CHAPTER 4

Quadratic Functions

"The only thing constant in life is change."

François de la Rochefoucauld

Nothing demonstrates this maxim better than the world of technology. The 8-track tape, which was cutting-edge music storage and distribution technology in the mid-1960s, gave way to the cassette tape by the mid-1970s. The cassette tape in turn gave way to the compact disc. The emergence of digital audio players, such as the Apple iPod, has, in turn, diminished the role of the compact disc. The rise and decline of different types of music storage and distribution media can be modeled by mathematical functions with variable rates of change.

SECTION 4.1 Variable Rates of Change

LEARNING OBJECTIVES

- Understand rates of change in a function model
- Calculate first and second differences of a table of data
- Determine the concavity and increasing/decreasing behavior of a function from a table or graph
- Interpret the meaning of inflection points in real-world contexts

GETTING STARTED

The average cost of a movie ticket in the United States has been continually rising. According to the website "Box Office Mojo" (www.boxofficemojo.com), there was an estimated 18-cent increase in the average price per ticket from 2006 to 2007. The increase from 1999 to 2000 was 31 cents per ticket. Notice that the increase in price each year has not been *constant* but has *varied*.

In this section we discuss the difference between constant and variable rates of change and see how to apply this knowledge to real-world data such as movie ticket prices. We use average rates of change to estimate unknown data values, to estimate the rate of change of a function at a single data point, and to analyze the *concavity* of a graph. We also use first and second differences and inflection points to describe the change in a function.

Variable Rates of Change

Table 4.1 gives the average movie ticket price in the United States at 5-year intervals, beginning in 1975. The difference in the successive values shown in Table 4.2 shows the increase in the average ticket price over each 5-year interval change.

Table 4.1

Year	Years Since 1975 t	Average Cost of a Movie Ticket (dollars) M
1975	0	2.05
1980	5	2.69
1985	10	3.55
1990	15	4.23
1995	20	4.35
2000	25	5.39
2005	30	6.40

Source: www.boxofficemojo.com

Table 4.2

Years Since 1975 t	Average Cost of a Movie Ticket (dollars) M	Change over 5 Years
0	2.05	
		\$2.69 − \$2.05 = \$0.64
5	2.69	
		\$3.55 − \$2.69 = \$0.86
10	3.55	
		\$4.23 − \$3.55 = \$0.68
15	4.23	
		\$4.35 − \$4.23 = \$0.12
20	4.35	
		\$5.39 − \$4.35 = \$1.04
25	5.39	
		\$6.40 − \$5.39 = \$1.01
30	6.40	

From Table 4.2 we first observe that over each 5-year interval, the average cost of a movie ticket always increases. A function such as this is known as an **increasing function** because the output values continually increase as the input values increase. Conversely, a function whose output values decrease as the input values increase is known as a **decreasing function**.

Increasing and Decreasing Functions

- An **increasing function** is a function whose output values *increase* as its input values increase.
- A **decreasing function** is a function whose output values *decrease* as its input values increase.

A function may or may not be increasing or decreasing over its entire domain. In fact, many functions increase on some intervals and decrease on others.

Next we observe that even though the average cost of a movie ticket is an increasing function, the amount of increase in the average ticket price over each 5-year period is not constant. Figure 4.1 shows the scatter plot of these data. Let's focus on two particular 5-year intervals: 1980–1985 and 1990–1995.

Over the 5-year period between 1980 and 1985, the ticket price increased from \$2.69 to \$3.55, an \$0.86 change. Between 1990 and 1995, the ticket price increased from \$4.23 to \$4.35, a \$0.12 change. Because the rate at which the ticket price is changing is not constant, we say the ticket price has a *variable rate of change*.

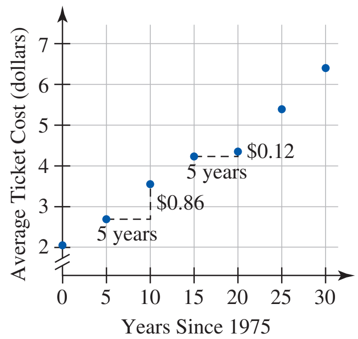


Figure 4.1

Variable Rate of Change

Any function whose rate of change varies (is not constant) is said to have a **variable rate of change**. All nonlinear functions have variable rates of change.

EXAMPLE 1 ■ Interpreting Increasing and Decreasing Functions

Figure 4.2 displays the median age of the first marriage for American women for every decade from 1900 to 2000. (*Source:* www.census.gov)

a. Determine between which years the median marriage age is increasing most rapidly. Then calculate the average annual change over that time period.

b. Determine between which years the median marriage age is decreasing most rapidly. Then calculate the average annual change over that time period.

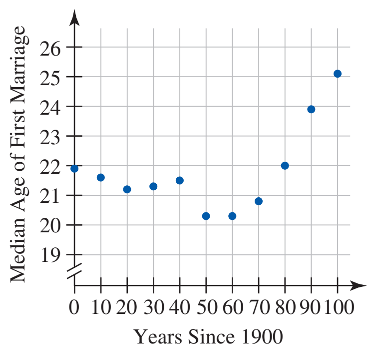


Figure 4.2

Solution Since the points of the scatter plot do not form a straight line, we know the function is nonlinear and has a variable rate of change.

a. We are looking for the two consecutive data points between which there is the greatest vertical increase. It appears the most pronounced vertical increase occurred between 1980 and 1990. Over that 10-year period, the median marriage age rose from 22 years to about 23.9 years.

$$\frac{23.9 - 22}{90 - 80} \ \frac{\text{years of age}}{\text{years since 1900}} = \frac{1.9}{10} \text{ year of age per year}$$
$$= 0.19 \text{ year of age per year}$$

Between 1980 and 1990, the average rate of change in the median marriage age of women was 0.19 year of age per year.

b. We are looking for the two consecutive data points between which there is the greatest vertical decrease. It appears the greatest vertical decrease occurred between 1940 and 1950. Over that 10-year period, the median marriage age appears to drop from about 21.5 years to roughly 20.3 years.

$$\frac{20.3 - 21.5}{50 - 40} \ \frac{\text{years of age}}{\text{years since 1900}} = -\frac{1.2}{10} \text{ year of age per year}$$
$$= -0.12 \text{ year of age per year}$$

Between 1980 and 1990, the average rate of change in the median marriage age of women was -0.12 year of age per year.

Average Rate of Change in Nonlinear Functions

When a function has a variable rather than a constant rate of change over a given interval, it is often helpful to determine the average rate of change of the function over the interval, as we did in Example 1. The average rate of change then can be used to fill in (interpolate) missing data over an interval or predict (extrapolate) unknown values outside of the domain of the function.

To see how this is done, consider the *line graph* (Figure 4.3) of the data from Example 1, along with the table of values (Table 4.3) that was used to produce the graph. (A **line graph** is a scatter plot with the data points connected by lines.)

Table 4.3

Years Since 1900 t	Median Age of First Marriage M
0	21.9
10	21.6
20	21.2
30	21.3
40	21.5
50	20.3
60	20.3
70	20.8
80	22.0
90	23.9
100	25.1

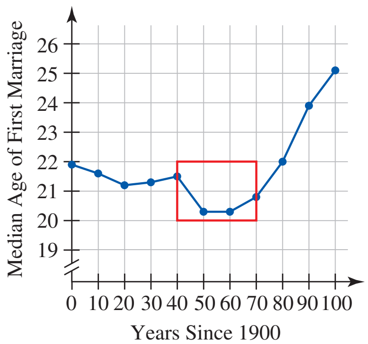


Figure 4.3

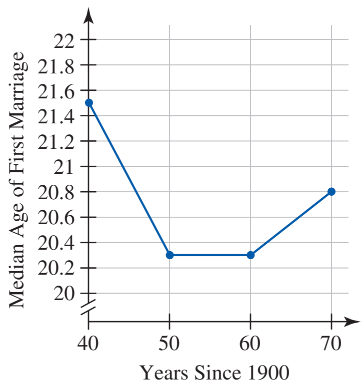


Figure 4.4

We can see that the rate of change over each 10-year interval of time varies. Focusing more closely on the 1940–1970 interval (boxed in Figure 4.3 and enlarged in Figure 4.4), we calculate the change over three 10-year intervals.

$$\frac{21.5 - 20.3}{40 - 50} = \frac{1.2}{-10}$$
$$= -0.12$$

$$\frac{20.3 - 20.3}{50 - 60} = \frac{0}{-10}$$
$$= 0$$

$$\frac{20.3 - 20.8}{60 - 70} = \frac{-0.5}{-10}$$
$$= 0.05$$

Between 1940 and 1950, the median marriage age decreased by 0.12 years of age per year. Between 1950 and 1960, the median marriage age was unchanged. Between 1960 and 1970, the median marriage age increased by 0.05 year of age per year.

What was the median marriage age in, for example, 1945, 1952, and 1968? We do not have data for these years but we can use the average rates of change to estimate. For example, in 1940 the median marriage age was 21.5 and decreased at an average rate of 0.12 year of age per year between 1940 and 1950. Since there are 5 years between 1940 and 1945, we have

$$21.5 + 5(-0.12) = 21.5 - 0.6$$
$$= 20.9$$

So we estimate that the median marriage age in 1945 was 20.9. Similarly, we can estimate the median marriage ages in 1952 and 1968.

1952

$$20.3 + 2(0) = 20.3$$

1968

$$20.3 + 8(0.05) = 20.3 + 0.4$$
$$= 20.7$$

We estimate the median marriage age was 20.3 years in 1952 and was 20.7 in 1968.

Rates of Change at an Instant

Let's now investigate how we can estimate the rate of change at a single instant using the average rate of change.

As a basketball passes through a hoop, the ball's height in relation to the basketball court decreases. As the ball falls, its speed increases due to the effects of gravity until the conflicting forces of friction and gravity cause the ball to fall at a constant rate (known as the *terminal velocity*).

To model this phenomenon, we dropped a basketball repeatedly through a hoop from a height of approximately 10 feet. We measured the height of the ball over time using a motion detector. Table 4.4 shows the average of the height readings collected every 0.2 seconds in four trials.

Table 4.4

Time (seconds) t	Height of Basketball (feet) H
0	9.98
0.2	9.34
0.4	7.49
0.6	4.42
0.8	0.13

We can estimate the velocity of the ball at 0.6 second by calculating the average rate of change in the height between 0.4 second and 0.6 second.

$$\begin{aligned}\frac{\Delta H}{\Delta t} &= \frac{4.42 - 7.49}{0.6 - 0.4}\ \frac{\text{feet}}{\text{second}}\\ &= \frac{-3.07}{0.2}\text{ feet per second} \qquad \textbf{The ball falls 3.07 feet over the 0.2 second}\\ &= -15.35\text{ feet per second}\end{aligned}$$

The ball falls at an average velocity of 15.35 feet per second *between* the 0.4 and 0.6 second marks.

How fast was the ball falling *right at* the 0.6 second mark? In other words, what was the velocity of the ball *at that instant in time*? Determining an answer to this question is problematic because two points are needed to find an average rate of change. Nevertheless, we can use the average rate of change concept to answer this question by making the time interval extremely small.

Table 4.5 provides additional information about the height of the ball near (before and after) the 0.6 second mark. If we use the time just before 0.6 second in the table—0.59 second—we can estimate the instantaneous rate of change at 0.60 second, which is the velocity of the ball at 0.6 second.

Table 4.5

Time (seconds) t	Height of Basketball (feet) H
0.57	4.96
0.58	4.78
0.59	4.60
0.60	4.42
0.61	4.23
0.62	4.04
0.63	3.85

$$\begin{aligned}\frac{\Delta H}{\Delta t} &= \frac{4.42 - 4.60}{0.60 - 0.59}\ \frac{\text{feet}}{\text{second}}\\ &= \frac{-0.18}{0.01}\text{ feet per second} \qquad \textbf{The ball falls 0.18 feet over the 0.01 second.}\\ &= -18\text{ feet per second}\end{aligned}$$

Thus we estimate that the ball is falling at a velocity of 18 feet per second at 0.60 second.

By picking a small Δt, we arrived at a reasonable estimate for how fast the ball was falling at a single point in time. The process of calculating an average rate of change with a small Δt is referred to as **estimating the instantaneous rate of change**.

HOW TO: ■ ESTIMATE THE INSTANTANEOUS RATE OF CHANGE

To estimate the instantaneous rate of change of a function at a point, determine the average rate of change of the function over a very small interval containing the point.

In other words, find $\frac{\Delta y}{\Delta x}$ for Δx close to 0.

Note that decreasing the value of Δx increased the accuracy of the estimate. It is customary to use the phrase "Δx approaches 0" and the notation $\Delta x \to 0$ to represent the idea of using ever decreasing positive values for Δx.

Successive Differences of Functions

Another way to analyze the behavior of a function with a variable rate of change is to use *successive differences*. When using successive differences, we look for patterns in the rates of change. To calculate **first differences** of a data table with equally spaced inputs, we calculate the difference in consecutive output values. The differences in consecutive first differences are referred to as **second differences**.

EXAMPLE 2 ■ Interpreting First and Second Differences

The per capita amount of money spent on prescription drugs, P, as a function of the years since 1990, t, can be modeled by Table 4.6, which was generated from the function $P(t) = 2.9t^2 - 2.6t + 158.7$. The first differences, ΔP, are shown in the table.

a. Explain the practical meaning of the first differences in this context.

b. Calculate the second differences. Then explain what the second differences tell us about the relationship between the per capita spending on prescription drugs and the years since 1990.

c. Using first and second differences, predict the per capita prescription drug spending in 1996.

Table 4.6

Years Since 1990 t	Per Capita Spending on Prescription Drugs (dollars) P	First Differences ΔP
0	158.7	
1	159.0	$159.0 - 158.7 = 0.3$
2	165.1	$165.1 - 159.0 = 6.1$
3	177.0	$177.0 - 165.1 = 11.9$
4	194.7	$194.7 - 177.0 = 17.7$
5	218.2	$218.2 - 194.7 = 23.5$

Source: Modeled from *Statistical Abstract of the United States, 2006*, Table 121

Solution

a. The first differences show us the annual rate of change in the per capita prescription drug spending in dollars per year. We observe that the first differences are increasing. That is, the annual rate of change in spending is increasing.

b. In Table 4.7, we calculate the differences of the first differences and see that the second differences are all 5.8.

Table 4.7

Years Since 1990 t	Per Capita Spending on Prescription Drugs (dollars) P	First Differences ΔP	Second Differences $\Delta(\Delta P)$
0	158.7		
1	159.0	0.3	
2	165.1	6.1	$6.1 - 0.3 = 5.8$
3	177.0	11.9	$11.9 - 6.1 = 5.8$
4	194.7	17.7	$17.7 - 11.9 = 5.8$
5	218.2	23.5	$23.5 - 17.7 = 5.8$

The second differences tell us that the annual rates of change in spending are increasing at a constant rate of 5.8 dollars per year each year.

c. Since the function has a constant second difference of 5.8, we can calculate the first difference between $t = 5$ and $t = 6$ by adding 5.8 to the first difference between $t = 4$ and $t = 5$ ($\Delta P = 23.5$).

$$23.5 + 5.8 = 29.3 \text{ dollars per year}$$

Between 1995 and 1996, the per capita spending on prescription drugs increased by \$29.30. Therefore, the function value at $t = 6$ will be 29.3 dollars more than the function value at $t = 5$ ($P = 218.2$).

$$218.2 + 29.3 = 247.5 \text{ dollars}$$

In 1996, the per capita prescription drug spending was 247.50 dollars.

Concavity and Second Differences

In general, second differences tell us about the concavity (curvature) of a nonlinear graph. When the second differences are positive, the function is *concave up* (curved upward). When the second differences are negative, the function is *concave down* (curved downward). Let's examine this concept within a real-world scenario.

In times of increasing gas prices, many drivers become concerned about the fuel efficiency of their vehicles. The U.S. Department of Energy reports that although each vehicle reaches its optimal fuel economy at a different speed, gas mileage usually increases up to speeds near 45 miles per hour and then decreases rapidly at speeds above 60 miles per hour. (*Source:* www.fueleconomy.gov)

Consider the fuel economy function F shown in Table 4.8 and Figure 4.5. Notice that the first differences decrease as speed increases. Between 10 and 20 miles per hour, gas mileage is increasing at a rate of 6.5 miles per gallon per additional 10 miles per hour of speed. Between 20 and 30 miles per gallon, gas mileage is still increasing but at the lesser rate of 4.5 miles per gallon per additional 10 miles per hour of speed. Between 40 and 50 miles per hour, the gas mileage is increasing at the much smaller rate of 0.6 miles per gallon per additional 10 miles per hour of speed.

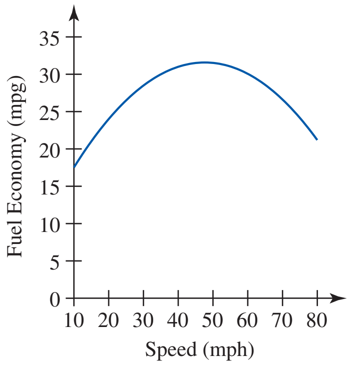


Figure 4.5

Table 4.8

Speed (mph) s	Fuel Economy (mpg) F	First Differences ΔF	Second Differences $\Delta(\Delta F)$
10	17.6		
		6.5	
20	24.1		−2.0
		4.5	
30	28.6		−2.0
		2.5	
40	31.1		−1.9
		0.6	
50	31.7		−2.1
		−1.5	
60	30.2		−2.0
		−3.5	
70	26.7		−1.9
		−5.4	
80	21.3		

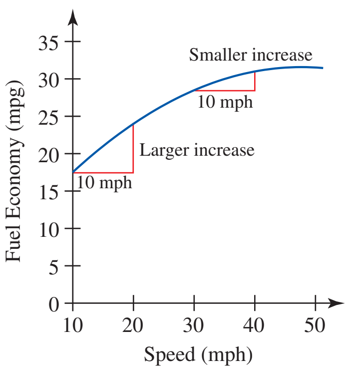


Figure 4.6

Let's look more closely at the first part of the graph, where the fuel economy is increasing as speed increases. See Figure 4.6. We can see although the fuel economy is increasing, it does not increase as much between 30 and 40 miles per hour as it does between 10 and 20 miles per hour. This is an example of a function that is *increasing* and *concave down.*

Table 4.8 shows that between 50 and 60 miles per hour, the gas mileage is decreasing by 1.5 miles per gallon per additional 10 miles per hour of speed. Between 70 and 80 miles per hour, the gas mileage is decreasing by 5.4 miles per gallon per 10 miles per hour of speed.

The magnitude of the decrease becomes greater as the speed increases, as shown in Figure 4.7. Since the first differences are decreasing (the second differences are negative), this part of the function is *decreasing* and *concave down.*

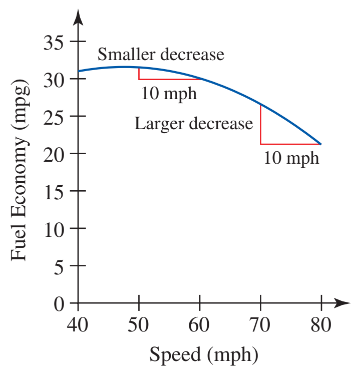


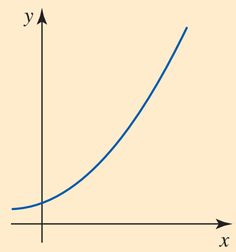
Figure 4.7

Notice that the second differences in Table 4.8 are all negative. This tells us that the first differences are decreasing. This means that the graph will be concave down on the entire domain of the function.

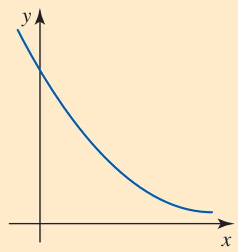
Concavity

- The graph of a function *f* is said to be **concave up** if its rate of change *increases* as the input values increase. Concave up functions curve upward.

