CHAPTER 3

Transformations of Functions

Getting lost in the back country is frightening and potentially life threatening. For most avid hikers, a topographic map and a compass are as important as an adequate supply of water.

Topographic maps use contour lines to show the variability of the terrain. Hikers are typically less interested in the elevation of a particular geographical feature (ridge, peak, saddle) than in the elevation change between their current position and the geographical feature. The concept of function transformations can help hikers transform map data into practical information and determine their best hiking route.

3.1 Vertical and Horizontal Shifts

3.2 Vertical and Horizontal Reflections

3.3 Vertical Stretches and Compressions

3.4 Horizontal Stretches and Compressions

STUDY SHEET

REVIEW EXERCISES

MAKE IT REAL PROJECT

SECTION 3.1 Vertical and Horizontal Shifts

LEARNING OBJECTIVES

- Identify what change in a function equation results in a vertical shift
- Identify what change in a function equation results in a horizontal shift

GETTING STARTED

With an elevation of 8850 meters (29,035 feet) at its highest point, Mount Everest is the tallest mountain in the world. Every year people gather in Nepal in organized expeditions to "summit"—climb to the peak of—Mount Everest. As the expeditions make their way up the mountain, climbers typically ascend to a certain elevation, spend some time resting, and then hike partway back down the mountain. In this way, their bodies can acclimatize to the extreme conditions of high elevation, low oxygen levels, physical strain, and cold.

In this section we explore the relationship between a function and a transformation of the function. Specifically, we investigate vertical and horizontal shifts and see how to model real-world situations such as the elevation of mountain climbers with such transformations. We also look at the symmetry of some function graphs.

Vertical Shifts

We know that functions may be represented by equations, graphs, tables, or words, but **transformations** (changes) in functions are often easiest to see using a graph. We will use the Mount Everest example to illustrate one of the most common transformations, the *vertical shift*.

On April 10, 2005, a Mount Everest expedition arrived at its base camp (elevation: 5165 meters). Between April 10 and May 30, 2005, the group followed the recommended pattern for acclimatization—spending periods of time resting and hiking up and down the mountain to ensure a successful summit. Table 3.1 and Figure 3.1 show the climbers' elevation, E (in meters), as a function of the days, d, since their arrival at base camp.

Table 3.1

Days Since April 10, 2005 d	Elevation (meters) $E(d)$
0	5165
4	5600
5	6363
6	5782
7	6400
15	7040
20	7680
21	5165
22	7040
24	7680
26	5165
33	6400
36	7040
39	7700
40	7040
41	6400
48	7040
49	7800
50	8850

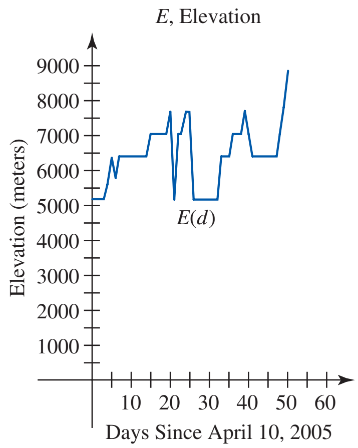


Figure 3.1

Source: www.project-himalaya.com

EXAMPLE 1 ■ Shifting a Graph Vertically

Based on the Mount Everest expedition data in Table 3.1 and Figure 3.1, create a table of values and draw the associated graph that represents the climbers' elevation relative to their base camp (elevation: 5165 meters) d days after April 10. How do your table and graph compare to the model E?

Solution To express the elevation relative to base camp, the elevation for any given day must reflect the difference between the actual elevation, $E(d)$, and the elevation at base camp, $B(d)$. For example, on April 25 ($d = 15$) the actual elevation is 7040 meters, which is 1875 meters higher than the elevation at base camp. Partial data showing this relationship is given in Table 3.2.

Table 3.2

Days Since April 10, 2005 d	Elevation (meters above sea level) $E(d)$	Elevation (meters above base camp) $B(d)$
0	5165	$5165 - 5165 = 0$
5	6363	$6363 - 5165 = 1198$
15	7040	$7040 - 5165 = 1875$
20	7680	$7680 - 5165 = 2515$
40	7040	$7040 - 5165 = 1875$
50	8850	$8850 - 5165 = 3685$

For every value of d, $B(d)$ is 5165 meters less than $E(d)$. This means that the graph of B is formed by shifting E downward 5165 units, as shown in Figure 3.2a. This vertical shift occurs because we have changed the vertical point of reference for the function. The graph of the new function is shown in Figure 3.2b.

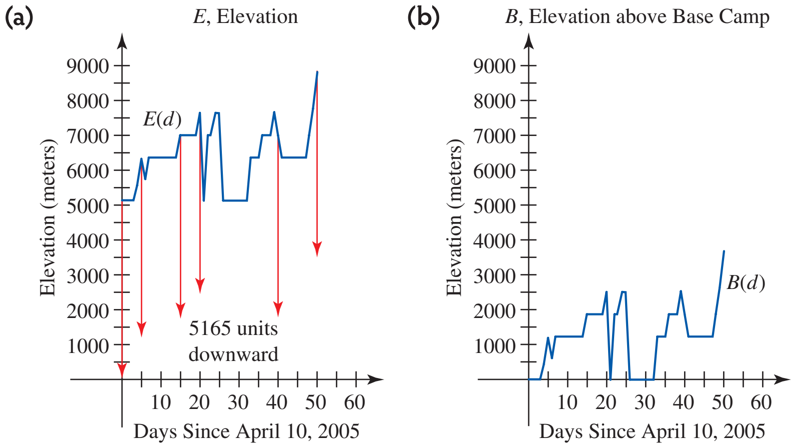


Figure 3.2

Note that the shifted graph in Figure 3.2b still possesses the basic properties of the original function. The relative changes in altitude from day to day remain the same, as do the number of days it took the expedition to summit the mountain. The only change is the vertical point of reference that results in the **vertical shift**.

VERTICAL SHIFTS

The graph of $g(x) = f(x) + k$ is the graph of f shifted vertically by $|k|$ units. If k is positive, the shift is upward. If k is negative, the shift is downward.

EXAMPLE 2 ■ Determining the Relationship between Two Functions That Differ by a Vertical Shift

Refer to the functions introduced in Example 1 and write an equation that relates $B(t)$ and $E(t)$.

Solution Recall that $B(t)$ was obtained by subtracting 5165 from $E(t)$. Therefore, $B(t) = E(t) - 5165$. The graph of B is the graph of E shifted downward by 5165.

Generalizing Transformations: Vertical Shifts

The key to understanding vertical shifts, including how the graphs of two functions are related, is to realize that we are defining the outputs of one function using the outputs of another function. In other words, we have a function defined such that we know its outputs for some set of inputs. This can be provided as a graph, a table, a description, or a formula. We then want to construct a new function, called the **image function**, based on the **parent function** (the function we already know).

For example, consider $g(x) = f(x) + 5$. We are implying there exists a function f whose outputs can be used to define the outputs of function g. For all values of x in the domain of f, we think of $g(x) = f(x) + 5$ as follows:

$$\underbrace{g(x)}_{\text{the output of } g \text{ at } x} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{f(x)}_{\text{the output of } f \text{ at } x} \quad \underbrace{+\ 5}_{\text{increased by 5}}$$

Because the basic idea is true no matter how the function f (the parent function) is given (description, table, graph, or formula), we can understand the relationship between the parent and image functions in all settings. With vertical shifts, the *pattern* of the outputs does not change. That is, the relationship between the outputs does not change even though their vertical placement on the graph varies.

EXAMPLE 3 ■ Using Vertical Shifts in a Real-World Context

Based on 2007 ticket prices, the cost of an adult Disney Park Hopper® Bonus Ticket can be modeled by

$$T(d) = -5.357d^2 + 59.04d + 28.20 \text{ dollars}$$

where d is the number of days the ticket authorizes entrance into Disneyland and Disney California Adventure. If Disney executives authorize a \$10 across-the-board increase in ticket prices for 2012, what will be the model for 2012 ticket prices?

Solution Since each ticket price is increased by \$10, the 2012 ticket price model will be the 2007 model plus \$10.

$$\begin{aligned} N(d) &= T(d) + 10 \\ &= (-5.357d^2 + 59.04d + 28.20) + 10 \\ &= -5.357d^2 + 59.04d + 38.20 \text{ dollars} \end{aligned}$$

Horizontal Shifts

Two functions may also be related by a shift to the left or right. In a **horizontal shift**, two functions will have identical output values but these output values will occur for different input values. We will illustrate this concept by returning to the Mount Everest expedition data.

EXAMPLE 4 ■ Shifting a Graph Horizontally

According to the electronic journal kept by members of the April 2005 Mount Everest expedition, the group was concerned their excursion might be delayed due to political instability in Nepal. To avoid potential delays, they sent their guides to base camp five days earlier than originally scheduled. (*Source:* www.project-himalaya.com)

a. Suppose the entire expedition had arrived at base camp five days early with their guides and then followed the pattern of climbing shown in Table 3.1 and Figure 3.1. How would we need to change the graph of E to model the elevation of the climbers?

b. Suppose political instability had delayed the expedition and the climbers reached their base camp a week later than planned but then followed the same pattern of climbing. How would we need to change the graph of E to model the elevation of the climbers?

Solution

a. If the climbers had arrived five days earlier with their guides, they would have reached base camp on April 5. Considering our original graph (Figure 3.1) and definition of d as "days since April 10," this corresponds to $d = -5$.

If we assume the climbers followed the same pattern of climbing from this date, then they would have reached the summit on May 25 instead of May 30. Likewise, the climbers would have reached *all* of the same elevations five days earlier, as shown in Table 3.3.

Table 3.3

d Value When Climbers Reached This Elevation if They Arrived 5 Days Early	Days Since April 10, 2005 d	Elevation (meters) $E(d)$
$0 - 5 = -5$	0	5165
$5 - 5 = 0$	5	6363
$15 - 5 = 10$	15	7040
$20 - 5 = 15$	20	7680
$40 - 5 = 35$	40	7040
$50 - 5 = 45$	50	8850

This change will cause the graph to shift to the left 5 units, as shown in Figures 3.3a and 3.3b.

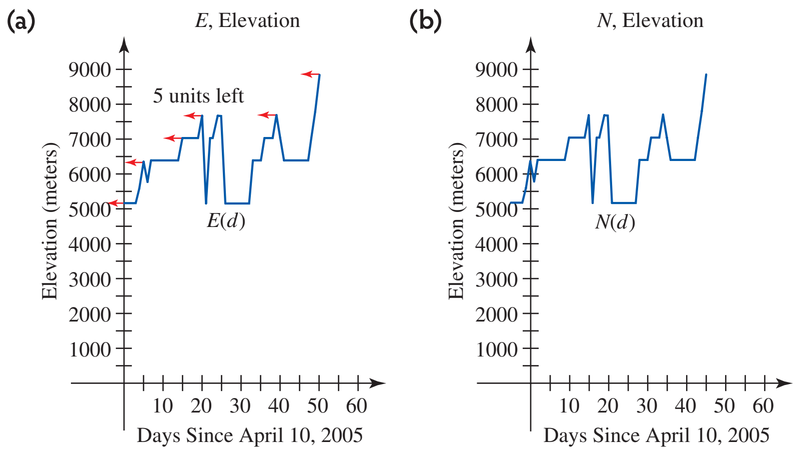


Figure 3.3

Observe that $N(d) = E(d + 5)$. We can verify this by checking a couple of points.

$$\begin{aligned} N(0) &= E(0 + 5) \\ &= E(5) \\ &= 6363 \end{aligned} \qquad \begin{aligned} N(35) &= E(35 + 5) \\ &= E(40) \\ &= 7040 \end{aligned}$$

The results are consistent with the graphs of N and E.

b. If the expedition began seven days late, the group would have reached base camp on April 17 instead of April 10. Again, the climbers would have reached all of the same elevations after the same number of days climbing but seven days later for each, as shown in Table 3.4.

Table 3.4

d Value When Climbers Reached This Elevation if They Arrived 7 Days Late	Days Since April 10, 2005 *d*	Elevation (meters) $E(d)$
$0 + 7 = 7$	0	5165
$5 + 7 = 12$	5	6363
$15 + 7 = 22$	15	7040
$20 + 7 = 27$	20	7680
$40 + 7 = 47$	40	7040
$50 + 7 = 57$	50	8850

We can model this new situation by shifting the original graph to the right 7 units, as shown in Figures 3.4a and 3.4b.

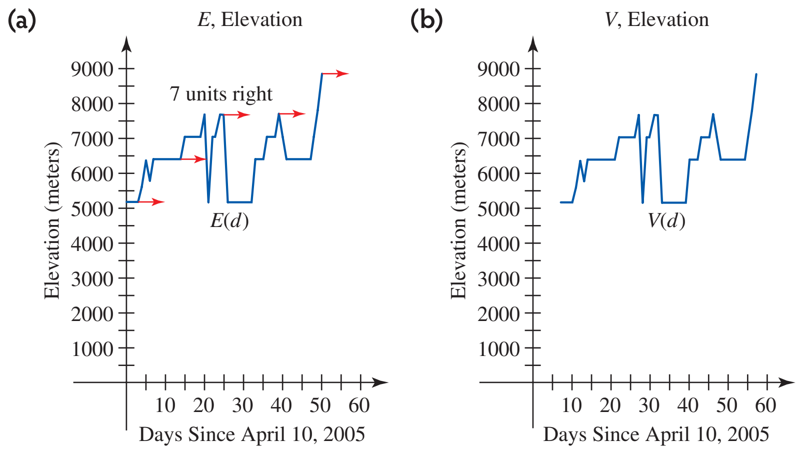


Figure 3.4

Observe that $V(d) = E(d - 7)$. We can verify this by checking a couple of points.

$$V(7) = E(7 - 7) \qquad V(50) = E(50 - 7)$$
$$= E(0) \qquad = E(43)$$
$$= 5165 \qquad = 6400$$

The results are consistent with the graphs of V and E.

We summarize our observations regarding horizontal shifts as follows.

Horizontal Shifts

The graph of $g(x) = f(x - h)$ is the graph of f shifted horizontally by $|h|$ units. If h is positive, the shift is to the right. If h is negative, the shift is to the left.

Generalizing Transformations: Horizontal Shifts

Again, the key to transformations is to remember that we are using the outputs of a parent function to define the outputs of an image function. Consider the function $g(x) = f(x - 2)$. We observe that g and f have identical *outputs*.

$$\underbrace{g(\ \ )}_{\text{the outputs of } g} \ \underbrace{=}_{\text{equal}} \ \underbrace{f(\ \ )}_{\text{the outputs of } f}$$

If $g(x) = f(x)$, then the functions are identical. If $g(x) = f(x - 2)$, the outputs are identical but they occur at *different inputs*. The expression $x - 2$ indicates that if we want to evaluate g for some input x, we need to look to the output of f when the input is two units less than x. For example, to find $g(-1)$ and $g(7)$, we do the following.

$$g(x) = f(x - 2) \qquad g(x) = f(x - 2)$$
$$g(-1) = f(-1 - 2) \qquad g(7) = f(7 - 2)$$
$$g(-1) = f(-3) \qquad g(7) = f(5)$$

The output of g at -1 is defined by the output of f at -3. **The output of g at 7 is defined by the output of f at 5.**

If f is given by the graph in Figure 3.5a, then g is defined by the graph in Figure 3.5b.

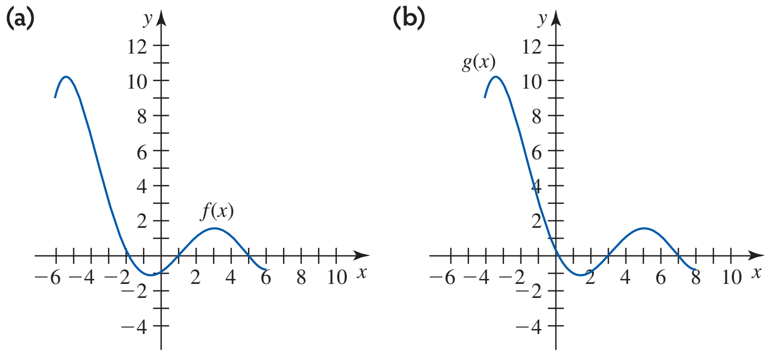


Figure 3.5

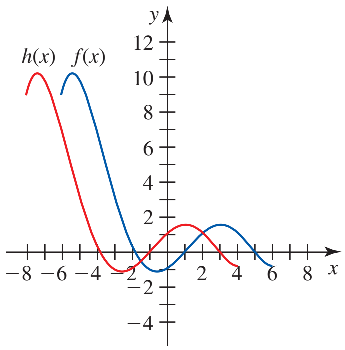


Figure 3.6

Note we can verify statements such as $g(7) = f(5)$ by using the graphs, where we can see that $g(7) = 0$ and $f(5) = 0$. If there is any confusion as to whether one function's graph is to the right or left of another function's graph, we simply look at the inputs in the definition. In the case of $g(x) = f(x - 2)$, x will always be larger than $x - 2$. Thus the same outputs occur in g for larger inputs than in f. This means the graph of g is to the right of the graph of f by 2 units (or the graph of f is to the left of the graph of g by 2 units).

The opposite is true for a relationship defined by $h(x) = f(x + 2)$. In this case the inputs in f are larger for the same output (since $x + 2$ is always larger than x). As shown in Figure 3.6, the graph of f is to the right of h (or the graph of h is to the left of f) by 2 units.

EXAMPLE 5 ■ **Graphing Function Transformations**

Use the graph of f shown in Figure 3.7 to draw the graph of $g(x) = f(x - 2) + 3$.

Solution As shown in Figure 3.8, the graph of g will be the graph of f shifted right 2 units and shifted upward 3 units since the outputs of g occur for larger inputs (x is larger than $x - 2$), and those outputs are then increased by 3. When completing multiple transformations like this we typically follow the order of operations.

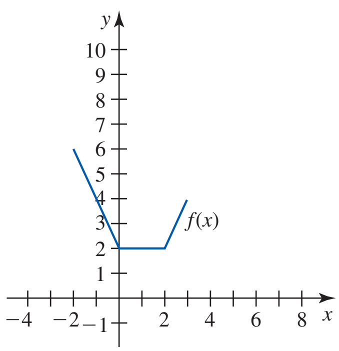


Figure 3.7

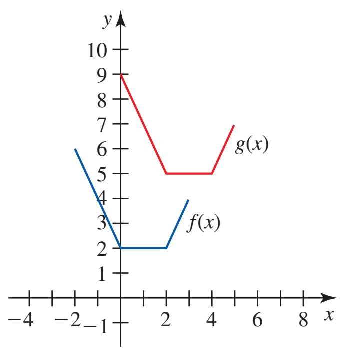


Figure 3.8

EXAMPLE 6 ■ Performing Shifts on a Table

The function f is defined by Table 3.5. Using this table,

a. Evaluate $f(x) + 6$ when $x = 2$.

b. Evaluate $f(x - 3)$ when $x = 1$.

c. Evaluate $f(0) - f(-3)$.

d. Create a table of values for $g(x)$ if $g(x) = f(x + 1) - 4$.

Table 3.5

x	$f(x)$
−3	4
−2	5
−1	7
0	11
1	19
2	28
3	46

Solution

a. $f(x) + 6 = f(2) + 6$ since $x = 2$

$= 28 + 6$ since $f(2) = 28$ from the table

$= 34$

b. $f(x - 3) = f(1 - 3)$ since $x = 1$

$= f(-2)$

$= 5$ from the table

c. $f(0) - f(-3) = 11 - 4$

$= 7$

d. To create the table of values for g, we could evaluate g for different x-values. For example,

$$\begin{aligned} g(-4) &= f(-4 + 1) - 4 \\ &= f(-3) - 4 \\ &= 4 - 4 \\ &= 0 \end{aligned}$$

This shows $(-4, 0)$ is a coordinate point for g. We could also find this point by taking the coordinate point $(-3, 4)$ from f and shifting it left 1 unit and downward 4 units. Using either of these procedures, we obtain the table of values for g shown in Table 3.6.

Table 3.6

x	$g(x)$
−4	0
−3	1
−2	3
−1	7
0	15
1	24
2	42

EXAMPLE 7 ■ Analyzing Shifts

The graph of $f(x) = x^3$ is given in Figure 3.9. Function g is a transformation of f and is shown in Figure 3.10. Describe how f was transformed to create g and write the formula for g in terms of f. Then simplify the result.

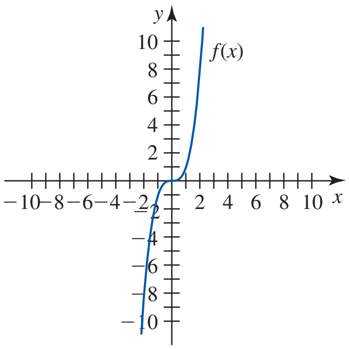


Figure 3.9

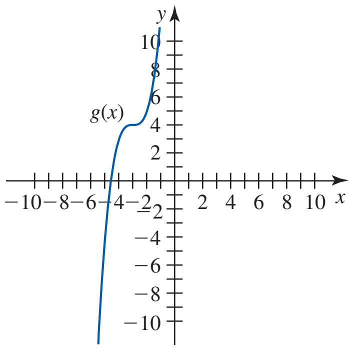


Figure 3.10

Solution As shown in Figure 3.11, function f has been shifted left 3 units and upward 4 units to create g.

The function notation for this relationship is

$$g(x) = f(x + 3) + 4$$

This shows that we find the outputs of g by using an input value 3 units less and then increasing the output of f by 4 units. We can now use this relationship to find the formula for g.

$$\begin{aligned} g(x) &= f(x + 3) + 4 \\ &= (x + 3)^3 + 4 \qquad \text{since } f(x) = x^3 \end{aligned}$$

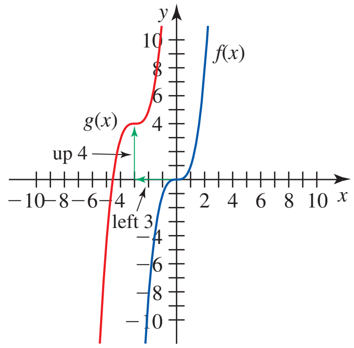


Figure 3.11

■ Aligning Data Horizontally

We often use a horizontal shift to *align data* before creating a mathematical model to reduce the size of the numbers used in the computation of model coefficient values. To align data, we determine the input value we want to use as a starting point. For example, if we want to model the population of the United States in the 21st century, we might choose to align the data to the year 2000, meaning $t = 0$ represents 2000, $t = 1$ represents 2001, and so on.

EXAMPLE 8 ■ Aligning Data Horizontally

The population of the United States increased early in the 21st century, as shown in Table 3.7. (*Source: Statistical Abstract of the United States, 2006*, Table 2)

a. Rewrite the population table as a function of t, where t is defined to be the number of years since 2000.

b. Describe the relationship between y and t, then write $P(t)$ in terms of y.

Table 3.7

Year y	U.S. Population (in thousands) $P(y)$
2000	282,402
2001	285,329
2002	288,173
2003	291,028
2004	293,907

Solution

a. The rewritten data is shown in Table 3.8.

Table 3.8

Years Since 2000 t	U.S. Population (in thousands) $P(t)$
0	282,402
1	285,329
2	288,173
3	291,028
4	293,907

b. Since y represents the year and t represents the number of years since 2000, $t = y - 2000$. Therefore, $P(t) = P(y - 2000)$.

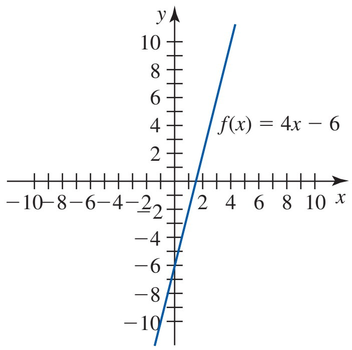
SUMMARY

In this section you learned how to use vertical and horizontal shifts to transform a function equation and its associated graph. You also learned how to determine the relationship between the equations of graphs that differ by a horizontal or vertical shift. Additionally, you learned how to align data to simplify the modeling process.

3.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, graph each transformation of f and write the formula for the transformed function in the form $y = mx + b$.



1. $f(x) - 2$
2. $f(x - 3)$
3. $f(x + 1) + 2.5$
4. $f(x - 3) + 2$

In Exercises 5–8,

a. Create a table of values for f and g.

b. Graph f and g.

c. Describe the relationship between the graph of f and the graph of g.

5. $f(x) = x^2, \quad g(x) = f(x + 2) + 7$
6. $f(x) = \sqrt{x}, \quad g(x) = f(x - 4) + 2$
7. $f(x) = |x| + 5, \quad g(x) = f(x - 1) - 3$
8. $f(x) = \dfrac{1}{x}, \quad g(x) = f(x + 6) + 1$

In Exercises 9–14, use Table A to evaluate each expression.

