \to \infty$.

Question 4: [20 marks]
(i) Let μ be a probability measure on $\mathbb{R}$ and φ be its characteristic function. Show that

$$\mu(\{a\}) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt, \quad \forall\, a \in \mathbb{R}.$$

(ii) Let X be a random variable with $P(X \in h\mathbb{Z}) = 1$ for some $h > 0$, where $\mathbb{Z}$ is the integer set. Let φ be the characteristic function of X. Prove that

$$P(X = x) = \frac{h}{2\pi} \int_{-\pi/h}^{\pi/h} e^{-itx}\varphi(t)dt \quad \text{for } x \in h\mathbb{Z}.$$

Question 5: [15 marks]
Let $X_1, X_2, \ldots$ be i.i.d. random variables. In statistical problems, likelihood ratios $U_n = \Pi_{i=1}^n g(X_i)/\Pi_{i=1}^n f(X_i)$ are encountered, where f, g are density functions, each being a candidate for the actual density of X_i. If g vanishes whenever f does, show that $\{U_n, n \geqslant 1\}$ is a martingale with respect to $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ when f is the true density.

Question 6: [10 marks]
Suppose $\{S_n, \mathcal{F}_n, n \geqslant 1\}$ is a martingale. Prove that for any finite stopping time T,

$$E|S_T| \leqslant \lim_{n\to\infty} E|S_n|.$$

Question 7: [15 marks]
Let $(X_i, Y_i), i \geqslant 1$ be i.i.d. L_2 random vectors with $EX_1 = EY_1 = 0$, and $\mathcal{F}_n = \sigma(X_1, Y_1, \ldots, X_n, Y_n)$, $S_n = \sum_{i=1}^n X_i$, $U_n = \sum_{i=1}^n Y_i$. Prove that for any integrable stopping time T w.r.t. $\{\mathcal{F}_n\}$, the identity $E(S_T U_T) = (ET)(EX_1Y_1)$ holds.

Question 8: [10 marks]
Let $X_n = b^n Y_n, n \geqslant 1, b > 1$, where $\{Y_n\}$ are bounded i.i.d. random variables. Prove that

$$\frac{1}{b_n}\sum_{i=1}^n X_i \to 0 \quad \text{a.s.}$$

provided $b_n/b^n \to \infty$.

Final 2016

March 10, 2019

STAT 5005

1. (1) $\pi - \lambda$ Theorem

   (2) $\mathcal{A}_1, \mathcal{A}_2$ are independent, and π-class. Prove: $\sigma(\mathcal{A}_1), \sigma(\mathcal{A}_2)$ are independent.

   (3) $X \in \mathcal{F}$, and $E|X| < \infty$, $E|XY| < \infty$. Prove:

   $$E(XY|\mathcal{F}) = XE(Y|\mathcal{F}).$$

   Hint: indicator method, suppose that X is indicator r.v., simple r.v., nonnegtive r.v. and general r.v. respectively.

2. Marcinkiewicz and Zygmund Theorm P72. Hint: $E|X| \leq \frac{1}{p}E|X|^p$

3. (1) Z has a standard normal distribution, to prove:

   $$P(|Z| \geq x) \leq e^{-\frac{x^2}{2}}$$

   (2) $X_1, X_2, \cdots \sim N(0,1)$ *i.i.d.*, $S_n = \sum_{i=1}^n X_i$, to prove:

   $$P(\max_{1 \leq k \leq n} |S_k| > x) \leq 2P(|S_n| > x)$$

   Hint:

   $$\begin{aligned} & \{\max_{1\leq k\leq n} |S_k| > x)\} = \cup_{k=1}^n \{|S_k| > x, |S_j| \leq x, j < k\} \\ & E1_{(S_k > x, S_j \leq x, j<k)} 1_{(|S_n| \geq x)} \\ \leq \; & E1_{(S_k > x, S_j \leq x, j<k)} 1_{(S_n - S_k < 0)} \\ \leq \; & E1_{(S_k > x, S_j \leq x, j<k)} E1_{(S_n - S_k < 0)} \\ = \; & E1_{(S_k > x, S_j \leq x, j<k)} E1_{(S_n - S_k \geq 0)} (By\ symmetric) \end{aligned}$$

   and

   $$\limsup_n \frac{S_n}{\sqrt{2nloglogn}} \leq 1 \; a.s.$$

Hint:

$$
\begin{aligned}
& \limsup_n \frac{S_n}{\sqrt{2nloglogn}} \le 1 \; a.s \\
\Leftarrow \quad & \forall \varepsilon > 0, P(\frac{S_n}{\sqrt{2nloglogn}} \ge 1+\varepsilon \; i.o.) = 0 \\
\Leftarrow \quad & \forall \varepsilon > 0, \sum_{n=1}^{\infty} P(\frac{S_n}{\sqrt{2nloglogn}} \ge 1+\varepsilon) < \infty \\
n = \theta^k, \theta > 1 \quad , \quad & \sum_{k=1}^{\infty} P(S_{\theta^k} \ge (1+\varepsilon)\sqrt{2\theta^k loglog(\theta^k)}) \\
= \quad & \sum_{k=1}^{\infty} exp(-(1+\varepsilon)^2 loglog(\theta^k)) \\
= \quad & \sum_{k=1}^{\infty} \frac{1}{[klog(\theta)]^{-(1+\varepsilon)^2}} < \infty \\
& P(\max_{0<n\le\theta^k} S_n \ge (1+\varepsilon)\sqrt{2\theta^k loglog(\theta^k)}) \\
\le \quad & 2P(S_{\theta^k} \ge (1+\varepsilon)\sqrt{2\theta^k loglog(\theta^k)}) \\
\Rightarrow \quad & \lim_k \max_{0<n\le\theta^k} \frac{S_n}{\sqrt{2\theta^k loglog(\theta^k)}} \le 1 \; a.s. \\
a(j) = \sqrt{2jloglog(j)} \quad , \quad & \frac{S_n}{a(n)} = \frac{S_n}{a(\theta^k)} \frac{a(\theta^k)}{\theta^k} \frac{\theta^k}{n} \frac{n}{a(n)} \\
\le \quad & a(1+\varepsilon) \; for \; \theta^{k-1} < n \le \theta^k \\
\Rightarrow \quad & P(\limsup_n \frac{S_n}{a(n)} \ge \theta(1+\varepsilon)) = 0 \\
\Rightarrow \quad & \limsup_n \frac{S_n}{a(n)} \le 1 \; a.s.
\end{aligned}
$$

4. (1) X_i ~Cauchy distribution, the density function is

$$p(x) = \frac{1}{\pi(1+x^2)},$$

To find the limit distribution of

$$\frac{\sum_{i=1}^n X_i}{n}.$$

(2) X_i are independent,

$$P(X_k = \pm 1) = \frac{1}{2} - \frac{1}{2k^2}, P(X_k = \pm k) = \frac{1}{2k^2}, B_n = \sum_{i=1}^n EX_i^2$$

To prove:

$$\frac{S_n}{B_n} \xrightarrow{d} N(0,1)$$

5. S_n is submartingale, τ is stopping time, to prove

   (1)
   $$E(S_{\tau\wedge n}) \le ES_n$$

   (2)
   $$P(\max_k S_k > x) \le E(|S_n| 1(\max_k S_k > x))$$

STAT 5030

1. Prove XGX^T is invariant of generalized inverse G of $X^T X$

2. Consider the model

   $$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, i = 1,2,3,4, j = 1,2,3,4,$$

   where ε_{ij} are independently distributed as $N(0, \sigma^2)$.

   (1) Let $\beta = (\mu, \tau_1, \tau_2, \tau_3, \tau_4)'$. Find a set of 4 linearly independent estimable functions of β.

   (2) Derive a test to test the null hypothesis $H_0 : \tau_1 - \tau_2 = \tau_3 - \tau_4$.

   (3) Is $\tau_1 + 2\tau_2$ estimable? Why?

3. There are two groups of data (Y_1, X_1), (Y_2, X_2) with

   $$Y_1 = X_1\beta + \varepsilon_1,$$

   $$Y_2 = X_2\beta + \varepsilon_2,$$

   X_1 and X_2 is not necessarily full-rank. And suppose that $\lambda^T\beta$ is estimable.

   (1) T_1 and T_2 are BLUE of $\lambda^T\beta$ for data (Y_1, X_1) and (Y_2, X_2), respectively. Give T_1 and T_2 and calculate $Var(T_1)$ and $Var(T_2)$

   (2) Let $T(\alpha) = \alpha T_1 + (1-\alpha)T_2$. Find α to minimize $Var(T(\alpha))$

   (3) Let $Y = (Y_1^T, Y_2^T)^T, X = (X_1^T, X_2^T)^T$, give the BLUE T_3 of $\lambda^T\beta$ for data (Y, X). And calculate $Var(T_3)$.

   (4) Explain $Var(T_3) \le Var(T(\alpha))$ with equality when $r(X_1) = 1$ or $r(X_2) = 1$

Some exercises of STAT 5005

Notation: $S_n = X_1 + \cdots + X_n$.

1. (i) If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that contains $\mathcal{P}$ then $\sigma(\mathcal{P}) \subset \mathcal{L}$.
   (ii) Let $\mathcal{P}$ be a π-system. If ν_1 and ν_2 are measures satisfying
   $$\nu_1(A) = \nu_2(A), \quad \forall A \in \mathcal{P}$$
   and there exists a sequence $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$ and $\nu_1(A_n) < \infty, \nu_2(A_n) < \infty$, then ν_1 and ν_2 agree on $\sigma(\mathcal{P})$.

