Project 4: Systems of Difference Equations with Maple: Zombie models

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Introduction

The following project observes the correlation between the populations of 4 groups of people: human susceptibles(S), infected humans(I), zombies(Z) and removed or dead people(R). As time goes on, some groups increase in population and some die off, depending on the initial values that are used in differential equations that define each population at each time stamp.

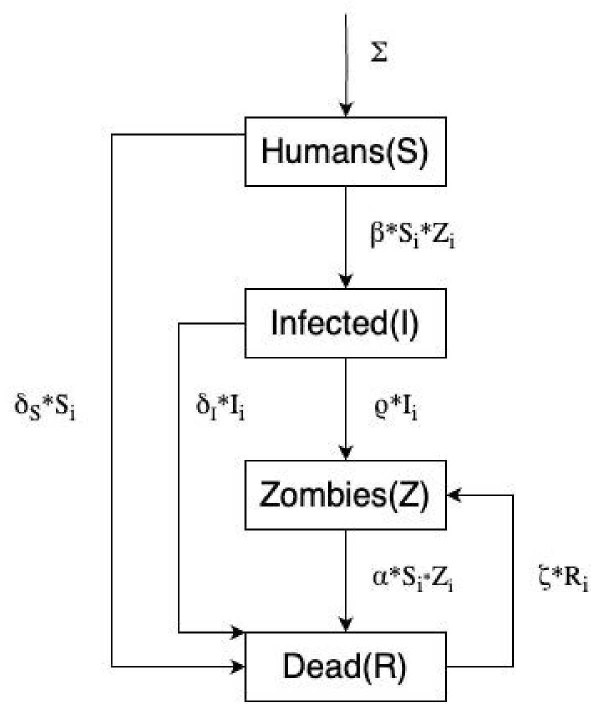
Here are the initial variables:

- d = time unit (delta t)
- Σ = number of humans arriving into the zombie area per time unit
- β = probability that a human is infected during human-zombie encounters in a time unit
- δ_S = probability that a susceptible human is killed (or dies) in a time unit
- δ_I = probability that an infected human is killed (or dies) in a time unit
- ρ = probability that an infected human turns into a zombie during a time unit
- α = probability that a zombie is killed during human-zombie encounters in a time unit
- ζ = probability that a removed individual turns into a zombie during a time unit

And here are the differential equations:

$$S_{i+1} = S_i + d(\Sigma - \delta_S * S_i)$$
$$I_{i+1} = I_i + d(\beta * S_i * Z_i - \rho * I_i - \delta_I * I_i)$$
$$Z_{i+1} = Z_i + d(\rho * I_i - \alpha * S_i * Z_i + \zeta * R_i)$$
$$R_{i+1} = R_i + d(\delta_S * S_i + \delta_I * I_i - \zeta * R_i + \alpha * S_i * Z_i)$$

The migration of people can be explained by the following diagram:



Task 1

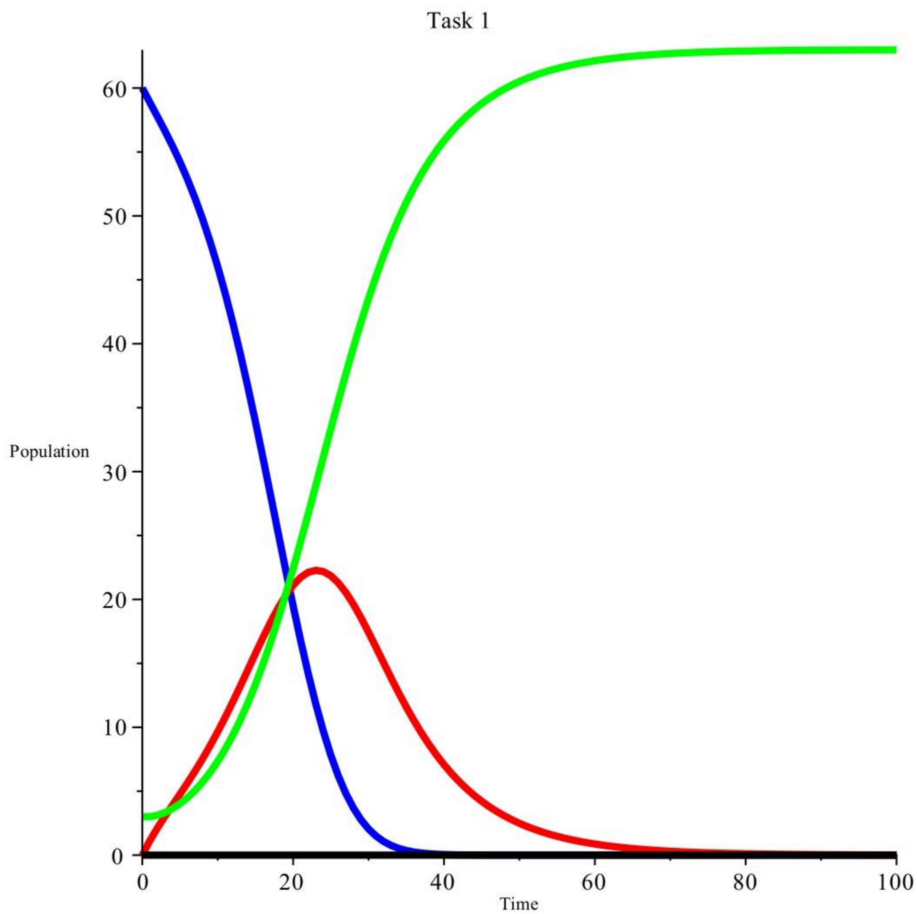
Given the initial values of

$d = 0.1;\ \Sigma = \alpha = \delta_S = \delta_I = \zeta = 0;\ \rho = 1;\ \beta = 0.0625$

and the initial conditions

$S_0 = 60;\ Z_0 = 3;\ I_0 = R_0 = 0$

we get the following graph:

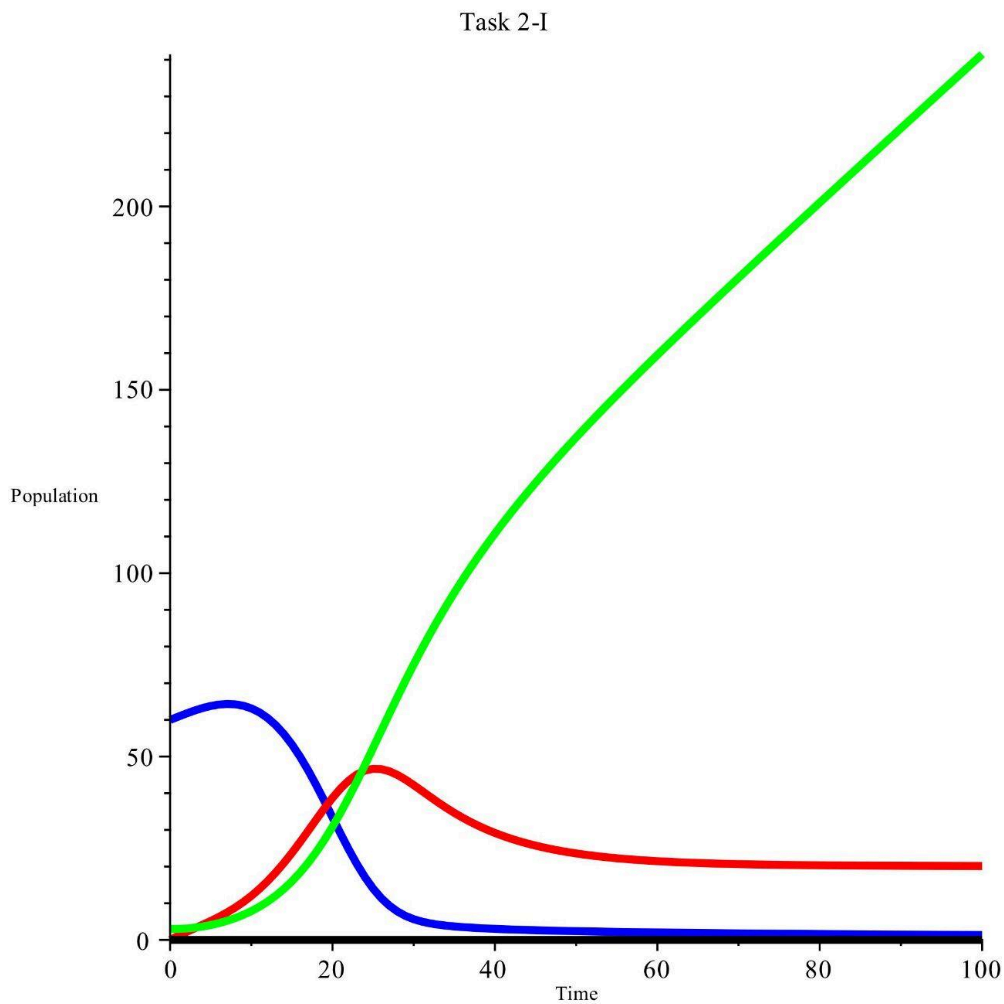


In all of the graphs humans are in blue, infected are in red, zombies are in green and the removed are in black.

As we can notice from this particular graph, everything changes drastically in the first 50 time units. All humans turn into zombies with a peak convertibility at approximately t=25. Since we defined above that nobody dies, the black line stays 0 for the entire duration of the process.

Task 2-I

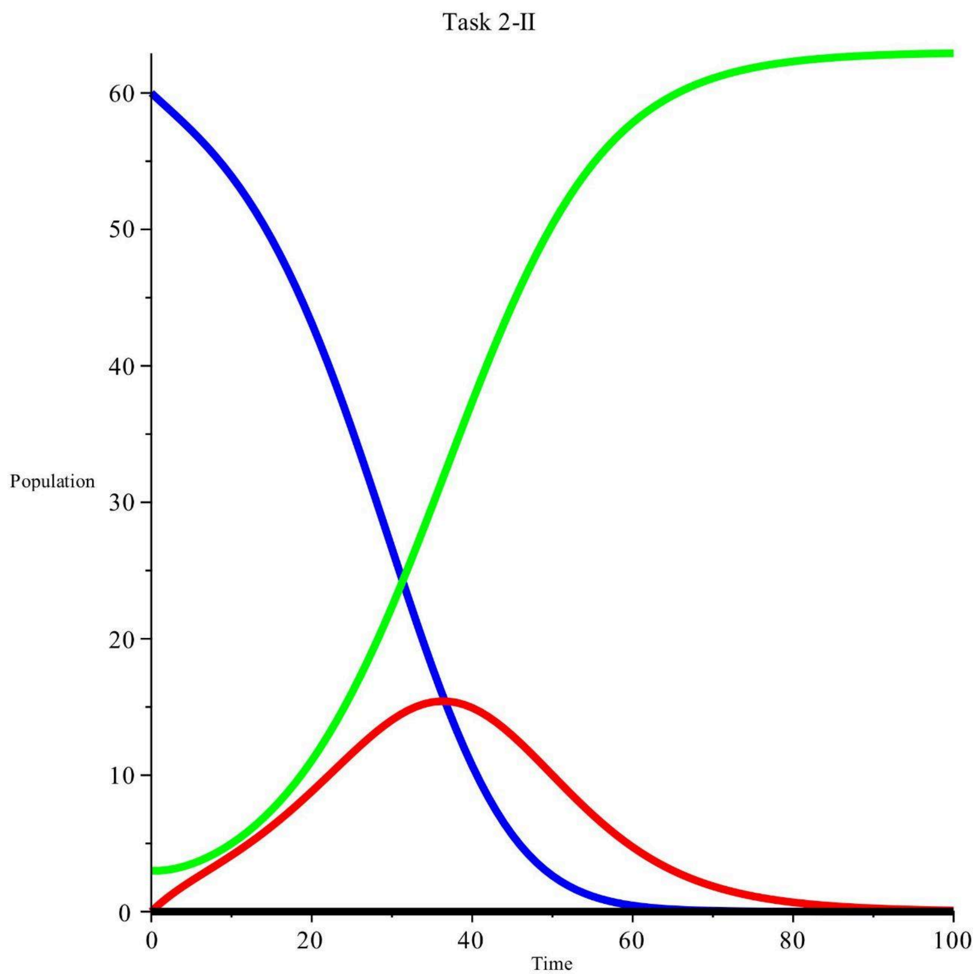
In subtask I the only value we change from the previous task is Σ which will be 20 instead of 0. Reminder: this is the number of newcomer humans that are entering the environment at each time unit.



