Department of Statistics, The Chinese University of Hong Kong
STAT5010 Advanced Statistical Inference (Term 1, 2020–21)

Mid-term Test on 2nd November 2020 (3:30 p.m.– 5:30 p.m. [+5 min grace period])

Instructions to Candidates

1. Attempt all five questions.

2. While every attempt is made to avoid defective questions, sometimes they do occur. If you believe a question is defective, the invigilators CANNOT give you any guidance beyond the instructions on the exam booklet.

3. The answer script in one single file should be submitted in `.pdf` format of size less than or equal to 5Mb to the collection box set up in Blackboard by 5:30 p.m. GMT+8 on 2nd November 2020. Please reserve at least 15 minutes of the exam time to save, convert and upload the file. No late submission or file in a wrong format will be entertained.

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1. Let $X_1, \dots, X_n$ be a random sample from a distribution with pdf $f(x;\theta)$. Consider a statistic $S(X)$.

   (a) What does it mean for S to be sufficient for θ?

   (b) What does it mean for S to be complete for the family of distributions $\{f_S(\cdot \mid \theta) : \theta \in \Omega\}$? Why are complete statistics important for finding uniformly minimum variance unbiased estimators (UMVUEs)?

2. (a) Let $X_1, \dots, X_n$ be i.i.d. from $N(0, 1)$, show that $\bar{X}$ and $(X_1 - \bar{X}, \dots, X_n - \bar{X})$ are independent.

   (b) Show that if $T(X)$ is a complete and sufficient statistic, then T is also minimal sufficient.

   (c) Let T be a sufficient statistic for $\mathcal{P} = \{P_\theta : \theta \in \Omega\}$, and let δ be an estimator of $g(\theta)$, and define $\eta(T) = E(\delta(X) \mid T)$. If $\theta \in \Omega$, $R(\theta, \delta) < \infty$ and $L(\theta, \cdot)$ is convex, where $R(\theta, \delta)$ denotes the risk function with respect to L, then $R(\theta, \eta) \leq R(\theta, \delta)$.

3. Let $X_1, \dots, X_n$ be a random sample from the $Gamma(3, \beta)$ distribution where the density of a Gamma (α, β) random variable is given by

$$f(x; \alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}.$$

   (a) Show that $E(X_1) = 3\beta^{-1}$.

   (b) Find the UMVUE of β. Calculate also the variance of your estimator.

4. Let $X_1, \dots, X_n$ be a random sample from the Poisson distribution with parameter λ. Find the UMVUE for $g(\lambda) = e^{-\lambda}$.

5. Let $X_1, \ldots, X_n$ be observations satisfying

$$X_i = \theta X_{i-1} + \epsilon_i, \quad i = 2, \ldots, n,$$

Where $X_1, \epsilon_2, \ldots, \epsilon_n$ are independent with $X_1 \sim N(0, (1-\theta^2)^{-1})$ and $\epsilon_i \sim N(0,1), i = 1, \ldots, n$. The parameter space is $\Omega = \{\theta : -1 < \theta < 1\}$. Show that the Cramér-Rao lower bound for unbiased estimator of θ is given by

$$B_n(\theta) = \frac{(1-\theta^2)^2}{2\theta^2 + (n-1)(1-\theta^2)}.$$