Hydrodynamics and Elasticity, Homework Sheet 1

Mikołaj Duch

12 października 2022

Problem 1

Let $a, b, c, x, y, z \in \mathbb{R}^3$ and denote

$$M := \begin{bmatrix} - & a & - \\ - & b & - \\ - & c & - \end{bmatrix}, \quad N := \begin{bmatrix} - & x & - \\ - & y & - \\ - & z & - \end{bmatrix}.$$

Recall that $\det(AB) = \det(A)\det(B)$ and $\det(A) = \det(A^T)$. Thus

$$\det(MN^T) = \det(M)\det(N) = [\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})] \cdot [\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})]$$

and also

$$MN^T = \begin{bmatrix} (a|x) & (a|y) & (a|z) \\ (b|x) & (b|y) & (b|z) \\ (c|x) & (c|y) & (c|z) \end{bmatrix}, \tag{1}$$

where $(\cdot|\cdot)$ denotes standard inner product in $\mathbb{R}^3$. Using tensor notation for $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ we can write

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i \epsilon_{ijk} b_j c_k$$

and analogously for $\mathbf{x}, \mathbf{y}, \mathbf{z}$. Thus

$$\det(MN^T) = (a_i \epsilon_{ijk} b_j c_k)(x_l \epsilon_{lmn} b_m c_n) = \epsilon_{ijk}\epsilon_{lmn} a_i b_j c_k x_l y_m z_n.$$

If we choose $\mathbf{a}, \mathbf{b}, \ldots$ to be $\mathbf{e}_i, \mathbf{e}_j, \ldots$ we obtain that

$$\det(MN^T) = \epsilon_{ijk}\epsilon_{lmn}.$$

On the other hand, using Equation 1 we get

$$\det(MN^T) = (a|x)\,(b|y)\,(c|z) + (a|y)\,(b|z)\,(c|x) + (a|z)\,(b|x)\,(c|y) - (a|z)\,(b|y)\,(c|x) - (a|y)\,(b|x)\,(c|z) - (a|x)\,(b|z)\,(c|y)\,.$$

Using the fact that $(e_i|e_j) = \delta_{ij}$ we get that

$$\det(MN^T) = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}$$

And therefore

$$\epsilon_{ijk}\epsilon_{lmn} = \delta_{il}\delta_{jm}\delta_{kn} + \delta_{im}\delta_{jn}\delta_{kl} + \delta_{in}\delta_{jl}\delta_{km} - \delta_{il}\delta_{jn}\delta_{km} - \delta_{in}\delta_{jm}\delta_{kl} - \delta_{im}\delta_{jl}\delta_{kn}.$$

Substituting $k = n$ in the above identity and using the fact that $\delta_{ik}\delta_{kj} = \delta_{ij}$ we obtain

$$\begin{aligned}\epsilon_{ijk}\epsilon_{lmk} &= 3\delta_{il}\delta_{jm} + \delta_{im}\delta_{jl} + \delta_{jl}\delta_{im} - \delta_{il}\delta_{jm} - \delta_{jm}\delta_{il} - 3\delta_{im}\delta_{jl} \\ &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.\end{aligned}$$

This implies

$$\epsilon_{ijk}\epsilon_{ijk} = \delta_{ii}\delta_{jj} - \delta_{ij}\delta_{ji} = \delta_{ii}\delta_{jj} - \delta_{ii} = 6.$$

1 Problem 2

a)

$$\nabla \cdot (\phi \mathbf{a}) = \frac{\partial}{\partial x^i}(\phi a^i(x)) = \left(\frac{\partial \phi}{\partial x^i}\right) a^i(x) + \phi \frac{\partial}{\partial x^i} a^i = (\nabla \phi) \cdot \mathbf{a}(x) + \phi \nabla \cdot \mathbf{a}.$$

b)

