CHAPTER 12

Sequences and Series

As humans, we constantly look for patterns. By recognizing patterns, we can more easily find order in the randomness of life. For example, people who are paid every two weeks know to expect a paycheck every fourteen days. Drivers who travel the same route to work every day soon memorize the route and find themselves making the trip with little mental effort.

Number patterns are intriguing because they frequently occur in real-world phenomena. Consider one of the most famous sequences of numbers: the Fibonacci sequence. The first few numbers of the sequence are 1, 1, 2, 3, 5, 8, 13, 21, 34, ... Each new number of the sequence is created by adding the two numbers before it: for example, $13 + 21 = 34$. This sequence is evident in the sequence of leaves on many plant stems. The spiral arrangement of sunflower seeds is also closely linked to the Fibonacci sequence.

SECTION 12.1 Sequences

LEARNING OBJECTIVES

- Determine whether a given sequence converges or diverges using graphs, tables, and symbols
- Find explicit and recursive formulas for sequences
- Use sequences to analyze real-world situations

GETTING STARTED

As many as one in five adults has high cholesterol. (*Source:* www.forcholessterol.com) People with high cholesterol are at a higher risk of heart disease (including heart attack and stroke) than those with healthier levels of cholesterol. Many people take prescription drugs to help lower the amount of cholesterol in their bloodstream. In most cases, this type of medicine is considered a maintenance medicine—users take the medicine every day for the rest of their lives. This brings up many interesting questions: "If I take the medicine every day, will the concentration of medicine in my blood continue to increase?" "Should I take a double dose if I forget to take the medicine at the right time?" "Is it possible for the amount of medicine in my body to reach unsafe levels?"

In this section we introduce the mathematical concept of sequences. With this concept we can address a number of real-world applications including the use of cholesterol-lowering medicine.

Sequences

Lipitor® is a common cholesterol-lowering medicine taken by about 18 million people in the United States. On average, people take a 20-milligram (mg) dosage once per day. Lipitor has a half-life of 14 hours. This is the amount of time it takes for half of the medicine to be eliminated from the body. (*Source:* www.medguides.medicines.org.uk) Will the amount of Lipitor in a person's body increase continuously if he continues to take the medicine everyday?

We can determine the amount of Lipitor left in the body by using the following formula.

$$\begin{aligned} \text{percent left} &= \left(\frac{1}{2}\right)^{\text{time/half life}} \\ &= \left(\frac{1}{2}\right)^{24/14} \\ &\approx 0.305 \end{aligned}$$

Since the half-life is 14 hours, roughly 30.5% of the dose is left in the body after 24 hours. We assume when the medicine is taken, it is instantly absorbed into the bloodstream. With these assumptions, we can compute how much medicine is in the body on the second day, assuming the medicine is taken at the same time each day. Initially, a 20-mg dose is taken. The second day 30.5% of the first 20-mg dose of medicine remains in the body when the second 20-mg dose is taken. Thus

$$(20)(0.305) + 20 = 26.1$$

After taking the second dose, there will be 26.1 mg of medicine in the body. By the time the third dose is taken, 30.5% of the 26.1 mg of medicine remains in the body, so

$$(26.1)(0.305) + 20 \approx 27.96$$

After the third dose, there is 27.96 mg of medicine in the body. If we keep track of the amount of medicine in the body each day, we generate a **sequence** of numbers.

$$20, 26.1, 27.96, \ldots$$

Sequence

A **sequence** a_n is a function whose domain is the set of natural numbers $\{1, 2, 3, \ldots\}$. The function values are called **terms** of the sequence. The terms are represented by $a_1, a_2, \ldots, a_n, \ldots$.

For the cholesterol drug example, the domain values represent the cumulative number of doses taken. The range values represent the amount of medicine remaining in the body at that time.

EXAMPLE 1 ■ Using Sequence Notation

Given the sequence 20, 26.1, 27.96, . . . , use proper notation to write the first three terms of the sequence. Then use sequence notation to define the function represented verbally by the instruction "multiply the previous number in the sequence by 0.305 (30.5% of the medicine remains in the body) and add 20 (the new dose is 20 mg of medicine)."

Solution The first term of the sequence corresponds with $n = 1$ and is written a_1. The second term of the sequence corresponds with $n = 2$ and is written a_2. In general, the ith term of the sequence corresponds with $n = i$ and is written a_i. Each of these terms represents the amount of medicine in the body after n doses.

$$a_1 = 20 \text{ mg}$$
$$a_2 = 26.1 \text{ mg}$$
$$a_3 = 27.96 \text{ mg}$$

We write the verbal instructions in sequence notation as

$$a_n = 0.305a_{n-1} + 20$$

In this formula, a_n represents any term of the sequence. The term a_{n-1} is the term right before a_n.

Even after analyzing the amount of medicine in the body after three doses, we see the amount of medicine is increasing by less and less each dose. The second dose adds 6.1 mg of medicine to the total amount in the body while the third dose only adds another 1.86 mg of medicine. This suggests that, over time, the amount of medicine in the body must reach a limit. This is called the **maintenance level**. In mathematical terms, we say a sequence that reaches a limit is a *converging* sequence. A sequence **converges** if the terms of the sequence get closer and closer to a particular finite value as the number of terms increases; it **diverges** otherwise.

Convergence of Sequences

A sequence $a_1, a_2, a_3, \ldots, a_n, \ldots$ **converges** (or reaches a **maintenance level**) if $a_n \to L$ as $n \to \infty$, where L is a finite number. Otherwise, we say the sequence **diverges**.

To determine the maintenance level in this case, we analyze the formula and graph for the sequence representing the amount of medicine in the body each day.

EXAMPLE 2 ■ Representing a Sequence Using a Graph

Lipitor is taken in 20-mg doses once per day. Each day, 69.5% of the medicine is eliminated from the body. Use a graph to find the maintenance level and explain what this value represents.

Solution Since 69.5% of the medicine is eliminated, 30.5% remains. The formula representing the situation is $a_n = 0.305a_{n-1} + 20$. We calculate the first nine terms of the sequence and graph them as shown in Figure 12.1.

$$20.00, 26.10, 27.96, 28.53, 28.70,$$
$$28.75, 28.77, 28.775, 28.776, \ldots$$

The graph suggests that the maintenance level is about 28.8 mg of medicine. This means that if 20 mg of Lipitor is taken each day, the amount of medicine in the body will reach 28.8 mg and is maintained at that amount. A Technology Tip detailing the process for creating graphs of sequences is given at the end of the section.

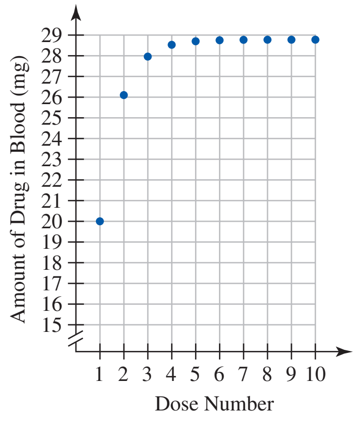


Figure 12.1

EXAMPLE 3 ■ Finding the Maintenance Level of a Sequence

An initial dose of 40 mg of Lipitor is taken. After this initial dose, a single 20-mg tablet is to be taken each day. Every 24 hours, 69.5% of the medicine is eliminated from the body. Find the maintenance level and explain what it means.

Solution We begin by writing the first four terms of the sequence. The first term is 40 mg. The following day, only 30.5% of the medicine remains in the body and a new dose of 20 mg is taken.

$$a_1 = 40$$
$$a_2 = (0.305)(40) + 20 = 32.2$$

This pattern continues for subsequent days.

$$a_3 = (0.305)(32.2) + 20 \approx 29.8$$
$$a_4 = (0.305)(29.8) + 20 = 29.1$$

We now have the first four terms of the sequence.

$$40, 32.2, 29.8, 29.1, \ldots$$

The sequence suggests that, over time, the amount of medicine in the body will become stable. A graph of the sequence will tell us more. In general, the nth term of the sequence may be expressed as a formula.

$$a_n = 0.305a_{n-1} + 20$$

Using $a_1 = 40$, we calculate additional terms and analyze a graph of the sequence shown in Figure 12.2. The graph shows that the amount of medicine in the body drops very quickly after the initial 40-mg dose and suggests that the maintenance level is about 28.8 mg. Additionally, since 69.5% of 28.8 mg is about 20 mg, the amount of medicine the body eliminates is the same amount as the next dose taken. In other

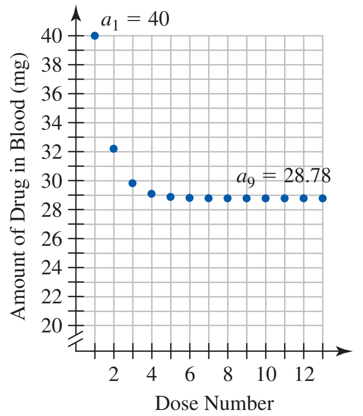


Figure 12.2

words, once the body has 28.8 mg of medicine, the amount eliminated (20 mg) and the new dose taken (20 mg) are the same. Therefore, the body is able to maintain 28.8 mg of medicine and unsafe levels will not build up.

An alternative way to find the maintenance level is to tackle the problem symbolically. The maintenance level k will have the property that

$$0.305k + 20 = k$$

Solving this equation for k yields

$$\begin{aligned} 0.305k + 20 &= k \\ 20 &= 0.695k \\ k &= \frac{20}{0.695} \\ &\approx 28.8 \end{aligned}$$

EXAMPLE 4 ■ Using Sequences

Atacand® (candesartan cilexetil) is a medicine for high blood pressure. According to the technical pamphlet for the drug, the half-life of candesartan is about 9 hours. The typical recommended starting dose is 16 mg once daily. A doctor may recommend the dose be taken twice daily; however, total daily doses should not exceed 32 mg. (*Source:* www.astrazeneca-us.com).

Suppose a doctor prescribes a patient to take a 16-mg dose once every 12 hours.

a. What is the maintenance level for the drug?

b. Suppose after reaching the maintenance level, the patient accidentally misses a dose and takes a double dose the next time he is scheduled to take the medicine. (We do not recommend doing this.) Assuming the drug is instantly absorbed and distributed evenly throughout the bloodstream, how much of the drug will be in the bloodstream immediately after the double dose is taken?

