Qualify Exam (June 10, 2014)

STAT5005+STAT5010 (09:00am-12:00pm)

STAT5005 Advanced Probability Theory

1.
   a) Prove that $W_n \xrightarrow{d} Z$ iff $\mathbb{E}f(Wn) \to \mathbb{E}f(Z)$ for any bounded continuous function f.
   b) $X_1, X_2, \ldots$ are iid with mean 0 and variance 1. Prove that

$$\lim_{n\to\infty} \frac{1}{n^{1/2}} \mathbb{E}\left[\sum_{i=1}^{n} |X_i|\right] = \sqrt{\frac{2}{\pi}}.$$

2.
   a) Prove the Marcinkiewicz-Zygmund strong law of large numbers ($\mathbb{E}(\mathrm{X}) = 0, 1 < p < 2$).
   b) $X_1, X_2, \ldots$ are iid with mean 0 and variance 1. $S_n = X_1 + \cdots + X_n$. Prove that

$$\limsup_{n\to\infty} \frac{S_n}{\sqrt{2n \log\log n}} = 1, a.s..$$

3.
   a) Prove that if A_1 and A_2 are independent, then $\sigma(A_1)$ and $\sigma(A_2)$ are independent.
   b) $X_1, X_2, \ldots$ are iid with $\mathbb{P}(X_i \le x) = e^{-x}$. $S_n = X_1 + \cdots + X_n$. Find the limiting distribution of $\sum_{i=1}^{n} I(X_i S_n > 1)$.
   c) $X_1, X_2, \ldots$ are iid Unif(0,1) random variables. $S_n = X_1 + \cdots + X_n$. Let $T = \inf\{n: S_n > 1\}$. Find $\mathbb{P}(T > n), \mathbb{E}(T), \mathbb{E}(S_T)$.

STAT5010 Advanced Statistical Inference

1. $X_1, \ldots, X_n$ sample from $N(\theta, \sigma^2)$
   a) If $\sigma^2 = \sigma_0^2$ known, prove $\bar{X}$ is UMVUE of θ
   b) If σ^2 is unknown, prove $\bar{X}$ is still UMVUE of θ by noting that $\bar{X}$ doesn't depend on σ_0^2.
   c) If $\theta = \theta_0$ known, show that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is not UMVUE of σ^2.
   d) If $\sigma^2 = \sigma_0^2$ known, find the UMVUE of $\mathbb{P}(\mathrm{X}_1 \ge 0)$.
2. Two-sample test with equal variance. Derive LRT for $H_0: \mu_X - \mu_Y \ge \delta \leftrightarrow H_1: \mu_X - \mu_Y < \delta$.