CHAPTER 6

Exponential and Logarithmic Functions

The term "exponential" is used widely in the media. Consider the following excerpts from actual news articles:

"The **exponential** amount of digitized content . . ."

" . . . demand continues to increase at an **exponential** pace . . ."

" . . . economic indicators continue their **exponential** rise . . ."

". . . [they] were given **exponential** worth in advertising dollars . . ."

As sophisticated as these phrases may sound, they are vague at best and senseless at worst. There is a clear need for an increased understanding of exponential relationships.

SECTION 6.1

Percentage Change

LEARNING OBJECTIVES

- Calculate change factors from tables and graphs
- Calculate percentage rates of change from tables, graphs, and change factors
- Recognize that functions with a constant percentage change are exponential functions

GETTING STARTED

Consistent advances in computer technology have produced vastly increased processing power since the early 1970s. In 1965, one of Intel's cofounders, Gordon Moore, made the empirical observation now known as Moore's Law: the number of transistors on an integrated circuit doubles every 2 years. (*Source:* www.intel.com/technology/mooreslaw) One way to analyze the doubling behavior of a situation such as this is to consider the growth percentage change that occurs over time.

In this section we distinguish between the rate of change we studied in previous chapters and percentage change. We also derive the standard equation for an exponential function (any function with a constant percentage change). Additionally, we show how to calculate growth and decay factors as well as percentage change. We use these concepts to "fill in the gaps" in data.

Patterns of Growth

The phrase "the population of the city of Gilbert, Arizona, is *increasing at a rate of 1000 people per month*" describes growth using an average rate of change, whereas the phrase "the population of Tempe, Arizona, is *increasing at 0.019% per year*" describes growth as a percentage change. Although these two familiar ways of describing growth sound similar and may have similar patterns in the short term, they are fundamentally different over the long term. Recall any function with a constant rate of change, such as the Gilbert population function, is a *linear function*. On the other hand, any function with a constant *percentage* rate of change, such as the population growth of Tempe, is an *exponential function*.

For example, consider the growth in the number of transistors on an integrated circuit. Table 6.1, generated using Moore's law, displays the predicted number of transistors over a 10-year period beginning in 1971.

Table 6.1

Years Since 1971 x	Number of Transistors T
0	2300
2	4600
4	9200
6	18,400
8	36,800
10	73,600

It is apparent from these data that the number of transistors is increasing over time. Thus, the function is an increasing function. From our prior study of polynomial functions, we know that to determine if a data set is linear or quadratic we need to find the successive differences over equal intervals of time. We see from Table 6.1 that the change in the years is consistently in equal intervals, so let's begin by looking at the first and second differences in Table 6.2.

Table 6.2

Years Since 1971 x	Number of Transistors T	First Differences ΔT	Second Differences $\Delta(\Delta T)$
0	2300	2300	2300
2	4600	4600	4600
4	9200	9200	9200
6	18,400	18,400	18,400
8	36,800	36,800	
10	73,600		

Since neither the first nor the second differences are constant, the function is neither linear nor quadratic.

A different way of quantifying the change in the number of transistors is to calculate the ratio (quotient) of successive output values for the equally spaced input values, as shown in Table 6.3. Notice that all of the successive ratios are equal. We say the data has a *2-year growth factor* of 2 since the output values are multiplied by the factor of 2 for each 2-year increase.

Table 6.3

Years Since 1971 x	Number of Transistors T_x	Ratio $\frac{T_{x+2}}{T_x}$
0	2300	$\frac{4600}{2300} = 2$
2	4600	$\frac{9200}{4600} = 2$
4	9200	$\frac{18{,}400}{9200} = 2$
6	18,400	$\frac{36{,}800}{18{,}400} = 2$
8	36,800	$\frac{73{,}600}{36{,}800} = 2$
10	73,600	

If the same pattern of growth continues, we can estimate the number of transistors in 2-year increments beyond year 10 by using the 2-year growth factor to expand the data as shown in the shaded portion of Table 6.4.

Table 6.4

Years Since 1971 x	Number of Transistors T_x	Tax-Year Growth Factor (Ratio) $\frac{T_{x+2}}{T_x}$
0	2300	2
2	4600	2
4	9200	2
6	18,400	2
8	36,800	2
10	73,600	2
12	147,200	2
14	294,400	2
16	588,800	

Exponential Functions

Finding and applying the growth factor works well if we continue to estimate the number of transistors at 2-year intervals. But to fill in the data (interpolate) for missing years or forecast the data (extrapolate) for future years, we need a function that calculates the number of transistors for any year.

From the preceding tables we see that the transistor function will have an initial value of 2300. That is, $f(0) = 2300$. Furthermore, by observing patterns we find that

$$\begin{aligned} f(2) &= 4600 \\ &= 2300(2) \\ &= 2300(2^1) \\ &= 2300(2^{\frac{1}{2}(2)}) \end{aligned} \qquad \begin{aligned} f(4) &= 9200 \\ &= 2300(4) \\ &= 2300(2^2) \\ &= 2300(2^{\frac{1}{2}(4)}) \end{aligned} \qquad \begin{aligned} f(6) &= 18{,}400 \\ &= 2300(8) \\ &= 2300(2^3) \\ &= 2300(2^{\frac{1}{2}(6)}) \end{aligned} \qquad \begin{aligned} f(8) &= 36{,}800 \\ &= 2300(16) \\ &= 2300(2^4) \\ &= 2300(2^{\frac{1}{2}(8)}) \end{aligned}$$

Thus it appears that $f(x) = 2300(2^{\frac{1}{2}x})$, where x is the number of years since 1971. With this function, we can calculate the number of transistors in 1972 (1 year after 1971) or 2012 (41 years after 1971).

Interpolation

$$\begin{aligned} f(1) &= 2300(2^{\frac{1}{2}(1)}) \\ &= 2300(2^{0.5}) \\ &\approx 2300(1.414) \\ &\approx 3252 \end{aligned}$$

Extrapolation

$$\begin{aligned} f(41) &= 2300(2^{\frac{1}{2}(41)}) \\ &= 2300(2^{20.5}) \\ &\approx 2300(1482910.4) \\ &\approx 3{,}410{,}693{,}921 \end{aligned}$$

We estimate there were 3252 transistors on an integrated circuit in 1972 and that there will be 3.4 billion transistors on an integrated circuit in 2012.

The **exponential function** $f(x) = 2300(2^{\frac{1}{2}x})$ can be rewritten as

$$\begin{aligned} f(x) &= 2300(2^{\frac{1}{2}x}) \\ &= 2300(2^{\frac{1}{2}})^x && \text{since } b^{mn} = (b^m)^n \\ &= 2300(\sqrt{2})^x && \text{since } b^{\frac{1}{2}} = \sqrt{b} \\ &\approx 2300(1.414)^x && \text{since } \sqrt{2} \approx 1.414 \end{aligned}$$

As stated earlier, 2300 is the initial value of the function. The value 1.414 is the annual growth factor or, more commonly, the **growth factor**.

EXPONENTIAL FUNCTION

A function of the form

$$y = ab^x$$

with $a \neq 0$, $b > 0$, and $b \neq 1$ is an **exponential function**.

a is the **initial value** of the function.

b is the **growth factor** if $b > 1$ and the **decay factor** if $b < 1$. (For convenience, we will refer to growth and decay factors as **change factors** when discussing strategies or concepts that apply to both types.)

EXAMPLE 1 ■ Writing the Equation for an Exponential Function

The annual growth factor for the population of the United States is approximately 1.09. In 2005, the population of the United States was estimated to be 298.2 million. (*Source:* World Health Organization) Write the equation of the exponential function that represents this situation.

Solution Let t represent the number of years since 2005 and P be the population of the United States (in millions). Since 2005 is 0 years after 2005, the initial value of the population function is 298.2 million. We have

$$P(t) = 298.2(b)^t$$

Since the annual growth factor is $b = 1.09$, the exponential function model equation is $P(t) = 298.2(1.09)^t$.

Finding Change Factors

If the input values of an exponential function are 1 unit apart, the *change factor* is simply equal to the ratio of consecutive output values. However, finding the change factor when the input values are more or less than 1 unit apart requires a bit more work.

For example, in the transistor scenario the input values were 2 units apart, so the ratio we calculated was the 2-year growth factor. Let's see how we can accurately find the annual (1-year) growth factor (which we found earlier by trial-and-error). Calling the annual growth factor b, we get

$$b \cdot b = 2 \text{ (2-year growth factor from Table 6.3)}$$
$$b^2 = 2$$
$$b = \sqrt{2} \approx 1.414$$

This agrees with our earlier result when we found the function $f(x) = 2300(1.414)^x$. We can now use the annual growth factor 1.414 to interpolate for years 1, 3, and 5, as shaded in Table 6.5.

Table 6.5

Years Since 1971 x	Number of Transistors T	Annual Growth Factor $\frac{T_{x+1}}{T_x}$	Two-Year Growth Factor $\frac{T_{x+2}}{T_x}$
0	2300	$\sqrt{2} \approx 1.414$	
1	3252	1.414	2
2	4600	1.414	
3	6502	1.414	2
4	9200	1.414	
5	13,001	1.414	2
6	18,400	1.414	

To generalize, consider the exponential function $y = ab^x$ that passes through the points (x_1, y_1) and (x_2, y_2). Calculating the ratio of the output values, we get

$$\frac{y_2}{y_1} = \frac{ab^{x_2}}{ab^{x_1}}$$
$$\frac{y_2}{y_1} = \frac{b^{x_2}}{b^{x_1}}$$
$$\frac{y_2}{y_1} = b^{x_2 - x_1}$$

Recall from the rules of rational exponents (see Just in Time in Section 6.4) that $(x^n)^{1/n} = x$. Therefore, we raise each side of the equation to the power $\frac{1}{x_2 - x_1}$ to solve for b.

$$b^{x_2 - x_1} = \frac{y_2}{y_1}$$
$$\left(b^{x_2 - x_1}\right)^{1/(x_2 - x_1)} = \left(\frac{y_2}{y_1}\right)^{1/(x_2 - x_1)}$$
$$b = \left(\frac{y_2}{y_1}\right)^{1/(x_2 - x_1)}$$

In words, the change factor is the ratio of the outputs raised to 1 over the difference in the inputs.

HOW TO: ■ CALCULATE CHANGE FACTORS

To calculate the change factor of an exponential function with points (x_1, y_1) and (x_2, y_2), raise the ratio of the output values to 1 over the difference in the input values. That is,

$$b = \left(\frac{y_2}{y_1}\right)^{1/(x_2 - x_1)}$$

Table 6.6

Years Since 2000 y	Number of Cremations C
0	629,362
5	778,025

Source: www.cremationassociation.org

EXAMPLE 2 ■ Finding Change Factors

According to the Cremation Association of North America (CANA), the number of people choosing to be cremated is increasing dramatically. Table 6.6 displays the number of people choosing cremation in the United States for the years 2000 and 2005.

a. Assuming these data have a common growth factor, determine the 5-*year* growth factor and the *annual* growth factor.

b. Use the annual growth factor to interpolate the number of cremations in 2001, 2002, 2003, and 2004.

Solution

a. To find the 5-year growth factor, we find the ratio by dividing the number of cremations for the year 2005 by the number of cremations for the year 2000.

$$\frac{C(5)}{C(0)} = \frac{778{,}025}{629{,}362} \qquad \frac{\text{number of cremations in 2005}}{\text{number of cremations in 2000}}$$

$$\approx 1.236 \qquad \text{5-year growth factor}$$

The 5-year growth factor is approximately 1.236.

The annual growth factor is given by

$$b = \left(\frac{778{,}025}{629{,}362}\right)^{1/(5-0)}$$

$$\approx (1.236)^{1/5}$$

$$\approx 1.043$$

The annual growth factor is approximately 1.043.

To check this, recall that the annual growth factor is the number we would have to multiply by itself 5 times to get the 5-year growth factor, as shown in Table 6.7.

Table 6.7

Years Since 2000 y	Number of Cremations C	Annual Growth Factor	Five-Year Growth Factor
0	629,362	b	
1	?	b	
2	?	b	
3	?	b	
4	?	b	1.236
5	778,025		

In other words, multiplying by 5 annual growth factors is equivalent to multiplying by the 5-year growth factor once. Letting b represent the annual growth factor, we can write this as

$$b \cdot b \cdot b \cdot b \cdot b = 1.236$$
$$b^5 = 1.236$$

Solving for b, we get

$$b^5 = 1.236$$
$$\sqrt[5]{b^5} = \sqrt[5]{1.236}$$
$$b = 1.043$$

Therefore, the annual growth factor is approximately 1.043, which confirms our earlier result.

b. We can now fill in Table 6.8 for 2001 to 2004. (Note that although we rounded b to 1.043 when we wrote it down, we used the more accurate estimate $b = 1.043322581$ to generate the table. All numbers have been rounded to the nearest whole number.)

Table 6.8

Years Since 2000 y	Number of Cremations C	Annual Growth Factor	Five-Year Growth Factor
0	629,362	1.043	
1	656,628	1.043	
2	685,074	1.043	
3	714,754	1.043	
4	745,719	1.043	1.236
5	778,025		

Percentage Change

So far in this text, we have described change in terms of the average rate of change or the change factor. However, change is often discussed in terms of a percentage change.

EXAMPLE 3 ■ Calculating a Percentage Change

One avenue people use to help generate adequate funds for retirement is the stock market. The Standard and Poor's 500 Index (S&P 500), considered by many to be the best indicator for the U.S. stock market as a whole, provides the benchmark by which other investments and portfolio managers are measured. (*Source:* www.zealllc.com) Table 6.9 shows how the S&P 500 Index changed between January 3, 1997, and January 3, 2007. Find the 10-year percentage change and use it to extrapolate the S&P 500 Index for the year 2017.

Table 6.9

Years Since January 3, 1997 y	S&P 500 Index S
0	748.03
10	1416.60

Source: finance.yahoo.com

Solution To find the 10-year percentage change we first calculate the 10-year growth factor, which is the ratio of the two indices for 1997 and 2007.

$$\begin{aligned}\text{10-year growth factor} &= \frac{1416.60}{748.03} \\ &\approx 1.89\end{aligned}$$

To convert the 10-year growth factor (1.89) to a percentage we multiply by 100%.

$$1.89(100\%) = 189\%$$

The 2007 index value is 189% of the 1997 value. If it were 100% of the 1997 index value, the index values would be the same. Therefore, the additional 89% represents the 10-year growth in the index value. Assuming the 2017 index value will be 189% of the 2007 index value, we predict the S&P 500 for the year 2017 to be

$$\begin{aligned}S(20) &= 1416.60(1.89) \\ &\approx 2677.37\end{aligned}$$

As we saw in Example 3, change factors and percentage rates of change are closely related.

Change Factors and Percentage Rates of Change

The **change factor**, b, of an exponential function is given by $b = 1 + r$, where r is the **percentage rate of change** (as a decimal).

- If $r > 0$, b is called a **growth factor** and r is called the **percentage growth rate**.
- If $r < 0$, b is called a **decay factor** and r is called the **percentage decay rate**.

EXAMPLE 4 ■ Finding an Exponential Function from a Percentage Rate

On March 31, 2007, USAA Federal Savings Bank advertised a 5-year certificate of deposit (CD) with an *annual percentage yield* of 5.0%. (**Annual percentage yield**, discussed in Section 6.3, is the percentage increase in the value of an investment over a 1-year period.) (*Source:* www.usaa.com)

Find an exponential function that models the value of a $1000 investment in the 5-year CD as a function of the number of years the money has been invested. Then calculate the value of the CD when it matures 5 years later.

Solution We convert the percentage growth rate into a growth factor.

$$\begin{aligned}b &= 1 + r \\ &= 1 + 0.05 && \text{since } 5\% = 0.05 \\ &= 1.05\end{aligned}$$

Since the initial investment is \$1000, the value of the investment after t years is given by $V(t) = 1000(1.05)^t$. To determine the value of the investment at maturity, we evaluate the function at $t = 5$.

$$\begin{aligned} V(5) &= 1000(1.05)^5 \\ &= 1276.28 \end{aligned}$$

A \$1000 investment into a CD with an annual percentage yield of 5% will be valued at \$1276.28 when it matures 5 years later.

Not only do exponential functions have constant change factors, they also have constant percentage rates of change. In fact, any function that is changing at a constant percentage rate is an exponential function.

Exponential Growth and Decay

Any function that *increases* at a constant percentage rate is said to demonstrate **exponential growth**. This growth may be very rapid (e.g., 44% per year) or very slow (e.g., 0.01% per year).

Any function that *decreases* at a constant percentage rate is said to demonstrate **exponential decay**. This decay may be very rapid (e.g., −50% per year) or very slow (e.g., −0.02% per year).

EXAMPLE 5 ■ Determining Decay Factors and Percentage Decay Rates

Although depreciation rates vary among vehicles, a typical car will lose about 15% to 20% of its value each year. (*Source:* www.kbb.com) Suppose you purchase a new Toyota Camry for \$21,500 and want to estimate its worth over the next 5 years. Assuming it will lose 15% of its value each year,

a. Determine the annual percentage decay rate, the annual decay factor, the 5-year decay factor, and the 5-year percentage decay rate.

b. Create a table of values for the car's value over this period.

Solution

a. Since the car's value is depreciating, the annual percentage decay rate will be a *negative* value (−15%) and is written as the decimal $r = -0.15$.

Since the decay factor is given by $b = 1 + r$, we have

$$b = 1.00 - 0.15 \qquad \textbf{decay factor = 1 − percentage decay}$$
$$b = 0.85 \qquad \textbf{annual decay factor}$$

The annual decay factor is 0.85.

To determine the 5-year decay factor, we raise the annual decay factor to the fifth power.

$$(0.85)^5 \approx 0.4437$$

The 5-year decay factor is 0.4437. That is, after 5 years, the car will be worth 44.37% of its original value.

To determine the 5-year decay rate, we subtract 1 from the 5-year decay factor since $r = b - 1$.

$$0.4437 - 1 = -0.5563$$

The car is depreciating at a rate of 55.63% every 5 years.

b. The values for the car over the time period are shown in Table 6.10.

Table 6.10

Age of Car a	Value Computation	Value of the Car V
0	$21{,}500(0.85)^0$	21,500.00
1	$21{,}500(0.85)^1$	18,275.00
2	$21{,}500(0.85)^2$	15,533.75
3	$21{,}500(0.85)^3$	13,203.69
4	$21{,}500(0.85)^4$	11,223.13
5	$21{,}500(0.85)^5$	9539.66

EXAMPLE 6 ■ Finding an Exponential Model

A news article titled "Study Indicates Volunteerism Rises Among Collegians" made the claim that "The number of college students volunteering grew more than 20 percent, from 2.7 million to 3.3 million, between 2002 and 2005." (*Source: East Valley Tribune*: October 16, 2006)

Find the annual growth factor, the annual percentage growth rate, and an exponential model for the number of college-aged volunteers. Use this to predict the number of volunteers in the year 2012.

Solution Since the percentage of college-aged volunteers grew by 20% over a 3-year interval, 20% is the 3-year percentage growth rate. We need to calculate the *annual* growth factor and the *annual* percentage growth rate. To do this, we first need to find the 3-year growth factor by using the fact that $b = 1 + r$.

$$b = 1 + 0.20$$
$$b = 1.20$$

Therefore, the 3-year growth factor is 1.20. We take the third root of the 3-year growth factor to find the annual growth factor.

$$\begin{aligned} b &= 1.20^{1/3} \\ &= \sqrt[3]{1.20} \\ &\approx 1.063 \end{aligned}$$

Since the annual growth factor is related to the annual percentage growth rate by the formula $b = 1 + r$, we have

$$\begin{aligned} 1.063 &= 1 + r \\ 1.063 - 1 &= r \\ 0.063 &= r \\ 0.063(100\%) &= r \\ 6.3\% &= r \end{aligned}$$

Therefore, the annual percentage growth rate is 6.3%.

Since the initial number of volunteers was 2.7 million and the annual growth factor is 1.063, the exponential model that represents the number of college-aged volunteers is $V(y) = 2.7(1.063)^y$, where V is the number of volunteers (in millions) and y is the number of years since 2002. We can use this to predict the number of college-aged volunteers in the year 2011 ($y = 9$).

$$V(9) = 2.7(1.063)^9$$
$$\approx 4.7 \text{ million student volunteers}$$

SUMMARY

In this section you learned how to distinguish between the rate of change and the percentage rate of change. You also learned that an exponential function has a constant percentage rate of change and the standard form $y = ab^x$. Additionally, you discovered how to calculate change factors and percentage rates of change and how to use these values to fill in the gaps in data sets.

6.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, determine whether the following tables are linear, quadratic, exponential, or none of these by calculating successive differences and/or change factors.

1.

x	y
0	80
1	40
2	20
3	10
4	5
5	2.5

2.

x	y
0	3
1	10
2	21
3	36
4	55
5	78

3.

x	y
0	0
1	3
2	16
3	45
4	96
5	175

4.

x	y
0	−5
1	−15
2	−45
3	−135
4	−405
5	−1215

5. Match the data sets in the following tables with the functions shown in A, B, and C.

a.

x	y
0	4
1	6
2	9
3	13.5
4	20.25

b.

x	y
1	80
2	64
3	51.2
4	40.96
5	32.768

c.

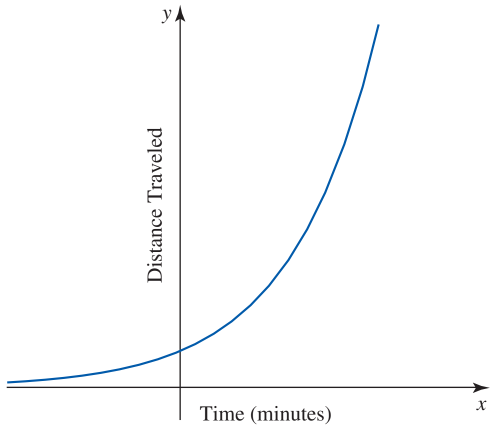
x	y
−5	0.69632
−4	1.7408
−3	4.352
−2	10.88
−1	27.2

A. $k(x) = 68(2.5)^x$
B. $f(x) = 4(1.5)^x$
C. $h(x) = 100(0.8)^x$

6. The table shows some values of an exponential function, f, and a linear function, g. Find the equation for $f(x)$ and $g(x)$ and use the functions to complete the table.

x	$f(x)$	$g(x)$
0	?	5
1	0.36	?
2	0.216	8.2
3	?	?
4	0.07776	?
5	?	13

7. After examining the following graph you determine that distance traveled increases at an increasing rate. Which of the following best describes the reasoning that can be used to determine this response?