Solution

a. We first determine how much of the drug is left after 12 hours.

$$\begin{aligned} \text{percent left} &= \left(\frac{1}{2}\right)^{\text{time/half life}} \\ &= \left(\frac{1}{2}\right)^{12/9} \\ &\approx 0.397 \end{aligned}$$

We calculate the first eight terms of the sequence representing the amount of medicine in the body. Since 39.7% remains after 12 hours, we have

$$\begin{aligned} a_1 &= 16 \\ a_2 &= (0.397)(16) + 16 = 22.4 \\ a_3 &= (0.397)(22.4) + 16 = 24.9 \\ a_4 &= (0.397)(24.9) + 16 = 25.9 \\ a_5 &= (0.397)(25.9) + 16 = 26.3 \\ a_6 &= (0.397)(26.3) + 16 = 26.4 \\ a_7 &= (0.397)(26.4) + 16 = 26.5 \\ a_8 &= (0.397)(26.5) + 16 = 26.5 \end{aligned}$$

The maintenance level seems to be approaching 26.5 mg and the graph shown in Figure 12.3 confirms this observation. To check our conclusion algebraically, we note that the maintenance level k will have the property that

$$0.397k + 16 = k$$

Solving this equation for k yields

$$\begin{aligned} 0.397k + 16 &= k \\ 16 &= 0.603k \\ k &= \frac{16}{0.603} \\ &\approx 26.5 \end{aligned}$$

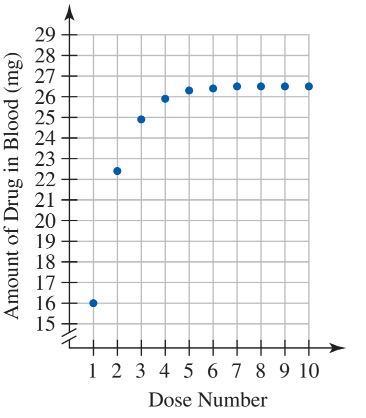


Figure 12.3

b. After reaching the maintenance level, 26.5 mg of medicine will remain in the body. If a new dose is not taken at the scheduled time, by the next scheduled time 10.5 mg [(0.397)(26.5) = 10.5] of medicine will remain in the body. If the patient takes two doses, there will be 10.5 mg + 16 mg + 16 mg = 42.5 mg of medicine in the body. This is noticeably higher than the maintenance level and may result in adverse side effects. However, if the patient does not have an adverse reaction, the amount of medicine in the body will return to the maintenance level over time. See Figure 12.4.

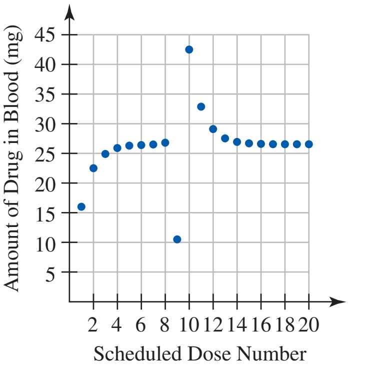


Figure 12.4

Not all sequences reach a maintenance level or a stable limit. Example 5 illustrates a sequence that diverges.

EXAMPLE 5 ■ Identifying a Divergent Sequence

An employee asks his employer to withhold \$1200 for retirement savings during the first year of employment and to increase the amount withheld by 2% in each subsequent year. If this continues for 20 years, how much money is withheld in the 20th year?

Solution We calculate the first four terms of the sequence.

$$\begin{aligned} a_1 &= 1200 \\ a_2 &= 1200 + (0.02)(1200) = 1224 \\ a_3 &= 1224 + (0.02)(1224) = 1248.48 \\ a_4 &= 1248.48 + (0.02)(1248.48) = 1273.45 \end{aligned}$$

These terms represent the amount of money withheld each of the first four years.

$$1200, 1224, 1248.48, 1273.45, \ldots$$

So far the amount withheld does not appear to be stabilizing. Using the Technology Tip at the end of this section to graph the sequence $a_n = 0.02a_{n-1} + a_{n-1} = 1.02a_{n-1}$ as shown in Figure 12.5, we determine the amount withheld in the 20th year will be \$1748.17. As we can see from the graph, this sequence does not have a limiting value. Therefore, this sequence diverges.

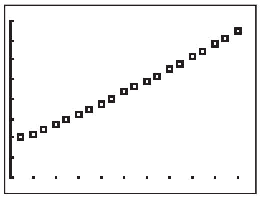


Figure 12.5

Each of the situations we have seen so far have been situations where the formula for the sequence was expressed using a *recursive formula*. A **recursive formula** defines the nth term of the sequence as a function of the previous term, a_{n-1}. Sometimes it is possible to express the formula for the nth term of a sequence *explicitly* in terms of the value of n. An **explicit formula**, which allows us to compute any term of the sequence easily, defines the nth term of the sequence as a function of n.

EXAMPLE 6 ■ Finding an Explicit Formula for a Sequence

In Example 5 we found a recursive formula for the situation where an employer withheld \$1200 for retirement savings during the first year of employment and increased the amount withheld by 2% each year. The recursive formula was

$$a_n = 0.02a_{n-1} + a_{n-1} = 1.02a_{n-1}$$

Write an explicit formula for the amount of money withheld in the nth year of employment.

Solution We write the first four terms of the sequence.

Value of term	1200	1224	1248.48	1273.45
Number of term	$n = 1$	$n = 2$	$n = 3$	$n = 4$

Notice we start with the term 1200, multiply it by 1.02, then multiply that answer by 1.02, and finally multiply that answer by 1.02.

	×1.02	×1.02	×1.02	
Value of term	1200	1224	1248.48	1273.45
Number of term	$n = 1$	$n = 2$	$n = 3$	$n = 4$

We express the idea of starting with a value (1200) and repeatedly multiplying by another value (1.02) using exponential notation:

$$a_n = 1200(1.02)^{n-1}$$

Using this explicit formula generates the same results as before.

$$\begin{aligned}
a_1 &= 1200(1.02)^{1-1} = 1200(1.02)^0 = 1200 \\
a_2 &= 1200(1.02)^{2-1} = 1200(1.02)^1 = 1224 \\
a_3 &= 1200(1.02)^{3-1} = 1200(1.02)^2 = 1248.48 \\
a_4 &= 1200(1.02)^{4-1} = 1200(1.02)^3 = 1273.45
\end{aligned}$$

EXAMPLE 7 ■ Writing Recursive and Explicit Formulas for a Sequence

A \$400 car payment is made each month. Write a recursive formula and an explicit formula for the total amount of money paid out in car payments over time. After making 60 payments (5 years), how much money has been paid out?

Solution We begin by writing the first four terms of the sequence, representing a running total of how much money is spent on car payments.

$$400, 800, 1200, 1600, \ldots$$

To get a particular term in the sequence, we add 400 to the previous term in the sequence. We express this as a recursive formula for the sequence.

$$a_n = a_{n-1} + 400$$

To get an explicit formula, we organize the sequence as follows.

Number of term	$n = 1$	$n = 2$	$n = 3$	$n = 4$
Function value	400	800	1200	1600

We see if we multiply the number of the term ($n = 1$, $n = 2$, and so on) by 400, we obtain the value of the term. Therefore, the explicit formula for this sequence is

$$a_n = 400n$$

We use the explicit formula to determine how much money has been paid out in car payments after 60 payments have been made. That is, we compute

$$a_{60} = (400)(60) = 24{,}000$$

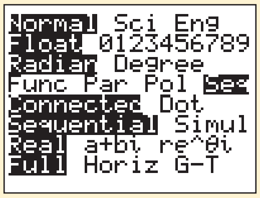
This means $24,000 has been paid through monthly car payments after 60 payments have been made.

SUMMARY

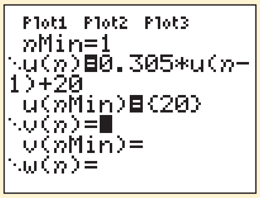
In this section you learned how to create a sequence of numbers to represent a particular situation as well as to determine whether the sequence converges or diverges. You also discovered how to represent sequences with graphs. Additionally, you learned how to express sequences using recursive and explicit formulas.

TECHNOLOGY TIP ■ GRAPHING A SEQUENCE

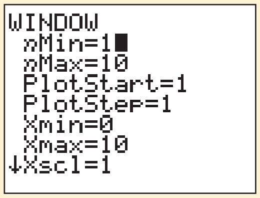
1. Press MODE. Change the mode to sequence mode by using the arrows to highlight `Seq` and pressing ENTER.

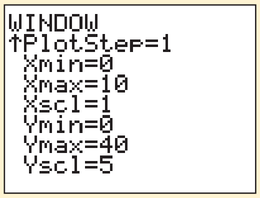


2. Edit the parameters using the Y = Editor. The parameter `nMin` refers to the minimum term number you choose to display. The parameter `u(n)` represents the formula. Access "u" by pressing 2nd and the number 7 (note the blue "u" above the 7 button). Access "n" by pressing the X,T,Θ,n button. The parameter `u(nMin)` represents the first term of the sequence.

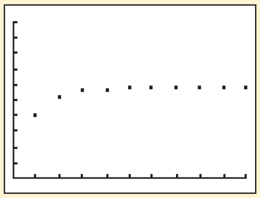


3. Edit the WINDOW parameters. The parameters `nMin` and `nMax` refer to the number terms in the sequence to be evaluated. **PlotStart** and **PlotStep** determine which term to plot first and the incremental value of n. The rest of the parameters are as usual, where x represents the term number and y represents the value in the sequence.

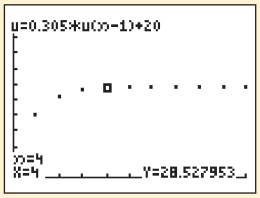




4. Press GRAPH.



5. Use TRACE as usual. You will see the term number, n, as well as the sequence value.



12.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5,

a. List the first six terms of the given sequence.

b. Predict whether the sequence will converge or diverge.

c. Check the accuracy of your prediction by using a graphing calculator to graph the first ten terms of the sequence.

1. $a_n = \dfrac{n}{3n+1}$

2. $a_n = \dfrac{n^2}{2^n - 1}$

3. $a_n = \dfrac{1}{2^n}$

4. $a_n = 4 + (-1)^n$

5. $a_n = \left(1 + \dfrac{1}{n}\right)^n$

In Exercises 6–14, find a formula for the general term of the sequence, a_n, and state whether the sequence converges. Assume the pattern in the given terms continues.

6. 3, 8, 13, 18, 23, . . .

7. 10, 5, 2.5, 1.25, . . .

8. $\dfrac{1}{2}, \dfrac{2}{5}, \dfrac{3}{10}, \dfrac{4}{17}, \ldots$

9. 2, 4, 8, 16, 32, . . .

10. 0, 5, 0, 5, 0, . . .

11. $\dfrac{1}{2}, -\dfrac{1}{4}, \dfrac{1}{8}, -\dfrac{1}{16}, \ldots$

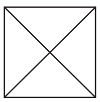
12. 0, 50, 87.5, 115.625, . . .

13. 100, 102, 104, 106, . . .

14. 300, 600, 900, 1200, . . .

15. Consider the general sequence $1, k, k^2, k^3, k^4, \ldots$. Determine for which values of k the sequence will converge and for which values of k the sequence will diverge.

16. Geometry A regular polygon is a geometric shape where each of the sides has the same length. For example, a square is a regular polygon since all four sides are the same length. A diagonal is a line segment drawn from one corner of a polygon to another. For example, a square has two diagonals, as shown in the figure.



Beginning with a triangle, we can express the number of diagonals as a sequence.

Triangle: $a_1 = 0$

Square: $a_2 = 2$

Write an explicit formula for the sequence giving the number of diagonals, a_n, for a regular polygon with n sides.

17. Fibonacci Sequence The Fibonacci sequence, named for Italian mathematician Leonardo Fibonacci, is the sequence 1, 1, 2, 3, 5, 8, 13, ...

a. Find the next five terms of the sequence. (*Hint:* For this sequence, a_n depends on a_{n-1} and a_{n-2}.)

b. Write a recursive formula for the sequence.

18. Using a calculator, repeatedly take the square root of 5. Express the decimal approximations of this as a sequence. That is, the sequence will be $\sqrt{5}, \sqrt{\sqrt{5}}, \sqrt{\sqrt{\sqrt{5}}}$, and so on. Does this sequence converge? If so, to what does it converge?

