CHAPTER 2

Linear Functions

Among all cancers, lung cancer is the number-one killer. Smoking is the leading cause of lung cancer. Fortunately, smoking rates in the United States have declined dramatically since 1950. The smoking rate in the United States can be represented by a linear function.

2.1 Functions with a Constant Rate of Change

2.2 Modeling with Linear Functions

2.3 Linear Regression

2.4 Systems of Linear Equations

2.5 Systems of Linear Inequalities

STUDY SHEET

REVIEW EXERCISES

MAKE IT REAL PROJECT

SECTION 2.1 Functions with a Constant Rate of Change

LEARNING OBJECTIVES

- Explain the real-world meaning of slope, horizontal intercept, and vertical intercept
- Write linear functions in slope-intercept form, point-slope, and standard form
- Graph linear functions

GETTING STARTED

In April 1896, the first modern Olympics were held in Athens, Greece. The U.S. Olympic Team Manager for those games, John Graham, was inspired by the marathon race he witnessed there. With the help of a Boston businessman, Graham planned the first annual Boston Marathon in April 1897. The Boston Marathon is now considered one of the premier distance races in the world, pitting the best runners against each other in a grueling run of over 26.2 miles. In 2009, the fastest runner was Ethiopian Deriba Merga, who completed the race in just over 2 hours and 8 minutes. His average speed, which was less than 5 minutes per mile, is an example of a constant rate of change. (*Source:* www .bostonmarathon.org)

In this section we investigate the concept of a constant rate of change, classify functions with constant rates of change as linear functions, and explore such functions with words, graphs, tables, and formulas. We develop strategies for graphing linear functions given in the three common forms—slope-intercept, point-slope, and standard.

Olaru Radian-Alexandru/Shutterstock.com

Constant Speed

If you have traveled in a car with the cruise control on, you have some sense of the concept of a constant speed, or constant rate of change in distance with respect to time. This mathematically powerful concept is closely linked to the linear function concept. Let's look at this relationship in the context of a marathon race.

Marathon courses have mile markers to help runners monitor their progress. Suppose that between two such mile markers, Deriba Merga ran at a *constant speed* of 17.6 feet per second (completing the mile in 5 minutes). This means that *for every second that elapsed since the time he passed the mile marker, he moved exactly 17.6 feet.* If Merga runs at this constant speed for 2 seconds, he will travel 35.2 feet past the mile marker: $17.6\,\frac{\text{feet}}{\text{second}}(2\text{ seconds}) = 35.2\text{ feet}$. If he runs at this constant speed for 14.9 seconds, he will travel 262.24 feet: $17.6\,\frac{\text{feet}}{\text{second}}(14.9\text{ seconds}) = 262.24\text{ feet}$. To generalize, if Merga runs at a constant speed of $17.6\,\frac{\text{feet}}{\text{second}}$ for t seconds after passing the mile marker, he will travel $17.6t$ feet past the mile marker.

$$\underbrace{d}_{\substack{\text{the distance (in feet)}\\ \text{that Merga runs}\\ \text{past the mile marker}}} \underbrace{=}_{\text{is}} \underbrace{17.6}_{\text{17.6 ft/sec}} \underbrace{\times}_{\text{for}} \underbrace{t}_{\substack{t\text{ seconds}\\ \text{since the time}\\ \text{he passed the}\\ \text{mile marker}}}$$

or

$$\underbrace{d}_{\substack{\text{the distance (in feet)}\\ \text{that Merga runs}\\ \text{past the mile marker}}} \underbrace{=}_{\text{is}} \underbrace{17.6t}_{17.6t\text{ ft}}$$

More broadly stated, if an object is traveling at a constant speed, then

$$\text{change in distance} = (\text{constant speed})(\text{time elapsed})$$

Calling the constant speed m, we say "the change in distance will always be m times the amount of time elapsed." For example, if the constant speed is 2.9 miles per hour, we say "the change in distance (in miles) will always be 2.9 miles per hour times the number of hours elapsed."

EXAMPLE 1 ■ **Interpreting Constant Speed**

Interpret each of the following constant speeds first with an example and then in general terms.

a. 3 miles per hour **b.** 6.14 meters per day

Solution

a. If an object travels at a constant speed of 3 miles per hour for 1 hour, the object travels 3 miles: $\left(3\,\dfrac{\text{miles}}{\text{hour}}\right)(1\text{ hour}) = 3$ miles. In general, if the object travels for h hours at a constant speed of 3 miles per hour it will travel $3h$ miles.

b. If an object travels at a constant speed of 6.14 meters per day for 1 day, the object travels 6.14 meters: $\left(6.14\,\dfrac{\text{meters}}{\text{day}}\right)(1\text{ day}) = 6.14$ meters. In general, if the object travels for d days at a constant speed of 6.14 meters per day, it will travel $6.14d$ meters.

■ Constant Rate of Change

The idea of constant speed can be generalized to make sense of a *constant rate of change.* A situation has a **constant rate of change** m if, whenever the input changes, the output changes by m times as much.

$$\text{change in output value} = m(\text{change in input value})$$

For example, let x be the input and y the output for some function. If that function has a constant rate of change of 5, then

$$\text{change in } y = 5(\text{change in } x)$$

Now suppose x changes by 8. By how much will y change?

$$\begin{aligned}\text{change in } y &= 5(\text{change in } x)\\ \text{change in } y &= 5(8)\\ \text{change in } y &= 40\end{aligned}$$

The value of y will change by 40. By how much will y change if x changes by, say, 0.0004?

$$\begin{aligned}\text{change in } y &= 5(\text{change in } x)\\ \text{change in } y &= 5(0.0004)\\ \text{change in } y &= 0.002\end{aligned}$$

If x changes by 0.0004, then the value of y will increase by 0.002.

EXAMPLE 2 ■ **Interpreting Constant Rates of Change**

Each of the following values of m is a constant rate of change for some function. Explain what each value tells about the relationship between the input and output variables of the function.

a. $m = -2.7$; input variable is x, output variable is y.

b. $m = 113.44$; input variable is t, output is $f(t)$.

Solution

a. change in $y = -2.7$(change in x)

b. change in $f(t) = 113.44$(change in t)

In mathematical notation, we use the Greek letter delta (Δ) to represent the phrase “change in.” Thus, for example, “change in $f(t) = 113.44$(change in t)” is written “$\Delta f(t) = 113.44\Delta t$.”

EXAMPLE 3 ■ Interpreting Constant Rates of Change

Explain what each of the following constant rates of change mean in the context of the given situation.

a. From 1990 to 2003, the concentration of carbon monoxide in the atmosphere had a near constant rate of change of –0.248 parts per million per year. (*Source: Statistical Abstract of the United States, 2006*, Table 359)

b. Based on data from 1974–2003, the death rate due to heart disease as a function of the percentage of people who smoke has a constant rate of change of 14.08 deaths per 100,000 people per percentage point. (*Source:* Modeled from CDC and Census Bureau data)

Solution

a. In this context, the dependent (output) variable is the concentration of carbon monoxide in parts per million, C, and the independent (input) variable is the year, t. A constant rate of change of -0.248 parts per million per year means

$$\begin{aligned}\text{change in carbon monoxide concentration} &= -0.248(\text{number of years that pass})\\ \Delta C &= -0.248\Delta t\end{aligned}$$

Note that the rate of change is negative. This means that as time passes, the carbon monoxide concentration decreases. Also, since this problem has a practical domain of years from 1990 to 2003, the change in years is restricted to this domain.

b. In this context, the dependent (output) variable is the death rate due to heart disease in deaths per 100,000 people, R, while the independent (input) variable is the percentage of people who smoke, p. A constant rate of change of 14.08 deaths per 100,000 people per percentage point means

$$\begin{aligned}\text{change in death rate} &= 14.08\begin{pmatrix}\text{change in percentage}\\ \text{of people who smoke}\end{pmatrix}\\ \Delta R &= 14.08\Delta p\end{aligned}$$

Note that the rate of change is positive. This means that as the percentage of people who smoke increases, the death rate due to heart disease increases.

In general, we have the following definition for the concept of the constant rate of change.

CONSTANT RATE OF CHANGE

A function has a **constant rate of change** m if, for any change in the independent variable, the dependent variable always changes by exactly m times as much.

EXAMPLE 4 ■ Calculating a Constant Rate of Change in Context

The number of Medicare enrollees between 1980 and 2004 can be modeled by a function with a constant rate of change. In 1980 there were 28.4 million Medicare enrollees. By 2004 the number of Medicare enrollees had increased to 41.7 million. (*Source: Statistical Abstract of the United States, 2006*, Table 132) Determine the constant rate of change by which the number of Medicare enrollees increased over time.

Solution Calling the constant rate of change m, we have

$$\begin{matrix}\text{change in Medicare}\\ \text{enrollees (in millions)}\end{matrix} = m\begin{pmatrix}\text{number of}\\ \text{years that pass}\end{pmatrix}$$

From 1980 to 2004, 24 years elapsed ($2004 - 1980 = 24$). During this time period, the number of Medicare enrollees increased by 13.3 million ($41.7 - 28.4 = 13.3$). Thus,

$$\text{13.3 million enrollees} = m\,(\text{24 years})$$

$$\frac{\text{13.3 million enrollees}}{\text{24 years}} = \frac{m\cancel{(\text{24 years})}}{\cancel{\text{24 years}}}$$

$$\left(\frac{13.3}{24}\right)\frac{\text{million enrollees}}{\text{year}} = m$$

$$m \approx \text{0.554 million enrollees per year}$$

This means that for each year that passed, the number of Medicare enrollees increased by about 0.554 million, or about 554,000. More generally, from 1980 to 2004 a change of t years means a change of approximately $0.554t$ million Medicare enrollees.

Example 4 illuminates two very important ideas. First, the units of the constant rate of change come from the units of the dependent and independent variables.

Units of a Rate of Change

The units of a rate of change are the units of the dependent variable divided by the units of the independent variable.

$$\text{units of rate of change} = \frac{\text{units of dependent variable}}{\text{unit of independent variable}}$$

We commonly write this as "units of output per unit of input."

The second idea from Example 4 is that we can create a formula to calculate a constant rate of change. Recall that there were 28.4 million Medicare enrollees in 1980 and 41.7 million Medicare enrollees in 2004. We can represent this information with the ordered pairs (1980, 28.4) and (2004, 41.7). Then

$$\begin{array}{c}\text{change in Medicare}\\ \text{enrollees (in millions)}\end{array} = m\left(\begin{array}{c}\text{number of years}\\ \text{since 1980}\end{array}\right)$$

$$(41.7 - 28.4) = m(2004 - 1980)$$

$$\frac{(41.7 - 28.4)}{(2004 - 1980)} = \frac{m\cancel{(2004 - 1980)}}{\cancel{(2004 - 1980)}}$$

$$m = \frac{41.7 - 28.4}{2004 - 1980}$$

Because the rate of change is constant (i.e., for any change in the input, the output always changes by exactly m times as much), m will be the same when comparing *any* ordered pairs for the function. Thus, if we say a function has a constant rate of change and that (x_1, y_1) and (x_2, y_2) are ordered pairs for the function, then

$$\text{change in } y = m(\text{change in } x)$$

$$\frac{\text{change in } y}{\text{change in } x} = \frac{m\cancel{(\text{change in } x)}}{\cancel{\text{change in } x}}$$

$$m = \frac{\text{change in } y}{\text{change in } x}$$

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$y_2 - y_1$ tells us how much y changes
$x_2 - x_1$ tells us how much x changes

HOW TO: ■ CALCULATE A CONSTANT RATE OF CHANGE

To calculate a constant rate of change m, divide the change in the dependent variable by the change in the independent variable.

$$m = \frac{\text{change in the dependent variable}}{\text{change in the independent variable}}$$

If given ordered pairs (x_1, y_1) and (x_2, y_2),

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

In Chapter 1 we calculated an average rate of change by finding the difference of two outputs divided by the difference in the corresponding inputs. We used the formula $m = \frac{y_2 - y_1}{x_2 - x_1}$, which we now know is the method for calculating a constant rate of change. This is not a coincidence—an average rate of change *is* a constant rate of change. Specifically, it is the constant rate of change necessary for the function values to change by the same amount for the same change in the input. Thus, when using an average rate of change, we are reasoning about a function *as if* it has a constant rate of change over a given interval.

To illustrate, we consider the graphs of two functions shown in Figures 2.1 and 2.2. We calculate the average rate of change in the functions over the intervals [3, 5], [3, 6], [3, 7], and [3, 8]. (The red line segments between each pair of points indicate which pairs of points are being used to calculate the average rate of change.)

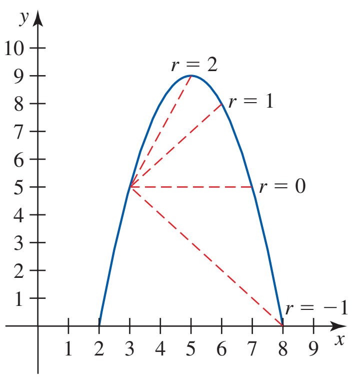


Figure 2.1

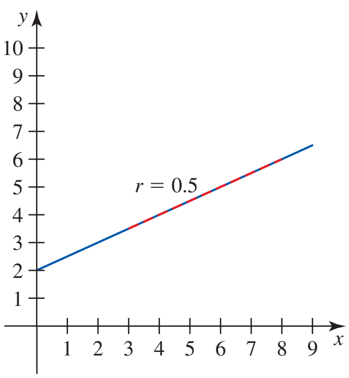


Figure 2.2

For the function in Figure 2.1, the average rate of change varies as soon as we vary the second point. For the intervals given, the average rates of change (r) are 2, 1, 0, and −1, respectively. The function in Figure 2.2 has a constant rate of change, so the average rate of change (r) over *any* interval is always 0.5. In fact, for any function with a constant rate of change, the average rate of change of the function over any interval is the same as the constant rate of change in the function.

EXAMPLE 5 ■ Using a Constant Rate of Change

Table 2.1 gives data points for a function with a constant rate of change. Find the value of a.

Table 2.1

x	y
−4	5.8
1	−0.2
2.5	a
3	−2.6

Solution To find a, we need to know the constant rate of change for this function. Since for any change in x, the value of y always changes by exactly m times as much, it does not matter which ordered pairs we choose. But for the sake of comparison, we show the calculations for several ordered pairs.

$(-4, 5.8)$ and $(3, -2.6)$	$(-4, 5.8)$ and $(1, -0.2)$	$(1, -0.2)$ and $(3, -2.6)$
$m = \dfrac{\text{change in } y}{\text{change in } x}$	$m = \dfrac{\text{change in } y}{\text{change in } x}$	$m = \dfrac{\text{change in } y}{\text{change in } x}$
$m = \dfrac{-2.6 - 5.8}{3 - (-4)}$	$m = \dfrac{-0.2 - 5.8}{1 - (-4)}$	$m = \dfrac{-2.6 - (-0.2)}{3 - 1}$
$m = \dfrac{-2.6 - 5.8}{3 + 4}$	$m = \dfrac{-0.2 - 5.8}{1 + 4}$	$m = \dfrac{-2.6 + 0.2}{3 - 1}$
$m = \dfrac{-8.4}{7}$	$m = \dfrac{-6}{5}$	$m = \dfrac{-2.4}{2}$
$m = -1.2$	$m = -1.2$	$m = -1.2$

Thus the constant rate of change for this function is -1.2. We now find a using the ordered pairs $(1, -0.2)$ and $(2.5, a)$ and the constant rate of change, $m = -1.2$.

$$
\begin{aligned}
m &= \frac{\text{change in } y}{\text{change in } x} \\
-1.2 &= \frac{-0.2 - a}{1 - (2.5)} \\
-1.2 &= \frac{-0.2 - a}{-1.5} \\
1.8 &= -0.2 - a \\
a + 1.8 &= -0.2 \\
a &= -2.0
\end{aligned}
$$

Linear Functions

A **linear function** is any function that has a constant rate of change. The constant rate of change is also called the **slope**. Linear functions are used extensively to model many real-world situations.

Linear Function

Any function with a constant rate of change is called a **linear function**.

Slope of a Linear Function

The constant rate of change of a linear function is called the **slope** of the function. The term *slope* is also used to refer to the steepness of the graph of a linear function.

EXAMPLE 6 ■ Creating a Linear Function from a Verbal Description

Based on data from 1990–2003, carbon monoxide pollutant concentrations in the United States have been decreasing at a rate of about 0.248 parts per million per year. In 1990, carbon monoxide pollutant concentrations were 6 parts per million. (*Source: Statistical Abstract of the United States, 2006*, Table 359) Create a function to model the carbon monoxide pollutant concentration in the United States as a function of years since 1990. Then estimate the carbon monoxide pollutant concentration for 2006.

Solution We are to find a pollutant concentration function $C(t)$ with the dependent variable, C, representing the carbon monoxide pollutant concentration (in parts per million) and the independent variable, t, representing the number of years since 1990.

We know $C(0) = 6$ because $t = 0$ corresponds to 1990, and in 1990 the pollutant concentration was 6 parts per million. We also know the pollutant concentration is decreasing by 0.248 parts per million each year. There are t years beyond 1990 so the expression $-0.248t$ will calculate the amount of change in the pollutant level over a period of t years. Therefore, we will add the change in pollutant level, $-0.248t$, to the pollutant level of 6 parts per million. The function is then

$$C(t) = 6 + (-0.248t)$$
$$C(t) = 6 - 0.248t$$

To estimate the pollutant concentration in 2006, we evaluate $C(t)$ at $t = 16$.

$$\begin{aligned} C(16) &= 6 - 0.248(16) \\ &= 2.032 \end{aligned}$$

We estimate the carbon monoxide pollutant concentration in 2006 to be 2.032 parts per million.

Slope-Intercept Form of a Linear Function

Linear functions may be represented in slope-intercept, point-slope, or standard form. Each form has its benefits and we will use them all.

Slope-Intercept Form of a Linear Function

A linear function with slope m and vertical intercept $(0, b)$ is written in **slope-intercept form** as

$$y = mx + b$$

The value b is commonly referred to as the **initial value**.

EXAMPLE 7 ■ Graphing a Function in Slope-Intercept Form

Graph the function $C(t) = -0.248t + 6$ from Example 6. Recall that C represents the carbon monoxide concentration (in parts per million) and t represents the number of years since 1990.

Solution We have $C(t) = -0.248t + 6$. We note the initial value is 6 so the vertical intercept is $(0, 6)$. Starting at $(0, 6)$, we increase the number of years since 1990. Since the slope is -0.248, each 1-year increase will result in a 0.248 parts per million *decrease* in the pollutant concentration. In other words, for each increase of t years, C changes by $-0.248t$. We calculate the concentration (C) values for the arbitrary values of $t = 4$, 10, and 13, then plot and connect the resultant ordered pairs. The graph is shown in Figure 2.3.

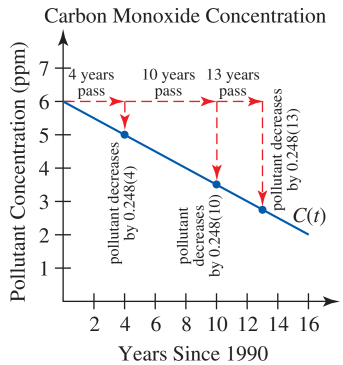


Figure 2.3

EXAMPLE 8 ■ Interpreting the Slope and Vertical and Horizontal Intercepts

The annual U.S. lumber imports from Canada from 1998 to 2003 can be approximated with the linear function $L(t) = 598.74t + 18{,}895$, where L is measured in millions of board feet and t is the number of years since 1998. (*Source:* Modeled from *Statistical Abstract of the United States, 2006*, Table 852) Find and interpret the slope, vertical intercept, and horizontal intercept of the linear function.

Solution The slope (constant rate of change) of the function is 598.74 million board feet per year since 1998. Thus, the United States increased its lumber imports from Canada by 598.74 million board feet per year between 1998 and 2003.

The vertical intercept is (0, 18,895). This means that in 1998 ($t = 0$), the United States imported 18,895 million board feet of lumber from Canada.

We find the horizontal intercept by letting $L(t) = 0$ and solving for t.

$$\begin{aligned} L(t) &= 598.74t + 18{,}895 \\ 0 &= 598.74t + 18{,}895 \\ -18{,}895 &= 598.74t \\ \frac{-18{,}895}{598.74} &= \frac{\cancel{598.74}t}{\cancel{598.74}} \\ -31.6 &\approx t \end{aligned}$$

The horizontal intercept is approximately (−31.6, 0). This tells us that 31.6 years before the end of 1998 (that is, mid-1957) the United States did not import any lumber from Canada. We are skeptical of this prediction because it lies well outside the domain used to create the model.

We saw that the graph of the function in Example 7 is a line. In fact, since every linear function has a constant rate of change, the graph of every linear function will be a line. This means that we only need two points to graph a linear function. Once we have plotted the two points, we can connect them with a line and know that every ordered pair that satisfies the function equation will lie on that line.

EXAMPLE 9 ■ Graphing a Function in Slope-Intercept Form

Graph $f(x) = -\dfrac{3}{2}x + 4$.

Solution To graph the line that represents this linear function, we use the vertical intercept (0, 4) as our first point. To find a second point, we evaluate the function at any other value of x. We choose $x = 6$.

$$\begin{aligned} f(x) &= -\frac{3}{2}x + 4 \\ f(6) &= -\frac{3}{2}(6) + 4 \\ &= -9 + 4 \\ &= -5 \end{aligned}$$

Thus a second point is (6, −5). The graph is shown in Figure 2.4

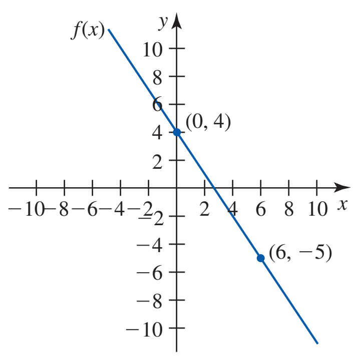


Figure 2.4

EXAMPLE 10 ■ Horizontal and Vertical Lines

Find the slope and equation for each line shown in Figure 2.5.

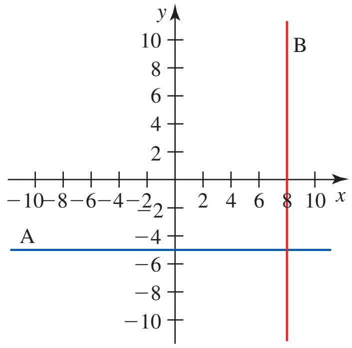


Figure 2.5

Solution We begin with A, the horizontal line. We pick two points on the line and determine the slope. We choose (0, −5) and (6, −5).

$$\begin{aligned} m &= \frac{-5-(-5)}{6-0} \\ &= \frac{-5+5}{6-0} \\ &= \frac{0}{6} \\ &= 0 \end{aligned}$$

The slope is zero. What does this mean? Let's refer back to our understanding of constant rate of change.

$$\begin{aligned} \text{change in } y &= m(\text{change in } x) \\ \text{change in } y &= 0(\text{change in } x) \\ \text{change in } y &= 0 \end{aligned}$$

Thus a slope of zero means that no matter what the change in x is, the change in y is zero. That is, y does not change. When x varies, y remains constant. Since the value of y is constant, the equation of the line does not depend on x but is defined entirely by that constant y value. So the equation of Line A in Figure 2.5 is $y = -5$.

We now consider B, the vertical line. We pick two points and determine the slope. We choose (8, 0) and (8, 2).

$$\begin{aligned} m &= \frac{2-0}{8-8} \\ &= \frac{2}{0} \rightarrow \text{undefined} \end{aligned}$$

In this case we do not get a numeric value and we say that the slope is *undefined*. In other words, the slope does not exist. Again, let's return to our understanding of a constant rate of change.

$$\begin{aligned} \text{change in } y &= m(\text{change in } x) \\ \text{change in } y &= m0 \\ \text{change in } y &= 0 \end{aligned}$$

This does not make sense because we know the change in y is not 0. In fact, no value of m can make the statement true. Thus, there is no defined slope for a vertical line.

For a vertical line, the x-value is constant but y can be any number. Consequently, the equation for a vertical line is of the form $x = a$ or, in this case, $x = 8$. Note that this is not a function—it fails the vertical line test (the same input has multiple outputs).

Equations for Horizontal and Vertical Lines

- A horizontal line with vertical intercept $(0, b)$ has equation $y = b$.
- A vertical line with horizontal intercept $(a, 0)$ has equation $x = a$.

Interpreting the Graphical Meaning of the Slope of a Line

The slope of a line tells us much about the graph of the linear function and, if the linear function represents a real-world situation, much about the underlying context.

RELATIONSHIP BETWEEN THE SLOPE OF A LINE AND ITS GRAPH

- A line with *positive* slope is *increasing* (going up) from left to right.
- A line with *negative* slope is *decreasing* (going down) from left to right.
- A line with *zero* slope is *horizontal*.
- A line with *undefined* slope is *vertical*.
- The greater the *magnitude* (absolute value) of the slope, the steeper the line.

Point-Slope Form of a Linear Function

The point-slope form of a line is most useful when we are given a point and a slope for an underlying context.

EXAMPLE 11 ■ Writing a Linear Function in Point-Slope Form

Between 1980 and 2004, the number of Medicare enrollees increased by approximately 0.554 million enrollees per year. By 2004 the number of Medicare enrollees had increased to 41.7 million. (*Source:* Modeled from *Statistical Abstract of the United States, 2006*, Table 132) Create a linear model for the number of Medicare enrollees as a function of the number of years since 2000.

Solution We first need to define our variables. We let E represent the number of Medicare enrollees (in millions) and t represent the number of years since 2000. Using these variables, we know the linear function has slope $m = 0.554$ and passes through the point $(4, 41.7)$ since $t = 4$ in 2004.

Observe that $E - 41.7$ represents the change in the Medicare enrollees since 2004 (in millions) and that $t - 4$ represents the change in years since 2004. From our earlier discussion of constant rate of change we know that

$$\text{change in output} = m(\text{change in input})$$

Applying that knowledge to this context, we have

$$E - 41.7 = 0.554(t - 4)$$

This form is referred to as *point-slope form*.

Given any constant rate of change m and ordered pair (x_1, y_1), we can write a formula to find any ordered pair (x, y) as follows.

$$\text{change in } y = m(\text{change in } x)$$
$$y - y_1 = m(x - x_1)$$

POINT-SLOPE FORM OF A LINEAR FUNCTION

A linear function with slope m and a point (x_1, y_1) is written in **point-slope form** as $y - y_1 = m(x - x_1)$.

When using technology to graph the line, we typically convert the point-slope form into the modified form $y = m(x - x_1) + y_1$, which is a format a graphing calculator accepts.

EXAMPLE 12 ■ Writing and Graphing Linear Functions in Point-Slope Form

The slope of a line is -2 and the line passes through the point $(3, -4)$. Write an equation of the line in point-slope form. Then graph the function.

Solution We know that the change in $y = m$(change in x). Applying that knowledge to this problem, we have

$$\begin{array}{c}\text{change from}\\ y = -4 \text{ to some}\\ \text{other } y\text{-value}\end{array} \quad = \quad -2 \times \left(\begin{array}{c}\text{change from}\\ x = 3 \text{ to some}\\ \text{other } x\text{-value}\end{array}\right)$$

$$\underbrace{y - (-4)}_{\substack{\text{the change from}\\ y\,=\,-4\text{ to some}\\ \text{other } y\text{-value}}} \quad \underbrace{=}_{\text{is equal to}} \quad \underbrace{-2 \times \overbrace{(x - 3)}^{\substack{\text{the change from}\\ x\,=\,3\text{ to some}\\ \text{other } x\text{-value}}}}_{\substack{\text{the rate of change}\\ \text{times the change in } x}}$$

$$y + 4 = -2(x - 3)$$

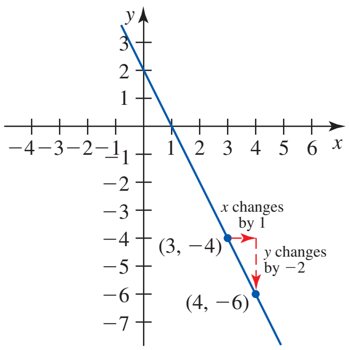


Figure 2.6

To graph the function, we plot the known point $(3, -4)$. Since the slope is -2, a 1-unit increase in x corresponds with a 2-unit decrease in y. So as x increases from 3 to 4, y decreases from -4 to -6. Thus, the point $(4, -6)$ is also on the line. We plot this point and draw a line passing through both points, as shown in Figure 2.6.

