Hydrodynamics and elasticity, Lecture notes

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Spis treści

1 Basic definitions

1.1 Organization

Course web page: https://www.fuw.edu.pl/~mklis/hydro2022.html

Requirements to obtain credit:

- Homework (30%),
- Midterm exam (35%),
- Written exam (35%),
- Oral exam (optional, only improves).

1.2 Basic laws

Example 1.2.1. Out of context Navier-Stokes equations:

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \underline{u},$$

$$\nabla \cdot \underline{u} = 0.$$

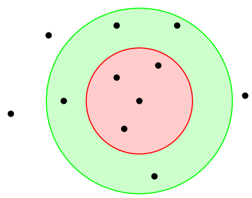
where $\underline{u}(\underline{r}, t)$ is a fluid velocity vector field, $\rho(\underline{r}, t)$ is a fluid density, $p(\underline{r}, t)$ is a pressure, ν is a kinematic viscosity.

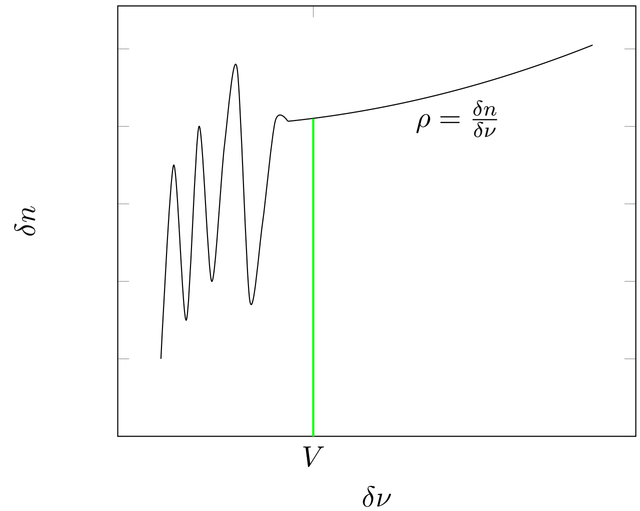
Continuum hypothesis states that

$$\rho = \frac{\delta \eta}{\delta \nu},$$

where η is a number of particles in a region and ν is a volume of this region. Of course if the volume ν is small enough ρ is may vary a lot (obviously it may not even be continuous). There is however such volume V which is „big enough", so that for $\nu > V$ ρ does not vary „that much".

Since matter is not continuous, we can't speak of a density at a point (in a mathematical sense) and thus, when we use the phrase „point" we mean „at a point for homogeneous physical system".



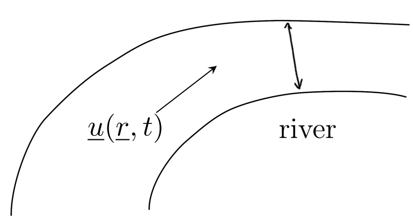


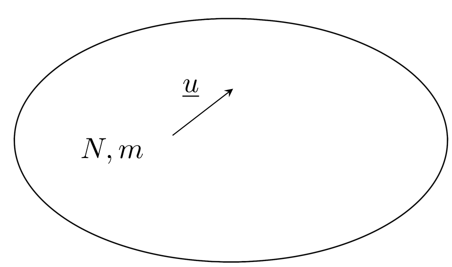
We also introduce length scale separation

$$\text{molecular length scale} \quad \ll \quad \nu^{\frac{1}{3}} \quad \ll \quad L.$$

We call those characteristic length scales „*micro*", „*muso*" and „*macro*" respectively.

1.2.1 Equilibrium thermodynamics





Let us examine an example of a water in a river. Assume that the water has some velocity $\underline{u}$, momentum $\underline{P}$ and also that it can exchange mass and heat, the latter given by $\mathrm{d}Q = T\mathrm{d}S$. Therefore an incremental change in the energy of this system can be expressed as

$$\mathrm{d}E = \underline{u} \cdot \mathrm{d}\underline{P} - p\mathrm{d}V + T\mathrm{d}S + \mu\mathrm{d}N,$$

where μ is the chemical potential of the system. Thus

$$E = E(\underline{P}, V, S, N).$$

This is the energy representation of a thermodynamical system. Comparing the equations above we obtain

$$\mathrm{d}E = \underbrace{\frac{\partial E}{\partial \underline{P}}}_{\underline{u}} \mathrm{d}\underline{P} + \underbrace{\frac{\partial E}{\partial V}}_{-p} \mathrm{d}V + \underbrace{\frac{\partial E}{\partial S}}_{T} \mathrm{d}S + \underbrace{\frac{\partial E}{\partial N}}_{\mu} \mathrm{d}N,$$

Those are called the Gibbs relation for E.

If we want to compute it for a fixed entropy we get

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}E - \frac{\underline{u}}{T}\mathrm{d}\underline{P} + \frac{p}{T}\mathrm{d}V - \frac{\mu}{T}\mathrm{d}N.$$

Thus

$$S = S(E, \underline{P}, V, N), \quad \mathrm{d}S = \underbrace{\frac{\partial S}{\partial E}}_{\frac{1}{T}} \mathrm{d}E + \underbrace{\frac{\partial S}{\partial \underline{P}}}_{-\frac{\underline{u}}{T}} \mathrm{d}\underline{P} + \underbrace{\frac{\partial S}{\partial V}}_{\frac{p}{T}} \mathrm{d}V + \underbrace{\frac{\partial S}{\partial N}}_{-\frac{\mu}{T}} \mathrm{d}N,$$

Those are Gibbs relations for S.

It is very tricky to control entropy — it's much easier to control the temperature. To obtain a description of our system when T is an independent variable we need to use another thermodynamical potential, which is Helmholz free energy. Transition is obtained by

$$(S \to T) \quad F = E - TS, \quad F = F(\underline{P}, V, T, N)$$

$$\mathrm{d}F = -S\mathrm{d}T - p\mathrm{d}V + \underline{u}\mathrm{d}\underline{P} + \mu\mathrm{d}N.$$

Now we can do the same to switch other variables. Thus we obtain

$$(V \to P) \quad H = E + pV, \quad H = H(\underline{P}, p, S, N),$$

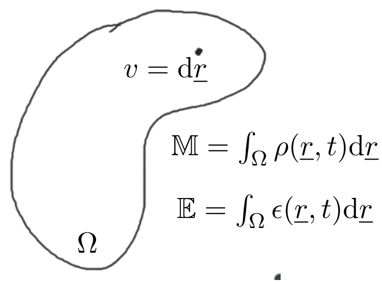
which is called an enthalpy,

$$(S \to T) \quad G = E + pV - TS, \quad G = G(\underline{P}, p, T, N),$$

which is called a Gibbs potential.

Now we want to consider if they are Galilean invariant

$$E(\underline{P}, V, S, N) = E_0(V, S, N) + \frac{\underline{P}^2}{2M},$$



Rysunek 1.3: Sample volume Ω. ρ stands for mass density, ε for density of the system.

where E_0 is the internal energy. For other potentials we obtain

$$H(\underline{P}, p, S, N) = H_0(p, S, N) + \frac{\underline{P}^2}{2M},$$

$$F(\underline{P}, V, T, N) = F_0(T, V, N) + \frac{\underline{P}^2}{2M},$$

$$G(\underline{P}, V, T, N) = G_0(T, p, N) + \frac{\underline{P}^2}{2M}.$$

Using $\underline{P} = M\underline{u}$ we get

$$E = E_0 + \frac{1}{2}M\underline{u}^2.$$

$$\mathrm{d}E = \mathrm{d}E_0 + \underline{u} \cdot \mathrm{d}\underline{P},$$

$$\mathrm{d}S = -\frac{1}{T}\underline{u} \cdot \mathrm{d}\underline{P} + \frac{1}{T}\mathrm{d}E + \frac{p}{T}\mathrm{d}V - \frac{\mu}{T}\mathrm{d}N,$$

$$\mathrm{d}S = \frac{1}{T}\mathrm{d}E_0 + \frac{p}{T}\mathrm{d}V - \frac{\mu}{T}\mathrm{d}N \quad \Longrightarrow \quad S(\underline{P}, E, V, N) = S(E_0, V, N).$$

Thus S is a Galilean invariant.

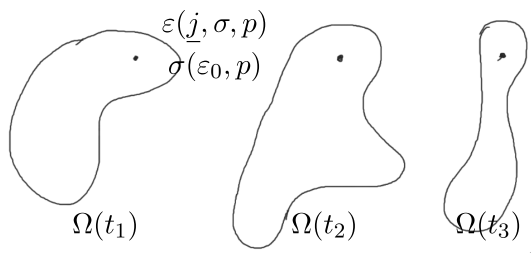
1.2.2 Heterogenous macroscopic system

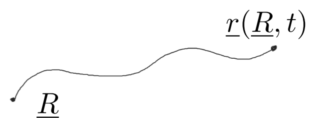
„Densities” are extensive properties per unit volume. Assume that $V = \text{const}$, $\mathrm{d}V = 0$. Example of densities

- $\rho = \frac{M}{V}$ mass density,
- $\underline{j} = \frac{P}{V}$ momentum density,
- $\varepsilon = \frac{E}{V}$ energy density,
- $\sigma = \frac{S}{V}$ entropy density.

$$\mathrm{d}E = \underline{u}\mathrm{d}\underline{P} - p\mathrm{d}V + T\mathrm{d}S + \mu\mathrm{d}N / \cdot \frac{1}{V}.$$

$$\mathrm{d}\varepsilon = \underline{u} \cdot \mathrm{d}\underline{j} + T\mathrm{d}\sigma + \mu\mathrm{d}n, \quad \mathrm{d}n = \frac{\mathrm{d}\rho}{m},$$





$$\rho = \frac{M}{V} = \frac{Nm}{V} = nm \quad \Longrightarrow \quad \mathrm{d}n = \frac{\mathrm{d}\rho}{m}.$$

Thus the energy fundamental representation in terms of densities can be written as

$$\varepsilon = \varepsilon(\underline{j}, \sigma, \rho).$$

After performing a Galilean transform we get

$$\varepsilon(\underline{j}, \sigma, \rho) = \varepsilon_0(\sigma, \rho) + \frac{1}{2}\rho \underline{u}^2.$$

We can do the same to represent entropy in terms of densities

$$\mathrm{d}S = \ldots \frac{1}{V} \quad \Longrightarrow \quad \mathrm{d}\sigma = \frac{1}{T}\mathrm{d}\varepsilon_0 - \frac{1}{T}\frac{\mu}{m}\mathrm{d}p.$$

Thus

$$\sigma = \sigma(\varepsilon_0, \rho),$$

which is also Galilean invariant.

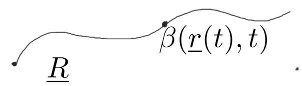
1.2.3 Flow of Heterogenous macroscopic system

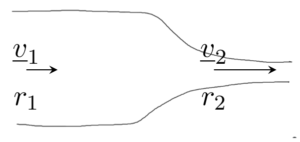
We are using thermodynamics equilibrium despite the fact that the system flows (which means that it is *not* in the equilibrium). However there is no contradiction if we assume local, thermodynamical equilibrium. The same assumption is made in Navier-Stokes equations.

From now one we use pseudostatic transition. The difference between quasi-static and pseudostatic is that pseudostatic need not to be reversible. Forces which acts during quasi-static transformation are those which keeps the system in equilibrium. Example of pseudostatic transition which is not quasi-static is a flow of viscose fluid.

1.2.4 Kinematics

We have two descriptions: Eulerian and Lagrangian. Transition between them is obtained by





the solutions to the initial problem

$$\frac{\mathrm{d}\underline{r}}{\mathrm{d}t} = \underline{u}(r, t), \quad \underline{r}(t = 0) = \underline{R}.$$

We want to find dependence of densities as the particle move, and those are

	Euler	Lagrange
velocity	$\underline{u}(r, t)$	$\underline{u}[r(t), t]$
density	$\rho(r, t)$	$\underline{\rho}[r(t), t]$

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} =?, \quad \frac{\mathrm{d}\rho}{\mathrm{d}t} =?.$$

To do that define $\beta = (\underline{u}, \rho, S, \sigma, \dots)$. We want to find how β change while following the motion of the particle $r(t)$.