Table A

x	$f(x)$	x	$f(x)$
−4	12	1	−3
−3	10	2	4
−2	7	3	5
−1	2	4	6
0	−1		

9. $f(x) + 2$ when $x = -3$
10. $f(x + 4)$ when $x = -1$
11. $f(x + 1) - 9$ when $x = -2$
12. $f(x - 5) - 4$ when $x = 4$
13. $f(-3) + f(1)$
14. $2f(0) - f(-2)$

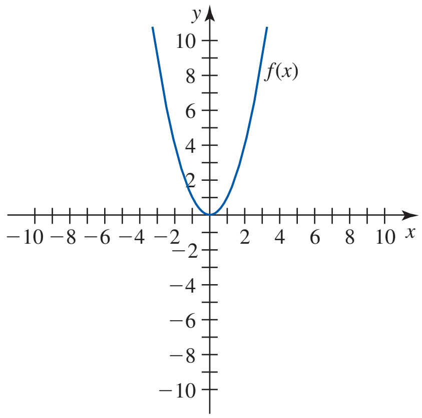
In Exercises 15–18, use Table A to solve each equation for x.

15. $f(x) = -3$
16. $f(x + 5) = 2$
17. $f(x) + 10 = 15$
18. $f(x - 1) + 2.5 = 14.5$

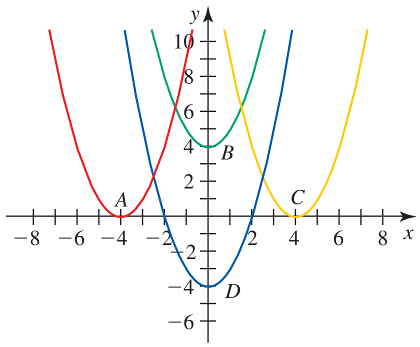
In Exercises 19–20, refer to Table A to answer each question.

19. Create a table of values for g given $g(x) = f(x - 4)$.
20. Create a table of values for k given $k(x) = f(x + 1) + 5$.

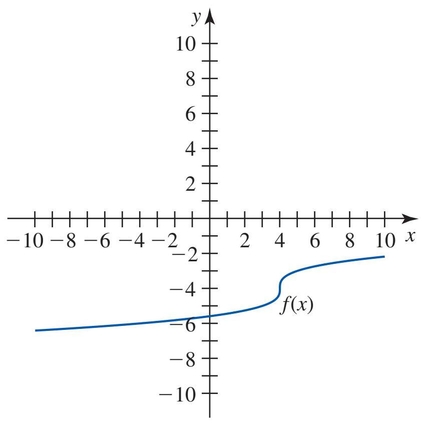
In Exercises 21–24, match the formula for each transformation of $f(x) = x^2$ with graph A, B, C, or D. Do not use a calculator.



21. $g(x) = (x + 4)^2$
22. $h(x) = (x - 4)^2$
23. $j(x) = x^2 - 4$
24. $k(x) = x^2 + 4$



In Exercises 25–30, draw the graph of each function as a transformation of the given function f.



25. $A(x) = f(x + 1)$

26. $B(x) = f(x) - 3$

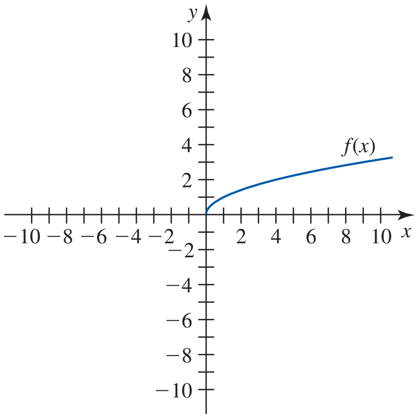
27. $C(x) = f(x - 3) + 2$

28. $D(x) = f(x + 4) - 1$

29. $E(x) = f(x) + \pi$

30. $G(x) = f\left(x - \sqrt{8}\right)$

In Exercises 31–34, you are given the graph of a transformation of $f(x) = \sqrt{x}$.



For each exercise,

a. Describe the transformations required on f to create the new function.

b. Use function notation to write each new function in terms of f.

c. Write the formula for the new function.

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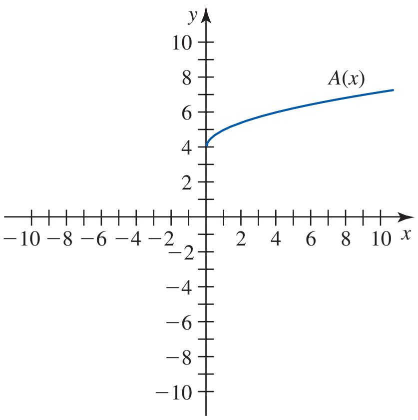

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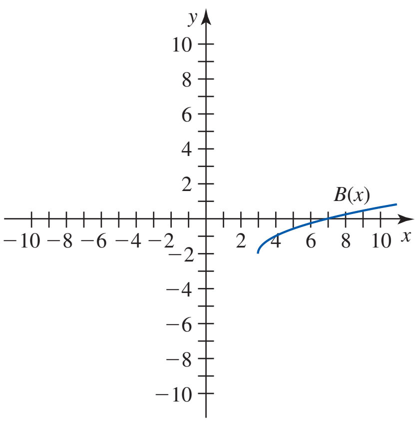

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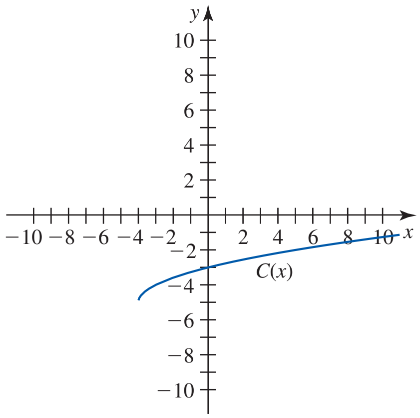

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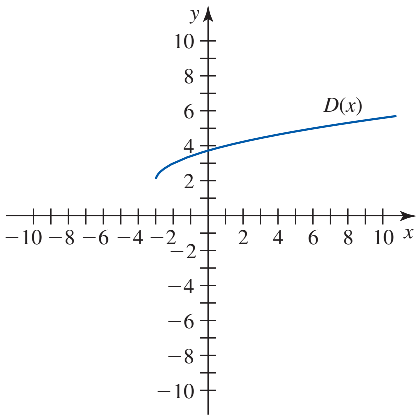


In Exercises 35–37, use the table to identify the shift(s) required to create the indicated function. Then write a formula for each transformed function in terms of f.

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
−1	−0.25	0.25	2.75	−1
0	0	2	3	0.75
1	0.25	6.75	3.25	1
2	2	16	5	1.25
3	6.75	31.25	9.75	3
4	16	54	19	7.75
5	31.25	85.75	34.25	17
6	54	128	57	32.25

35. $g(x)$

36. $h(x)$

37. $j(x)$

SHOW YOU KNOW

38. Explain why $f(x) + k$ shifts the graph of f vertically instead of horizontally and why $f(x - h)$ shifts the graph of f horizontally instead of vertically.

39. Sketch a graph of a nonlinear function on graph paper, then pick two points on your graph and calculate the average rate of change between these points. Shift your function vertically and note where the two points you chose previously are now located. Use this example to explain why vertical shifts do or do not affect the average rate of change of a function.

40. After completing Exercise 39, one of your classmates repeats the exercise using a horizontal shift and claims that horizontal shifts change the average rate of change of a function. A second classmate disagrees and says the average rate of change remains the same. Explain how each of these students could reach his or her conclusion.

41. One of your classmates tells you that horizontal shifts seem backward. He does not understand why $f(x - 4)$ would shift f to the right while $f(x + 4)$ would shift f to the left. Write an explanation for your classmate.

MAKE IT REAL

42. Satellite Television In January 2007, satellite television service through DirecTV was offered at a monthly rate of \$55.97 for 140 channels, high-definition access, and DVR service. The high definition DVR unit cost an additional \$299. **(*Source:* DirecTV)**

a. Write a function to model the total cost for this satellite television service for m months of subscription.

b. A company salesman offered a \$100 rebate on the high definition DVR. Show how to transform the original function to accommodate this rebate. Write the formula for the new function.

43. Professional Football From 1985 to 2006 the average weight of professional football players increased from about 225 to 248 pounds. **(*Source:* www.palmbeachpost.com)** The average weight of a professional football player can be modeled by

$$W(t) = 225(1.00465)^t \text{ pounds}$$

where t is the years since 1985. Assuming football pads weigh about 13 pounds, what is the function that gives the average weight of a professional football player in full pads?

44. Baseball Salaries Based on data from 1990 to 2006, the average annual salary of a major league baseball player can be modeled by

$$A(t) = 0.1474t + 0.4970 \text{ million dollars}$$

where t is the number of years since 1990. **(*Source:* Modeled from CBSSportsline.com, MLB Baseball 2006 Salaries)**

a. According to the model, what was the average salary in 1990?

b. Create a new function $I(t)$ that gives the *increase* in the average salary over the 1990 salary level.

c. If the meaning of t was changed from *years since 1990* to *years since 1900*, how would the function $A(t) = 0.1474t + 0.4970$ need to change for the results to still make sense?

In Exercises 45–48,

a. Create a table of values of aligned data.

b. Describe the horizontal shift required to align the data.

c. If the original function is f and the aligned function is g, use function notation to write g in terms of f.

45. Align the data to 1980.

Year	Number of Public Airports in the U.S.		Year	Number of Public Airports in the U.S.
1980	4814		1995	5415
1985	5858		2000	5317
1990	5589		2002	5286

***Source: Statistical Abstract of the United States, 2006*, Table 1062**

46. Align the data to 2000.

Year	Retail Sales of Indoor Houseplants (in millions)
2000	1332
2001	1784
2002	2128
2003	1571
2004	1495

***Source: Statistical Abstract of the United States, 2006*, Table 1231**

47. Align the data to 1998.

Year	Number of Doctorate Degrees Awarded in Astronomy
1995	173
1998	206
1999	159
2000	185
2001	186
2002	144
2003	167

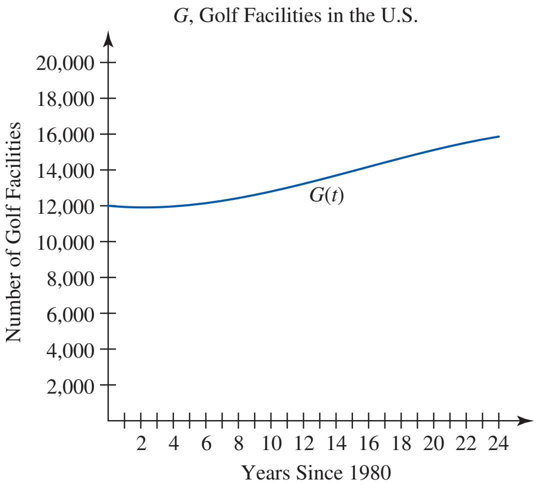
***Source: Statistical Abstract of the United States, 2006*, Table 784**

48. Align the data to 1995.

Year	Reported Burglaries (in thousands)		Year	Reported Burglaries (in thousands)
1989	3168		1997	2641
1991	3157		1999	2101
1993	2835		2001	2117
1995	2594		2003	2153

***Source: Statistical Abstract of the United States, 2006*, Table 293**

49. Golf Courses The number of golf facilities in the United States as a function of years since 1980 can be modeled by function *G*.

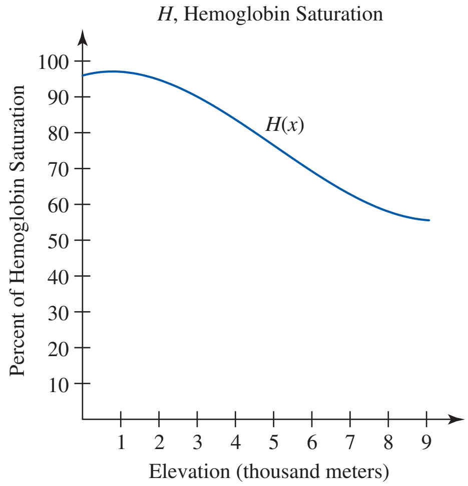


a. Suppose you shifted this function to the right 5 units. What would your new graph represent? Use function notation to write your new function in terms of *G*.

b. Suppose you shifted this function to the left 10 units. What would your new graph represent? Use function notation to write your new function in terms of *G*.

c. Suppose you shifted this function downward 12,000 units. What would your new graph represent? Use function notation to write your new function in terms of *G*.

50. Oxygen Levels Function *H* shows the approximate percent of hemoglobin saturation in the blood at an elevation of x thousand meters above sea level. Hemoglobin is the protein in red blood cells that transports oxygen throughout the body. Thus, this function shows how efficiently the human body can absorb and distribute oxygen at various elevations. (***Source:*** **Modeled from www.dr-amy.com/rich/oxygen)**



a. The human body requires a minimum of 87% hemoglobin saturation to function normally. At approximately what elevation is this saturation reached? Use this information to discuss the health dangers of climbing to Mount Everest's summit.

b. Research your city's elevation. About what percent of hemoglobin saturation would you expect your city's citizens to have? (If you can only find your city's elevation in feet, the conversion is 1000 feet = 304.8 meters.)

c. Use a vertical shift to create a new graph with the vertical axis representing the hemoglobin saturation percentage above the safe level of 87%.

d. Write the formula for the graph from part (c) in terms of *H*.

51. Atmospheric Pressure The first table shows the common atmospheric pressure for different altitudes.

Altitude (meters) ***a***	**Atmospheric Pressure (psi)** ***P*(*a*)**
0	14.70
1000	13.04
2000	11.53
3000	10.17
4000	8.94
5000	7.83
6000	6.84
7000	5.96
8000	5.16

a. Use a graphing calculator to plot the points.

b. Find a linear regression model for this data. Is it a reasonable fit? Explain.

c. The next table represents a transformation of the function *P*. Describe the transformation, then explain what the new function shows and how the table should be labeled. Then write the formula for the new function.

−4000	14.70
−3000	13.04
−2000	11.53
−1000	10.17
0	8.94
1000	7.83
2000	6.84
3000	5.96
4000	5.16

d. The next table represents a different transformation of the function P. Describe the transformation, then explain what the new function shows and how the table should be labeled. Then write the formula for the new function.

0	0
1000	−1.66
2000	−3.17
3000	−4.53
4000	−5.76
5000	−6.87
6000	−7.86
7000	−8.74
8000	−9.54

52. Sprinter's Time and Distance The function D models the distance a 100-meter sprinter has traveled t seconds after the beginning of a race.

a. Sketch a graph that could represent this situation. Be sure to label your axes appropriately.

b. In some races, such as junior high track meets, volunteers time runners with a handheld stopwatch. However, the volunteers may not all begin their timers precisely with the start of the race. Suppose it takes a volunteer 0.75 seconds after the race begins to start the stopwatch. Draw a graph for the sprinter's distance traveled as a function of the time since the volunteer started the stopwatch.

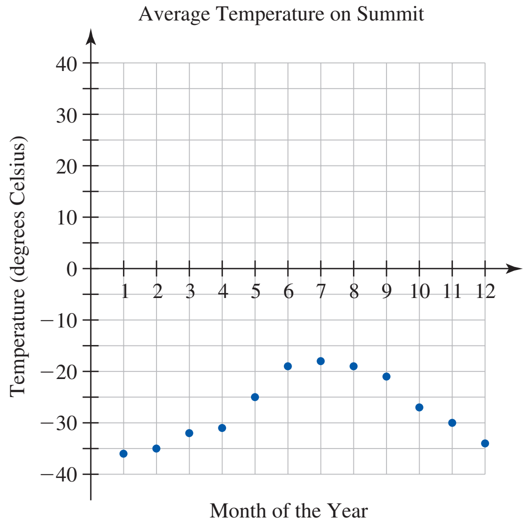
c. If possible, use transformations to describe the relationship between D and your new function, then discuss how this relates to the domain and range of your new function.

d. Repeat parts (b) and (c) for the following situations.

i. The volunteer accidentally started the stopwatch 0.25 seconds before the race actually began.

ii. The volunteer accidentally stopped the stopwatch halfway through the race and then took 2 full seconds to start it again.

53. Mount Everest Summit Temperatures The graph shows the average temperature, f, on the summit of Mount Everest during the m^{th} month of the year.



Source: **www.mounteverest.net**

Interpret the transformations $f(m + 12)$ and $f(m - 12)$ in this context.

■ STRETCH YOUR MIND

Exercises 54–57 are intended to challenge your understanding of vertical and horizontal shifts.

54. Use the table to evaluate the expression

$$f(x - f(x)) + f(f(x + 4) - x)$$

when $x = -1$.

x	$f(x)$
−4	12
−3	10
−2	7
−1	2
0	−1
1	−3
2	4
3	5
4	6

55. The function $P = f(n)$ describes the price you pay for buying n items. Suppose you have a coupon for 15% off the total price of your purchase. Can you model this using only shifts of the original function? If so, show how. If not, explain why it is impossible.

56. You are told the function f has the following property: $f(x) = f(x + 5n)$ for any integer values of n. Describe any characteristics this function will possess.

57. Create the graphs of a function and its inverse. Explain how and why a vertical shift and horizontal shift of the original function affect the inverse function.

SECTION 3.2

Vertical and Horizontal Reflections

LEARNING OBJECTIVES

- Identify what change in a function equation results in a horizontal reflection
- Identify what change in a function equation results in a vertical reflection
- Understand the concept of symmetry and determine if a function is even, odd, or neither

GETTING STARTED

One person's debt payment is another person's income. Although taking on excessive debt as a consumer is dangerous financially, it is beneficial financially for a consumer to receive debt payments from others. Whether debt is viewed as financially positive or negative depends upon whether you owe money or are owed money. This idea can be explained mathematically using the concept of reflections.

In this section we explore mathematical reflections in equations, graphs, and tables.

Vertical Reflections

In January 2003, one of the authors took out a \$13,460 loan to buy a new car. Over the next few years, he monitored the loan balance and total mileage. The function A, graphed in Figure 3.12, models the situation where the amount owed, A, is in thousands of dollars and the miles driven, m, are in thousands of miles.

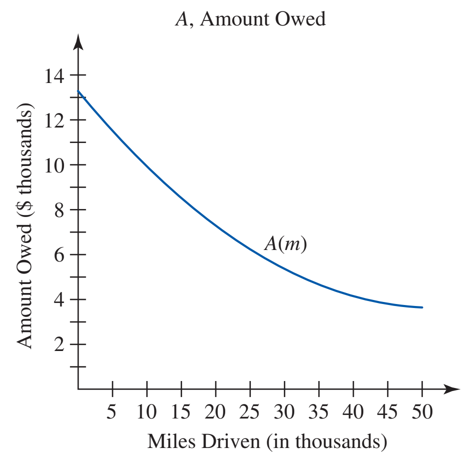


Figure 3.12

The equation for the model is $A(m) = 0.003569m^2 - 0.3715m + 13.29$. This formula was created from the actual data by using *quadratic regression*, a technique we will address in Chapter 4.

EXAMPLE 1 ■ Reflecting a Function Vertically

Net worth is the difference between a person's assets and liabilities. In short, *assets* are the things you own that have a dollar value and *liabilities* are the debts you owe to others. Assuming your assets remain constant, a reduction in your liabilities increases your net worth. Using this definition of net worth and the graph shown in Figure 3.12,

a. Draw a graph for the function N, the effect the car loan has on the author's net worth as a function of the miles he has driven the car. Explain how the graphs of A and N are related.

b. Use function notation to write N in terms of the function A.

c. Write a formula for $N(m)$.

Solution

a. If the car loan amount is \$13,460, the loan *reduces* the author's net worth by \$13,460. We represent this idea of reduction with −\$13,460. The car loan has a negative effect on the net worth. In other words, when the loan function A is positive, the effect on net worth function N is negative. Table 3.9 shows the relationship between A and N.

Table 3.9

Miles Driven (thousands) m	Amount Owed (\$ thousands) $A(m)$	Effect on Net Worth (\$ thousands) $N(m)$
4	11.86	−11.86
7.5	10.70	−10.70
15	8.52	−8.52
22.5	6.74	−6.74
0	13.46	−13.46

The graph for $N(m)$ is shown in Figure 3.13. This change demonstrates a **vertical reflection** (also called a **reflection about the horizontal axis**). As shown in Figure 3.14, each of the output values is the same number of units from the horizontal axis, but the values are positive for A and negative for N.

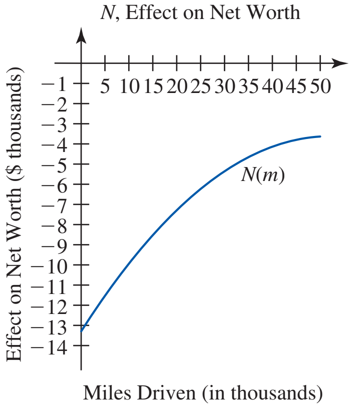


Figure 3.13

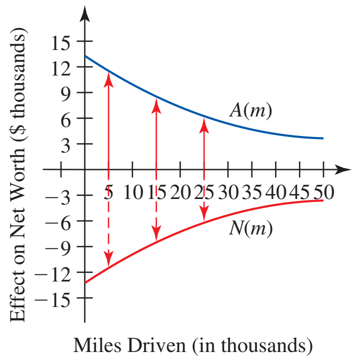


Figure 3.14

b. The function $N(m) = -A(m)$ describes the relationship between N and A. For example, to find the car loan's effect on the net worth after the author drove the car 15,000 miles, we can use the function relationship.

$$\begin{aligned} N(m) &= -A(m) \\ N(15) &= -A(15) \\ &= -(8.57) \\ &= -8.57 \end{aligned}$$

After he drove the car 15,000 miles, the car loan reduced the author's net worth by \$8570.

c. Since $N(m) = -A(m)$ and $A(m) = 0.002994m^2 - 0.3421m + 13.03$, we have

$$\begin{aligned} N(m) &= -A(m) \\ N(m) &= -(0.002994m^2 - 0.3421m + 13.03) \\ &= -0.002994m^2 + 0.3421m - 13.03 \end{aligned}$$

VERTICAL REFLECTIONS

The graph of $g(x) = -f(x)$ is the graph of f reflected vertically about the horizontal axis.

Generalizing Transformations: Vertical Reflections

Returning to the idea that in a transformation we use the outputs of one function to define the outputs of a new function, we generalize vertical reflections as follows.

$$\underbrace{g(x)}_{\text{the output of } g \text{ at } x} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{-}_{\text{the opposite of}} \underbrace{f(x)}_{\text{the output of } f \text{ at } x}$$

EXAMPLE 2 ■ Graphing a Vertical Reflection

Given the graph of f in Figure 3.15, graph $g(x) = -f(x)$.

Solution As shown in Figure 3.16, to obtain the function g, we make all positive values of f negative and all negative values of f positive.

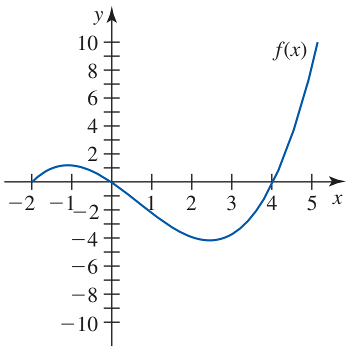


Figure 3.15

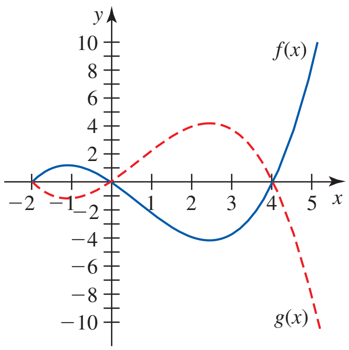


Figure 3.16

EXAMPLE 3 ■ Determining the Equation of a Vertically Reflected Graph

The graph of $f(x) = x^4 - 4x^2$ is shown in Figure 3.17 together with the graph of a function g. What is the function equation for g?

Solution From the graph, we see g is a vertical reflection of f. Therefore,

$$\begin{aligned} g(x) &= -f(x) \\ &= -(x^4 - 4x^2) \qquad \text{since } f(x) = x^4 - 4x^2 \\ &= -x^4 + 4x^2 \end{aligned}$$

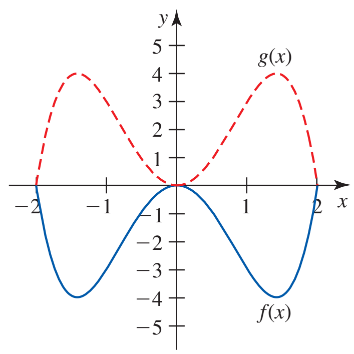


Figure 3.17

Horizontal Reflections

When a person travels underwater, the person's body bears the weight of the water above it. The pressure of the water on the person's body increases rapidly as she descends below the surface. For each 33 feet she descends underwater, the pressure on her body increases by 14.7 pounds per square inch (psi). (*Source:* www.onr.navy.mil)

EXAMPLE 4 ■ **Reflecting a Function Horizontally**

Based on the pressure information just given,

a. Write a function for P, the increased pressure on a human body d feet under the surface of the ocean. Then create a table of values and graph the function.

b. Let x represent a person's elevation (feet above sea level). Then graph a new function I to model the increased pressure at elevation x.

c. Use function notation to write function I in terms of function P.

Solution

a. Since the rate of change is constant, P will be a linear function. For each 33-foot increase in depth, the pressure increases by 14.7 pounds. The rate of change is given by

$$\frac{14.7 \text{ psi}}{33 \text{ feet}} \approx 0.45 \text{ psi per foot}$$

So $P(d) = 0.45d$. The table and graph of the model are shown in Table 3.10 and Figure 3.18.