2. Suppose $X_1, \ldots, X_n$ are independent with $EX_i = 0$ and $\mathrm{var}(X_i) < \infty$. Then
   $$P\left(\max_{1\le k\le n} |S_k| \ge x\right) \le \frac{\mathrm{var}(S_n)}{x^2}$$
   Furthermore, let X_m be s submartingale, for any $\lambda > 0$ define $A = \{\max_{0\le m\le n} X_m^+\}$. Then
   $$\lambda P(A) \le EX_n I(A) \le EX_n^+.$$

3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables, then
   $$E|X_1| < \infty \Rightarrow \tfrac{S_n}{n} \to EX_1, a.s.$$
   $$E|X_1| = \infty \Rightarrow \limsup_{n\to\infty} \tfrac{|S_n|}{n} = \infty, a.s.$$

3′. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed random variables with $E|X_1| = \infty$. Let $\{a_n\}$ be a sequence of positive numbers satisfying the condition $a_n/n \uparrow$. Then we have
   $$\limsup_n |S_n|/a_n = 0 \text{ a.s., or } = \infty \text{ a.s.}$$
   according as
   $$\sum_n P(|X_n| \ge a_n) < \infty \text{ or } = \infty$$

4. If $X_1, X_2, \ldots$ are independent and identically distributed sysmetric r.v.'s then for every $x \ge 0$,
   $$P(|S_n| > x) \ge \frac{1}{2} P(\max_{1\le k\le n} |X_k| > x) \ge \frac{1}{2}[1 - e^{-nP(|X_1|>x)}].$$

4′. If $X_1, X_2, \ldots$ are independent and symmetric, try to prove for any $x > 0$,
   $$P(\max_{1\le i\le n} |S_i| > x) \le 2P(|S_n| > x).$$

5. Let $\{X_n\}$ be a sequence of independent zero-mean random variables. Then
   $$\limsup_n \frac{S_n}{\sqrt{2s_n^2 \log\log s_n}} = 1, a.s.$$
   where $s_n = \mathrm{var}(S_n)$.

6. Show that $X_n \xrightarrow{d} X$ if and only if $Ef(X_n) \to Ef(X)$ for all bounded continuous function f.

7. Let $X_1, \ldots, X_n$ are independent and identically distributed exponential random variables with mean 1. Try to find the limit distribution of $\sum_{i=1}^n I(X_i S_n > 1)$ as $n \to \infty$.

Some exercise answer

Chaojie Wang

June 8, 2015

1.(a) refer to π-λ theorem proof.

(b) Let $l(\mathcal{P})$ be the smallest λ-system containing $\mathcal{P}$. Using (i), then $\sigma(\mathcal{P}) \subset l(\mathcal{P})$. It suffices to prove ν_1 and ν_2 agree on $l(\mathcal{P})$. Let $\mathcal{A} = \{A : \nu_1(A) = \nu_2(A)\}$. It suffices to prove $l(\mathcal{P}) \subset \mathcal{A}$, that is $\mathcal{A}$ is λ-system containing $\mathcal{P}$.

Since for any $A \in \mathcal{P}$, then $\nu_1(A) = \nu_2(A)$. So $\mathcal{P} \subset \mathcal{A}$. Then check $\mathcal{A}$ is λ-system.

(i) Since there exists a sequence $A_n \in \mathcal{P}$ with $A_n \uparrow \Omega$, then

$$\nu_1(\Omega) = \lim_{n\to\infty} \nu_1(A_n) = \lim_{n\to\infty} \nu_2(A_n) = \nu_2(\Omega)$$

Hence $\Omega \in \mathcal{A}$.

(ii) If $A \in \mathcal{A}$, then $\nu_1(A) = \nu_2(A)$. So

$$\nu_1(A^c) = \nu_1(\Omega) - \nu_1(A) = \nu_2(\Omega) - \nu_2(A) = \nu_2(A^c)$$

Hence $A^c \in \mathcal{A}$.

(iii) If a disjoint countable sequence $A_i \in \mathcal{A}$, then

$$\nu_1(\cup_i A_i) = \sum_i \nu_1(A_i) = \sum_i \nu_2(A_i) = \nu_2(\cup_i A_i)$$

Hence $\cup_i A_i \in \mathcal{A}$. Above all, $\mathcal{A}$ is λ-system. Proven.

2. Since $EX_i = 0$, then $Var(S_n) = ES_n^2$. Let $A_k = \{|S_k| \geq x \ but \ |S_j| < x \ for \ j < k\}$,

$$\begin{aligned}
ES_n^2 = \int S_n^2 dP &\geq \sum_{k=1}^n \int_{A_k} ES_n^2 dP = \sum_{k=1}^n \int_{A_k} E(S_n - S_k + S_k)^2 dP \\
&= \sum_{k=1}^n \int_{A_k} E(S_n - S_k)^2 + ES_k^2 + 2E(S_n - S_k)S_k dP \\
&\geq \sum_{k=1}^n \int_{A_k} ES_k^2 + 2E(S_n - S_k)S_k dP
\end{aligned}$$

Since $S_n - S_k$ and S_k are independent, then $E(S_n - S_k)S_k = E(S_n - S_k)ES_k = 0$. Thus,

$$ES_n^2 \geq \sum_{k=1}^{n} \int_{A_k} ES_k^2 dP \geq \sum_{k=1}^{n} \int_{A_k} x^2 dP = x^2 \sum_{k=1}^{n} P(A_k)$$
$$= x^2 P(\cup_{k=1}^{n} A_k) = x^2 P(\max_{1\leq k\leq n} |S_k| \geq x)$$

Thus, $P(\max_{1\leq k\leq n} |S_k| \geq x) \leq Var(S_n)/x^2$.

Let $N = \inf\{m : X_m \geq \lambda \ or \ m = n\}$. Since $X_N \geq \lambda$ on A, then $\lambda P(A) \leq EX_N 1_A$. Since $X_N \leq X_n$ on A and $X_N = X_n$ on A^c. then $EX_N 1_A \leq EX_n 1_A$.

And $EX_n 1_A \leq EX_n^+$ is trivial, thus $\lambda P(A) \leq EX_n 1(A) \leq EX_n^+$.

3. If $E|X_1| < \infty$, refer to strong law of large numbers. If $E|X_1| = \infty$, let $X_n^M = X_n \wedge M$, and $S_n^M = X_1^M + \cdots + X_n^M$, since $E|X_n^M| < \infty$, using strong law of large numbers,

$$\limsup_{n\to} \frac{|S_n|}{n} \geq \limsup_{n\to} \frac{S_n}{n} \geq \lim_{n\to} \frac{S_n^M}{n} = EX_1^M$$

Since $E|X_1| = \infty$, without loss of generality, assume $EX_1^+ = \infty$ and $EX_1^- < \infty$.

As $M \uparrow \infty$, $EX_1^M = E(X_1^M)^+ - E(X_1^M)^- \uparrow \infty$. Thus, $\limsup_{n\to} |S_n|/n = \infty$.

3'. If $\sum_n P(X_n \geq a_n) = \infty$, since $a_n/n \uparrow$, then $a_{kn} \geq ka_n$ for any integer k. Thus,

$$\sum_{n=1}^{\infty} P(|X_1| \geq ka_n) \geq \sum_{n=1}^{\infty} P(|X_1| \geq a_{kn}) \geq \frac{1}{k} \sum_{m=k}^{\infty} P(|X_1| \geq a_m) = \infty.$$

Thus, $\limsup_n |X_n|/a_n = \infty$ a.s.

And $\max\{|S_n|, |S_{n-1}|\} \geq |X_n|/2$, then $\limsup_n |S_n|/a_n = \infty$ a.s.

If $\sum_n P(X_n \geq a_n) < \infty$, note that

$$\sum_{m=1}^{\infty} mP(a_{m-1} \leq |X_i| < a_m) = \sum_{n=1}^{\infty} P(|X_i| \geq a_{n-1})$$

Let $Y_n = X_n 1_{|X_n|<a_n}$. Since $\sum_n P(Y_n \neq X_n) = \sum_n P(|X_n| \geq a_n) < \infty$, then $P(Y_n \neq X_n \ i.o.) = 0$. Thus, it suffices to investigate T_n. Let $a_0 = 0$, then

$$\sum_{n=1}^{\infty} Var(Y_n/a_n) \leq \sum_{n=1}^{\infty} EY_n^2/a_n^2 = \sum_{n=1}^{\infty} a_n^{-2} \sum_{m=1}^{n} \int_{(a_{m-1},a_m]} y^2 dF(y)$$
$$= \sum_{m=1}^{\infty} \int_{(a_{m-1},a_m]} y^2 dF(y) \sum_{n=m}^{\infty} a_n^{-2}$$

Since $a_n \geq na_m/m$ for $n \geq m$, then $\sum_{n=m}^{\infty} a_n^{-2} \leq m^2/a_m^2 \sum_{n=m}^{\infty} n^{-2} \leq Cma_m^{-2}$, thus

$$\sum_{n=1}^{\infty} Var(Y_n/a_n) \leq C \sum_{m=1}^{\infty} m \int_{(a_{m-1},a_m]} dF(y) = C \sum_{n=1}^{\infty} P(|X_i| \geq a_{n-1}) < \infty$$

So, $\sum_{n=1}^{\infty} Y_n/a_n$ converge a.s. Since $a_n \uparrow \infty$, then $a_n^{-1} \sum_{m=1}^{n} (Y_m - EY_m) \to 0$.

It suffices to prove $ET_n/a_n \to 0$ where $T_n = Y_1 + \cdots + Y_n$.

Since $E|X_1| = \infty$, $\sum_{n=1}^{\infty} P(|X_1| \geq a_n) < \infty$, and $a_n/n \uparrow$, we must have $a_n/n \uparrow \infty$. Thus, for any fixed N,

$$|a_n^{-1} \sum_{m=1}^{n} EY_m| \leq a_n^{-1} \sum_{m=1}^{n} E(|X_m|; |X_m| < a_m) \leq \frac{na_N}{a_n} + \frac{n}{a_n} E(|X_i|; a_N \leq |X_i| < a_n)$$

Since $a_n/n \uparrow \infty$, the first term $\downarrow 0$. Since $m/a_m \downarrow$, the second term,

$$\frac{n}{a_n} E(|X_i|; a_N \leq |X_i| < a_n) \leq \sum_{m=N+1}^{n} \frac{m}{a_m} E(|X_i|; a_{m-1} \leq |X_i| < a_m)$$
$$\leq \sum_{m=N+1}^{n} mP(a_{m-1} \leq |X_i| < a_m) \to 0$$

4. The first inequality, let $\tau = \inf\{j : |X_j| \geq x\}$. We have

$$P(|S_n| \geq x) = \sum_{j=1}^{n} P(|S_n| \geq x, \tau = j)$$

Now, by symmetry, since for every $i = 1, \cdots, n$, $(-X_1, \cdots, -X_{j-1}, X_J, -X_{j+1}, \cdots, -X_n)$ has the same distribution as $(X_1, \cdots, X_n)$ and $\{\tau = j\}$ only depends on $|X_1|, \cdots, |X_j|$, we have

$$P(|S_n| \geq x) = \sum_{j=1}^{n} P(|X_j - T_j| \geq x, \tau = j)$$

where $T_j = S_n - X_j, j \leq n$. Then summing the two probability, since $|S_n| + |X_j - T_j| \geq |S_n + X_j - T_j| \geq 2|X_j|$, then

$$2P(|S_n| \geq x) \geq \sum_{j=1}^{n} P(\tau = j) = P(\max_{1 \leq j \leq n} |X_j| \geq x)$$

The second inequality is easy, since $e^{-P(|X_1| \geq x)} \geq 1 - P(|X_1| \geq x)$, then

$$P(\max_{1 \leq j \leq n} |X_j| \geq x) = 1 - P(\max_{1 \leq j \leq n} |X_j| < x) = 1 - (P(|X_j| < x))^n$$
$$= 1 - (1 - P(|X_j| \geq x))^n \geq 1 - e^{-nP(|X_1| \geq x)}$$

4' Should be similar, no detail.

