

Hydrodynamics and elasticity, Lecture notes

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1 | Basic definitions

1.1 Organization

Course web page: <https://www.fuw.edu.pl/~mklis/hydro2022.html>

Requirements to obtain credit:

- Homework (30%),
- Midterm exam (35%),
- Written exam (35%),
- Oral exam (optional, only improves).

1.2 Basic laws

Example 1.2.1. Out of context Navier-Stokes equations:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -\frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u},$$

$$\nabla \cdot \mathbf{u} = 0.$$

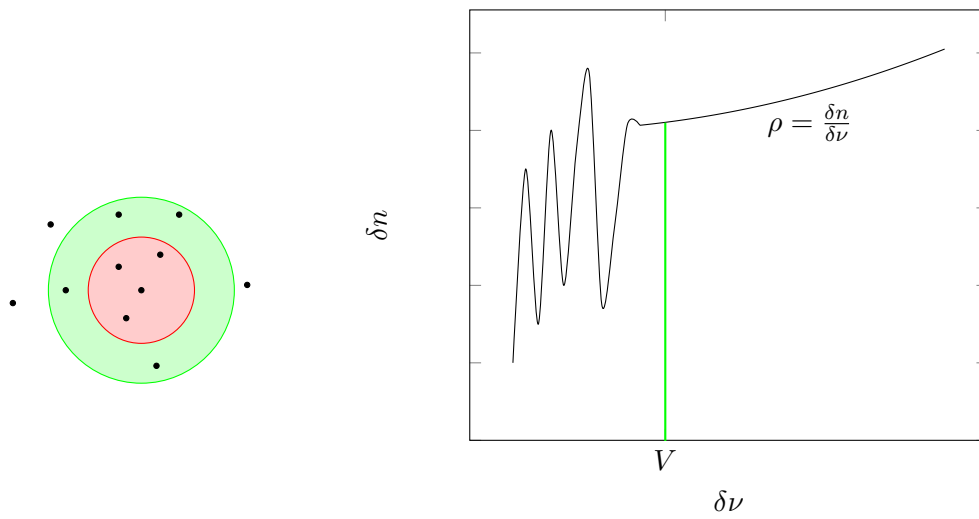
where $\mathbf{u}(\mathbf{r}, t)$ is a fluid velocity vector field, $\rho(\mathbf{r}, t)$ is a fluid density, $p(\mathbf{r}, t)$ is a pressure, ν is a kinematic viscosity.

Continuum hypothesis states that

$$\rho = \frac{\delta \eta}{\delta \nu},$$

where η is a number of particles in a region and ν is a volume of this region. Of course if the volume ν is small enough ρ may vary a lot (obviously it may not even be continuous). There is however such volume V which is „big enough”, so that for $\nu > V$ ρ does not vary „that much”.

Since matter is not continuous, we can't speak of a density at a point (in a mathematical sense) and thus, when we use the phrase „point” we mean „at a point for homogeneous physical system”.

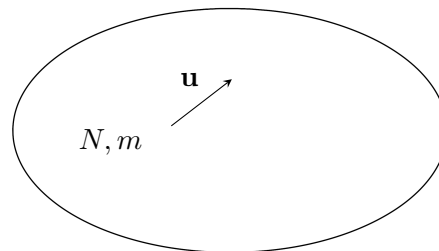
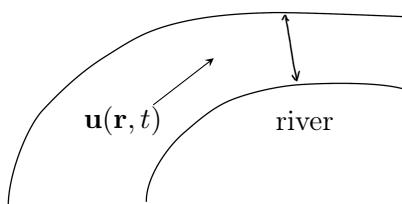


We also introduce length scale separation

$$\text{molecular length scale} \ll \nu^{\frac{1}{3}} \ll L.$$

We call those characteristic length scales „*micro*”, „*muso*” and „*macro*” respectively.