Table 3.10

Depth (feet) d	Increased Pressure (psi)
20	9
40	18
60	27
80	36

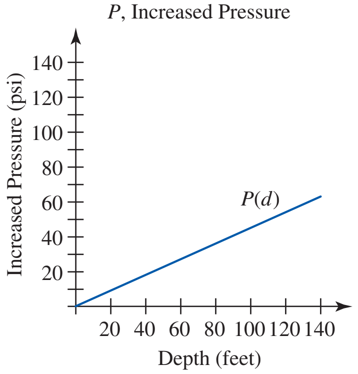


Figure 3.18

b. Because the surface of the ocean has an elevation of 0 feet, negative numbers represent elevations below sea level and positive numbers represent elevations above sea level. When the depth below the surface is d feet, the elevation is $x = -d$. Table 3.11 is a partial table of values.

If we plot the points from the table using elevation as the independent variable instead of depth, we get the graph in Figure 3.19.

Table 3.11

Depth (feet) d	Elevation (feet) $x = -d$	Increased Pressure (psi)
20	−20	9
40	−40	18
60	−60	27
80	−80	36

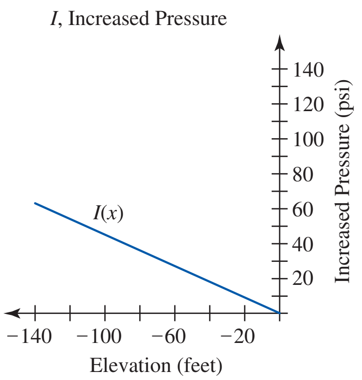


Figure 3.19

c. Figures 3.18 and 3.19 show that the basic relationship between the distance below the surface and the increased pressure is the same whether we determine it using depth or elevation. Because the pressure increases as *depth increases*, when we use depth as the independent variable the pressure change function has a positive rate of change. Because the pressure increases as the *elevation decreases*, when we use elevation as the independent variable the pressure change function has a negative rate of change. Thus, the new function I that models the increased pressure as a function of elevation is $I(x) = -0.45x$.

The key to writing the relationship between these functions is to see that $x = -d$ and $d = -x$. The same depth will be represented by numbers of the same magnitude but with opposite signs in each function. Thus

$$P(d) = P(-x) \quad \text{and} \quad I(x) = I(-d) \quad \text{where } x = -d$$

In addition, both functions will output the same values for increased pressure at the same depths as long as the appropriate input values are used. We can say that

$$P(d) = I(-d) \quad \text{and} \quad I(x) = P(-x)$$

In other words, using the opposite input value in each function will return the same output value.

To see this, consider the increased pressure 40 feet below the surface of the ocean. This is a d value of 40 or an x value of -40.

$$\begin{aligned} P(d) &= I(-d) \\ P(40) &= I(-40) \\ &= -0.45(-40) \\ &= 18 \text{ psi} \end{aligned} \qquad\qquad \begin{aligned} I(x) &= P(-x) \\ I(-40) &= P(-(-40)) \\ &= P(40) \\ &= 0.45(40) \\ &= 18 \text{ psi} \end{aligned}$$

The pressure at an elevation of -40 feet is identical to the pressure 40 feet below the surface of the ocean.

We can also use this relationship to verify the formula we wrote for $I(x)$.

$$\begin{aligned} I(x) &= P(-x) \\ &= 0.45(-x) \\ &= -0.45x \end{aligned}$$

We summarize our observations on horizontal reflections as follows.

Horizontal Reflections

The graph of $g(x) = f(-x)$ is the graph of f reflected horizontally about the vertical axis.

Generalizing Transformations: Horizontal Reflections

In general, we think of horizontal reflections as follows.

$$\underbrace{g(x)}_{\text{the output of } g \text{ at } x} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{f(-x)}_{\text{the output of } f \text{ at the opposite of } x}$$

EXAMPLE 5 ■ Reflecting a Graph Horizontally

Use the graph of f shown in Figure 3.20 to draw the graph of $g(x) = f(-x)$.

Solution As shown in Figure 3.21, the graph of g will be the graph of f reflected horizontally about the vertical axis. If the point (a, b) is on the graph of f, the point $(-a, b)$ will be on the graph of g.

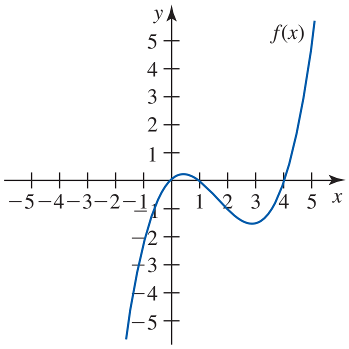


Figure 3.20

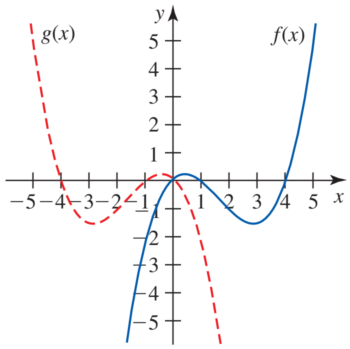


Figure 3.21

■ Combining Shifts and Reflections

Sometimes changes in a situation will require that we combine shifts and reflections to modify an original function to create our new model.

EXAMPLE 6 ■ Combining Shifts and Reflections

The value 14.7 psi is the pressure exerted by Earth's atmosphere on the human body at sea level. Recall that when a person descends 33 feet underwater, the pressure increases by 14.7 psi of pressure. Thus the pressure on a person 33 feet underwater is twice that of the pressure on a person at sea level.

a. Use a table and graph to model the total amount of pressure, T, exerted on a person underwater at elevation t.

b. In Example 4, we found the function showing increased pressure as a function of depth in feet, $P(d) = 0.45d$. How can this function be transformed to create the function in part (a)?

c. Use function notation to write the function in part (b) in terms of the original function P and the variable x.

d. Write the formula for $T(x)$.

Solution

a. The pressure at sea level is 14.7 psi, and the pressure increases by 14.7 psi for each additional 33 feet of underwater descent. Using the rate of pressure change we determined in Example 4, about 0.45 psi per foot, we can fill in values that are not multiples of 33 in Table 3.12. The graph is shown in Figure 3.22.

Table 3.12

Elevation (feet above sea level) x	Total Pressure (psi) $T(x)$
0	14.7
−10	14.7 + 0.45(10) = 19.2
−20	14.7 + 0.45(20) = 23.7
−33	14.7 + 14.7 = 29.4
−40	14.7 + 0.45(40) = 32.7
−50	14.7 + 0.45(50) = 37.2
−66	14.7 + 14.7(2) = 44.1
−99	14.7 + 14.7(3) = 58.8

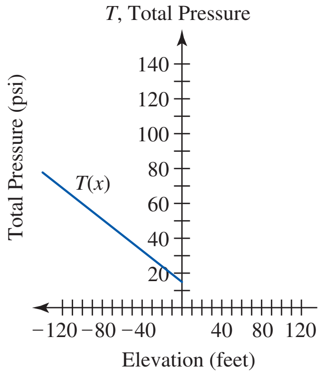


Figure 3.22

b. As shown in Figure 3.23, the function T is a result of two transformations from the original function P. First, T is a horizontal reflection of P since the input values are *elevation* instead of *depth*. The resultant function has also been shifted upward 14.7 units. This shift is required since the output values are for *total pressure* instead of *increased pressure* because the pressure at sea level is equal to 14.7 psi.

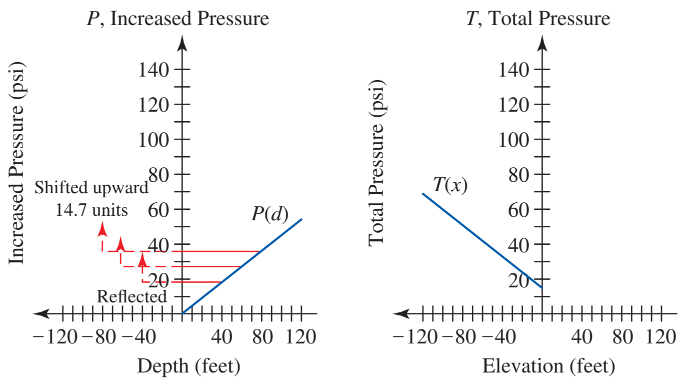


Figure 3.23

c. To write the function T in terms of P we need to incorporate the two transformations just described. Since $d = -x$, $P(d) = P(-x)$. The transformation $P(-x)$ reflects the graph of $P(x)$ horizontally. The total pressure is 14.7 psi greater than P, so $T(x)$ may be written as

$$T(x) = P(-x) + 14.7$$

d. From Example 4 we know that $P(d) = 0.45d$, so $P(-x) = 0.45(-x)$. Thus

$$\begin{aligned} T(x) &= P(-x) + 14.7 \\ &= 0.45(-x) + 14.7 \\ &= -0.45x + 14.7 \end{aligned}$$

EXAMPLE 7 ■ Identifying and Defining Multiple Transformations

The graph of $y = f(x)$ is given in Figure 3.24.

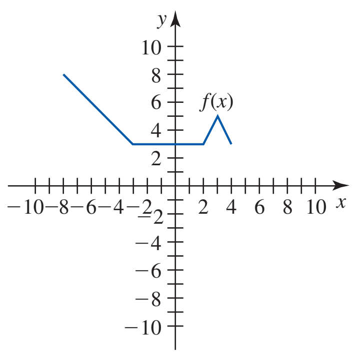
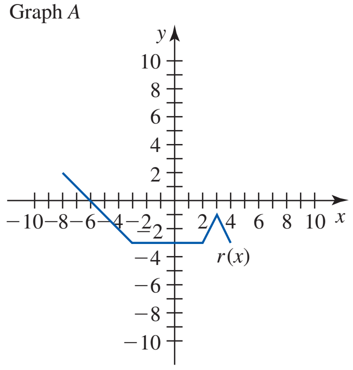


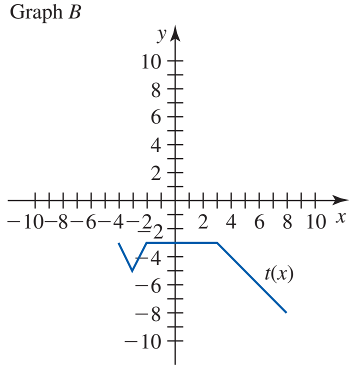
Figure 3.24

For each of the following functions, describe the transformation(s) used on the function $y = f(x)$ to create the new function, then use function notation to write each new function in terms of f.

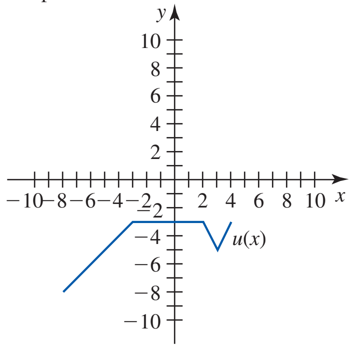
a. Graph *A*



b. Graph *B*



c. Graph *C*



Solution

a. In Graph *A*, the original function has been shifted downward 6 units. We express r in terms of f as

$$r(x) = f(x) - 6$$

b. In Graph *B*, the original function has been reflected in both the vertical and horizontal directions. We express t in terms of f as

$$t(x) = -f(-x)$$

c. In Graph *C*, the original function has been reflected vertically. We express u in terms of f as

$$u(x) = -f(x)$$

EXAMPLE 8 ■ Writing Formulas for Transformed Functions

The graph of $f(x) = x^2 - 3$ is shown in Figure 3.25.

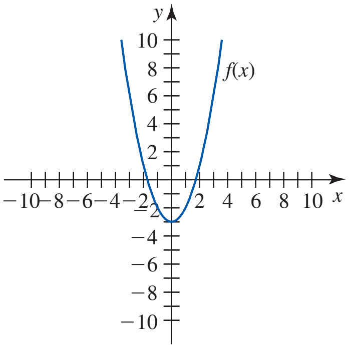
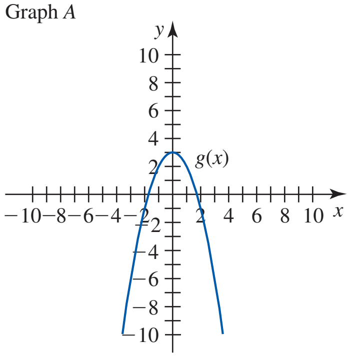


Figure 3.25

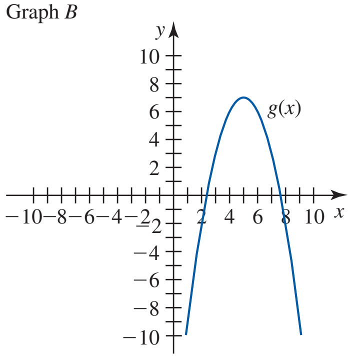
Match each of the following formulas to the appropriate transformed graph, then write the formula of the transformed function.

a. $g(x) = f(x + 4) - 3$ **b.** $g(x) = -f(x)$ **c.** $g(x) = -f(x - 5) + 4$

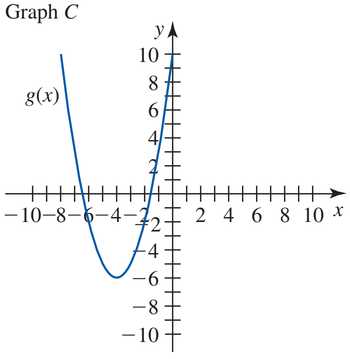
Graph *A*



Graph *B*



Graph *C*



Solution

a. $g(x) = f(x + 4) - 3$. This relationship shows that shifting the original function f to the left 4 units and downward 3 units will yield the graph of g, which is displayed in Graph C. The formula for $y = g(x)$ is

$$\begin{aligned} g(x) &= f(x + 4) - 3 \\ &= [(x + 4)^2 - 3] - 3 \qquad \text{since } f(x) = x^2 - 3 \\ &= [(x^2 + 8x + 16) - 3] - 3 \\ &= (x^2 + 8x + 13) - 3 \\ &= x^2 + 8x + 10 \end{aligned}$$

b. $g(x) = -f(x)$. From this relationship we can see g is the vertical reflection of the original function f. This is shown in Graph A. The formula for $y = g(x)$ is

$$\begin{aligned} g(x) &= -f(x) \\ &= -(x^2 - 3) \qquad \text{since } f(x) = x^2 - 3 \\ &= -x^2 + 3 \end{aligned}$$

c. $g(x) = -f(x - 5) + 4$. This relationship shows g to be the result of a vertical reflection of f that was then shifted to the right 5 units and upward 4 units, which we can see in Graph B. The formula for $y = g(x)$ is

$$\begin{aligned} g(x) &= -f(x - 5) + 4 \\ &= -[(x - 5)^2 - 3] + 4 \qquad \text{since } f(x) = x^2 - 3 \\ &= -[(x^2 - 10x + 25) - 3] + 4 \\ &= -(x^2 - 10x + 22) + 4 \\ &= -x^2 + 10x - 22 + 4 \\ &= -x^2 + 10x - 18 \end{aligned}$$

Symmetry

The concept of symmetry is closely related to the concept of reflections. Artists, architects, fashion designers, and many others use symmetry—a characteristic of objects that have two identical halves—to create products that appeal to our sense of beauty and interest. In fact, human beings are creatures of symmetry. Except for slight imperfections and the placement of some internal organs, our bodies are essentially symmetrical.

Functions can display symmetry as well. In mathematics we most often use symmetry to describe shapes or graphs that are unchanged after being reflected across a straight line, called a **line of symmetry**. Some examples of figures and graphs with lines of symmetry are shown in Figure 3.26.

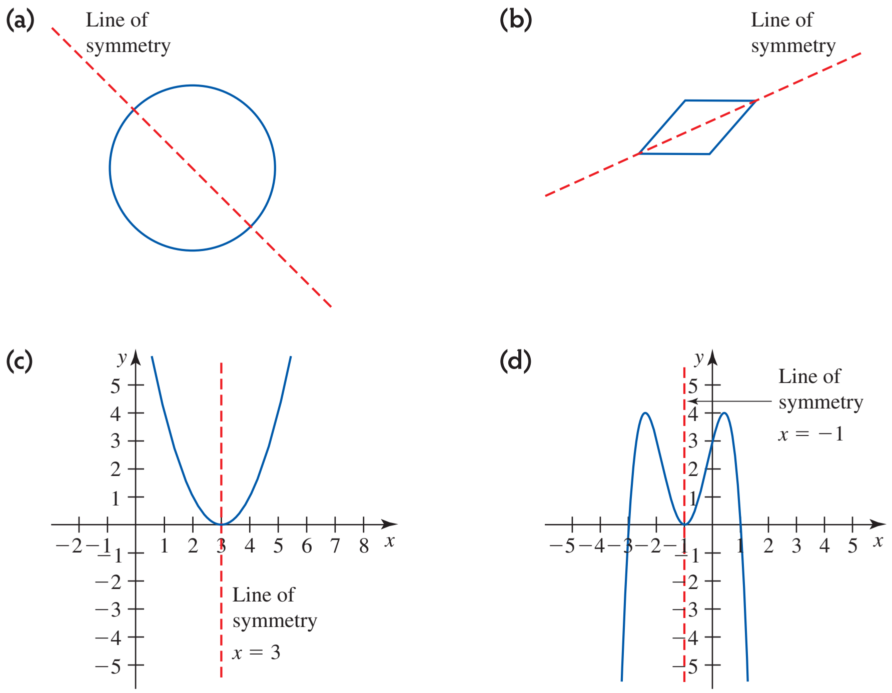


Figure 3.26

Functions that are symmetric with respect to the vertical axis are called **even functions** and those that are symmetric with respect to the origin are called **odd functions**.

Even Functions

A function f is an **even function** if $f(-x) = f(x)$. Even functions are symmetric with respect to the vertical axis, meaning the vertical axis is the line of symmetry.

Odd Functions

A function f is an **odd function** if $f(-x) = -f(x)$. Odd functions are symmetric with respect to the origin.

In practical terms, if reflecting a graph f about the vertical axis results in the same graph as that obtained by reflecting the graph f about the horizontal axis, then the function is an odd function.

EXAMPLE 9 ■ Testing for Even and Odd Symmetry

Determine if each of the following functions display even symmetry, odd symmetry, or neither.

a. $f(x) = \frac{1}{3}x$

b. $f(x) = 3x^2 + 3x$

c. $f(x) = x^4 - 2x^2$

Solution

a. First, let's examine the graph shown in Figure 3.27. It appears this function will have odd symmetry. If the function is reflected horizontally (Figure 3.28) or vertically (Figure 3.29) it will yield the same function.

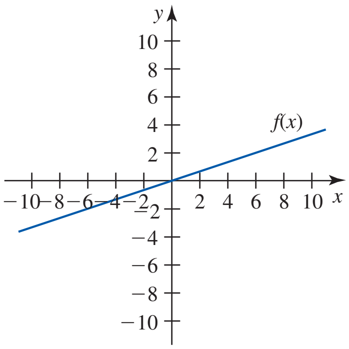


Figure 3.27

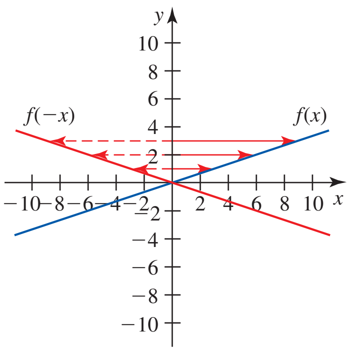


Figure 3.28

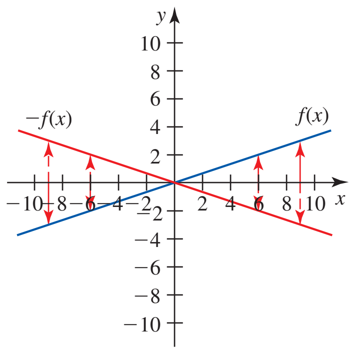


Figure 3.29

We can verify this by finding the formula for the function $f(-x)$ and showing that this is the same as the function for $-f(x)$:

$$f(x) = \frac{1}{3}x \qquad\qquad f(x) = \frac{1}{3}x$$

$$f(-x) = \frac{1}{3}(-x) \qquad -f(x) = -\left(\frac{1}{3}x\right)$$

$$= -\frac{1}{3}x \qquad\qquad = -\frac{1}{3}x$$

$$f(-x) = -f(x)$$

Therefore, this function has odd symmetry (is symmetrical with respect to the origin).

b. First, let's examine the graph shown in Figure 3.30. The function does not have a line of symmetry at $x = 0$, so it will not have even symmetry. Furthermore, it will not generate the same function when reflected vertically (Figure 3.31) and horizontally (Figure 3.32) so it does not have odd symmetry.

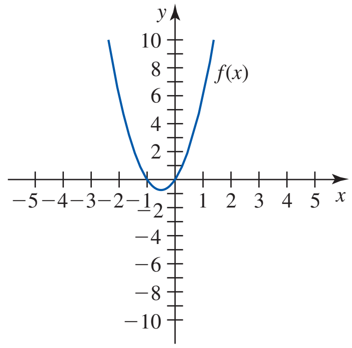


Figure 3.30

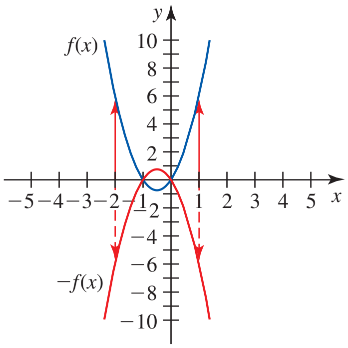


Figure 3.31

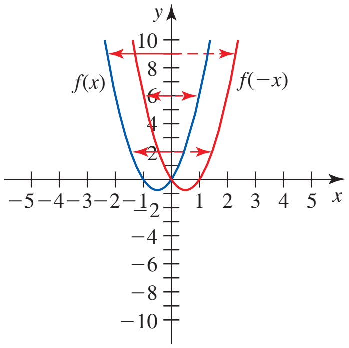


Figure 3.32

We can verify this observation algebraically as well.

$$\begin{aligned} f(x) &= 3x^2 + 3x \\ f(-x) &= 3(-x)^2 + 3(-x) \\ &= 3(x^2) - 3x \\ &= 3x^2 - 3x \end{aligned} \qquad \begin{aligned} f(x) &= 3x^2 + 3x \\ -f(x) &= -(3x^2 + 3x) \\ &= -3x^2 - 3x \end{aligned}$$

Since $f(x) \neq f(-x)$, the function does not have even symmetry. Also, since $f(-x) \neq -f(x)$, the function does not have odd symmetry.

c. Again, let's begin by looking at the graph in Figure 3.33. This function appears to have even symmetry. We can verify this by comparing $f(x)$ and $f(-x)$.

$$\begin{aligned} f(x) &= x^4 - 2x^2 \\ f(-x) &= (-x)^4 - 2(-x)^2 \\ &= x^4 - 2(x^2) \\ &= x^4 - 2x^2 \end{aligned}$$

$$f(x) = f(-x)$$

This function has even symmetry (is symmetrical with respect to the vertical axis).

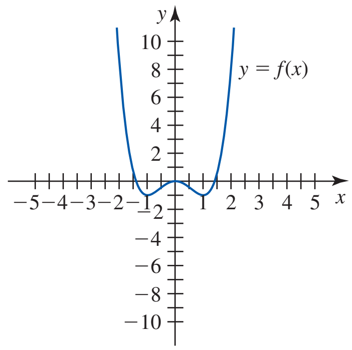


Figure 3.33

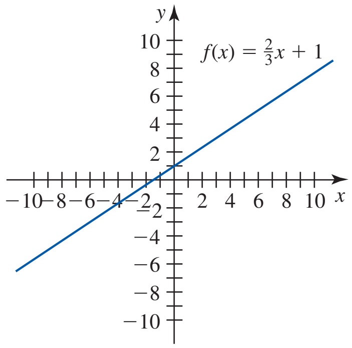
SUMMARY

In this section you learned how to transform functions with vertical and horizontal reflections. You also discovered how to determine if a function has even or odd symmetry.

3.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, graph each transformation of f and write the formula for the transformed function in the form $y = mx + b$.



1. $-f(x)$
2. $f(-x)$
3. $-f(x - 3)$
4. $f(-x) - 6$

In Exercises 5–10, use the values shown in Table B to evaluate each expression.

Table B

x	$f(x)$
−8	26
−6	20
−4	13
−2	5
0	−4
2	−14
4	−25
6	−37
8	−50

5. $-f(x)$ when $x = -6$
6. $f(-x)$ when $x = 2$
7. $f(-x - 2) + 1$ when $x = 4$
8. $-f(-x) - 3.3$ when $x = -6$
9. $-f(-2) + f(6)$
10. $f(-8) - 3f(2)$

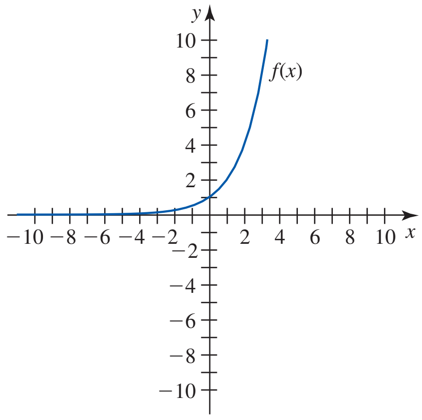
In Exercises 11–14, solve each equation for x using Table B.

