MAT151 - Unit 2 Notes

Module 6 - Exponential Functions

- Google Slide Notes

Observing Rate of Change

Change in t	Years since 2010, t	US Sales for McDonald's, in billions of \$, M	Change in M
+1	7	37.4807	+ 0.558
+1	8	38.0387	+ 0.558
+1	9	38.5967	+ 0.558
+1	10	39.1547	+ 0.558
+1	11	39.7127	+ 0.558
+1	12	40.2707	+ 0.558
+1	13	40.8287	+ 0.558
	14	41.3867	

0.6478(0.0483)

Change in t	Years since 2010, t	US Sales for In-N-Out Burger, in billions of \$, B	Change in B
+1	7	0.6478	+0.03129
+1	8	0.67909	+0.03280
+1	9	0.711890	+0.03438
+1	10	0.74627	+0.03604
+1	11	0.78232	+0.03779
+1	12	0.82010	+0.03961
+1	13	0.85971	+0.04152
	14	0.90124	

- Notice that the rate of change of McDonald's US sales with respect to time is constant.
- Notice that the rate of change of In-N-Out US sales with respect to time is increasing.
  - This should be your first indicator to check for percentage increases and possibly an exponential function.
- First you need to find the percentage that it is increasing by. This is done by getting the rate of change between each output, and then dividing rate of change by **y1**.
  - **To get the percent change: y2 - y1 /y1**
  - **To get the change factor: y2/y 1**
- The table on the left is **Linear** and the table on the right is **Exponential**.
- The general formula for an exponential function is: **y = a * bx**

	Years since 2010, t	US Sales for In-N-Out Burger, in billions of \$, B	
+1	7	0.6478	* 1.0483
+1	8	0.67909	* 1.0483
+1	9	0.711890	* 1.0483
+1	10	0.74627	* 1.0483
+1	11	0.78232	* 1.0483
+1	12	0.82010	* 1.0483
+1	13	0.85971	* 1.0483
	14	0.90124	

Breakdown

Change in time From Ref Input

$$B(t) = 0.6478(1.0483)^{t-7}$$

Output of Ref Point

Change Factor = 1.0483

Percent Change = 4.83%

Years since 2010, t	US Sales for In-N-Out Burger, in billions of \$, B
7	0.6478
8	0.67909
9	0.711890
10	0.74627
11	0.78232
12	0.82010
13	0.85971
14	0.90124

+1 +1 +1 +1 +1 +1 +1

* 1.0483 * 1.0483 * 1.0483 * 1.0483 * 1.0483 * 1.0483 * 1.0483

How To Solve

b. Define a function that gives US sales of In-N-Out Burger, in billions with respect to t, the number of years since 2010.

Years since 2010, t	US Sales for In-N-Out Burger in billions of \$, B
5	
6	
7	0.6478
8	0.67909
9	0.711890
10	0.74627
11	0.78232
12	0.82010
13	0.85971
14	0.90124

$+3$

+1 +1 +1 +1 +1 +1 +1

$\div 1.0483$ $\div 1.0483$

$*1.0483$ $*1.0483$ $*1.0483$ $*1.0483$ $*1.0483$ $*1.0483$ $*1.0483$

$$\underbrace{\underbrace{\underbrace{0.6478(1.0483)}_{\text{Sales in YR 8}}(1.0483)}_{\text{Sales in YR 9}}(1.0483)}_{\text{Sales in YR 10}}$$

$t=10$: $0.6478(1.0483)^{3}$ ← change in time from $t=7$ to $t=10$: $10-7$

$t=20$: $0.6478(1.0483)^{13}$ ← change in time from $t=7$ to $t=20$: $20-7$

t : $0.6478(1.0483)^{t-7}$ ← change in time from $t=7$ to t: $t-7$

$$B = 0.6478(1.0483)^{t-7}$$

Change factor

% change : 4.83%

ref pt: $(7, 0.6478)$

$y = croc(\underbrace{x - x_{ref}}_{\Delta \text{input}}) + y_{ref}$

$B = 0.6478(1.0483)^{5-7}$

$0.6478(1.0483)^{-2}$

$\dfrac{0.6478}{1.0483^2}$

Change Factor & Percent Change

Although previously *change* was discussed in terms of *amount of change* or *average rate of change*, exponential change is discussed in terms of a *change factor* or in terms of *percentage change*.