1.2.1 Equilibrium thermodynamics



Let us examine an example of a water in a river. Assume that the water has some velocity \mathbf{u} , momentum \mathbf{P} and also that it can exchange mass and heat, the latter given by $dQ = TdS$. Therefore an incremental change in the energy of this system can be expressed as

$$dE = \mathbf{u} \cdot d\mathbf{P} - pdV + TdS + \mu dN,$$

where μ is the chemical potential of the system. Thus

$$E = E(\mathbf{P}, V, S, N).$$

This is the **energy representation** of a thermodynamical system. Comparing the equations above we obtain

$$dE = \underbrace{\frac{\partial E}{\partial \mathbf{P}}}_{\mathbf{u}} d\mathbf{P} + \underbrace{\frac{\partial E}{\partial V}}_{-p} dV + \underbrace{\frac{\partial E}{\partial S}}_T dS + \underbrace{\frac{\partial E}{\partial N}}_{\mu} dN,$$

Those are called the Gibbs relation for E .

If we want to compute it for a fixed entropy we get

$$dS = \frac{1}{T} dE - \frac{\mathbf{u}}{T} d\mathbf{P} + \frac{p}{T} dV - \frac{\mu}{T} dN.$$

Thus

$$S = S(E, \mathbf{P}, V, N), \quad dS = \underbrace{\frac{\partial S}{\partial E}}_{\frac{1}{T}} dE + \underbrace{\frac{\partial S}{\partial \mathbf{P}}}_{-\frac{\mathbf{u}}{T}} d\mathbf{P} + \underbrace{\frac{\partial S}{\partial V}}_{\frac{p}{T}} dV + \underbrace{\frac{\partial S}{\partial N}}_{-\frac{\mu}{T}} dN,$$

Those are Gibbs relations for S .

It is very tricky to control entropy — it's much easier to control the temperature. To obtain a description of our system when T is an independent variable we need to use another thermodynamical potential, which is Helmholtz free energy. Transition is obtained by

$$(S \rightarrow T) \quad F = E - TS, \quad F = F(\mathbf{P}, V, T, N)$$

$$dF = -SdT - pdV + \mathbf{u}d\mathbf{P} + \mu dN.$$

Now we can do the same to switch other variables. Thus we obtain

$$(V \rightarrow P) \quad H = E + pV, \quad H = H(\mathbf{P}, p, S, N),$$

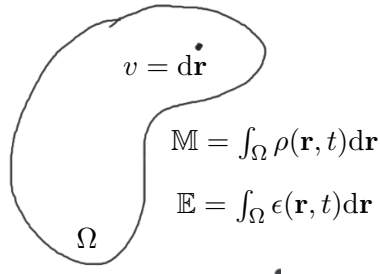
which is called an enthalpy,

$$(S \rightarrow T) \quad G = E + pV - TS, \quad G = G(\mathbf{P}, p, T, N),$$

which is called a Gibbs potential.

Now we want to consider if they are Galilean invariant

$$E(\mathbf{P}, V, S, N) = E_0(V, S, N) + \frac{\mathbf{P}^2}{2M},$$



Rysunek 1.3: Sample volume Ω . ρ stands for mass density, ϵ for density of the system.

where E_0 is the **internal energy**. For other potentials we obtain

$$H(\mathbf{P}, p, S, N) = H_0(p, S, N) + \frac{\mathbf{P}^2}{2M},$$

$$F(\mathbf{P}, V, T, N) = F_0(T, V, N) + \frac{\mathbf{P}^2}{2M},$$

$$G(\mathbf{P}, V, T, N) = G_0(T, p, N) + \frac{\mathbf{P}^2}{2M}.$$

Using $\mathbf{P} = M\mathbf{u}$ we get

$$E = E_0 + \frac{1}{2}M\mathbf{u}^2.$$

$$dE = dE_0 + \mathbf{u} \cdot d\mathbf{P},$$

$$dS = -\frac{1}{T}\mathbf{u} \cdot d\mathbf{P} + \frac{1}{T}dE + \frac{p}{T}dV - \frac{\mu}{T}dN,$$

$$dS = \frac{1}{T}dE_0 + \frac{p}{T}dV - \frac{\mu}{T}dN \implies S(\mathbf{P}, E, V, N) = S(E_0, V, N).$$

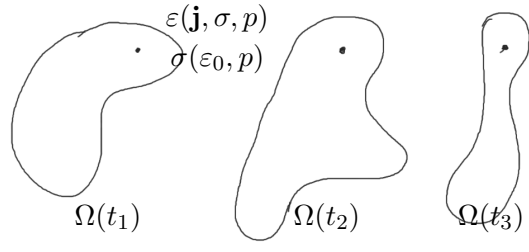
Thus S is a Galilean invariant.