11. $f(-x) = 13$
12. $-f(x) = 25$
13. $-f(x + 6) - 12 = 13$
14. $f(-x + 4) + 7 = 3$

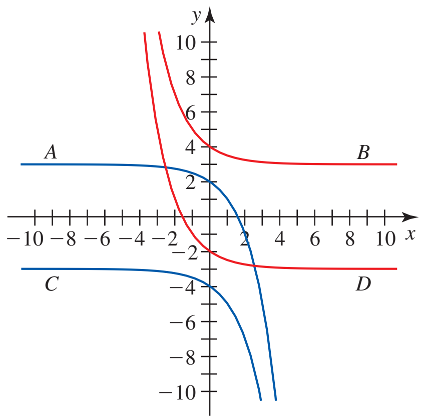
In Exercises 15–16, refer to Table B to answer each question.

15. Create a table of values for j if $j(x) = -f(-x)$.
16. Create a table of values for k if $k(x) = -f(x + 6) - 4$.

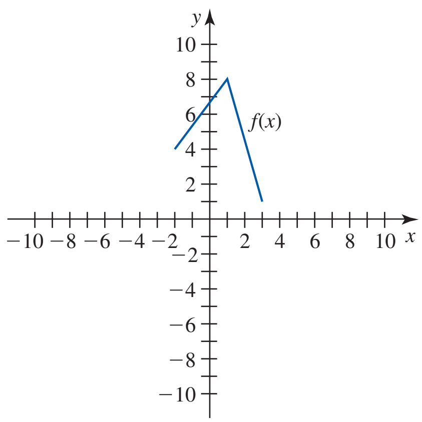
In Exercises 17–20, match the formula for the transformation of $f(x) = 2^x$ with its Graph A, B, C, or D. Do not use a calculator.



17. $g(x) = 2^{-x} + 3$
18. $h(x) = 2^{-x} - 3$
19. $j(x) = -2^x + 3$
20. $k(x) = -2^x - 3$

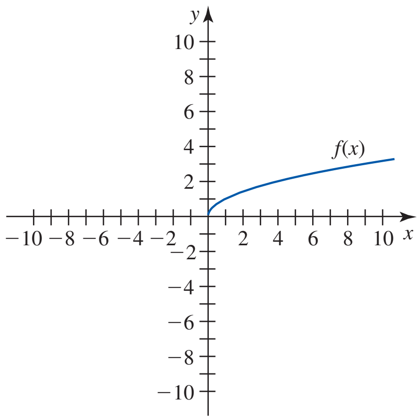


In Exercises 21–24, draw the graph of each function as a transformation of f.



21. $A(x) = -f(x)$
22. $C(x) = f(-x) + 3$
23. $D(x) = -f(x - 5)$
24. $E(x) = -f(-x) + 1$

In Exercises 25–28, you are given the graph of a transformation of $f(x) = \sqrt{x}$.



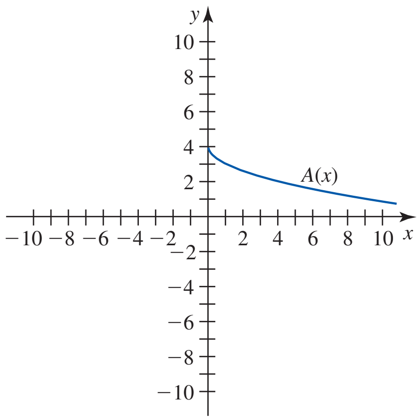
For each exercise,

a. Describe the transformations required on f to create the new function.

b. Use function notation to write each new function in terms of f.

c. Write the formula for the new function.

25.



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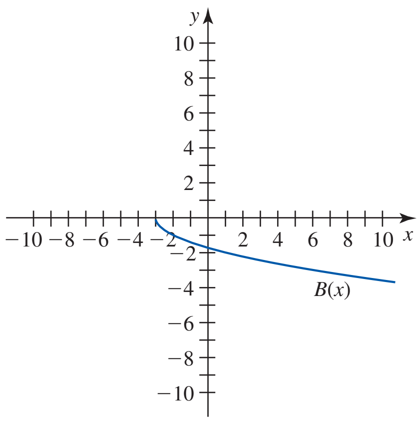
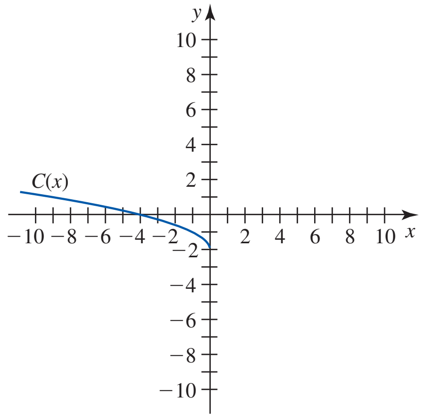


27.



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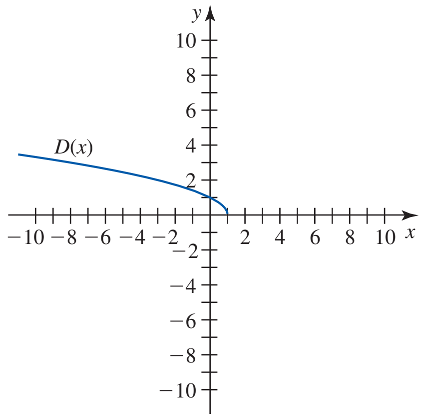


In Exercises 29–31, use the table to identify the transformation(s) on f required to create the indicated function. Then write a formula for each transformed function in terms of f.

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
−4	0	3	0	3
−3	3	1	2	1
−2	5	0	1	2
−1	6	−3	−1	4
0	4	−5	−4	7
1	1	−6	−6	9
2	−1	−4	−5	8
3	−2	−1	−3	6
4	0	1	0	3

29. $g(x)$

30. $h(x)$

31. $j(x)$

For Exercises 32–35,

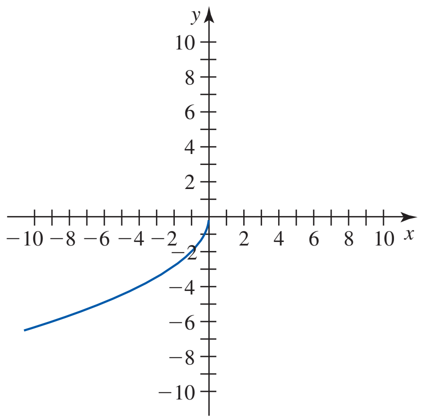
a. Complete the graph to make it an odd function.

b. Complete the graph to make it an even function.

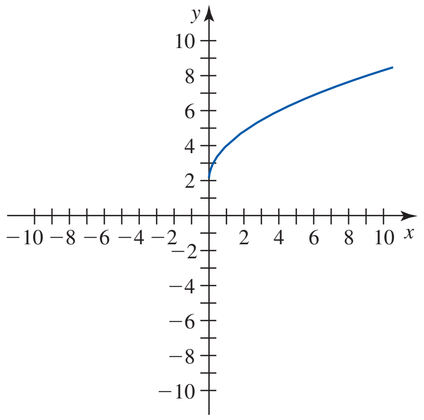
c. Complete the graph to make it neither even nor odd.

Use only continuous functions. If it is not possible to create a graph of the indicated type, say not possible *and explain why.*

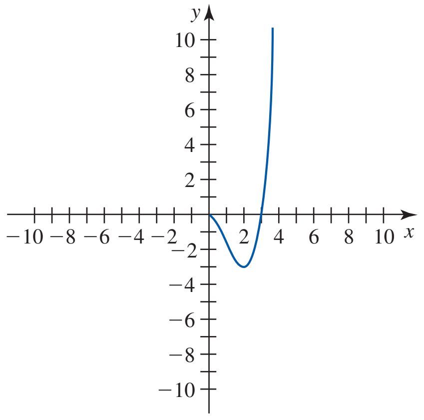
32.



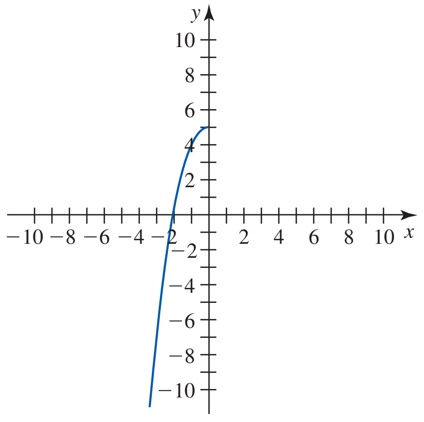
33.



34.



35.



In Exercises 36–40, determine whether each of the following functions is even, odd, or neither.

36. $a(x) = \dfrac{1}{x^2}$

37. $b(x) = \sqrt[3]{x}$

38. $c(x) = 4x^3 - x$

39. $d(x) = x^2 - 2x + 1$

40. Create a table that shows a function with

a. Even symmetry.

b. Odd symmetry.

c. No symmetry.

SHOW YOU KNOW

41. Refer to your work on Exercises 32–35. What appears to be a condition for a function to have odd symmetry? Use the definition of odd function to explain why this condition must exist, then explain how to use this fact to quickly identify functions that could not be odd.

42. How do each of the following transformations affect the rate of change, m, and vertical intercept, b, of a linear function $y = mx + b$?

a. Horizontal shift

b. Vertical shift

c. Horizontal reflection

d. Vertical reflection

43. Given the function $f(x) = mx + b$, determine what values of m and b are required for the function to have the indicated symmetry. If it is not possible for the function to have the indicated symmetry, say *not possible* and explain why.

a. Odd

b. Even

c. Both odd and even

d. Neither odd nor even

44. Function f is even and function g is odd. Which transformations have the ability to change these symmetries and which do not?

45. Given $g(x) = f(-x)$, we know that g will be a horizontal reflection of f. Explain why this change to the function causes this type of transformation.

46. Explain why there are no functions (other than $y = 0$) that possess symmetry with respect to the horizontal axis.

MAKE IT REAL

47. Mortgage Payments The table shows mortgage payments as a function of the total amount borrowed on a 30-year fixed loan at 7% annual interest. (Note: We do not factor in additional costs such as taxes, mortgage insurance, etc.)

Amount Borrowed ($ thousands) b	**Monthly Payment (dollars)** $P(b)$
100	665.30
200	1330.60
225	1496.93
280	1862.85
330	2195.50

a. Is this a linear relationship? Defend your answer.

b. Write a formula for this situation.

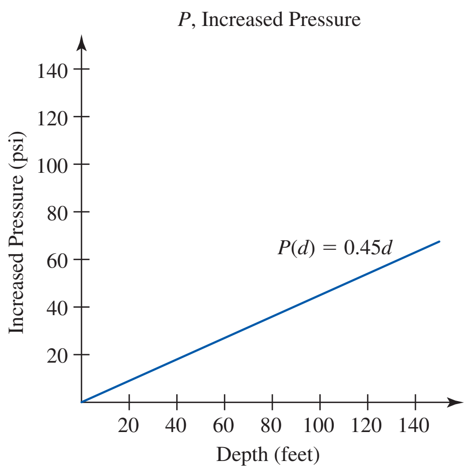
c. Reflect your function horizontally. Draw and label the new function and explain how you could interpret your new graph.

d. Reflect your function vertically. Draw and label the new function and explain how you could interpret your new graph.

e. Would a horizontal shift have any practical meaning in this situation? Explain.

f. Would a vertical shift have any practical meaning in this situation? Explain.

For Exercises 48–49, use the graph of P, which shows the increased pressure in psi as a function of the underwater depth in feet.



48. Pressure Underwater Maroon Lake, located in Colorado's Rocky Mountains, is at an elevation of about 2920 m (9580 ft). At this elevation the atmospheric pressure is about 10.3 psi.

a. Create a table, graph, and formula for L, the total pressure on the human body r feet below the surface of Maroon Lake.

b. Explain how to transform the function P to create L.

49. Pressure Underwater Many experts recommend that recreational divers avoid underwater depths of greater than 100 feet.

a. Construct a table and graph showing B, the increased pressure on the human body z feet less than the maximum recommended safe diving depth.

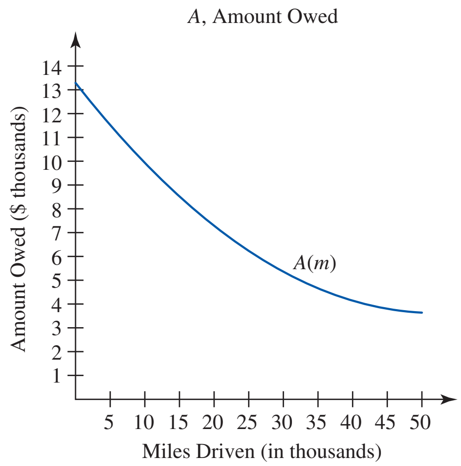
b. Explain the transformations on the function P that we use to create B.

c. Use an equation to show the relationship between d and z.

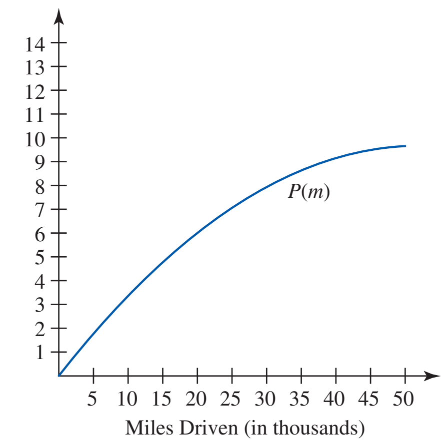
d. Use function notation to write B in terms of P.

50. Car Loans For this exercise, recall the function A, which modeled the amount owed on the author's car loan in thousands of dollars after driving m thousand miles:

$$A(m) = 0.003569m^2 - 0.3715m + 13.29$$



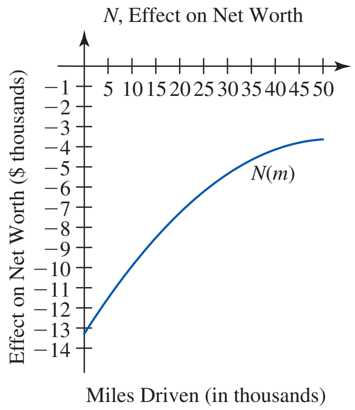
a. Explain the transformations on A that can be used to create function P.



b. What does the new function represent and how should we label the vertical axis?

c. Write the formula for the function P.

d. P can also be thought of as a transformation of N, the effect on the author's net worth in thousands of dollars after driving m thousand miles.



Use function notation to write N in terms of P.

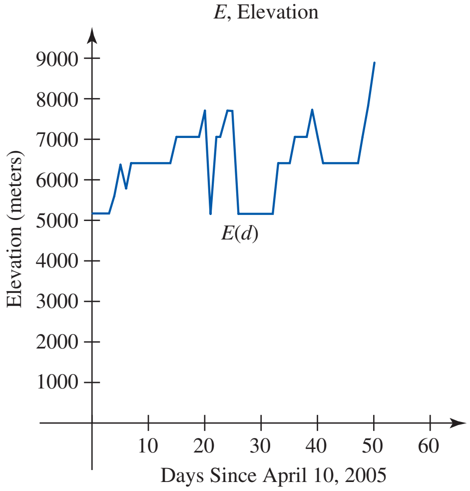
51. Atmospheric Temperatures As altitude increases, temperature tends to decrease according to the model $T(A) = -6.505A + 15.01$, where T is the temperature in degrees Celsius and A is the altitude above sea level in kilometers.

a. Construct the table of values and graph for R, the difference between room temperature (about 22°C) and the actual temperature at a specific elevation.

b. Describe how the function T can be transformed to generate the graph of R.

c. Use function notation to write R in terms of T, then write the formula for R.

52. Mount Everest For this exercise, use the graph of E, the climbers' elevation d days after April 10, 2005.



a. Construct the graph for S, the difference in elevation between the summit and the climbers' elevation d days after April 10.

b. Describe how the function E, which modeled the climbers' elevation d days after April 10, can be transformed to generate the graph of S.

c. Use function notation to write S in terms of E.

■ STRETCH YOUR MIND

Exercises 53–56 are intended to challenge your understanding of reflections and symmetry.

53. Is it possible for a function to have "translational symmetry," meaning that a horizontal or vertical shift of the function could yield an identical function? If so, give an example. If not, explain why it is not possible.

54. Explain any differences and similarities in the following functions: $p(x) = -f(x) + k$, $q(x) = -[f(x) + k]$, and $r(x) = -f(x) - k$.

55. Is it possible for transformations of a function to create a relationship that is no longer a function? Explain.

56. Create the graphs of a function and its inverse. Explain how and why horizontal and vertical reflections in the original function affect the inverse function.

SECTION 3.3 Vertical Stretches and Compressions

LEARNING OBJECTIVES

- Identify what change in a function equation results in a vertical stretch
- Identify what change in a function equation results in a vertical compression

GETTING STARTED

According to www.caloriecontrol.org, about 65% of adults in the United States are overweight. Extra weight can increase the risk of heart disease and diabetes. Overweight Americans spend billions of dollars a year on diet books and diet programs, yet most diet counselors give the same advice: to lose weight you must burn more calories than you consume. A 3500-calorie difference between calories consumed and calories burned will result in 1 pound of weight lost. If you decide to burn calories by running, how far would you have to run to burn 200 extra calories each day? We will answer questions such as these in this section.

In this section we expand our understanding of function transformations by introducing vertical and horizontal stretches and compressions. We also see how such transformations can be used to model real-world problems such as exercise plans for weight loss.

■ Vertical Stretches and Compressions

The number of calories you burn depends on a number of factors including the type of exercise you do, how strenuously you exercise, and your body weight. Table 3.13 shows the average number of calories a 155-pound person burns while running.

Table 3.13

Distance (miles) d	Calories Burned (155-pound person) $C(d)$
5	563
6	704
7	809
8	950
9	1056
10	1126

Source: www.nutristrategy.com

EXAMPLE 1 ■ Stretching and Compressing a Function Vertically

Using Table 3.13,

a. Use linear regression to model C, the calories burned by a 155-pound person who runs d miles. Then explain what the slope of the model represents.

b. A 190-pound person will burn about 22.6% more calories per mile than a 155-pound person. If H represents the calories burned for a 190-pound person running d miles, create a table, formula, and graph for H and use function notation to write H in terms of C. (Note: A 22.6% increase means the number of calories burned is larger by a factor of 1.226.)

c. A 130-pound person will burn about 83.8% as many calories per mile as a 155-pound person. If L represents the calories burned for a 130-pound person running d miles, create a table, formula, and graph for L and use function notation to write L in terms of C. (Note: If a number is 83.8% of another number, then it is 0.838 times as large.)

Solution

a. Using linear regression on our graphing calculator, we find $C(d) = 114.63d + 8.2857$, which is graphed in Figure 3.34. The slope is 114.63 calories per mile. That is, for each mile a 155-pound person runs, the total calories burned increases by almost 115 calories.

b. To represent "22.6% more calories," we need to multiply the output values from C by a factor of 1.226, as shown in Table 3.14. The points are plotted in Figure 3.35.

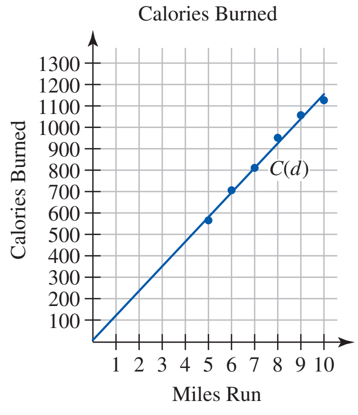


Figure 3.34

Table 3.14

Distance (miles) d	Calories Burned (155-pound person) $C(d)$	Calories Burned (190-pound person) $H(d)$
5	563	$1.226(563) \approx 690$
6	704	$1.226(704) \approx 863$
7	809	$1.226(809) \approx 992$
8	950	$1.226(950) \approx 1165$
9	1056	$1.226(1056) \approx 1295$
10	1126	$1.226(1126) \approx 1380$

The table shows that the outputs for each value of d are greater for H than they are for C. However, as indicated on Figure 3.35, the difference between each output is not constant: as the output values in C get larger, the amount of increase to the next output value of H also gets larger. The outputs do not change by a constant *amount* (as in a vertical shift) but are related by a constant *factor*: each output for H is 1.226 times the corresponding output for C. This is very clear when we compare the graphs. The calculations used to fill in the table for $H(d)$ yield the following relationship:

$$H(d) = 1.226C(d)$$

Now we can use this relationship to find the formula for H and graph the function, as shown in Figure 3.36.

$$\begin{aligned} H(d) &= 1.226C(d) \\ &= 1.226(114.63d + 8.2857) \\ &= 140.5d + 10.16 \end{aligned}$$

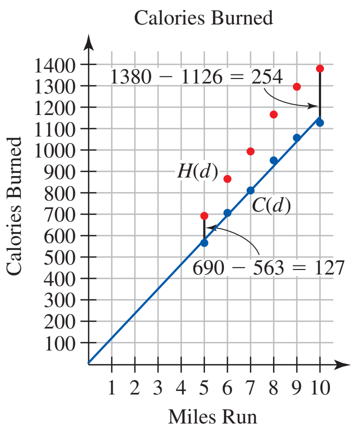


Figure 3.35

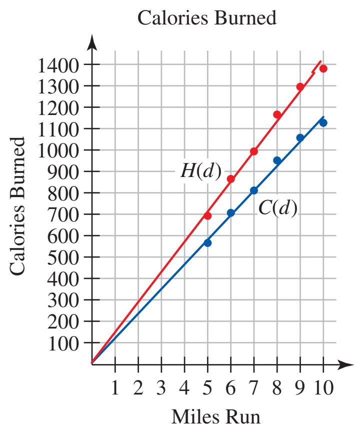


Figure 3.36

The difference in the slopes of the two functions solidifies our earlier point. The 190-pound person is burning about 22.6% more calories per hour than the 155-pound person, and thus the difference between the total calories they burn will continue to increase as the distance run increases.

c. We can model the calories burned by a 130-pound person (83.8% of those burned by a 155-pound person) by multiplying all of the outputs of C by 0.838, as shown in Table 3.15.

Table 3.15

Distance (miles) d	Calories Burned (155-pound person) $C(d)$	Calories Burned (130-pound person) $L(d)$
5	563	$0.838(563) \approx 472$
6	704	$0.838(704) \approx 590$
7	809	$0.838(809) \approx 678$
8	950	$0.838(950) \approx 796$
9	1056	$0.838(1056) \approx 885$
10	1126	$0.838(1126) \approx 944$

Again, although the difference between the outputs is not constant, the values of L and C are related: each output of L is 0.838 times the corresponding output in C. Table 3.15 yields the following relationship between L and C:

$$L(d) = 0.838C(d)$$

Now we can use this relationship to find the formula for L and graph the function, as shown in Figure 3.37.

$$\begin{aligned} L(d) &= 0.838\ C(d) \\ &= 0.838(114.63d + 8.2857) \\ &= 96.06d + 6.943 \end{aligned}$$

In Figure 3.37 we can again see how the percentage difference in the number of calories burned per hour yields different slopes in the two functions. This means the difference between the calories burned will continue to grow larger as the number of miles run increases.

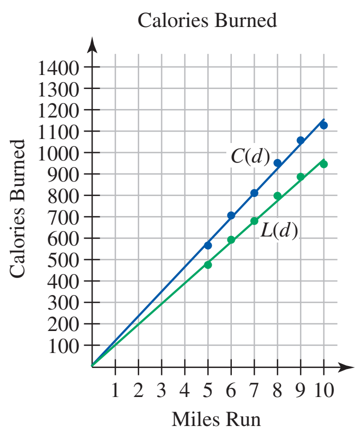


Figure 3.37

The transformations in Example 1 represent a *vertical stretch* (part b) and a *vertical compression* (part c). This type of transformation occurs any time the outputs of two functions differ by a constant *factor*. When the magnitude (absolute value) of the constant factor is greater than 1 as in part (b), the result is a **vertical stretch**. We call it a stretch because the graph appears distorted as if someone physically stretched the graph to be farther from the horizontal axis. When the magnitude (absolute value) of the constant factor is less than 1 as in part (c), the result is a **vertical compression**. We call it a compression because the graph appears distorted as if someone physically pushed the graph closer to the horizontal axis. We summarize our observations of vertical stretches and compressions as follows.

Vertical Stretches and Compressions

The graph of $g(x) = af(x)$ is the graph of f stretched or compressed vertically by a factor of $|a|$ units. If $|a| > 1$, the transformation is a **vertical stretch**. If $0 < |a| < 1$, the transformation is a **vertical compression**.

Generalizing Transformations: Vertical Stretches and Compressions

As in previous sections, we can generalize vertical stretches and compressions by remembering that transformations use the outputs of the parent function to define the outputs of the image function. In the case of vertical stretches and compressions, we multiply the outputs of the parent function by some constant factor for each input to define the image function. For example, consider $g(x) = 4f(x)$ and $g(x) = \frac{1}{3}f(x)$.

$$\underbrace{g(x)}_{\text{the output of } g \text{ at } x} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{4}_{\text{four times}} \quad \underbrace{f(x)}_{\text{the output of } f \text{ at } x}$$

$$\underbrace{h(x)}_{\text{the output of } h \text{ at } x} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{\frac{1}{3}}_{\text{one-third}} \quad \underbrace{f(x)}_{\text{the output of } f \text{ at } x}$$

EXAMPLE 2 ■ Interpreting and Graphing Function Transformations

Determine the relationship between the function $f(x) = x^3 - 3x^2$ and $g(x) = 0.5(x^3 - 3x^2)$. Then graph both functions.

