# Proof of the Volume of a Sphere

Gabriel-Andrew Pollo-Guilbert

June 7, 2015

The volume of a sphere can be defined with the equation, where $r \in \mathbb{R}$:

$$V_{sphere}(r) = \frac{4\pi r^3}{3}$$

This equation can be proven using calculus. We know that a circle is defined with the following equation, where $x, y, r \in \mathbb{R}$:

$$x^2 + y^2 = r^2$$

With a little bit of algebra, we can get a function that gives the half of the circle:

$$\begin{aligned}
& x^2 + y^2 = r^2 \\
\Leftrightarrow \quad & y^2 = r^2 - x^2 \\
\Leftrightarrow \quad & \sqrt{y^2} = \sqrt{r^2 - x^2} \\
\Leftrightarrow \quad & |y| = \sqrt{r^2 - x^2} \\
\Rightarrow \quad & y = \sqrt{r^2 - x^2}
\end{aligned}$$

If we rotate the half-circle around the x axis, we get a sphere. We can, thus, calculate the volume of a sphere using a solid of revolution. Since the function is defined in $x \in [-r, r]$, we can use the disk method:

$$\begin{aligned}
V_{sphere}(r) &= \int_{-r}^{r} \pi \left[\sqrt{r^2 - x^2}\right]^2 \mathrm{d}x \\
&= \int_{-r}^{r} \pi \left[r^2 - x^2\right] \mathrm{d}x \\
&= \pi \int_{-r}^{r} \left[r^2 - x^2\right] \mathrm{d}x
\end{aligned}$$

This integral can be easily solved:

$$\begin{aligned}
\pi \int_{-r}^{r} \left[r^2 - x^2\right] \mathrm{d}x &= \pi \int_{-r}^{r} r^2 \mathrm{d}x - \pi \int_{-r}^{r} x^2 \mathrm{d}x \\
&= \pi r^2 \int_{-r}^{r} \mathrm{d}x - \pi \int_{-r}^{r} x^2 \mathrm{d}x \\
&= \pi r^2 \cdot x \Big|_{-r}^{r} - \pi \cdot \frac{x^3}{3} \Big|_{-r}^{r} \\
&= \pi r^2 \cdot 2r - \pi \cdot \frac{2r^3}{3}
\end{aligned}$$

This is a now a simple matter of simplifying the expression:

$$\begin{aligned} \pi r^2 \cdot 2r - \pi \cdot \frac{2r^3}{3} &= 2\pi r^3 - \frac{2\pi r^3}{3} \\ &= \frac{6\pi r^3}{3} - \frac{2\pi r^3}{3} \\ &= \frac{4\pi r^3}{3} \end{aligned}$$

Thus, the volume of a sphere follows the predicted equation:

$$V_{sphere}(r) = \int_{-r}^{r} \pi \left[\sqrt{r^2 - x^2}\right]^2 \mathrm{d}x = \frac{4\pi r^3}{3}$$