■ Standard Form of a Linear Function

In some situations we do not have a designated independent and dependent variable. Instead, there are two variable quantities that relate to each other. In these circumstances, slope-intercept and point-slope forms are not natural representations. Instead, we use the *standard form* of a linear function.

STANDARD FORM OF A LINEAR FUNCTION

A linear function can be written in **standard form** as

$$Ax + By = C$$

where A, B, and C are real numbers.

EXAMPLE 13 ■ Writing a Formula for a Linear Function in Standard Form

The American Heart Association recommends that adults eat 25 to 30 grams of fiber each day. (*Source:* www.americanheart.org) Metamucil Orange Coarse Milled Fiber Supplement provides 3 grams of fiber per tablespoon and a cup of Kashi GOLEAN Crunch! Cereal provides 8 grams of fiber. (*Source:* www.metamucil.com and www.kashi.com) Suppose an adult male wants to use these products to get 15 grams of fiber each day. (The rest of the suggested fiber amount will come from other food sources.) Construct a linear function to model how much Metamucil supplement and Kashi cereal the man needs to consume each day to meet this goal.

Solution Let c be the tablespoons of Metamucil the man takes in a day and k be the cups of Kashi the man eats in a day. Since Metamucil has 3 grams of fiber per tablespoon and Kashi has 8 grams of fiber per cup, we have

$$3\,\frac{\text{grams}}{\text{tablespoon}} \cdot c \text{ tablespoons} + 8\,\frac{\text{grams}}{\text{cup}} \cdot k \text{ cups} = 15 \text{ grams}$$
$$3\,\frac{\text{grams}}{\cancel{\text{tablespoon}}} \cdot c\ \cancel{\text{tablespoons}} + 8\,\frac{\text{grams}}{\cancel{\text{cup}}} \cdot k\ \cancel{\text{cups}} = 15 \text{ grams}$$
$$3c \text{ grams} + 8k \text{ grams} = 15 \text{ grams}$$
$$3c + 8k = 15$$

To show that the function $3c + 8k = 15$ is a linear function, we can solve for one of the variables.

$$3c + 8k = 15$$
$$8k = -3c + 15$$
$$k = \frac{-3c + 15}{8}$$
$$k = -\frac{3}{8}c + \frac{15}{8}$$

Since there is a constant rate of change ($-\frac{3}{8}$ cups of Kashi per tablespoon of Metamucil), the function is linear.

EXAMPLE 14 ■ Graphing a Linear Function in Standard Form

We choose to plot c on the horizontal axis and k on the vertical axis for the function $3c + 8k = 15$ from Example 13. Find the vertical and horizontal intercepts of the function and interpret their meanings in the given context. Then graph the function.

Solution To find the vertical intercept, we set $c = 0$ and solve for k.

$$3c + 8k = 15$$
$$3(0) + 8k = 15$$
$$0 + 8k = 15$$
$$8k = 15$$
$$k = \frac{15}{8}$$

The vertical intercept $\left(0, \frac{15}{8}\right)$ tells us that if the man only eats Kashi, he will need to eat $\frac{15}{8}$ cups to obtain 15 grams of fiber.

To find the horizontal intercept, we set $k = 0$ and find the value of c.

$$3c + 8k = 15$$
$$3c + 8(0) = 15$$
$$3c + 0 = 15$$
$$3c = 15$$
$$c = 5$$

The horizontal intercept (5, 0) tells us that if the man only takes Metamucil, he must take 5 tablespoons to obtain 15 grams of fiber.

Using the two intercept points, we construct the graph shown in Figure 2.7. Each point on the line shows combinations of Metamucil and Kashi the man can take to obtain 15 grams of fiber.

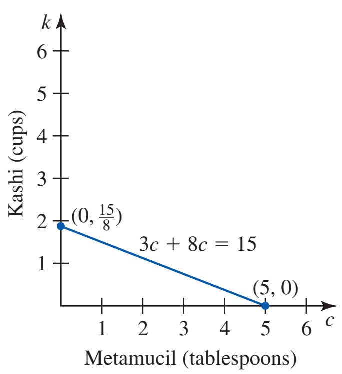


Figure 2.7

Since the intercepts for a linear function in standard form are so convenient for graphing the function, let's streamline the process of finding them.

$$\begin{aligned} Ax + By &= C & \qquad Ax + By &= C \\ Ax + B(0) &= C & A(0) + By &= C \\ Ax &= C & By &= C \\ x &= \frac{C}{A} & y &= \frac{C}{B} \end{aligned}$$

Thus the horizontal intercept is $\left(\frac{C}{A}, 0\right)$ and the vertical intercept is $\left(0, \frac{C}{B}\right)$.

HOW TO: ■ GRAPH A LINEAR FUNCTION IN STANDARD FORM

To graph the linear function $Ax + By = C$,

1. Plot the horizontal intercept $\left(\frac{C}{A}, 0\right)$.
2. Plot the vertical intercept $\left(0, \frac{C}{B}\right)$.
3. Connect the two intercepts with a straight line.

EXAMPLE 15 ■ Graphing a Linear Equation in Standard Form

Graph $2x - 6y = 10$.

Solution We begin by finding the intercepts.

$$x = \frac{10}{2} \qquad y = \frac{10}{-6}$$

$$x = 5 \qquad y = -\frac{5}{3}$$

The horizontal intercept is (5, 0) and the vertical intercept is $\left(0, -\frac{5}{3}\right)$. The graph is shown in Figure 2.8.

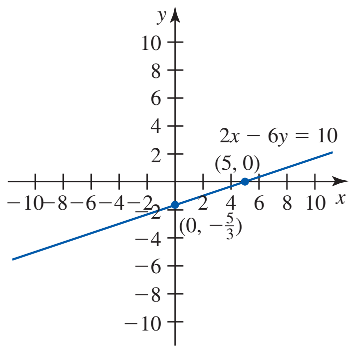


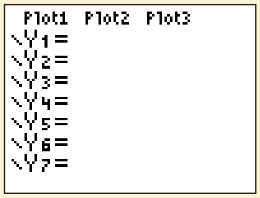
Figure 2.8

SUMMARY

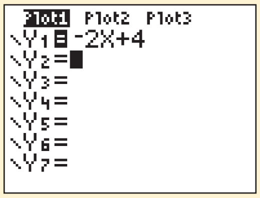
In this section, you learned the meaning of constant rate of change and discovered that any function with a constant rate of change is a linear function. You learned how to create and graph linear functions given in words, graphs, tables, and formulas. Additionally, you learned how to use the three common forms of linear functions: slope-intercept, point-slope, and standard.

TECHNOLOGY TIP ■ GRAPHING A FUNCTION

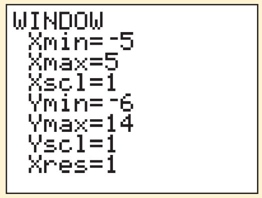
1. Bring up the graphing list by pressing the [Y=] button.



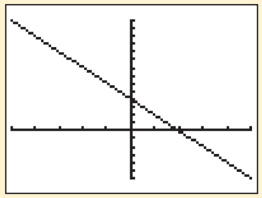
2. Type in the function(s) using the [X,T,θ,n] button for the variable.



3. Specify the size of the viewing window by pressing the [WINDOW] button and editing the parameters. The `Xmin` is the minimum x-value, `Xmax` is the maximum x-value, `Ymin` in the minimum y-value, and `Ymax` is the maximum y-value. The `Xscl` and `Yscl` are used to specify the spacing of the tick marks on the graph. `Xres` refers to the resolution of the graph. Changing this value will change the speed at which the graph is drawn.



4. Draw the graph by pressing the [GRAPH] button.



2.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10, you are given information about a linear function. Determine the slope of the linear function and its vertical and horizontal intercepts.

1. $y = -4x + 10$
2. $y = 0.5x - 19$
3. $y = -\frac{4}{5}(x - 1) - \frac{1}{2}$
4. $y = (x + 6) + 7$
5. The line passing through (4, 0), (0, 8)
6. The line passing through (4, 4), (9, 9)
7. The line passing through (2, 9), (5, 4)
8. The line passing through (−15, 4), (21, 4)
9. $4x + 2y = -10$
10. $3x - 8y = 14$

In Exercises 11–16, determine the constant rate of change (slope) of the linear function and explain what it means in each context.

11. Annual fees, which are presently \$55, are projected to increase by \$5 per year for the next decade.
12. The car, originally valued at \$12,800, has been decreasing in value at a constant rate over the past eight years. It is now worth \$8200.
13. At a local community college, a student's tuition bill was \$1008 for 9 credit hours. The student adds 4 credit hours and his tuition increases to \$1456.
14. A high school basketball team notices that attendance at its games changes at a constant rate based on the number of losses the team has suffered. When the team had lost six games, 275 people attended the next game. When the team had lost 11 games, 180 people attended the next game.
15. Due to a salary freeze, my salary will remain the same for the next 3 years.

16. The store starts its retail workers at \$7.00 per hour but guarantees fixed-value raises every 6 months. The manager says I will be making \$10.00 per hour after working for the company for 3 years. (*Hint:* The wage is a linear function of the number of 6-month periods worked.)

In Exercises 17–26, you are given information about a linear function. For each exercise,

a. Graph the linear function by hand.

b. Determine the vertical and horizontal intercepts of the graph algebraically and plot them on the graph.

17. $y = -4x + 12$

18. $y = x + 3$

19. $y = 2x$

20. $y = 6$

21. $y = 1.5(x - 2) + 7$

22. $y = -3(x + 5) - 1$

23. $9x - 4y = 36$

24. $11x + 2y = 10$

25. The line passing through (3, 1) and (5, −1)

26. The line passing through (4, 2) and (6, 5)

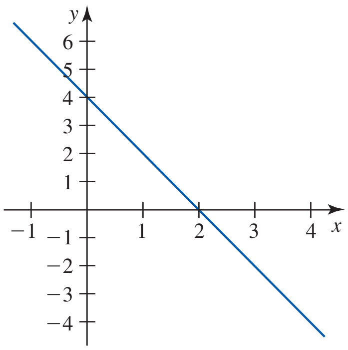
In Exercises 27–33, you are given a pair of coordinates, a graph, or a verbal description of a linear function. For each exercise, find a formula that defines the function.

27. (0, 3), (4, 7)

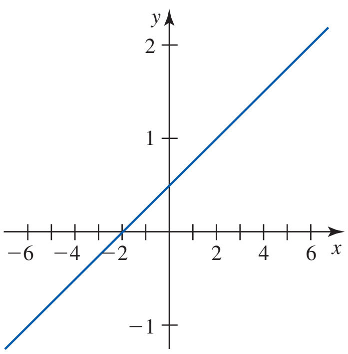
28. (−5, 13), (5, 17)

29. (4, 27), (21, 27)

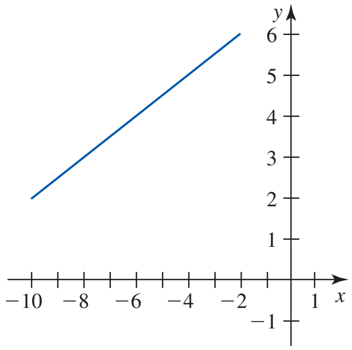
30.



31.



32.



33. Vehicle Speed A driver is presently driving 50 miles per hour and begins accelerating at a constant rate of 1 mile per hour per second. Model speed as a function of seconds elapsed since acceleration began.

In Exercises 34–37, you are given a table of values for a linear function. For each table, find the missing value, a.

34.

x	y
−2	7
10	a
14	55

35.

x	y
3	−2
6	−12
8	a

36.

x	y
6	87
a	83
11	77

37.

x	y
a	10.5
−9	15
−7	18

SHOW YOU KNOW

38. Explain why a horizontal line has zero slope and a vertical line has undefined slope.

39. Describe the connection between the often-used interpretation of slope as "rise over run" and the interpretation of slope as a number relating the relative changes in the output to the relative changes in the input.

40. When graphing linear functions, why does a greater magnitude slope create a steeper line? (Assume the scale of the graph remains unchanged.)

41. A classmate states that the slope-intercept form of a linear function can be thought of as a special case of the point-slope form of a linear function. Explain what your classmate means.

42. A classmate claims all linear functions have a vertical intercept. Is your classmate correct? Explain.

43. A classmate erroneously claims the terms *no slope*, *zero slope*, and *undefined slope* all mean the same thing. Explain the difference between each of the terms.

MAKE IT REAL

In Exercises 44–49,

a. Determine the vertical and horizontal intercepts of the graph of the implied linear function algebraically.

b. Using the intercepts from part (a), graph each of the linear functions by hand.

c. Explain the meaning of the intercepts in part (a) within the given context.

44. Orange Prices A devastating freeze in January 2007 destroyed roughly 75% of California Central Valley's orange crop. Market analysts predicted that as a result of the freeze, an orange that cost \$0.50 before the freeze would cost \$1.50 after the freeze. Model the price of an orange as a linear function of the percentage of the crop that was destroyed. (*Hint:* When 0% of the crop was destroyed, the price was \$0.50.)

45. Death Rate from Heart Disease In 1980, the age-adjusted death rate due to heart disease was 412.1 deaths per 100,000 people. Between 1980 and 2003, the death rate decreased at a near-constant rate. In 2003, the death rate was 232.1 deaths per 100,000 people. (***Source:*** ***Statistical Abstract of the United States, 2006*****, Table 106**) Model the death rate due to heart disease as a linear function of years since 1980.

46. McDonald's Dividends Between 2000 and 2002, McDonald's Corporation dividends per share increased at a constant rate of \$0.01 per year. In 2000, the dividend per share was \$0.215 (***Source:*** **McDonald's Investor Fact Sheet, January 2006**) Model the dividend per share as a linear function of years since 2000.

47. Highway Signs Many cross-country highway travelers are accustomed to seeing road signs warning of a steep downward slope such as the 8% decline shown in the figure. An 8% decline means that the road descends 8 feet for each 100 feet of horizontal distance traveled.

Given that the elevation of the road adjacent to the sign is 8240 feet, model the road elevation as a function of the horizontal distance from the sign. (Assume we are measuring the elevation as the road descends away from the sign.)

48. Book Prices In 2009, bestselling fiction books in mass-market paperback editions sold for about \$8. At the same time, bestselling fiction hardcover editions sold for about \$27. A reader wanted to spend about \$100 on new books in some combination of paperback and hardcover editions. (***Source:*** **www.publishersweekly.com**)

49. Fiber Intake One cup of Total Whole Grain cereal contains 4 grams of fiber. One scoop of ProFiber fiber supplement contains 6 grams of fiber. A person wants to consume 10 grams of fiber at breakfast using some combination of these products. (***Source:*** **product packaging**)

For Exercises 50–56, define the variables and then write the linear function for each real-world situation described.

50. Voting Age Population Between 1980 and 2000, the number of U.S. residents of voting age increased at a nearly constant rate of 2.1 million people per year. In 1992, there were 189.5 million U.S. residents of voting age. Model the number of U.S. residents of voting age as a function of the number of years since 1980. (***Source:*** ***Statistical Abstract of the United States, 2006*****, Table 407**)

51. Civil Service Retirement System Between 1980 and 2000, the number of participants in the Federal Civil Servant Retirement System changed at a nearly constant rate. In 1995, there were 3.73 million participants. In 1999, there were 3.36 million participants. Model the number of Federal Civil Servant Retirement System participants as a function of years since 1980. (***Source:*** ***Statistical Abstract of the United States, 2006*****, Table 539**)

52. Fiber Intake Experts recommend eating several servings of fresh fruit each day to receive tremendous health benefits. Suppose you want to increase your daily fiber intake by 10 grams by consuming fresh blueberries and blackberries. Blueberries contain about 2 grams of fiber per serving and blackberries contain about 5 grams of fiber per serving. Model the combinations of blueberry and blackberry servings you could consume to meet your goal. (***Source:*** **www.uhs.wisc.edu**)

53. Income Tax Returns Between 1990 and 2000, the number of personal income tax returns filed in the United States increased at a nearly constant rate of 1,555,000 returns per year. In 1990, the number of personal income tax returns that were filed was 109,868,000. Model the number of personal income tax returns filed as a function of years since 1990. (***Source:*** ***Statistical Abstract of the United States, 2006*****, Table 471**)

54. Protein Intake You are planning a meal of oven-roasted chicken breast with macaroni and cheese, and you want the meal to contain 8 grams of protein. Oven-roasted chicken breast contains about 0.17 grams of protein per gram of chicken and macaroni and cheese contains 0.03 grams of protein per gram. Model the combinations of grams of chicken and macaroni and cheese you could consume to meet your goal. (***Source:*** **www.highproteinfoods.com**)

55. Retirement Systems
During a 13-year period, the number of beneficiaries for state and local retirement systems changed at a nearly

constant rate compared to the amount of employee contributions. When there were \$13.9 billion in employee contributions, there were 4.03 million beneficiaries. When there were \$28.8 billion in employee contributions, there were 6.49 million beneficiaries. Model the number of beneficiaries as a function of the employee contributions. (*Source: Statistical Abstract of the United States, 2006*, Table 541)

56. Orthodontist Bill One of the authors set up the following payment plan with his daughter's orthodontist.

- Overall Treatment fee: \$4950
- Down payment, insurance payment, and records fee paid by author: \$1980
- Balance financed at 0% interest: \$2970
- Monthly payment: \$135

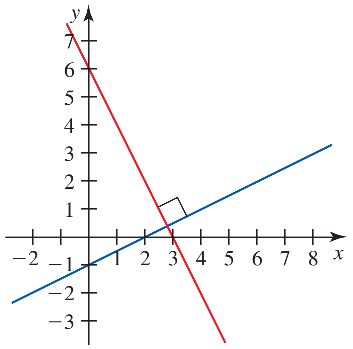
Write a formula for the loan balance as a function of the number of monthly payments that have been made.

STRETCH YOUR MIND

Exercises 57–61 are intended to challenge your understanding of linear functions.

57. Two lines are said to be *parallel* if they do not intersect at any point. If $f(x) = mx + b$ and $g(x) = nx + c$ are parallel lines, what conclusions can you draw about m, n, b, and c?

58. Two lines are said to be *perpendicular* if they intersect at a 90° angle (see the figure).



a. What is the relationship between the slopes of the two perpendicular lines in the figure?

b. Based on your result in part (a), what do you predict is the relationship between the slopes of any two perpendicular lines?

c. Using a ruler and graph paper, draw several examples of perpendicular lines to test your prediction in part (b). Did your prediction hold true? Explain.

59. Explain why the equation of a vertical line cannot be written in slope-intercept form.

60. Do all lines represent functions? Explain.

61. Slope Rating in Golf *Slope rating* is a measurement of the difficulty of a golf course that makes it possible for poorer golfers to compete against better golfers in head-to-head matchups. The *handicap* of a golfer is a measurement of how good the golfer is compared to par (the number of strokes a round is expected to take without any mistakes).

Imagine two golfers:

- Golfer A is a very good golfer with a handicap of 3. This means that Golfer A would expect to earn a score of 3 over par. For example, on a par 75 course, she would expect to score 78.
- Golfer B is a recreational golfer who only occasionally breaks a score of 100 so he has a handicap of 25 for a par 75 course.

On a course of average difficulty (slope rating of 113), both golfers would be expected to shoot near their handicaps. But as the course difficulty increases, Golfer B's scores will rise faster than Golfer A's. On the very challenging Pebble Beach–Spyglass Hill Golf Course (slope of 147), Golfer B will need some help if the two golfers are to have a competitive match against each other. Otherwise Golfer A will most likely win easily.

The number of shots a golfer is expected to shoot over par, E, is equal to the ratio of the course slope, s, to the slope of an average course (113) multiplied by the person's handicap, h,

$$E(s, h) = \left(\frac{s}{113}\right)(h)$$

For each of the following questions, assume that a golfer has a handicap of 25 ($h = 25$).

a. What score will the golfer most likely shoot at Pebble Beach–Spyglass Hill Golf Course (par 72 and slope of 147)?

b. What score will the golfer most likely shoot if he plays the relatively easy Greenfield Lakes Golf Club in Gilbert, Arizona (par 62 and slope of 91)?

c. Solve $E(s, 25) = 30$ for s and then explain what your answer means in the context of golf courses.

d. Explain how a golfer playing on a course with slope 110 could use the idea of constant rate of change to anticipate how his score will change on this course in comparison to playing an average course with slope 113.

(*Hint:* Make use of the $\frac{s}{113}$ part in the $E(s, h)$ formula.)

SECTION 2.2

LEARNING OBJECTIVES

- Determine if two quantities are directly proportional
- Construct linear models of real-world data sets and use them to predict results
- Find the inverse of a linear function and interpret its meaning in a real-world context

Modeling with Linear Functions

GETTING STARTED

One challenge for international travelers is having to convert back and forth between metric and English units of measurement. For example, if the posted speed limit is 50 kilometers per hour, what is the speed in miles per hour? Knowing how to use a linear function to convert to miles per hour could save you the cost of a speeding ticket.

In this section, we continue our discussion of linear functions, refining the process of constructing linear functions from real-world data sets and scenarios. We also revisit the concept of inverse functions and demonstrate how to find the inverse of a linear function algebraically.

■ Recognizing When to Use a Linear Model

Several key phrases alert us to the fact that a linear model may be used to model a data set. Table 2.2 details how to interpret the mathematical meaning of some commonly occurring phrases.

Table 2.2

Phrase	Mathematical Meaning
Increasing at a rate of 20 people per year	The constant rate of change of the linear function is $20\ \frac{\text{people}}{\text{year}}$.
Tickets cost \$37 per person	The constant rate of change is $37\ \frac{\text{dollars}}{\text{person}}$.
Sales decrease by 100 tickets for every \$1 increase in price	The constant rate of change is $m = \frac{-100 \text{ tickets}}{1 \text{ dollar}}$ $= -100$ tickets per dollar.
There are 350 students today. The number of students is increasing by 10 students per month.	The initial value is 350 students and the constant rate of change is $10\ \frac{\text{students}}{\text{month}}$.
The price is \$12.25 and is decreasing at a constant rate of \$0.02 per day.	The initial value is \$12.25 and the constant rate of change is $-0.02\ \frac{\text{dollars}}{\text{per day}}$.

Some of the simplest linear models to construct are those that model direct proportionalities, where one quantity is a constant multiple of another quantity.

Direct Proportionality

Two quantities are **directly proportional** when one quantity is a constant multiple of the other. That is,

$$y = kx \quad \text{for a constant } k$$

k is called the **constant of proportionality**.

Solving the equation $y = kx$ for k yields $k = \frac{y}{x}$. Thus another way to define direct proportionality is to say that two quantities are directly proportional if the output divided by the input is a constant.

EXAMPLE 1 ■ Constructing a Linear Model for Directly Proportional Quantities

The 1-month average retail price for gasoline in Texas was \$3.219 per gallon on February 26, 2011. (*Source:* www.TexasGasPrices.com) Write a function that will give the total cost of a gasoline purchase as a function of the number of gallons purchased at this average retail price. Then calculate the cost of 20 gallons of gasoline.

Solution Let C be the total cost (in dollars) and g be the number of gallons purchased. Since the price per gallon is constant, the total cost will be the product of the price per gallon and the number of gallons purchased. That is,

$$C(g) = 3.219g$$

To determine the cost of 20 gallons of gasoline, we evaluate this function at $g = 20$.

$$\begin{aligned} C(20) &= 3.219(20) \\ &= \$64.38 \end{aligned}$$

Twenty gallons of gasoline cost \$64.38.

One of the most well-known examples of direct proportionality is the relationship between the diameter of a circle, d, and its circumference, C. Scholars and builders in antiquity knew that the circumference of a circle was directly proportional to its diameter, that is $C = kd$, yet the value of the constant of proportionality, k, eluded them for centuries. Ultimately, the number $\pi \approx 3.1416$ was discovered to be the constant of proportionality k, giving us the familiar formula $C = \pi d$. (See the Peer into the Past for details on the history of π.)

Direct proportions can also be used to convert from English to metric units of measure and vice versa. One such conversion is demonstrated in Example 2.

EXAMPLE 2 ■ Using Direct Proportionality to Convert Units of Measure

The speed of 100 kilometers per hour is roughly equivalent to 62.14 miles per hour. Additionally, 0 kilometers per hour is equivalent to 0 miles per hour. Find a function that converts kilometers per hour into miles per hour. Then calculate the speed in miles per hour that is equivalent to 50 kilometers per hour.

Solution We let the variable m represent miles per hour and the variable x represent kilometers per hour. Since x is the independent variable and m is the dependent variable, ordered pairs will be of the form (x, m). We have two points: (100, 62.14) and (0, 0). We calculate the slope of the line between the two points.

PEER INTO THE PAST

THE NUMBER π

From ancient times, it has been known that the ratio of the circumference of a circle to its diameter, π, is a very special value. Estimating the value of π has quite a history and has even had a competitive flair. As far back as 2000 B.C. the Babylonians and Egyptians had estimates for π of $\frac{25}{8} \approx 3.125$ and $\frac{256}{81} \approx 3.160$, respectively. The Chinese ($\approx$1200 B.C.) as well as the Israelites ($\approx$550 B.C.) estimated π as 3 (see 1 Kings 7:23 in the Bible). Archimedes ($\approx$300 B.C.) estimated an average of 3.1463, and Fibonacci (A.D. 1220) arrived at 3.141818. In more modern times yet prior to computer technology, Ludolph van Ceulen (A.D. 1596) calculated π to 35 correct decimal places, Machin (A.D. 1766) to 100, Richter (A.D. 1855) to 500, and Ferguson (A.D. 1947) to 808 correct decimal places. Since the advent of supercomputers, we now know π to more than 10 billion decimal places. (*Source:* http://mathforum.org/isaac/problems/pi2.html)

On a humorous note, March 14 (the date 3/14 and Albert Einstein's birthday) has been dubbed Pi Day. Mathematics educators, students, and enthusiasts around the country take some time to celebrate π. Fun festivities include pie-eating and pi-memorization contests. For information on Pi Day go to http://en.wikipedia.org/wiki/Pi_day.

$$\text{slope} = \frac{62.14 - 0}{100 - 0} \quad \frac{\text{miles per hour}}{\text{kilometers per hour}}$$

$$= 0.6214 \text{ miles per kilometer} \qquad \text{The "per hour" terms cancel out.}$$

Since the vertical intercept is (0, 0), the linear model in condensed function notation is

$$m(x) = 0.6214x$$

Consequently, kilometers per hour and miles per hour are directly proportional.

To determine the speed in miles per hour that is equivalent to 50 kilometers per hour, we evaluate this function at $x = 50$.

$$\begin{aligned} m(50) &= 0.6214(50) \\ &\approx 31 \end{aligned}$$

Therefore, 50 kilometers per hour is approximately equal to 31 miles per hour.

As pointed out earlier, identifying key phrases in a verbal expression helps us recognize when linear relationships exist. The next example focuses on recognizing this.