19. A sequence is considered **strictly monotone** if it is either always increasing or always decreasing from term to term. Explain whether each of the following sequences is strictly monotonic or not. Then, determine if the sequence converges or diverges.

a. $a_n = \dfrac{1}{2}n$

b. $a_n = \dfrac{1}{n^2}$

c. $a_n = \dfrac{(-1)^n}{n^2}$

SHOW YOU KNOW

20. Explain what it means for a sequence to converge. Use graphs, numbers, symbols, and words in your explanation.

21. Create two sequences, both of which diverge. Explain why the sequences diverge and include graphs to support your explanation.

22. Create two sequences, both of which converge. Explain why the sequences converge and include graphs to support your explanation.

23. Using the idea of rate of change, describe the shape of a graph of a sequence that is ever increasing (strictly monotonic) but converges. Then, describe the shape of a graph of a sequence that is ever decreasing but converges.

MAKE IT REAL

24. Inflation Inflation is an increase in the price of a particular product. For example, suppose the value of a MP3 player increases by the May 2007 inflation rate of 2.69% per year for the next 5 years. (***Source:*** **www.inflationdata.com**) If the MP3 player costs \$90, write its predicted price for the next 5 years as a sequence. Does this sequence converge? Explain.

25. Chain Letter One chain letter scam, called the "Make Money Fast Pyramid," claims that by mailing \$1 to five people, you can make thousands of dollars. (***Source:*** **hoaxbusters.ciac.org**) It allegedly works like this: You receive a letter with five names. You mail a copy of the letter to five friends and send \$1 to the top name on the list. Before you mail the letter to your friends, you remove the name at the top of the list and add your name to the bottom of the list.

a. If the chain is not broken, how much money will you receive before your name is removed from the list?

b. What if there were only three names on the list instead of five?

c. What if there are n names on the list? Write your answer as a sequence where C_n is the name of the sequence.

26. **Dog Years** Depending on the type of breed, 1 year for a human is equivalent to 7 years for a dog. A baby human and a puppy are born on the same day. Write a sequence showing the age of the dog (in dog years) as a function of the baby's age for the first 10 years of the baby's life. Then write an explicit formula for this sequence.

27. **Purchasing a Car** As a rule of thumb, a car depreciates in value by 15% every year. (*Source:* www.bankrate.com) The price for a 2007 Toyota Camry Hybrid 4-door Sedan was \$25,900 on June 19, 2006. (*Source:* www.nada.com) If this car was purchased and kept indefinitely, does the value of the car converge or diverge? If it converges, to what value does it converge?

28. **Pharmaceuticals—Aspirin** Aspirin washes out of the bloodstream rapidly. The amount of effective aspirin left in the bloodstream is reduced by 50% approximately every 3 hours. (*Source:* www.rxlist.com)

Suppose a patient takes 650 mg of aspirin every 4 hours, as recommended. (Assume the aspirin is absorbed into the bloodstream immediately.)

a. Generate the first five terms of the sequence that represent this situation. Explain why this situation can be represented with sequences.

b. Write an implicit formula giving the medicine level in the patient's bloodstream after each dose is taken and absorbed.

c. Create a graph that shows units of medicine as a function of number of doses.

d. How many doses can the patient safely take? Support your answer graphically, numerically, symbolically, and verbally.

e. Discuss the consequences of missing a dose. Does taking a double dose make up for the missed dose? Explain.

29. **Forestry Management** In its 5-year Urban Forestry Management Plan, the city of Muskego, Wisconsin, planned to increase the number of trees in city parks. According to the plan, there were 500 trees in city parks and the city planned to increase the number of trees by 50% over 5 years. The plan also indicated that the city expected 10% of the newly planted trees to die during the 5-year period. Additionally, the city planned to remove 85 trees over the same time period. (*Source:* City of Muskego Urban Forestry Management Plan at www.ci.muskego.wi.us) Assuming the city will implement the same 5-year plan indefinitely,

a. Write a sequence of numbers that represents the number of trees in city parks for the first four 5-year periods.

b. Based on the sequence from part (a), write a rule for the number of trees in city parks after n years.

c. Represent this situation graphically.

d. Does repeated application of the 5-year plan make sense in the long run? Justify your answer.

e. Will the forest size stabilize? If so, in how many years and with how many trees? Justify your answer.

30. **Semiconductor Manufacturing** In the manufacturing of semiconductors, highly toxic chemicals are used. When the need arises, environmental engineers wear protective suits designed to neutralize specific toxic chemicals. However, the ability of a suit to neutralize a chemical is usually reduced with prolonged exposure. Suppose an engineer has put on a new, clean protective suit and begins to clean up a chemical spill in the factory. During each hour of cleanup, the suit neutralizes 30% of the toxic chemical on it and accumulates an additional 12 micrograms of the chemical.

a. Make a table of the amounts of unneutralized chemical on the suit during each hour of the cleanup. Describe the growth of these values.

b. Write the first five terms of the sequence related to this situation. Then write a formula and create a graph for the sequence.

c. The suit is safe to use until it has accumulated 35 micrograms or more of the toxic chemical. How long can the engineer safely continue to clean up the spill before the suit must be changed?

d. Reanalysis of the toxicity of the chemicals indicates that the suit can be used safely with up to 40 micrograms of accumulated chemical. Does this make a significant difference in the amount of time the engineer can continue to work on the chemical spill? Explain.

31. **Pharmaceuticals—Lamisil®** Lamisil tablets are taken by people with fingernail and toenail fungus problems. Typically, 250 mg tablets are taken once daily for six weeks for people with fingernail fungus and for 12 weeks for people with toenail fungus. With a half-life of 36 hours, 37% of the medicine is removed from the bloodstream each day. (*Source:* www.drugs.com)

How much medicine will be in the bloodstream at the end of each treatment period for each of the two treatment groups? How close to the maintenance level did each group get?

32. **Pharmaceuticals—Singulair®** Singulair is medicine taken by people for the long-term management of asthma. People aged 15 and over are typically prescribed a single dose of 10 mg of Singulair each day. Children aged 2 to 5 years of age are typically prescribed a single dose of 4 mg of Singulair each day. With a half-life of 4 hours, about 98.4% of the medicine is removed from the bloodstream every day. (*Source:* www.drugs.com)

Use what you have learned in this section to analyze the effects of the two different dosage levels. Include graphs and formulas and indicate what the maintenance levels are in each case.

33. **Pharmaceuticals—Prilosec®** Prilosec is used to eliminate acid reflux disease in people. Typically, people take a 20-mg dose once daily. Studies have shown that in healthy people, Prilosec has a half-life of about one-half hour. This is equivalent to having 99.9% of the medicine eliminated from the bloodstream every day. The study also showed that in people with unhealthy livers, the half-life is 3 hours. This is equivalent to having 99.51% of the medicine elimi-

nated every day. (*Source:* www.drugs.com) Analyze the effects of the two different half-lives. Include graphs and formulas and indicate what the maintenance levels are in each case.

34. Pharmaceuticals—Percocet® Percocet is a pain relief medicine that contains acetaminophen and oxycodone, which is highly addictive. Research has shown it is extremely unhealthy for a person to maintain a level of 15 mg of Percocet in the bloodstream. Dosing depends on the severity of the pain that a person is experiencing. Percocet has a half-life of about 2 hours. (*Source:* en.wikipedia.org)

Determine the maximum amount of Percocet a doctor should prescribe to a patient who will take the medicine every 4 hours. What will be the maximum dosage in this situation? Use graphs, formulas, and tables to justify your answers.

35. Pharmaceuticals—Claritin® Claritin tablets relieve nasal congestion, runny nose, and watering eyes due to allergies. The half-life of Claritin depends on urinary pH: The half-life is approximately 4 hours when the urinary pH is 5 and 12 hours when the urinary pH is 8. Typically, people take a 240-mg dose of Claritin once daily. (*Source:* www.drugs.com)

With a half-life of 4 hours, the body eliminates approximately 98.4% of the medicine every day. With a half-life of 12 hours, the body eliminates approximately 75% of the medicine every day. Analyze the effects of the two different half-lives. Include graphs and formulas and indicate what the maintenance levels are in each case.

STRETCH YOUR MIND

Exercises 36–41 are intended to challenge your understanding of sequences.

36. Using a calculator, a student starts with a number greater than 1 and repeatedly presses the square root button. If each successive result represents the term of a sequence of numbers, write a formula for the nth term of this sequence. Does this sequence converge? If so, to what value does it converge?

37. Using a calculator, a student starts with a positive number less than 1 and repeatedly presses the square root button. If each successive result represents the term of a sequence of numbers, write a formula for the nth term of this sequence. Does this sequence converge? If so, to what value does it converge?

38. The Fibonacci sequence (1, 1, 2, 3, 5, 8, 13, ...) can be defined recursively by the formula $a_n = a_{n-1} + a_{n-2}$. How does the ratio $\frac{a_{n+2}}{a_{n+1}}$ compare to $\frac{a_n}{a_{n+1}}$ for $n \geq 1$?

39. Investigate the sequence of numbers defined by the formula $a_n = \frac{4\sin(n)}{n}$. Does the sequence converge? If so, to what value does it converge?

40. A sequence is considered monotonic if the terms always increase (that is, $a_{n+1} > a_n$) or always decrease (that is, $a_{n+1} < a_n$) for all $n \geq 1$. Determine whether the sequence $a_n = \frac{n}{n^2+1}$ is monotonic. Justify your answer.

41. The following sequence is known as the harmonic sequence:

$$a_n = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Determine whether or not the sequence defined by $b_n = \frac{a_n}{a_{n-1}}$ converges or diverges. Justify your answer.

SECTION 12.2 Arithmetic and Geometric Sequences

LEARNING OBJECTIVES

- Determine whether a sequence is arithmetic or geometric
- Use arithmetic and geometric sequences to analyze real-world problems

GETTING STARTED

The cost of tuition is a financial concern for many students. At the University of Minnesota–Duluth, undergraduate students paid $253.50 per credit during the 2006–2007 academic year. (*Source:* www.d.umn.edu) To determine the total cost of tuition, students add an additional $253.50 for every additional credit hour for which they enroll.

Students who graduate from college expect that the investment in tuition costs will pay off. In many cases, it does. A recent survey showed that new college graduates with a degree in chemical engineering earned an average of $59,361. (*Source:* money.cnn.com) This represents a 5.4% increase from the previous year. Assuming a 5.4% increase each year, the starting salary for chemical engineering majors will increase at an ever-increasing rate with a growth factor of 1.054.

In this section we investigate the types of sequences that can be used to represent these two situations: arithmetic and geometric sequences.

Arithmetic Sequences

Suppose we plan to take classes at the University of Minnesota–Duluth. We can express the total cost for tuition as a sequence where the value of n represents the number of credits taken and a_n is the total tuition cost.

$$a_1 = 253.50$$
$$a_2 = 507.00$$
$$a_3 = 760.50$$
$$a_4 = 1014.00$$

Each subsequent term in the sequence is obtained by adding the constant amount of \$253.50 to the previous term. This is an example of an **arithmetic sequence**.

Arithmetic Sequence

An **arithmetic sequence** is a sequence in which each subsequent term of the sequence is found by adding a common difference, d, to the previous term. The formula for the nth term of an arithmetic sequence is

$$a_n = a_1 + (n - 1)d$$

EXAMPLE 1 ■ Determining the Explicit Formula of an Arithmetic Sequence

The cost of tuition at the University of Minnesota–Duluth is \$253.50 per credit. Write an explicit formula for the cost of enrolling for n credits. What is the total tuition cost if a student enrolls for 12 credits?

