Unit 3 - Function Transformations, Quadratic, Polynomial and Rational Functions

General Notes

Module 10 - Function Transformation & Quadratic Functions

Transformations Activity

- Overall Guide

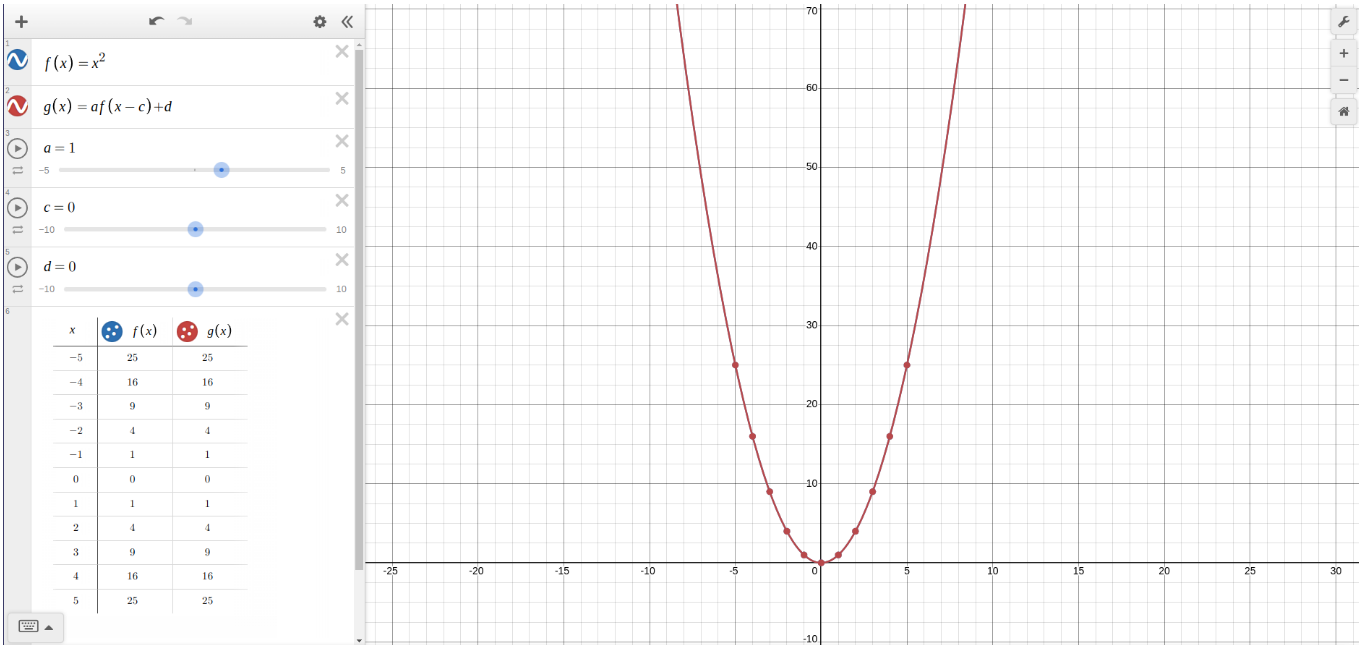
Functions can be transformed in various ways:

- Stretched or compressed vertically or horizontally
- Shifted up or down, or left or right

Transformation Activity: https://www.desmos.com/calculator/fneiikwfiy

Two defined functions:

1. ***f*(x) = x²**
2. ***g*(x) = a*f*(x - c) + d**



- **a**, **c**, and **d** are parameters.
- In this scenario, **f** is the **parent function** and **g** is the **transformed / image function**.
- **d** moves the vertical offset by **d** and shifts the graph up or down.
- **a** is the slope and stretches or compresses the graph vertically by a factor of **a** and can create a **vertical reflection**.
  - **1 < a:** The graph is stretched vertically (narrower than original)
  - **0 < a < 1:** The graph is compressed vertically (wider than original graph)
  - **a < 0:** Creates a vertical reflection.
- **c** moves the horizontal intercept by **c** and shifts the graph left or right.
  - **c < 0:** The shift is to the left.
  - **c > 0:** The shift is to the right.
  - In a table, the output of the parent function is moved by **c** places down or up.

Writing a Function In Terms of a Function

When setting *a* = 2:

x	*f*(x)	*g*(x)
-5	25	50
-4	16	32
-3	9	18
-2	4	8
-1	1	2
0	0	0
1	1	2
2	4	8
3	9	18
4	16	32
5	25	50

- A formula for the function **g** in terms of **f**:
  $g(x) = 2f(x)$
- Given that $f(x) = x^2$, a formula for the function **g** in terms of **f**:
  $g(x) = 2x^2$

When setting $d = 2$:

x	$f(x)$	$g(x)$
-5	25	27
-4	16	18
-3	9	11
-2	4	6
-1	1	3
0	0	2
1	1	3
2	4	6
3	9	11
4	16	18
5	25	27

- A formula for the function **g** in terms of **f**:
  $g(x) = f(x) + 2$
- Given that $f(x) = x^2$, a formula for the function **g** in terms of **f**:
  $g(x) = x^2 + 2$

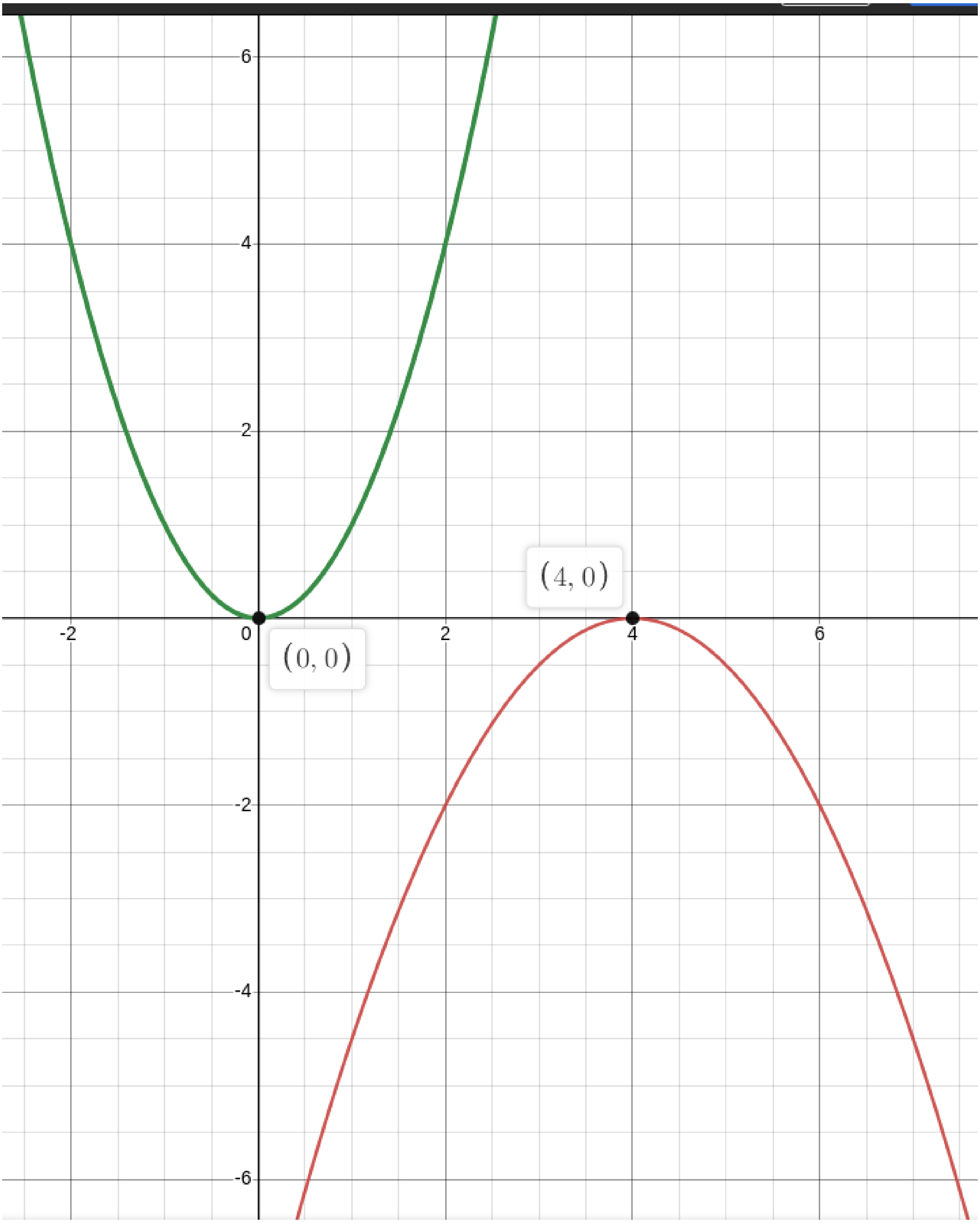
When setting *c = 1*:

x	*f*(x)	*g*(x)
-5	25	36
-4	16	25
-3	9	16
-2	4	9
-1	1	4
0	0	1
1	1	0
2	4	1
3	9	4
4	16	9
5	25	16

- A formula for the function **g** in terms of **f**:
  $g(x) = f(x - 1)$
- Given that $f(x) = x^2$, a formula for the function **g** in terms of **f**:
  $g(x) = (x - 1)^2$

Vertex of a Function

The vertex is the point where the maximum or minimum occurs of the function.



Vertex Example 1

***g*(x) = 2*f*(x) + 3**

- Vertical stretch by a factor of **2**
- Vertical stretch up **3**

- Vertex: **(0, 3)**

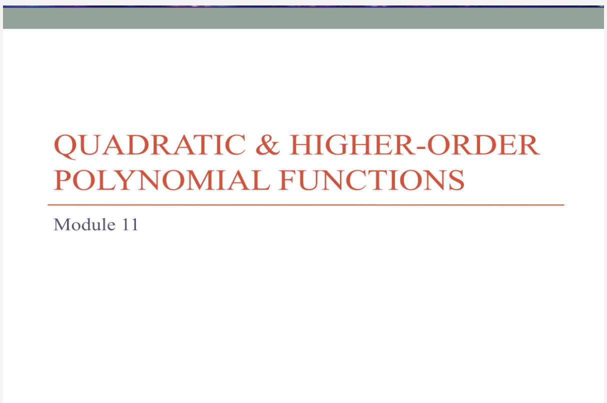
Vertex Example 2

***h*(x) = *ƒ*(x + 2) - 1**

- Horizontal shift left **2**
- Vertical shift down **1**
- Vertex: **(-2, -1)**

Module 11 - Quadratic & Higher-Order Polynomial Functions

Module 11 - Google Slides

> - **Google Slides:**
>
> [Link to slides]
>
> 
>

Quadratic Functions

Basic quadratic function: $\boldsymbol{f}$**(x) = $\mathbf{x^2}$**

- Graph is called **parabolic**
- Shape is open *up* if $\boldsymbol{f}$**(x) = $\mathbf{x^2}$**
- Shape is open *down* if $\boldsymbol{f}$**(x) = $\mathbf{-x^2}$**
- Both ends point in the same direction
- The graph will have a maximum or a minimum
  - This occurs at the vertex

- They are symmetric
  - The axis of symmetry passes through the vertex

There are two types of parabolas:

- **Concave down**
  - Has a maximum value at the vertex
  - Does not have a minimum
- **Concave up**
  - Has a minimum value at the vertex
  - Does not have a maximum

Vertex Form of a Quadratic Function

$y = a(x - h)^2 + k$

- The function above with **a != 0** is in **vertex form**.
- The point *(h, k)* is called the **vertex** of the parabola.
- The effect of the different variables:
  - **a** is a vertical stretch/compression/reflection
  - **h** is a horizontal shift left/right
  - **k** is a vertical shift up/down
- **h** is the input reference and **k** is the output reference
  - *(h, k)* is the reference point

Different Forms of a Quadratic Function

- In some forms, the **vertex** of a function is easier to determine.
- In other forms, vertical/horizontal intercept are easier to determine.

Standard Form

> **$y = ax^2 + bx + c$**

- In this form, **c** will always be the **vertical intercept**.

Factored Form

> $y = a(x - x_1)(x - x_2)$

- In this form, $x - x_1 = 0$ and $x - x_2 = 0$ can be used to find the **horizontal intercepts** *(or **zeros**)* of the graph.

Vertex Form

> $y = a(x - h)^2 + k$

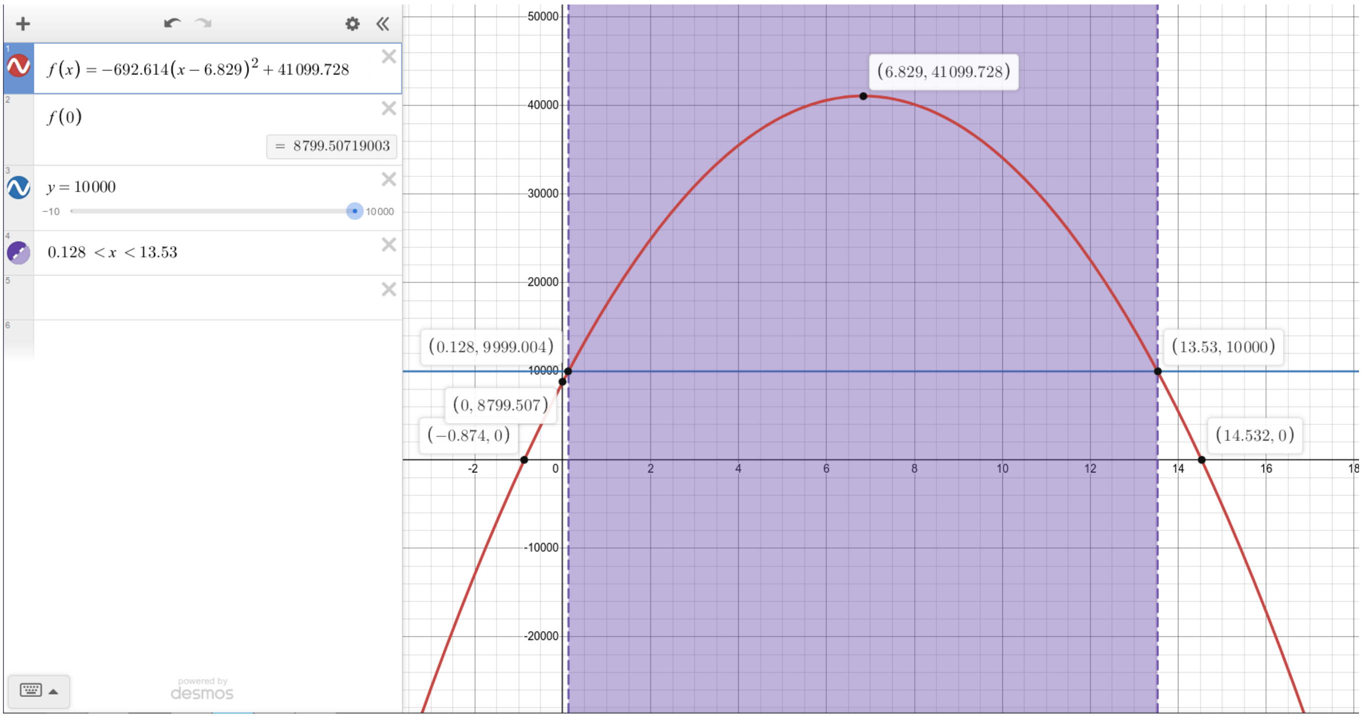
- In this form, **(h, k)** will always be the **vertex** of the function and either the **minimum** or the **maximum** depending on if the graph is **concave up** or **concave down**.
  - The concavity can be determined by the sign of **a**.
    - **-x** is concave down
    - **x** is concave up

Problems In Different Forms

Problem in Vertex Form

According to the CDC, the number of AIDS cases among people aged 21 or younger in the US is finally starting to decline after seeing many years of increase.

The function that models the number of cases with respect to the number of years since 2004 is:
$N(x) = -692.614(x - 6.829)^2 + 41099.728$



1. Determine approximately when the number of AIDS cases was at its peak. What was the maximum number?
    1. Identify whether the graph is concave up or concave down based off if **a** is negative. **It's concave down**.
    2. Determine the maximum (because it's concave down) by finding the vertex
    3. Vertex is **[6.829, 41099.728]**, so the answer is **6.829**
2. Identify and explain the vertical intercept of the function.
    - **Option 1**
        1. Graph the function
        2. Create another entry using **0** as ***ƒ*(x)**: ***ƒ*(0)**
    - **Option 2**
        1. Substitute **0** into the equation, because it turns into:

           $\boldsymbol{f(0) = -692.614(x - 6.829)^2 + 41099.728}$
        2. $\boldsymbol{f(0) = -692.614(0 - 6.829)^2 + 41099.728}$
        3. $\boldsymbol{f(0) = -692.614(6.829)^2 + 41099.728}$
        4. $\boldsymbol{f(0) = -692.614(46.635) + 41099.728}$
        5. $\boldsymbol{f(0) = -32300.221 + 41099.728}$
        6. $\boldsymbol{f(0) = 8799.507 \rightarrow 8799.51}$
3. Determine when the model would predict AIDS cases is zero (Horizontal intercept).
    - **Option 1**
        1. Graph the function

2. Find the horizontal intercepts
3. Answer: **~1 year before 2004 and ~14.5 years after 2004**

- **Option 2**
  1. Set the equation equal to **0**, turning it into:

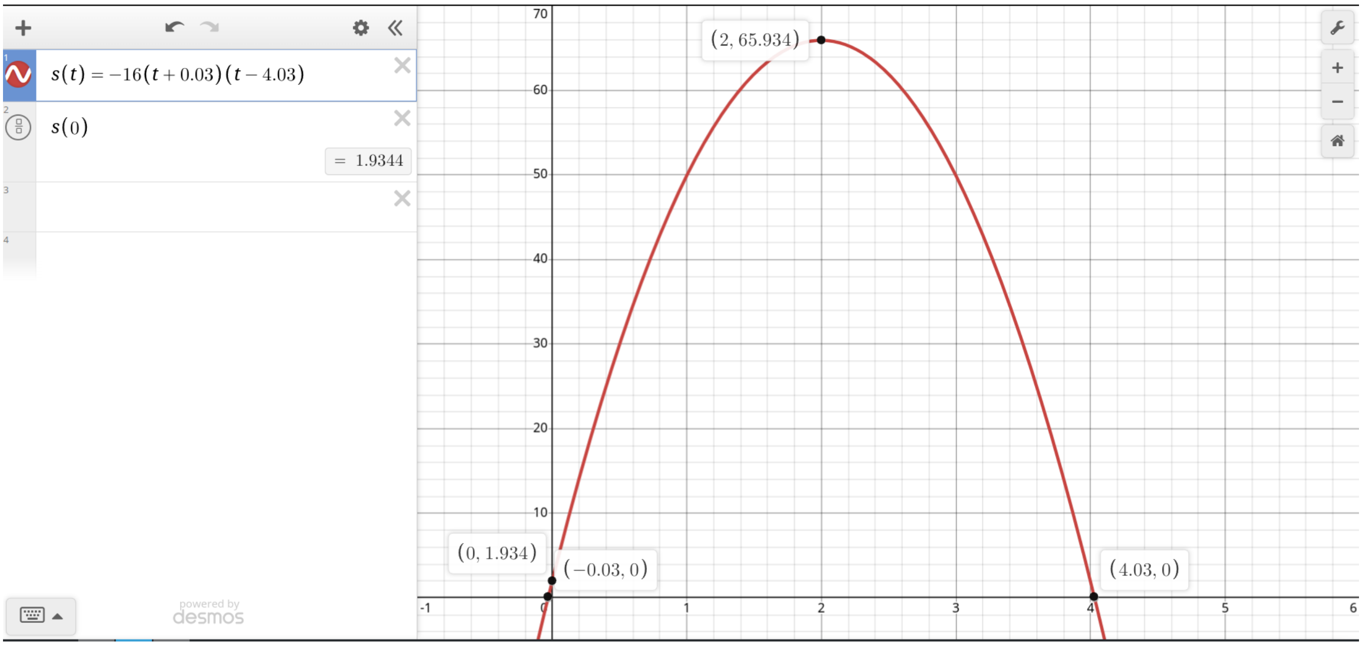
     $$0 = -692.614(x - 6.829)^2 + 41099.728$$
  2. $-41099.728 = -692.614(x - 6.829)^2$
  3. $^{-41099.728}/_{-692.614} = (x - 6.829)^2$
  4. $\sqrt{(^{-41099.728}/_{-692.614})} = x - 6.829$
  5. $\sqrt{(^{-41099.728}/_{-692.614})} + 6.829 = x$
  6. Solve
  7. $x = 14.53224731 \rightarrow 14.53$