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\mathrm{d}\beta[\underline{r}(t), t]}{\mathrm{d}t} = \frac{\partial \beta}{\partial t} + \frac{\mathrm{d}r_1}{\mathrm{d}t}\frac{\partial \beta}{\partial r_1} + \frac{\mathrm{d}r_2}{\mathrm{d}t}\frac{\partial \beta}{\partial r_2} + \frac{\mathrm{d}r_3}{\mathrm{d}t}\frac{\partial \beta}{\partial r_3}$$
$$= \frac{\partial \beta}{\partial t} + \underline{u}(t) \cdot \nabla \beta = \frac{\partial \beta}{\partial t} + \mathrm{d}\beta(\underline{u}(t)).$$

Equation

$$\frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\partial \beta}{\partial t} + \underline{u}(t) \cdot \nabla \beta, \tag{1.1}$$

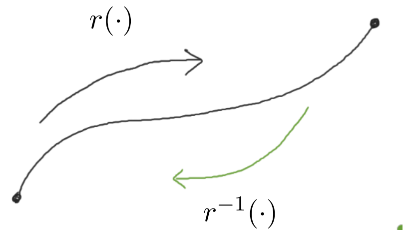
shows a relationship between Eulerian and Lagrangian world, because $\frac{\mathrm{d}\beta}{\mathrm{d}t}$ is a typical Lagrangian while fields (like $\underline{u}(t)$) are common in Euler's description. (Fields are Eulerian objects). We introduce a total (material) derivative as

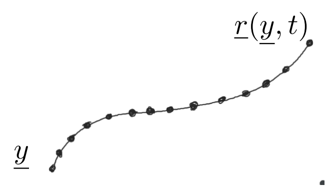
$$\frac{\mathrm{d}\dots}{\mathrm{d}t} = \underbrace{\frac{\partial \dots}{\partial t}}_{\text{local derivative}} + \underbrace{\underline{u} \cdot \nabla(\dots)}_{\text{advective derivative}} .$$

Acceleration of fluid particle

$$\beta = \underline{u} \quad \Longrightarrow \quad \frac{\mathrm{d}\beta}{\mathrm{d}t} = \frac{\mathrm{d}\underline{u}}{\mathrm{d}t}.$$

Consider converging chanell with a stationary flow, i.e. $\frac{\partial \underline{u}}{\partial t} = 0$.





Rysunek 1.4: Streakline is a line made of all particles that for time s, $0 \le s \le t$ passed through a fixed point $\underline{y}$.

With that in mind

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = \underline{u} \cdot \nabla \underline{u}.$$

The term $\underline{u} \cdot \nabla \underline{u}$ should be interpreted as follows. Treat $\underline{u}$ as a map $\underline{u}(\underline{r}) : \mathbb{R}^3 \to \mathbb{R}^3$. Thus $\nabla \underline{u}$ is just a map $D\underline{u} : \mathbb{R}^3 \to \mathbb{R}^3$ expressed by a matrix

$$\underline{u}(\underline{r}) = \begin{bmatrix} u_1(r_1, r_2, r_3) \\ u_2(r_1, r_2, r_3) \\ u_3(r_1, r_2, r_3) \end{bmatrix}, \quad D\underline{u} = \begin{bmatrix} \frac{\partial u_1}{\partial r_1} & \frac{\partial u_1}{\partial r_2} & \frac{\partial u_1}{\partial r_3} \\ \frac{\partial u_2}{\partial r_1} & \frac{\partial u_2}{\partial r_2} & \frac{\partial u_2}{\partial r_3} \\ \frac{\partial u_3}{\partial r_1} & \frac{\partial u_3}{\partial r_2} & \frac{\partial u_3}{\partial r_3} \end{bmatrix}.$$

Therefore inner product $\underline{u} \cdot \nabla \underline{u}$ really means

$$\underline{u} \cdot \nabla \underline{u} = (D\underline{u})(\underline{u}).$$

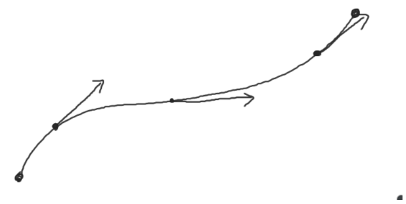
At least I think so...

The term $\underline{u} \cdot \nabla \underline{u}$ is called a convective acceleration. Although the flow is stationary, the particle experiences acceleration related to movement along the flow lines.

We can interpret the fluid flow as a mapping which takes one point and maps it to the other.

Streakline Imagine a cigarette and assume that there is no diffusion. The smoke is made out of small, fluid particles. The flow line is made out of fluid particles which were passing through $\underline{y}$ in the time interval $0 \le s \le t$. Suppose that we froze time at $t = t_0$. Choose point A. Streakline through the point A is a curve made of all particles (in the given moment) that have passed through the point A at some $t < t_0$.

$$\underline{r}[\underline{R}(y, s), t], \quad 0 \le s \le t.$$



Rysunek 1.5: Streamline is just a flow of a vector field for at a fixed time t.

Streamline Streamline is a integral curve of a vector field $X(t)$ at a given time $t = t_0$. They do not intersect neither with each other nor with themselves. Equation:

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \underline{u}(\underline{r}, t).$$

Trajectory Trajectory is a path traced by a chosen particle.

For the stationary flow the streakline and streamline are the same.

For stationary flows i.e. $(\frac{\partial u}{\partial t} = 0)$. Trajectory $\equiv$ streakline $\equiv$ streamline.

1.3 Balance equations

$\underline{u}(\underline{r}, t)$ — velocity field, $\rho(\underline{r}, t)$ — density flow. They are not completely independent since the mass has to be conserved.

$$M = \int_V \rho(\underline{r}, t)\mathrm{d}\underline{r},$$

$$\frac{\partial M}{\partial t} = \frac{\partial}{\partial t} \in V \rho(\underline{r}, t) = \int_V \frac{\partial \rho(\underline{r}, t)}{\partial t}\mathrm{d}\underline{r}.$$

$$\frac{\partial M}{\partial t} = \int_V \frac{\partial \rho}{\partial t}\mathrm{d}\underline{r} = -\int_{\partial V} \rho\underline{u} \cdot \underline{n}\mathrm{d}a,$$

using Stokes theorem

$$= -\int_V \nabla \cdot (\rho\underline{u})\mathrm{d}\underline{r}.$$

In other words

$$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\underline{u})\right] \mathrm{d}\underline{r} = 0,$$

and, since V is arbitrary,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\underline{u}) = 0.$$

Which is called the continuity equation. This is the general mass conservation, fluid can change density and so on.

2 Lecture 2

Reminder Material derivative

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla \cdot$$

Mass conservation implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0.$$

New stuff Expanding the above equation we obtain

$$\underbrace{\frac{\partial \rho}{\partial t} + \underline{u} \cdot \nabla \rho}_{\frac{\mathrm{d}\rho}{\mathrm{d}t}} + \rho \nabla \cdot \underline{u} = 0,$$

and thus

$$\frac{1}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\nabla \cdot \underline{u}.$$

If we introduce specific volume $\nu = 1/\rho$ we obtain

$$\frac{1}{\nu} \frac{\mathrm{d}\nu}{\mathrm{d}t} = \nabla \cdot \underline{u}.$$

We introduced that because we want to study incompressible flow. If the flow is incompressible we express it by saying that

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \quad \text{or} \quad \frac{\mathrm{d}\nu}{\mathrm{d}t} = 0.$$

From the continuity equation incompressibility of the flow implies that

$$\nabla \cdot \underline{u} = 0.$$

For the incompressible flow the $\underline{u}$ is divergence-free or solonoidal (TODOI didn't hear well).

If $\nabla \cdot \underline{u} = 0$ and $f(\underline{r}, t) = f_0 = \text{const}$ then[1]

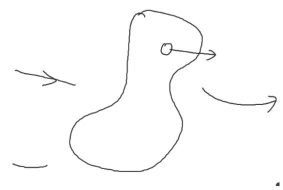
$$\forall t \quad f(\underline{r}, t) = f_0 = \text{const}.$$

Positive divergence implies expansion, negative implies compression.

[1] It follows from the continuity equation.

2.1 Newton's second law (Momentum balance)

Consider a closed system (volume $V(t)$) which is comprised of the same fluid particles (it flows with a fluid).



We want to calculate the momentum of such material volume. It is obviously an integral

$$\underline{P}(t) = \int_{V(t)} \rho(\underline{r}, t)\underline{u}(\underline{r}, t)\mathrm{d}\underline{r}.$$

That is a linear momentum of the material volume. We want to state the Newton second law:

$$\frac{\mathrm{d}\underline{P}(t)}{\mathrm{d}t} = \underline{F},$$

where $\underline{F}$ is the net force.

$$\frac{\mathrm{d}\underline{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho(\underline{r}, t)\underline{u}(\underline{r}, t)\mathrm{d}\underline{r} = ?.$$

Here is a theorem (i.e. fancy name for Leibniz rule):

Theorem 2.1.1 (Raynold's transport theorem)**.**

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \beta(\underline{r}, t)\mathrm{d}\underline{r} = \int_V \left[\frac{\partial\beta}{\partial t} + \nabla\cdot(\beta\underline{u})\right]\mathrm{d}\underline{r} = \int_V \left[\frac{\partial\beta}{\partial t} + \underline{u}\cdot\nabla\beta + \beta\nabla\cdot\underline{u}\right],$$

where V is a fixed quantity, called control volume (i.e. any volume that coincides with $V(t)$).

The things that contribute to this change can be interpreted as

1. local change $\frac{\partial\beta}{\partial t}$,
2. advection i.e. $\underline{u}\cdot\nabla\beta$,
3. changing volume i.e. $\beta\nabla\cdot\underline{u}$.

Applying RTT to the momentum we get

$$\frac{\mathrm{d}\underline{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho\underline{u}\mathrm{d}\underline{r} = \int_V \left[\frac{\partial\rho\underline{u}}{\partial t} + \nabla\cdot(\rho\underbrace{\underline{u}\underline{u}}_{\underline{u}\otimes\underline{u}})\right]$$

Homework Show that if $\beta = \rho b$, then

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho b \mathrm{d}\underline{r} = \int_V \rho \frac{\mathrm{d}b}{\mathrm{d}t}\mathrm{d}\underline{r},$$

using RTT (Raynold's transport theorem) and the continuity equation.

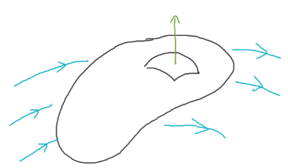
$$\frac{\xi}{x} = \int_V \mathrm{d}V$$

$$\frac{\mathrm{d}\underline{P}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \underline{u} \mathrm{d}\underline{r} = \int_V \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t}\mathrm{d}\underline{r} = \underline{F},$$

and thus the integral's form of Newton second law

$$\int_V \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t}\mathrm{d}\underline{r} = \underline{F}.$$

2.2 Further consequences of RTT



$$M = \int_V \rho \mathrm{d}\underline{r} \quad \Longrightarrow \quad \frac{\partial M}{\partial t} = \frac{\partial}{\partial t}\int_V \rho \mathrm{d}\underline{r} = \int_V \frac{\partial \rho}{\partial t}\mathrm{d}\underline{r} = -\int_V \nabla \cdot (\rho \underline{u})\mathrm{d}\underline{r} = -\int_{\partial V} \rho \underline{u} \cdot \hat{n}\mathrm{d}S.$$

To note: material volume is the volume that flows with the fluid.

