Qualifying Exam. (Linear Models) Dec. 2013

1. Consider a regression model,

$$Y = X\beta + \epsilon,$$

where X, $n \times p$, is full column rank. Let $Y = (Y_1, ..., Y_n)'$, $X' = (x_1, x_2, ..., x_n)$ and let $Y_{(i)}$ be the corresponding Y vector and $X_{(i)}$ be the corresponding X matrix after deleting the i-th case. Further, assume that $Var(\epsilon) = \sigma^2 I$. Let e_i be the ith residual and h_i be the ith diagonal element of the hat matrix. Let $\hat{\beta}$ and $\hat{\beta}_{(i)}$ be the least squares estimate of β with and without the ith case included in the data respectively. Also, we let SSE and $SSE_{(i)}$ be the error Sum of squares with and without the ith case included in the data respectively.

You are given the result that

$$\hat{\beta} - \hat{\beta}_{(i)} = \frac{(X'X)^{-1}x_i e_i}{1 - h_i}.$$

(a) Show that

$$Y'_{(i)}Y_{(i)} = Y'Y - Y_i^2$$

(b) Show that

$$Y'_{(i)}X_{(i)}\hat{\beta}_{(i)} = Y'X\hat{\beta} - y_i^2 + \frac{e_i^2}{1 - h_i}$$

(c) Find the value of $SSE_{(i)}$ if $SSE = 20$, $e_i = 0.8$ and $h_i = 0.2$.

2. Consider a linear regression model,

$$y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \cdots + \beta_{p-1} x_i^{p-1} + \epsilon_i,$$

$i = 1, ..., n$. Also, $\epsilon_1, \epsilon_2, ..., \epsilon_n$ are i.i.d. $N(0, \sigma^2)$. Let $P_k(x)$ be a polynomial of order k. Then the above model could be rewritten as

$$y_i = \alpha_0 P_0(X_i) + \alpha_1 P_1(X_i) + \cdots + \alpha_{p-1} P_{p-1}(X_i) + \epsilon_i,$$

Assume that

$$\sum_{i=1}^{n} P_l(x_i) P_m(x_i) = 0, \quad l \neq m, \quad \text{for all } l \text{ and } m,$$

(a) Derive the least squares estimator of α_j, $j = 0, 1, ..., p-1$.

(b) Derive the test for testing the null hypothesis H_0: $\alpha_j = 0$.