Homework 2

Štěpán Skalka
02QIC

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Assignment 1

(a) From general formula for matrix multiplication $C_{ij} = A_{ik}B_{ki}$ it follows:

$$(\rho^2)_{11} = \rho_{11}^2 + \rho_{12}\,\rho_{21} = |\alpha|^4 + p^2|\alpha|^2|\beta|^2, \tag{1}$$

$$(\rho^2)_{22} = \rho_{22}^2 + \rho_{21}\,\rho_{12} = |\beta|^4 + p^2|\alpha|^2|\beta|^2. \tag{2}$$

Therefore:

$$\mathrm{Tr}(\hat{\rho}^2) = |\alpha|^4 + |\beta|^4 + 2p^2|\alpha|^2|\beta|^2 \tag{3}$$

(b) For $p = 1$ the purity γ is:

$$\gamma = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 = (|\alpha|^2 + |\beta|^2)^2 = 1, \tag{4}$$

this follows from the fact that $|\alpha|^2 + |\beta|^2$ is exactly the trace of $\hat{\rho}$ and hence is equal to 1.

(c) For $\alpha = \beta = \frac{1}{\sqrt{2}}$, the purity $\gamma = \mathrm{Tr}(\hat{\rho}^2)$ takes the form of:

$$\gamma = \mathrm{Tr}(\hat{\rho}^2) = \tfrac{1}{2}(1 + p^2). \tag{5}$$

The state is pure if and only if $\gamma = 1$, assuming that p is real, that leads to $p = \pm 1$.

(d) For $\alpha = 1$ and $\beta = 0$ purity does not depend on the parameter p. This can be obtained either from the fact that $\gamma = 1$ or from the density matrix, which for these α and β takes the form $\hat{\rho} = |0\rangle\langle 0|$ combined with the fact, that the density matrix composed of only one projector always represents the pure state.

Assignment 2

Density matrix $\hat{\rho}$ from task can be rewritten as:

$$\begin{aligned}\hat{\rho} &= \tfrac{1}{2}\,|0\rangle\langle 0| + \tfrac{1}{2}\,|+\rangle\langle +| \\ &= \tfrac{1}{2}\,|0\rangle\langle 0| + \tfrac{1}{4}(|0\rangle + |1\rangle)(\langle 0| + \langle 1|) \\ &= \tfrac{3}{4}\,|0\rangle\langle 0| + \tfrac{1}{4}\,|1\rangle\langle 1| + \tfrac{1}{4}\,|0\rangle\langle 1| + \tfrac{1}{4}\,|1\rangle\langle 0|\end{aligned}$$

And therefore in matrix representation:

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{6}$$

Von Neuman entropy can be expanded into eigenbasis $\{\psi_i\}$ as follows:

$$\begin{aligned} S &= -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -\mathrm{Tr}(\textstyle\sum_i \lambda_i \ln\lambda_i \left|\psi_i\right\rangle\left\langle\psi_i\right|) = -\sum_i \lambda_i \ln\lambda_i \mathrm{Tr}(\left|\psi_i\right\rangle\left\langle\psi_i\right|) \\ &= -\textstyle\sum_i \lambda_i \ln\lambda_i \end{aligned} \tag{7}$$

Eigenvalues of (6) are $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{2}}{4}$ and entropy hence equals

$$S = -(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4})\ln(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4}) - (\tfrac{1}{2} - \tfrac{\sqrt{2}}{4})\ln(\tfrac{1}{2} - \tfrac{\sqrt{2}}{4}) = 0.42 \tag{8}$$

Assignment 3

(a) Using spectral decomposition of pauli operators from table Tab. 1 we will rewrite $\hat{\sigma}_z \otimes \hat{\sigma}_x$.

Vl. číslo	σ_x	σ_y	σ_z
$\lambda = +1$	$\left\|+\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\i\end{bmatrix}$	$\left\|0\right\rangle = \begin{bmatrix}1\\0\end{bmatrix}$
$\lambda = -1$	$\left\|-\right\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$	$\frac{1}{\sqrt{2}}\begin{bmatrix}1\\-i\end{bmatrix}$	$\left\|1\right\rangle = \begin{bmatrix}0\\1\end{bmatrix}$

Table 1: Tabulka popisující spektrum Pauliho operátorů.

Firstly we express $\left|+\right\rangle\left\langle+\right|$ and $\left|-\right\rangle\left\langle-\right|$ in computional basis:

$$\left|+\right\rangle\left\langle+\right| = \tfrac{1}{2}\Big(\left|0\right\rangle + \left|1\right\rangle\Big)\Big(\left\langle 0\right| + \left\langle 1\right|\Big) \tag{9}$$

$$= \tfrac{1}{2}\Big(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| + \left|1\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right|\Big) \tag{10}$$

$$\left|-\right\rangle\left\langle-\right| = \tfrac{1}{2}\Big(\left|0\right\rangle - \left|1\right\rangle\Big)\Big(\left\langle 0\right| - \left\langle 1\right|\Big) \tag{11}$$

$$= \tfrac{1}{2}\Big(\left|0\right\rangle\left\langle 0\right| + \left|1\right\rangle\left\langle 1\right| - \left|1\right\rangle\left\langle 0\right| - \left|0\right\rangle\left\langle 1\right|\Big). \tag{12}$$

Now we can finally write $\hat{\sigma}_z \otimes \hat{\sigma}_x$ in terms of projectors on the computational basis. Big endian notation for shortened multiple qubit bras and kets is used.

$$\hat{\sigma}_z \otimes \hat{\sigma}_x = \Big(\left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\Big) \otimes \Big(\left|+\right\rangle\left\langle+\right| - \left|-\right\rangle\left\langle-\right|\Big) \tag{13}$$

$$= \Big(\left|0\right\rangle\left\langle 0\right| - \left|1\right\rangle\left\langle 1\right|\Big) \otimes \Big(\left|1\right\rangle\left\langle 0\right| + \left|0\right\rangle\left\langle 1\right|\Big) \tag{14}$$

$$= \left|00\right\rangle\left\langle 01\right| + \left|01\right\rangle\left\langle 00\right| - \left|10\right\rangle\left\langle 11\right| - \left|11\right\rangle\left\langle 10\right| \tag{15}$$

(b) Now we have to evaluate expectation value of this operator in state $\phi^- = \frac{1}{\sqrt{2}}(\left|00\right\rangle - \left|11\right\rangle)$:

$$\left\langle\phi^-\right|\hat{\sigma}_z \otimes \hat{\sigma}_x\left|\phi^-\right\rangle = \tfrac{1}{2}(\left\langle 00\right| - \left\langle 11\right|)\ \hat{\sigma}_z \otimes \hat{\sigma}_x\ (\left|00\right\rangle - \left|11\right\rangle) = 0 \tag{16}$$

It is zero, because operator $\hat{\sigma}_z \otimes \hat{\sigma}_x$ does not contain any terms, that would be non zero. Non zero terms would have to be projectors formed only using kets from $\{\left|00\right\rangle, \left|11\right\rangle\}$ and bras from $\{\left\langle 00\right|, \left\langle 11\right|\}$.