Consider a material volume $V(t)$ and its mass given by

$$M = \int_{V(t)} \rho \mathrm{d}\underline{r}.$$

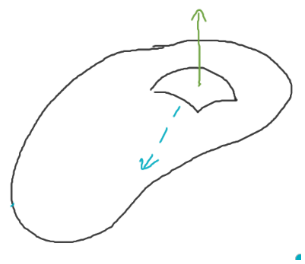
The mass conservation means that

$$\frac{\mathrm{d}M}{\mathrm{d}t} = 0.$$

We calculate

$$\frac{\mathrm{d}M}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho \mathrm{d}\underline{r} = \int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u})\right]\mathrm{d}\underline{r} = 0.$$

The above equation states that the mass of the material volume which travels with the flow remains constant.



Force model In the following equation

$$\int_V \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} \mathrm{d}\underline{r} = \underline{F},$$

we do not know what $\underline{F}$ is and therefore need a model for it.

Consider that the fluid acts on its surface element $\mathrm{d}a$ (with a normal vector $\hat{n}$). Let $\underline{t}$ be a force per unit area, and $\mathrm{d}\underline{F} = \underline{t}\mathrm{d}a$.

$$\underline{F} \overset{\text{model}}{=} \int_{\partial V} \mathrm{d}\underline{F} = \int_{\partial V} \underline{t}\mathrm{d}a = -\int_V \nabla p \mathrm{d}\underline{r}.$$

Assume that $\underline{t} = -p\hat{n}$, where p is a pressure and ∇p a pressure field.

$$\int_V \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} \mathrm{d}\underline{r} = -\int_V \nabla p \mathrm{d}\underline{r},$$

$$\int_V \left[\rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} + \nabla p\right] \mathrm{d}\underline{r} = 0 \quad \Longrightarrow \quad \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = -\nabla p.$$

We may also write it as

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \cdot \underline{u}\right) = -\nabla p.$$

This is the Euler model of the ideal fluid (ideal fluid without dissipation).

2.3 Equilibrium

Equilibrium is obtained when $\underline{u} = 0$ and thus $\nabla p = 0$ (especially if $p = \text{const}$).

- ideal fluid $\underline{t} = -p\hat{n}$
- In general $\underline{t} = \underline{\underline{\Sigma}}^T \cdot \underline{n}$, where Σ is a Cauchy stress tensor (second order tensor).

In general case force can have a form

$$\underline{F} = \int_{\partial V} \underline{t}\mathrm{d}a = \int_{\partial V} \underline{\underline{\Sigma}}^T \cdot \hat{n}\mathrm{d}a = \int_V \nabla \cdot \underline{\underline{\Sigma}}\mathrm{d}\underline{r},$$

with the stress tensor

$$\underline{\underline{\Sigma}} = -p \cdot \underline{\underline{1}} + \underline{\underline{\Sigma}}'.$$

Newton's second law

$$\int_V \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} \mathrm{d}\underline{r} = \int \nabla \cdot \underline{\underline{\Sigma}} \mathrm{d}\underline{r}.$$

$$\underline{\underline{\Sigma}} = \underbrace{-p\underline{\underline{1}}}_{\text{ideal term}} + \underbrace{\underline{\underline{\Sigma}}}_{\text{deviatory part}}$$

Deviatoric part vanishes in equilibrium.

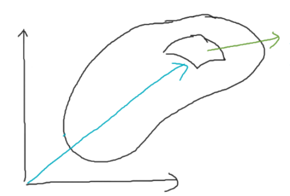
Ideal fluid model $\underline{\underline{\Sigma}}' = 0$, $\underline{\underline{\Sigma}} = -p\underline{\underline{1}}$.

$$\nabla \cdot \Sigma = \nabla \left(-p\underline{\underline{1}}\right) = -\nabla p.$$

Summary Til now we formulated

- Continuity equation
$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$$
- Newton's law
$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \cdot \underline{u} \right) = \nabla \cdot \underline{\underline{\Sigma}}$$
- What's next? Angular momentum.

2.4 Angular momentum



Consider a material volume $V(t)$, with density ρ and a point at $\underline{r}$ moving with a velocity $\underline{u}$.

$$\underline{L}(t) = \int_{V(t)} \underline{r} \times \rho \underline{u} \mathrm{d}\underline{r} = \int_{V(t)} \rho (\underline{r} \times \underline{u}) \mathrm{d}\underline{r} = \int_{V(t)} \rho \underline{l} \mathrm{d}\underline{r},$$

where $\underline{l} = \underline{r} \times \underline{u}$ is the angular momentum per unit mass.

The law of the change of the angular momentum

$$\frac{\mathrm{d}\underline{L}}{\mathrm{d}t} = \underline{N},$$

where $\underline{N}$ is a net torque acting on $V(t)$.

$$\frac{\mathrm{d}\underline{L}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \underline{l} \mathrm{d}\underline{r} \overset{\text{RTT + cont}}{=\!=} \int_V \rho \frac{\mathrm{d}\underline{l}}{\mathrm{d}t} \mathrm{d}\underline{r}.$$

We calculate $\underline{N}$ by

$$\underline{N} = \int_{\partial V} \underline{r} \times \underline{t} \mathrm{d}a.$$

$$\int_V \rho \frac{\mathrm{d}\underline{l}}{\mathrm{d}t} \mathrm{d}\underline{r} = \int_{\partial V} \underline{r} \times \underline{t} \mathrm{d}a = \int_{\partial V} (\underline{r} \times \Sigma^T) \cdot \hat{n} \mathrm{d}a$$

using divergence theorem

$$= \int_V \nabla \cdot \left[(\underline{r} \times \Sigma^T)^T \right] \mathrm{d}\underline{r}. \tag{2.1}$$

where the divergence theorem reads

$$\int_{\partial V} T \cdot \hat{n} \mathrm{d}a = \int_V \nabla \cdot T^T \mathrm{d}\underline{r}.$$

Going back to 2.1

$$= \int_V \left[\underline{r} \times \nabla \cdot \Sigma - 2\underline{\sigma} \right] \mathrm{d}\underline{r},$$

where $\underline{\sigma}$ is the axial vector associated with Σ. Thus

$$\int_V \rho \frac{\mathrm{d}\underline{l}}{\mathrm{d}t} \mathrm{d}\underline{r} = \int_V \left[\underline{r} \times \nabla \cdot \Sigma - 2\underline{\sigma} \right] \mathrm{d}\underline{r},$$

and since the volume V can be anything we get

$$\rho \frac{\mathrm{d}\underline{l}}{\mathrm{d}t} = \underline{r} \times \nabla \cdot \Sigma - 2\underline{\sigma}. \tag{2.2}$$

This is too complicated, we need to simplify it.

$$\rho \frac{\mathrm{d}\underline{l}}{\mathrm{d}t} = \rho \frac{\mathrm{d}\underline{r} \times \underline{u}}{\mathrm{d}t} = \rho \underline{r} \times \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} + \rho \frac{\mathrm{d}\underline{r}}{\mathrm{d}t} \times \underline{u} = \underline{r} \times \underbrace{\rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t}}_{\nabla \cdot \underline{\underline{\Sigma}}} = \underline{r} \times \nabla \cdot \underline{\underline{\Sigma}}.$$

Substituting it to 2.2 we get

$$\underline{\sigma} = 0.$$

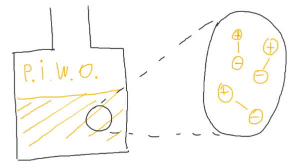
Thus the Σ is symmetric.

The stress tensor need not to be symmetric for magnetic fluids (it works for simple fluids).

„Complex" fluids Consider a magnetic fluid in a bottle, with magnetic dipoles. Assume that we apply a magnetic field, so there is a reorientation and an internal torque appears.

$$\underline{N} = \int_{\partial V} \underline{r} \times \underline{t} \mathrm{d}a + \int_V \underline{b} \mathrm{d}\underline{r},$$

where $\underline{b}$ is the internal torque.



2.5 Energy conservation

Consider a material volume V and a small surface element da. The energy is given by

$$E(t) = \int_{V(t)} \rho(\underline{r}, t) e(\underline{r}, t) \mathrm{d}\underline{r},$$

where $e(\underline{r}, t)$ is the energy for unit mass.

The „law” of change

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}W}{\mathrm{d}t} + \frac{\mathrm{d}Q}{\mathrm{d}t},$$

where W is a mechanical work and Q is a heat.

Using RTT and continuity we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho e \mathrm{d}\underline{r} = \int_V \rho \frac{\mathrm{d}e}{\mathrm{d}t} \mathrm{d}\underline{r},$$

$$\frac{\mathrm{d}W}{\mathrm{d}t} = \int_{\partial V} \underline{t} \cdot \underline{u} \mathrm{d}a = \int_{\partial V} (\underline{\underline{\Sigma}}^T \cdot \underline{u}) \cdot \hat{n} \mathrm{d}a = \int_V \nabla \cdot (\underline{\underline{\Sigma}} \cdot u) \mathrm{d}\underline{r}.$$

$$\frac{\mathrm{d}Q}{\mathrm{d}t} = -\int_{\partial V} \underline{q} \cdot \hat{n} \mathrm{d}a = -\int_V \nabla \cdot \underline{q} \mathrm{d}\underline{r},$$

where $\underline{q}$ is the heat flow per unit surface per unit time.

Summing up we get

$$\rho \frac{\mathrm{d}e}{\mathrm{d}t} = \nabla \cdot (\Sigma \cdot \underline{u}) - \nabla \cdot \underline{q}. \tag{2.3}$$

Let's introduce the separation

$$e = e_0 + \frac{1}{2} \underline{u}^2.$$

Substituting it into 2.3 we obtain

$$\rho \frac{\mathrm{d}e_0}{\mathrm{d}t} = -\rho \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \underline{u}^2 \right) + \nabla \cdot (\Sigma \cdot \underline{u}) - \nabla \cdot \underline{q}.$$

$$\rho \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \underline{u}^2 \right) = \underline{u} \cdot \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \underline{u} \cdot (\nabla \cdot \Sigma) = \nabla \cdot (\Sigma \cdot u) - \Sigma : \nabla \underline{u}$$

For the internal energy e_0:

$$\rho \frac{\mathrm{d}e_0}{\mathrm{d}t} = \Sigma : \nabla \underline{u} - \nabla \cdot \underline{q}.$$

For $\Sigma = -p\underline{\underline{1}} + \Sigma'$

$$\Sigma : \nabla \underline{u} = -p(\nabla \cdot \underline{u}) + \Sigma' : \nabla \underline{u}.$$

2.6 Ideal fluid approximation

We assume no deviatoric stress, and no heat flows

$$\Sigma' = 0$$
$$\underline{q} = 0.$$

For the ideal fluid

$$\rho \frac{\mathrm{d}e_0}{\mathrm{d}t} = -p(\nabla \cdot \underline{u}).$$

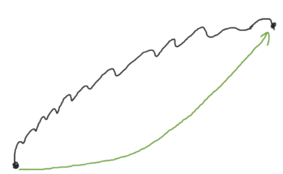
This approximation is a result of assuming that the particles move all together, so there is no viscosity and also no way to conduct a heat.

For a compressible flow

- $\nabla \cdot \underline{u} > 0 \implies \frac{\mathrm{d}e_0}{\mathrm{d}t} < 0$ (expansion),
- $\nabla \cdot \underline{u} < 0 \implies \frac{\mathrm{d}e_0}{\mathrm{d}t} > 0$ (compression),

and for the incompressible flow there is no way to change internal energy, $\frac{\mathrm{d}e_0}{\mathrm{d}t} = 0$. In ideal fluid there is no dissipation. However the real fluids do.

