CHAPTER 8

Matrices

One way to achieve increased wealth is to use your money to make more money. Investment options range from low-risk savings accounts to high-risk penny stocks. Taking too little or too much risk can result in an unacceptable return on your investment. By putting your money in a number of different types of investments, you can reduce your overall risk and increase your likelihood of financial success.

SECTION 8.1 Using Matrices to Solve Linear Systems

LEARNING OBJECTIVES

- Write systems of linear equations in augmented matrices
- Use technology to reduce augmented matrices to reduced row echelon form
- Solve linear systems using matrices and interpret the meaning of the results in a real-world situation

GETTING STARTED

Financial advisors typically encourage investors to diversify their investment portfolios. By investing in a variety of different types of investments, investors can reduce their risk while increasing the probability of a good financial return on their investment. Determining how much money to place in each type of investment account is not an exact science; however, a system of linear equations can be used to help decide which mix of investments is best.

In this section we introduce matrix notation and demonstrate how matrices are used to represent and solve systems of linear equations. We also show how to use technology to simplify an augmented matrix to reduced row echelon form.

Modeling with a System of Three Equations

An investor with \$30,000 in an existing mutual fund IRA wants to roll over her investment into a new retirement plan with TIAA-CREF. The company offers a variety of accounts with varying returns, as shown in Table 8.1.

Table 8.1

CREF Variable Annuity Accounts	Average Annual Return				
	Unit Value	1-year	5-year	10-year	Since Inception
Bond Market	\$79.67	5.80%	4.39%	5.86%	6.95%
Equity Index	\$103.79	19.53%	11.05%	7.29%	11.22%
Global Equities	\$112.52	23.70%	13.41%	6.54%	10.01%
Growth	\$72.53	18.33%	8.23%	3.02%	8.19%
Inflation-Linked Bond	\$46.72	3.53%	5.58%	6.33%	6.24%
Money Market	\$24.33	5.02%	2.50%	3.68%	4.72%
Social Choice	\$131.54	13.08%	8.71%	6.97%	9.99%
Stock	\$269.02	22.19%	12.71%	7.55%	10.67%

Source: www.tiaa-cref.com; as of 6/30/07

The investor looks at the long-term performance of each account as an indicator of a likely long-term return. She decides to invest her money in the Bond Market, Equity Index, and Social Choice accounts. After analyzing her *risk tolerance*, she decides to invest twice as much money in the Bond Market account as in the Equity Index account because it appears to be more stable. (A person with a high risk tolerance is more comfortable with the possibility of large fluctuations in the annual percentage return than a person with a low risk tolerance.) The investor wants to earn a 7% return on her investments. She assumes she will be able to earn the 10-year average annual return.

To determine how much money she should invest in each account, we begin by defining variables to represent the amount invested in each account. We let x be the amount of money invested in the Bond Market account, y the amount invested in the Equity Index account, and z the amount invested in the Social Choice account. Since she has \$30,000 to invest, we have

$$x + y + z = 30{,}000$$

Since she plans to invest twice as much money in the Bond Market account as in the Equity account, we have

$$x = 2y$$
$$x - 2y = 0$$

The Bond Market account has a 5.86% return rate, the Equity Index account has a 7.29% return rate, and the Social Choice account has a 7.55% return rate. Since she wants to earn a 7% return on her total investment, we have

$$0.0586x + 0.0729y + 0.0755z = 0.07(30{,}000)$$
$$0.0586x + 0.0729y + 0.0755z = 2100$$

Thus we need to solve the following system of equations.

$$\begin{aligned} x + y + z &= 30{,}000 \\ x - 2y &= 0 \\ 0.0586x + 0.0729y + 0.0755z &= 2100 \end{aligned}$$

Matrix Notation

Solving a system of three equations with three unknowns is somewhat complex. However, with the mathematical machinery of matrices, we will be able to solve such systems efficiently. We begin by introducing the necessary notation.

