1.(a) $\mu \sim N(\mu_0, \Sigma_0)$. $f(\mu|y) \propto f(y|\mu) f(\mu) \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)'\Sigma^{-1}(y_i-\mu) + (\mu-\mu_0)'\Sigma_0^{-1}(\mu-\mu_0)\right\}$

$\propto \exp\left\{-\frac{1}{2}\left[\mu'(n\Sigma^{-1}+\Sigma_0^{-1})\mu - 2\mu'(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)\right]\right\}$

$\mu|y \sim N(\mu^* = \Sigma^*(\Sigma^{-1}y + \Sigma_0^{-1}\mu_0),$ $\propto \exp\left\{-\frac{1}{2}\left[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)\right]'(n\Sigma^{-1}+\Sigma_0^{-1})\left[\mu-(n\Sigma^{-1}+\Sigma_0^{-1})^{-1}(\Sigma^{-1}y+\Sigma_0^{-1}\mu_0)\right]\right\}$

$\Sigma^* = (n\Sigma^{-1}+\Sigma_0)^{-1})$

(b) $f(\mu)=1$. $\mu|y \sim N(\bar{y}, \frac{1}{n}\Sigma)$

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(c) from (a), $\mu^{(1)}|\mu^{(2)}, y \sim N(\mu_1^* + \Sigma_{12}^*\Sigma_{22}^{*-1}(\mu_2^{(2)}-\mu_2^*), \Sigma_{11}^* - \Sigma_{12}^*\Sigma_{22}^{*-1}\Sigma_{21}^*)$

1. Assume that y is a $p \times 1$ random vector, with the multivariate normal distribution

d) $f(y|\mu,\Sigma) = |\Sigma|^{-\frac{n}{2}}\exp\left\{-\frac{\sum}{2}(y_i-\mu)'\Sigma^{-1}(y_i-\mu)\right\} = |\Sigma|^{-\frac{n}{2}}\exp\left\{-\frac{1}{2}tr(\Sigma^{-1}[n(\bar{x}-\mu)(\bar{x}-\mu)'+S])\right\}$ $S=\sum(x_i-\bar{x})(x_i-\bar{x})'$

$$y|\mu, \Sigma \sim N(\mu, \Sigma),$$

$f(\mu|\Sigma) \propto \exp\left\{-\frac{K_0}{2}(\mu-\mu_0)'\Sigma^{-1}(\mu-\mu_0)\right\}|\Sigma|^{-\frac{1}{2}}$

$= \exp\left\{-\frac{K_0}{2}tr\,\Sigma^{-1}(\mu-\mu_0)(\mu-\mu_0)'\right\}$

where μ is a $p \times 1$ vector and Σ is a $p \times p$ covariance matrix, which is symmetric and positive definite.

$f(\Sigma) \propto |\Sigma|^{-\frac{1}{2}(\rho_0+p+1)}\exp\left\{-\frac{1}{2}tr\,\Sigma^{-1}\Sigma_0\right\}$

(a) When Σ is known, specify a conjugate prior distribution for μ and derive the corresponding posterior ~~distribution.~~

(b) When Σ is known, specify a noninformative prior distribution for μ and derive the corresponding posterior distribution.

$\Sigma, y \sim N(\frac{n\bar{y}+K_0\mu_0}{n+K_0}, \frac{\Sigma}{n+K_0})$

(c) If μ is partitioned into subvectors $\mu^{(1)}$ and $\mu^{(2)}$, derive the posterior conditional distribution $p(\mu^{(1)}|\mu^{(2)}, y)$ with a conjugate prior and known Σ.

$|y \sim IW((\frac{S}{p} + \Sigma_0 + \frac{nK_0}{n+K_0}(\bar{x}-\mu_0)(\bar{x}-\mu_0)')^{-1}, n+\rho_0)$

(d) When Σ is unknown, a joint prior distribution for (μ, Σ) is assigned as

$f(y|\mu,\Sigma)$

$$\begin{aligned} \Sigma &\sim IW(\rho_0, \Sigma_0^{-1}) \\ \mu|\Sigma &\sim N(\mu_0, \Sigma/\kappa_0), \end{aligned}$$

$\mu|\Sigma \sim N(m_0, \Sigma)$

$\Lambda_{K_0} = m_0$

$H_0 y_K = I$

where $IW(\cdot, \cdot)$ denotes the inverse Wishart distribution, and ρ_0, Σ_0, μ_0, and κ_0 are hyperparameters. Derive the joint posterior distribution of (μ, Σ). $\varphi_{q|K} = \sigma^2 V$

$f(\mu,\Sigma|y) \propto f(y|\mu,\Sigma) f(\mu|\Sigma) f(\Sigma)$

2. A generalized linear mixed effect model (GLMM) is defined as

$$g(\mu_{it}) = \mathrm{x}_{it}^T\beta + \mathrm{z}_{it}^T\mathrm{u}_i,$$

where $g(\cdot)$ is a link function, $\mu_{it} = E(y_{it}|\mathrm{u}_i)$, y_{it} is the observation for subject i at time t, x_{it} and z_{it} are vectors of explanatory variables, u_i is a $q \times 1$ vector of subject-specific random effects, and $\mathrm{u}_i \sim N(0, \Sigma)$.

(a) Explain why the GLMM is commonly used in the analysis of longitudinal data.

(b) If $g(\cdot)$ is a log link, $q = 1$, $z_{it} = 1$, and $u_i \sim N(0, \sigma^2)$, show that

$$cov(y_{it}, y_{is}) = \exp(\mathrm{x}_{it}^T\beta + \mathrm{x}_{is}^T\beta)\{\exp(\sigma^2)(\exp(\sigma^2) - 1)\}.$$

$\log \mu_{it} = x_{it}^T\beta + u_i$

$Cov(y_{it}, y_{is}) = \mathbb{E}[Cov(y_{it}, y_{is}|u_i)] + Cov(\mathbb{E}(y_{it}|u_i), \mathbb{E}(y_{is}|u_i))$

$\parallel$ 0 $\parallel$ $Cov(\mu_{it}, \mu_{is})$

Given u_i, $y_{it} \perp y_{is}$

$= e^{x_{it}^T\beta + x_{is}^T\beta} \cdot Var(e^{u_i})$

$\parallel$

$\mathbb{E}[e^{2u_i}] - (\mathbb{E}[e^{u_i}])^2$

$\parallel$

$e^{2\sigma^2} - e^{\sigma^2}$