Problem from STAT5030

1. Consider the linear model

$$Y_{n\times 1} = X\beta + \epsilon, \qquad E(\epsilon) = 0, \quad Cov(\epsilon) = \sigma^2 V$$

with $\beta = (\beta_1, ..., \beta_p)'$ and sample data $(X, Y_{n\times 1})$. Let V be a known positive definite matrix.

$(X'V^{-1}X)^{-1}X'V^{-1}Y$

(a) What is the Generalized Least Squares estimator of β?

(b) (Prediction) Let

$$X_0 = \begin{bmatrix} X_{n+1,1} & X_{n+1,2} & \cdots & X_{n+1,p} \\ X_{n+2,1} & X_{n+2,2} & \cdots & X_{n+2,p} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ X_{n+m,1} & X_{n+m,2} & \cdots & X_{n+m,p} \end{bmatrix}$$

$Xa =$ $\quad X'P = X_0'$

∃可逆 $XA = X_0$

$XP = X_0$

and Y_0 be the values that we are interested to predict given X_0. The column space of X_0' is a subset of the column space of X'. Assume that ∃A s.t. $X'A = X_0'$

$$Y_0 = X_0\beta + \epsilon_0, \qquad E(\epsilon_0) = 0, \quad Cov(\epsilon_0) = \sigma^2 V_0$$

$A'VA$

with V_0 be a known positive definite matrix. In addition, assume that $Cov(\epsilon, \epsilon_0) = 0$ and $X_0\beta$ is estimable. $= \sigma^2[X_0(X'V^{-1}X)^{-1}X_0' + V_0]$

$X_0(X'V^{-1}X)^{-1}X'V^{-1}Y$

i. What is the prediction of Y_0 (denoted by $\hat{Y}_0$)? $\triangleq \hat{Y}_0$ $\quad Cov(\hat Y_0 - Y_0) = X_0(X'V^{-1}X)^{-1}X'V^{-1}V V^{-1}X(X'V^{-1}X)^{-1}X_0' + \sigma^2 V_0$

ii. What is $Cov(\hat{Y}_0 - Y_0)$? $\hat Y_0 - Y_0 = X_0(X'V^{-1}X)^{-1}X'V^{-1}(X\beta+\epsilon) - X_0\beta - \epsilon_0 = X_0(X'V^{-1}X)^{-1}X'V^{-1}\epsilon - \epsilon_0$

(c) Let X_0 be the same as given in Part (b), and the model is also the same EXCEPT that $Cov(\epsilon, \epsilon_0) = \sigma^2 W$.

Now, let $Y_0^* = CY$ be a linear unbiased predictor of Y_0. Define the prediction mean squared error (PMSE) of Y_0^* as $\quad Y_0^* - Y_0 = CY - Y_0 = CX\beta + C\epsilon - X_0\beta - \epsilon_0 \sim ((CX - X_0)\beta,$

$$E(Y_0^* - Y_0)'A(Y_0^* - Y_0)$$

$\sigma^2 tr(ACW)$

where A is a positive definite matrix.

$\beta'X'C'AX_0\beta$ $\quad Cov(C\epsilon - \epsilon_0) =$

i. Prove that the PMSE of Y_0^* is

$$\beta'(CX - X_0)'A(CX - X_0)\beta + \sigma^2 tr[A(CVC' + V_0 - 2CW)]$$

$\sigma^2[CVC' + V_0 - 2CW]$

ii. The best (minimum PMSE) linear unbiased estimator of Y_0 is

$$Y_0^* = \hat{Y}_0 + D.$$

Find D. $\quad Y_0^* = X_0(X'V^{-1}X)^{-1}X'V^{-1}Y + D \triangleq CY + D$; $\quad Y_0^* - Y_0 \sim (D, \sigma^2[CVC' + V_0 - 2CW])$

PMSE of $Y_0^* = D'AD + \sigma^2 tr[A(CVC' + V_0 - 2CW)]$

attains minimum at $D = 0$

2. Let

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}, \qquad Y = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

and assume that X and X_1 have full column rank. Consider the linear model

$$Y = X\beta + \epsilon$$

$- X_2'X_2(X_1'X_1+X_2'X_2)^{-1}$
$(X_1'Y_1+X_2'Y_2)$

where $\epsilon \sim N(0, \sigma^2 I)$. Let $\hat{\beta}$ be the least squares estimator of β and $\hat{Y} = X\hat{\beta} = (\hat{Y}_1, \hat{Y}_2)'$. Further, for the linear model

$$Y_1 = X_1\beta^* + \epsilon^*$$

$X_1'X_1(X_1'X_1+X_2'X_2)^{-1}(X_1'Y_1+X_2'Y_2)$

where $\epsilon^* \sim N(0, \sigma^2 I)$, the least squares estimator of β^* is $\hat{\beta}^*$. Let

$= X_1'X_1 + X_2'Y_2$

$$\hat{Y}^* = X\hat{\beta}^* = \begin{bmatrix} \hat{Y}_1^* \\ \hat{Y}_2^* \end{bmatrix}$$

$\hat\beta = (X'X)^{-1}(X'Y)$

$= (X_1'\ X_2')\begin{pmatrix} X_1 \\ X_2 \end{pmatrix}(X_1'Y_1+X_2'Y_2)$

$= (X_1'X_1+X_2'X_2)^{-1}$

Define

$$Y - \hat{Y} = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - \begin{bmatrix} \hat{Y}_1 \\ \hat{Y}_2 \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

$$Y - \hat{Y}^* = \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} - \begin{bmatrix} \hat{Y}_1^* \\ \hat{Y}_2^* \end{bmatrix} = \begin{bmatrix} e_1^* \\ e_2^* \end{bmatrix}$$

$\hat{Y} = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = \begin{pmatrix} X_1\hat\beta \\ X_2\hat\beta \end{pmatrix}$ $\quad e_1 = Y_1 - X_1\hat\beta$

(a) Prove that

$$\hat{\beta} - \hat{\beta}^* = M_1^{-1}X_2'e_2$$ ✓ How

where $M_1 = X_1'X_1$.

$(X_1\ )$

(b) Express e_2 in terms of e_2^* and rewrite the expression of $\hat{\beta} - \hat{\beta}^*$ in Part (a)

X_1' ( )
X_2'

(c) The following is a data set with sample size = 7 $\quad -3 \quad M_1 \quad \hat\beta^* = (X_1'X_1)^{-1}(X_1'Y_1)$

x	-3	-2	-1	0	1	2	3
y	14	7	.	.	.	.	-2

$X_1'Y_1$

For the above data and with a simple linear regression model, the parameter estimate $\hat{\beta}^* = (6, -2)'$.

$\hat\beta^*$

$X_2'Y_2 - X_2'X_2(X_1'X_1+X_2'X_2)^{-1}$
$(X_1'Y_1+X_2'Y_2)$

Suppose an additional observation $(x, y) = (4, 4)$ is obtained (You now have 8 pairs of (x, y) in your updated dataset), compute the new parameter estimate $\hat{\beta}$. (Hint: use Parts (a) and (b))

$X_1 = x \quad Y_1 = y$
$X_2 = 4 \quad Y_2 = 4$ ✓

$e_2 = Y_2 - X_2\hat\beta$

$e_2^* = Y_2 - X_2\hat\beta^*$
$= Y_2 - X_2(X_1'X_1)^{-1}(X_1'Y_1)$
$\hat\beta = (X_1'X_1)^{-1}(X_1')(Y - Y_2).$