Report on Mathematical Modelling Project 1 (Goldilocks and the three bears)

By Kristian Don, Vladimir Filatov, Andrew Mcgivney, Nuno Ursu
Year 1 students of the Mathematical Sciences course of Technical University of Dublin

October 2023

Table of Contents

Introduction

In the classic story, Goldilocks entered the bear's house and found three bowls of porridge, one for Daddy Bear, Mommy Bear, and Baby Bear. She guessed the bears were out for a walk and would return when Daddy Bear's porridge cooled. Goldilocks, not acting recklessly, enjoyed Mummy Bear's porridge. Daddy Bear's was too hot and Baby Bear's was too cold so Goldilocks mixed them and it was just right .

The aim of this project is to find out when the bears will be returning and if Goldilocks will have enough time to eat the rest of the porridge, take naps, and break their chairs.

To answer these questions we'll need to make some assumptions about the missing information and apply Newton's Law of Cooling. The equation is typically written as:

$$T(t) = T(s) + (T(0) - T(s)) * e^{-kt}$$

- T(t) - temperature at t time
- T(s) - surrounding temperature
- T(0) - initial temperature
- k - coefficient of cooling

Measurement assumptions

In this case, we are assuming that k is the same for all three porridge bowls, despite their different sizes. Let's start by making some reasonable assumptions.

Rather than picking specific numbers, our group decided to pick ranges of measurements that look accurate.

- Room temperature - 18 to 25 (a)
- Daddy’s initial porridge temperature - 90 to 96 (bD)
- Mommy’s initial porridge temperature - 80 to 90 (bM)
- Baby’s initial porridge temperature - 70 to 80 (bB)
- Ideal temperature for eating - 63 to 68 (c)
- Time it takes her to eat Mommy’s porridge - 2 to 5 (d)
- Ratio of Baby’s porridge to Daddy’s porridge - 2:1 to 4:1 (f)
- Too cold - at least -5 degrees from ideal
- Too hot - at least +5 degrees from ideal

Ideal room temperature for a human is 22º Celcius. As the bears live in a hut in the middle of the forest, it is wise to assume that the temperature can get down to 18 and as high up as 25 on a sunny day (assuming it's summer). Eating a bowl of porridge can sometimes not be quick, but the problem states: “She gulped down Mommy Bear’s porridge (which took a few minutes)”, which implies a range of around 2 to 5 minutes. Talking about porridge portion sizes, our team assumed this ratio range as it is the most realistic if the Baby has a small bowl, as babies do not usually eat much, while grown male adults do. The problem sheet also mentioned “pretty numbers”, for that reason we included a rule for the temperatures of the porridges that are too cold and too hot. The 5º difference in temperature for food may not be a lot, but it is what our team chose to be the prettiest, as raising the value further would let down the rest of the “pretty numbers”.

Body of Report

Mommy’s porridge

The text states that Goldilocks eats Mommy’s porridge when its temperature is “just right” or ideal. Using Newton’s formula of cooling we get, $c = a + (bM - a)e^{-kt}$. This means that $kt = -ln((c - a)/(bM - a))$.

Baby’s porridge, Daddy’s porridge and their mixture

Using the same formula we need to find the temperatures of the other two porridges after she eats the first one.

$TD = a + (bD - a)e^{-k(t+d)}$ and $TB = a + (bB - a)e^{-k(t+d)}$. Since there is f times more of Daddy’s porridge than there is of Baby’s porridge, the formula for finding the temperature of their mixture is $(fTD + TB)/(f + 1) = c$. When we combine the three equations together we get:

$$fa + f(bD - a)e^{-k(t+d)} + a + (bB - a)e^{-k(t+d)} = c(f + 1)$$
$$e^{-k(t+d)} * (f(bD - a) + bB - a) = c(f + 1) - a(f + 1)$$
$$e^{-k(t+d)} = (c - a)(f + 1)/(f(bD - a) + bB - a)$$
$$- kt - dk = ln((c - a)(f + 1)/(f(bD - a) + bB - a))$$
$$k = (ln((c - a)(f + 1)/(f(bD - a) + bB - a)) + kt)/ - d$$

This is valid since in the previous paragraph we got a formula for kt using our input variables.

Now, using these formulas to get k and t (t is kt/k) from our input values we can determine when the bears will come and how much free time she has for each unique input set of values.