5, 6, 7 refer to Qualify 2014

1 Measure Theory and Expectation

1. (Complete probability space) Given a probability space $(\Omega, \mathcal{F}, P)$, let

$$\bar{\mathcal{F}} = \{E \subset \Omega : E \triangle F \subset N; F, N \in \mathcal{F}, P(N) = 0\},$$

and define $\bar{P}(E) = P(F)$. Prove $(\Omega, \bar{\mathcal{F}}, \bar{P})$ is a probability space.

2. Let θ be uniformly distributed on $[0, 1]$. For each d.f. F, define $G(y) = \sup\{x : F(x) \le y\}$. Then $G(\theta) \sim F$.

3. (Some basic inequalities)

a) Jensen's Inequality: Suppose φ is convex, then

$$E(\varphi(X)) \ge \varphi(EX).$$

b) Hölder's inequality: If $p, q \in [1, \infty]$ with $1/p + 1/q = 1$ then

$$E|XY| \le \|X\|_p \|Y\|_q,$$

where $\|X\|_p = (E|X|^p)^{1/p}$.

c) Chebyshev's inequality: Suppose $\varphi : R \to R$ has $\varphi \ge 0$, let $A \subset R$ and let $a = \inf\{\varphi(y), y \in A\}$,

$$aP(X \in A) \le E(\varphi(X); X \in A) \le E\varphi(X).$$

Specially,

$$a^2 P(|X| \ge a) \le EX^2, \quad P(X > a) \le e^{-ta} E e^{tX}, t > 0.$$

4. If $\{X_n\}$ is a sequence of identically distributed r.v.'s with finite mean, then

$$\lim_n \frac{1}{n} E(\max_{1 \le m \le n} |X_m|) = 0.$$

5. If X and Y are independent, $E|X|^p < \infty$ for some $p \ge 1$, and $EY = 0$, then

$$E(|X + Y|^p) \ge E|X|^p.$$

2 Law of Large numbers

$$S_n = \sum_{i=1}^{n} X_i.$$

1. Let $\{X_n\}$ be pairwise independent with a common d.f. such that

   i) $E(X; |X| \leq n) = o(1)$,

   ii) $nP(|X| > n) = o(1)$;

   then $S_n/n \to 0$ in probability.

2. If $X_1, X_2, ...$ are independent and symmetric, try to prove for any $\epsilon > 0$,

$$P(\max_{1 \leq i \leq n} |S_i| > \epsilon) \leq 2P(|S_n| > \epsilon).$$

3. Let $X_1, X_2, ...$ be iid, and $\mu_n = E(X_1; |X_1| \leq n)$. Use the previous inequality to prove $S_n/n - \mu_n \to 0$ in probability if and only if $xP(|X| > x) \to 0$ as $x \to \infty$.

4. Suppose $X_1, X_2, ...$ are iid Cauchy r.v.'s. Suppose $b_n = c_n n$ and $c_n \uparrow \infty$. Show that $S_n/b_n \to 0$ in probability.

5. Suppose X_n are iid Poisson r.v.'s with rate $\lambda > 0$. Prove that

$$\limsup_n \frac{X_n \log \log n}{\log n} = 1 \quad a.s.$$

6. Suppose $X_1, X_2, ...$ are iid with mean 1 and a_n are bounded real numbers. Then, $\frac{1}{n}\sum_{i=1}^{n} a_i \to 1$ if and only if $\frac{1}{n}\sum_{i=1}^{n} a_i X_i \to 1$ a.s..

7. Suppose $X_1, X_2, ...$ are iid with $E|X|^p < \infty$ for some $0 < p < 2$. Then

   i) $\sum_{n=1}^{\infty} [X_n - EX_n]/n^{1/p} < \infty$ $a.s.$ for $1 < p < 2$;

   ii) $\sum_{n=1}^{\infty} X_n/n^{1/p} < \infty$ $a.s.$ for $0 < p < 1$.

8. Suppose $X_1, X_2, ...$ are independent with mean μ_n and variance σ_n^2 such that $\mu_n \to 0$ and $\sum_{i=1}^{n} \sigma_i^2 \to \infty$. Show that

$$\frac{\sum_{i=1}^{n} X_i/\sigma_i^2}{\sum_{i=1}^{n} \sigma_j^{-2}} \to 0 \quad a.s.$$

3 Central limit theorems

1. Homeworks.

2. Let X_n be a sequence of r.v.'s; let $\mathcal{F}_n$ be the σ-field generated by $\{X_k, 1 \leq k \leq n\}$, and $\mathcal{F}'_n$ that by $\{X_k, k > n\}$. The sequence is called *m-dependent* if there exists an integer $m \geq 0$ such that for every n the fields $\mathcal{F}_n$ and $\mathcal{F}'_{n+m}$ are independent. Suppose X_n is a sequence of m-dependent, uniformly bounded r.v.'s such that
$$\frac{\sigma(S_n)}{n^{1/3}} \to \infty$$
as $n \to \infty$, where $\sigma(S_n) = \sqrt{\mathrm{Var}(S_n)}$. Then $(S_n - ES_n)/\sigma(S_n)$ converges to a standard normal distribution.

3. Let $X_1, X_2, \ldots$ be iid r.v.'s with mean 0 and variance 1. Let $\{N_n, n \geq 1\}$ be a sequence of r.v.'s taking only strictly positive integer values such that
$$\frac{N_n}{n} \to c \quad \text{in prob.}$$
where $c > 0$ is a constant. Then $S_{N_n}/\sqrt{N_n}$ converges to a standard normal distribution.

Other important things: $\pi - \lambda$ theorem, subsequence method, Poisson convergence.

$S_1 = 0$ and … ependent random
… pose for each $i \geqslant 1$, $E(X_i) = 0$ and

$$S_n = \sum_{1 \leqslant i < j \leqslant n} a_{ij} X_i X_j,$$

where $\{a_{ij} : 1 \leqslant i < j < \infty\}$ are constants.

(i) Prove that $\{S_n, \mathcal{F}_n : n \geqslant 1\}$ is a martingale.

(ii) Prove that

$$E[\max_{1 \leqslant i \leqslant n} (S_i^+)^2] \leqslant 4E(S_n^2)$$

(iii) Compute $E(S_n^2)$.

Question 3: [15 marks] Let $X_1, X_2, \ldots$ be a sequence of 1-dependent random variables, that is, for any integer $j \geqslant 1$, $\{X_i : i \leqslant j\}$ is independent of $\{X_i : i \geqslant j+2\}$. Suppose for each $i \geqslant 1$, $E(X_i) = 0$ and $E(X_i^2) = \sigma_i^2 < \infty$. Let $S_k = \sum_{i=1}^{k} X_i$. Prove that for any $x > 0$,

① 与 ②.

① ② j $j+1$ $j+2$

$$P(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant x) \leqslant \frac{4}{x^2} \sum_{1 \leqslant k \leqslant n} \sigma_i^2.$$

Question 4: [20 marks] Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables. They may not have finite expectation. EX 可能 $= \infty$. Let $S_n = X_1 + \cdots + X_n$. Fix a constant $0 < p < 1$. Prove that $E(|X_1|^p) < \infty$ if and only if as $n \to \infty$,

$$\frac{S_n}{n^{1/p}} \to 0 \quad a.s.$$

(2.5.8 2)

Question 5: [20 marks] Let $\{S_n, n \geqslant 1\}$ be a one-dimensional random walk.

(i) Prove that if $P(S_1 \neq 0) > 0$, then for any finite interval $[a, b]$ there exists an $\epsilon < 1$ such that

and constant $C > 0$.

$$P\{S_j \in [a, b], 1 \leqslant j \leqslant n\} \leqslant \epsilon^n . C$$

(ii) Prove that if $P(S_n > 0 \text{ for all } n \geqslant 1) > 0$, then

$$\sum_{n=1}^{\infty} P(S_n \leqslant 0, S_{n+1} > 0) < \infty.$$

$\hookrightarrow P(S_n \leq 0, S_{n+1} > 0 \text{ i.o.}) = 0.$

$\Rightarrow (?)\ P(S_n > 0 \text{ i.o.}) = 1$

~~or $P(S_n < 0$ i.o.$) = 1$~~

$P(S_n - S_{n-1}$ i.o.$) = 1.$

π-λ Theorem Proof

Chaojie Wang

May 20, 2015

$\pi-\lambda$ **Theorem:** If $\mathcal{P}$ is a π-system and $\mathcal{L}$ is a λ-system that containing $\mathcal{P}$ then $\sigma(\mathcal{P}) \subset \mathcal{L}$

Proof: Let $l(\mathcal{P})$ be the smallest λ-system containing $\mathcal{P}$. Then $l(\mathcal{P}) \subset \mathcal{L}$. It suffices to prove $\sigma(\mathcal{P}) \subset l(\mathcal{P})$, that is to prove $l(\mathcal{P})$ is σ-field containing $\mathcal{P}$.

To prove $l(\mathcal{P})$ is σ-field, we have following lemma.

Lemma: If $\mathcal{F}$ is π-system and λ-system, then $\mathcal{F}$ is σ-field.

Proof: (i) If $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$ since $\mathcal{F}$ is λ-system.

(ii) If $A_i \in \mathcal{F}$ are countable sequence, let $B_i = A_i/A_{i-1}$ for $i \geq 2$ and $B_1 = A_1$, then B_i are countable disjoint sequence with $\cup_i B_i = \cup_i A_i$. Since $B_i = A_i/A_{i-1} = A_i \cap A_{i-1}^c$, and $A_i, A_{i-1}^c \in \mathcal{F}$, $\mathcal{F}$ is π-system, then $B_i \in \mathcal{F}$. Since $B_i \in \mathcal{F}$ are disjoint countable sequence, and $\mathcal{F}$ is λ-system, then $\cup_i B_i \in \mathcal{F}$. Thus $\cup_i A_i = \cup_i B_i \in \mathcal{F}$. Proven.