1.2.2 Heterogenous macroscopic system

TODO Fig#5

„Densities” are extensive properties per unit volume. Assume that $V = \text{const}$, $dV = 0$.
Example of densities

- $\rho = \frac{M}{V}$ mass density,
- $\mathbf{j} = \frac{\mathbf{P}}{V}$ momentum density,
- $\epsilon = \frac{E}{V}$ energy density,
- $\sigma = \frac{S}{V}$ entropy density.



$$dE = \mathbf{u} d\mathbf{P} - p dV + T dS + \mu dN / \cdot \frac{1}{V}.$$

$$d\varepsilon = \mathbf{u} \cdot d\mathbf{j} + T d\sigma + \mu dn, \quad dn = \frac{d\rho}{m},$$

$$\rho = \frac{M}{V} = \frac{Nm}{V} = nm \implies dn = \frac{d\rho}{m}.$$

Thus the energy fundamental representation in terms of densities can be written as

$$\varepsilon = \varepsilon(\mathbf{j}, \sigma, \rho).$$

After performing a Galilean transform we get

$$\varepsilon(\mathbf{j}, \sigma, \rho) = \varepsilon_0(\sigma, \rho) + \frac{1}{2} \rho \mathbf{u}^2.$$

We can do the same to represent entropy in terms of densities

$$dS = \dots \frac{1}{V} \implies d\sigma = \frac{1}{T} d\varepsilon_0 - \frac{1}{T} \frac{\mu}{m} dp.$$

Thus

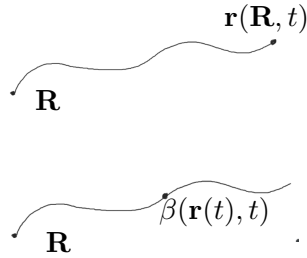
$$\sigma = \sigma(\varepsilon_0, \rho),$$

which is also Galilean invariant.

1.2.3 Flow of Heterogenous macroscopic system

We are using thermodynamics equilibrium despite the fact that the system flows (which means that it is *not* in the equilibrium). However there is no contradiction if we assume local, thermodynamical equilibrium. The same assumption is made in Navier-Stokes equations.

From now on we use pseudostatic transition. The difference between quasi-static and pseudostatic is that pseudostatic need not to be reversible. Forces which acts during quasi-static transformation are those which keeps the system in equilibrium. Example of pseudostatic transition which is not quasi-static is a flow of viscose fluid. TODO Fig#6.



1.2.4 Kinematics

We have two descriptions: Eulerian and Lagrangian. Transition between them is obtained by the solutions to the initial problem

$$\frac{d\mathbf{r}}{dt} = \mathbf{u}(r, t), \quad \mathbf{r}(t=0) = \mathbf{R}.$$

We want to find dependence of densities as the particle move, and those are

	Euler	Lagrange
velocity	$\mathbf{u}(r, t)$	$\mathbf{u}[r(t), t]$
density	$\rho(r, t)$	$\rho[r(t), t]$

$$\frac{d\mathbf{u}}{dt} = ?, \quad \frac{d\rho}{dt} = ?.$$

To do that define $\beta = (\mathbf{u}, \rho, S, \sigma, \dots)$. We want to find how β change while following the motion of the particle $r(t)$.

$$\begin{aligned} \frac{d\beta}{dt} &= \frac{d\beta[\mathbf{r}(t), t]}{dt} = \frac{\partial\beta}{\partial t} + \frac{dr_1}{dt} \frac{\partial\beta}{\partial r_1} + \frac{dr_2}{dt} \frac{\partial\beta}{\partial r_2} + \frac{dr_3}{dt} \frac{\partial\beta}{\partial r_3} \\ &= \frac{\partial\beta}{\partial t} + \mathbf{u}(t) \cdot \nabla\beta = \frac{\partial\beta}{\partial t} + d\beta(\mathbf{u}(t)). \end{aligned}$$