Increasing/Concave Up

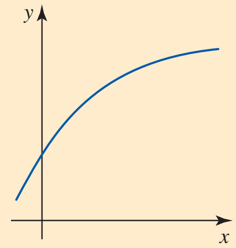


Decreasing/Concave Up

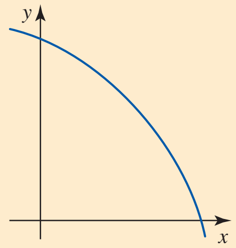


- The graph of a function *f* is said to be **concave down** if its rate of change *decreases* as the input values increase. Concave down functions curve downward.

Increasing/Concave Down



Decreasing/Concave Down



■ Inflection Points

Many graphs are concave up on portions of their domain and concave down on others. We refer to points on a graph where the concavity changes as **inflection points**. To show where the concavity changes on the graph in Figure 4.8, we mark the inflection points.

The first inflection point occurs where the graph changes from concave up to concave down. The second inflection point occurs where the graph changes from concave down to concave up.

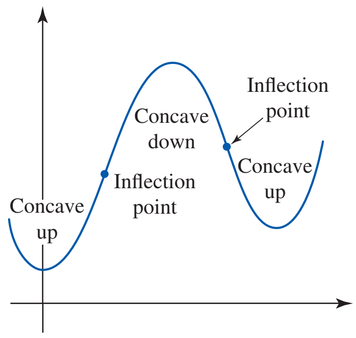


Figure 4.8

Inflection Point

The point on a graph where the function changes concavity is called an **inflection point**. The inflection point is the point where the instantaneous rate of change is locally maximized or minimized.

EXAMPLE 3 ■ Interpreting Inflection Points in a Real-World Context

Savvy investors in multifamily properties (apartment buildings) closely monitor the markets in which they invest. Marcus and Millichap Real Estate Investment Services helps investors by providing in-depth reports on various sectors of the U.S. rental market. Based on data from 2003 to 2007, the average price per unit (apartment) for multifamily properties in Columbus, Ohio, can be modeled by

$$p(t) = -2.417t^3 + 14.14t^2 - 20.65t + 48.99 \text{ thousand dollars}$$

where t is the number of years since 2003. (*Source:* Modeled from data in Marcus and Millichap's 2008 National Apartment Report) A graph of the model is shown in Figure 4.9.

a. Estimate the intervals over which the function is increasing, decreasing, concave up, and concave down.

b. Determine if there are any inflection points on the graph.

c. Explain what the answers in parts (a) and (b) tell us about multifamily housing prices in Columbus, Ohio, between 2003 and 2007.

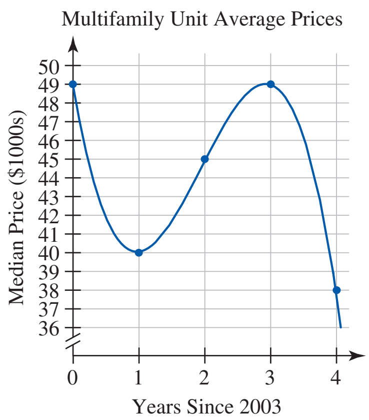


Figure 4.9

Solution

a. The graph appears to increase between $t = 1$ and $t = 3$. On the intervals [0, 1] and [3, 4], the graph appears to be decreasing. The graph is concave up from $t = 0$ to $t = 2$ and concave down from $t = 2$ to $t = 4$.

b. The inflection point appears to be (2, 45).

c. Between 2003 and 2004 median unit prices were decreasing; however, the rate of decrease lessened as time moved forward. Between 2004 and 2005, prices were increasing at an increasing rate. Between 2005 and 2006 prices continued to increase but at a lesser rate. That is, the rate of increase lessened as time moved forward. Between 2006 and 2007, prices again decreased; however, the magnitude of the rate of decrease became greater as time moved forward.

The inflection point indicates that in 2005 the median price was about $45,000 per unit. This was the time when the instantaneous rate of change was locally maximized; that is, when prices were increasing most rapidly.

SUMMARY

In this section you learned the difference between constant and variable rates of change. You used average rates of change to estimate unknown data values, to estimate the rate of change of a function at a single data point, and to analyze the concavity of a graph. You also used first and second differences to help describe the change in a function.

4.1 EXERCISES

SKILLS AND CONCEPTS

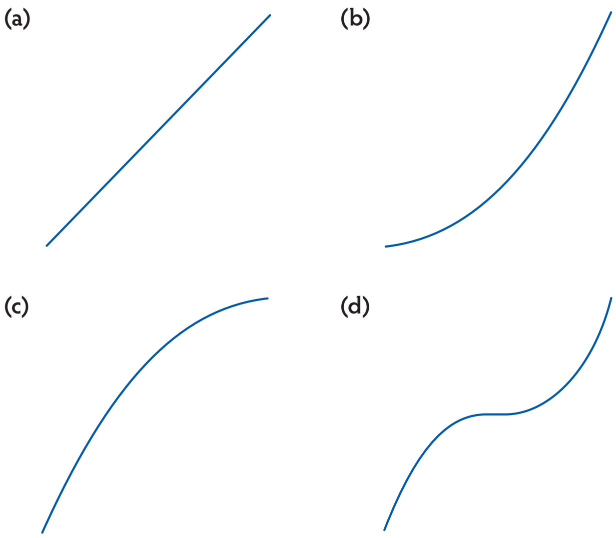
For Exercises 1–10, create a table of values and an associated graph that

1. Are increasing and concave up.
2. Are decreasing and concave down.
3. Are increasing and concave down.
4. Are decreasing and concave up.
5. Are increasing at a constant rate.
6. Are decreasing at a constant rate.
7. Are constant.
8. Are increasing, with a point of inflection.
9. Have two points of inflection.
10. Are concave up twice and concave down once.
11. Each of the functions f, g, and h in Table A are increasing, but each increases in a different way. Which of the graphs in Figure A best fits each function?

Table A

x	$f(x)$	$g(x)$	$h(x)$
1	23	10	2.2
2	24	20	2.5
3	26	29	2.8
4	29	37	3.1
5	33	44	3.4
6	38	50	3.7

Figure A

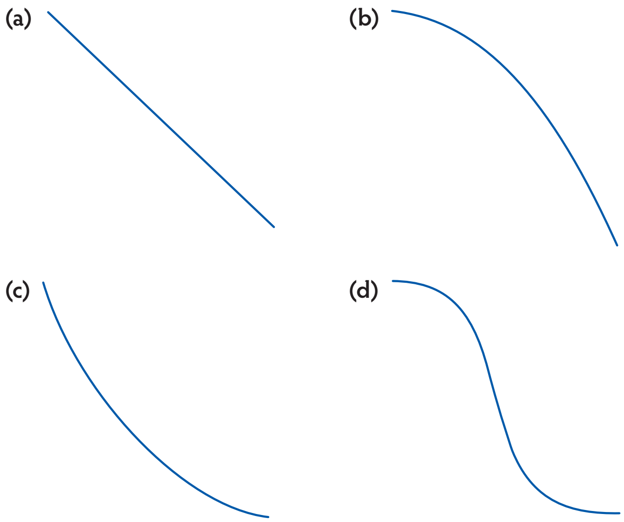


12. Each of the functions f, g, and h in Table B are decreasing, but each decreases in a different way. Which of the graphs in Figure B best fits each function?

Table B

x	$f(x)$	$g(x)$	$h(x)$
2	−4	−2	−8
5	−6	−12.5	−17
7	−12	−24.5	−26
11	−27	−60.5	−35
12	−32	−72	−44
18	−38	−162	−53

Figure B



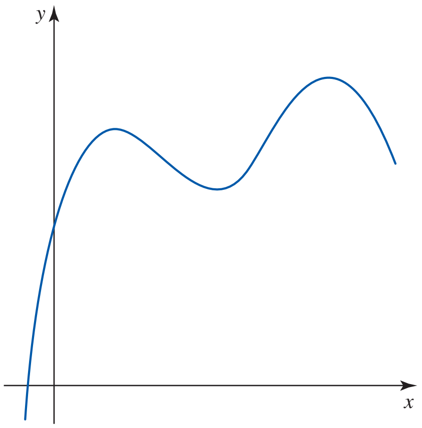
13. Determine whether each function f, g, and h in the table has a constant rate of change or a variable rate of change.

x	$f(x)$	$g(x)$	$h(x)$
−4	14	18	39
−2	19	26	36
0	22	38	33
2	23	56	30
4	20	83	27

14. In the following table, there are missing data values. Use the information you have been given to fill in the missing data.

x	y	First Differences Δy	Second Differences $\Delta(\Delta y)$
−3			
		17	
−2	−16		−6
−1			−6
		5	
0	0		−6
1			−6
		−7	
2	−8		−6
3	−21		

15. Label points A, B, C, D, E, and F on the graph shown.



a. Point A is a point on the curve where the instantaneous rate of change is negative.

b. Point B is a point on the curve where the value of the function is positive.

c. Point C is a point on the curve where the instantaneous rate of change is the most positive.

d. Point D is a point on the curve where the instantaneous rate of change is 0.

e. Points E and F are different points on the curve where the instantaneous rate of change is about the same.

For Exercises 16–19, use the table to answer each question.

x	$f(x)$	$g(x)$
−2	0	5
−1	3	3
0	4	2
1	−1	1
2	6	−1
3	−2	0

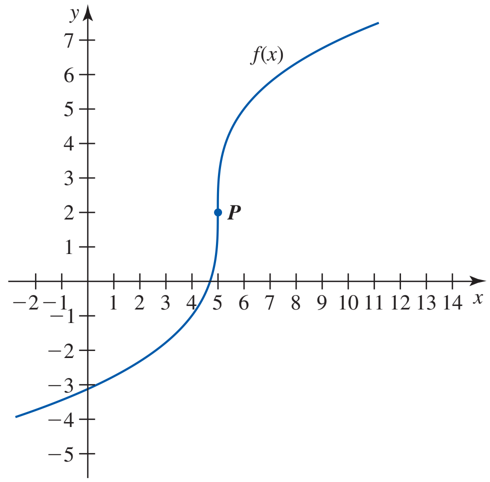
16. Compute the average rate of change of f from $x = -2$ to $x = 3$.

17. Compute the average rate of change of g from $x = -1$ to $x = 2$.

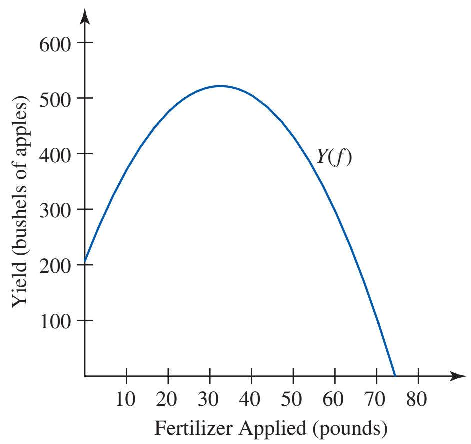
18. Estimate the instantaneous rate of change of g at $x = -1$.

19. Estimate the instantaneous rate of change of f at $x = 0$.

20. Using the following graph, explain the behavior of function f on the interval from $x = 0$ to $x = 15$. State over which intervals f is increasing or decreasing and concave up or concave down. What does the point ***P*** indicate?



21. Fertilizer The following figure shows the yield, Y, of an apple orchard (in bushels) as a function of the amount of fertilizer, f (in pounds), used on the orchard.

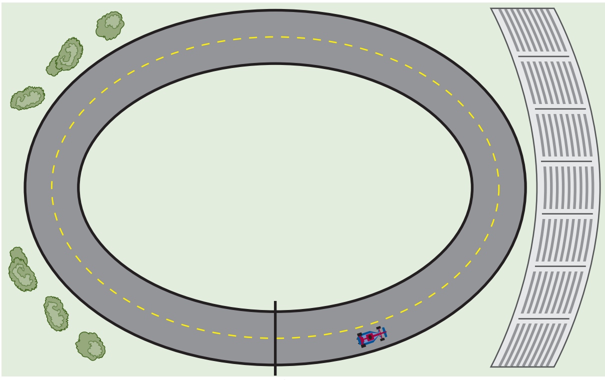


a. Over what interval(s) is the function increasing? Decreasing?

b. Discuss the concavity of the function and then explain what that tells about the apples and fertilizer.

22. Sprinter A poorly conditioned sprinter starts a 400-meter race at a rapid pace; however, as the race progresses his speed decreases. By the time he reaches the finish line, he is walking. Sketch a graph of the sprinter's distance traveled as a function of time since the start of the race.

23. Racing A race car is being driven around an oval racetrack at a constant speed.



Construct a rough sketch of a function that represents the *shortest* distance between the car and the starting line while imagining the car moving around the track at a constant rate. Identify any important aspects of the graph such as increasing, decreasing, concave up, concave down, and points of inflection.

SHOW YOU KNOW

For Exercises 24–27, determine if each function has a constant or a variable rate of change over the interval provided. Explain or show how you know.

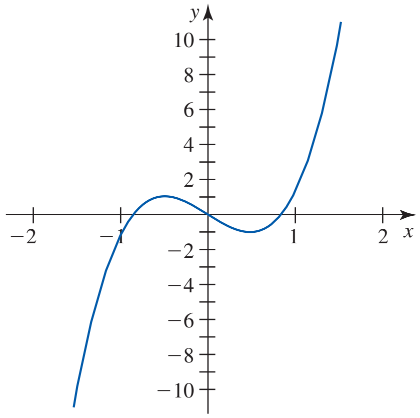
24.

x	y
−8	80
−7	63
−6	48
−5	35
−4	24
−3	15
−2	8
−1	3

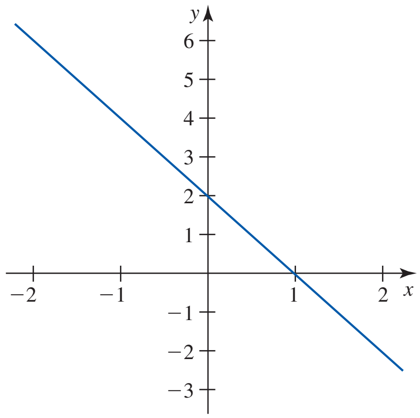
25.

x	y
0	1
2	5
4	6
6	12
8	16
10	19
12	20
14	25
16	29

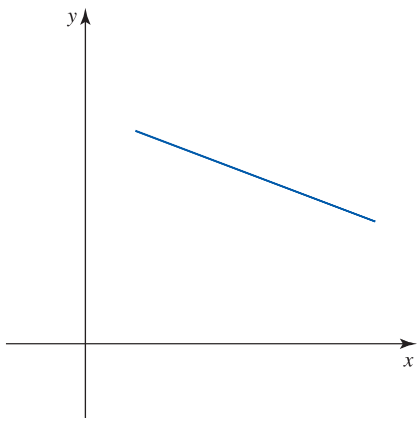
26.



27.

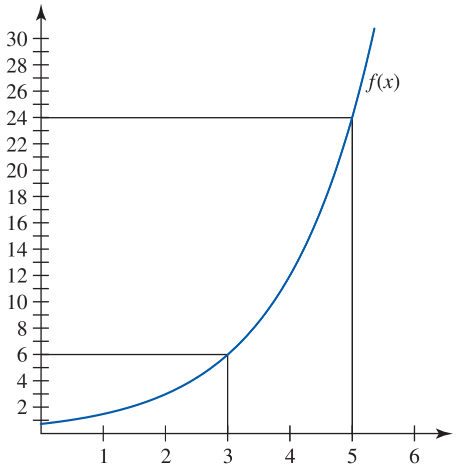


28. The figure shows a portion of the graph of a function. If it is possible, explain whether the function has a constant or variable rate of change. If it is impossible to determine, explain why.



29. Given the following table and graph of the function f, a classmate claims the average rate of change of f between $x = 3$ and $x = 5$ is 18. You know that this response is incorrect. What does the classmate's response tell you about his thinking? Explain.

x	0	1	2	3	4	5	6
$f(x)$	0.75	1.5	3	6	12	24	48

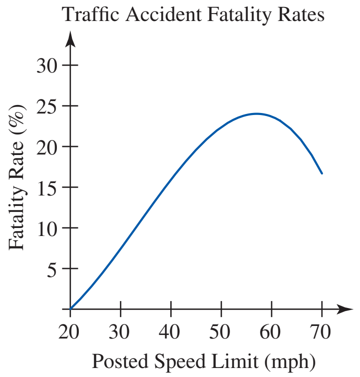


MAKE IT REAL

30. Marathon Runner After recovering from an injury, an athlete begins training again for a marathon. As she gets stronger, she is able to run faster. Although her run times improve each week, they improve by a lesser amount as time goes on. She runs a full marathon once a week. Sketch a graph of the number of minutes it takes her to run a marathon as a function of the number of weeks since her recovery from the injury. Label the independent and dependent axes.

Eric Gevaert/Shutterstock.com

31. Traffic Fatalities A model F representing the percentage of traffic accidents that are fatal, F, as a function of the posted speed limit, s, is shown in the figure.



***Source*: Modeled from www-nrd.nhtsa.dot.gov**

a. For what speeds is F increasing and for what speeds is it decreasing?

b. Is F primarily concave up or concave down? What does this tell us about the fatality rates?

32. Phoenix High Temperatures In the summer of 1990, the temperatures in Arizona reached an all-time high (so high, in fact, that some airlines decided it might be unsafe to land their planes there). The daily temperatures in Phoenix for June 19–29, 1990 are given in the table.

Date: June 1990 (day of month)	Temperature (°F)
19	109
20	113
21	114
22	113
23	113
24	113
25	120
26	122
27	118
28	118
29	108

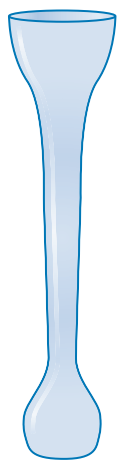
***Source:* www.weather.com**

a. Identify over what time intervals the function is constant, increasing, and decreasing.

b. Between June 24 and June 26, is the graph of the data concave up or concave down? Explain.

c. In practical terms, what does the concavity of the data set tell about the Phoenix temperature between June 24 and June 26?

33. The Yard House The Yard House restaurant opened in 1996 in Long Beach, California. It derives its name from a 3-foot-tall glass (called a yard-of-ale glass) originally designed for stagecoach drivers. The elongated neck made it possible to hand the stagecoach driver his drink without climbing up onto or down from the coach. The basic shape of the glass is displayed in the figure. (***Source:* www.yardhouse.com**)



As a beverage is poured into the glass at a constant rate until the glass is full, the height of the liquid will rise in the glass over time.

a. Sketch a graph of the height of the liquid in the glass as a function of time.

b. As you sketch the graph of the function, is it important to consider that the liquid is being poured into the glass at a constant rate? Explain.

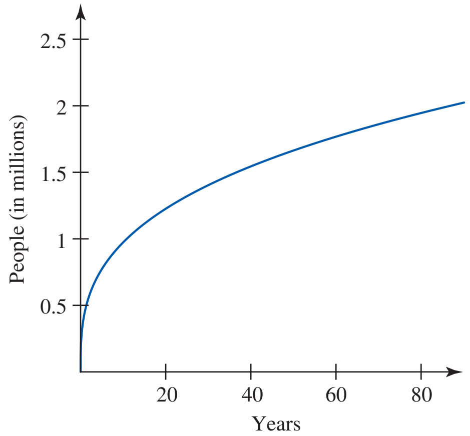
34. Identity Theft A January 26, 2006, headline in *The Wall Street Journal* read, "ID Theft Complaints Still Rising, but Rate of Increase Slows."

a. Write a description of what this headline means.