4. Determine the range of years we would expect the number of cases to be at least 10,000.
   1. Graph the function
   2. Plot a separate line that is 10,000
   3. Determine where the line intersects with the graph
   4. *Optional: Use inequalities to create a section visualizing it.*
   5. **Answer: 0.128 <= x <= 13.53**

Problem in Factored Form

A water rocket can be purchased at many toy stores. One company claims that the height above the ground (in feet) for their rocket *t* seconds after it is launched can be modeled by:

> **$s(t) = -16(t + 0.03)(t - 4.03)$**



1. Determine the vertical intercept of the function and explain its meaning (if any) in the context of the problem.
    1. Graph the chart
    2. Find the vertical intercept
    3. **[0, 1.934]**
    4. At the start of the launch, before it has taken off, it is **1.934** feet above the ground.
2. Determine the horizontal intercepts of the function and explain their meaning (if any) in the context of the problem.
    - **Option 1:** ***s*(t)** will equal 0 when any of the factors equal 0, so when **t + 0.03 = 0** or **t - 4.03 = 0**:
        1. Get the intercepts from the formula.
        2. **+0.03** → **-0.03** and **4.03**
        3. **[-0.03, 0]** and **[4.03, 0]**
    - **Option 2**
        1. Graph the chart
        2. Find the horizontal intercepts
        3. **[-0.03, 0]** and **[4.03, 0]**
    - The first intercept has no meaning ***[-0.03, 0]***, the second intercept means that the rocket will hit the ground **4.03 seconds** after launching.
3. Determine the vertex of the function and explain its meaning in the context of the problem.
    - **Option 1**
        1. Graph the function

2. Find the vertex
3. **[2, 65.934]**
4. After 2 seconds from launching, the rocket will be at its highest
   point
   of 65.934 feet in the air.
5. Due to symmetry, the vertex will occur halfway between the two zeros:
   1. $\mathbf{{}^{4.03-0.03}/_{2} = 2}$
   2. ***s*****(2) = 65.934** is the vertex.

- **Option 2**
  1. Due to symmetry, the vertex will occur halfway between the 2 zeros
     and
     can be found by:

     $\mathbf{{}^{4.03 + (-0.03)}/_{2} = 2}$
  2. ***s*****(2) = 65.934** ← *After plugging* ***2*** *into the formula*
  3. **[2, 65.934]** is the vertex

The horizontal intercepts (**[0.03, 0]** and **[4.03, 0]**) are often referred
to as the **zeros** of the function.

Problem in Standard Form (Converting to Standard Form)

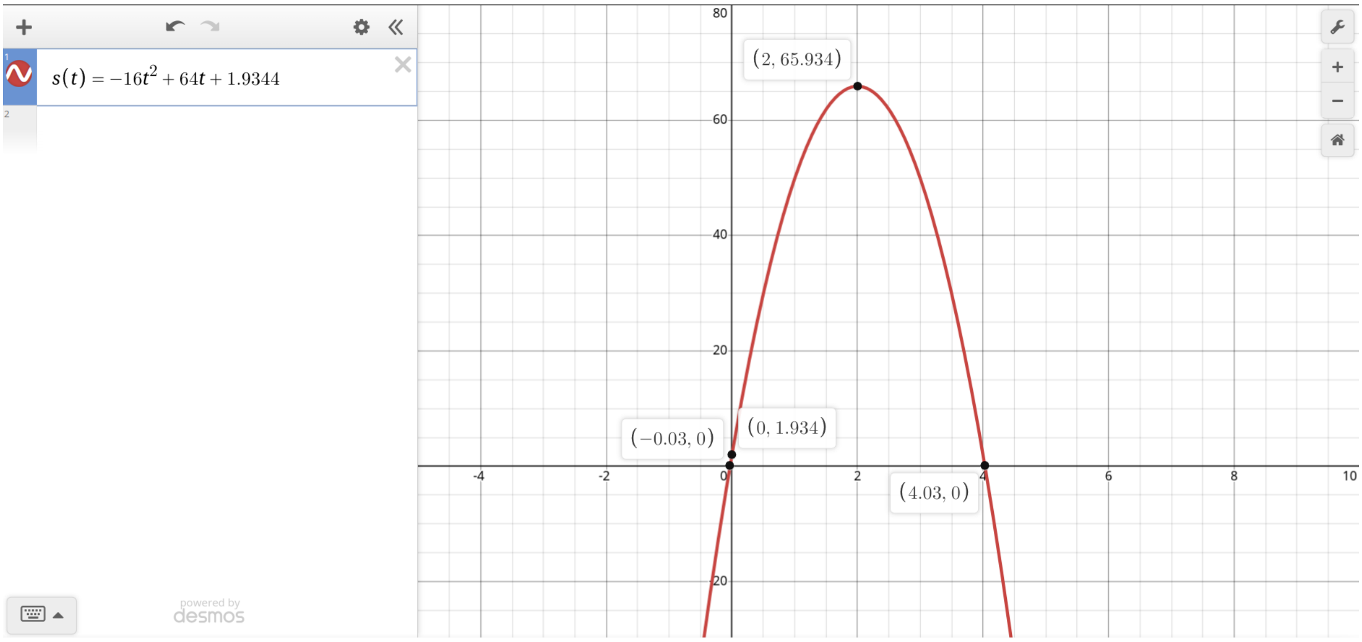
First, convert the previous Factored form to standard form:

1. $\boldsymbol{s\mathrm{(t)} = -16(\mathrm{t} + 0.03)(\mathrm{t} - 4.03)}$
2. $\boldsymbol{s\mathrm{(t)} = -16(\mathrm{t}^2 - 4.03\mathrm{t} + 0.03\mathrm{t} - 0.1209)}$
3. $\boldsymbol{s\mathrm{(t)} = -16\mathrm{t}^2 + 64\mathrm{t} + 1.9344}$

Getting the vertical intercept ***s*****(0):**

The **c** value is always the output of the vertical intercept. In this
case **1.9344**, so: **[0, 1.9344]**

You can determine the horizontal intercept and vertex by graphing the function:



Converting to Vertex Form

To convert back to vertex form, you need to know the vertex. You can either graph it or to solve algebraically, you can **complete the square**.

1. The forms for a polynomial function:
    - > **Standard Form:**
      >
      > **$f(x) = ax^2 + bx + c$**
    - > **Vertex Form:**
      >
      > **$f(x) = a(x - h)^2 + k$**
2. **$s(t) = -16t^2 + 64t + 1.9344$**
3. Factor out the **16t** from the first two terms.
    - When possible, you can also factor it out of the last term if it goes into it.
4. **$s(t) = -16(t^2 - 4t)\ 1.9344$**
5. Complete the square by adding and subtracting the square of half of the co-efficient of the ***b*** term: **$(^b/_2)^2$**
    1. **$(^{-4}/_2)^2$**
    2. **$(-2)^2$**
    3. **4**

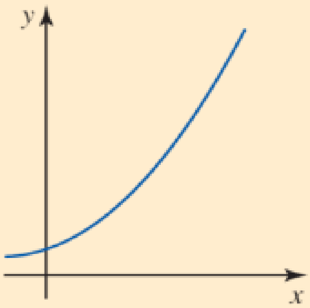
6. $s(t) = -16(t^2 - 4t + 4 - 4) - 1.9344$
7. Group the first three terms and factor it as a square of a binomial
8. $s(t) = -16((t^2 - 4t + 4) - 4) - 1.9344$
9. $s(t) = -16((t - 2)^2 - 4) + 1.9344$
10. Distribute and simplify
11. $s(t) = -16(t - 2)^2 - 16(-4) + 1.9344$
12. $s(t) = -16(t - 2)^2 + 64 + 1.9344$
13. $s(t) = -16(t - 2)^2 + 65.9344$

Concavity

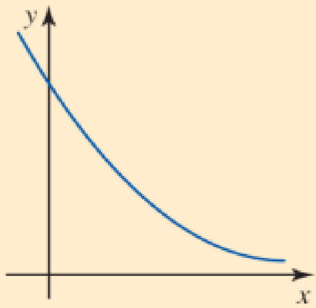
CONCAVITY

- The graph of a function *f* is said to be **concave up** if its rate of change *increases* as the input values increase. Concave up functions curve upward.

Increasing/Concave Up

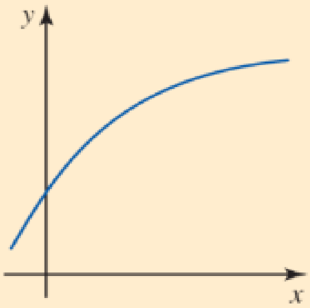


Decreasing/Concave Up

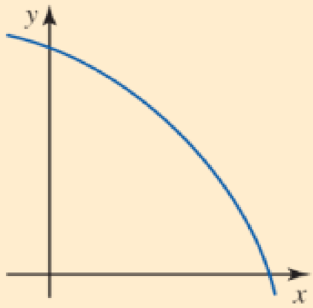


- The graph of a function *f* is said to be **concave down** if its rate of change *decreases* as the input values increase. Concave down functions curve downward.

Increasing/Concave Down



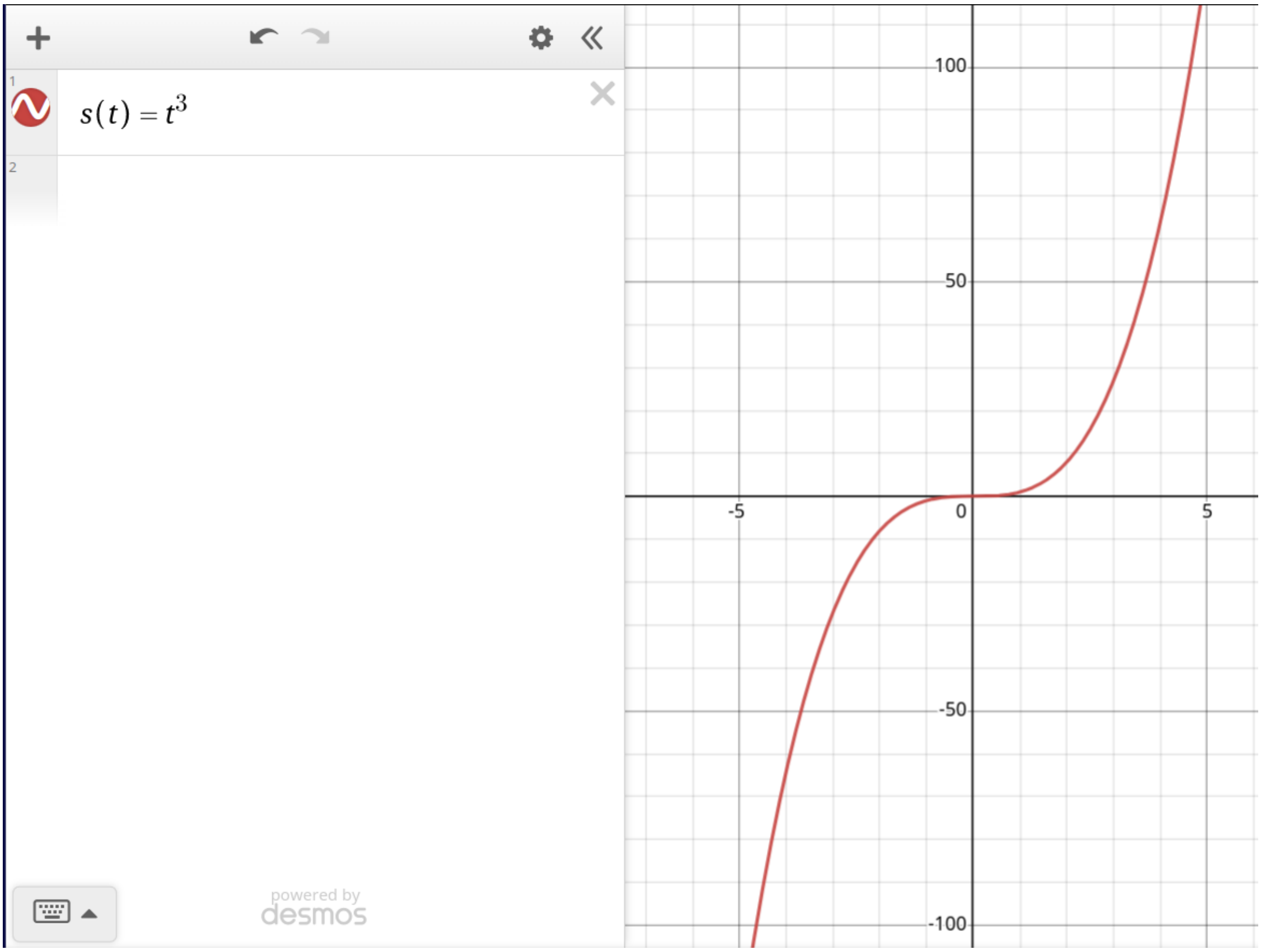
Decreasing/Concave Down



Inflection Point

The point on a graph where the function changes concavity is called the inflection point.

Consider the following: $\mathbf{x^3}$



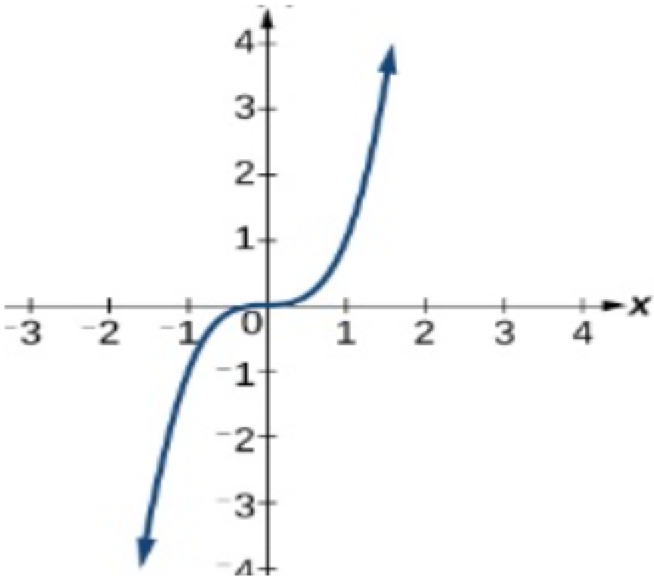
- An inflection point is a point where the curve changes from bending one way to bending the other way *(like the top of a hill)*.
- The point where rate of change of the function changes from increasing to decreasing, or from decreasing to increasing.

Cubic Functions

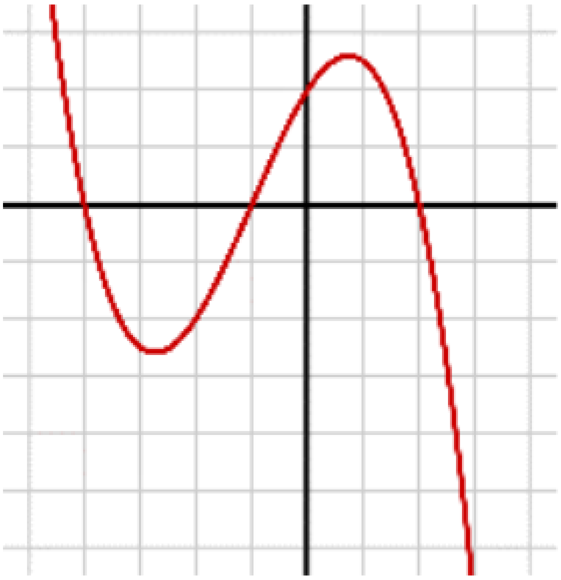
Cubic functions are functions that have two concavities and one inflection point.

Cubic Function Example 1

Cubic Function Example 2



Cubic Function Example 3



Polynomial Functions

Linear, quadratic, and cubic are all types of **polynomial functions**, defined as follows:

POLYNOMIAL FUNCTION

For whole number n, a function of the form

$$y = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \cdots + a_1 x + a_0$$

with $a_n \neq 0$ is called a **polynomial function of degree *n***. Each $a_i x^i$ is called a **term**. The a_i are real-number values called the **coefficients** of the terms.

- $\mathbf{a_n x^n}$ is the **leading term** of the polynomial.
- **n** is the **degree** of the polynomial.
- $\mathbf{a_n}$ is the **leading coefficient** of the polynomial.
- The leading term is ***always*** the term with the largest exponent.