Now it suffice to prove $l(\mathcal{P})$ is π-system, i.e. if $A, B \in l(\mathcal{P})$ then $A \cap B \in l(\mathcal{P})$

Construct $G_A = \{B : B \cap A \in l(\mathcal{P})\}$ with $A \in l(\mathcal{P})$, to prove $l(\mathcal{P}) \subset G_A$. Since $l(\mathcal{P})$ is the smallest λ-system containing $\mathcal{P}$, that is to prove G_A is λ-system containing $\mathcal{P}$.

To prove G_A is λ-system, check (i) $\Omega \in G_A$ since $\Omega \cap A = A \in l(\mathcal{P})$;

(ii) If $B \in G_A$ then $B \cap A \in l(\mathcal{P})$. So $B^c \cap A = A - A \cap B$. Since $A, A \cap B \in l(\mathcal{P})$ and $A \cap B \subset A$, $l(\mathcal{P})$ is λ-system then $B^c \cap A = A - A \cap B \in l(\mathcal{P})$. Thus $B^c \in G_A$

(iii) If $B_i \in G_A$ are disjoint countable sequence, then $(\cup_i B_i) \cap A = \cup_i (B_i \cap A) \in l(\mathcal{P})$ since $l(\mathcal{P})$ is λ-system and $B_i \cap A \in l(\mathcal{P})$ are disjoint countable sequence. Thus $\cup_i B_i \in G_A$.

Thus, G_A is λ-system. Then, check G_A containing $\mathcal{P}$ with $A \in l(\mathcal{P})$.

If $A \in \mathcal{P}$, then for any $B \in \mathcal{P}$ then $A \cap B \in \mathcal{P}$ since $\mathcal{P}$ is π-system. Thus $A \cap B \in \mathcal{P} \subset l(\mathcal{P})$ and $\mathcal{P} \subset G_A$. Then G_A is λ-system containing $\mathcal{P}$, so $l(\mathcal{P}) \subset G_A$. Thus, if $A \in \mathcal{P}$ and $B \in l(\mathcal{P})$ then $A \cap B \in l(\mathcal{P})$. By symmetry, if $A \in l(\mathcal{P})$ and $B \in \mathcal{P}$ then $A \cap B \in l(\mathcal{P})$. Thus, if $A \in l(\mathcal{P})$, then $\mathcal{P} \subset G_A$. Proven.

Strong Law of Large Numbers Proof

Chaojie Wang

May 21, 2015

Strong Law of Large Numbers: Let $X_1, X_2 \cdots$ be pairwise independent identically distributed random variables with $E|X_i| < \infty$. Let $EX_i = \mu$ and $S_n = X_1 + \cdots + X_n$. Then $S_n/n \to \mu$ a.s. as $n \to \infty$

Proof: First we truncated $Y_k = X_k 1_{(|X_k| \leq k)}$, let $T_n = Y_1 + \cdots + Y_n$.

Since $\sum_{k=1}^{\infty} P(X_k \neq Y_k) = \sum_{k=1}^{\infty} P(|X_k| > k) \leq \int_0^{\infty} P(|X_k| > x)dx = E|X_K| < \infty$, then $P(X_k \neq Y_k \ i.o.) = 0$. It implies that $|S_n(\omega) - T_n(\omega)| \leq R(\omega) < \infty$. Thus, it suffices to prove $T_n/n \to \mu$ a.s.

Let $k(n) = [\alpha^n]$ with $\alpha > 1$, using subsequence method,

$$\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) \leq \sum_{n=1}^{\infty} \epsilon^{-2}[k(n)]^{-2} E(T_{k(n)} - ET_{k(n)})^2$$
$$= \sum_{n=1}^{\infty} \epsilon^{-2}[k(n)]^{-2} Var(T_{k(n)}) = \epsilon^{-2} \sum_{n=1}^{\infty} [k(n)]^{-2} \sum_{m=1}^{k(n)} Var(Y_m)$$
$$= \epsilon^{-2} \sum_{m=1}^{\infty} Var(Y_m) \sum_{n:k(n) \geq m}^{\infty} [\alpha^n]^{-2} \leq 4\epsilon^{-2}(1-\alpha^{-2})^{-1} \sum_{m=1}^{\infty} Var(Y_m)/m^2$$

Note that $[\alpha^n] \geq \alpha^n/2$, then

$$\sum_{n:k(n) \geq m}^{\infty} [\alpha^n]^{-2} \leq 4 \sum_{n:k(n) \geq m}^{\infty} \alpha^{-2n} = 4m^{-2}(1-\alpha^{-2})^{-1}$$

Now it suffices to prove $\sum_{m=1}^{\infty} Var(Y_m)/m^2 < \infty$.

Lemma 1: $\sum_{m=1}^{\infty} Var(Y_m)/m^2 \leq 4E|X_1| < \infty$

Proof:

$$\sum_{m=1}^{\infty} Var(Y_m)/m^2 \leq \sum_{m=1}^{\infty} EY_m^2/m^2 = \sum_{m=1}^{\infty} \int_0^{\infty} 2yP(|Y_m| > y)dy/m^2$$
$$= \sum_{m=1}^{\infty} \int_0^{\infty} 1_{(y<m)} 2yP(|X_m| > y)dy/m^2 = \int_0^{\infty} \{\sum_{m=1}^{\infty} 1_{(y<m)} \frac{1}{m^2}\} 2yP(|X_m| > y)dy$$
$$= \int_0^{\infty} \{\sum_{m>y}^{\infty} \frac{1}{m^2} 2y\} P(|X_m| > y)dy \leq 4 \int_0^{\infty} P(|X_m| > y)dy = 4E|X_1| < \infty$$

Lemma 2: If $y \geq 0$ then $\sum_{m>y}^{\infty} \frac{2y}{m^2} \leq 4$

Proof: If $y \geq 1$ then $[y]+1 \geq 2$. Thus

$$\sum_{m>y}^{\infty} \frac{2y}{m^2} = \sum_{m=[y]+1}^{\infty} \frac{2y}{m^2} \leq \int_{[y]}^{\infty} \frac{2y}{x^2} dx = \frac{2y}{[y]} \leq 4$$

If $0 < y < 1$ then

$$\sum_{m>y}^{\infty} \frac{2y}{m^2} = 2y + \sum_{m=2}^{\infty} \frac{2y}{m^2} = 2y(1 + \sum_{m=2}^{\infty} \frac{1}{m^2}) \leq 4$$

Thus, the subsequence $(T_{k(n)} - ET_{k(n)})/k(n) \to 0$ a.s. Since the dominated convergence theorem implies that $EY_k \to EX_1$ as $k \to \infty$. So $T_{k(n)}/k(n) \to EX_1$ a.s.

If $k(n) \leq m < k(n+1)$,

$$\frac{T_{k(n)}}{k(n+1)} \leq \frac{T_m}{m} \leq \frac{T_{k(n+1)}}{k(n)} \Rightarrow \frac{T_{k(n)}}{k(n)} \frac{k(n)}{k(n+1)} \leq \frac{T_m}{m} \leq \frac{T_{k(n+1)}}{k(n+1)} \frac{k(n+1)}{k(n)}$$

Since $k(n) = [\alpha^n]$ then $k(n+1)/k(n) \to \alpha$. Thus

$$\frac{1}{\alpha} EX_1 \leq \liminf_{m\to\infty} T_m/m \leq \limsup_{m\to\infty} T_m/m \leq \alpha EX_1$$

Since $\alpha > 1$ is arbitrary, then $\lim_{m\to\infty} T_m/m = EX_1$. Proven.

STAT 5005 Syllabus

Chaojie Wang

June 2, 2015

1 Measure Theory

1.1 Probability Space

1. σ-field: (i) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$. (ii) countable $A_i \in \mathcal{F} \Rightarrow \cup_i A_i \in \mathcal{F}$

2. measure: (i) $\mu(A) \geq 0$ (ii) disjoint contable $A_i \in \mathcal{F} \Rightarrow \mu(\cup_i A_i) = \sum_i \mu(A_i)$;

Probability measure: additional (iii) $\mu(\Omega) = 1$

3. measure space $(\Omega, \mathcal{F})$; probability space $(\Omega, \mathcal{F}, P)$

4. Theorem 1.1.1: properties for μ on $(\Omega, \mathcal{F})$: (i) monotonicity

(ii) subadditivity: $A \subset \cup_{m=1}^{\infty} A_m \Rightarrow \mu(A) \leq \sum_{m=1}^{\infty} \mu(A_m)$

(iii) continuity from below (iv) continuity from above

5. Borel sets $\mathcal{R}^d$, the smallest σ-field containing all the open sets.

6. semialgebra: (i) $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$ (ii) $A \in \mathcal{F} \Rightarrow A^c$ is finite disjoint union

algebra: (i) $A, B \in \mathcal{F} \Rightarrow A \cup B \in \mathcal{F}$ (ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$;

$$semialgebra \xrightleftharpoons[\text{must}]{\text{lemma 1.1.3}} algebra \xrightleftharpoons[\text{must}]{\text{unnecessary e.g.1.1.4}} \sigma\text{-}algebra$$

1.2 Distribution

1. check $\mu(A) = P(X \in A)$ is probability measure.

2. Theorem 1.2.1: distribution function $F(x) = P(X \leq x)$ properties:

(i) nondecreasing; (ii) $\lim\limits_{x \to \infty} F(x) = 1$ and $\lim\limits_{x \to -\infty} F(x) = 0$;

(iii) right continuous; (iv) If $F(x-) = \lim_{y \uparrow x} F(y)$ then $F(x-) = P(X < x)$;

(v) $P(X = x) = F(x) - F(x-)$

3*. Theorem 1.2.2: If F satisfies (i) (ii) (iii), then it is the distribution function of some random variable.

Hint: $X(\omega) = \sup\{y : F(y) < \omega\}$. Check $\{\omega : X(\omega) \leq x\} = \{\omega : \omega \leq F(x)\}$

4. Theorem 1.2.3: For $x > 0$,

$$(x^{-1} - x^{-3})exp(-x^2/2) \le \int_x^\infty exp(-y^2/2)dy \le x^{-1}exp(-x^2/2)$$

Hint: left side $= \int_x^\infty (1 - 3y^{-4})exp(-y^2/2)dy$; right side let $y = z + x$

1.3 Random Variables

1. measurable map from $(\Omega, \mathcal{F})$ to $(S, \mathcal{S})$,

$$X^{-1}(B) = \{\omega : X(\omega) \in B\} \in \mathcal{F} \quad for\ all\ B \in \mathcal{S}$$

If $(S, \mathcal{S}) = (\mathbf{R}^d, \mathcal{R}^d)$ and $d > 1$, X is random vector. If $d = 1$, X is random variable.