Solution The cost of enrolling for 1 credit is \$253.50, therefore, $a_1 = 253.50$. With a cost of \$253.50 per credit, the common difference is $d = 253.50$. The explicit formula for the total cost is

$$a_n = 253.50 + (n - 1)(253.50)$$
$$a_n = 253.50 + 253.50n - 253.50$$
$$a_n = 253.50n$$

We want to know the cost for 12 credits, so we need to compute the value of a_{12}.

$$a_{12} = (253.50)(12)$$
$$a_{12} = 3042.00$$

The total tuition cost for 12 credits is \$3042.

Arithmetic Sequences and Linear Functions

There is a strong connection between arithmetic sequences and linear functions. The common difference, d, corresponds to the constant rate of change, or slope. The value of a_1 of the sequence corresponds to the initial value of the linear function. However, unlike linear functions, the domain of an arithmetic sequence is limited to the natural numbers. (In the tuition example, we assume students may not enroll for a fraction of a credit.)

EXAMPLE 2 ■ Graphing an Arithmetic Sequence

Graph the sequence where $a_1 = 10$ and $d = 1.5$. Explain why this is an arithmetic sequence and how it is similar to a linear function.

Solution We first write a formula for this sequence.

$$a_n = a_1 + (n-1)d$$
$$a_n = 10 + (n-1)(1.5)$$
$$a_n = 10 + 1.5n - 1.5$$
$$a_n = 1.5n + 8.5$$

We then plot the sequence shown in Figure 12.6 by creating several values of a_n.

$$a_1 = (1.5)(1) + 8.5 = 10$$
$$a_2 = (1.5)(2) + 8.5 = 11.5$$
$$a_3 = (1.5)(3) + 8.5 = 13$$

Since each subsequent term is computed by adding the constant 1.5, the sequence follows a linear trend, but is discrete in nature.

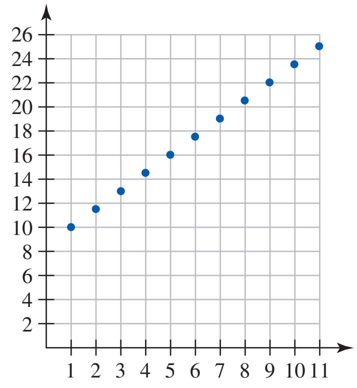


Figure 12.6

■ Geometric Sequences

Suppose the starting salary for a chemical engineer is \$59,361 and is increasing by 5.4% per year. This salary schedule can be expressed as a sequence where the value of n represents the number of years into the future and a_n is the starting salary. We begin with \$59,361 and multiply by 1.054 (because increasing by 5.4% is equivalent to multiplying by 1.054).

$$a_1 = 59{,}361.00$$
$$a_2 = 62{,}566.49$$
$$a_3 = 65{,}945.08$$
$$a_4 = 69{,}506.12$$

Each subsequent term in the sequence is obtained by multiplying the previous term by the constant factor 1.054. The ratio of a term to the previous term is also equal to 1.054. In this example of a **geometric sequence**, observe that

$$a_1 = 59{,}361.00$$
$$a_2 = 59{,}361.00 \cdot 1.054$$
$$a_3 = (59{,}361.00 \cdot 1.054) \cdot 1.054 = 59{,}361.00 \cdot 1.054^2$$
$$a_4 = [(59{,}361.00 \cdot 1.054) \cdot 1.054] \cdot 1.054 = 59{,}361.00 \cdot 1.054^3$$

GEOMETRIC SEQUENCE

A **geometric sequence** is a sequence in which each subsequent term of the sequence is found by multiplying the previous term by a *common ratio*, r. The formula for the nth term of a geometric sequence is

$$a_n = a_1 r^{n-1}$$

The ratio is called a common ratio because if we take any term in the sequence and divide it by the previous term, we get a constant.

EXAMPLE 3 ■ Using a Geometric Sequence

The starting salary for a new chemical engineer is \$59,361. Assuming this salary is increasing by 5.4% annually, write an explicit formula for the salary amount after n years. What will the salary be in 5 years?

Solution We know $a_1 = \$59{,}361$ and $r = 1.054$. The salary schedule is a geometric sequence with the explicit formula

$$a_n = a_1 r^{n-1}$$
$$a_n = 59{,}361(1.054)^{n-1}$$

To find the starting salary when $n = 5$, we need to compute the value of a_5.

$$a_5 = 59{,}361(1.054)^{5-1}$$
$$a_5 = 59{,}361(1.054)^4$$
$$a_5 = 73{,}259.45$$

The starting salary for a chemical engineer in 5 years is predicted to be \$73,259.45.

Geometric Sequences and Exponential Functions

There is a strong connection between geometric sequences and exponential functions. The common ratio, r, corresponds to the constant change factor. The value of a_1 of the sequence corresponds to the initial value of the exponential function. As was the case with arithmetic sequences, the domain of a geometric sequence is limited to the positive integers.

EXAMPLE 4 ■ Graphing a Geometric Sequence

Graph the sequence where $a_1 = 50$ and $r = 0.75$. Explain why this is a geometric sequence and how it is similar to an exponential function.

Solution We first write a formula for this sequence.

$$a_n = a_1 r^{n-1}$$
$$a_n = 50(0.75)^{n-1}$$

We then plot the sequence shown in Figure 12.7 by creating several values of a_n.

$$a_1 = 50(0.75)^{1-1} = 50$$
$$a_2 = 50(0.75)^{2-1} = 37.5$$
$$a_3 = 50(0.75)^{3-1} = 28.125$$
$$a_4 = 50(0.75)^{4-1} = 21.09375$$

The graph of the exponential function $y = 50(0.75)^x$ has the same basic shape as $a_n = 50(0.75)^{n-1}$; however, the exponential function is defined for all real numbers while the geometric sequence is only defined for positive integers.

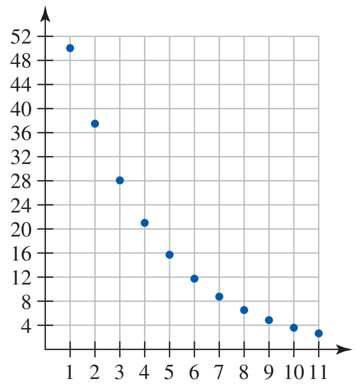


Figure 12.7

EXAMPLE 5 ■ **Distinguishing between Arithmetic and Geometric Sequences**

For each scenario, decide whether an arithmetic or geometric sequence will best model the situation. Explain how you know, then write an explicit formula for the sequence.

a. Each time a picture is taken with a digital camera, 1.5 megabytes of memory is used.

b. One gallon of milk cost \$0.23 in 1950. Milk has increased in price by an average of 2.2% annually. (*Sources:* data.bls.gov, www.cbo.gov)

c. Each time you fold a piece of paper with a thickness of 0.0125 inches, the thickness of the stack doubles.

d. Each time a field-goal kicker makes a field goal, 3 points are added to his total points scored for the season.

Solution

a. Since the amount of memory or storage space used increases by a constant amount of 1.5 megabytes per picture, this situation is best modeled by an arithmetic sequence with a constant difference, d, of 1.5. Assuming there were no pictures stored on the camera initially, we have the sequence 0, 1.5, 3, 4.5, ... This is represented by the following equation.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_n &= 0 + (n - 1)(1.5) \\ a_n &= 1.5n - 1.5 \end{aligned}$$

b. Since the cost of a gallon of milk is increasing by a constant factor of 2.2%, the situation is best modeled by a geometric sequence. The first term is 0.23 and the common ratio, r, of consecutive terms is 1.022 so

$$a_n = 0.23(1.022)^{n-1}$$

c. Doubling a small amount of thickness will not add much. But, doubling larger and larger amounts will add larger and larger amounts to the thickness. Therefore, this situation is best represented by a geometric sequence that has a constant ratio of 2.

$$a_n = 0.0125(2)^{n-1}$$

d. Each time a field goal is successfully attempted, three points are added to the total points scored by the kicker. An arithmetic sequence with a common difference of 3 best represents this situation. We assume the total points scored starts at the beginning of the season, so $a_1 = 0$. We have the sequence 0, 3, 6, 9, . . . , which is represented by the following equation.

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_n &= 0 + (n - 1)(3) \\ a_n &= 3n - 3 \end{aligned}$$

SUMMARY

In this section you learned the difference between arithmetic and geometric sequences and how these are related to linear and exponential functions, respectively. You also learned how to use arithmetic and geometric sequences to represent and analyze real-world situations.

12.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10, determine if the sequence is arithmetic, geometric, or neither. Then write a formula for the nth term of the sequence.

1. $3, 5, 7, 9, \ldots$

2. $2, 3, 4.5, 6.75, \ldots$

3. $40, 10, 2.5, 0.625, \ldots$

4. $105, 125, 145, 165, \ldots$

5. $1, -1, 1, -1, \ldots$

6. $10, 5, 0, -5, \ldots$

7. $\frac{1}{2}, \frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \ldots$

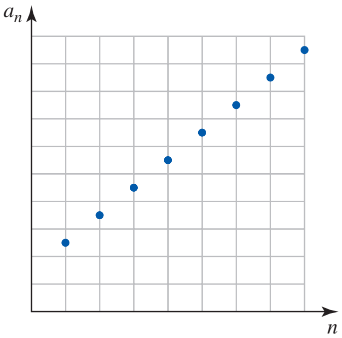
8. $\frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \ldots$

9. $5, 2.5, 0, -2.5, \ldots$

10. $27, -3, 1, -\frac{1}{3}, \ldots$

In Exercises 11–15, determine if the graph represents an arithmetic or a geometric sequence. If the sequence is arithmetic, determine whether the value of d is positive or negative. If it is geometric, determine whether the value of r is greater than 1 or between 0 and 1. For all the graphs, assume both the horizontal and vertical axes use the same scale.

11.



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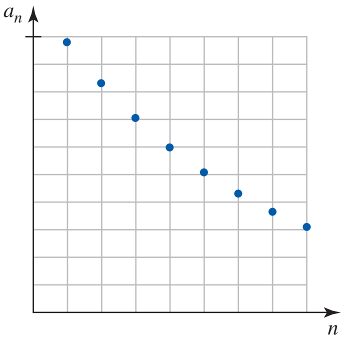

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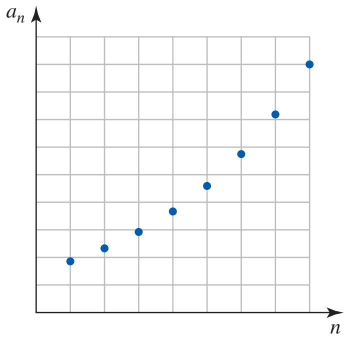

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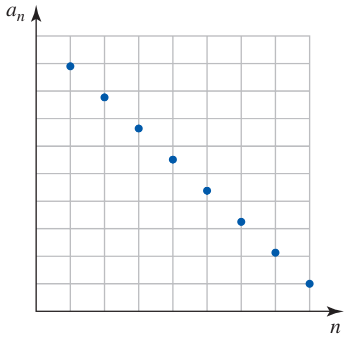

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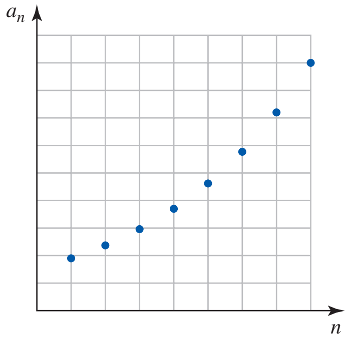


In Exercises 16–25, two terms of an arithmetic sequence are given. Write an explicit formula for the sequence and find the indicated term.