Polynomial Function Examples

Polynomial Example 1

> ***f*(x) = $2x^2 + 3x - 10$**
>
> - **$2x^2$** is the leading term
> - **2** is the LC (Leading Coefficient)
> - ^{2} is the degree

Polynomial Example 2

> ***g*(x) = $2x^5 + 4x^4 - 13x^2 + 8$**
>
> - **$2x^5$** is the leading term
> - **2** is the LC
> - ^{5} is the degree

Polynomial Example 3

> ***h*(x) = $-8x^2 + 3x^5 - 7x^7 - 9$**
>
> - **$-7x^7$** is the leading term
> - **-7** is the LC
> - ^{7} is the degree

Polynomial Example 4

> ***j*(x) = $14x^2 - 6x^2 - x$**
>
> - **$14x^2$** is the leading term
> - **14** is the LC
> - ^{2} is the degree

Polynomial Example 5

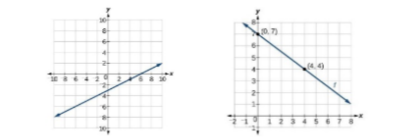
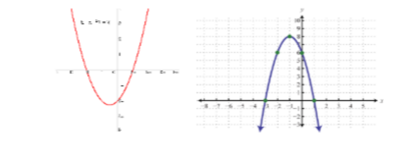
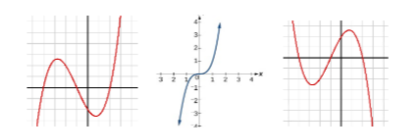
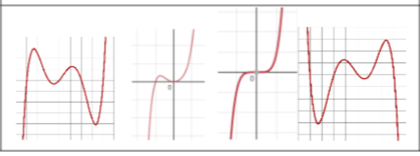
> ***k*(x) = 2 + 6x + 18x^2 - 2x^6**
>
> - **-2x^6** is the leading term
> - **-2** is the LC
> - **^{6}** is the degree

Graphs of Polynomial Functions

Polynomial Summary

[Polynomial Summary PDF]

Polynomial Function Patterns Summary

	Degree and Constant Differences	Concavity & Inflection Points & Extrema & Zeros	Sample Graph	End Behavior if leading coefficient > 0	End Behavior if leading coefficient < 0
Linear	1st degree constant 1st differences	0 concavity 0 inflection points 0 extrema 1 zero		As $x \to \infty$ $f(x) \to \infty$ As $x \to -\infty$ $f(x) \to -\infty$	As $x \to \infty$ $f(x) \to -\infty$ As $x \to -\infty$ $f(x) \to \infty$
Quadratic	2nd degree constant 2nd differences	1 concavity 0 inflection points 1 extrema up to 2 zeros		As $x \to \infty$ $f(x) \to \infty$ As $x \to -\infty$ $f(x) \to \infty$	As $x \to \infty$ $f(x) \to -\infty$ As $x \to -\infty$ $f(x) \to -\infty$
Cubic	3rd degree constant 3rd differences	2 concavities 1 inflection point 0 or 2 extrema up to 3 zeros		As $x \to \infty$ $f(x) \to \infty$ As $x \to -\infty$ $f(x) \to -\infty$	As $x \to \infty$ $f(x) \to -\infty$ As $x \to -\infty$ $f(x) \to \infty$
Quartic	4th degree constant 4th differences	1 or 3 concavities 0 or 2 inflection points 1 or 3 extrema up to 4 zeros		As $x \to \infty$ $f(x) \to \infty$ As $x \to -\infty$ $f(x) \to \infty$	As $x \to \infty$ $f(x) \to -\infty$ As $x \to -\infty$ $f(x) \to -\infty$
Quintic	5th degree constant 5th differences	2 or 4 concavities 1 or 3 inflection points 0, 2 or 4 extrema up to 5 zeros		As $x \to \infty$ $f(x) \to \infty$ As $x \to -\infty$ $f(x) \to -\infty$	As $x \to \infty$ $f(x) \to -\infty$ As $x \to -\infty$ $f(x) \to \infty$

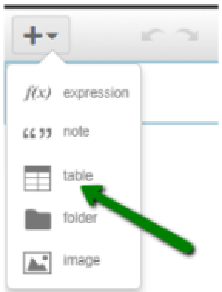
Regression Usng Desmos

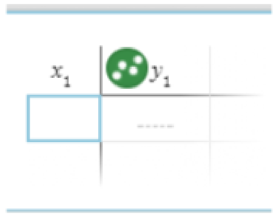
Regression Using Desmos

1. **Enter the Data by adding a table**

x_1 is the input data

y_1 is the output data





2. **Create the regression equation**
   a. The overall syntax is:

$y_1 \sim$ (function choice with any letter choice parameters a, b, c,...) and x_1 as the input variable

Samples:

Linear

$y_1 \sim ax_1 + b$ **or** $y_1 \sim hx_1 + z$

Quadratic

$y_1 \sim ax_1^2 + bx_1 + c$

Cubic

$y_1 \sim ax_1^3 + bx_1^2 + cx_1 + d$

Logistic

$$y_1 \sim \frac{a}{\left(1 + be^{(-cx_1)}\right)}$$

Quartic

$y_1 \sim ax_1^4 + bx_1^3 + cx_1^2 + dx_1 + f$

Do NOT use "e" as a parameter; desmos reserves "e" as 2.71828...
Use any other letter

Exponential

$y_1 \sim a \cdot b^{x_1}$

Log Mode

Click the "Log Mode" button

Power

$y_1 \sim a \cdot x_1^b$

The parameters and correlation coefficient will populate below the model of choice as shown:

Correlation Coefficient

STATISTICS

$R^2 = 0.7857$

RESIDUALS

e_1 plot

PARAMETERS

$a = 0.0959797$ $b = -0.422125$

$c = 0.107432$ $d = 2.21871$

Model Parameters

- Because polynomial functions are fairly predictable, we can summarize the characteristics and appearance of the graphs of polynomial functions of the first through fifth degree.

End Behavior of Polynomial Functions

For any polynomial function, as **x** approaches **±∞**, $\boldsymbol{f}$**(x)**
approaches **±∞**.

- As the magnitude (absolute value) of x gets larger and larger, the magnitude of the function values will also get larger and larger.
- Symbolically, we write:

> **as x → ±∞, $\boldsymbol{f}$(x) → ±∞**

Determining End-Behavior

Rather than graphing out each individual polynomial function, instead locate the leading term and determine the end-behavior based off it.

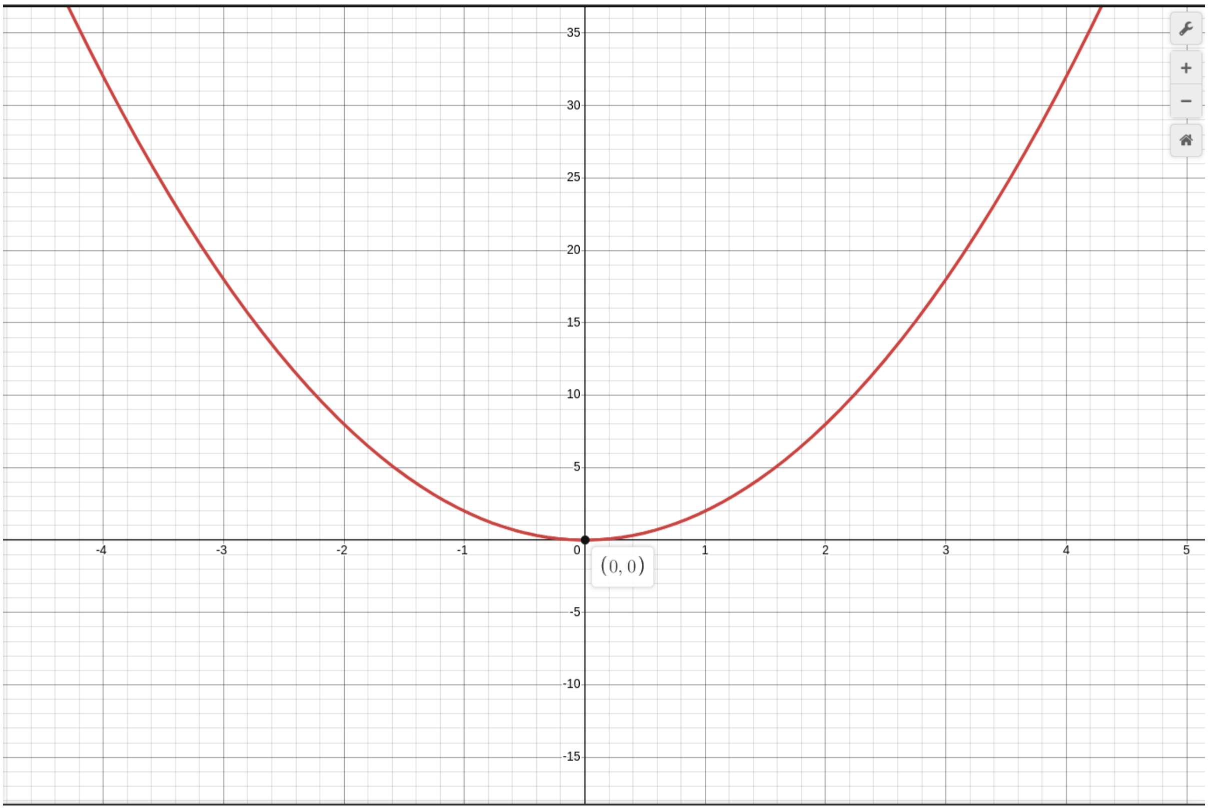
Some general rules for the end behavior of polynomial functions:

- If the degree of the polynomial is even, the end behavior will be the same, going to either positive or negative infinity, based on the sign of the leading coefficient.
- If the degree of the polynomial is odd, the function will have opposite end-behavior of the sign of the leading coefficient, going to either positive or
  negative infinity, based on the sign of the leading coefficient.

The value of the output is impacted the most by the leading term of the function.

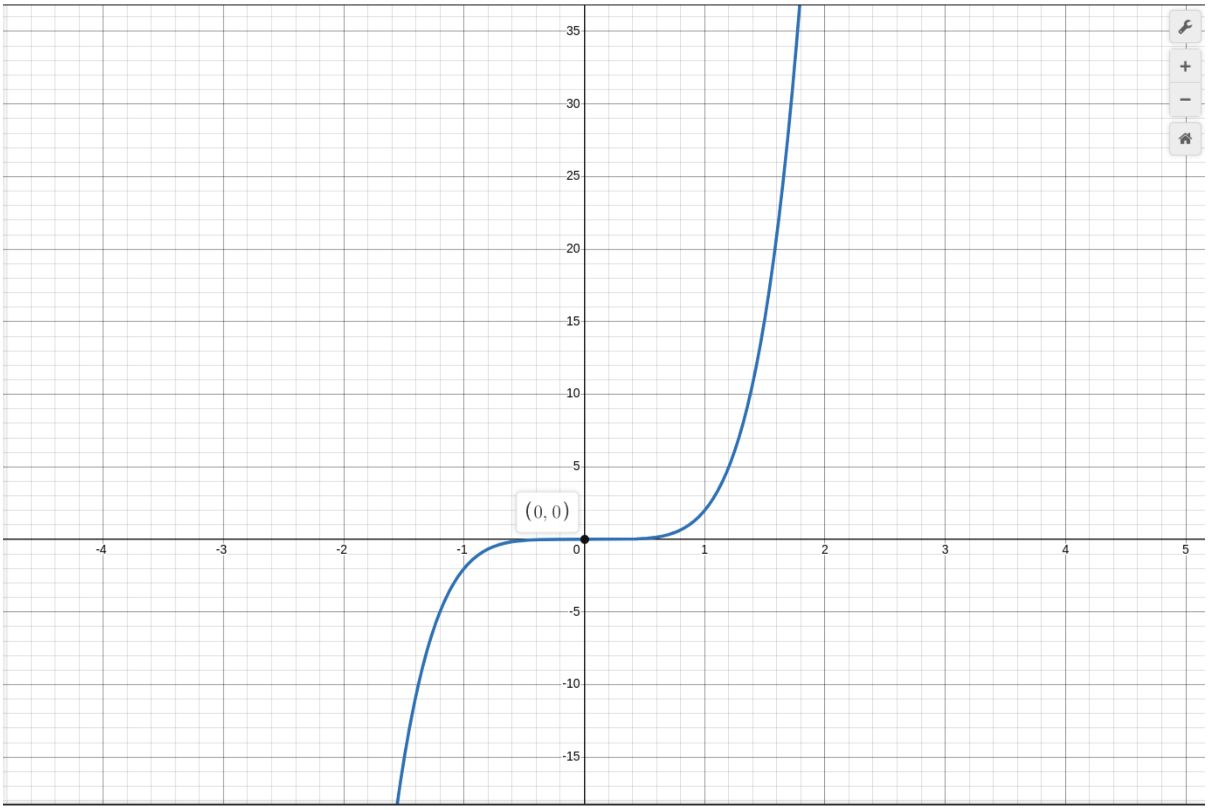
- Because of this, **you only need to graph the leading term**.
- The end-behaviors are referring to each concavity and how the end-behaviors will be either the same or opposite of each-other.

Function 1



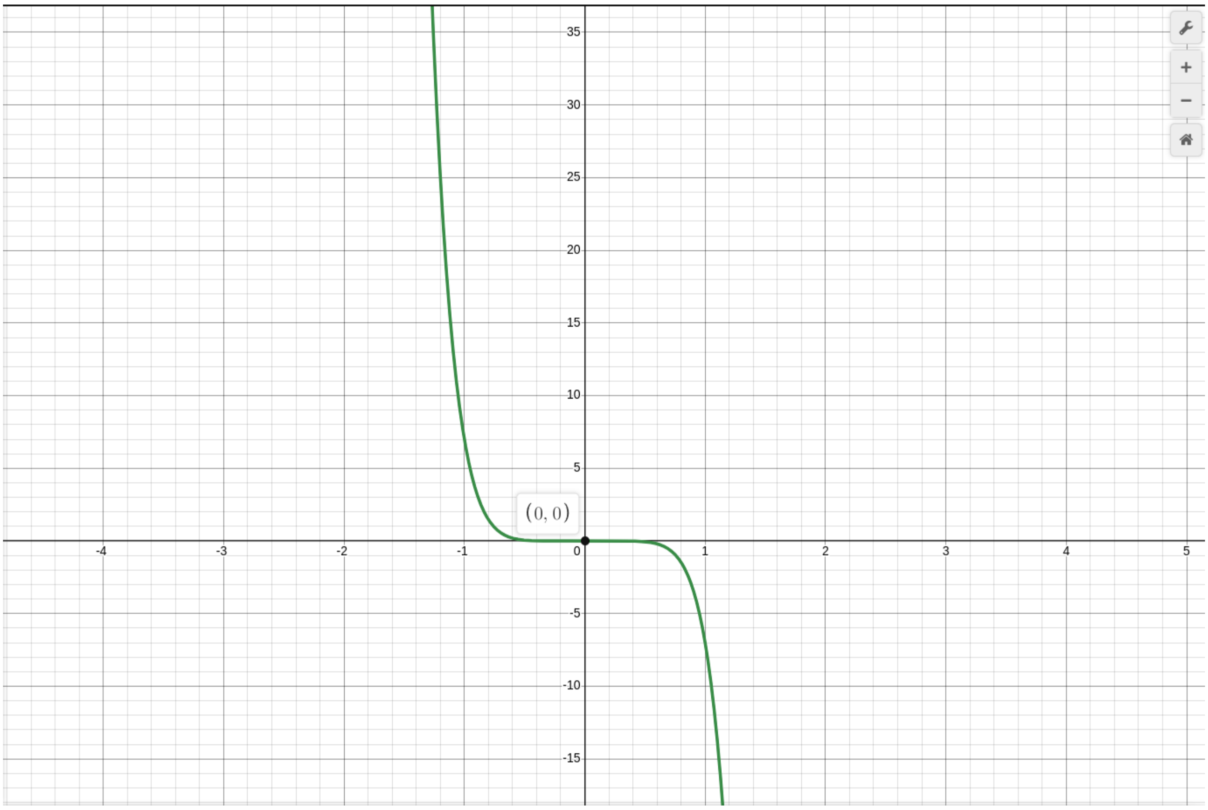
> ***f*(x) = $2x^2$ + 3x - 10**
>
> > As **x** → **±∞, *f*(x) ≈ $2x^2$**
> >
> > As **x** → **±∞, *f*(x)** → **∞**
>
> - Only **$2x^2$** was graphed because it is the leading term.

Function 2



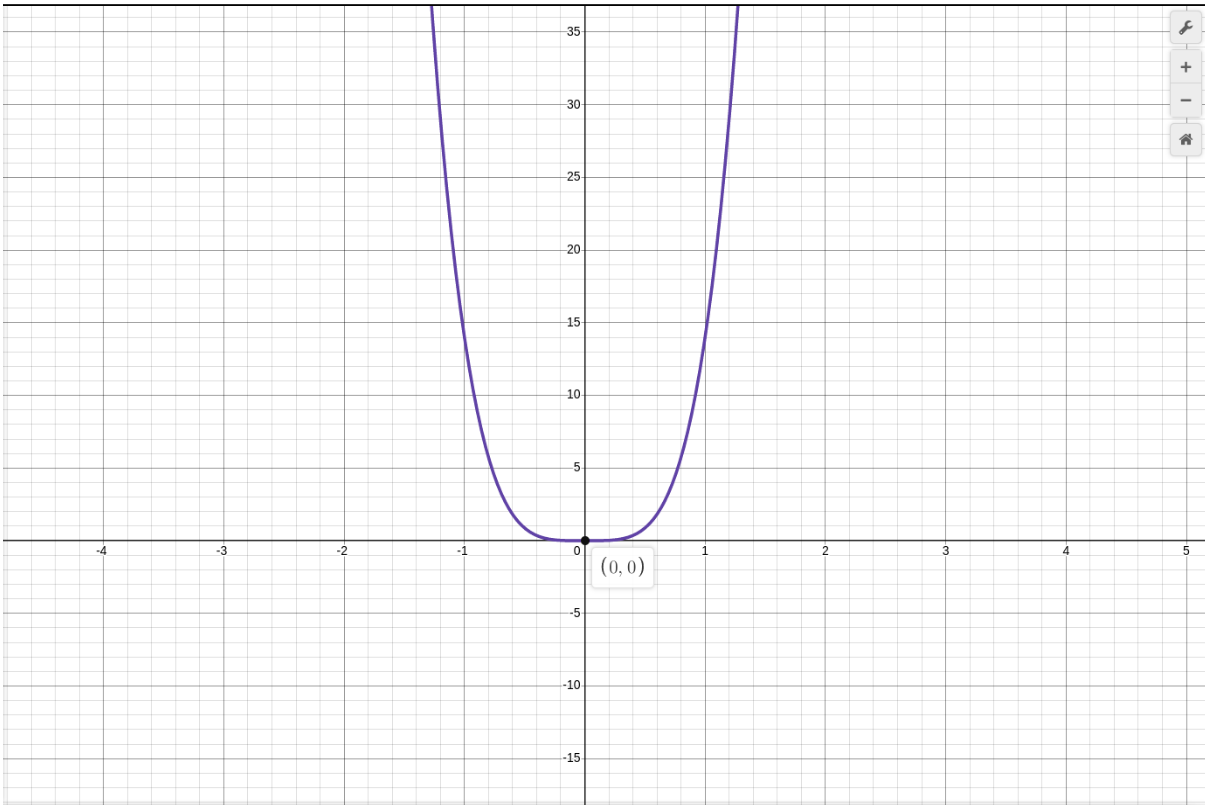
> $g(x) = 2x^5 + 4x^4 - 13x^2 + 8$
>
> > As $x \to \pm\infty$, $g(x) \approx 2x^5$
> >
> > As $x \to +\infty$, $g(x) \to \infty$
> >
> > As $x \to -\infty$, $g(x) \to -\infty$
>
> - Only $2x^5$ was graphed because it is the leading term.