EXAMPLE 3 ■ Constructing a Linear Model from a Verbal Description

T-Mobile's Even More™ 500 Talk plan cost \$39.99 per month in 2011. The plan included free evening and weekend minutes and 500 whenever minutes. Additional minutes used cost \$0.45 per minute. (*Source:* www.t-mobile.com)

a. Write a function that will give the monthly cell phone cost as a function of the number of additional minutes used over 500 minutes. (T-Mobile defines *additional minutes* to be the minutes used beyond a subscriber's monthly allotment.)

b. Calculate the cost of using a total of 600 non-weekend minutes.

Solution In solving any word problem, identifying the meaning of all variables is critical. We let c be the monthly cell phone cost (in dollars) and m be the number of additional minutes used. Since the monthly cost depends on the number of additional minutes used, c is the dependent variable and m is the independent variable.

a. The first indicator that this is a linear function is the constant rate of change: \$0.45 per minute. This will be the slope of the linear function. The \$39.99 fixed value is the initial value of our function. We have

$$c(m) = 0.45m + 39.99$$

b. We are asked to find the cost of using 600 non-weekend minutes. (Since weekend minutes are free, they will not affect the monthly cell phone cost.) Since 500 minutes are included with the plan, 100 additional minutes will be used. We evaluate the function at $m = 100$.

$$\begin{aligned} c(100) &= 0.45(100) + 39.99 \\ &= 45 + 39.99 \\ &= 84.99 \end{aligned}$$

It costs \$84.99 to use 600 non-weekend minutes. Note that although only 20% more minutes were used than the plan allowed, the monthly cost more than doubled.

EXAMPLE 4 ■ Constructing a Linear Model from a Table of Data

Table 2.3 shows the total undergraduate nonresident tuition and fees cost per semester for students enrolled in the Golf Management Program at Arizona State University—Polytechnic in 2006–2007.

Table 2.3

Nonresident	Enrolled Hours				
	7	8	9	10	11
Nonresident Undergraduate Tuition	\$4592	\$5248	\$5904	\$6560	\$7216
Program Tuition	600	600	600	600	600
Financial Aid Trust	22	22	22	22	22
Association of Students of AZ	1	1	1	1	1
Total Undergraduate Nonresident Tuition & Fees	\$5215	\$5871	\$6527	\$7183	\$7839

Source: www.asu.edu

a. Find a function that models the total undergraduate nonresident tuition and fees as a function of the number of enrolled hours for students taking 7–11 credits.

b. Predict the cost of enrolling in 10.5 credit hours.

Solution We first identify the variables, letting t represent the total tuition and fees cost and h represent the enrolled hours. Since the total tuition and fees cost depends on the number of enrolled hours, t is the dependent variable and h is the independent variable.

a. Observe that the program tuition, financial aid trust, and Association of Students of Arizona fees do not change as the number of enrolled hours increases. These fees total \$623 regardless of whether a student enrolls in 7 or 11 credit hours. Note also that the nonresident undergraduate tuition increases. If the tuition increases at a constant rate, this rate will be the slope of a linear function. We construct Table 2.4 to determine the rate of change.

Table 2.4

Enrolled Hours	Nonresident Undergraduate Tuition	Change in Nonresident Undergraduate Tuition
7	\$4592	
		\$5248 − \$4592 = \$656
8	\$5248	
		\$5904 − \$5248 = \$656
9	\$5904	
		\$6560 − \$5904 = \$656
10	\$6560	
		\$7216 − \$6560 = \$656
11	\$7216	

Table 2.4 shows that the tuition increases at a constant rate of \$656 per enrolled credit hour. Consequently, the total nonresident undergraduate tuition and fees may be modeled by a linear function.

When we add the \$623 fee to the \$4592 tuition cost for 7 credit hours, we get \$5215. Thus, the point (7, 5215) is on the line. With this point and the slope of \$656 per credit hour, we can construct the linear model.

$$\underbrace{t - 5215}_{\text{tuition and fees above \$5215}} = \underbrace{656}_{\text{tuition increase per credit hour}} \underbrace{(h - 7)}_{\text{credit hours above 7}}$$

Since we are going to use this model to forecast a value of t, we solve for t.

$$t = 656(h - 7) + 5215$$

Note that the practical domain for this model is $7 \le h \le 11$ since the data only included values of h between 7 and 11. Forecasting the cost of tuition and fees for credit hours outside of this range may not be accurate.

b. To determine the cost of 10.5 credit hours, we evaluate this function at $h = 10.5$.

$$\begin{aligned} t(h) &= 656(h - 7) + 5215 \\ t(10.5) &= 656(10.5 - 7) + 5215 \\ &= 7511 \end{aligned}$$

We predict it costs \$7511 to enroll in 10.5 credit hours of classes. (This assumes that tuition is prorated for partial credits.)

Inverses of Linear Functions

In Chapter 1 we introduced the concept of inverse functions. Recall the domain of a function f was the range of its inverse function f^{-1} and the range of the function f was the domain of its inverse function f^{-1}. This notion is represented as follows.

$$x \to \boxed{f} \to y = f(x)$$

$$f^{-1}(y) = x \leftarrow \boxed{f^{-1}} \leftarrow y$$

The phrase "y is the function of x" corresponds with the phrase "x is the inverse function of y." Symbolically, $y = f(x)$ is related to $x = f^{-1}(y)$. Recognizing this relationship between a function and its inverse is critical for a deep understanding of inverse functions, especially in a real-world context.

Let's investigate inverse functions in the context of the tuition and fees equation we created in Example 4. If we let $t = f(h)$, we can write the tuition and fees equation in slope-intercept form as

$$t = 656h + 623$$

which we can represent as

$$\textit{enrolled hours} \to \boxed{f} \to \textit{tuition and fees}$$

We would like to find a function that reverses the process. That is, we want a function that converts *tuition and fees* into *enrolled hours*. This function will be the inverse function f^{-1}, as represented here:

$$\textit{enrolled hours} \leftarrow \boxed{f^{-1}} \leftarrow \textit{tuition and fees}$$

Fortunately, we can easily find the inverse function f^{-1} by solving the equation $t = 656h + 623$ for h.

$$\begin{aligned} t &= 656h + 623 \\ t - 623 &= 656h \\ h &= \frac{t - 623}{656} \\ h &= \frac{1}{656}t - \frac{623}{656} \end{aligned}$$

Notice the independent variable of this new function is t and the dependent variable is h. In inverse function notation, we write the inverse as

$$f^{-1}(t) = \frac{1}{656}t - \frac{623}{656} \qquad \text{since } h = f^{-1}(t)$$

Let's check to see if this function does indeed convert *tuition and fees* into *enrolled hours*. From Table 2.3, we know that 8 enrolled hours cost \$5871 so we expect our inverse function to give us the same result. We write the equation in inverse function notation and evaluate the function at $t = 5871$.

$$f^{-1}(5871) = \frac{1}{656}(5871) - \frac{623}{656}$$
$$f^{-1}(5871) = \frac{5871}{656} - \frac{623}{656}$$
$$f^{-1}(5871) = \frac{5248}{656}$$
$$f^{-1}(5871) = 8$$
$$t = 8 \qquad \text{since } f^{-1}(h) = t$$

Thus, \$5871 will cover the cost of nonresident undergraduate tuition and fees for 8 credit hours. This agrees with the information presented in Table 2.3.

Many students struggle with the concept of inverse functions. To help you get a better grasp on this concept, we will work a few straightforward examples before summarizing the process of finding an inverse of a linear function.

EXAMPLE 5 ■ Finding the Inverse of a Linear Function

Find the inverse of the function $y = 4x - 2$.

Solution In the function we are given, x is the independent variable and y is the dependent variable. We solve this equation for x.

$$y = 4x - 2$$
$$y + 2 = 4x$$
$$\frac{y + 2}{4} = x$$
$$x = \frac{1}{4}y + \frac{1}{2}$$

In this new equation, y is the independent variable and x is the dependent variable. We can write the original function as

$$f(x) = 4x - 2 \qquad \text{since } y = f(x)$$

Similarly, we can write the inverse function as

$$f^{-1}(y) = \frac{1}{4}y + \frac{1}{2} \qquad \text{since } x = f^{-1}(y)$$

The function $f^{-1}(y) = \frac{1}{4}y + \frac{1}{2}$ is the inverse of $f(x) = 4x - 2$.

EXAMPLE 6 ■ Finding the Inverse of a Linear Function

We can convert degrees Celsius into degrees Fahrenheit using the equation

$$F = \frac{9}{5}C + 32$$

where F is degrees Fahrenheit and C is degrees Celsius. Given that $F = f(C)$, find the function f^{-1}. Then calculate $f^{-1}(50)$ and interpret what it means in its real-world context.

Solution We solve the equation $F = \frac{9}{5}C + 32$ for C.

$$F = \frac{9}{5}C + 32$$
$$F - 32 = \frac{9}{5}C$$
$$C = \frac{F - 32}{\frac{9}{5}}$$
$$C = \frac{5}{9}F - \frac{160}{9}$$
$$f^{-1}(F) = \frac{5}{9}F - \frac{160}{9} \qquad \text{since } f^{-1}(F) = C$$

The input to f^{-1} is degrees Fahrenheit and the output is degrees Celsius. $f^{-1}(50)$ will convert 50 degrees Fahrenheit into degrees Celsius.

$$\begin{aligned} f^{-1}(50) &= \frac{5}{9}(50 - 32) \\ &= \frac{5}{9}(18) \\ &= 10 \end{aligned}$$

Thus 50 degrees Fahrenheit is the same as 10 degrees Celsius.

We now summarize the process of finding the inverse of a linear function. Observe that since dividing by 0 is undefined, this process only works for linear functions with nonzero slopes.

HOW TO: ■ FIND THE INVERSE OF A LINEAR FUNCTION

1. Write the function $f(x) = mx + b$ as $y = mx + b$.
2. Solve the function equation for x.

$$y = mx + b$$
$$y - b = mx$$
$$x = \frac{y - b}{m}$$
$$x = \frac{1}{m}y - \frac{b}{m}$$

3. Write the inverse function using the notation

$$f^{-1}(y) = \frac{1}{m}y - \frac{b}{m}$$

Recall that horizontal lines can be written in the form $y = b$. The inverse of a horizontal line is a vertical line $x = a$; however, a vertical line is not a function. Therefore, only linear functions with nonzero slopes have an inverse *function*.

It is customary in many textbooks to write the inverse of a function $y = f(x)$ as $y = f^{-1}(x)$. Some teach that to find the inverse function we interchange the x and y variables and solve the resultant equation for y. But this strategy does not make sense when working with inverses in a real-world context. For example, we stated earlier that the function $f(C) = \frac{9}{5}C + 32$ converts degrees Celsius into degrees Fahrenheit and the function $f^{-1}(F) = \frac{5}{9}F - \frac{160}{9}$ converts degrees Fahrenheit into degrees Celsius. What does $f^{-1}(C) = \frac{5}{9}C - \frac{160}{9}$ do? Nothing except confuse us! It has no meaning in a real-world context because the input to the inverse function must be degrees Fahrenheit, not degrees Celsius. By replacing the variable F with the variable C in the inverse function, we completely obscured the real-world meaning of the functions. For this reason, we have chosen to use a less-traditional, more meaningful approach to working with inverses.

SUMMARY

In this section you refined your ability to construct linear functions from real-world data sets and scenarios. You also learned how to calculate and interpret average rates of change. Finally, you learned how to find the inverse of a linear function algebraically.

2.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, find the inverse of the function $y = f(x)$. Write your solution in inverse function notation, $f^{-1}(y)$.

1. $y = -4x + 10$
2. $y = 0.5x - 19$
3. $y = 9x - 18$
4. $y = 6x - 15$
5. $y = \frac{1}{3}x + 4$

SHOW YOU KNOW

6. What are some key phrases that indicate that a verbal description can be modeled by a linear function?
7. What does it mean to say two quantities are *directly proportional*?
8. There is one type of linear function that does not have an inverse function. What type of linear function is it and why does it not have an inverse function?
9. Explain what it means for one function to be the inverse of another function. Use the terms *domain* and *range* as a part of your explanation. Use diagrams as appropriate.
10. Why don't we find the inverse function by interchanging the x and the y and solving for y when working with functions in a real-world context?

MAKE IT REAL

In Exercises 11–18, determine if a linear function will be a good model for the data set or verbal description. If the data appears to be linear or nearly linear, state why and find a linear function that models the data set. Otherwise, explain why a linear model will not be a good fit.

11. **Gas Prices** On January 20, 2007, gasoline in Ozark, Alabama, was priced at \$1.889 per gallon. (***Source:*** **www.alabamagasprices.com**)

    Model the total cost of the gasoline as a function of the number of gallons purchased.

12. **Gas Prices** On January 20, 2007, gasoline in Queens, New York, was priced at \$2.339 per gallon. (***Source:*** **www.newyorkgasprices.com**)

    Model the total cost of the gasoline as a function of the number of gallons purchased.

13. **Retail vs. Wholesale** A six-pack of Stylish Plaid Capri Pants featured at wholesaleclothingmart.com cost \$103.50 in 2007. In the retail clothing industry, it is customary to mark up the wholesale price of an item by 100%. (That is, the retail price is double the wholesale price.)

    Model the total retail revenue in 2007 as a function of the number of pairs of pants sold at retail price.

14. **Retail vs. Wholesale** A six-pack of Stylish Pleated Maternity Tops with Lace Trim featured at wholesaleclothingmart.com cost \$93 in 2007. In the retail clothing industry, it is customary to mark up the wholesale price of an item by 100%. (That is, the retail price is double the wholesale price.)

    Model the total retail revenue from tops in 2007 as a function of the number of tops sold at retail price.

15. **Restaurant Expenses** The total company restaurant expenses incurred by Burger King Holdings, Inc., between 2004 and 2006 are given in the table. Model the expenses as a function of the fiscal year.

Fiscal Year	Total Company Restaurant Expenses ($ millions)
2004	1087
2005	1195
2006	1296

Source: **Burger King Annual Report, 2006**

16. McDonald's Restaurants The total revenues earned by McDonald's restaurants between 2000 and 2005 are given in the table. Model total revenues as a function of the fiscal year.

Fiscal Year	Total Revenues ($ millions)
2000	14,243
2001	14,870
2002	15,406
2003	17,140
2004	19,065
2005	20,460

Source: **McDonald's Investor Fact Sheet, Jan. 2006**

17. Children in Madagascar Model the number of children under 5 years of age as a function of years since 1990.

Years Since 1990 t	Children under 5 (thousands) C
9:0	2120
7	2630
8	2707
9	2787
10	2859
11	2946
12	3036

Source: **World Health Organization**

18. Prescription Drug Spending Model the per capita prescription drug spending as a function of years since 1990.

Years Since 1990 t	Per Capita Spending on Prescription Drugs (dollars) P
0	158
5	224
8	311
9	368
10	423
11	485
12	552
13	605

***Source: Statistical Abstract of the United States, 2006*, Table 121**

In Exercises 19–20, assume that the real-world context can be accurately modeled with a linear function.

19. Aging U.S. Population In 1950, the number of people age 65 and older who lived in the United States was 12 million. By 2005, that number had grown to 37 million people. **(*Source: Health United States 2006*, p. 16)**

Model the number of people who are age 65 and older as a linear function of the number of years since 1950.

20. Life Expectancy in the United States Between 1900 and 2003, the life expectancy for men and women in the United States increased dramatically. For men, life expectancy increased from 48 to 75 years. For women, life expectancy increased from 51 to 80 years. **(*Source: Health United States 2006*, p. 30)**

Model male life expectancy as a function of female life expectancy.

21. Frozen Oranges A devastating freeze in California's Central Valley in January 2007 wiped out approximately 75% of the state's citrus crop. According to an Associated Press news report, 40-pound boxes of oranges that were selling for \$6 before the freeze were selling for \$22 after the freeze.

A linear model for the price of a box of oranges as a function of the percentage of the citrus crop that was frozen is given by

$$P = f(F) = \frac{16}{75}F + 6$$

where P is the price of a 40-pound box of oranges and F is the percentage of the citrus crop that was frozen.

Find the inverse function. Then determine the value of $f^{-1}(14)$ and interpret the real-world meaning of the result.

22. Leaning Tower of Pisa Construction of the Tower of Pisa was completed in 1360. By 1990, the tilt of the tower was so severe that it was closed for renovation. Renovators were able to reduce the tower's 1990 tilt by 17 inches. The resultant tower leans 13.5 feet (162 inches) off the perpendicular. When the tower was reopened in 2001, officials forecast that it would take 300 years for the tower to return to its 1990 tilt **(*Source: TIME Magazine*, June 25, 2001)**

a. Construct a linear formula that models the lean of the renovated tower, where l is the number of inches from the perpendicular and t is the number of years since 2001.

b. Use the formula from part (a) to predict the lean of the tower in 2100.

23. Smoking and Heart Disease Based on data from 1974 to 2003, the death rate due to heart disease in the United States (in deaths per 100,000 people) can be modeled by

$$D = f(p) = 14.08p - 53.87$$

where p is the percentage of people who smoke (***Source:*** **Modeled from CDC and Census Bureau data)**

Find the inverse function. Then determine the value of $f^{-1}(101)$ and interpret the real-world meaning of the result.

24. Snow Runoff Based on data from June 2006, the forecast for maximum 5-day snow runoff volumes for the American River at Folsom, CA, can be modeled by

$$v = f(t) = -2.606t + 131.8$$

thousand acre-feet, where t is the number of days since the end of May 2006. (***Source:*** **Modeled from National Weather Service data)** That is, the model forecasts the snow runoff for the 5-day period beginning on the selected day of June 2006.

Find the inverse function. Then determine the value of $f^{-1}(61.4)$ and interpret the real-world meaning of the result.

25. Marketing Labor Costs Based on data from 1990 to 2003, the marketing labor cost for farm foods may be modeled by

$$L = f(t) = 10.47t + 146.8 \text{ billion dollars}$$

where t is the number of years since 1990. (***Source:*** **Modeled from *Statistical Abstract of the United States, 2006*, Table 842)**

Find the inverse function. Then determine the value of $f^{-1}(251.5)$ and interpret the real-world meaning of the result.

■ STRETCH YOUR MIND

Exercises 26–29 are intended to challenge your understanding of linear function models and inverse functions.

26. A linear function model passes through every point of a data set. Will the model perfectly forecast unknown function values? Explain.

27. A classmate claims function modeling is a waste of time since models do not always accurately predict the future. Provide a convincing argument to refute this claim.

28. A classmate claims $f^{-1}(x)$ is equivalent to $\dfrac{1}{f(x)}$. Do you agree? Explain.

29. What are some techniques you can use to quickly determine whether or not a data table represents a linear or nearly linear function?

SECTION 2.3 Linear Regression

LEARNING OBJECTIVES

- Use linear regression to find the equation of the line of best fit
- Use a linear regression model to make predictions
- Explain the meaning of the correlation coefficient (r) and the coefficient of determination (r^2)

GETTING STARTED

In 1938, the U.S. Congress passed the Fair Labor Standards Act, which was signed into law by President Franklin Roosevelt. The intent of this act was to eliminate "labor conditions detrimental to the maintenance of the minimum standards of living necessary for health, efficiency and well-being of workers." (*Source:* www.dol.gov) The federal minimum wage, begun as part of this act, required that employers pay their workers \$0.25 per hour in 1938. Over time, the minimum wage has increased in an attempt to keep up with the rising costs of goods and services. In 2011, the federal minimum wage was \$7.25 per hour. (*Source:* www.dol.gov) Information such as this can be modeled using linear regression.

In this section we use linear regression to determine the equation of the linear function that best fits a corresponding data set. We then use linear regression models to make predictions. We also discuss ways to determine how well the linear model represents a data set.

Linear Regression

The federal minimum wage has increased since its inception in 1938. Table 2.5 gives the hourly minimum wage each decade beginning in 1950.

Table 2.5

Year	Years Since 1950 t	Federal Minimum Wage (dollars) W
1950	0	0.75
1960	10	1.00
1970	20	1.60
1980	30	3.10
1990	40	3.80
2000	50	5.15

Source: www.dol.gov

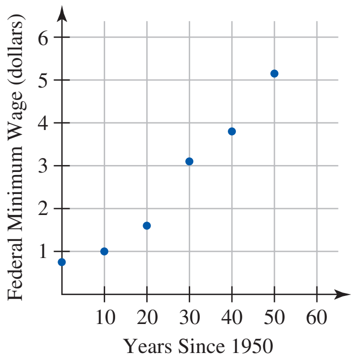


Figure 2.9

Let's investigate the relationship between the quantities given in the table—namely, the value of t (years since 1950) and W (federal minimum wage). We begin by examining the scatter plot of these data shown in Figure 2.9.

We see that a relationship between the two quantities does exist: As the number of years since 1950 increases, the federal minimum wage increases. We can also describe this relationship quantitatively using a linear model, as in Example 1.

EXAMPLE 1 ■ Selecting a Linear Function to Model a Data Set

Determine which of the three linear function graphs in Figure 2.10 best models the federal minimum wage data. Then use the graph that best models the data to predict the minimum wage in 2010.

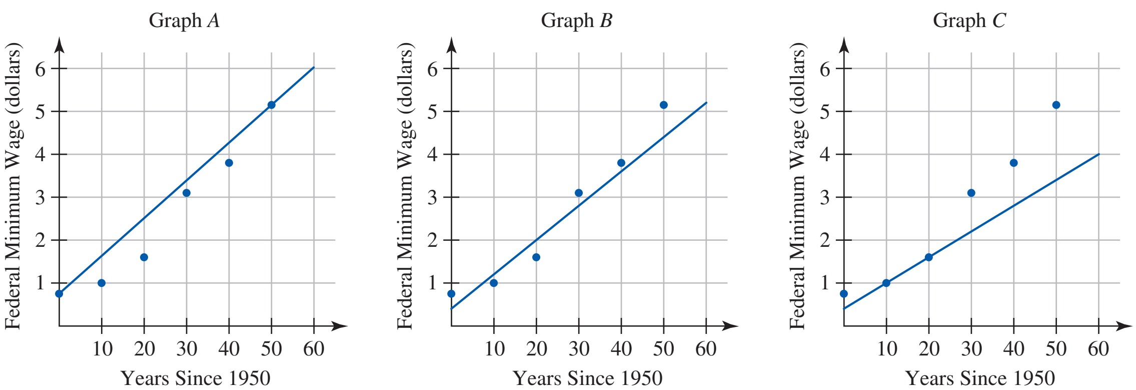


Figure 2.10

Solution We are looking for a linear function that "goes through" the middle of the scatter plot as much as possible. Graph A connects the first and last data points but does not represent the other data points well. Graph C connects the second and third data points but also does not represent the other data points well. It appears as though Graph B is the linear function that best models the federal minimum wage data.

To use the model to predict the minimum wage in 2010, we must find the value of W when $t = 60$ since 2010 is 60 years after 1950. Using Graph B as shown in Figure 2.11, we estimate the minimum wage to be \$5.20.

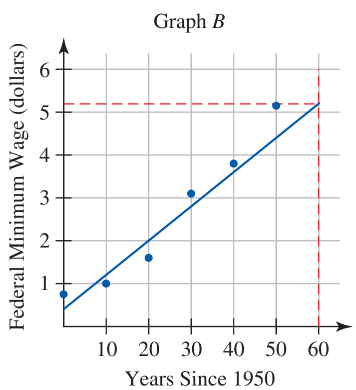


Figure 2.11

Least Squares

Our choice of Graph B in Example 1 was based on a visual analysis only. To mathematically determine which is truly the line of best fit, we compute the *sum of squares*. The **sum of squares** is calculated by finding the difference between each output of the data set and the corresponding "predicted" value from the linear model, squaring each of these differences, and then adding them up. (We must square the differences since some of the differences will be positive and some will be negative. Adding them without squaring them would result in cancellation of positive and negative values.) Each of these individual differences is called a **residual** because it measures the "residue" in our prediction (the function value) from the actual data value of the given data. The line whose sum of squares is closest to 0 is called the **line of best fit**.

To illustrate, we calculate the sum of squares for each of the three graphs of Example 1 by using the raw data and the equation of each line. The results are shown in Tables 2.6, 2.7, and 2.8.

Table 2.6 Graph *A*

Years Since 1950 t	Federal Minimum Wage (dollars) W	Linear Model (dollars) $A(t) = 0.088t + 0.75$	Residual $(W - A)$	Square of Residual $(W - A)^2$
0	0.75	0.75	0	0
10	1.00	1.63	−0.63	0.3969
20	1.60	2.51	−0.91	0.8281
30	3.10	3.39	−0.29	0.0841
40	3.80	4.27	−0.47	0.2209
50	5.15	5.15	0	0
			Total Sum of Squares	**1.5300**

Table 2.7 Graph *B*

Years Since 1950 t	Federal Minimum Wage (dollars) W	Linear Model (dollars) $C(t) = 0.0783t + 0.50$	Residual $(W - B)$	Square of Residual $(W - B)^2$
0	0.75	0.50	0.25	0.0625
10	1.00	1.28	−0.28	0.0784
20	1.60	2.07	−0.47	0.2209
30	3.10	2.85	0.25	0.0625
40	3.80	3.63	0.17	0.0289
50	5.15	4.42	0.73	0.5329
			Total Sum of Squares	**0.9861**

Table 2.8 Graph C

Years Since 1950 t	Federal Minimum Wage (dollars) W	Linear Model (dollars) $C(t) = 0.06t + 0.40$	Residual $(W - C)$	Square of Residual $(W - C)^2$
0	0.75	0.40	0.35	0.1225
10	1.00	1.00	0	0.0000
20	1.60	1.60	0	0.0000
30	3.10	2.20	0.90	0.8100
40	3.80	2.80	1.00	1.0000
50	5.15	3.40	1.75	3.0625
			Total Sum of Squares	**4.9950**

Of the three sums of the squares of the residuals, Graph B has the smallest sum (0.9861). Thus Graph B is the most accurate of the three graphs.

Interpolation and Extrapolation

One reason we want to find the linear model that most accurately represents the actual data is so that our predictions will be as accurate as possible.

Interpolation and Extrapolation

- **Interpolation** is the process of predicting the output value for an input value that lies between the maximum and minimum input values of the data set.
- **Extrapolation** is the process of predicting the output value for an input value that comes before the minimum input value or after the maximum input value of a data set.

To highlight these concepts, we use the three models from Graph A, Graph B, and Graph C from Example 1 to predict the federal minimum wage in 1966 (interpolation) and in 2010 (extrapolation). In comparing the values generated by the three models, we will see the importance of determining the most accurate model.

EXAMPLE 2 ■ Predicting Values Using a Linear Model

Using each of the linear models for the federal minimum wage given in Tables 2.6–2.8, predict the minimum wage in 1966 and in 2010. Then compare the values produced by each model.

Solution Since 1966 is 16 years after 1950, we substitute $t = 16$ into each function to predict the federal minimum wage in 1966. Since 2010 is 60 years after 1950, we substitute $t = 60$ into each function to predict the federal minimum wage in 2010. The differing results are shown in Table 2.9.

Table 2.9

Years Since 1950 t	Model A $A(t) = 0.088t + 0.75$	Model B $B(t) = 0.0783t + 0.50$	Model C $C(t) = 0.06t + 0.40$
16	2.16	1.75	1.36
60	6.03	5.20	4.00

Model A projects the highest minimum wage and Model C predicts the lowest minimum wage in 1966 and 2010. Since Model B was the line of best fit, we conclude that Model B is the most accurate predictor of the three models.

Linear Regression Model

To find the line of best fit, we could do as we have done so far in this section: draw several lines that we think best fit the scatter plot of the data, compute the total sum of the squares of the differences, and choose the model that produces the smallest sum. But how would we know we had found the best line? We do not want to go through the least squares process for every possible model! Fortunately, we can use technology to easily find the line of best fit, which is also called the **linear regression model**. The Technology Tip at the end of this section details the steps needed to find this linear model.