16. $a_1 = 2, a_5 = 10, a_{10} = ?$

17. $a_1 = 5, a_6 = 12, a_9 = ?$

18. $a_1 = -2, a_7 = 19, a_{15} = ?$

19. $a_1 = \frac{1}{2}, a_4 = -\frac{1}{2}, a_8 = ?$

20. $a_2 = 6, a_3 = 9, a_9 = ?$

21. $a_3 = 5, a_5 = 15, a_{15} = ?$

22. $a_2 = -1, a_6 = 7, a_{20} = ?$

23. $a_5 = 12, a_9 = 22, a_2 = ?$

24. $a_8 = \frac{25}{4}, a_{10} = \frac{83}{12}, a_{11} = ?$

25. $a_3 = \frac{9}{4}, a_6 = \frac{3}{2}, a_{16} = ?$

In Exercises 26–35, two terms of a geometric sequence are given. Write an explicit formula for the sequence and find the indicated term.

26. $a_1 = 2, a_3 = 8, a_6 = ?$

27. $a_1 = 60, a_4 = 7.5, a_7 = ?$

28. $a_1 = 5, a_2 = -5, a_{10} = ?$

29. $a_1 = 81, a_5 = 16, a_6 = ?$

30. $a_2 = -2, a_3 = 4, a_7 = ?$

31. $a_3 = 50, a_5 = 0.5, a_7 = ?$

32. $a_2 = 10, a_4 = 12.1, a_8 = ?$

33. $a_5 = 81, a_9 = 6561, a_2 = ?$

34. $a_8 = \frac{49}{2}, a_{10} = \frac{1}{2}, a_{11} = ?$

35. $a_3 = 100, a_6 = -0.1, a_2 = ?$

36. Two parents decided to put \$20 per week into a coffee can beginning on the day their baby boy was born. They continued this practice until the baby turned 18. At that time, they gave the money to their son to help pay for college.

- **a.** Is this situation best modeled by a geometric or an arithmetic sequence? Why?
- **b.** Write an explicit formula for the *n*th term of the sequence.
- **c.** Use the formula from part (b) to determine how much money the child was given on his 18th birthday.
- **d.** Suppose the son earns a scholarship and does not need the money that his parents had saved. Instead, he takes the money he receives on his 18th birthday and deposits it into an account that earns 4% annual interest. Is this situation best expressed using a geometric or an arithmetic sequence? Why?
- **e.** Write an explicit formula for the *n*th term of the sequence.
- **f.** Use the formula from part (e) to determine how much money the son will have on his 25th birthday.

SHOW YOU KNOW

37. What are the similarities and differences between arithmetic and geometric sequences?

38. Explain the relationship between an arithmetic sequence and a linear function.

39. Explain the relationship between a geometric sequence and an exponential function.

MAKE IT REAL

40. Starting Salaries According to a poll, the starting salary of a newly hired accountant in 2007 was \$46,718. (***Source:*** **money.cnn.com**) This was a 2.3% increase over the previous year.

- **a.** Explain why this situation can be represented by a geometric sequence. What information do the terms of the sequence provide?
- **b.** Write an explicit formula for the *n*th term of the sequence.
- **c.** Compute a_5 and explain what it means.

41. Tuition Payment Tuition at the University of Hawaii in 2007 was \$214 per credit for the first 11 credit hours. (***Source:*** **www.hawaii.edu**)

- **a.** Explain why this situation is best represented by an arithmetic sequence. What information do the terms of the sequence provide?
- **b.** Write an explicit formula for the *n*th term of the sequence.
- **c.** Compute a_{11} and explain what it means.

42. Rebounding Ball In a classroom experiment conducted by students in one of the authors' classrooms, it was discovered that when a racquetball is dropped on the classroom floor, it rebounds about $\frac{2}{3}$ of the height from which it was dropped. Imagine that we drop a ball and it always rebounds to $\frac{2}{3}$ of the previous height.

- **a.** Explain why this situation is best represented by a geometric sequence. What information do the terms of the sequence provide?
- **b.** Write an explicit formula for the *n*th term of the sequence.
- **c.** Compute a_5 and explain what it means.
- **d.** Will $a_n = 0$ for some value of n? Explain why or why not.

43. World's Tallest Champagne Fountain The tallest champagne fountain in the world was built by Luuk Broos in December 1999. (***Source:*** **timesofindia.indiatimes.com**) The 56-story tower of champagne glasses was on display at the Steigenberger Kurhaus Hotel, Scheveningen, Netherlands. The tower was quite complicated with many sections. For one section, there was one champagne glass at the top of the section with one additional champagne glass added to each subsequent row.

- **a.** Is this situation best expressed using a geometric or an arithmetic sequence? Why?
- **b.** Write an explicit formula for the *n*th term of the sequence.

STRETCH YOUR MIND

Exercises 44–47 are intended to challenge your understanding of arithmetic and geometric sequences.

44. A classmate tells you that she knows of an arithmetic sequence where $a_5 = 50$, $a_{10} = 55$, and $a_{20} = 65$. Write a convincing argument to support or refute her claim.

45. A classmate tells you that he knows of a geometric sequence where $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{4}$, and $a_{16} = 55$. Write a convincing argument to support or refute his claim.

46. Construct an arithmetic sequence of the form $a_n = a_1 + (n - 1)d$ and a geometric sequence of the form $b_n = b_1 r^{n-1}$ such that $a_n = b_n$ for at least one value of n. If you think it is not possible, clearly explain why you think so.

47. Create a geometric sequence of the form $a_n = a_1 r^{n-1}$ that oscillates between positive and negative values as $n \to \infty$. That is, for some arbitrarily large values of n, the values of a_n should be positive, then negative, then positive, and so on. If you think it is not possible, clearly explain why you think so.

SECTION 12.3

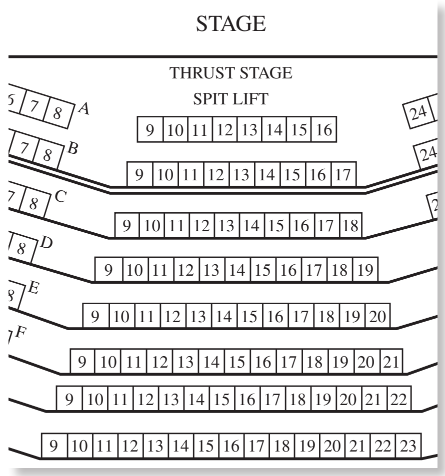
Series

LEARNING OBJECTIVES

- Determine whether a series is arithmetic or geometric
- Determine whether a series converges or diverges using graphs, tables, and symbols
- Compute the partial sums of arithmetic and geometric series
- Use series to solve real-world problems

GETTING STARTED

The Port Theatre in Nanaimo, British Columbia, Canada, was built to "stimulate and enhance artistic, cultural and economic activity of central Vancouver Island." (*Source:* www.porttheatre.nisa.com) As shown in the seating chart, the number of seats in each row of the front center section increases by one seat per row from the first to the eighth row. That is, there are 8 seats in the front row, 9 seats in the second row, 10 seats in the third row, and so on. To find how many seats are in this particular section, we could count the individual chairs, but if we use a mathematical *series* we obtain the answer more quickly.



Used by permission of The Port Theatre, Nanaimo, BC

In this section we investigate mathematical series, that is, the sums of the terms in a sequence, and see how to use them in real-world scenarios.

■ The Sum of the Terms in an Arithmetic Sequence

We can express the number of seats in the specified section of the Port Theatre as a sequence, where each term represents the number of chairs in the given row.

$$8, 9, 10, 11, 12, 13, 14, 15$$

To determine the total number of chairs, we sum the terms in the sequence. Since the sequence has eight terms, we write this sum using the notation S_8.

$$S_8 = 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$

In the Port Theatre situation, it is relatively easy to sum the eight terms in the sequence.

$$S_8 = 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15 = 92$$

However, if we were adding hundreds or thousands of terms, we would want a more efficient method.

Notice that if we pair up the first and last terms, the second and seventh terms, the third and sixth terms, and so on, each pair adds to 23.

$$S_8 = 8 + 9 + 10 + 11 + 12 + 13 + 14 + 15$$

Since there are four pairs of terms that sum to 23, the sum of the eight terms is $23 \times 4 = 92$. We use this observation to develop a formula for the sum of an arithmetic sequence.

If we take an arbitrary arithmetic sequence with n terms, the sum of the first and last terms is $a_1 + a_n$. The sum of the second and second-to-last terms is also $a_1 + a_n$ since $a_2 + a_{n-1} = (a_1 + d) + (a_n - d) = a_1 + a_n$. In fact, working inward from the ends, each pair of terms will sum to $a_1 + a_n$. How many of these pairs are there? Since there are n terms, there are $\frac{n}{2}$ pairs.

Sum of a Finite Arithmetic Sequence

The sum of the first n terms of a finite arithmetic sequence with first term a_1 and common difference d is given by

$$S_n = \frac{n}{2}(a_1 + a_n)$$

EXAMPLE 1 ■ Using a Finite Arithmetic Sequence

The Port Theatre has a section of seats in the front and center that has 8 seats in the first row, 9 seats in the second row, 10 seats in the third row, and so on. Find the number of seats in the eight rows in this section.

Solution We need to find the sum of the first eight terms of the sequence whose first term is $a = 8$ and whose common difference is $d = 1$. We first write the formula for the nth term of the arithmetic sequence as

$$\begin{aligned} a_n &= a_1 + (n - 1)d \\ a_n &= 8 + (n - 1)(1) \\ a_n &= 8 + n - 1 \\ a_n &= 7 + n \end{aligned}$$

We now find the eighth term of the sequence:

$$\begin{aligned} a_8 &= 7 + 8 \\ a_8 &= 15 \end{aligned}$$

Now we can find the sum of the first eight terms of the sequence.

$$\begin{aligned} S_8 &= \frac{8}{2}(8 + 15) \\ &= 4(23) \\ &= 92 \end{aligned}$$

There are 92 seats in this section of the Port Theatre.

EXAMPLE 2 ■ Computing the Sum of an Arithmetic Sequence

In the Nintendo Wii bowling game, the player tries to roll a series of strikes (knock down all pins in one roll of the ball) with more and more pins in the alley at each level. The final level challenges the player to knock down pins that are arranged so that there is one pin in the first row, two pins in the second row, and so on. Given there are 13 rows of pins, how many pins will the player need to knock down to complete the level?

Solution We know the first term of the arithmetic series is $a_1 = 1$, the last term is $a_{13} = 13$, and the common difference is $d = 1$. We use the formula for the sum of a finite arithmetic sequence to determine the sum of the 13 terms:

$$S_n = \frac{n}{2}(a_1 + a_n)$$

$$S_{13} = \frac{13}{2}(1 + 13)$$

$$= \frac{13}{2}(14)$$

$$= 91$$

There are 91 pins to be knocked down.

Geometric Series

Azithromycin is an antibiotic used to fight diseases such as strep throat. A typical prescription requires taking two 250-mg tablets initially followed by a single 250-mg tablet each day for four consecutive days. The half-life of azithromycin is 68 hours. (*Source:* www.drugs.com) This means that 50% of the medicine is eliminated from the body every 68 hours. (This is equivalent to eliminating about 22% of the medicine from the body daily.) We will investigate this situation to determine how much medicine will be in the body when the final dose is taken.