Function 3



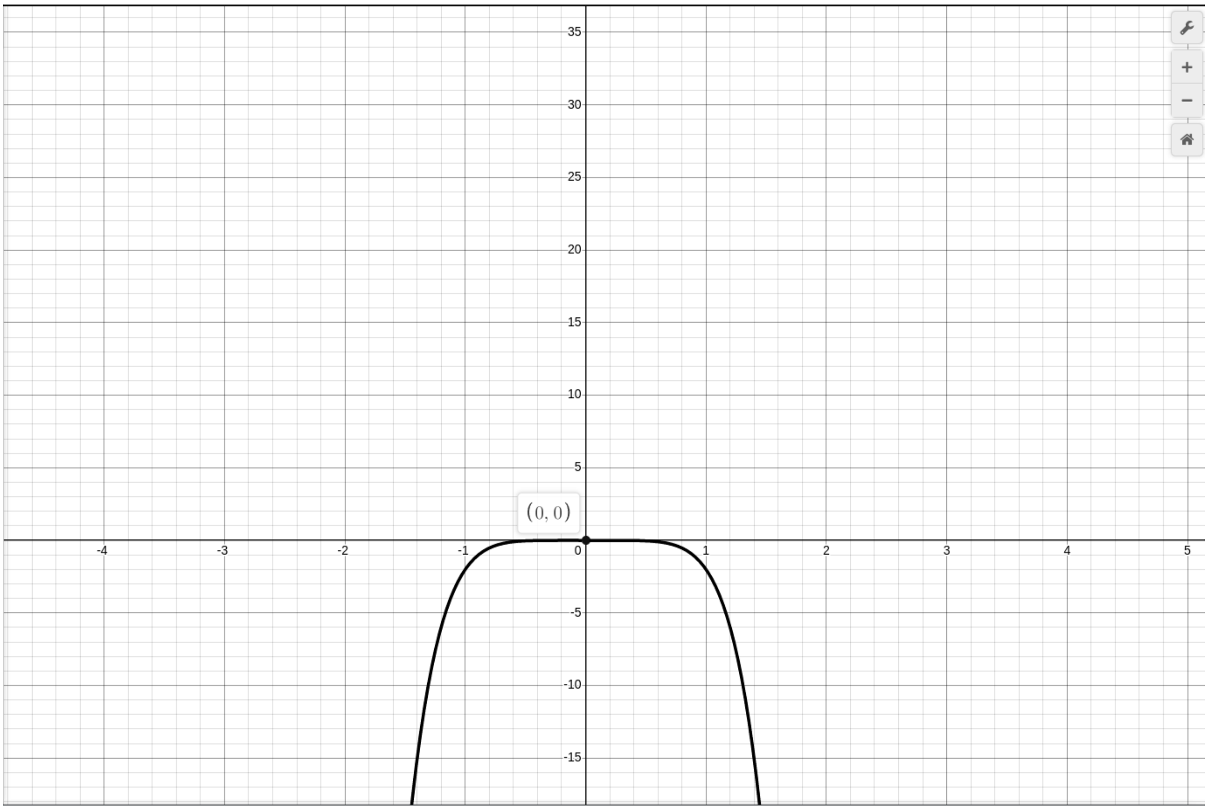
> ***h*(x) = -8x^2 - 3x^5 - 7x^7 - 9**
>
> > As **x** → **±∞, *h*(x) ≈ -7x^7**
> >
> > As **x** → **+∞, *g*(x)** → **–∞**
> >
> > As **x** → **–∞, *g*(x)** → **∞**
>
> - Only **7x^7** was graphed because it is the leading term.

Function 4



> ***j*(x) = $14x^4 - 6x^2 - x$**
>
> > As **x** → **±∞, *j*(x) ≈ $14x^4$**
> >
> > As **x** → **±∞, *j*(x)** → **∞**
>
> - Only **$14x^4$** was graphed because it is the leading term.

Function 5



> ***k*(x) = 2 + 6x + 18x^2 - 2x^6**
>
> > As **x** → **±∞, *k*(x) ≈ -2x^6**
> >
> > As **x** → **±∞**, ***k*(x)** → **−∞**
>
> - Only **2x^6** was graphed because it is the leading term.

Relative and Absolute Extrema of Polynomial Functions

- The term **relative extrema** is used to refer to maxima and minima simultaneously.
- The graph of a polynomial function of degree ***n*** will have at most ***n* - *1*** relative extrema but it may have fewer.
- An **odd** function will **never** have an absolute max or min
- An **even** function will **always** have an absolute max or min

Relative Extrema of Polynomial Functions

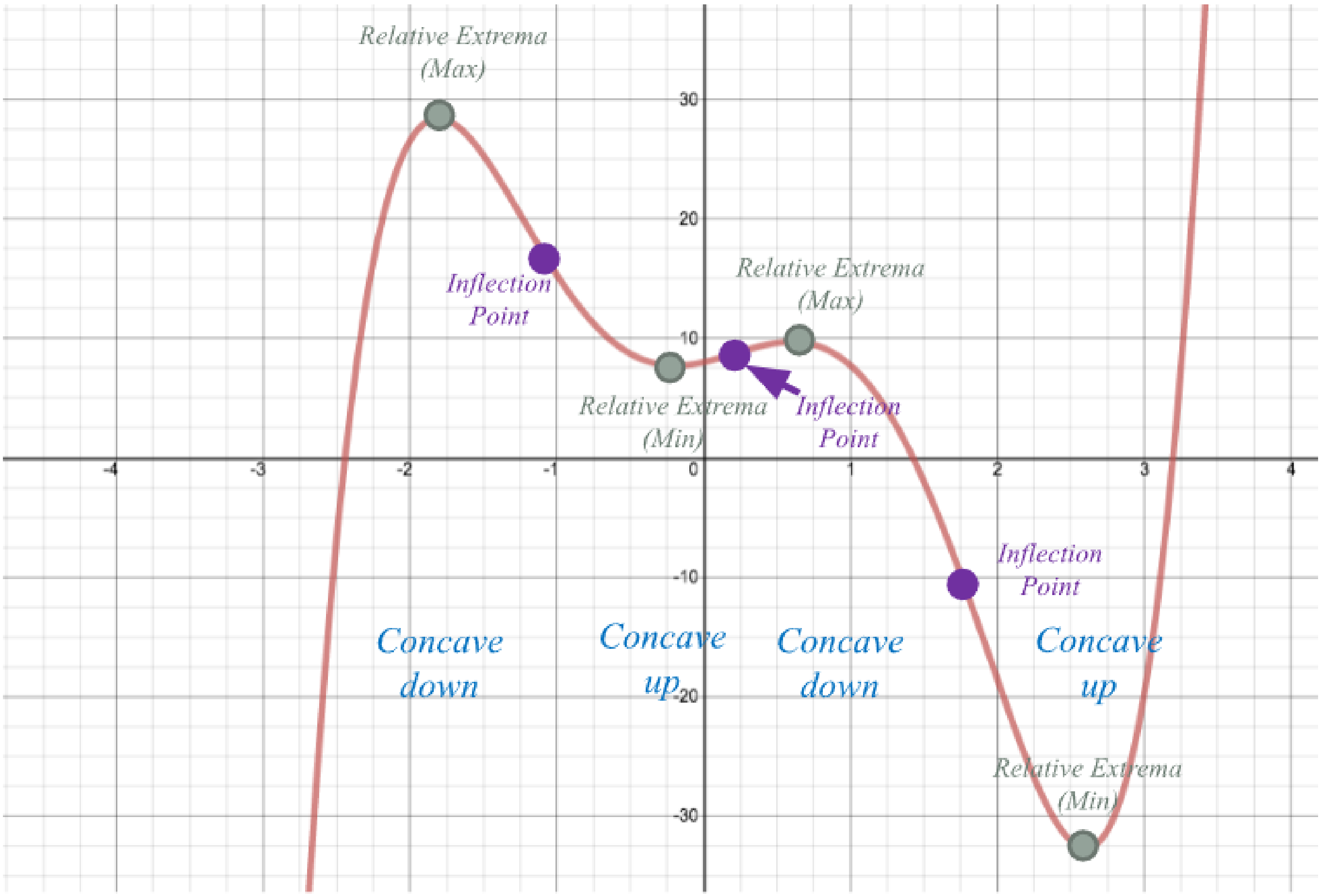
- A **relative maximum** occurs at the point where a graph changes from increasing to decreasing.
- A **relative minimum** occurs at the point where a graph changes from decreasing to increasing.

Absolute Extrema of Polynomial Functions

- A relative maximum is called an **absolute maximum** if the function value is never larger than at this point for all inputs.
- A relative minimum is called an **absolute minimum** if the function value is never smaller than at this point for all inputs.

Identifying All Points On a Graph

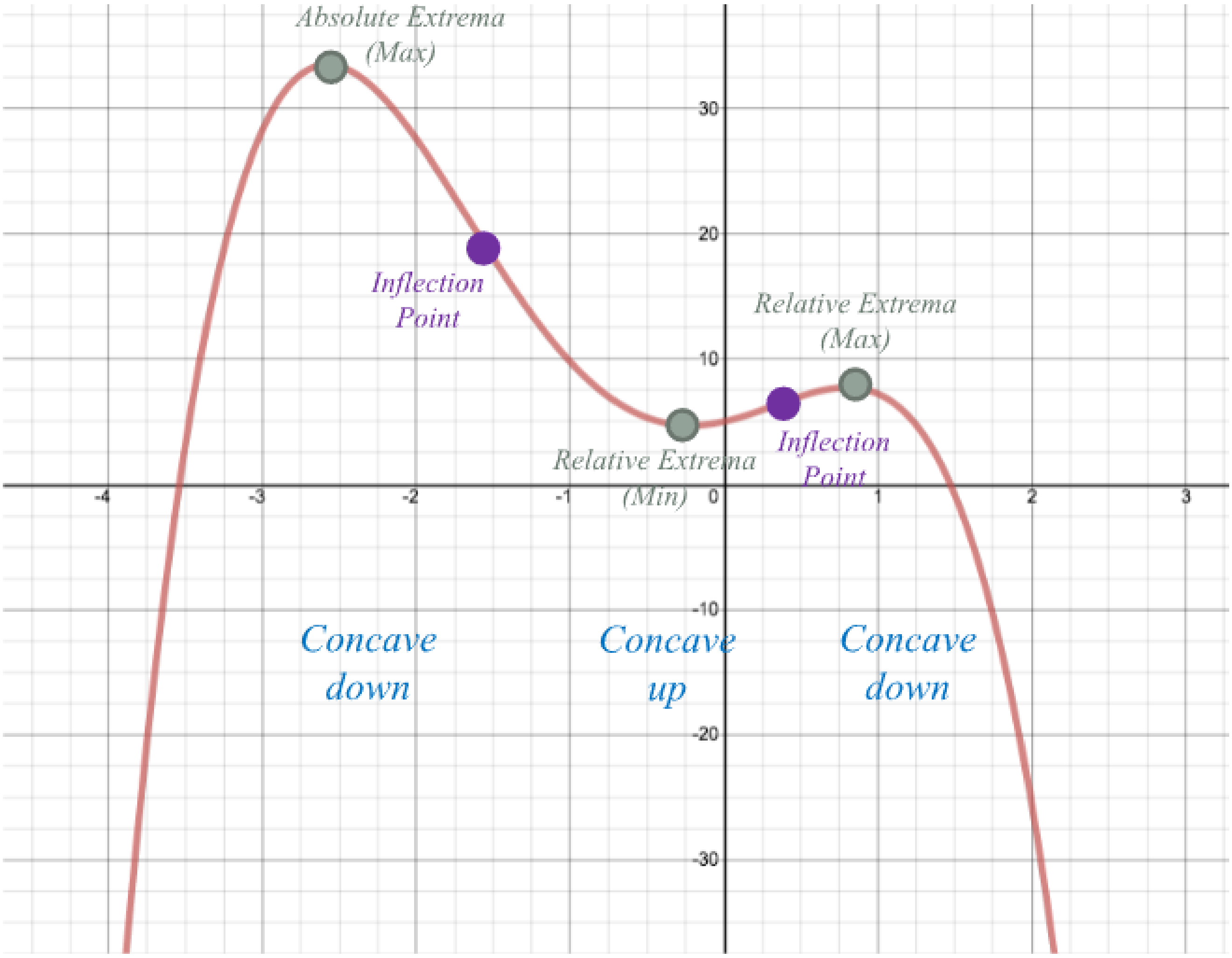
Example 1



- This could be a 5th degree polynomial because the function has opposite end behavior (so the degree must be odd), there are 4 concavities, 3 inflection points, and 4 relative extrema.

- The inflection points are halfway between each extrema

Example 2

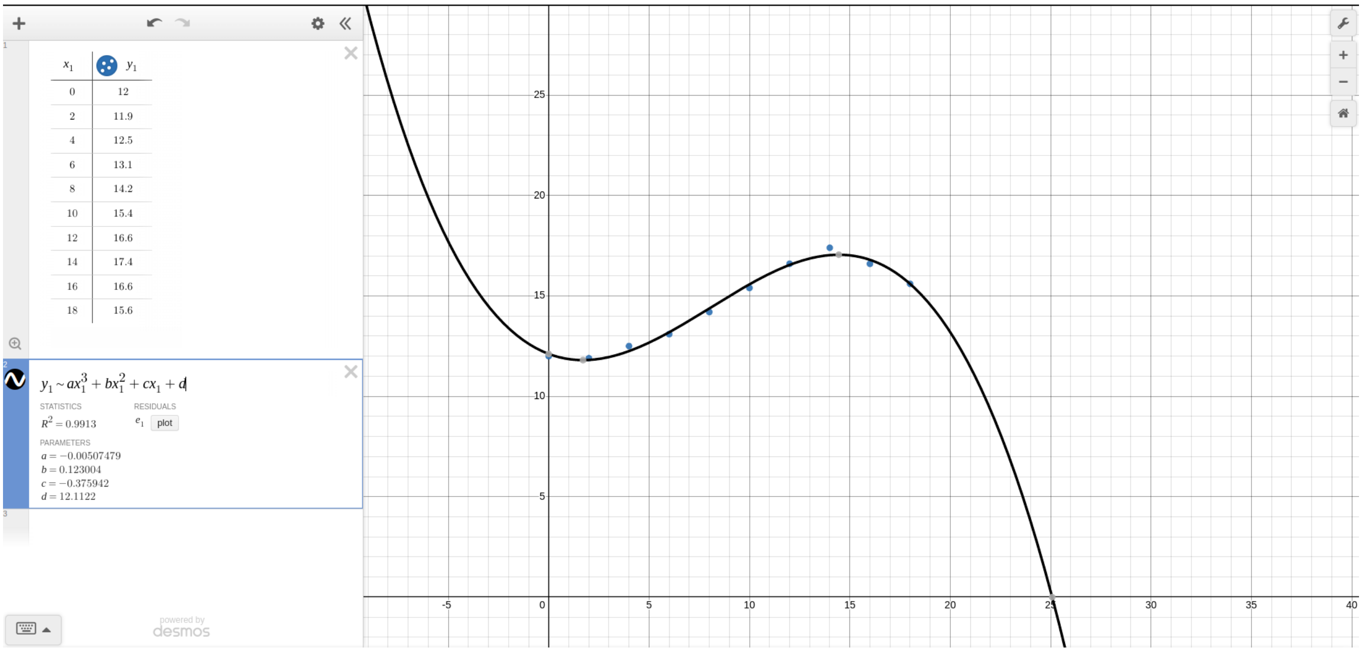


- This could be a 4th degree polynomial because the function has the same end behavior (so the degree must be even), there are 3 concavities, 2 inflection points, and 3 relative extrema (one absolute)

Computing and Graphing a Cubic Function

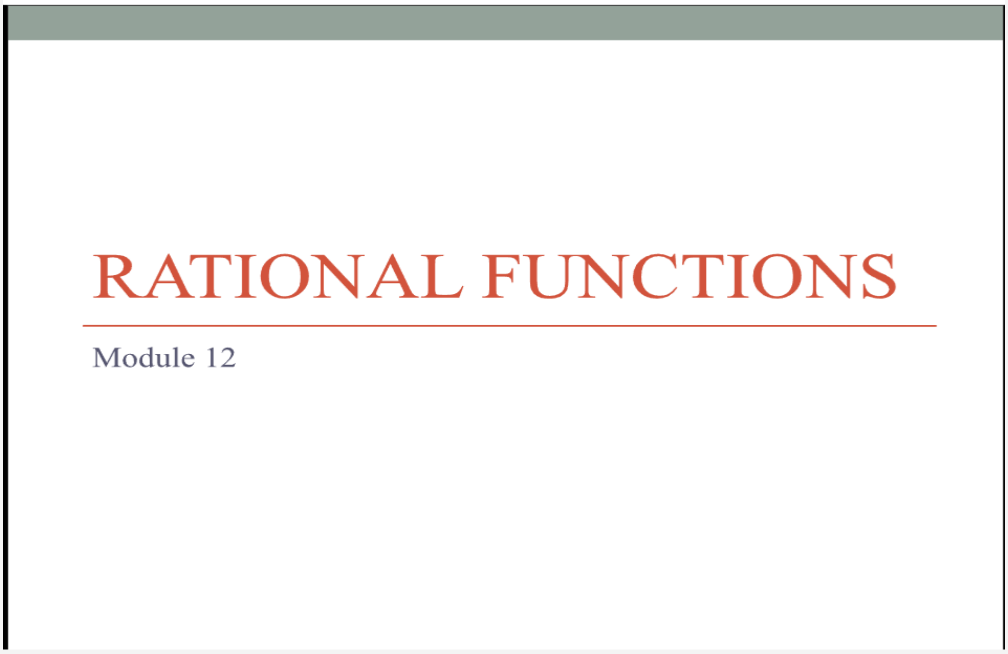
Use the following formula to graph a cubic function:

> **$y_1 \sim ax_1^3 + bx_1^2 + cx_1 + d$**



Module 12 - Rational Functions

Module 12 - Google Slides

> **Google Slides:**
>
> [Slides Link]
>
> 
>

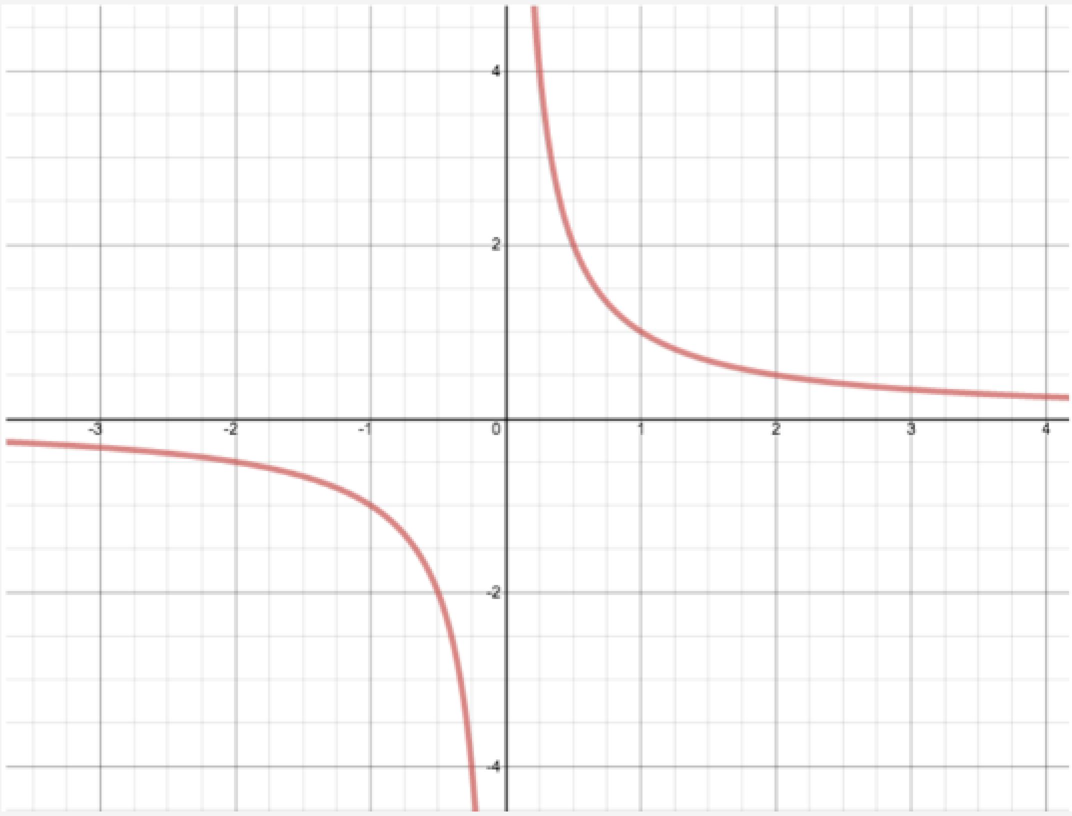
Rational Functions

When one polynomial function is divided by another, a **rational function** is created:

> $f(x) = {}^{p(x)}/_{q(x)}$

- Where ***p*(x)** and ***q*(x)** are polynomial functions with ***q*(x) ≠ 0**

The most basic form of a Rational Function:

> $f(x) = {}^{1}/_{x}$
>
> 
>

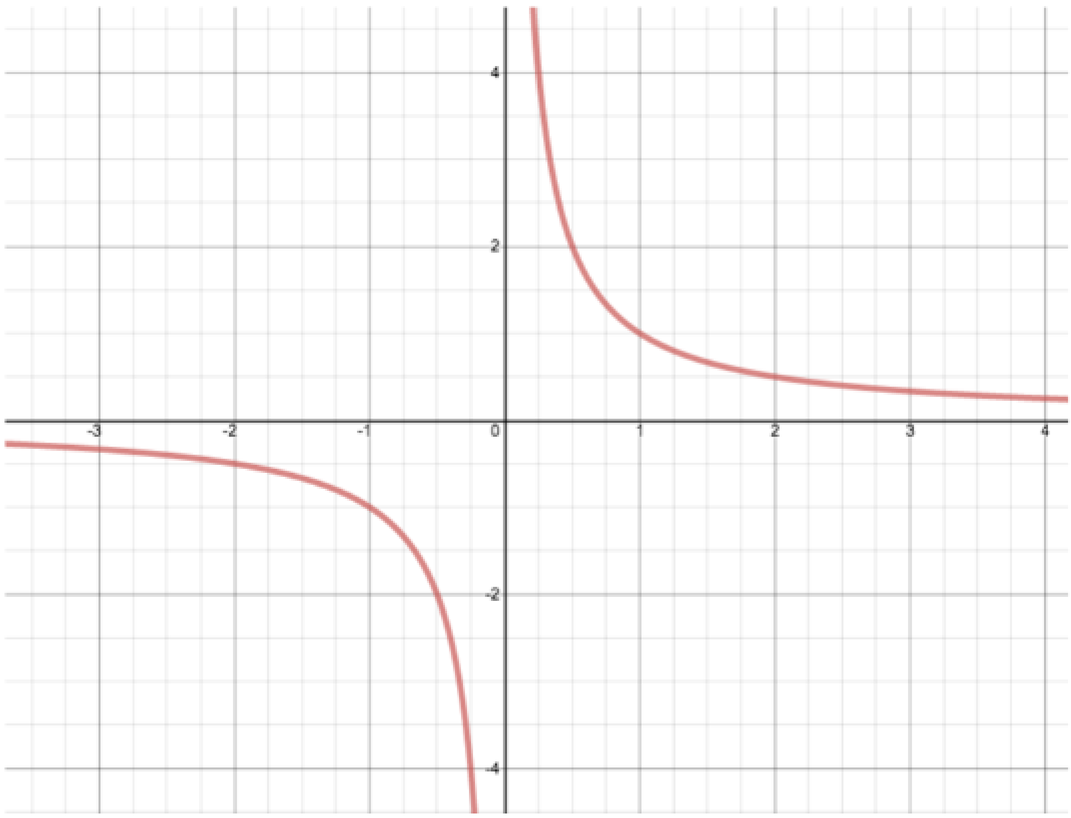
- At **x = 0**, ***f*(x)** is undefined.
- As **x → 0**, the outputs will get larger and larger in both the positive negative directions.
    - Since $f(x) = {}^{1}/_{x}$, as **x** gets smaller and smaller, ${}^{1}/_{x}$ gets larger and larger.
- There is a **vertical asymptote** at ***x* = 0**

Vertical Asymptotes

A **vertical asymptote** of a function ***ƒ*(x)** is a vertical line, **x = a**, that the graph of ***ƒx*(x)** approaches, but does not cross.

- More formally, as ***x*** approaches ***a***, ***ƒ*(x)** approaches **±∞**
  - Symbolically, we write this as **x → a**, ***ƒ*(x) → ±∞**
- Vertical asymptotes are not part of the graph of a rational function.
  - They are often drawn still because they are helpful in describing how the function behaves.
- Remember that a vertical asymptote is a line *(not a value)*, so you need to express an asymptote using the formula for a line.
  - *I.e. x = some number*

Vertical Asymptote Graph Example



- As **x → ±∞, *ƒ*(x) → 0**
- Since ***ƒ*(x) = $^1⁄_x$**, as **x** gets larger and larger, $^1⁄_x$ gets smaller and smaller.

Horizontal Asymptotes

The **horizontal asymptote** of a rational function is a horizontal line, **$y = b$**, that the function approaches as the independent variables approaches **$-\infty$** or **∞**.

- A rational function can have both a **horizontal asymptote** and a **vertical asymptote**.
- To find horizontal asymptotes, you must know how the output values of the function behave as the input values approach **$\pm\infty$**.
- **Important:** The graph of a rational function never crosses a vertical asymptote. However, the graphs of some rational functions do cross their horizontal asymptotes.

Finding The Vertical and Horizontal Asymptote

- In general, a **vertical asymptote** occurs at values of ***x*** that make the denominator equal to zero.
  - You can determine the asymptotes using the function by setting the denominator equal to zero and solving for the input variable.
- In general, a **horizontal asymptote** can be determined by looking at the end behavior of the function.
  - If the value of the output approaches a specific value ***b*** as **$x \to \infty$** or **$x \to -\infty$**, then the function has a horizontal asymptote at **$y = b$**.
  - **Not all rational functions will have a horizontal asymptote.**

Finding The Asymptotes Example 1

Given the function:

> $g(x) = \frac{3}{x - 2}$

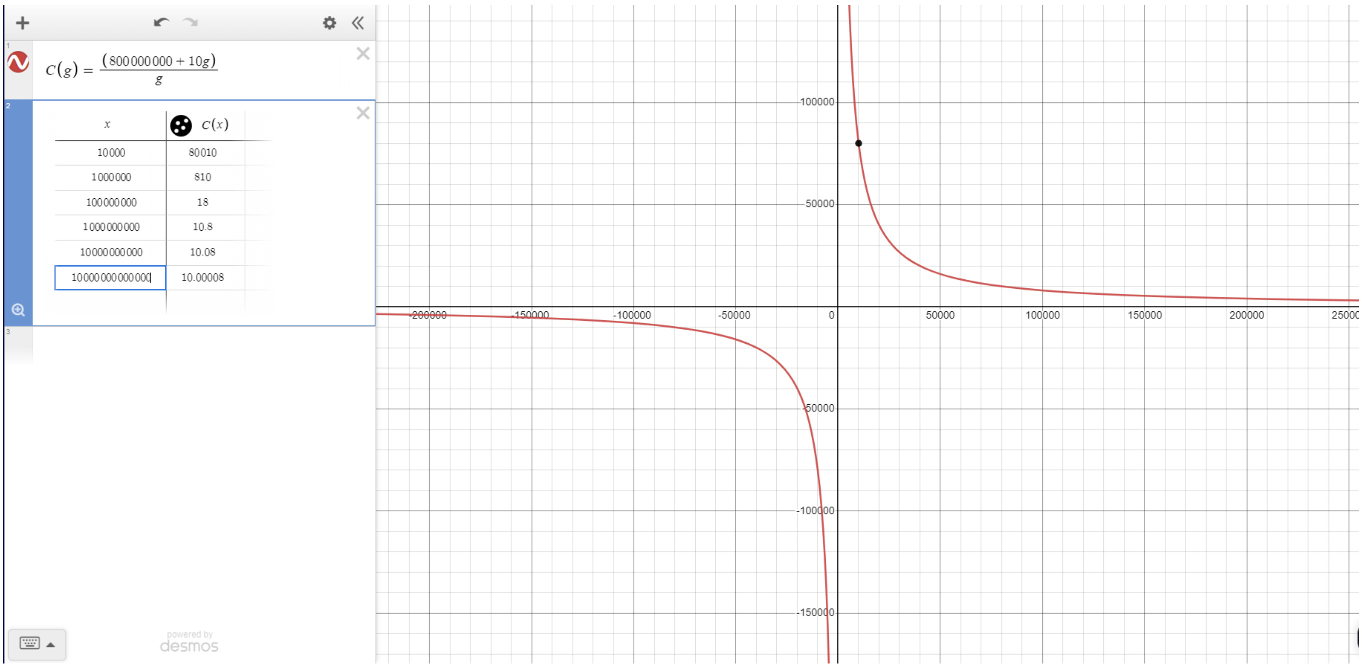
- The *vertical asymptote* is **$x = 2$**, because as **$x \to 2$**, **$g(x) \to \pm\infty$** and because **$x = 2$** is not in the domain (the denominator cannot equal **0**, and **$2 - 2 = 0$**).

- The *Horizontal asymptote* is **y = 0** because as **x** → ∞, ***g*(x)** → **0**
- Using transformations:
    - The function ***g*** has been stretched vertically by a factor of **3** and shifted right **2**, compared to ***f*(x) =** $^1/_x$
    - The vertical stretch does not impact either asymptote, but the horizontal shift will move the vertical asymptote right **2** units (from **x = 0**) to **x = 2**.

Finding The Asymptotes Example 2

Given the function, which gives the average cost (in dollars per gram) of producing **g** grams of a new drug:

> ***C*(g) =** $^{\mathbf{800,000,000 + 10g}}/_{\mathbf{g}}$



- The **vertical asymptote** is **g = 0**, because if **g = 0**, ***C*(g)** is undefined. **

- The **horizontal asymptote** is ***C*(g) = 10**

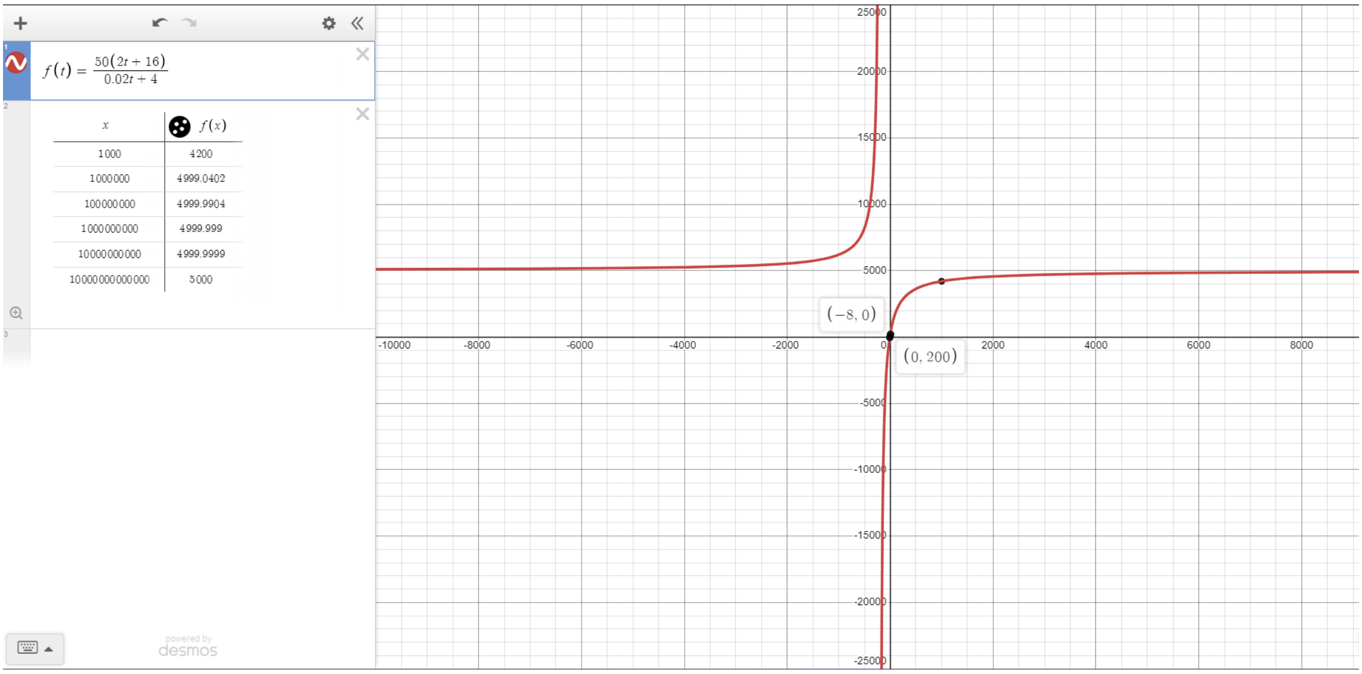
    - As we look at end behavior, we see that the output will get closer and closer to 10.

Finding The Asymptotes Example 3

> *Context:* A national park research team noticed a dramatic reduction in the deer population ina 150,000-acre protected area. In order to increase the population of deer, the park services introudced 125 additional deer into the area. The researchers' population model predicts that after this, the expected number of deer **N** with respect to time ***t*** in years since 1990 will be described by the function below.

Given the function:

> **N = f(t) = $^{50(2t + 16)}/_{0.02t + 4}$**



- **Vertical Intercept: [0, 200]**
    1. Replace the input with **0**
    2. **f(0) = $^{50(2(0) + 16)}/_{0.02(0) + 4}$**
    3. Solve
    4. **Vertical Intercept = [0, 200]**
    5. **Meaning:** In 1990, there were 200 deer.
- **Horizontal Intercept: [-8, 0]**
    1. **0 = $^{50(2t + 16)}/_{0.02t + 4}$**
    2. Multiply both sides by the denominator

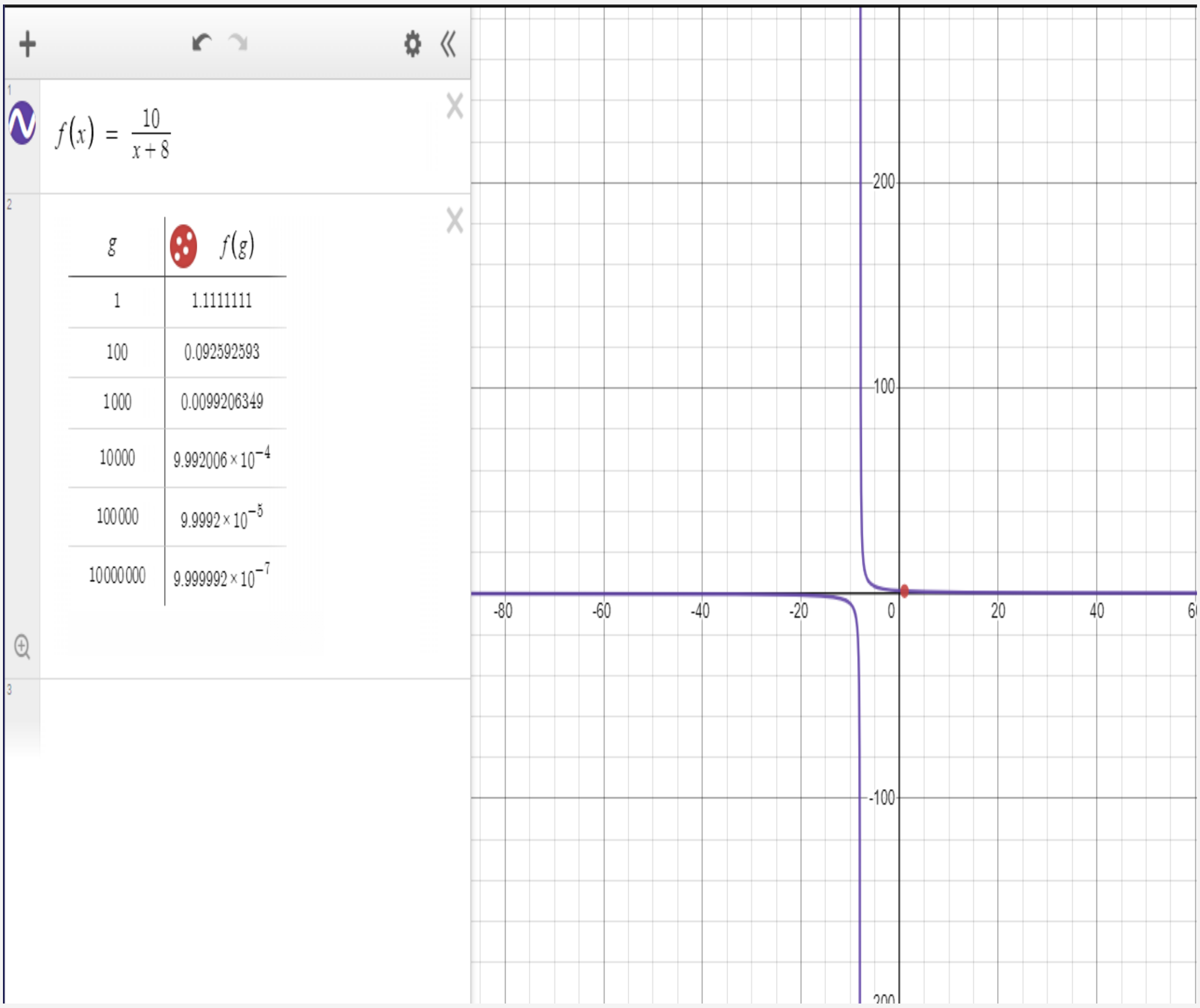
3. **0 = 50(2t + 16)****
4. Solve
5. **Horizontal Intercept = [0, 8]**
6. **Meaning:** There is no meaning.

- **Vertical Asymptote: x = 5000**
  1. Set the denominator equal to **0**.
  2. **0 = 0.02t + 4**
  3. Solve
  4. **t = -200**
  5. **Meaning:** There is no meaning.
- **Horizontal Asymptote: y = -16**
  1. Use a table in [Desmos] to determine, *or* use the end behavior of the function *(using the leading terms)*.
  2. As **t** → **±∞**, **N** → **5000**
  3. **N = 5000**
  4. **Meaning:** As time increases, the number of deer will approach 5000.