An $m \times n$ Matrix

An $m \times n$ matrix A is an array of numbers with m rows and n columns. The plural of matrix is *matrices.*

Capital letters are typically used to represent matrices. The following examples show matrices of various dimensions.

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 5 & 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ -1 \\ 6 \end{bmatrix}, \quad C = \begin{bmatrix} 2.1 & 1.9 \\ 0.1 & 0.8 \end{bmatrix}, \quad D = \begin{bmatrix} 2 & 7 \end{bmatrix}$$

Matrix A is a 2×3 matrix because it has two rows and three columns. Similarly, B is a 3×1 matrix, C is a 2×2 matrix, and D is a 1×2 matrix.

A matrix that consists of a single row of numbers, such as matrix D, is called a **row matrix**. A matrix that consists of a single column of numbers, such as matrix B, is called a **column matrix**. A matrix that has the same number of rows as columns, such as matrix C, is called a **square matrix**.

The numbers (or variables) inside the matrix are called the **entries** of the matrix. The entries of an $m \times n$ matrix A may be represented as

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The subscript of each entry indicates its row and column position. For example a_{21} refers to the entry in the second row and first column. In general, an entry a_{ij} is the term in row i and column j.

EXAMPLE 1 ■ Determining the Dimensions and Entry Values of a Matrix

Determine the dimensions of the matrix A and the value of the entries a_{12}, a_{21}, and a_{24}.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

Solution The matrix has three rows and four columns so it is a 3×4 matrix. For this matrix $a_{12} = 2$, $a_{21} = 5$, and $a_{24} = 8$.

■ Representing a System of Linear Equations with an Augmented Matrix

To represent the system of linear equations

$$\begin{aligned} 2x + 3y &= 8 \\ 4x - y &= 2 \end{aligned}$$

with a matrix, we write

$$\left[\begin{array}{cc|c} 2 & 3 & 8 \\ 4 & -1 & 2 \end{array}\right]$$

The matrix is called an **augmented matrix** because the *coefficient matrix*, $\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, is augmented (added on to) with the *column matrix*, $\begin{bmatrix} 8 \\ 2 \end{bmatrix}$. The vertical bar between the last two columns of numbers indicates that the matrix is an augmented matrix. The first column of the matrix contains the coefficients of the x-terms. In this case, the x-terms $2x$ and $4x$ have coefficients 2 and 4, respectively. The second column contains the coefficients of the y-terms. In this case, the y-terms $3y$ and $-y$ have coefficients 3 and -1, respectively.

To solve a system of equations with a matrix, we must first write the matrix in *reduced row echelon form*, using matrix row operations or technology.

■ Row Operations

When we discussed the elimination method in Section 2.4, we identified three operations that yielded an equivalent system of equations.

1. Interchange (change the position of) two equations.
2. Multiply an equation by a nonzero number.
3. Add a nonzero multiple of one equation to a nonzero multiple of another equation.

We can apply similar operations, called **row operations**, to augmented matrices.

Row Operations

For any augmented matrix of a system of equations, the following row operations yield an augmented matrix of an equivalent system of equations.

1. Interchange (change the position of) two rows.
2. Multiply a row by a nonzero number.
3. Add a nonzero multiple of one row to a nonzero multiple of another row and replace either row with the result.

Before the technology that is available today, mathematicians had to repeatedly apply row operations to convert a matrix into *reduced row echelon form*. A matrix is said to be in **reduced row echelon form** if it meets the following criteria.

Reduced Row Echelon Form

An augmented matrix is said to be in reduced row echelon form if it satisfies each of the following conditions.

1. The leading entry (first nonzero entry) in each row is a 1.
2. The leading entry of each row is the only nonzero entry in its corresponding column.
3. The leading entry in each row is to the right of the leading entry in the row above it.
4. All rows of zeros are at the bottom of the matrix.

A matrix in reduced row echelon form is also referred to as a **reduced matrix**.