Once again, the zombies prevail, this time at a much faster rate. The green line(zombies) doesn't stop growing due to the fact that humans keep coming and “feeding” the zombie population. We can actually notice a small increase in humans before t=10 because there are still very few zombies and they can't grow as fast, but then, at t=20 everything flips and all the humans start getting infected(rising red line). And once again, nobody dies, the black line is flat.

Task 2-II

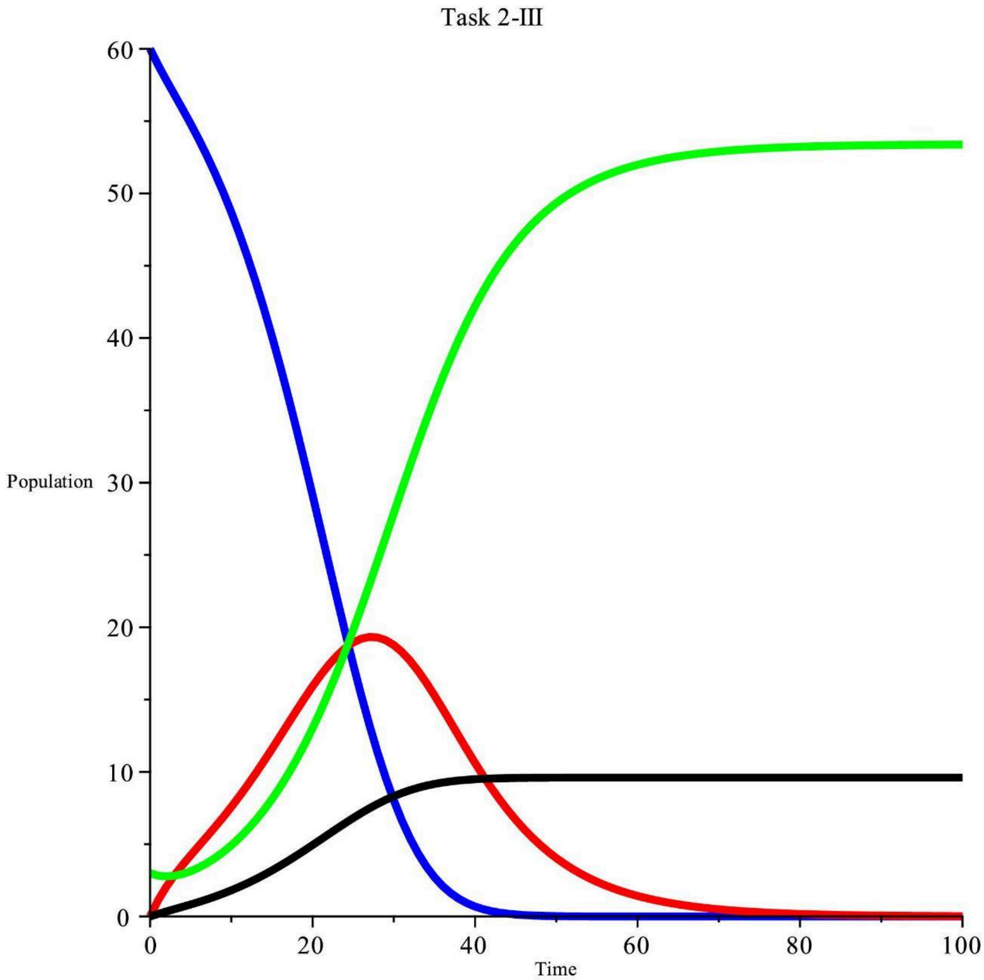
This time the changed value is β, going from 0.0625 to 0.03. It affects the probability that a human is infected during human-zombie encounters, giving an advantage to humans another time.



Another case-another loss for the humans. Due to the change in probability we "stretched" the graph on the x axis, since the ratio of humans to zombies didn't drop as fast as it did in task 1. This time humans lasted 20 days more (compared to task 1). Nobody dies just like the previous times, everybody gets zombified.