Solution We observe $g(x) = 0.5f(x)$. That is, the image function g has all of the same input values as the parent function f, but the output values of g are one-half (0.5) as big as those of f. Thus the graph of g is the graph of f compressed vertically by a factor of 0.5. The graphs of both functions are shown in Figure 3.38.

Notice that $(1, -2)$ on f becomes $(1, -1)$ on g, $(2, -4)$ on f becomes $(2, -2)$ on g, and so on. The horizontal intercepts of each function are the same because half of 0 is still 0.

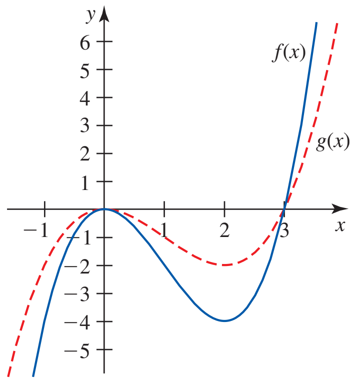


Figure 3.38

EXAMPLE 3 ■ Determining the Effect of Stretches and Compressions on Rates of Change

The British rail system, called BritRail, has a special offer for tourists called a BritRail Pass that provides unlimited train travel for a specified number of days. The pass must be purchased prior to arriving in the country, and the price is the same regardless of how many days it is actually used (up to the maximum).

In 2004, one of the authors went to Scotland on his honeymoon and traveled in the country via passenger train. The BritRail Freedom of Scotland Pass, which cost \$189 in 2004, gave the author unlimited travel on trains in Scotland for up to 8 days. The same pass was priced at \$292 in 2007. (*Source:* www.acprailnet.com)

a. Create a formula and graph for the average cost per day of each pass given that the pass is used for exactly u days.

b. Write the 2007 average daily cost as a function of the 2004 average daily cost. Then explain how the graphs of the two functions are related.

c. Write the 2004 average daily cost as a function of the 2007 average daily cost. Then explain how the graphs of the two functions are related.

Solution

a. We obtain the average cost per day by dividing the ticket cost by the number of days the ticket is used, u. Thus we have

$$C(u) = \frac{189}{u} \quad \text{and} \quad T(u) = \frac{292}{u}$$

where $C(u)$ is the average daily cost of the \$189 pass (in dollars per day) and $T(u)$ is the average daily cost of the \$292 pass (in dollars per day).

Since the passes are valid for 8 days, we graph the functions on the interval [0, 8]. Furthermore, since the passes may only be used for whole numbers of days, we plot only the points that correspond with whole days. See Figure 3.39. This type of graph is called a *discrete* graph since it is defined for a finite number of values.

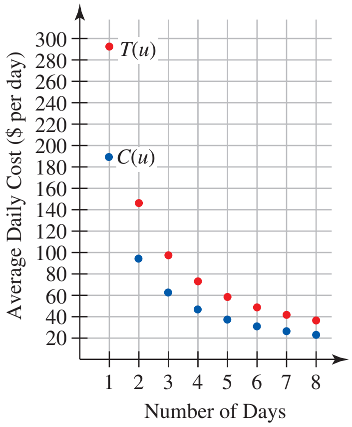


Figure 3.39

b. From part (a), we know $C(u) = \frac{189}{u}$ gives the 2004 average cost and $T(u) = \frac{292}{u}$ gives the 2007 average cost. Since both functions are a multiple of $\frac{1}{u}$, we can write one function in terms of the other.

$$\begin{aligned} T(u) &= \frac{292}{u} \cdot \frac{189}{189} && \text{Multiply by 1.} \\ &= \frac{292}{189} \cdot \frac{189}{u} && \text{Regroup terms.} \\ &= \frac{292}{189}(C(u)) && \text{since } C(u) = \frac{189}{u} \\ &\approx 1.54C(u) \end{aligned}$$

The graph of T is the graph of C *stretched* vertically by a factor of 1.54. That is, the output values of T are 1.54 times as big as the output values of C for any value of u.

c. From part (a), we know $C(u) = \frac{189}{u}$ gives the 2004 average cost and $T(u) = \frac{292}{u}$ gives the 2007 average cost. Since both functions are a multiple of $\frac{1}{u}$, we can write one function in terms of the other.

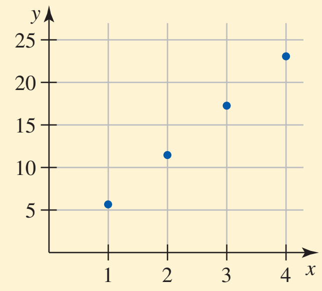
$$\begin{aligned} C(u) &= \frac{189}{u} \cdot \frac{292}{292} && \text{Multiply by 1.} \\ &= \frac{189}{292} \cdot \frac{292}{u} && \text{Regroup terms.} \\ &= \frac{189}{292}(T(u)) && \text{since } T(u) = \frac{292}{u} \\ &\approx 0.65T(u) \end{aligned}$$

The graph of C is the graph of T *compressed* vertically by a factor of 0.65. The output values of C are 0.65 times as big as the output values of T for any value of u.

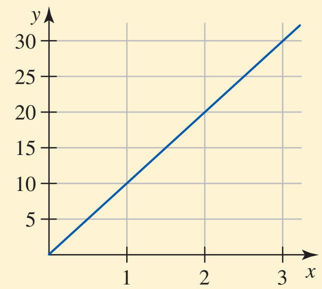
JUST IN TIME ■ DISCRETE VERSUS CONTINUOUS FUNCTIONS

Functions may be classified as either *discrete* or *continuous*.

A **discretely defined function** has a finite number of input values. For example, if a movie costs \$5.75 per person, then the linear function $P(x) = 5.75x$ models the total cost for x people to attend the movie. This function is defined only for nonnegative integer values of x (such as 0, 1, 2, 3, 4, etc.). The graph is a series of disconnected points.



A **continuously defined function** is defined for every value within the interval of its domain. For example, if $d(t)$ models the depth of water in a bathtub t minutes after turning the water on and if it takes 3 minutes to completely fill the tub, we can evaluate $d(t)$ for any value of t between 0 and 3. All values of t (such as 1.3, 2.0459, 2.9999, and so on) exist and provide us with a meaningful output.



■ Using Stretches and Compressions to Change Units

To make numbers less cumbersome, we often change the units used in a data set. For example, consider Table 3.16, which shows the same relationship between values using different units.

Table 3.16

Year t	Total Consumer Debt (in billions of dollars) $C(t)$	Total Consumer Debt (in dollars) $D(t)$
1985	593.00	593,000,000,000
1990	789.10	789,100,000,000
1995	1095.80	1,095,800,000,000
2000	1556.25	1,556,250,000,000
2005	2175.25	2,175,250,000,000

Source: Federal Reserve Board

The numbers under *billions of dollars* are much easier to work with. Changing output units in this way is an example of a *vertical compression* or *stretch* since, for the same input value, the output values differ by a constant factor. We can show the relationship between D and C as follows:

$$D(t) = 1{,}000{,}000{,}000C(t)$$

or

$$C(t) = 0.000000001D(t)$$

■ Combining Transformations

We can use vertical stretches and compressions with other types of transformations. We just need to reason through the transformation by considering the relative inputs and outputs of the related functions. In doing so, we will follow the order of operations.

EXAMPLE 4 ■ Using Multiple Transformations to Change a Function

The graph of g is the graph of f reflected vertically, stretched vertically by a factor of 2, and shifted downward 3 units. Write the equation of g in terms of f.

Solution The equation is $g(x) = -2f(x) - 3$. The -2 stretches the graph of f vertically by a factor of 2 and reflects the resultant graph vertically. The -3 reduces the resultant output values by 3 units, resulting in a vertical shift downward by 3 units.

EXAMPLE 5 ■ Combining Transformations

Explain the relationship between the graph of g and the graph of f given that the function $f(x) = \sqrt{x}$ and the function $g(x) = 3f(x + 2)$. Then graph both functions to verify the result.

Solution Since $x + 2$ is larger than x, the related outputs occur at larger inputs in g than they do in f, telling us that g is 2 units to the left of f (see the green curve in Figure 3.40). The outputs we get from f are then multiplied by 3 to get the final output for g (the red curve). Therefore, the graph of g is the graph of f shifted left by 2 units and then stretched vertically by a factor of 3.

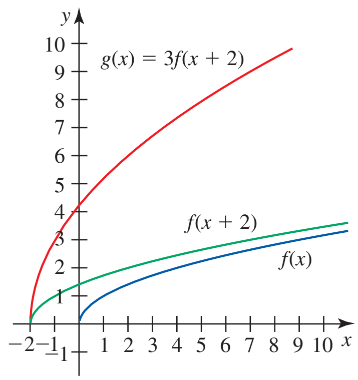


Figure 3.40

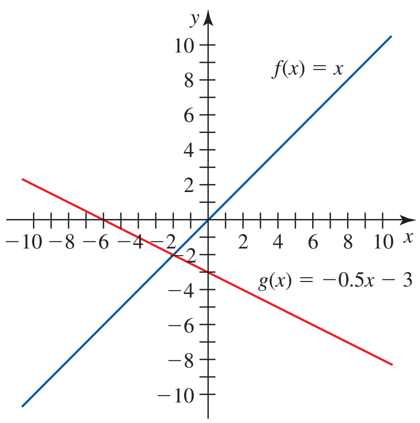
SUMMARY

In this section you learned how to transform functions using vertical stretches and compressions. You also learned how to use vertical stretches and compressions in modeling real-world data sets.

3.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–6, graph each transformation of f or g and find the formula for the transformed function.



1. $y = 4f(x) + 1$
2. $y = \frac{3}{4}f(-x)$
3. $y = 3g(x + 2)$
4. $y = -\frac{2}{3}g(x)$
5. $y = -0.5f(x - 5)$
6. $y = 6g(-x) - 4.5$

In Exercises 7–10, explain the transformations on f required to create g. It is not necessary to graph the functions.

7. $f(x) = x^3, \quad g(x) = 8(x - 2)^3$
8. $f(x) = x, \quad g(x) = \frac{4}{3}x + 3$
9. $f(x) = 3x^5, \quad g(x) = 3(-x)^5$
10. $f(x) = (x + 5)^3, \quad g(x) = -\frac{1}{7}(x + 5)^3$

In Exercises 11–16, use the values shown in Table C to evaluate each expression.

Table C

x	$f(x)$
−4	7
−3	13
−2	29
−1	20
0	6
1	−1
2	−4
3	−13
4	−28

11. $-\frac{2}{3}f(x)$ when $x = -3$

12. $4f(x-2)$ when $x = 1$

13. $-3f(-x) + 7$ when $x = -2$

14. $3.5f(-x) - 2.1$ when $x = 3$

15. $4f(-3) + \frac{1}{2}f(0)$

16. $1.7f(4) - 3.25f(0)$

In Exercises 17–20, solve each equation for x using Table C.

17. $-2f(x) = -40$

18. $-\frac{1}{4}f(x) - 6 = 1$

19. $5f(x-2) + 3 = -62$

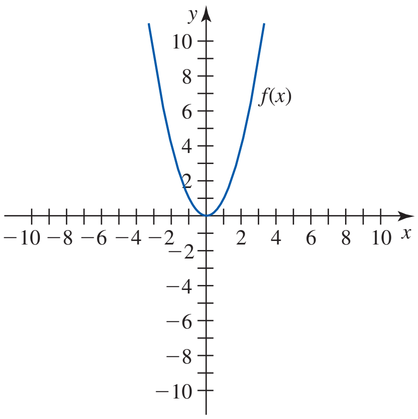
20. $\frac{1}{2}f(x-4) + 10 = 13.5$

In Exercises 21–22, refer to Table C to answer each question.

21. Create a table of values for h if $h(x) = -\frac{1}{2}f(x)$.

22. Create a table of values for j if $j(x) = 2f(-x)$.

For Exercises 23–26, draw the graph of each transformation of the function $f(x) = x^2$, then write the formula for the transformed function.



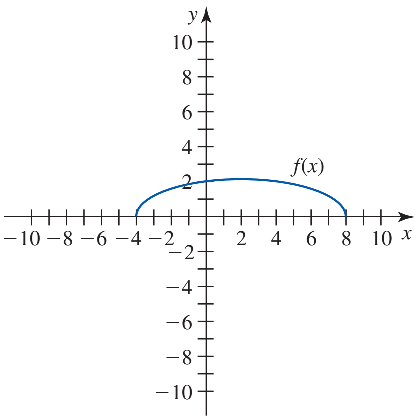
23. $g(x) = -f(x) + 1$

24. $h(x) = 3f(x+1)$

25. $j(x) = \frac{1}{3}f(x-2) + 5$

26. $k(x) = -\frac{3}{4}f(x) - 5$

For Exercises 27–30, draw the graph of each transformation of the given function.



27. $g(x) = f(-x) - 3$

28. $h(x) = \frac{1}{4}f(x-2) + 1$

29. $j(x) = -6f(-x) - 2$

30. $k(x) = \frac{5}{3}f(x-3) + 4$

In Exercises 31–33, use the table to identify the transformation(s) on f required to create the indicated function. Then use function notation to write each transformed function in terms of f.

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
−1	−12	−48	−6	36
0	−2	−8	−1	6
1	5.5	22	2.75	−16.5
2	4	16	2	−12
3	−1.5	−6	−0.75	4.5
4	3	12	1.5	−9
5	7	28	3.5	−21
6	−1	−4	−0.5	3

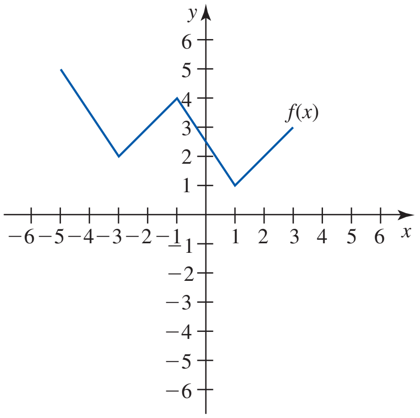
31. $g(x)$

32. $h(x)$

33. $j(x)$

For Exercises 34–35, list at least three coordinate points that lie on each transformation of f.

34.



a. $g(x) = -3f(x) + 4$

b. $h(x) = 2f(x - 3)$

c. $k(x) = -4f(x - 6) + 5$

35. $f(x) = 6x^2$

a. $g(x) = 2f(x + 3)$

b. $h(x) = -3.5f(x) - 2$

c. $k(x) = 4.25f(-x)$

SHOW YOU KNOW

36. The function D models the cost of admission for n adults at Disneyland. Explain how vertical stretches or compressions can be used to model the cost of admission if a group has a coupon giving them 15% off each ticket.

37. The function C models the average cost of electricity for a single-family home in Chicago during month m of the year. Explain how vertical stretches or compressions can be used to model the average cost after a 3% increase in the cost of electricity.

38. When you perform a vertical stretch or compression, what happens to the vertical intercept? Why?

39. When you perform a vertical stretch or compression, what happens to the horizontal intercept(s)? Why?

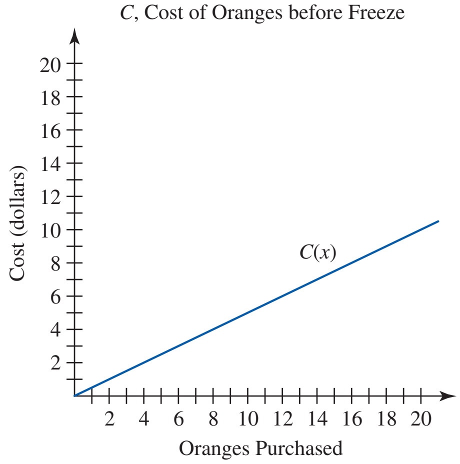
40. How do vertical stretches and compressions affect the rate of change, m, and vertical intercept, b, of a linear function $y = mx + b$?

41. A classmate says any linear function can be created by performing transformations on the function $f(x) = x$. Is your classmate correct? Explain.

42. Sketch a graph of a nonlinear function on graph paper, then pick two points on your graph and calculate the average rate of change between these points. Stretch or compress your function vertically and note where the two points you chose previously are now located. Use this example to explain the effect a vertical stretch or compression has on the average rate of change of a function.

MAKE IT REAL

43. Orange Prices A devastating freeze in January 2007 destroyed roughly 75% of California Central Valley's orange crop. Market analysts predicted this freeze would cause the price of an orange in supermarkets to triple. The graph shows the function C, the cost in dollars of purchasing x oranges before the 2007 freeze. (In this model, we assume no discounts are given for large orders of oranges.)



a. What is the slope of C and what does it represent in the situation?

b. Find a function F that models the cost of oranges after the freeze. Then write F in terms of C.

c. What is the slope of F and how is it related to the slope of C?

44. Value of the Dollar Based on data from 1980 to 2005, the value of the dollar based on producer prices can be modeled by

$$V(t) = -0.00004785t^3 + 0.02314t^2 - 0.04774t + 1.137$$

where t is the number of years since 1980. **(*Source:* Modeled from *Statistical Abstract of the United States, 2007*, Table 705)**

Write the formula for $P(t)$ given $P(t) = 100V(t)$. What does the function P represent in this situation?

45. Travel Costs In February 2007, a BritRail 15-day Flexipass sold for \$644. This pass allowed a traveler unlimited travel on British trains for 15 days during a 60-day period. **(*Source:* www.raileurope.com)**

a. Write a formula to model the per-day cost if a traveler purchased one of these passes and rode the train d days. Then draw a graph of this function.

b. Travelers can also purchase a BritRail 15-day Consecutive Pass, which allows unlimited travel over a 15-day period. In February 2007 this pass sold for \$499. Explain how this situation can be modeled by transforming the function created in part (a). Then graph the transformed function.

c. Which option is cheaper for the same number of days of travel? Why do you think BritRail sells this pass for less?

46. **Travel Costs** A Eurail Select Pass allows travelers to customize their rail passes by choosing the number of countries they want to visit and the number of travel days they want to use. In February 2007, a first-class 3-country, 10-day unlimited-use pass sold for $609. (***Source:*** **www.raileurope.com**)

a. Write a formula to model the per-day cost of using this pass if the traveler rides the train on d days. Then graph the function.

b. In the same month, a 5-country, 10-day unlimited-use pass sold for $705. Explain how the per-day cost for the 5-country ticket can be found by transforming the function created in part (a). Then draw the transformed graph.

c. In February 2007, Eurail offered a special promotion. If travelers purchased a 10-day ticket they would receive an additional day of travel for free. Can you use transformations to model this special promotion? Explain.

Exercises 47–52 focus on vertical stretches and compressions used to change the units for functions.

47. **Registered Vehicles** The table shows the total number of registered vehicles in the United States as a function of the year.

Year t	Number of Registered Vehicles $N(t)$
1980	155,796,000
1990	188,798,000
1995	201,530,000
2000	221,475,000
2001	230,428,000
2002	229,620,000
2003	231,390,000

Source: ***Statistical Abstract of the United States, 2006*****, Table 1078**

a. Function M represents the approximate number of registered vehicles in the United States in *millions* as a function of the year. Describe how N can be transformed to create M, then use function notation to write M in terms of N.

b. The function L can be created by shifting M to the left 1970 units. Explain what L models, then use function notation to write L in terms of N.

48. **Speed of Sound** The table shows the speed of sound at sea level in feet per second for different Celsius temperatures.

Temperature (°C) t	Speed of Sound (feet per second) $S(t)$
0	1087.003
5	1096.907
10	1106.722
15	1116.450
20	1126.095
25	1135.658
30	1145.141

Source: **www.digitaldutch.com/atmoscalc**

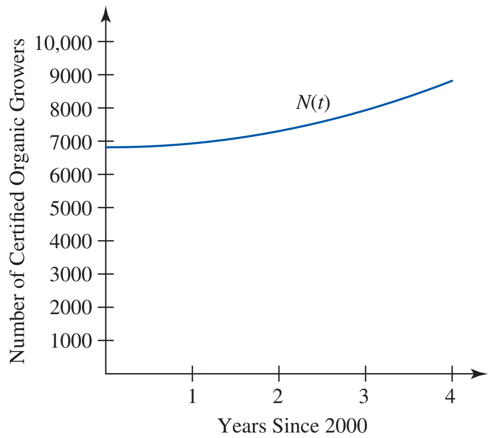
a. Without calculating or graphing, does it appear that a linear model will fit this data well? Explain.

b. What is the average rate of change of S between 0°C and 30°C?

c. The conversion between feet and meters is 1 foot = 0.3048 meters. Explain the transformation needed on S so that it models the speed of sound in meters per second.

d. Celsius and kelvin are temperature measurements such that a change of one degree Celsius is the same change in temperature as one kelvin, but they have a different point of reference: 0°C = 273.15 K. Use function notation to represent the relationship between S and P, the speed of sound at sea level in meters per second for a temperature of m kelvin.

49. **Organic Growers** Based on data from 2000 to 2003, the number of certified growers of organic crops and animals can be modeled by N, as shown in the graph.

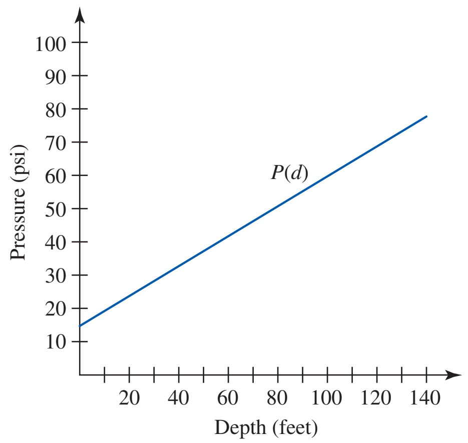


Source: **Modeled from** ***Statistical Abstract of the United States, 2006*****, Table 803**

a. A function R models the number of certified organic growers in *thousands*, where t is the number of years since 2000. Explain the transformation of N required to create R, then use function notation to write N in terms of R.

b. Draw the graph of R.

50. Pressure and Depth The graph shows the function P, the total pressure in pounds per square inch at a depth of d feet.



An *atmosphere* is another unit for measuring pressure, where 1 atmosphere is equal to the pressure of the atmosphere at sea level (14.7 pounds per square inch).

a. Explain how P can be transformed to create R, the total pressure in atmospheres at a depth of d feet.

b. Draw the graph of R.

51. Salaries Based on data from 1993 to 2003, the median annual salary in *thousands of dollars* for individuals in the science and engineering field can be modeled by $S(t) = 1.899t + 41.75$, where t is the number of years since 1990. **(*Source:* Modeled from *Science and Engineering Indicators 2006*, National Science Foundation, Table 3-8)**

a. A function $L(t)$ models the median annual salary in *dollars* for individuals in the science and engineering field t years since 1990. Use function notation to write L in terms of S, then write the formula for L.

b. Function Y is defined in terms of S as $Y(t) = 0.001\ S(t)$. Describe what Y represents, then write the formula for Y.

52. Nursing Home Care Based on data from 1960 to 2004, the cost of nursing home care can be modeled by

$$N(t) = 0.06759t^2 - 0.2958t + 0.0624$$

where t is years since 1960 and N is in *billions of dollars*. **(*Source:* Modeled from *Statistical Abstract of the United States, 2007*, Table 120)**

a. A function H models the cost of nursing home care in *dollars*, where t is years since 1960. Use function notation to write H in terms of N, then write the formula for H.

b. Function C in terms of N is defined as $C(t) = 1000N(t)$. Describe what C represents, then write the formula for C.

■ STRETCH YOUR MIND

Exercises 53–57 are intended to challenge your understanding of vertical stretches and compressions.

53. Gas Prices The EPA estimates a 2007 PT Cruiser has a fuel efficiency of about 19 miles per gallon for city driving. **(*Source:* www.fueleconomy.gov)**

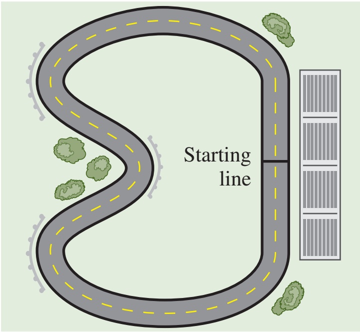
a. Write a formula to show $T(c)$, the cost of driving c miles in the city in a 2007 PT Cruiser if the driver filled up in Fairbanks for \$2.19 per gallon.

b. Explain how you could transform T to determine the cost of driving c miles in the city if the driver filled up in Amarillo for \$2.44 per gallon, then use function notation to write your new function in terms of T.

c. The EPA estimates a 2007 PT Cruiser has a fuel efficiency of about 26 miles per gallon on the highway. **(*Source:* www.fueleconomy.gov)** Explain how you could transform T to determine the cost of driving h miles on the highway if the driver filled up in Fairbanks, then use function notation to write your new function in terms of T.

d. Repeat part (c) given that the driver filled up in Amarillo.