2.7 Entropy balance



Consider a closed system and assume that a heat has been transfered to the system. We can move between S and $S + \mathrm{d}S$ by reversible and irreversible paths. For the reversible one we have

$$\mathrm{d}S = \frac{\mathrm{d}Q}{T},$$

and for an irreversible proces

$$\mathrm{d}S \geq \frac{\mathrm{d}Q}{T}.$$

$$S(t) = \int_{V(t)} \rho(\underline{r}, t) s(\underline{r}, t) \mathrm{d}\underline{r},$$

where $s(\underline{r}, t)$ is entropy per unit volume. Entropy balance implies

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}_e S}{\mathrm{d}t} + \frac{\mathrm{d}_i S}{\mathrm{d}t}$$

$$\mathrm{d}_e S = \frac{\mathrm{d}Q}{T}.$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}\int_{V(t)} \rho s \mathrm{d}\underline{r} = \int -\rho \frac{\mathrm{d}s}{\mathrm{d}t}\mathrm{d}\underline{r},$$

$$\frac{\mathrm{d}_e S}{\mathrm{d}t} = -\int_{\partial V} \frac{\underline{\underline{q}}}{T} \cdot \hat{n} \mathrm{d}a = -\int_V \nabla \cdot \left(\frac{\underline{\underline{q}}}{T}\right) \mathrm{d}\underline{r}$$

$$\frac{\mathrm{d}_i S}{\mathrm{d}t} = \int_V \theta \mathrm{d}\underline{r},$$

where θ is the entropy production per unit volume per unit time. Thus the entropy balance equation is

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = -\nabla \cdot \left(\frac{\underline{\underline{q}}}{T}\right) + \theta, \quad \theta \geq 0.$$

For ideal fluid (no internal efects) $\theta = 0$, and the change of entropy

$$\frac{\mathrm{d}s}{\mathrm{d}t} = 0.$$

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Recall

$$\begin{aligned}\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0, \\ \frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}t} = \nabla \cdot \underline{u}, \quad \nu = \frac{1}{\rho}, \\ \rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \underline{u} \right) = \nabla \cdot \Sigma + \rho g,\end{aligned} \tag{3.1}$$

angular momentum balance

$$\Sigma^T = \Sigma.$$

energy per unit mass

$$\rho \frac{\mathrm{d}e_0}{\mathrm{d}t} = -p(\nabla \cdot \underline{u}) + \Sigma' : \nabla \underline{u} - \nabla \cdot \underline{q} \tag{3.2}$$

entropy per unit mass

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = -\nabla \cdot \left(\frac{\underline{q}}{T} \right) + \theta,$$

where θ is the entopy production. The second law of thermodynamics says that $\theta > 0$.

Those are complete system of balance equations and in fact quite general ones.

Local thermodynamics equilibrium approximation

$$\theta = \rho \frac{\mathrm{d}s}{\mathrm{d}t} + \nabla \cdot \left(\frac{\underline{q}}{T} \right),$$

$$s = s(e_0, \nu),$$

and the Gibbs relation

$$\mathrm{d}s = \frac{1}{T}\mathrm{d}e_0 + \frac{p}{T}\mathrm{d}\nu.$$

$\mathrm{d}e_0/\mathrm{d}t$ comes form the flow of a energy and the $\mathrm{d}\nu/\mathrm{d}t$ part comes from continuity equation. We can write it as

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{T} \underbrace{\rho \frac{\mathrm{d}e_0}{\mathrm{d}t}}_{\text{Eq } 3.2} + \frac{p}{T} \underbrace{\frac{1}{\nu}\frac{\mathrm{d}\nu}{\mathrm{d}t}}_{\text{Eq } 3.1},$$

$$\rho \frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{T}\left[-p(\nabla \cdot \underline{u}) + \Sigma' : \nabla \underline{u} - \nabla \cdot \underline{q} \right] + \frac{p}{T}(\nabla \cdot \underline{u}).$$

$$\nabla \cdot \left(\frac{\underline{q}}{T}\right) = \frac{1}{T}\nabla \cdot \underline{q} - \frac{1}{T^2}\underline{q} \cdot \nabla T,$$

$$\theta = -\cancel{\frac{p}{T}(\nabla \cdot \underline{u})} + \frac{1}{T}\Sigma' : \nabla \underline{u} - \cancel{\frac{1}{T}\nabla \cdot \underline{q}} + \cancel{\frac{p}{T}(\nabla \cdot \underline{u})} + \cancel{\frac{1}{T}\nabla \cdot \underline{q}} - \frac{1}{T^2}\underline{q} \cdot \nabla T.$$

Thus the entropy production is given by the formula

$$\theta = -\frac{1}{T^2}\underbrace{\underline{q} \cdot \nabla T}_{\text{heat flux}} + \frac{1}{T}\underbrace{\Sigma' : \nabla \underline{u}}_{\text{momentum flux}} \geq 0.$$

Imagine that you have a flow and a temperature gradient — then the fluid will flow from the hotter part to the colder. Note that both terms have to be positive, because when one is missing, the other one must be non-negative.

How does those equations simplify in the ideal fluid model?

3.1 Ideal fluid model

Recall that for the ideal fluid we have

$$\underline{q} = 0, \quad \Sigma' = 0 \quad \Longrightarrow \quad \theta = 0,$$

and therefore there is no entropy production.

Hydrodynamics of ideal fluids:

1. $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0,$
2. $\rho \frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = -\nabla p + \rho \underline{g},$
3. $\rho \frac{\mathrm{d}e_0}{\mathrm{d}t} = -p(\nabla \cdot \underline{u}),$
4. $\frac{\mathrm{d}s}{\mathrm{d}t} = 0.$

Since the 2^{nd} equation is a vector one we have 6 equations. Unfortunately, there are 7 unknowns, which means we are one equation short. The one that is missing is the thermodynamical equation of state (for pressure).

3.2 Thermodynamics

In the equation

$$e_0 = e_0(s, \nu)$$

the specific volume as a variable is not that interesting, since it cannot be easily controlled. It would be much better to use a pressure instead of ν. To do that we switch the thermodynamical potential, from energy to enthalpy

$$h_0 = e_0 + p\nu = e_0 + \frac{p}{\rho}, \quad h_0 = h_0(s, p),$$

$$\mathrm{d}h_0 = T\mathrm{d}s + \nu \mathrm{d}p = \underbrace{T\mathrm{d}s}_{=0} + \frac{\mathrm{d}p}{\rho},$$

if we work in a regime where s is fixed. Thus

$$\mathrm{d}h_0 = \frac{\mathrm{d}p}{\rho}, \quad h_0 = h_0(p), \quad \frac{\partial h_0}{\partial p}|_s = \frac{1}{\rho}.$$

Therefore $\rho = \rho(p)$ or $p = p(\rho)$ and that's what we've been missing. This equation is called the equation of state.

For an ideal gas

$$p(\rho) = C\rho^{\gamma}, \quad \gamma = \frac{c_p}{c_v}.$$

For an ideal fluid

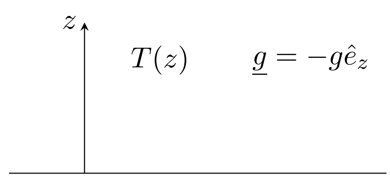
$$\rho = \rho_0 = \text{const.}$$

3.3 Hydrostatics

Now that we have a complete set of equations, we can solve physical problems. Assume for now that $\underline{u} = 0$ which leads to

$$\nabla p = \rho \underline{g}, \quad p = p(\rho).$$

Assume that we work with an ideal gas and know the temperature profile i.e. T(height).



$$\frac{\mathrm{d}p}{\mathrm{d}z} = -\rho g, \quad p(z) = ?,$$

For an ideal gas $p = \rho RT$, thus

$$\int_{p(z=)}^{p(z)} \frac{\mathrm{d}p'}{p'} = -\int_0^z \frac{g\mathrm{d}z}{RT(z)} \quad \Longrightarrow \quad p(z) = p(0)\exp\left(-\int_0^z \frac{g\mathrm{d}z}{RT(z)}\right),$$

where we've used

1. $T(z) \to p(z)$
2. $\rho(z)$ from the equation $\rho = \frac{p}{RT}$

Sound waves Fluid in equilibrium (neglecting gravity)

$$\underline{u} = 0,$$

and $p = p_{og} = \text{const.}$, $\rho = \rho_{og} = \text{const.}$, $s = s_{og} = \text{const.}$. A sound wave is just a „small" perturbation of the equilibrium state

$$\underline{u}(\underline{r}, t) = 0 + \underline{u}'(\underline{r}, t) \quad (|\underline{u}'| \ll c)$$

$$p(\underline{r}, t) = p_{eq} + p'(\underline{r}, t) \quad (p' \ll p_{eq})$$
$$\rho(\underline{r}, t) = \rho_{eq} + \rho'(\underline{r}, t) \quad (\rho' \ll \rho_{eq}),$$
$$s(\underline{r}, t) = s_{eq} = \text{const.}$$

Now we want to find primed variables. How? By using the equations of an ideal fluid. We have

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0, \tag{3.3}$$

$$\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) = -\nabla p, \tag{3.4}$$

$$s = s_{eq} = \text{const.} \tag{3.5}$$

The sound velocity squared

$$\nabla p = \underbrace{\left(\frac{\mathrm{d}p}{\mathrm{d}\rho} \right)}_{c^2} \nabla \rho = c^2 \nabla \rho.$$

Those equations are very hard to solve mathematically, since they are non-linear. However, since we just want to solve an easier case, those can be linearly approximated. Omitting all second order terms, we obtain

$$\frac{\partial (\rho_{eq} + \rho')}{\partial t} + \nabla \cdot \left[(\rho_{eq} + \rho') \underline{u} \right] = 0,$$

and the linearized continuity equation

$$\nabla \cdot \underline{u}' = -\frac{1}{\rho_{eq}} \frac{\partial \rho'}{\partial t}. \tag{3.6}$$

Now we take care of the second equation (3.4)

$$(\rho_{eq} + \rho') \left[\frac{\partial \underline{u}'}{\partial t} + \underbrace{\underline{u}' \cdot \nabla \underline{u}'}_{\approx 0} \right] = -c^2 \nabla (\rho_{eq} + \rho'),$$

and obtain the linearized equation of motion

$$\frac{\partial \underline{u}'}{\partial t} = -\frac{c^2}{\rho_{eq}} \nabla \rho'.$$

We have the following system

$$\nabla \cdot \underline{u}' = -\frac{1}{\rho_{eq}} \frac{\partial \rho'}{\partial t},$$
$$\frac{\partial \underline{u}'}{\partial t} = -\frac{c^2}{\rho_{eq}} \nabla \rho'.$$

Differentiating the first equation with respect to time and using divergence operator on the second one we get

$$\frac{\partial^2(\rho')}{\partial t^2} = c^2 \nabla^2 \rho'.$$

So the speed of sound

$$c^2 = \left(\frac{\partial p}{\partial \rho}\right)_s.$$

For $s = \text{const}$, $p(\rho) = C\rho^\gamma$, it can be approximated as

$$\left(\frac{\partial p}{\partial \rho}\right)_s = \gamma \frac{p}{\rho} \approx \gamma \frac{p_{eq}}{\rho_{eq}}.$$

$$c^2 \approx \gamma \frac{p_{eq}}{\rho_{eq}} = \gamma R T_{eq}.$$

1-D wave equation

$$\frac{\partial^2(\rho')}{\partial t^2} = c^2 \frac{\partial^2(\rho')}{\partial x^2}, \quad \rho'(x,t) = ?. \tag{3.7}$$

To solve it we write ρ' in a Fourier representation

$$\rho'(x,t) = \int \mathrm{d}k \int \mathrm{d}\omega \rho'(k,\omega) e^{ikx} e^{-i\omega t}.$$

Substituting the above into 3.7 we get

$$(-i\omega)^2 \rho'(k,\omega) = c^2 (ik)^2 \rho'(k,\omega),$$

$$\omega^2 = c^2 k^2 \quad \Longrightarrow \quad \omega = \pm ck.$$

The above equation is called the dispersion relation. Using it we get

$$\exp\left[i(kx - \omega t)\right] = \exp\left[ik(x - ct)\right].$$

Finally

$$\rho'(x,t) = \int \mathrm{d}k \rho_1'(x) \exp\left(ik(x-ct)\right) + \int \mathrm{d}k \rho_2' \exp(ik(x+ct)),$$

which are two families of perturbations travelling in the opposite directions.