Finding The Asymptotes Example 4

Given the function:

$f(x) = {}^{10}/_{x + 8}$

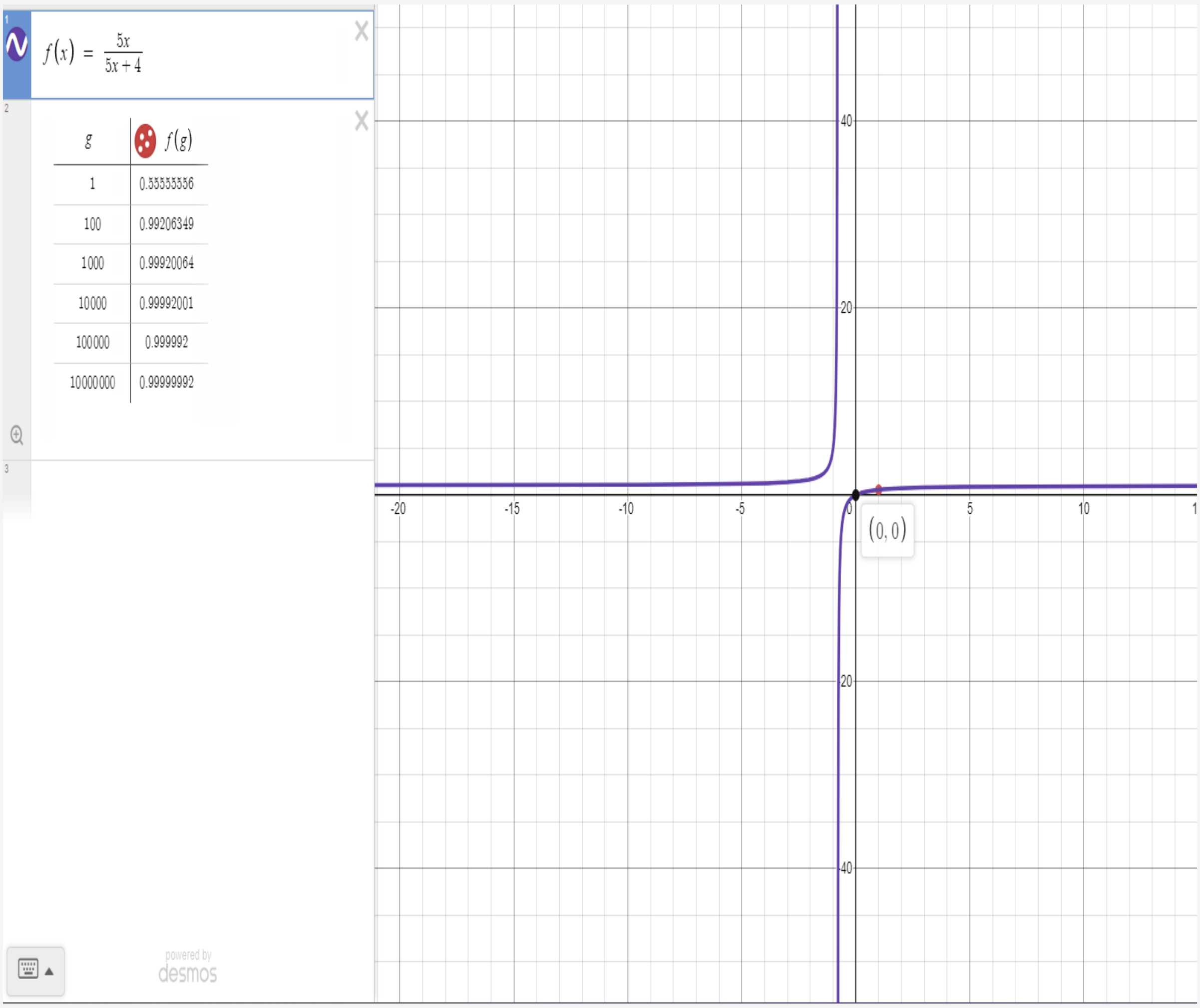


- **Vertical Asymptote: x = -8**
- **Horizontal Asymptote: y = 10**

Finding The Asymptotes Example 5

Given the function:

> $g(x) = {}^{5x}\!/_{5x + 4}$

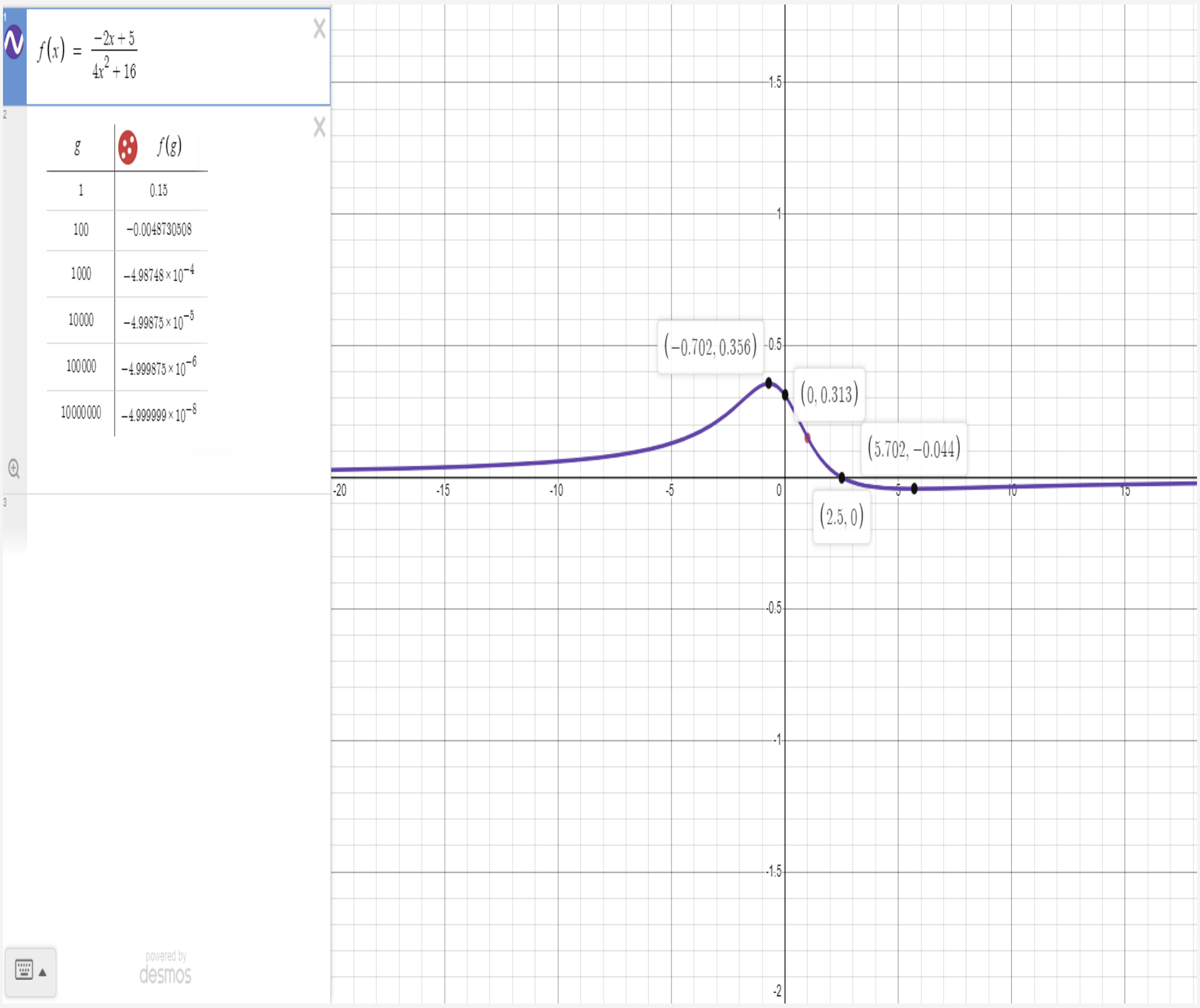


- **Vertical Asymptote: x = -$^{4}/_{5}$**
- **Horizontal Asymptote: y = 1**

Finding The Asymptotes Example 6

Given the function:

> **$h(x) = {}^{-2x + 5}/_{4x^2 + 16}$**

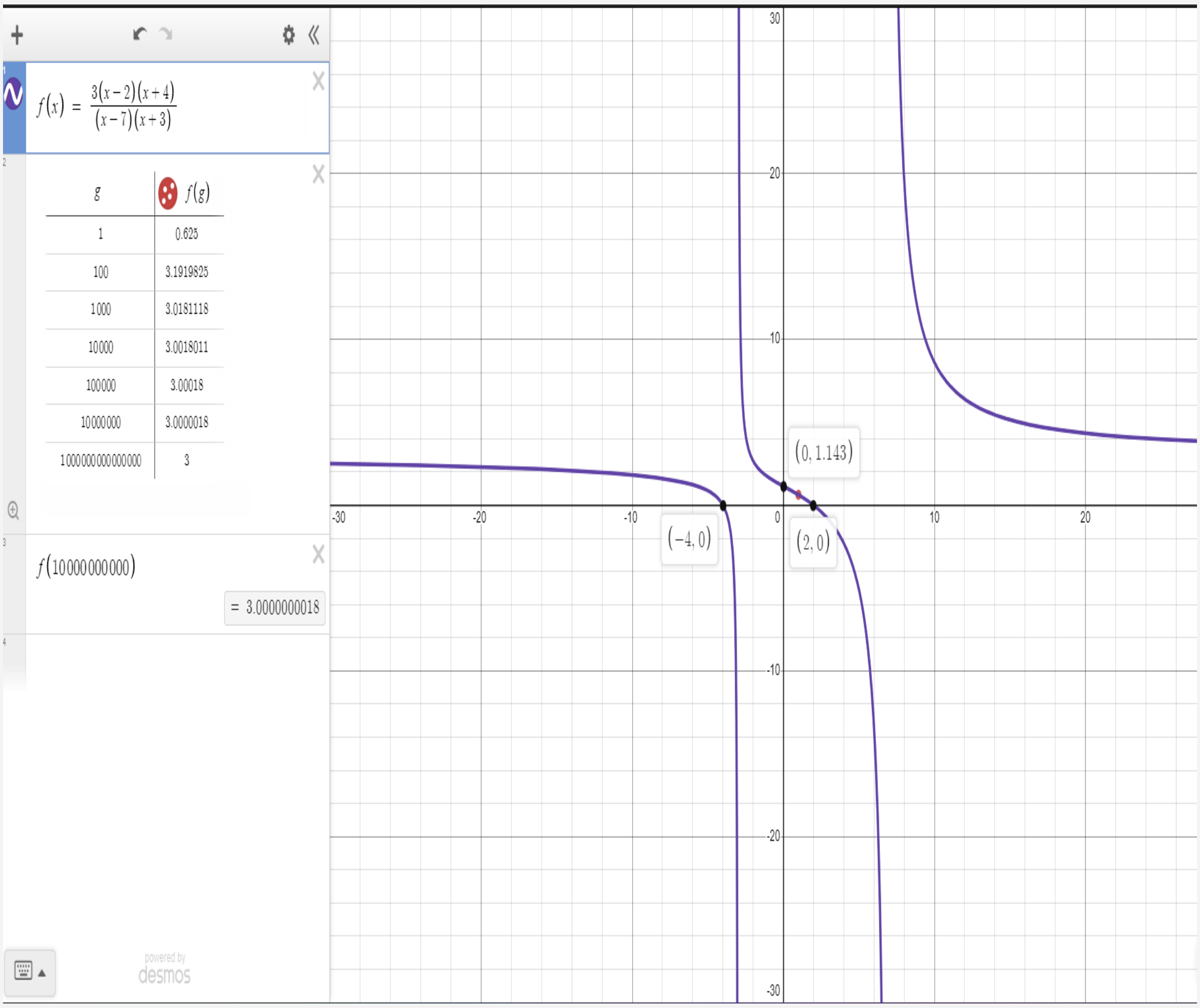


- **Vertical Asymptote: None**
  - There is no square root of **-16**.
- **Horizontal Asymptote: y = -5**

Finding The Asymptotes Example 7

Given the function:

$j(x) = {}^{3(x - 2)(x + 4)}/_{(x - 7)(x + 3)}$

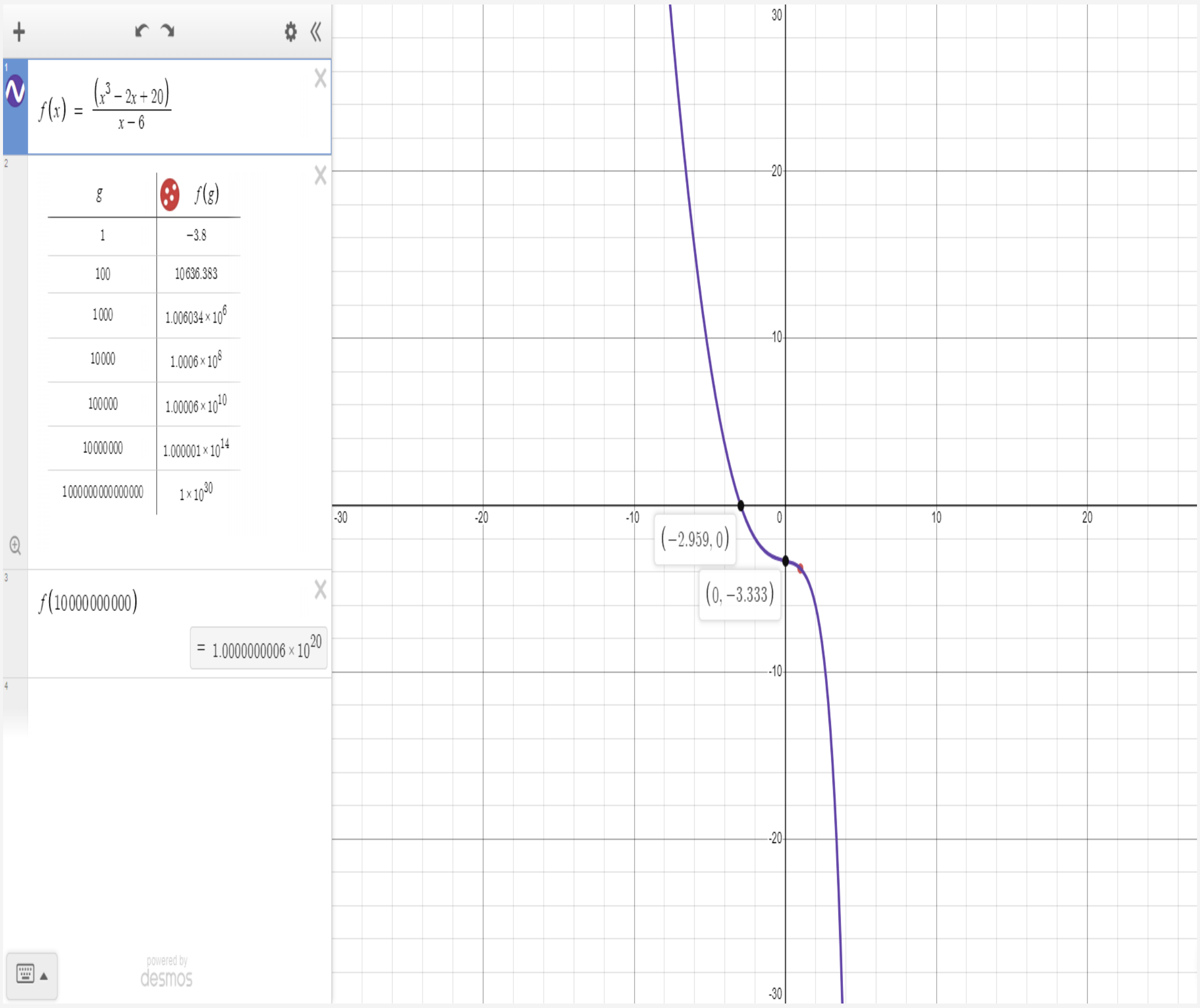


- **Vertical Asymptote: x = 7** & **x = -3**
- **Horizontal Asymptote: y = 3**

Finding The Asymptotes Example 8

Given the function:

> ***k*(x) = ${}^{x^3 - 2x + 20}/_{x - 6}$**



- **Vertical Asymptote: x = 6**
- **Horizontal Asymptote: None**
  - The output gets bigger and bigger as input gets bigger and bigger

---

Finding Horizontal Asymptotes With Leading Terms

Remember that the end behavior of a **Polynomial Function** is determined by the leading term of the polynomial and that **horizontal asymptotes are talking about the end behavior**.

Example:

> ***k*(x) = 8x^5 + 12x^4 - 4x^8 + 12**

- The leading term is **$-4x^8$**, so as **$x \to \pm\infty$**,
  ***k*(x) → $-4x^8$**

Because a Rational Function is the ratio of 2 polynomials, end behavior of a Rational Function can be determined by the ratio of the leading terms.