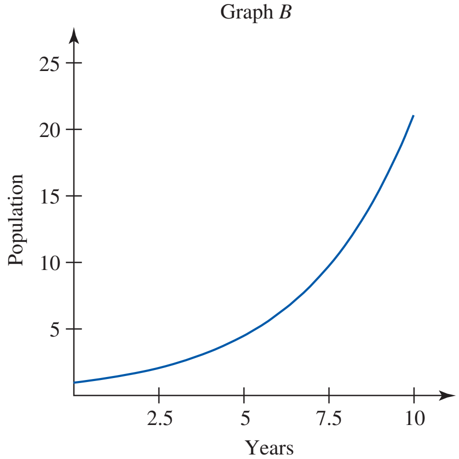
b. Sketch a possible graph of the number of ID theft complaints as a function of time. Be sure the shape of the graph matches what the headline states.

c. Imagine you are a newspaper reporter analyzing the following graphs. Write a headline that would represent accurately each graph. Make sure your headline clearly describes the special characteristics of the graph.

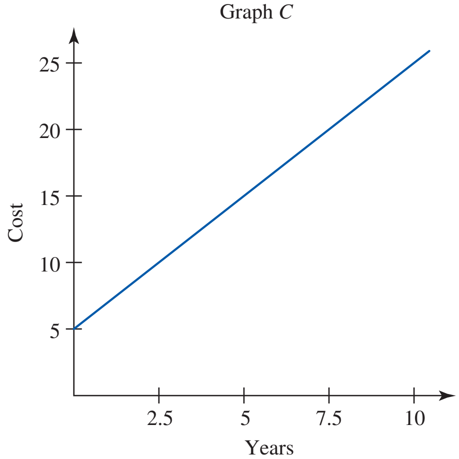
Graph *A*



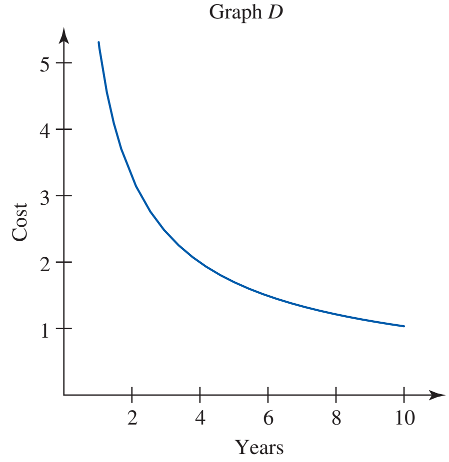
Graph *B*



Graph *C*



Graph *D*



35. Home-Field Advantage A common belief in athletics is that the home team has an advantage due to hometown fans, familiar surroundings, and short travel times. To test this hypothesis, two Georgia Southern University professors analyzed data collected from Major League Baseball. As expected, the results indicated that a home-field advantage does exist in the major leagues, but only under certain circumstances. Specifically, the strength of the home-field advantage varies with the number of runs scored by the home team. A claim made in their published research article stated, "The probability of a home team winning a game increases as it scores more runs, but it increases at a decreasing rate." **(*Source:* "An Analysis of the Home-Field Advantage in Major League Baseball Using Logit Models: Evidence from the 2004 and 2005 Seasons," Levernier & Barilla, J. Quant. Analysis in Sports, www.bepress.com/jqas/vol3/iss1/1/)**

a. Sketch a graph of a function that could reasonably approximate the researchers' claim. Make sure to properly label the independent and dependent variables.

b. Describe in your own words what their claim, "The probability of a home team winning a game increases as it scores more runs, but it increases at a decreasing rate," means in terms of the likelihood of the home team winning baseball games.

c. Explain why you think the claim the researchers make could be true in terms of a baseball contest.

36. Home Foreclosures A headline in the *East Valley Tribune* (February 10, 2007) stated, "Numbers are Rising in the Valley, but Not as Bad as 2002: Foreclosure Fears." The accompanying article stated "foreclosures shot up in some East Valley communities in 2006 but remain far below the number of foreclosures four years ago." Using the information in the newspaper's article, complete the following table with reasonable values that could model how the number of foreclosures may have changed as described from 2002 to 2006.

Years Since 2002 d	Number of Foreclosures F
0	1563
1	
2	
3	
4	314

37. Per Capita Income The per capita personal income of each resident of the United States from 1960 to 2000 is given in the table. The dependent variable P represents the per capita income (in dollars) and t represents the number of years since 1960.

Years Since 1960 t	Per Capita Income (dollars) P
0	1,832
10	4,334
20	9,865
30	18,425
40	30,013

Source: U.S. Department of Commerce

a. Using *averages*, estimate $P(5)$ and explain what the value of the answer means in the real-world context.

b. Using the *average rate of change* from $t = 0$ to $t = 10$, estimate $P(5)$ and explain what the value of the answer means in the real-world context.

c. If you were to estimate $P(13)$, would it be possible to use ***either*** averages ***or*** the average rate of change to arrive at the estimate? Justify your answer.

d. Using *successive differences*, estimate $P(50)$ and explain what the value of the answer means in the real-world context.

38. Fast-Food Sales Actual and predicted fast-food restaurant sales from 1990 to 2010 are given in the table.

Years Since 1990 t	Fast-Food Restaurant Sales ($ millions) S
0	69,840
2	82,433
4	96,341
6	112,882
8	133,372
10	159,126
12	191,461
14	231,693
16	281,138
18	341,112
20	412,932

Source: Statistical Abstract of the United States, 2006, Table 1269

a. Using *averages*, estimate $S(11)$ and explain what the value of the answer means in the real-world context.

b. Using the *average rate of change* from $t = 0$ to $t = 10$, estimate $S(11)$ and explain what the value of the answer means in the real-world context.

c. If you were to estimate $S(13)$, would it be possible to use ***either*** averages ***or*** the average rate of change to arrive at the estimate? Justify your answer.

d. Using *successive differences*, estimate $S(22)$ and explain what the value of the answer means in the real-world context.

39. Cable Television The following table gives the number of basic cable television subscribers (in thousands).

Years Since 1970 d	Number of Basic Cable TV Subscribers (in thousands) F
5	9,800
10	17,500
15	35,440
20	50,520
25	60,550
30	66,250
31	66,732
32	66,472
33	66,050

Source: Statistical Abstract of the United States, 2007, Table 1134

a. Use the table to estimate the point of inflection and interpret the meaning of the result.

b. From the table, estimate the instantaneous rate of change in 2002 and interpret the practical meaning of the result (including units).

40. Abortions The table gives the number of reported abortions in South Carolina from 1984 to 2004.

Years Since 1984 t	Number of Reported Abortions A
0	11,704
2	12,174
4	14,133
6	13,285
8	11,008
10	10,992
12	9,326
14	8,801
16	7,527
18	6,657
20	6,565

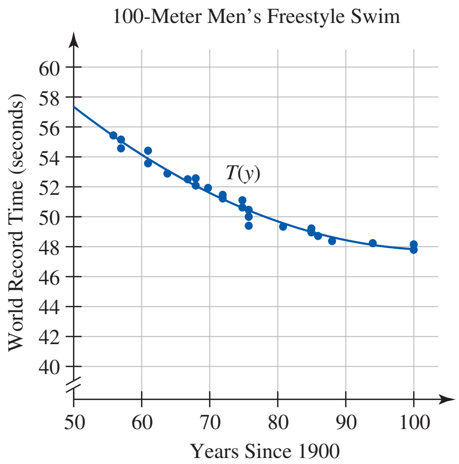
Source: Citizen magazine, January 2007

a. Find the average rate of change in the number of abortions between 1984 and 1990. Then interpret the meaning of the result, including units.

b. Find the average rate of change in the number of abortions between 1990 and 2004. Then interpret the meaning of the result, including units.

c. In 1990, South Carolina passed a law requiring parental consent for minors to have an abortion and a 1-hour waiting period after abortion counseling. Does it appear from the data that the law had an effect on the number of reported abortions? Refer to the first differences and average rates of change in your response.

41. World Records The year of a world record set in the men's 100-meter freestyle swim, y, and the record time in seconds, T, is given in the scatter plot along with a function model $T(y)$.



Source: **www.duelinthepool.com/USASWeb**

a. Is the graph of T increasing or decreasing? Explain what this tells you about the world record times.

b. Is the graph of T concave up or concave down? Explain what the concavity tells you about the decline in the world record times.

42. In a race for political office, an incumbent politician claims, "As long as I have been in office, the crime rate has dropped!" In the same campaign, the incumbent's opponent claims, "We need to vote Mrs. X out because crime continues to rise!" Is it possible that both political candidates are telling the truth? Justify your answer using a graph.

43. World Record Time for the Mile Since 1913, the world record time for the runners of the mile has decreased, as shown in the table.

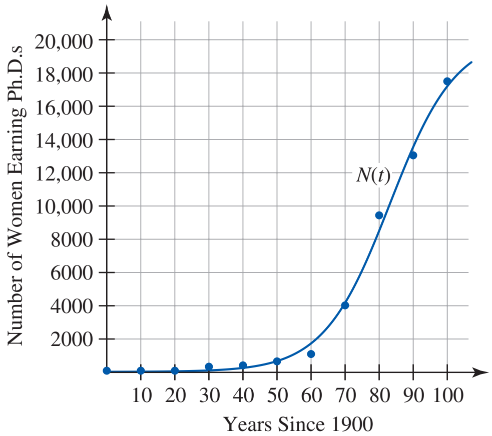
Year	Record Time (minutes)		Year	Record Time (minutes)
1913	4.24		1965	3.89
1923	4.17		1975	3.82
1933	4.13		1985	3.77
1943	4.04		1993	3.74
1954	3.99		1999	3.72

Source: **International Association of Athletics Federation**

a. Between 1913 and 1999, what was the average annual decrease in the world record time?

b. Draw a scatter plot of the data set. Then describe the practical meaning of the concavity of the graph.

44. Women Earning Ph.D.s The function shown in the figure estimates the number of women who earned a Ph.D. in the 20th century.

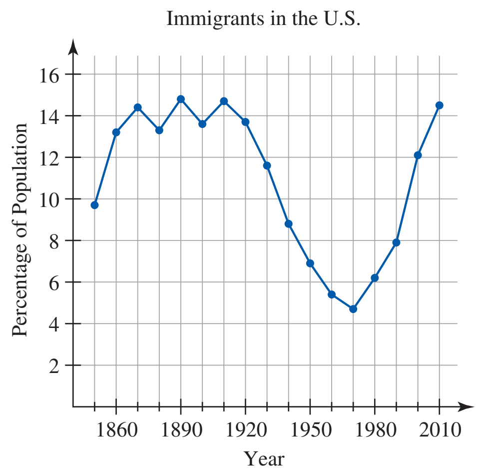


a. Over what interval(s) is the function increasing and concave up? Over what interval(s) is it increasing and concave down?

b. What does the concavity indicate about the rate of increase in the number of women earning Ph.D.s?

c. Estimate the average rate of change from $t = 1950$ to $t = 1960$, from $t = 1960$ to $t = 1970$, and from $t = 1970$ to $t = 1980$.

45. Immigration The percentage of the total U.S. population made up of immigrants from 1850 to 2006 is shown in the graph.

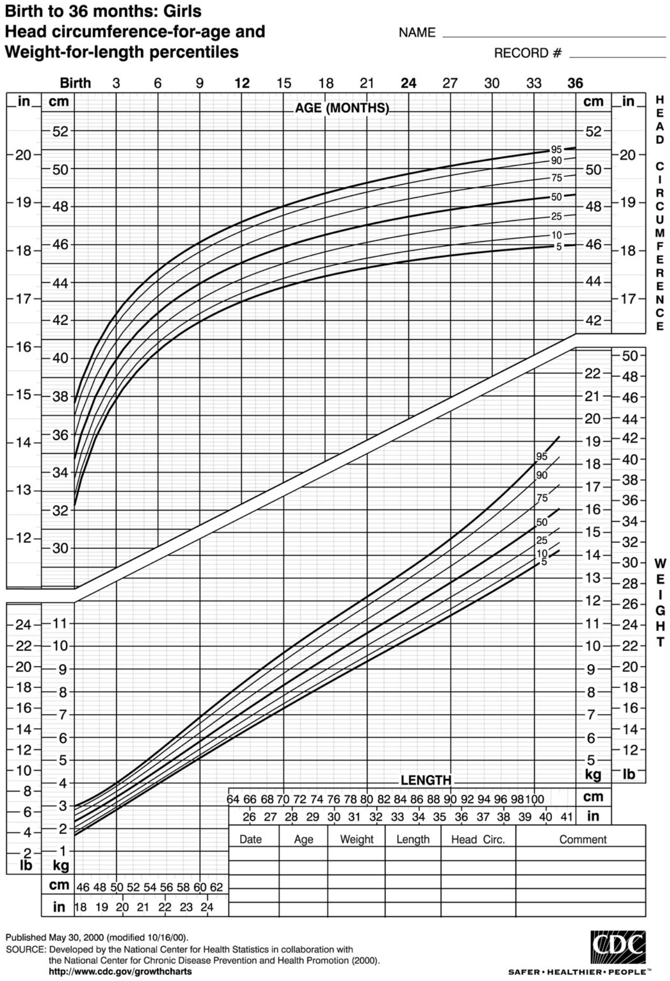


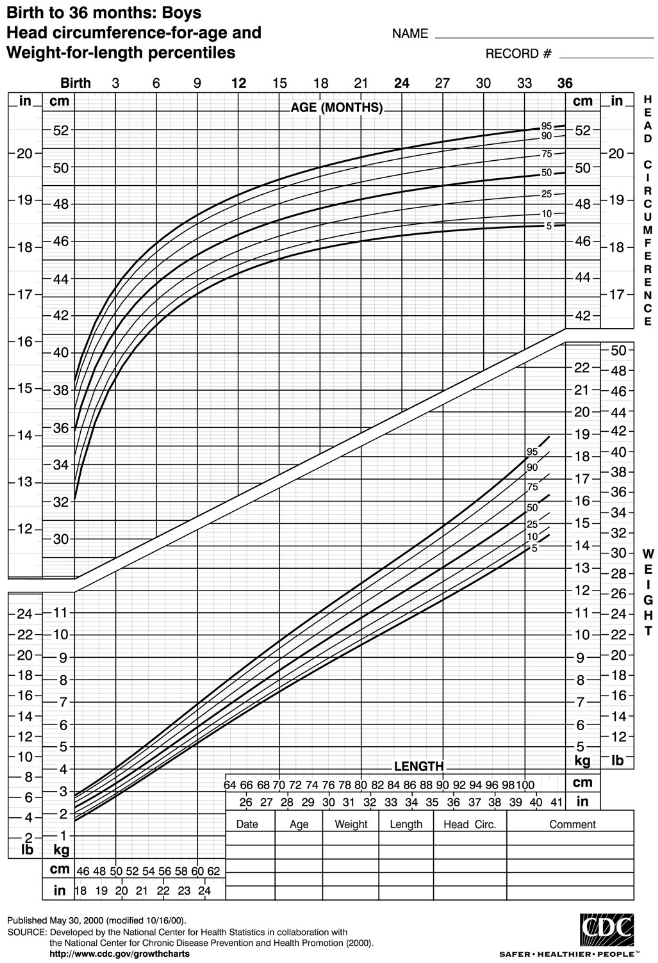
Source: **www.census.gov**

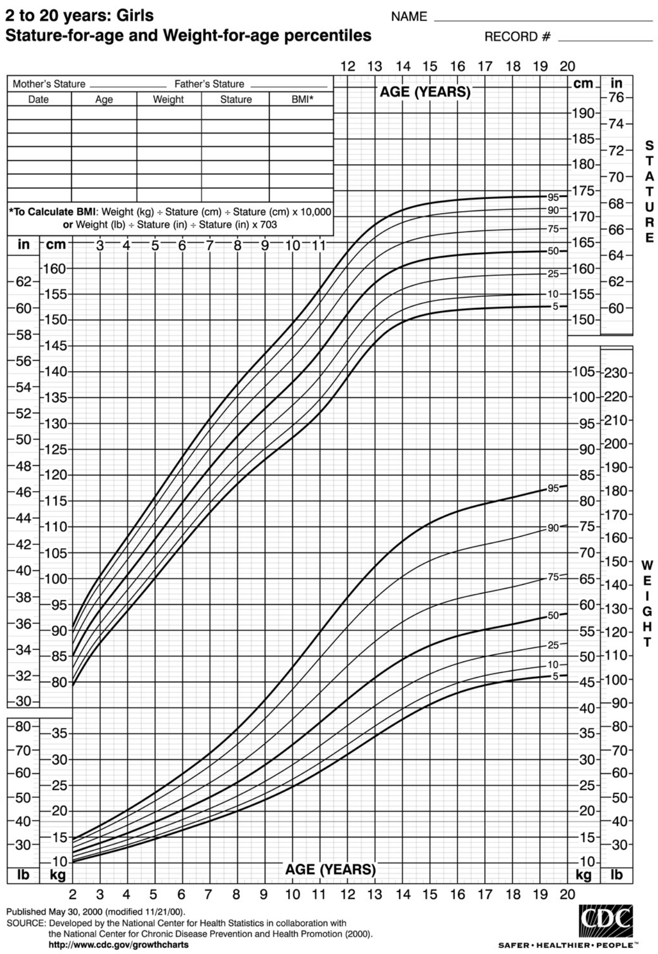
a. Describe what the graph indicates about the percentage of immigrants in the United States. Refer to the concepts of *increasing*, *decreasing*, and *concavity* as appropriate.

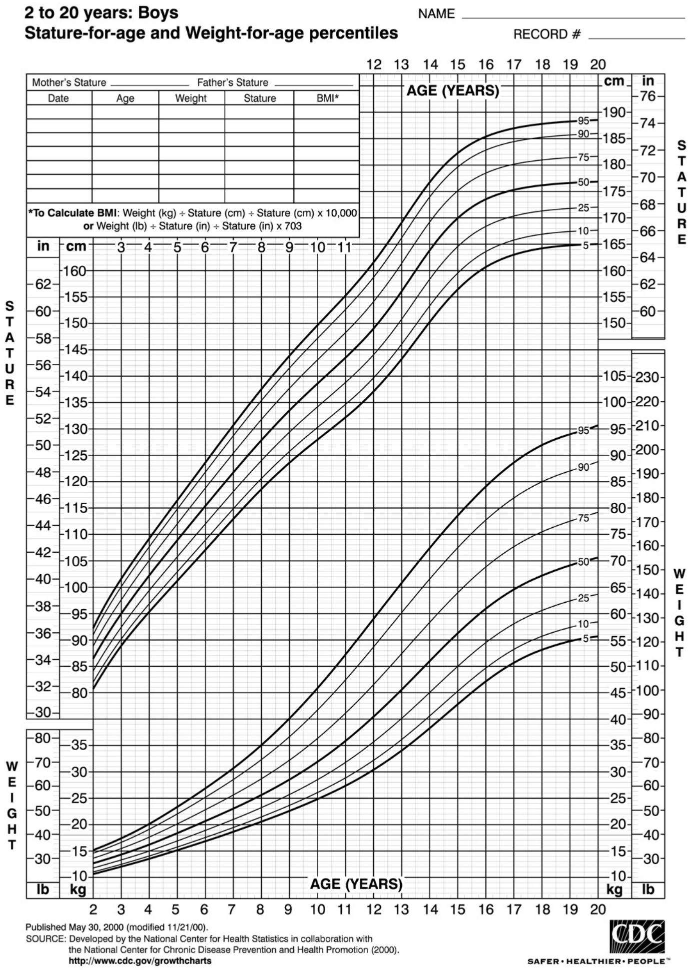
b. Give possible historical reasons for why there have been such drastic changes in the percentage of immigrants in the United States.

46. Growth Charts The U.S. Centers for Disease Control produces growth charts to help pediatricians and parents assess the health and growth patterns of children. (*Source:* **www.cdc.gov**) For each chart shown, write a description of what the chart indicates about boys and girls of various ages. Include a description of the nature of the function in terms of whether it is increasing or decreasing, its concavity, and inflection points.









