Homework 1
AAE 512, Computational Aerodynamics
Purdue University
Due: 2022-01-24

1. Problem 2.12 in the textbook. Classify the following system of equations as elliptic, hyperbolic, or parabolic:
$$\beta^2 \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$
2. Problem 2.13. Classify the following equation as elliptic, hyperbolic, or parabolic:
$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = -e^{-kt}$$
Classify this, too:
$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 4$$
3. Problem 2.27. Solve the following partial differential equation:
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, 0 \le x \le 1$$
$$u(0,t) = 0, u(1,t) = 0, u(x,0) = \sin(2\pi x)$$
4. Nonlinear pendulum problem. The equation of motion for a simple, nonlinear pendulum is given by:
$$\frac{d^2\phi}{dt^2} + \frac{g}{R}\sin\phi = 0$$
$$\phi = \phi_0, \frac{d\phi}{dt} = 0, \text{for } t = 0$$
where R is the length of the pendulum, g is the acceleration of gravity, and ϕ is the angle from the vertical. To solve this equation numerically, recast it as two first-order equations:
$$\frac{du_1}{dt} = -\frac{g}{R}\sin u_2$$
$$\frac{du_2}{dt} = u_1$$
$$u_1(0) = 0, u_2(0) = \phi_0$$
For this problem, take $\phi_0 = 1$, $\frac{g}{R} = 1$, and integrate numerically out to $t = 10$. Use the following first order integration scheme: $u(t + \Delta t) \approx u(t) + \frac{du}{dt}\Delta t$. Do the integration out with $\Delta t = 10^0, 10^{-1}, 10^{-2}, \ldots, 10^{-8}$; that is, with ten steps up to one billion steps. Using your best solution as a reference, plot the error $|u_2(10) - u_{2,\text{ref}}(10)|$ versus Δt on a log-log plot. What is the slope of the error curve? **Write your program in C, C++, or Fortran!**

Problem 1

PTA Problem 2.12

$$\beta^2 \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

$$\begin{bmatrix} \beta^2 & 0 \\ 0 & 1 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \frac{\partial}{\partial y} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$A \frac{\partial w}{\partial x} + C \frac{\partial w}{\partial y} = F$$

$$A = \begin{bmatrix} \beta^2 & 0 \\ 0 & 1 \end{bmatrix}, \quad |A| = \beta^2$$

$$C = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad |C| = -1$$

$$|B| = \begin{vmatrix} \beta^2 & -1 \\ 0 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 0 \\ -1 & 1 \end{vmatrix} = 0$$

$$D = |B|^2 - 4|A||C| = 0^2 - 4(\beta^2)(-1) = 4\beta^2 > 0 \Rightarrow \text{hyperbolic}$$

Problem 2

PTA Problem 2.13

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = -e^{-kt}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = -\frac{\partial u}{\partial x} - e^{-kt}$$

$a=1$ $c=1$

$b=0$

$b^2 - 4ac = 0 - 4(1)(1) = -4 < 0 \Rightarrow$ elliptic

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = 4$$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} = -\frac{\partial u}{\partial y} + 4$$

$a=1$ $b=-1$

$c=0$

$b^2 - 4ac = (-1)^2 - 4(1)(0) = 1 > 0 \Rightarrow$ hyperbolic

Problem 3

PTA Problem 2.27

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

$u(0,t)=0, \quad u(1,t)=0$

$u(x,0) = \sin(2\pi x)$

try $u(x,t) = f(t)\,g(x)$

$\frac{\partial u}{\partial t} = \dot{f} g \quad , \quad \frac{\partial^2 u}{\partial x^2} = f g''$

$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$

$\dot{f} g - f g'' = 0 \;\Rightarrow\; \dot{f} g = f g'' \;\Rightarrow\; \frac{\dot{f}}{f} = \frac{g''}{g}$ [If we were being very careful, we would consider the case $f=0$ or $g=0$.]

The only way the two functions can be equal for all x and t is to be equal to a constant.