54. Racing Speeds The following racetrack is being used for a professional race car event.



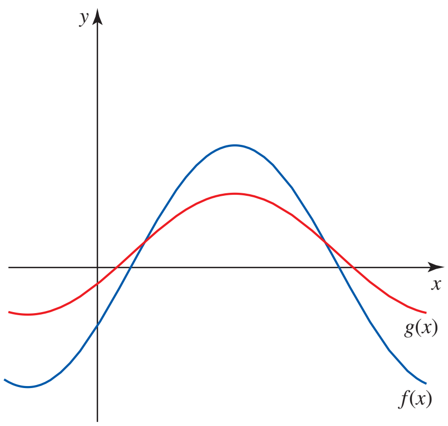
a. Assuming racers will slow down for a turn and speed up for a straightaway, sketch a graph that could model a racer's speed versus time for one lap of the race. Label this function as $s(t)$.

b. Using your answer to part (a), sketch a graph for $d(t)$, the total distance traveled throughout one lap of the race as a function of time.

c. To encourage families to come to the main event, race organizers constructed a scale model of the racetrack that kids can race around in go-carts. The racetrack is a one-tenth scale replica of the original, and the go-carts have been designed to have a maximum speed of one-tenth that of a real race car. Explain the relationship between the time it takes the professional drivers to finish one lap versus the time it takes for a go-cart to finish one lap. Do not take into consideration the skill level of the drivers.

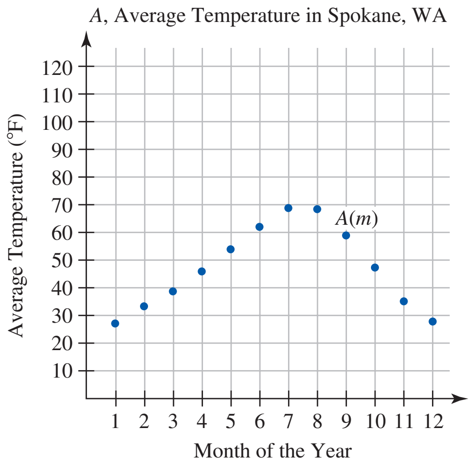
d. Use the transformations of the functions in parts (a) and (b) to create models for the speed and distance traveled of a go-cart over one lap.

55. Examine the functions f and g in the figure.



Is it possible to create g by transforming f using a vertical compression? Explain.

56. Average Temperatures The average temperature, A, in Spokane, Washington, in degrees Fahrenheit as a function of m, the month of the year, is shown in the graph.



Source: www.cityrating.com

a. The following formula can be used to convert Fahrenheit temperatures to Celsius temperatures: $C = \frac{5}{9}(F - 32)$. Use this information to describe how to transform function A to represent the average temperature in Spokane in degrees Celsius. (*Hint: Be careful!*)

b. Draw the graph of the transformed function.

57. The function P gives the U.S. per capita spending on prescription drugs in dollars t years after 1970. A classmate says a vertical stretch factor equal to the U.S. population will create a function that shows the total amount spent on prescription drugs in the United States. Explain the error in your classmate's thinking.

SECTION 3.4 Horizontal Stretches and Compressions

LEARNING OBJECTIVES

- Identify what change in a function equation results in a horizontal stretch
- Identify what change in a function equation results in a horizontal compression

GETTING STARTED

When a person takes medicine, the body immediately begins to break down the drug. Consequently, medicine has a limited time of peak effectiveness before an additional dose is required. An important measurement in determining dosage size and schedule is the *half-life* of the drug, or the amount of time it takes the body to eliminate half of the medicine in its system.

In this section we demonstrate how horizontal stretches and compressions can be used to model real-world applications such as half-lives. We also look at the effect of horizontal stretches and compressions on graphs, tables, and function equations.

Horizontal Stretches and Compressions

The half-life of a drug may vary from person to person, depending on a number of factors. Let's see how horizontal transformations can be used to determine times between doses for various half-lives.

EXAMPLE 1 ■ **Stretching a Function Horizontally**

Cefotetan (SEF oh tee tan), a prescription antibiotic, has a normal half-life of about 4 hours and is usually prescribed in 2-gram doses. (*Source:* www.merck.com)

a. Create a table of values and a graph for A, the amount of Cefotetan present in the body t hours after taking a 2-gram dose (assuming no further doses are taken).

b. Some people process the drug more slowly. In their bodies, Cefotetan's half-life may be up to 5 hours. Create a table of values and a graph for S, the amount of Cefotetan present in the body t hours after taking a 2-gram dose for these people (assuming no further doses are taken).

c. Explain the connection between S and A. Then use function notation to write S in terms of A.

Solution

a. A half-life of 4 hours tells us that for every 4 hours that passes the amount of Cefotetan will be reduced by half. This is shown in Table 3.17.

Table 3.17

Time Since Taking One Dose (hours) t	Amount of Cefotetan Remaining with a 4-Hour Half-Life (grams) $A(t)$
0	2
4	1
8	0.5
12	0.25
16	0.125

The graph of these values is shown in Figure 3.41. Each of the points is connected by a smooth curve since we may calculate the amount of Cefotetan at any time.

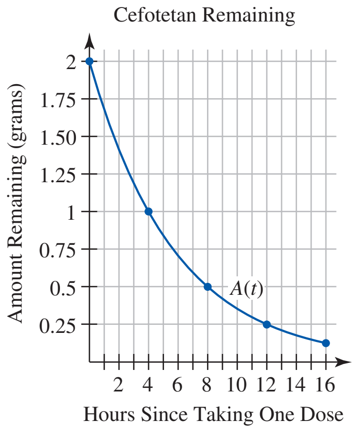


Figure 3.41

b. Using the same idea with a 5-hour half-life gives us the values shown in Table 3.18 and the graph in Figure 3.42.

Table 3.18

Time Since Taking One Dose (hours) t	Amount of Cefotetan Remaining with a 5-Hour Half-Life (grams) $S(t)$
0	2
5	1
10	0.5
15	0.25
20	0.125

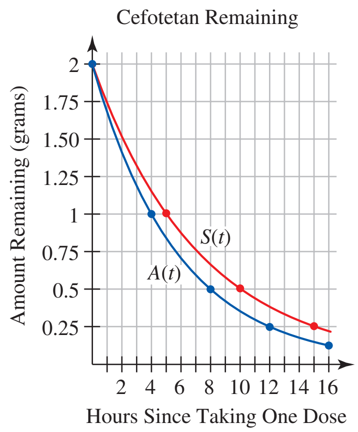


Figure 3.42

c. For any $t > 0$, we can see that $S(t) > A(t)$. With a longer half-life, the drug is not eliminated as quickly. Thus there will always be more Cefotetan remaining in S than in A for the same value of t (when $t > 0$). We see the following relationships:

i. $S(5) = A(4)$. In both scenarios the patient has 1 gram remaining after one half-life period, but this is 5 hours in S and 4 hours in A.

ii. $S(10) = A(8)$. In both scenarios the patient has 0.5 gram remaining after two half-life periods, but this is 10 hours in S and 8 hours in A.

iii. $S(15) = A(12)$. In both scenarios the patient has 0.25 gram remaining after three half-life periods, but this is 15 hours in S and 12 hours in A.

Notice that the difference between corresponding inputs is not constant; however, they are related. We can see this relationship when the inputs are compared as a ratio. Notice that each ratio is equivalent to $\frac{4}{5}$.

$$\frac{4}{5} \qquad \frac{8}{10} = \frac{4}{5} \qquad \frac{12}{15} = \frac{4}{5}$$

To find $S(t)$, we have to use an input in A that is $\frac{4}{5}$ of t. That is,

$$S(t) = A\left(\frac{4}{5}t\right)$$

To verify this relationship, let's use it to find $S(20)$

$$\begin{aligned} S(t) &= A\left(\frac{4}{5}t\right) \\ S(20) &= A\left(\frac{4}{5}(20)\right) \\ &= A\left(\frac{80}{5}\right) \\ &= A(16) \end{aligned}$$

Using Tables 3.17 and 3.18, we see that $A(16) = 0.125$ gram and $S(20) = 0.125$ gram.

The relationship $S(t) = A\left(\frac{4}{5}t\right)$ is an example of a *horizontal stretch* of A; the graph appears to have been *stretched* so that it is farther from the vertical axis. In a **horizontal stretch**, the same output values occur for input values that are different by a constant *factor*. (Recall that to be a horizontal *shift*, the input values must change by a constant *amount*.) A common mistake is to think of this relationship as a *compression* because the fraction $\frac{4}{5}$ is less than 1. However, if we recall that $S(5) = A(4)$, $S(10) = A(8)$, and $S(15) = A(12)$, we can see the same output values are occurring *later* in S than in A. So this is a stretch. To determine the stretch factor, let's take a close-up look at the graphs of these functions in Figure 3.43.

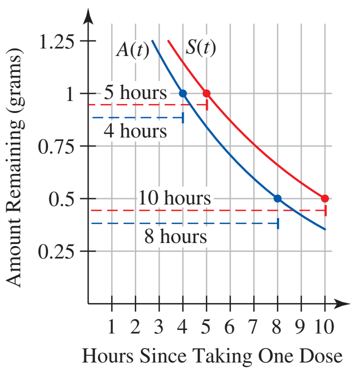


Figure 3.43

The input values of S are $\frac{5}{4}$ of the corresponding inputs in A (the inputs giving the same amount of medicine remaining). So, $S(t) = A\left(\frac{4}{5}t\right)$ shows a horizontal stretch of A by a factor of $\frac{5}{4}$.

EXAMPLE 2 ■ Compressing a Function Horizontally

As noted in Example 1, Cefotetan has a normal half-life of 4 hours. However, some people process the drug more quickly. In their bodies Cefotetan's half-life may be as short as 3 hours.

a. Create a table of values and a graph for F, the amount of Cefotetan present in the body t hours after taking a 2-gram dose for a person who processes the drug more quickly (assuming no further doses are taken).

b. Explain the connection between F and A, then use function notation to write F in terms of A.

Solution

a. A half-life of 3 hours tells us that for every 3 hours that passes the amount of Cefotetan will be reduced by half. This is shown in Table 3.19 and Figure 3.44.

Table 3.19

Time Since Taking One Dose (hours) t	Amount of Cefotetan Remaining with a 3-Hour Half-Life (grams) $F(t)$
0	2
3	1
6	0.5
9	0.25
12	0.125

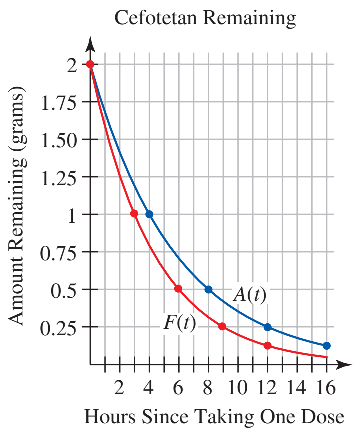


Figure 3.44

b. For any $t > 0$, we can see $F(t) < A(t)$. With a shorter half-life, the drug is eliminated more quickly. Thus there will always be less Cefotetan remaining in F than in A for the same value of t (when $t > 0$). We see the following relationships between F and A:

i. $F(3) = A(4)$. In both scenarios the patient has 1 gram remaining after one half-life period, but this is 3 hours in F and 4 hours in A.

ii. $F(6) = A(8)$. In both scenarios the patient has 0.5 gram remaining after two half-life periods, but this is 6 hours in F and 8 hours in A.

iii. $F(9) = A(12)$. In both scenarios the patient has 0.25 gram remaining after three half-life periods, but this is 8 hours in F and 12 hours in A.

The relationship between the inputs in A and F may be compared as a ratio. Notice that each ratio is equivalent to $\frac{4}{3}$.

$$\frac{4}{3} \qquad \frac{8}{6} = \frac{4}{3} \qquad \frac{12}{9} = \frac{4}{3}$$

To find $F(t)$, we must use an input in A that is $\frac{4}{3}$ of t. That is,

$$F(t) = A\left(\frac{4}{3}t\right)$$

To verify this relationship, let's use it to find $F(12)$:

$$\begin{aligned} F(t) &= A\left(\frac{4}{3}t\right) \\ F(12) &= A\left(\frac{4}{3}(12)\right) \\ &= A\left(\frac{48}{3}\right) \\ &= A(16) \end{aligned}$$

Using Tables 3.17 and 3.19, we see that $A(16) = 0.125$ gram and $F(12) = 0.125$ gram.

The relationship $F(t) = A\left(\frac{4}{3}t\right)$ discussed in Example 2 is a *horizontal compression* of A because the graph appears to have been squeezed closer to the vertical axis. In a **horizontal compression**, the same output values occur for input values that are different by a constant *factor*. A common mistake is to think of this relationship as a *stretch* because the fraction $\frac{4}{3}$ is greater than 1. However, if we recall that $F(3) = A(4)$, $F(6) = A(8)$, and $F(9) = A(12)$, we can see the same output values are occurring *earlier* in F than in A. So this is a compression. To determine the compression factor, let's take a close-up look at the graphs of these functions in Figure 3.45.

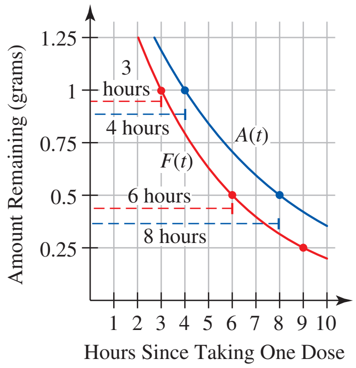


Figure 3.45

The input values of F are $\frac{3}{4}$ the magnitude of the corresponding inputs in A (the inputs giving the same amount of medicine remaining). Thus, $F(t) = A\left(\frac{4}{3}t\right)$ shows a horizontal compression of A by a factor of $\frac{3}{4}$.

Notice that the stretch or compression factor is always the reciprocal of the coefficient of the input variable. If the reciprocal is less than 1, the transformation is a horizontal compression. If the reciprocal is greater than 1, the transformation is a horizontal stretch. We summarize our observations as follows.

HORIZONTAL STRETCHES AND COMPRESSIONS

The graph of $g(x) = f(bx)$ is the graph of f stretched or compressed horizontally by a factor of $\left|\frac{1}{b}\right|$.

- If $\left|\frac{1}{b}\right| < 1$, the transformation is a **horizontal compression**.
- If $\left|\frac{1}{b}\right| > 1$, the transformation is a **horizontal stretch**.

Generalizing Transformations: Horizontal Stretches and Compressions

Horizontal compressions are the perhaps the trickiest of the transformations even when we remember that transformations use the outputs of a parent function to define the outputs of an image function. In the case of horizontal stretches and compressions, we look for a relationship between inputs of the two functions that will result in the same output values. Consider the relationship $g(x) = f(3x)$. This tells us the two functions will have *identical* outputs.

$$\underbrace{g(\quad)}_{\text{the outputs of } g} \underbrace{=}_{\text{equal}} \underbrace{f(\quad)}_{\text{the outputs of } f}$$

However, the expression $3x$ tells us these outputs result from different inputs. If we want to evaluate g for some input x, we need to look to the output of function f when its input is 3 times as great as x (a larger magnitude input).

$$\underbrace{g(x)}_{\text{the outputs of } g \text{ at } x} \quad \underbrace{=}_{\text{equal}} \quad \underbrace{f(3x)}_{\text{the outputs of } f \text{ at 3 times } x}$$

To illustrate, let's find $g(-2)$ and $g(1)$.

$$\underbrace{\begin{aligned} g(x) &= f(3x) \\ g(-2) &= f(3 \cdot -2) \\ g(-2) &= f(-6) \end{aligned}}_{\text{The output of } g \text{ at } -2 \text{ is defined by the output of } f \text{ at } -6.} \qquad \underbrace{\begin{aligned} g(x) &= f(3x) \\ g(1) &= f(3 \cdot 1) \\ g(1) &= f(3) \end{aligned}}_{\text{The output of } g \text{ at 1 is defined by the output of } f \text{ at 3.}}$$

If f is given by the values in Table 3.20, then g has the values in Table 3.21.

Table 3.20

x	$f(x)$
−12	20
−6	15
−1	12
0	11
3	10
5	9
9	6

Table 3.21

x	$g(x)$
−4	20
−2	15
−1/3	12
0	11
1	10
5/3	9
3	6

If f is defined by the graph in Figure 3.46a, then g is defined by the graph in Figure 3.46b.

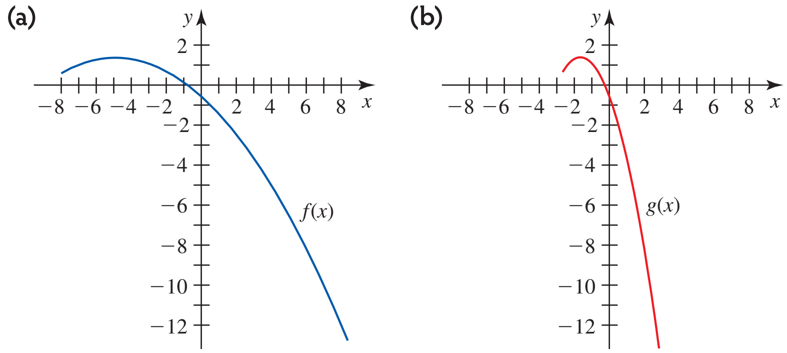


Figure 3.46

We can verify statements such as $g(1) = f(3)$ using either the tables or the graphs. We see that this is a *horizontal compression* by a factor of $\frac{1}{3}$.

To determine which function has outputs that occur for inputs closer to zero (and thus determine whether the image function is a compression or a stretch), we simply need to look at the inputs in the definition. In the case of $g(x) = f(3x)$, x will always be closer to zero than $3x$, so the same outputs occur in g for inputs closer to zero than in f. The opposite is true for a relationship defined by a statement such as $h(x) = f(0.5x)$. In this case the inputs in f are closer to zero for the same output.

$$\underbrace{h(x)}_{\text{the outputs of } h \text{ at } x} \quad \underbrace{=}_{\text{equal}} \quad \underbrace{f(0.5x)}_{\text{the outputs of } f \text{ at half of } x}$$

This is a *horizontal stretch* by a factor of 2 (because $\frac{1}{0.5} = 2$).

EXAMPLE 3 ■ **Determining a Compression from an Equation**

Describe the graphical relationship between $f(x) = 5x^2$ and $g(x) = 5(2x)^2$. Then graph both functions to confirm the accuracy of your conclusion.

Solution Notice $g(x) = f(2x)$. Since the reciprocal of the coefficient of the input variable is $\frac{1}{2}$, the graph of g is the graph of f compressed horizontally by a factor of $\frac{1}{2}$. The graphs are shown in Figure 3.47.

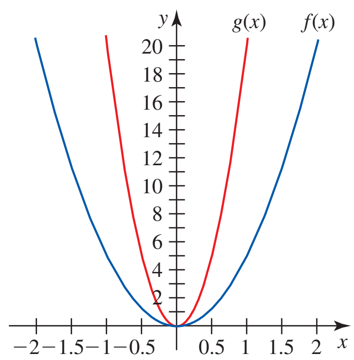


Figure 3.47

■ Using Stretches and Compressions to Change Units

Horizontal stretches and compressions are often used to change the units on the input variable. For example, consider Table 3.22 showing the amount of taxes to be paid by married couples filing a joint tax return for the given income amounts in two different functions.

Table 3.22

Income (\$ thousands) n	Tax Paid (dollars) $T(n)$	Income (dollars) $d = 1000n$	Tax Paid (dollars) $A(d)$
20	2,249	20,000	2,249
30	3,749	30,000	3,749
40	5,249	40,000	5,249
50	6,749	50,000	6,749
60	8,249	60,000	8,249
70	10,621	70,000	10,621
80	13,121	80,000	13,121
90	15,621	90,000	15,621

Source: 2006 IRS Tax Table, www.irs.gov

Function A is a *horizontal stretch* of T because the same output values now occur for input values 1000 times as great. We can also say T is a *horizontal compression* of A because the same output values occur for input values 0.001 times as great. Since $d = 1000n$ and $n = 0.001d$, we can relate A and $T(n)$ as follows:

$$A(d) = T(0.001d)$$

or

$$T(n) = A(1000n)$$

■ Combining Transformations

Horizontal stretches and compressions may be combined with the other transformations we have studied: horizontal and vertical shifts, horizontal and vertical reflections, and vertical stretches and compressions.

EXAMPLE 4 ■ Combining Transformations

For each of the following, draw the graph of $g(x)$.

a. The function f is graphed in Figure 3.48. Draw $g(x) = f(-2x) - 6$.

b. Given $f(x) = x^2$, graph $g(x) = -\frac{1}{4}f\left(\frac{1}{3}x\right) + 2$.

Solution

a. The function $g(x) = f(-2x) - 6$ is the combination of three transformations on f: a horizontal compression by $\frac{1}{2}$, a horizontal reflection, and a vertical shift downward 6 units. Using the graph of f to find the points, we do the stretches, compressions, and reflections first and the shifts last, as shown in Table 3.23 and Figure 3.49.

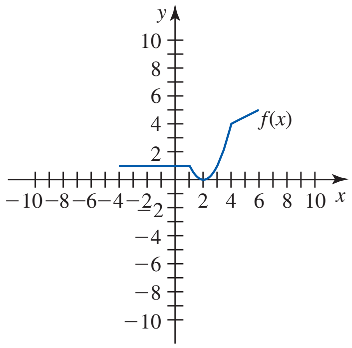


Figure 3.48

Table 3.23

x	$f(x)$ (Figure 3.49a)	Horizontal Compression by $\frac{1}{2}$ $f(2x)$ (Figure 3.49b)	Horizontal Reflection $f(-2x)$ (Figure 3.49c)	Vertical Shift Downward 6 $f(-2x) - 6$ (Figure 3.49c)	Coordinate Points for $g(x)$ (Figure 3.49c)
−4	1	undefined	undefined	undefined	undefined
−3	1	undefined	$f(-2(-3)) = f(6) = 5$	$5 - 6 = -1$	$(-3, -1)$
−2	1	$f(2(-2)) = f(-4) = 1$	$f(-2(-2)) = f(4) = 4$	$4 - 6 = -2$	$(-2, -2)$
−1	1	$f(2(-1)) = f(-2) = 1$	$f(-2(-1)) = f(2) = 0$	$0 - 6 = -6$	$(-1, -6)$
0	1	$f(2(0)) = f(0) = 1$	$f(-2(0)) = f(0) = 1$	$1 - 6 = -5$	$(0, -5)$
1	1	$f(2(1)) = f(2) = 0$	$f(-2(1)) = f(-2) = 1$	$1 - 6 = -5$	$(1, -5)$
2	0	$f(2(2)) = f(4) = 4$	$f(-2(2)) = f(-4) = 1$	$1 - 6 = -5$	$(2, -5)$
3	1	$f(2(3)) = f(6) = 5$	undefined	undefined	undefined
4	4	undefined	undefined	undefined	undefined
5	4.5	undefined	undefined	undefined	undefined
6	5	undefined	undefined	undefined	undefined

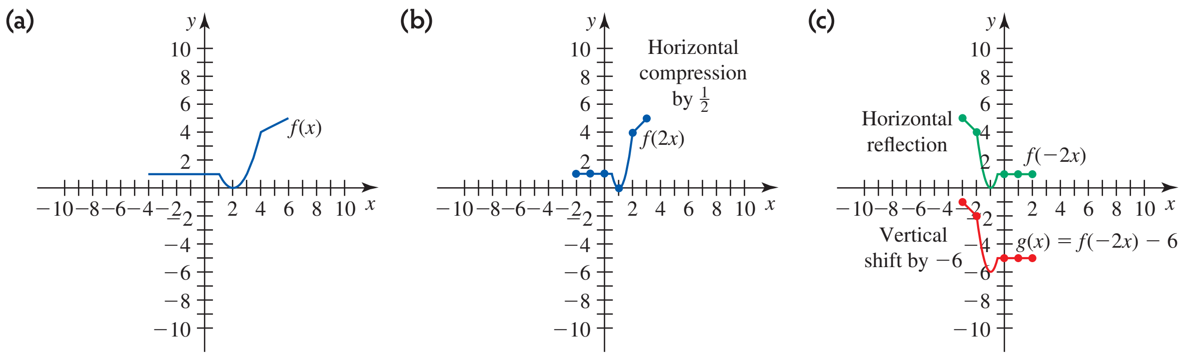


Figure 3.49

b. We are given $f(x) = x^2$ and asked to graph $g(x) = -\frac{1}{4}f\left(\frac{1}{3}x\right) + 2$. Function g is the combination of four transformations on f: a horizontal stretch by a factor of 3, a vertical compression by $\frac{1}{4}$, a vertical reflection, and a vertical shift upward 2 units. We can again examine how these transformations affect the coordinates of the original function, as shown in Table 3.24. To do this we will take a slightly different approach than in the previous tables to give us another way of approaching transformations. This method keeps track of what happens to individual points as the function transforms instead of looking at what happens at fixed values of x. The graphical progression of the transformations is shown in Figure 3.50.