■ STRETCH YOUR MIND

Exercises 47–50 are intended to challenge your understanding of rates of change and concavity of graphs.

47. Sketch the graph of a function f with all of the following properties:

$f(0) = 3$.

$f(x)$ is decreasing for $0 \le x \le 4$.

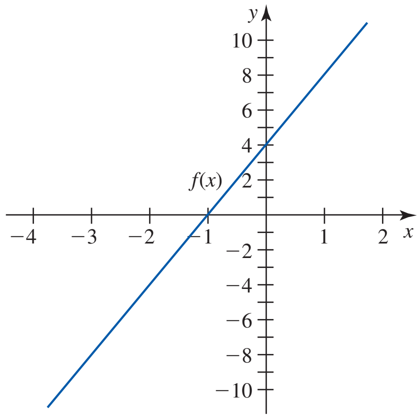
$f(x)$ is increasing for $4 \le x \le 6$.

$f(x)$ is decreasing for $x > 6$.

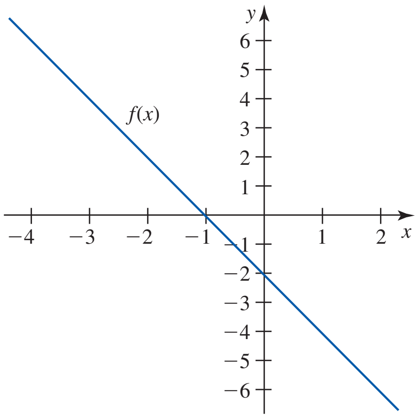
$f(x) \to 11$ as $x \to \infty$.

For Exercises 48–49, the graph of f is provided. Sketch the graph of a function g in which the output values of $g(x)$ represent the instantaneous rate of change for $f(x)$ at each value of x.

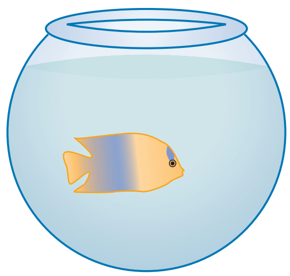
48.



49.



50. Water is poured into a spherical fish bowl at a constant rate.



a. Sketch a graph of the function that best models the height of water in the spherical bowl as a function of the amount of water (volume) in the bowl.

b. Discuss what the concavity of the graph represents in this context.

SECTION 4.2 Modeling with Quadratic Functions

LEARNING OBJECTIVES

- Recognize the relationship between a quadratic equation and its graph
- Use second differences to determine if a quadratic equation represents a data set
- Construct and use quadratic models to predict unknown results and interpret these findings in a real-world context

GETTING STARTED

Successful business executives know how to make money in an ever-changing marketplace. For companies in the retail business, customer loyalty is one of the hallmarks of financial success. The *quantity discount*—where an item's price is reduced for customers who buy large quantities of the item—is one marketing strategy used by successful retailers to attract and retain customers.

In this section we investigate quadratic functions and their applications. We look at equations, data tables, and graphs of these functions. We determine how to find a quadratic function from a data table by using quadratic regression. We apply these concepts to a number of real-world situations, including the business strategy of the quantity discount.

■ The Quadratic Equation in Standard Form

In 2007, National Pen Company offered the promotion shown in Table 4.9 for the Dynagrip Pen in their online catalog.

Table 4.9

Paid Order Size	Price per Pen
50	\$0.79
100	\$0.77
150	\$0.75
250	\$0.73
500	\$0.71

Source: www.pens.com

Notice that as the order size increases, the company reduces the price per pen, employing the quantity discount strategy. When using such pricing strategies, the company must be aware of the impact the price reductions will have on its revenue.

From the table we see that the first \$0.02 price reduction is offered for an order of 50 additional pens (100 pens), and the next for an order of 50 more (150 pens). We would expect each following discount to be given for orders of 50 more each time, but the table shows this is not the case. The next discount is given instead for an order of 100 more than the previous order (250 pens) and the final discount for an order of 250 more (500 pens).

To see why the company does not offer the \$0.02 discount for each additional 50 pens ordered, we begin by considering the hypothetical pricing structure shown in Table 4.10, which assumes that for every increase of 50 pens in the order, the price per pen decreases by \$0.02. To make our calculations simpler, we rewrite the pricing structure in terms of 50-pen sets and price per 50-pen set. See Table 4.11.

Table 4.10

Paid Order Size	Price per Pen
50	\$0.79
100	\$0.77
150	\$0.75
250	\$0.71
500	\$0.61

Table 4.11

Number of 50-Pen Sets x	Price per Set p
1	\$39.50
2	\$38.50
3	\$37.50
5	\$35.50
10	\$30.50

The revenue from pen sales is the product of the price per 50-pen set and the number of sets sold. That is,

$$R(x) = px$$

where R is the revenue (in dollars), p is the price (in dollars per set), and x is the number of sets sold. We update our data table to show the revenue generated (Table 4.12) and plot revenue as a function of the number of 50-pen sets in Figure 4.10.

Table 4.12

Number of 50-Pen Sets x	Price per Set p	Revenue R
1	\$39.50	\$39.50
2	\$38.50	\$77.00
3	\$37.50	\$112.50
5	\$35.50	\$177.50
10	\$30.50	\$305.00

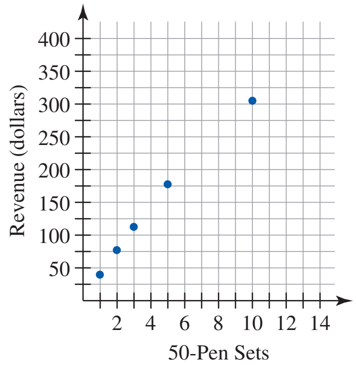


Figure 4.10

So far, it appears that this pricing strategy will continue to increase the company's revenue. But this will not always be the case. Since the price per set decreases by \$1 for each additional set sold, 15 sets will sell for \$25.50 each, 20 sets will sell for \$20.50 each, 30 sets will sell for \$10.50 each, and 40 sets will sell for \$0.50 each. As shown in Figure 4.11, the corresponding revenues are \$382.50, \$410.00, \$315.00, and \$20.00.

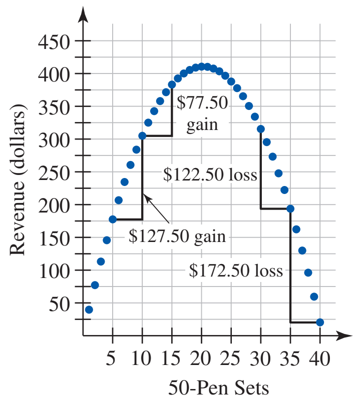


Figure 4.11

What happened? Between 0 and 20 sets revenue is increasing although at a decreasing rate. Beyond 20 sets, revenue is decreasing and at an increasingly rapid rate. So at what point should the company stop offering the additional \$0.02 per pen discount? It appears from the graph that the additional discount should not be offered for orders consisting of more than 20 50-pen sets. Since revenue does not take into account the cost of producing the pens, the company may need to further adjust the price reduction limit when additional factors are taken into consideration.

Notice that the revenue function depends on price and quantity sold. Is there a way to write revenue as a function of quantity sold only? Yes. We return to the pen-set hypothetical pricing data table, repeated here as Table 4.13.

Table 4.13

Number of 50-Pen Sets x	Price per Set P
1	\$39.50
2	\$38.50
3	\$37.50
5	\$35.50
10	\$30.50

Recall that each 1-set increase corresponds with a \$1 decrease in price per set. Since the rate of change in price is constant, price as a function of sets is a linear function with slope $m = -1$. We substitute in (1, 39.50) to determine the initial value.

$$p(x) = -1x + b$$
$$39.50 = -1(1) + b$$
$$b = 40.50$$

Therefore,

$$p(x) = -x + 40.50$$

We can now write the revenue function $R(x) = px$ exclusively in terms of x.

$$\begin{aligned} R(x) &= px \\ &= (-x + 40.50)x \\ &= -x^2 + 40.50x \end{aligned}$$

The graph of the revenue function is the parabola shown in Figure 4.12. A function equation of this form is called a *quadratic equation in standard form.*

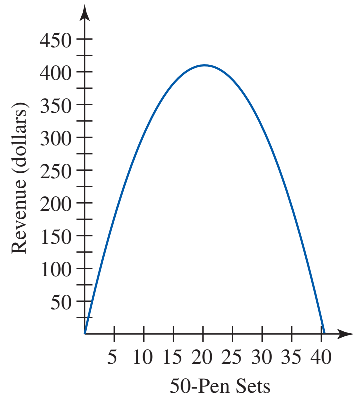


Figure 4.12

Quadratic Equation in Standard Form

A function equation of the form

$$y = ax^2 + bx + c$$

with constants a, b, and c and with $a \neq 0$ is called a **quadratic equation in standard form**. The graph of a quadratic equation is a **parabola**.

To discover the meaning of the *parameters* a, b, and c in the quadratic equation $y = ax^2 + bx + c$, let's use the revenue function $R(x) = -x^2 + 40.50x$ and first determine the units of the parameters. In this equation, $a = -1$, $b = 40.50$, and $c = 0$. The output of the revenue function is *dollars*, so the units of each of the terms of the quadratic equation must be *dollars*. That is, the units of $-1x^2$, $40.50x$, and 0 must all be *dollars*. But the input of the function is *50-pen sets*, so the units of x are pen sets, not dollars. Thus the units of the coefficients of each term must compensate for this. We have

$$\begin{aligned} (\text{units of } a)x^2 &= \text{dollars} \\ (\text{units of } a)(\text{pen sets})^2 &= \text{dollars} \\ \text{units of } a &= \frac{\text{dollars}}{(\text{pen sets})^2} \end{aligned} \qquad \begin{aligned} (\text{units of } b)x &= \text{dollars} \\ (\text{units of } b)(\text{pen sets}) &= \text{dollars} \\ \text{units of } b &= \frac{\text{dollars}}{\text{pen set}} \end{aligned}$$

So the units of a are *dollars per pen set squared*, the units of b are *dollars per pen set*, and the units of c are *dollars*. We now use this information to help define the meanings of the parameters.

The c is the initial value of the function. That is, $f(0) = c$. In this case, $f(0) = 0$, so the vertical intercept of the function's parabola is (0, 0). In other words, when 0 pens have been sold, 0 dollars of revenue have been earned.

From its units (dollars per pen set), we know that b is a rate of change. But what does it represent? Let's evaluate $R(x) = -x^2 + 40.50x$ at $x = 0$ and $x = h$, where h is some value of x "close" to 0. We have $R(0) = 0$ and $R(h) = -h^2 + 40.50h$. Let's calculate the average rate of change between these two values.

$$\begin{aligned} \text{average rate of change} &= \frac{R(h) - R(0)}{h - 0} \; \frac{\text{dollars}}{\text{pen sets}} \\ &= \frac{(-h^2 + 40.50h) - 0}{h} \\ &= -h + 40.50 \text{ dollars per pen set} \end{aligned}$$

Observe as the value of h gets close to 0, the average rate of change approaches 40.50. Thus the *instantaneous* rate of change at $x = 0$ is 40.50 dollars per pen set. On the graph, the slope of the parabola at the vertical intercept is 40.50. In other words, when 0 pen sets have been sold, revenue is increasing at a rate of 40.50 dollars per pen set.

The value of a relates to how much the rate of change itself is changing. As is clear from the graph and our previous discussion, the rate of change is not constant and thus has its own rate of change. The value for a is half of the rate of change in the rate of change. (We'll show why later.) In this case, $a = -1$ so the rate of change in the rate of change is -2 dollars per pen set for each pen set. That is, the revenue per pen set decreases by 2 dollars for each additional pen set sold. Since the revenue per pen set is itself decreasing, the parabola is concave down.

We summarize our conclusions as follows.

The Meaning of a, b, and c in a Quadratic Equation

In the quadratic equation $y = ax^2 + bx + c$, the parameters a, b, and c represent the following:

a = one half of the rate of change in the rate of change

b = the instantaneous rate of change at $x = 0$ (initial rate of change)

c = the value of y at $x = 0$ (initial value)

EXAMPLE 1 ■ Interpreting the Meaning of the Parameters in a Quadratic Equation

In 1962, Sam Walton opened the first Walmart store. The chain grew rapidly to 24 stores by 1967. In that year, the company generated \$12.6 million in sales. Today Walmart is one of the world's premier retailers, generating \$312 billion in net sales in 2006. (*Source:* walmartstores.com) Based on data from 1996 to 2006, the net sales of Walmart can be modeled by the quadratic function

$$s(t) = 0.8636t^2 + 14.39t + 84.72 \text{ billion dollars}$$

where t is the number of years since the end of 1996. Explain the meaning of the parameters in the model in their real-world context. Then explain the graphical meaning of the parameters.

Solution Since $c = 84.72$ billion dollars, the model estimates that Walmart earned 84.72 billion dollars in revenue in 1996. Since $b = 14.39$ billion dollars per year, the model estimates that at the end of 1996, Walmart sales revenue was increasing at a rate of 14.39 billion dollars per year. Since $a = 0.8636$, the model estimates that the increase in revenue is increasing at a rate of 1.73 ($2 \cdot 0.8636 \approx 1.73$) billion dollars per year each year. For example, since revenue was increasing at a rate of 14.39 billion dollars per year in 1996, we expect that in 1997 revenue will be increasing at a rate of about 16.12 billion dollars per year ($14.39 + 1.73 = 16.12$).

Since $c = 84.72$, the vertical intercept of the parabola is (0, 84.72). Since $b = 14.39$, the slope of the parabola at the vertical intercept is 14.39. Since $a = 0.8636$ is a positive number, the rate of change is itself increasing, which means the parabola is concave up. See Figure 4.13.

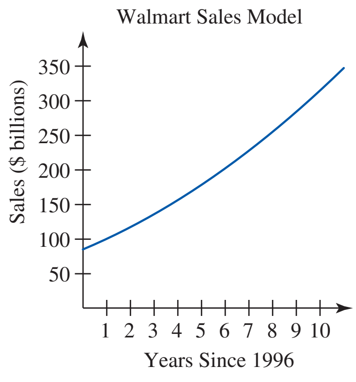


Figure 4.13

Although the model in Example 1 is quadratic, only a portion of the parabola is used to model the data set. This is often the case with quadratic models of real-world data sets.

Determining If a Data Set Represents a Quadratic Function

In Section 4.1 we defined first differences to be the set of differences in outputs for equally spaced inputs, and second differences to be the set of differences in the first differences. We also saw that since linear functions have a constant rate of change, they have constant first differences.

To see how we can use successive differences to determine if a data set represents a quadratic function, let's return to the pen-set revenue function. First we reconstruct the table of values for the function and then calculate the first differences. See Table 4.14.

Table 4.14

x	$R(x) = -x^2 + 40.50x$	First Differences
0	0	
		39.50
1	39.50	
		37.50
2	77.00	
		35.50
3	112.50	

From Table 4.14 we note that the first differences in the quadratic function are not constant. But we see from our calculations in Table 4.15 that the *second* differences of the function *are* constant.

Table 4.15

x	$R(x) = -x^2 + 40.50x$	First Differences	Second Differences
0	0		
		39.50	
1	39.50		−2
		37.50	
2	77.00		−2
		35.50	
3	112.50		

We can also see this by looking at Figure 4.14, which shows that the first differences decrease by 2 for each 1-unit increase in the number of 50-pen sets.

Another way to observe the constant second differences is to graph the rate-of-change function, as shown in Figure 4.15. That function is a line with a slope of −2 and a vertical intercept of 40.50, the initial rate of change in the revenue function.

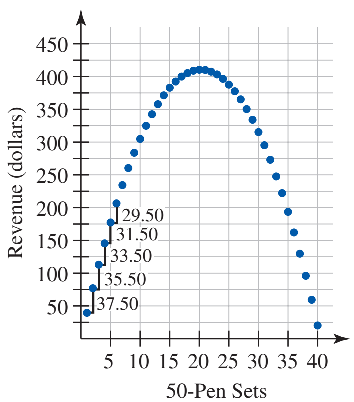


Figure 4.14

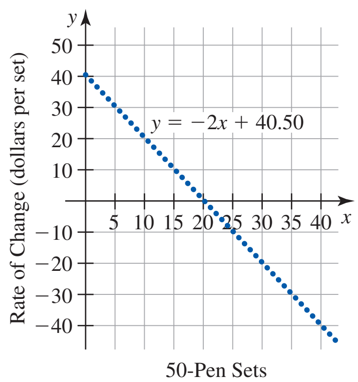


Figure 4.15

Will the second differences always be constant for a quadratic function? That is, will the rate of change always be changing at a constant rate? We investigate this idea using the quadratic function $y = ax^2 + bx + c$. We evaluate this function at five input values, each spaced 1 unit apart. (Note: Each value of y in Table 4.16 has been simplified algebraically.)

Table 4.16

x	$y = ax^2 + bx + c$	First Differences	Second Differences
x_1	$y = ax_1^2 + bx_1 + c$		
		$2ax_1 + (a + b)$	
$x_1 + 1$	$y = ax_1^2 + (2a + b)x_1 + (a + b + c)$		$2a$
		$2ax_1 + (3a + b)$	
$x_1 + 2$	$y = ax_1^2 + (4a + b)x_1 + (4a + 2b + c)$		$2a$
		$2ax_1 + (5a + b)$	
$x_1 + 3$	$y = ax_1^2 + (6a + b)x_1 + (9a + 3b + c)$		$2a$
		$2ax_1 + (7a + b)$	
$x_1 + 4$	$y = ax_1^2 + (8a + b)x_1 + (16a + 4b + c)$		

So yes, the second differences are always constant in a quadratic function and do not depend on the value of x_1. Thus, as stated earlier, we can use second differences to determine if a data set represents a quadratic function.

Difference Properties of Quadratic Functions Given in Tables

For equally spaced input values, quadratic functions have *linear* first differences and *constant* second differences.

EXAMPLE 2 ■ Determining If a Table Represents a Quadratic Function

A golden rectangle is said to be the most aesthetically pleasing of all rectangles. Artists and architects have incorporated the shape into drawings, buildings, and works of art such as the canvas of Salvador Dali's *The Sacrament of the Last Supper* shown here. Table 4.17 shows the width and area in centimeters (cm) of various golden rectangles.

Determine if the area of a golden rectangle is a quadratic function of its width.

Table 4.17

Width (cm)	Area (cm^2)
10	161.80
20	647.21
30	1456.23
40	2588.85
50	4045.08

Solution We construct Table 4.18 to calculate first and second differences.

Table 4.18

Width (cm)	Area (cm^2)	First Differences	Second Differences
10	161.80		
		485.41	
20	647.21		323.61
		809.02	
30	1456.23		323.61
		1132.62	
40	2588.85		323.61
		1456.23	
50	4045.08		

Since the second differences are constant, the area of a golden rectangle is a quadratic function of its width. (In the exercises at the end of the chapter, we will further investigate the properties of golden rectangles.)

Using Quadratic Regression to Find a Quadratic Function of Best Fit

Many real-world data sets have second differences that are not constant but are nearly so. Such data sets can still be modeled with a quadratic function. Just as we used linear regression to find a line of best fit in Chapter 2, we can use quadratic regression to find the quadratic function that best fits a data set. Although the model of best fit will not pass through every point of the data set, it is often sufficiently accurate to describe the relationship between the values of the data set.

EXAMPLE 3 ■ Using Quadratic Regression to Find a Model of Best Fit

The number of hospital beds in the United States has been *decreasing* since 1990 despite the fact that the population of the United States has been increasing. Table 4.19 shows the number of hospital beds in the United States by year.

Use quadratic regression to find the quadratic function model for the data set. Then explain the meaning of the function parameters.