A. The amount of distance traveled is greater for each successive second.

B. As you move to the right on the graph the slope gets steeper.

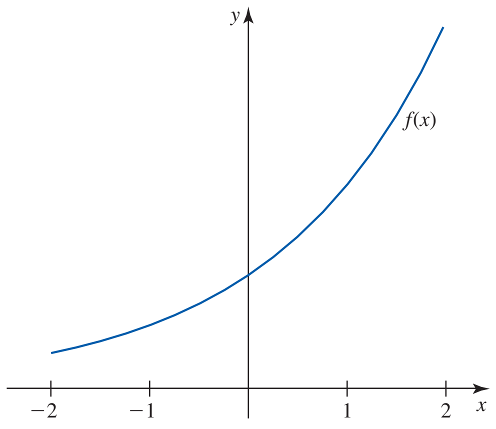
C. The rate is greater for each successive second.

D. All of the above.

E. None of the above.

8. Allowance Riddle Suppose a child is offered two choices to earn an increasing weekly allowance: the first option he can choose begins at 1 cent and doubles each week, while the second option begins at $1 and increases by $1 each week. How much allowance would the child earn in 4 weeks, 8 weeks, and a year? Which option is the best choice for the child?

In Exercises 9–12, use the graph of f(x) below to answer each question.



9. Considering the information provided by the graph of $f(x)$, is it possible to determine if the function is quadratic or exponential? Why or why not?

10. In the following table, find values of $f(x)$ that will make the function quadratic.

x	$f(x)$
−2	
−1	
0	
1	
2	

11. In the following table, find values of $f(x)$ that will make the function exponential.

x	$f(x)$
−2	
−1	
0	
1	
2	

12. Give a possible equation for $f(x)$.

SHOW YOU KNOW

13. Explain what we mean when we talk about *percentage growth* or *percentage decay*.

14. Write an explanation comparing *constant percentage change* to *constant rate of change*.

15. Microorganism Growth A microbiologist generated a model that describes the number of bacteria in a culture after t days but has just updated the model from $P(t) = 7(2)^t$ to $P(t) = 7(3)^t$. Which of the following implications can you draw from this information? Defend your choice(s).

A. The final number of bacteria is three times as much instead of two times as much as the original amount.

B. The initial number of bacteria is three instead of two.

C. The number of bacteria triples every day instead of doubling every day.

D. The growth rate of the bacteria in the culture is 30 percent per day instead of 20 percent per day.

16. Jack and the Beanstalk Jack plants a 5-centimeter-tall beanstalk in his backyard. When the beanstalk is 5-centimeters tall it grows by about 15% per day for the next month. What formula represents the height of the beanstalk as a function of the number of days since it was transplanted? Explain your choice.

A. $H(t) = 5 + 0.15t$

B. $H(t) = 5.15t$

C. $H(t) = 5(1.15)^t$

D. $H(t) = 5.15^t$

E. $H(t) = 5 + 1.15t$

17. Salary Adjustments A person's salary is reduced by 5% one year due to necessary budget cuts. The next year (after business has improved) the person is given a 5% raise. Is the employee's income back to where it was originally? Explain your reasoning and support your argument with at least one of the following: equations, graphs, or tables.

MAKE IT REAL

18. U.S. Immigrants The number of immigrants coming into the United States increased from 385,000 in 1975 to 1,122,000 in 2005. (***Source: Statistical Abstract of the United States, 2007*, Table 5**) Calculate the average rate of change, the 30-year growth factor, the annual growth factor, the 30-year percentage change, and the average annual percentage change.

19. Stock Market According to the 2004 Andex Chart, the average return of a $1 investment made in 1925—with no acquisition costs or taxes, and all income reinvested into the S&P 500—would have grown to $2641 by 2005. (***Source:* www.andexcharts.com**)

a. What is the 80-year growth factor?

b. What is the 80-year percentage growth rate?

c. What is the average annual growth factor?

d. What is the average annual percentage change?

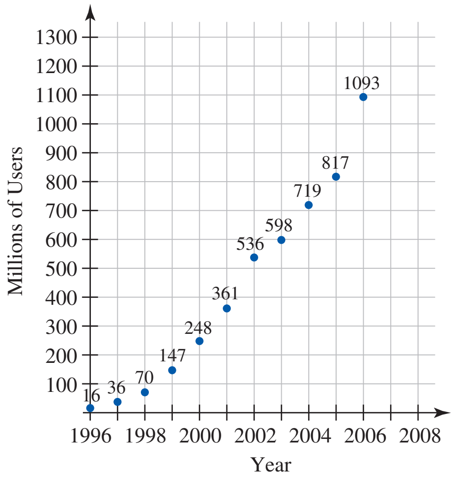
e. Write an equation for the function $I(y)$, which would model the value of the initial $1 investment, I, as a function of the number of years since 1925.

f. Evaluate $I(90)$ and explain the meaning of the numerical value in the context of the problem.

20. Golf Course Management Managers of golf pro shops anticipate the total revenue for an upcoming year based on the predicted total number of rounds that will be played at their course. They do this by looking at the total number of rounds played in prior years. Assume there were 33,048 rounds played 3 years ago and 38,183 rounds played this year. Calculate the average rate of change and the annual percentage change, and use each to predict the number of rounds to be played 2 years from now.

Stephen Mcsweeny/ Shutterstock.com

21. Internet Usage The graph displays the incredible growth in worldwide Internet usage from 1996 to 2006. Using successive differences and ratios, determine whether a linear, quadratic, or exponential function would be the best mathematical model.



***Source:* www.internetworldstats.com/emarketing.htm**

22. Professional Baseball Salaries Professional athletes are some of the highest paid people in the world. The average major league baseball player's salary climbed from $1,998,000 in 2000 to $2,866,500 in 2006. (***Source:* www.sportsline.com**)

a. Assuming *linear* growth in the average players' salaries, find a formula $L(y)$ for the average salary L as a function of the year since 2000, y. Explain what the slope and vertical intercept mean in the real-world context. Use $L(y)$ to predict this year's average salary.

b. Assuming *exponential* growth in the average players' salaries, find a formula $E(y)$ as a function of the year since 2000, y. Explain what the initial value and growth factor mean in the real-world context. Use $E(y)$ to predict this year's average salary.

23. Purchasing Power of the Dollar The purchasing power of the dollar as measured by consumer prices from 1980 to 2005 is given in the table. (Assume that in 1982, $1 was worth $1.)

Years Since 1980 y	Value of the Dollar V
0	1.215
5	0.928
10	0.766
15	0.656
20	0.581
25	0.512

***Source: Statistical Abstract of the United States, 2007*, Table 705**

a. Use the data to calculate the change factor for the purchasing power of the dollar for each of the following time intervals: (i) 1980 through 2005, (ii) 1990 through 1995, and (iii) 2000 through 2005.

b. Express the changes in part (a) as percentage changes. Write a sentence interpreting each answer in the real-world context.

c. Calculate the average rate of change and the percentage change in purchasing power of the dollar from 1985 through 2005.

d. What is the practical meaning to consumers if the percentage change in the purchasing power of the dollar is negative? Positive? Zero?

24. **Tuition at American Private Universities** The table shows the average yearly tuition and required fees charged by American private universities in the school year beginning in 2000.

Years Since 2000 y	Average Tuition (dollars) T
0	16,072
1	17,377
2	18,060
3	18,950
4	20,045
5	21,235

Source: Annual Survey of Colleges, **The College Board, New York**

a. Using a rate of change and percentage change, show why these data can be modeled by a linear function or an exponential function.

b. Plot the data points and add the graph of the linear function that best models $T(y)$.

c. What is the slope for the linear function modeling tuition and fees for private universities? What does this mean in the real-world context?

d. What prediction does this formula give for average tuition and fees at American private universities for the academic year beginning in 2011?

25. **Tuition at American Public Universities** The table shows the average yearly in-state tuition and required fees charged by American public universities in the school year beginning in 2000.

Years Since 2000 y	Average Tuition (dollars) T		Years Since 2000 y	Average Tuition (dollars) T
0	3508		3	4645
1	3766		4	5126
2	4098		5	5491

Source: Annual Survey of Colleges, **The College Board, New York**

a. Using a rate of change and percentage change, show why these data can be modeled by a linear function or an exponential function.

b. What is the slope for the linear function modeling tuition and fees for public universities? What does this mean in the real-world context?

c. Explain what the slope of the linear function model tells you about the rate of increase in tuition at public versus private institutions.

26. **Celebrity Spending Power** Johnny Depp is estimated to have earned \$18 million in 2006. In contrast, the average American citizen made roughly \$30,000. To get a sense of what it feels like to make \$18 million a year, the prices of a number of common products are listed together with the "feels like" price.

Product	At This Price (dollars)	Feels Like (dollars)
house	275,000.00	458.33
car	20,000.00	33.33
laptop	2000.00	3.33
boom box	300.00	0.50
hotel room	100.00	0.17
nice meal	40.00	0.07
hamburger	2.29	0.00
soda	1.00	0.00

Source: **www.bankrate.com/brm/news/financial_literacy/movies_salary.asp?web=brm&athlete=163&submit=submit**

Using percentages, explain how the table converts the actual product price for the average American citizen to the "feels like" price for Johnny Depp.

27. **U.S. Metro Populations** According to the U.S. Census Bureau, Atlanta, Georgia, added more people than any other metropolitan area from 2000 to 2006. The population of Atlanta experienced a 6-year percentage growth rate of 21%. The New Orleans area, still recovering from Hurricane Katrina, suffered a 6-year percentage decay rate of 22.2% over the same time period. (***Source:*** **Arizona Republic, April 5, 2007**) Give the annual percentage growth rate for Atlanta and the annual percentage decay rate for New Orleans from 2000 to 2006.

28. **Computer Transistors** In the Getting Started feature of this section, we stated that in 1965 Moore's Law predicted that the number of transistors on a circuit would double every 2 years. Assuming this is true, we can model the growth of the number of transistors as $T(y) = 2300(\sqrt{2})^y$, where y is the number of years since 1971. The table shows the actual number of transistors for different Intel computers. Do these data provide evidence for or against using the mathematical model $T(y)$ and Moore's Law?

Intel Product Name	Years Since 1971 y	Number of Transistors on Integrated Circuit T
4004 Microprocessor	0	2300
8008 Microprocessor	1	3500
8080 Microprocessor	3	6000
8086 Microprocessor	7	29,000
286 Microprocessor	11	134,000
386 Microprocessor	14	275,000
486 Microprocessor	18	1,200,000
Pentium	22	3,100,000
Pentium II	26	7,500,000
Pentium III	28	9,500,000
Pentium IV	29	42,000,000
Itanium 2	31	220,000,000
Dual Core Itanium 2	35	1,700,000,000

Source: Intel: www.intel.com

29. **Jeans Sale** In 2007, Lucky Brand Jeans advertised their Socialite Jean for $118. (*Source:* www.luckybrandjeans.com) If a pair of jeans costs $118 today, how much will a pair cost in w weeks if the price is reduced by
   a. $10 per week?
   b. 10% per week?

30. **Photocopies** Most photocopy machines allow for enlarging or reducing the size of the original. Suppose you have a chart you wish to photocopy for a report. The chart is 8 inches by 5 inches. To reduce the size of the copy by 20%, you set the machine to reduce the chart to 80% of its original dimensions.
   a. What will be the dimensions of the photocopy?
   b. What is the percentage reduction in the *area* of the photocopy?
   c. If the chart must fit into a space 3 inches high in your report, how many reductions must you make with the photocopier set at 80% to reduce the height of the image from the original 5 inches to 3 inches?
   d. If you were to actually perform ten 80% reductions (photocopies of photocopies), what will the dimensions of your chart be?
   e. Find formulas for the width, w, and height, h, of the chart in terms of the number of 80% reductions, r.

31. **In the News** Find a newspaper article that claims something is "growing exponentially" and either support or refute that claim. Do you think the reporter simply meant the data is increasing rapidly or was the reference truly to a constant percentage change?

32. **Hotel Revenue** In 2006, hotels in Mesa and Chandler, Arizona, raised room rates by 8.7% over the prior year's rates and still filled more rooms than the year before. The average daily rate of a room in 2005 was $81.77. (*Source: East Valley Tribune*, April 12, 2007)
   a. To what price did the room rates rise in 2006?
   b. If the room rates continue to rise at 8.7% a year for the next few years, write a function, $R(y)$, to determine the room rate, R, in dollars for a given year, y, since 2005.
   c. How long do you think the room rates could continue to rise at 8.7% a year? Why?

33. **Radioactive Fallout** During the late 1950s and early 1960s, atmospheric tests of nuclear weapons became a global political concern because of the radioactive substances they released into the air (fallout). The most problematic of these substances was iodine-131, a radioactive isotope of iodine. Iodine-131 can settle on grass, be consumed by cows, become concentrated in milk, and ultimately end up in the thyroid glands of human beings who drink the milk. (*Source:* www.answers.com) The half-life of iodine-131 is about eight days, meaning half of the original quantity is left after eight days. What fraction of the iodine-131 released in an atmospheric nuclear test would be left after 30 days?

34. **Coca-Cola Production** The 2001 annual report of the Coca-Cola Company stated:

   > *"Our worldwide unit case volume increased 4 percent in 2001, on top of a 4 percent increase in 2000. The increase in unit case volume reflects consistent performance across certain key operations despite difficult global economic conditions. Our business system sold 17.8 billion unit cases in 2001."* (*Source:* Coca-Cola Company 2001 Annual Report, p. 46)

   A unit case is equivalent to 24 8-ounce servings of finished beverage.
   a. Would a linear or exponential function best model the unit case volume of Coca-Cola? Explain.
   b. Based on the given data, it appears unit case volume has an annual percentage growth of 4%. If the unit case volume is modeled with an exponential function, what will be the annual growth factor?
   c. Find the exponential function that best models the unit case volume of Coca-Cola.
   d. Use the exponential model from part (c) to forecast the unit case volume sold in 2003 and 2004.
   e. According to Coca-Cola's 2004 Annual Report, the company sold "approximately 19.8 billion unit cases of [their] products in 2004 and approximately 19.4 billion cases in 2003." (*Source:* Coca-Cola Company 2004 Annual Report, p. 47) How accurate was the exponential model from part (c) at forecasting the 2003 and 2004 results? Explain.

35. **Family Trees** Have you ever constructed, or looked at, your family tree? Ignoring divorces, second marriages, and adoptions, answer each of the following questions, assuming the people would be alive.
   a. How many parents do you have?
   b. How many direct ancestors two generations before your generation (grandparents) do you have?

c. How many direct ancestors three generations before your generation (great-grandparents) do you have?

d. How many direct ancestors four generations before your generation do you have?

e. Find a formula for the exponential function that would give you the number of ancestors, A, based on the number of prior generations, g, you go back.

36. Airplane Takeoffs There are many factors that can require aircraft to allow for longer takeoff distances, called takeoff rolls. To ensure a safe takeoff, it is the responsibility of the pilot to assess all factors and conditions to calculate the cumulative total distance required for takeoff. The table indicates the possible cumulative effect of some takeoff conditions on takeoff rolls for a certain light airplane.

Additional Takeoff Condition	Cumulative Takeoff Roll (meters)
normal	400
tail wind (+20%)	$400(1.20) = 480$
weight (+20%)	$480(1.20) = 576$
temperature (+20%)	
grass landing strip (+25%)	
altitude (+10%)	
upslope (+10%)	

Source: **www.auf.asn.au/index.html**

a. Given that the normal run for the airplane is 400 meters, complete the table of takeoff values, accounting for the additional percentages as each additional takeoff condition is added.

b. What is the overall percentage increase in takeoff roll when all six takeoff conditions exist?

37. Atmospheric Pressure The physics of atmospheric pressure is well known—air resistance is lower at higher altitudes. Therefore, baseballs hit at Denver's Coors Field carry farther than in any other stadium in the country. Research confirms that a homerun that travels 400 feet in Miami would travel 420 feet in Denver. Atmospheric pressure decreases approximately exponentially with increasing height above sea level, at a rate of about 0.4% every 100 feet. (***Source:*** **www.aip.org/dbis/stories/2006/15254.html**) By what percentage is air pressure reduced by moving from sea level to Denver?

38. Home Values The median price of a home in Las Vegas, Nevada dropped from \$312,346 in 2006 to \$306,100 in 2007. Let t be the number of years since 2006.

a. Assume the decrease in housing prices has been linear. Give an equation for the line representing price, P, in terms of t. What would the value of the home be in years 2 and 3?

b. If instead the housing prices have been falling exponentially, find an equation, $V(t)$, of the form $y = ab^x$ to represent housing prices. What would the value of the home be in years 2 and 3?

c. On the same set of axes, sketch the functions $P(t)$ and $V(t)$.

d. Which model for the price growth do you think is more realistic? Explain.

39. Rumor Spread The spread of a rumor can be modeled mathematically using an exponential function. Assume one person knows something about another person and tells two people and this process continues, with each person telling two more. Find the equation for a function that gives the number of people, n, who have heard the rumor after a number of iterations, x.

40. Chain Letters A once-common but now-illegal money-making scheme is the *chain letter.* Each new recipient is to send a small sum of money (typically ranging from \$1 to \$5) to the first person on the list. The new recipient must then remove the first name on the list, move the remaining names up, and add his name at the bottom of the list. The recipient must then copy the letter and mail it out to five or ten more people. The hope is this procedure will continue indefinitely with each recipient receiving a large sum of money. The success of such ventures rests solely on the notion of exponential growth in new recipients and the increasing number at each successive layer.

Show that within a few mailings the entire global population would need to participate in order for those listed on the letter to earn any income, and therefore the majority of people participating will lose their invested money.

41. Swimming Pool Imagine a swimming pool that is 32 feet by 16 feet, or 512 square feet. Now suppose the owner has neglected maintaining the pool and algae begins to grow on the bottom surface. The first day the algae has covered about 4 square inches of the bottom of the pool, or $\frac{1}{16}$ square foot.

a. Show that if the area covered by the algae doubles each day, then before the end of the 16th day it will have covered the entire bottom of the pool.

b. Suppose you visited the pool when it was half covered by algae. How much time do you have to act before the pool is completely covered? Provide the reasoning behind your answer.

■ STRETCH YOUR MIND

The following exercise is intended to challenge your understanding of percentage change.

42. Population Growth According to the *East Valley Tribune*, Gilbert, AZ, had an estimated population of 161,059 residents through June 2004, while Tempe, AZ, had an estimated 162,652 residents. Gilbert adds 1000 new residents a month, while Tempe calculates its population by using an annual percentage growth rate of 0.019%.

a. According to these models, will Gilbert's population ever exceed Tempe's? Explain using a table, equation, or graph of two functions that model each city's growth rates.

b. Explain why even though Tempe's population is described as having a percentage growth rate, it is not growing very rapidly.

SECTION 6.2 Exponential Function Modeling and Graphs

LEARNING OBJECTIVES

- Construct exponential models algebraically from tables or words
- Use exponential regression to model real-world data sets
- Graph exponential functions given in equations, tables, or words

GETTING STARTED

Although virtually nonexistent in the United States, mosquito-transmitted malaria is a major killer of children in Africa. There currently is no vaccine, but a number of strategies can reduce the transmission of the disease, including the use of insecticide treated nets (ITNs). The nets can reduce malaria transmission by more than half. In recent years, the distribution of nets in Africa has increased exponentially.

In this section we focus on creating exponential models of real-world situations such as the distribution of ITNs. We use both algebraic and technological methods to create the models from data given in tables or words. We also discover how the growth factor and initial value affect the shape of the exponential function graph.

■ Modeling with Constant Percentage Rates of Change

As discussed in the previous section, anything that grows or decays at a constant percentage rate can be modeled by an exponential function. Consequently, a reference to a percentage of growth in a verbal description often signals underlying exponential behavior.

EXAMPLE 1 ■ Creating an Exponential Model from a Verbal Description

According to the World Health Organization, 538 thousand ITNs were distributed in the African region in 1999. In 2003, 9485 thousand nets were distributed. Between 1999 and 2003, net distribution increased at a nearly constant percentage rate. Assuming net distribution will increase at a constant percentage rate, find the function that models the distribution and forecast the number of nets that will be distributed in 2011. Then explain whether or not the estimate is realistic.

Solution Let t be the number of years since 1999 and let n be the number of nets distributed (in thousands). Since the distribution of nets is anticipated to increase at a constant percentage rate, we can use an exponential model. Since $t = 0$ corresponds with 1999, the initial value is 538. So far we have $n(t) = 538(b)^t$. Although the growth factor is not readily apparent, we can calculate it by substituting the second data point into the equation. Since 2003 corresponds with $t = 4$, we have

$$\begin{aligned} 9485 &= 538(b)^4 \\ 17.63 &= b^4 \\ (17.63)^{1/4} &= (b^4)^{1/4} \\ 2.049 &= b \end{aligned}$$

Thus the exponential model is $n(t) = 538(2.049)^t$.

To determine the net distribution level in 2011, we evaluate this function at $t = 12$.

$$n(12) = 538(2.049)^{12}$$
$$\approx 2{,}946{,}000 \text{ (accurate to 4 significant digits)}$$

According to the model, 2,946,000 thousand (2.946 billion) nets will be distributed in 2011.

Although the model gives a good estimate for years near the original data set, the further we move away from 2003, the less confident we are in the prediction because few things can sustain exponential growth indefinitely. Since 2011 is relatively far away from the last year in the data set (2003), we question the accuracy of the forecast.

EXAMPLE 2 ■ **Comparing a Rate of Change to a Percentage Change Rate**

The Netto Extra Treated Net was launched by Netto Manufacturing Co., Ltd., in April 2005 to meet the high demand for insecticide treated nets. According to a certificate of analysis issued with the net, the initial deltamethrin (insecticide) content in the net is 50.40 milligrams per square meter. After six washes, the deltamethrin residue was measured to be 34.72 milligrams per square meter. (*Source:* www.nettogroup.com)

Calculate the average rate of change and percentage rate of change. Which one more accurately represents this situation?

Solution To calculate the average rate of change, we use the average rate of change formula from Chapter 4.

$$\frac{50.40 - 34.72}{0 - 6}\ \frac{\text{mg per square meter}}{\text{washes}} \approx -2.613 \text{ mg per square meter per wash}$$

On average, each wash removes 2.613 milligrams of deltamethrin per square meter.

To determine the percentage change, we must first determine the decay factor.

$$\left(\frac{y_2}{y_1}\right)^{\frac{1}{x_2 - x_1}} = \left(\frac{50.40}{34.72}\right)^{\frac{1}{0-6}}$$
$$\approx 0.9398$$

We subtract 1 from the decay factor to determine the percentage rate of change.

$$0.9398 - 1 = -0.0602$$

Approximately 6.02% of the deltamethrin residue is removed with each wash.

We expect that as the quantity of deltamethrin residue available in the net decreases, the amount that is removed in each subsequent wash will also decrease. For this reason, the percentage rate of change seems to more accurately represent what is going on in this situation.

■ Modeling Half-Life and Doubling Time

In November 2006, Alexander Litvinenko, a former Russian spy living in Britain, was killed by poisoning while investigating the death of a Russian journalist. The poison used was polonium-210, a radioactive substance. The ensuing investigation led to international finger-pointing; however, as of April 2011, the case remained unsolved.