- **A function is exponential if: For equal changes in the input, the output changes by the *same factor* (or the same percent).**
  - Another way to describe this: **For equal changes in input, the ratio of consecutive outputs is constant (and that ratio is the Change Factor).**

Factor Change vs Percent Change (Independent Research)

A factor change is a ratio of the new value to the old value, while a percent change is the difference between the new value and the old value, expressed as a percentage of the old value.

A factor change can be expressed as follows:

- **factor change = new value / old value**

For example, if the *old value* is **100** and the *new value* is **150**, the factor change would be **150/100 = 1.5**.

A percent change can be expressed as follows:

- **percent change = (new value - old value) / old value * 100%**

For example, if the *old value* is **100** and the *new value* is **150**, the percent change would be **(150 - 100) / 100 * 100% = 50%**.

- Percent change is often used to measure the relative change in a quantity, while factor change is used to measure the absolute change in a quantity.
- In general, percent change is more useful when comparing changes in different quantities, while factor change is more useful when comparing changes in the same quantity.

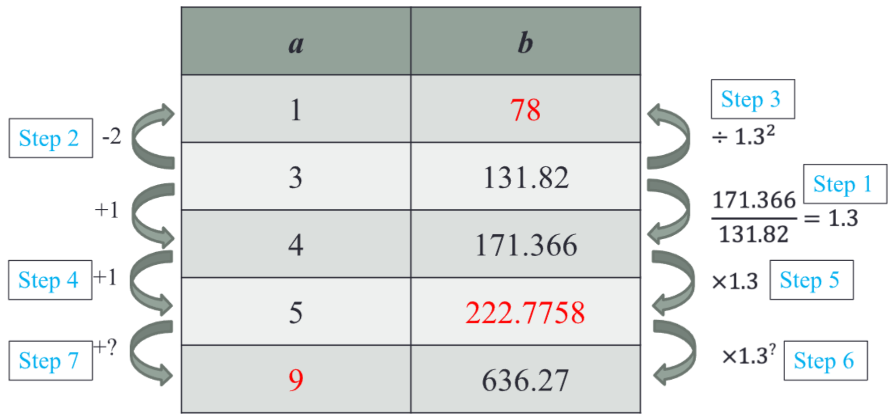
Example

x	y
0	8
1	6
2	4.5
3	3.375

- There is a constant ratio of 0.75 between each output, therefore it is exponential.
- **Change Factor: 0.75**

- **Percent Change: -25%**
- **y = 8(0.75)x**

Completing a Table



a	*b*
1	78
3	131.82
4	171.366
5	222.7758
9	636.27

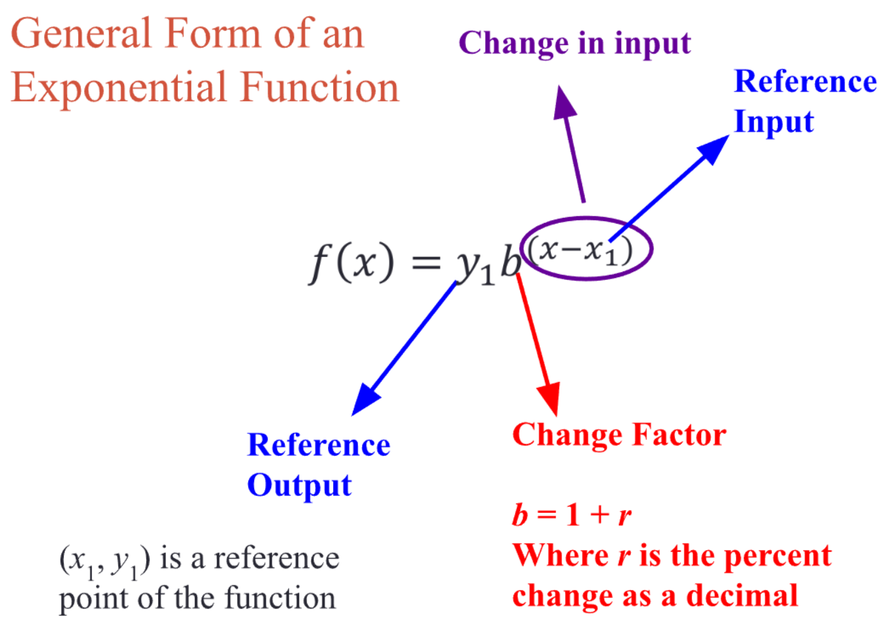
$222.7758 \times 1.3 \times 1.3 \times 1.3 \times 1.3 = 636.27$

$222.7758 \times 1.3^4 = 636.27$

The number of factors needed to multiply is 4, which means the change in input is 4

- The final input was found by repeatedly multiplying **222.7758** by **1.3** until the value lined up with **636.27**.