Table 4.19

Years Since 1990 t	Hospital Beds (in thousands) h
0	1213
4	1128
8	1013
10	984
12	976
14	956

Source: Health, United States, 2006

Solution The full development of quadratic regression is beyond the scope of this text. Instead, we use a graphing calculator and the Technology Tip at the end of this section. We determine that the quadratic equation that best models the number of hospital beds in the United States is $H(t) = 0.9002t^2 - 31.67t + 1220$ thousand beds, where t is the number of years since 1990.

According to the model, there were 1220 thousand beds in 1990 and at that time the number of beds was decreasing at a rate of 31.57 thousand beds per year. However, the rate of change will increase (become less negative) by 1.8004 thousand beds per year each year ($2(0.9002) = 1.8004$).

EXAMPLE 4 ■ Using Quadratic Regression to Find a Model of Best Fit

Use quadratic regression to find the quadratic function that best models the data set in Table 4.20, the per capita spending on prescription drugs in the United States. Interpret the meaning of the parameters of the model. Then predict the per capita spending for 2006.

Table 4.20

Years Since 1990 t	Per Capita Spending on Prescription Drugs (dollars) P
5	224
9	368
10	423
11	485
12	552
13	605

Source: Statistical Abstract of the United States, 2006, Table 121

Solution Using a graphing calculator and the Technology Tip at the end of this section, we determine that the quadratic model is $P(t) = 2.76t^2 - 1.11t + 159$.

According to the model, the per capita prescription drug spending was \$159 in 1990 and was decreasing at a rate of \$1.11 per year. The rate of change itself was increasing at a rate of \$5.52 per year each year.

Since the year 2006 corresponds with $t = 16$, we have

$$\begin{aligned} P(16) &= 2.76(16)^2 - 1.11(16) + 159 \\ &\approx 848 \end{aligned}$$

According to the model, per capita prescription drug spending in the United States was about \$848 in 2006.

EXAMPLE 5 ■ Using Quadratic Regression to Model a Data Set

Chipotle Mexican Grill achieved remarkable financial results between 2001 and 2005, as shown in Table 4.21.

Table 4.21

Year t	Franchise Royalties and Fees (\$1000s) f	Restaurant Sales (\$1000s) s
2001	267	131,331
2002	753	203,892
2003	1493	314,027
2004	2142	468,579
2005	2618	625,077

Source: Chipotle Mexican Grill, Inc., 2005 Annual Report, p. 24

Draw a scatter plot of restaurant sales as a function of franchise royalties and fees. If the data appears to be concave up or concave down, use quadratic regression to find the quadratic model that best fits the data. Then use the model to estimate restaurant sales when franchise royalties and fees reach \$3 million.

Solution The scatter plot shown in Figure 4.16 appears to be concave up. Using a graphing calculator and the Technology Tip at the end of this section, we determine that the quadratic model for sales (accurate to 4 significant digits) is

$$s(f) = 0.05196f^2 + 54.89f + 120{,}000 \text{ thousand dollars}$$

where f is the amount of franchise royalties and fees (in thousand dollars). To verify the accuracy of our work, we graph the model and the scatter plot together in Figure 4.17.

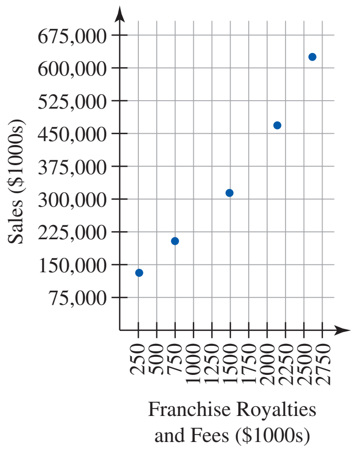


Figure 4.16

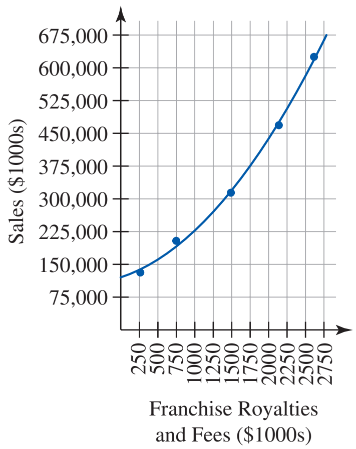


Figure 4.17

We are to forecast sales when franchise royalties and fees reach \$3 million, which is equivalent to 3000 thousand dollars. Since the franchise royalties and fees are given in thousands, we must evaluate $s(3000)$.

$$\begin{aligned} s(f) &= 0.05196f^2 + 54.89f + 120{,}000 \\ s(3000) &= 0.05196(3000)^2 + 54.89(3000) + 120{,}000 \\ &= 752{,}310 \\ &= 752{,}300 \text{ (accurate to 4 significant digits)} \end{aligned}$$

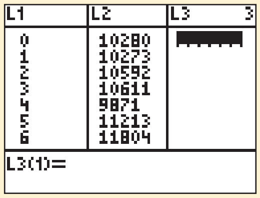
We estimate that when franchise royalties and fees reach \$3 million, sales will reach \$752.3 million.

SUMMARY

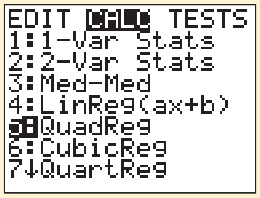
In this section you learned the standard form of the quadratic equation and discovered the meaning of its parameters. You also learned how to use first and second differences to determine if a data set represents a quadratic function. Additionally, you learned how to use quadratic regression to find the quadratic function of best fit for a data set and how to interpret the model in its real-world context.

TECHNOLOGY TIP ■ QUADRATIC REGRESSION

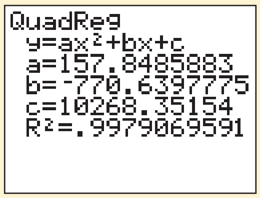
1. Enter the data using the Statistics Menu List Editor.



2. Bring up the Statistics Menu Calculate feature by pressing STAT and using the blue arrows to move to the `CALC` menu. Then select item `5:QuadReg`, and press ENTER.



3. If you want to automatically paste the regression equation into the Y = Editor so that you can easily graph the model, press the key sequence VARS: `Y-VARS: Function:` $\texttt{Y}_1$ and press ENTER. Otherwise press ENTER.



4.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, calculate the first and second differences of the data table. Then indicate whether the data represents a linear or quadratic function or neither.

1.

x	y
0	1
1	3
2	9
3	19
4	33

2.

x	y
2	−4
4	−16
6	−36
8	−64
10	−100

3.

x	y
5	2
10	4
15	6
20	8
25	10

4.

x	y
4	5
8	10
12	20
16	40
20	80

5.

x	y
−3	−1
0	−10
3	−1
6	26
9	71

SHOW YOU KNOW

6. Given that the quadratic function $s(t) = at^2 + bt + c$ represents the distance of a car from Orlando (in miles) after t hours of travel, explain what a, b, and c represent including units.
7. How can you tell if a data set with equally spaced inputs represents a quadratic function?
8. What does it mean to say that a data set has linear first differences?
9. What does a second difference indicate?
10. What is the relationship between the second difference of a quadratic equation and its associated graph?
11. In terms of a rate of change, what does it mean for a parabola to be concave down?
12. In terms of a rate of change, what does it mean for a parabola to be concave up?
13. A concave up parabola passes through the point (0, 3). From this information, what do you know about the parameters of $y = ax^2 + bx + c$?
14. In the equation $y = ax^2 + bx + c$, what is the *graphical* significance of the value of b?
15. In the equation $y = ax^2 + bx + c$, what is the *graphical* significance of the value of c?

Exercises 16–20 focus on the relationship between a quadratic model equation and the situation being modeled.

16. What do we mean when we say the "initial value" of a quadratic function model?
17. What do we mean when we say the "initial rate of change" of a quadratic function model?
18. If $a < 0$ in the quadratic model $y = ax^2 + bx + c$, what do we know about the rate of change of the model?
19. If $a > 0$ in the quadratic model $y = ax^2 + bx + c$, what do we know about the rate of change of the model?
20. Why can't $a = 0$ for a quadratic function model $y = ax^2 + bx + c$? Use the concept of rate of change in your explanation.

MAKE IT REAL

In Exercises 21–30, explain the real-world meaning of the parameters a, b, and c of the quadratic function model.

21. **Medicare Enrollees** Based on data from 1980 to 2004, the number of Medicare enrollees (in millions) can be modeled by

$$M(t) = -0.00472t^2 + 0.663t + 28.4$$

where t is the number of years since 1980. (***Source:*** **Modeled from *Statistical Abstract of the United States, 2006*, Table 132)**

22. **Prescriptions** Based on data from 1990 to 2003, the amount of money spent on prescription drugs (per capita) can be modeled by

$$P(t) = 2.889t^2 - 2.613t + 158.7$$

dollars, where t is the number of years since 1990. (***Source: Statistical Abstract of the United States, 2006*, Table 121)**

23. **Children in Madagascar** Based on data from 1990 to 2002, the number of children under 5 in Madagascar can be modeled by

$$C(t) = 1.046t^2 + 60.82t + 2152$$

thousand children, where t is the number of years since 1990. (***Source:*** **Modeled from World Health Organization data)**

24. **Malaria Cases** Based on data from 1998 to 2002, the number of clinical malaria cases reported in children under 5 years of age in Ghana can be modeled by

$$C(t) = -140{,}281t^2 + 658{,}186t + 583{,}452$$

cases, where t is the number of years since 1998. (***Source:*** **www.afro.who.int)**

25. **USAA Membership** Based on data from 2003 to 2007, the number of members of the USAA (an insurance and financial services company) can be modeled by

$$M(t) = 0.05t^2 + 0.15t + 5.0$$

million members, where t is the number of years since 2003. (***Source:*** **Modeled from USAA 2007 Report to Members, p. 21)**

26. **U.S. Population** Based on data from 1990 to 2004, the population of the United States can be modeled by

$$P(t) = -19.56t^2 + 3407t + 250{,}100$$

thousand people, where t is the number of years since 1990. (***Source:*** **Modeled from *Statistical Abstract of the United States, 2006*, Table 2)**

27. **Online School Enrollment** Based on data from 2003–2004 through 2006–2007, the number of students enrolled in the Arizona Virtual Academy can be modeled by

$$s(x) = 141.25x^2 + 358.75x + 318.75$$

students, where x is the number of years since 2003–2004. (***Source:*** **Modeled from Arizona Virtual Academy Fact Sheet)**

28. **Consumer Spending on Books** Based on data from 2004 to 2005 and projections for 2006 to 2009, the amount of money spent by consumers on books classified as *adult trade* can be modeled by

$$b(x) = -31.15x^2 + 556.1x + 14970$$

million dollars, where x is the number of years since 2004. (**Source:** Modeled from ***Statistical Abstract of the United States, 2007***, Table 1119)

29. U.S. Oil Production vs. Imports Based on data from 1985 to 2004, the difference between U.S. oil field production and net oil imports can be modeled by

$$b(t) = 4.294t^2 - 278.3t + 2251$$

million barrels, where t is the number of years since 1985. (**Source:** Modeled from ***Statistical Abstract of the United States, 2007***, Table 881)

30. Yogurt Production Based on data from 1997 to 2005, the amount of yogurt produced in the United States annually can be modeled by

$$y(x) = 14.99x^2 + 62.14x + 1555$$

million pounds, where x is the number of years since 1997. (**Source:** Modeled from ***Statistical Abstract of the United States, 2007***, Table 846)

In Exercises 31–35, use quadratic regression and a graphing calculator to find the quadratic function that best fits the data set. Then use the model to forecast the value of the function at the indicated point.

31. Live Births by Race

Years Since 1990 t	Total Live Births (in thousands) b	Live Births to Women Racially Classified as *White* (in thousands) w
0	1165	670
5	1254	785
9	1308	840
10	1347	866
11	1349	880
12	1366	904
13	1416	947
14	1470	983

***Source: Statistical Abstract of the United States, 2007*, Table 83**

Model *white* births as a function of total live births. How many *white* births will there be when live births reach 1500 thousand?

32. Abortions

Years Since 1985 x	Abortions (per 1000 live births) a
0	422
5	389
10	350
15	324
16	325
17	319

***Source: Statistical Abstract of the United States, 2007*, Table 96**

What was the abortion rate in 2005?

33. Manufacturing Employees

Years Since 2000 x	Computer and Electronic Products Industry Employees (in thousands) e
0	1820
2	1507
3	1355
4	1323
5	1320

***Source: Statistical Abstract of the United States, 2007*, Table 980**

How many computer and electronic products industry employees were there in 2009?

34. Manufacturing Employees

Years Since 1990 x	Aerospace Products and Parts Industry Employees (in thousands) e
0	841
10	517
12	470
13	442
14	442
15	456

***Source: Statistical Abstract of the United States, 2007*, Table 980**

How many aerospace products and parts industry employees were there in 2009?

35. NFL Player Salaries

Years Since 2000 x	NFL Player Average Salary ($1000s) s
0	787
1	986
2	1180
3	1259
4	1331
5	1400

***Source: Statistical Abstract of the United States, 2007*, Table 1228**

What was the NFL player average salary in 2008?

■ STRETCH YOUR MIND

Exercises 36–38 are intended to challenge your understanding of quadratic functions.

36. Show there does not exist a quadratic function that passes through all of the following points: (0, 4), (3, 13), (10, 34).

37. A classmate claims that the following table has a constant second difference of 8. Do you agree? Explain.

x	y
0	-1
2	9
4	27
8	53
12	87

38. The vertex of a certain concave down parabola is (0, 5). Explain as completely as possible what you know about the parameters of its equation $y = ax^2 + bx + c$.

SECTION 4.3 Forms and Graphs of Quadratic Functions

LEARNING OBJECTIVES

- Recognize and use the vertex, standard, and factored forms of quadratic functions
- Determine the vertex, horizontal intercepts, and vertical intercept of a quadratic function from its equation, data table, or graph
- Use the quadratic formula to solve real-world problems

GETTING STARTED

The rectangle is said to be the most common geometric shape we encounter in our daily lives. Whether in art or architecture, the properties of rectangles are fascinating.

Numerous applications of mathematics are represented using the concept of rectangles. The fact that the area of a rectangle may be written as a quadratic function of the length of its shortest side has relevance in settings seemingly unrelated to geometry.

In this section we introduce several forms of the quadratic equation. We also discuss the parabolic graphs of quadratic functions. By knowing the concavity, horizontal and vertical intercepts, and the vertex of a parabola, we are able to better understand the meaning of quadratic function models such as that for a rectangle.

■ Vertex Form of a Quadratic Function

In Section 4.2 we learned the standard form of a quadratic function, $y = ax^2 + bx + c$, with initial value c and initial rate of change b. We also noted that a was half of the rate of change in the rate of change. Quadratic functions may also be written in *vertex form.* This form is especially useful for graphing and relies heavily on the concept of function transformations.

EXAMPLE 1 ■ Constructing a Quadratic Equation in Vertex Form

A square is to be cut out of the middle of a 12-inch by 12-inch matting board to make a frame. The mat frame will be placed over a square picture and should overlap the picture by 0.5 inch on each side, as shown in Figure 4.18.

Write a quadratic equation in vertex form that represents the area of the mat frame after the square in the middle is removed. Then use the equation to calculate the mat frame area for a 6-inch by 6-inch picture and an 8-inch by 8-inch picture. Finally, graph the parabolas representing the area of any x-inch by x-inch picture and the area of the corresponding mat frame.

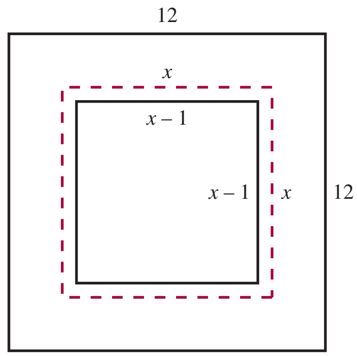


Figure 4.18

Solution Since both sides of the matting board are 12 inches long, the area of the mat before the center is removed is given by

$$\begin{aligned} M &= 12 \times 12 \\ &= 144 \text{ square inches} \end{aligned}$$

The area of the picture is $P = x^2$. The square opening of the mat is to overlap a picture by 0.5 inch on each side. That is, the length of each side of the square opening must be reduced by 1 inch (0.5 inch from each side). The area of the square that is removed from the center of the mat is given by

$$C = (x - 1)^2$$

The area of the mat frame will be the difference in the area of the original mat and the area of the square that is removed. That is,

$$\begin{aligned} A &= M - C \\ &= 144 - (x - 1)^2 \\ &= -(x - 1)^2 + 144 \end{aligned}$$

So the area of the mat frame is $A(x) = -(x - 1)^2 + 144$. A quadratic function written this way is said to be in **vertex form**.

Using the function, we find that the area of the mat frame for a 6-inch by 6-inch picture is

$$\begin{aligned} A(6) &= -(6 - 1)^2 + 144 \\ &= -25 + 144 \\ &= 119 \text{ square inches} \end{aligned}$$

The area of the mat frame for an 8-inch by 8-inch picture is

$$\begin{aligned} A(8) &= -(8 - 1)^2 + 144 \\ &= -49 + 144 \\ &= 95 \text{ square inches} \end{aligned}$$

The graphs representing the area of any x-inch by x-inch square picture and its corresponding mat frame are shown in Figure 4.19.

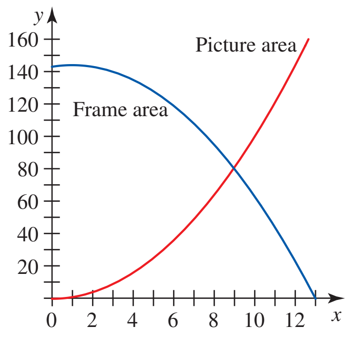


Figure 4.19

Vertex Form of a Quadratic Function

The equation of a parabola written in the form

$$y = a(x - h)^2 + k$$

with $a \neq 0$ is said to be in **vertex form**. The point (h, k) is called the **vertex** of the parabola.

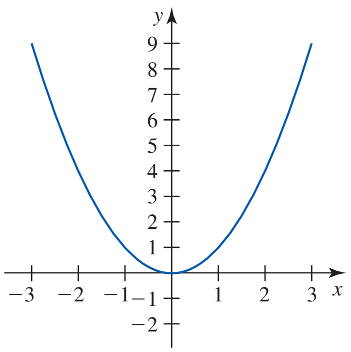


Figure 4.20

The **vertex** of a parabola is the point on the graph where the rate of change is equal to zero. For the basic quadratic function, $y = x^2$ (which is equivalent to $y = x^2 + 0x + 0$), we know the rate of change is equal to zero at (0, 0) since the initial value is $c = 0$ and the initial rate of change is $b = 0$. From the graph of the function shown in Figure 4.20, we see the vertex is the "turning point" of the graph. To the left of the vertex, the graph is decreasing. To the right of the vertex, the graph is increasing. In terms of a rate of change, the rate of change is negative to the left of the vertex, 0 at the vertex, and positive to the right of the vertex.

The minimum or maximum value of the function occurs at the vertex of the parabola.

Maximum and Minimum Values of a Quadratic Function

- The *maximum* value of a *concave down* parabola occurs at the vertex. A concave down parabola does not have a minimum value.
- The *minimum* value of a *concave up* parabola occurs at the vertex. A concave up parabola does not have a maximum value.

Thus when working with quadratic functions, the statement "find the maximum (minimum) value of the function" is equivalent to "find the y-coordinate of the vertex."

How does the graph of $y = a(x - h)^2 + k$ compare to $y = x^2$? From our understanding of transformations, we know that a vertically stretches or compresses the graph of $y = x^2$ by a factor of a. If $a < 0$, the graph will also be reflected vertically about the horizontal axis. Recall also that h shifts the graph horizontally $|h|$ units and k shifts the graph vertically $|k|$ units. Table 4.22 gives several quadratic functions in vertex form and the corresponding effects of a, h, and k.