$\frac{\dot{f}}{f} = \frac{g''}{g} = -k^2$

$\dot{f} + k^2 f = 0$

try $f = Ae^{\lambda t}$, $\dot{f} = \lambda f \;\Rightarrow\; \lambda f + k^2 f = 0 \Rightarrow \lambda = -k^2 \Rightarrow f = Ae^{-k^2 t}$

$g'' + k^2 g = 0$

try $g = Ae^{i\lambda x}$, $g' = i\lambda g$, $g'' = -\lambda^2 g \;\Rightarrow\; -\lambda^2 g + k^2 g = 0 \Rightarrow \lambda^2 = k^2 \Rightarrow \lambda = \pm k$

$g = Ae^{\pm ikx}$

general solution: $u(x,t) = e^{-k^2 t}\left[Ae^{ikx} + Be^{-ikx}\right]$

or $u(x,t) = e^{-k^2 t}\left[A\cos(kx) + B\sin(kx)\right]$

$u(0,t) = e^{-k^2 t}\left[A\overset{1}{\cancel{\cos(0)}} + B\overset{0}{\cancel{\sin(0)}}\right] = 0 \Rightarrow A = 0$

$u(1,t) = e^{-k^2 t}\left[\overset{0}{\cancel{A}}\cos(k) + B\sin(k)\right] = 0 \Rightarrow \sin k = 0 \Rightarrow k = \pm n\pi,\ n = 0,1,2,\ldots$

$u(x,t) = Be^{-n^2\pi^2 t}\sin(n\pi x)$

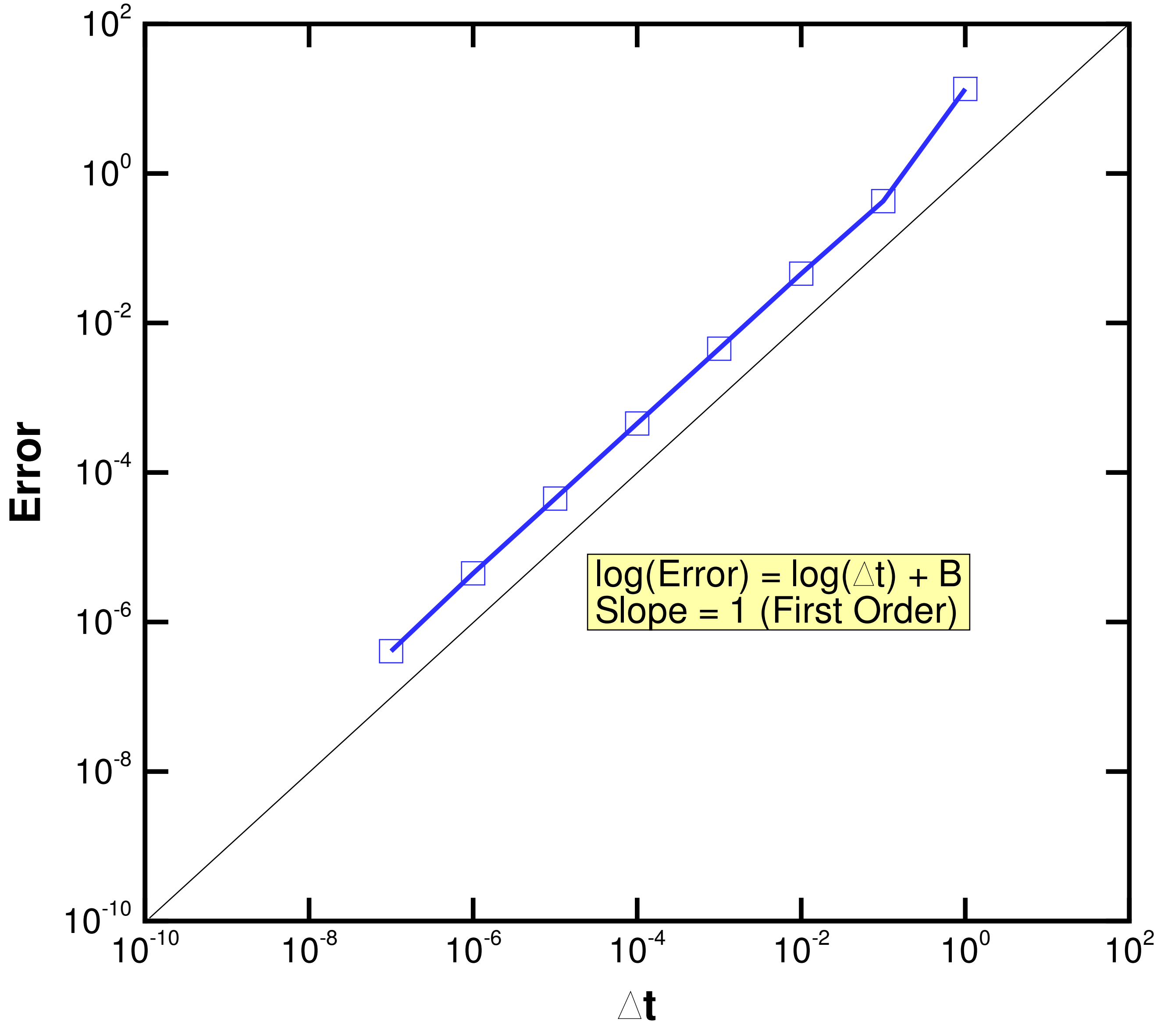
$u(x,0) = \sin 2\pi x \Rightarrow Be^{0}\sin(n\pi x) = \sin(2\pi x) \Rightarrow B = 1,\ n = 2$