Radioactive substances such as polonium-210 decay exponentially. The *half-life* of polonium-210 is 138.376 days (138 days, 9 hours, 1 minute, and 26 seconds). The **half-life** of a substance is the amount of time it takes for half of the initial amount of the substance to remain. Half-lives are used widely in chemistry when comparing various radioactive elements.

EXAMPLE 3 ■ Determining a Percentage Rate of Decay from a Half-Life

Polonium-210 has a half-life of 138.376 days. What percentage of the substance decays each day?

Solution Since the substance is decaying exponentially, we can model the amount remaining by $y = ab^t$. Since half of the initial value remains after 138.376 days, we have

$$\frac{1}{2}a = ab^{138.376}$$
$$\frac{1}{2} = b^{138.376}$$
$$\left(\frac{1}{2}\right)^{1/138.376} = (b^{138.376})^{1/138.376}$$
$$0.9950 = b$$

Since $b = 1 + r$, $r = -0.005$. The amount of polonium-210 remaining is decreasing at a rate of 0.5% per day.

Doubling time is the amount of time it takes for something that is growing to double. As was the case with half-life, doubling time is independent of the initial value of the exponential function.

EXAMPLE 4 ■ Determining the Percentage Rate of Growth from a Doubling Time

The median price of a home in the United States was about $250,000 in March 2007. (*Source:* www.zillow.com) At what annual percentage rate would property values have to increase for the median price to double by March 2017?

Solution Since we are assuming a constant percentage growth, we can use an exponential function to model the value of the investment.

$$2(250{,}000) = 250{,}000(1 + r)^{10}$$
$$2 = (1 + r)^{10}$$
$$(2)^{1/10} = [(1 + r)^{10}]^{1/10}$$
$$1.072 \approx 1 + r$$
$$0.072 \approx r$$

A doubling time of 10 years corresponds with an annual percentage rate of about 7.2%.

EXAMPLE 5 ■ Determining If Data Can Be Represented by an Exponential Model

Table 6.11 shows the average price for brand-name prescription drugs from 2000 to 2004. Determine if the data set has a constant or nearly constant percentage rate of change. If it does, model the data set with an exponential function and estimate the average drug price in 2006.

Solution Since the input values are equally spaced, we need only determine if consecutive output values have a constant ratio. See Table 6.12.

Table 6.11

Years Since 2000	Average Brand-Name Drug Price (dollars)
0	65.29
1	69.75
2	77.49
3	85.57
4	95.86

Source: Statistical Abstract of the United States, 2006, Table 126

Table 6.12

Years Since 2000	Average Brand-Name Drug Price (dollars)	Ratio of Consecutive Output Values
0	65.29	1.06831061
1	69.75	1.11096774
2	77.49	1.10427152
3	85.57	1.12025242
4	95.86	

The ratios are all approximately equal to 1.1, so they are nearly constant. Thus an exponential model is appropriate for this data set. Using 65.29 as the initial value, we construct an exponential model

$$p(t) = 65.29(1.1)^t \text{ dollars}$$

where t is the number of years since 2000.

To determine the average drug price in 2006, we evaluate the function at $t = 6$.

$$\begin{aligned} p(6) &= 65.29(1.1)^6 \\ &\approx 115.7 \end{aligned}$$

According to the model, the average brand-name drug price in 2006 was $115.70.

Using Regression to Find an Exponential Model

In Example 5, we constructed an exponential model for a data set with nearly constant ratios. By using exponential regression, we can find the exponential model that best fits the data set. Using the Technology Tip at the end of this section, we determine the exponential model of best fit for the brand-name prescription drug price is

$$p(t) = 64.25(1.102)^t \text{ dollars}$$

where t is the number of years since 2000. Using the model of best fit, we get a slightly different estimate for the average brand-name prescription drug price in 2006.

$$\begin{aligned} p(6) &= 64.25(1.102)^6 \\ &\approx 115.10 \end{aligned}$$

EXAMPLE 6 ■ Using Exponential Regression to Model a Data Set

Find the equation of the exponential function that best fits the data set shown in Table 6.13. Then forecast the U.S. exports to Colombia in 2010.

Table 6.13

Years Since 1990 *t*	U.S. Exports to Colombia ($ millions) *C*
0	119
10	415
11	452
12	520
13	512
14	593
15	677

Source: Statistical Abstract of the United States, 2007, Table 827

Solution Using the Technology Tip at the end of this section, we obtain

$$C(t) = 122.8(1.122)^t \text{ million dollars}$$

where t is the number of years since the end of 1990.

To forecast exports in 2010, we evaluate the function at $t = 20$.

$$\begin{aligned} C(20) &= 122.8(1.122)^{20} \\ &= 1228 \text{ million dollars} \end{aligned}$$

We estimate that U.S. exports to Colombia will be $1228 million in 2010.

Graphing Exponential Functions

The growth factor plays a significant role in determining the shape of the graph of an exponential function. Let's investigate its effect in the context of economic forecasting.

The Congressional Budget Office (CBO) is given the responsibility of forecasting the economic future of the U.S. government. In a March 2007 report entitled *The Uncertainty of Budget Projections: A Discussion of Data and Methods*, the CBO explained that "uncertainty increases as the projections extend into the future" (Preface).

For example, the CBO estimated the *nominal gross domestic product* (GDP) was \$13,235 billion in 2006 and that it would increase by 4.3% in 2007. (*Source:* www.cbo.gov) (The GDP of a country is the market value of its final goods and services produced within a year.) Assuming growth at a constant percentage rate in the future, we can construct an exponential function model for the GDP. Since the initial value is \$13,235 billion and the annual growth rate is 4.3%, we have

$$G(t) = 13{,}235(1.043)^t \text{ billion dollars}$$

where t is the number of years since 2006. The graph of the function is shown in Figure 6.1.

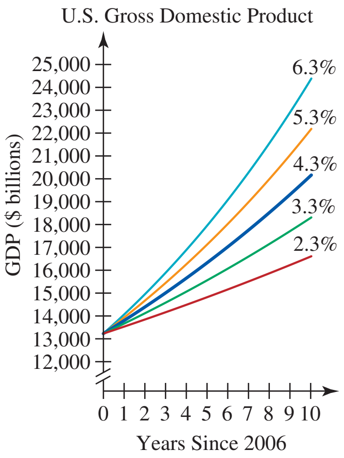


Figure 6.1

From the graph, we see that the projected GDP in 2016 ($t = 10$) is just over \$20,000 billion. However, with any projection there is a level of uncertainty. To address this concern, let's look at the additional projections shown in Figure 6.1 for rates somewhat near 4.3%—2.3%, 3.3%, 5.3%, and 6.3%. This will give us an idea of possible ranges of values. We see that the larger the percentage rate, the steeper the graph. That is, the higher the percentage rate, the higher the rate of change in the GDP. We also see that although the difference between the projections for 2016 is substantial, the difference in the projections for 2007 is relatively small.

Thus, the change factor tells us much about the graph. Recall that the change factor b is equivalent to 1 plus the percentage change rate. That is, $b = 1 + r$.

The Graphical Significance of the Change Factor

The change factor, b, controls the steepness and increasing/decreasing behavior of the exponential function $y = ab^x$. For positive a,

- if $b > 1$, the graph is increasing, and increasing the value of b will make the graph increase more rapidly.
- if $0 < b < 1$, the graph is decreasing, and decreasing the value of b will make the graph decrease more rapidly.

EXAMPLE 7 ■ Identifying an Exponential Function Based on Its Graph

Figure 6.2 shows population models for Cambodia, Kazakhstan, Rwanda, and Hungary. Based on data from 1995 to 2004, the population percentage rates of change for the countries are 2.0%, −0.7%, 5.0%, and −0.2%, respectively. (*Source:* World Health Statistics, 2006) Identify which graph corresponds to each country. Then estimate the 2005 population for each country.

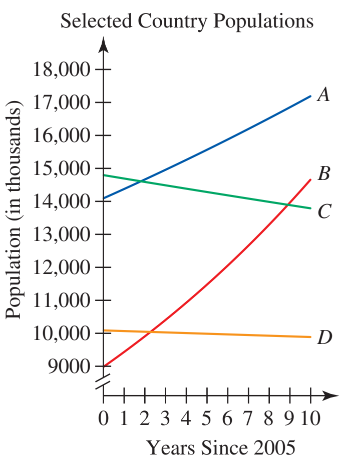


Figure 6.2

Solution Both Cambodia and Rwanda have positive growth rates, so their graphs will be increasing. However, Rwanda has a higher percentage rate of growth so its graph will be steeper. Graph B corresponds with Rwanda and Graph A with Cambodia.

Both Kazakhstan and Hungary have a negative growth rate so their populations are decreasing. Since Hungary is decreasing at a less negative percentage rate, its graph will be less steep. So Graph D corresponds with Hungary and Graph C with Kazakhstan.

The vertical intercept of each graph is the initial population of the corresponding country. The initial populations (in thousands) are approximately

Kazakhstan: 14,800

Cambodia: 14,100

Hungary: 10,100

Rwanda: 9000

In Example 7, we stated that the vertical intercept of the graph corresponded with the initial value of the exponential function. To see why, let's consider a generic exponential function, $y = ab^x$. To find the vertical intercept, we set x equal to zero.

$$\begin{aligned} y &= ab^0 \\ &= a(1) && \text{since } b^0 = 1 \text{ for } b \neq 0 \\ &= a \end{aligned}$$

So the vertical intercept is $(0, a)$ and the initial value of the exponential function corresponds with the vertical intercept.

Graphical Meaning of the Initial Value

The exponential function $y = ab^x$ has vertical intercept $(0, a)$, where a is the initial value of the function.

EXAMPLE 8 ■ Interpreting Exponential Function Graphs

Inflation refers to the increase in prices that occurs over time. For example, if the annual inflation rate is 3% then an item that costs \$100 today will cost $100 + 100(0.03) = \$103$ a year from now.

Consider the price of the chic leather boots for women shown in the ad below. In 1911, they cost \$3.79. A comparable boot, shown next to the ad, retailed for \$57.00 in 2011.

a. Determine the average annual inflation rate for the boots.

b. Graph an exponential function model for the price of the boots.

c. Estimate the price of the boots in 1963 and 2003 from the graph.

Solution

a. We first need to determine the ratio of the prices.

$$\frac{57}{3.79} = 15.04$$

The 2011 price was about 15 times more than the 1911 price. To determine the annual growth factor, we raise this number to 1 over the length of the period between 1911 and 2011 (100 years).

$$(15.04)^{(1/100)} \approx 1.0275$$

We subtract 1 from the growth factor to obtain the annual percentage change rate.

$$\begin{aligned} 1.0275 - 1 &= 0.0275 \\ &= 2.75\% \end{aligned}$$

The average annual rate of inflation on the boots was about 2.75%.

b. A model for the price of the boots t years after 1911 is $p(t) = 3.79(1.0275)^t$. The graph is shown in Figure 6.3. Although the percentage change rate is constant (2.75%), the annual increase in price increases as the price itself increases, so the graph is concave up.

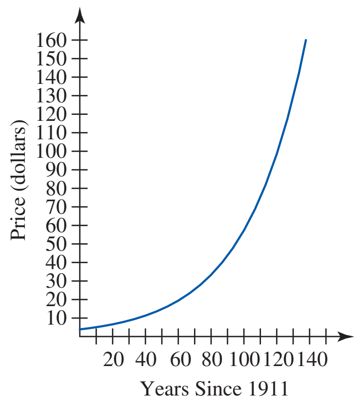


Figure 6.3

c. In 1963, $t = 52$. From the graph it appears as if the boot price was approximately \$15 in 1963. In 2003, $t = 92$. From the graph it appears as if the boot price was approximately \$46 in 2003.

All exponential functions have a *horizontal asymptote* at the horizontal axis. To see why, suppose we have a decreasing exponential function with initial value 100 and constant percentage change −10%. This means for each 1 unit increase in the input, the output decreases by 10%. Table 6.14 shows the values of the function for the first four nonnegative integer values of the input.

Table 6.14

x	y	10% of y
0	100	$0.1(100) = 10$
1	$100 - 10 = 90$	$0.1(90) = 9$
2	$90 - 9 = 81$	$0.1(81) = 8.1$
3	$81 - 8.1 = 72.9$	$0.1(72.9) = 7.29$

Notice as the value of y becomes smaller, 10% of y also becomes smaller. That is, the amount by which y is decreasing is getting smaller as x increases. Also notice each function value is 90% of the value before it (e.g., 90 is 90% of 100, 81 is 90% of 90, and so on). This pattern will continue with the y-values becoming smaller while remaining positive. In other words, as $x \to \infty$, $y \to 0$. Thus a horizontal asymptote occurs at the horizontal axis. A similar argument applies for increasing exponential functions. In that case, as $x \to -\infty$, $y \to 0$.

In general, the graph of an exponential function will take on one of the four basic shapes shown in Figure 6.4. Notice in each case the line $y = 0$ (the horizontal axis) is a horizontal asymptote. Notice also that, as was the case with quadratic functions, the value of a controls the concavity of the graph. If $a > 0$, the graph is concave up. If $a < 0$, the graph is concave down.

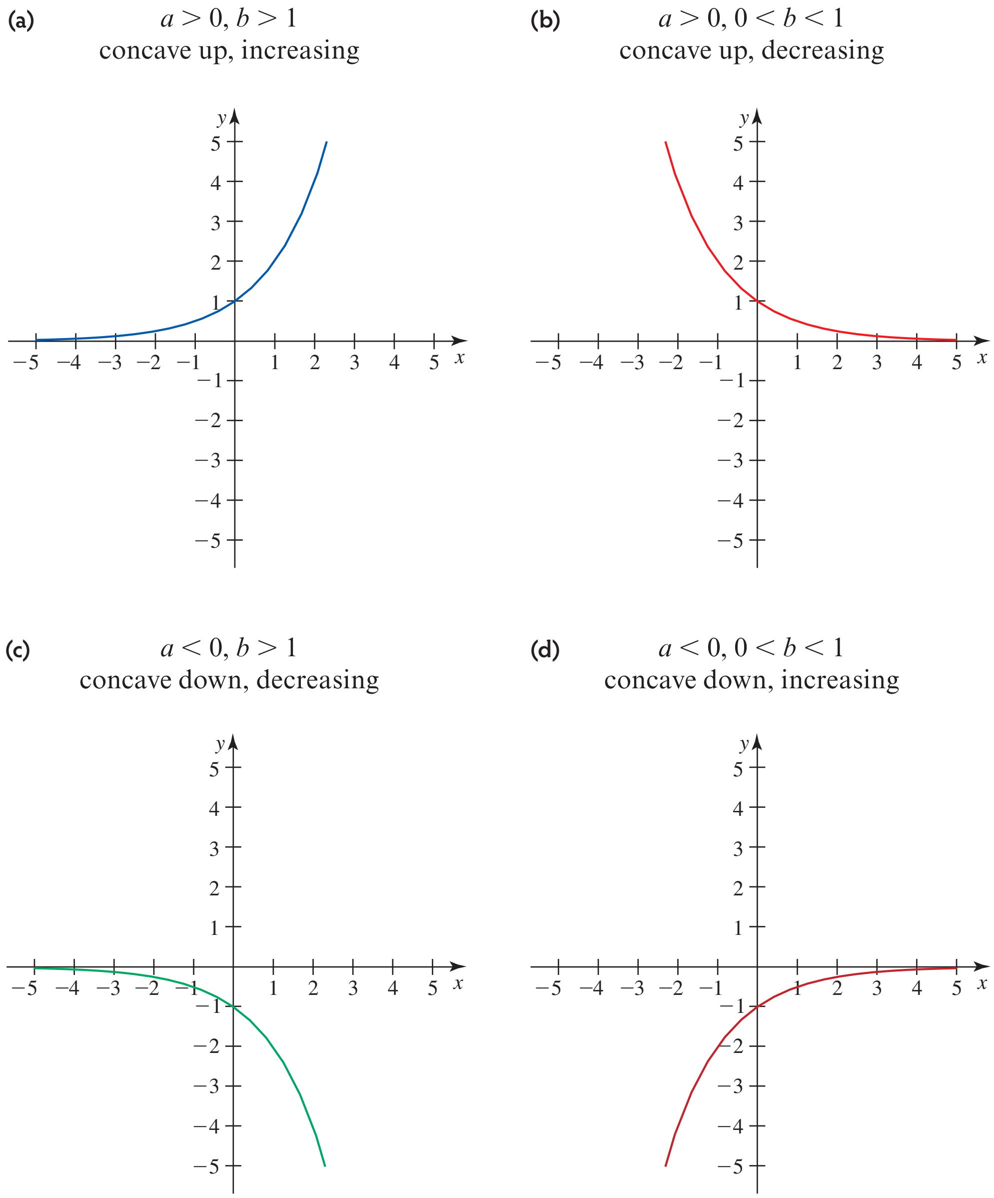


Figure 6.4

As was the case with other functions we have investigated, exponential function graphs may be altered using transformations, as shown in the next example.

EXAMPLE 9 ■ Using Exponential Regression to Model an Aligned Data Set

Table 6.15 shows the amount of land in farms in the United States between 1978 and 1997.

Table 6.15

Years Since 1978 t	Land in Farms (million acres) F		Years Since 1978 t	Land in Farms (million acres) F
0	1014.8		14	945.5
4	986.8		19	931.8
9	964.5			

Source: Statistical Abstract of the United States, 2001, Table 796

Find a function model for the data set. Then use the model to forecast the land in farms in 2007.

Solution We draw the scatter plot of the data, shown in Figure 6.5, to get an idea of what type of function may fit the data set. The data set appears to be concave up and decreasing so an exponential model may fit the data set well. We use the Technology Tip at the end of this section to find the exponential model of best fit, $F(t) = 1008(0.9956)^t$, and graph the resultant function in Figure 6.6.

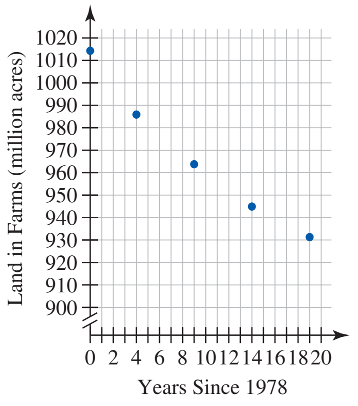


Figure 6.5

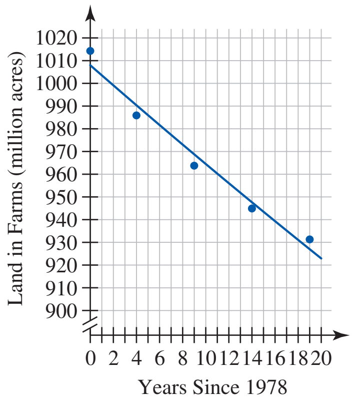


Figure 6.6

The graph, which appears nearly linear, does not fit the data set. The coefficient of determination from the graphing calculator, $r^2 = 0.976$, is not as close to 1 as we had expected. What happened? Recall the calculator assumes that an exponential function has a horizontal asymptote at $y = 0$, but this particular data set is "bending" too quickly to have a horizontal asymptote at $y = 0$. We can visually estimate that this data set will have a horizontal asymptote at $y = 900$ and create an aligned set of data by subtracting 900 from each of the farm size values as shown in Table 6.16.

Table 6.16

Years Since 1978 t	Land in Farms (million acres) F	Aligned Data $F - 900$
0	1014.8	114.8
4	986.8	86.8
9	964.5	64.5
14	945.5	45.5
19	931.8	31.8

Now we repeat the regression using the aligned data as the dependent variable. The resultant model, $A(t) = 115.1(0.9352)^t$, has a coefficient of determination much closer to 1 ($r^2 = 0.9990$). Because this value is close to 1, we see that our guess that the horizontal asymptote is $y = 900$ was a good one. To vertically shift this model to the position of the farm land data, we add back 900 to get $L(t) = 115.1(0.9352)^t + 900$. We graph the new model along with the scatter plot and the original model in Figure 6.7. We see that the new model fits the data much better than did the original.

To forecast the 2007 land in farms, we evaluate $L(t)$ at $t = 29$.

$$L(t) = 115.1(0.9352)^t + 900$$

$$L(29) = 115.1(0.9352)^{29} + 900$$

$$\approx 916.5$$

We estimate in 2007 there were 916.5 million acres of farm land in the United States.

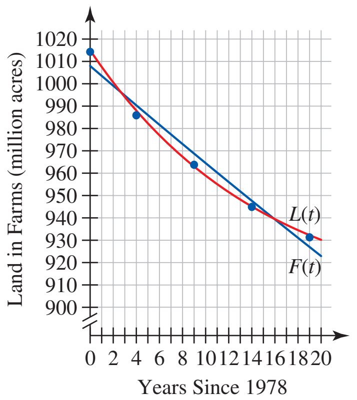


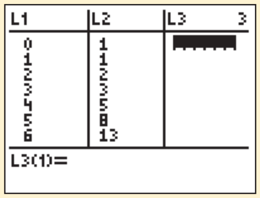
Figure 6.7

SUMMARY

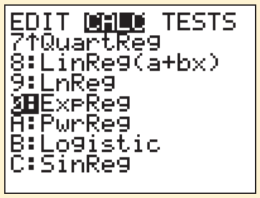
In this section you learned how to construct an exponential function model from tables and words. You discovered how to use exponential regression to find the exponential model that best fits a data set. You also studied the various characteristics of an exponential function's graph, including the effect of the initial value and growth factor on the graph. Finally, you learned that sometimes it is necessary to align a data set to find the best model.

TECHNOLOGY TIP ■ EXPONENTIAL REGRESSION

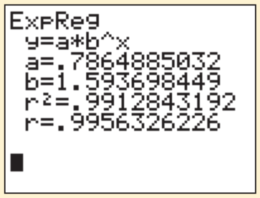
1. Enter the data using the Statistics Menu List Editor.



2. Bring up the Statistics Menu Calculate feature by pressing STAT and using the blue arrows to move to `CALC`. Then select item `0: ExpReg` and press ENTER.



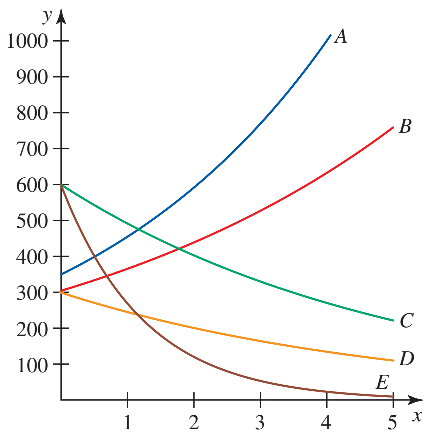
3. If you want to automatically paste the regression equation into the Y = Editor, press the key sequence VARS; `Y-VARS`; `Function`; `Y1` and press ENTER. Otherwise press ENTER.



6.2 EXERCISES

■ SKILLS AND CONCEPTS

In Exercises 1–5, identify which of the following exponential graphs corresponds with each given equation.