Task 2-III

This time the changed value is α , which corresponds to the probability that a zombie is killed during human-zombie encounters, rising up to 0.01(previously 0).



Instantly we notice an obvious difference compared to earlier graphs and that is the black line, which looks like x=y^3 and not y=0. We can also notice that this time the final number of zombies is around 53 and not 63, which is represented by the black line, stabilizing at y=10. Other than that, the graph looks about the same with some small numeral corrections.

Task 3

As for task 3, it was the most exotic one, with a whole different system of differential equations and humans-zombies interaction pattern. This time humans are prepared for zombies and attack them in an organized manner at defined time stamps.
Here are the new differential equations with a small change, adding a new variable to the preexisting alpha:

$$S_{i+1} = S_i + d(\Sigma - \beta * S_i * Z_i - \delta_S * S_i)$$

$$I_{i+1} = I_i + d(\beta * S_i * Z_i - \rho * I_i - \delta_I * I_i)$$

$$Z_{i+1} = Z_i + d(\rho * I_i - (\alpha + \omega_i) * S_i * Z_i + \zeta * R_i)$$

$$R_{i+1} = R_i + d(\delta_S * S_i + \delta_I * I_i - \zeta * R_i + (\alpha + \omega_i) * S_i * Z_i)$$

Where:

$$\omega_i = \alpha \sum_{j=0}^{m} exp(-\frac{1}{2}(\frac{d*i-T_j}{\sigma})^2)$$

Where:
T(j)- point at which the attack is going to or not going to happen and σ is the length of the attack.

For Task 3 here are the initial values:

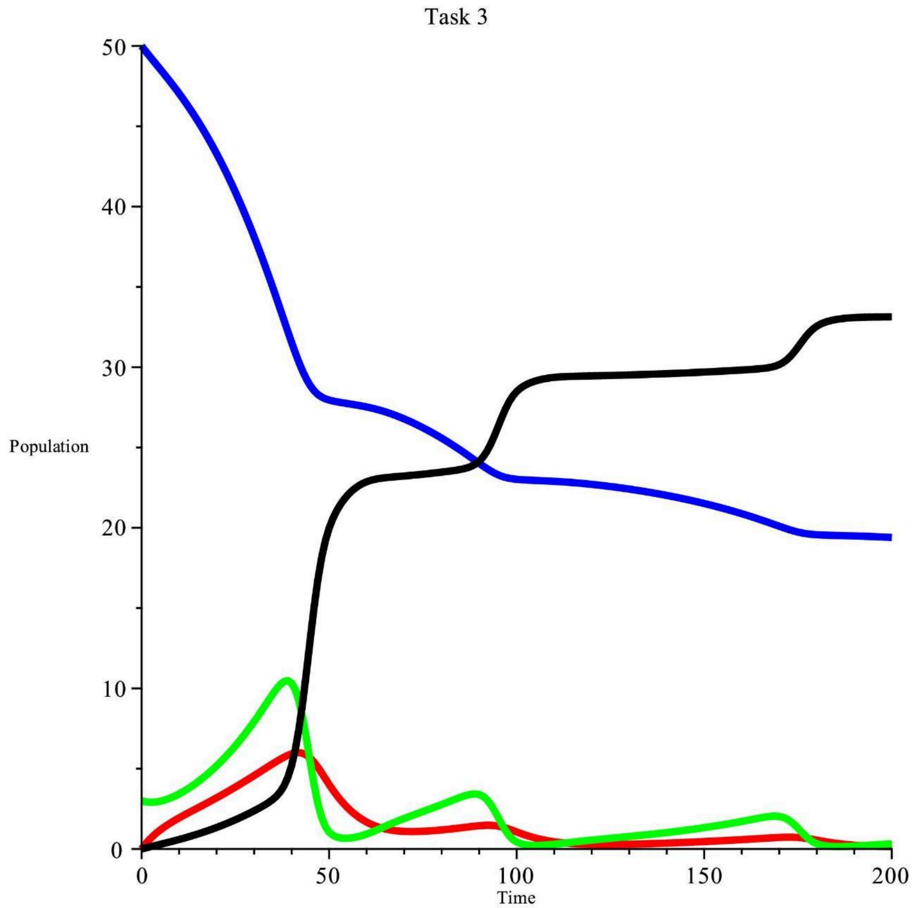
$d = 0.1;\ \Sigma = \delta_S = \delta_I = \zeta = 0;\ \rho = 1;\ \beta = 0.02;\ \alpha = 0.2\beta;\ a = 10\beta;\ \sigma = 0.5$

and the initial conditions:

