# Gaussian Integral With $\alpha = 1$

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The Gaussian integral can be defined as the improper integral in $\mathbb{R}$:

$$G(\alpha) = \int_{-\infty}^{+\infty} \mathrm{e}^{-\alpha x^2} \mathrm{d}x \tag{1}$$

A special case of this integral arises when $\alpha = 1$, which yields the equality:

$$G(1) = \int_{-\infty}^{+\infty} \mathrm{e}^{-x^2} \mathrm{d}x = \sqrt{\pi} \tag{2}$$

This equality can be proven using various methods. The most intuitive proof is using Cartesians coordinates on half of the integral. We can get these coordinates by squaring the integral and posing $x = y$ in the right integral:

$$\begin{aligned} H(1) &= \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\ \Rightarrow [H(1)]^2 &= \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \cdot \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \\ &= \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \cdot \int_{0}^{\infty} \mathrm{e}^{-y^2} \mathrm{d}y \end{aligned} \tag{3}$$

We now have two integrals using differents variables. Thus, we can simplify (3) by insering one integral into the other:

$$\begin{aligned} \int_{0}^{\infty} \mathrm{e}^{-x^2} \mathrm{d}x \cdot \int_{0}^{\infty} \mathrm{e}^{-y^2} \mathrm{d}y &= \int_{0}^{\infty} \mathrm{e}^{-x^2} \left[ \int_{0}^{\infty} \mathrm{e}^{-y^2} \mathrm{d}y \right] \mathrm{d}x \\ &= \int_{0}^{\infty} \left[ \int_{0}^{\infty} \mathrm{e}^{-x^2} \cdot \mathrm{e}^{-y^2} \mathrm{d}y \right] \mathrm{d}x \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-x^2} \cdot \mathrm{e}^{-y^2} \mathrm{d}y \, \mathrm{d}x \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-x^2 - y^2} \mathrm{d}y \, \mathrm{d}x \\ &= \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-(x^2 + y^2)} \mathrm{d}y \, \mathrm{d}x \end{aligned} \tag{4}$$

Because $x, y \in \mathbb{R}$, we can define $y = s\,x$ and $\mathrm{d}y = x\,\mathrm{d}s$, where $s \in \mathbb{R}$:

$$\begin{aligned} \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-(x^2 + y^2)} \mathrm{d}y \, \mathrm{d}x &= \int_{0}^{\infty} \int_{0}^{\infty} \mathrm{e}^{-(x^2 + s^2 x^2)} x \, \mathrm{d}s \, \mathrm{d}x \\ &= \int_{0}^{\infty} \int_{0}^{\infty} x \, \mathrm{e}^{-x^2(1+s^2)} \mathrm{d}s \, \mathrm{d}x \\ &= \int_{0}^{\infty} \int_{0}^{\infty} x \, \mathrm{e}^{-x^2(1+s^2)} \mathrm{d}x \, \mathrm{d}s \end{aligned} \tag{5}$$

The inner integral can be easily solved by posing $u = x^2(1+s^2)$ and $\mathrm{d}u = 2x(1+s^2)\,\mathrm{d}x$:

$$\begin{aligned}
\int_0^\infty x\,\mathrm{e}^{-x^2(1+s^2)}\mathrm{d}x &= \frac{1}{2(1+s^2)}\int_0^\infty 2(1+s^2)\,\mathrm{e}^{-x^2(1+s^2)}\mathrm{d}x \\
&= \frac{1}{2(1+s^2)}\int_0^\infty \mathrm{e}^{-u}\mathrm{d}u \\
&= \frac{1}{2(1+s^2)}\lim_{b\to\infty}\left[-e^{-u}\right]_0^b \\
&= \frac{1}{2(1+s^2)}
\end{aligned} \tag{6}$$

Thus, the inner integral in (5) can be replaced by the result given by (6):

$$\begin{aligned}
\int_0^\infty\int_0^\infty x\,\mathrm{e}^{-x^2(1+s^2)}\mathrm{d}x\,\mathrm{d}s &= \int_0^\infty \frac{1}{2(1+s^2)}\mathrm{d}s \\
&= \frac{1}{2}\int_0^\infty \frac{1}{1+s^2}\mathrm{d}s
\end{aligned} \tag{7}$$

The remaining integral is a case of the inverse trigonometric function of the tangent. The primitive will be of the following form where $a \in \mathbb{R}$ and $u$ is a real function:

$$\int \frac{1}{a^2+u^2}\mathrm{d}u = \frac{1}{a}\arctan\left(\frac{u}{a}\right) \tag{8}$$

By solving the final part of the integral, we get:

$$\frac{1}{2}\int_0^\infty \frac{1}{1+s^2}\mathrm{d}s = \frac{1}{2}\lim_{b\to\infty}\left[\arctan(s)\right]_0^b = \frac{\pi}{4} \tag{9}$$

Thus:

$$\begin{aligned}
&\left[\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x\right]^2 = \frac{\pi}{4} \\
\Rightarrow &\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \frac{\sqrt{\pi}}{2}
\end{aligned} \tag{10}$$

We need to calculate the other part of the integral. By looking at its graph and knowing that $e^{-(x)^2} = e^{-(-x)^2}$, we can assume that the integral is even. Thus, we can double the half to obtain the final result:

$$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\mathrm{d}x = 2\int_0^\infty \mathrm{e}^{-x^2}\mathrm{d}x = \sqrt{\pi} \tag{11}$$