General Form of Exponential Function



The Exponential Formula and Graphs

f(x) = abx

Effect 1

- As the value of the function of **x -> infinity** (gets larger and larger):
  - **f(x)** gets closer and closer to **0**
  - As **x -> infinity**, **f(x) -> 0**
- As the value of the function of **x -> -infinity** (gets more and more negative):
  - **f(x)** gets larger and larger
  - As **x -> -infinity**, **f(x) -> infinity**

Effect 2

- The effect that **a** has on the function of the graph:
  - The **vertical intercept** output value
  - If **a < 0** then there is a vertical reflection.
- The effect that **b** has on the function of the graph:
  - Determines growth or decay:
    * If **b > 1** then growth
    * If **0 < b < 1** then decay

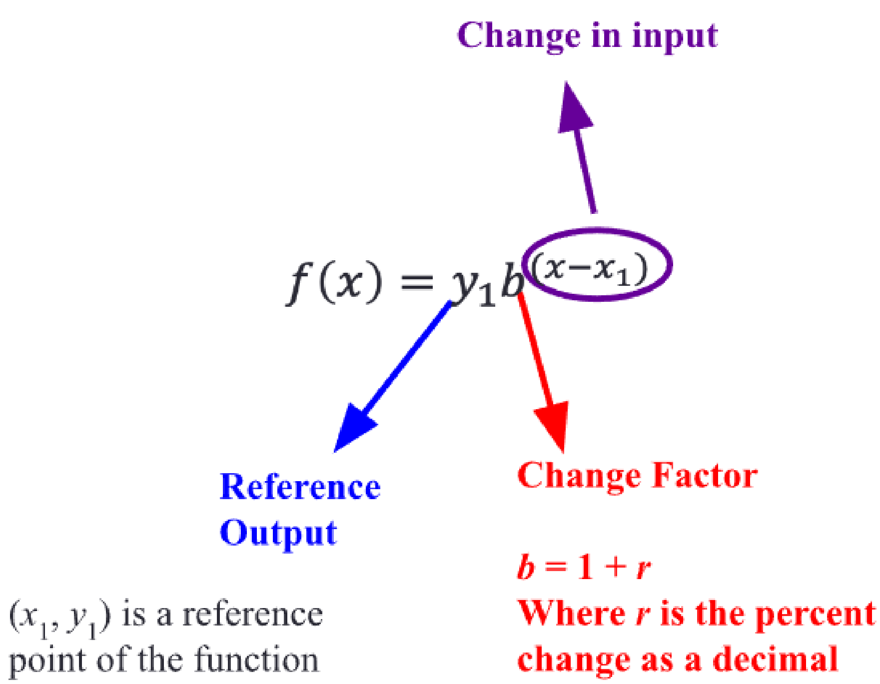
Effect 3

- If the coefficient **a** in an exponential formula is 0, it means that the exponential function will always evaluate to 0, regardless of the value of the exponent.
  - For example, if the exponential function is **f(x) = 0 * b^x**, then **f(x) = 0** for any value of **x**. This means that the exponential function is not increasing or decreasing, but stays constant at a value of **0**.
- If the base **b** is also **0**, then the exponential function is undefined, since it would involve dividing by **0**.

