STAT5005 QE (2019-2020).

1. We first prove if $\frac{S_n}{n^{1/p}} \to 0$ a.s., then $E(|X_1|^p)<\infty$.

$$\frac{X_n}{n^{1/p}} = \frac{S_n}{n^{1/p}} - \frac{S_{n-1}}{(n-1)^{1/p}} \cdot \frac{(n-1)^{1/p}}{n^{1/p}}.$$

Since $\frac{S_n}{n^{1/p}} \to 0$ a.s., $\frac{X_n}{n^{1/p}} \to 0$ a.s.

Then we have

$$P\left(\left|\frac{X_n}{n^{1/p}}\right| > 1 \text{ i.o.}\right) = 0$$

By Borel-Cantelli lemma, we have

$$\sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{n^{1/p}}\right|>1\right) < \infty$$

$$E|X_1|^p = \int_0^\infty P(|X_1|^p > x)\,dx \le 1 + \sum_{n=1}^{\infty} P(|X_n|^p > n) < \infty$$

Then we prove the reverse. To prove $\frac{S_n}{n^{1/p}} \to 0$ a.s., it suffices to show

$$\sum_{i=1}^{n} \frac{X_i}{i^{1/p}} \text{ converges a.s.}$$, by Kronecker's lemma.

Set $A=1$, $Y_n = \frac{X_n}{n^{1/p}} \mathbb{1}\left\{\left|\frac{X_n}{n^{1/p}}\right| \le A\right\}$.

i) $$\sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{n^{1/p}}\right| > A\right) = \sum_{n=1}^{\infty} P\left(\left|\frac{X_n}{n^{1/p}}\right| > 1\right)$$

$$= \sum_{n=1}^{\infty} P(|X_n|^p > n)$$

$$= \sum_{n=1}^{\infty} P(|X_1|^p > n)$$, by iid condition.

$$\le E|X_1|^p$$

$$< \infty$$

ii) $$\sum_{n=1}^{\infty} E|Y_n| = \sum_{n=1}^{\infty} E\left|\frac{X_n}{n^{1/p}} \mathbb{1}\left\{\left|\frac{X_n}{n^{1/p}}\right| \le 1\right\}\right|$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^{1/p}} E|X_n \mathbb{1}\{|X_n| \le n^{1/p}\}|$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{1}{n^{1/p}} E|X_n \mathbb{1}\{(j-1)^{1/p} \le |X_n| \le j^{1/p}\}|$$

$$= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{1}{n^{1/p}} E|X_1 \mathbb{1}\{(j-1)^{1/p} \le |X_1| \le j^{1/p}\}|$$, by iid condition

$$\le \sum_{j\ge1}^{\infty} C_p j^{1-\frac{1}{p}} \int_{j^{1/p}}^{(j-1)^{1/p}} P(|X_1|>x)\,dx$$

$$\le \sum_{j=1}^{\infty} \frac{C_p}{p} \int_{(j-1)^{1/p}}^{j^{1/p}} p x^{p-1} P(|X_1|>x)\,dx$$

$$= \frac{C_p}{p} E|X_1|^p < \infty$$

iii) $$\sum_{n=1}^{\infty} Var(Y_n) \le \sum_{n=1}^{\infty} E(Y_n^2)$$

$$= \sum_{n=1}^{\infty} E\left(\frac{X_n^2}{n^{2/p}} \mathbb{1}\{|X_n| \le n^{1/p}\}\right)$$

$$= \sum_{n=1}^{\infty} \sum_{j=1}^{n} \frac{1}{n^{2/p}} E(X_n^2 \mathbb{1}\{(j-1)^{1/p} \le |X_n| \le j^{1/p}\})$$

$$= \sum_{j=1}^{\infty} \sum_{n=j}^{\infty} \frac{1}{n^{2/p}} E(X_1^2 \mathbb{1}\{(j-1)^{1/p} \le |X_1| \le j^{1/p}\})$$, by iid assumption.

$$\le \sum_{j=1}^{\infty} C_p j^{1-\frac{2}{p}} \int_{(j-1)^{1/p}}^{j^{1/p}} x P((j-1)^{1/p} \le |X| \le j^{1/p})\,dx$$

$$\le \sum_{j=1}^{\infty} \frac{C_p}{p} \int_{(j-1)^{1/p}}^{j^{1/p}} p\, 2x \cdot x^{p-2} P((j-1)^{1/p} \le |x| \le j^{1/p})\,dx$$

$$\le \frac{2C_p}{p} E|X_1|^p < \infty$$

By Kolmogorov's Three-series Theorem, we have

$$\sum_{n=1}^{\infty} \frac{X_n}{n^{1/p}} \text{ converges a.s.}$$

Then the desired result follows.