1. $y = 350(1.3)^x$
2. $y = 305(1.2)^x$
3. $y = 600(0.82)^x$
4. $y = 300(0.82)^x$
5. $y = 600(0.45)^x$

In Exercises 6–10, describe the graph of the function without drawing the graph. Use the terms increasing, decreasing, concave up, concave down, vertical intercept, and horizontal asymptote, as appropriate.

6. $f(x) = 21(0.9)^x$
7. $g(x) = -4(2.9)^x$
8. $h(x) = -0.8(1.9)^x$
9. $f(t) = 4(1.8)^t + 17$
10. $f(t) = 0.25(4)^t - 2$

■ SHOW YOU KNOW

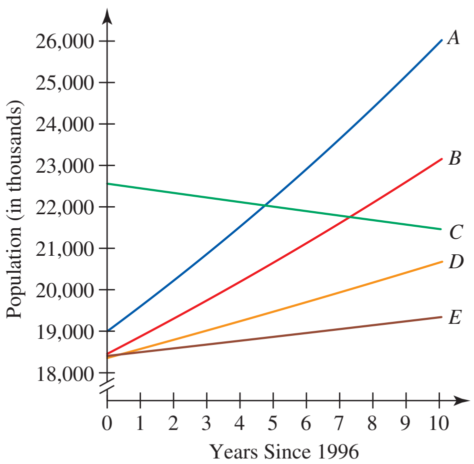
11. How can you tell if a table of data represents an exponential function?

12. A classmate claims that any two points define a unique exponential function. Do you agree? Explain.
13. What are some key words in a verbal description that indicate an exponential function can be used to model the situation? What is meant by such words?
14. What does the term *half-life* mean?
15. What does the term *doubling time* mean?

MAKE IT REAL

In Exercises 16–20, determine which of the following graphs corresponds with a population model for each of the given countries. Then estimate the 2005 population of the country.

Country Populations



Source: **Modeled from World Health Organization data**

16. **Australia Population** Between 1996 and 2006, the population of Australia grew at a rate of 1.2% annually.
17. **Ghana Population** Between 1996 and 2006, the population of Ghana grew at a rate of 2.3% annually.
18. **Romania Population** Between 1996 and 2006, the population of Romania *decreased* at a rate of 0.5% annually.
19. **Sri Lanka Population** Between 1996 and 2006, the population of Sri Lanka grew at a rate of 0.5% annually.
20. **Afghanistan** Between 1996 and 2006, the population of Afghanistan grew at a rate of 3.2% annually.

In Exercises 21–30, model the data with an exponential function, if appropriate, and answer the given questions. If an exponential function model is not appropriate for the situation, explain why. Do not use regression for these exercises.

21. **Population of the United States** According to the World Health Organization, the population of the United States was 298,213 thousand in 2005. Between 1995 and 2004, the population grew at an average rate of 0.9% annually. **(*Source:* World Health Statistics, 2006)** Assuming the percentage growth rate will remain the same in the future, model the U.S. population as a function of years since 2005.
22. **Population of China** According to the World Health Organization, the population of China was 1,323,345 thousand in 2005. Between 1995 and 2004, the population grew at an average rate of 0.7% annually. **(*Source:* World Health Statistics, 2006)** Assuming the population will continue to grow at this percentage rate, model the population of China as a function of years since 2005. What is the projected population of China in 2011?

grafica/Shutterstock.com

23. **Crude Oil Production** Based on data from 1970 to 2004, the production of crude oil in the United States has decreased at a rate of roughly 0.27 quadrillion BTU per year. In 1970, the production level was 20.4 quadrillion BTU. **(*Source: Statistical Abstract of the United States, 2007,* Table 895)**
24. **National Health Expenditures** Between 1960 and 2004, the total national expenditure on health costs increased by roughly 10.1% annually. In 1960, national health expenditures were $28 billion. **(*Source: Statistical Abstract of the United States, 2007,* Table 120)**
25. **Research and Development Spending** In 2003, Microsoft Corporation was ranked number 1 in research and development spending. That year, the company spent $7779 million on research and development. The 10th-ranked company was Glaxo Smith Kline, which spent $4910 million on research and development. According to the list of the top 10 companies, for each increase in rank number, the amount of spending decreased at a nearly constant percentage rate. **(*Source: Science and Engineering Indicators 2006,* National Science Foundation, Table 4-6).**

    Use the model to predict how much money was spent on research and development by the 5th-ranked company (Toyota Motor). Then compare your model result to the actual amount spent ($6210 million).
26. **Insurance Expenditures for Health Care** Between 1960 and 2004, insurance company expenditures for health care increased at an ever-increasing rate. In 1960, $6 billion was spent on health care. In 2004, $659 billion was spent on health care. **(*Source: Statistical Abstract of the United States, 2007,* Table 120)**
27. **Atmospheric Pressure** At 0 meters elevation, atmospheric pressure is 14.696 pounds per square inch. At 8500 meters, atmospheric pressure is 4.801 pounds per square inch. As elevation increases, atmospheric pressure decreases less rapidly. **(*Source:* Modeled from Digital Dutch 1976 Standard Atmosphere Calculator)** According to the model, what is the atmospheric pressure at 6000 meters?
28. **Salary for Engineers and Scientists** Between 1993 and 2003, annual median salaries for science and engineering occupations increased from $48,000 to $66,000. Over that time period, median salaries increased at a rate of roughly $1800 per year. **(*Source: Science and Engineering Indicators 2006*, National Science Foundation, Table 3-8)**
29. **Adults with Only an Elementary Education** In 1910, 23.8% of the adults in the United States had only an elementary education. By 1950 that number had fallen to 11.1%, and by 1990 it had dropped to 2.4%. **(*Source: Digest of Education Statistics, 2005*, National Center for Education Statistics)**
30. **Coca-Cola Production** In its 2001 Annual Report, the Coca-Cola Company reported

    "Our worldwide unit case volume increased 4 percent in 2001, on top of a 4 percent increase in 2000. The increase in unit case volume reflects consistent

performance across certain key operations despite difficult global economic conditions. Our business system sold 17.8 billion unit cases in 2001." (*Source:* Coca-Cola Company 2001 Annual Report, p. 46)

A unit case is equivalent to 24 8-ounce servings of finished beverage.

In Exercises 31 to 40, use exponential regression to find a model for the data set. Then use the model to answer the given question. (As appropriate, align the data before finding the model equation.)

31. School Expenditures

School Year Since 1990–1991 *t*	Expenditure per Pupil (dollars) *E*
0	4902
2	5160
4	5529
6	5923
8	6508
10	7380
12	8044

Source: National Center for Education Statistics

What are the projected school expenditures per pupil in 2010–2011?

32. Atmospheric Pressure

Altitude (meters) *a*	Atmospheric Pressure (psi) *p*
0	14.696
1000	13.035
2000	11.530
3000	10.168
4000	8.940
5000	7.835
6000	6.843
7000	5.955
8000	5.163

Source: Digital Dutch 1976 Standard Atmosphere Calculator

What is the atmospheric pressure at 4500 meters?

33. National Health Spending

Years Since 1995 *t*	Health Expenditures ($ billions) *H*		Years Since 1995 *t*	Health Expenditures ($ billions) *H*
0	1020		5	1359
1	1073		6	1474
2	1130		7	1608
3	1196		8	1741
4	1270		9	1878

Source: Statistical Abstract of the United States, 2007, Table 120

At what percentage rate are health expenditures increasing?

34. Brand-Name Drug Prices

Years Since 1995 *t*	Average Brand-Name Drug Price (dollars) *d*
0	40.22
2	49.55
3	53.51
4	60.66
5	65.29
6	69.75
7	77.49
8	85.57
9	95.86

Source: Statistical Abstract of the United States, 2006, Table 126

What do the initial value and growth factor of the model represent in this real-world context?

35. Highway Accidents

Years Since 2000 *t*	Accidents Resulting in Injuries (percent) *p*
0	49.9
1	48.0
2	46.3
3	45.6
4	45.1

Source: Statistical Abstract of the United States 2007, Table 1047

What does the change factor tell about highway accident injuries?

36. Municipal Governments

Number of Local Municipal Governments (in thousands) *m*	Years Since 1965 *Y*
18.048	2
18.517	7
18.862	12
19.076	17
19.200	22
19.279	27
19.372	32
19.429	37

Source: Statistical Abstract of the United States, 2007, Table 415

In what year is the number of municipal governments projected to reach 20 thousand?

37. Government Expenditures

Years Since 1990 y	Expenditures (\$ billions) E
0	1872.6
5	2397.6
10	2886.5
11	3061.6
12	3240.8
13	3424.7
14	3620.6
15	3877.2

Source: Statistical Abstract of the United States, 2007, Table 418

What are the projected government expenditures for 2010?

38. Bottled Water

Years Since 1980 t	Per Capita Bottled Water Consumption (gallons) w
0	2.4
5	4.5
10	8.0
14	10.7
15	11.6
16	12.5
17	13.1
18	16.0
19	18.1

Source: Statistical Abstract of the United States, 2001, Table 204

Who could benefit from the per capita bottled water consumption model?

39. Professional Basketball Salaries

Years Since 1980 t	NBA Average Salary (\$1000s) b
0	170
5	325
10	750
15	1900
16	2000
17	2200
18	2600

Source: Statistical Abstract of the United States, 2001, Table 1324

Will the average NBA salary be more or less than \$3 million in 2010?

40. U.S. Gross Domestic Product

Years Since 1930 t	GDP (\$ billions) g
0	91.3
10	101.3
20	294.3
30	527.4
40	1039.7
50	2795.6
60	5803.2
70	9872.9

Source: www.lycos.com

What will be the gross domestic product of the United States in 2010?

■ STRETCH YOUR MIND

Exercises 41–45 are intended to challenge your understanding of exponential functions.

41. Given that an exponential function $y = ab^x$ passes through the points (c, ab^c) and $(c + h, ab^{c+h})$,

a. Create a formula that calculates the average rate of change of the exponential function over the interval $[c, c + h]$. (*Hint:* $[c, c + h]$ means $c \le x \le c + h$.)

b. Describe the relationship between the formula for the average rate of change of an exponential function and the equation of the exponential function.

42. Explain why a function with a constant percentage change cannot be linear.

43. Explain why an exponential function $y = ab^x$ does not have a horizontal intercept.

44. A classmate claims that two exponential functions, $f(x) = ab^x$ and $g(x) = cd^x$, will never intersect if $a > c$ and $b > d$. Do you agree? Justify your conclusion.

45. A classmate incorrectly claims any function that is concave up and has a horizontal asymptote at the x-axis is an exponential function. Give the equation of a function that could be used to persuade your classmate that he is incorrect.

SECTION 6.3

Compound Interest and Continuous Growth

LEARNING OBJECTIVES

- Use the compound interest formula to calculate the future value of an investment
- Construct and use continuous growth models
- Use exponential models to predict and interpret unknown results

GETTING STARTED

Saving for retirement is an important goal for many people, but Americans are saving less money than they should. When planning for retirement, the earlier you can start saving and investing the better off you are, thanks to the benefits of compound interest.

In this section we investigate the compound interest formula and its importance in financial planning. In addition, we learn about continuous growth and the number *e*. These concepts are used in real-world situations as diverse as saving for retirement, population growth, and radioactive half-life.

■ Compound Interest

Banks make a profit by loaning money to people and charging interest for this service. Some of the money that banks lend comes from customers who deposit money with the bank. To encourage customers to deposit their money with the bank, a bank pays its customers *interest*, calculated as a percentage of the amount of money the customer has in the bank during each interest period.

Consider the following situation. In April 2007, Heritage Bank offered a 1-year certificate of deposit (CD) with a minimum investment of \$1000 at a *nominal interest rate* of 4.42% compounded quarterly. (*Nominal* means "in name only.") (*Source:* www.bankrate.com) We will use this interest rate to calculate the future value of a \$1000 investment in the certificate of deposit after 1 year. The future value relies on the interest rate and the **compounding frequency**, the number of times per year that interest is paid. This particular CD is compounded quarterly (four times a year). To find the impact this will have on the future value of the CD after 1 year, we need to calculate the quarterly **periodic rate**, or the interest rate that will be applied to the CD every 3 months. We do this by dividing the nominal interest rate by the compounding frequency.

$$\frac{4.42\%}{4} = 1.105\%$$

Thus, every 3 months Heritage Bank will pay interest equivalent to 1.105% of the money in the account.

PERIODIC RATE

The **periodic rate** is calculated as

$$\text{periodic rate} = \frac{\text{nominal interest rate}}{\text{compounding frequency}}$$

We can now find the future value of the CD after 1 year by using a quarterly growth factor of 1.01105 (an increase of 1.105%) as shown in Table 6.17.

Table 6.17

End of Quarter	Value Calculation	End-of-Quarter Value
1 (3 months)	1000(1.01105)	\$1011.05
2 (6 months)	1011.05(1.01105)	\$1022.22
3 (9 months)	1022.22(1.01105)	\$1033.52
4 (12 months)	1033.52(1.01105)	\$1044.94

Since the nominal rate was 4.42%, we may have expected the final value to be \$1044.20 instead of \$1044.94. Why the \$0.74 difference? To find out, let's look more closely at the values of the CD at the end of each of the four quarters of the year. See Table 6.18.

Table 6.18

Quarter	End-of-Quarter Value	Interest Payment
0 (initial deposit)	\$1000.00	
1	\$1011.05	\$11.05
2	\$1022.22	\$11.17
3	\$1033.52	\$11.30
4	\$1044.94	\$11.42

Note that the interest payments continue to increase throughout the year. This is due to **compound interest**, which occurs when interest is paid on the initial amount invested as well as all previously earned interest.

The Compound Interest Formula

The preceding method to find a future value is not very efficient. Let's use our knowledge of exponential functions to determine a formula that will allow us to quickly and easily calculate future value. Since the initial value of the CD was \$1000 and the quarterly growth factor was 1.0115, the value of the CD after q quarters is given by

$$V(q) = 1000(1.0115)^q \text{ dollars}$$

In general, the equation will be of the form

$$\begin{aligned}\text{future value} &= (\text{initial amount}) \cdot (\text{periodic growth factor})^{(\text{total number of compoundings})}\\ &= (\text{initial amount}) \cdot (1 + \text{periodic rate})^{(\text{total number of compoundings})}\end{aligned}$$

The final value is usually described in terms of the number of years that have passed:

$$\begin{aligned}\text{future value} &= (\text{initial amount}) \cdot (1 + \text{periodic rate})^{(\text{total number of compounding})}\\ &= (\text{initial amount}) \cdot \left(1 + \frac{\text{nominal rate}}{\text{compoundings per year}}\right)^{\left(\frac{\text{compoundings}}{\text{year}} \cdot \text{years}\right)}\end{aligned}$$

To simplify the notation, we represent each of the quantities with a variable when we write the compound interest formula.

Compound Interest Formula

The future value A of an initial investment P is given by

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

where n is the compounding frequency (the number of times interest is paid per year), t is the number of years the money is invested, and r is the nominal interest rate in decimal form $\left(r = \frac{\text{nominal rate}}{100\%}\right)$.

Table 6.19 shows compounding frequencies and associated formulas for compound interest.

Table 6.19

Compounding Frequency	Number of Compoundings per Year	Compound Interest Formula
annually	1	$A = P\left(1 + \frac{r}{1}\right)^{1t}$
semiannually	2	$A = P\left(1 + \frac{r}{2}\right)^{2t}$
quarterly	4	$A = P\left(1 + \frac{r}{4}\right)^{4t}$
monthly	12	$A = P\left(1 + \frac{r}{12}\right)^{12t}$
daily	365	$A = P\left(1 + \frac{r}{365}\right)^{365t}$

EXAMPLE 1 ■ Comparing Future Values

In March 2007, MetLife Bank advertised a 5-year CD with a 4.64% nominal interest rate compounded daily. (*Source:* www.bankrate.com) Assuming someone could continue to invest their money at this rate and compounding frequency up until retirement at age 65, compare the future value of a \$5000 investment of a 25-year-old to the future value of a \$10,000 investment for a 45-year-old. Assume no additional deposits or withdrawals are made.

Solution The 25-year-old will be accumulating interest 365 days a year for 40 years (the time until she retires), while the 45-year-old will be accumulating interest 365 days a year for 20 years (the time until he retires). Using the compound interest formula, we have

The 25-year-old

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= 5000\left(1 + \frac{0.0464}{365}\right)^{365 \cdot 40} \\ &= 5000(1 + 0.00013)^{14,600} \\ &= 5000(1.00013)^{14,600} \\ &= \$31,986.69 \end{aligned}$$

The 45-year-old

$$\begin{aligned} A &= P\left(1 + \frac{r}{n}\right)^{nt} \\ &= 10,000\left(1 + \frac{0.0464}{365}\right)^{365 \cdot 20} \\ &= 10,000(1 + 0.00013)^{7300} \\ &= 10,000(1.00013)^{7300} \\ &= \$25,292.96 \end{aligned}$$

Deducting the amount of the initial investment, we see that the 25-year-old earned \$26,986.69 in interest while the 45-year-old earned \$15,292.96 in interest.

EXAMPLE 2 ■ Comparing the Effects of Different Interest Rates

In March 2007, Third Federal Savings and Loan offered a 5-year CD at a 5.15% nominal interest rate compounded quarterly. At the same time, Chase Bank offered a 5-year CD at a 3.68% nominal interest rate compounded daily. (*Source:* www.bankrate.com) If you want the future value of the CD to be \$5000 in 5 years, determine which account would require you to invest the least amount.

Solution In this case we know the future value A and need to find the initial investment P.

Third Federal S&L

$$\begin{aligned}
A &= P\left(1 + \frac{r}{n}\right)^{nt} \\
5000 &= P\left(1 + \frac{0.0515}{4}\right)^{4(5)} \\
5000 &= P(1 + 0.013)^{20} \\
5000 &= P\,(1.013)^{20} \\
5000 &= P(1.292) \\
\frac{5000}{1.292} &= P \\
\$3871.27 &= P
\end{aligned}$$

Chase Bank

$$\begin{aligned}
A &= P\left(1 + \frac{r}{n}\right)^{nt} \\
5000 &= P\left(1 + \frac{0.0368}{365}\right)^{365(5)} \\
5000 &= P(1 + 0.0001)^{1825} \\
5000 &= P(1.0001)^{1825} \\
5000 &= P(1.202) \\
\frac{5000}{1.202} &= P \\
\$4159.72 &= P
\end{aligned}$$

The Third Federal S&L CD requires an initial deposit \$288.45 less (\$4159.72 − \$3871.27) than the Chase Bank CD.

Compound Interest as an Exponential Function

Since compound interest causes an account to grow by a fixed percent each year, we may categorize the compound interest formula as an exponential function. First, recall the following property of exponents.

$$(x^a)^b = x^{ab} \quad \text{or} \quad x^{ab} = (x^a)^b$$

From this property we can rewrite the compound interest formula as follows.

$$\begin{aligned}
A &= P\left(1 + \frac{r}{n}\right)^{nt} \\
&= P\left[\left(1 + \frac{r}{n}\right)^{n}\right]^{t}
\end{aligned}$$

If we then let $b = \left(1 + \frac{r}{n}\right)^n$, we can rewrite the compound interest formula in the form $y = ab^x$.

$$A = Pb^t$$

For example, a 4.4% CD compounded monthly with an initial deposit of \$1000 could be written as

$$\begin{aligned} A &= P\left[\left(1+\frac{r}{n}\right)^n\right]^t \\ &= 1000\left[\left(1+\frac{0.044}{12}\right)^{12}\right]^t \\ &= 1000[(1+0.00367)^{12}]^t \\ &= 1000[(1.00367)^{12}]^t \\ &= 1000(1.0449)^t \qquad \text{Evaluate } (1.00367)^{12}. \end{aligned}$$

Written in this form, the 1.0449 is the annual growth factor and the corresponding rate (4.49%) is the *annual percentage yield*. The **annual percentage yield (APY)** is the *actual* amount of interest earned during the year and takes into account not only the nominal rate but also the compounding frequency. The Truth in Savings Act of 1991 requires banks to publish the APY for all savings accounts and CDs so consumers can more easily compare offers from different banks and make informed financial decisions.

Annual Percentage Yield (APY)

The **annual percentage yield** of an investment earning a nominal interest rate r compounded n times per year is given by

$$\begin{aligned} \text{APY} &= \left(1+\frac{r}{n}\right)^n - 1 \\ &= \text{annual growth factor} - 1 \end{aligned}$$

EXAMPLE 3 ■ Modeling Compound Interest with Exponential Functions

An investor makes a \$3000 initial deposit in a CD with a 3.68% interest rate compounded weekly.

a. Use the compound interest formula to find an exponential function to model the future value of the investment.

b. Determine the annual percentage yield for the investment.

Solution

a.

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 3000\left(1+\frac{0.0368}{52}\right)^{52t} \qquad \text{1 year} = \text{52 weeks} \\ &= 3000(1+0.0007)^{52t} \\ &= 3000(1.0007)^{52t} \\ &= 3000[(1.0007)^{52}]^t \\ &= 3000(1.0375)^t \end{aligned}$$

So $A(t) = 3{,}000(1.0375)^t$ models the future value of the CD.

b. From the exponential model, we can see the annual growth factor is 1.0375 and the corresponding annual percentage yield is 3.75% $(1.0375 - 1)$.

Does increasing the compounding frequency always increase the future value of an investment? In Example 4, we explore this question.

EXAMPLE 4 ■ Finding the Continuous Growth Factor

Using the compound interest formula, examine what happens to \$1 invested in an account with a 100% interest rate if interest is compounded an increasing number of times per year. Discuss what you discover.

Solution Based on the examples we have seen thus far, it appears that the value of the account will increase as we increase the compounding frequency. To see whether compounding every hour, minute, or second will dramatically increase the account value, consider Table 6.20.

Table 6.20

Compounding Method	Compounding Frequency n	Growth Factor $\left(1+\frac{1}{n}\right)^n$	Annual Percentage Yield (APY)	Account Value after 1 Year
annually	1	2	100%	\$2
semiannually	2	2.25	125%	\$2.25
quarterly	4	2.441	144.1%	\$2.44
monthly	12	2.613	161.3%	\$2.61
weekly	52	2.693	169.3%	\$2.69
daily	365	2.715	171.5%	\$2.71
hourly	8760	2.718	171.8%	\$2.72
every minute	525,600	2.718	171.8%	\$2.72
every second	31,536,000	2.718	171.8%	\$2.72
ten times per second	315,360,000	2.718	171.8%	\$2.72

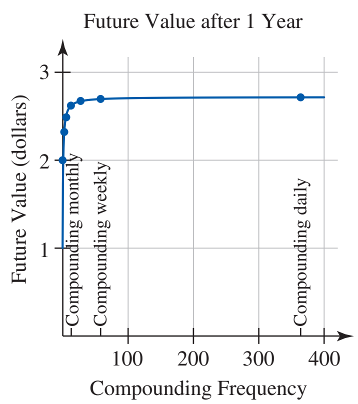


Figure 6.8

The table reveals an interesting result. While it is initially true that the account value increases as the compounding frequency increases, the account value then levels off. In other words, there is a *limiting value* to the future value of the account as the compounding frequency increases. It appears the account will never be worth more than \$2.72 after 1 year, and the maximum annual growth (APY) will never be more than 171.8% (a growth factor of approximately 2.718). The graph in Figure 6.8 demonstrates the impact of the compounding frequency on the future value of the account and clearly shows this limiting value.

