1. (a) $\mu \sim N(\mu_0, \Sigma_0)$. $f(\mu|y) \propto f(y|\mu)f(\mu) \propto \exp\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu) + (\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)\}$

$\propto \exp\{-\frac{1}{2}[\mu'(n\Sigma^{-1}+\Sigma_0^{-1})\mu - 2\mu'(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)]\}$

$\mu|y \sim N(\mu^* = \Sigma^*(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0),$ $\Sigma^* = (n\Sigma^{-1}+\Sigma_0)^{-1})$

$\propto \exp\{-\frac{1}{2}[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)]'(n\Sigma^{-1}+\Sigma_0^{-1})[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)]\}$

(b) $f(\mu)=1$. $\mu|y \sim N(\bar{y}, \frac{1}{n}\Sigma)$

# Problem from STAT5020

(c) from (a) $\mu^{(1)}|\mu^{(2)},y \sim N(\mu_1^* + \Sigma_{12}^*\Sigma_{22}^{*-1}(\mu^{(2)}-\mu_2^*), \Sigma_{11}^*-\Sigma_{12}^*\Sigma_{22}^{*-1}\Sigma_{21}^*)$

1. Assume that y is a $p \times 1$ random vector, with the multivariate normal distribution

d) $f(y|\mu,\Sigma) \propto |\Sigma|^{-\frac{n}{2}}\exp\{-\frac{1}{2}\sum(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\} = |\Sigma|^{-\frac{n}{2}}\exp\{-\frac{1}{2}tr(\Sigma^{-1}[n(\bar{x}-\mu)(\bar{x}-\mu)'+S])\}$

$$\mathrm{y}|\mu, \Sigma \sim N(\mu, \Sigma),$$

$S=\sum(x_i-\bar{x})(x_i-\bar{x})'$

$f(\mu|\Sigma) \propto \exp\{-\frac{\kappa_0}{2}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\}|\Sigma|^{-\frac{1}{2}}$

where $\mu$ is a $p \times 1$ vector and $\Sigma$ is a $p \times p$ covariance matrix, which is symmetric and positive definite.

$= \exp\{-\frac{\kappa_0}{2}tr\,\Sigma^{-1}(\mu-\mu_0)(\mu-\mu_0)'\}$

(a) When $\Sigma$ is known, specify a conjugate prior distribution for $\mu$ and derive the corresponding posterior distribution.

$f(\Sigma) \propto |\Sigma|^{-\frac{1}{2}(\rho_0+p+1)}\exp\{-\frac{1}{2}tr\,\Sigma^{-1}\Sigma_0\}$

(b) When $\Sigma$ is known, specify a noninformative prior distribution for $\mu$ and derive the corresponding posterior distribution.

$\mu|\Sigma, y \sim N(\frac{n\bar{y}+\kappa_0\mu_0}{n+\kappa_0}, \frac{\Sigma}{n+\kappa_0})$

(c) If $\mu$ is partitioned into subvectors $\mu^{(1)}$ and $\mu^{(2)}$, derive the posterior conditional distribution $p(\mu^{(1)}|\mu^{(2)}, \mathrm{y})$ with a conjugate prior and known $\Sigma$.

$\Sigma|y \sim IW((\rho+\Sigma_0+\frac{n\kappa_0}{n+\kappa_0}(\bar{x}-\mu_0)(\bar{x}-\mu_0)')^{-1}, n+\rho)$

(d) When $\Sigma$ is unknown, a joint prior distribution for $(\mu, \Sigma)$ is assigned as

$\mu|\Sigma \sim N(m_0, \Sigma)$

$f(y|\mu,\Sigma)$

$$\Sigma \sim IW(\rho_0, \Sigma_0^{-1})$$
$$\mu|\Sigma \sim N(\mu_0, \Sigma/\kappa_0),$$

$\Lambda_{\kappa_0} = m_0$

$H_0 y_K = I$

where $IW(\cdot, \cdot)$ denotes the inverse Wishart distribution, and $\rho_0$, $\Sigma_0$, $\mu_0$, and $\kappa_0$ are hyperparameters. Derive the joint posterior distribution of $(\mu, \Sigma)$.

$\varphi_{y|c} = \sigma^2 V$

$f(\mu,\Sigma|y) \propto f(y|\mu,\Sigma)f(\mu|\Sigma)f(\Sigma)$

2. A generalized linear mixed effect model (GLMM) is defined as

$$g(\mu_{it}) = \mathrm{x}_{it}^T\beta + \mathrm{z}_{it}^T\mathrm{u}_i,$$

where $g(\cdot)$ is a link function, $\mu_{it} = E(y_{it}|\mathrm{u}_i)$, $y_{it}$ is the observation for subject $i$ at time $t$, $\mathrm{x}_{it}$ and $\mathrm{z}_{it}$ are vectors of explanatory variables, $\mathrm{u}_i$ is a $q \times 1$ vector of subject-specific random effects, and $\mathrm{u}_i \sim N(0, \Sigma)$.

(a) Explain why the GLMM is commonly used in the analysis of longitudinal data.

(b) If $g(\cdot)$ is a log link, $q = 1$, $z_{it} = 1$, and $u_i \sim N(0, \sigma^2)$, show that

$$cov(y_{it}, y_{is}) = \exp(\mathrm{x}_{it}^T\beta + \mathrm{x}_{is}^T\beta)\{\exp(\sigma^2)(\exp(\sigma^2) - 1)\}.$$

$\log \mu_{it} = x_{it}^T\beta + u_i$

$Cov(y_{it}, y_{is}) = \mathbb{E}[\underbrace{Cov(y_{it}, y_{is}|u_i)}_{=0}] + \underbrace{Cov(\mathbb{E}(y_{it}|u_i), \mathbb{E}(y_{is}|u_i))}_{=Cov(\mu_{it}, \mu_{is})}$

Given $u_i$, $y_{it} \perp y_{is}$

$= e^{x_{it}^T\beta + x_{is}^T\beta} \cdot Var(e^{u_i})$



$Var(e^{u_i}) = \mathbb{E}[e^{2u_i}] - (\mathbb{E}[e^{u_i}])^2 = e^{2\sigma^2} - e^{\sigma^2}$