$S_0 = 50;\ Z_0 = 3;\ I_0 = R_0 = 0$

Assuming the attacks take place at:

$T_0 = 5,\ T_1 = 10\ and\ T_2 = 18.$



The first thing that meets the eye is that for the first time humans have beaten the zombies! The zombie population was rising until it met the planned attack at t=5(the graph depicts 50 due to d=0.1), when the zombie population fell almost to zero and the human population stopped dying at such a high rate. Of course, this first major attack(by number of humans and zombies) came with the fastest rise in mortality, leveling at a whopping 24. The second battle coming at t=10 wasn't as legendary, but still was much needed due to the fact that the zombie population started to regrow after the initial blow. Killing all but 1 zombie it seems, the attack had a mortality of around 5. And once again, the descent of the human population graph became less steep and will reach a final number of ~20 after the third attack, which, as it seems, killed all of the zombies.

Something that the differential equations don't take into account is that there cannot be a non-natural number of people, whether they are infected, zombified, removed or alive. So naturally, it seems it is not possible to completely eliminate all of the population, zombies, in this case.

Task 4 Introduction

For task 4, our group decided to adjust some of the features that we thought would make the project more realistic:

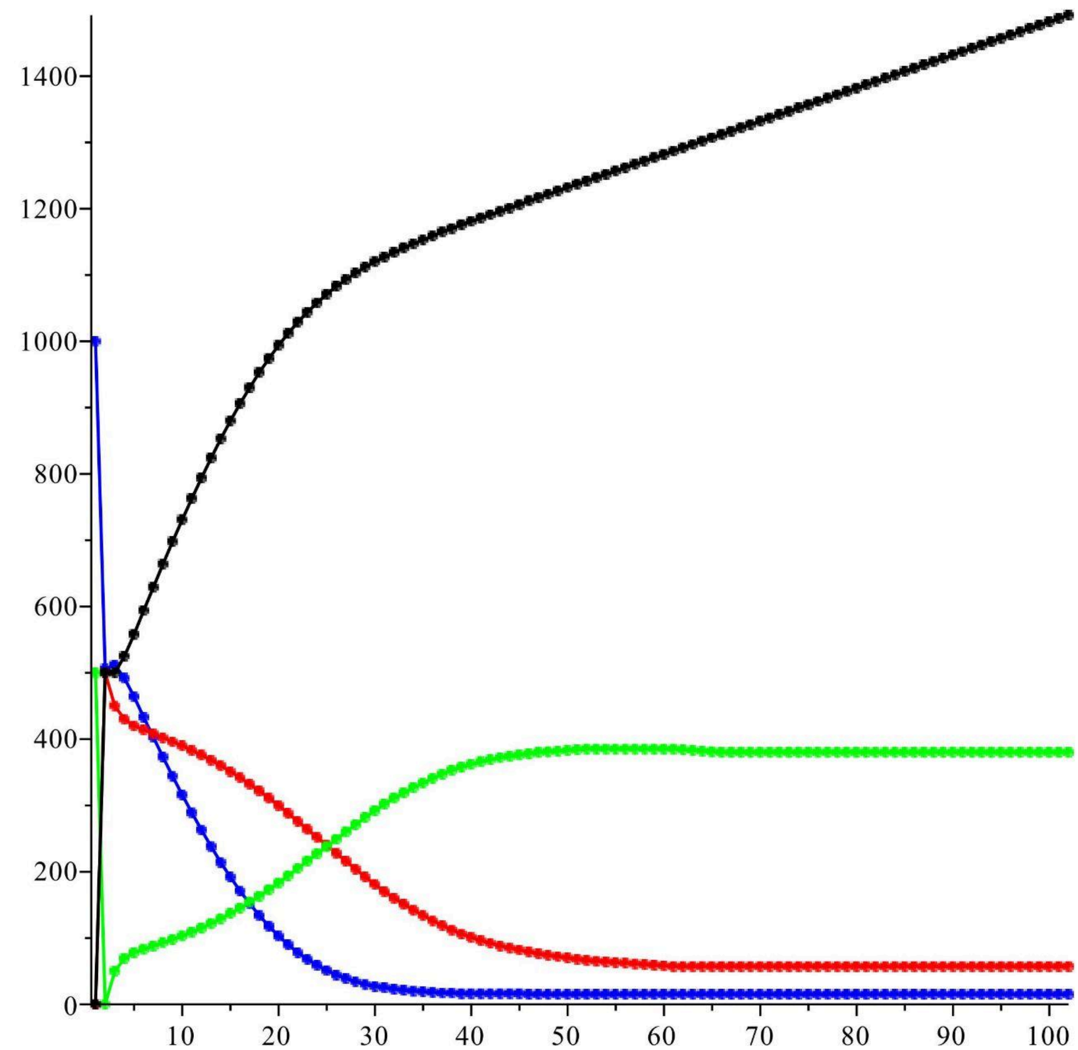
1. Fix the issue stated in the conclusion of task 3. For this, we simply rounded each population to the lower side, so there wouldn't be any fractions, as well as artificially growing populations.
2. The graphs will look more discrete, since only whole numbers are allowed. For this, we used the dataplot function instead of plot.
3. Since small numbers will look ugly on the discrete graphs, we bumped up the numbers for the initial populations, giving a more global view of the situation.