Equation

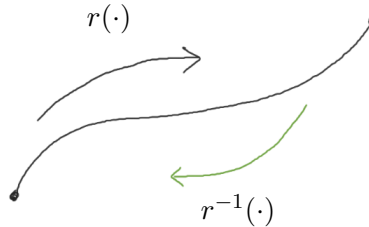
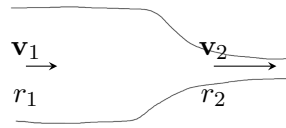
$$\frac{d\beta}{dt} = \frac{\partial\beta}{\partial t} + \mathbf{u}(t) \cdot \nabla\beta, \quad (1.1)$$

shows a relationship between Eulerian and Lagrangian world, because $\frac{d\beta}{dt}$ is a typical Lagrangian while fields (like $\mathbf{u}(t)$) are common in Euler's description. (Fields are Eulerian objects). We introduce a **total (material) derivative** as

$$\frac{d\dots}{dt} = \underbrace{\frac{\partial\dots}{\partial t}}_{\text{local derivative}} + \underbrace{\mathbf{u} \cdot \nabla(\dots)}_{\text{advective derivative}}.$$

Acceleration of fluid particle

$$\beta = \mathbf{u} \implies \frac{d\beta}{dt} = \frac{d\mathbf{u}}{dt}.$$



Consider converging channel with a stationary flow, i.e. $\frac{\partial \mathbf{u}}{\partial t} = 0$.

With that in mind

$$\frac{d\mathbf{u}}{dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = \mathbf{u} \cdot \nabla \mathbf{u}.$$

The term $\mathbf{u} \cdot \nabla \mathbf{u}$ should be interpreted as follows. Treat \mathbf{u} as a map $\mathbf{u}(\mathbf{r}) : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. Thus $\nabla \mathbf{u}$ is just a map $D\mathbf{u} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ expressed by a matrix

$$\mathbf{u}(\mathbf{r}) = \begin{bmatrix} u_1(r_1, r_2, r_3) \\ u_2(r_1, r_2, r_3) \\ u_3(r_1, r_2, r_3) \end{bmatrix}, \quad D\mathbf{u} = \begin{bmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} & \frac{\partial u_1}{\partial r_3} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} & \frac{\partial u_2}{\partial r_3} \\ \frac{\partial u_3}{\partial r_1} & \frac{\partial u_3}{\partial r_2} & \frac{\partial u_3}{\partial r_3} \end{bmatrix}.$$

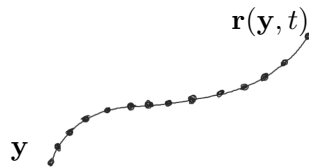
Therefore inner product $\mathbf{u} \cdot \nabla \mathbf{u}$ really means

$$\mathbf{u} \cdot \nabla \mathbf{u} = (D\mathbf{u})(\mathbf{u}).$$

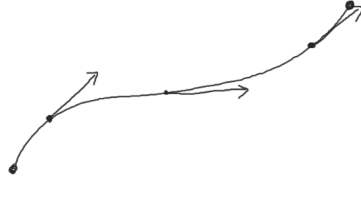
At least I think so...

The term $\mathbf{u} \cdot \nabla \mathbf{u}$ is called a **convective acceleration**. Although the flow is stationary, the particle experiences acceleration related to movement along the flow lines.

We can interpret the fluid flow as a mapping which takes one point and maps it to the other.



Rysunek 1.4: Streakline is a line made of all particles that for time s , $0 \leq s \leq t$ passed through a fixed point \mathbf{y} .



Rysunek 1.5: Streamline is just a flow of a vector field for at a fixed time t .

Streakline Imagine a cigarette and assume that there is no diffusion. The smoke is made out of small, fluid particles. The flow line is made out of fluid particles which were passing through \mathbf{y} in the time interval $0 \leq s \leq t$. Suppose that we froze time at $t = t_0$. Choose point A . **Streakline** through the point A is a curve made of all particles (in the given moment) that have passed through the point A at some $t < t_0$.

$$\mathbf{r}[\mathbf{R}(y, s), t], \quad 0 \leq s \leq t.$$

Streamline Streamline is a integral curve of a vector field $X(t)$ at a given time $t = t_0$. They do not intersect neither with each other nor with themselves. Equation:

$$\frac{d\mathbf{r}}{ds} = \mathbf{u}(\mathbf{r}, t).$$

Trajectory **Trajectory** is a path traced by a chosen particle.

For the stationary flow the streakline and streamline are the same.

For stationary flows i.e. $(\frac{\partial \mathbf{u}}{\partial t} = 0)$. Trajectory \equiv streakline \equiv streamline.