From the table and graph in Example 4, we observe $\left(1+\frac{1}{n}\right)^n \to \approx 2.718$ as the compounding frequency $n \to \infty$. This number (≈ 2.718) is so important in mathematics that it has its own name: e.

THE NUMBER e

e is the irrational number 2.718281828459 . . . .

Since e is reserved to represent this special number, using e to represent a variable quantity in a function should be avoided. See the *Peer into the Past* sidebar on page 361 for more information about the number e.

Let's see how the number e emerges from the compound interest formula as we let the compounding frequency become infinitely large. Recall $\left(1 + \frac{1}{x}\right)^x \to e$ as $x \to \infty$. We let $x = \frac{n}{r}$. Observe that if r is any positive constant, then as $n \to \infty$, $x \to \infty$. Since $x = \frac{n}{r}$, $n = rx$. Let's return to the compound interest formula, $A = P\left(1 + \frac{r}{n}\right)^{nt}$. We can rewrite this as

$$\begin{aligned} A &= P\left[\left(1 + \frac{r}{n}\right)^n\right]^t \\ &= P\left[\left(1 + \frac{1}{x}\right)^{rx}\right]^t && \text{since } \frac{r}{n} = \frac{1}{x} \text{ and } n = rx \\ &= P\left[\left(1 + \frac{1}{x}\right)^x\right]^{rt} \end{aligned}$$

But $\left(1 + \frac{1}{x}\right)^x \to e$ as $x \to \infty$. So, as the compounding frequency, n, gets infinitely large, x also gets infinitely large and the compound interest formula becomes $A = Pe^{rt}$. We represent the notion of infinitely large n by using the term **continuous compounding**.

Continuous Compound Interest

The future value A of an initial investment P earning a **continuous compound interest** rate r is given by

$$A = Pe^{rt}$$

where t is the number of years after the initial investment is made. The annual percentage yield is $\text{APY} = e^r - 1$.

To understand how we determined the APY, observe how the continuous compound interest formula can be converted to the compound interest formula:

$$\begin{aligned} A &= Pe^{rt} \\ &= P(e^r)^t \\ &= Pb^t && \text{Let } b = e^r. \\ &= P(1 + \text{APY})^t && \text{since } b = 1 + \text{APY} \end{aligned}$$

Notice $e^r = 1 + \text{APY}$ or $\text{APY} = e^r - 1$.

Thus, the growth factors for continuous compound interest can be found using e as follows:

100% interest rate, compounded continuously:	$e^1 \approx 2.7183$
5% interest rate, compounded continuously:	$e^{0.05} \approx 1.0513$
10% interest rate, compounded continuously:	$e^{0.10} \approx 1.1052$

Notice the annual growth factor for continuous compound interest is e^r.

EXAMPLE 5 ■ Continuous Compound Interest

Find the future value after 4 years of $2000 invested in an account with a 5.3% nominal interest rate compounded continuously.

Solution Using the formula for continuous compound interest, we have

$$\begin{aligned} A &= Pe^{rt} \\ &= 2000e^{0.053(4)} \\ &= 2000e^{0.212} \\ &= 2000(1.236) \\ &= \$2472.30 \end{aligned}$$

After 4 years, the account will be worth $2472.30.

e: The Natural Number

PEER INTO THE PAST

HISTORY OF *e*

The number *e* indirectly appears in mathematics literature in the 1600s without being given a special name. Despite its vital importance to logarithms, its connections to graphs of hyperbolas, and its powerful implications in calculus, the formalized discovery of *e* took place much as you have seen it in this section—from the compound interest formula.

The renowned mathematician Wilhelm Liebnitz was the first to use *e* as a number in writing in 1690, although he called it *b* at that time. It was Euler, the man who is responsible for much of modern algebraic notation, who named it *e*.

(*Sources:* http://www-groups.dcs.st-and.ac.uk/~history/PrintHT/e.html; Eli Maor, *e: The Story of a Number*, Princeton University Press, 1994.)

Is *e* only helpful for continuous compound interest situations? The answer is a resounding *no*! In fact, *e* has so many amazing uses and appears in such a wide variety of contexts that it is often called the *natural number*. When working with exponential functions, it is very common to use e^k as the growth factor when an exponential function models something that is growing or decaying continuously instead of periodically.

Consider the following situation. The total annual health-related costs in the United States in billions of dollars can be modeled by the function

$$H(t) = 30.917(1.1013)^t$$

where t is the number of years since 1960. (*Source: Statistical Abstract of the United States, 2007*, Table 120). The annual growth rate is 10.13%, but is that growth *periodic* or *continuous*? In other words, do health-related costs only increase once a year, twice a year, every month, and so on, or are the costs increasing all of the time? Since the function models total costs for all people in the United States, every time health-related costs increase for *any person*, the annual costs for the country as a whole will increase. Given that there are so many products, medicines, medical researchers, and medical facility construction projects that are considered to be health-related costs, we can assume that the growth is continuous. Based on this explanation, we see that this exponential growth is more closely related to continuous compound interest than it is to periodically compounded interest.

The function H is similar to continuous compound interest, so we could use an exponential function involving the number e to model the annual health-related expenditures. Since we are not dealing with an interest-bearing account, we will use the more generalized form of $f(x) = ae^{kx}$ instead of the more specific $A = Pe^{rt}$. By finding the value of k such that $e^k = b$, we can rewrite the exponential function using the number e.

$$\begin{aligned} e^k &= b \\ e^k &= 1.1013 \end{aligned}$$

We use a system of equations and a calculator to find the value of k, as shown in Figure 6.9.

$$\begin{aligned} y_1 &= e^k \\ y_2 &= 1.1013 \end{aligned}$$

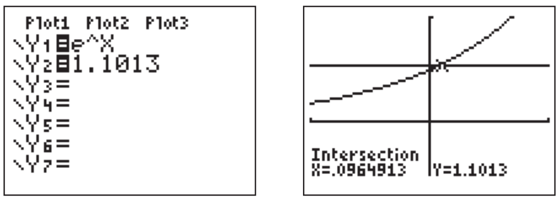


Figure 6.9

We see that $k = 0.09649$, so $e^{0.09649} = 1.1013$. Thus, we rewrite H as

$$\begin{aligned} H(t) &= 30.917(1.1013)^t \\ &= 30.917(e^{0.09649})t \\ &= 30.917e^{0.09649t} \end{aligned}$$

The functions $H(t) = 30.917(1.1013)^t$ and $H(t) = 30.917e^{0.09649t}$ are equivalent, provided we disregard round-off error.

General Form of an Exponential Function Using *e*

Exponential functions of the form $f(x) = ab^x$ can be written in the form

$$f(x) = ae^{kx}$$

where a is the initial value, k is the continuous growth rate, and $b = e^k$.

EXAMPLE 6 ■ Converting $f(x) = ae^{kx}$ to $f(x) = ab^x$

Rewrite the function $f(x) = 1000e^{0.07x}$ in the form $f(x) = ab^x$.

Solution Since $b = e^k$, we can find the growth factor b by evaluating $e^{0.07}$.

$$\begin{aligned} f(x) &= 1000e^{0.07x} \\ &= 1000(e^{0.07})^x \\ &= 1000(1.0725)^x \end{aligned}$$

■ Continuous Growth (Decay) Rate

We know that functions of the form $f(x) = ab^x$ exhibit exponential growth when $b > 1$ and exponential decay when $b < 1$. Since $b = e^k$, what effect does k have on the value of b? Let's consider the results when $k = 0.1$ and when $k = -0.1$.

$$\begin{aligned} b &= e^k \\ &= e^{0.1} \\ &\approx 1.105 \end{aligned} \qquad \begin{aligned} b &= e^k \\ &= e^{-0.1} \\ &\approx 0.905 \end{aligned}$$

It appears that $f(x) = ae^{kx}$ has exponential growth when $k > 0$ and exponential decay when $k < 0$.

Continuous Growth (Decay) Rate

The **continuous growth (decay) rate** of an exponential function is the value of k that makes the following relationship true.

$$e^k = b$$

where b is the growth (decay) factor of an exponential function.

If $k > 0$, then $b > 1$ and the function is growing.

If $k < 0$, then $0 < b < 1$ and the function is decaying.

SUMMARY

In this section you learned how to calculate the future value of a compound interest account. You also learned that continuous compound interest creates a continuous exponential function and relies on the natural number e. Finally, you learned that e may be used as part of the base of an exponential function using the relationship $e^k = b$.

6.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–6, choose the approximate value of each expression from the following list: {1.4, 1.7, 2.7, 5.4, 5.7, 7.4}. Do not use a calculator.

1. e **2.** $e - 1$

3. $e + 3$ **4.** $2e$

5. $\frac{e}{2}$ **6.** e^2

In Exercises 7–10, use <, >, or = to accurately describe the relationship between each number, then explain how you know this to be true. Do not use a calculator.

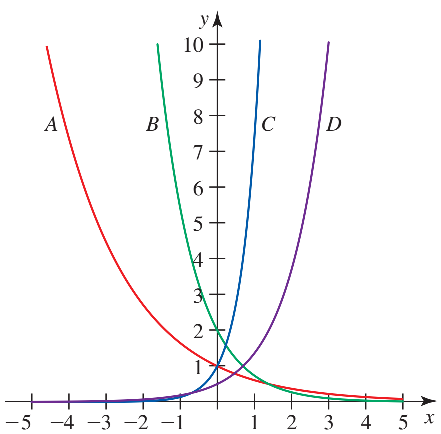
7. e^3 ___ 27 **8.** e^0 ___ 1

9. e^{-2} ___ 0.25 **10.** $\sqrt{e}$ ___ 2

11. Match each function with its graph.

a. $y = e^{2x}$ **b.** $y = e^{-0.5x}$

c. $y = 2e^{-x}$ **d.** $y = 0.5e^{x}$



In Exercises 12–15, you are given a continuous growth rate for an account that compounds interest continuously.

a. What is the APY of the account?

b. What nominal rate would be necessary for an account with *quarterly* compound interest to have the same APY?

12. Continuous rate of 3%

13. Continuous rate of 0.9%

14. Continuous rate of 2.75%

15. Continuous rate of 5.55%

In Exercises 16–19, you are given a nominal rate for an account. For each, find the continuous growth rate that has the same APY.

16. Nominal rate of 4% compounded annually

17. Nominal rate of 5.5% compounded monthly

18. Nominal rate of 2.75% compounded semiannually

19. Nominal rate of 3.61% compounded quarterly

SHOW YOU KNOW

20. Explain what $\frac{r}{12}$ and $12t$ represent in the compound interest formula $A = P\left(1 + \frac{r}{12}\right)^{12t}$.

21. Is there a limit to the future value of $4000 invested at a nominal rate of 3.5% if interest is compounded an increasing number of times? Explain.

22. Is the APY of an account always higher than the nominal rate? Explain.

23. Suppose you invest $2000 evenly in two different company stocks. Over 3 years, the first stock increases in value with an average of 3% APY while the second stock decreases in value by an average of 3% APY. At the end of 3 years have you gained money, lost money, or broken even? Explain.

24. Consider the functions $y = 2^x$, $y = 3^x$, and $y = e^x$.

a. For what values of x is $2^x < e^x < 3^x$? Explain how you know this.

b. For what values of x is $3^x < e^x < 2^x$? Explain how you know this.

MAKE IT REAL

Certificates of Deposit

In Exercises 25–30, use the given information to determine the future value of each CD when it matures, assuming the minimum amount is invested.

	Institution	Maturity	Nominal Rate	Compounding Method	Minimum Investment
25.	Bank of America	5 years	3.64%	monthly	\$1000
26.	First Arizona Savings	3 years	4.91%	quarterly	\$500
27.	Discover Bank	4 years	5.07%	daily	\$2500
28.	World Savings Bank	2.5 years	4.31%	daily	\$1000
29.	Queens County Savings Bank	6 months	3.96%	semiannually	\$2500
30.	Guarantee Bank	6 months	4.67%	quarterly	\$1000

Quoted rates are from www.bankrate.com and were accurate as of April 2007.

Certificates of Deposit

In Exercises 31–34, use the given information to answer the questions for each CD.

a. Each time interest is compounded, what is the percentage gain in the CD's value?

b. What is the future value of the CD when it matures, assuming the minimum amount is invested?

c. What is the average rate of change of the investment from the time when the CD is purchased until the time it matures?

d. What is the APY for the CD?

e. What initial investment would be required for the CD to be worth \$5000 when it matures?

f. How much more money would the CD be worth if the interest was compounded continuously instead of periodically?

	Institution	Maturity	Nominal Rate	Compounding Method	Minimum Investment
31.	State Bank & Trust	2 years	4.75%	annually	\$500
32.	Bank of America	1 year	2.86%	monthly	\$1000
33.	Bank of Albuquerque	3 years	4.18%	quarterly	\$1000
34.	Crescent State Bank	6 months	3.05%	daily	\$500

Quoted rates are from www.bankrate.com and were accurate as of April 2007.

In Exercises 35–40, use the given information to determine which CD will have a higher annual percentage yield (APY). Quoted rates are from www.bankrate.com and were accurate as of April 2007.

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35. Certificates of Deposit ING Direct offers a nominal rate of 5.01% compounded annually and BankDirect offers a nominal rate of 4.88% compounded daily.

36. Certificates of Deposit Fireside Bank offers a nominal rate of 5.10% compounded monthly and Discover Bank offers a nominal rate of 5.07% compounded daily.

37. Certificates of Deposit Eastern Savings Bank offers a nominal rate of 5.03% compounded monthly and Intervest National Bank offers a nominal rate of 5.02% compounded daily.

38. Certificates of Deposit National Bank of Kansas City offers a nominal rate of 5.07% compounded daily and Flagstar Bank offers a nominal rate of 5.11% compounded quarterly.

39. Certificates of Deposit Nova Savings Bank offers a nominal rate of 4.93% compounded daily and NBC Bank offers a nominal rate of 5.00% compounded semiannually.

40. Certificates of Deposit Flagstar Bank offers a nominal rate of 5.10% compounded quarterly and California First National Bank offers a nominal rate of 5.05% compounded monthly.

In Exercises 41–42, population growth models are given for countries based on the World Health Organization's 2006 World Health Statistics. In each model, the population is in millions t years after 2006.

a. Find the annual growth/decay rate of the population.

b. Rewrite the model in the form $f(t) = ab^t$.

c. State whether the population is increasing or decreasing and explain how you know.

41. Cambodia: $C(t) = 9.038e^{0.0198t}$

42. Kazakhstan: $K(t) = 14.825e^{-0.007t}$

In Exercises 43–44, population growth models are given for countries based on the World Health Organization's 2006 World Health Statistics. In each model the population is in millions t years after 2006.

a. Find the continuous growth/decay rate of the population.

b. Rewrite the model in the form $f(t) = ae^{kt}$.

c. State whether the population is increasing or decreasing and explain how you know.

43. Rwanda: $R(t) = 9.038(1.05)^t$

44. Hungary: $H(t) = 10.098(0.998)^t$

In Exercises 45–46, use the following information. Oxaprozin is a medicine prescribed to manage osteoarthritis and rheumatoid arthritis. After the drug reaches peak concentration in the bloodstream, the body begins to reduce the amount of medicine present according to the formula $A(t) = 600e^{-0.0173t}$, where A is the amount of medicine (in milligrams) left in the body t hours after peak concentration. **(*Source:* Modeled from www.merck.com data)**

45. Prescription Medication What is the continuous decay rate? What does this tell you about the situation?

46. Prescription Medication Rewrite the formula in the form $A(t) = ab^t$. What does b represent in this situation?

47. Available Light Underwater When light hits water, like the surface of the ocean, some of the light is reflected, some is absorbed, and some is scattered, resulting in a decreasing amount of surface light available as depth increases. In clear ocean water, the percent of surface light P available at a depth of d meters can be modeled by $P(d) = 100e^{-0.0307d}$. **(*Source:* www.oceanexplorer.noaa.gov)** Also, in clear water, photosynthesis can occur at depths of up to 200 meters. **(*Source:* www.waterencyclopedia.com)**

a. What percent of surface light is required for photosynthesis to occur?

b. At what depth is 50% of surface light available? 1% of surface light?

c. The model can also be used for murkier water by changing the value of k. Explain how you think k would change to represent water that is not very clear.

d. Write P in the form $P(d) = ab^d$ and explain what b represents.

■ STRETCH YOUR MIND

Exercises 48–52 are intended to challenge your understanding of compound interest and continuous growth.

48. How long will it take for \$12,000 invested at a nominal rate of 5.12% compounded every second to have the same value as \$14,000 invested at a nominal rate of 5.01% compounded every minute?

49. \$2000 is invested at a nominal rate of 3% compounded monthly. \$1000 is invested in an account compounding interest monthly. If after 8 years the two investments have the same value, how many times greater is the nominal rate of the second account than that of the first account?

50. An investment doubles over 13 years. How often was interest compounded if the nominal rate was 5.3374%?

51. Population Growth In 1869, Argentina's population was about 1,800,000. By 1947, its population was about 16,000,000. Write an exponential model for Argentina's population growth in the form $P(t) = ae^{kt}$. **(*Source:* www.yale.edu/ynhti)**

52. Explain difficulties you may face in finding the inverse of an exponential function.

SECTION 6.4 Solving Exponential and Logarithmic Equations

LEARNING OBJECTIVES

- State and use the rules of logarithms
- Solve exponential equations using logarithms and interpret the real-world meaning of the results
- Solve logarithmic equations using exponentiation and interpret the real-world meaning of the results

GETTING STARTED

According to the World Health Organization's 2006 World Health Statistics, Bangladesh's population was 141.8 million in 2005 and had been growing at an average annual rate of 1.8% over the previous decade. We can model this population (in millions) with $p(t) = 141.8(1.018)^t$, where t is the number of years since 2005. We can use this model to predict the population for any value of t in the practical domain. But what if we want to algebraically determine the year the model predicts the population to reach 250 million?

In this section we discuss logarithmic functions, which are the inverses of exponential functions, and see how they are used in real-world contexts such as population predictions. We also explore the rules of logarithms and learn how they can be used to solve exponential and logarithmic equations.

Logarithms

Suppose we want to know the year in which the population of Bangladesh should reach 250 million, assuming the current growth rate continues. Using the model $P(t) = 141.8(1.018)^t$, we predict the population over the next 50 years, as shown in Table 6.21 and graphed in Figure 6.10.

Table 6.21

Years Since 2005 t	Population (in millions) $p(t)$
0	141.8
10	169.5
20	202.6
30	242.2
40	289.5
50	346.0

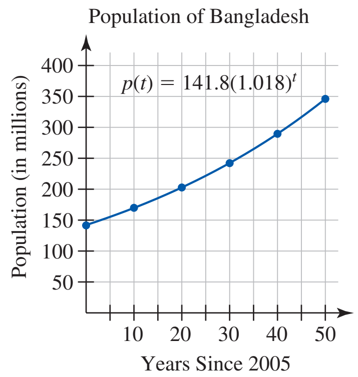


Figure 6.10

In the original function, t is the input and p is the output. But since we know the population, p, (250 million) and want to find the value of t, we need to work with the *inverse* of the exponential function to input a p-value and receive t as the output. Let's first look at the table of values (Table 6.22) and graph (Figure 6.11) of the inverse of the original relationship.

Table 6.22

Population (in millions) p	Years Since 2005 $t(p)$
141.8	0
169.5	10
202.6	20
242.2	30
289.5	40
346.0	50

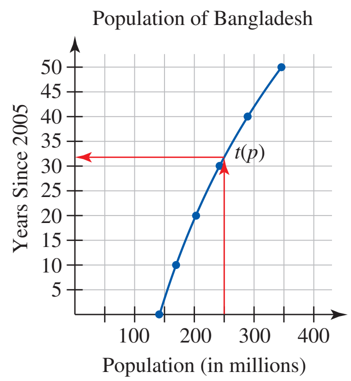


Figure 6.11

From the table, it appears Bangladesh will have 250 million people sometime between 2035 and 2045 ($30 < t < 40$). The graph gives a little better estimate, showing that the population should reach 250 million in about 2037 ($t = 32$).

To algebraically verify this value, we return to the original exponential function $p(t) = 141.8(1.018)^t$ and solve for t when $p = 250$ (which represents 250,000,000):

$$250 = 141.8(1.018)^t$$

$$\frac{250}{141.8} = \frac{141.8(1.018)^t}{141.8}$$

$$1.763 = (1.018)^t$$

Now, to solve for t in the exponent, we must use the inverse of the exponential function. Observe that t is the exponent we place on 1.018 to get 1.763. We use the symbol *log* (which stands for *logarithm*) to represent the phrase "the exponent we place on."

t is	the exponent we place on	1.018	to get 1.763
$t =$	log	$_{1.018}$	1.763

$$t = \log_{1.018}(1.763)$$

We read this as "t equals log base 1.018 of 1.763." Using properties and methods we will soon discuss, we can find $t \approx 31.785$, or a little less than 32 years after 2005. Thus, $t \approx 31.785$ is the answer to the question "What exponent on 1.018 is required to get 1.763?" In other words, $1.018^{31.785} \approx 1.763$.

Logarithmic Functions

We can generalize the preceding discussion to apply to any exponential situation. The symbol **log** is short for **logarithm**, and the two are used interchangeably. The equation $y = \log_b(x)$, which we read "y equals log base b of x," means "y is the exponent we place on b to get x." We can also think of it as answering the question "What exponent on b is necessary to get x?" That is, y is the number that makes the equation $b^y = x$ true.

LOGARITHMIC FUNCTIONS

Let b and x be real numbers with $b > 0$, $b \neq 1$, and $x > 0$. The function

$$y = \log_b(x)$$

is called a **logarithmic function**. The value b is called the **base** of the logarithmic function. We read the expression $\log_b(x)$ as "log base b of x."

A logarithmic function is the inverse of an exponential function. So if x is the independent variable and y is the dependent variable for the *logarithmic* function, then y is the independent variable and x is the dependent variable for the corresponding *exponential* function.

INVERSE RELATIONSHIP BETWEEN LOGARITHMIC AND EXPONENTIAL FUNCTIONS

$y = \log_b(x)$ is equivalent to $b^y = x$

For example, $y = \log_6(36)$ answers the question "What exponent do we place on 6 to get 36?" In other words, what value of y makes $6^y = 36$ true? Since $6^2 = 36$, $y = 2$. Symbolically, we write $\log_6(36) = 2$. Thus, "the exponent we place on 6 to get 36" is 2 and "log base 6 of 36 is 2."

JUST IN TIME ■ PROPERTIES OF EXPONENTS

If b, m, and n are real numbers with $b > 0$, then the following properties hold.