We assume the medicine is absorbed into the bloodstream immediately after taking a dose. Initially, 500 mg of medicine is taken. We represent this as $M_1 = 500$ mg. On day 2, a second dose of 250 mg is taken. At that time, the amount of medicine that will be in the body is about

$$M_2 = 250 + 500(0.78) = 640 \text{ mg}$$

This includes the 250-mg second dose and the 78% of the 500-mg first dose that remains. On day 3, a third dose of 250 mg is taken. At that time, the amount of medicine that will be in the body is

$$M_3 = 250 + 640(0.78) \approx 749 \text{ mg}$$

This includes the 250-mg third dose and 78% of the 640 mg that remains from the first two doses.

To better see a pattern develop, consider the following nonsimplified version of the computations:

$$\begin{aligned} M_3 &= 250 + (0.78)(M_2) \\ &= 250 + (0.78)[250 + (500)(0.78)] \\ &= 250 + (250)(0.78) + (500)(0.78)^2 \\ &\approx 749 \text{ mg} \end{aligned}$$

On day 4, a fourth dose of 250 mg is taken. At that time, the amount of medicine that will be in the body is

$$\begin{aligned} M_4 &= 250 + (0.78)(M_3) \\ &= 250 + (0.78)[250 + (250)(0.78) + (500)(0.78)^2] \\ &= 250 + (250)(0.78) + (250)(0.78)^2 + (500)(0.78)^3 \\ &\approx 834 \text{ mg} \end{aligned}$$

With the exception of the final term, we see a sum in the form

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

Expressions of this form are known as a finite geometric series. A **finite geometric series** is a sum in which each term in the sum is found by multiplying the previous term by a constant. In other words, a geometric series is the sum of the terms of a geometric sequence.

FINITE GEOMETRIC SERIES

A finite geometric series with n terms can be expressed as

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1}$$

where a is the first term of the series and r is the *common ratio* or *constant multiple*.

In our example, each term (except for the last) is multiplied by 0.78 since 78% of the medicine remains in the body upon taking the next dose. We can express this sum as an explicit formula once we recognize the pattern.

250	250(0.78)	$250(0.78)^2$	$250(0.78)^3$	$500(0.78)^4$
$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$

Expressing each term in the series as a_n, we have

$$a_1 = 250$$
$$a_2 = 250(0.78)$$
$$a_3 = 250(0.78)^2$$
$$a_4 = 250(0.78)^3$$
$$a_5 = 500(0.78)^4$$

We use sigma notation to express the first four terms of this sum. The Greek letter sigma (Σ) means to create a sum.

$$\sum_{n=1}^{4} 250(0.78)^{n-1}$$

In this case, the terms of the sum are formed by letting $n = 1$, $n = 2$, $n = 3$, $n = 4$. Of course, in this case we also have to add the final term.

$$a_5 = 500(0.78)^4$$

Therefore, the amount of azithromycin in the body after taking the fifth dose is

$$\sum_{n=1}^{4} 250(0.78)^{n-1} + a_5 = 250 + (250)(0.78) + (250)(0.78)^2 + (250)(0.78)^3 + (500)(0.78)^4$$
$$= 900.8 \text{ mg}$$

EXAMPLE 3 ■ **Using a Finite Geometric Series**

A person deposits \$2000 every year into an account that earns 8% compounded annually. Using sigma notation, express the amount of money that will be in the account after 8 years, then determine the amount.

Solution Let A_n represent the amount of money in the account immediately after the \$2000 is deposited into the account for the nth time. Thus,

$$A_1 = 2000$$
$$A_2 = 2000 + (2000)(1.08) = 4160$$

We find A_2, the amount of money in the account after the second year's deposit is made, by adding the new \$2000 deposit to the original \$2000 plus the interest the original deposit earned. We continue this pattern. After the third year's deposit, we will have \$2000, plus the previous years' deposits and the interest they have earned.

$$\begin{aligned} A_3 &= 2000 + (4160)(1.08) \\ &= 2000 + (2000 + (2000)(1.08))(1.08) \\ &= 2000 + (2000)(1.08) + (2000)(1.08)^2 \\ &= 6492.80 \end{aligned}$$

We investigate what happens after the deposit is made for year 4 and then look for a pattern to help us write the sum in sigma notation.

$$\begin{aligned} A_4 &= 2000 + (6492.80)(1.08) \\ &= 2000 + [2000 + (2000)(1.08) + (2000)(1.08)^2](1.08) \\ &= 2000 + (2000)(1.08) + (2000)(1.08)^2 + (2000)(1.08)^3 \\ &= 9012.22 \end{aligned}$$

We see we have a geometric series that begins with \$2000. Each subsequent term is multiplied by the common multiple 1.08. Using sigma notation, we have

$$\begin{aligned} &\sum_{n=1}^{8} 2000(1.08)^{n-1} \\ &\quad = 2000 + (2000)(1.08) + (2000)(1.08)^2 + (2000)(1.08)^3 + \cdots + (2000)(1.08)^7 \\ &\quad = 21{,}273.26 \end{aligned}$$

After the eighth deposit, the account will have \$21,273.26.

Finding the Formula for a Finite Geometric Series

In the previous examples, we needed to add a finite number of terms of the particular geometric sequence. In the first case, we added a total of five terms. In the second case, we added a total of eight terms. If we needed to add 100 terms, our current method would be tedious and inefficient.

To develop a more efficient method, we begin with a general finite geometric sequence and represent the sum of the first n terms using S_n.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

Next, we multiply each side of this equation by r.

$$rS_n = ar + ar^2 + ar^3 + \cdots + ar^n$$

Now we subtract rS_n from S_n.

$$\begin{aligned} S_n - rS_n &= (a + ar + ar^2 + \cdots + ar^{n-1}) - (ar + ar^2 + ar^3 + \cdots + ar^{n-1} + ar^n) \\ &= (a + ar + ar^2 + \cdots + ar^{n-1}) - ar - ar^2 - ar^3 - \cdots - ar^{n-1} - ar^n \\ &= a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} - \cancel{ar} - \cancel{ar^2} - \cancel{ar^3} - \cdots - \cancel{ar^{n-1}} - ar^n \\ &= a - ar^n \end{aligned}$$

We see that all of the terms except the first and the last ones cancel out. Since our goal is to find a simplified way to express S_n, we solve this equation for S_n.

$$\begin{aligned} S_n - rS_n &= a - ar^n \\ S_n(1 - r) &= a - ar^n \\ S_n &= \frac{a - ar^n}{1 - r} \\ &= \frac{a(1 - r^n)}{1 - r} \end{aligned}$$

Sum of a Finite Geometric Sequence

The sum of the first n terms of a finite geometric sequence with first term a and common ratio r is

$$S_n = \frac{a(1 - r^n)}{1 - r}$$

provided that $r \neq 1$.

EXAMPLE 4 ■ Finding the Sum of a Finite Geometric Sequence

Use the formula for the sum of a finite geometric sequence to confirm that the amount of azithromycin in the body will be about 901 mg after the fifth dose. Recall that 22% of the medicine is eliminated from the body each day and that the prescription calls for an initial dose of 500 mg followed by daily doses of 250 mg.

Solution We first need to find the sum of the first four terms of the finite geometric sequence where $a = 250$ mg and $r = 0.78$ (since 22% is eliminated, 78% remains).

$$\begin{aligned} S_4 &= \frac{250 - 250(0.78)^4}{1 - 0.78} \\ &= 715.7 \end{aligned}$$

Next, we determine how much of the original dose is remaining in the body.

$$500(0.78)^4 = 185.1$$

We add these results to get the final amount of medicine in the body after the fifth dose is taken.

$$715.7 + 185.1 = 900.8 \text{ mg}$$

EXAMPLE 5 ■ Finding the Sum of a Finite Geometric Sequence

How much azithromycin will be in the body if the medicine is taken for 10 days? Recall that the prescription calls for an initial dose of 500 mg followed by daily doses of 250 mg and that 22% of the medicine is eliminated from the body each day.

Solution We need to find the sum of the first nine terms of the geometric sequence where $a = 250$ and $r = 0.78$.

$$\begin{aligned} S_9 &= \frac{250 - 250(0.78)^9}{1 - 0.78} \\ &= 1014.9 \text{ mg} \end{aligned}$$

We also need to consider the amount of the initial 500-mg dose that is left in the body.

$$500(0.78)^9 = 53.4 \text{ mg}$$

We add these results to get the final amount of medicine in the body after the tenth dose is taken.

$$1014.9 + 53.4 = 1068.3 \text{ mg}$$

EXAMPLE 6 ■ **Using the Sum of a Finite Geometric Sequence**

A person deposits \$1500 every year into an account that earns 12% compounded annually. How much money will be in this account after 20 years? After 40 years?

Solution We have a finite geometric series where $a = \$1500$ and $r = 1.12$.

$$S_{20} = \frac{1500 - 1500(1.12)^{20}}{1 - 1.12}$$
$$= 108{,}078.66$$

After 20 years, the account will contain over \$108,000.

We can use the same formula to find the amount of money in the account after 40 years.

$$S_{40} = \frac{1500 - 1500(1.12)^{40}}{1 - 1.12}$$
$$= 1{,}150{,}637.13$$

After 40 years, the account will contain over \$1 million.

Convergence and Infinite Geometric Series

When we look at the examples involving medicine and saving money, we see that their end behavior differs. The amount of medicine in the bloodstream stabilized over time, but the amount of money in the savings account continued to grow at an increasing rate. If we were to let the number of terms summed in each series go to infinity, we would see two very different results. A series that stabilizes (reaches a maintenance level) is said to **converge**. A series that does not stabilize is said to **diverge**.

We now consider the sum of an infinite geometric series for different cases of r to determine under what circumstances the infinite sum will converge or diverge.

***Case 1: r* = 1**

Recall that we are working with the geometric series of the form $S_n = a + ar + ar^2 + \cdots + ar^{n-1}$. If $r = 1$, then

$$S_n = a + a + a + \cdots + a$$
$$= na$$

If $n \to \infty$, then $S_n \to \infty$ and the series diverges.

***Case 2: r* > 1**

If $r > 1$, then subsequent terms in the sequence will get larger and larger (since we are multiplying a by values larger than 1). As $n \to \infty$, the sum gets larger and larger and thus diverges. This is what happens in the case of the money account.

***Case 3: r* < −1**

If $r < -1$, then we multiply a by positive values when the power of r is even and negative values when the power of r is odd. Since $|r| > 1$, $|r^n| > |r^{n-1}|$ for all values of n. Therefore, the sum will oscillate and become larger and larger in magnitude. Again, the infinite sum diverges.

***Case 4:* $-1 < r < 1$**

The only remaining case is a series with values of r that are between -1 and 1 as in the case of the azithromycin, where $r = 0.22$. If we analyze the formula for a finite geometric sum, we recognize that the series will stabilize.

$$S_n = \frac{a - ar^n}{1 - r}$$
$$= \frac{a(1 - r^n)}{1 - r}$$

As $n \to \infty$, $r^n \to 0$ since $-1 < r < 1$. Therefore,

$$\text{as } n \to \infty,\ S_n \to \frac{a(1 - 0)}{1 - r} = \frac{a}{1 - r}$$

In general, we have the following.

Sum of an Infinite Geometric Series

The sum of the first n terms of an infinite geometric series

$$S_n = a + ar + ar^2 + \cdots + ar^{n-1}$$

converges to

$$S = \frac{a}{1 - r}$$

as $n \to \infty$ provided that $-1 < r < 1$. Otherwise, the series diverges.

We can verify the formula for the sum of an infinite geometric series if we consider the sum for very large values of n.