Table 4.22

Equation	Effect of a	Effect of h	Effect of k
$y = x^2$	none	none	none
$y = (x - 1)^2$	none	shift right 1 unit	none
$y = 3(x + 2)^2 - 8$	vertically stretch by a factor of 3	shift left 2 units	shift downward 8 units
$y = -x^2$	reflect about horizontal axis	none	none
$y = -x^2 + 4$	reflect about horizontal axis	none	shift upward 4 units
$y = -3(x - 2)^2 + 6$	reflect about horizontal axis, vertically stretch by a factor of 3	shift right 2 units	shift upward 6 units

Note: When graphing a function from its equation, *reflections, stretches, and compressions must be done* ***before*** *horizontal and vertical shifts.*

EXAMPLE 2 ■ **Graphing a Quadratic Equation in Vertex Form by Hand**

Graph the function $y = -0.5(x + 2)^2 + 4$ by plotting points.

Solution The vertex is $(-2, 4)$. We create Table 4.23, a table of values for the quadratic equation $y = -0.5(x + 2)^2 + 4$ by selecting x values near $x = -2$. We then plot the points and connect them with a smooth curve, as shown in Figure 4.21.

Table 4.23

x	y
-5	-0.5
-4	2
-3	3.5
-2	4
-1	3.5
0	2
1	-0.5

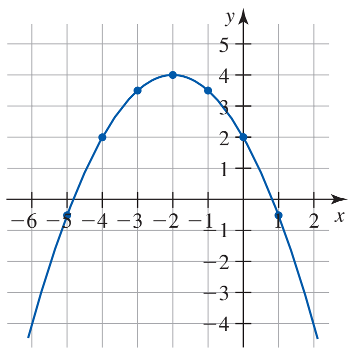


Figure 4.21

JUST IN TIME ■ THE ABSOLUTE VALUE FUNCTION, $|x|$

The absolute value function of a number is the distance between the number and 0. For example, $|-5| = 5$ and $|5| = 5$. The absolute value function is formally defined as a *piecewise* function:

$$|x| = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$$

This notation says if $x \geq 0$, the function value is equal to x but if $x < 0$, the function value is equal to $-x$. We'll use this formal definition to evaluate $|-5|$.

$$|-5| = -(-5) \text{ since } -5 < 0$$
$$|-5| = 5$$

As a sort of shorthand, we sometimes write $|x| = \pm x$.

JUST IN TIME ■ $\sqrt{x^2}$

Is $\sqrt{x^2} = x$? Consider the following table of values

x	x^2	$\sqrt{x^2}$
−2	4	2
−1	1	1
0	0	0
1	1	1
2	4	2

Observe that $\sqrt{x^2} = -x$ when $x < 0$ and $\sqrt{x^2} = x$ when $x \geq 0$. That is,
$\sqrt{x^2} = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \geq 0 \end{cases}$, which is equivalent to $\sqrt{x^2} = |x|$.

■ Finding the Horizontal Intercepts of a Quadratic Function in Vertex Form

We can determine the location of the horizontal intercepts of a quadratic function by setting the vertex form equal to zero and solving.

$$0 = a(x - h)^2 + k$$
$$-a(x - h)^2 = k$$
$$(x - h)^2 = -\frac{k}{a}$$
$$\sqrt{(x - h)^2} = \sqrt{-\frac{k}{a}}$$
$$|x - h| = \sqrt{-\frac{k}{a}}$$
$$\pm(x - h) = \sqrt{-\frac{k}{a}}$$

The plus-or-minus sign indicates that this breaks out into two separate cases as follows:

$$x - h = \sqrt{-\frac{k}{a}} \qquad\qquad -(x - h) = \sqrt{-\frac{k}{a}}$$

$$x = h + \sqrt{-\frac{k}{a}} \qquad\qquad x - h = -\sqrt{-\frac{k}{a}}$$

$$x = h - \sqrt{-\frac{k}{a}}$$

Combining both results, we have $x = h \pm \sqrt{-\frac{k}{a}}$. The horizontal intercepts lie $\sqrt{-\frac{k}{a}}$ units to the left and to the right of the x-coordinate of the vertex, h.

Horizontal Intercepts of a Quadratic Function in Vertex Form

The horizontal intercepts of a quadratic function

$$y = a(x - h)^2 + k$$

with $a \neq 0$ occur at $x = h \pm \sqrt{-\frac{k}{a}}$ provided $-\frac{k}{a} > 0$. If $-\frac{k}{a} < 0$, the parabola does not have any horizontal intercepts.

EXAMPLE 3 ■ **Modeling a Real-World Situation with a Quadratic Function**

As of 2011, the highest jump (68 inches) by a dog was achieved by Cinderella May (a greyhound) at the Purina Dog Chow Incredible Dog Challenge show in 2006. (*Source:* www.guinessrecords.com)

Write a quadratic equation in vertex form to model the vertical height of the dog t seconds after leaping into the air. Determine the horizontal intercepts of the function and interpret what they mean in this context. Then graph the height function to verify your conclusions. (*Hint:* The rate of change in the vertical velocity is referred to as *acceleration due to gravity* and is approximately equal to −32 feet per second per second on Earth.)

Solution We are to write the function in the form $y = a(t - h)^2 + k$. Recall a is half of the rate of change in the rate of change. In this case, a is half of the acceleration due to gravity (−32 feet per second per second), or −16 feet per second per second. Since the maximum height occurs at the y-coordinate of the vertex, k is the maximum height obtained, 68 inches. However, before substituting the value for k into the equation, we need to convert the value from inches to feet.

$$68 \text{ inches} \cdot \frac{1 \text{ foot}}{12 \text{ inches}} = 5.67 \text{ feet}$$

Now we can write

$$y = -16(t - h)^2 + 5.67$$

We also know that at time $t = 0$, the dog was on the ground ($y = 0$). Substituting this point into the equation, we solve to find the value of h.

$$\begin{aligned} 0 &= -16(0 - h)^2 + 5.67 \\ 0 &= -16h^2 + 5.67 \\ 16h^2 &= 5.67 \\ h^2 &= 0.3542 \\ h &\approx 0.595 \end{aligned}$$

Thus the model for the height of the dog t seconds after leaping into the air is

$$y = -16(t - 0.595)^2 + 5.67$$

The graph of this function, shown in Figure 4.22, is a concave down parabola with vertex (0.595, 5.67). The horizontal intercepts are determined by

$$x = h \pm \sqrt{-\frac{k}{a}}$$

$$t = h \pm \sqrt{-\frac{k}{a}}$$

$$\approx 0.595 \pm \sqrt{-\frac{5.67}{(-16)}}$$

$$\approx 0.595 \pm 0.595$$

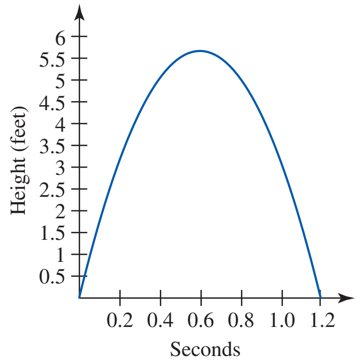


Figure 4.22

So $t = 0.595 - 0.595 = 0$ seconds or $t = 0.595 + 0.595 = 1.190$ seconds. In this context, the first horizontal intercept indicates that at 0 seconds the dog had not left the ground. The second horizontal intercept indicates that at 1.190 seconds the dog had returned to the ground after her leap.

EXAMPLE 4 ■ **Determining the Vertex Form of a Quadratic Equation from a Graph**

The graph in Figure 4.23 models the height of a ball that is propelled into the air at a rate of 32 feet per second from a height of 3 feet. Determine the vertex form of the quadratic equation represented by the graph. Then describe the practical meaning of the graph.

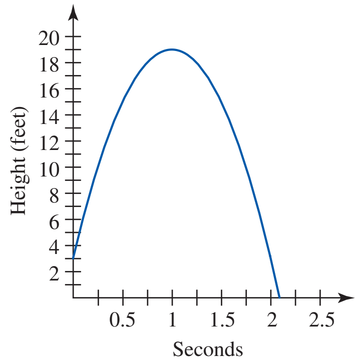


Figure 4.23

Solution From the graph, we can see the vertex occurs at (1, 19). Consequently, the vertex form of the quadratic equation will be $h(t) = a(t - 1)^2 + 19$. To determine the value of a, we must substitute in another point from the graph. For ease of computation, we choose (0, 3).

$$3 = a(0 - 1)^2 + 19$$

$$3 = a + 19$$

$$a = -16$$

So the vertex form of the quadratic equation is $h(t) = -16(t - 1)^2 + 19$.

From the graph, we can see the initial height of the ball is 3 feet above the ground. Its height increases for the first second although at a decreasing rate. At the end of the first second, the ball reaches its maximum height (19 feet). At that instant, its velocity (rate of change) is 0 feet per second. The ball then begins to fall toward the ground as a result of gravity. It strikes the ground shortly after 2 seconds of flight time.

■ Comparing the Vertex and Standard Forms

So far in this section we have focused exclusively on the *vertex form* of the equation of a parabola. We can readily see the relationship between the vertex form and standard form discussed in Section 4.2 by expanding the vertex form.

$$\begin{aligned} y &= a(x-h)^2 + k \\ &= a(x-h)(x-h) + k \\ &= a(x^2 - 2hx + h^2) + k \\ &= ax^2 - 2ahx + ah^2 + k \\ &= ax^2 + (-2ah)x + (ah^2 + k) \\ &= ax^2 + bx + c \qquad \text{Let } b = -2ah \text{ and } c = ah^2 + k. \end{aligned}$$

This conversion between forms yields a significant result:

$$b = -2ah$$

$$h = -\frac{b}{2a}$$

That is, the x-coordinate of the vertex is equal to $-\frac{b}{2a}$.

x-Coordinate of the Vertex

The x-coordinate of the vertex of a quadratic function in standard form, $y = ax^2 + bx + c$, is $x = -\frac{b}{2a}$.

EXAMPLE 5 ■ Finding the Vertex of a Parabola from an Equation

Although virtually nonexistent in the United States, malaria is one of the primary killers of children in Africa, claiming over a million lives annually. International efforts to help curb the spread of the disease have made some inroads yet there remains a great need for a long-term solution.

Based on data from 1998 to 2002, the number of clinical malaria cases in children under 5 years of age in Ghana can be modeled by

$$C(t) = -140{,}281t^2 + 658{,}186t + 583{,}452 \text{ cases}$$

where t is the number of years since the end of 1998.

According to the model, when will the number of cases reach a maximum? At that time, how many cases will there be?

Solution Since $-140{,}281 < 0$, this parabola will be concave down. The maximum value of the function will occur at the vertex.

$$\begin{aligned} t &= -\frac{b}{2a} \\ &= -\frac{658{,}186}{2(-140{,}281)} \\ &\approx 2.346 \end{aligned}$$

The number of cases reached a maximum roughly 2.346 years after the end of 1998, or 4 months into 2001. (The 2 gets us to the end of 2000. The 0.346 is a portion of the year 2001. We convert it to months as shown here: $0.346 \text{ year} \cdot \frac{12 \text{ months}}{1 \text{ year}} \approx 4 \text{ months}$.)

To determine the number of cases at that time, we evaluate the quadratic function at $t = 2.346$.

$$\begin{aligned} C(2.346) &= -140{,}281(2.346)^2 + 658{,}186(2.346) + 583{,}452 \\ &\approx 1{,}355{,}490 \end{aligned}$$

According to the model, in the 1-year period ending at the end of April 2001, there were 1,355,490 clinical malaria cases in Ghana in children under the age of 5 years.

Recall that a quadratic function in *vertex form* has horizontal intercepts at $x = h \pm \sqrt{-\frac{k}{a}}$. To find the horizontal intercepts for a quadratic function in *standard form* we write h and $-\frac{k}{a}$ in terms of a, b, and c. Recall $h = -\frac{b}{2a}$ and $c = ah^2 + k$. Thus,

$$c = ah^2 + k$$

$$k = c - ah^2$$

$$\frac{k}{-a} = \frac{c - ah^2}{-a}$$

$$-\frac{k}{a} = -\frac{c - ah^2}{a}$$

$$-\frac{k}{a} = \frac{ah^2 - c}{a}$$ **Distribute the negative sign and reorder terms in numerator.**

$$-\frac{k}{a} = h^2 - \frac{c}{a}$$ **Divide both terms in numerator by a.**

$$-\frac{k}{a} = \left(-\frac{b}{2a}\right)^2 - \frac{c}{a}$$ **since $h = -\frac{b}{2a}$**

$$-\frac{k}{a} = \left(\frac{b^2}{4a^2}\right) - \frac{c}{a}$$

$$-\frac{k}{a} = \frac{b^2}{4a^2} - \frac{c}{a} \cdot \frac{4a}{4a}$$ **Create a common denominator.**

$$-\frac{k}{a} = \frac{b^2 - 4ac}{4a^2}$$ **Write as a single fraction.**

Therefore,

$$x = h \pm \sqrt{-\frac{k}{a}}$$

$$= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{4a^2}}$$ **since $h = -\frac{b}{2a}$ and $-\frac{k}{a} = \frac{b^2 - 4ac}{4a^2}$**

$$= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{\sqrt{4a^2}}$$

$$= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

This result is referred to as the **quadratic formula**.

Quadratic Formula

The horizontal intercepts of $y = ax^2 + bx + c$ with $a \neq 0$ occur at

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

If the *discriminant* $b^2 - 4ac$ is negative, the function does not have any horizontal intercepts.

It is customary in many textbooks to write the quadratic formula in the equivalent form of

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

One drawback of this equivalent form is that it obscures the fact that the horizontal intercepts are equidistant from the x-coordinate of the vertex, $-\frac{b}{2a}$.

EXAMPLE 6 ■ Finding the Horizontal Intercepts of a Quadratic Function

The Space Needle is a popular tourist attraction in Seattle, Washington. The observation deck stands 520 feet above the ground. If a water droplet falls from the observation deck, how long will it take to reach the ground? The height of the water droplet above the ground t seconds after it is dropped is given by $h(t) = -16t^2 + 520$. (This model neglects air resistance.) Graph the function to verify your results.

Solution The droplet will be on the ground when $h = 0$, so we need to find the horizontal intercepts of the function. Using the quadratic formula, we have

$$\begin{aligned} t &= -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \\ &= -\frac{0}{2(-16)} \pm \frac{\sqrt{(0)^2 - 4(-16)520}}{2(-16)} \\ &= 0 \pm \frac{\sqrt{33280}}{-32} \\ &\approx \pm 5.7 \text{ seconds} \end{aligned}$$

In this context, it only makes sense to talk about a positive value of t. According to the model, the water droplet will strike the ground 5.7 seconds after it is dropped. The graph in Figure 4.24 verifies this result.

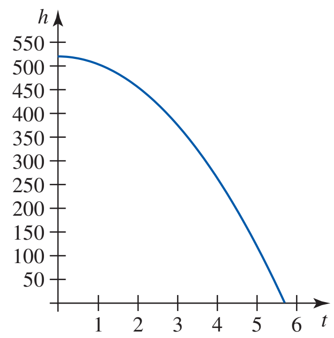


Figure 4.24

■ Factored Form of a Quadratic Function

When working with graphs, the *factored form* of a quadratic function is especially useful for determining the location of horizontal intercepts.

FACTORED FORM OF A QUADRATIC FUNCTION

The equation of a parabola written in the form

$$y = a(x - x_1)(x - x_2)$$

with $a \neq 0$ is said to be in **factored form**.

- The horizontal intercepts of the parabola are $(x_1, 0)$ and $(x_2, 0)$.
- The vertex of the parabola lies halfway between the horizontal intercepts at $x = \frac{x_1 + x_2}{2}$.
- The vertical intercept is $(0, ax_1x_2)$.

We saw earlier how the vertex and standard forms of a quadratic function were related. To see how the factored form is related to the standard form, we multiply out the factored form.

$$\begin{aligned} y &= a(x - x_1)(x - x_2) \\ &= a(x^2 - x_1x - x_2x + x_1x_2) \\ &= a[x^2 - (x_1 + x_2)x + x_1x_2] \\ &= ax^2 - a(x_1 + x_2)x + ax_1x_2 \\ &= ax^2 + bx + c \qquad \text{Let } b = -a(x_1 + x_2) \text{ and } c = ax_1x_2 \end{aligned}$$

We see that the vertical intercept c is the product of the horizontal intercepts and a, whereas b is the sum of the horizontal intercepts times the opposite of a.

EXAMPLE 7 ■ Working with a Quadratic Function in Factored Form

By summing the lengths of the sides, we can see that the rectangle in Figure 4.25 has a perimeter of 8 units. The equation for the area of this rectangle is

$$\begin{aligned} A &= (8 - 2x)(2x - 4) \\ &= -2(-4 + x)(2(x - 2)) \\ &= -4(x - 4)(x - 2) \end{aligned}$$

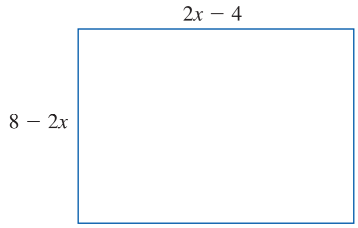


Figure 4.25

a. Describe the appearance of the graph based on its equation.

b. Graph the function to verify your answer to part (a).

c. Determine the practical domain for the area function.

d. Explain the practical meaning of the vertex.

Solution

a. Since $-4 < 0$, the parabola is concave down. It has horizontal intercepts at $x = 4$ and at $x = 2$. The vertex lies halfway between these horizontal intercepts, at $x = \frac{2 + 4}{2} = 3$. Evaluating the function at $x = 3$ yields

$$\begin{aligned} A(3) &= -4(3 - 4)(3 - 2) \\ &= 4 \end{aligned}$$

The vertex is (3, 4).

b. The graph in Figure 4.26 verifies the results.

c. We know the area of a rectangle cannot be negative. We see from the graph that $A \geq 0$ when $2 \leq x \leq 4$. So the practical domain of the function is $2 \leq x \leq 4$.

d. The vertex shows that the area is maximized at 4 square units when $x = 3$. When $x = 3$, the rectangle has sides of length 2 since $8 - 2(3) = 2$ and $2(3) - 4 = 2$.

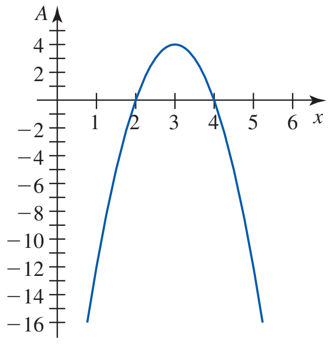
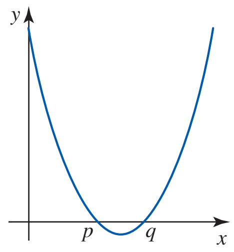


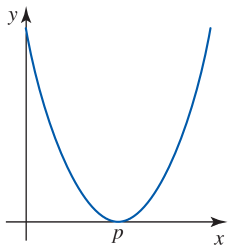
Figure 4.26

■ Factoring Quadratic Functions

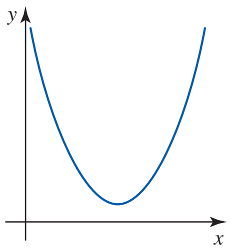
Once you have become skilled at reading and interpreting quadratic function graphs, you can quickly determine if a quadratic equation is factorable and how it factors. The horizontal intercepts of a quadratic function graph correspond directly with the factors of the quadratic function, as shown in Figure 4.27. Every factorable quadratic function can be written in one of the two forms shown in the figure.



(a) two x-intercepts
$y = a(x - p)(x - q)$



(b) one x-intercept
$y = a(x - p)^2$



(c) no x-intercepts
not factorable

Figure 4.27

EXAMPLE 8 ■ Determining the Factored Form of the Quadratic Equation from a Graph

Determine the factored form of the quadratic equation whose graph is shown in Figure 4.28.