2*. Theorem 1.3.1: If $\{\omega : X(\omega) \in A\} \in \mathcal{F}$ for all $A \in \mathcal{A}$ and $\mathcal{A}$ generates $\mathcal{S}$, then X is measurable. Remark: $\{(-\infty, x] : x \in \mathbf{R}\}$ generate $\mathcal{R}$.

Method: construct $\mathcal{B} = \{B : \{\omega : X(\omega) \in B\} \in \mathcal{F}\}$, to prove $\mathcal{B} \subset \mathcal{S}$. Since $\mathcal{S} = \sigma(\mathcal{A})$, it suffices to prove $\mathcal{B}$ is σ-algebra containing $\mathcal{A}$.

3. σ-field generated by X: $\sigma(X) = \{\{X \in B\} : B \in \mathcal{S}\}$

4. Theorem 1.3.2: If $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ and $f : (S, \mathcal{S}) \to (T, \mathcal{T})$ are measurable maps, then $f(X)$ is a measurable map from $(\Omega, \mathcal{F})$ to $(T, \mathcal{T})$

$\Rightarrow$ Theorem 1.3.3: If $X_1, \cdots, X_n$ are r.v. and $f : (\mathbf{R}^n, \mathcal{R}^n) \to (\mathbf{R}, \mathcal{R})$ is measurable, then $f(X_1, \cdots, X_n)$ is a r.v..

$\Rightarrow$ Theorem 1.3.4: If $X_1, \cdots, X_n$ are r.v., then $X_1 + \cdots + X_n$ is a r.v..

5. Theorem 1.3.5: If $X_1, \cdots, X_n$ are r.v., then so are

$$\inf_n X_n, \quad \sup_n X_n, \quad \limsup_n X_n = \inf_n(\sup_{m \ge n} X_m), \quad \liminf_n X_n = \sup_n(\inf_{m \ge n} X_m).$$

1.4 Integration

Step 1: Simple functions: $\varphi = \sum_{i=1}^m a_i 1_{A_i}$, define $\int \varphi d\mu = \sum_{i=1}^m a_i \mu(A_i)$

(i) If $\varphi \ge 0$ a.e. then $\int \varphi d\mu \ge 0$; (ii) for any $a \in \mathbf{R}$, $\int a\varphi d\mu = a \int \varphi d\mu$;

(iii) $\int \varphi + \psi d\mu = \int \varphi d\mu + \int \psi d\mu$; (iv) If $\varphi \le \psi$ a.e., then $\int \varphi d\mu \le \int \psi d\mu$

(v) If $\varphi = \psi$ a.e., then $\int \varphi d\mu = \int \psi d\mu$; (vi) $|\int \varphi d\mu| \le \int |\varphi| d\mu$

Step 2: Bounded function: $\varphi \le f \le \psi$ with $|f| < M$ and φ, ψ are simple functions, define and check $\int f d\mu = \sup_{\varphi \le f} \int \varphi d\mu = \inf_{f \le \psi} \int \psi d\mu$, check (i)-(vi)

Hint: construct $E_k = \{x \in E : \frac{kM}{n} \ge f(x) > \frac{(k-1)M}{n}\}$ for $-n \le k \le n$.

Let $\varphi_n(x) = \sum\limits_{k=-n}^{n} \frac{(k-1)M}{n} 1_{E_k}$ and $\psi_n(x) = \sum\limits_{k=-n}^{n} \frac{kM}{n} 1_{E_k}$

Step 3: Nonnegative function $f \geq 0$: define

$$\int f d\mu = \sup\{\int h d\mu : 0 \leq h \leq f, h \text{ is bounded and } \mu(\{x : h(x) > 0\}) < \infty\}$$

Lemma 1.4.4*: Let $E_n \uparrow \Omega$ have $\mu(E_n) < \infty$ and let $a \wedge b = \min(a, b)$. Then

$$\int_{E_n} f \wedge n d\mu \uparrow \int f d\mu \quad as\ n \to \infty$$

Hint: $n > M$, $\int_{E_n} f \wedge n d\mu \geq \int_{E_n} h d\mu = \int h d\mu - \int_{E_n^c} h d\mu \to \int h d\mu$, then check (i)-(vi).

(iii) $\int_{E_n} (f + g) \wedge n d\mu \leq \int_{E_n} f \wedge n d\mu + \int_{E_n} g \wedge n d\mu$

Step 4: General functions: f is integrable if $\int |f| d\mu < \infty$, define

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

Lemma 1.4.6: If $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ and $\int f_i d\mu \leq \infty$, then

$$\int f d\mu = \int f_1 \mu - \int f_2 d\mu$$

then check (i)-(vi).

1.5 Properties of Integral

1. Theorem 1.5.1: Jensen's Inequality

2. Theorem 1.5.2: Hölder's Inequality; Specially, Cauchy-Schwarz inequality.

Hint: $xy \leq x^p/p + y^q/q$

3*. Conditions that guarantee exchange of limits and integral under $f_n \to f$ a.s.

(a) Theorem 1.5.3: Bounded convergence theorem. ($|f_n| \leq M$ and $\mu(E) < \infty$)

Hint: $|\int f_n d\mu - \int f d\mu| \to 0$

(b)* Theorem 1.5.4: Fatou's lemma ($f_n \geq 0$)

Hint: $g_n = \inf_{m \geq n} f_m$: $g_n \uparrow g = \liminf f_n$, $g_n \leq f_n$, $(g_n \wedge m)1_{E_m} \uparrow (g \wedge m)1_{E_m}$

(c) Theorem 1.5.5: Monotone convergence theorem ($f_n \geq 0$ and $f_n \uparrow f$)

(d) Theorem 1.5.6: Dominated convergence theorem ($f_n \leq g$ and g is integrable)

1.6 Expected Value

1*. Theorem 1.6.4: Chebyshev's inequality ($\varphi \geq 0$). Specially, Markov's inequality

Hint: $i_A = \inf\{\varphi(y) : y \in A\}$

2. Exercise 1.5.3: Minkowski inequality: $||f + g||_p \leq ||f||_p + ||g||_p$ for $p \in (1, \infty)$

Hint: $|f + g|^p \leq |f + g|^{p-1}|f| + |f + g|^{p-1}|g|$

3. Theorem 1.6.8: Suppose $X_n \to X$ a.s. Let g, h be continuous functions with

(i) $g \geq 0$ and $g(x) \to \infty$ as $|x| \to \infty$, (ii) $|h(x)|/g(x) \to 0$ as $|x| \to \infty$

(iii) $Eg(X_n) \leq K < \infty$ for all n. Then $Eh(X_n) \to Eh(X)$

Hint: $\bar{X}_n = X_n 1_{(|X| \leq M)}$ with large M $P(|X| \geq M) = 0$ and triangle inequality

$E|h(\bar{Y}) - h(Y)| \leq E(|h(\bar{Y})|) = E(\frac{|h(\bar{Y})|}{g(Y)} g(Y)) \leq \epsilon_M Eg(Y)$

4. Theorem 1.6.9: Change of variables formula: $Ef(X) = \int_S f(y)\mu(dy)$

Hint: Indicator $\xrightarrow{linear\ extend}$ Simple $\xrightarrow{f_n = [2^n f]/2^n \wedge n}$ Nonnegative $\xrightarrow{E = Ef^+ - Ef^-}$ integral

1.7 Product measure, Fubini's Theorem

1°. Theorem 1.7.2 Fubini's Theorem: If $f \geq 0$ or $\int |f| d\mu < \infty$ then

$$\int_X \int_Y f(x,y)\mu_2(dy)\mu_1(dx) = \int_{X \times Y} f d\mu = \int_Y \int_X f(x,y)\mu_2(dx)\mu_1(dy)$$

2 Laws of Large Numbers

2.1 Independence

1. Event independent; random variables independent; σ-fields independent; collections of sets independent.

2*. Theorem 2.1.2: $\pi - \lambda$ Theorem

Hint: π-system: $A, B \in \mathcal{F} \Rightarrow A \cap B \in \mathcal{F}$; λ-system: (i) $\Omega \in \mathcal{F}$,

(ii) $A, B \in \mathcal{F}, A \subset B \Rightarrow B - A \in \mathcal{F}$, (iii) $A_n \in \mathcal{F}, A_n \uparrow A \Rightarrow A \in \mathcal{F}$

(ii) $A \in \mathcal{F} \Rightarrow A^c \in \mathcal{F}$, (iii) disjoint countable sequence $A_i \in \mathcal{F} \Rightarrow \cup_i A_i \in \mathcal{F}$

$l(\mathcal{P})$ the smallest λ-system containing $\mathcal{P}$; $\pi + \lambda \Rightarrow \sigma$-field; $G_A = \{B : A \cap B \in l(\mathcal{P})\}$ with $A \in l(\mathcal{P})$; G_A contain $\mathcal{P}$.

3*. Theorem 2.1.3: suppose $\mathcal{A}_1, \cdots, \mathcal{A}_n$ are independent and each $\mathcal{A}_i$ is a π-system. Then $\sigma(\mathcal{A}_1), \cdots, \sigma(\mathcal{A}_n)$ are independent.

Hint: $A_i \in \mathcal{A}_i$, $F = A_2 \cap \cdots \cap A_n$, $\mathcal{L} = \{A : P(F)P(A) = P(A \cap F)\}$ is λ-system

(a) Theorem 2.1.4: $X_1, \cdots, X_n$ are independent if for all $x_1, \cdots, x_n \in (-\infty, \infty]$,

$$P(X_1 \leq x_1, \cdots, X_n \leq x_n) = \Pi_{i=1}^n P(X_i \leq x_i)$$

Hint: $\mathcal{A}_i = \{X_i \leq x_i\}$; $\mathcal{A}_i$ is π-system; $\sigma(\mathcal{A}_i)$ independent; $\sigma(\mathcal{A}_i) = \sigma(X_i)$

(b) Theorem 2.1.5: $\mathcal{F}_{i,j}$ independent, $\mathcal{G}_i = \sigma(\cup_j \mathcal{F}_{i,j}) \Rightarrow \mathcal{G}_1, \cdots, \mathcal{G}_n$ are independent.