3D case Using Fourier representation

$$\rho'(\underline{r},t) = \int \mathrm{d}\underline{k} \int \mathrm{d}\omega \rho'(\underline{k},\omega) \exp\left[i(\underline{k}\cdot\underline{r} - \omega t)\right],$$

$$\frac{\partial^2(\rho')}{\partial t^2} = c^2 \nabla^2 \rho',$$

$$(-i\omega)^2 \rho'(\underline{k},\omega) - c^2 (i\underline{k})^2 \rho'(\underline{k},\omega) \quad \Longrightarrow \quad \omega = \pm c\left|k\right|,$$

$$\exp\left[i(\underline{k}\cdot\underline{r} - \omega t)\right] = \exp\left[ik(\hat{k}\cdot\underline{r} - ct\right].$$

Note that c doesn't depend on ω. If it would, we will call such wave dispersive. Nondispersive waves travel without changing shape.

We know $\rho'(\underline{r}, t)$. What about $p'(\underline{r}, t)$ and $\underline{u}'(\underline{r}, t)$?

By substituting into

$$\mathrm{d}p = \left(\frac{p}{\rho}\right)_s \mathrm{d}\rho,$$

relations $p'(\underline{r}, t) = c^2\rho'(\underline{r}, t)$, we can calculate $p'(\underline{r}, t)$.

To obtain velocity $\underline{u}$ we note that

$$\frac{\partial \underline{u}'}{\partial t} = -\frac{c^2}{\rho_{eq}}\nabla\rho'$$

and, by using Fourier transform, we obtain

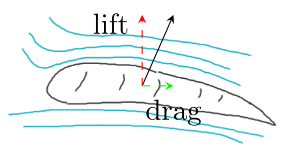
$$\underline{u}'(\underline{k}, \omega) = \frac{c^2}{\rho_{eq}\omega}\underline{k}\rho'(\underline{k}, \omega) = \pm\frac{c^2}{\rho_{eq}ck}\underline{k}\rho'(\underline{k}, \omega)$$

$$\underline{u}' = \pm\frac{c}{\rho_{eq}}\rho'\hat{k}.$$

Sound waves are longitudinal (particles move in the same way the waves propagates).

3.4 Aerodynamics

Imagine that there is a wing profile and consider a wind tunel configuration.



Question: what is the force that the flow exerts on the object? The force can be decomposed into two parts: the lift and the drag.

$$\underline{F} = -\int_S p\hat{n}\mathrm{d}s, \quad p(\underline{r}, t) = ?$$

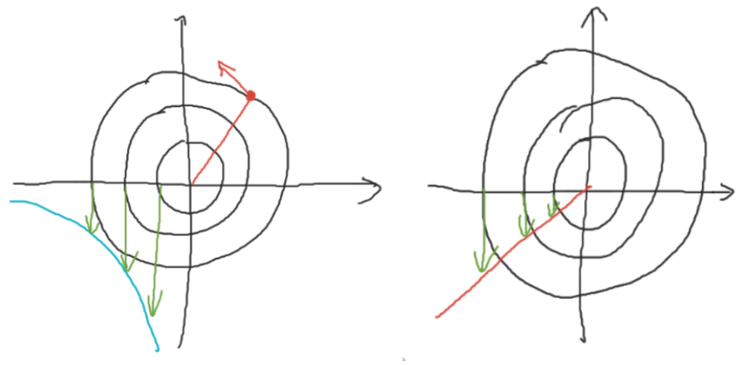
To obtain p, $\underline{u}$ and $\underline{F}$ we have to solve Euler's equations

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g}$$

$$\frac{\partial p}{\partial t} + \rho\cdot(\rho\underline{u}) = 0.$$

Those are most often almost impossible to solve. We should try to simplify the mathematical problem here.

First we can try choosing a different variable. We change $\underline{u}$ into vorticity $\underline{\xi} = \nabla\times\underline{u}$.



Euler's equations

$$\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \cdot \underline{u} = -\frac{1}{\rho}\nabla p + \underline{g}$$

1. $\underline{g} = -\nabla\chi$
2. $$\underline{u} \cdot \nabla \cdot \underline{u} = \nabla\left(\frac{\underline{u}^2}{2}\right) + (\nabla \times \underline{u}) \times \underline{u} = \nabla\left(\frac{\underline{u}^2}{2}\right) + \underline{\xi} \times \underline{u}$$
3. $\frac{1}{\rho}\nabla p = \nabla h_0$

$$\frac{\partial \underline{u}}{\partial t} + \nabla \times \underline{u} = -\nabla h_0 - \nabla\left(\frac{\underline{u}^2}{2}\right) - \nabla\chi = -\nabla\left(h_0 + \frac{1}{2}\underline{u}^2\chi\right) =: -\nabla\mathfrak{h}, \tag{3.8}$$

where

$$\mathfrak{h} = h_0 + \frac{\underline{u}^2}{2} + \chi.$$

Then Equation 3.8 reads

$$\frac{\partial \underline{u}}{\partial t} + \underline{\xi} \times \underline{u} = -\nabla\mathfrak{h}.$$

By taking the rotation of both sides

$$\frac{\partial \overbrace{\nabla \times \underline{u}}^{\xi}}{\partial t} + \nabla \times (\underline{\xi} \times \underline{u}) = 0,$$

we obtain

$$\nabla \times (\underline{\xi} \times \underline{u}) = (\underline{u} \cdot \nabla)\underline{\xi} - (\underline{\xi} \cdot \nabla)\underline{u} + \xi(\nabla \cdot \underline{u}) - \underline{u}(\nabla \cdot \underline{\xi}) =$$

if the flow is incompressible

$$\frac{\partial \underline{\xi}}{\partial t} + \underline{u} \cdot \nabla\underline{\xi} = \underline{\xi} \cdot \nabla \cdot \underline{u}$$

$$\frac{\mathrm{d}\underline{\xi}}{\mathrm{d}t} = \underline{\xi} \cdot \nabla\underline{\xi}.$$

If $\nabla\underline{u} = 0$ the ξ is an invariant of a motion.

4 | Lecture 4

4.1 Recall what we already know

For the ideal fluid the stress tensor consists only of p and an identity tensor. However, real fluids are not ideal. Newton's second law for real fluid

$$\rho\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = -\nabla p + \underline{f}, \quad \rho\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \nabla \cdot \underline{\underline{\Sigma}} + \underline{f}.$$

It would be perfect if we knew the heat equation i.e. $p = p(\rho, T)$, but for now we only have the equation of state $p(\rho)$.

Other equations that we have

1. momentum equation
2. mass conservation
3. equation of state
4. continuity equation

For an incomprehensible fluid

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \quad \Longrightarrow \quad \rho(\nabla \cdot \underline{u}) = 0.$$

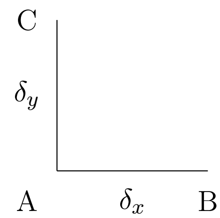
$$\frac{\partial \underline{u}}{\partial t} + (\nabla \times \underline{u}) \times \underline{u} = -\nabla(\frac{\underline{u}^2}{2} + \varphi + \psi),$$

where $\underline{f} = -\nabla\varphi$, $\psi = \frac{p}{\rho}$. By introducting vorticity $\xi = \nabla \times \underline{u}$ we can write it as

$$\frac{\partial \underline{u}}{\partial t} + \xi \times \underline{u} = -\nabla(\frac{\underline{u}^2}{2} + \varphi + \psi)$$

$\frac{1}{2}\xi$ represents the averege angular velocity of this initially $\perp$ segments in the fluid. Consider a 2D fluid $\underline{\xi} = \xi\hat{e}_z$, $\xi = \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}$, and look on two small, perpendicular segments. We will consider a difference between their components velocities in the y direction.

$$u_y(B) - u_y(A) = u_y(x+\delta, y) - u_y(x,y) = \cancel{u_y(x,y)} + \frac{\partial u_y}{\partial x}\delta x - \cancel{u_y(x,y)} = \frac{\partial u}{\partial x}\delta x.$$



This is an instantaneous angular velocity of AB around the $\perp$ axis through A. Computing the same for rotation along C axis we get

$$u_x(C) - u_x(A) = \frac{\partial u_x}{\partial y}\delta_y.$$

So the vorticity at a point informs us about how much will rotate two, initially close, points. Vorticity is a measure of rotation, but rather nonintuitive.

Example. Take rigid body motion $\underline{u} = \underline{\Omega} \times \underline{r}$.

$$\underline{\xi} = \nabla \times \underline{u}, \quad \xi_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x^j} = \epsilon_{ijk}\epsilon_{klm}\Omega_l \overbrace{\frac{\partial r_m}{\partial x^j}}^{\delta_{jm}} = \epsilon_{ijk}\epsilon_{klj}\Omega_l = -\left(\delta_{il} - 3\delta_{il}\right)\Omega_l = 2\Omega_i.$$

Thus, for this particular flow

$$\underline{\xi} = 2\underline{\Omega}.$$

Note that the left side refers to local rotation, and right refers to global rotation.

Nonintuitive case Consider bathtub vortex. It can be represented as

$$\underline{u} = \frac{k}{r}\hat{e}_\theta.$$

Trick

$$\nabla \times \underline{u} = \frac{1}{r}\begin{bmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & ru_\theta & u_z \end{bmatrix} = 0.$$

It means that the amount of global and local rotation perfectly cancel out, and the local vorticity meter shows nothing.

4.2 Irrotational flows

A flow with $\underline{\xi} = 0$ is called irrotational.

$$\left.\begin{matrix} \nabla \times \underline{u} = 0 \\ \nabla \times \nabla\chi = 0 \end{matrix}\right\} \quad \underline{u} = \nabla\chi,$$

which is called potential flow. χ can be defind as

$$\chi(\underline{r}) = \int_{r_0}^{\underline{r}} \underline{u}\mathrm{d}\underline{r}', \quad \chi(\underline{r} + \mathrm{d}\underline{r}) - \chi(\underline{r}) = \nabla\chi \cdot \mathrm{d}\underline{r} = \underline{u} \cdot \mathrm{d}\underline{r},$$

Note. χ is determined uniquely for simply connected domains (with no holes).

From the stokes theorem

$$\oint_{1-2} \underline{u}\cdot \mathrm{d}\underline{r} = \int \mathrm{d}S(\nabla\times\underline{u}) = 0,$$

since $\nabla\times\underline{u}$.

4.3 Bernoulli theorem

Euler's equation

$$\rho\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = -\nabla p + \underline{f}.$$

We make the following assumptions

1. potential form $\underline{f} = -\nabla\varphi$
2. incomprehensible flow $\rho = \mathrm{const}$

Note that

$$\frac{\mathrm{d}\underline{u}}{\mathrm{d}t} = \frac{\partial\underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = \frac{\partial\underline{u}}{\partial t} + (\nabla\times\underline{u})\times\underline{u} + \nabla\left(\frac{1}{2}\underline{u}^2\right),$$

thus we can rewrite

$$\frac{\partial\underline{u}}{\partial t} + \cancelto{0}{\underline{\xi}\times\underline{u}} = -\nabla\left(\frac{\underline{u}^2}{2} + \varphi + \psi\right), \quad \psi = \frac{p}{\rho},$$

since flow is irrotational. For a barotropic fluid

$$\frac{1}{\rho}\nabla p = \nabla\psi, \quad \psi = \int\frac{\mathrm{d}p'}{\rho(p')}.$$

Now we can write

$$\frac{\partial\underline{u}}{\partial t} = \frac{\partial\nabla\chi}{\partial t} = \nabla\frac{\partial\chi}{\partial t}.$$

Gathering everything on a one side we get

$$\nabla\left(\frac{\partial\chi}{\partial t} + \frac{\underline{u}^2}{2} + \psi + \varphi\right) = 0,$$

and also

$$\frac{\partial\chi}{\partial t} + \frac{\underline{u}^2}{2} + \psi + \varphi = C(t),$$

which is Cauchy first integral of the Euler equation for an irrotational form.