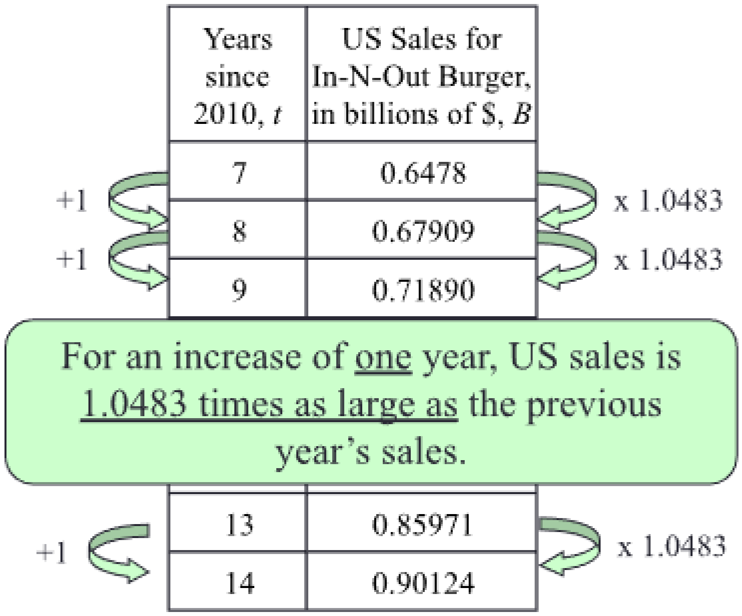
Module 7 - Exponential Functions Continued

Google Slides

Recall - General Form of an Exponential Function



One-Year Change Factor



Years since 2010, t	US Sales for In-N-Out Burger, in billions of \$, B
7	0.6478
8	0.67909
9	0.71890
13	0.85971
14	0.90124

Three-Year Change Factor (Multi-year Change Factor / Percent Change)

Years since 2010, t	US Sales for In-N-Out Burger, in billions of \$, B
7	0.6478
8	0.67909
9	0.71890
10	0.74627
11	0.78232
12	0.82010
13	0.85971
14	0.90124

$+3$

$\times(1.0483)^3 = 1.1520$

Every 3 years, sales increase by 3 factors of 1.0483, or by 1 factor of 1.1520.

1.1520 is called a 3-year change factor, because every 3 years, sales are 1.1520 times as large as they were, or are 15.20% larger (this is the 3-year percent change)

*The change in input tells you the factor size.

- You get the change factor of several years worth by getting the one-year change factor first (**1.0483**), and then putting it to the power of the change in input.

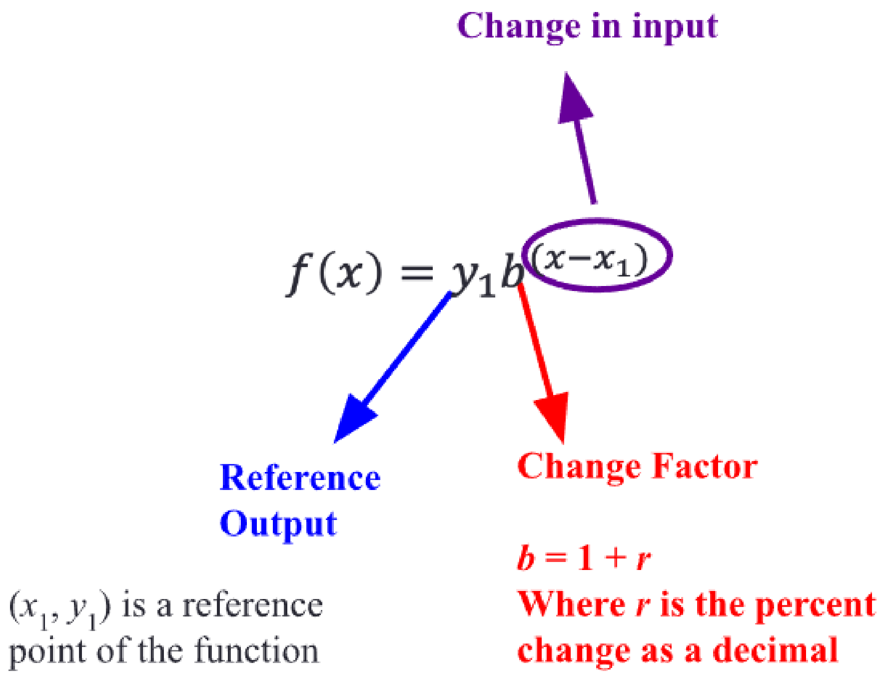
In order to use the 3-year change factor of **1.1520** to get the 6-year and the 10-year changes:

1. **6-year: 1.15202 = (1.04833)2 = 1.3271**
2. **10-year: 1.152010/3 = (1.15201/3)10 = 1.6027**