Table 3.24

Coordinate Points for $f(x) = x^2$ (Fig. 3.50a; blue)	Horizontal Stretch by 3 $f\left(\frac{1}{3}x\right)$ (Fig. 3.50b; purple)	Vertical Compression by $\frac{1}{4}$ and Vertical Reflection $-\frac{1}{4}f\left(\frac{1}{3}x\right)$ (Fig. 3.50a; green)	Vertical Shift Upward 2 $-\frac{1}{4}f\left(\frac{1}{3}x\right) + 2$ (Fig. 3.50b)	Coordinate Points for $g(x)$ (Fig. 3.50b)
$(-3, 9) \rightarrow$	$(-9, 9) \rightarrow$	$\left(-9, -\frac{9}{4}\right) = (-9, -2.25) \rightarrow$	$(-9, -0.25)$	$(-9, -0.25)$
$(-2, 4) \rightarrow$	$(-6,4) \rightarrow$	$\left(-6, -\frac{4}{4}\right) = (-6, -1) \rightarrow$	$(-6, 1)$	$(-6, 1)$
$(-1, 1) \rightarrow$	$(-3, 1) \rightarrow$	$\left(-3, -\frac{1}{4}\right) = (-3, -0.25) \rightarrow$	$(-3, 1.75)$	$(-3, 1.75)$
$(0, 0) \rightarrow$	$(0, 0) \rightarrow$	$(0, 0) \rightarrow$	$(0, 2)$	$(0, 2)$
$(1, 1) \rightarrow$	$(3, 1) \rightarrow$	$\left(3, -\frac{1}{4}\right) = (3, -0.25) \rightarrow$	$(3, 1.75)$	$(3, 1.75)$
$(2, 4) \rightarrow$	$(6, 4) \rightarrow$	$\left(6, -\frac{4}{4}\right) = (6, -1) \rightarrow$	$(6, 1)$	$(6, 1)$
$(3, 9) \rightarrow$	$(9, 9) \rightarrow$	$\left(9, -\frac{9}{4}\right) = (9, -2.25) \rightarrow$	$(9, -0.25)$	$(9, -0.25)$

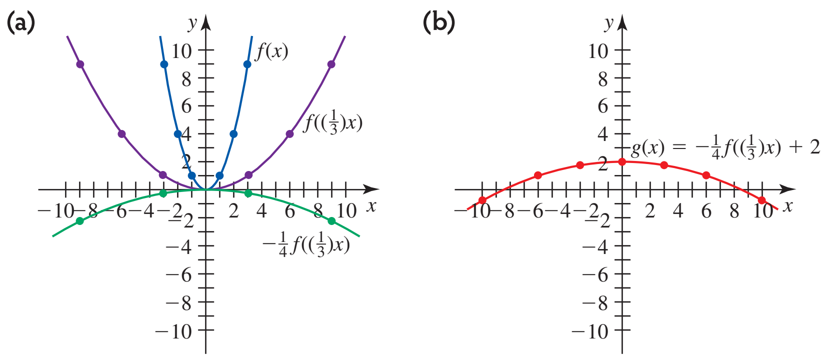


Figure 3.50

EXAMPLE 5 ■ **Solving a Transformed Function Equation**

Given the Table 3.25 for $f(x)$, solve the transformed function equation $f(2x) = 1$.

Table 3.25

x	$f(x)$
0	8
1	12
2	10
3	6
4	1

Solution Since $f(2x) = 1$ and $f(4) = 1$, $2x = 4$. Dividing each side by 2 yields the solution $x = 2$.

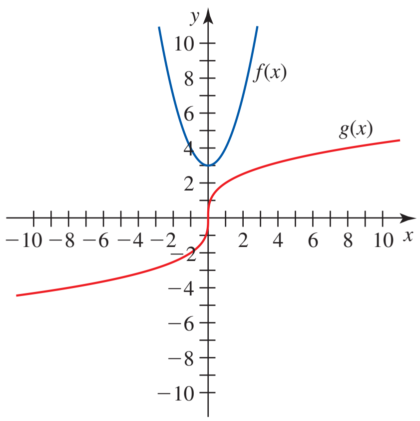
SUMMARY

In this section you learned how to transform functions using horizontal stretches and compressions. You also learned how to use horizontal stretches and compressions in real-world contexts.

3.4 EXERCISES

■ SKILLS AND CONCEPTS

In Exercises 1–4, graph each transformation of f or g.



1. $y = f\left(\frac{1}{4}x\right)$

2. $y = g(-2x)$

3. $y = -g(0.5x) - 5$

4. $y = -\frac{1}{2}f(3x)$

In Exercises 5–10, use the values shown in Table D to evaluate each expression.

Table D

x	$f(x)$		x	$f(x)$		x	$f(x)$
−4	3		−1	10		2	−5
−3	6		0	−2		3	24
−2	1		1	15		4	−9

5. $f(3x)$ when $x = 1$

6. $f\left(-\frac{2}{3}x\right)$ when $x = 3$

7. $f(0.5x) + 2$ when $x = 4$

8. $2f(2x)$ when $x = -2$

9. $-10f\left(-\frac{4}{3}x\right) - 2.5$ when $x = -3$

10. $7f\left(\frac{1}{6}(x - 5)\right) - 1$ when $x = 11$

In Exercises 11–14, solve each equation for x using Table D.

11. $f\left(\frac{1}{3}x\right) = 24$

12. $3f(4x) = -15$

13. $2f(2.5x) - 7 = 5$

14. $\frac{6}{5}f(4(x + 2)) = -\frac{12}{5}$

In Exercises 15–18, refer to Table D to answer each question.

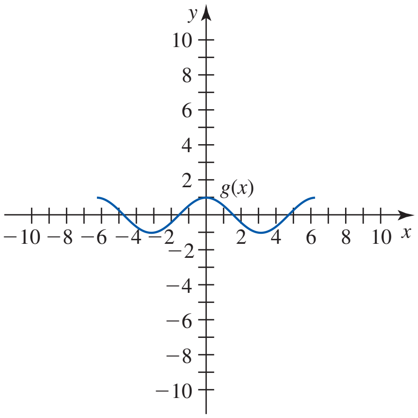
15. Create a table of values for g if $g(x) = f(4x)$.

16. Create a table of values for h if $h(x) = f\left(\frac{1}{3}x\right)$.

17. Create a table of values for j if $j(x) = -2f(3x)$.

18. Create a table of values for k if $k(x) = f\left(-\frac{1}{2}x\right) + 3$.

In Exercises 19–24, match each graph with the appropriate transformation of the given function.



a. $A(x) = g(2x) + 3$

b. $B(x) = g\left(\frac{1}{4}x\right) + 3$

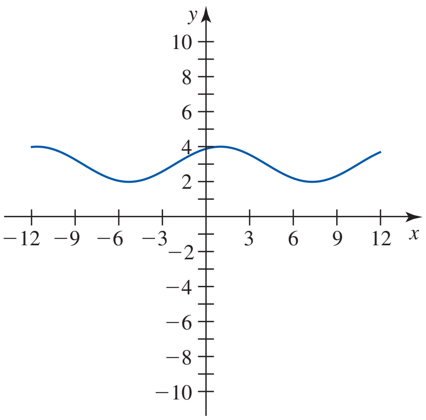
c. $C(x) = g\left(\frac{1}{2}x\right) + 3$

d. $D(x) = g(4x) + 3$

e. $E(x) = g(2(x - 1)) + 3$

f. $F(x) = g\left(\frac{1}{2}(x - 1)\right) + 3$

19.



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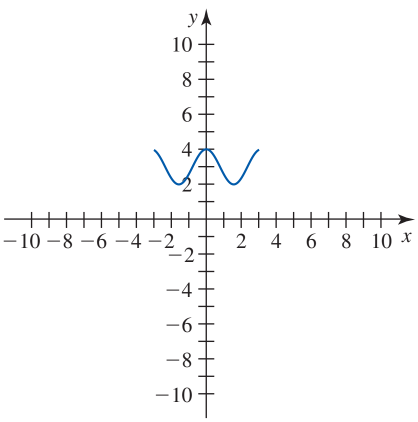

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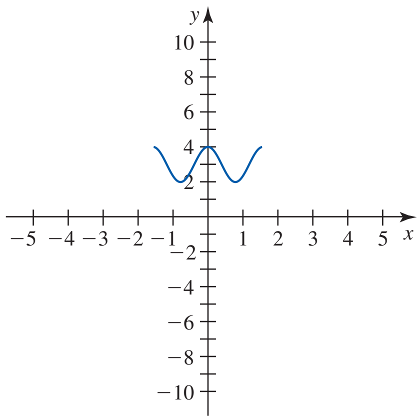

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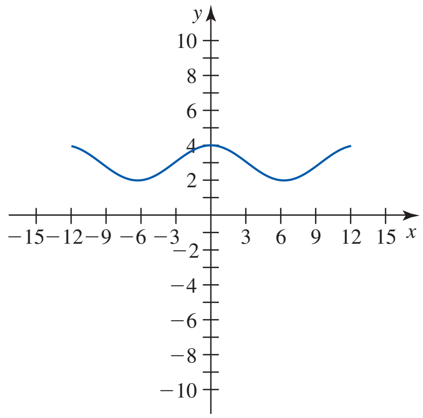

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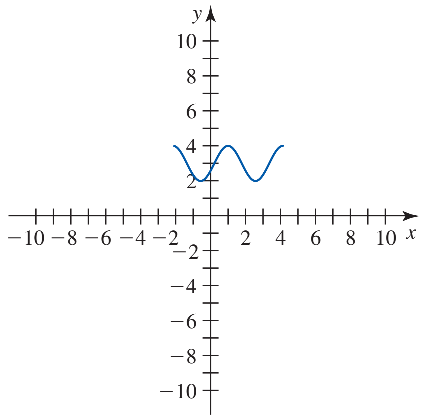

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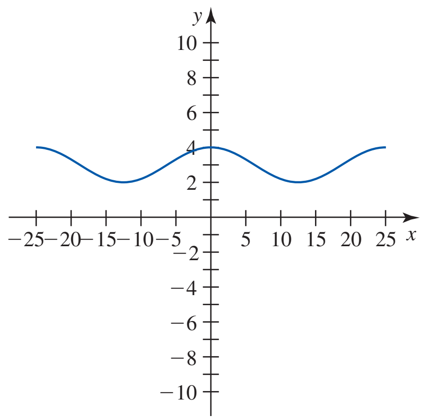
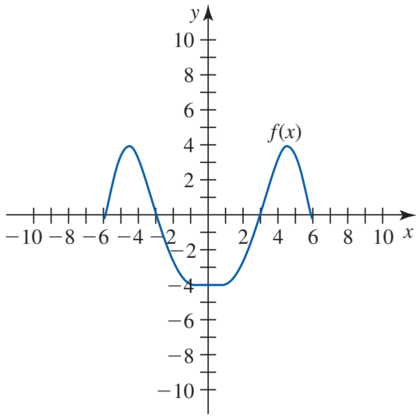


For Exercises 25–28, draw the graph of each transformation of the function f.



25. $g(x) = f(2x)$

26. $h(x) = f(-3x)$

27. $j(x) = f\left(\frac{1}{4}x\right) + 2$

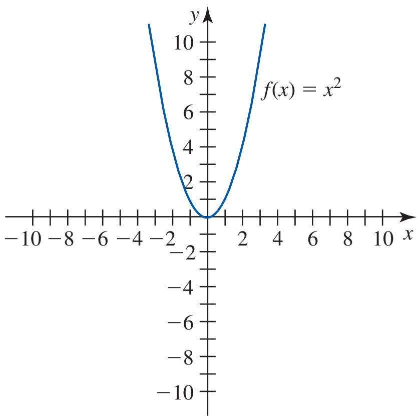
28. $k(x) = f\left(-\frac{3}{4}x\right) - 2$

For Exercises 29–31,

a. Describe the horizontal transformations required on f to create the new function.

b. Use function notation to write each new function in terms of f.

c. Write the formula for the new function. (You may wish to check your work using a graphing calculator.)



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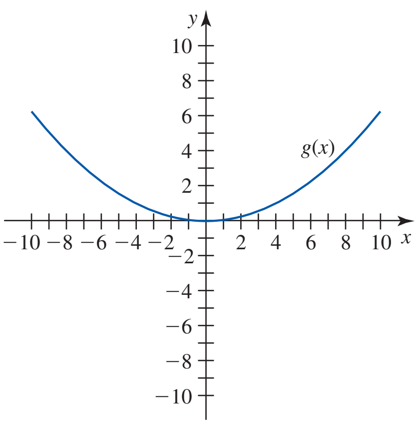

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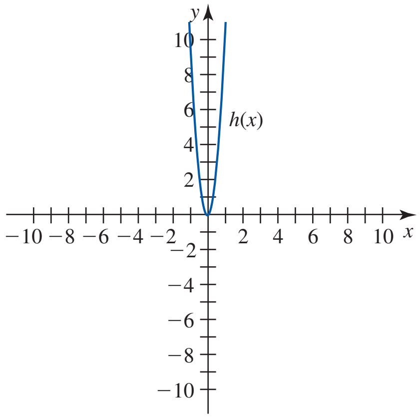

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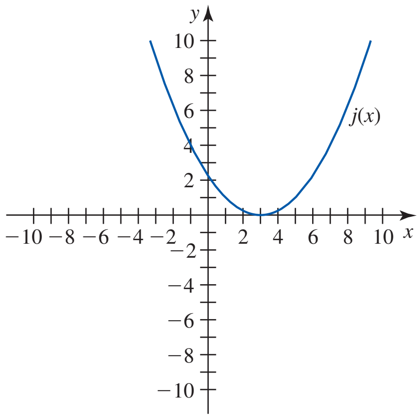
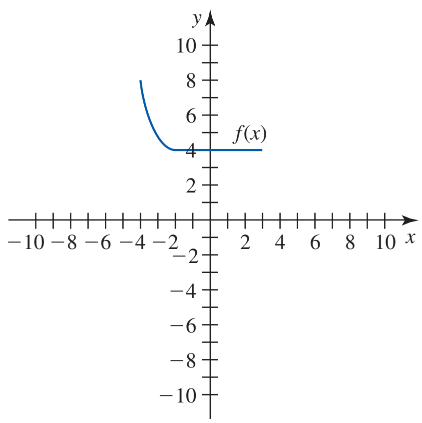


For Exercises 32–35, draw the graph of each transformation of the given function f.



32. $g(x) = 4f(-0.5x)$

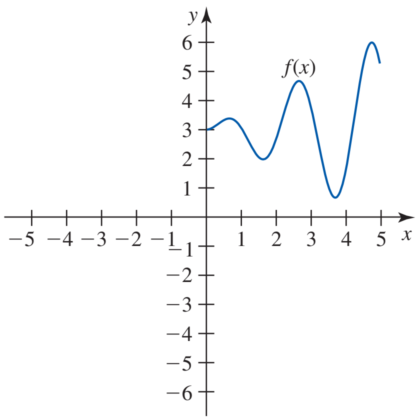
33. $h(x) = \frac{1}{2}f\left(\frac{1}{2}x\right)$

34. $j(x) = -2f(-1.75x)$

35. $k(x) = \frac{3}{5}f\left(-\frac{2}{3}x\right)$

For Exercises 36–37, list three coordinate points that lie on each transformation of the given function f.

36.



a. $g(x) = f(3x)$

b. $h(x) = -f\left(\frac{1}{5}x\right) - 3$

c. $k(x) = -6f\left(\frac{2}{3}x\right)$

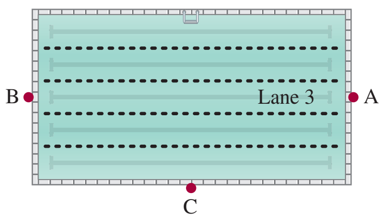
37. $f(x) = \frac{1}{x}$

a. $g(x) = f(2x)$

b. $h(x) = f\left(-\frac{1}{4}x\right) + 7$

c. $k(x) = -2f(1.75x)$

38. Swimming Competition During a race a swimmer swims at a constant speed from A to B and back in lane 3 of the swimming pool shown in the figure. Judges standing at positions A, B, and C are observing the race.



Schmid Christophe/
Shutterstock.com

a. Sketch a graph for D_A, the distance between the swimmer and Judge A as a function of the time since the race began. Assume the race involves two laps and that the swimmer begins the race on Judge A's side of the pool.

b. Sketch a graph of D_B, the distance between the swimmer and Judge B as a function of the time since the race began. Is the graph of D_B a transformation of D_A? Explain.

c. Sketch a graph of D_C, the distance between the swimmer and Judge C as a function of the time since the race began.

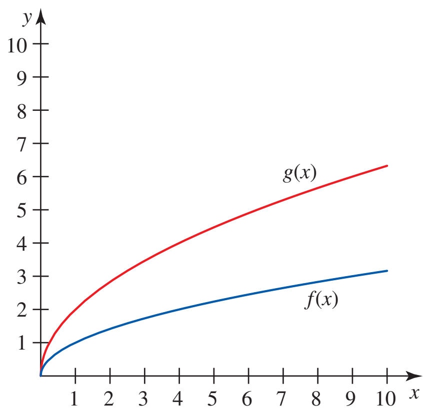
d. In the next race, the swimmer in lane 3 swims twice as fast as the previous swimmer. How will this affect the graphs of the functions in parts (a) through (c)? Explain.

In Exercises 39–40, the graphs of functions f and g are given.

a. Explain how f may be transformed vertically to create g, then use function notation to write g in terms of f.

b. Explain how f may be transformed horizontally to create g, then use function notation to write g in terms of f.

39.



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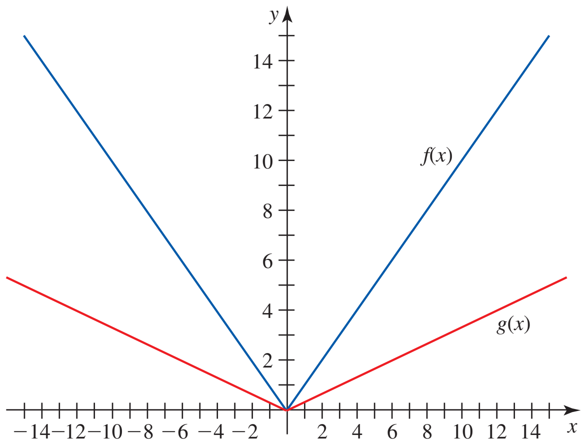
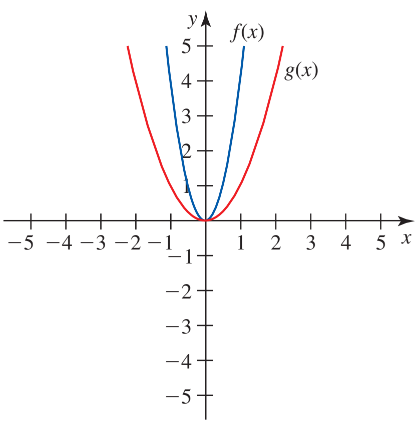


SHOW YOU KNOW

41. When you perform a horizontal stretch or compression, what happens to the vertical intercept?

42. When you perform a horizontal stretch or compression, what happens to the horizontal intercept(s)?

43. In the figure, function g is a transformation of $f(x) = x^2$.



One of your classmates says g is created by vertically compressing f by a factor of $\frac{1}{4}$, while another classmate says it is a horizontal stretch by a factor of 2. Which classmate is correct? Explain.

44. Is it always true that a function defined by using a vertical stretch or compression can also be defined by using a horizontal stretch or compression instead, and vice versa? Explain.

45. Function g is a transformation of f such that $g(x) = af(bx)$. When transforming f to create g, which transformation should be done first? Explain.

46. We have looked at six types of transformations: horizontal and vertical shifts, horizontal and vertical reflections, and horizontal and vertical stretches/compressions. Which of these transformations may impact the following function characteristics? List all that apply.

a. Vertical intercept

b. Horizontal intercept(s)

c. Average rate of change from $x = 1$ to $x = 3$

47. Sketch a graph of a nonlinear function on graph paper, then pick two points on the graph and calculate the average rate of change between the points. Stretch or compress the function vertically and note where the two points previously chosen are now located. Use this example as an aid to explain the effect a horizontal stretch or compression has on the average rate of change of a function.

MAKE IT REAL

48. Medicine Rifampin (rif AM pin) is a drug used to help manage active tuberculosis and has a half-life of about 4 hours in many people. (***Source:*** **www.merck.com**)

a. Create a table of values to model the amount of Rifampin present in a person's body as a function of the time since taking a 300-milligram dose. (Assume no further doses are taken.)

b. In some patients the half-life can be as short as 3 hours. Explain how to transform the function created in part (a) to model the amount of the drug present in a person's body if the half-life is 3 hours.

49. Medicine Gentamicin (jen ta MYE sin) is an antibiotic with a half-life of about 3 hours in many people. (***Source:*** **www.merck.com**)

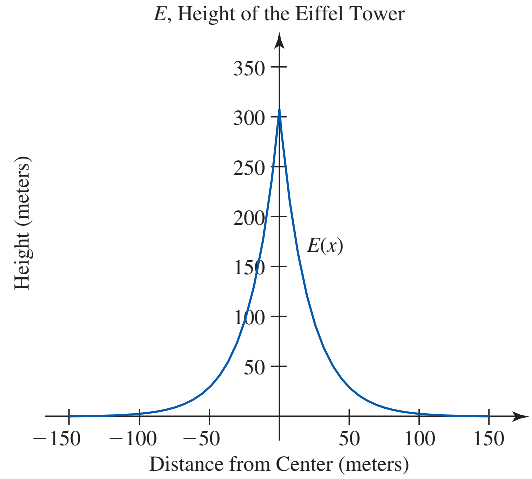
a. Create a table of values to model the amount of Gentamicin present in a person's body as a function of the time since taking a 150-milligram dose. (Assume no further doses are taken.)

b. In some patients the half-life can be as short as 2 hours. Explain how to transform the function created in part (a) to model the amount of the drug present in a person's body if the half-life is 3 hours.

c. In people with advanced kidney disease, the half-life of the drug can be as long as 70 hours. Explain how to transform the function created in part (a) to model the amount of the drug present in a person's body if the half-life is 70 hours.

50. Scale Models E approximates the shape of the Eiffel Tower. Its output is the height in meters of the tower x meters from its center, as shown in the graph. (A "negative" distance is interpreted to mean a distance measured from the center to the left side of the tower.)

Jose Ignacio Soto/Shutterstock.com

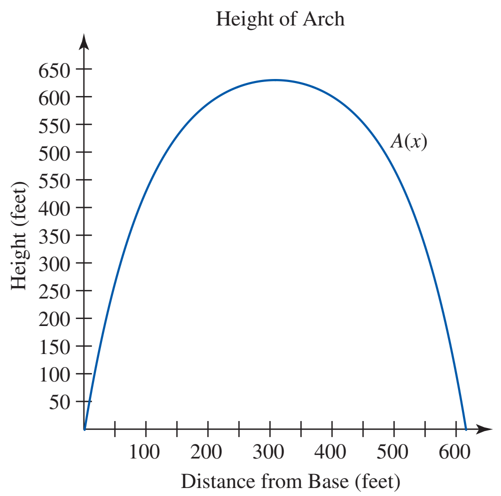


a. Kings Island Amusement Park in Mason, Ohio, has a one-third scale model of the Eiffel Tower. Explain the transformations on E necessary to create a graph showing K, the height of Kings Island's replica in meters as a function of the distance from its center.

b. Paris Las Vegas Hotel in Las Vegas, Nevada, originally planned to create a full-scale replica of the Eiffel Tower, but had to create a half-scale replica because of the hotel's proximity to the Las Vegas airport. Explain the transformations on K needed to create the graph for the Paris Las Vegas's replica.

51. Scale Models The Gateway Arch (Jefferson National Expansion Memorial) in St. Louis, Missouri was built in the 1960s to celebrate the westward expansion of the United States. Its shape can be modeled by the function A, the height of the arch in feet x feet from the base of its left leg.

Paperlandmarks sells paper kits that hobbyists assemble to create scale replicas of famous landmarks. For the Gateway Arch, the company sells 1/600th scale replicas and 1/1000th scale replicas. (***Source:*** **www.paperlandmarks.com**)



a. Approximately how tall will each scale model be (in feet)?

b. For each kit, describe how A must be transformed to create a graph of the height of each model as a function of the distance from the base of its left leg.

c. Use function notation to write each of the functions in part (b) in terms of A.

d. Since it is uncommon to measure models of this size in feet, rewrite the functions in part (c) so that the dimensions of the models are measured in inches.

Exercises 52–55 focus on stretches and compressions used to change the units for functions.

52. Mortgage Payments The following tables show principal and interest payments as a function of the total amount borrowed for a 30-year fixed loan at 6% annual interest.

Amount Borrowed (dollars) w	Monthly Payment (dollars) $Y(w)$
100,000	599.55
200,000	1199.10
225,000	1348.99
280,000	1678.74
330,000	1978.52

Amount Borrowed ($ thousands) b	Monthly Payment (dollars) $P(b)$
100	599.55
200	1199.10
225	1348.99
280	1678.74
330	1978.52

a. Explain the transformation required on Y to create the function P, then use function notation to demonstrate this relationship.

b. Given that $T(b) = 0.001P(b)$, explain what the function T models.

c. Explain the similarities and differences in the rates of change for functions Y and P.

53. Minimum Wage The first table shows the minimum wage mandated by federal law t years after 1950.

Years Since 1950 t	Federal Minimum Wage (dollars) W
0	0.75
10	1.00
20	1.60
30	3.10
40	3.80
50	5.15

Source: www.dol.gov

The next table shows a transformation of the minimum wage table.

