Homework 1

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October 7, 2024

Assignment 1

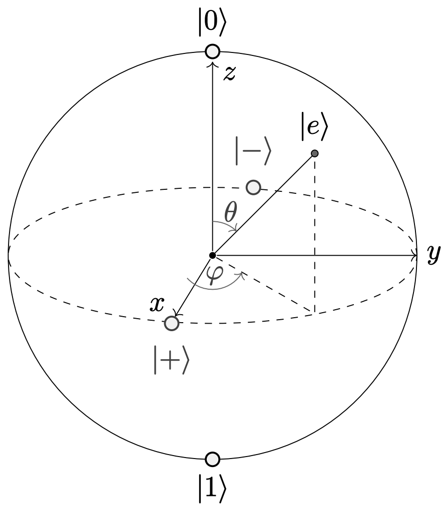
Consider general orthogonal states $|e\rangle$ and $|f\rangle$:

$$|e\rangle = \alpha\,|0\rangle + \beta\,|1\rangle\,, \quad |f\rangle = \beta^*\,|0\rangle - \alpha^*\,|1\rangle\,. \tag{1}$$

Using Bloch sphere parametrization one can obtain:

$$|e\rangle = \cos\tfrac{\theta}{2}\,|0\rangle + e^{i\varphi}\sin\tfrac{\theta}{2}\,|1\rangle \tag{2}$$

$$|f\rangle = e^{-i\varphi}\sin\tfrac{\theta}{2}\,|0\rangle - \cos\tfrac{\theta}{2}\,|1\rangle \tag{3}$$



Using global phase irrelevance and trigonometric identities it is possible to express state $|f\rangle$ as follows

$$\begin{aligned}
|f\rangle &= e^{-i\varphi}\sin\tfrac{\theta}{2}\,|0\rangle - \cos\tfrac{\theta}{2}\,|1\rangle && (4)\\
&= \sin\tfrac{\theta}{2}\,|0\rangle - e^{i\varphi}\cos\tfrac{\theta}{2}\,|1\rangle && (5)\\
&= \sin\tfrac{\theta}{2}\,|0\rangle + e^{i\varphi}\cos(-\tfrac{\theta}{2})\,|1\rangle && (6)\\
&= \cos(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|0\rangle + e^{i\varphi}\sin(-\tfrac{\theta}{2} + \tfrac{\pi}{2})\,|1\rangle && (7)\\
&= \cos(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|0\rangle - e^{i\varphi}\sin(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|1\rangle && (8)\\
&= \cos(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|0\rangle + e^{i\pi}e^{i\varphi}\sin(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|1\rangle && (9)\\
&= \cos(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|0\rangle + e^{i(\varphi+\pi)}\sin(\tfrac{\theta}{2} - \tfrac{\pi}{2})\,|1\rangle && (10)
\end{aligned}$$

Therefore, on Bloch sphere, state orthogonal to the one with coordinates (θ, φ) has coordinates $(\theta - \pi, \varphi + \pi)$ and hence is pointing in the opposite direction.

Assignment 2

(a) Density matrix $\hat{\rho}$ from task can be rewritten as:

$$\begin{aligned}\hat{\rho} &= \tfrac{1}{2}\left|0\right\rangle\left\langle 0\right| + \tfrac{1}{2}\left|+\right\rangle\left\langle +\right| \\ &= \tfrac{1}{2}\left|0\right\rangle\left\langle 0\right| + \tfrac{1}{4}(\left|0\right\rangle + \left|1\right\rangle)(\left\langle 0\right| + \left\langle 1\right|) \\ &= \tfrac{3}{4}\left|0\right\rangle\left\langle 0\right| + \tfrac{1}{4}\left|1\right\rangle\left\langle 1\right| + \tfrac{1}{4}\left|0\right\rangle\left\langle 1\right| + \tfrac{1}{4}\left|1\right\rangle\left\langle 0\right|\end{aligned}$$