Linear Regression Model

The equation of the line that best fits a data set, as determined by the least value of the total sum of the squares of the residuals, is known as the **linear regression model** or the **least squares regression line**.

EXAMPLE 3 ■ Determining a Linear Regression Model

Use linear regression to determine the equation of the line of best fit for the federal minimum wage data. Describe the relationship between the quantities t (years since 1950) and W (federal minimum wage, in dollars) using this regression model.

Solution First, we use a graphing calculator to determine the equation of the line of best fit. (Refer to the Technology Tip at the end of this section for detailed instructions on computing the linear regression model.) Screen shots from the steps are shown in Figures 2.12 and 2.13.

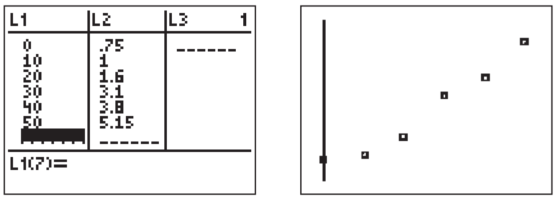


Figure 2.12

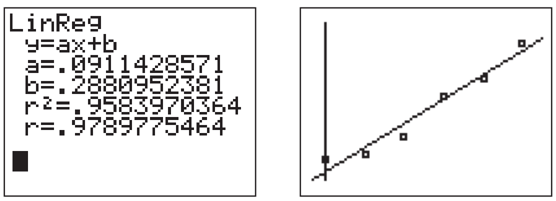


Figure 2.13

We find $W(t) = 0.091t + 0.288$ is the equation of the line of best fit. Using this model, we predict that in 1950 the minimum wage was \$0.288 and that it increases by \$0.091 each year.

In Table 2.10 we confirm that the line in Example 3 has the least squares value when compared to those found in Tables 2.6–2.8 for Models A–C.

PEER INTO THE PAST

REGRESSION

Why is the process of finding the line of best fit called "linear regression"? During the 1870s, Sir Francis Galton investigated the relationship between the average height of parents and that of their offspring. What Galton observed and recorded was that the offspring of particularly tall parents were also tall—but not as tall as their parents. The offspring of particularly short parents were also short—but not as short as their parents. That is, the offspring of these parents tended to be less tall or less short; they *regressed* toward the mean height of the population.

We see this "regression toward the mean" in many real-world situations. For example, if a basketball player scores an extraordinarily high number of points in one game, he most likely will not score as many points in the next game. The number of points will "regress toward the mean" or be closer to the player's average number of points per game.

Sir Francis Galton
1822–1911

Table 2.10

Years Since 1950 t	Federal Minimum Wage (dollars) W	Linear Model (dollars) $F(t) = 0.091t + 0.288$	Residual $(W - F)$	Square of Residuals $(W - F)^2$
0	0.75	0.29	0.46	0.2116
10	1.00	1.20	−0.20	0.0400
20	1.60	2.11	−0.51	0.2601
30	3.10	3.02	0.08	0.0064
40	3.80	3.93	−0.13	0.0169
50	5.15	4.84	0.31	0.0961
			Total Sum of Squares of Residuals	**0.6311**

We see this linear regression model produce the smallest least squares value when compared to those computed for the linear models representing Graph *A* (1.5300), Graph *B* (0.9861), and Graph *C* (4.995).

Coefficient of Determination

You may have noticed that the graphing calculator outputs a value, r^2, when computing a linear regression model, as shown in Figure 2.14.

This value, known as the **coefficient of determination**, describes the strength of the fit of a linear regression model to a set of data. The stronger the fit, the closer this value, r^2, is to 1.

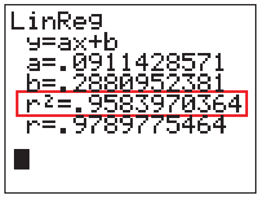


Figure 2.14

Coefficient of Determination

The **coefficient of determination**, r^2, is a value that describes the strength of fit of a linear regression model to a set of data. The closer the value of r^2 is to 1, the stronger the fit.

EXAMPLE 4 ■ Computing a Linear Regression Model and Coefficient of Determination

Private philanthropy is the act of donating money to support a charitable cause. For example, many colleges and universities accept private philanthropy to fund scholarships for financially needy students.

Table 2.11 shows the amount of money donated by U.S. residents, corporations, and foundations for philanthropic purposes between 2000 and 2003.

Table 2.11

Years Since 2000 t	Funds ($ billion) F
0	227.7
1	229.0
2	234.1
3	240.7

Source: Statistical Abstract of the United States, 2006, Table 570

a. Find the linear regression model for the data.

b. Interpret the meaning of the slope and the vertical intercept of the model in terms of private philanthropy funds, *F*, and years since 2000, *t*.

c. Use the model to predict how much money in private philanthropy funds will be given in 2008.

d. State the coefficient of determination.

Solution

a. We use a graphing calculator to find the linear regression model, as shown in Figure 2.15.

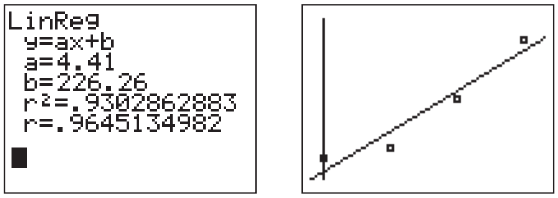


Figure 2.15

The linear regression model is $P(t) = 4.41t + 226.26$.

b. The model suggests that in 2000 ($t = 0$), \$226.26 billion in private philanthropy funds was given and the amount increases each year at a rate of \$4.41 billion per year.

c. Since 2008 is 8 years after 2000, we substitute $t = 8$ into the regression model.

$$\begin{aligned} P(t) &= 4.41t + 226.26 \\ P(8) &= 4.41(8) + 226.26 \\ &= 261.54 \end{aligned}$$

We predict that \$261.54 billion in private philanthropy funds will be given in 2008.

d. The coefficient of determination is approximately 0.93.

Correlation Coefficient

Another way to measure the strength of fit of a linear regression model to a data set involves the **correlation coefficient**, r. When using a graphing calculator for linear regression, the value of r is shown below the value of r^2 (see Figure 2.14). The computation of this number requires statistical knowledge and will not be addressed here, but we will explain how to interpret it.

The correlation coefficient, r, is a value such that $-1 \le r \le 1$. If $r = 1$, we have a perfectly linear data set with a positive slope. If $r = -1$, we have a perfectly linear data set with a negative slope. If $r = 0$, we say there is no correlation between the input and output.

Correlation Coefficient

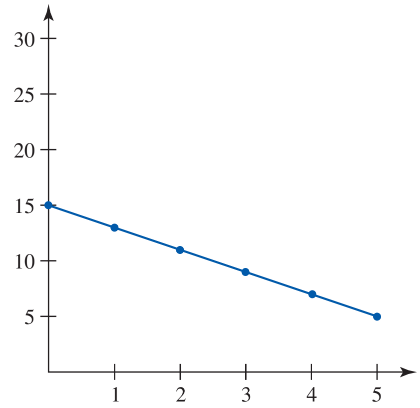
The **correlation coefficient**, r, is a value that describes the strength of fit of a linear regression model to a set of data. The closer the value of $|r|$ is to 1, the better the fit. When $r > 0$, the regression line is increasing. When $r < 0$, the regression line is decreasing.

EXAMPLE 5 ■ **Interpreting the Correlation Coefficient**

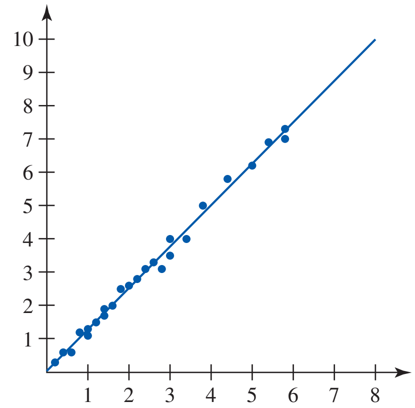
Match each graph with the correct correlation coefficient value and explain your reasoning.

A. $r = 1$ **B.** $r = -1$ **C.** $r = 0$ **D.** $r = 0.996$ **E.** $r = -0.995$ **F.** $r = -0.945$

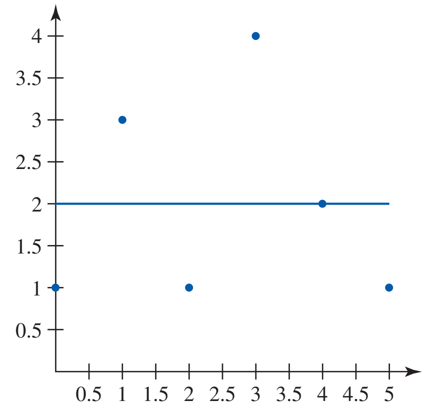
I.



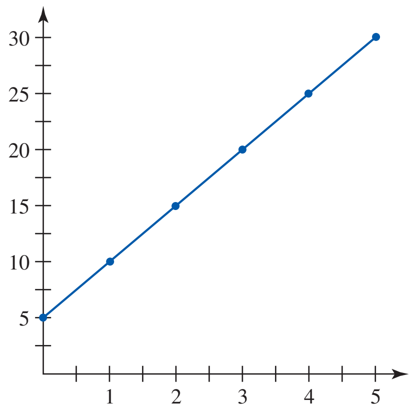
II.



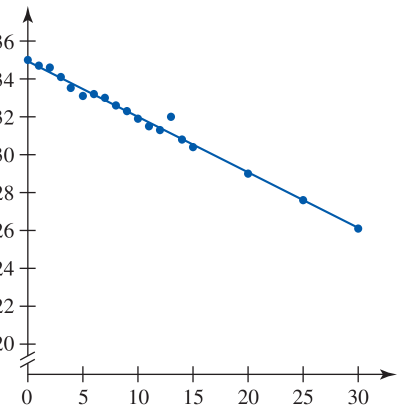
III.



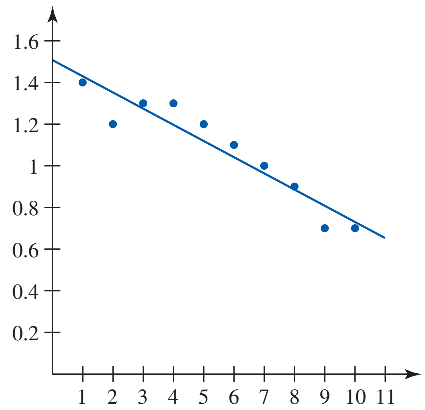
IV.



V.



VI.



Solution I—B, II—D, III—C, IV—A, V—E, VI—F

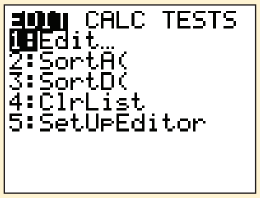
Graph I is perfectly linear with a negative slope so the correlation coefficient is $r = -1$. Graph II, although not perfectly linear, seems to have a strong linear fit with a positive slope. Therefore, the correlation coefficient must be $r = 0.996$. Graph III seems to have no correlation. As the dependent variable value increases, the independent variable value neither increases (indicating a positive correlation) nor decreases (indicating a negative correlation). Therefore, the correlation coefficient must be $r = 0$. Graph IV is perfectly linear with a positive slope so the correlation coefficient is $r = 1$. Graph V shows a negative correlation, as does Graph VI. However, Graph V seems to be more strongly fit to the linear model shown than Graph VI. Therefore, Graph V must have a correlation coefficient of $r = -0.995$ and Graph VI must have a correlation coefficient of $r = -0.945$.

SUMMARY

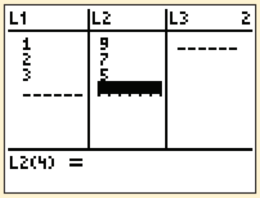
In this section you learned how to use linear regression to find the line of best fit, or linear regression model, for a data set. You also learned that the process of predicting an unknown data value using a linear model is called interpolation or extrapolation, depending on the value of the input variable. And you discovered that both the coefficient of determination and the correlation coefficient indicate the strength of fit of the line of best fit to the data set.

TECHNOLOGY TIP ■ CREATING LISTS OF VALUES FOR A DATA SET

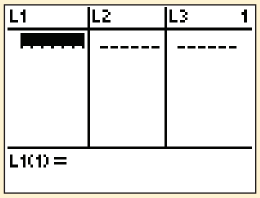
1. Bring up the Statistics Menu by pressing the STAT button.



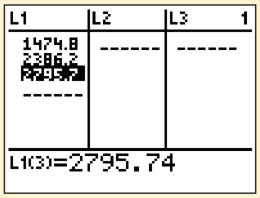
2. Bring up the List Editor by selecting `1:EDIT` and pressing ENTER.



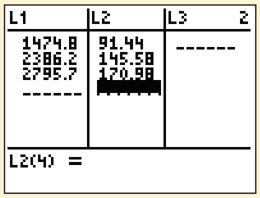
3. Clear the lists. If there exists data in the list, use the arrows to move the cursor to the list heading, `L1`. Press the CLEAR button and press ENTER. This clears all of the list data. Repeat for each list with data. (Warning: Be sure to use CLEAR instead of DELETE. DELETE removes the entire column.)



4. Enter the numeric values of the *inputs* in list `L1` and press ENTER after each entry.

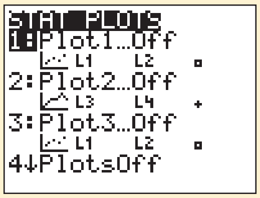


5. Enter the numeric values of the *outputs* in list `L2` and press ENTER after each entry. When complete, press 2ND then MODE to exit out of the List Editor.

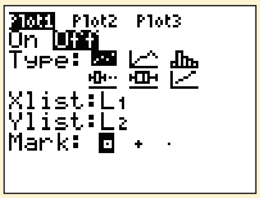


TECHNOLOGY TIP ■ DRAWING A SCATTER PLOT

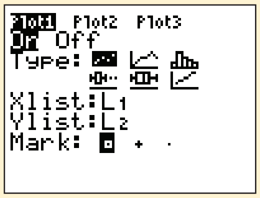
1. Bring up the Statistics Plot Menu by pressing the 2ND button then the Y= button.



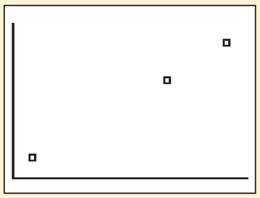
2. Open `Plot1` by pressing ENTER.



3. Turn on `Plot1` by moving the cursor to `On` and pressing ENTER. Confirm that other menu entries are as shown.

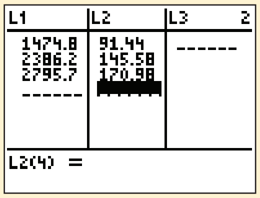


4. Graph the scatter plot by pressing ZOOM and scrolling to `9:ZoomStat`. Press ENTER. This will graph the entire scatter plot along with any functions in the Graphing List. The ZoomStat feature automatically adjusts the viewing window so that all of the data points are visible.

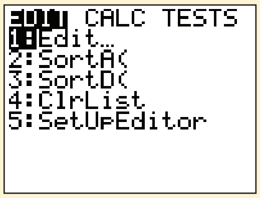


TECHNOLOGY TIP ■ LINEAR REGRESSION

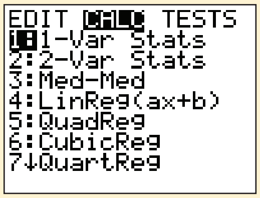
1. Enter the numeric values of the *inputs* and *outputs* of the data set using the List Editor.



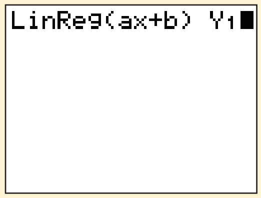
2. Return to the Statistics Menu by pressing the STAT button.



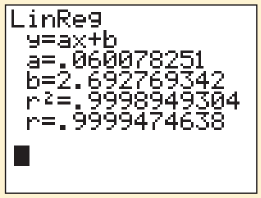
3. Bring up the Calculate Menu by using the arrows to select CALC.



4. Select `4:LinReg(ax+b)`. To automatically paste the regression equation into the Y= Editor, press VARS, `Y-VARS`, `1:Function`, `1:Y1`.



5. Press ENTER. The line of best fit is $y = 0.6008x + 2.693$ and has coefficient of determination $r^2 = 0.9999$.



Optional Step:
If the correlation coefficient and coefficient of determination do not appear, do the following:
Press 2ND then 0, scroll to `DiagnosticOn` and press ENTER twice. This will ensure that the correlation coefficient r and the coefficient of determination r^2 will appear the next time you do a regression.

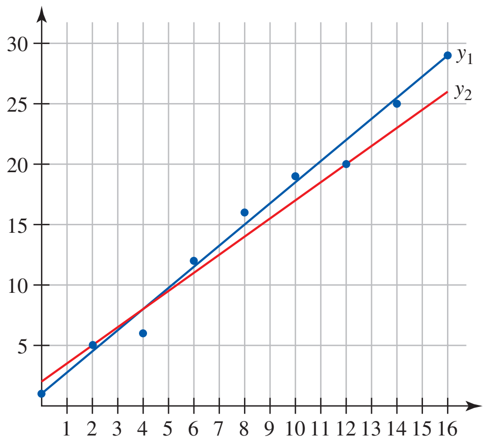
2.3 EXERCISES

■ SKILLS AND CONCEPTS

In Exercises 1–5, compute the total sum of the squares of the differences between the data points and each of the linear models given. Then use the result to determine which linear model, y_1 or y_2, best fits the data.

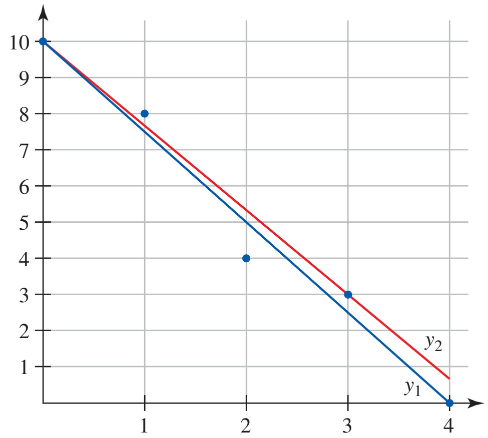
1. $y_1 = \frac{7}{4}x + 1$ or $y_2 = \frac{3}{2}x + 2$

x	y
0	1
2	5
4	6
6	12
8	16
10	19
12	20
14	25
16	29



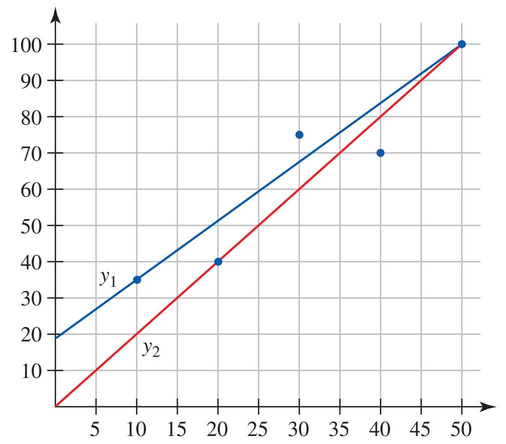
2. $y_1 = -\frac{5}{2}x + 10$ or $y_2 = -\frac{7}{3}x + 10$

x	y
0	10
1	8
2	4
3	3
4	0



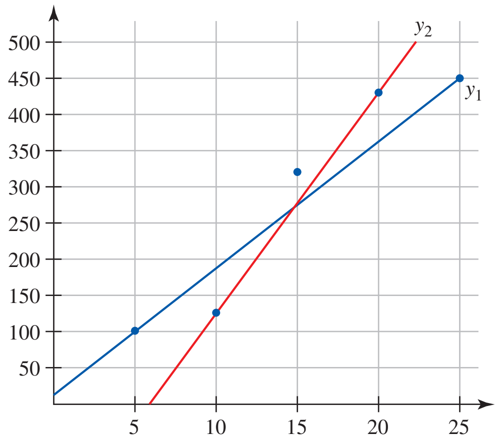
4. $y_1 = \frac{13}{8}x + \frac{75}{4}$ or $y_2 = 2x$

x	y
10	35
20	40
30	75
40	70
50	100



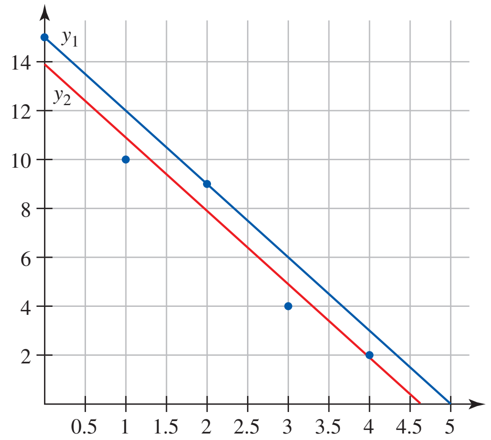
3. $y_1 = 17.5x + 12.5$ or $y_2 = 30.5x - 180$

x	y
5	100
10	125
15	320
20	430
25	450



5. $y_1 = -3x + 15$ or $y_2 = -3x + 13.9$

x	y
0	15
1	10
2	9
3	4
4	2



SHOW YOU KNOW

6. What is linear regression?
7. Explain what the value of the coefficient of determination indicates.
8. Explain the difference between interpolation and extrapolation.
9. Why is it important to look at the coefficient of determination or correlation coefficient before using a linear model to interpolate or extrapolate?
10. If the coefficient of determination of a model is 1, what is the relationship between the model and the original data set?

MAKE IT REAL

11. **Taxable Income** The data in the table show the amount of taxes to be paid by married couples filing a joint tax return for the given taxable income amounts.

Taxable Income ($ thousands) *I*	Tax Paid (dollars) *T*
20	2249
30	3749
40	5249
50	6749
60	8249
70	10,621
80	13,121
90	15,621

Source: 2006 IRS Tax Table, www.irs.gov

a. Create a linear regression model for these data.
b. Using the model found in part (a), interpret the practical meaning of the slope and vertical intercept of the model.
c. Use the linear regression model to predict the amount of tax paid if a married couple's income is $72,000. Is this an example of interpolation or extrapolation? Explain.
d. Use the linear regression model to predict the amount of tax paid if a married couple's income is $110,000. Is this an example of interpolation or extrapolation? Explain.
e. Based on the coefficient of determination, do you think the interpolation or extrapolation values you found are accurate?

12. **Households with Televisions** The data in the table show the number of households in the United States with televisions.

a. Create a linear regression model for these data.
b. Using the model found in part (a), interpret the practical meaning of the slope and vertical intercept of the model.
c. Use the linear regression model to predict the number of households with televisions in 1985. Is this an example of interpolation or extrapolation? Explain.
d. Use the linear regression model to predict the number of households with televisions in 2010. Is this an example of interpolation or extrapolation? Explain.

Years Since 1980 *t*	Number of Households (in millions) *H*
0	76
10	92
15	95
17	97
18	98
20	101
21	102
22	106
23	107

Source: Statistical Abstract of the United States, 2006, Table 1117

e. Based on the coefficient of determination, do you think the interpolation or extrapolation values you found are accurate?

13. **Carbon Monoxide Pollutant Concentrations** The data in the table show the carbon monoxide pollutant concentration, in parts per million, for years since 1990. The data are based on 359 monitoring stations from various locations in the United States.

Years Since 1990 *t*	Carbon Monoxide Pollutant Concentration (parts per million) *P*
0	6
5	4.7
9	3.9
10	3.4
11	3.2
12	3
13	2.8

Source: Statistical Abstract of the United States, 2006, Table 359

a. Create a linear regression model for these data.
b. Using the model found in part (a), interpret the practical meaning of the slope and vertical intercept of the model.
c. Use the linear regression model to predict the carbon monoxide pollutant concentration in 1992. Is this an example of interpolation or extrapolation? Explain.
d. Use the linear regression model to predict the carbon monoxide pollutant concentration in 2010. Is this an example of interpolation or extrapolation? Explain.
e. Based on the coefficient of determination, do you think the interpolation or extrapolation values you found are accurate?

14. Home Run Hitting The table shows the cumulative number of home runs hit by Luis Gonzalez up to the All-Star break in 2006.

Game Number g	Cumulative Number of Home Runs Hit H
4	1
5	2
12	3
13	4
16	5
76	6
87	7

Source: www.mlb.com

a. Create a linear regression model for these data.

b. Using the model found in part (a), interpret the practical meaning of the slope and vertical intercept of the model.

c. Use the linear regression model to predict the number of home runs hit by Luis Gonzales by the 100th game of the season. Is this an example of interpolation or extrapolation? Explain.

d. Use the linear regression model to predict the number of home runs hit by Luis Gonzales by the final game of the season, game 162. Is this an example of interpolation or extrapolation? Explain.

e. Based on the coefficient of determination, do you think the interpolation or extrapolation values you found are accurate?

15. Famous Skyscrapers The data in the table show the number of stories and the height of the 10 tallest skyscrapers in the world.

Skyscraper	Number of Stories n	Height (feet) h
Taipei 101	101	1667
Petronas Tower 1	88	1483
Willis Tower	110	1451
Jin Mao Building	88	1381
Two International Finance Centre	88	1362
CITIC Plaza	80	1283
Shun Hing Square	69	1260
Empire State Building	102	1250
Central Plaza	78	1227

Source: www.infoplease.org

a. Create a linear regression model for these data.

b. Using the model found in part (a), interpret the practical meaning of the slope of the model.

c. Use the linear regression model to predict the height of a new building that has 50 stories. Is this an example of interpolation or extrapolation? Explain.

d. Use the linear regression model to predict the height of a new building that has 120 stories. Is this an example of interpolation or extrapolation? Explain.

e. Based on the coefficient of determination, do you think the interpolation or extrapolation values you found are accurate?

16. Manatee Population Because the Florida manatee population is threatened, the Florida Manatee Sanctuary Act of 1978 was enacted to protect the species. Scientists interested in the relationship between the number of manatee deaths and time collected the data shown in the table.

a. Create a scatter plot of these data.

b. Find the linear regression model for these data.

c. Referring to the coefficient of determination, r^2, and the correlation coefficient, r, explain whether or not the linear model represents the situation well.

d. Use the linear regression model to predict the year in which the number of manatee deaths will be 450.

Year	Manatee Deaths	Year	Manatee Deaths
1974	7	1991	174
1975	25	1992	163
1976	62	1993	146
1977	114	1994	192
1978	84	1995	201
1979	77	1996	416
1980	63	1997	242
1981	116	1998	232
1982	114	1999	269
1983	81	2000	272
1984	128	2001	325
1985	119	2002	305
1986	122	2003	380
1987	114	2004	276
1988	133	2005	396
1989	168	2006	416
1990	206		

Source: research.myfwc.com

17. Manatee Population Because many manatee deaths can be attributed to a boating incident, scientists are interested in the relationship between the number of manatee deaths and the number of registered boats in Florida. Use the data set to respond to the items that follow.

Year	Registered Boats (in thousands)	Manatee Deaths
1990	716	206
1991	715	174
1992	710	163
1993	730	146
1994	748	192
1995	766	201
1996	789	416
1997	803	242
1998	824	232

Source: research.myfwc.com

a. Create a scatter plot of the number of manatee deaths as a function of the number of registered boats (in thousands).

b. Find the linear regression model for these data.

c. Explain the practical meaning of the slope and vertical intercept of the model.

d. Referring to the coefficient of determination, r^2, and the correlation coefficient, r, explain whether or not the linear model represents the situation well.

e. Suppose the number of registered boats increased to 3 million boats. Use the linear regression model to predict the number of manatee deaths. Discuss the accuracy of your result.