This can be further simplified by setting

$$\chi' = \chi + \int C(t)\mathrm{d}t, \quad \nabla\chi' = \nabla\chi$$

then

$$\frac{\partial\chi'}{\partial t} + \frac{\underline{u}^2}{2} + \varphi + \psi = 0.$$

This works everywhere in the fluid and is called Bernoulli's theorem. If the flow if steady then $\frac{\partial \chi'}{\partial t} = 0$ and

$$\frac{\underline{u}^2}{2} + \varphi + \psi = \text{const.}$$

This is Bernoulli's theorem for steady irrotational flow.

4.4 Bernoulli's theorem for rotational forms

Lamb's form

$$\frac{\partial \underline{u}}{\partial t} + \underline{\xi} \times \underline{u} = -\nabla \left(\frac{\underline{u}^2}{2} + \varphi + \psi \right),$$

but there is no velocity potential. Consider steady flow and perform scalar multiplication by $\underline{u}$. We get

$$0 = \underline{u} \cdot (\underline{\xi} \times \underline{u}) = -\underline{u} \cdot \nabla \left(\frac{\underline{u}^2}{2} + \varphi + \psi \right) = \underline{u} \cdot \nabla H, \quad H = \frac{1}{2}\underline{u}^2 + \varphi + \psi.$$

The quantity H is constatn along streamlines!

Evangelista Torricelli in 1664 asked the following problem. A barrel of wine has a little spout at the bottom. If we remove a plag we see a stream of fluid. How long will it take for the barrel to drain?

Assuming that we deal with an incompressible fluid, $\psi = \frac{p}{\rho}$, and $\varphi = gz$ Choose a streamline and two points on it A, B. By using second Bernoulli equation we can calculate a velocity of an outgoing fluid. It will be equal

$$H_A = gh + \frac{p_0}{\rho} + \frac{1}{2} \cdot 0^2,$$

$$H_B = 0 + \frac{p_0}{\rho} + \frac{1}{2}\underline{u}^2.$$

Comparing those two we obtain

$$\frac{p_0}{\rho} + gh = \frac{p_0}{\rho} + \frac{1}{2}\underline{u}^2 \qquad \Longleftrightarrow \qquad V = \sqrt{2gh}.$$

For a bottle with diameter 1 m and height 2 m, and $a = 5$ cm, $V = 6.3 \ \frac{\text{m}}{\text{s}}$.

4.5 Vorticity equation

$$\frac{\partial \underline{u}}{\partial t} + \underline{\xi} \times \underline{u} = -\nabla H$$

$$\frac{\partial \underline{\xi}}{\partial t} + \nabla \times (\underline{\xi} \times \underline{u}) = -\nabla \times \nabla H = 0$$

$$\nabla \times (\underline{\xi} \times \underline{u}) = (\underline{u} \cdot \nabla)\underline{\xi} + \cancel{\underline{\xi}(\nabla \cdot \underline{u})} - \cancel{(\nabla \cdot \underline{\xi})}\underline{u} - (\underline{\xi} \cdot \nabla)\underline{u}.$$

thus

$$\frac{\partial \underline{\xi}}{\partial t} + (\underline{u} \cdot \nabla)\underline{\xi} = (\underline{\xi} \cdot \nabla)\underline{u} \quad \Longrightarrow \quad \frac{\mathrm{d}\underline{\xi}}{\mathrm{d}t} = (\underline{\xi} \cdot \nabla)\underline{u}.$$

For a 2D flow

$$\underline{u} = \begin{pmatrix} u_x \\ u_y \\ 0 \end{pmatrix}, \quad \underline{\xi} = \begin{pmatrix} 0 \\ 0 \\ \xi \end{pmatrix}, \quad (\underline{\xi} \cdot \nabla)\underline{u} = 0,$$

thus

$$\frac{\mathrm{d}\underline{\xi}}{\mathrm{d}t} = 0.$$

1. For a 2D flow vorticity of a fluid element is conserved.
2. In a 3D, if at any time t_0 $\underline{\xi} = 0$ then for $t > t_0$ it will remain 0. (Persistance of irrotational flows, Cauchy-Lagrange theorem)
3. Consider a steady flow. Then
$$(\underline{u} \cdot \nabla)\underline{\xi} = 0.$$

4.6 Circulation

An ideal fluid is sometimes called inviscid. Consider force field $\underline{g} = -\nabla g$ and consider a material curve made of fluid elements.

Define a circulation along a curve $c(t)$, which is equal to

$$\Gamma(t) = \oint_{c(t)} \underline{u} \cdot \mathrm{d}\underline{r}.$$

Kelvin Circualtion theorem: $\Gamma(t) = \mathrm{const}$

Dowód. We calculate the change on the circulation while the $c(t)$ is changing. Let's look at it in an Euler picture.

$$\delta\Gamma(c(t), t) = \Gamma(c(t+\delta t), t+\delta t) - \Gamma(c(t), t) = \oint_{c(t+\delta t)} \underline{u}(r', t+\delta t)\mathrm{d}\underline{r}' - \oint_{c(t+\delta t)} \underline{u}(r')\mathrm{d}\underline{r}'$$

$$\delta\Gamma(t) = \oint_{c(t)} \underline{u}(\underline{r} + \underline{u}(\underline{r}, t)\delta t, t + \delta t) \cdot (\mathrm{d}\underline{r} + (\mathrm{d}\underline{r} \cdot \nabla)\underline{u}\delta t - \oint_{c(t)} \underline{u}(\underline{r}, t)\mathrm{d}\underline{r} =$$

expand to first order in δt

$$= \oint_{c(t)} \left\{ \frac{\partial \underline{u}}{\partial t} + (\underline{u}(\underline{r}, t) \cdot \nabla)\underline{u}\delta t) \cdot \mathrm{d}\underline{r} + \underline{u}(\underline{r}, t) \cdot (\mathrm{d}\underline{r} \cdot \nabla)\underline{u}\delta t \right\} = \int_{c(t)} (-\nabla \frac{p}{\rho} - \nabla\varphi + \frac{1}{2}\nabla\underline{u}^2\mathrm{d}\underline{r} = \int \nabla \ldots \text{TODO}.$$

□

5 Lecture 5

5.1 Magnus effect

TODOFig0 Consider a flow around a circular cylinder of a radius a with

$$u_r = 0, \quad u_\theta = -2U\sin\theta.$$

Points where velocity is equal to 0 are called stagnation points. We denote them by S_1, S_2. Recall that there is no force acting on the circle! (D'Alambert paradox).

We will do something artificial to obtain the force. Assme that the cylinder rotates. A circulation is a flow which flow lines are circles. A free vortex is

$$\underline{u}_{\text{vortex}} = \frac{\Gamma}{2\pi r}\hat{e}_\theta,$$

where Γ is just

$$\Gamma = \oint_C \underline{u} \cdot \mathrm{d}\underline{r},$$

i.e. a circulation associated with the flow. We have to satisfy the boundary conditions and they happen to be satisfied when we superpose original flow with the artificial one.

$$\nabla \times \underline{u}_{\text{vortex}} = \underline{0},$$

$$\underline{u}_{\text{vortex}} = \nabla\phi_{\text{vortex}}.$$

Thus

$$\frac{1}{r}\frac{\partial\Phi_{\text{vortex}}}{\partial\theta} = u_\theta = \frac{\Gamma}{2\pi r}, \quad \frac{\partial\Phi_{\text{vortex}}}{\partial\theta} = \frac{\Gamma}{2\pi}, \quad \Phi_{\text{vortex}} = \frac{\Gamma}{2\pi}\theta.$$

$$\Phi = U\left(r + \frac{a^2}{r}\right)\cos\theta + \frac{\Gamma}{2\pi}\theta.$$

Velocity field $\underline{u} = \nabla\Phi$

$$u_r = \frac{\partial\Phi}{\partial r} = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta,$$

$$u_\theta = \frac{1}{r}\frac{\partial\Phi}{\partial\theta} = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta + \frac{\Gamma}{2\pi r}.$$

For $r = a$ we have $u_r = 0$, $u_\theta = -2U\sin\theta + \Gamma/(2\pi a)$. The stagnation points move downwords.

TODOFig1

Stagnation poins are at points that satisfy

$$u_\theta = 0 = 2U\sin\theta - \frac{\Gamma}{2\pi a} \quad \Longrightarrow \quad \sin\theta = \frac{\Gamma}{4\pi U a} = \frac{y_s}{a}.$$

Thus $y_s = -a$ implies $\Gamma = -4\pi U a$. If we increase U even more the saturation points may not be on the surface of the circle.

TODOFig2 TODOFig3

Lets calculate the force

$$\underline{F} = -\int_{\partial V} p(r = a, \theta)\hat{n} dS,$$

and Bernoulli

$$\frac{p}{\rho} + \frac{\underline{u}^2}{2} = \text{const.} \quad \Longrightarrow \quad p = \text{const.} - \frac{\rho \underline{u}^2}{2}.$$

Therefore,

$$p(r = 0, \theta) = \text{const} - \frac{1}{2}\left[2U\sin\theta - \frac{\Gamma}{2\pi a}\right]^2,$$

$$\hat{n} = \cos\theta\hat{e}_x + \sin\theta\hat{e}_y, \quad \mathrm{d}S = aL\mathrm{d}\theta,$$

where L is the length of the cylinder. Due to the symmetry we expect that $\underline{F} \sim \hat{e}_y$.

$$F_x = \frac{1}{2}\rho\int_0^{2\pi}\left[2U\sin\theta - \frac{\Gamma}{2\pi a}\right]^2 \cos\theta aL\mathrm{d}\theta = 0,$$

$$F_y = \frac{1}{2}\rho\int_0^{2\pi}\left[2U\sin\theta - \frac{\Gamma}{2\pi a}\right]^2 \sin\theta aL\mathrm{d}\theta = -\rho U\Gamma.$$

It is called the Magnus effect.