1.3 Balance equations

$\mathbf{u}(\mathbf{r}, t)$ — velocity field, $\rho(\mathbf{r}, t)$ — density flow. They are not completely independent since the mass has to be conserved.

$$\begin{aligned} M &= \int_V \rho(\mathbf{r}, t) d\mathbf{r}, \\ \frac{\partial M}{\partial t} &= \frac{\partial}{\partial t} \int_V \rho(\mathbf{r}, t) d\mathbf{r} = \int_V \frac{\partial \rho(\mathbf{r}, t)}{\partial t} d\mathbf{r}. \\ \frac{\partial M}{\partial t} &= \int_V \frac{\partial \rho}{\partial t} d\mathbf{r} = - \int_{\partial V} \rho \mathbf{u} \cdot \mathbf{n} da, \end{aligned}$$

using Stokes theorem

$$= - \int_V \nabla \cdot (\rho \mathbf{u}) d\mathbf{r}.$$

In other words

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] d\mathbf{r} = 0,$$

and, since V is arbitrary,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

Which is called the **continuity equation**. This is the general mass conservation, fluid can change density and so on.

2 | Lecture 2

Reminder Material derivative

$$\frac{d}{dt} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla.$$

Mass conservation implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0.$$

New stuff Expanding the above equation we obtain

$$\underbrace{\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \mathbf{u}}_{\frac{d\rho}{dt}} + \mathbf{u} \cdot \nabla \rho = 0,$$

and thus

$$\frac{1}{\rho} \frac{d\rho}{dt} = -\nabla \cdot \mathbf{u}.$$

If we introduce **specific volume** $\nu = 1/\rho$ we obtain

$$\frac{1}{\nu} \frac{d\nu}{dt} = \nabla \cdot \mathbf{u}.$$

We introduced that because we want to study incompressible flow. The flow is incompressible then we say that

$$\frac{d\rho}{dt} = 0, \quad \text{or} \quad \frac{d\nu}{dt} = 0.$$

From the continuity equation incompressibility of a flow implies that

$$\nabla \cdot \mathbf{u} = 0.$$

For the incompressible flow the \mathbf{u} is divergence-free or solenoidal? **TODO**

If $\nabla \cdot \mathbf{u} = 0$ and $f(\mathbf{r}, t) = f_0 = \text{const}$ then

$$\forall t \quad f(\mathbf{r}, t) = f_0 = \text{const}.$$

Positive divergence implies expansion, negative implies compression.

2.1 Newton's second law (Momentum balance)

Material volume. Consider closed system (volume $V(t)$) which is comprised of the same fluid particles. It flows with a fluid. **TODO** Fig0. We want to calculate a momentum of such **material volume**. It is obviously an integral

$$\mathbf{P}(t) = \int_{V(t)} \rho(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t) d\mathbf{r}.$$

That is a linear momentum of the material volume. We want to state the Newton second law:

$$\frac{d\mathbf{P}(t)}{dt} = \mathbf{F},$$

where \mathbf{F} is a net force.

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt} \int_{V(t)} \rho(\mathbf{r}, t) \mathbf{u}(\mathbf{r}, t) d\mathbf{r} = ?.$$

Here is a theorem (i.e. fancy name for Leibniz rule):

Theorem 2.1.1 (Raynold's transport theorem).

$$\frac{d}{dt} \int_{V(t)} \beta(\mathbf{r}, t) d\mathbf{r} = \int_V \left[\frac{\partial \beta}{\partial t} + \nabla \cdot (\beta \mathbf{u}) \right] d\mathbf{r} = \int_V \left[\frac{\partial \beta}{\partial t} + \mathbf{u} \cdot \nabla \beta + \beta \nabla \cdot \mathbf{u} \right] d\mathbf{r},$$

where V is a fixed quantity, called control volume (i.e. any volume that coincides with $V(t)$).

The things that contribute to this change can be interpreted as

1. local change $\frac{\partial \beta}{\partial t}$,
2. advection i.e. $\mathbf{u} \cdot \nabla \beta$,
3. changing volume i.e. $\beta \nabla \cdot \mathbf{u}$.