18. Chipotle Mexican Grill The table shows Chipotle Mexican Grill's costs for food, beverage, and packaging along with labor costs for the indicated years.

Years Since 2000 t	Food, Beverage, and Packaging Costs ($ thousands) f	Labor Costs ($ thousands) L
1	45,236	46,048
2	67,681	66,515
3	104,921	94,023
4	154,148	139,494
5	202,288	178,721

Source: Chipotle Mexican Grill, Inc., 2005 Annual Report, p. 24

a. Create a scatter plot of the labor costs as a function of food, beverage, and packaging costs.

b. Find the linear regression model for these data. Use the model to describe the labor costs and the food, beverage, and packaging costs.

c. Referring to the coefficient of determination, r^2, and the correlation coefficient, r, explain whether or not the linear model represents the situation well.

d. Suppose the labor costs for Chipotle Mexican Grill were 200,000 ($ thousands). Predict the food, beverage, and packaging costs associated with this labor cost. Discuss the accuracy of this result.

19. Cigarette Smoking The data in the table give the percentage of people who smoke cigarettes for selected years.

a. Create a scatter plot of the percentage of people who smoke as a function of the year.

b. Find the linear regression model for these data.

c. Referring to the coefficient of determination, r^2, and the correlation coefficient, r, explain whether or not the linear model represents the situation well.

d. In what year does the model predict that the percentage of people who smoke will be 0%? Discuss the accuracy of this result.

Year t	People Who Smoke Cigarettes (percent) P
1974	36.9
1979	33.1
1983	31.6
1985	30
1987	28.8
1988	28.1
1990	25.4
1991	25.8
1992	26.3
1993	24.7
1994	24.9
1995	24.5
1997	24
1998	23.4
1999	22.7
2000	22.6
2001	22
2002	21.4
2003	21.1

Source: www.cdc.gov

20. Heart Disease and Cigarette Smoking The data in the table give the percentage of people who smoke cigarettes and the death rate (deaths per 100,000 people) caused by heart disease.

Year t	People Who Smoke Cigarettes (percent) P	Heart Disease Death Rate (per 100,000) D
1974	36.9	458.8
1979	33.1	401.6
1983	31.6	388.9
1985	30	375
1987	28.8	355.9
1988	28.1	352.5
1990	25.4	321.8
1991	25.8	313.8
1992	26.3	306.1
1993	24.7	309.9
1994	24.9	299.7
1995	24.5	296.3
1997	24	280.4
1998	23.4	272.4
1999	22.7	267.8
2000	22.6	257.6
2001	22	247.8
2002	21.4	240.8
2003	21.1	232.1

Source: www.cdc.gov and *Statistical Abstract of the United States, 2006*, Table 106)

a. Create a scatter plot of heart disease death rate as a function of the percentage of people who smoke.

b. Find the linear regression model for these data.

c. Explain the practical meaning of the slope of the model.

d. Use the linear regression model to predict the death rate due to heart disease if 10% of people smoke cigarettes.

21. Golf Ball Collecting An Arizona man living in a home alongside the fairway of a golf course collected the balls he found in his yard from January 2008 to January 2009 and created the following table of data.

Months Since Jan. 2008 *m*	Average Number of Golf Balls Found per Day *G*
0	5.80
1	5.25
2	7.93
3	12.33
4	13.32
5	12.53
6	13.75
7	14.94
8	16.28
9	15.38
10	13.33
11	13.07
12	12.77

Source: Data collected by Jim Simpson, Gilbert, Arizona

a. Create a scatter plot of average number of golf balls found as a function of the months since January 2008.

b. Find the linear regression model for these data.

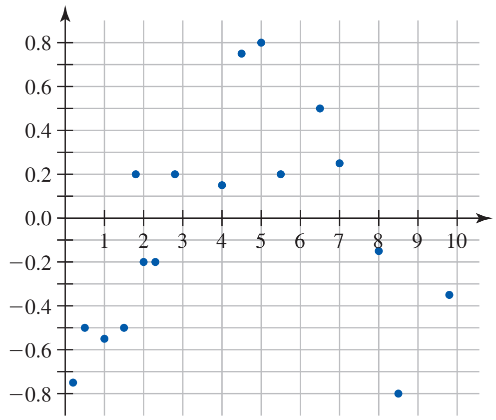
c. Explain the practical meaning of the slope of the model.

d. Use the linear regression model to predict the average number of golf balls the man will find in March 2010.

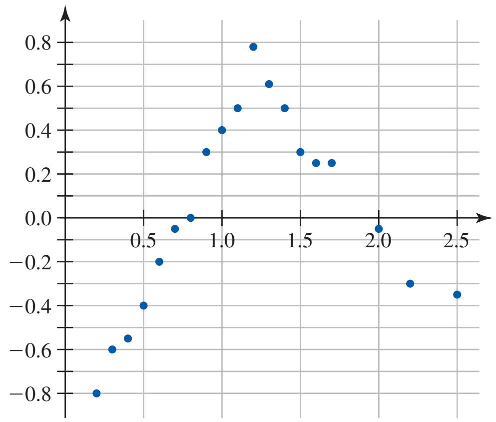
■ STRETCH YOUR MIND

For Exercises 22–26, the residual plot for a linear model is shown. A residual plot has the same independent variable as the scatter plot of the original data set; however, the dependent variable is the residual (the difference in the actual output and the predicted output). Based on the graph of the residual plot, explain whether or not the linear model fits the original data set well.

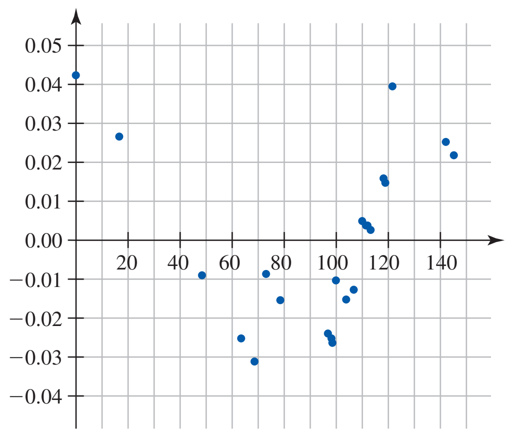
22.



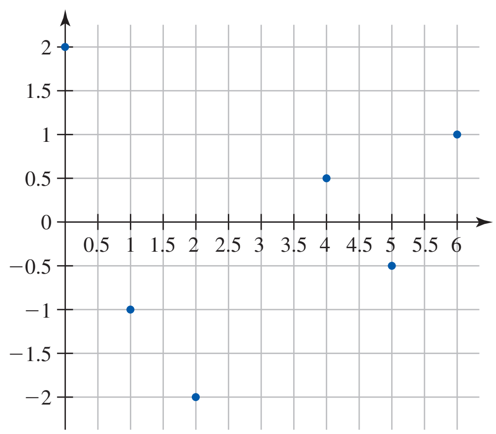
23.



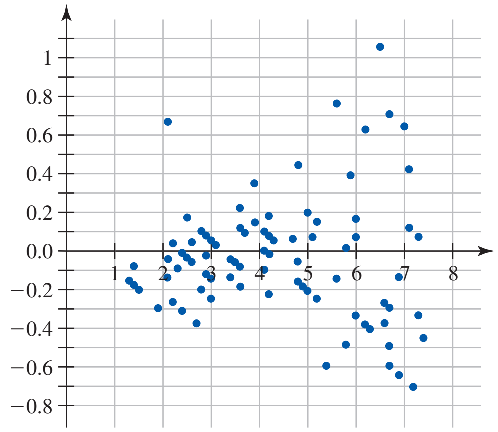
24.



25.



26.



SECTION 2.4 Systems of Linear Equations

LEARNING OBJECTIVES

- Determine the solution to a system of equations algebraically, graphically, and using technology and interpret the real-world meaning of the results
- Use the substitution and elimination methods to solve linear systems that model real-world scenarios
- Determine if systems of linear equations are dependent or inconsistent and explain the real-world meaning of the results

GETTING STARTED

In many real-world contexts, it is common to work with more than one equation at the same time. For example, a snack company combines several ingredients to create a batch of a trail mix of a specific volume. The volume of the mix is equal to the sum of the volumes of the individual ingredients, and the relationship between the volumes can be represented by an equation. The company may also have nutritional targets related to the fat, protein, and sugar content of the mix. The company can create equations that relate the nutritional value of the ingredients to the nutritional value of the mix. To determine how much of each ingredient to put in the mix, the company would solve a volume equation and a nutritional-value equation as a system of equations.

In this section we solve systems of linear equations using a variety of methods. We also discuss how to determine if a system is dependent or inconsistent and discover the real-world meaning of those terms.

■ Systems of Linear Equations

A **system of equations** is a group of two or more equations. To solve a system of equations means to find values for the variables that satisfy all of the equations in the system. Systems of equations can involve any number of equations and variables; however, we will limit ourselves to situations containing two variables in this section.

> **SYSTEM OF LINEAR EQUATIONS IN TWO VARIABLES**
>
> A **system of linear equations** in two variables is a group of two or more linear equations that use the same variables.

To develop an understanding of systems of linear equations, let's look at some variables that affect the cost of a taxi ride. Taxis play a key role in cities around the world, ferrying tourists, taking people to and from airports, and even saving lives by keeping intoxicated people from driving. The cost of hiring a taxi varies from city to city and usually depends on how far a person travels and how much time the taxi spends waiting (in traffic or for a client).

Table 2.12 compares the costs for hiring a taxi in Seattle, Washington, and Dallas, Texas, in July 2009.

Table 2.12

City	First Mile	Cost per Additional Mile	Cost per Minute Waiting
Seattle	\$2.50	\$2.00	\$0.50
Dallas	\$4.00	\$1.80	\$0.30

Source: www.taxifarefinder.com

We will simplify our discussion by assuming that a typical taxi ride involves 5 minutes of wait time. Using this assumption, the wait-time cost in Seattle is \$2.50 and in Dallas is \$1.50.

EXAMPLE 1 ■ Exploring a System of Equations Using a Table of Values

Suppose we hired a taxi in Seattle and in Dallas. Using the fare data in Table 2.12, create a table of values and estimate the number of additional miles that must be driven for the trip costs to be equal.

Solution We begin with the fixed costs.

Seattle: first mile cost + wait-time cost = \$2.50 + \$2.50 = \$5.00

Dallas: first mile cost + wait-time cost = \$4.00 + \$1.50 = \$5.50

Each additional mile costs \$2.00 in Seattle and \$1.80 in Dallas. So the variable costs are \$2.00 × additional miles in Seattle and \$1.80 × additional miles in Dallas. To create a table of values, we note the calculations necessary to determine the total cost of a trip. In Seattle, we begin with a \$5.00 charge and add an additional \$2.00 for each mile traveled. For example, suppose we traveled 3 additional miles. Then

$$\begin{aligned}\text{Seattle taxi cost} &= \$5.00 + \frac{\$2.00}{1\text{ mi}} \cdot 3\text{ mi}\\ &= \$5.00 + \frac{\$2.00}{1\,\cancel{\text{mi}}} \cdot 3\,\cancel{\text{mi}}\\ &= \$5.00 + \$2.00(3)\\ &= \$11.00\end{aligned}$$

Using similar calculations for Dallas, we create Table 2.13.

Table 2.13

Additional Miles Traveled	Cost in Seattle	Cost in Dallas
0	\$5.00	\$5.50
1	\$7.00	\$7.30
2	\$9.00	\$9.10
3	\$11.00	\$10.90
4	\$13.00	\$12.70
5	\$15.00	\$14.50

We notice the cost for hiring a taxi in Seattle, which is less expensive for a trip of less than 2 additional miles, becomes more expensive by the time we have traveled 3 additional miles. However, it is not clear exactly how far we must travel for the cost in Seattle to equal the cost in Dallas. We can get a better idea by creating a table for values between 2 and 3 additional miles, as shown in Table 2.14. To create the table, we use the cost equation for Seattle, Cost = 5.00 + 2(additional miles), and the cost equation for Dallas, Cost = 5.50 + 1.80(additional miles).

Table 2.14

Additional Miles Traveled	Cost in Seattle	Cost in Dallas
2	\$9.00	\$9.10
2.2	\$9.40	\$9.46
2.4	\$9.80	\$9.82
2.5	\$10.00	\$10.00
2.6	\$10.20	\$10.18
2.8	\$10.60	\$10.54
3.0	\$11.00	\$10.90

Now we can see that at 2.5 additional miles the cost in Seattle and the cost in Dallas are both \$10.00. The *solution to the system of linear equations* is (2.5, 10.00).

Solution to a System of Equations in Two Variables

The **solution to a system of linear equations** in two variables (if it exists) is a pair of values (x, y) such that when x is used as the input, each equation returns the same y-value. Graphically this will appear as a point (x, y) where the lines of all the equations intersect.

Solving a System Using Graphs

One common approach to solving a system of equations is to graph all of the equations simultaneously and find the point of intersection. We do this in Example 2.

EXAMPLE 2 ■ Solving a System of Equations Graphically

Write a system of linear equations to represent the cost of hiring a taxi in Seattle and the cost of hiring a taxi in Dallas as functions of the number of additional miles traveled. Then graph the functions on the same set of axes and discuss what information the graphs reveal.

Solution We let S represent the cost in dollars of hiring a taxi in Seattle, D represent the cost in dollars of hiring a taxi in Dallas, and x represent the number of additional miles traveled. Combining the fixed and variable costs for each city, we get

$$\text{Seattle: } S(x) = 2.00x + 5.00$$
$$\text{Dallas: } D(x) = 1.80x + 5.50$$

We now graph these functions as shown in Figure 2.16. The graphs have been restricted to domains of $x \geq 0$ because the number of additional miles traveled must be nonnegative to make sense.

The two lines appear to intersect at (2.5, 10). We estimate this to be the solution of the system; however, we can use algebra to verify if this is an exact solution. We will do this in Example 3.

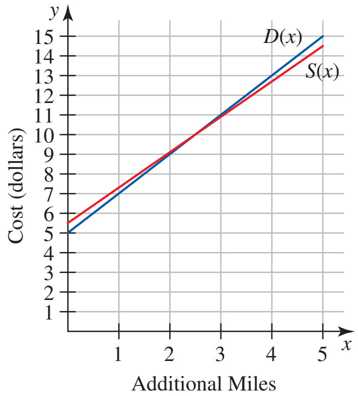


Figure 2.16

■ Solving a System Using the Substitution Method

For many systems of equations, the solution includes “messy” numbers that are difficult to determine from a table or graph. Fortunately, most graphing calculators have an *intersect* command that allows us to quickly determine the point of intersection of two graphs. We detail this process in the Technology Tip at the end of the section.

EXAMPLE 3 ■ Solving a System Using the Substitution Method

Use algebra to solve the system of equations in Example 2.

Solution From Example 2, we have the system

$$S(x) = 2.00x + 5.00$$
$$D(x) = 1.80x + 5.50$$

We also know a solution to this system means that hiring a taxi in either city costs exactly the same for some number of additional miles traveled. This tells us that we want to find the value for x such that $S(x) = D(x)$. In other words, we want

$S(x)$	$=$	$D(x)$
the cost of hiring a taxi in Seattle for x miles beyond the first mile	is the same as	the cost of hiring a taxi in Dallas for x miles beyond the first mile

$$S(x) = D(x)$$
$$2.00x + 5.00 = 1.80x + 5.50$$

This method of solving a system of equations is known as the **substitution method** because we are substituting the expressions $2.00x + 5.00$ and $1.80x + 5.50$ for S and D in the relationship $S(x) = D(x)$. We solve the equation.

$$2.00x + 5.00 = 1.80x + 5.50$$
$$0.20x + 5.00 = 5.50$$
$$0.20x = 0.50$$
$$x = 2.5$$

A hired taxi traveling 2.5 additional miles will cost the same in either city. We verify the answer by substituting this value into each equation of the original system.

$$S(x) = 2.00x + 5 \qquad D(x) = 1.80x + 5.50$$
$$S(2.5) = 2.00(2.5) + 5 \qquad D(2.5) = 1.80(2.5) + 5.50$$
$$= 10.00 \qquad = 10.00$$

The solution to the system is (2.5, 10.00). When the taxi travels 2.5 additional miles the cost is $10.00 in both cities.

HOW TO: ■ USE THE SUBSTITUTION METHOD

To solve a system of linear equations of the form

$$y = ax + b$$
$$y = cx + d$$

using the substitution method,

1. Replace the value of y in the first equation with $cx + d$. This creates a new equation $cx + d = ax + b$.
2. Solve the new equation $cx + d = ax + b$ for x. The solution will be $x = \frac{c - a}{b - d}$.
3. Substitute the value of x back into either the first or second equation and solve for y.

If the system of equations models real-world data, you should also complete Steps 4 and 5.

4. Ask yourself if the mathematical solution makes sense in its real-world context.
5. Reevaluate the functions, as necessary, and verbally express the real-world meaning of the result.

We have demonstrated how to solve a system of equations using a table, a graph, and algebraic methods. There is another clever approach worth considering. We observe there is a \$0.50 difference (\$5.50 − \$5.00 = \$0.50) in the cost for the first mile in each city. We also note the cost for each additional mile differs by \$0.20 per mile (\$2.00 − \$1.80 = \$0.20). For the total cost to be the same, the taxi must travel enough additional distance that the difference in fares will make up for the difference in initial costs. Therefore, we divide \$0.50 by \$0.20 per mile.

$$\frac{\$0.50}{\$0.20/\text{mile}} = 2.5 \text{ miles}$$

Note this approach does not require a table, graph, or formal equations. However, it does require that we have a solid understanding of the situation.

We have just seen how each of the four representations of a function—table, graph, formula, and verbal description (with a few easy calculations)—can be used to find the solution to a system. Each method has its own advantages and disadvantages. Being flexible enough to work with all four representations gives us more ways to approach a problem and provides us with a deeper understanding of what the solution means in a given context. Note also that in many real-world scenarios, an exact solution to a system of equations is not necessary; an estimate from a table of values or a graph is often good enough.

EXAMPLE 4 ■ Solving a System of Equations

Solve the following system of equations using the substitution method. Then check your answer by graphing.

$$y = -3x + 2$$
$$y = 0.5x - 34.75$$

Solution We know we want the outputs of both functions to be the same for a given input. For sake of clarity, we label the equations, making them $y_1 = -3x + 2$ and $y_2 = 0.5x - 34.75$. We want to know the value of x that makes $y_1 = y_2$. Then

$$\begin{aligned} y_1 &= y_2 \\ -3x + 2 &= 0.5x - 34.75 \\ 2 &= 3.5x - 34.75 \\ 36.75 &= 3.5x \\ x &= 10.5 \end{aligned}$$

We substitute 10.5 for x in either original equation to get the y-value that completes the solution. It doesn't matter which equation we use. To demonstrate, we use both.

$$\begin{aligned} y &= -3x + 2 & y &= 0.5x - 34.75 \\ y &= -3(10.5) + 2 & y &= 0.5(10.5) - 34.75 \\ y &= -31.5 + 2 & y &= 5.25 - 34.75 \\ y &= -29.5 & y &= -29.5 \end{aligned}$$

Thus, the solution to this system of equations is (10.5, −29.5). We graph the system of equations to check our work, as shown in Figure 2.17.

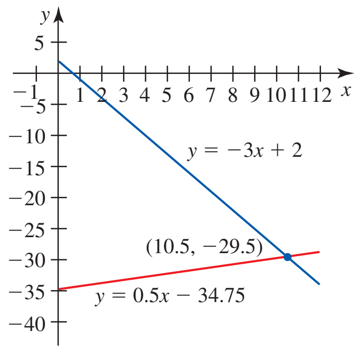


Figure 2.17

The graph confirms that our solution is correct. (Note: If you are using a calculator and the intersection point does not appear, adjust your viewing window to make sure that the solution is included in the domain and range of the viewing window.)

Business Applications of Linear Systems

Business analysts are often interested in profit, revenue, and costs of a company. In evaluating a business plan, it is important to know at what sales level revenue and costs are expected to be equal. This sales level is referred to as the **break-even point** and is the point at which the company begins to turn a profit.

Any business has two types of costs: *fixed costs* and *variable costs.* **Fixed costs** are those that remain constant regardless of production levels. For example, building rent, product research and development, and advertising are usually fixed costs. **Variable costs** are those that vary with the level of production. For example, raw materials and production-line worker wages are variable costs.

Break-even analysis may be applied to large companies or small home-based businesses. An example of one such business is given in Example 5.

EXAMPLE 5 ■ Determining a Break-Even Point

An artisan wants to sell her handmade craft angels online. She estimates her material cost for each angel to be \$3.50. As of January 2007, the online merchant craftmall.com charged a \$14.95 per month fee for a Premier account featuring up to 25 products. Comparing her craft to similar crafts on the market, the artisan estimates she can sell the craft angel for \$9.95. How many angels will she have to sell each month to break even? At that production level, what will be her production cost, revenue, and profit?

Solution The cost equation for the craft angels is the sum of the variable cost, \$3.50 per angel, and the fixed cost, \$14.95. Let a be the number of angels sold in a month. The cost equation is

$$C(a) = 3.50a + 14.95 \text{ dollars}$$

The revenue equation is

$$R(a) = 9.95a \text{ dollars}$$

We want to determine when her revenue will equal her cost. In other words, we want to find the value of a such that $R(a) = C(a)$. Graphing the two functions simultaneously results in the graphs shown in Figure 2.18.

It appears the graphs intersect near (2.3, 23) at the intersection point, $R(a) = C(a)$. We can find the exact point of intersection by using the substitution method.

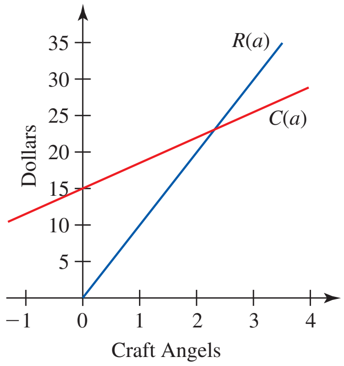


Figure 2.18

$$\begin{aligned} R(a) &= C(a) \\ 9.95a &= 3.50a + 14.95 \\ 6.45a &= 14.95 \\ a &= \frac{14.95}{6.45} \\ a &\approx 2.318 \end{aligned}$$

We evaluate $R(a)$ at $a = 2.318$ and determine that

$$\begin{aligned} R(2.318) &= 9.95(2.318) \\ &\approx 23.06 \end{aligned}$$

Thus the break-even point is roughly (2.318, 23.06). In the context of the problem, though, it does not make sense to talk about 2.318 angels. So we conclude she must sell 3 angels per month to cover her costs.

The cost to produce and advertise 3 angels is

$$\begin{aligned} C(3) &= 3.50(3) + 14.95 \\ &= \$25.45 \end{aligned}$$

The revenue from the sale of 3 angels is

$$\begin{aligned} R(3) &= 9.95(3) \\ &= \$29.85 \end{aligned}$$

She will profit \$4.40 if she sells 3 angels (\$29.85 − \$25.45 = \$4.40).

It is essential when solving a real-world problem (such as the ones shown in Examples 1–3 and 5) to make sure the solution makes sense in the context of the problem. A common error among students is to accept a mathematically correct answer (e.g., (2.318, 23.06)) without verifying that it makes sense in the context of the problem.

EXAMPLE 6 ■ Solving a System in Standard Form Using Substitution

Suppose in January 2009 a local community college ordered replacement computers for employees in two departments, math and history. For the math department, a purchase of \$17,427.89 was made that consisted of 6 Dell OptiPlex 360 desktops and 5 Dell Latitude E4200 laptops. For the history department, a purchase of \$13,211.91 was made that consisted of 6 Dell OptiPlex 360 desktops and 3 Dell Latitude E4200 laptops. (*Source:* www.dell.com)

a. Write a system of equations for this situation and explain what each term in the equations represents.

b. Solve the system algebraically and explain what the answer means in its real-world context.

Solution

a. The unknowns are the prices per computer for the desktop and laptop models purchased. Let d be the price in dollars for a Dell OptiPlex 360 desktop computer and let t be the price in dollars for a Dell Latitude E4200 laptop. We use the information given to create the following equations.

$$\underbrace{6d}_{\text{the total cost of 6 desktops}} \underbrace{+}_{\text{plus}} \underbrace{5t}_{\text{the total cost of 5 laptops}} \underbrace{=}_{\text{must equal}} \underbrace{17{,}427.89}_{\text{the total cost of the order for the math department}}$$

$$\underbrace{6d}_{\text{the total cost of 6 desktops}} \underbrace{+}_{\text{plus}} \underbrace{3t}_{\text{the total cost of 3 laptops}} \underbrace{=}_{\text{must equal}} \underbrace{13{,}211.91}_{\text{the total cost of the order for the history department}}$$

b. Solving this system will tell us the prices the community college paid for each type of computer. To use the substitution method, we will solve the first equation for t and substitute this expression into the second equation.

$$\begin{aligned} 6d + 5t &= 17{,}427.89 \\ 5t &= 17{,}427.89 - 6d \\ t &= \frac{17{,}427.89 - 6d}{5} \end{aligned}$$

Now we use the second equation and substitute the above expression for t.

$$\begin{aligned} 6d + 3t &= 13{,}211.91 \\ 6d + 3\left(\frac{17{,}427.89 - 6d}{5}\right) &= 13{,}211.91 && \textbf{Substitute } \frac{17{,}427.89 - 6d}{5} \textbf{ for } t. \\ 6d + \frac{3(17{,}427.89) - 3(6d)}{5} &= 13{,}211.91 \\ 6d + \frac{52{,}283.67 - 18d}{5} &= 13{,}211.91 \\ 5(6d) + 5\left(\frac{52{,}283.67 - 18d}{5}\right) &= 5(13{,}211.91) && \textbf{Multiply through by 5 to eliminate the fraction.} \\ 30d + \cancel{5}\left(\frac{52{,}283.67 - 18d}{\cancel{5}}\right) &= 66{,}059.55 \\ 30d + 52{,}283.67 - 18d &= 66{,}059.55 \\ 12d + 52{,}283.67 &= 66{,}059.55 \\ 12d &= 13{,}775.88 \\ d &= 1147.99 \end{aligned}$$

This tells us the college paid \$1147.99 for each desktop. We substitute this value for d in either original equation to determine the price for a laptop.

$$\begin{aligned} 6d + 5t &= 17{,}427.89 \\ 6(1147.99) + 5t &= 17{,}427.89 \\ 6887.94 + 5t &= 17.427.89 \\ 5t &= 10{,}539.95 \\ t &= 2107.99 \end{aligned}$$

Therefore, the solution is $d = 1147.99$ and $t = 2107.99$. This means each Dell OptiPlex 360 Desktop cost \$1147.99 and each Dell Latitude E4200 laptop cost \$2107.99.

Solving a System Using the Elimination Method

Example 6 demonstrates the challenges of using substitution to solve a system of equations in standard form. However, there is another method that works much better for such systems.

To illustrate, let's return to the situation in Example 6. The math department and history department both ordered 6 desktops. We assume they paid the same price per computer, so the total price for each order was different only because the number of laptops purchased was different. The difference in order totals was $4215.98, and there was a difference of 2 laptops purchased. This means that 2 laptops must cost $4215.98. Therefore,

$$\begin{aligned} 2t &= 4215.98 \\ t &= 2107.99 \end{aligned}$$

Plugging this answer back into one of the original equations will tell us the price for a desktop, d, and we arrive at the same solution found in Example 6 much more efficiently.