And therefore in matrix representation:

$$\begin{pmatrix} \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix} \tag{11}$$

(b) Bloch vector represents projectoin of density operator (minus identity) on Pauli vector:

$$\hat{\rho} = \hat{I} + \vec{n} \cdot \hat{\vec{\sigma}} = \frac{1}{2}\begin{pmatrix} 1 + n_z & n_x - in_y \\ n_x + in_y & 1 - n_z \end{pmatrix} \tag{12}$$

After comparison with (11) we obtain:

$$1 + n_z = \tfrac{3}{2}, \quad 1 - n_z = \tfrac{1}{2}, \quad n_x = \tfrac{1}{2}, \quad n_y = 0, \tag{13}$$

which has the solution:

$$\vec{n} = \begin{pmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{pmatrix} \tag{14}$$

Assignment 3

(a) From general formula for matrix multiplication $C_{ij} = A_{ik}B_{ki}$ it follows:

$$(\rho^2)_{11} = \rho_{11}^2 + \rho_{12}\,\rho_{21} = |\alpha|^4 + p^2|\alpha|^2|\beta|^2, \tag{15}$$

$$(\rho^2)_{22} = \rho_{22}^2 + \rho_{21}\,\rho_{12} = |\beta|^4 + p^2|\alpha|^2|\beta|^2. \tag{16}$$

Therefore:

$$\mathrm{Tr}(\hat{\rho}^2) = |\alpha|^4 + |\beta|^4 + 2p^2|\alpha|^2|\beta|^2 \tag{17}$$

(b) For $p = 1$ the purity γ is:

$$\gamma = |\alpha|^4 + |\beta|^4 + 2|\alpha|^2|\beta|^2 = (|\alpha|^2 + |\beta|^2)^2 = 1, \tag{18}$$

this follows from the fact that $|\alpha|^2 + |\beta|^2$ is exactly the trace of $\hat{\rho}$ and hence is equal to 1.

(c) For $\alpha = \beta = \frac{1}{\sqrt{2}}$, the purity $\gamma = \mathrm{Tr}(\hat{\rho}^2)$ takes the form of:

$$\gamma = \mathrm{Tr}(\hat{\rho}^2) = \tfrac{1}{2}(1 + p^2). \tag{19}$$

The state is pure if and only if $\gamma = 1$, assuming that p is real, that leads to $p = \pm 1$.

(d) For $\alpha = 1$ and $\beta = 0$ purity does not depend on the parameter p. This can be obtained either from the fact that $\gamma = 1$ or from the density matrix, which for these α and β takes the form $\hat{\rho} = \left|0\right\rangle\left\langle 0\right|$ combined with the fact, that the density matrix composed of only one projector always represents the pure state.

Assignment 4

Von Neuman entropy can be expanded into eigenbasis $\{\psi_i\}$ as follows:

$$\begin{aligned} S &= -\mathrm{Tr}(\hat{\rho}\ln\hat{\rho}) = -\mathrm{Tr}(\textstyle\sum_i \lambda_i \ln\lambda_i \left|\psi_i\right\rangle\left\langle\psi_i\right|) = -\textstyle\sum_i \lambda_i \ln\lambda_i \mathrm{Tr}(\left|\psi_i\right\rangle\left\langle\psi_i\right|) \\ &= -\textstyle\sum_i \lambda_i \ln\lambda_i \end{aligned} \tag{20}$$

Eigenvalues of (11) are $\lambda_{1,2} = \frac{1}{2} \pm \frac{\sqrt{2}}{4}$ and entropy hence equals

$$S = -(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4})\ln(\tfrac{1}{2} + \tfrac{\sqrt{2}}{4}) - (\tfrac{1}{2} - \tfrac{\sqrt{2}}{4})\ln(\tfrac{1}{2} - \tfrac{\sqrt{2}}{4}) = 0.42 \tag{21}$$