Using it to the momentum we get

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt} \int_{V(t)} \rho \mathbf{u} d\mathbf{r} = \int_V \left[\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \underbrace{\mathbf{u} \mathbf{u}}_{\mathbf{u} \otimes \mathbf{u}}) \right] d\mathbf{r}$$

Homework Show that if $\beta = \rho b$, then

$$\frac{d}{dt} \int_{V(t)} \rho b d\mathbf{r} = \int_V \rho \frac{db}{dt} d\mathbf{r},$$

using RTT (Raynold's transport theorem) and the continuity equation.

$$\frac{d\mathbf{P}}{dt} = \frac{d}{dt} \int_{V(t)} \rho \mathbf{u} d\mathbf{r} = \int_V \rho \frac{d\mathbf{u}}{dt} d\mathbf{r} = \mathbf{F},$$

and thus the integral's form of Newton second law

$$\int_V \rho \frac{d\mathbf{u}}{dt} d\mathbf{r} = \mathbf{F}.$$

2.2 Further consequences of RTT

TODOFig1.

$$M = \int_V \rho d\mathbf{r} \implies \frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \int_V \rho d\mathbf{r} = \int_V \frac{\partial \rho}{\partial t} d\mathbf{r} = - \int_{\partial V} \rho \mathbf{u} \cdot \hat{n} dS = - \int_V \nabla \cdot (\rho \mathbf{u}) d\mathbf{r}.$$

To note: material volume is the volume that flows with the fluid.

TODOFig2

Consider material volume $V(t)$ and its mass given by

$$M = \int_{V(t)} \rho d\mathbf{r}.$$

Thus the mass conservation means that

$$\frac{dM}{dt} = 0.$$

We calculate

$$\frac{dM}{dt} = \frac{d}{dt} \int_{V(t)} \rho d\mathbf{r} = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) \right] d\mathbf{r} = 0.$$

In the following equation

$$\int_V \rho \frac{d\mathbf{u}}{dt} d\mathbf{r} = \mathbf{F},$$

we do not know what \mathbf{F} is and therefore need a model for it.

TODOFig3

Consider that the fluid acts on its surface element da (with normal vector \hat{n}). Let \mathbf{t} be a force per unit area, and $d\mathbf{F} = \mathbf{t} da$.

$$\mathbf{F} \stackrel{\text{model}}{=} \int_{\partial V} d\mathbf{F} = \int_{\partial V} \mathbf{t} \cdot da = - \int_V \nabla p d\mathbf{r}.$$

Assume that $\mathbf{t} = -p\hat{n}$, where p is a pressure and ∇p a **pressure field**.

$$\begin{aligned} \int_V \rho \frac{d\mathbf{u}}{dt} d\mathbf{r} &= \mathbf{F} = - \int_V \nabla p d\mathbf{r}, \\ \int_V \left[\rho \frac{d\mathbf{u}}{dt} + \nabla p \right] d\mathbf{r} &= 0 \implies \rho \frac{d\mathbf{u}}{dt} = -\nabla p. \end{aligned}$$

We may also write it as

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \cdot \mathbf{u} \right) = -\nabla p.$$

This is the **Euler model of the ideal fluid** (ideal fluid without dissipation).

2.3 Equilibrium

Equilibrium is obtained when $\mathbf{u} = 0$ and thus $\nabla p = 0$. Especially if $p = \text{const.}$

- ideal fluid $\mathbf{t} = -p\hat{\mathbf{n}}$
- In general $\mathbf{t} = \Sigma^T \cdot \mathbf{n}$, where Σ is **Cauchy stress tensor** (second order tensor).

In general case force can have a

$$\underline{\underline{\Sigma}} = -p\underline{\underline{1}} + \underline{\underline{\Sigma'}}$$

$$\mathbf{F} = \int_{\partial V} \mathbf{t} da = \int_{\partial V} \Sigma^T \cdot \hat{\mathbf{n}} da = \int_V \nabla \cdot \Sigma d\mathbf{r}.$$

Newton's second law

$$\int_V \rho \frac{d\mathbf{u}}{dt} d\mathbf{r} = \nabla \cdot \Sigma d\mathbf{r}.$$

$$\Sigma = \underbrace{-p\underline{\underline{1}}}_{\text{ideal term}} + \underbrace{\underline{\underline{\Sigma}}}_{\text{deviatoric part}}$$

Deviatoric part vanishes in equilibrium.

Ideal fluid model $\Sigma' = 0$, $\Sigma = -p\underline{\underline{1}}$.

$$\nabla \cdot \Sigma = \nabla \cdot (-p\underline{\underline{1}}) = -\nabla p.$$

Summary Til now we formulated

•

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

•

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \cdot \mathbf{u} \right) = \nabla \cdot \Sigma$$

- What's next?

