Indian Institute of Technology Indore
MA-204 Numerical Methods
Assignment 3 : System of Linear Equations

1. Determine the convergence factors $\{\mu, \eta\}$ for the Jacobi and Gauss-Seidel methods for the system

$$\begin{bmatrix} 4 & 0 & 2 \\ 0 & 5 & 2 \\ 5 & 4 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}.$$

Further, for both the methods find the minimum number of iterations needed to obtain the approximate solution correct up to 4 decimal places.

2. Use Gershgorin's theorem to show that all the eigenvalues of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 2 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ lie within the interval $[-2, 5]$.

3. Find QR decomposition of the matrix $A = \begin{bmatrix} -3 & -5 & -8 \\ 6 & 4 & 1 \\ -6 & 2 & 5 \end{bmatrix}$. Further use QR algorithm to find the first two approximations for the eigenvalues of the matrix A.

4. Use Power method to find the first two approximations to the eigen-pair corresponding to the eigenvalue of maximum modulus of the matrix

$$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix},$$

using the initial vector $z = [1, 2]^t$ and $u = [1, 0]^t$. Also find an approximatation to the eigenvector of the other eigenvalue.

5. Let $A = [a_{ij}]$ be a 3×3 real matrix. Show that

$$\|A\|_\infty = \max_{1 \le i \le 3} \left(\sum_{j=1}^{3} |a_{ij}| \right).$$

6. Give an example of a 2×2 matrix A such that
   (a) Symmetric Positive Definite matrix A for which $\eta \geq 1$.
   (b) Symmetric matrix A for which $\mu \geq 1$.

Practice Problems

1. Let the coefficient matrix A of the system $Ax = b$ be invertible and $\overline{x}$ is an approximate solution to the true solution x. Suppose $\gamma = b - A\overline{x}$. show that

$$\frac{1}{\text{cond}\,(A)}\frac{\|\gamma\|}{\|b\|} \leq \frac{\|x-\overline{x}\|}{\|x\|} \leq \text{cond}\,(A)\frac{\|\gamma\|}{\|b\|}.$$

2. Let the coefficient matrix A in the system $AX = b$ be non-singular and δA and δb be perturbations in A and b respectively. If $A + \delta A$ is non-singular and $(A + \delta A)(X + \delta X) = b + \delta b$, then show that

$$\frac{\|\delta X\|}{\|X\|} \leq \frac{\text{cond}\,(A)}{1 - \text{cond}\,(A)\dfrac{\|\delta A\|}{\|A\|}}\left\{\frac{\|\delta A\|}{\|A\|} + \frac{\|\delta b\|}{\|b\|}\right\}.$$

3. State and prove Gershgorin's theorem.

4. Let A be a strictly row diagonally dominant matrix of order $n \times n$. If $x^{(k)}$ denote the kth Gauss-Jacobi approximation of the solution of the system $Ax = b$, show that **[6]**

$$\|x^{(k)} - x\| \leq \frac{\mu^{(k)}}{1-\mu}\|x^{(1)} - x^{(0)}\|.$$

5. Let A and B be any two square matrices of order $n \times n$. Show that if A is invertible and

$$\|A - B\| < \frac{1}{\|A^{-1}\|},$$

then B is also invertible.

6. Prove that if $\mu < 1$, then $\eta \leq \mu$

7. Show that a strictly row diagonally dominant matrix is invertible.