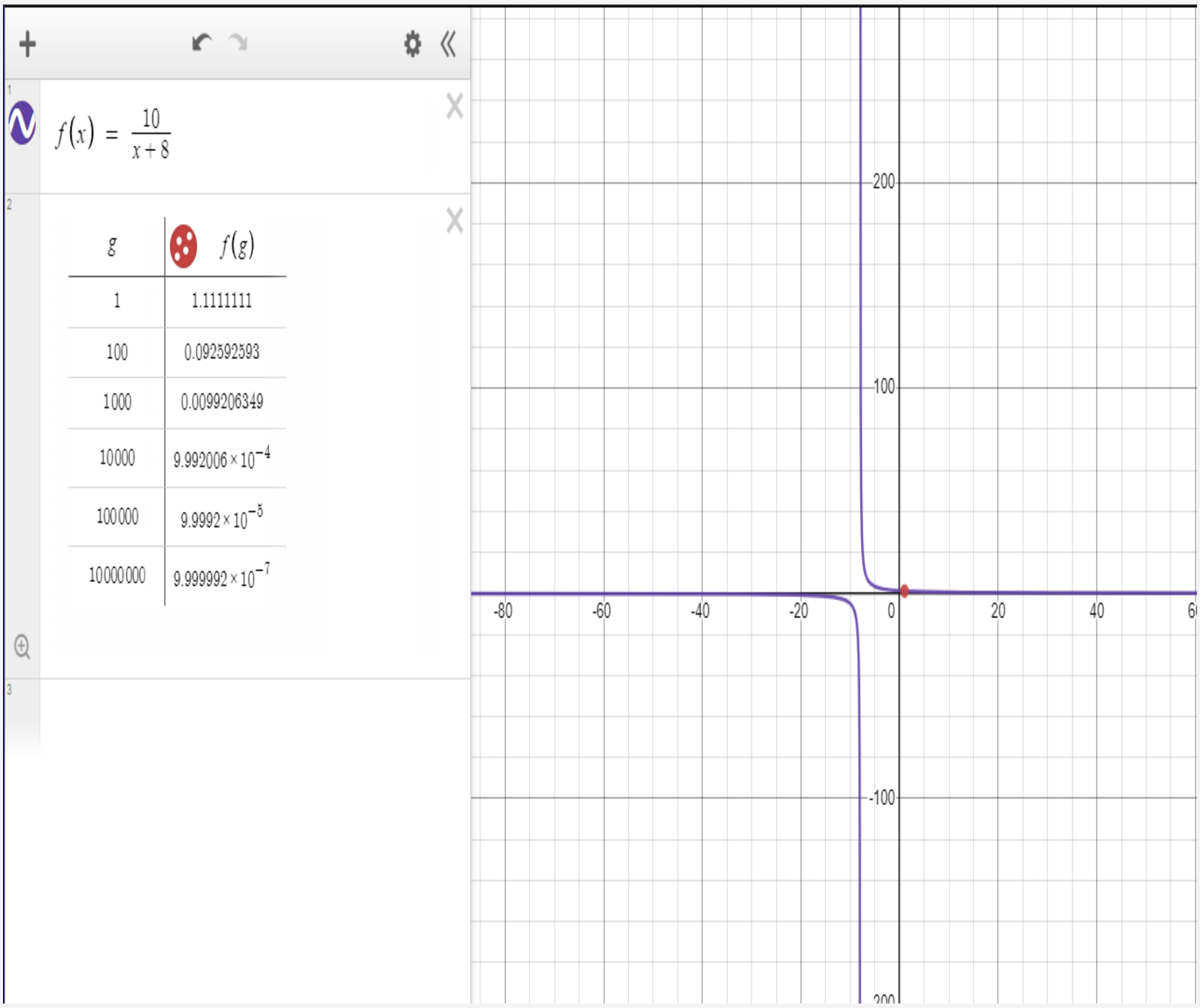
HOW TO: FIND HORIZONTAL ASYMPTOTES

For a rational function $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{ax^n + \cdots}{bx^m + \cdots}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator,

- If $n < m$ (i.e., if the degree of the numerator is less than that of the denominator), a horizontal asymptote occurs at $y = 0$.
- If $n = m$ (i.e., if the degree of the numerator is equal to that of the denominator), a horizontal asymptote occurs at $y = \dfrac{a}{b}$.
- If $n > m$ (i.e., if the degree of the numerator is greater than that of the denominator), the function does not have a horizontal asymptote.

Horizontal Asymptote - Leading Term Example 1

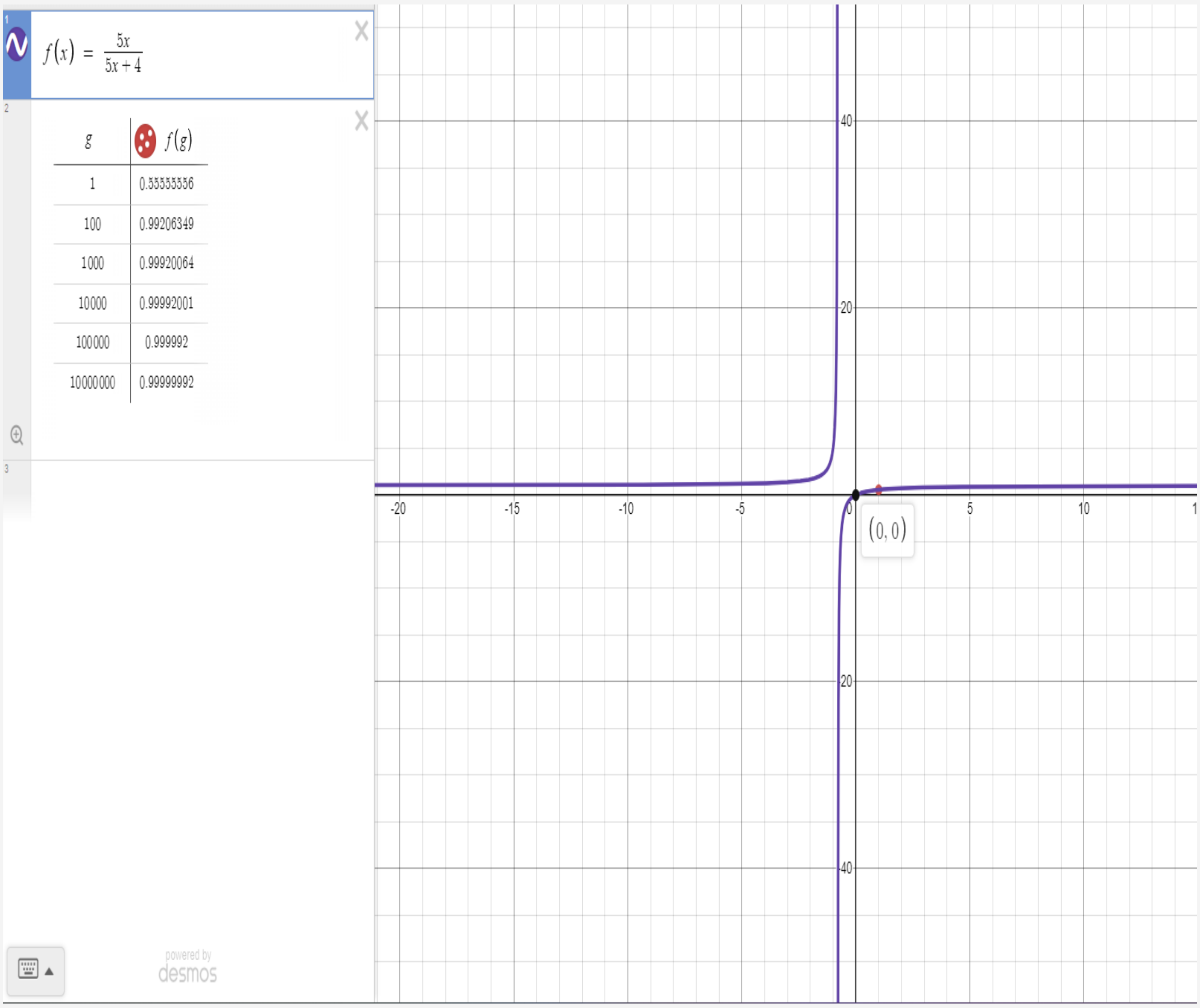
> **$f(x) = {}^{10}\!/_{x+8}$**



- As **x** → **±∞**, **f(x)** → **${}^{10}\!/_{x}$** → **0**
  - So HA is **y = 0**

Horizontal Asymptote - Leading Term Example 2

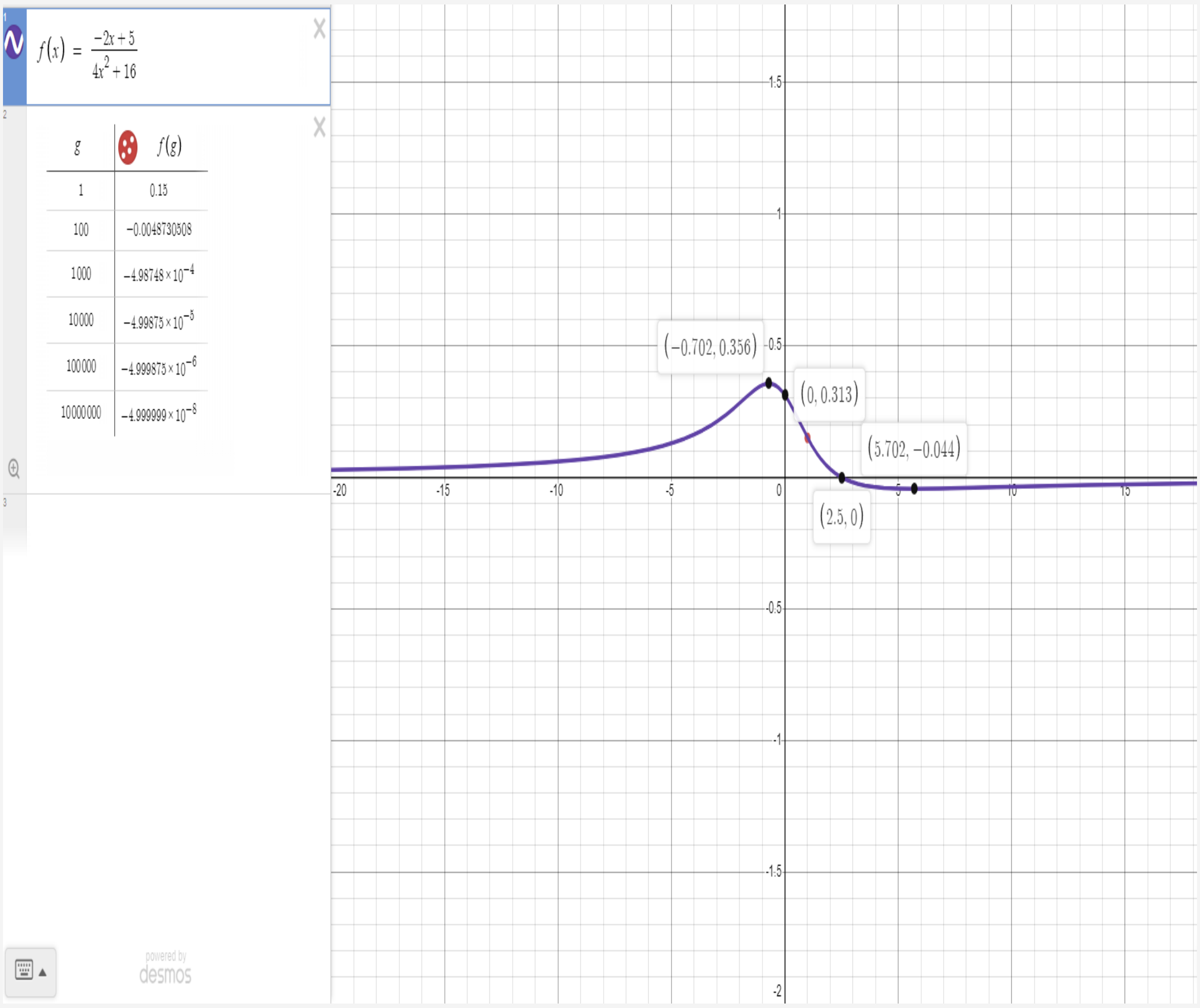
> $g(x) = \frac{5x}{5x + 4}$



- As $x \to \pm\infty$, $g(x) \to \frac{5x}{5x} \to 1$
  - So HA is $y = 1$

Horizontal Asymptote - Leading Term Example 3

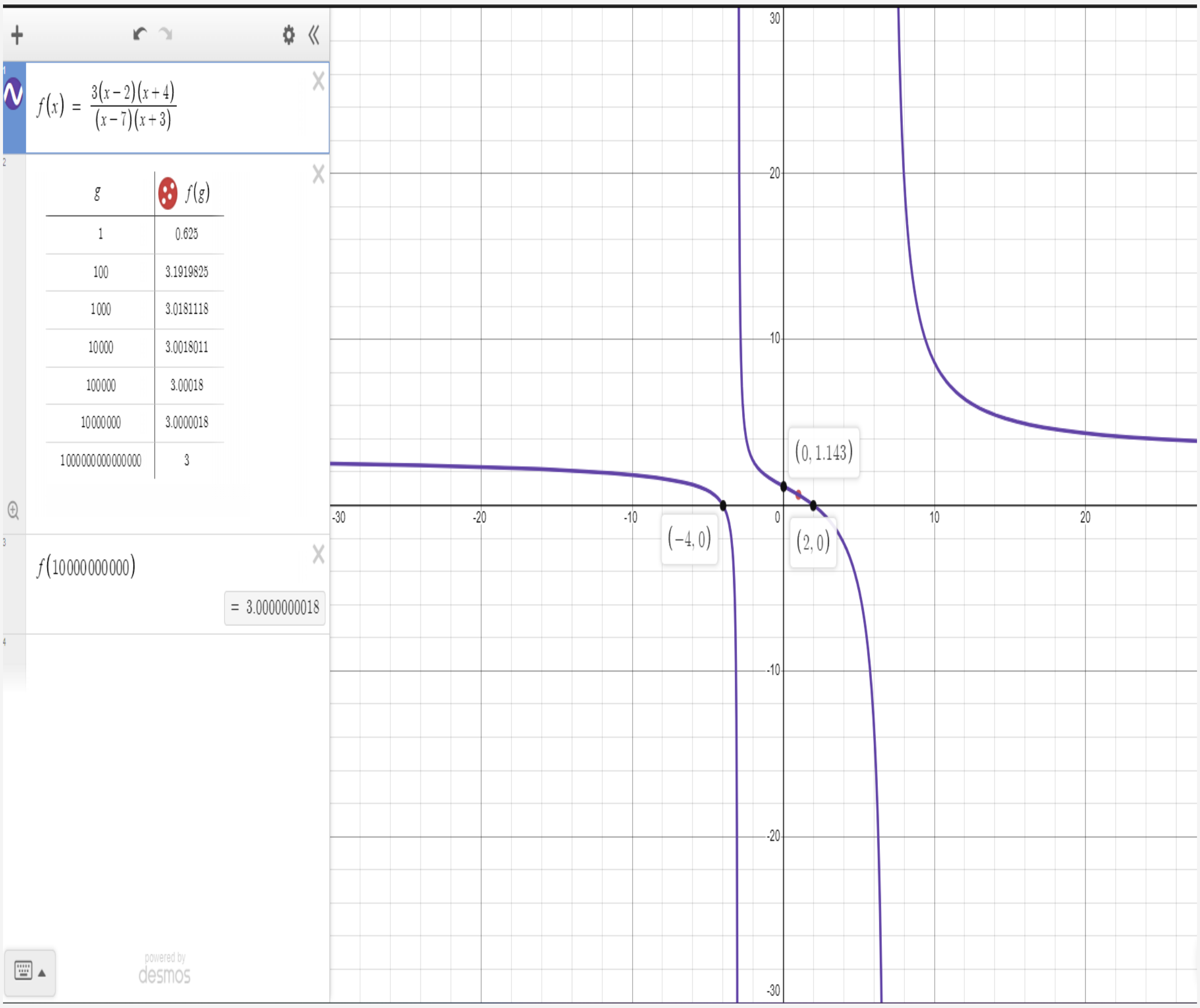
> ***h*(x) = $^{-2x + 5}/_{4x^2 + 16}$**



- As **x** ⟶ **±∞**, ***h*(x)** ⟶ **$^{-2x}/_{4x^2}$** ⟶ **0**
  - So HA is **y = 0**

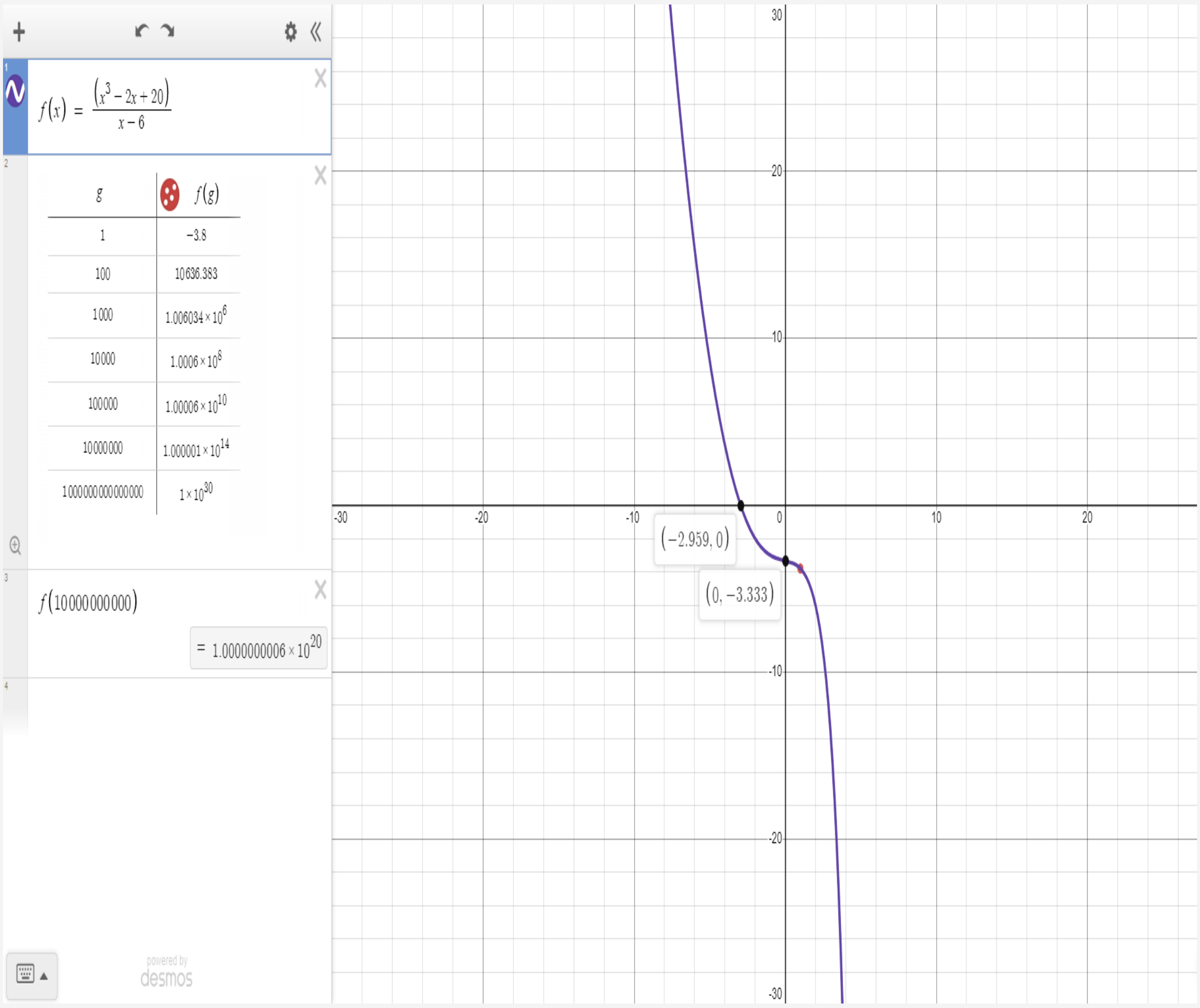
Horizontal Asymptote - Leading Term Example 4

> ***j*(x) = $^{3(x - 2)(x + 4)}/_{(x - 7)(x + 3)}$**



- As **x** → **±∞**, ***j*(x)** →
  $^{3x^2}/_{x^2}$ → 3
  - So HA is **y = 3**
  - The leading terms were gotten by multiplying (**3 * x * x**) and (**x * x**)

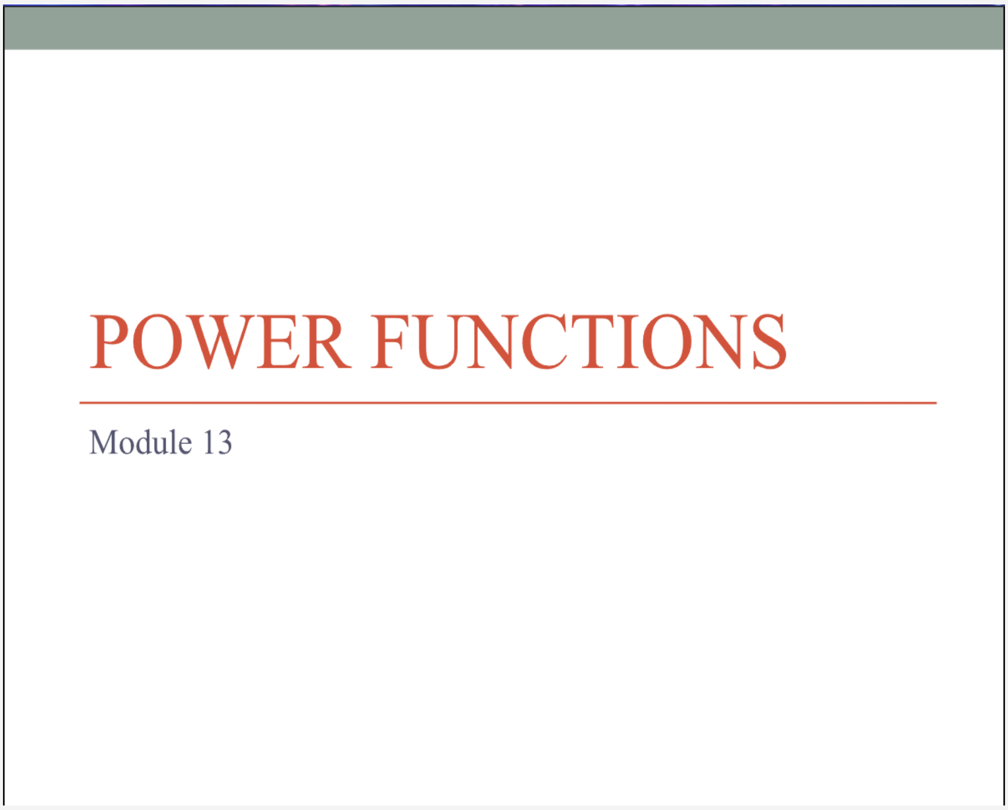
Horizontal Asymptote - Leading Term Example 5

> $k(x) = x^3 - 2x + 20/x - 6$
>
> 
>

- As $x \to \pm\infty$, $k(x) \to$ $x^3/x = x^2$, which is not a line.
    - So there is no HA.