$$\boxed{u(x,t) = e^{-4\pi^2 t}\sin(2\pi x)}$$

checking: $u_t = -4\pi^2 u$, $u_x = 2\pi e^{-4\pi^2 t}\cos(2\pi x)$, $u_{xx} = -4\pi^2 e^{-4\pi^2 t}\sin(2\pi x) = -4\pi^2 u$

$u_t - u_{xx} = (-4\pi^2 u) - (-4\pi^2 u) = 0$

Problem 4



Error
Δt
10^{2}
10^{0}
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-10}
10^{-8}
10^{-6}
10^{-4}
10^{-2}
10^{0}
10^{2}
log(Error) = log(Δt) + B
Slope = 1 (First Order)

```c
//
// AAE 512, Computational Aerodynamics
// Purdue University, Spring 2016
// Problem 4, Homework 1
//
// to compile: gcc -Wall -O -o chw1p4 chw1p4.c -lm
//

#include<stdlib.h>
#include<stdio.h>
#include<math.h>

const int nequations = 3;

void uinit(double *uu) {
  // initial conditions
  uu[0] = 0.; // time
  uu[1] = 0.; // u1
  uu[2] = 1.; // u2
}

void uprime(double *dudt, double *uu) {
  // derivatives
  dudt[0] = 1.;           // dt/dt = 1
  dudt[1] = -sin(uu[2]); // du1/dt
  dudt[2] = uu[1];       // du2/dt
}

int main(void) {
  double u[nequations], up[nequations];
  const double tmax = 10.;
  const int logn_max = 8;

  printf("Variables=\"<greek>D</greek>t\",\"Error\"\n");

  for (int logn = 0; logn <= logn_max; logn++) {
    double dt = pow(10.,-(double)logn);
    int nsteps = (int)(tmax/dt);

    uinit(u);

    for (int istep = 1; istep <= nsteps; istep++) {
      uprime(up, u);
```

```
      for (int i = 0; i < nequations; i++) {
        u[i] += up[i]*dt;
      }
    }

    const double u_reference = -9.98949860e-01;
    printf("%.8E %.8E\n", dt, fabs(u[2]-u_reference));
  }

  return EXIT_SUCCESS;
}
```

```
!
! AAE 512, Computational Aerodynamics
! Purdue University, Spring 2016
! Problem 4, Homework 1
!
! to compile: gfortran -Wall -fdefault-real-8 -O -o fhw1p4 fhw1p4.f90
!

module problem4
  implicit none
  integer, parameter :: nequations = 3
contains
  subroutine uinit(uu)
    ! initial conditions
    real, intent(out) :: uu(0:nequations)
    uu(0) = 0. ! time
    uu(1) = 0. ! u1
    uu(2) = 1. ! u2
  end subroutine uinit

  subroutine uprime(dudt, uu)
    ! derivatives
    real, intent(in) :: uu(0:nequations)
    real, intent(out) :: dudt(0:nequations)
    dudt(0) = 1.           ! dt/dt = 1
    dudt(1) = -sin(uu(2)) ! du1/dt
    dudt(2) = uu(1)       ! du2/dt
  end subroutine uprime
end module problem4



program mainprog
  use iso_fortran_env
  use problem4
  implicit none
  real :: u(0:nequations), up(0:nequations)
  real, parameter :: tmax = 10., u_reference = -9.98949860e-01
  integer, parameter :: logn_max = 8
  integer :: i, istep, logn, nsteps
  real :: dt

  write(output_unit,'(a)') 'Variables="<greek>D</greek>t","Error"'
```

```fortran
  do logn = 0, logn_max
     dt = 10.**(-logn)
     nsteps = int(tmax/dt)

     call uinit(u)

     do istep = 1, nsteps
        call uprime(up, u)

        do i = 0, nequations-1
           u(i) = u(i) + up(i)*dt
        end do
     end do

     write(output_unit,'(es14.8,1x,es14.8)') dt, abs(u(2)-u_reference)
  end do
end program mainprog
```