Task 4-I

Firstly, we wanted to see the graph reach a point of equilibrium, when the incoming humans would balance the forces and the ratio of humans to zombies would stay the same for infinity. We used the differential equations from tasks 1-3(no attack formula) and the following initial values:

$d = 0.1;\ \Sigma = 60;\ \alpha = 0.01;\ \delta_S = \delta_I = \zeta = 0;\ \rho = 1;\ \beta = 0.01$

$S_0 = 1000;\ Z_0 = 500;\ I_0 = R_0 = 0$



Right from the start we notice dramatic changes in numbers, both the human and the zombie population decline by around 500. This actually causes the zombies to go instinct for one period of time, the reason it goes back up is due to the big number of infected humans that will convert. By t=40 the situation stabilizes with 15 humans, 57 infected and 380 zombies, while the number of the dead grows by 5 each time. Of course, this turn of events is very unfavorable for humans as at some point people will stop coming into the environment and the humans inside will die off. But in this hypothetical version the numbers will stay the same forever(we have run the program up to $t = 10^6$, or 1 million).

Task 4-II

After reading the provided article "Escaping the Zombie Threat by Mathematics"by Hans Petter Langtangen, Kent-Andre Mardal and Pål Røtnes, our group chose to conduct a similar experiment with 3 stages of social attitude, except use numbers that we thought would be appropriate, instead of analyzing a piece of cinematography. Our main assumptions are as follows:

- The initial spread of zombies is through some widespread source, like water or wheat, which causes a large number of infected very suddenly
- Zombies are slow, yet quite contagious
- The dead cannot become zombies
- The zombie virus only affected a small isolated town of 7000 persons
- 3 stages: mass panic after the outburst, mass isolation in hideouts, counter attack with reinforcements

1st stage:

$d = 0.013;\ \Sigma = -20;\ \alpha = 0.007;\ \delta_S = 0.007;\ \delta_I = 0.07;\ \zeta = 0;\ \rho = 1;\ \beta = 0.02$

$S_0 = 6300;\ Z_0 = 300;\ I_0 = 400;\ R_0 = 0$

2nd stage changes:

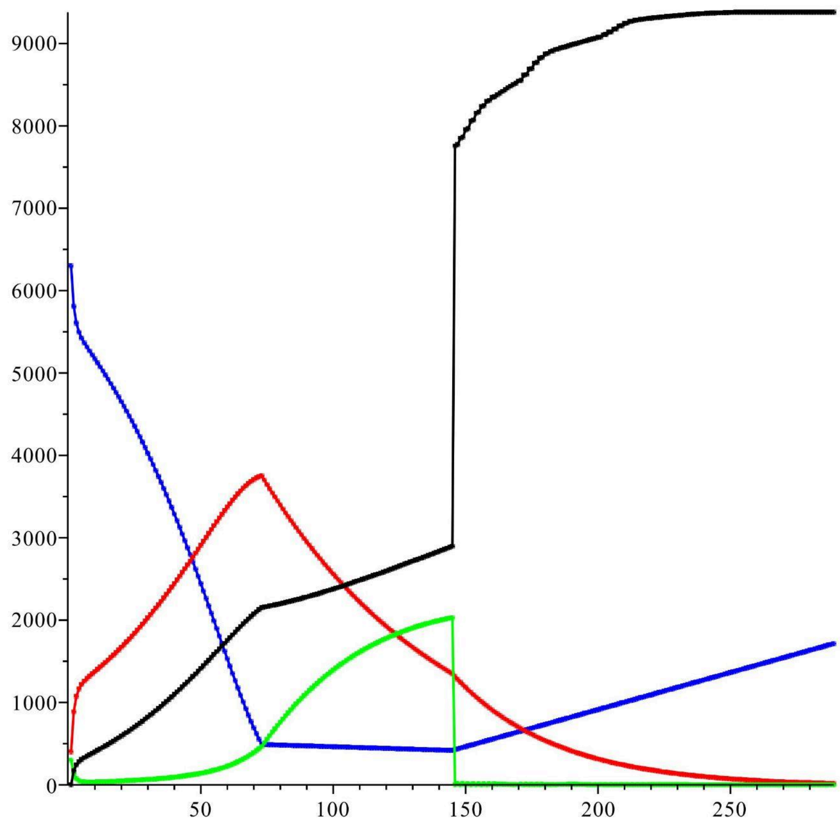
$\Sigma = 0;\ \alpha = 0.001;\ \delta_S = 0.01;\ \delta_I = 0.07;\ \beta = 0.00001$

3rd stage changes:

$\Sigma = 700;\ \alpha = 0.05;\ \delta_S = 0.00001 = \delta_I = 0.9;\ \beta = 0;\ a = 0.8;\ \sigma = 5 * d;$

$T_0 = 150 * d; T_1 = 173 * d; T_3 = 205 * d$

Some of these numbers are very much logical in our hypothetical interpretation of the story, while a few could only be explained as "the graph will look nicer this way". So without further ado, here is the graph for task4-II:



First off, we notice the 2 time points at which the graph changes dramatically: t=72 and t=144. Since the dt was chosen as 20 minutes, at t=72 the second 24h stage begins and the same for t=144, except stage 3 continues until the defeat of the zombies.

The first stage, being the stage of the mass panic, saw a pronounced exchange from the human population to the infected population and an obvious growth in death rate. But the zombie population did not seriously start growing yet.

The next day, when everything calmed down, people stopped leaving the town or even their houses,the infected started converting into zombies. Some would die due to the humans around figuring out the patterns that come with infection and getting rid of them. The humans wouldn't die or get infected much, let's assume this type of zombies is the slow one, and they are not prone to breaking into houses or sniffing out their prey.

In the morning, the military finally arrived at the town, shown by the tremendous incline of the blue line and coordinated numerous attacks. With the first one being the most deadly one, we notice the black line going almost vertically up by a very shocking amount, and then two more small bumps signaled by the next attacks.

Finally, in two days the army managed to get rid of most zombies and infected(leaving ~20 unfound) and left.

Conclusion

Mathematics is a powerful tool when dealing with such statistical models, but the tricky part is setting up the best version of the differential equations that take a lot into account and are precise with the data.

Following the given system has proven that one small change in the initial data can cause staggering change in the outputted graph, while sometimes it's the opposite: you think you are changing a number significantly, but in reality you don't notice it. This shows that some input values have stronger influence than others, even with simple equations like the ones we had in tasks 1-3.

Contributions

Vova	Calculations, report
Kristian	Task4-I values, presenting
Nuno	Attendance
Andrew	Attendance
Abdullah	Presentation