Let's examine this approach in a more formal algebraic way. By taking the difference of the number of laptops ordered and the difference of the order totals we were able to create the equation $2t = 4215.98$, which was easy to solve.

$$\begin{array}{rcrcl} 6d + 5t = 17{,}427.89 & \rightarrow & 6d + 5t &=& 17{,}427.89 \\ \underline{-(6d + 3t = 13{,}211.91)} & \rightarrow & \underline{-6d - 3t} &\underline{=}& \underline{-13{,}211.91} \\ & & \underbrace{0d + 2t}_{\substack{\text{There is no difference in the}\\ \text{number of desktops purchased,}\\ \text{but there is a difference of}\\ \text{2 laptops.}}} &=& \underbrace{4215.98}_{\substack{\text{There was a \$4215.98}\\ \text{difference in price between}\\ \text{the two orders.}}} \end{array}$$

$$\begin{aligned} 0d + 2t &= 4215.98 & & \\ 2t &= 4215.98 & & \text{The difference of 2 laptops is responsible for the \$4215.98 difference in price.} \\ t &= 2107.99 & & \text{Each laptop costs \$2107.99 (then use this to find } d\text{).} \end{aligned}$$

This approach is called the **elimination method** because it eliminates one of the variables to get an equation that is easier to solve. In the example, we found the difference between the two equations, thereby eliminating the variable d and leaving us with the equation $2t = 4215.98$, which was very easy to solve.

HOW TO: ■ USE THE ELIMINATION METHOD WITH TWO EQUATIONS AND TWO VARIABLES

To solve a system of two equations using the elimination method,

1. Write both equations in standard form.
2. Vertically align the variables in one equation with the corresponding variables in the other equation.
3. If one of the variables has the same coefficient in both equations (or the same magnitude coefficient with the opposite sign), move on to Step 4. If not, multiply one (or both) equations by a constant to create this situation.
4. Combine the equations using addition or subtraction. This will eliminate one of the variables.
5. Solve the resulting equation for the remaining variable.
6. Substitute the value from Step 5 back into one of the original equations and solve for the other variable.

EXAMPLE 7 ■ Using the Elimination Method

Suppose a business is purchasing new cell phones from Verizon Wireless for some of the company's executives and office staff. The company executives asked for 4 Blackberry 8830 Smartphones and 2 Samsung Knack cell phones, which totaled \$518.82. The office staff requested 2 Blackberry 8830 Smartphones and 7 Samsung Knack cell phones, which totaled \$518.79. (Prices accurate as of 2009.)

a. Write a system of equations to represent this situation and explain what each term in your equations represents.

b. Solve the system using the elimination method.

Solution

a. The unknowns in this situation are the prices for the individual cell phones. Let b be the price in dollars of a Blackberry 8830 Smartphone and let k be the price in dollars of a Samsung Knack.

$$\underbrace{4b}_{\text{The total cost of 4 Blackberry phones}} \quad \underbrace{+}_{\text{plus}} \quad \underbrace{2k}_{\text{the total cost of 2 Samsung phones}} \quad \underbrace{=}_{\text{must equal}} \quad \underbrace{518.82}_{\text{the total cost of the order for the company executives}}$$

$$\underbrace{2b}_{\text{The total cost of 2 Blackberry phones}} \quad \underbrace{+}_{\text{plus}} \quad \underbrace{7k}_{\text{the total cost of 7 Samsung phones}} \quad \underbrace{=}_{\text{must equal}} \quad \underbrace{518.79}_{\text{the total cost of the order for the office staff}}$$

b. Applying the elimination method to this system creates an immediate problem. If we find the difference between the orders, we will see that there is a difference of two Blackberry phones and five Samsung phones. This means that subtracting the two equations will not eliminate one of the variables. Thus, it appears that the elimination method will not help us here.

But let's play a game of "what if?" Suppose instead of ordering 2 Blackberry phones and 7 Samsung phones for the office staff, we double the order. This means the total price for the order will also double: 4 Blackberry phones, 14 Samsung phones, and a total cost of \$1037.58.

$$2b + 7k = 518.79 \qquad \text{Start with the original equation.}$$

$$2(2b + 7k = 518.79) \qquad \text{Multiply all of the terms by 2.}$$

$$4b + 14k = 1037.58 \qquad \text{Get a new equation without affecting the values of } b \text{ and } k.$$

We now have a system with the equations $4b + 2k = 518.79$ and $4b + 14k = 1037.58$. Although this no longer exactly represents the original situation, we have done nothing to alter the prices of each phone (the values of b and k are unaffected by this change). We have created an **equivalent system of equations** that has the same solution as the original system but is easier to solve.

$$4b + 2k = 518.82$$
$$4b + 14k = 1037.58$$

We now apply the elimination method to find the solution.

$$\begin{array}{rcrcr} 4b + 2k = 518.82 & \rightarrow & 4b + 2k & = & 518.82 \\ \underline{-(4b + 14k = 1037.58)} & \rightarrow & \underline{-4b - 14k} & \underline{=} & \underline{-1037.58} \\ & & \underbrace{0b - 12k}_{\text{There is no difference in the number of Blackberry phones purchased, but there is a difference of 12 Samsung phones (the negative means that the second order had more phones).}} & = & \underbrace{-518.76}_{\text{There was a \$518.76 difference in price between the two orders (the negative means that the second order was more expensive).}} \end{array}$$

$$0b - 12k = -518.76$$

$$-12k = -518.76 \quad \text{The difference of 12 Samsung phones is responsible for the \$518.76 difference in price.}$$

$$k = 43.23 \quad \text{Each Samsung phone costs \$43.23.}$$

We now use the value of k to find b.

$$\begin{aligned} 4b + 2k &= 518.82 \\ 4b + 2(43.23) &= 518.82 \\ 4b + 86.26 &= 518.82 \\ 4b &= 432.36 \\ b &= 108.09 \end{aligned}$$

Thus, the solution is $b = 108.09$ and $k = 43.23$. The company paid \$108.09 for each Blackberry 8830 Smartphone and \$43.23 for each Samsung Knack.

Equivalent Systems of Equations

A system of equations can be modified to create an **equivalent system of equations** that has the same solution as the original system. The following operations on a system will not affect the solution.

1. Interchange (change the position of) two equations.
2. Multiply an equation by a nonzero number.
3. Add (or subtract) a nonzero multiple of one equation to a nonzero multiple of another equation.

EXAMPLE 8 ■ Using the Elimination Method to Solve a System of Equations

A 1-cup serving of oil-roasted, salted peanuts contains 37.94 grams of protein and 27.26 grams of carbohydrates. A 1-cup serving of seedless raisins (not packed) contains 4.67 grams of protein and 114.74 grams of carbohydrates. (*Source:* www.nutri-facts.com) GORP (Good Ol' Raisins and Peanuts) is a popular snack food that provides short-term energy from carbohydrates and long-term energy from protein. How many cups of peanuts and how many cups of raisins are needed to create a 70-cup mix of GORP containing 1325 grams of protein?

Solution We let p be the number of cups of peanuts and r be the number of cups of raisins in the mix. Since the mix contains 70 cups of GORP, we create the following equation.

$$\underbrace{P}_{\text{the number of cups of peanuts}} \underbrace{+}_{\text{plus}} \underbrace{r}_{\text{the number of cups of raisins}} \underbrace{=}_{\text{must equal}} \underbrace{70}_{\text{70 cups total}}$$

Although both protein and carbohydrate information is provided, only the protein information is needed to solve this problem. Since each cup of peanuts contains 37.94 grams of protein, the peanuts contribute $37.94p$ grams of protein to the mix. Similarly, since each cup of raisins contains 4.67 grams of protein, the raisins contribute $4.67r$ grams of protein to the mix. We want the total amount of protein to be 1325 grams, so we create the following equation.

$$\underbrace{37.94p}_{\text{the total grams of protein in } p \text{ cups of peanuts}} \underbrace{+}_{\text{plus}} \underbrace{4.67r}_{\text{the total grams of protein in } r \text{ cups of raisins}} \underbrace{=}_{\text{must equal}} \underbrace{1325}_{\text{1325 total grams of protein in the mix}}$$

This yields the following system of equations.

$$p + r = 70$$
$$37.94p + 4.67r = 1325$$

We first create an equivalent system of equations and then solve the system using the elimination method.

$$37.94(p + r = 70) \rightarrow 37.94p + 37.94r = 2655.8$$ **Create an equivalent equation.**

$$\begin{array}{r} 37.94p + 37.94r = 2655.8 \\ -(37.94p + 4.67r \;\; = 1325) \\ \hline 0p + 33.27r = 1330.8 \end{array}$$ **Solve the equivalent system of equations.**

$$0p + 33.27r = 1330.8$$
$$33.27r = 1{,}330.8$$
$$r = 40$$

The mixture contains 40 cups of raisins. Since the mixture contains a total of 70 cups, there must be 30 cups of peanuts in the mixture. Therefore, 30 cups of peanuts and 40 cups of raisins are needed for a 70-cup mixture of GORP to have 1325 grams of protein.

EXAMPLE 9 ■ Determining When a System of Equations Has No Solution

Find the solution to the following system of linear equations.

$$4x - 5y = 20$$
$$8x - 10y = 30$$

Solution We use the elimination method. We multiply the first equation by -2 with the aim of eliminating x.

$$\begin{array}{rcr} -2(4x - 5y = 20) & \rightarrow & -8x + 10y = -40 \\ 8x - 10y = 30 & \rightarrow & 8x - 10y = \;\;\; 30 \\ \hline & & 0x + 0y \;\; = -10 \end{array}$$

$$0x + 0y = -10$$
$$0 = -10$$

But $0 \neq -10$. Since the system of equations led to a contradiction, the system does not have a solution. This can readily be seen by writing both equations in slope-intercept form.

$$y = 0.8x - 4$$
$$y = 0.8x - 3$$

The lines have the same slope but different y-intercepts, so they are parallel. Therefore, the two lines do not intersect and the system of equations does not have a solution. Graphing both lines as shown in Figure 2.19 validates our conclusion.

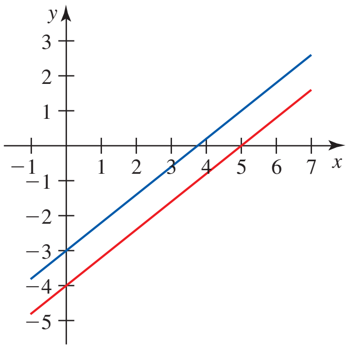


Figure 2.19

EXAMPLE 10 ■ Solving a System of Linear Equations with Infinitely Many Solutions

One of the authors is a fan of Nature Valley's Sweet and Salty Nut Granola Bars. According to the package labeling, the Peanut and the Almond bars have the nutritional values shown in Table 2.15.

Table 2.15

Variety	Calories from Fat	Protein (in grams)	Fat (in grams)	Carbohydrates (in grams)
Peanut	80	4	9	19
Almond	60	3	7	22

If the author wants to consume exactly 26 grams of protein and 520 calories from fat, how many bars of each type should he eat?

Solution We begin by identifying our variables. We let p be the number of Peanut bars eaten and a be the number of Almond bars eaten. Since the question does not ask about grams of fat or carbohydrates, we ignore those table values.

We need to set up an equation for *protein* and an equation for *calories from fat*. The amount of protein consumed by eating p Peanut bars is $4p$ since each bar contains 4 grams of protein. Similarly, the amount of protein from eating a Almond bars is $3a$ since each bar contains 3 grams of protein. The total amount of protein we want to consume is 26 grams. Therefore,

$$\text{protein: } 4p + 3a = 26$$

The calories from fat from eating p Peanut bars is $80p$ since each peanut bar contains 80 calories from fat. Similarly, the calories from fat from eating a Almond bars is $60a$ since each almond bar contains 60 calories from fat. The total number of calories from fat we want to consume is 520. Therefore,

$$\text{calories from fat: } 80p + 60a = 520$$

Since we want both equations to be simultaneously true, we combine them into a system of equations.

$$\begin{aligned} 4p + 3a &= 26 \\ 80p + 60a &= 520 \end{aligned}$$

Since we have focused on the elimination method in recent examples, let's use the substitution method for more practice. We choose to solve the first equation $4p + 3a = 26$ for p.

$$\begin{aligned} 4p + 3a &= 26 \\ 4p &= -3a + 26 \\ p &= -0.75a + 6.5 \end{aligned}$$

Substituting this value of p into the second equation yields

$$\begin{aligned} 80p + 60a &= 520 \\ 80(-0.75a + 6.5) + 60a &= 520 \\ (-60a + 520) + 60a &= 520 \\ 520 &= 520 \end{aligned}$$

What happened? The variable a dropped out and the equation resulted in a true statement, $520 = 520$. This indicates that the system of equations is *dependent*. Looking again at the original equations, we can see the second equation is equivalent to 20 times the first equation. In other words, multiplying the first equation by 20 will yield an equivalent system involving two identical equations. Therefore, any solution that satisfies the first equation will satisfy the second equation. We generate a table of a few values for the first equation, Table 2.16.

Table 2.16

a	$p = -0.75a + 6.5$
0	6.5
1	5.75
2	5
3	4.25
4	3.5
5	2.75
6	2

Whole-number solutions include $a = 2$, $p = 5$ and $a = 6$, $p = 2$. That is, eating two Almond bars and five

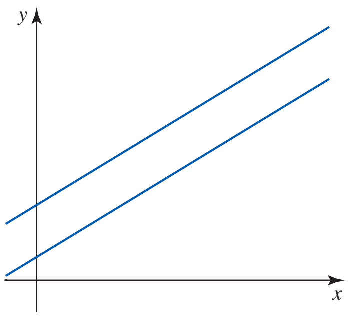
Peanut bars or eating six Almond bars and two Peanut bars will result in the desired amounts of protein and calories from fat.

In addition to these whole-number solutions, there are infinitely many real-number solutions, some of which are shown in the table. In fact, any point on this line is a solution to the system of equations. However, the only solutions that make sense in the context of this real-world application are solutions for which both a and p are nonnegative.

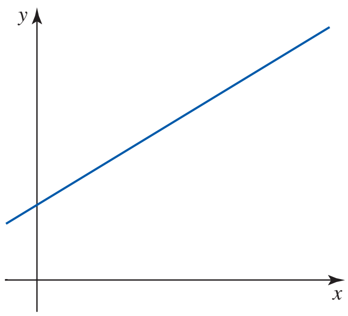
Solutions to Systems of Equations

Systems of linear equations will have 0, 1, or infinitely many solutions.

- A system of equations without a solution is said to be **inconsistent** (Figure 2.20a).
- A system of equations with infinitely many solutions is said to be **dependent** (Figure 2.20b).



(a) Inconsistent system
Two parallel lines
No solution



(b) Dependent system
Both linear equations give the same line
Infinitely many solutions

Figure 2.20

■ Solving a System of Three or More Equations

For a system of three or more linear equations to have a solution, all of the lines must intersect at the same point. The fact that two lines intersect at a point (a, b) does not ensure that the third line will intersect the first two lines at the same point.

EXAMPLE 11 ■ Solving a System of Three Equations

Solve the following system of equations.

$$\begin{aligned} 2x - y &= 5 \\ 3x + 2y &= 11 \\ 4x - 4y &= 8 \end{aligned}$$

Solution We will find the point of intersection of the first two equations and then check to see if the point is a solution to the third equation. The first equation may be written as $y = 2x - 5$. Substituting this value in for y in the second equation and solving for x yields

$$\begin{aligned} 3x + 2y &= 11 \\ 3x + 2(2x - 5) &= 11 \\ 3x + 4x - 10 &= 11 \\ 7x - 10 &= 11 \\ 7x &= 21 \\ x &= 3 \end{aligned}$$

We substitute this value of x into $y = 2x - 5$ and solve.

$$\begin{aligned} y &= 2(3) - 5 \\ &= 1 \end{aligned}$$

The point of intersection of the first two lines is (3, 1). We will check to see if this point satisfies the third equation.

$$\begin{aligned} 4x - 4y &= 8 \\ 4(3) - 4(1) &= 8 \\ 12 - 4 &= 8 \\ 8 &= 8 \end{aligned}$$

Since the resultant statement is true, the solution to the system of equations is $x = 3$ and $y = 1$. (A false statement would have shown that the system was inconsistent.) We confirm the result in Figure 2.21.

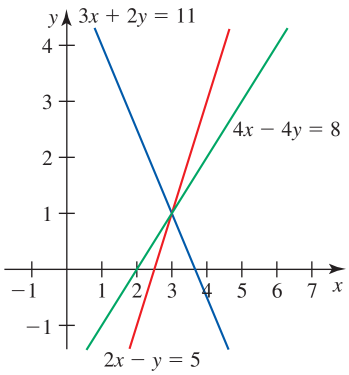


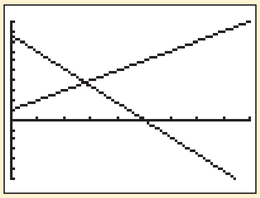
Figure 2.21

SUMMARY

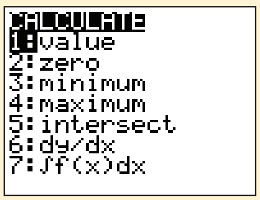
In this section you learned how to find the solution to a system of linear equations using tables, graphs, and algebraic approaches. You learned two algebraic methods for solving linear systems of equations: substitution and elimination. You also learned systems may have no solution, one solution, or infinitely many solutions.

TECHNOLOGY TIP ■ FINDING A POINT OF INTERSECTION

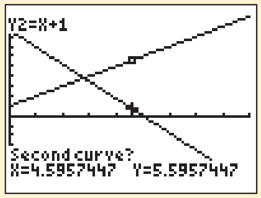
1. Simultaneously graph both functions in the system of linear equations. Adjust the window as necessary to make the point of intersection visible.



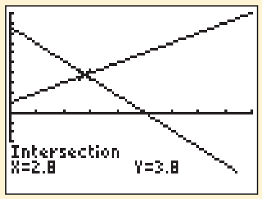
2. Press 2ND TRACE to bring up the Calculate Menu.



3. Select `5: intersect`. The calculator asks `"First curve?"` Use the blue arrow buttons to select either curve then press ENTER. The calculator asks `"Second curve?"` Again, use the arrow buttons to select the second line and press ENTER.



4. The calculator asks `"Guess?"` If there is more than one point of intersection, move the cursor near the desired point of intersection and press ENTER. Otherwise, just press ENTER. The point of intersection is highlighted on the graph and its coordinate is displayed.



2.4 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10,

a. Solve the system of equations algebraically. If a solution does not exist, so state.

b. Use a graphing calculator to find the point(s) of intersection of the lines. Compare your solutions to those obtained in part (a). (*Hint:* Remember that to graph a linear equation given in standard form on your graphing calculator, you will need to solve the equation for y.)

1. $y = 5x - 9$
$y = 2x - 3$

2. $-5x + 2y = 8$
$y = -3x + 26$

3. $y = 5$
$y = -3.21x + 4.32$

4. $-26x + 10y = 112$
$21x + 100y = 1000$

5. $-98x + 10y = -76$
$-98x + 10y = -28$

6. $y = 0.5x - 9$
$y = 0.5x + 10$

7. $y = 2x - 9$
$y = -3x - 9$
$y = \frac{1}{2}x - 9$

8. $y = 0.25x + 1$
$4x + y = 18$
$y = 2$

9. $y = 4.8x + 6$
$y = 1.9x - 2.1$
$y = 2x + 1.6$

10. $y = 11x - 12$
$y = 15x - 21$
$y = 15x - 12$

In Exercises 11–16, solve the system of equations. Verify your solution by graphing the system of equations or by substituting your solution back into the original equations. If the system of equations is inconsistent or dependent, so state.

11. $2x + 2y = 20$
$3x + 2y = 27$

12. $4x + 5y = 61$
$-4x - 6y = -74$

13. $3x + 2y = -50$
$9x - y = -80$

14. $4x - y = 5$
$12x - 3y = 0$

15. $x + y = 0.9$
$2x + 3y = 4.2$

16. $9x - 2y = 648$
$7x + 5y = 504$

In Exercises 17–22, you are given incomplete tables of values for two linear functions. For each exercise,

a. Determine the constant rate of change for each function and use it to complete the table of values.

b. Write a formula for each linear function.

c. Solve the system, then use the table of values to justify that your solution makes sense.

d. Graph the system to verify your solution in part (c).

17.

x	$f(x)$	$g(x)$
−2	1	
−1		
0	7	2
1		0
2		

18.

x	$f(x)$	$g(x)$
−2		2
0	34	4
2		
4	18	
6		

19.

x	$f(x)$	$g(x)$
−5	4	
0		−13
1		
4		7
5	−6	

20.

x	$f(x)$	$g(x)$
−10		
−5	−3	
0	7	
3		−2
7		0

21.

x	$f(x)$	$g(x)$
−2	3.36	11.15
−0.5		
0		
2.5		4.4
3	5.36	

22.

x	$f(x)$	$g(x)$
−3		
−1.5	−0.16	
2.5		12.04
3.5	−14.16	
4		14.59

SHOW YOU KNOW

23. What is the relationship between the solution to a system of equations and the graph of the system?

24. While solving a system of equations algebraically, we get part of the solution first, such as $x = 3.5$, then substitute this value back into one of the original equations to find the other part of the solution. Why doesn't it matter which original equation we use for this final step?

25. Can a system of equations be simultaneously inconsistent and dependent? Explain.

26. The graph of a system of linear equations consists of a single line. How many solutions are there to the system? Explain.

27. Summarize the basic ideas behind the substitution method.

28. Summarize the basic ideas behind the elimination method.

MAKE IT REAL

29. Diabetes The incidence of diabetes in men and women aged 45–64 years increased between 1994 and 1999. The incidence rate for women can be modeled by

$$F(t) = 3.75t + 131 \text{ people per thousand}$$

and the incidence rate for men can be modeled by

$$M(t) = 20t + 109 \text{ people per thousand}$$

where t is the number of years after 1994. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 109)**

a. Solve the system of equations algebraically and explain what the solution means.

b. Check your solution by graphing the system or substituting your solution back into the original equations.

30. Gasoline Prices Based on data from 1990 to 2000, the retail cost of unleaded regular gasoline can be modeled by

$$R(t) = 0.035t + 1.16 \text{ dollars per gallon}$$

and the retail cost of unleaded premium gasoline can be modeled by

$$P(t) = 0.034t + 1.35 \text{ dollars per gallon}$$

where t is the number of years after 1990. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 704)**

a. Solve the system of equations algebraically and explain what your solution means.

b. Based on your experience of purchasing gasoline, does your solution seem reasonable? Explain.

31. Weekly Food Cost Based on data from 1990 and 2000, the weekly food cost for a 15- to 19-year-old male in a four-person family t years after 1990 can be modeled by one of the following functions, depending on the family's food spending behavior. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 705)**

Thrifty Plan
$T(t) = 0.56t + 21.40$ dollars

Moderate-Cost Plan
$M(t) = 1.06t + 36.80$ dollars

Liberal Plan
$L(t) = 1.21t + 42.60$ dollars

According to the models, will the food cost of all three plans ever be the same? Explain.

32. Weekly Food Cost Based on data from 1990 and 2000, the weekly food cost for a 12- to 19-year-old female in a four-person family t years after 1990 can be modeled by one of the follow functions, depending on the family's food spending behavior. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 705)**

Hannamariah/Shutterstock.com

Thrifty Plan
$T(t) = 0.55t + 20.80$ dollars

Moderate-Cost Plan
$M(t) = 0.85t + 30.10$ dollars

Liberal Plan
$L(t) = 1.04t + 36.30$ dollars

According to the models, will the food cost of all three plans ever be the same? Explain.

33. Bread and Gasoline Prices Based on data from 1990 to 2000, the retail cost of unleaded regular gasoline can be modeled by

$$R(t) = 0.035t + 1.16 \text{ dollars per gallon}$$

where t is the number of years since 1990. Based on data from 1993 to 2000, the price of a loaf of whole wheat bread may be modeled by

$$C(t) = 0.0404t + 0.991$$

where t is the number of years since 1990. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Tables 704, 706)** Rounded up to the nearest year, when will a loaf of whole wheat bread cost more than a gallon of unleaded regular gasoline?

34. Entertainment Revenues Based on data from 1998 and 1999, the amount of revenue brought in by amusement and theme parks can be modeled by

$$A(t) = 177t + 7335 \text{ million dollars}$$

where t is the number of years since the end of 1998. The amount of revenue brought in by racetracks may be modeled by

$$R(t) = 507t + 4599 \text{ million dollars}$$

(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 1231)

a. Solve the system of equations algebraically and explain what the solution means.

b. Check your solution by graphing the system or by substituting your solution back into the original equations.

35. High School Athletics The table shows the number of men and women participating in high school sports from 1990 to 2002 (the given year is when the school year began).

Years Since 1990	Men (in millions)	Women (in millions)
0	3.41	1.89
2	3.42	2.00
4	3.54	2.24
6	3.71	2.47
8	3.83	2.65
10	3.92	2.78
12	3.99	2.86

***Source: Statistical Abstract of the United States, 2006*, Table 1237**

a. Use linear regression to create a system of equations that models the high school athletics participation of men and women. Round the initial value and slope of each model to 2 decimal places.

b. What are the constant rates of change used in the models and what do they represent?

c. Solve the system algebraically and explain what the solution represents. Check your answer by graphing the system or by substituting your solution back into the original equations.

d. What assumption(s) do we make when solving this system of equations?

36. Chicken and Fish Consumption The table shows the per capita chicken and fish consumption in pounds per person in the United States from 1980 to 2003.

Years Since 1980	Per Capita Chicken Consumption (pounds per person)	Per Capita Fish Consumption (pounds per person)
0	32.7	12.4
5	36.4	15
10	42.4	15
15	48.2	14.8
20	54.2	15.2
22	56.8	15.6
23	57.5	16.3

***Source:* *Statistical Abstract of the United States, 2006*, Table 202**

a. Use linear regression to create a system of equations that models the per capita consumption of chicken and fish. Round your models off to 2 decimal places.

b. What are the constant rates of change in your models? What do they represent?

c. Solve the system algebraically and explain what the solution represents. Check your answer by graphing the system or by substituting your solution back into the original equations.

d. Explain whether or not you think the solution in part (c) is reasonable. How could you check whether this solution matches what really happened?

37. GORP Mix In Example 8 we learned that a cup of oil-roasted, salted peanuts has 37.94 grams of protein and 27.26 grams of carbohydrates and a cup of seedless raisins contains 4.67 grams of protein and 114.74 grams of carbohydrates. (***Source:*** **www.nutri-facts.com**) If you created a 30-cup mixture with 2130 grams of carbohydrates, how many cups of peanuts and how many cups of raisins are in the mixture?

For Exercises 38–41, refer to the following table of values for taxi fares in various cities. The term flag drop *refers to the cost of the first portion of a mile that is included in the initial fare. Depending on the city, the included portion may be as little as 0.10 mile or as much as 1 mile.*

City	Flag Drop	Cost per Additional Mile	Cost per Minute Waiting
New York City	\$2.50	\$2.00	\$0.40
Baltimore	\$1.80	\$2.20	\$0.40
Atlanta	\$2.50	\$2.00	\$0.35
Seattle	\$2.50	\$2.00	\$0.50

Source: **www.taxifarefinder.com**

For each exercise,

a. Write a system of equations to represent the total costs of hiring a taxi in each indicated city as functions of the number of additional miles traveled beyond the flag drop. Be sure to identify what your variables represent.

b. Solve the system of equations using algebra. Verify your solution by graphing the system or by substituting your solution back into the original equations. If no solution exists, so state.

c. Explain what your answer in part (b) means and explain whether it makes sense in the context of the problem.

38. Taxi Fares Two passengers, one in New York City and the other in Baltimore, each hire a taxi. Assume each encounters 4 minutes of wait time during their rides.

39. Taxi Fares Two passengers, one in Baltimore and the other in Seattle, each hire a taxi. Assume each encounters 2 minutes of wait time during their rides.