The Technology Tip at the end of the section shows how to use a graphing calculator to write an augmented matrix in reduced row echelon form. In subsequent examples we will use this technology, but first we will demonstrate the manual process to help you grasp the concepts.

Suppose we are given the system of equations

$$2x + y = 11$$
$$4x + 3y = 27$$

The corresponding matrix is $\left[\begin{array}{cc|c} 2 & 1 & 11 \\ 4 & 3 & 27 \end{array}\right]$ and the corresponding graph is shown in Figure 8.1. Note that the two lines intersect at (3, 5).

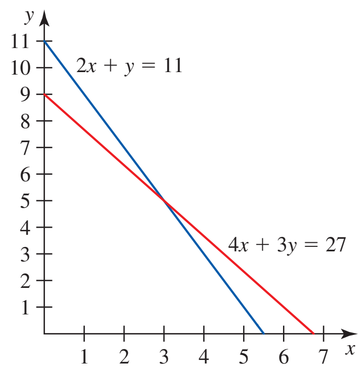


Figure 8.1

For the first step of the row reduction process, we multiply the first row by 2, subtract the second row, and place the result in the second row. This row operation eliminates the 4 in the first column. The resultant matrix is $\left[\begin{array}{cc|c} 2 & 1 & 11 \\ 0 & -1 & -5 \end{array}\right]$ and the corresponding graph is shown in Figure 8.2. Notice that these two lines intersect at the same point, (3, 5), as the first pair of lines.

We next add the first two rows and place the result in the first row. This eliminates the 1 in the first row and second column. The resultant matrix is $\left[\begin{array}{cc|c} 2 & 0 & 6 \\ 0 & -1 & -5 \end{array}\right]$ and the corresponding graph is shown in Figure 8.3. This pair of lines also intersects at (3, 5).

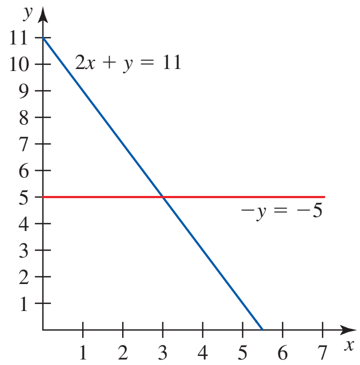


Figure 8.2

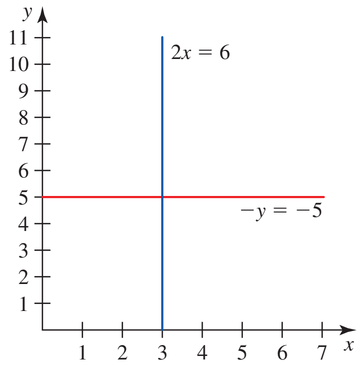


Figure 8.3

The equations $2x = 6$ and $-y = -5$ simplify to $x = 3$ and $y = 5$, respectively. The corresponding final matrix is $\left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & 5 \end{array}\right]$. Now we can read the intersection point directly from the matrix.

Let's now return to the investment scenario introduced at the start of the section. Recall that we had created the following system of equations.

$$\begin{aligned} x + y + z &= 30{,}000 \\ x - 2y &= 0 \\ 0.0586x + 0.0729y + 0.0755z &= 2100 \end{aligned}$$

In the equations, x is the amount of money invested in the Bond Market account, y is the amount invested in the Equity Index account, and z is the amount invested in the Social Choice account. The corresponding augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30{,}000 \\ 1 & -2 & 0 & 0 \\ 0.0586 & 0.0729 & 0.0755 & 2100 \end{array}\right]$$

Using technology, we reduce the matrix to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 9065.93 \\ 0 & 1 & 0 & 4532.97 \\ 0 & 0 & 1 & 16{,}401.10 \end{array}\right]$$

The reduced matrix corresponds with the system of equations

$$\begin{aligned} 1x + 0y + 0z &= 9065.93 \\ 0x + 1y + 0z &= 4532.97 \\ 0x + 0y + 1z &= 16{,}401.10 \end{aligned}$$

which simplifies to

$$\begin{aligned} x &= 9065.93 \\ y &= 4532.97 \\ z &= 16{,}401.10 \end{aligned}$$

These equations show that the investor should invest \$9065.93 in the Bond Market account, \$4532.97 in the Equity Index account, and \$16,401.10 in the Social Choice account.

