Shajed Ahmed | Problem 1 | 01/28/2022

$$\beta^2 \frac{du}{dx} - \frac{dv}{dy} = 0$$

$$\frac{dv}{dx} - \frac{du}{dy} = 0$$

let $A = \begin{bmatrix} \beta^2 & -1 \\ 0 & -1 \end{bmatrix}$

$w = \begin{bmatrix} u \\ v \end{bmatrix}$

$$\frac{\partial w}{\partial x} + A \frac{\partial w}{\partial y} = 0$$

taking determinant of A

$$\det[A] = \beta^2 - 0$$

$$\det[A] = \beta^2$$

since $\det[A] > 0$

$\boxed{\text{hyperbolic}}$

Problem 2

general form

$$a\frac{\partial^2\phi}{\partial x^2} + b\frac{\partial^2\phi}{\partial x\partial y} + c\frac{\partial^2\phi}{\partial y^2} = H\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \phi, x, y\right)$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} = -e^{kt}$$

$$u_{tt} + u_{xx} + u_x = -e^{kt}$$

let $t = y$

$$u_{yy} + u_{xx} + u_x = -e^{kt}$$

or $u_{xx} + u_{yy} + u_x = -e^{kt}$

$a = 1 \quad b = 0 \quad c = 1$

$B^2 - 4AC$

$0 - 4(1)(1) = -4$

so $b^2 - 4ac < 0$ $\boxed{\text{elliptic}}$

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} = 4$$

$$u_{xx} - u_{xy} + u_y = 4$$

so $a = 1 \quad b = -1 \quad c = 0$

$B^2 - 4AC$

$(-1)^2 - 4(1)(0)$

$= 1$

so $B^2 - 4AC > 0$ $\boxed{\text{hyperbolic}}$

Problem 3

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \qquad 0 \le x \le 1$$

<u>Given:</u> $u(0,t) = 0$
$u(1,t) = 0$

$u(x,0) = \sin(2\pi x)$

let $u(x,0) = \sin(2\pi x) = \phi(x)$

assume $u(x,t) = X(x)\,T(t)$

find $\frac{\partial u}{\partial t}$ & $\frac{\partial^2 u}{\partial x^2}$

$$\frac{\partial u}{\partial t} = X\frac{dT}{dt} \qquad \frac{\partial^2 u}{\partial x^2} = T\frac{\partial^2 X}{dx^2}$$

Sub into OG PDE

$$X\frac{\partial T}{dt} = T\frac{\partial^2 X}{dx^2} \xRightarrow{\text{separate variables}} \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2}$$

let both halves equal a <u>constant</u> $(\lambda^2, 0, -\lambda^2)$

Case ① Positive constant $= \lambda^2$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2$$

$$\frac{1}{T}\frac{dT}{dt} = \lambda^2 \Rightarrow \frac{dT}{dt} = \lambda^2 T \Rightarrow T = Ae^{\lambda^2 t}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = \lambda^2 \Rightarrow \frac{d^2X}{dx^2} - \lambda^2 X = 0$$

$$X = B\cosh(\lambda x) + C\sinh(\lambda x)$$

$$u = XT = Ae^{\lambda^2 t}\left[B\cosh(\lambda x) + C\sinh(\lambda x)\right]$$

but $t \to \infty$, $u \to \infty$ unless $A = 0$ or $B = C = 0$

<u>Trivial Soln</u>

Case ② Const = 0

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx} = 0$$

$$\frac{1}{T}\frac{dT}{dt} = 0 \Rightarrow \frac{dT}{dt} = 0 \rightarrow T = A$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = 0 \Rightarrow \frac{d^2X}{dx^2} = 0 \rightarrow X = Bx + C$$

$$u = XT = A(Bx + C)$$

@ $x = 0 \quad u = 0 = AC$

$x = L \quad u = 0 = ABL \quad$ but $L \neq 0$ so $AB = 0$

Trivial Solution

Case ③ Constant negative $= -\lambda^2$

$$\frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2$$

$$\frac{1}{T}\frac{dT}{dt} = -\lambda^2 \Rightarrow \frac{dT}{dt} = -\lambda^2 T \rightarrow T = Ae^{-\lambda^2 t}$$