40. Taxi Fares Two passengers, one in New York City and the other in Atlanta, each hire a taxi. Assume each encounters 6 minutes of wait time during their rides.

41. Taxi Fares Two passengers, one in Seattle and the other in Atlanta, each hire a taxi. Assume neither passenger encounters any wait time during their rides.

42. Computer Prices A graphic design firm placed two orders for computers from Apple Inc. For their web design department, they purchased 2 15-inch MacBook Pro laptops and 3 Mac Pro Quad-core desktops for a total order price of \$10,895. For their graphic arts department they purchased 4 15-inch MacBook Pro laptops and 3 Mac Pro Quad-core desktops for a total order price of \$14,293. (***Source:*** **www.apple.com, June 2009**) Assume the prices of the computers did not change between orders.

a. Create a system of equations to model this situation. Be sure to identify what your variables represent.

b. Solve the system of equations. How much did each type of computer cost?

43. Computer Prices A university placed two orders for computers from Hewlett-Packard. The order for the science department totaled \$11,449.87, and consisted of 7 HDX 16t Premium Series laptops and 6 Pavilion dv7t Series laptops. The order for the English department totaled \$9,699.89 and consisted of 6 HDX 16t Premium Series laptops and 5 HP Pavilion dv7t Series laptops. (***Source:*** **www.hp.com, June 2009**) Assume the prices of the computers did not change between orders.

a. Create a system of equations to model this situation. Be sure to identify what your variables represent.

b. Solve the system of equations. How much did each type of computer cost?

44. Mixing Cereals One of the authors enjoys mixing his cereal, eating 2 cups of a combination of Cheerios and Kashi GOLEAN! Crunch each morning. Cheerios contain 3 grams of fiber per cup while Kashi GOLEAN! Crunch contains 8 grams of fiber per cup. (***Source:*** **product labels**)

The author wants to eat 2 cups of cereal, consisting of a combination of Cheerios and Kashi, and wants his bowl of cereal to contain 13 grams of fiber.

a. What are the unknowns in this situation?

b. Create a system of equations to model this situation.

c. Solve the system of equations and explain what your solution means.

d. Verify your solution by graphing the system or by substituting the solution into the original equations.

45. Trail Mix Planters Honey Nut Medley Trail Mix contains 4 grams of protein per 1-ounce serving. Planters Dry Roasted Peanuts contain 8 grams of protein per 1-ounce serving. (***Source:*** **product labels**) Suppose you want to create an 8-ounce mixture of trail mix and peanuts that has a combined total of 42 grams of protein.

a. What are the unknowns in this situation?

b. Create a system of equations to model this situation.

c. Solve the system of equations and explain what your solution means.

d. Verify your solution by graphing the system or by substituting the solution into the original equations.

For Exercises 46–49, refer to the following table of values for taxi fares in various cities.

City	Flag Drop	Cost per Additional Mile	Cost per Minute Waiting
Chicago	\$2.25	\$1.80	\$0.33
St. Louis	\$2.50	\$1.70	\$0.37
Phoenix	\$2.50	\$1.80	\$0.33
San Francisco	\$3.10	\$2.25	\$0.45

Source: **www.taxifarefinder.com, updated June 2008**

For each exercise,

a. Write a system of equations to represent the situation described. Be sure to explain what your variables represent.

b. Solve the system of equations. Verify your solution by graphing the system or by substituting your solution back into the original formulas. If the system is inconsistent or dependent, so state.

c. Explain what your answer in part (b) means and explain whether it makes sense in the context of the problem.

46. Taxi Fares Two passengers hired taxis, one in San Francisco and the other in Chicago, and each rode the same additional distance beyond the flag drop and had the same wait time during their rides. The passenger in San Francisco paid a total of \$17.05, while the passenger in Chicago paid a total of \$13.38.

47. Taxi Fares Two passengers hired taxis, one in St. Louis and the other in Phoenix, and each rode the same additional distance beyond the flag drop and had the same wait time during their rides. The passenger in St. Louis paid a total of \$6.53, while the passenger in Phoenix paid a total of \$6.52.

48. Taxi Fares Two passengers hired taxis, one in San Francisco and the other in St. Louis, and each rode the same additional distance beyond the flag drop and had the same wait time during their rides. The passenger in San Francisco paid a total of \$11.65, while the passenger in St. Louis paid a total of \$9.02.

49. Taxi Fares Two passengers hired taxis, one in Chicago and the other in Phoenix, and each rode the same additional distance beyond the flag drop and had the same wait time during their rides. The passenger in Chicago paid a total of \$10.44, while the passenger in Phoenix paid a total of \$10.69.

STRETCH YOUR MIND

Exercises 50–53 are intended to challenge your understanding of systems of linear equations.

50. Part-Time Employment A full-time student works two jobs. The first job pays \$6.75 per hour and the second job pays \$8.00 per hour. The student plans to work 20 hours per week. The first job offers 4-hour shifts and is near the student's apartment so she uses a total of 0.25 gallon of gas for a round trip to that job. The second job also offers 4-hour shifts; however, it is several miles from her home so she uses 1 gallon of gas for a round trip to the job. How many hours should she work in each job if she wants to work 20 hours, earn \$155, and use exactly 2 gallons of gas?

51. A system of linear equations consists of the functions $f(x) = mx + b$ and $g(x) = \frac{1}{m}x - \frac{b}{m}$. For what values of m and b will the system be inconsistent or dependent? If the system has a unique solution, what will it be?

52. Solving Systems with Three or More Variables A system of equations with more than two variables can be solved using the elimination method. For example, a system with three equations and three variables can be solved by taking a pair of equations and using the elimination method to eliminate one of the variables, then taking a different pair of equations and eliminating the same variable. The resulting two equations with two variables can then be solved as we have seen in this section. Using this approach, solve each of the following systems.

a.
$$\begin{aligned} x - y + z &= 6 \\ x + y + z &= 8 \\ x - y - z &= -4 \end{aligned}$$

b.
$$\begin{aligned} 3x - 3y + z &= 5 \\ x + y + z &= 11 \\ 2x + 2z &= 10 \end{aligned}$$

c.
$$\begin{aligned} -2x - y + 4z &= 8 \\ 2x + y - 4z &= -8 \\ x + y + z &= 6 \end{aligned}$$

d. $x + y + z + w = 1$
$x - y + z - w = 2$
$x - y - z + w = 3$
$x - y - z - w = 4$

53. A system of equations has more variables than it has equations. Is it possible for such a system to have exactly one solution? Explain.

SECTION 2.5 Systems of Linear Inequalities

LEARNING OBJECTIVES

- Graph linear inequalities given in slope-intercept or standard form
- Determine the corner points of a solution region of a system of linear inequalities
- Explain the practical meaning of solutions of linear inequalities in real-world contexts

GETTING STARTED

Many students work multiple part-time jobs to finance their education. Often the jobs pay different wages and offer varying hours. Suppose a student earns \$10.50 per hour delivering pizza and \$8.00 per hour working in a campus computer lab. If the student can only work 30 hours per week and must earn \$252 in that period, how many hours must he spend at each job to meet his earnings goal? A system of linear inequalities can be used to answer this question.

In this section we demonstrate how to graph linear inequalities and systems of linear inequalities. We also show that the solution region of a system of linear inequalities is the intersection of the graphs of the individual inequalities.

■ Linear Inequalities

In many real-world problems we are interested in a range of possible solutions instead of a single solution. For example, when you buy a house, lenders will calculate the maximum amount of money they are willing to loan you, but you do not have to borrow the maximum amount. You may borrow any amount up to the maximum. In mathematics, we use inequalities to represent the range of possible solutions that meet the given criteria.

Inequality Notation

$x \leq y$ is the set of all values of x less than or equal to y.
$x \geq y$ is the set of all values of x greater than or equal to y.
$x < y$ is the set of all values of x less than but not equal to y.
$x > y$ is the set of all values of x greater than but not equal to y.

The inequalities $x < y$ and $x > y$ are called **strict inequalities** since the two variables cannot ever be equal. Although strict inequalities have many useful applications, we will focus on the nonstrict inequalities in this section.

An easy way to keep track of the meaning of an inequality is to remember the inequality sign always points toward the smaller number (the number that is furthest to the left on the number line). Consider these everyday examples of inequalities:

- You must be at least 16 years old to get a driver's license. ($16 \le a$) or ($a \ge 16$)
- You must be at least 21 years old to legally buy alcohol. ($21 \le a$) or ($a \ge 21$)
- The maximum fine for littering is \$200. ($200 \ge f$) or ($f \le 200$)
- Your carry-on bag must be no more than 22 inches long. ($22 \ge l$) or ($l \le 22$)

A linear inequality looks like a linear equation with an inequality sign in the place of the equal sign. Thus linear inequalities can be manipulated algebraically in the same way as linear equations with one major exception: when we multiply or divide each side of an inequality by a negative number, we must reverse the direction of the inequality sign. For example, if we multiply each side of $2 < 3$ by -1 we get $-2 > -3$, not $-2 < -3$.

Graphing Linear Inequalities

The graph of a linear inequality is a region bordered by a line called a **boundary line**. The **solution region** of a linear inequality is the set of all points (including the boundary line) that satisfy the inequality.

Consider the inequality $x + 2y \ge 4$. As shown in Figure 2.22, the boundary line of the solution region is $x + 2y = 4$ since the points that satisfy this linear equation also satisfy the inequality.

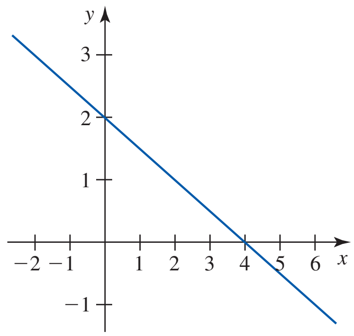


Figure 2.22

Now we need to find the remaining points (x, y) that satisfy the inequality. To find which points off the line satisfy the inequality we pick a few points and test them, as shown in Table 2.17. To satisfy the inequality, $x + 2y$ must be at least 4.

Table 2.17

x	y	$x + 2y$	In solution region?
−1	1	1	no
0	1	2	no
1	3	7	yes
2	2	6	yes
3	0	3	no
5	2	9	yes
6	1	8	yes

The points from the table are plotted in Figure 2.23, along with the boundary line. Graphically speaking, what do the four points in the solution region have in common? They are all on the same side of the boundary line. In fact, all points on or above this boundary line satisfy the inequality. We represent this notion by shading the region above the boundary line, as shown in Figure 2.24. Although we checked multiple points in this problem, we really need to check only one point not on the boundary line to determine which region to shade.

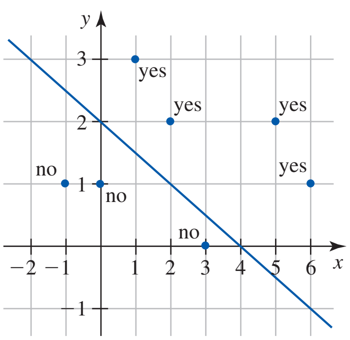


Figure 2.23

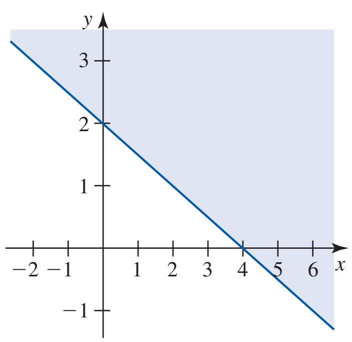


Figure 2.24

The linear inequality graphing process is summarized as follows.

HOW TO: ■ GRAPH A LINEAR INEQUALITY

To graph the solution region of the linear inequality $ax + by \le c$ (or $ax + by \ge c$),

1. Graph the boundary line $ax + by = c$.
2. Select a point on one side of the boundary line. (If the line does not pass through the origin, (0, 0) is an excellent choice for easy computations.)
3. Substitute the point into the linear inequality and simplify. If the simplified statement is true, the point and all other points on the same side of the line are in the solution region. If the statement is false, all points on the opposite side of the line are in the solution region.
4. Shade the solution region.

EXAMPLE 1 ■ Graphing the Solution Region of a Linear Inequality

Graph the solution region of the linear inequality $2x + y \le 4$.

Solution We can easily find the x-intercept of the boundary line $2x + y = 4$ by dividing the constant term by the coefficient on the x-term.

$$x = \frac{4}{2} = 2$$

The point (2, 0) is the x-intercept.

We find the y-intercept of the boundary line by dividing the constant term by the coefficient on the y-term.

$$y = \frac{4}{1} = 4$$

The point (0, 4) is the y-intercept. As shown in Figure 2.25, we graph the x- and y-intercepts and then draw the line through the intercepts.

Next we pick the point (0, 0) to plug in to the inequality to see which side to shade.

$$2(0) + 0 \le 4$$
$$0 \le 4$$

The statement is true, so all points on the same side of the line as (0, 0) are in the solution region. Figure 2.26 shows the shaded solution region.

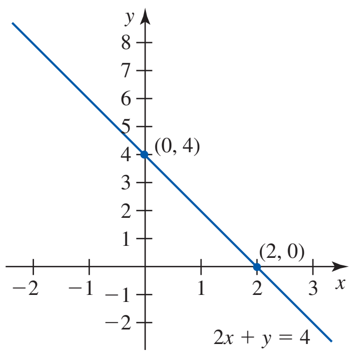


Figure 2.25

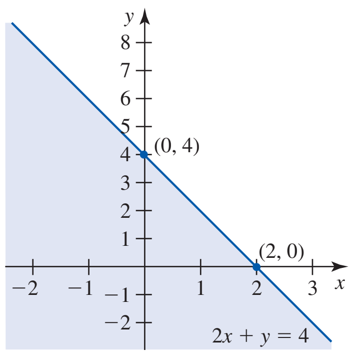


Figure 2.26

If you choose to convert lines from standard to slope-intercept form before graphing, the following properties will help you to quickly identify the solution region without having to check a point.

Solution Region of a Linear Inequality in Slope-Intercept Form

- The solution region of a linear inequality $y \geq mx + b$ contains the line $y = mx + b$ and the shaded region ***above*** the line.
- The solution region of a linear inequality $y \leq mx + b$ contains the line $y = mx + b$ and the shaded region ***below*** the line.

Graphing Systems of Linear Inequalities

Just as we can graph systems of linear equations, we can graph systems of linear inequalities. The solution region of a system of linear inequalities is the intersection of the solution regions of the individual inequalities. When graphing a solution region by hand, we typically place arrows on the boundary lines to indicate which side of the line satisfies the given inequality. Once all the inequality graphs have been drawn, we shade the region that has arrows from all sides pointing into the interior of the region.

EXAMPLE 2 ■ Graphing the Solution Region of a System of Linear Inequalities

Graph the solution region of the following system of linear inequalities.

$$\begin{aligned} 3x + 2y &\leq 5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$$

Solution We rewrite the first linear inequality in slope-intercept form.

$$\begin{aligned} 3x + 2y &\leq 5 \\ 2y &\leq -3x + 5 \\ y &\leq -\tfrac{3}{2}x + \tfrac{5}{2} \\ y &\leq -1.5x + 2.5 \end{aligned}$$

The boundary line is a line with slope -1.5 and y-intercept $(0, 2.5)$. Since y is less than or equal to the expression $-1.5x + 2.5$, we place arrows pointing toward the region below the line, as shown in Figure 2.27.

The next two inequalities, $x \geq 0$ and $y \geq 0$, limit the solution region to positive values of x and y. The line $x = 0$ is the vertical axis. The line $y = 0$ is the horizontal axis. Therefore, as shown in Figure 2.28, the solution region of the system of inequalities is the triangular region to the right of the line $x = 0$, above the line $y = 0$, and below the line $y = -1.5x + 2.5$.

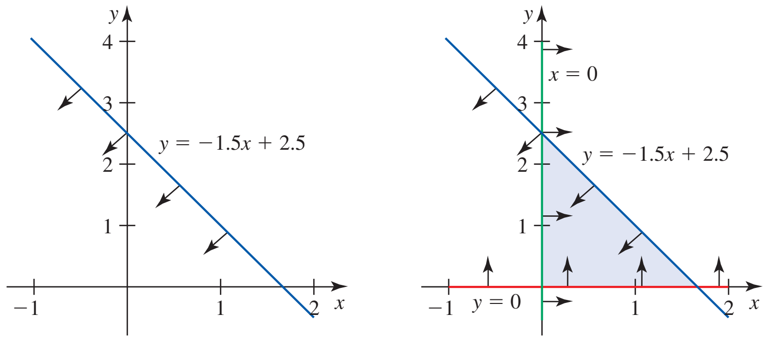


Figure 2.27 Figure 2.28

If it is possible to draw a circle around the solution region, the solution region is **bounded**. If no circle will enclose the entire solution region, the solution region is **unbounded**. The solution region in Example 2 is bounded.

EXAMPLE 3 ■ Graphing a System of Linear Inequalities with an Unbounded Solution Region

Graph the solution region of the following system of linear inequalities.

$$\begin{aligned} 4x + y &\geq 4 \\ -x + y &\geq 1 \end{aligned}$$

Solution The x-intercept of the boundary line $4x + y = 4$ is (1, 0) and the y-intercept is (0, 4). Plugging in the test point (0, 0), we get

$$\begin{aligned} 4(0) + (0) &\geq 4 \\ (0) &\geq 4 \end{aligned}$$

Since the statement is false, we graph $4x + y = 4$ and place arrows pointing toward the side of the line not containing the origin.

The x-intercept of the boundary line $-x + y = 1$ is $(-1, 0)$ and the y-intercept is (0, 1). Plugging in the test point (0, 0), we get

$$\begin{aligned} -(0) + (0) &\geq 1 \\ 0 &\geq 1 \end{aligned}$$

Since the statement is false, we graph $-x + y = 1$ and place arrows pointing toward the side of the line not containing the origin. Then we shade the overlapping solution regions, as shown in Figure 2.29.

Note that we cannot draw a circle around the solution region because it is not bounded above the line $y = x + 1$ or above the line $y = -4x + 4$. Consequently, the solution region is unbounded and the system has infinitely many solutions.

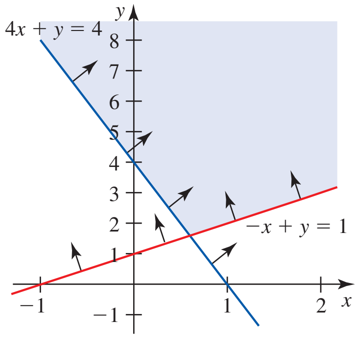


Figure 2.29

EXAMPLE 4 ■ Graphing a System of Linear Inequalities with No Solution

Graph the solution region of the following system of linear inequalities.

$$\begin{aligned} -2x + 2y &\geq 6 \\ -x + y &\leq 1 \end{aligned}$$

Solution We graph the boundary lines by first rewriting the inequalities in slope-intercept form. Solving the first inequality for y, we get $y \geq x + 3$ so the arrows point toward the region above the line $y = x + 3$. Solving the second inequality for y, we get $y \leq x + 1$ so the arrows point toward the region below the line $y = x + 1$. There is no overlap, so we shade no region, as shown in Figure 2.30.

Because the lines have the same slope ($m = 1$), they are parallel and will never intersect. As seen in the graph, the two solution regions also never intersect. Therefore, this system of linear inequalities has no solution. That is, no ordered pair exists that satisfies both inequalities.

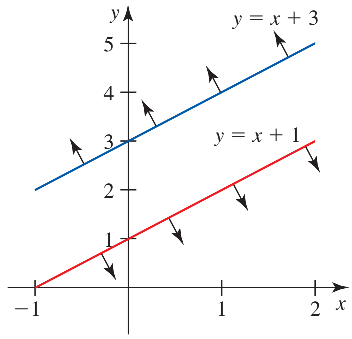


Figure 2.30

EXAMPLE 5 ■ Using Systems of Linear Inequalities in a Real-World Context

The American Diabetes Association and the American Dietetic Association make recommendations regarding food intake in their "Exchange Lists for Meal Planning" guide. As a service to diabetic consumers who must carefully monitor their diets, some food producers provide this *exchange* information on product packaging.

For example, Kashi Sales, LLC, distributes TLC Tasty Little Crackers (a 7-grain snack cracker) and TLC Tasty Little Chewies (a chewy granola bar). According to the product packaging, a 15-cracker serving counts as 1.5 Carbohydrates and 0.5 Fat. Similarly, one granola bar counts as 1 Carbohydrate and 1 Fat. (*Source:* Kashi product packaging)

A woman with diabetes wants to eat some crackers and granola bars and needs to keep her Carbohydrates to at most 4 units and her Fat to at most 3 units. Determine three different combinations of crackers and granola bars that will allow her to stay within her dietary guidelines.

Solution Let c be the number of 15-cracker servings of crackers and g the number of granola bars the woman eats. We construct one inequality for Carbohydrates and another inequality for Fat.

$$\text{Carbohydrates: } 1.5c + 1g \le 4$$
$$\text{Fat: } 0.5c + 1g \le 3$$

In the context of the problem, we also know $c \ge 0$ and $g \ge 0$ since consuming negative quantities of food does not make sense. As shown in Figure 2.31, we graph each of the lines and shade the solution region.

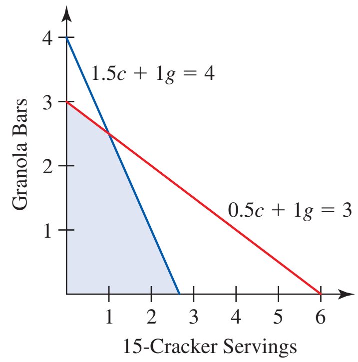


Figure 2.31

Among the several different solutions that meet her dietary requirements are

0 servings of crackers and 3 granola bars	3 Carbohydrates, 3 Fat
1 serving of crackers and 2 granola bars	3.5 Carbohydrates, 2.5 Fat
2 servings of crackers and 1 granola bar	4 Carbohydrates, 2 Fat

Notice none of the solutions we listed maximized both Carbohydrates and Fat. That's okay; we wanted *at most* 4 Carbohydrates units and *at most* 3 Fat units.

■ Corner Points

Sometimes it is helpful to know the values of the corners of a solution region, or the **corner points**. To find the coordinates of a corner point, we solve the system of equations formed by the two intersecting boundary lines that form the corner.

Corner Points

The points of intersection of the boundary lines of a system of linear inequalities bordering the shaded solution region are called **corner points**.

EXAMPLE 6 ■ Finding the Corner Points of a Solution Region

According to the 2010–2011 catalog, Brigham Young University requires students to complete a minimum of 120 credit hours to earn a bachelor's degree. Suppose a student has a 3.25 cumulative grade point average and hopes to raise it to at least 3.50 by the time he has completed 60 credit hours. What grade must he earn in future credit hours to achieve his goal? (Assume the highest grade that can be earned in any course is 4.0.)

Set up the system of linear inequalities that models this situation, draw the solution region, and find the coordinates of the corner points.

Solution Let c be the number of credit hours the student has *completed* and f be the number of credit hours the student *will take in the future.* Since he must earn at least 120 credit hours for the degree, we have

$$\text{credit hours: } c + f \geq 120$$

The student has already earned $3.25c$ grade points since his cumulative grade point average is 3.25 and he has completed c credit hours. The highest possible grade he can earn in any of the $60 - c$ credit hours that remain before he has exceeded 60 credit hours is 4.0. Therefore,

$$\begin{aligned}
3.25c + 4.0(60 - c) &\geq 3.5(60) \\
3.25c + 240 - 4c &\geq 210 \\
-0.75c + 240 &\geq 210 \\
-0.75c &\geq -30 \\
c &\leq \frac{30}{0.75} = 40
\end{aligned}$$

Remember that dividing by a negative reverses the inequality sign.

We have the following system of linear inequalities.

$$\begin{aligned}
c + f &\geq 120 \\
c &\leq 40 \\
c &\geq 1 \\
f &\geq 0
\end{aligned}$$

We added the inequality $f \geq 0$ because it does not make sense to talk about a negative number of credit hours. We added the inequality $c \geq 1$ since the student had to have already taken at least one credit in order to have a grade point average of 3.25. The graph of the solution region is shown in Figure 2.32.

We see the solution region is unbounded with two corner points. From the graphs of the boundary lines, we can determine the coordinates of the corner points, (1, 119) and (40, 80), but we will verify our conclusion algebraically.

The first corner point occurs at the intersection of $c = 1$ and $c + f = 120$.

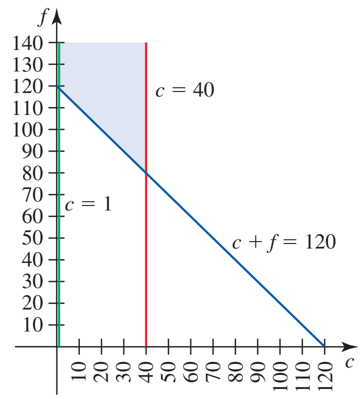


Figure 2.32

$$\begin{aligned}
c + f &= 120 \\
1 + f &= 120 \qquad \text{since } c = 1 \\
f &= 119
\end{aligned}$$

For the second corner point, we need to determine where the line $c = 40$ and the line $c + f = 120$ intersect. In this case, it is easy.

$$\begin{aligned}
c + f &= 120 \\
(40) + f &= 120 \qquad \text{since } c = 40 \\
f &= 80
\end{aligned}$$

Therefore, the second corner point is (40, 80) as we saw on the graph.

It may have seemed superfluous to calculate the coordinates of the corner points in Example 6 when the coordinates were readily apparent from the graph of the solution region. However, Example 7 illustrates the hazards of relying solely on a graph.

EXAMPLE 7 ■ **Finding the Corner Points of a Solution Region**

Graph the solution region for the following system of inequalities and determine the coordinates of the corner points.

$$\begin{aligned} x + y &\le 5 \\ -5x + 5y &\le 6 \\ y &\ge 2 \end{aligned}$$

Solution The graph of the solution region is shown in Figure 2.33. From the graph, it appears the corner points of the region are at or near (2, 3), (3, 2), and (0.75, 2). Let's check this algebraically.

We find the coordinates of the first corner point by solving the system of equations for the boundary lines that form the corner.

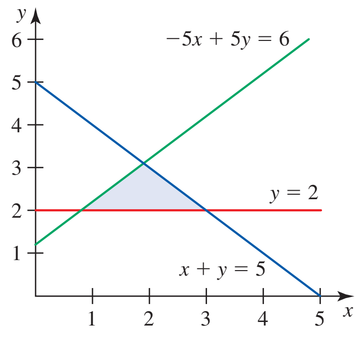


Figure 2.33

$$\begin{aligned} x + y &= 5 \\ -5x + 5y &= 6 \end{aligned}$$

Using the elimination method, we add five times the first equation to the second equation to get

$$\begin{aligned} 0x + 10y &= 31 \\ y &= 3.1 \end{aligned}$$

We substitute this y-value back into the first equation to find the value of x.

$$\begin{aligned} x + y &= 5 \\ x + (3.1) &= 5 \\ x &= 1.9 \end{aligned}$$

Thus the coordinates of the first corner point are (1.9, 3.1), not (2, 3) as we estimated from the graph.

Again, we find the coordinates of the second corner point by solving the system of equations for the boundary lines that form that corner.

$$\begin{aligned} x + y &= 5 \\ y &= 2 \end{aligned}$$

Since the second equation tells us $y = 2$, we need only substitute this value into the first equation to find x.

$$\begin{aligned} x + y &= 5 \\ x + (2) &= 5 \\ x &= 3 \end{aligned}$$

The coordinates of the second corner point are (3, 2), which agrees with our graphical conclusion.