$c = a + (bD - a)e^{-k(t+d+FT)}$ FT- free time

$$- k(FT + d + t) = ln((c - a)/(bD - a))$$
$$FT =- ln((c - a)/(bD - a))/k - d - t$$

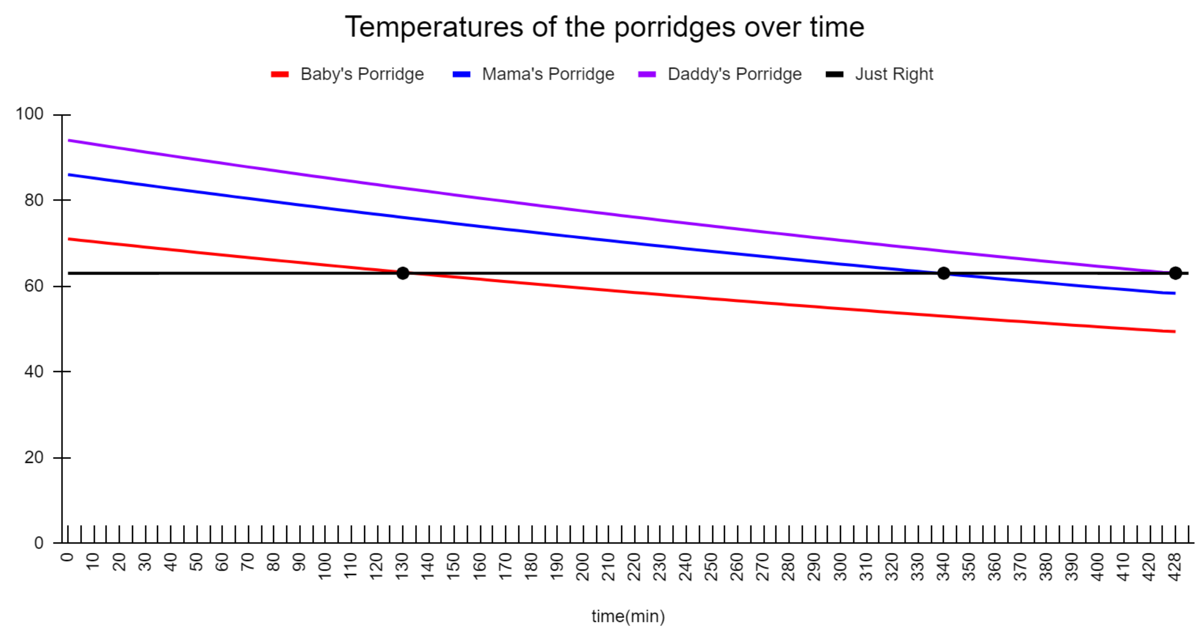
Now we use Python programming language and Jupyter Notebook to get a range of values that are “pretty” and work with the formula.

```
timel = 87
import math
for a in range(18, 25): #room temp
    for bD in range(90,96): #dads initial temp
        for bM in range(80,90): #moms initial temp
            for bB in range(70,80): #babys initial temp
                if bD-bM <5 or bM-bB < 5:
                    break
                for c in range(63,68): #ideal temp
                    for d in range(2, 5): #time to eat moms porridge
                        for f in range(2,4): #ratio dads to babys
                            kt = -math.log((c-a)/(bM-a))
                            upperfrac = ((f+1)*c) - (a*(f+1))
                            lowerfrac = (f*(bD - a)) +(bB - a)
                            k = (math.log(upperfrac/lowerfrac)+kt)/-d
                            if k == 0:
                                break
                            t = kt/k
                            Dtime = ( -math.log((c-a)/(bD-a)) )/k  #when the bears come
                            toohot = (a+(bD-a)*math.e**(-k*(t+d))-c)
                            toocold = (a+(bB-a)*math.e**(-k*(t+d))-c)
                            if Dtime-d -t> timel and toohot > 5 and toocold < -5:
                                print('| a=',a,'| bD=',bD,'| bM=',bM,'| bB=',bB,'|
c=',c,'|d=',d,'|f=',f,'| time left:'
                                      ,round(Dtime-d-t,2),'| hot:',round(toohot,2),'|
cold:',round(toocold,2))
                                print(k,t)
print('done')
```

Where timel is a variable to filter the longest FT’s possible. The output is:

```
| a= 18 | bD= 95 | bM= 87 | bB= 72 | c= 64 |d= 4 |f= 2 | time left: 87.05 | hot: 5.09 |
cold: -10.17
0.0012048216089872166 336.53538837919905
| a= 18 | bD= 95 | bM= 87 | bB= 72 | c= 65 |d= 4 |f= 2 | time left: 87.05 | hot: 5.2 | cold:
-10.39
0.0012048216089872305 318.6852725939697
| a= 18 | bD= 95 | bM= 87 | bB= 72 | c= 66 |d= 4 |f= 2 | time left: 87.05 | hot: 5.31 |
cold: -10.62
0.0012048216089872305 301.21097678055906
| a= 18 | bD= 95 | bM= 87 | bB= 72 | c= 67 |d= 4 |f= 2 | time left: 87.05 | hot: 5.42 |
cold: -10.84
0.0012048216089872305 284.0970015256927
done
```

All of these sets of values work but we can see that the last one has the smallest t so it's more realistic. That means that the answer to the main question is roughly 87 minutes or 1 hour and 27 minutes. That is how much time Goldilocks has until the bears return after she has eaten Mommy's porridge.