Property	Example
1. $b^{-n} = \dfrac{1}{b^n}$	**1.** $2^{-3} = \dfrac{1}{2^3}$
2. $b^m \cdot b^n = b^{m+n}$	**2.** $4^3 \cdot 4^2 = (4 \cdot 4 \cdot 4)(4 \cdot 4) = 4^{3+2} = 4^5$
3. $\dfrac{b^m}{b^n} = b^{m-n}$	**3.** $\dfrac{7^5}{7^3} = \dfrac{7 \cdot 7 \cdot 7 \cdot 7 \cdot 7}{7 \cdot 7 \cdot 7} = 7^{5-3} = 7^2$
4. $b^{mn} = (b^m)^n = (b^n)^m$	**4.** $5^6 = (5 \cdot 5 \cdot 5)(5 \cdot 5 \cdot 5) = (5^3)^2$ or $5^6 = (5 \cdot 5)(5 \cdot 5)(5 \cdot 5) = (5^2)^3$
5. $\sqrt[m]{b} = b^{1/m}$	**5.** $\sqrt[3]{6} = 6^{1/3}$

EXAMPLE 1 ■ Evaluating a Logarithm

a. Find the value of y given that $y = \log_3(81)$.

b. Estimate the value of y given that $y = \log_4(50)$.

Solution

a. $y = \log_3(81)$ answers the question: "What exponent do we place on 3 to get 81?" That is, what value of y makes the equation $3^y = 81$ true? Since $3^4 = 81$, $y = 4$. Symbolically, we write $\log_3(81) = 4$ and say the "log base 3 of 81 is 4."

b. $y = \log_4(50)$ answers the question: "What exponent do we place on 4 to get 50?" That is, what value of y makes the equation $4^y = 50$ true? The answer to this question is not a whole number. Since $4^2 = 16$ and $4^3 = 64$, we know y is a number between 2 and 3.

■ Rules of Logarithms

When equations involving logarithms become more complex, we use established rules of logarithms to manipulate or simplify the equations. There are two keys to understanding these rules. First, *logarithms are exponents*. Thus, the rules of logarithms come from the properties of exponents. Second, any number may be written as an exponential of any base.

Consider the number 81. Using exponents, we can say $3^4 = 81$ or $9^2 = 81$. Thus, we can substitute either 3^4 or 9^2 in any equation or formula that contains the number 81. When we apply this concept to logarithms, we see the benefit of this insight.

EXAMPLE 2 ■ Solving Logarithmic Equations

Solve each of the following equations for y.

a. $y = \log_3(81)$ **b.** $y = \log_9(81)$

Solution

a. We know that $y = \log_3(81)$ means $3^y = 81$. Since $3^4 = 81$, $y = 4$. Another way to approach this problem is to rewrite 81 as 3^4 in the equation.

$$\begin{aligned} y &= \log_3(81) \\ y &= \log_3(3^4) \end{aligned}$$

Now the equation says "What exponent on 3 gives you 3^4?" The answer to this question is more obvious: 4. Thus,

$$\begin{aligned} y &= \log_3(3^4) \\ &= 4 \end{aligned}$$

b. We translate $y = \log_9(81)$ into the equation $9^y = 81$. Since $9^2 = 81$, $y = 2$. Using the alternative approach, we have

$$\begin{aligned} y &= \log_9(81) \\ &= \log_9(9^2) && \text{Substitute } 9^2 \text{ for 81.} \\ &= 2 && \text{2 is the exponent on 9 that gives } 9^2. \end{aligned}$$

We generalize our results from Example 2 in the following rule.

LOGARITHM RULE 1

$$\log_b(b^m) = m$$

There are two immediate results of Log Rule 1 that will prove extremely useful. They are $\log_b 1 = 0$ and $\log_b b = 1$. Recall that $b^0 = 1$ for all nonzero values of b and $b^1 = b$ for all b. Therefore,

$$\begin{aligned} \log_b(b^m) &= m && \textbf{Log Rule 1} \\ \log_b(b^0) &= 0 && \textbf{Set } m = 0. \\ \log_b(1) &= 0 && \textbf{since } b^0 = 1 \end{aligned}$$

So log base b of 1 is always 0 no matter the value of b. Similarly,

$$\begin{aligned} \log_b(b^m) &= m && \textbf{Log Rule 1} \\ \log_b(b^1) &= 1 && \textbf{Set } m = 1. \\ \log_b(b) &= 1 && \textbf{since } b^1 = b \end{aligned}$$

So log base b of b is always 1 no matter the value of b.

EXAMPLE 3 ■ Using the Rules of Logarithms

Calculate the value of y in each of the following equations.

a. $y = \log_7(49)$

b. $y = \log_{100}(0.01)$

c. $y = \log_2(0.25)$

Solution

a.

$$\begin{aligned} y &= \log_7(49) \\ &= \log_7(7^2) && \textbf{since } 49 = 7^2 \\ &= 2 && \textbf{Log Rule 1} \end{aligned}$$

So 2 is the exponent we place on 7 to get 49.

b.

$$\begin{aligned} y &= \log_{100}(0.01) \\ &= \log_{100}\left(\frac{1}{100}\right) && \textbf{since } 0.01 = \frac{1}{100} \\ &= \log_{100}(100^{-1}) && \textbf{since } \frac{1}{x^n} = x^{-n} \\ &= -1 && \textbf{Log Rule 1} \end{aligned}$$

So -1 is the exponent we place on 100 to get 0.01.

c.

$$\begin{aligned} y &= \log_2(0.25) \\ &= \log_2(\tfrac{1}{4}) && \textbf{since } 0.25 = \tfrac{1}{4} \\ &= \log_2(\tfrac{1}{2^2}) && \textbf{since } 4 = 2^2 \\ &= \log_2(2^{-2}) && \textbf{since } \frac{1}{x^n} = x^{-n} \\ &= -2 && \textbf{Log Rule 1} \end{aligned}$$

So -2 is the exponent we place on 2 in order to get 0.25.

Using what we just practiced, we can develop other rules involving logarithms. For Log Rule 2, recall that $b^y = m$ means the same thing as $\log_b(m) = y$. Thus we have

$$\begin{aligned} b^y &= m \\ b^{\log_b(m)} &= m && \textbf{since } y = \log_b(m) \end{aligned}$$

LOGARITHM RULE 2

$$b^{\log_b(m)} = m$$

For Log Rule 3, let's begin with the following equation: $y = \log_2(16 \cdot 64)$. One way to approach this problem is to multiply 16 and 64 first.

$$\begin{aligned} y &= \log_2(16 \cdot 64) \\ &= \log_2(1024) && \textbf{Multiply } 16 \cdot 64. \\ &= \log_2(2^{10}) && \textbf{Substitute } 2^{10} \textbf{ for 1024.} \\ &= 10 && \textbf{Log Rule 1} \end{aligned}$$

Another way to approach this relies on the properties of exponents. Observe that the equation $y = \log_2(16 \cdot 64)$ is a base 2 logarithm. This tells us that we want to be sure to get a base 2 exponential inside of the parentheses.

$$\begin{aligned} y &= \log_2(16 \cdot 64) \\ &= \log_2(2^4 \cdot 2^6) && \textbf{Substitute } 2^4 \textbf{ for 16 and } 2^6 \textbf{ for 64.} \\ &= \log_2(2^{4+6}) && \textbf{Exponent Property 2: } x^a \cdot x^b = x^{a+b} \\ &= 4 + 6 && \textbf{Log Rule 1} \end{aligned}$$

Before adding the 4 and 6 together to get 10, let's consider what these numbers represent. The 4 came from the fact that $2^4 = 16$, so $4 = \log_2 16$. The 6 came from the fact that $2^6 = 64$, so $6 = \log_2 64$. Thus, $4 + 6$ may be rewritten as $\log_2(16) + \log_2(64)$. In other words,

$$\log_2(16 \cdot 64) = \log_2 16 + \log_2 64$$

and both expressions are equal to 10.

Let's repeat this process to generalize the rule. We start with $y = \log_b(m \cdot n)$. We have

$$\begin{aligned} y &= \log_b(m \cdot n) \\ b^y &= m \cdot n && \textbf{relationship between logs and exponentials} \\ b^y &= (b^{\log_b m})(b^{\log_b n}) && \textbf{Log Rule 2} \\ b^y &= (b^{\log_b m + \log_b n}) && \textbf{Exponent Property 2: } x^n \cdot x^m = x^{n+m} \\ y &= \log_b m + \log_b n && \textbf{Equal exponentials with same bases have equal exponents.} \end{aligned}$$

But $y = \log_b(mn)$, so $\log_b(mn) = \log_b m + \log_b n$.

Logarithm Rule 3

$$\log_b(mn) = \log_b(m) + \log_b(n)$$

For the next rule of logarithms, let's consider the equation $y = \log_2\left(\frac{64}{16}\right)$. One way to approach this problem is to first divide 64 by 16.

$$\begin{aligned} y &= \log_2\left(\frac{64}{16}\right) \\ &= \log_2(4) && \textbf{Divide 64 by 16.} \\ &= \log_2(2^2) && \textbf{Substitute } 2^2 \textbf{ for 4.} \\ &= 2 && \textbf{Log Rule 1} \end{aligned}$$

Alternatively, we can solve the problem using a more cumbersome approach that will lead us to Log Rule 4.

$$\begin{aligned} y &= \log_2\left(\frac{64}{16}\right) \\ &= \log_2\left(\frac{2^6}{2^4}\right) \\ &= \log_2(2^6 \cdot 2^{-4}) \\ &= \log_2(2^6) + \log_2(2^{-4}) && \text{Log Rule 3} \\ &= \log_2(2^6) + (-4) && \text{Log Rule 1} \\ &= \log_2(2^6) - (4) \\ &= \log_2(2^6) - \log_2(2^4) && \text{Log Rule 1} \end{aligned}$$

The key relationship we want to recognize is that $\log_2\left(\frac{2^6}{2^4}\right)$ is equal to $\log_2(2^6) - \log_2(2^4)$.

LOGARITHM RULE 4

$$\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$$

The final rule of logarithms is one of the most useful in solving exponential equations. Consider $\log_3(8^5)$. We have

$$\begin{aligned} \log_3(8^5) &= \log_3(8 \cdot 8 \cdot 8 \cdot 8 \cdot 8) \\ &= \log_3(8) + \log_3(8) + \log_3(8) + \log_3(8) + \log_3(8) && \text{Log Rule 3} \\ &= 5\log_3(8) && \text{There are 5 } \log_3 8 \text{ terms.} \end{aligned}$$

This observation yields the fifth rule of logarithms.

LOGARITHM RULE 5

$$\log_b(m^n) = n\log_b(m)$$

Although not explicitly a logarithm rule, "taking the log" of both sides is a common mathematical technique used in solving exponential equations. Assume $a = b$. Let x be the exponent we place on 10 to get a and let y be the exponent we place on 10 to get b. That is, $10^x = a$ and $10^y = b$. Since $a = b$, $10^x = 10^y$. But if $10^x = 10^y$ then $x = y$. We represent the verbal descriptions of x and y with logarithmic notation: $x = \log_{10}(a)$ and $y = \log_{10}(b)$. Since $x = y$, $\log_{10}(a) = \log_{10}(b)$. Therefore, if $a = b$, $\log_{10}(a) = \log_{10}(b)$. When we "take the log of both sides" we are simply applying this observation in a problem-solving situation. We use this technique in Example 4.

EXAMPLE 4 ■ Solving an Exponential Equation Using Logarithms

The total annual health-related costs in the United States in billions of dollars may be modeled by the function $H(t) = 30.917(1.1013)^t$, where t is the number of years since 1960. (*Source: Statistical Abstract of the United States, 2007*, Table 120) According to the model, when will health-related costs in the United States reach 250 billion dollars?

Solution

$$
\begin{aligned}
250 &= 30.917(1.1013)^t \\
8.086 &\approx 1.1013^t \\
\log_{10}(8.086) &= \log_{10}(1.1013)^t && \text{Take the log of both sides.} \\
\log_{10}(8.086) &= t\log_{10}(1.1013) && \text{Log Rule 5} \\
\frac{\log_{10}(8.086)}{\log_{10}(1.1013)} &= t \\
21.66 &= t && \text{Evaluate with a calculator.}
\end{aligned}
$$

According to the model, 21.66 years after 1960 (8 months into 1982), the health-related costs in the United States reached 250 billion dollars.

We summarize the rules of logarithms as follows.

Rules of Logarithms

Let b, m, and n be real numbers with $b > 0$, $b \neq 1$, $m > 0$, and $n > 0$. Under these constraints, the following rules are always true.

Rule 1: $\log_b(b^m) = m$

Rule 2: $b^{\log_b(m)} = m$

Rule 3: $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$

Rule 4: $\log_b\left(\dfrac{m}{n}\right) = \log_b(m) - \log_b(n)$

Rule 5: $\log_b(m^n) = n\log_b(m)$

■ Common and Natural Logarithms

Although any positive number other than 1 may be used as a base for a logarithm, there are two bases that are used so frequently that they have special names. A base-10 logarithm, $\log_{10}(x)$, is called the **common log**. When writing the common log, it is customary to omit the "10" and simply write $\log(x)$. Thus, $y = \log(x)$ means "What exponent on 10 gives us x?" The answer is "y is the exponent on 10 that gives us x ($x = 10^y$)."

A base-e logarithm, $\log_e(x)$, is called the **natural log** (so named because $e \approx 2.71828$ is called the *natural number*). When writing the natural log we write $\ln(x)$ instead of $\log_e(x)$. Thus, $y = \ln(x)$ asks "What exponent on e gives us x?" and the answer is "y is the exponent on e that gives us x ($x = e^y$)."

Common and Natural Logarithms

The **common logarithm**, $y = \log_{10}(x)$, is typically written as

$$y = \log(x)$$

This is equivalent to $x = 10^y$.

The **natural logarithm**, $y = \log_e(x)$, is typically written as

$$y = \ln(x)$$

This is equivalent to $x = e^y$.

EXAMPLE 5 ■ Using Common and Natural Logs

Evaluate each of the following expressions:

a. $\log(1000)$

b. $\log(0.1)$

c. $\ln(e^5)$

d. $\ln\left(\frac{1}{\sqrt[4]{e}}\right)$

Solution

a.	$\log(1000)$	
	$\log(10^3)$	$10^3 = 1000$
	3	3 is the exponent on 10 that gives us 10^3.
b.	$\log(0.1)$	
	$\log\left(\frac{1}{10}\right)$	$0.1 = \frac{1}{10}$
	$\log(10^{-1})$	$\frac{1}{10} = 10^{-1}$
	-1	-1 is the exponent on 10 that gives us 10^{-1}.
c.	$\ln(e^5)$	
	5	5 is the exponent on e that gives us e^5.
d.	$\ln\left(\frac{1}{\sqrt[4]{e}}\right)$	
	$\ln\left(\frac{1}{e^{1/4}}\right)$	$e^{1/4} = \sqrt[4]{e}$
	$\ln(e^{-1/4})$	$e^{-1/4} = \frac{1}{e^{1/4}}$
	$-\frac{1}{4}$	$-\frac{1}{4}$ is the exponent on e that gives us $e^{-1/4}$.

■ Finding Logarithms Using a Calculator

When logarithms do not evaluate to integers, a calculator can help us find accurate decimal approximations. We use the $\boxed{\text{LOG}}$ and $\boxed{\text{LN}}$ buttons on the calculator to find the common and natural logs of any number.

EXAMPLE 6 ■ Calculating Exact Logarithms

Use a calculator to find log(600) and ln(100). Then interpret and check your answers.

```
log(600)
          2.77815125
ln(100)
         4.605170186
```

Figure 6.12

Solution Using a calculator, we get the results shown in Figure 6.12. This tells us that $10^{2.778} \approx 600$ and $e^{4.605} \approx 100$. Since we used rounded numbers in these statements, when we check our answers we see they are slightly off. Our accuracy may be improved by using the exact values stored in the calculator, as shown in Figure 6.13.

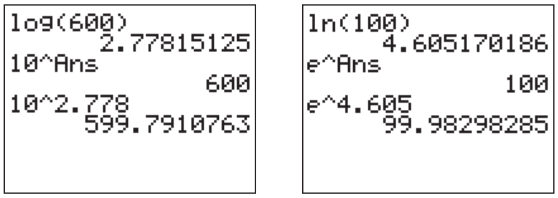


Figure 6.13

Unfortunately, some calculators are not programmed to calculate logarithms of other bases, such as $\log_3(17)$. However, using the rules of logarithms we can *change the base* of any logarithm and make it a common or natural logarithm.

Change of Base Formula

For all $x > 0$, $y = \log_b(x)$ may be written as

$$y = \frac{\log(x)}{\log(b)} \text{ or } y = \frac{\ln(x)}{\ln(b)}$$

In either of these forms, the logarithm may be evaluated or graphed with a calculator.

The origin of this formula is not entirely obvious. However, with a little creativity we can use the existing rules of logarithms to show it holds true.

Suppose that $y = \log_b(x)$. Recall that $y = \log_b(x)$ is equivalent to $b^y = x$. Now consider $\frac{\log(x)}{\log(b)}$.

$$\begin{aligned}
\frac{\log(x)}{\log(b)} &= \frac{\log(b^y)}{\log(b)} && \textbf{since } b^y = x \\
&= \frac{y\log(b)}{\log(b)} && \textbf{Log Rule 5} \\
&= y && \textbf{since } \frac{\log(b)}{\log(b)} = 1 \\
&= \log_b(x) && \textbf{since } y = \log_b(x)
\end{aligned}$$

Thus $\log_b(x) = \dfrac{\log(x)}{\log(b)}$.

EXAMPLE 7 ■ Changing the Base of a Logarithm

Using the rules of logarithms, solve the logarithmic equation $y = \log_3(17)$.

Solution We know from the definition of a logarithm $y = \log_3(17)$ means $3^y = 17$. We also know $2 < y < 3$ since $3^2 = 9$, $3^3 = 27$, and $9 < 17 < 27$. To determine the exact value for y we apply the change of base formula.

$$y = \log_3(17)$$

$$y = \frac{\log(17)}{\log(3)} \qquad \text{change of base formula}$$

$$y \approx 2.579 \qquad \text{Evaluate with a calculator.}$$

This tells us that $\log_3(17) \approx 2.579$, so $3^{2.579} \approx 17$. The same formula can be used with natural logs, yielding the same result.

$$y = \log_3(17)$$

$$y = \frac{\ln(17)}{\ln(3)} \qquad \text{change of base formula}$$

$$y \approx 2.579 \qquad \text{Evaluate with a calculator.}$$

Solving Exponential and Logarithmic Equations

Now let's use some of the rules and techniques we have learned in this section to solve equations.

EXAMPLE 8 ■ Solving an Exponential Equation

The populations of India, I, and China, C, can be modeled using the exponential functions $I(t) = 1{,}103{,}371(1.015)^t$ and $C(t) = 1{,}323{,}345(1.007)^t$, where t is in years since 2005. According to these models, when will the populations of the two countries be equal?

Solution We set the model equations equal to each other and solve for t.

$$1{,}103{,}371(1.015)^t = 1{,}323{,}345(1.007)^t$$

$$\frac{1{,}103{,}371(1.015)^t}{1{,}103{,}371} = \frac{1{,}323{,}345(1.007)^t}{1{,}103{,}371}$$

$$(1.015)^t = 1.199(1.007)^t$$

$$\frac{(1.015)^t}{(1.007)^t} = \frac{1.199(1.007)^t}{(1.007)^t}$$

$$\left(\frac{1.015}{1.007}\right)^t = 1.199$$

$$(1.008)^t = 1.199$$

$$\ln(1.008)^t = \ln(1.199)$$

$$t\ln(1.008) = \ln(1.199)$$

$$t = \frac{\ln(1.199)}{\ln(1.008)}$$

$$t \approx 22.97$$

Almost 23 years after 2005, or just before the end of 2028, we expect China and India to have the same population. (*Note:* As in other examples and exercises in this text, we are keeping the actual values in our calculator even though we round the values when we write them down. By waiting to round until we have our final answer, we obtain a more accurate result.)

EXAMPLE 9 ■ **Solving a Logarithmic Equation**

Solve the equation $\log_5(2x + 3) = 3$ for x.

Solution

$$\begin{aligned} \log_5(2x + 3) &= 3 \\ 5^3 &= 2x + 3 \\ 125 &= 2x + 3 \\ 122 &= 2x \\ x &= 61 \end{aligned}$$

For some students, logarithms are confusing. The following box gives a list of common student errors when using logarithms.

COMMON ERRORS WITH LOGARITHMS

Each of the following represent common errors when calculating with logarithms:

$$\log_b(m + n) \neq \log_b(m) + \log_b(n)$$

$$\log_b(m - n) \neq \log_b(m) - \log_b(n)$$

$$\log_b\left(\frac{m}{n}\right) \neq \frac{\log_b(m)}{\log_b(n)}$$

$$\log_b(m \cdot n) \neq (\log_b(m))(\log_b(n))$$

$$\log_b(m \cdot n^p) \neq p \log_b(m \cdot n)$$

SUMMARY

In this section you discovered the inverse relationship of logarithmic functions and exponential functions and learned that a logarithm is an exponent. You also developed and used the rules of logarithms to simplify logs and to solve exponential and logarithmic equations.

6.4 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, rewrite each statement in exponential form.

1. $\log_7(343) = 3$

2. $\log_5\left(\frac{1}{15{,}625}\right) = -6$

3. $\log(600) \approx 2.778$

4. $\ln(x) = y$

In Exercises 5–8, rewrite each statement in logarithmic form.

5. $4^7 = 16{,}384$

6. $5.5^{-2} = \frac{1}{30.25}$

7. $e^{3.2} \approx 24.53$

8. $10^y = x$

In Exercises 9–20, evaluate each expression without using a calculator.

9. $\log_4(64)$

10. $\log_5(25)$

11. $\log_2(64)$

12. $\log_8\left(\frac{1}{64}\right)$

13. $\log(10^4)$

14. $\log(1000)$

15. $\log\left(\frac{1}{100}\right)$

16. $\log(0.0001)$

17. $\ln(e^{0.2})$

18. $\ln(1)$

19. $\ln\left(\frac{1}{e^{1.3}}\right)$

20. $\ln(\sqrt{e})$

In Exercises 21–24, state the two integer values between which each expression falls. Then use the change of base formula to evaluate the expression exactly.

21. $\log_2(20)$

22. $\log_4\left(\frac{1}{50}\right)$

23. $\ln(10)$

24. $\log(0.003)$

In Exercises 25–28 you will examine common mistakes students make when simplifying logarithms. Use $m = 10$ and $n = 100$ to explore each of the following.

25. Show $\log(m + n) \neq \log(m) + \log(n)$.

26. Show $\log\left(\frac{m}{n}\right) \neq \frac{\log(m)}{\log(n)}$.

27. Show $\log(m \cdot n) \neq (\log(m))(\log(n))$.

28. Show $\log(m \cdot n^2) \neq 2\log(m \cdot n)$.

In Exercises 29–41, use the inverse relationship between logarithmic and exponential functions to solve each equation for x. Simplify your answers using the rules of logarithms and the change of base formula.