EXAMPLE 7 ■ Using an Infinite Series

A person is given a prescription that requires a single 500-mg dose of medicine every day. Each day, 25% of the medicine is eliminated from the body. How much medicine is in the body after 10 days? 50 days? 100 days? What if the person takes the medicine indefinitely?

Solution We recognize this to be a situation involving a finite geometric series with $a = 500$ and $r = 0.75$ (since 25% of the medicine is eliminated, 75% remains). We use the formula for the sum of a finite geometric series to find the amount of medicine in the body after 10, 50, and then 100 days.

$$S_{10} = \frac{500 - 500(0.75)^{10}}{1 - 0.75}$$
$$= 1887.37 \text{ mg}$$

After the 10th dose, there will be 1887.37 mg in the body.

$$S_{50} = \frac{500 - 500(0.75)^{50}}{1 - 0.75}$$
$$= 2000.00 \text{ mg}$$

After the 50th dose, there will be 2000 mg of medicine in the body. It appears as though we are reaching the maintenance level. Let's verify by finding the amount of medicine in the body after the 100th dose.

$$\begin{aligned} S_{100} &= \frac{500 - 500(0.75)^{100}}{1 - 0.75} \\ &= 2000 \text{ mg} \end{aligned}$$

The amount of medicine is being maintained at 2000 mg. If the medicine was to be taken indefinitely, then we could say that $n \to \infty$. We could then find the sum of the infinite geometric series where $a = 500$ mg and $r = 0.75$.

$$\begin{aligned} S &= \frac{500}{1 - 0.75} \\ &= 2000 \text{ mg} \end{aligned}$$

As predicted, the amount of medicine in the body will stabilize at 2000 mg.

EXAMPLE 8 ■ Using an Infinite Series

Attracting large events such as the Super Bowl to a city can boost the local economy. The boost comes in the form of money spent by visitors who pay for lodging, food, transportation, souvenirs, and so on. Economists estimate that about 70% of every dollar introduced into the local economy is spent again locally. Then, 70% of that money is spent again locally. The process continues with 70% of the money spent in the community being spent again locally. It is estimated that during the 1998 Super Bowl in San Diego, visitors contributed well over \$125 million in direct spending to the region while they were there. (*Source:* www.nfl.com) What was the total impact, in dollars, on the local San Diego economy considering how the money was re-spent locally?

Solution We begin with the \$125 million that was spent by visitors to San Diego. Of that amount, 70% was spent again locally, and this pattern continues indefinitely. We can express the economic impact as the infinite geometric series, S.

$$\begin{aligned} S &= 125 + (125)(0.70) + (125)(0.70)^2 + (125)(0.70)^3 + \cdots \\ &= \sum_{n=1}^{\infty} 125(0.70)^{n-1} \end{aligned}$$

Thus we can find the sum of this infinite geometric series where $a = 125$ and $r = 0.70$.

$$\begin{aligned} S &= \frac{125}{1 - 0.70} \\ &= 416.67 \end{aligned}$$

We estimate the total economic impact of the Super Bowl on the San Diego area to be \$416,670,000.

Finding a Rational Number Given a Repeating Decimal

Many people know the repeating decimal 0.3333 . . . represents the fraction $\frac{1}{3}$. But what fraction does the repeating decimal 0.82222 . . . represent? We can use the ideas from this section to find a rational number that is equivalent to a given repeating decimal.

EXAMPLE 9 ■ Writing a Repeating Decimal as a Rational Number

Express the repeating decimal 0.82222 . . . as a rational number.

Solution We begin by rewriting the repeating decimal as a sum.

$$0.8 + 0.02 + 0.002 + 0.0002 + \cdots$$

After the first term of 0.8, we see a geometric series where the first term is 0.02 and each subsequent term is found by multiplying by $\frac{1}{10} = 0.1$.

$$0.8 + \underbrace{0.02 + 0.002 + 0.0002 + \cdots}_{\text{Geometric series}}$$

We can find the sum of the infinite geometric series.

$$\begin{aligned} S &= \frac{0.02}{1 - 0.1} \\ &= \frac{0.02}{0.9} \\ &= \frac{2}{90} && \text{Multiply by } \frac{100}{100}. \\ &= \frac{1}{45} \end{aligned}$$

We keep the calculations in fraction form since we ultimately want to express the repeating decimal in fraction form. We now add the initial 0.8 to the fraction $\frac{1}{45}$ to obtain the final rational number.

$$\begin{aligned} 0.82222\ldots &= 0.8 + \frac{1}{45} \\ &= \frac{8}{10} + \frac{1}{45} \\ &= \frac{(8)(9)}{90} + \frac{(1)(2)}{90} \\ &= \frac{72 + 2}{90} \\ &= \frac{74}{90} \\ &= \frac{37}{45} \end{aligned}$$

SUMMARY

In this section you learned how to find the sum of finite arithmetic and geometric sequences. You also learned how to determine whether a series converges or diverges. Additionally, you saw how to calculate the sum of a convergent infinite geometric series and use these series in real-world situations.

12.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, determine whether the series is arithmetic, geometric, or neither. Explain how you know.

1. $\sum_{n=1}^{10} 500\left(\frac{1}{2}\right)^n$

2. $\sum_{n=1}^{10} 500 + \frac{1}{2}n$

3. $\sum_{n=1}^{20} 2n^2$

4. $\sum_{n=1}^{5} (4 + 2n)$

5. $\sum_{n=1}^{5} 4(2)^n$

In Exercises 6–10, find the sum of the first 20 terms in the arithmetic series.

6. $3 + 6 + 9 + 12 + \cdots + 3n + \cdots$

7. $-5 + -1 + 3 + 7 + \cdots + (4n - 9) + \cdots$

8. $10 + 9.5 + 9 + 8.5 + \cdots + (9.5 + 0.5k) + \cdots$

9. $\frac{1}{4} + \frac{3}{8} + \frac{1}{2} + \frac{5}{8} + \cdots + \left(\frac{1}{8} + \frac{1}{8}k\right) + \cdots$

10. $7 + 9 + 11 + 13 + \cdots + (5 + 2n) + \cdots$

11. A pyramid of logs has 2 logs in the top row, 4 logs in the second row from the top, 6 logs in the third row from the top, and so on, until there are 200 logs in the bottom row.

a. Write and interpret the first 10 terms of this sequence of numbers.

b. Provide the formula for the nth term of the sequence and use it to find the number of logs in the 76th row.

c. Use an arithmetic series to determine the total number of logs in the pyramid.

In Exercises 12–16, find the sum of the first four terms in the geometric series. Then determine a formula for the sum of the first n terms in the series. Determine whether the series converges or diverges. If the series converges, state its sum.

12. $\frac{3}{4^0} + \frac{3}{4^1} + \frac{3}{4^2} + \frac{3}{4^3} + \cdots + \frac{3}{4^{n-1}} + \cdots$

13. $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \frac{1}{24} + \cdots + \frac{1}{3 \cdot 2^{n-1}} + \cdots$

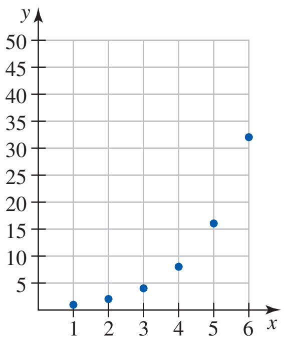
14. $2 + \frac{2}{2^{1/2}} + \frac{2}{3^{1/2}} + \frac{2}{4^{1/2}} + \cdots + \frac{2}{k^{1/2}} + \cdots$

15. $10 + 1 + \frac{1}{10} + \frac{1}{100} + \cdots + \frac{10}{10^{k-1}} + \cdots$

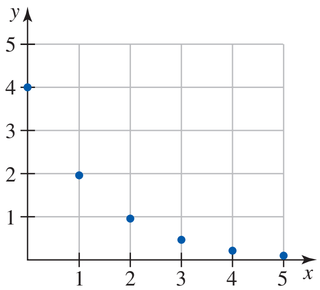
16. $2 + 4 + 6 + 8 + \cdots + 2k + \cdots$

In Exercises 17–21, match the given series with the appropriate graph (A–E) of the partial sums.

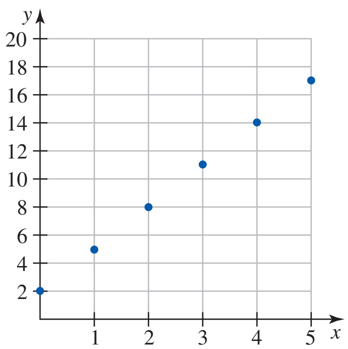
A.



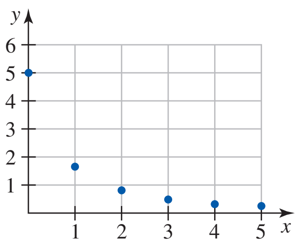
B.



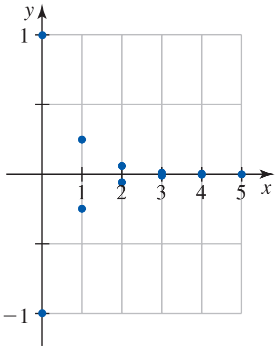
C.



D.



E.



17. $\sum_{n=1}^{\infty} 4\left(\frac{1}{2}\right)^n$

18. $\sum_{n=1}^{\infty} \frac{1}{2}(2^n)$

19. $\sum_{n=1}^{\infty} \left(-\frac{1}{4}\right)^n$

20. $\sum_{n=1}^{\infty} (2 + 3n)$

21. $\sum_{n=1}^{\infty} \frac{10}{(n+1)(n+2)}$

In Exercises 22–26, an infinite geometric series is given. Explain how you can tell if the infinite series will converge or diverge simply by looking at the expression. Do not determine the sum of the infinite series.

22. $\sum_{n=1}^{\infty} \frac{1}{5}(5^n)$

23. $\sum_{n=1}^{\infty} 5\left(\frac{1}{5}\right)^n$

24. $\sum_{n=1}^{\infty} 5\left(-\frac{1}{5}\right)^n$

25. $\sum_{n=1}^{\infty} 1\left(\frac{2}{3}\right)^n$

26. $\sum_{n=1}^{\infty} \frac{2}{3}(1^n)$

In Exercises 27–31, express each repeating decimal as a rational number.

27. 0.2222 . . .

28. 0.9999 . . .

29. 2.242424 . . .

30. 1.7373737 . . .

31. 0.181818 . . .

In Exercises 32–41, find the missing value.

32. $a = 6, d = 5, S_{20} = 1070, a_{20} = ?$

33. $a = 250, d = -2, S_{50} = 10{,}050, a_{50} = ?$

34. $a_{100} = 304, d = 3, S_{100} = 15{,}550, a = ?$

35. $a_{200} = 89.5, d = 0.5, S_{200} = 15{,}900, a = ?$

36. $r = 1.5, a = 5, S_5 = ?$

37. $r = 0.25, a = 40, S_{10} = ?$

38. $r = 3, S_6 = 1164.8, a = ?$

39. $r = 2, S_5 = 310, a = ?$

40. $r = 4, S_7 = 2730.5, a = ?$

41. $r = 1.5, S_{12} = 100, a = ?$

■ SHOW YOU KNOW

42. Write a finite geometric series where the common ratio is between −1 and 1. Explain why the series you created is a geometric series. Find the total sum of your finite geometric series.

