Altitude of a Triangle Given the Side Lengths

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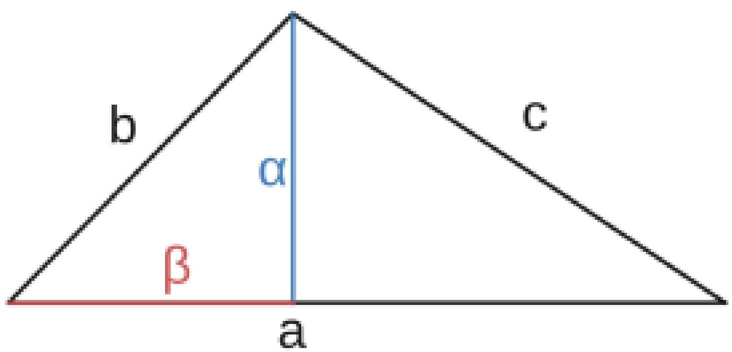
1 Theorem



Let Δ be the triangle with side lengths a, b, c. Denote the length of its altitude by α. Let β be the length from the vertex opposite to c to the altitude. Then:

$$\alpha^2 = \frac{-(a^2 + b^2 - c^2)}{4a^2} + b^2$$

2 Proof Outline

Since there are two right triangles in the setup, we can set up two equations using Pythagoras' Theorem. Then, we just need to isolate for α

3 Proof

□

By Pythagoras' theorem, we get the two equations:

$$b^2 = \beta^2 + \alpha^2 \tag{1}$$

$$c^2 = (a-\beta)^2 + \alpha^2 \tag{2}$$

Subtract 1 from 2 to get:

$$b^2 - c^2 = \beta^2 - (a-\beta)^2$$

Expand:

$$b^2 - c^2 = \beta^2 - a^2 + 2a\beta - \beta^2$$

Cancelling out term:

$$a^2 + b^2 - c^2 = 2a\beta$$

$$\beta = \frac{a^2 + b^2 - c^2}{2a} \tag{3}$$

We can substitute 3 back into 1 to get:

$$b^2 = \left(\frac{a^2 + b^2 - c^2}{2a}\right)^2 + \alpha^2$$

This gives the desired result:

$$\alpha^2 = \frac{-(a^2 + b^2 - c^2)^2}{4a^2} + b^2 \tag{4}$$

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4 A Bit of Flavor

When I was in high school, I discovered that I had a passion for mathematics at around grade 11. This theorem is the first one that I discovered on my own, after toying around with basic geometry.