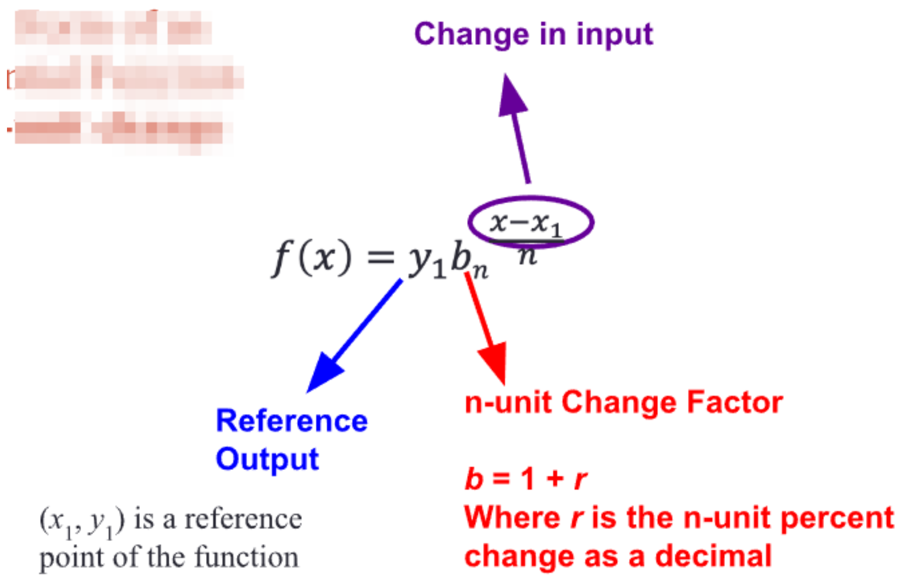
- *It's important to note that there can be a difference in value when using multi-year change factors due to rounding.*

General Exponential Forms

Exponential Function Using One-Unit Change Factor



Exponential Function Using n-Unit Change Factor



Half Life

The **half-life** of a substance is the amount of time it takes for half of the intial amount of the substance to remain.

- When given the half-life, the n-unit change factor is **always 0.5** and n is always the size of the half-life.

Example

A car has a value of **$24,000** with a half-life of **7 years**.

Calculating the depreciation rate per-year

1. **(0.5)1/7**
2. **Change Factor = 0.9057**
3. **Percent Change = -9.43%**

Calculating the value after 10 years

1. **24000(0.5)10/7**
2. **8915.97 dollars**

Example 2

After starting with 78 micrograms, the mass of bacteria decreases by 35% every 2 hours.

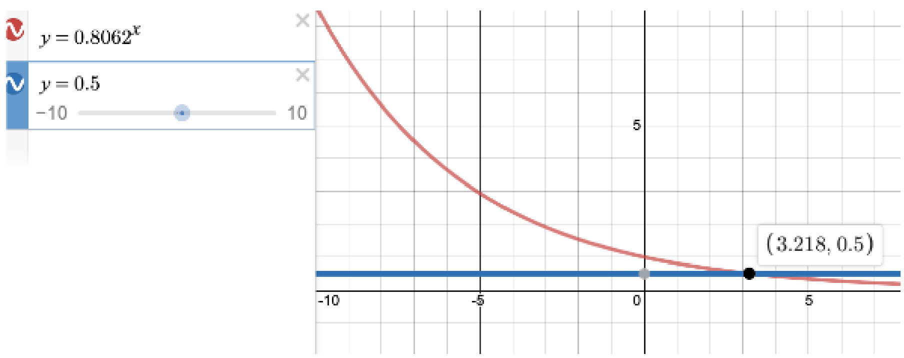
Determine the half life of the bacteria.

Method 1

- The 2-hour change factor: **0.65**
- The 1-hour change factor: $\mathbf{(0.65)1/2 = 0.8062}$

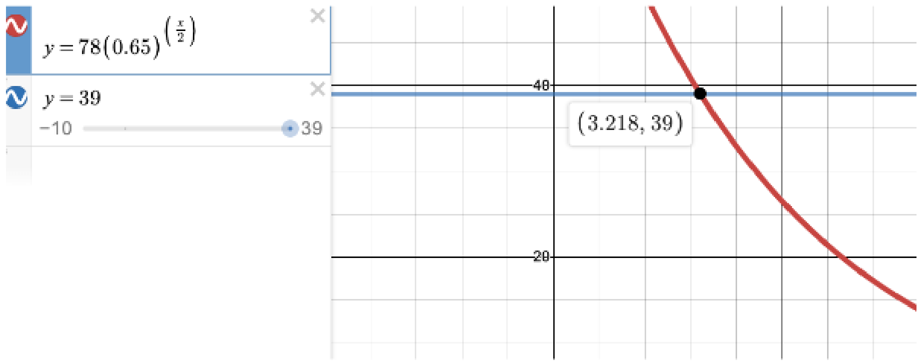
The formula: $\mathbf{(0.8062)x = 0.5}$

We will solve this graphically:



Method 2

1. Define variables
   - $\boldsymbol{t}$ **= time elapsed in hours**
   - $\boldsymbol{m}$ **= mass of bacteria in micrograms**
2. Write a formula: $\mathbf{m = 78(0.65)}\boldsymbol{t}\mathbf{/2}$
3. Find half the amount of the original amount: $\mathbf{78(0.5) = 39}$
4. Substitute the found value into the formula: $\mathbf{39 = 78(0.65)}\boldsymbol{t}\mathbf{/2}$
5. Solve it graphically



Doubling Time

Doubling time is the amount of time it takes for something that is growing to double.

- When given the doubling time, the n-unit change factor is **always** 2 and n is always the size of the doubling time.

Example

Suppose you have bacteria with a mass of 12 micrograms and that the doubling time of this bacteria was 8 hours.

1. Determine the percent that the mass of bacteria increase by each hour.
2. Define a formula for the function that gives the mass of bacteria m in micrograms after t hours.

Since the doubling time is 8 hours, the 8-hour change factor is **2**.

- The 1-hour change factor: **(2)1/8= 1.0905**
- the 1-hour percent change: **9.05%**.

The formula would be: **m = 12(2)t/8**

Exponential Regression

In the real world, there are datasets that aren't quite exponential but are close. When this happens, we create an **exponential model** *(an exponential funciton that models the data)*.

- We do this using **Exponential Regression**.

How to do exponential regression in desmos: https://www.youtube.com/watch?v=XnOzmfdbaXU

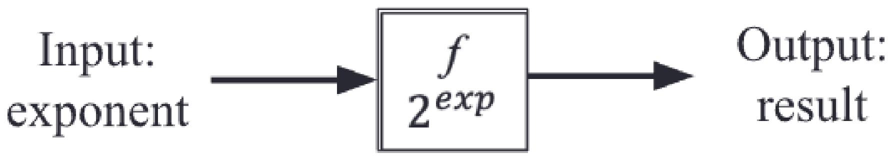
- Be sure to select **Log Mode** on Desmos for Exponential Regression. If you forget to do so, your models will be slightly different.

Module 8 - Logarithmic Functions

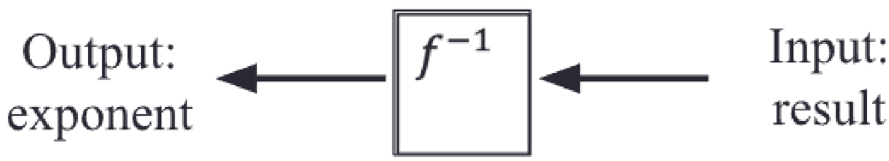
- Google Slides
- **log** is short for **logarithm**

For solving for the double time, graphing was used before using, as an example, **(1.0420)x = 2** with it being graphed to find the number. To solve this algebraically, you would need to use a **logarithmic function**.

When given a function like ***f*(exp) = 2exp**:



In order to solve it, you need to find the inverse:



Solving for the Inverse Function Using Log

Example 1

Assume we know the output of ***f*** is 8 and are looking to solve **2exp = 8**, use the inverse function, **f-1(8) = 3**.

- This is because **23 = 8**

Another example:

1. ***2exp = 32***
2. Find how many times **2** goes into **32**
3. The inverse: **f-1(32) = 5**
    - Because **25 = 32**

Example 2

Solve: $2.1(1.57)^{\frac{x-3}{2}}+4 = 12$ for x.

$$2.1(1.57)^{\frac{x-3}{2}} = 8$$
$$(1.57)^{\frac{x-3}{2}} = \frac{8}{2.1}$$
$$\frac{x-3}{2} = log_{1.57}\left(\frac{8}{2.1}\right)$$
$$\frac{x-3}{2} = 2.97$$
$$x - 3 = 5.94$$
$$x = 8.94$$

Re-write in log form

- Try to keep things as fractions to preserve accuracy

Example 3

In 2000, the population of Tucson was 489,355. In 2013, it was 526,116. Assuming exponential growth, predict when the population will reach 600,000 people.

