STAT5005 final exam 2021/22
[Totally 100 marks] (2:30-5:30pm, 9 December 2021)

Instructions:

1. Turn off all the communication devices during the examination.
2. This is a closed book examination. Only one A4-sized help sheet is allowed.
3. Cheating is a serious offence. Students who commit the offence may score no mark in the examination. Furthermore, more serious penalty may be imposed.

Question 1: [20 marks] (a) Prove that two random variables X and Y are independent if and only if $E[g(X)h(Y)] = E[g(X)]E[h(Y)]$ for all bounded and continuous functions g and h.

(b) Let X be a random variable with finite variance $\sigma^2 > 0$. Prove that there exist constants $A > 0$ and $\delta > 0$ such that the characteristic function $\varphi(t)$ of X satisfies

$$|\varphi(t)| \leqslant 1 - At^2 \quad \text{for } |t| \leqslant \delta.$$

[For complex numbers, $|a + bi| := \sqrt{a^2 + b^2}$. Hint: Consider mean zero random variables first.]

Question 2: [20 marks] Let $S_n = \sum_{j=1}^n X_j$ where $\{X_j, 1 \leqslant j \leqslant n\}$ are independent random variables with $EX_j = 0, EX_j^2 = \sigma_j^2$ and $s_n^2 = \sum_{j=1}^n \sigma_j^2 > 0$. Prove that for $\lambda > \gamma > 1$,

$$P(\max_{1\leqslant j\leqslant n} |S_j| \geqslant \lambda s_n) \leqslant \frac{\gamma^2}{\gamma^2 - 1} P(|S_n| \geqslant (\lambda - \gamma)s_n).$$

[Hint: Let $T = \inf\{1 \leqslant j \leqslant n : |S_j| \geqslant \lambda s_n\}$ and $T = n + 1$ otherwise. First argue that for $1 < \gamma < \lambda$,

$$P(\max_{1\leqslant j\leqslant n} |S_j| \geqslant \lambda s_n) \leqslant P(|S_n| \geqslant (\lambda-\gamma)s_n) + \sum_{j=1}^{n-1} P(T = j)P(|S_n - S_j| \geqslant \gamma s_n).$$

Then apply Chebychev's inequality.]

Question 3: [15 marks] Let $A_1, A_2, \dots$ be a sequence of independent events such that

$$\phi(n) := \sum_{i=1}^n P(A_i) \to \infty, \text{ as } n \to \infty.$$

Let $X_n = \sum_{i=1}^n 1_{A_i}$. Fix a positive integer k and let

$$T = \inf\{n \geqslant 1 : X_n = k\}.$$

That is, T is the first time k of the events have occurred. Prove that

$$(a) \quad T < \infty \ a.s.$$

and

$$(b) \quad E[\phi(T)] = k.$$

Question 4: [15 marks] Suppose for each $n \geqslant 1$, $\{X_{n1}, X_{n2}, \ldots, X_{nn}\}$ are independent random variables with

$$P(X_{ni} = 1) = p_{ni}, \quad P(X_{ni} = 0) = 1 - p_{ni}.$$

Assume that there exists a positive constant λ such that as $n \to \infty$, (i) $\sum_{i=1}^n p_{ni} \to \lambda$ and (ii) $\max_{1 \leqslant i \leqslant n} p_{ni} \to 0$. Prove that $\sum_{i=1}^n X_{ni}$ converges in distribution to $Poi(\lambda)$, the Poisson distribution with parameter λ. [Assume that we know the characteristic function of $Poi(\lambda)$ is $e^{\lambda(e^{it}-1)}$.]

Question 5: [15 marks] Let Z_n denote the number of particles in a population at time $n = 1, 2, 3, \ldots$. Note that if $Z_n = 0$ at some time n, the population becomes extinct (i.e., $X_{n+m} = 0$ for all $m \geqslant 0$). Suppose that for every integer $N > 0$, there exists $\delta > 0$ such that for all n, the conditional probability

$$P(X_{n+1} = 0 | X_1 = x_1, \ldots, X_n = x_n) \geqslant \delta, \text{ if } x_n \leqslant N.$$

Let F be the event of extinction, i.e., $F = \cup_{n=1}^{\infty} \{X_n = 0\}$. Let G be the event $\{X_n \to \infty\}$. Prove that $P(F) + P(G) = 1$, that is, the population eventually becomes either extinct or explode.

Question 6: [15 marks] Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables. Let $p + q = 1$ with $p, q \in (0, 1)$, $p \neq q$ and suppose that

$$P(X_i = 1) = p, \quad P(X_i = -1) = q.$$

Set $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$ and note that $\mathcal{F}_n, n \geqslant 1$ is a filtration. The asymmetric random walk is the stochastic process

$$S_n = \sum_{i=1}^n X_i, \quad n \geqslant 1.$$

Define $M_n = (q/p)^{S_n}$. Prove that $M_n, n \geqslant 1$ is a martingale.