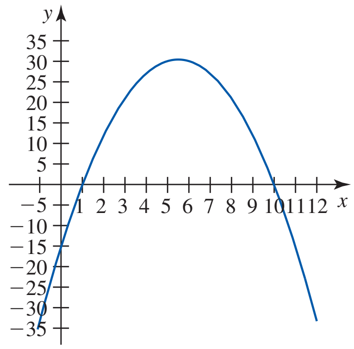


Figure 4.28

Solution We can see from the graph the horizontal intercepts are (1, 0) and (10, 0). So the factored form of the equation will be $y = a(x - 1)(x - 10)$. It also appears the graph passes through $(0, -15)$. We use this point to determine the value of a.

$$-15 = a(0 - 1)(0 - 10)$$
$$-15 = a(-1)(-10)$$
$$-15 = 10a$$
$$a = -1.5$$

The factored form of the quadratic equation is $y = -1.5(x - 1)(x - 10)$.

To determine how to factor $y = x^2 + bx + c$ algebraically, we observe $(x + p)(x + q) = x^2 + (p + q)x + pq$. If we let $b = p + q$ and $c = pq$, we have determined the factored form for every factorable quadratic function of the form $y = x^2 + bx + c$.

EXAMPLE 9 ■ Factoring a Quadratic Function Algebraically

Factor $y = x^2 + 8x + 12$ algebraically.

Solution We are looking for values for p and q that add together to equal 8 and multiply together to equal 12. We begin by creating a list of number pairs that multiply together to be 12. We then add together each number in the pair. See Table 4.24.

Table 4.24

p	q	Product	Sum
1	12	12	13
2	6	12	8
3	4	12	7
−1	−12	12	−13
−2	−6	12	−8
−3	−4	12	−7

Notice the factors that add to be 8 are 2 and 6. Therefore, $y = x^2 + 8x + 12$ in factored form is $y = (x + 2)(x + 6)$.

As we have seen throughout this section, the vertex, standard, and factored forms all have their advantages. Some situations are better suited for one form than another.

EXAMPLE 10 ■ Using Quadratic Function Graphs in a Real-World Context

A homeowner plans to enclose a portion of her backyard for a garden as depicted in Figure 4.29. The area of her yard is 2700 square feet. She has 70 feet of fencing and wants to enclose an area as large as possible for the garden. She will use the existing block wall for two sides of her garden but will need to fence the remaining two sides. What is the maximum area that she will be able to enclose with the 70 feet of fencing?

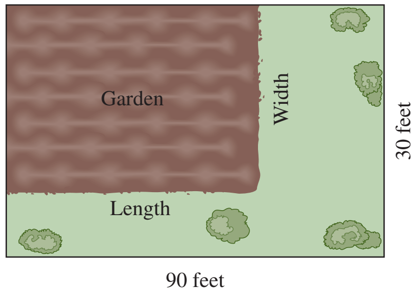


Figure 4.29

Solution We let l represent the length of the garden and w represent the width. The amount of fencing that will be used is $l + w = 70$. That is, $l = -w + 70$. The area of the garden is

$$\begin{aligned} A &= lw \\ &= (-w + 70)w \\ &= -(w - 70)w && \text{factored form} \\ &= -w^2 + 70w && \text{standard form} \end{aligned}$$

This is a concave down parabola. From the factored form, we can tell the vertex will be at $w = 35$ since 35 lies halfway between the horizontal intercepts of $w = 0$ and $w = 70$. We get the same result if we calculate $-\frac{b}{2a}$ in standard form.

$$\begin{aligned} w &= -\frac{b}{2a} \\ &= -\frac{70}{2(-1)} \\ &= 35 \end{aligned}$$

If the width is 35 feet, then the length of the garden is $l = -35 + 70 = 35$ feet. At first glance, it appears a 35-foot by 35-foot garden will maximize the area. However, if we look closely, we recognize that we cannot have a garden 35 feet wide because the width of the yard itself is 30 feet. The graph of the function in Figure 4.30 will help us see what is going on.

The red dashed line indicates the maximum width of the garden that is possible in the yard: 30 feet. From the graph, we see a width of 30 feet will yield the maximum area for gardens of width 0 feet to 30 feet. We calculate the corresponding length and area.

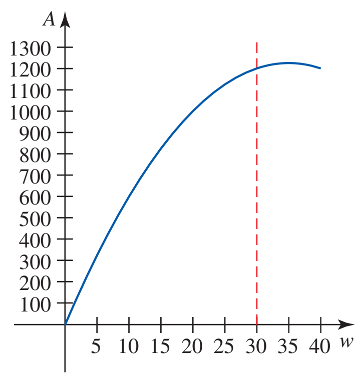


Figure 4.30

$$\begin{aligned} l &= -w + 70 & A &= lw \\ &= -30 + 70 & &= 40(30) \\ &= 40 & &= 1200 \end{aligned}$$

For this backyard, a garden of width of 30 feet and length of 40 feet will yield the maximum area that can be enclosed with 70 feet of fencing. The maximum area is 1200 square feet.

EXAMPLE 11 ■ Using Quadratic Function Graphs in a Real-World Context

A rancher plans to build three adjacent corrals for his flock of sheep. The adult rams (males) will be placed in the first corral, the lambs 4 months old and older will be placed in the second corral, and the adult ewes (females) and their lambs younger than four months old will be placed in the third corral. See Figure 4.31.

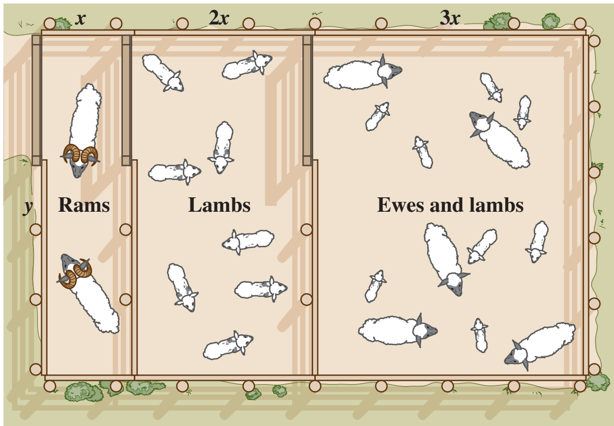


Figure 4.31

The perimeter of the ram corral must be 200 feet. The total enclosed area of all three corrals must be at least 6000 square feet. What is the minimum amount of fencing that will be needed?

Solution Since the perimeter of the ram corral is 200, we have $2x + 2y = 200$.

$$\begin{aligned} 2x + 2y &= 200 \\ x + y &= 100 \\ y &= 100 - x \end{aligned}$$

The combined width of the corrals is $x + 2x + 3x = 6x$. The combined area is

$$\begin{aligned} A &= (\text{width})(\text{length}) \\ &= (6x)(y) \\ &= 6x(100 - x) \qquad \textbf{since } y = 100 - x \\ &= -6x(x - 100) \end{aligned}$$

From the factored form, we can see the area function has horizontal intercepts (0, 0) and (100, 0). The vertex will be halfway between, at $x = 50$. The graph of the parabola is shown in Figure 4.32.

The red dashed line is drawn on the graph at 6000 square feet, the minimum enclosed area desired. The width of the corral must be of sufficient size as to generate an area value at or above that horizontal line. To find where the graph of the parabola and the line intersect, we set the equation of the horizontal line equal to the equation of the parabola.

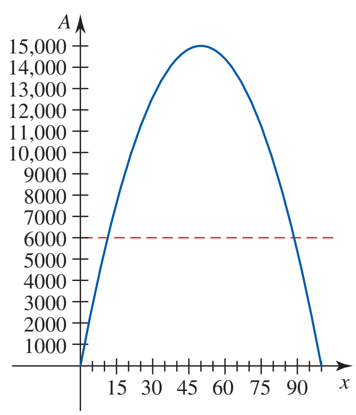


Figure 4.32

$$\begin{aligned} -6x^2 + 600x &= 6000 \\ -x^2 + 100x &= 1000 \\ x^2 - 100x + 1000 &= 0 \end{aligned}$$

We determine the intersection points by using the quadratic formula.

$$\begin{aligned}
x &= -\frac{-100}{2(1)} \pm \frac{\sqrt{(-100)^2 - 4(1)(1000)}}{2(1)} \\
&= 50 \pm \frac{\sqrt{6000}}{2} \\
&= 50 \pm \frac{\sqrt{400(15)}}{2} \\
&= 50 \pm \frac{20\sqrt{15}}{2} \\
&= 50 \pm 10\sqrt{15}
\end{aligned}$$

So $x \approx 11.3$ or $x \approx 88.7$. If the width of the ram corral is between 11.3 feet and 88.7 feet, the enclosed area will be at least 6000 square feet.

The total amount of fencing needed is found by adding the lengths of all of the sides.

$$\begin{aligned}
\text{total fencing} &= 12x + 4y \\
&= 12x + 4(100 - x) \\
&= 12x + 400 - 4x \\
&= 8x + 400
\end{aligned}$$

We want this number to be as small as possible. We can see the smaller the value of x, the smaller the amount of fencing will be needed. Consequently, the value of x that minimizes the amount of fence needed while still meeting the square footage requirement is $x = 11.3$. The corresponding value of y is 88.7 since $y = 100 - 11.3$. The minimum amount of fencing needed is $8(11.3) + 400 = 490.4$ feet.

Inverses of Quadratic Functions

Do quadratic functions have inverse functions? Let's consider the function $y = f(x) = x^2$ and solve this function for x.

$$\begin{aligned}
y &= x^2 \\
\sqrt{y} &= \sqrt{x^2} \\
\sqrt{y} &= |x| \\
\sqrt{y} &= \pm x \\
x &= \pm\sqrt{y} \\
f^{-1}(y) &= \pm\sqrt{y}
\end{aligned}$$

We know that in a function, each input value must correspond with exactly one output value. In this case, we see that each positive input value will correspond with two output values instead of one. For example, $f^{-1}(4)$ equals 2 and -2. Therefore, this quadratic function does not have an inverse function. In general, ***quadratic functions do not have inverse functions***.

SUMMARY

In this section you learned how to work with the graphs of quadratic functions with equations given in vertex, standard, and factored forms and saw how these forms are related. You also learned strategies for finding the horizontal intercepts and the vertex of a parabola using these forms. Additionally, you learned how to use and interpret these forms and graphs in real-world situations.

4.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–15, determine the coordinates of the vertex and horizontal intercepts of the parabola. If no horizontal intercepts exist, so state.

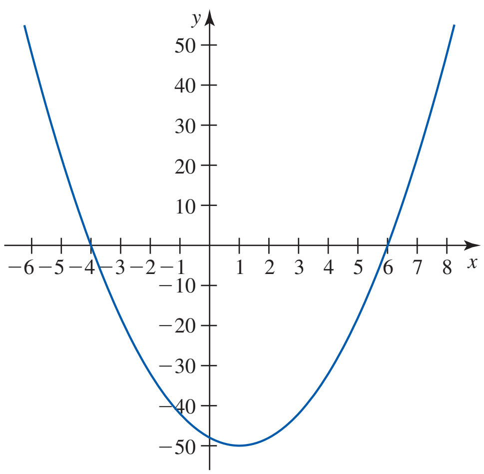
1. $f(x) = 4(x - 3)^2 + 2$
2. $f(x) = -9(x + 12)^2 + 18$
3. $f(x) = 0.25(x - 2)^2 - 1$
4. $g(x) = -2(x - 5)^2 + 10$
5. $f(x) = 8(x - 0.5)^2 - 2$
6. $y = -2x^2 + 4x - 8$
7. $y = 0.1x^2 + 0.4x - 1.2$
8. $y = x^2 + 4x + 5$
9. $y = 9x^2 - 4$
10. $y = 3x^2 + 4x + 1$
11. $f(x) = 6(x - 4)(x + 2)$
12. $g(x) = -5x(x - 10)$
13. $f(x) = -3(x + 7)(x + 11)$
14. $g(x) = 0.2(x - 4)(x + 12)$
15. $f(x) = -7(x + 1)(x - 1)$

In Exercises 16–20, sketch the graph of the quadratic function by hand.

16. $g(x) = -2(x - 5)^2 + 10$
17. $y = -2x^2 + 4x - 8$
18. $y = 9x^2 - 4$
19. $g(x) = -5x(x - 10)$
20. $g(x) = 0.2(x - 4)(x + 12)$

In Exercises 21–25, determine the equation of the quadratic function from its graph.

21.



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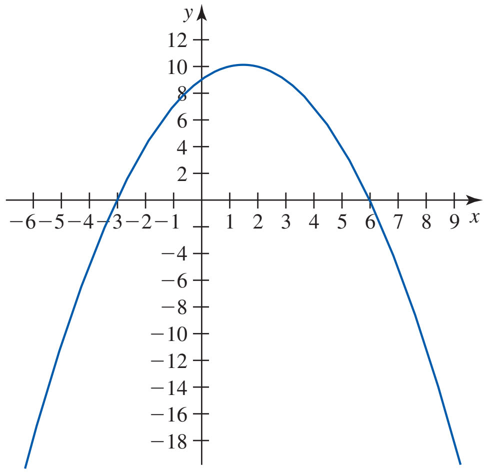

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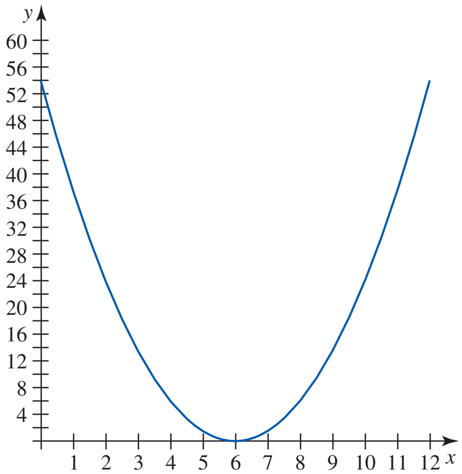
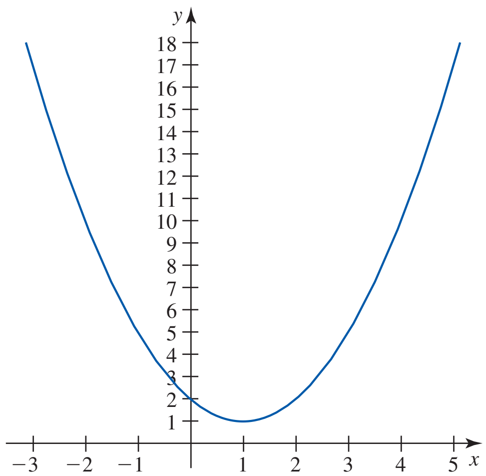
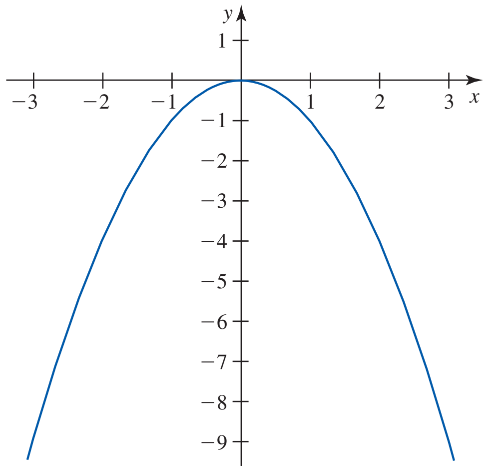


24.



25.



In Exercises 26–30, factor the quadratic function.

26. $y = x^2 - 11x + 24$

27. $y = x^2 + 2x + 1$

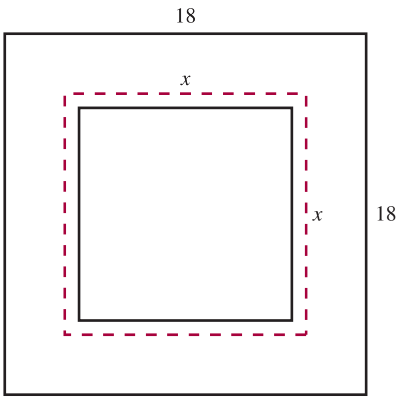
28. $y = x^2 - 16$

29. $y = x^2 - 2x - 24$

30. $y = x^2 - 5x - 6$

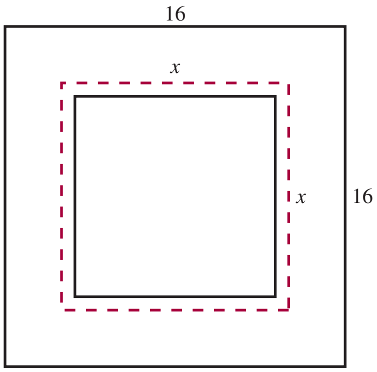
In Exercises 31–34, create a quadratic function to model the situation. Then answer all given questions.

31. Picture Frame A square is to be cut out of the middle of an 18-inch by 18-inch matting board to make a frame. The mat frame will be placed over a square picture and should overlap the picture by 1 inch on each side, as shown in the figure.



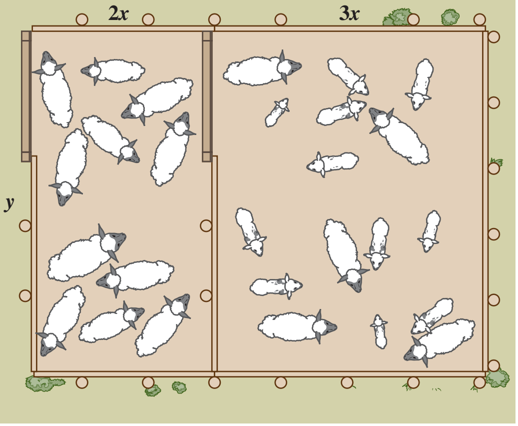
What will be the mat frame area for a 12-inch by 12-inch picture and a 15-inch by 15-inch picture?

32. Picture Frame A square is to be cut out of the middle of a 16-inch by 16-inch matting board to make a frame. The mat frame will be placed over a square picture and should overlap the picture by 1 inch on each side, as shown in the picture.



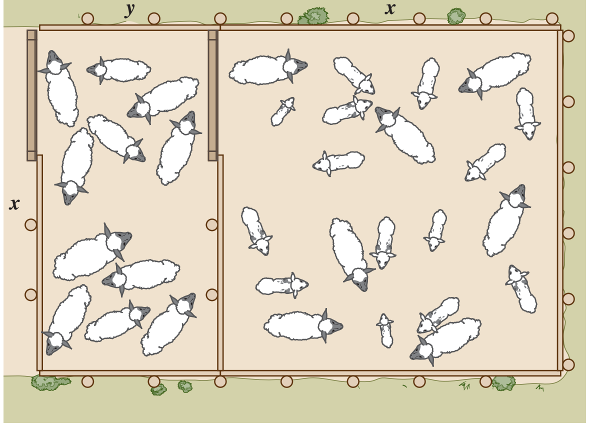
What will be the mat frame area for a 10-inch by 10-inch picture and a 14-inch by 14-inch picture?

33. Livestock Pens A rancher has 500 feet of fencing to construct two adjacent rectangular pens (see figure).



What are the dimensions of the pens with maximum combined area?

34. Livestock Pens A rancher has 600 feet of fencing to construct two adjacent rectangular pens (see figure).



What are the dimensions of the pens with maximum combined area?

SHOW YOU KNOW

35. Describe the advantages and disadvantages of each form of a quadratic function: standard, vertex, factored.

36. How is the *vertex* form of a quadratic function related to the *standard* form?

37. How is the *factored* form of a quadratic function related to the *standard* form?

38. A quadratic function cannot have an inverse function. Why not?

39. Describe how you can quickly find the second horizontal intercept of a quadratic function if you know the first horizontal intercept and the vertex.

