Department of Statistics, The Chinese University of Hong Kong
Qualifying Examination in June 2020

STAT5010 Advanced Statistical Inference

Instruction: All questions are compulsory. Throughout, the abbreviations 'i.i.d', 'pdf/pmf' and 'MLE' stand for 'independent and identically distributed', 'probability density/mass function' and 'maximum likelihood estimator', respectively. A normal distribution in $\mathbb{R}^d$ with mean vector $\boldsymbol{\mu}$ and covariance matrix Σ is denoted by $N_d(\boldsymbol{\mu}, \Sigma)$ and $N(\boldsymbol{\mu}, \Sigma)$ corresponds to the univariate case $d = 1$. For observations X arising from a parametric model $\{f(\cdot, \theta) : \theta \in \Theta\}$, $\Theta \subseteq \mathbb{R}$, the quadratic risk of a decision rule $\delta(X)$ is defined to be $R(\delta, \theta) = E_\theta(\delta(X) - \theta)^2$.

1. Consider an i.i.d. sample $X_1, \ldots, X_n$ arising from the model

$$\{f(\cdot, \theta) : \theta \in \mathbb{R}\}, \quad f(x, \theta) = \frac{1}{2} e^{-|x-\theta|}, \quad x \in \mathbb{R},$$

of *Laplace distributions*. Assuming n to be odd for simplicity, find the MLE. Discuss what happens when n is even. Calculate also the Fisher information.

2. Given $X_1, \ldots, X_n$ i.i.d. random variables such that $E(X_1) = 0$, $E(X^2) \in (0, \infty)$, the *Student t-statistic* is given by

$$t_n = \frac{n^{1/2} \bar{X}_n}{S_n}, \quad \bar{X}_n = \frac{1}{n} \sum_{i=1}^n X_i, \quad S_n^2 = \frac{1}{(n-1)^2} \sum_{i=1}^n (X_i - \bar{X}_n)^2.$$

Show that $t_n \xrightarrow{d} N(0, 1)$ as $n \to \infty$. Assuming now $E(X_1) = \mu \in \mathbb{R}$, deduce an asymptotic level $1 - \alpha$ confidence interval for μ.

3. (a) Consider estimation of $\theta \in \Theta = [0, 1]$ in a $Bin(n, \theta)$ model under quadratic risk.
   i. Find the unique minimax estimator $\tilde{\theta}$ of θ and deduce that the maximum likelihood estimator $\hat{\theta}$ of θ is *not* minimax for fixed sample size $n \in \mathbb{N}$.
   ii. Show, however, that the maximum likelihood estimator dominates $\tilde{\theta}_n$ in the large sample limit by proving that, as $n \to \infty$,

$$\lim_{n\to\infty} \frac{\sup_\theta R(\hat{\theta}_n, \theta)}{\sup_\theta R(\tilde{\theta}_n, \theta)}$$

   and that

$$\lim_{n\to\infty} \frac{R(\hat{\theta}_n, \theta)}{R(\tilde{\theta}_n, \theta)} < 1 \quad \text{for all } \theta \in [0, 1], \theta \neq \frac{1}{2}.$$

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   (b) Consider $X_1, \ldots, X_n$ i.i.d. from a $N(\theta, 1)$-model where $\theta \in \Theta = [0, \theta)$. Show that the sample mean $\bar{X}_n$ is inadmissible for quadratic risk, but that it is still minimax. What happens if $\Theta = [a, b]$ for some $0 < a < b < \infty$.

4. Let $(X, X_n : n \in \mathbb{N})$ be random vectors in $\mathbb{R}^k$.
   (a) Suppose $E\|X_n - X\| \to 0$ as $n \to \infty$ where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^k$. Deduce that $X_n \xrightarrow{p} X$ as $n \to \infty$.
   (b) Show that the converse in (b) is false.

5. Suppose that $X_1, \ldots, X_n$ are independent random variables, and $X_i \sim N(\theta_i, \sigma^2)$ for $i = 1, \ldots, n$, where σ^2 is a known constant. Find a size α UMP test for

$$H_0 : \theta_1 = \ldots = \theta_n = 0 \quad \text{versus} \quad H_1 : \theta_i = \theta_{i0}, \quad i = 1, \ldots, n,$$

where $\theta_{10}, \ldots, \theta_{n0}$ are given constants. You are required to identify the test statistic and the rejection region.