$\Rightarrow$ Theorem 2.1.6: If for $1 \leq i \leq n$, $1 \leq j \leq m(i)$, $X_{i,j}$ independent and $f_i : \mathbf{R}^{m(i)} \to \mathbf{R}$ are measurable then $f_i(X_{i,1}, \cdots, X_{i,m(i)})$ are independent.

Hint: $\mathcal{F}_{i,j} = \sigma(X_{i,j})$, $f(X_{i,1}, \cdots, X_{i,m(i)}) \in \mathcal{F}_i$

4. Theorem 2.1.7: X_i independent with distribution $\mu_i \Rightarrow (X_1, \cdots, X_n)$ has $\mu_1 \times \cdots \mu_n$

5. Theorem 2.1.8: X, Y independent, $h(x,y) = f(x)g(y) \Rightarrow Eh(X,Y) = Ef(X)Eg(Y)$

6. Theorem 2.1.9: $X_i \geq 0$ or $E|X_i| < \infty$, X_i independent $\Rightarrow E(\prod_{i=1}^n X_i) = \prod_{i=1}^n EX_i$

7. Convolution: Theorem 2.1.10 for distribution function, 2.1.11 for density function

2.2 Weak Laws of Large Numbers

1. $Y_n \to Y$ in probability: if for all $\epsilon > 0$, $P(|Y_n - Y| > \epsilon) \to 0$ as $n \to \infty$

2. Lemma 2.2.2: If $p > 0$ and $E|Z_n|^p \to 0$ then $Z_n \to 0$ in probability. (Chebyshev)

Remark: $E|X_n - X|^p \to 0$ implies $X_n \to X$ in L^p

(a) Theorem 2.2.3: L^2 weak law (X_i uncorrelated, $Var(X_i) \leq C < \infty$)

(b) Theorem 2.2.4: $\frac{S_n - \mu_n}{b_n} \to 0$ in probability. ($\mu_n = ES_n$, $\sigma_n^2 = Var(S_n)$, $\sigma_n^2/b_n^2 \to 0$)

3. $\sum\limits_{m=1}^n \frac{1}{m} \geq \int_1^n \frac{1}{x}dx \geq \sum\limits_{m=2}^n \frac{1}{m} \Rightarrow \log n \leq \sum\limits_{m=1}^n \frac{1}{m} \leq 1 + \log n$

4*. Theorem 2.2.6: Weak law for triangular arrays. For each n let $X_{n,k}$, $1 \leq k \leq n$ be independent. Let $b_n > 0$ and $b_n \to \infty$ and let $\bar{X}_{n,k} = X_{n,k}1_{(|X_{n,k}| \leq b_n)}$, suppose that $n \to \infty$

(i) $\sum_{k=1}^n P(|X_{n,k}| > b_n) \to 0$ (ii) $b_n^{-2} \sum_{k=1}^n E\bar{X}_{n,k}^2 \to 0$

If we let $S_n = X_{n,1} + \cdots + X_{n,n}$ and put $a_n = \sum_{k=1}^n E\bar{X}_{n,k}$ then

$$(S_n - a_n)/b_n \to 0 \textit{ in probability}$$

Hint: $P(|\frac{S_n - a_n}{b_n}| > \epsilon) \leq P(|\frac{\bar{S}_n - a_n}{b_n}| > \epsilon) + P(S_n \neq \bar{S}_n)$

$\Rightarrow$ Theorem 2.2.7: Weak law of large numbers.

(i) X_i are i.i.d; (ii) $xP(|X_i| > x) \to 0$ as $x \to \infty$;

Let $\mu_n = E(X_1 1_{|X_1| \leq n})$, then $S_n/n - \mu_n \to 0$ in probability.

Hint: Lemma 2.2.8: $E(Y^p) = \int_0^\infty py^{p-1}(Y > y)dy$;

$E\bar{X}_1^2/n \leq \frac{1}{n}\int_0^n 2yP(|X_1| > y)dy = \int_0^1 g_n(y)dy$ with $g_n(y) \to 0$ a.s.

$\Rightarrow$* Theorem 2.2.9: $S_n/n \to \mu$ in probability (X_i are i.i.d, $E|X_i| < \infty$)

Hint: $xP(|X_1| > x) \leq E(|X_1|1_{(|X_1| > x)}) \to 0$

2.3 Borel-Cantelli Lemmas

1. $X_n \to X$ a.s.: $P(\lim\limits_{n\to\infty} X_n = X) = 1$;

2. $\limsup A_n$; $\liminf A_n$; $X_n \to 0$ a.s. $\Leftrightarrow$ for all $\epsilon > 0$, $P(|X_n| > \epsilon \; i.o.) = 0$

$\limsup X_n \geq c$ a.s $\Leftrightarrow P(X_n \geq c - \epsilon \; i.o.) = 1$

3*. Theorem 2.3.1: Borel-Cnatelli Lemma: $\sum_{n=1}^\infty P(A_n) < \infty \Rightarrow P(A_i \; i.o.) = 0$

(a) Theorem 2.3.2: $X_n \to X$ in probability $\Leftrightarrow$ every subsequence $X_{n(m)}$ there is a further subsequence $X_{n(m_k)}$ then $X_{n(m_k)} \to X$ a.s.

Hint: $\epsilon_k \downarrow 0$, $P(|X_{n(m_k)} - X| > \epsilon_k) \le 2^{-k}$ summable;

Conversely, Theorem 2.3.3: $y_{n(m_k)} = P(|X_{n(m_k)} - X| > \epsilon_k) \to 0 \Rightarrow y_n \to 0$

$\Rightarrow$ Theorem 2.3.4: If f is continuous, $X_n \to X$ in probability $\Rightarrow f(X_n) \to f(X)$ in probability. If f is bounded in addition, $Ef(X_n) \to Ef(X)$

(b) Theorem 2.3.5: If X_i are i.i.d and $EX_i^4 < \infty$, then $S_n/n \to \mu$ a.s.

Hint: $ES_n^4 = EX_1^4 + 3(n^2 - n)(EX_1^2)^2 \le Cn^2$

4. Theorem 2.3.6: The second Borel-Cantelli lemma: A_n are independent,

$\sum_{n=1}^{\infty} P(A_n) = \infty \Rightarrow P(A_n\ i.o.) = 1$

Hint: $1 - P(A_n\ i.o.) = \lim_{n\to\infty} P(\cap_{m\ge n} A_m^c) = \lim_{n\to\infty} \prod_{m\ge n}(1 - P(A_m)) \le \lim_{n\to\infty} e^{-\sum P(A_m)}$

(a) Theorem 2.3.7: If $E|X_i| = \infty$, then $P(|X_n| \ge n\ i.o.) = 1$. So, $P(\lim S_n/n\ exist) = 0$.

Hint: $C \equiv \{\omega : \lim S_n/n\ exist\}$. On $C \cap \{\omega : X_n \ge n\ i.o.\}$, $|\frac{S_n}{n} - \frac{S_{n+1}}{n+1}| > \frac{2}{3}\ i.o.$

Remark: it shows $E|X_i| < \infty$ is necessary for the strong law of large number

5*. Theorem 2.3.8: If $\sum_{n=1}^{\infty} P(A_n) = \infty$, then $\sum_{m=1}^{n} 1_{A_m} / \sum_{i=1}^{n} P(A_m) \to 1$ a.s.

Hint: $n_k = \inf\{n : ES_n \ge k^2\}$, $k^2 \le ET_k \le ET_{k+1} \le (k+1)^2 + 1$

Remark: subsequence method

2.4 Strong Law of Large Numbers

1*. Theorem 2.4.1: Strong law of large numbers. (first proof)

Hint: truncated $Y_k = X_k 1_{(|X_k| \le k)}$, $P(X_k \ne Y_k\ i.o.) = 0$, $|S_n(\omega) - T_n(\omega)| \le R(\omega) < \infty$

$k(n) = [\alpha^n]$, $\sum_{n=1}^{\infty} P(|T_{k(n)} - ET_{k(n)}| > \epsilon k(n)) < \infty$,

$\sum_{n=1}^{\infty} \frac{1}{k(n)^2} \sum_{m=1}^{k(n)} Var(Y_m) = \sum_{m=1}^{\infty} Var(Y_m) \sum_{n:k(n)\ge m} \frac{1}{k(n)^2}$,

$EY_k \to EX_1$ and $ET_{k(n)}/k(n) \to EX_1$

Lemma 2.4.2 $\sum_{m=1}^{\infty} Var(Y_m)/m^2 \le 4E|X_1| < \infty$

Lemma 2.4.3 If $y > 0$ then $\sum_{k>y} k^{-2} \le 4$

2. Theorem 2.4.5: $EX_i^+ = \infty$ and $EX_i^- < \infty$ then $S_n/n \to \infty$ a.s.

Hint: $X_i^M = X_i \wedge M$, $S_n^M/n \to EX_1^M$, $E(X_i^M)^+ \uparrow EX_1^+ = \infty$

3. Theorem 2.4.6: $N_t = \sup\{n : T_n \le t\}$, then $N_t/t \to 1/\mu$ a.s.

Hint: $\frac{T_{N_t}}{N_t} \le \frac{t}{N_t} \le \frac{T_{N_t+1}}{N_t+1} \frac{N_t+1}{N_t}$

4. Theorem 2.4.7: The Glivenko-Cantelli Theorem: $\sup_x |F_n(x) - F(x)| \to 0$ a.s.

Hint: $x_{i,k} = \inf\{y : F(y) \ge j/k\}$ for $1 \le j \le k-1$. $F(x_{j,k}-) - F(x_{j-1,k}) \le 1/k$

$|F_n(x_{j,k}) - F(x_{j,k})| < 1/k$, $|F_n(x_{j,k}-) - F(x_{j,k}-)| < 1/k$

2.5 Convergence of Random Series

1. tail σ-field: $\mathcal{T} = \cap_n \mathcal{F}'_n$ where $\mathcal{F}'_n = \sigma(X_n, X_{n+1} \cdots)$

$A \in \mathcal{T} \Leftrightarrow$ changing a finite number of values does not affect the occurence of the event.

Example 2.5.1: If $B_n \in \mathcal{R}$ then $\{X_n \in B_n \; i.o.\} \in \mathcal{T}$

Example 2.5.2: (i) $\{\lim_{n\to\infty} S_n \; exist\} \in \mathcal{T}$ (ii) $\{\limsup_{n\to\infty} S_n > 0\} \notin \mathcal{T}$

(iii) $\{\lim_{n\to\infty} S_n/c_n > x\} \in \mathcal{T}$ if $c_n \to \infty$

2. Theorem 2.5.1: Kolmogorov's 0-1 law: $A \in \mathcal{T}$ then $P(A) = 0$ or 1

Hint: $A \in \sigma(X_1, \cdots, X_k)$ and $B \in \sigma(X_{k+1}, X_{k+2}, \cdots)$ independent.