0	0.75
1	1.00
2	1.60
3	3.10
4	3.80
5	5.15

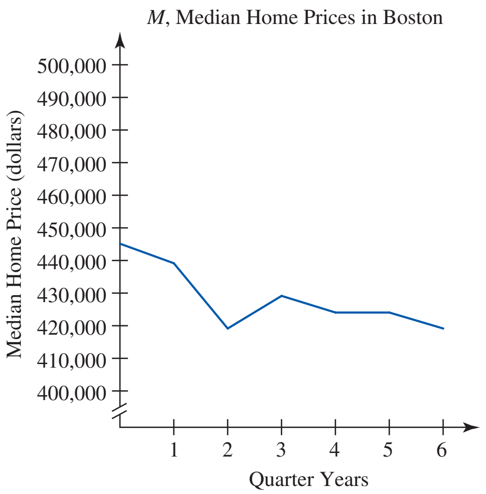
a. Explain the transformation on W required to create the new function.

b. Use function notation to write the new function in terms of W.

c. Place appropriate labels on the second table to describe what it models.

d. Should these functions be graphed on the same set of axes? Explain.

54. Home Prices The graph below shows M, the median price of homes in the Boston area. The value $x = 0$ corresponds to the third quarter of 2005 (Jul–Sep 2005), and x is in quarters of a year (3-month periods).



Source: www.housingtracker.com

a. Create an estimated table of values for a new function that models the median home price in *thousands of dollars* as a function of the number of quarter years after the third quarter of 2005.

b. Use function notation to write the new function in terms of M.

c. Create an estimated table of values for a new function that models the median home price in *thousands of dollars* as a function of the *number of years* after the third quarter of 2005.

d. Use function notation to show the relationship between the new function in part (c) and M.

55. Atmospheric Pressure Based on data from sea level to 8500 meters, the atmospheric pressure can be modeled by $P(a) = 14.96(0.9998)^a$ pounds per square inch, where a is the altitude in meters. **(*Source:* Modeled from Digital Dutch 1976 Standard Atmosphere Calculator)**

a. Function $R(k)$ models the atmospheric pressure in pounds per square inch where k is the altitude in *kilometers* (1000 meters = 1 kilometer). Explain the transformation on P required to create R, then use function notation to write R in terms of P.

b. Write the formula for R as a function of k.

c. Function S models the atmospheric pressure in pounds per square inch where c is the altitude in *centimeters* (1 meter = 100 centimeters). Explain the transformation on P required to create S, then use function notation to write S in terms of P.

d. Write the formula for S as a function of c.

■ STRETCH YOUR MIND

Exercises 56–60 are intended to challenge your understanding of vertical and horizontal stretches and compressions.

56. When performing transformations, not every point of a function has to change. Explain the characteristics of a point if it did not change after performing the indicated transformation.

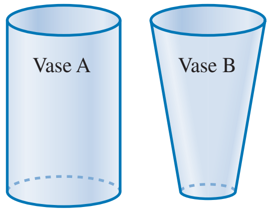
a. Horizontal stretch/compression

b. Vertical stretch/compression

c. Horizontal reflection

d. Vertical reflection

57. Water is poured into two vases at a constant rate.

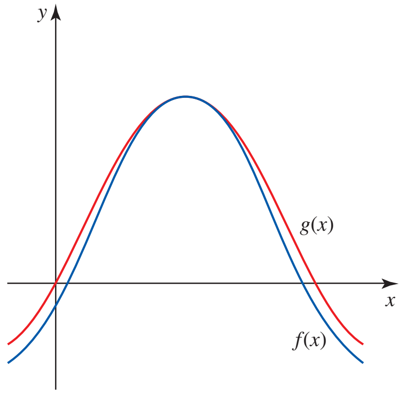


a. Sketch graphs showing the height of the water in each vase as a function of time.

b. If the width of each vase was increased, what would happen to the graphs drawn in part (a)? Explain.

c. If the rate at which the water was poured into the vases increased, what would happen to the graph drawn in part (a)? Explain.

58. Examine the following functions f and g.



Is it possible to create g by transforming f using only a horizontal stretch or compression? Explain.

59. Create the graphs of a function and its inverse. Explain how a vertical stretch or compression of the original function will affect the inverse function.

60. Create the graphs of a function and its inverse. Explain how a horizontal stretch or compression of the original function will affect the inverse function.

CHAPTER 3 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 3.1

1. What happens numerically, graphically, and symbolically when a function is shifted vertically? Why?
2. What happens numerically, graphically, and symbolically when a function is shifted horizontally? Why?

SECTION 3.2

3. What happens numerically, graphically, and symbolically when a function is reflected either horizontally or vertically? Why?
4. What is meant by even and odd symmetry? How can we test for even or odd symmetry?

SECTION 3.3

5. What is a vertical stretch or compression?
6. What impact do vertical stretches and compressions have on the rate of change?

SECTION 3.4

7. What is a horizontal stretch or compression?
8. What impact do horizontal stretches and compressions have on rate of change?
9. What effect does a negative stretch or compression have on a function?

REVIEW EXERCISES

■ SECTION 3.1 ■

For each given function in Exercises 1–3,

a. Graph the function.

b. Describe the transformations necessary on f to create g.

c. Draw the graph for g by performing the necessary transformations.

d. List at least three coordinate points on the graph of g.

e. Write the formula for g.

1. $f(x) = x^3, \quad g(x) = f(x - 1) + 6$

2. $f(x) = 4\sqrt{x}, \quad g(x) = f(x + 2) - 5$

3. $f(x) = \frac{1}{2}x + 3, \quad g(x) = f(x - 4) - 2$

In Exercises 4–6, use the table to determine the transformations on f required to create the indicated function. Then use function notation to write each transformed function in terms of f.

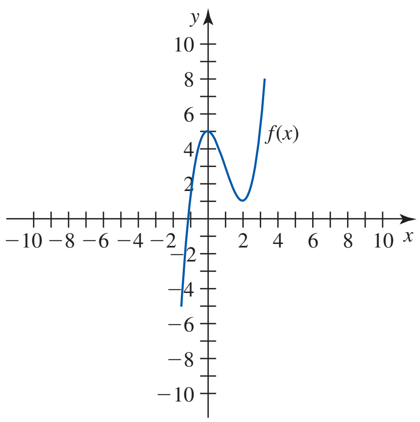
x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
−4	2	1	11	10
−3	4	1	5	12
−2	7	2	1	15
−1	11	4	0	19
0	5	7	−1	13
1	1	11	−3	9
2	0	5	−7	8
3	−1	1	−12	7
4	−3	0	−20	5

4. $g(x)$

5. $h(x)$

6. $j(x)$

In Exercises 7–8, describe the transformation on f required to create each function. Then use function notation to write each transformed function in terms of f.



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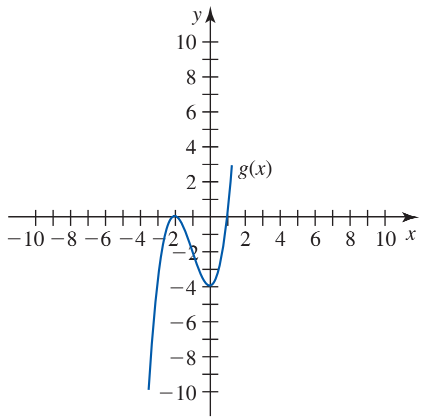

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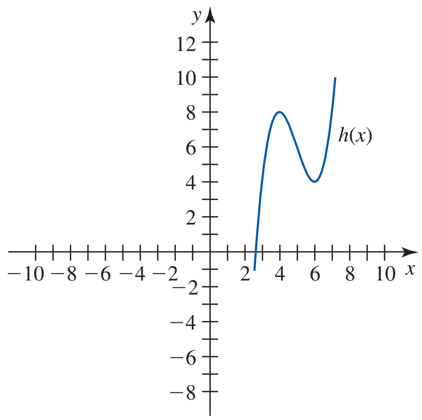


In Exercises 9–10,

a. Create a table of values for the aligned data.

b. Describe the horizontal translation required to align the data.

c. If the original function is f and the aligned function is g, use function notation to write g in terms of f.

9. Align the data to 1992.

Year	Median Family Income in Constant (2003) Dollars
1980	44,452
1985	45,223
1990	48,248
1992	46,992
1994	47,615
1996	49,378
1998	52,675
2000	54,191

***Source: Statistical Abstract of the United States, 2006*, Table 679**

10. Align the data to 1999.

Year	Individual Income Tax Returns Filed (in millions)
1996	116.1
1997	118.4
1998	120.3
1999	122.5
2000	124.9
2001	127.1
2002	129.4
2003	130.3
2004	130.1

Source: Statistical Abstract of the United States, 2006, Table 471

11. Refer to the table in Exercise 9.

a. Create a table that shows the increase in median family income since 1980.

b. Describe the transformation required to create the table in part (a).

c. If the original function is f and the transformed function is g, use function notation to write g in terms of f.

12. Refer to the table in Exercise 10.

a. Create a table that shows the increase in individual tax returns filed since 1998.

b. Describe the transformation required to create the table in part (a).

c. If the original function is f and the transformed function is g, use function notation to write g in terms of f.

In Exercises 13–16, use the values shown in Table E to evaluate each expression.

Table E

x	$f(x)$
-3	-11
-2	-8
-1	-1
0	3
1	5
2	4
3	2

13. $f(x) - 6$ when $x = 2$

14. $f(x) + 2.5$ when $x = 0$

15. $f(x + 1)$ when $x = -3$

16. $f(x - 5)$ when $x = 3$

In Exercises 17–18, solve each equation for x using Table E.

17. $f(x) + 1 = 6$

18. $f(x - 4) = -1$

In Exercises 19–20, refer to Table E to answer each question.

19. Create a table of values for g if $g(x) = f(x + 2)$.

20. Create a table of values for h if $h(x) = f(x) - 4.5$.

SECTION 3.2

For each given function in Exercises 21–23,

a. Graph the function.

b. Describe the transformations necessary on f to create g.

c. Draw the graph for by performing the necessary transformations.

d. List at least three coordinate points on the graph of g.

e. Write the formula for g.

21. $f(x) = x^3, \quad g(x) = f(-x) + 2$

22. $f(x) = 2|x|, \quad g(x) = -f(x + 1)$

23. $f(x) = \frac{2}{5}x - 7, \quad g(x) = -f(-x) - 6$

24. Falling Objects The function $d(t) = 16t^2$ can be used to model the distance an object travels in feet t seconds after it has been dropped.

a. Draw a graph for d.

b. Suppose you dropped an object and it fell for 1.6 seconds before hitting the ground. How high did it fall from?

c. Suppose you drop an object from your hand 4 feet above ground. How long will it take to hit the ground? How long will it take to reach a height of 2 feet off the ground?

d. Imagine dropping an object off the roof of a 100-foot-tall building. Graph the function h, the height of the object above ground t seconds after being dropped.

e. Describe how you could have created the graph for function h by performing transformations on d. Then use function notation to demonstrate the relationship between the functions.

25. Business Expenses The table shows the labor expenses for Chipotle Mexican Grill in years after 2000.

Year Since 2000 t	Labor Costs (\$ thousands) $L(t)$
1	46,048
2	66,515
3	94,023
4	139,494
5	178,721

Source: Chipotle Mexican Grill, Inc., 2005 Annual Report, p. 24

a. Why are labor costs considered negative numbers when determining their effect on profit? Explain how L can be transformed to represent this.

b. Use function notation to write N, the effect on profit of the labor costs, in terms of L.

c. Describe what function S represents if $S(y) = N(y - 2000)$.

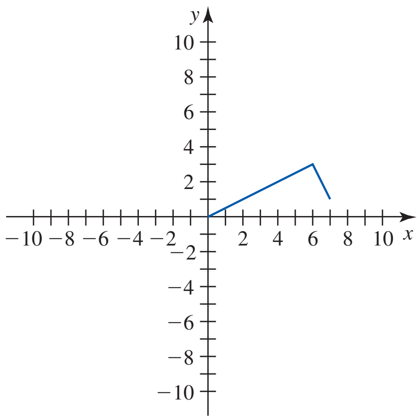
d. Create a table of values for S. Would a linear function be a good model for S? Explain why or why not.

In Exercises 26–27, complete each of the following for the given graph.

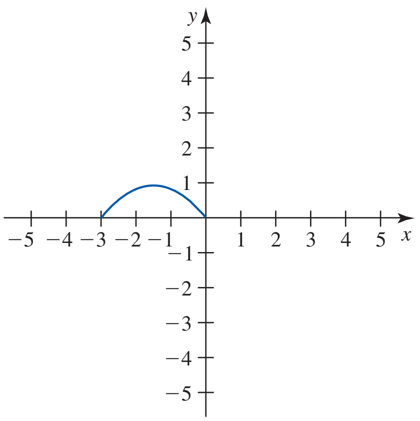
a. Explain what it means for a function to display even symmetry. Then complete the graph so that the function displays even symmetry.

b. Explain what it means for a function to display odd symmetry. Then complete the graph so that the function displays odd symmetry.

26.



27.



In Exercises 28–30, determine whether each of the following functions has even symmetry, odd symmetry, or neither type of symmetry.

28. $f(x) = 4x + 3$

29. $f(x) = \dfrac{2}{x^2 + 1}$

30. $f(x) = 2.3x^5 + x^3$

■ SECTION 3.3 ■

For each given function in Exercises 31–33,

a. Graph the function.

b. Describe the transformations necessary on f to create g.

c. Draw the graph for g by performing the necessary transformations.

d. List at least three coordinate points on the graph of g.

e. Write the formula for g.

31. $f(x) = 2x, \quad g(x) = 5f(x - 4)$

32. $f(x) = \dfrac{1}{x^2}, \quad g(x) = -0.75f(x)$

33. $f(x) = \sqrt{x} - 1, \quad g(x) = 1.5f(-x) + 8$

34. Complete the table of values as much as possible. You will not have enough information to fill in every missing number.

x	$f(x)$	$\frac{1}{2}f(x)$	$4f(-x)$	$-2f(x-3)$
−9	2			
−6	4			
−3	7			
0	11			
3	5			
6	1			
9	0			

In Exercises 35–40, use the values shown in Table F to evaluate each expression.

Table F

x	$f(x)$
−3	14
−2	11
−1	10
0	12
1	16
2	22
3	31

35. $4f(x)$ when $x = -1$

36. $-f(x + 4)$ when $x = -2$

37. $\frac{4}{3}f(x - 2) + 1$ when $x = 0$

38. $3.25\,f(-x) + 7$ when $x = 3$

39. $9f(-2) + 0.5f(0)$

40. $\frac{5}{2}f(1) - 3\,f(3)$

In Exercises 41 –42, solve each equation for x using Table F.

41. $3f(-x) = 30$

42. $\frac{1}{3}f(x - 5) + 6 = 10$

In Exercises 43–44, refer to Table F to answer each question.

43. Create a table of values for g if $g(x) = -f(-x)$.

44. Create a table of values for h if $h(x) = 2.5f(x) - 4$.

45. Member Benefits In January 2007, one of the authors visited a Barnes & Noble bookstore with a list of hardcover books he was considering for purchase. Let R represent the retail cost of purchasing of these books.

a. The author is a Barnes & Noble member, which entitles him to 20% off the cost of hardcover books. Use function notation to show the relationship between the author's cost, A, of purchasing x of the books on his list and the retail cost.

b. When he visited the store, the author had a \$50 gift card that he intended to use toward the purchase of these books. Use function notation to write G, the author's cost to purchase x books from his list as a member with the gift card, in terms of R.

c. In performing the transformations on R to create G, which transformation must occur first? Explain why this order makes sense in the given situation.

d. What do you know about this situation if function G's output is negative?

46. Baseball Salaries The average salary of a major league baseball player in millions of dollars can be modeled by $A(t) = 0.1474t + 0.4970$, where t is the years since 1990. (*Source:* **http://sportsline.com/mlb/salaries/avgsalaries**)

a. A function S models the average salary of a major league baseball player in *dollars*. Explain how to transform A to create S. Then use function notation to write S in terms of A.

b. Write the formula for S.

c. V models the average salary in dollars of a major league baseball player in year y. Use function notation to write V in terms of S.

d. Write the formula for V.

47. Military Spending The table shows the total spent for military payroll in thousands (i.e., \$15,375 represents \$15,375,000) between 1990 and 2003.

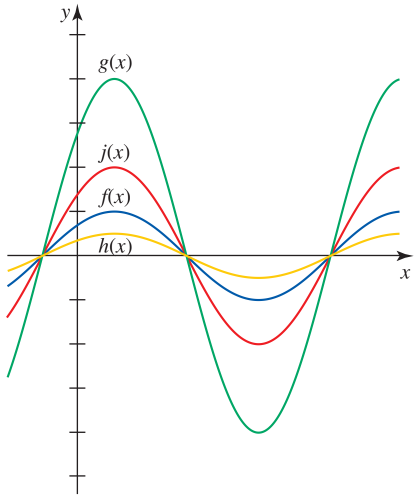
Year y	Total Military Payroll Expenses (\$ thousands) $P(y)$
1990	88,650
1995	98,396
2000	103,447
2001	106,013
2002	114,950
2003	122,270

Source: Statistical Abstract of the United States, 2006, **Table 496.**

a. S represents the total military payroll expenses in *dollars*. Explain how to transform P to create S. Then use function notation to write S in terms of P.

b. M represents the total military payroll expenses in *millions of dollars*. Explain how to transform P to create M. Then use function notation to write M in terms of P.

48. The graph shows f and three functions created by performing vertical stretches or compressions on f. Estimate the stretch or compression factor necessary to create each function.



SECTION 3.4

For each given function in Exercises 49–51,

a. Graph the function.

b. Describe the transformations necessary on f to create g.

c. Draw the graph for g by performing the necessary transformations.

d. List at least three coordinate points on the graph of g.

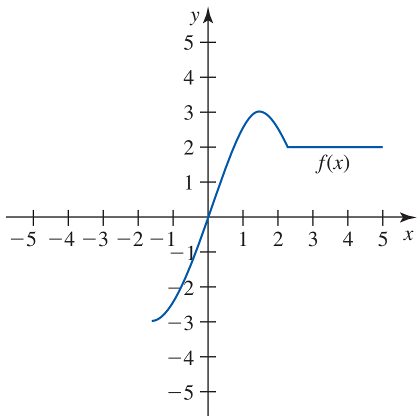
e. Write the formula for g.

49. $f(x) = 5x^2, \quad g(x) = f(0.25x) - 10$

50. $f(x) = x + 4, \quad g(x) = -f(2x) + 6$

51. $f(x) = \sqrt{x}, \quad g(x) = f(-3x)$

In Exercises 52–55, describe the transformation on f required to create each function. Then graph each transformed function.



52. $g(x) = f\left(\frac{1}{3}x\right) - 2$

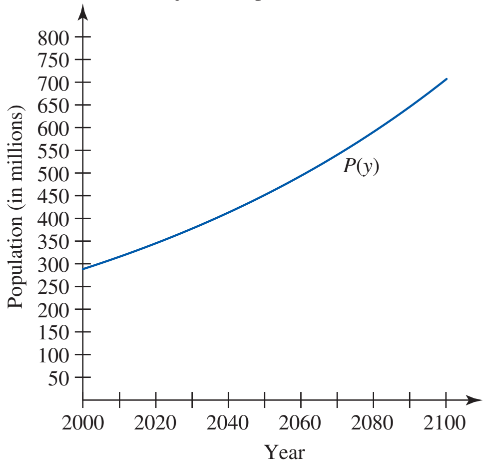
53. $g(x) = -\frac{1}{2}f(x - 2) + 4$

54. $g(x) = 2f(-x)$

55. $g(x) = f(2(x + 4))$

56. U.S. Population The following graph of P is the projected population of the United States (in millions) throughout the 21st century, assuming that the growth rate from 1995–2004 continues.

P, Projected Population of the U.S.



Source: **Modeled from World Health Organization**

a. The function A models the U.S. population in millions t years after 2000. Describe how to transform P to create A. Then use function notation to write A in terms of P.

b. The function N represents the population of the United States in millions x decades after 2000. Use function notation to write N in terms of P and describe the transformations involved.

c. Explain what M models if

$$M(x) = 1{,}000{,}000P(10x + 2000)$$

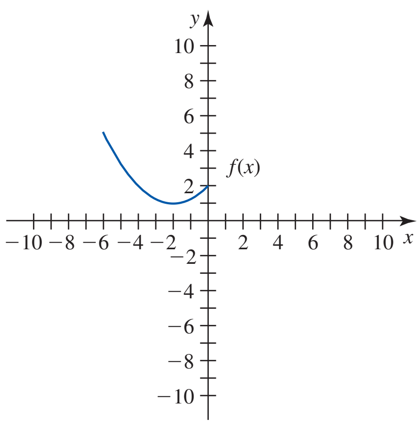
57. Real Estate The function $V(T) = 0.096345T + 15.099$ approximates the number of vacant housing units (in thousands) in a state that contains T thousand total housing units. (This function model is valid for up to 5900 thousand total housing units.)

a. Explain what f models if

$$f(x) = 1000V(0.001x)$$

b. Write the formula for f.

In Exercises 58–60, describe the transformation required on the given function f to create each function. Then use function notation to write each transformed function in terms of f.



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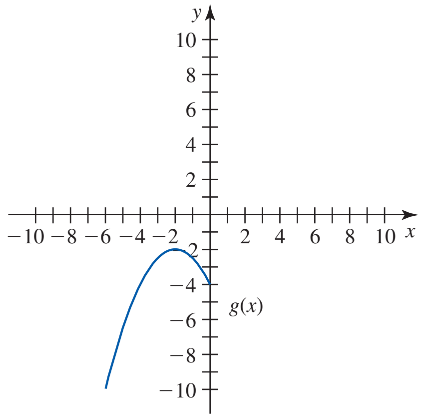

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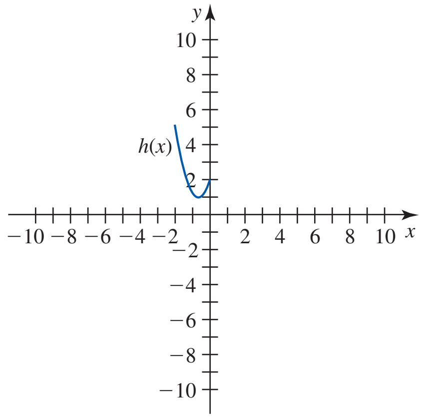
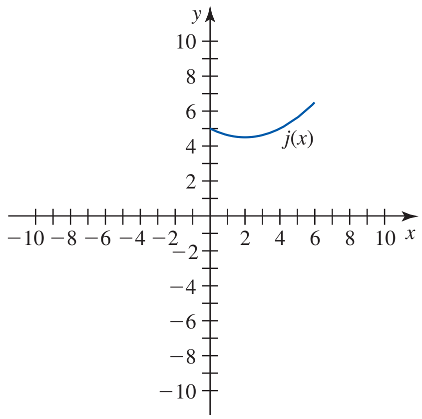


60.



61. The function $g(x) = \dfrac{1}{3}x$ is a transformation of the function $f(x) = x$. Is $g(x)$ the result of a vertical compression by a factor of $\dfrac{1}{3}$ or a horizontal stretch by a factor of 3? Explain.

62. Master's Degrees Awarded The table shows the number of master's degrees awarded (in thousands) in the United States between 1960 and 2000.

Decades Since 1960 t	Number of Master's Degrees Awarded (in thousands) $M(t)$
0	75
0.5	121
1	209
1.5	293
2	298
2.5	286
3	325
3.5	398
4	457

***Source: Statistical Abstract of the United States, 2006*, Table 286.**

a. $D(y)$ models the number of master's degrees awarded y years after 1960. Explain how to transform M to create D. Then use function notation to write d in terms of M.

b. Explain what N represents if

$$N(y) = 1000D(y)$$

63. U.S. Exports Based on data from 1990 to 2005, the value of U.S. goods exported to Colombia may be modeled by

$$C(t) = 122.81(1.22)^t \text{ million dollars}$$

where t is the number of years since 1990. **(*Source:* Modeled from *Statistical Abstract of the United States, 2006*, Table 1293)**

a. E models the U.S. exports to Colombia d decades after 1990. Explain how to transform C to create E. Then use function notation to write E in terms of C.

b. Write the formula for E.

c. Explain what P represents if

$$P(d) = 1{,}000{,}000E(d)$$

In Exercises 64–67, explain the effect, if any, of each type of transformation on the given function characteristic.

a. Vertical translation
b. Horizontal translation
c. Vertical reflection
d. Horizontal reflection
e. Vertical stretch/compression
f. Horizontal stretch/compression

64. Horizontal intercept(s)

65. Vertical intercept

66. Sign of the rate of change (positive vs. negative)

67. Magnitude of the rate of change

Make It Real Project

What to Do

1. From an area of personal interest, find a set of data that is a function of year.
2. Pick a year and align the data to that year. Describe the transformation of the data needed to achieve this alignment.
3. Perform a vertical shift on the data. Describe what the vertical shift represents.
4. Starting over with the original data set, use a vertical stretch or compression to change the units. Explain what the transformed data set represents.
5. Use a horizontal stretch or compression to organize the data by decades instead of years. Explain what the transformed data set represents.
6. Would performing a vertical or horizontal reflection on the data set have any real-world meaning? Explain.

Where to Find Data

The following websites may contain data that may be helpful for this project.

InfoPlease

- www.infoplease.com/almanacs.html

Information from sports to education to science and health

U.S. Census Bureau

- www.census.gov

A variety of information on various characteristics of the U.S. population

BBall Sports

- www.bballsports.com/

A database of statistics from professional baseball, basketball, hockey, and football

United Nations Statistics Division

- unstats.un.org/unsd/databases.htm

Demographics and business statistics from countries around the world