STAT5005 final exam 2018/19
[Totally 100 marks] (3:30-6:30pm, 6 December 2018)

Instructions:

1. Turn off all the communication devices during the examination.
2. This is a closed book examination. Only one A4-sized help sheet is allowed.
3. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.

Question 1: [30 marks]

(a) Let $\{A_{n,j} : n \geqslant 1, 1 \leqslant j \leqslant n\}$ be a triangular array of events. Suppose for any n, $\{A_{n,j} : 1 \leqslant j \leqslant n\}$ are independent. Suppose further that

$$\sum_{j=1}^{n} P(A_{n,j}) \to 0 \quad \text{as } n \to \infty.$$

Prove that

$$P(\cup_{j=1}^{n} A_{n,j}) \sim \sum_{j=1}^{n} P(A_{n,j}),$$

where $a_n \sim b_n$ means $a_n/b_n \to 1$ as $n \to \infty$ and $0/0$ is understood here to be equal to 1.

(b) Let $\{X_j, j \geqslant 1\}$ be independent, identically distributed random variables with mean 0 and finite variance $\sigma^2 > 0$. What are the limiting distributions of

$$\frac{\sum_{j=1}^{n} X_j}{\sqrt{\sum_{j=1}^{n} X_j^2}} \quad \text{and} \quad \frac{\sqrt{n}\sum_{j=1}^{n} X_j}{\sum_{j=1}^{n} X_j^2}$$

as $n \to \infty$? Justify the answer.

(c) Use the $\pi - \lambda$ theorem to prove that if $\mathcal{A}_1, \ldots, \mathcal{A}_n$ are independent, and each $\mathcal{A}_i$ is a π-system, then $\sigma(\mathcal{A}_1), \ldots, \sigma(\mathcal{A}_n)$ are independent.

Question 2: [15 marks] Let $X_1, X_2, \ldots$ be a sequence of independent random variables. Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Suppose for each $i \geqslant 1$, $E(X_i) = 0$ and $E(X_i^2) = \sigma_i^2 < \infty$. Let $S_1 = 0$ and

$$S_n = \sum_{1 \leqslant i < j \leqslant n} a_{ij} X_i X_j,$$

where $\{a_{ij} : 1 \leqslant i < j < \infty\}$ are constants.

(i) Prove that $\{S_n, \mathcal{F}_n : n \geqslant 1\}$ is a martingale.

(ii) Prove that

$$E[\max_{1 \leqslant i \leqslant n} (S_i^+)^2] \leqslant 4E(S_n^2)$$

(iii) Compute $E(S_n^2)$.

Question 3: [15 marks] Let $X_1, X_2, \ldots$ be a sequence of 1-dependent random variables, that is, for any integer $j \geqslant 1$, $\{X_i : i \leqslant j\}$ is independent of $\{X_i : i \geqslant j+2\}$. Suppose for each $i \geqslant 1$, $E(X_i) = 0$ and $E(X_i^2) = \sigma_i^2 < \infty$. Let $S_k = \sum_{i=1}^{k} X_i$. Prove that for any $x > 0$,

$$P(\max_{1 \leqslant k \leqslant n} |S_k| \geqslant x) \leqslant \frac{4}{x^2} \sum_{1 \leqslant i \leqslant n} \sigma_i^2.$$

Question 4: [20 marks] Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables. They may not have finite expectation. Let $S_n = X_1 + \cdots + X_n$. Fix a constant $0 < p < 1$. Prove that $E(|X_1|^p) < \infty$ if and only if as $n \to \infty$,

$$\frac{S_n}{n^{1/p}} \to 0 \quad a.s.$$

Question 5: [20 marks] Let $\{S_n, n \geqslant 1\}$ be a one-dimensional random walk.

(i) Prove that if $P(S_1 \neq 0) > 0$, then for any finite interval $[a, b]$ there exists $\epsilon < 1$ and $C > 0$ such that for all $n \geqslant 1$,

$$P\{S_j \in [a, b], 1 \leqslant j \leqslant n\} \leqslant C\epsilon^n.$$

(ii) Prove that if $P(S_n > 0 \text{ for all } n \geqslant 1) > 0$, then

$$\sum_{n=1}^{\infty} P(S_n \leqslant 0, S_{n+1} > 0) < \infty.$$