29. $5^x = 625$

30. $2^t = 0.25$

31. $3^x = 13$

32. $15^x = 2$

33. $e^x = 3$

34. $13^{5x} - 4 = 39$

35. $\log_3(x) = 5$

36. $\log_4(x) = 0$

37. $\log(x) = 3$

38. $\ln(x) = -5$

39. $\log_6(24x) = 3$

40. $\ln\left(\frac{5}{x}\right) = -2$

41. $ab^x = y$

SHOW YOU KNOW

42. The domain of the function $y = \log_b(x)$ is $x > 0$. Explain why the negative real numbers and zero are not in the domain of the function.

43. One of the rules of logarithms is $b^{\log_b(x)} = x$. Use the example $10^{\log(x)} = x$ to explain why this is true.

44. Can you combine $\log(8) + \ln(2)$ into a single logarithm? Defend your answer.

45. Explain how you can use the information $\log_3(10) \approx 2.1$ and $\log_3(4) \approx 1.26$ to approximate $\log_3(400)$.

MAKE IT REAL

In Exercises 46–47, use the following information. The population of India is currently growing according to the formula $P(t) = 1{,}103.4e^{0.0149t}$, where P is the population in millions and t is the number of years since 2005.

46. Population According to the model, in what year should the population of India reach 1,500,000,000 people (be careful with the units!)?

47. Population Find the inverse function of $P(t)$, then explain what it models.

In Exercises 48–49, use the following information. The antibiotic clarithromycin is eliminated from the body according to the formula $A(t) = 500e^{-0.1386t}$, where A is the amount remaining in the body (in milligrams) t hours after the drug reaches peak concentration.

48. Medicine How much time will pass before the amount of drug in the body is reduced to 100 milligrams?

49. Medicine Find the inverse of $A(t)$ and explain what the inverse function models.

In Exercises 50–51 you are given models for the population (in millions) of different countries t years after 2005. For each exercise, determine the year in which the models predict the populations will be equal. **(*Source:* World Health Organization's 2006 World Health Statistics)**

50. Population

Rwanda: $R(t) = 9.04(1.05)^t$

Hungary: $H(t) = 10.1(0.98)^t$

51. Population

Cambodia: $C(t) = 14.07(1.02)^t$

Kazakhstan: $K(t) = 14.83(0.93)^t$

In Exercises 52–53 you are given information on different CDs. For each exercise, determine how long it would take for the values of each CD to be equal. **(*Source:* www.bankrate.com, April 2007)**

52. Certificates of Deposit

- \$4000 invested at Charter One Bank with a nominal rate of 3.31% compounded quarterly
- \$3000 invested at Dearborn Federal Savings Bank with a nominal rate of 4.95% compounded annually

53. Certificates of Deposit

- \$2000 invested at Wachovia Bank with a nominal rate of 3.73% compounded daily
- \$2500 invested at Bank of America with a nominal rate of 3.252% compounded monthly

In Exercises 54–55, use the following information. A pH reading is used to measure the relative acidity of a substance. A pH of 7 is considered neutral (the pH of distilled water), a pH of less than 7 is acidic, and a pH above 7 is alkaline (basic). The pH, P, is measured by finding the concentration of hydrogen ions x in moles per liter in the substance, using the formula $P(x) = -\log(x)$. **(*Source:* waterontheweb.org)**

54. Chemistry Healthy human blood should have hydrogen ion concentrations between 4.467×10^{-8} and 3.548×10^{-8} moles per liter. **(*Source:* www.trans4mind.com)** What is the pH range of healthy human blood? Is this acidic or alkaline?

55. Chemistry The pH values of some common substances are given in the table on the next page. For each, find the hydrogen ion concentration.

Substance	pH P
battery acid	0.3
orange juice	4.3
sea water	8.0
bleach	12.6

Source: **waterontheweb.org**

■ STRETCH YOUR MIND

Exercises 56–60 are intended to challenge your understanding of logarithms and logarithmic functions.

56. If $b > c > 1$, for what values of x is $\log_b(x) > \log_c(x)$?

57. Solve each of the following equations. *Hint: After finding potential solutions, make sure they work in the original equation.*

a. $\log(x) + \log(x + 21) = 2$

b. $\log_6(x + 4) + \log_6(x - 1) = 1$

c. $\log_2(x - 2) + \log_2(3x + 1) = 3$

58. For $\log_b(x) = 2 \log_c(x)$, what must be the relationship between b and c?

59. Health Insurance The formula $E(t) = \dfrac{3328.1008}{1 + 247.4153e^{-0.0929t}}$ can be used to model insurance companies' total annual expenditures on health care costs in billions of dollars t years after 1960. (***Source: Statistical Abstract of the United States, 2007*, Table 120**) During what year did the total annual costs reach \$330 billion? (Solve algebraically without graphing.)

60. Without using the change of base formula and a calculator, sketch a graph of $y = \log_x(4)$. Explain why the function behaves as it does.

SECTION 6.5 Logarithmic Function Modeling

LEARNING OBJECTIVES

- Graph logarithmic functions from equations and tables
- Use logarithmic regression to model real-world data sets
- Use logarithms to linearize exponential data to find an exponential model

GETTING STARTED

Excessive government spending increases the deficit and may increase the tax burden on a country's citizens. In 1990, the United States government spent \$1872.6 billion. In 2005, the government spent \$3877.2 billion—more than twice the 1990 level of spending. (*Source: Statistical Abstract of the United States, 2007*) Economists monitoring government spending are interested not only in projecting the spending for future years but also in predicting when government spending will reach certain levels.

In this section we look at logarithmic function modeling and see how it can be used to analyze issues such as government spending. We also investigate logarithmic function graphs and use logarithms to linearize a data set to find an exponential model.

■ Graphing Logarithmic Functions

Using the techniques addressed earlier in the chapter, we model United States government spending with the exponential function

$$s(t) = 1872.6(1.0497)^t$$

where s represents the spending (in billion dollars) and t represents years since 1990. Using the model, we can determine the amount of government spending in a particular year. By solving the equation for t, we can create a model that will give us the year in which a particular level of spending is projected to occur.

$$s = 1872.6(1.0497)^t$$

$$\frac{s}{1872.6} = 1.0497^t$$

$$t = \log_{1.0497}\left(\frac{s}{1872.6}\right) \qquad \text{Rewrite in logarithmic form.}$$

$$t = \frac{\ln\left(\frac{s}{1872.6}\right)}{\ln(1.0497)} \qquad \text{change of base formula}$$

$$t = \frac{\ln(s) - \ln(1872.6)}{\ln(1.0497)} \quad \text{Log Rule 4}$$
$$t = \frac{\ln(s)}{\ln(1.0497)} - \frac{\ln(1872.6)}{\ln(1.0497)}$$
$$t = 20.62\ln(s) - 155.3$$

The function $t(s) = 20.62\ln(s) - 155.3$ models the number of years since 1990, t, in which government spending will be s billion dollars. This logarithmic function is the inverse of the exponential function $s(t) = 1872.6(1.0497)^t$.

HOW TO: ■ FIND THE INVERSE OF AN EXPONENTIAL FUNCTION

The inverse of the exponential function $p = ab^t$ is the logarithmic function

$$t = \log_b\left(\frac{p}{a}\right)$$
$$= \frac{\ln(p) - \ln(a)}{\ln(b)}$$
$$= \frac{\ln(p)}{\ln(b)} - \frac{\ln(a)}{\ln(b)}$$

To further understand the logarithmic model $t(s) = 20.62\ln(s) - 155.3$, we graph the equation in Figure 6.14. The graph is increasing and concave down with a horizontal intercept at the initial 1990 level of government spending ($1872.6 billion). When did government spending reach $3000 billion? From the graph, we see that approximately 10 years after 1990 government spending reached $3000 billion.

By learning the basic shapes of logarithmic function graphs, we can quickly determine from a scatter plot if a logarithmic model is appropriate for a particular real-world situation. The shape of a logarithmic function graph depends on the base of the logarithm. However, regardless of the base, the graph will have a vertical asymptote at the vertical axis, as shown in Figure 6.15.

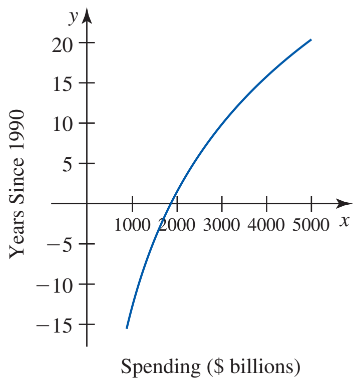


Figure 6.14

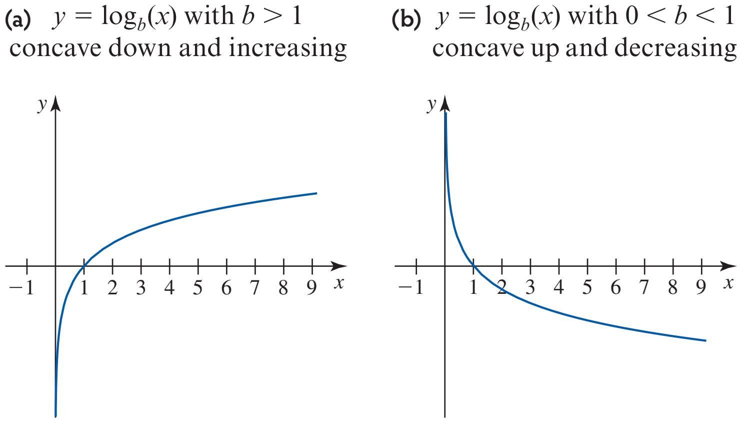


Figure 6.15

EXAMPLE 1 ■ Using Regrecvssion to Find a Logarithmic Model for a Data Set

The data set in Table 6.23 and scatter plot in Figure 6.16 show the inflation rates of the top 10 countries with the lowest rates of inflation. Determine if a logarithmic function model is appropriate for this situation. If a logarithmic function is appropriate, use regression to find the logarithmic model.

Table 6.23

Top 10 Countries C	Percent Inflation P
1. Nauru	−3.6
2. Vanautu	−1.6
3. San Marino	−1.5
4. N. Mariana Islands	−0.8
5. Barbados	−0.5
6. Dominica	−0.1
7. Israel	−0.1
8. Niger	0.2
9. Japan	0.3
10. Kiribati	0.5

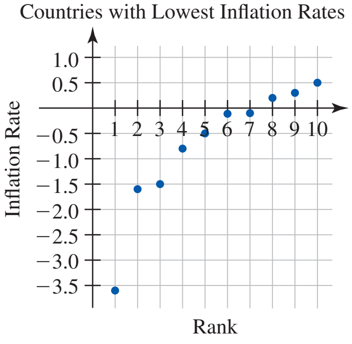


Figure 6.16

Source: CIA–The World Factbook, 2006

Solution The data set and scatter plot appear to be more or less increasing and concave down. Since the countries are listed in rank order, we know as the rank number increases the inflation rate will also increase (or remain the same). A logarithmic model is appropriate for this situation.

Using the Technology Tip at the end of this section, we determine the logarithmic equation of best fit is $P(c) = -3.237 + 1.666\ln(c)$. A graph of the model and the data is shown in Figure 6.17.

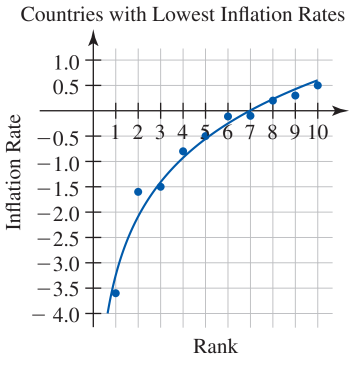


Figure 6.17

■ Finding an Exponential Model Using Logarithms

Recall that an exponential data set is characterized by a constant ratio for equally spaced values. Another way to detect if a data set is exponential is to take the logarithm of the output values, as shown in Example 2.

EXAMPLE 2 ■ Using Logarithms to Linearize Data

Complete Table 6.24 by calculating the logarithm of each of the output values. Then identify the mathematical relationship between the resultant values of $y = \log(3(2^t))$.

Table 6.24

t	$f(t) = 3(2^t)$	$y = \log(3(2^t))$		t	$f(t) = 3(2^t)$	$y = \log(3(2^t))$
−2	0.75			1	6	
−1	1.5			2	12	
0	3					

Solution We complete the table as shown in Table 6.25 and then look for a pattern by calculating the average rates of change.

Table 6.25

t	$f(t)= 3(2^t)$	$y = \log(3(2^t))$	Average Rate of Change
−2	0.75	$\log(0.75) \approx -0.1249$	$\frac{0.1761 - (-0.1249)}{-1 - (-2)} \approx 0.30$
−1	1.5	$\log(1.5) \approx 0.1761$	$\frac{0.4771 - 0.1761}{0 - (-1)} \approx 0.30$
0	3	$\log(3) \approx 0.4771$	$\frac{0.7782 - 0.4771}{1 - 0} \approx 0.30$
1	6	$\log(6) \approx 0.7782$	$\frac{1.079 - 0.7782}{2 - (1)} \approx 0.30$
2	12	$\log(12) \approx 1.079$	

Since $y = \log(3(2^t))$ has a constant rate of change, it must be a linear function. To write $y = \log(3(2^t))$ as a linear function, observe that

$$\begin{aligned} \log(3(2^t)) &= \log(3) + \log(2^t) && \text{since } \log(ab) = \log(a) + \log(b) \\ &= \log(3) + t\log(2) && \text{since } \log(b)^n = n\log(b) \end{aligned}$$

Since $\log(3) \approx 0.4771$ and $\log(2) \approx 0.3010$, $\log(3(2^t)) \approx 0.3010t + 0.471$. If we let $y = \log(3(2^t))$, then $y \approx 0.3010t + 0.4771$. We readily recognize that y is an increasing linear function with slope 0.3010 and initial value 0.4771. We complete the table again (Table 6.26), this time using the alternate form, $y = \log(3) + t\log(2)$.

Table 6.26

t	$f(t) = 3(2^t)$	$y + \log(3) + t\log(2)$	Average Rate of Change
−2	0.75	$\log(3) - 2\log(2) \approx -0.1249$	$\log(2) \approx 0.30$
−1	1.5	$\log(3) - 1\log(2) \approx 0.1761$	$\log(2) \approx 0.30$
0	3	$\log(3) - 0\log(2) \approx 0.4771$	$\log(2) \approx 0.30$
1	6	$\log(3) + 1\log(2) \approx 0.7782$	$\log(2) \approx 0.30$
2	12	$\log(3) + 2\log(2) \approx 1.079$	

The results agree with our earlier computations.

As shown in Example 2, we can use logarithms to linearize an exponential data set. This approach provides yet another way to find an exponential model, as demonstrated in Example 3.

EXAMPLE 3 ■ Finding an Exponential Model from a Linearized Data Set

Table 6.27 shows the number of insecticide treated nets (ITNs) sold or distributed in the African region in the fight against malaria.

Table 6.27

Years Since 1999 t	Total Number ITNs Sold or Distributed (in thousands) N
0	538
1	886
2	2228
3	4346
4	9485

Source: www.afro.who.int

a. Calculate $\log(N)$ at each data point.

b. Use regression to find the linear equation that relates t and $\log(N)$.

c. Use the result from part (b) to find the exponential equation that relates t and N.

Solution

a. We create Table 6.28 to calculate $\log(N)$.

Table 6.28

Years Since 1999 t	Total Number of ITNs Sold or Distributed (in thousands) N	$\log(N)$
0	538	2.731
1	886	2.947
2	2228	3.348
3	4346	3.638
4	9485	3.977

b. Using linear regression on the data in columns t and $\log(N)$, we obtain $\log(N) = 0.3183t + 2.692$.

c. Rewriting $\log(N) = 0.3183t + 2.692$ in exponential form yields

$$\begin{aligned} N &= 10^{0.3183t+2.692} \\ &= 10^{0.3183t}10^{2.692} && \text{since } b^{n+m} = b^n b^m \\ &= (2.081)^t(492.0) && \text{since } b^{nt} = (b^n)^t \\ &= 492.0(2.081)^t \end{aligned}$$

So the exponential function model is $N(t) = 492.0(2.081)^t$ thousand ITNs, where t is the number of years since 1999.

The steps to find an exponential model by linearizing a data set are summarized below.

HOW TO: ■ FIND AN EXPONENTIAL MODEL USING LOGARITHMS

To find an exponential model for t as a function of p,

1. Calculate $\log(p)$ for each value of p.
2. Use linear regression on the data set with input value t and output value $\log(p)$.
3. Write the linear regression equation in the form $\log(p) = mt + b$.
4. Rewrite the logarithmic equation in exponential form:

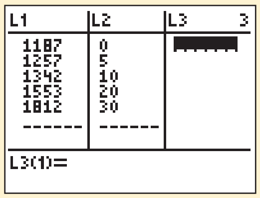
$$\begin{aligned} p &= 10^{mt+b} \\ &= (10^{mt})(10^{b}) \\ &= (10^{b})(10^{m})^{t} \end{aligned}$$

SUMMARY

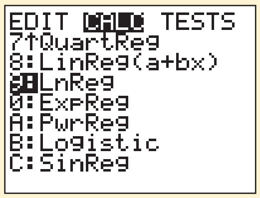
In this section you learned how to use logarithmic functions to model real-world data sets. You also learned how to use logarithms to linearize a data set and find an exponential function model.

TECHNOLOGY TIP ■ LOGARITHMIC REGRESSION

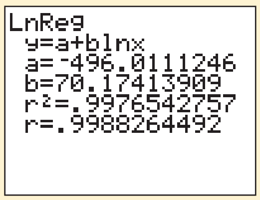
1. Enter the data using the Statistic Menu List Editor.



2. Bring up the Statistics Menu Calculate feature by pressing STAT and using the blue arrows to move to `CALC`. Then select item `9:LnReg`, and press ENTER.



3. If you want to automatically paste the regression equation into the Y = Editor so that you can easily graph the model, press the key sequence VARS; `Y-VARS`; `Function`; `Y1` and press ENTER. Otherwise press ENTER.



6.5 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, determine algebraically the equation of the linear function that passes through points of the form (t, log(p)).

1.

t	p
1	1
2	2
3	4
4	8

2.

t	p
1	6
2	18
3	54
4	162

3.

t	p
0	1000
1	1500
2	2250
3	3375

4.

t	p
−2	4000
−1	2000
1	500
3	125

5.

t	p
2	200
3	20
4	2
5	0.2

6. Given a logarithmic function, $f(x) = \ln(x)$, write a formula that calculates the average rate of change in the function between a point $(a, f(a))$ and $(a + 1, f(a) + 1))$.

7. Using the formula from Exercise 6, calculate the average rate of change for the values of a given in the table.

a	Average Rate of Change
1	
10	
100	
1000	

SHOW YOU KNOW

8. Explain how exponential and logarithmic functions are related.

9. What is the relationship between the initial value of an exponential function and the graph of its logarithmic function inverse?

10. Describe the appearance of the graph of the logarithmic function $y = \log_b(x)$ with $0 < b < 1$.

11. Describe the appearance of the graph of the logarithmic function $y = \log_b(x)$ with $1 < b$.

12. Describe the appearance of a scatter plot that may be effectively modeled with a logarithmic function.

13. Describe in words what happens to the average rate of change of $f(x) = \ln(x)$ as $x \to \infty$.

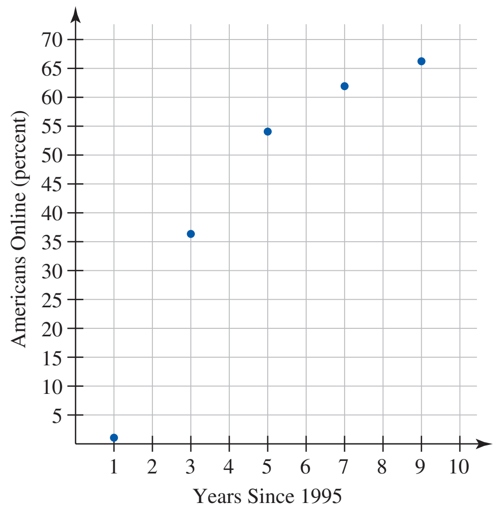
14. In calculus, we learn the *instantaneous* rate of change of $f(x) = \ln(x)$ is $\frac{1}{x}$. Describe what happens to the instantaneous rate of change of $f(x) = \ln(x)$ as $x \to \infty$ and as $x \to 0$.

15. Describe what the graph of the function $f(x) = \ln(x)$ tells you about the instantaneous rate of change of the function.

MAKE IT REAL

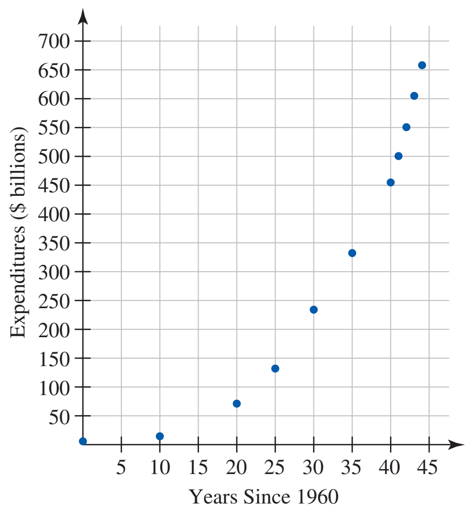
In Exercises 16–20, determine from the scatter plot if a logarithmic function model is a good fit for the data. Explain your reasoning.

16. Americans Who Go Online



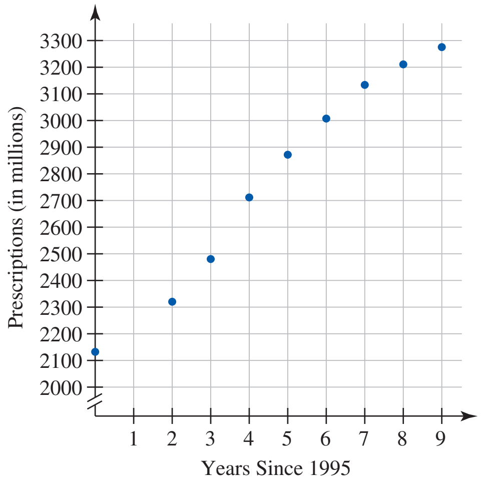
Source: Science and Engineering Indicators 2006, **National Science Foundation, Table 7-8**

17. Insurance Expenditures



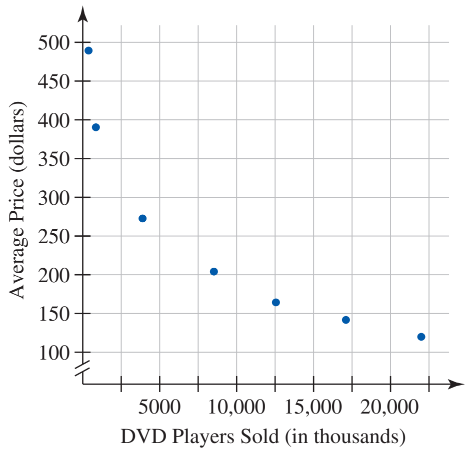
Source: Statistical Abstract of the United States, 2007, **Table 120**

18. Drug Prescriptions



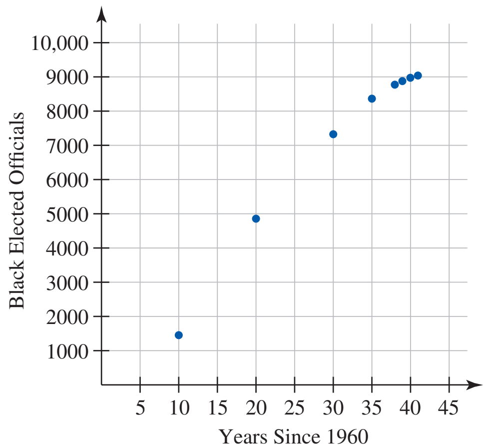
***Source: Statistical Abstract of the United States, 2006*, Table 126**

19. DVD Player Sales



***Source:* Consumer Electronics Association**

20. Elected Officials



***Source: Statistical Abstract of the United States, 2007*, Table 403**

In Exercises 21–28, find a logarithmic model for the data set algebraically.