2.4 Angular momentum

Fig4

Consider material volume $V(t)$, density ρ and a point at \mathbf{r} moving with a velocity \mathbf{u} .

$$\mathbf{L}(t) = \int_{V(t)} \mathbf{r} \times \rho \mathbf{u} d\mathbf{r} = \int_{V(t)} \rho (\mathbf{r} \times \mathbf{u}) d\mathbf{r},$$

where $\mathbf{l} = \mathbf{r} \times \mathbf{u}$ is the **angular momentum per unit mass**.

The law of the change of angular momentum

$$\frac{d\mathbf{L}}{dt} = \mathbf{N},$$

where \mathbf{N} is a **net torque** acting on $V(t)$.

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt} \int_{V(t)} \rho \mathbf{l} d\mathbf{r} \stackrel{\text{RTT} + \text{cont}}{=} \int_V \rho \frac{d\mathbf{l}}{dt} d\mathbf{r}.$$

We calculate \mathbf{N} by

$$\mathbf{N} = \int_{\partial V} \mathbf{r} \times \mathbf{t} da.$$

$$\int_V \rho \frac{d\mathbf{l}}{dt} d\mathbf{r} = \int_V \mathbf{r} \times \mathbf{t} da = \int_{\partial V} (\mathbf{r} \times \Sigma^T) \cdot \hat{n} da$$

using divergence theorem

$$= \int_V \nabla \cdot [(\mathbf{r} \times \Sigma^T)^T] d\mathbf{r}. \quad (2.1)$$

where divergence theorem:

$$\int_{\partial V} T \cdot \hat{n} da = \int_V \nabla \cdot T^T d\mathbf{r}.$$

Going back to 2.1

$$= \int_V [\mathbf{r} \times \nabla \cdot \Sigma - 2\sigma] d\mathbf{r},$$

where σ is the axial vector associated with Σ . Thus

$$\int_V \rho \frac{d\mathbf{l}}{dt} d\mathbf{r} = \int_V [\mathbf{r} \times \nabla \cdot \Sigma - 2\sigma] d\mathbf{r},$$

and since the volume V can be anything we get

$$\rho \frac{d\mathbf{l}}{dt} = \mathbf{r} \times \nabla \cdot \Sigma - 2\sigma. \quad (2.2)$$

This is too complicated, we need to simplify it.

$$\rho \frac{d\mathbf{l}}{dt} = \rho \frac{d\mathbf{r} \times \mathbf{u}}{dt} = \rho \mathbf{r} \times \frac{d\mathbf{u}}{dt} + \rho \frac{d\mathbf{r}}{dt} \times \mathbf{u} = \mathbf{r} \times \underbrace{\rho \frac{d\mathbf{u}}{dt}}_{\nabla \Sigma} = \mathbf{r} \times \nabla \cdot \Sigma.$$

Substituting it to 2.2 we get

$$\sigma = 0.$$

Thus the Σ is symmetric.

The stress tensor need not to be symmetric for magnetic fluids (it works for the simple fluids).

„Complex” fluids TODOFig5

Consider a magnetic fluid in a bottle, with magnetic dipoles. Assume that we apply a magnetic field, so there is a reorientation and the internal torque appears.

$$\mathbf{N} = \int_{\partial V} \mathbf{r} \times \mathbf{t} da + \int_V \mathbf{b} d\mathbf{r},$$

where \mathbf{b} is the internal torque.

2.5 Energy conservationTODOFig6

Consider material volume V and a small surface element da . The energy is given by

$$E(t) = \int_{V(t)} \rho(\mathbf{r}, t) e(\mathbf{r}, t) d\mathbf{r},$$

where $e(\mathbf{r}, t)$ is the energy for unit volume.

The „law” of change

$$\frac{dE}{dt} = \frac{dW}{dt} + \frac{dQ}{dt},$$

where W is a mechanical work and Q is a heat.