2. (a) $X_1, X_2, \cdots$ are independent, and obviously for all $n = 1, 2, \cdots$,

$$E(X_n) = 0$$

By Theorem in lecture notes, when $\sum_{n=1}^{\infty} E(X_n^2) < \infty$, $\sum_{i=1}^{\infty} X_i$ converges almost surely.

$$E(X_n^2) = a_n^2$$

Therefore, when $\sum_{n=1}^{\infty} a_n^2 < \infty$, $\sum_{n=1}^{\infty} X_n$ converges a.s.

(b) It's not true.

Suppose $Z \sim N(0,1)$, and $X_n = Z$, $Y_n = -Z$, $X = Y = Z$, then it's obviously,

$$X_n \xrightarrow{d} X \text{ and } Y_n \xrightarrow{d} Y.$$

However, $X_n + Y_n = 0$, $X + Y = 2Z$, then $X_n + Y_n$ doesn't converge to $X+Y$ in distribution.

When $\{X, X_1, \cdots\}$ are independent with $\{Y, Y_1, \cdots\}$, it's true. The characteristic function for $(X_n + Y_n)$ is,

$$\varphi_{X_n+Y_n}(t) = \varphi_{X_n}(t) \cdot \varphi_{Y_n}(t) \text{, by independence.}$$

$$\to \varphi_X(t)\, \varphi_Y(t). \text{ since } X_n \xrightarrow{d} X \text{ implies } \varphi_{X_n}(t) \to \varphi_X(t),\ \forall t \in \mathbb{R}.$$

$$= \varphi_{X+Y}(t) \quad \text{similar for } Y.$$

Then $X_n + Y_n \xrightarrow{d} X + Y$.

3. (a) i) $E|Y_n| < \infty$ since $Y_n \le 1$ a.s.

ii) It's obviously that $R_n \in \mathcal{F}_n$. ~~Therefore~~ $Y_n = f(R_n)$, therefore $Y_n \in \mathcal{F}_n$.

iii) $E(Y_n \mid \mathcal{F}_{n-1})$

$$= E\left(\frac{\sum_{i=1}^{n-1} X_i + X_n}{r+b+n} \,\middle|\, \mathcal{F}_{n-1} \right) \text{, where } X_i = \{\text{the drawn ball is red at } i\text{th operation}\}.$$

$$= \frac{\sum_{i=1}^{n-1} X_i}{r+b+n} + E\left(\frac{X_n}{r+b+n} \,\middle|\, \mathcal{F}_{n-1} \right)$$

$$= \frac{\sum_{i=1}^{n-1} X_i}{r+b+n} + \frac{1}{r+b+n} \cdot \frac{Y_{n-1}}{r+b+(n-1)}$$

$$= Y_{n-1}$$

Therefore, $\{Y_n\}$ is a martingale.

and $\{Y_n\}$ is a martingale and ~~is~~ also submartingale.

b) $0 \le E\{Y_n\} \le 1$. Therefore by Martingale Convergence Theorem,

$\{Y_n\}$ converges almost surely to a limit Y with $E|Y| < \infty$.

c) $T = \inf\{n : r+b+n-R_n > 1\}$.

$\{T=n\} \in \mathcal{F}_n$, $\forall n=1,2,\dots$, hence T is a stopping time.

$$E\left(\frac{T}{T+2}\right) = E(Y_T)$$

$= Y_0$, since $\{Y_n\}$ is bounded martingale

$= \frac{1}{2}$

(d) $P(Y_n \ge \frac{3}{4}$ for some $n)$

$= P(\max_{1 \le k \le n} Y_n \ge \frac{3}{4})$

$\le \dfrac{E(Y_n)}{\frac{3}{4}}$ - by Doob's inequality

$= \dfrac{(\frac{1}{2})}{(\frac{3}{4})}$

$= \frac{2}{3}$.