21. Municipal Governments

Years Since 1965 Y	Number of Local Municipal Governments (in thousands) m
2	18.048
7	18.517
12	18.862
17	19.076
22	19.200
27	19.279
32	19.372
37	19.429

***Source: Statistical Abstract of the United States, 2007*, Table 415**

22. U.S. Government Spending

Expenditures ($ billions) E	Year y		Expenditures ($ billions) E	Year y
1872.6	1990		3240.8	2002
2397.6	1995		3424.7	2003
2886.5	2000		3620.6	2004
3061.6	2001		3877.2	2005

***Source: Statistical Abstract of the United States, 2007*, Table 418**

23. Elected Officials

Years Since 1960 Y	Number of Black Elected Officials N
10	1469
20	4890
30	7335
35	8385
38	8830
39	8896
40	9001
41	9061

***Source: Statistical Abstract of the United States, 2007*, Table 403**

24. Pressure and Altitude

Atmospheric Pressure (psi) P	Altitude (meters) a		Atmospheric Pressure (psi) P	Altitude (meters) a
14.696	0		7.835	5000
13.035	1000		6.843	6000
11.530	2000		5.955	7000
10.168	3000		5.163	8000
8.940	4000			

***Source:* Digital Dutch 1976 Standard Atmosphere Calculator**

25. Purchasing Power of the Dollar

Years Since 1975 t	Value of the Dollar (using Consumer Price Index, 1982 = \$1) v
5	1.215
10	0.928
15	0.766
20	0.656
25	0.581
30	0.512

Source: Statistical Abstract of the United States, 2007, Table 705

26. Births to White Mothers

Year t	Total Live Births (in thousands) b	Live Births to Women Racially Classified as *White* (in thousands) w
1990	1165	670
1995	1254	785
1999	1308	840
2000	1347	866
2001	1349	880
2002	1366	904
2003	1416	947
2004	1470	983

Source: Statistical Abstract of the United States, 2007, Table 83

Model live births to white mothers as a function of total live births.

27. Disneyland Tickets—Adult

Days in Park d	2007 Park Hopper® Bonus Ticket Cost (dollars) T
1	83
2	122
3	159
4	179
5	189

Source: www.disneyland.com

28. Disneyland Tickets—Child

Days in Park d	2007 Park Hopper® Bonus Ticket Cost (dollars) T
1	73
2	102
3	129
4	149
5	159

Source: www.disneyland.com

Frank C. Wilson

In Exercises 29–36, use regression to find the logarithmic model of best fit for the data set. Compare your results to the corresponding exercise in Exercises 21–28.

29. Municipal Governments

Years Since 1965 Y	Number of Local Municipal Governments (in thousands) m
2	18.048
7	18.517
12	18.862
17	19.076
22	19.200
27	19.279
32	19.372
37	19.429

Source: Statistical Abstract of the United States, 2007, Table 415

30. U.S. Government Spending

Expenditures (\$ billions) E	Year y	Expenditures (\$ billions) E	Year y
1872.6	1990	3240.8	2002
2397.6	1995	3424.7	2003
2886.5	2000	3620.6	2004
3061.6	2001	3877.2	2005

Source: Statistical Abstract of the United States, 2007, Table 418

31. Elected Officials

Years Since 1960 Y	Number of Black Elected Officials N
10	1469
20	4890
30	7335
35	8385
38	8830
39	8896
40	9001
41	9061

Source: Statistical Abstract of the United States, 2007, Table 403

32. Pressure and Altitude

Atmospheric Pressure (psi) P	Altitude (meters) a	Atmospheric Pressure (psi) P	Altitude (meters) a
14.696	0	7.835	5000
13.035	1000	6.843	6000
11.530	2000	5.955	7000
10.168	3000	5.163	8000
8.940	4000		

Source: Digital Dutch 1976 Standard Atmosphere Calculator

33. Purchasing Power of the Dollar

Years Since 1975 t	Value of the Dollar (using Consumer Price Index, 1982 = \$1) v
5	1.215
10	0.928
15	0.766
20	0.656
25	0.581
30	0.512

Source: Statistical Abstract of the United States, 2007, Table 705

34. Births to White Mothers

Year t	Total Live Births (in thousands) b	Live Births to Women Racially Classified as *White* (in thousands) w
1990	1165	670
1995	1254	785
1999	1308	840
2000	1347	866
2001	1349	880
2002	1366	904
2003	1416	947
2004	1470	983

Source: Statistical Abstract of the United States, 2007, Table 83

Model live births to white mothers as a function of total live births.

35. Disneyland Tickets—Adult

Days in Park d	2007 Park Hopper® Bonus Ticket Cost (dollars) T
1	83
2	122
3	159
4	179
5	189

Source: www.disneyland.com

36. Disneyland Tickets—Child

Days in Park d	2007 Park Hopper® Bonus Ticket Cost (dollars) T
1	73
2	102
3	129
4	149
5	159

Source: www.disneyland.com

In Exercises 37–42,

a. Linearize the data.

b. Use linear regression to find a model of the form $\log(p) = mt + b$.

c. Solve the equation in part (b) for p and rewrite the result in the standard form of an exponential function.

37. School Expenditures

School Year Since 1990–1991 t	Expenditure per Pupil (dollars) E
0	4902
2	5160
4	5529
6	5923
8	6508
10	7380
12	8044

Source: National Center for Education Statistics

38. Atmospheric Pressure

Altitude (meters) a	Atmospheric Pressure (psi) P
0	14.696
1000	13.035
2000	11.530
3000	10.168
4000	8.940
5000	7.835
6000	6.843
7000	5.955
8000	5.163

Source: Digital Dutch 1976 Standard Atmosphere Calculator

39. National Health Spending

Years Since 1995 t	Health Expenditures (\$ billions) H
0	1020
1	1073
2	1130
3	1196
4	1270
5	1359
6	1474
7	1608
8	1741
9	1878

Source: Statistical Abstract of the United States, 2007, Table 120

40. Brand-Name Drug Prices

Years Since 1995 t	Average Brand-Name Drug Price (dollars) d
0	40.22
2	49.55
3	53.51
4	60.66
5	65.29
6	69.75
7	77.49
8	85.57
9	95.86

Source: Statistical Abstract of the United States, 2006, Table 126

41. Highway Accidents

Years Since 2000 t	Accidents Resulting in Injuries (percent) p
0	49.9
1	48.0
2	46.3
3	45.6
4	45.1

Source: Statistical Abstract of the United States, 2007, Table 1047

42. Municipal Governments

Number of Local Municipal Governments (in thousands) m	Years Since 1965 Y
18.048	2
18.517	7
18.862	12
19.076	17
19.200	22
19.279	27
19.372	32
19.429	37

Source: Statistical Abstract of the United States, 2007, Table 415

■ STRETCH YOUR MIND

Exercises 43–46 are intended to challenge your understanding of logarithmic function modeling.

43. A data set contains positive and negative domain (input) values. A classmate claims that it is impossible to model the data with a logarithmic function. Do you agree or disagree? Explain.

44. As with other functions, logarithmic functions may be transformed using shifts, reflections, stretches, and compressions. Describe how the graph of $h(x) = -2\log(4(x+1)) - 3$ looks in comparison to $g(x) = \log(x)$.

45. What types of asymptotes do logarithmic functions have? Explain.

46. Where will the graph of $f(x) = 3((0.5)^x)$ and the graph of $f^{-1}(x) = \dfrac{\ln(x)}{\ln(0.5)} - \dfrac{\ln(3)}{\ln(0.5)}$ intersect?

CHAPTER 6 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which focus on the "big ideas" of this chapter.

SECTION 6.1

1. What do we mean when we talk about percentage growth or percentage decay?
2. What is a growth factor and what is a decay factor?
3. What is the relationship between a change factor and a change rate?
4. How do you determine from an equation whether the exponential function is growing or decaying?

SECTION 6.2

5. What key verbal indicators suggest an exponential model may be appropriate for a particular real-world situation?
6. How can you tell if a data set represents an exponential function?
7. What effect does changing the initial value and change factor of an exponential function have on the graph of the function?
8. In terms of a rate of change, what does the concavity of an exponential function tell about the function?

SECTION 6.3

9. What is compound interest?
10. What is a periodic growth rate?
11. As the compounding frequency increases, what happens to the growth factor in the compound interest formula?
12. What is the difference between a nominal interest rate and an annual percentage yield?
13. What is the difference between periodic growth and continuous growth?

SECTION 6.4

14. What is a logarithm?
15. How are exponential and logarithmic functions related?
16. How are logarithms used in solving exponential equations?
17. How are exponential functions used in solving logarithmic equations?

SECTION 6.5

18. In terms of a rate of change, what does the concavity of a logarithmic function tell us about the function?
19. How are logarithmic function graphs distinguished from other function graphs?
20. What features of a scatter plot indicate a logarithmic function model may be appropriate?

REVIEW EXERCISES

■ SECTION 6.1 ■

In Exercises 1–5, determine if the table of values is linear, quadratic, exponential, or none of these by calculating successive differences and/or change factors.

1.

x	y
10	25
20	55
30	85
40	115
50	145
60	175

2.

x	y
−3	−0.125
−2	−0.25
−1	−0.5
0	−1
1	−2
2	−4

3.

x	y
−8	−81
−7	−64
−6	−49
−5	−36
−4	−25
−3	−16

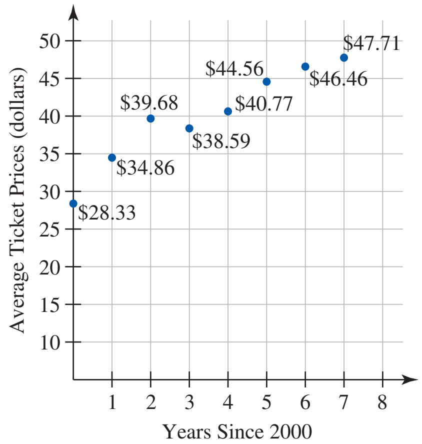
4.

x	y
11	5.55
12	5.60
13	5.65
14	5.70
15	5.75
16	5.80

5.

x	y
−2	81
−1	9
0	1
1	0.11111
2	0.12345
3	0.00137

6. Professional Baseball Tickets Team Marketing Report released its Fan Cost Index, an annual survey of how expensive it is to attend a game at each of the 30 major league baseball parks. (The index takes into account the price of four tickets, food, drink, parking, and souvenirs.) As usual, Boston's Fenway Park topped the list: On average it cost a family of four \$313.83 to take in a game there in the 2007 season. Ticket prices from 2000 to 2007 for the Boston Red Sox are shown in the following scatter plot. Calculate the growth factor and the percentage increase for each year from 2000 to 2007.



Source: **www.teammarketing.com**

7. Salary of Professors The average annual percentage increase in salaries for full professors from universities reporting to the American Association of University Professors (AAUP) was 3.55% from 2000 to 2007. The average salary for a full professor with a doctorate was \$115,475 in 2007. Estimate the average salaries for the years 2008 and 2009 by assuming the percentage change of 3.55% continues.

8. Disinfecting Swimming Pools Chlorine is used to disinfect swimming pools. The chlorine concentration should be between 1.5 and 2.5 parts per million (ppm). On sunny, hot days, up to 30% of the chlorine dissipates into the air or combines with other chemicals. Therefore, the chlorine concentration, C, in parts per million in a particular pool after s sunny days can be modeled by $C(s) = 2.5(0.7)^s$.

a. What is the initial concentration of chlorine in the swimming pool?

b. Sketch a graph of $C(s)$.

c. Evaluate $C(4)$ and explain what the value means in the real-world context.

d. Estimate when more chlorine should be added.

9. Investment Returns From January 3, 1957 to January 3, 2007 (50 years), the percentage increase for the S&P 500 was 2940%. (***Source:*** **finance.yahoo.com**)

a. What is the annual percentage increase and what is the annual growth factor?

b. Using the 50-year percentage increase of 2940%, determine the 2007 value of a \$10,000 initial investment in 1957 that grew at the same rate as the S&P 500.

SECTION 6.2

In Exercises 10–12, construct an exponential function model for the situation and answer any given questions.

10. Investment Value Based on data from August 1, 1952 through May 31, 2007, the average annual return on an investment in the CREF Stock Account was 10.72%. (*Source:* www.tiaa-cref.com)

According to the model, how much would a \$1000 investment made in the account on August 1, 1952 be worth on August 1, 2007?

11. Investment Value Based on data from March 1, 1990 through May 31, 2007, the average annual return on an investment in the CREF Social Choice Account was 10.14%. (*Source:* www.tiaa-cref.com)

According to the model, how much would a \$5000 investment made in the account on March 1, 1990 be worth on March 1, 2010?

12. Population Growth In 2005, Qatar had a population of 813,000 and was growing at an average annual rate of 4.0%. (*Source:* World Health Organization)

According to the model, what will be the population of Qatar in 2015?

In Exercises 13–15, describe in words the appearance of the graph from the equation, using the terms increasing, decreasing, concave up, concave down, vertical intercept, and horizontal asymptote, as appropriate. Then check your work by using a calculator to graph the function.

13. $y = 2(1.5)^x$

14. $y = 0.5(2.3)^x$

15. $y = 17(0.8)^x$

In Exercises 16–18, use exponential regression to find the model of best fit for the Consumer Price Index data. (The Consumer Price Index is used to measure the increase in prices over time. In each of the following tables, the index is assumed to have the value 100 in the year 1984.)

Then use the model to predict the value of the function when $t = 25$ and interpret the real-world meaning of the result.

16. Price of Dental Services

Years Since 1980 t	Price Index I
0	78.9
5	114.2
10	155.8
15	206.8
20	258.5

Source: Statistical Abstract of the United States, 2001, Table 694

17. Price of a Television Set

Years Since 1980 t	Price Index I
0	104.6
5	88.7
10	74.6
15	68.1
20	49.9

Source: Statistical Abstract of the United States, 2001, Table 694

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18. Price of Admission to Entertainment Venues

Years Since 1980 t	Price Index I
0	83.8
5	112.8
10	151.2
15	181.5
20	230.5

Source: Statistical Abstract of the United States, 2001, Table 694

SECTION 6.3

Certificates of Deposit

In Exercises 19–21, use the given information to answer the questions for each CD.

a. Each time interest is compounded, what is the percentage gain in the CD's value?

b. What is the future value of the CD when it matures, assuming the minimum amount is invested?

c. What is the average rate of change of the investment from when the CD is purchased until the CD matures?

d. What is the APY for the CD?

e. What initial investment would be required for the CD to be worth \$5000 when it matures?

f. How much more money would the CD be worth if the interest was compounded continuously instead of periodically?

	Institution	Maturity	Nominal Rate	Compounding Method	Minimum Investment
19.	National City Bank of Kentucky	5 years	4.40%	monthly	\$2500
20.	Regions Bank	3 years	3.55%	quarterly	\$500

	Institution	Maturity	Nominal Rate	Compounding Method	Minimum Investment
21.	Countrywide Bank, FSB	3 months	6.16%	daily	$10,000

Source: Quoted rates are from www.bankrate.com and were accurate as of April 2007.

In Exercises 22–23, you are given a continuous growth rate for an account that compounds interest continuously.

a. What is the APY of the account?

b. What nominal rate would be necessary for an account with quarterly compound interest to have the same APY?

22. Continuous rate of 2%

23. Continuous rate of 1.83%

In Exercises 24–25, you are given a periodic growth rate for an account. For each, find the continuous growth rate that has the same APY.

24. Nominal rate of 6% compounded monthly

25. Nominal rate of 3.25% compounded daily

In Exercises 26–27, you are given population growth models based on information from www.census.gov. In each model the population is in millions t years after 2006.

a. Find the annual growth/decay rate of the population.

b. Rewrite the model in the form $f(t) = ab^t$.

26. Population Saudi Arabia: $S(t) = 26.42e^{0.0227t}$

27. Population Bulgaria: $B(t) = 7.45e^{-0.009t}$

In Exercises 28–29, you are given population growth models based on information from www.census.gov. In each model the population is in millions t years after 2006.

a. Find the continuous growth/decay rate of the population.

b. Rewrite the model in the form $f(t) = ae^{kt}$.

28. Population Egypt: $E(t) = 77.5(1.018)^t$

29. Population Lithuania: $L(t) = 3.6(0.997)^t$

In Exercises 30–31, use the following information. Celecoxib is a medicine prescribed to manage osteoarthritis and rheumatoid arthritis. After the drug reaches peak concentration in the blood stream, the body begins to reduce the amount of medicine present according to the model $A(t) = 200(0.9389)^t$, where A is the amount of medicine (in milligrams) left in the body t hours after peak concentration. (*Source:* Modeled using information from www.merck.com)

30. Medicine Find the continuous decay factor, explain what it represents, and rewrite the model in the form $A(t) = ae^{kt}$.

31. Medicine Estimate the half-life of celecoxib, then explain how you found your answer.

SECTION 6.4

In Exercises 32–33, rewrite each statement in logarithmic form.

32. $8^3 = 512$

33. $10^{-4} = \dfrac{1}{10{,}000}$

In Exercises 34–35, rewrite each statement in exponential form.

34. $\log_2(128) = 7$

35. $\ln(19) \approx 2.944$

In Exercises 36–39, evaluate each expression without using a calculator.

36. $\log_3(81)$

37. $\log_2\left(\dfrac{1}{32}\right)$

38. $\log(100{,}000)$

39. $4^{\log_4(6)}$

In Exercises 40–41, state the two integer values between which each expression falls. Then check your estimate using the change of base formula.

40. $\log(49)$

41. $\log_5(600)$

In Exercises 42–43,

a. Determine the year in which the model predicts each country's population will be 18 million.

b. Find the inverse of the population model and explain what the inverse function is used to find.

42. Population Ecuador's population in millions can be modeled by $E(t) = 13.36(1.0124)^t$, where t is in years since 2006. (*Source:* www.census.gov)

43. Population Guatemala's population in millions can be modeled by $G(t) = 12.18e^{0.02205t}$, where t is in years since 2006. (*Source:* www.census.gov)

44. Astronomy The function $A(m) = 0.6735e^{0.423m}$ models the aperture size A (in millimeters) required to see an object with a magnitude of m. (*Source:* Modeled using data from www.ayton.id.au) Find the inverse of this function and explain what it models.

45. Certificates of Deposit Use the compound interest formula to write an exponential function to model the value of $8000 invested in a 3-year CD from UFBDirect.com with a nominal rate of 4.75% compounded monthly. (*Source:* www.bankrate.com in April 2007) Then find the inverse of your function and explain what the inverse models.

46. Population Sudan's population (in millions) can be modeled by $S(t) = 40.19(1.026)^t$ and Ukraine's population by $U(t) = 46.96(0.9925)^t$, where t is in years since 2005. (*Source:* www.census.gov) In what year do these models predict the two countries will have the same population?

47. Certificates of Deposit Determine how long it would take for the following investments to have the same value: $5000 invested in a Capital One CD with a nominal rate of 4.88% compounded daily and $5500 invested in an Integra Bank CD with a nominal rate of 3.96% compounded semiannually. (*Source:* www.bankrate.com in April 2007)

In Exercises 48–50, use the following information. The brightest celestial body seen from Earth appears to be the sun, but this is because the sun is relatively close to Earth, not because the sun is a very bright star in the universe. For this reason, scientists sometimes refer to an object's absolute magnitude, which measures how bright an object would appear if it was located 10 parsecs (or 32.6 light years) away. The formula $A(d) = 11 - 5\log(d)$ models the absolute magnitude of an object with an apparent magnitude of 6 (barely visible with the naked eye) if that object is d parsecs from Earth. **(*Source:* www.astro.northwestern.edu)**

48. Astronomy Find $A(12)$ and explain what your answer represents.

49. Astronomy Find d if $A(d) = 2.3$ and explain what your answer represents.

50. Astronomy Find the inverse of $A(d)$ and explain what the inverse function models.

■ SECTION 6.5 ■

In Exercises 51–53, describe the appearance of the graph based on its equation.

51. $y = \log_2(x)$

52. $y = \log_{0.4}(x)$

53. $y = 3\log_4(x)$

In Exercises 54–56, find the logarithmic function that best fits the data. Then answer the given question.

54. Percentage of TV Homes with a VCR

Years Since 1984 t	Homes with a VCR (percent) V
1	20.8
3	48.7
5	64.6
7	71.9
8	75.0
9	77.1
10	79.0
11	81.0
12	82.2
13	84.2
14	84.6

***Source: Statistical Abstract of the United States, 2001*, Table 1126**

Evaluate $V(4)$ and explain what the solution means in this real-world context.

55. Price of Wine Consumed at Home

Consumer Price Index i	Years Since 1980 t
89.5	0
100.2	5
114.4	10
133.6	15
151.6	20

***Source: Statistical Abstract of the United States, 2001*, Table 694**

Evaluate $t(140)$ and explain what the solution means in this real-world context.

56. Price of a Television Set

Consumer Price Index i	Years Since 1980 t
104.6	0
88.7	5
74.6	10
68.1	15
49.9	20

***Source: Statistical Abstract of the United States, 2001*, Table 694**

Evaluate $t(90)$and explain what the solution means in this real-world context.

Make It Real Project

What to Do

1. Find a set of at least six data points from an area of personal interest. Choose data that appear to increase or decrease by a constant percentage rate.
2. Draw a scatter plot of the data and explain why an exponential growth or exponential decay function might best model the situation.
3. Find a regression model for your data.
4. Using rate of change ideas, describe the relationship between the two quantities.
5. Explain the practical meaning of the initial value and change factor of the model.
6. Use the model to predict the value of the function at an unknown point and explain why you think the prediction is accurate or not accurate.
7. Determine the inverse of the exponential model.
8. Interpret the meaning of the inverse function in the context of the situation.
9. Explain how a consumer and/or a businessperson could benefit from the model and from the inverse of the model.