Kutta-Joukowski law states that, for any configuration of the boundary with the rotating „artificial" flow, the force is given by

$$F_x = 0, \quad F_y = -\rho U\Gamma.$$

5.2 Flow around the plane wing

We will do so by using the conformal mapping form a cylinder to a wing. But first, we need some mathematical tools We assume that our problem is two-dimensional, $\nabla\times\underline{u} = 0$, $\nabla\cdot\underline{u} = 0$. Thus we have the potential Φ such that $\underline{u} = \nabla\Phi$, $\nabla^2\Phi = 0$. Also, we have Ψ such that $\underline{u} = \nabla\times\underline{\psi}\hat{e}_z$, $\nabla^2\Psi = 0$. Thus, for $\underline{u} = (u, v)$

$$u = \frac{\partial\Phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad v = \frac{\partial\Phi}{\partial y} = -\frac{\partial\Psi}{\partial x}.$$

Consider a complex plane with $z = x + iy$ and an analytic funciton

$$w(z) = \Phi(x, y) + i\Psi(x, y),$$

called complex velocity potential. Thus, complex velocity

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \lim_{\Delta z \to 0} \frac{w(z+\Delta z) - w(z)}{\Delta z}.$$

$$\frac{\mathrm{d}w}{\mathrm{d}z} = u - iv, \quad \left|\frac{\mathrm{d}w}{\mathrm{d}z}\right|^2 = u^2 + v^2.$$

Example. Consider a uniform flow $\underline{u} = U\hat{e}_x = (U, 0)$. Thus

$$\frac{\partial \Phi}{\partial x} = U, \quad \Phi = Ux,$$
$$\frac{\partial \Psi}{\partial y} = U, \quad \Psi = Uy,$$
$$w(z) = Ux + iUy.$$

Free vortex

$$\underline{u} = \frac{\Gamma}{2\pi r}\hat{e}_\theta, \quad w(z) = ?,$$
$$\frac{1}{r}\frac{\partial \Phi}{\partial \theta} = \frac{\Gamma}{2\pi r} \quad \Longrightarrow \quad \Phi = \frac{\Gamma}{2\pi}\theta,$$
$$-\frac{\partial \Psi}{\partial r} = \frac{\Gamma}{2\pi r} \quad \Longrightarrow \quad \Psi = -\frac{\Gamma}{2\pi}\log r,$$
$$w(z) = \Phi + i\Psi = \frac{\Gamma}{2\pi}\theta - i\frac{\Gamma}{2\pi}\log r = \frac{\Gamma}{2\pi}(\theta - i\log r).$$

Substituting $z = re^{i\theta}$ we obtain

$$w(z) = -i\frac{\Gamma}{2\pi}\log z.$$

Cylinder without circluation

$$\phi = U\left(r + \frac{a^2}{r}\right)\cos\theta,$$

$$\Psi = U\left(r - \frac{a^2}{r}\right)\sin\theta,$$

$$w(z) = U\left(z + \frac{a^2}{z}\right).$$

Cylinder with circulation We just superpose the free vortex one with the cylinder without circulation to obtain

$$w(z) = U\left(z + \frac{a^2}{z}\right) - i\frac{\Gamma}{2\pi}\log z$$

Plane wing Assume that we know the solution for the problem when the wing is just a cylinder and also that we know the mapping from cylindrical problem into the physical one. Assume that $z = z(z_1)$ and also we know the inverse $z_1 = z_1(z)$. Then, our solution is given by

$$w(z) = w_1[z_1(z)].$$

Joukowski transformation Consider

$$Z = z_1 + \frac{a^2}{z_1}, \quad a \in \mathbb{R}$$

and

$$z_1 = \rho e^{i\phi}, \quad \rho \neq a.$$

Then,

$$Z = x + iy = \rho e^{i\varphi} + \frac{a^2}{\rho} e^{-i\varphi} = \left(\rho + \frac{a^2}{\rho}\right)\cos\varphi + i\left(\rho - \frac{a^2}{\rho}\right)\sin\varphi.$$

TODOFigNext (XD)

We want to calculate a force on a plate that forms an angle α. Consider a cylinder with

$$w_1(z_1) = U\left(z + \frac{a^2}{z_1}\right).$$

To obtain a plate we first need to rotate the cylinder by α, thus multiply by $z_1 e^{i\alpha}$. We get $z_2 = z_1 e^{i\alpha}$. Next we squeeze using the Joukowski transformation

$$z = z_2 + \frac{a^2}{z2}.$$

Using the formula for the derivative of a composite function we obtain

$$\frac{\mathrm{d}w}{\mathrm{d}z} = \frac{\mathrm{d}w_1}{\mathrm{d}z_1}\frac{\mathrm{d}z_1}{\mathrm{d}z_2}\frac{\mathrm{d}z_2}{\mathrm{d}z} = \frac{\mathrm{d}w_1}{\mathrm{d}z_1} e^{-i\alpha}\left[1 - \frac{a^2}{z_2^2}\right]^2.$$

To obtain the force we use Kutta-Joukowski theorem. That requires calculating the circulation first. But we used only conformal mapping, thus the circulation is the same as in the case of the cylinder. Thus the force is 0 again XD.

Note that we have a singularity for $z_1 = a$. It is called a trailing edge. To solve this problem we say that $\mathrm{d}w_1/\mathrm{d}z_1 = 0$ and the trailing edge is a stagnation point.

We introduce an artificial circulation to make $z = a$ a stagnation point. It is called a Kutta condition. The circulation that we have to put is equal to

$$\alpha = -\theta_s, \quad \sin\theta_s = \frac{\Gamma}{4\pi U a}.$$

Now the circulation is non-zero and thus

$$\Gamma = -4\pi U a \sin\alpha, \quad F = 4\pi U a \rho \sin\theta.$$

TODOloads of figs

A stall is a phenomenon when α is big enough that the force starts to decrease. In case of real fluids we also have the turbulent wake.

6 Lecture 6

Demonstration of a difference between water and corn syroup. The latter has a high viscosity.

A viscous fluid starts flowing under shear. Shear stress means naprężenia ścinające. A Newtonian fluid resist its changing of form proportionally to the rate of deformation.

The rate of deformation We consider two points, one at r_0 and the other at $r_0 + \mathrm{d}r_0$, and consider their motion to $\underline{r}(r_0 + \mathrm{d}r_0, t)$ and $\underline{r}(r_0, t)$ respectively. We have

$$\mathrm{d}\underline{r} = \underline{r}(\underline{r}_0 + \mathrm{d}\underline{r}_0, t) - \underline{r}(\underline{r}_0, t).$$

How fast does $\mathrm{d}\underline{r}$ change while moving along the trajectory of motion? To calculate the velocity we need a material derivative of $\underline{r}$. We have (in the Lagrange picture)

$$\begin{aligned}\frac{D}{Dt}\mathrm{d}\underline{r} &= \frac{\partial}{\partial t}\underline{r}(\underline{r}_0 + \mathrm{d}\underline{r}, t) - \frac{\partial}{\partial t}\underline{r}(\underline{r}_0, t)\\ &= \underline{v}(\underline{r}_0 + \mathrm{d}\underline{r}_0, t) - \underline{v}(\underline{r}_0, t)\end{aligned}$$

switching to Euler

$$\begin{aligned}\frac{D}{Dt}\mathrm{d}\underline{r} &= \underline{v}(\underline{r} + \mathrm{d}\underline{r}, t) - \underline{v}(\underline{r}, t)\\ &= (\nabla\underline{v}) \cdot \mathrm{d}\underline{r}.\end{aligned}$$

The object $\nabla\underline{v}$ is a velocity gradient, with coordinates given by

$$\frac{D}{Dt}(\mathrm{d}r)_i = (\nabla \cdot \underline{v})_i^j \mathrm{d}r_j,$$

thus

$$\frac{D}{Dt}\mathrm{d}\underline{r} = (\nabla\underline{v}) \cdot \mathrm{d}\underline{r}.$$

We build rate of deformation tensor as

$$\hat{D} = (\nabla\underline{v})^S = \frac{1}{2}\left[(\nabla\underline{v}) + (\nabla\underline{v})^T\right].$$

We want to connect the stress in the fluid to the deformation. Recall

$$\underline{\underline{\Sigma}} = -p\underline{\underline{1}}.$$

We assume that stress is linear to deformation (in a Newtonian fluid) and the fluid is isotropic. We would like to know what is the most general, linear relationship between them. It is given by

$$\Sigma_{ij} = C_{ijkl} D_{kl}.$$

Fun fact: there is only one isotropic, symmetric tensor of order 4 in dimension 3

$$C_{ijkl} = \alpha \delta_{ij}\delta_{lk} + \beta \delta_{ik}\delta_{jl} + \gamma \delta_{il}\delta_{jk}.$$

Thus

$$\Sigma_{ij} = \alpha \delta_{ij} D_{lk} + \beta D_{ij} + \gamma D_{ji} = \alpha \delta_{ij} D_{ll} + (\beta + \gamma) D_{ij}.$$

Let us denote $\lambda = \alpha_{ij}$, $2\mu = \beta + \gamma$. Thus

$$\underline{\underline{\Sigma}}' = \lambda(\mathrm{tr}(\underline{\underline{D}}))\underline{\underline{1}} + 2\mu \underline{\underline{D}},$$

$$\mathrm{tr}\underline{\underline{\Sigma}}' = (3\lambda + 2\mu)\mathrm{tr}(D) = 3\xi \mathrm{tr}(D),$$

where $\xi = \lambda + \frac{2}{3}\mu$ and is called a bulk viscosity and μ is called a shear viscosity.

The equations of motion (momentum eq.)

$$\rho \frac{D\underline{v}}{Dt} = \nabla \cdot \underline{\underline{\Sigma}} + \underline{f}.$$

We know something about the relation between Σ and $\frac{D\underline{v}}{Dt}$ so we can calculate the closed form.

$$(\nabla \cdot (-p\underline{1}))_j = \frac{\partial}{\partial x^i}(-p\delta_{ij}) = -\frac{\partial p}{\partial x^j}.$$

Barotropic fluids We want to calculate $\nabla \cdot \underline{\underline{\Sigma}}'$.

$$(\nabla \cdot \underline{\underline{\Sigma}}')_i = \frac{\partial \Sigma'_{ij}}{\partial x^j} = 2\mu \frac{\partial D_{ij}}{\partial x^j} + \lambda \delta_{ij} \frac{\partial D_{kk}}{\partial x^j}$$

inserting

$$D_{ij} = \frac{1}{2}\left(\frac{\partial v_j}{\partial x^i} + \frac{\partial v_i}{\partial x^j}\right)$$

into the above equation we obtain

$$(\nabla \cdot \underline{\underline{\Sigma}}')_i = \mu \left(\frac{\partial^2 v_j}{\partial x^j \partial x^i} + \frac{\partial^2 v_k}{\partial x^j \partial x^k}\right) + \lambda \frac{\partial^2 v_k}{\partial x^j \partial x^k}\delta_{ij}.$$

Thus we have

$$(\nabla \cdot \underline{\underline{\Sigma}}')_i = \mu \nabla^2 v_i + (\lambda + \mu)\frac{\partial^2 v_j}{\partial x^j \partial x^i} = \mu \nabla^2 + (\lambda + \mu)[\nabla(\nabla \cdot \underline{v})]_i$$

Using the equations

$$\rho \left(\frac{\partial \underline{v}}{\partial t} + (\underline{v} \cdot \nabla)\underline{v}\right) = -\nabla p + \mu \nabla^2 \underline{v} + (\xi + \frac{1}{3}\mu)\nabla(\nabla \cdot \underline{v}) + \underline{f}$$

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0, \quad p = p(\rho).$$

Example	ρ kg/m^3	μ / Pa·s	ν m^3/s
hydrogen	0.084	$8.8 \cdot 10^{-5}$	10^{-4}
air	1.18	$1.8 \cdot 10^{-5}$	$1.5 \cdot 10^{-5}$
water	1000	10^{-3}	10^{-6}

Intuition Assume a unidirectional flow along x, varying along y with

$$\underline{v} = [v_x(y), 0, 0].$$

Lot of things simplifies, for example

$$\nabla \cdot \underline{v} = 0.$$

Let's calculate the deformation tensor

$$\underline{\underline{D}} = (\nabla \underline{v})^S = \frac{1}{2}\begin{pmatrix} 0 & \frac{\partial v_x}{\partial y} & 0 \\ \frac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \underline{\underline{\Sigma}}' = \lambda(\mathrm{tr}\underline{\underline{D}})\underline{1} + 2\mu\underline{\underline{D}} = \mu\begin{pmatrix} 0 & \frac{\partial v_x}{\partial y} & 0 \\ \frac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

What is the interpretation of such results?

$$\underline{f} = \underline{\underline{\Sigma}} \cdot \underline{n}, \quad f_i = \Sigma_{ij}\underline{n}_j,$$

with $\underline{n}$ as the normal vector, we get

$$\underline{f} = \underline{f}_n + \underline{f}_s,$$

where the first is the normal component and the second is shear component. Choose $\underline{n} = \hat{e}_y$. We have

$$\underline{f} = \underline{\underline{\Sigma}}\hat{e}_j = \mu\begin{pmatrix} 0 & \frac{\partial v_x}{\partial y} & 0 \\ \frac{\partial v_x}{\partial y} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \mu\frac{\partial v_x}{\partial y} \\ 0 \\ 0 \end{pmatrix}.$$

This means that there is purely shear stress, with the relation between rate of change of velocity between subsequent fluid layer given by

$$\Sigma_{xy} = \frac{\partial v_x}{\partial y}.$$

We point out that $[\mu] = \mathrm{Pa} \cdot S$. μ is a dynamic viscosity. We can write $N \cdot S$ as

$$\frac{D\underline{v}}{Dt} = \frac{1}{\rho}\nabla p + \nu\nabla^2\underline{v} + \frac{\underline{f}}{\rho},$$

where $\nu = \mu/\rho$ and is called a kinematic viscosity.