MAKE IT REAL

In Exercises 40–45, create a quadratic function to model the situation. Then answer all given questions.

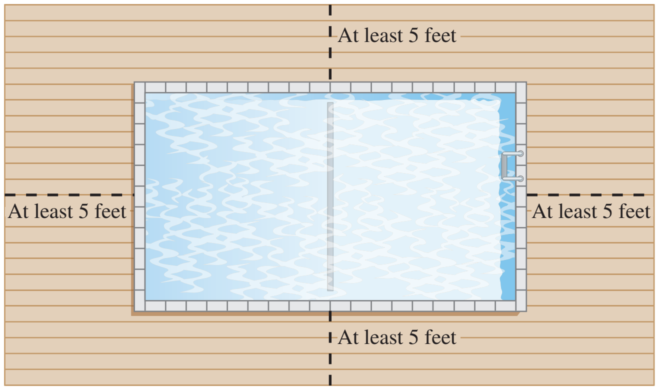
40. Men's High Jump Record As of 2011, the men's outdoor high jump record was held by Javier Sotomayor of Cuba. On July 27, 1993, he jumped 2.45 meters (8 feet 1/2 inch) into the air. Assuming air resistance was negligible and that he landed on a cushion 3 feet above the height from which he jumped, how long was he airborne? (*Hint:* The height above the ground of a person t seconds after he jumps into the air can be modeled by $s(t) = -16(t - h)^2 + k$ feet, where h is the time he reaches his maximum height and k is the maximum height (in feet).)

41. Women's High Jump Record As of 2011, the women's outdoor high jump record was held by Stefka Kostadinova of Bulgaria. On August 30, 1987, she jumped 2.09 meters (6 feet 10 inches) into the air. Assuming air resistance was negligible and that she landed on a cushion 3 feet above the height from which she jumped, how long was she airborne? (*Hint:* The height above the ground of a person t seconds after she jumps into the air can be modeled by $s(t) = -16(t - h)^2 + k$ feet, where h is the time she reaches her maximum height and k is the maximum height (in feet).)

42. Longest Vertical Drop on Earth Mount Thor in Canada is said to have the longest vertical drop (4100 feet) on Earth. (*Source:* www.wikipedia.com) How long would it take for a pebble dropped off the cliff to reach the ground? (For the purpose of this model, neglect air resistance.) (*Hint:* The height of a falling object above the ground t seconds after it is dropped can be modeled by $s(t) = -16t^2 + s_0$, where s_0 is the initial height.)

43. Swimming Pool Design Many counties require in-ground swimming pools to be set back at least 5 feet from the fence line and from all buildings.

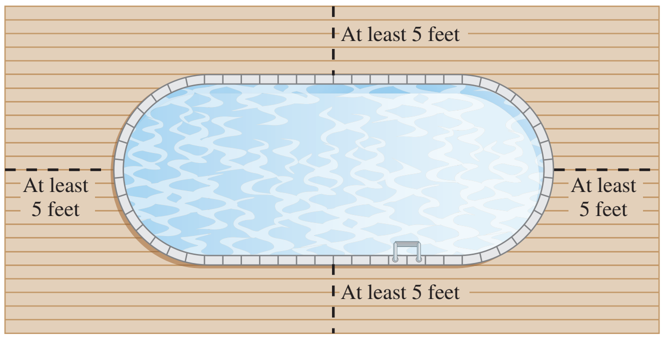
A family plans to install a rectangular pool into their 30-foot by 60-foot backyard. They want the pool to have a 110-foot perimeter and contain as much area as possible. One pool shape they are considering is a rectangle.



What will be the dimensions of the rectangular pool with maximum surface area?

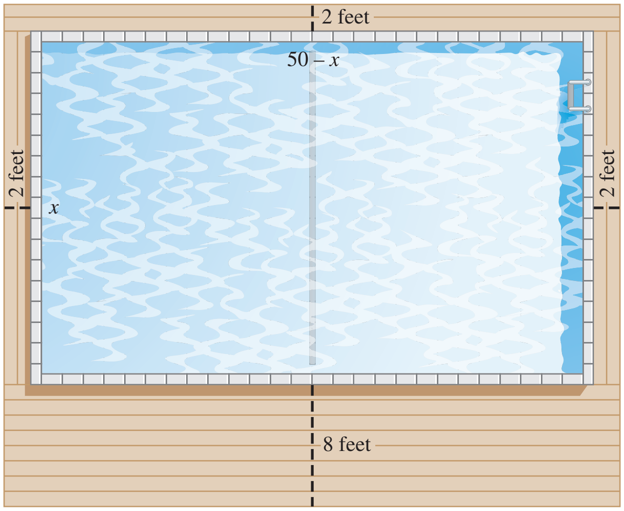
44. Swimming Pool Design Many counties require in-ground swimming pools to be set back at least 5 feet from any fence line and from all buildings.

A family plans to install a pool into their 30-foot by 60-foot backyard. They want the pool to have a 110-foot perimeter and contain as much area as possible. One pool shape they are considering is a rectangle with two semi-circles attached at each end.



What will be the dimensions of the pool of this shape with maximum surface area? (*Hint:* Recall that the area of a circle is $A = \pi r^2$ and the circumference of a circle is $C = 2\pi r$.)

45. Pool Decking Single-family, in-ground swimming pools are typically bordered with pool decking. It is customary to have a limited amount of decking on the far side of the pool and a larger amount of decking on the near side of the pool (to accommodate deck furniture). A rectangular pool with a 100-foot perimeter is bordered on three sides by 2 feet of decking and on one side by 8 feet of decking. The pool surface area is to be at least 500 square feet. (See figure on next page.)



What pool dimensions will require the least amount of decking?

■ STRETCH YOUR MIND

Exercises 46–50 are intended to challenge your understanding of quadratic functions.

46. Water Rockets As a youth, the favorite toy of one of the authors was a red water rocket, similar to that shown in the photo.

Assuming air resistance is negligible, the rocket is launched from a height of 2 feet, and the maximum height attained by the water rocket is 66 feet, determine the initial velocity of the rocket. That is, find the velocity of the rocket at the moment it is launched.

47. Water Rockets As of March 2007, the world record for the greatest height reached by a water rocket was reported to be 2088 feet. (*Source:* **www.wikipedia.com**) Assuming the rocket was launched from a height of 3 feet and air resistance was negligible, determine how long it took for the rocket to reach its maximum height and its total flight time from launch to landing.

48. Show how to convert a quadratic function in factored form, $y = a(x - x_1)(x - x_2)$, into a quadratic function in vertex form, $y = a(x - h^2) + k$. Specify the relationship between x_1 and x_2 and h and k.

49. Explain why some quadratic functions ***cannot*** be written in factored form.

50. How can you tell if a quadratic function in vertex form can be rewritten in factored form?

CHAPTER 4 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 4.1

1. What is meant by *increasing* and *decreasing* functions?
2. What does the term *variable rate of change* mean?
3. How can average rates of change be used to fill in gaps in a table of data?
4. What does it mean to find the rate of change at an instant? How is the rate of change at an instant calculated? How can the rate of change at an instant be visualized on a graph?
5. How are rates of change and successive differences used to describe the concavity of a graphical model?
6. How can rates of change be used to describe inflection points?

SECTION 4.2

7. What distinguishes a quadratic function from a linear function? Use the language of rate of change in your response.
8. Interpret the parameters of a quadratic model, $y = ax^2 + bx + c$. What does each mean in a modeling situation?

SECTION 4.3

9. What do the parameters a, h, and k represent in a quadratic equation in vertex form, $y = a(x - h)^2 + k$?
10. How can you find the horizontal intercepts of a quadratic equation in vertex form?
11. How can you find the vertex of a quadratic model in standard form? What does the vertex represent?
12. What is the quadratic formula used for? What is a discriminant?

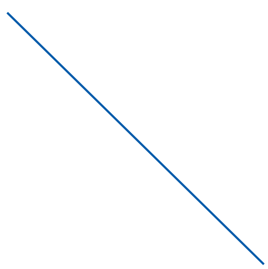
REVIEW EXERCISES

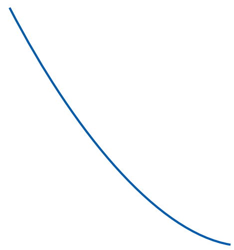
SECTION 4.1

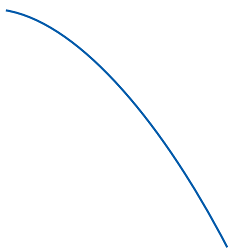
For Exercises 1–3, create a table of values and an associated graph that

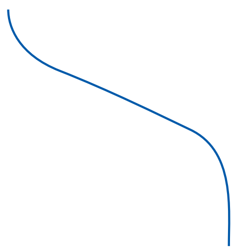
1. Is decreasing and concave down.
2. Has a point of inflection.
3. Is concave down twice and concave up once.
4. Each of the functions f, g, h, and j shown in the table are decreasing, but each decreases in a different way. Which of the graphs best fits each function? Explain how you made your choice.

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
−6	34.3	9.4	30	17.49
−4	32.7	7.9	10	8.55
−2	30.8	6.4	−2	4.18
0	28.5	4.9	−4	2.05
2	25.7	3.4	−5	1
4	22.2	1.9	−6	0.49
6	17.2	0.4	−8	0.24
8	8.5	−0.9	−20	0.12

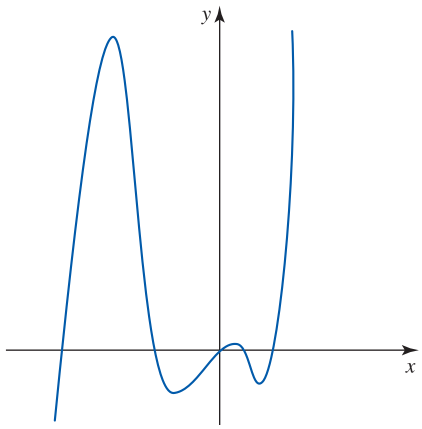
a. 

b. 

c. 

d. 

5. Identify any inflection points in the functions of Exercise 4. Then explain how to find an inflection point from a table and how to find an inflection point from a graph.
6. Label points A, B, C, D, E, and F on the graph of f according to the following descriptions of each point.
   a. Point A is a point on the curve where the instantaneous rate of change is positive.
   b. Point B is a point on the curve where the value of the function is positive.
   c. Point C is a point on the curve where the instantaneous rate of change is the most negative.
   d. Point D is a point on the curve where the instantaneous rate of change is 0.
   e. Points E and F are different points on the curve where the instantaneous rates of change have the same magnitude but opposite signs.



7. The table shows the number of fighter aircraft produced by three of the major powers in World War II from 1940 to 1944. For each country,
   a. Describe the concavity of the function, including identifying any possible inflection points. Then interpret your findings in the context of the situation.
   b. Sketch a graph that could model each function and label the independent and dependent axes.

Year y	Germany $G(y)$	USSR $U(y)$	United Kingdom $K(y)$
1940	2,746	4,574	4,283
1941	3,744	7,086	7,064
1942	5,515	9,924	9,849
1943	10,898	14,590	10,727
1944	26,326	17,913	10,730

***Source:** World War II: The Encyclopedia of Facts and Figures*, **John Ellis, Table 93**

For Exercises 8–12, refer to the fighter aircraft production data given in Exercise 7.

8. Compute the average rate of change of G from 1942 to 1944, give the units on this rate, and interpret what this number means.
9. Compute the average rate of change of U from 1940 to 1943, give the units on this rate, and interpret what this number means.
10. Estimate the instantaneous rate of change of K at $y = 1941$, give the units on this rate, and interpret what this number means.

11. Estimate the instantaneous rate of change of G at $y = 1943$, give the units on this rate, and interpret what this number means.

12. Using *successive differences*, predict what each country's production would have been in 1945 (assuming the war continued through the entire year).

13. The table shows the temperatures in Seattle, Washington, from 11 P.M. on March 13, 2007 to 3 P.M. on March 14, 2007.

Hours Since 11 P.M. March 13, 2007 t	Temperature in Seattle, WA (°F) $F(t)$
0	43
2	42
4	41
6	41
8	39.9
10	37
12	39
14	42
16	42.1

Source: **weather.noaa.gov**

a. Plot the points and connect them with a smooth curve to model the temperature over this time period.

b. Describe the intervals where the function is increasing, decreasing, and constant. Then explain what this tells us about temperatures in Seattle over these intervals.

c. Describe the intervals where the function is concave up and concave down. Then explain what this tells us about temperatures in Seattle over these intervals.

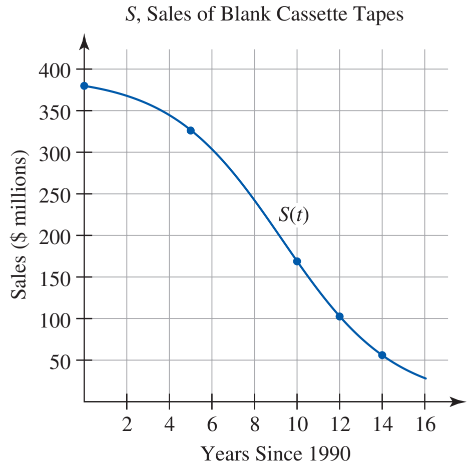
d. Use *averages* to find $F(3)$, $F(9)$, and $F(13)$.

e. Use *average rates of change* to find $F(3)$, $F(9)$, and $F(13)$. Then compare these results to part (d).

f. What would $t = 6.75$ represent? Could you use either averages or the average rate of change to find $F(6.75)$? Explain.

14. Blank Cassette Tape Sales Over the past couple of decades, sales of blank cassette tapes have declined, as shown in the table and graph.

Years Since 1990 t	Blank Cassette Tape Sales ($ millions) $S(t)$
0	376
5	334
10	162
12	98
14	66



a. Describe the intervals over which the function is concave up and concave down. Then explain what this tells about the sales of blank cassette tapes.

b. Use *averages* to estimate $S(11)$ and $S(13)$.

c. Use *average rates of change* to estimate $S(11)$ and $S(13)$. Then compare these results to part (b).

d. Use *successive differences* to predict $S(15)$ and $S(16)$.

15. Five years ago Coach Anderson took over as head coach of the worst basketball team in the league, at which time the team began to improve. After this season, however, some of the fans want Coach Anderson fired. They claim his early gains have not continued, while the coach's supporters say the team's record has improved each year under Coach Anderson. Using a graph and a discussion of concavity, explain how both groups can be correct.

16. Over the last three years, a company has begun to lose money. The CEO says the losses are not too severe and that the company should be able to keep doing business, but analysts have predicted big trouble for this company in the near future. Is the graph of the company's profits concave up or concave down? Explain.

SECTION 4.2

Exercises 17–19 focus on a conceptual understanding of quadratic functions.

17. Describe how to determine if a table of data, a graph, or an equation represents a quadratic function.

18. If the second differences of a table of data are negative, what do you know about its graph?

19. Describe two different methods that can be used to find a quadratic model for a table of data.

In Exercises 20–21, explain the meaning of the parameters of the quadratic function.

20. Nuclear Power Consumption Based on data from 1970 to 2004, the consumption of nuclear power in the United States can be modeled by

$$N(t) = -0.003084t^2 + 0.3512t + 0.04516$$

quadrillion BTUs, where t is the number of years since 1970. (*Source:* Modeled from *Statistical Abstract of the United States, 2007*, Table 895)

21. Consumer Spending on Farm Foods Based on data from 1990 to 2004, consumer expenditure for farm foods can be modeled by

$$E(t) = 1.008t^2 + 9.951t + 450.5$$

billion dollars, where t is the number of years since 1990. (*Source:* Modeled from *Statistical Abstract of the United States, 2007*, Table 818)

In Exercises 22–23, use a graphing calculator and quadratic regression to find the quadratic model that best fits the data set.

22. U.S. Wine Imports

Years Since 1990 t	Wine Imports (in thousands of hectoliters) W
0	2510
5	2781
10	4584
12	5655
13	6214
14	6549
15	7262

Source: Statistical Abstract of the United States, 2007, Table 819

23. U.S. Exports to Canada

Years Since 1990 t	Exports to Canada ($ millions) C
0	4217
10	7640
11	8121
12	8660
13	9313
14	9741
15	10570

Source: Statistical Abstract of the United States, 2007, Table 827

SECTION 4.3

In Exercises 24–26, find the vertex and horizontal intercepts of the quadratic function.

24. $y = -10(x + 1)^2 - 20$

25. $y = 4x^2 + 16x + 30$

26. $y = -6(x + 20)(x - 8)$

In Exercises 27–29, graph the quadratic function by hand. (Hint: Refer to your answers in Exercises 24–26.)

27. $y = -10(x + 1)^2 - 20$

28. $y = 4x^2 + 16x + 30$

29. $y = -6(x + 20)(x - 8)$

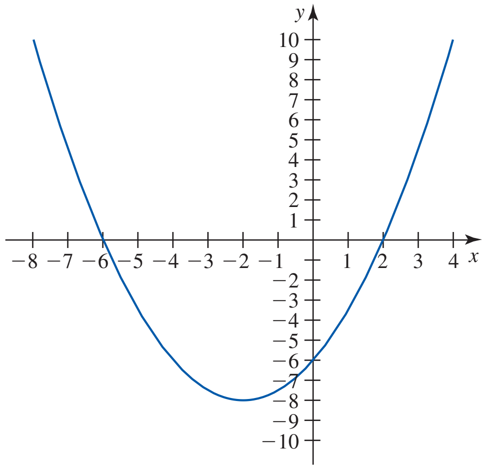
In Exercises 30–31, create a quadratic function to model the situation. Then answer all given questions.

30. Golden Rectangle A golden rectangle is a rectangle with the property that the ratio of its length to its width is equal to the ratio of the sum of its length and width to its length. That is, $\frac{l}{w} = \frac{l + w}{l}$. The width of a particular golden rectangle is 2 meters. What is the length and area of the rectangle?

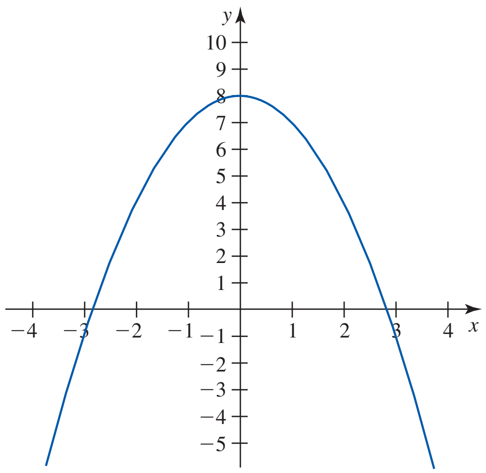
31. Orange Prices On March 15, 2007, Safeway.com offered individual large Navel oranges for $1.61 per pound. A 4-pound bag of oranges was offered for $4.99. (*Source:* www.safeway.com) Assuming that the total cost of purchasing n pounds of oranges can be modeled by a quadratic function, calculate the price the store should charge for a 10-pound bag of oranges and the corresponding price per pound.

In Exercises 32–33, determine the equation of the quadratic function from the graph.

32.



33.



Make It Real Project

What to Do

1. Find a set of at least six data points from an area of personal interest. Choose data that appear to be concave up or concave down.
2. Draw a scatter plot of the data and explain why you do or do not believe a quadratic model would fit the data well.
3. Find a quadratic regression model for your data.
4. Interpret the physical meaning of the parameters, a, b, and c.
5. Use the model to predict the value of the function at an unknown point and explain why you do or do not think the prediction is accurate.
6. Describe the transformations (shifts, stretches, compressions, reflections) that your regression model shows compared to a simple quadratic equation,
$y = x^2$.
7. Write your quadratic model in vertex form and interpret the meaning of the parameters.
8. Describe your function using the language of rate of change (increasing, decreasing, concave up, concave down). Be sure to explain what these terms mean in the context of your data set.