$$\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda^2 \Rightarrow \frac{d^2X}{dx^2} + \lambda^2 X = 0$$

$$\Rightarrow X = B\cos(\lambda x) + C\sin(\lambda x)$$

$$u = XT = Ae^{-\lambda^2 t}\left[B\cos(\lambda x) + C\sin(\lambda x)\right]$$

let $AB = c_1 \quad AC = c_2$

so, $u = e^{-\lambda^2 t}\left[c_1\cos(\lambda x) + c_2(\lambda x)\right]$

apply BCs

$x = 0, u = 0$

$0 = e^{-\lambda^2 t}\left[c_1\cos(0) + c_2\sin(0)\right]$

$c_1 = 0$

so, $u = c_2 e^{-\lambda^2 t}\sin(\lambda x)$

$x = L, u = 0$

$$0 = c_2 e^{-\lambda^2 t}\underbrace{\sin(\lambda L)}$$

must $\sin(\lambda L) = 0$

$\lambda L = n\pi$

$\lambda_n = \frac{n\pi}{L}$

where $n =$ positive integer

$n = 1, 2, 3, \ldots$

$$u_n = c_2 e^{-\lambda_n^2 t} \sin(\lambda_n, x)$$

linear combination $\hookrightarrow$ $$u = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \sin(\lambda_n, x)$$

apply IC

$$t = 0, \quad u = \phi(x)$$

$$\phi(x) = \sum_{n=1}^{\infty} a_n e^0 \sin(\lambda_n, x) = \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)$$

apply sturm-liouville theorem (multiply by $\sin(\lambda_m x)$)

$$\int_0^L \phi(x) \sin(\lambda_m x)\,dx = \int_0^L \sin(\lambda_m x) \sum_{n=1}^{\infty} a_n \sin(\lambda_n x)\,dx$$

$$\int_0^L \phi(x)\sin(\lambda_m x)\,dx = \sum_{n=1}^{\infty} \int_0^L a_n \sin(\lambda_m x)\sin(\lambda_n x)\,dx$$

so if $\int_0^L \sin(\lambda_m x)\sin(\lambda_n x)\,dx = 0$

therefore $m = n$, $\lambda_m = \lambda_n$

means all terms in summation cancel except when $m=n$

$$\int_0^L \phi(x)\sin(\lambda_m x)\,dx = \underbrace{\int_0^L a_m \sin^2(\lambda_m x)\,dx}_{a_m \cdot \frac{L}{2}}$$

$$\text{So,} \quad a_m = a_n = \frac{2}{L}\int_0^L \phi(x)\sin(\lambda_n x)\,dx$$

recall $\phi(x) = \sin(2\pi x)$

$$a_n = \frac{2}{L}\int_0^L \sin(2\pi x)\sin(\lambda_n x)\,dx$$

<u>Soln</u>:

$$u = \sum_{n=1}^{\infty} a_n e^{-\lambda_n^2 t} \sin(\lambda_n x)$$

where, $\lambda_n = \frac{n\pi}{L}$

n = positive integer

where

$$a_n = \frac{2}{L}\int_0^L \sin(2\pi x)\sin(\lambda_n x)\,dx$$

Problem 4

C++ Provided .txt Outputs

Δt	$u_2(10)$
10^0	2.066
10^{-1}	1.83896
10^{-2}	1.66883
10^{-3}	1.6638
10^{-4}	1.66329
10^{-5}	1.66324
10^{-6}	1.66324
10^{-7}	1.66324
10^{-8}	1.66324

[Note: 10^{-8} iteration was 24.8GB!]

[Note: t is first column
u_2 is second column
u_1 is third column
for output file]

[Note: After 10^{-6} the results are the same.]

"best solution" is 10^{-8} steps

$$u_{2,ref}(10) = 1.66324$$

plotted $|u_2(10) - u_{2,ref}(10)|$ vs Δt

, in attached excel file