Module 13 - Power Functions

Module 13 - Google Slides

> **Google Slides:**
>
> Slides | PDF
>
> 
>

Power Functions

A **Power Function** is a function that has the form:

> **$y = ax^b$**

- Where *a* and *b* are constants, is called a **Power Function**.

Power Functions vs Polynomial Functions

A single term polynomial is a special case of a Power Function.

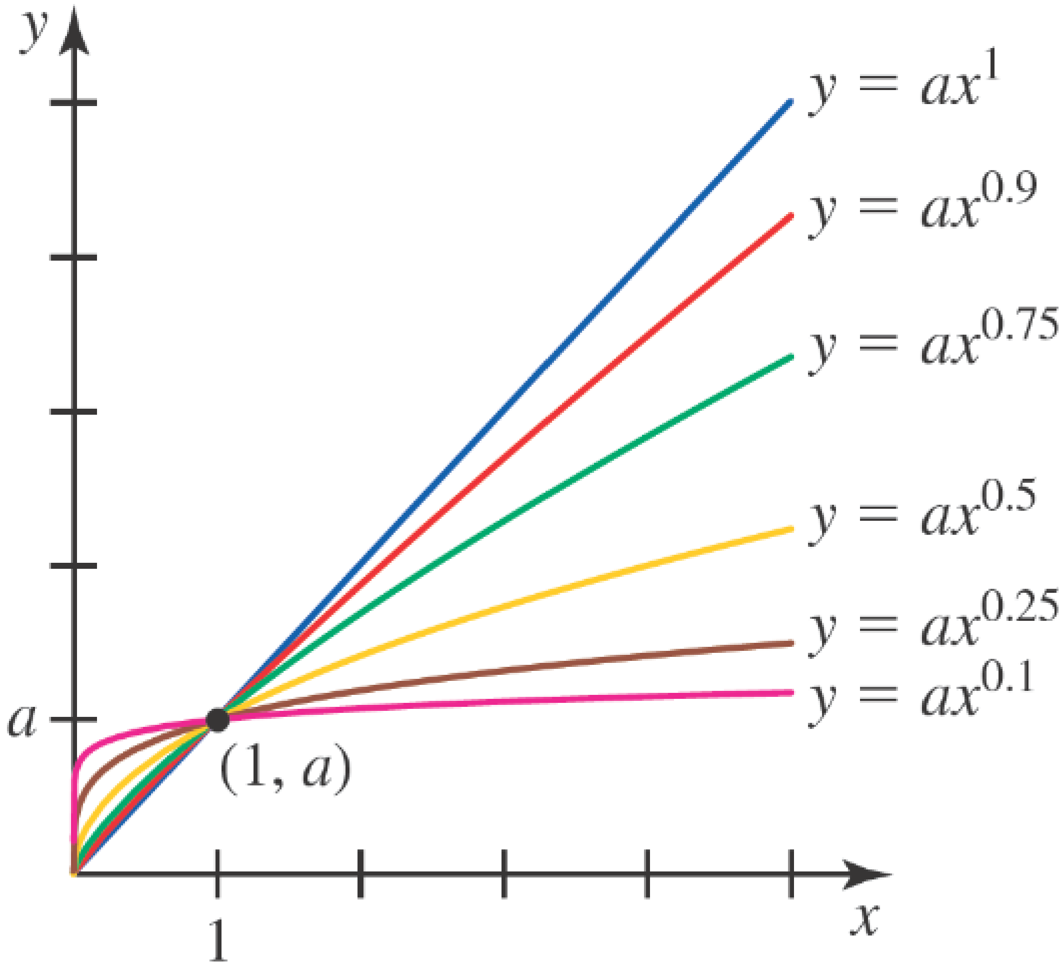
- There are two main differences between Power Functions and Polynomial Functions:

- A *power function* is a **single-term** function.
- A *polynomial function* may have **multiple terms**.

- In a power function the exponent, *b*, can be any **real-number value**. But in a polynomial function the exponent, *n*, must be a **non-negative integer**.

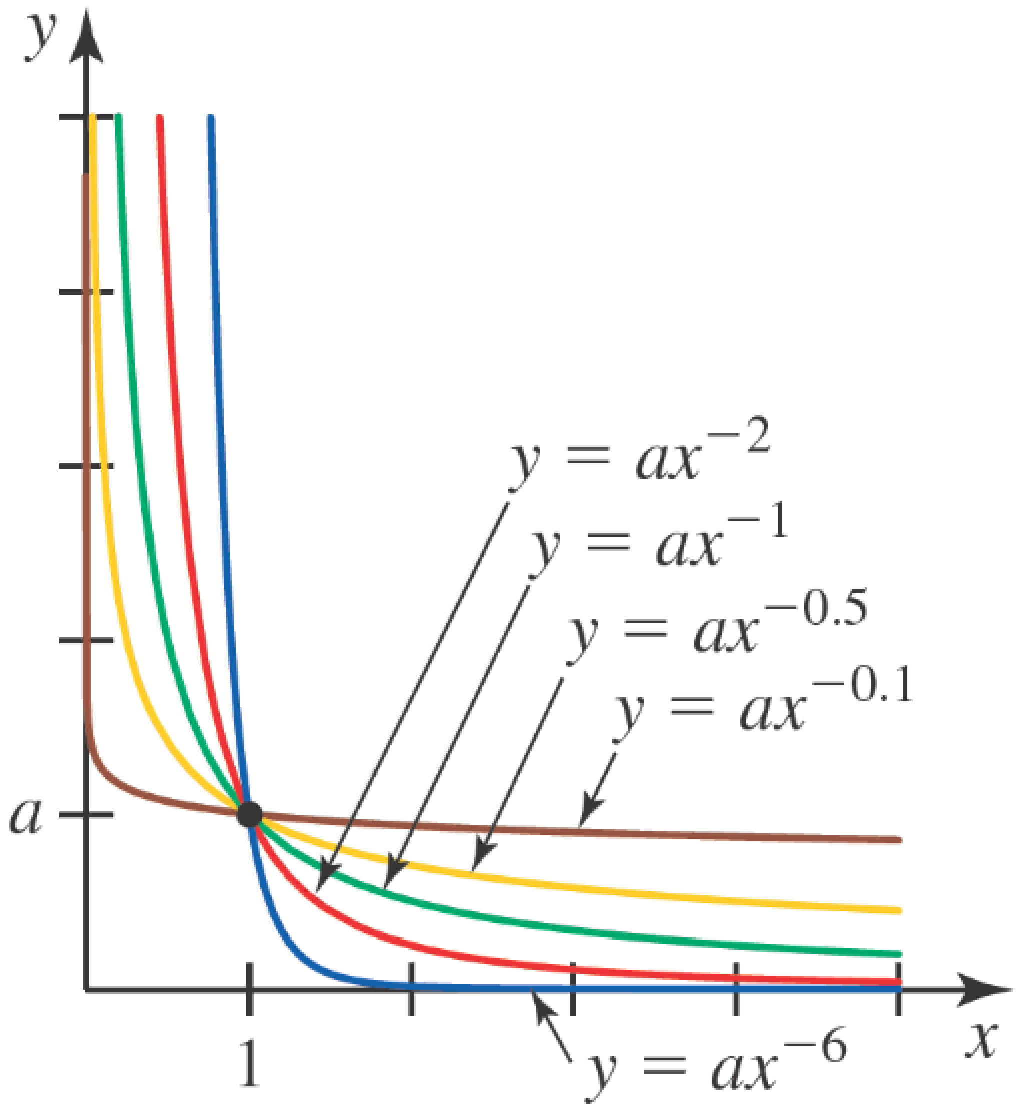
Power Function Graphs

Power Function Graph - `x > 0` and `0 < b < 1`



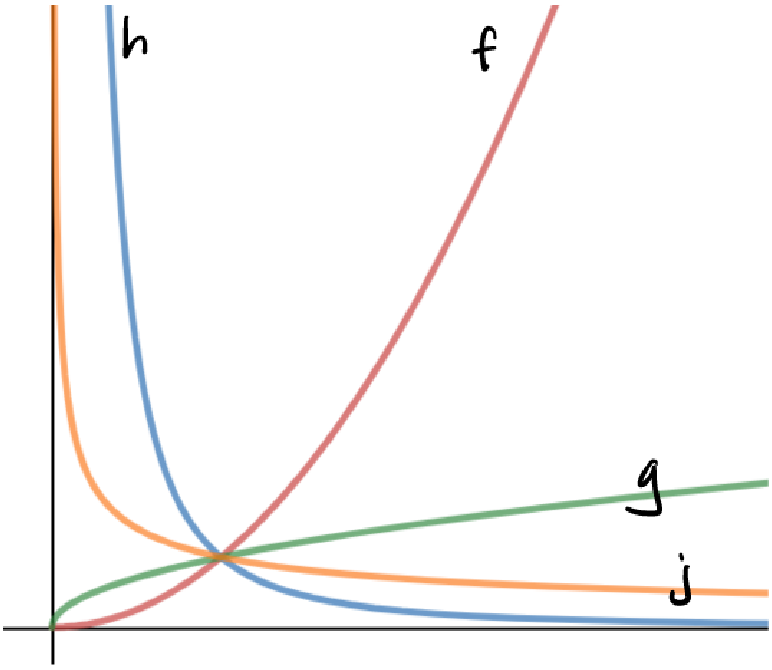
- Illustrates the behavior of several power functions $y = ax^b$ with the characteristic that **x > 0** and **0 < b < 1**.

Power Function Graph - `x > 0` and `b < 0`



- Illustrates the behavior of several power functions **y = ax^{b}** with the characteristic that **x > 0** and **b < 0**.

Power Function Graph - Multiple Power Functions



h: ax^{-n}

f: ax^{n}

g: $ax^{\frac{1}{n}}$

j: $ax^{-\frac{1}{n}}$

- ***h*(x) = ax^{-n}**
  - **ax^{-n}** is decreasing at an increasing rate.
  - $^{a}\!/\!_{x^n}$ will approach **0** faster than $^{a}\!/\!_{x^{1/n}}$.
- ***f*(x) = ax^{n}**
  - **ax^{n}** is increasing at an increasing rate.
- ***g*(x) = $ax^{1/n}$**
  - **$ax^{1/n}$** is increasing at a decreasing rate.
- ***j*(x) = $ax^{1/-n}$**
  - **$ax^{1/-n}$** is decreasing at a decreasing rate.

Solving Power Functions

Solving Equations of Power Functions - Example 1

Migration Speed of Runners:

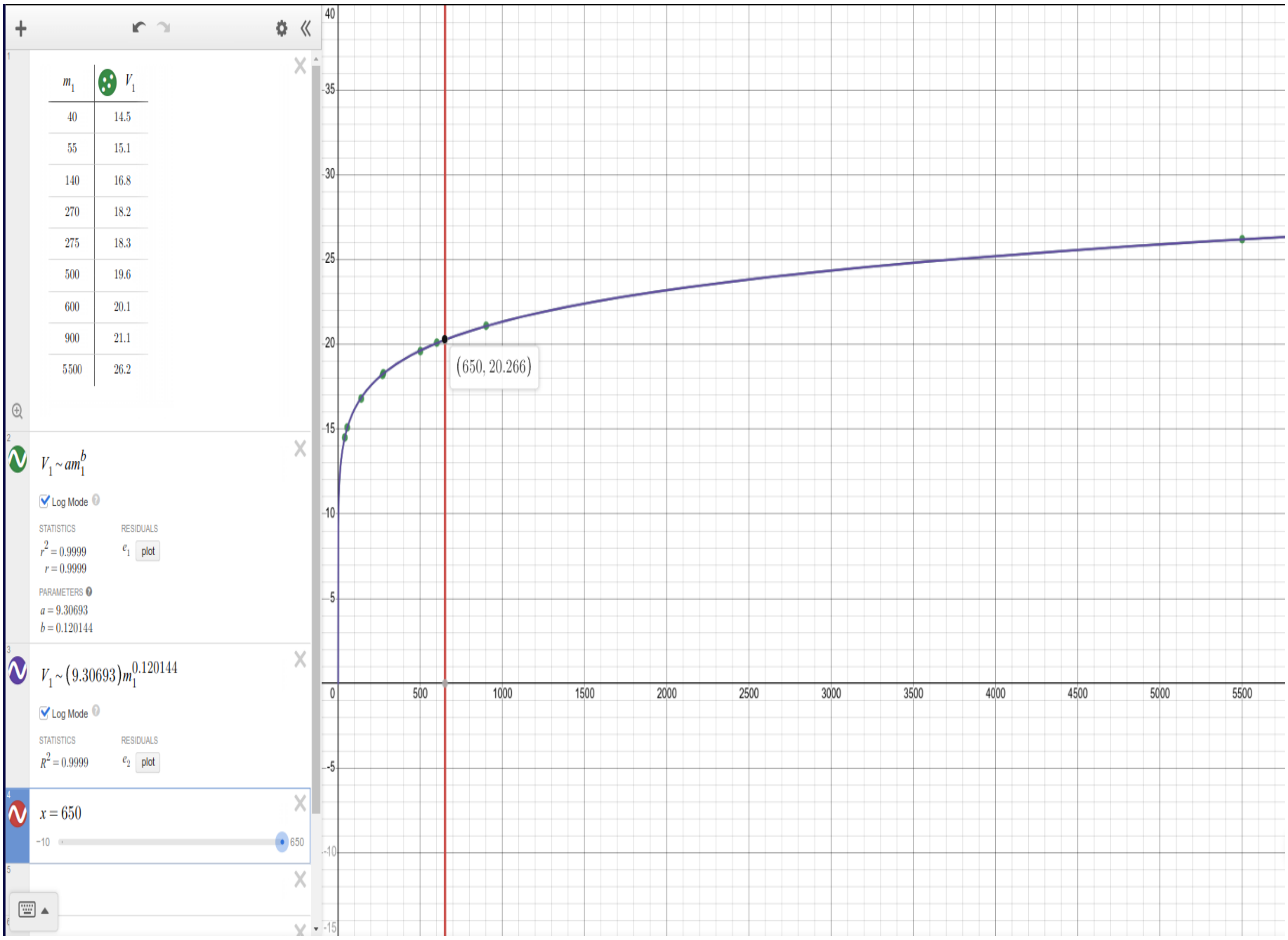
Animal	Mass (kg) *m*	Speed (km/day) *V*
Grey wolf	40	14.5
Gazelle	55	15.1
Dall sheep	140	16.8
Zebra	270	18.2
Wildebeest	275	18.3
Polar bear	500	19.6
Caribou	600	20.1
American buffalo	900	21.1
African elephant	5500	26.2

1. Plot the table data on a Desmos.
2. Use regression to get the equation of the line of best fit.
   - The regression algorithm:

     $\mathbf{V_1 \sim am_1^b}$
3. You get the values:
   - **a = 9.30693**
   - **b = 0.120144**
4. Use the values to get the regression model:
   1. $\boldsymbol{V(m) \sim 9.30693m_1^{0.120144}}$
   2. $\boldsymbol{V(m) \sim 9.31m_1^{0.12}}$
5. Convert the regression model to a power function:
   - $\boldsymbol{V(m) = 9.31m_1^{0.12}}$
6. Get the value of a specific input using **x =** and finding the intersection of the line and the custom input.

- ***m*** is the mass of the animal in kilograms.
- ***V*** is the speed of the animal in kilometers per day.

Example 1 Graph



Solving Equations of Power Functions - Example 2

Use the following formula: $\mathbf{(x^b)^{1/b} = x}$

Equation: $\mathbf{2x^{3/2} = 10}$

1. $\mathbf{2x^{3/2} = 10}$
2. $\mathbf{x^{3/2} = 5}$
3. $\mathbf{(x^{3/2})^{2/3} = 5^{2/3}}$
4. $\mathbf{x = 5^{2/3} \approx 2.92}$
5. $\mathbf{x \approx 2.92}$

Solving Equations of Power Functions - Example 3

Use the following formula: $\mathbf{(x^b)^{1/b} = x}$

Equation: $\mathbf{1.5m^{0.65} = 25}$

1. $\mathbf{1.5m^{0.65} = 25}$

2. $m^{0.65} = {}^{25}\!/_{1.5}$
3. $m = ({}^{25}\!/_{1.5})^{1/0.65} \approx 75.81712417$
4. $m \approx 75.82$

Solving Equations of Power Functions - Example 4

Use the following formula: $(x^b)^{1/b} = x$

Equation: ${}^{5}\!/_{w^{2.5}} = 0.5$

1. ${}^{5}\!/_{w^{2.5}} = 0.5$
2. $5 = (0.5)(w^{2.5}) \rightarrow 5 = 0.5w^{2.5}$
3. ${}^{5}\!/_{0.5} = w^{2.5}$
4. $({}^{5}\!/_{0.5})^{1/2.5} = w$
5. $2.511885432 \approx w$
6. $w \approx 2.51$