Homework 3

Štěpán Skalka
02QIC

October 22, 2024

Assignment 1

General 2-qubit state can be expressed as:

$$|\psi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle\,. \tag{1}$$

If the state is separable following must also hold:

$$\begin{aligned}|\psi\rangle &= (\alpha|0\rangle + \beta|1\rangle)\otimes(\gamma|0\rangle + \delta|1\rangle) && (2)\\ &= \alpha\gamma|00\rangle + \alpha\delta|01\rangle + \beta\gamma|10\rangle + \beta\delta|11\rangle && (3)\end{aligned}$$

Comparing with (1) leads to system of equation that is solvable iff

$$ad - cd = 0. \tag{4}$$

This is therefore sufficient (and due to binary nature of information about separability also necessary) condition for separability.

(a) $ad - bc = 0 - 0 = 0$, state is **separable**.

(b) $ad - bc = 1 - 1 = 0$, state is **separable**.

(c) $ad - bc = -9 + 1 = -8$, state is **entangled**.

Assignment 2

Reduced density operators $\rho_A = \mathrm{Tr}_B(|\psi\rangle\langle\psi|)$ and and $\rho_B = \mathrm{Tr}_A(|\psi\rangle\langle\psi|)$ can be easily obtained from projector $|\psi\rangle\langle\psi|$:

$$\begin{aligned}|\psi\rangle\langle\psi| = \Big[\, &9|00\rangle\langle00| + |01\rangle\langle01| + |10\rangle\langle10| + 9|11\rangle\langle11|\\ &+ 3|00\rangle\langle01| - 3|00\rangle\langle10| - 9|00\rangle\langle00|\,11\\ &+ 3|01\rangle\langle00| - |01\rangle\langle10| - 3|01\rangle\langle11|\\ &- 3|10\rangle\langle00| - |10\rangle\langle01| + 3|10\rangle\langle11|\\ &- 9|11\rangle\langle00| - 3|11\rangle\langle01| + 3|11\rangle\langle10|\Big].\end{aligned}$$

Taking the projections needed to get traces one obtains ρ_A:

$$\begin{aligned}\rho_A &= \sum_i \langle i_B|\Big(|\psi\rangle\langle\psi|\Big)|i_B\rangle && (5)\\ &= \tfrac{1}{20}\Big[10|0\rangle\langle0| + 10|1\rangle\langle1| - 6|0\rangle\langle1| - 6|1\rangle\langle0|\Big] && (6)\end{aligned}$$

and ρ_B:

$$\begin{aligned}\rho_B &= \sum_i \langle i_B| \Big(|\psi\rangle \langle\psi| \Big) |i_B\rangle \\ &= \tfrac{1}{20}\Big[10\,|0\rangle\langle 0| + 10\,|1\rangle\langle 1| + 6\,|0\rangle\langle 1| + 6\,|1\rangle\langle 0|\Big].\end{aligned} \tag{7}$$

Which translates to:

$$\rho_A = \begin{pmatrix} \frac{1}{2} & -\frac{3}{10} \\ -\frac{3}{10} & \frac{1}{2} \end{pmatrix}, \quad \rho_B = \begin{pmatrix} \frac{1}{2} & \frac{3}{10} \\ \frac{3}{10} & \frac{1}{2} \end{pmatrix}. \tag{8}$$

Theese reduced density matricies can indeed be used to determine whether the state is entangled or not. State is separable if and only if reduced density matricies are density matricies of pure state.

$$\mathrm{Tr}(\rho_A^2) = \mathrm{Tr}(\rho_B^2) = \tfrac{68}{100} \neq 1 \tag{9}$$

Trace of the square of the reduced matricies is not equal to 1. Therefore they are not pure density matricies and state $|\psi\rangle$ is entangled.

Theorem above is easily proved. Consider Schmidt decomposition of state $|\psi\rangle$:

$$|\psi\rangle = \sum_i \lambda_i \,|i\rangle_A \,|i\rangle_B\,. \tag{10}$$

Density matrix then takes form

$$\rho = |\psi\rangle\langle\psi| = \sum_i \lambda_i^2 \,|i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B|\,. \tag{11}$$

Reduced density matrix is then calculated as:

$$\begin{aligned}\rho_A = \mathrm{Tr}_B(\rho) &= \sum_i \lambda_i^2\, \mathrm{Tr}_B(\,|i_A\rangle\langle i_A| \otimes |i_B\rangle\langle i_B|\,) \\ &= \sum_i \lambda_i^2\, |i_A\rangle\langle i_A|\, \mathrm{Tr}_B(\,|i_B\rangle\langle i_B|\,) = \sum_i \lambda_i^2\, |i_A\rangle\langle i_A|\,,\end{aligned} \tag{12}$$

for ρ_B:

$$\rho_B = \mathrm{Tr}_A(\rho) = \sum_i \lambda_i^2\, |i_B\rangle\langle i_B|\,. \tag{13}$$

If state $|\psi\rangle$ is separable, then it has Schmidt number 1. Therefore only one of λ_i is not zero. Thus $\rho_A = |i_A\rangle\langle i_A|$, $\rho_B = |i_B\rangle\langle i_B|$ and so resuced density matricies represent pure states.

On the other hand, if $\rho_A = |i_A\rangle\langle i_A|$ and $\rho_B = |i_B\rangle\langle i_B|$ are pure, it implies $\rho_A = |i_A\rangle\langle i_A|$, $\rho_B = |i_B\rangle\langle i_B|$. From equations above it follows that, the schmidt number must be 1 and state is therefore separable.