$A \in \sigma(X_1, X_2, \cdots,)$ and $B \in \mathcal{T}$ independent. $\mathcal{T} \subset \sigma(X_1, X_2, \cdots,)$

3*. Theorem 2.5.2: Kolmogorov's maximal inequality

$$P(\max_{1\le k\le n} |S_k| \ge x) \le x^{-2} Var(S_n)$$

Hint: $A_k = \{|S_k| \ge x \; but \; |S_j| < x \; for \; j < k\}$. $ES_n^2 \ge \sum_{k=1}^n \int_{A_k} ES_n^2 dP$

$\Rightarrow$ Theorem 2.5.3: $EX_n = 0$. If $\sum_{n=1}^\infty Var(X_n) < \infty$, then $\sum_{n=1}^\infty X_n(\omega)$ converge a.s.

Hint: $P(\sup_{m,n\ge M} |S_m - S_n|) \to 0$ as $M \to \infty$

4. Theorem 2.5.4: Kolmogorov's three-series theorem: Let $A > 0$, $Y_i = X_i 1_{(|X_i|\le A)}$.

$\sum_{n=1}^\infty X_n$ converge a.s. $\Leftrightarrow$ (i) $\sum_{n=1}^\infty P(|X_n| > A) < \infty$ (ii) $\sum_{n=1}^\infty EY_n$ converge

(iii) $\sum_{n=1}^\infty Var(Y_n) < \infty$

Remark: if $A = 1$, then (ii) implies (iii)

5. Theorem 2.5.5: Kronecker's Lemma: If $a_n \uparrow$, $\sum_{n=1}^\infty x_n/a_n$ converge then

$$\frac{1}{a_n} \sum_{m=1}^n x_m \to 0$$

Hint: $b_m = \sum_{k=1}^m x_k/a_k$

Remark: Theorem 2.5.3 and 2.5.5 are combo.

$\Rightarrow$ Theorem 2.5.6: The strong law of large numbers (second proof)

Hint: $\sum_{m=1}^\infty Var(Y_m)/m^2 < \infty \Rightarrow \sum_{m=1}^\infty Y_m/m$ converge $\Rightarrow \frac{1}{n}\sum_{m=1}^\infty Y_m \to 0$

6. Theorem 2.5.7: $S_n/n^{1/2}(\log n)^{1/2+\epsilon} \to 0$ a.s.

Remark*: Kolmogorov's test: $\limsup S_n/n^{1/2}(\log\log n)^{1/2} = \sigma\sqrt{2}$ a.s.

7*. Theorem 2.5.8: Marcinkiewicz-Zygmund Strong Law: $E|X_1|^p < \infty$ where $1 < p < 2$, then $S_n/n^{1/p} \to 0$ a.s.

Hint: $Y_k = X_k 1_{(|X_k|\le k^{1/p})}$, $\sum_{m=1}^\infty Var(Y_m/m^{1/p}) < \infty$, $\sum_{m=n}^\infty \frac{1}{m^{2/p}} \le Cy^{p-2}$

$\mu_m = -E(X_i; |X_i| > m^{1/p})$, $|\mu_m| \le m^{1/p-1} E(|X_i|^p; |X_i| > m^{1/p})$

8. Theorem 2.5.9: $E|X_i| = \infty$, a_n a positive sequence a_n/n increasing.

Then $\limsup |S_n|/a_n = 0$ or ∞ according to $\sum_n P(|X_1| \le a_n) < \infty$ or $= \infty$.

2.6 Large Deviation

3 Central Limit Theorems

3.1 The De Moivre-Laplace Theorem

1. Stirling formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$
2. Theorem 3.1.3: De Moivre-Laplace Theorem

3.2 Weak Convergence

1. $F_n \Rightarrow F$: if $F_n(y) \to F(y)$ for all y that are continuity points of F
2. Theorem 3.2.2: $F_n \Rightarrow F \Rightarrow$ there exist Y_n with distribution F_n so that $Y_n \to Y$ a.s.

$\Rightarrow$ Theorem 3.2.3: $X_n \Rightarrow X \Leftrightarrow$ every bounded continuous function g, $Eg(X_n) \to Eg(X)$

$\Rightarrow$ Theorem 3.2.4: Continuous mapping theorem. If $X_n \Rightarrow X_\infty$ and $P(X_\infty \in D_g) = 0$ then $g(X_n) \Rightarrow g(X_\infty)$. If g is bounded then $Eg(X_n) \to Eg(X_\infty)$

3. Exercise 3.1.12: If $X_n \to X$ in probability then and $X_n \Rightarrow X$. Conversely, if $X_n \Rightarrow c$ then $X_n \to c$ in probability.

Exercise 3.1.13: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, then $X_n + Y_n \Rightarrow X + c$

Exercise 3.1.14: If $X_n \Rightarrow X$ and $Y_n \Rightarrow c$, then $X_n Y_n \Rightarrow cX$

3.3 Characteristic Functions

1. Ch.f.: $\varphi(t) = Eexp(itX) = E\cos(tX) + iE\sin(tX)$
2. Theorem 3.3.1: (a) $\varphi(0) = 1$; (b) $\varphi(-t) = \bar{\varphi}(t)$; (c) $|\varphi(t)| = |Ee^{itX}| \le E|e^{itX}| = 1$; (d) $|\varphi(t+h) - \varphi(t)| \le E|e^{itX} - 1|$; (e) $Ee^{it(aX+b)} = e^{itb}\varphi(at)$
3. Theorem 3.3.2: X_1 and X_2 independent $\Rightarrow X_1 + X_2$ has ch.f. $\varphi_1(t)\varphi_2(t)$
4. Lemma 3.3.3: $\lambda_1 + \cdots + \lambda_n = 1$ then $\sum_{i=1}^n \lambda_i F_i$ has ch.f. $\sum_{i=1}^n \lambda_i \varphi_i$
5. Theorem 3.3.4: The inversion formula; If $a < b$,

$$\lim_{T\to\infty} (2\pi)^{-1} \int_{-T}^{T} \frac{e^{-ita} - e^{-itb}}{it} \varphi(t)dt = \mu(a,b) + \frac{1}{2}\mu(\{a,b\})$$

Hint: $R(\theta, T) = \int_{-T}^{T} \frac{\theta t}{t} dt = \pi, \theta > 0; = -\pi, \theta < 0; = 0, \theta = 0$

6. Theorem 3.3.5: If $\int |\varphi(t)|dt < \infty$,

$$f(y) = \frac{1}{2\pi} \int e^{-ity}\varphi(t)dt$$

7°. Theorem 3.3.6: Continuity Theorem

8. Lemma 3.3.7: $|e^{ix} - \sum_{m=0}^n \frac{(ix)^m}{m!}| \le \min(\frac{|x|^{n+1}}{(n+1)!}, \frac{2|x|^n}{n!})$

9. Theorem 3.3.8: If $E|X|^2 < \infty$ then

$$\varphi(t) = 1 + itEX - t^2 E(X^2)/2 + o(t^2)$$

where $o(t^2) \le t^2 E(|t| \cdot |X|^3 \wedge 2|X|^2)$

10. Lemma 3.3.9: If $\limsup_{h \downarrow 0} \{\varphi(h) - 2\varphi(0) + \varphi(-h)\}/h^2 > -\infty$ then $E|X|^2 < \infty$

3.4 Central Limit Theorems

1. Theorem 3.4.1: $(S_n - n\mu)/\sigma n^{1/2} \Rightarrow \chi$

Remark: characteristic function method

2. Theorem 3.4.2: If $c_n \to c \in \mathbf{C}$ then $(1 + c_n/n)^n \to e^c$

Proof: Lemma 3.4.3: Let $z_1, \cdots, z_n$ and $w_1, \cdots, w_n$ be complex number of modulud $\le \theta$, then

$$|\prod_{m=1}^{n} z_m - \prod_{m=1}^{n} w_m| \le \theta^{n-1} \sum_{m=1}^{n} |z_m - w_m|$$

Lemma 3.4.4: If b is a complex number with $|b| \le 1$ then $|e^b - (1+b)| \le |b|^2$

Hint: $|e^b - (1+b)| = \frac{b^2}{2!} + \cdots \le \frac{|b|^2}{2}(1 + \frac{1}{2} + \frac{1}{2^2}) = |b|^2$

Remark: characteristic function method and Lemma 3.4.3 are combo

3. Theorem 3.4.5: The Lindeberg-Feller theorem: suppose (a) $\sum_{m=1}^{n} EX_{n,m}^2 \to \sigma^2 > 0$; (b) $\lim\limits_{n\to\infty} \sum_{m=1}^{n} E(|X_{n,m}|^2; |X_{n,m}| > \epsilon) = 0$. Then $S_n \Rightarrow \sigma\chi$ as $n \to \infty$

Hint: $|\varphi_{n,m}(t) - (1 - t^2\sigma_{n,m}^2/2)| \le E(|tX_{n,m}|^3 \wedge 2|tEX_{n,m}|^2)$

$\sigma_{n,m}^2 \le \epsilon^2 + E(|X_{n,m}|^2, |X_{n,m}| > \epsilon) \to 0$

3.5 Local Limit Theorem

3.6 Poisson Convergence

1. Theorem 3.6.1: For each n let $X_{n,m}$, $1 \le m \le n$ be independent random variables with $P(X_{n,m} = 1) = p_{n,m}$, $P(X_{n,m} = 0) = 1 - p_{n,m}$. Suppose

(i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$; (ii) $\max_{1 \le m \le n} p_{n,m} \to 0$.

If $S_n = X_{n,1} + \cdots, X_{n,n}$ then $S_n \Rightarrow Z$ where Z is $Poisson(\lambda)$

Hint: Lemma 3.4.4, ch.f. $exp(\lambda(e^{it} - 1))$

$\Rightarrow$ Theorem 3.6.6: Let $X_{n,m}$, $1 \le m \le n$ be independent nonnegative integer valued random variable with $P(X_{n,m} = 1) = p_{n,m}$, $P(X_{n,m} \ge 2) = \epsilon_{n,m}$.

