To prove:

$$\frac{S_n}{B_n} \xrightarrow{d} N(0,1)$$

5. S_n is submartingale, τ is stopping time, to prove

   (1)
   $$E(S_{\tau\wedge n}) \leq ES_n$$

   (2)
   $$P(\max_k S_k > x) \leq E(|S_n| 1(\max_k S_k > x))$$

STAT 5030

1. Prove XGX^T is invariant of generalized inverse G of $X^T X$

2. Consider the model

   $$y_{ij} = \mu + \tau_i + \varepsilon_{ij}, i = 1,2,3,4, j = 1,2,3,4,$$

   where ε_{ij} are independently distributed as $N(0, \sigma^2)$.

   (1) Let $\beta = (\mu, \tau_1, \tau_2, \tau_3, \tau_4)'$. Find a set of 4 linearly independent estimable functions of β.

   (2) Derive a test to test the null hypothesis $H_0 : \tau_1 - \tau_2 = \tau_3 - \tau_4$.

   (3) Is $\tau_1 + 2\tau_2$ estimable? Why?

3. There are two groups of data (Y_1, X_1), (Y_2, X_2) with

   $$Y_1 = X_1\beta + \varepsilon_1,$$

   $$Y_2 = X_2\beta + \varepsilon_2,$$

   X_1 and X_2 is not necessarily full-rank. And suppose that $\lambda^T\beta$ is estimable.

   (1) T_1 and T_2 are BLUE of $\lambda^T\beta$ for data (Y_1, X_1) and (Y_2, X_2), respectively. Give T_1 and T_2 and calculate $Var(T_1)$ and $Var(T_2)$

   (2) Let $T(\alpha) = \alpha T_1 + (1-\alpha)T_2$. Find α to minimize $Var(T(\alpha))$

   (3) Let $Y = (Y_1^T, Y_2^T)^T, X = (X_1^T, X_2^T)^T$, give the BLUE T_3 of $\lambda^T\beta$ for data (Y, X). And calculate $Var(T_3)$.

   (4) Explain $Var(T_3) \leq Var(T(\alpha))$ with equality when $r(X_1) = 1$ or $r(X_2) = 1$

(5) A and B are symmetric and nonnegtive matrix and a is in the vector space of A and B, that's to say, there exist x and y s.t. $a = Ax = By = (A+B)z$, then

$$a^T A^- a a^T B^- a \geq a^T (A+B)^- a (a^T A^- a + a^T B^- a)$$

with equality if $r(A) = 1$ or $r(B) = 1$.

Hint:

$$\begin{aligned}
a^T A^- a &= x^T A x = x^T P P^T x \\
a^T B^- a &= y^T B y = y^T Q Q^T y \\
a^T (A+B)^- a &= z^T (A+B) z = z^T A z + z^T B z \\
(z^T A x)^2 = (z^T P P^T x)^2 &\leq (z^T P P^T z)(x^T P P^T x) = (z^T A z)(x^T A x) \\
(z^T B y)^2 = (z^T Q Q^T y)^2 &\leq (z^T Q Q^T y)(x^T Q Q^T y) = (z^T B z)(y^T B y) \\
z^T A z + z^T B z &\geq (z^T (A+B) z)^2 (\frac{1}{x^T A x} + \frac{1}{y^T B y}) \\
z^T A z + z^T B z &\leq \frac{x^T A x y^T B y}{x^T A x + y^T B y}
\end{aligned}$$

If $r(A) = 1$, then $A = uu^T$, where u is a vector. Then

$$(z^T u u^T x)^2 = (z^T u u^T z)(x^T u u^T x)$$

$$\begin{aligned}
By &= Ax = uu^T x = ku \\
Bz &= Ax - Az = (k-l)u
\end{aligned}$$

where $k = u^T x, l = u^T z$. Thus, we have $Bz = QQ^T z = \frac{k-l}{k} QQ^T y$, then $Q^T z = \frac{k-l}{k} Q^T y$ s.t.

$$(z^T Q Q^T y)^2 = (z^T Q Q^T y)(x^T Q Q^T y).$$

4. About ridge regression and LASSO.