$\frac{526116}{489355} = 1.0751$ which is the 13-year change factor.

Write the formula for the function:

let t = number of years since 2000 and P = population of Tucson.

$$P = 489355(1.0751)^{t/13}$$

Then solve:

$$600000 = 489355(1.0751)^{t/13}$$
$$\frac{600000}{489355} = (1.0751)^{t/13}$$
$$\frac{t}{13} = log_{1.0751}\left(\frac{600000}{489355}\right) = 2.815$$
$$t = 36.595 \text{ (so in 2037)}$$

Re-write in log form

- Try to keep things as fractions to preserve accuracy

Example 4

1. **g(x) = 19(0.77)3x**
2. **g(x) = 19(0.773)x**

3. **0.773** is the change factor

The inverse function is known as the **Logarithmic Function**:

$$f^{-1}(result) = exponent$$

$$log_2(result) = exponent$$

Text Versions:

- **f-1(result) = exponent**
- **log2(result) = exponent**

You would read **log2(8) = 3** as "log base 2 of 8 equals 3".

- The *result* is the input and the *exponent* on the base of 2 is the output.

Find the inverse function of *any* exponential function:

- **y = logb(x)** is the inverse function for **by = x**

$$y = log_b(x) \leftrightarrow b^y = x$$

log form exponential form

- Think of it as "What exponent is necessary on **b** to get **x**?"

Common and Natural Logarithms

Common and Natural logarithms are logarithms that are used so frequently that they have special names:

- **Common Log: log10(x)**
  - It's common to omit the **10** and write it as **log(x)**
  - **y = log(x)** means "What exponent on 10 gives us **x**?"
  - **x = 10y**
- **Natural Log: loge(x)**
  - **e ~= 2.71828** also called the *natural number*
  - When writing the natural log, you write **ln(x)** instead of **loge(x)**.

Common and Natural Logarithms

The **common logarithm**, $y = \log_{10}(x)$, is typically written as

$$y = \log(x)$$

This is equivalent to $x = 10^y$.

The **natural logarithm**, $y = \log_e(x)$, is typically written as

$$y = \ln(x)$$

This is equivalent to $x = e^y$.

Change of Base Formula

$$\log_b(x) = \frac{\log(x)}{\log(b)} = \frac{\ln(x)}{\ln(b)}$$

Logarithm Rules

Logarithm product rule

$$\log_b(x \times y) = \log_b(x) + \log_b(y)$$

Logarithm quotient rule

$$\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$$

Logarithm power rule

$$\log_b(x^y) = y \times \log_b(x)$$

Logarithm base switch rule

$$\log_b(c) = \frac{1}{\log_c(b)}$$

Logarithm change of base rule

$$\log_b(x) = \frac{\log_c(x)}{\log_c(b)}$$

Graphic Logarithmic Functions

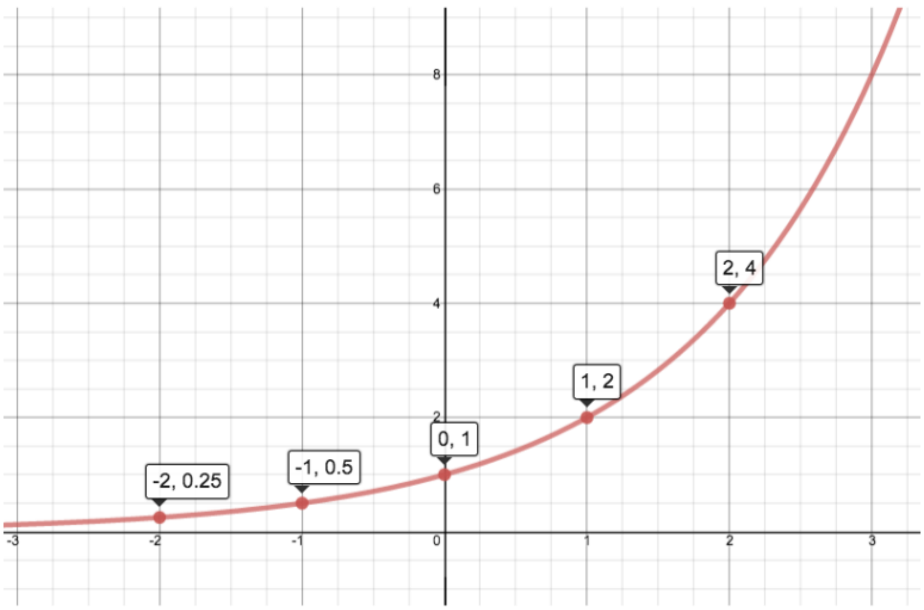
- Great for determining from a scatter plot if a logarithmic model is appropriate for a particular real-world situation.