6.1 Boundary conditions

1) On the surface of a solid body $\underline{v} = 0$, $\underline{v}_n = 0$, $\underline{v}_t = 0$. Those are called stick (or not slim) boundary conditions. Otherwise infinite gradients.

2) On a free surface $\underline{v}_n = 0$. A tangential velocity is not necesserily zero, but is not determined a priori. The normal stresses have to be continuous, thus $\underline{\underline{\Sigma}} \cdot \underline{n}$ are continuous.

6.2 Bulk viscosity

For an ideal fluid

$$\begin{aligned}
\underline{\underline{\Sigma}} &= -p\underline{\underline{1}} + \underline{\underline{\Sigma}}^T \\
&= -p\underline{\underline{1}} + (\xi - \frac{2}{3}\mu)(\nabla \cdot \underline{v})\underline{\underline{1}} + \mu[(\nabla \underline{v}) + (\nabla \underline{v})^T] \\
&= -p\underline{\underline{1}} + \xi(\nabla \cdot \underline{v})\underline{\underline{1}} + \mu[(\nabla \underline{v}) + (\nabla \underline{v})^T - \frac{2}{3}(\nabla \cdot \underline{v})\underline{\underline{1}}] \\
&= 0,
\end{aligned}$$

thus $\underline{\underline{\Sigma}}$ is symmetric and stressless tensor.

For a viscous fluid

$$p^* = -\frac{1}{3}\mathrm{tr}\underline{\underline{\Sigma}} = p - \xi(\nabla \cdot \underline{v}).$$

The part p is a pressure and $\xi(\nabla \cdot \underline{v})$ is a dynamic pressure. The second part is important only when the fluid is compressible (for example in sound waves).

6.3 Reynolds number

Assume incompressible fluid. We introduce one number to rule them all — the Reynolds number. We performed an experiment TODOFig. Laminar flow — continuous, paralell streamlines. After increasing the flow speed he observed irregularities. At some point it starts to go crazy, for example completely mix with a fluid.

Consider a steady flow of a viscous fluid.

$$\begin{aligned}
\rho(\underline{v} \cdot \nabla)\underline{v} &= -\nabla p + \mu \nabla^2 \mu \\
\nabla \cdot \underline{v} &= 0.
\end{aligned}$$

Consider a flow with characteristic length L and velocity U. Those scale as

$$\rho(\underline{v} \cdot \nabla)\underline{v} \sim \rho U \cdot \frac{U}{L}$$

$$\mu \nabla^2 \underline{v} \sim \mu \frac{U}{L^2}.$$

Thus the Reynolds number

$$R_e = \frac{\text{inertia}}{\text{viscosity}} = \frac{\rho \frac{U^2}{L}}{\mu \frac{U}{L^2}} = \frac{\rho U L}{\mu} = \frac{UL}{\nu}.$$

We have the following possibilities

- $R_e \ll 1$ inertia $\ll$ viscosity, laminar flow (Stokes flow),
- $R_e \approx 1$ can still be laminar,
- $R_e \gg 1$ inertia dominated flow — turbulent flow.

The exact number for which there occurs transition from laminar to turbulent depends on the problem. Typically it is around 1000. For water 10^{-6}, for a bacteria $R_e = 10^{-6}$, for honey on toast $R_e = 10^{-3}$, for a swimmer in a pool

7 Lecture 7

Test 8.12, 8:15 (3h).

7.1 Pipe flow

We assume that the flow is stationary, incompressible and viscous, through a circular pipe of radius R, length L. The flow is driven by a pressure difference Δp.

TODOpipe fig

We introduce a system of coordinates when the z axis coincides with the axis of symmetry of cylinder (steady = stationay). For now we use cylindrical coordinates. Due to symmetry of a problem we have

$$\underline{u} = [0, 0, u(\underline{r})], \quad r^2 = x^2 + y^2.$$

The flow is incompressible, thus

$$\nabla \cdot \underline{u} = 0 = \frac{\partial u(r)}{\partial z}.$$

Navier-Stokes equations reads

$$(\underline{u} \cdot \nabla)\underline{u} = u \frac{\partial}{\partial z} u = 0$$

and the laplacian

$$\nabla^2 \underline{u} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u(\underline{r}) \hat{e}_z = \hat{e}_z \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right).$$

Thus Navier-Stokes equations

$$\begin{array}{ccc} x & 0 & = & -\frac{1}{\rho}\frac{\partial p}{\partial x} \\ y & 0 & = & -\frac{1}{\rho}\frac{\partial p}{\partial y} \\ z & 0 & = & -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu \nabla^2 u \end{array}$$

Since

$$\underbrace{\frac{1}{\mu}\frac{\mathrm{d}p}{\mathrm{d}z}}_{\text{only on } z} = \underbrace{\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u}_{\text{only on } x,y},$$

both sides must be equal to a constant. Therefore $p = az + b$ thus

$$p(z) = -\frac{\Delta p}{L} z + p_0 + \Delta p$$

Now we need to solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -\frac{\Delta p}{\mu L}.$$

Plugging $u = u(r)$ we obtain

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r}.$$

The second derivative

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{x^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right).$$

Analogously for y. The sum

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u = \frac{2}{r}\frac{\partial u}{\partial r} + \frac{x^2+y^2}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial u}{\partial r}\right) = \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial r^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right).$$

Summing up we get

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) = -\frac{\Delta p}{\mu L} \quad \Longleftarrow \quad u(r) = -\frac{\Delta p}{4\mu L} r^2 + c_1 \log(r) + c_2.$$

We know that $c_1 = 0$ since inside the pipe the velocity must be finite. Stick boundary condition implies $u(R) = 0$ thus $c_2 = \frac{\Delta p}{4\mu L} R^2$. Thus the flow in a pipe is given by

$$\underline{u}(\underline{r}) = \frac{\Delta p}{4\mu L}(R^2 - r^2).$$

A volumetric flux (discharge/wydatek), flow rate

$$Q = \int_0^R u(r) \cdot 2\pi r \mathrm{d}r = \frac{\Delta p \pi R^4}{8\mu L}.$$

This is called Hagen-Poiseuille formula and the flow Hagen-Poiseuille flow. We can also use $|\nabla p| = \Delta p / L$, and then

$$\langle u \rangle = \frac{Q}{A} = \frac{R^2}{8\mu}\frac{\Delta p}{L} = \frac{R^2}{8\mu}\nabla p$$

This is failry simmilar to electric resistance.

Reynolds number Reminder (TODOwhat is U)

$$R_e = \frac{UL\mu}{\rho} = \frac{UL}{\nu},$$

pipe flow

$$R_e = \frac{Ud}{\nu}.$$

Typical critical $R_e \approx 2300$.

Ostwald viscometer It is a device that determines the viscosity without any moving parts. TODOfig We measure the time needed for emptying the higher bowl. Estimate for ν, $G = \rho_0 g$

$$\nu = \frac{G\pi R^4}{8\rho_0 Q} = \frac{\pi R^4 g}{8V} T.$$

7.2 Dissipation of energy

We assume no external forces, just viscosity. Also, our fluids are incompressible, $\rho = \text{const}$. Total kinetic energy of the fluid

$$\varepsilon_{kin} = \int_\Omega \mathrm{d}V \left(\frac{\rho \underline{u}^2}{2}\right).$$

The question is $\frac{\mathrm{d}\varepsilon_{kin}}{\mathrm{d}t} =?$. Using Navier-Stokes equations we can calculate

$$\frac{\partial}{\partial t}\left(\frac{\rho \underline{u}^2}{2}\right) = \rho \underline{u}\left[-(\underline{u}\cdot\nabla)\underline{u} - \frac{\nabla p}{\rho} + \frac{1}{\rho}\nabla\cdot \underline{\underline{\Sigma}}''\right].$$

Note that $\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{u} = (\underline{u}\cdot\nabla)(\underline{u}^2/2)$ is a scalar. Thus

$$\frac{\partial}{\partial t}\left(\frac{\rho \underline{u}^2}{2}\right) = -\rho(\underline{u}\cdot\nabla)\left[\frac{\underline{u}^2}{2} + \frac{p}{\rho}\right] + \underline{u}\nabla\cdot\underline{\underline{\Sigma}}'.$$

We can simplify the last summand by

$$\underline{u}\cdot\nabla\cdot\underline{\underline{\Sigma}}' = \frac{\partial}{\partial x^j}\left(u_j \Sigma'_{ji}\right) - \Sigma'_{ji}\frac{\partial u_i}{\partial x^j} = \nabla\cdot(\underline{u}\cdot\underline{\underline{\Sigma}}') - \underline{\underline{\Sigma}}' : (\nabla\underline{u}).$$

Reminder

$$\underline{\underline{A}} : \underline{\underline{B}} = A_{ij}B_{ji}.$$

Since $\underline{u}$ is divergence-free we have $\underline{u}\cdot\nabla\varphi = \nabla\cdot(\varepsilon\underline{u})$, where φ is any scalar function.

Thus the final form of energy dissipation (we use the assumption that the fluid is Newtonian)

$$\rho\frac{\partial}{\partial t}\left(\frac{\underline{u}^2}{2}\right) = -\nabla\cdot\left[\rho\underline{u}\left(\frac{\underline{u}^2}{2} + \frac{p}{\rho}\right) - \underline{u}\cdot\underline{\underline{\Sigma}}'\right] - \mu\left[(\nabla\underline{u}) + (\nabla\underline{u})^T\right] : (\nabla\underline{u}).$$

Since we integrate a divergence over a surface integral on which velocity vanishes, the square bracket vanishes. Thus

$$\frac{\partial\varepsilon_{kin}}{\partial t} = -\mu\int_\Omega \mathrm{d}V\left[(\nabla\underline{u}) + (\nabla\underline{u})^T\right] : (\nabla\underline{u})$$

$$= -\mu\int_\Omega \mathrm{d}V\left[(\nabla\underline{u}) + (\nabla\underline{u})^T\right] : \frac{1}{2}[(\nabla\underline{u}) + (\nabla\underline{u})^T] = -\frac{\mu}{2}\int_\Omega \mathrm{d}V\left(\frac{\partial u_i}{\partial x^j} + \frac{\partial u_j}{\partial x^i}\right)^2.$$

Since $\dot{\varepsilon}_{kin} < 0 \implies \mu > 0$. This is the energy loss due to the internal friction in a fluid (which is 0 in eulerian fluid).

7.3 Time dependant flow

The ideal fluid model is a good approximation when we are far away from boundaries.

We want to solve a problem of oscillatory motions. Imagine that you have a half-space filled with viscous fluid (μ) and the surface oscillates with

$$\underline{u} = u\hat{e}_y, \quad u = u_0 \cos(\omega t).$$

What is the resulting flow? By symmetry the flow should not depend on z. We propose

$$\underline{u}(x,t) = [0, u(x,t), 0], \quad \nabla \cdot \underline{u} = 0, \quad (\underline{u} \cdot \nabla)\underline{u} = 0.$$

Now we write the Navier-Stokes equations

$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \underline{u}.$$

The x component (vertical axis) is constant, the y component

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial x^2}.$$

This is the diffusion equation. Let's try a solution of the form

$$u(x,t) = u_0 \exp(i(kx - \omega t)).$$

After substituting it into the original equation we obtain

$$k^2 = \frac{i\omega}{\nu}.$$