Would she have time to eat the rest of the porridge?

Before answering this question, some more assumptions need to be made. If Baby has B amount of porridge, we know that Daddy has F*B amount of porridge. Let's assume then that Mommy has F*B/2 amount of porridge. This seems like a fair assumption as Mommy will always have as much or somewhat more porridge than Baby depending on the ratio. We only know how much time it took Goldilocks to eat Mommy's porridge (4 minutes), so that is what we are working with.

$$Mommy's = M$$
$$Daddy's = 2 * M$$
$$Baby's = 2 * M/F$$

$$Mommy's = 4\,mins$$
$$Daddy's = 4 * 2 = 8\,mins$$
$$Baby's = 4 * 2/2 = 4\,mins$$

With 87 minutes left, Goldilocks definitely can spare 12 minutes to eat the mixture of Baby's and Daddy's porridge. This ultimately leaves her with 75 minutes.

Would she have time to break their chairs and take a nap in their beds?

Taking the 75 minutes that Goldiocks has left after eating the porridge, it is safe to assume that she would have plenty of time. Assuming each chair is of different size and weight Goldilocks would take 4 minutes 30 seconds to break the chairs. Baby Bear's being the lightest could easily be smashed in 30 seconds. Mommy Bear's is slightly larger and heavier and would prove more challenging for Goldilocks. However, after 1 minute Mommy Bear's chair would be broken. Finally, the tallest and heaviest chair. Daddy Bear's chair would be difficult for little Goldilocks to break but with some pushing and kicking it would take her 3 minutes.

Taking those values away from the initial 75 minutes, leaves Goldilocks with 70 minutes 30 seconds. In the story Goldilocks prefers Baby Bear's bed over any other, so we can assume she would nap there the longest. A long nap typically takes 30 minutes so we'll go with that. Mommy Bear's bed is larger and less comfortable, so we can assume a shorter nap of 20 minutes. Lastly, since Daddy Bear's bed is the least comfortable, a 10 minute nap seems reasonable.

After subtracting 60 minutes of nap time, Goldilocks is left with 10 minutes and 30 seconds to leave the house and safely escape the bears. This proves that Goldilocks would have more than enough time to both break the chairs, and nap in their beds.

Optional Task(different k values)

Measurement assumptions

- Room temperature - 18 to 25 (a)
- Initial porridge temperature - 80 to 96 (b)
- Ideal temperature for eating - 63 to 68 (c)
- Time it takes her to eat Mommy's porridge - 2 to 5 (d)
- Ratio of Baby's porridge to Daddy's porridge - 2:1 to 4:1 (f)
- Too cold - at least -5 degrees from ideal
- Too hot - at least +5 degrees from ideal
- Mommy's porridge - f/2*baby's porridge

The bigger the k value, the faster it cools. So the smaller the bowl, the bigger the k value.

$Kb = Kb$
$Kd = Kb/f$
$Km = 2Kb/f$

From now on every k will mean Kb, or Baby's K value.

Mommy's porridge

$c = a + (b - a)e^{-2kt/f}$
$(c - a)/(b - a) = e^{-2kt/f}$
$kt =- f/2 * ln((c - a)/(b - a))$

Baby's porridge, Daddy's porridge and their mixture

$$TD = a + (b - a)e^{-k(t+d)/f}$$
$$TD = a + (b - a)e^{-k(t+d)}$$

Once again mixing the two values(considering the ratio) we get:

$$fa + f(b - a)e^{-k(t+d)/f} + a + (b - a)e^{-k(t+d)} = c(f + 1)$$

In order to solve for k we need to have -k(t+d) on one side of the equation. This raises a problem as we are dealing with two monomials of type a^b and a^(b/c). It is not algebraically possible to combine them both to deduct -k(t+d).

Conclusions

In conclusion, Goldilocks has 87 minutes before the bears arrive, she has time to eat the remaining porridge, ---. Solving for different k values resulted in an error based on the incapacity of solving the mixture equation with the available resources and algebraical limits. While solving the problem to answer the first question, our team encountered the fact that Mommy's porridge cooled down for 337 minutes or more than 5 and a half hours. Although the team is sure that all the calculations are correct and that all the assumed measurements are valid, this number is too big to be real.

References

- Ideal porridge temperature
- Initial porridge temperature

Contributions

Vova	Calculations, python code, formatting, optional task
Kristian	Calculations, graph, extra questions, proofreading
Nuno	Introduction, presentation work
Andrew	Presentation work, presenting