Using RTT and continuity we get

$$\frac{dE}{dt} = \frac{d}{dt} \int_{V(t)} \rho e d\mathbf{r} = \int_V \rho \frac{de}{dt} d\mathbf{r},$$

$$\frac{dW}{dt} = \int_{\partial V} \mathbf{t} da \cdot \mathbf{u} = \int_{\partial V} (\Sigma^T \cdot \mathbf{u}) \cdot \hat{n} da = \int_V \nabla \cdot (\underline{\Sigma} \cdot \mathbf{u}) d\mathbf{r}.$$

$$\frac{dQ}{dt} = - \int_{\partial V} \mathbf{q} \cdot \hat{n} da = - \int_V \nabla \cdot \mathbf{q} d\mathbf{r},$$

where \mathbf{q} is the heat flow per unit surface per unit time.

Summing up we get

$$\rho \frac{de}{dt} = \nabla \cdot (\Sigma \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}. \quad (2.3)$$

Let's introduce the separation

$$e = e_0 + \frac{1}{2} \mathbf{u}^2.$$

Substituting it into 2.3 we obtain

$$\rho \frac{de_0}{dt} = -\rho \frac{d}{dt} \left(\frac{1}{2} \mathbf{u}^2 \right) + \nabla \cdot (\Sigma \cdot \mathbf{u}) - \nabla \cdot \mathbf{q}.$$

$$\rho \frac{d}{dt} \left(\frac{1}{2} \mathbf{u}^2 \right) = \mathbf{u} \cdot \rho \frac{d\mathbf{u}}{dt} = \mathbf{u} \cdot (\nabla \cdot \Sigma) = \nabla \cdot (\Sigma \cdot \mathbf{u}) - \Sigma : \nabla \mathbf{u}$$

For the internal energy e_0 :

$$\rho \frac{de_0}{dt} = \Sigma : \nabla \mathbf{u} - \nabla \cdot \mathbf{q}.$$

For $\Sigma = -p\mathbf{1} + \Sigma'$

$$\Sigma : \nabla \mathbf{u} = -p(\nabla \cdot \mathbf{u}) + \Sigma' : \nabla \mathbf{u}.$$

2.6 Ideal fluid approximation

We assume no deviatoric stress, and no heat flows

$$\begin{aligned} \Sigma' &= 0 \\ \mathbf{q} &= 0. \end{aligned}$$

The for the ideal fluid

$$\rho \frac{de_0}{dt} = -p(\nabla \cdot \mathbf{u}).$$

This approximation is a result of assuming that the particles move all together, so there is no viscosity and also no way to conduct heat.

For a compressible flow

- $\nabla \cdot \mathbf{u} > 0 \implies \frac{de_0}{dt} < 0$ (expansion),
- $\nabla \cdot \mathbf{u} < 0 \implies \frac{de_0}{dt} > 0$ (compression),

and for the incompressible flow there is no way to change internal energy, $\frac{de_0}{dt} = 0$. In ideal fluid there is no dissipation. However the real fluids do.

2.7 Entropy balance

TODO Fig7

Consider closed system and assume that a heat has been transferred to the system. We can move between S and $S + dS$ by reversible and irreversible paths. For the reversible one we have

$$dS = \frac{dQ}{T},$$

and for irreversible proces

$$dS \geq \frac{dQ}{T}.$$

$$S(t) = \int_{V(t)} \rho(\mathbf{r}, t) s(\mathbf{r}, t) d\mathbf{r},$$

where $s(\mathbf{r}, t)$ is entropy per unit volume. Entropy balance implies

$$\frac{dS}{dt} = \frac{d_e S}{dt} + \frac{d_i S}{dt}$$

$$d_e S = \frac{dQ}{T}.$$

$$\begin{aligned}\frac{dS}{dt} &= \frac{d}{dt} \int_{V(t)} \rho s d\mathbf{r} = \int -V \frac{ds}{dt} d\mathbf{r}, \\ \frac{d_e S}{dt} &= - \int_{\partial V} \frac{\mathbf{q}}{T} \cdot \hat{n} da = - \int_V \nabla \cdot \left(\frac{\mathbf{q}}{T} \right) d\mathbf{r} \\ \frac{d_i S}{dt} &= \int_V \theta d\mathbf{r},\end{aligned}$$

where θ is the entropy production per unit volume per unit time. Thus the **entropy balance equation** is

$$\rho \frac{ds}{dt} = -\nabla \cdot \left(\frac{\mathbf{q}}{T} \right) + \theta, \quad \theta \geq 0.$$

For ideal fluid (no internal effects) $\theta = 0$, and the change of entropy

$$\frac{ds}{dt} = 0.$$