Function: $\mathbf{y = 2x}$

Table:

x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

Graph:



- The function has a ***horizontal* asymptote at y = 0**
  - As **x** gets larger and larger in the negative direction, the value of y gets closer and closer to 0.

The inverse

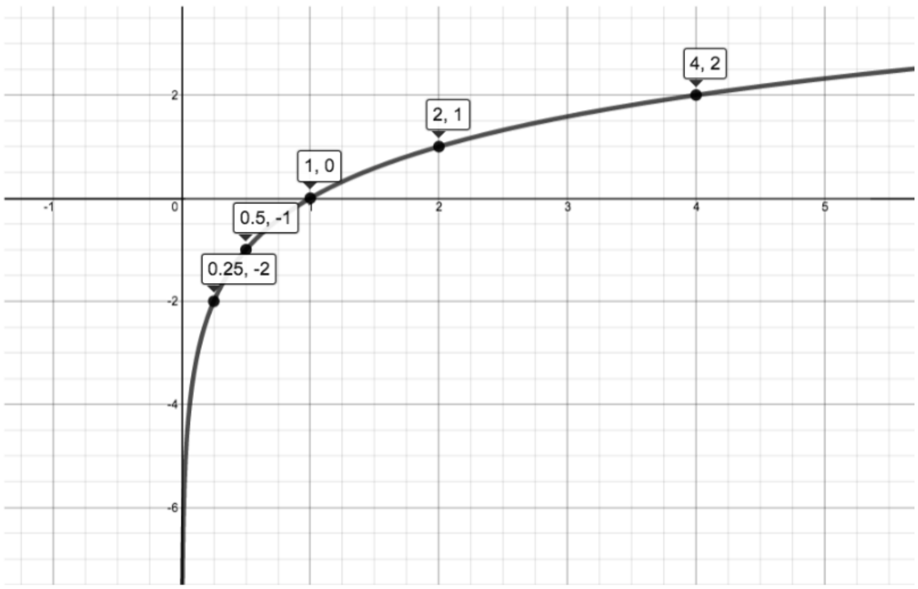
x = log2y

Table:

y	x
1/4	-2
1/2	-1
1	0
2	1
4	2

y	x

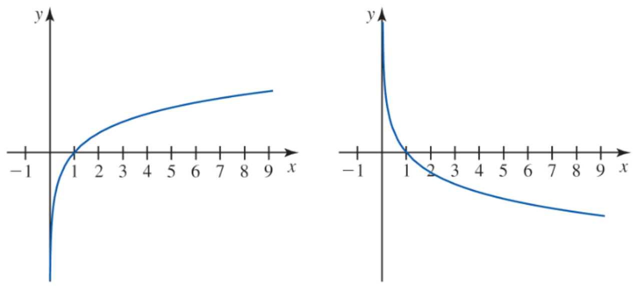
Graph:



- The function has a ***vertical* asymptote at 0**.
  - As the input approaches 0, the output starts to reach positive or negative infinity.

Examples

Regardless of the base, the graph will have a vertical asymptote at the vertical axis.



$y = \log_b(x)$ with $b > 1$
concave down and increasing

$y = \log_b(x)$ with $0 < b < 1$
concave up and decreasing

Summary

- Summary of all log notes

LOGARITHMIC FUNCTIONS

Let b and x be real numbers with $b > 0$, $b \neq 1$, and $x > 0$. The function

$$y = \log_b(x)$$

is called a **logarithmic function**. The value b is called the **base** of the logarithmic function. We read the expression $\log_b(x)$ as "log base b of x."

A logarithmic function is the inverse of an exponential function.

INVERSE RELATIONSHIP BETWEEN LOGARITHMIC AND EXPONENTIAL FUNCTIONS

$y = \log_b(x)$ is equivalent to $b^y = x$