(i) $\sum_{m=1}^{n} p_{n,m} \to \lambda \in (0, \infty)$; (ii) $\max_{1 \le m \le n} p_{n,m} \to 0$; (iii) $\sum_{m=1}^{n} \epsilon_{n,m} \to 0$.

If $S_n = X_{n,1} + \cdots, X_{n,n}$ then $S_n \Rightarrow Z$ where Z is $Poisson(\lambda)$

$\Rightarrow$ Theorem 3.6.7: Let $N(s,t)$ be the number of arrival in the time intervala $(s,t]$. If

(i) the numbers of arribals in disjoint intervals are independet;

(ii) the distribution of $N(s,t)$ only depends on $t-s$;

(iii) $P(N(0,h)=1)=\lambda h+o(h)$; (iv) $P(N(0,h)\geq 2)=o(h)$ holds

then $N(0,t)$ has a Poisson distribution with mean λt

Hint: $P(S_n\neq S_n')\to 0$, converging together lemma

2. Poisson process with rate λ: N_t satisfying

(i) If $0=t_0<t_1\cdots<t_n$, $N(t_k)-N(t_{k-1})$, $1\leq k\leq n$ are independent.

(ii) $N(t)-N(s)$ is $Poisson(\lambda(t-s))$

Hint: $X_{n,m}=N(\frac{(m-1)t}{n},\frac{mt}{n})$

3. Another method to construct Poisson process: ξ_i are independent, $P(\xi_i>t)=e^{-\lambda t}$, $N_t=\sup\{n:T_n\leq t\}$ is Poisson distribution.

Hint: $T_n\sim gamma(n,\lambda)$, $f_{T_n}(t)=\frac{\lambda^n s^{n-1}}{(n-1)!}e^{-\lambda s}$ for all $s\geq 0$

3.7 Stable Laws

3.8 Infinitely Bivisible Distribution

3.9 Limit Theorem in $\mathbf{R}^d$

4 Random Walks

4.1 Stopping Times

1. Theorem 4.1.2: One of following has probability one:

(i) $S_n=0$ for all n; (ii) $S_n\to\infty$; (iii) $S_n\to-\infty$;

(iv) $-\infty=\liminf S_n<\limsup S_n=\infty$

2. Stopping Time: If for every $n<\infty$, $\{N=n\}\in\mathcal{F}_n$ where $\mathcal{F}_n=\sigma(X_1,\cdots,X_n)$

Example: the hitting time of A, $N=\inf\{n:S_n\in A\}$

3°. Theorem 4.1.3: Conditional on $\{N<\infty\}$, $\{X_{N+n},n\geq 1\}$ is independent of $\mathcal{F}_N$ and has the same distribution as the original sequence.

(a) Theorem 4.1.5: Wald's Equation: $ES_N=EX_1EN$

(b) Theorem 4.1.6: Wald's Second Equation: $ES_T^2=\sigma^2ET$

4.2 Recurrence

1. Theorem .2.2 For any random walk, following are equivalent:

(i) $P(\tau_1 < \infty) = 1$; (ii) $P(S_m = 0 \ i.o.) = 1$; (iii) $\sum_{m=0}^{\infty} P(S_m = 0) = \infty$

4.3 Visit to 0, Arcsine Law

4.4 Renewal Theory

5 Martingales

5.1 Conditional Expectation

1. $Y = E(X|\mathcal{F})$ is r.v. (i) $Y \in \mathcal{F}$; (ii) for all $A \in \mathcal{F}$, $\int_A X dP = \int_A Y dP$

Remark: if $A = \Omega$, then $E(E(X|\mathcal{F})) = EX$

2. Lemma 5.1.1: Y is integrable. Hint: $E|Y| \leq E|X|$

3. Theorem 5.1.2: (a) $E(aX + Y|\mathcal{F}) = aE(X|\mathcal{F}) + E(Y|\mathcal{F})$

(b) If $X \leq Y$ then $E(X|\mathcal{F}) \leq E(Y|\mathcal{F})$ Hint: $\{A = E(X|\mathcal{F}) - E(Y|\mathcal{F}) \geq \epsilon > 0\}$

(c) If $X_n \geq 0$ and $X_n \uparrow X$ with $EX < \infty$, then $E(X_n|\mathcal{F}) \uparrow E(X|\mathcal{F})$

Hint: $\int_A E(X - X_n|\mathcal{F}) dP = 0$ for all A

4. Theorem 5.1.3: Jensen's Inequality: $\varphi(E(X|\mathcal{F})) \leq E(\varphi(X)|\mathcal{F})$.

$\Rightarrow$ Theorem 5.1.4: $E(|E(X|\mathcal{F})|^p) \leq E|X|^p$

5. Theorem 5.1.5: If $\mathcal{F} \subset \mathcal{G}$ and $E(X|\mathcal{G}) \in \mathcal{F}$ then $E(X|\mathcal{F}) = E(X|\mathcal{G})$

$\Rightarrow$ Theorem 5.1.6: If $\mathcal{F}_1 \subset \mathcal{F}_2$ then (i) $E(E(X|\mathcal{F}_1)|\mathcal{F}_2) = E(X|\mathcal{F}_1)$

(ii) $E(E(X|\mathcal{F}_2)|\mathcal{F}_1) = E(X|\mathcal{F}_1)$ Hint: if $X \in \mathcal{F}$ then $E(X|\mathcal{F}) = X$

Remark: the smaller σ-field always wins

6*. Theorem 5.1.7: If $X \in \mathcal{F}$ and $E|X|, E|XY| < \infty$, then $E(XY|\mathcal{F}) = XE(Y|\mathcal{F})$

Hint: indicator→simple→nonnegative→general

$\Rightarrow$ Theorem 5.1.8: $EX^2 < \infty \Rightarrow E(X|\mathcal{F})$ is the variable $Y \in \mathcal{F}$ that minimize $E(X-Y)^2$

Hint: $E(Z(X - E(X|\mathcal{F}))) = 0$ for $Z \in \mathcal{F}$

5.2 Martingales, Almost Sure Convergence

1. Martingale: (i) $E|X_n| < \infty$ (ii) $X_n \in \mathcal{F}$ for all n (iii) $E(X_{n+1}|\mathcal{F}_n) = X_n$ for all n;

Supermartingale: (iii) $E(X_{n+1}|\mathcal{F}_n) \leq X_n$; Submartingale: (iii) $E(X_{n+1}|\mathcal{F}_n) \geq X_n$

2. Theorem 5.2.1: X_n is supermartingale $\Rightarrow$ for $n > m$, $E(X_n|\mathcal{F}_m) \leq X_m$

$\Rightarrow$ Theorem 5.2.2: (i) X_n is submartingale $\Rightarrow$ for $n > m$, $E(X_n|\mathcal{F}_m) \geq X_m$

(ii) X_n is martingale $\Rightarrow$ for $n > m$, $E(X_n|\mathcal{F}_m) = X_m$

3. Theorem 5.2.3: X_n is martingale and φ is convex function $\Rightarrow \varphi(X_n)$ is submartingale

$\Rightarrow$ Theorem 5.2.4: X_n is submartingale and φ is increasing convex function $\Rightarrow \varphi(X_n)$ is submartingale

4. Predictable sequence H_n: If $H_n \in \mathcal{F}_{n-1}$ for all $n \geq 1$.

Remark: $(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$

5. Theorem 5.2.5: X_n is supermartingale. If $H_n \geq 0$ is predictable and each H_n is bounded then $(H \cdot X)_n$ is a supermartingale

$\Rightarrow$ Theorem 5.2.6: X_n is supermartingale $\Rightarrow X_{N\wedge n}$ is supermartingale

Hint: $H_n = 1_{N\geq n}$, $(H \cdot X)_n = X_{N\wedge n} = X_{N\wedge n} - X_0$

6*. Theorem 5.2.7: Upcrossing Inequality: Define $N_0 = -1$,

$N_{2k-1} = \inf\{m > N_{2k-2} : X_m \leq a\}$, $N_{2k} = \inf\{m > N_{2k-1} : X_m \geq b\}$

$U_n = \sup\{k : N_{2k} \leq n\}$. If X_m is submartingale then

$$(b-a)EU_n \leq E(X_n - a)^+ - E(X_0 - a)^+$$

Hint: $H = 1$ if $N_{2k-1} < m \leq N_{2k}$ and $= 0$ otherwise. $Y_n = a + (X_n - a)^+$, $K_m = 1 - H_m$

$(b-a)U_n \leq (H \cdot Y)_n$, $Y_n - Y_0 = (K \cdot Y)_n + (H \cdot Y)_n$

$\Rightarrow$ Theorem 5.2.8: Martingale Convergence Thereom (submartingale): X_n is submartingale with $EX^+ < \infty$, then X_n converge a.s. to X with $E|X| < \infty$.

Hint: $\cup_{a,b\in\mathbf{Q}}\{\liminf X_n < a < b < \limsup X_n\}$ has probability 0; Fatou's lemma

$\Rightarrow$ Theorem 5.2.9: Martingale Convergence Thereom (supermartingale): If $X_n \geq 0$ is supermartingale then $X_n \to X$ a.s. and $EX \leq Ex_0$

8. Theorem 5.2.10: Doob's Decomposition: Any submartingale X_n can be written a unique $X_n = M_n + A_n$ where M_n is martingale and A_n is predicatable increasing sequence with $A_0 = 0$.

Hint: $E(X_n|\mathcal{F}_{n-1}) = E(M_n + A_n|\mathcal{F}_{n-1}) = M_{n-1} + A_n = X_{n-1} + A_n - A_{n-1}$

5.3 Examples

5.4 Doob's Inequality, Convergence in L^p

1. Theorem 5.4.1: N is stoppint time with $P(N \leq k) = 1$, X_n is submartingale $\Rightarrow$ $EX_0 \leq EX_N \leq EX_k$

Hint: $X_{N\wedge n}$ is submartingale

2. Theorem 5.4.2: Doob's Inequality: X_n is submartingale, $\bar{X}_n = \max_{1\leq m\leq n} X_m^+$, $A = \{\bar{X}_n \geq \lambda\}$ then

$$\lambda P(A) \leq EX_n 1_A \leq EX_n^+$$

3. Theorem 5.4.7: Conditional variance formula:

$$Var((X_n - X_m)^2|\mathcal{F}_m) = E(X_n^2|\mathcal{F}_m) - X_m^2$$

5.5 Uniform Inequality, Convergence in L^1

5.6 Backwards Martingales

5.7 Optional Stopping Theorems

6 Question

1. metric, tight?
2. 3.2.9 why not directly?
3. Lemma 3.3.3 proof.