Finally, we find the coordinates of the third corner point by solving the system of equations for the boundary lines that form that corner.

$$\begin{aligned} -5x + 5y &= 6 \\ y &= 2 \end{aligned}$$

Since the second equation tells us $y = 2$, we substitute this value into the first equation to find x.

$$\begin{aligned} -5x + 5y &= 6 \\ -5x + 5(2) &= 6 \\ -5x + 10 &= 6 \\ -5x &= -4 \\ x &= 0.8 \end{aligned}$$

The coordinates of the third corner point are (0.8, 2). Our estimate from the graph, (0.75, 2), was close but not precise.

EXAMPLE 8 ■ Using a Linear System of Inequalities to Find the Ideal Work Schedule

A student earns \$8.00 per hour working in a campus computer lab and \$10.50 per hour delivering pizza. If she has only 30 hours per week to work and must earn at least \$252 in that period, how many hours can she spend at each job to meet her income goal?

Solution Let c be the number of hours she works in the computer lab and p be the number of hours she works delivering pizza. She can work at most 30 hours. This is represented by the inequality

$$c + p \le 30$$ The maximum number of work hours is 30.

The amount she earns working in the lab is $8.00c$ and the amount of money she earns delivering pizza is $10.50p$. Her total income must be at least \$252. That is,

$$8c + 10.5p \ge 252$$ The minimum amount of income is \$252.

Solving the inequalities for p in terms of c, we get the following system of inequalities. (We add the restrictions $p \ge 0$ and $c \ge 0$ since she cannot work a negative number of hours at either job.)

$$p \le -c + 30$$
$$p \ge -\tfrac{16}{21}c + 24$$ In decimal form, $p \ge -0.7619c + 24$ approximately.
$$p \ge 0$$
$$c \ge 0$$

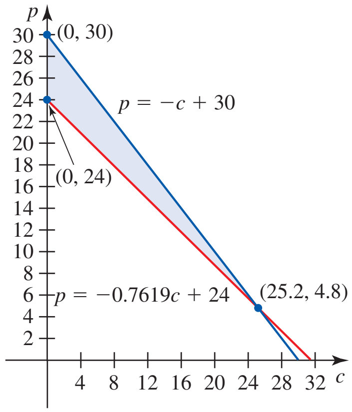


Figure 2.34

The graph of the solution region is shown in Figure 2.34. Every point of the solution region represents a combination of hours at the two jobs that will result in earnings of at least \$252.

The corner points of the solution region are (0, 24), (0, 30), and (25.2, 4.8). (We obtained the first two points from the graph and found the third point by calculating the intersection of the two boundary lines.) We use Table 2.18 to calculate her weekly earnings at the corner points and a few interior points to check our solution region.

Table 2.18

	Lab Hours	Pizza Hours	Weekly Earnings
corner point	0	24	\$252.00
corner point	0	30	\$315.00
corner point	25.2	4.8	\$252.00
interior point	5	25	\$302.50
interior point	10	20	\$290.00
interior point	15	14	\$267.00

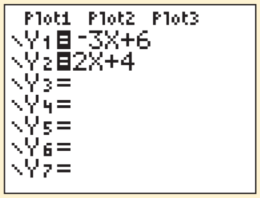
We see that although the weekly earnings vary, in every case the number of work hours is less than or equal to 30 hours and the earnings are greater than or equal to \$252.

SUMMARY

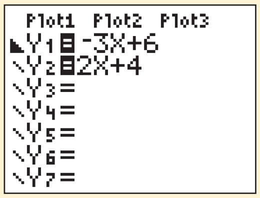
In this section you learned how to graph linear inequalities. You discovered that the solution region to a system of linear inequalities is the intersection of the graphs of the solution regions of the individual inequalities. You also discovered how to find the corner points of a solution region.

TECHNOLOGY TIP ■ GRAPHING A SYSTEM OF LINEAR INEQUALITIES

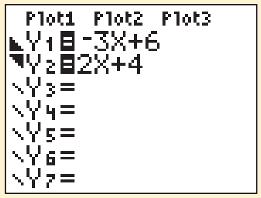
1. Enter the linear equations that are the boundary lines associated with each inequality by using the Y= Editor. (We will use the system $y \leq -3x + 6$ and $y \geq 2x + 4$ for this example.)



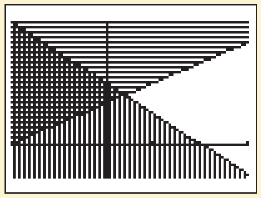
2. Move the cursor to the \ at the left of the Y1 and press ENTER repeatedly. This will cycle through several graphing options. We want to shade the region below the line so we will pick the lower triangular option.



3. Move the cursor to the \ at the left of the Y2 and press ENTER repeatedly. We want to shade the region above the line so we will pick the upper triangular option.



4. Press GRAPH to draw each of the shaded regions. The region with the cross-hatched pattern is the solution region.



2.5 EXERCISES

SKILLS AND CONCEPTS

*In Exercises 1–10, graph the solution region of the linear inequality. Then use the graph to determine if the given point **P** is in the solution region.*

1. $2x + y \leq 6$; $\boldsymbol{P} = (2, 4)$
2. $4x + y \leq 0$; $\boldsymbol{P} = (1, 1)$
3. $x + 5y \leq 10$; $\boldsymbol{P} = (0, 0)$
4. $5x + 6y \leq 30$; $\boldsymbol{P} = (0, 5)$
5. $-2x + 4y \geq -2$; $\boldsymbol{P} = (1, 2)$
6. $x - y \leq 10$; $\boldsymbol{P} = (5, -5)$
7. $5x - 4y \leq 0$; $\boldsymbol{P} = (1, 0)$
8. $-3x - 3y \leq 9$; $\boldsymbol{P} = (2, -1)$
9. $2x - y \geq 8$; $\boldsymbol{P} = (-3, 2)$
10. $7x - 6y \geq 12$; $\boldsymbol{P} = (0, -1)$

In Exercises 11–25, graph the solution region of the system of linear inequalities. If there is no solution, explain why.

11. $\begin{aligned} -4x + y &\geq 2 \\ -2x + y &\geq 1 \\ x &\leq 0 \end{aligned}$

12. $\begin{aligned} -5x + y &\geq 0 \\ 2x + y &\leq 4 \\ y &\geq 0 \end{aligned}$

13. $\begin{aligned} -2x + 6y &\leq 8 \\ 4x - 12y &\leq -6 \end{aligned}$

14. $\begin{aligned} 10x - y &\geq 12 \\ 9x - 2y &\geq 2 \end{aligned}$

15. $\begin{aligned} 3x - 2y &\leq 4 \\ 11x - 20y &\geq 2 \end{aligned}$

16. $\begin{aligned} 9x - 6y &\leq 0 \\ 4x + 5y &\leq 23 \end{aligned}$

17. $\begin{aligned} x - y &\leq -5 \\ 9x + y &\leq 25 \end{aligned}$

18. $\begin{aligned} 2x + 5y &\leq 2 \\ 3x - 5y &\leq 3 \end{aligned}$

19. $\begin{aligned} 6x + 2y &\leq 10 \\ -x - 2y &\geq -5 \\ x &\geq 0 \\ y &\geq 0 \end{aligned}$

20. $\begin{aligned} x - y &\geq 3 \\ 6x + 7y &\leq 44 \\ 6x - 7y &\leq 16 \end{aligned}$

21. $\begin{aligned} 2x - 4y &\geq 16 \\ 9x + y &\leq -4 \\ -3x + 6y &\leq -24 \end{aligned}$

22. $\begin{aligned} 2x - 2y &\geq 0 \\ 3x + y &\leq 4 \\ 5x - y &\geq 5 \end{aligned}$

23. $\begin{aligned} 8x - y &\geq 3 \\ x + 2y &\leq 11 \\ 9x + y &\leq 14 \end{aligned}$

24. $\begin{aligned} -4x + y &\geq 2 \\ -2x + y &\geq 1 \\ y &\geq 1 \\ x &\leq 1 \end{aligned}$

25. $\begin{aligned} -5x + y &\geq 0 \\ 2x + y &\leq 4 \\ y &\leq 1 \\ x &\leq 1 \end{aligned}$

In Exercises 26–27, set up the system of linear inequalities that can be used to solve the problem. Then perform the indicated tasks.

26. Student Wages A student earns \$15.00 per hour designing web pages and \$9.00 per hour supervising a campus tutoring center. She has at most 30 hours per week to work and needs to earn at least \$300. Graph the region showing all possible work-hour allocations that meet her time and income requirements.

27. Wages A salaried employee earns \$900 per week managing a copy center. He is required to work a minimum of 35 hours but no more than 45 hours weekly. As a side business, he earns \$25 per hour designing brochures for local business clients. To maintain his current standard of living, he must earn \$1100 per week. To maintain his quality of life, he limits his workload to 50 hours per week. Given that he has no control over the number of hours he has to work managing the copy center, will he be able to consistently meet his income and workload goals? Explain.

■ SHOW YOU KNOW

28. What is a corner point and how is it related to a system of linear inequalities?

29. What is the difference between a bounded and an unbounded solution region?

30. How many solutions may a system of linear inequalities have?

31. Given the linear inequality $ax + by \leq c$, explain under what conditions the solution region will contain the origin.

32. A system of linear inequalities that contains exactly three inequalities can have at most how many corner points? Explain.

■ MAKE IT REAL

In Exercises 33–35, set up the system of linear inequalities that can be used to solve the problem. Then perform the indicated tasks.

33. Nutritional Content A 32-gram serving of Skippy® Creamy Peanut Butter contains 150 milligrams of sodium and 17 grams of fat. A 56-gram serving of Bumble Bee® Chunk Light Tuna in Water contains 250 milligrams of sodium and 0.5 gram of fat. (*Source:* **product labeling**) Health professionals advise that a person on a 2500 calorie diet should consume no more than 2400 milligrams of sodium and 80 grams of fat. Graph the region showing all possible serving combinations of peanut butter and tuna that a person could eat and still meet the dietary guidelines.

34. Nutritional Content A Nature Valley® Strawberry Yogurt Chewy Granola Bar contains 130 milligrams of sodium and 3.5 grams of fat. A Nature's Choice® Multigrain Strawberry Cereal Bar contains 65 milligrams of sodium and 1.5 grams of fat. (*Source:* **product labeling**) Health professionals advise that a person on a 2500 calorie diet should consume no more than 2400 milligrams of sodium and 80 grams of fat. Graph the region showing all possible serving combinations of granola bars and cereal bars that a person could eat and still meet the dietary guidelines.

35. Commodity Prices A 25-pound carton of peaches holds 60 medium peaches or 70 small peaches. In August 2002, the wholesale price for local peaches in Los Angeles was \$9.00 per carton for medium peaches and \$10.00 per carton for small peaches. (*Source:* **Today's Market Prices**) A fruit vendor has budgeted up to \$100 to spend on peaches. He estimates that weekly demand for peaches is least 420 peaches but no more than 630 peaches. He wants to buy enough peaches to meet the minimum estimated demand but no more than the maximum estimated demand. Graph the region showing which small and medium peach carton combinations meet his demand and budget restrictions.

■ STRETCH YOUR MIND

Exercises 36–45 are intended to challenge your understanding of graphs of linear inequalities.

36. Graph the solution region of the following system of linear inequalities and identify the coordinates of the corner points.

$$\begin{aligned} 2x + 3y &\leq 6 \\ -2x + 4y &\geq 4 \\ -5x + y &\leq 15 \\ x &\leq 5 \\ y &\geq 2 \end{aligned}$$

37. Graph the solution region of the following system of linear inequalities and identify the coordinates of the corner points.

$$\begin{aligned} -2x + y &\leq 4 \\ 7x + 2y &\geq 8 \\ x &\leq 0 \end{aligned}$$

38. Graph the solution region of the following system of linear inequalities and identify the coordinates of the corner points.

$$\begin{aligned} -x + y &\leq 0 \\ -x - y &\geq -4 \\ y &\geq 2 \end{aligned}$$

39. Write a system of inequalities whose solution region has corner points (0, 0), (1, 3), (3, 5), and (2, 1).

40. Write a system of inequalities whose solution region has corner points (1, 1), (1, 3), (5, 3), and (2, 1).

41. Write a system of inequalities whose *unbounded* solution region has corner points (0, 5), (2, 1), and (5, 0).

42. Write a system of inequalities whose *unbounded* solution region has corner points (0, 5), (4, 4), and (5, 0).

43. A student concludes that the corner points of a solution region defined by a system of linear inequalities are (0, 0), (1, 1), (0, 2), and (2, 2). After looking at the graph of the region, the instructor immediately concludes that the student is incorrect. How did the instructor know?

44. Is it possible to have a *bounded* solution region with exactly one corner point? If so, give a system of inequalities whose solution region is bounded and has exactly one corner point.

45. Is it possible to have an *unbounded* solution region with exactly one corner point? If so, give a system of inequalities whose solution region is unbounded and has exactly one corner point.

CHAPTER 2 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 2.1

1. What distinguishes linear functions from other types of functions?
2. What does it mean to say a function has a constant rate of change?
3. How do you determine if a data set, graph, equation, or verbal expression represents a linear function?
4. What is an efficient way to graph a linear function whose equation is given in standard form? In slope-intercept form? In point-slope form?

SECTION 2.2

5. What key phrases indicate that a linear function can model a phenomenon that is being described verbally?

SECTION 2.3

6. What does it mean for a line to be the line of best fit for a data set?
7. What is the relationship between the phrases *least squares* and *line of best fit*?
8. What does the coefficient of determination represent?
9. How do you interpret the value of the correlation coefficient?
10. What is linear regression used for?

SECTION 2.4

11. Describe the different ways to solve a system of equations and describe the benefits of each.
12. What is the relationship between the algebraic solution to a system of linear equations and the point of intersection of the graphs of the equations in the system?
13. Explain, using examples, the meaning of the terms *inconsistent* and *dependent* system of equations.

SECTION 2.5

14. Why are systems of linear inequalities useful in problem solving?
15. In a real-world context, what does the graphical solution region for a system of linear inequalities represent?
16. What are corner points and how are they used in solving systems of linear inequalities?
17. In the context of systems of linear inequalities, what is meant by bounded and unbounded solution regions?

REVIEW EXERCISES

SECTION 2.1

In Exercises 1–3, determine the slope of the linear function that passes through each point and then give its vertical and horizontal intercept.

1. (1, 7) and (6, 2)

2. (2, 9) and (4, 3)

3. (−3, −4) and (0, −2)

In Exercises 4–6, determine the slope and any horizontal or vertical intercepts of each linear function.

4. $y = -5x + 10$

5. $y = -3x + 18$

6. $y = 2x - 12$

In Exercises 7–12, linear functions are represented numerically, symbolically, or verbally.

a. Graph each linear function by hand.

b. Determine the slope and any vertical or horizontal intercepts of the graph algebraically and plot them on your graph.

c. For real-world contexts, explain the meaning of the slope and horizontal and vertical intercepts. If any of the values do not make sense in the given context, so state.

7. $y = 5x + 4$

8. $y = -9$

9. The line passing through (5, 6), (−3, −9)

10. The line that passes through (−8, 2), (0, 5)

11. Medical Research Medical researchers have found that there is a linear relationship between a person's blood pressure and their weight. In males 35 years of age, for every 5-pound increase in the person's weight there is generally an increase in the systolic blood pressure of 2 millimeters of mercury (mmHg). Moreover, for a male 35 years of age and 190 pounds the preferred systolic blood pressure is 125 mmHg.

12. Business Costs The weekly payroll cost C (in dollars) of running a cell phone company is related to the number n of salespersons. Suppose there are fixed costs of \$4800 per week, and each salesperson costs the company \$1100 per week. For example, if there are 10 salespersons, then the weekly cost is \$11,000.

13. Car Sales A car company has found a linear relationship between the amount of money it spends on advertising and the number of cars it sells. Suppose when it spent \$50,000 on advertising, it sold 500 cars. Moreover, assume for each additional \$5000 spent, it will sell 20 more cars.

a. Find a formula for c, the number of cars sold, as a linear function of the amount spent on advertising, a.

b. What is the slope of the linear function and its meaning in this context?

c. What is the vertical intercept of the function and its meaning in this context?

SECTION 2.2

In Exercises 14–16, find a model for the data and answer the given questions.

14. Nutrition The Recommended Daily Allowance (RDA) for fat for a person on a 2000 calorie per day diet is less than 65 grams. A McDonald's Big 'N' Tasty™ sandwich contains 32 grams of fat. A super size order of French fries contains 29 grams of fat. (***Source:*** **www.mcdonalds.com**)

a. Write the equation for fat grams consumed as a function of super size orders of French fries eaten.

b. Write the equation for fat grams consumed as a function of Big 'N' Tasty sandwiches eaten.

c. How many combination meals (Big 'N' Tasty sandwich and super size order of French fries) can a person eat without exceeding the RDA for fat?

15. Used Car Value In 2011, the average price of a 2007 Toyota Prius was \$14,800. The average retail price of a 2010 Toyota Prius was \$21,950. (***Source:*** **www.nadaguides.com**) Find a linear model for the value of a Prius in 2011 as a function of its production year.

16. Used Car Value In 2001, the average retail price of a 1998 Mercedes-Benz Roadster two-door SL500 was \$51,400. The average retail price of a 2000 Mercedes-Benz Roadster two-door SL500 was \$66,025. (***Source:*** **www.nadaguides.com**)

a. Find a linear model for the value of a Mercedes-Benz Roadster two-door SL500 in 2001 as a function of its production year.

b. Use your linear model to predict the 2001 value of a 1999 Mercedes-Benz Roadster two-door SL500.

c. A 1999 Mercedes-Benz Roadster two-door SL500 had an average retail price of \$58,500 in 2001. How good was your linear model at predicting the value of the vehicle?

In Exercises 17–18, you are given an incomplete table of values for a real-world function. Determine the average rate of change over the given interval, then use it to estimate the remaining values in the table.

17. Admissions to Spectator Amusements

Year	Admissions to Spectator Amusements ($ billions)
2003	\$36.0
2004	
2005	
2006	\$39.9

***Source: Statistical Abstract of the United States, 2008*, Table 1193**

18. Total Attendance at National Football League Games

Year	Total Attendance at NFL Games (thousands of people)
2003	21,709
2004	
2005	
2006	22,256

Source: Statistical Abstract of the United States, 2008, Table 1204

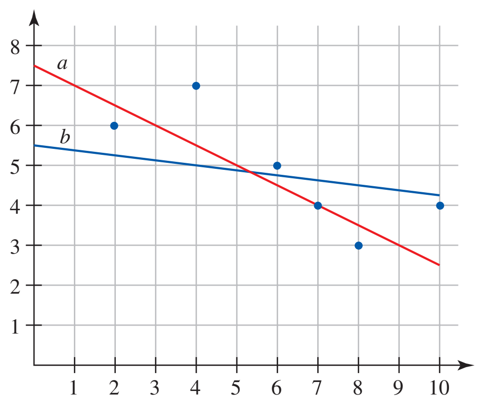
■ SECTION 2.3 ■

19. Consider the number of hours of sleep per day for a baby relative to the age of the baby (starting with newborn). Would the correlation coefficient of a linear model of this situation be positive, negative, or zero? Explain.

20. Consider the average winning speed of cars competing in the Indianapolis 500 relative to the year in which the race took place (starting in 1912). Would the correlation coefficient in this situation be positive, negative, or zero? Explain.

21. Researchers collected data on 100 individuals' grade point averages and the number of hours per week spent watching television. In the linear regression done on this data, the correlation coefficient, r, was -0.95. What does this tell about the relationship between grade point averages and television watching in this study? Explain what the negative sign as well as the 0.95 numerical value mean in terms of this study.

22. Which line (a or b) is the line of best fit in the following figure? Write a verbal explanation of why the line you chose is the line of best fit.



23. Sleep Hours The table shows the recommended number of hours of sleep each day for the typical child as given by Lucile Packard Children's Hospital at Stanford.

Age (months) a	Total Sleep Hours per Day H
0	16
1	15.5
3	15
6	14
9	14
12	14
18	13.5
24	13

Source: www. lpch.org

a. Create a scatter plot of the total sleep hours per day as a function of the age.

b. Find the linear regression model for these data.

c. Does the linear model fit the data well? Explain. (Refer to the correlation coefficient or coefficient of determination in your explanation.)

d. Use the model to predict the total sleep hours per day for a 3-year-old child.

24. Net Sales vs. Cost of Sales The cost of sales and net sales for Apple Computer, Inc. between 2000 and 2005 is given in the table.

Years Since 2000	Cost of Sales ($ millions)	Net Sales ($ millions)
1	4128	5363
2	4139	5742
3	4499	6207
4	6020	8279
5	9888	13,931

Source: www.apple.com

a. Which of the following is best modeled by a linear function?

i. cost of sales as a function of year

ii. net sales as a function of year

iii. net sales as a function of cost of sales

b. Find the equation of the linear model that best fits the data set you identified in part (a).

c. According to the model, what will be the cost of sales when Apple Computer, Inc. generates 20,000 million dollars in net sales?

25. Height and Weight of a Professional Basketball Team The table shows the height and weight of the players on the Phoenix Suns 2006–2007 roster.

Height (inches) H	Weight (pounds) W	Height (inches) H	Weight (pounds) W
74	200	82	250
75	188	75	195
77	210	79	215
83	250	80	215
80	230	82	245
80	220	81	235
80	230	74	200
79	228		

Source: **www.nba.com**

a. For the data provided, find the equation for the line of best fit and label it $w(h)$.

b. Interpret the meaning of the slope and vertical intercept in the context of this problem.

c. Find $w(70)$ and interpret the meaning of the answer in the context of this problem.

d. Solve $w(h) = 200$ and interpret the meaning of the answer in the context of this problem.

■ SECTION 2.4 ■

In Exercises 26–29,

a. Solve the system of equations algebraically. If a solution does not exist, so state.

b. Use a graphing calculator to find the point(s) of intersection of the lines. Compare your solutions to those obtained in part (a). (*Hint:* Remember that to graph a standard form equation using your graphing calculator, you will need to solve the equation for y.)

26. $y = 3x + 4$
$y = 1.7x + 5.3$

27. $y = 4.6x + 10$
$y = -10x + 44$

28. $y = 7.1x - 41.5$
$y = -1.9x + 3.5$

29. $y = 2.5x - 1$
$y = -0.25x + 10$

30. Artist and Athlete Earnings In 1999 the number of paid employees in the independent artist, writer, and performer industry surpassed the number of paid employees in the sports team industry. However, the annual payroll for the artist industry still trailed the sports team industry payroll by \$4.1 billion. Based on data from 1998 and 1999, the payroll for the sports team industry can be modeled by

$$S(t) = 990t + 5718 \text{ million dollars}$$

and the payroll for the independent artist industry can be modeled by

$$A(t) = -35t + 3494 \text{ million dollars}$$

where t is the number of years since 1998. **(*Source:* Modeled from *Statistical Abstract of the United States, 2001*, Table 1232)**

a. According to the models, when were the payrolls for the two industries equal? Check your solution by graphing the system or by substituting the solution back into the original equations.

b. Does this solution seem reasonable? Explain.

31. Pork and Beef Consumption The table shows the per capita retail consumption for pork and beef in pounds per person in the United States from 1981 to 2001.

Years Since 1980	Retail Pork Consumption (pounds per person)	Retail Beef Consumption (pounds per person)
1	54.7	78.3
8	52.5	72.7
11	50.2	66.7
16	48.5	67.3
21	50.3	66.3

Source: **www.agmanager.info**

a. Use linear regression to create a system of equations that models the per capita retail pork and beef consumption. Round off to 2 decimal places.

b. What are the constant rates of change in your models? What do they represent?

c. Solve the system algebraically and explain what the solution represents. Check your answer by graphing the system or by substituting your solution back into the original equations.

d. What assumption(s) do we make when solving this system of equations?

In Exercises 32–33, solve the system of equations. Verify your solution by graphing the system or by substituting your solution back into the original equations. If the system of equations is inconsistent or dependent, so state.

32. $2x + 2y = 20$
$3x + 2y = 27$

33. $4x + 5y = 61$
$-4x - 6y = -74$

34. Computer Prices A small business placed two orders for computers from Hewlett-Packard. The order for the human resources department totaled \$12,576.24 and consisted of 8 HDX 16t Premium Series laptops and 5 Pavilion dv7t Series laptops. The order for the marketing department totaled \$10,524.30 and consisted of 7 HDX 16t Premium Series laptops and 4 HP Pavilion dv7t Series laptops. **(*Source:* www.hp.com, June 2009)** Assume that the prices of the computers did not change between orders.

a. Create a system of equations to model this situation. Be sure to identify what your variables represent.

b. Solve the system of equations. How much did each type of computer cost?

35. Trail Mix Planters Honey Nut Medley Trail Mix contains 8 grams of protein per 2-ounce serving. Planters Dry Roasted Peanuts contain 16 grams of protein per 2-ounce serving. **(*Source:* product labels)** Suppose you want to create a 16-ounce mixture of trail mix and peanuts that has a combined total of 84 grams of protein.

a. What are the unknowns in this situation?

b. Create a system of equations to model this situation.

c. Solve the system of equations and explain what your solution means.

d. Verify your solution by graphing the system or by substituting the solution into the original equations.

■ SECTION 2.5 ■

In Exercises 36–39, graph the solution region of the linear inequality. Then use the graph to determine if the given point P is a solution.

36. $4x - 2y \le 6$; $\boldsymbol{P} = (2, 9)$

37. $3x + 5y \le 0$; $\boldsymbol{P} = (1, 7)$

38. $9x - 8y \le 12$; $\boldsymbol{P} = (8, 9)$

39. $7x + 6y \le 42$; $\boldsymbol{P} = (3, 4)$

In Exercises 40–43, graph the solution region of the system of linear inequalities. If there is no solution, explain why.

40. $$\begin{aligned} -4x + 3y &\ge 2 \\ -3x + 2y &\ge 1 \end{aligned}$$

41. $$\begin{aligned} -10x + y &\ge 0 \\ 2x + y &\le 4 \end{aligned}$$

42. $$\begin{aligned} 2x + 4y &\le 8 \\ 6x - 2y &\le -6 \end{aligned}$$

43. $$\begin{aligned} x + y &\le 8 \\ -x + y &\le 0 \\ 4x - 2y &\le 8 \end{aligned}$$

In Exercise 44, set up the system of linear inequalities that can be used to solve the problem. Then graph the solution region.

44. Wages A salaried employee earns \$800 per week managing a retail store. She is required to work a minimum of 40 hours but no more than 50 hours weekly. As a side business, she earns \$30 per hour designing web sites for local business clients. To maintain her current standard of living, she must earn \$1000 per week. To maintain her quality of life, she limits her workload to 50 hours per week. Given that she has no control over the number of hours she will have to work managing the retail store, will she be able to consistently meet her workload and income goals? Explain.

Make It Real Project

What to Do

1. Find a set of at least six data points from an area of personal interest.
2. Draw a scatter plot of the data and explain why you think a linear model would or would not fit the data well.
3. Find the equation of the line of best fit for the data.
4. Interpret the physical meaning of the slope and y-intercept of the model.
5. Use the model to predict the value of the function at an unknown point. Do you think the prediction is accurate? Explain.
6. Explain how a consumer and/or a businessperson could benefit from the model.

Where to Find Data

Box Office Guru

www.boxofficeguru.com

Look at historical data on movie revenues.

Nutri-Facts

www.nutri-facts.com

Compare nutritional content of common foods based on serving size.

Quantitative Environmental Learning Project

www.seattlecentral.org/qelp

Look at environmental information in easy-to-access charts and tables.

U.S. Census Bureau

www.census.gov

Look at data on U.S. residents ranging from Internet usage to family size.

Local Gas Station or Supermarket

Track an item's price daily for a week.

School Registrar

Ask for historical tuition data.

Utility Bills

Look at electricity, water, or gas usage.

Employee Pay Statements

Look at take-home pay or taxes.