EXAMPLE 2 ■ Creating a System of Equations and Solving It with Matrices

A college student works three jobs. The first job pays \$7 per hour, the second job pays \$8 per hour, and the last job pays \$10 per hour. Between the three jobs, the student works 30 hours a week. The student enjoys the first job (since it is related to his field of study) and hates the third job even though it pays the most. The student needs to earn \$250 a week. How many hours should the student work in each job?

Solution We begin by defining variables to represent the number of hours worked in each job. We let f represent the number of hours spent working in the first job, s represent the number of hours spent working the second job, and t represent the number of hours working the third job.

Since the student works 30 hours a week, we know that

$$f + s + t = 30$$

The total amount of pay for a job is determined by multiplying the pay rate by the number of hours worked. Since the student must earn \$250, we have

$$7f + 8s + 10t = 250$$

Using each equation as a row in the augmented matrix, we rewrite the system of equations as

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 30 \\ 7 & 8 & 10 & 250 \end{array}\right]$$

Using technology, we reduce the augmented matrix to

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & -10 \\ 0 & 1 & 3 & 40 \end{array}\right]$$

We then rewrite the matrix as a system of equations.

$$\begin{aligned} 1f + 0s - 2t &= -10 \\ 0f + 1s + 3t &= 40 \end{aligned}$$

This is not too meaningful as written, but if we rewrite each equation, we can make sense of it.

$$\begin{aligned} f &= 2t - 10 \\ s &= 40 - 3t \end{aligned}$$

The first equation tells us the number of hours spent in the first job is 10 hours less than twice the number of hours worked in the third job. The second equation tells us the number of hours spent in the second job is 40 hours minus three times the number of hours worked in the third job.

Since the student hates the third job, we want t to be as small as possible while keeping f and s nonnegative. Let's try $t = 5$ hours.

$$f = 2(5) - 10 = 0$$
$$s = -3(5) + 40 = 25$$

One solution to the system of equations is to work 0 hours at the first job, 25 hours at the second job, and 5 hours at the third job. (In fact, the student cannot work fewer than 5 hours at the third job because it would make f a negative number of hours.) Let's double-check to make sure that this adds up to 30 hours and will earn the required \$250.

$$0 + 25 + 5 = 30$$
$$7(0) + 8(25) + 10(5) = 250$$

This solution allows the student to make the needed \$250 while working the least amount of hours possible at the third job. It also shows the student doesn't need the first job—the one he likes most.

Dependent Systems of Equations

Although we chose exactly one solution in Example 2, the system of equations actually has infinitely many solutions. Recall from Section 2.4 that a system of equations with infinitely many solutions is said to be **dependent**. To find a few more solutions for the system from Example 2,

$$f = 2t - 10$$
$$s = 40 - 3t$$

we construct Table 8.2 with different values of t and calculate f and s.

Table 8.2

Hours in Third Job t	Hours in First Job $f = 2t - 10$	Hours in Second Job $s = 40 - 3t$
6	2	22
8	6	16
10	10	10
12	14	4

Any of these combinations of hours will result in 30 hours of work and \$250 in earnings.

To ensure all variables are nonnegative, the following conditions must be met.

$$\begin{aligned} 2t - 10 &\geq 0 \\ 2t &\geq 10 \\ t &\geq 5 \end{aligned} \qquad \begin{aligned} 40 - 3t &\geq 0 \\ 40 &\geq 3t \\ 13\tfrac{1}{3} &\geq t \end{aligned}$$

So any value of t between 5 and $13\frac{1}{3}$ hours can be used to calculate a solution that will make sense in the context of our problem.