The i-th component

$$\begin{aligned}[\nabla \times (\nabla \times \mathbf{a})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x^j}(\epsilon_{klm}\frac{\partial}{\partial x^l} a^m) = \epsilon_{ijk}\epsilon_{lmk}\frac{\partial^2}{\partial x^j \partial x^l} a^m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial^2}{\partial x^j \partial x^l} a^m = \frac{\partial}{\partial x^i}\frac{\partial a^j}{\partial x^j} - \frac{\partial^2 a^i}{\partial (x^j)^2} \\ &= \frac{\partial}{\partial x^i}(\nabla \cdot \mathbf{a}) - [\nabla^2 \mathbf{a}]_i.\end{aligned}$$

Thus

$$[\nabla \times (\nabla \times \mathbf{a})] = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}.$$

c)

$$\begin{aligned}\nabla \cdot (\mathbf{v} \times \mathbf{u}) &= \frac{\partial}{\partial x^i}(\epsilon_{ijk} v^j u^k) = \epsilon_{ijk}\frac{\partial v^j}{\partial x_i} u^k + \epsilon_{ijk}\frac{\partial u^k}{\partial x_i} v^j = \left[\epsilon_{kij}\frac{\partial v^j}{\partial x_i}\right] u^k + \left[\epsilon_{kij}\frac{\partial u^k}{\partial x_i}\right] v^j \\ &= [\nabla \times \mathbf{v}]_k\, u^k + [\nabla \times \mathbf{u}]_k\, v^k = \mathbf{u} \cdot [\nabla \times \mathbf{v}] + \mathbf{v} \cdot [\nabla \times \mathbf{u}].\end{aligned}$$

d)

The i-th component

$$[\mathrm{Div}(\phi \mathbf{T})]_i = \frac{\partial \phi T^{ij}}{\partial x^j} = \underbrace{\frac{\partial \phi}{\partial x^j}}_{\nabla \phi} T^{ij} + \phi \frac{\partial T^{ij}}{\partial x^j} = [\mathbf{T} \cdot (\nabla \phi)]_i + [\phi \mathrm{Div} \mathbf{T}]_i.$$

2 Problem 3

a)

$$\mathbf{T} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad \mathbf{T}^S = \frac{1}{2}(\mathbf{T} + \mathbf{T}^T) = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix}, \quad \mathbf{T}^A = \frac{1}{2}(\mathbf{T} - \mathbf{T}^T) = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}.$$

b)

Axial vector of $\mathbf{T}^A$ is a vector $\omega \in \mathbb{R}^3$ which satisfies

$$\forall \mathbf{a} \in \mathbb{R}^3 \quad T^A(\mathbf{a}) = \omega \times \mathbf{a}.$$

Using index notation for fixed index i we obtain

$$\epsilon_{ijk}\omega_j a_k = T_{ik}a_k \quad \implies \quad \epsilon_{ijk}\omega_j = T_{ik} \iff \epsilon_{kij}\omega_j = -T_{ki}.$$

Substituting for example $i = 1, k = 2$ we obtain

$$-T_{12} = \epsilon_{12j}\omega_j = \omega_3.$$

Analogously we obtain that $\omega_1 = -T_{23}, \omega_2 = T_{13}$. Thus the axial vector of $\mathbf{T}^A$ is

$$\omega = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

c)

Let us show first that $\mathbf{v} \cdot \mathbf{R}^A \cdot \mathbf{v} = 0$.

$$\mathbf{v} \cdot \mathbf{R}^A \cdot \mathbf{v} = \frac{1}{2}v^i(R_{ij} - R_{ji})v^j = \frac{1}{2}v^i R_{ij} v^j - \frac{1}{2}v^j R_{ij} v^i = \frac{1}{2}(\mathbf{v} \cdot \mathbf{R} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{R} \cdot \mathbf{v}) = 0$$

Thus

$$\mathbf{v} \cdot \mathbf{R}^S \cdot \mathbf{v} = \mathbf{v}(\mathbf{R} - \mathbf{R}^A) \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{R} \cdot \mathbf{v} - 0 = \mathbf{v} \cdot \mathbf{R} \cdot \mathbf{v}.$$