43. Explain why the formula for the sum of a finite geometric series includes the stipulation that $r \neq 1$.

44. Explain why the formula for the sum of an infinite geometric series requires that $-1 < r < 1$ in order for the infinite series to converge.

45. Write an infinite geometric series that will converge. Explain how you know that the series you created will converge. Find the total sum of your infinite series.

46. Explain why the idea of an infinite geometric series can be used to find a rational number that is equivalent to a repeating decimal. Then, write your own repeating decimal and then express your number as a rational number.

■ MAKE IT REAL

47. Economics According to the Federal Reserve Bank of Boston, \$180 million of counterfeit money was put into the economy in 1999. (*Source:* www.bos.frb.org) Suppose each time this counterfeit money is used, 26% of it is seized. What is the total dollar impact of the counterfeit money from 1999 on the economy?

48. Economic Impact of an Event Officials from the City of Indianapolis predicted that up to \$45 million was going to be spent by visitors who were in town for the NCAA Final Four basketball tournament in 2006. It was expected that much of this money would be spent on national chain restaurants and hotels and therefore may not be spent again locally. (*Source:* www.courier-journal.com) Suppose only 40% of the money is spent again locally. What is the total economic impact of the Final Four basketball tournament in Indianapolis?

49. Pharmaceuticals—Azithromycin Azithromycin is an antibiotic often prescribed to treat strep throat. The half-life of azithromycin is 68 hours. Suppose 753.3 mg of azithromycin is in the bloodstream after the fifth dose is taken. If the person stops taking the medicine at this time, how long will it take before the medicine is eliminated from the bloodstream? (*Source:* www.drugs.com)

■ STRETCH YOUR MIND

Exercises 50–56 are intended to challenge your understanding of series.

50. A ball is dropped from a height of 15 meters. It rebounds one-half of the vertical height from which it just fell. It continues to rebound in this way indefinitely. Find the total distance that the ball traveled.

51. The transcendental number e can be expressed as an infinite series. The number e can be approximated using the infinite series

$$e \approx 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots$$

Write this infinite series using sigma notation and verify that the partial sums approach e as the number of terms approaches ∞.

52. The transcendental number π can be expressed as an infinite series. The number π may be approximated using the infinite series

$$\pi \approx \frac{4}{1} - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} + \frac{4}{9} + \frac{4(-1)^{n+1}}{2n - 1} + \cdots$$

Write this infinite series using sigma notation and verify that the partial sums approach π as the number of terms approaches ∞.

53. Verify, using multiple representations, that the following series converges to 1.

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$$

54. What is the value(s) of p if $\sum_{n=1}^{\infty} (2 + p)^{-2n} = \frac{1}{8}$?

55. Write a discussion revealing the errors in reasoning in the following "proof."

$$0 = 0 + 0 + 0 + \cdots$$
$$0 = (1 - 1) + (1 - 1) + (1 - 1) + \cdots$$
$$0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$
$$0 = 1 + (-1 + 1) + (-1 + 1) + \cdots$$
$$0 = 1 + 0 + 0 + \cdots$$
$$0 = 1$$

56. Construct an argument that explains that if the series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, then the series $\sum_{n=1}^{\infty} \frac{1}{n - 5}$ must also diverge.

CHAPTER 12 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 12.1

1. What is a sequence and what does it mean to say that a sequence converges?
2. What is the difference between recursive and explicit formulas for sequences?

SECTION 12.2

3. What is the difference between a geometric sequence and an arithmetic sequence?
4. What is the relationship between an arithmetic sequence and a linear function? What is the relationship between a geometric sequence and an exponential function?

SECTION 12.3

5. What is the difference between a sequence and a series?
6. How do you find the sum of a finite arithmetic series and why does this procedure make sense?
7. What does it mean to say that we can find the sum of an infinite series?
8. How do you find the sum of an infinite geometric series?

REVIEW EXERCISES

SECTION 12.1

In Exercises 1–4,

a. List the first six terms of the given sequence.

b. Use a graphing calculator to graph the sequence.

c. State whether the sequence appears to converge or diverge.

1. $a_n = \dfrac{n^2 + 1}{n}$

2. $a_n = \dfrac{10n}{3^n}$

3. $a_n = 0.5a_{n-1} + 2$

4. $a_n = 1.5a_{n-1} + 1$

In Exercises 5–8, find a formula for the general term of the sequence, a_n, and state whether the sequence converges. Assume the pattern in the given terms continues.

5. $-2, 1, 4, 7, \ldots$

6. $18, 6, 2, \dfrac{1}{3}, \ldots$

7. $\dfrac{2}{3}, \dfrac{4}{5}, \dfrac{6}{7}, \dfrac{8}{9}, \ldots$

8. $2, 3, \dfrac{9}{2}, \dfrac{27}{4}, \dfrac{81}{8}, \ldots$

9. Heartworm Medicine Dog owners give their dogs heartworm medicine once every month to prevent heartworm disease. One type of heartworm medicine, Interceptor Flavor Tabs, contains 11.5 mg of medicine and has a half-life of about 36 hours. (***Source:*** **www.beaconforhealth.org**)

a. Generate the first five terms of the sequence that would represent the amount of heartworm medicine in the bloodstream every 36 hours.

b. Create an explicit formula representing the amount of medicine in the bloodstream.

c. Estimate the amount of medicine in the bloodstream at the end of 2 weeks.

d. How much medicine will be in the bloodstream at the time the next dose is administered one month later? Use a graph to confirm your answer.

10. Pharmaceuticals—Naproxen Naproxen is an anti-inflammatory medicine often used by patients with arthritis to reduce swelling in joints. With a half-life of 15 hours, about 42.6% of the medicine is eliminated from the bloodstream every 12 hours. A typical prescription calls for the patient to take one 500-mg tablet every 12 hours. (***Source:*** **www.drugs.com**)

a. Generate the first five terms of the sequence that would represent the amount of medicine in the bloodstream every 12 hours.

b. What is the maintenance level in this situation?

c. How long does it take to reach the maintenance level? Use a graph to confirm your answer.

SECTION 12.2

In Exercises 11–15, two terms of an arithmetic sequence are given. Write an explicit formula for the sequence and find the indicated term.

11. $a_1 = -5, a_5 = 3, a_{10} = ?$

12. $a_1 = 10, a_8 = 45, a_{18} = ?$

13. $a_5 = 1, a_8 = -14, a_2 = ?$

14. $a_{20} = 100, a_{25} = 102.5, a_{65} = ?$

15. $a_8 = 2, a_{11} = \dfrac{11}{4}, a_1 = ?$

In Exercises 16–20, two terms of a geometric sequence are given. Write an explicit formula for the sequence and find the indicated term.

16. $a_1 = 10, a_3 = 2.5, a_6 = ?$

17. $a_1 = -1, a_5 = -16, a_{10} = ?$

18. $a_2 = 1, a_6 = 81, a_1 = ?$

19. $a_5 = 8, a_7 = \dfrac{1}{8}, a_3 = ?$

20. $a_{10} = \dfrac{128}{25}, a_{15} = \dfrac{4096}{25}, a_{38} = ?$

21. Starting Salaries According to a poll, the starting salary of a newly hired college English major in 2007 was $32,553. (***Source:*** **money.cnn.com**) This was a 5.3% increase over the previous year.

a. Explain why this situation is best represented by a geometric sequence. What information do the terms of the sequence provide?

b. Write an explicit formula for the nth term of the sequence.

c. Compute a_5 and explain what it means.

22. Skydiving In November 2003, a group of 70 skydivers performed one of the largest canopy formation jumps ever. (***Source:*** **www.boeing.com**) The canopy formation is created, after the parachutes are opened, so that there is one person at the top, two people below the top person, then three people below them, and so on. There are nine rows following this pattern before the pattern is inverted. For this exercise, only consider the first nine rows of the formation.

a. Is this situation better expressed using a geometric or an arithmetic sequence? Why?

b. Write an explicit formula for the nth term of the sequence.

SECTION 12.3

In Exercises 23–26, determine whether the series is arithmetic, geometric, or neither. Justify your answer.

23. $\displaystyle\sum_{n=1}^{10} 2 + 3n$

24. $\displaystyle\sum_{n=1}^{5} 10\left(\frac{1}{14}\right)^{n-1}$

25. $\displaystyle\sum_{n=1}^{15} n$

26. $\displaystyle\sum_{n=1}^{5} 2n^2 + 1$

In Exercises 27–31, find the sum of the first 20 terms of the arithmetic series.

27. $5 + 10 + 15 + 20 + \cdots + 5n + \cdots$

28. $-2 + 0 + 2 + 4 + \cdots + (2n - 4) + \cdots$

29. $10 + \frac{41}{4} + \frac{21}{2} + \frac{43}{4} + \cdots + \left(\frac{1}{4}n + \frac{39}{4}\right) + \cdots$

30. $100 + 103 + 106 + 109 + \cdots + (3n + 97) + \cdots$

31. $(-5) + (-9) + (-13) + (-17) + \cdots + (-4n - 1) + \cdots$

In Exercises 32–36, find the sum of the first four terms of the geometric series. Determine whether the series converges or diverges. If the series converges, state its sum.

32. $\frac{1}{5^0} + \frac{1}{5^1} + \frac{1}{5^2} + \frac{1}{5^3} + \cdots + \frac{1}{5^{n-1}} + \cdots$

33. $\frac{1}{7} + \frac{2}{7} + \frac{4}{7} + \frac{8}{7} + \cdots + \frac{2^{n-1}}{7} + \cdots$

34. $3 + \frac{3}{\sqrt{2}} + \frac{3}{2} + \frac{3}{2\sqrt{2}} + \frac{3}{4} + \cdots + \frac{3}{(\sqrt{2})^{k-1}} + \cdots$

35. $5 + 1 + \frac{1}{5} + \frac{1}{25} + \cdots + \frac{5}{5^{k-1}} + \cdots$

36. $3 + 6 + 12 + 24 + \cdots + 3(2)^{k-1} + \cdots$

In Exercises 37–40, express each repeating decimal as a rational number.

37. $0.1111\ldots$

38. $0.123123\ldots$

39. $0.1818\ldots$

40. $0.23\overline{1}\ldots$

41. Pharmaceuticals—Naproxen Suppose a maintenance level of 871.1 mg of naproxen is in the bloodstream. If the person stops taking the medicine at this time, how long before the medicine is eliminated from the bloodstream? The half-life of naproxen is 15 hours. (*Source:* www.drugs.com)

42. Pharmaceuticals—Singulair® Suppose a maintenance level of 10 mg of Singulair is in the bloodstream. If the person stops taking the medicine at this time, how long before the medicine is eliminated from the bloodstream? The half-life of Singulair is 4 hours. (*Source:* www.drugs.com)

Make It Real Project

What to Do

1. Choose a prescription drug that you are interested in researching. Find the half-life for the drug and a typical dose.
2. Write the general term of the sequence describing the amount of the drug in the bloodstream for a person who takes the typical dose.
3. Write the general term of the geometric series describing the amount of the drug in the bloodstream.
4. Provide a full analysis for the drug including its maintenance level using graphs, tables, and symbols.
5. Describe the effects of missing a dose. Determine the effect of taking a double dose the next time the medication is to be taken.
6. Write a report summarizing your findings.

Where to Find Data

Data for this project can be found at the following sites, among others. Look under "Metabolism," "Pharmacokinetics," or "Elimination" in the technical information section for half-life data.

- www.drugs.com
- www.rxmed.com
- www.astrazeneca-us.com

At these sites, you will also see the type of information that is included concerning information to the consumer.