EXAMPLE 3 ■ Using Matrices to Solve a Dependent System of Equations

According to the product packaging, Trader Joe's Semi Sweet Chocolate Chips contain 4 grams of fat, 1 gram of protein, and 1 gram of fiber in each 15-gram serving. (One tablespoon of chocolate chips weighs about 15 grams.) A 30-gram serving of shelled walnuts contains 20 grams of fat, 5 grams of protein, and 2 grams of fiber. A 30-gram serving of Diamond Premium Almonds contains 15 grams of fat, 6 grams of protein, and 4 grams of fiber. (One-fourth cup of walnuts or almonds weighs 30 grams.)

A hiker plans to make a trail mix with chocolate chips, walnuts, and almonds. She wants 150 grams of the mix and wants the mix to contain 16 grams of fiber. How many servings of chocolate chips, walnuts, and almonds should she use to make the mix?

Solution We first define variables, letting c be the number of servings of chocolate chips, w be the number of servings of walnuts, and a be the number of servings of almonds.

The first equation of the system will be related to the weight of the mix. Recall that a serving of chocolate weighs 15 grams and a serving of nuts weighs 30 grams. We need 150 grams total in the mix.

$$\text{Grams equation: } 15c + 30w + 30a = 150$$

We readily notice each coefficient is a multiple of 15, so we simplify the equation by dividing each side by 15.

$$\text{Grams equation: } c + 2w + 2a = 10$$

The second equation will be related to the fiber content in the mix. Recall that a serving of chocolate has 1 gram of fiber, a serving of walnuts has 2 grams of fiber, and a serving of almonds has 4 grams of fiber. We need 16 grams of fiber in the mix.

$$\text{Fiber equation: } 1c + 2w + 4a = 16$$

We consolidate these equations into a system of equations and write the augmented matrix.

$$\begin{aligned} c + 2w + 2a &= 10 \\ 1c + 2w + 4a &= 16 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 1 & 2 & 4 & 16 \end{array}\right]$$

Using technology, we reduce the matrix to

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

We then write the resultant system of equations

$$\begin{aligned} c + 2w + 0a &= 4 \\ 0c + 0w + 1a &= 3 \end{aligned}$$

which simplifies to

$$\begin{aligned} c + 2w &= 4 \\ a &= 3 \end{aligned}$$

The mix requires 3 servings of almonds but we have flexibility in the amount of chocolate chips and walnuts to include. We can rewrite the equation relating these two quantities as $c = -2w + 4$ and construct Table 8.3 to show some of the possible options for chocolate and walnuts.

Table 8.3

Servings of Walnuts w	Servings of Chocolate Chips $c = -2w + 4$
0	4
0.5	3
1	2
2	0

Inconsistent Systems of Equations

Not every system of equations has a solution. Recall from Section 2.4 that a system of equations with no solution is said to be **inconsistent**. When a system of equations has no solution, a contradiction will occur in the augmented matrix. This is illustrated in Example 4.

EXAMPLE 4 ■ Using Matrices with an Inconsistent System of Equations

Continuing with the trail mix scenario in Example 3, we add an additional constraint. We want the trail mix to contain exactly 2 servings of almonds. How many servings of chocolate chips, walnuts, and almonds should we include in the mix?

Solution The new constraint is given by

$$\text{Almond equation: } 0c + 0w + 1a = 2$$

We add this equation to the other two equations from Example 3.

$$\begin{aligned} c + 2w + 2a &= 10 \\ 1c + 2w + 4a &= 16 \\ 0c + 0w + 1a &= 2 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & 2 & 10 \\ 1 & 2 & 4 & 16 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Using technology, we reduce the matrix to

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

This is equivalent to the following system of equations.

$$\begin{aligned} c + 2w &= 0 \\ a &= 0 \\ 0 &= 1 \end{aligned}$$

Notice the bottom equation states that 0 is the same as 1, which is a false statement. As a result, we know the system of equations does not have a solution. In other words, if we use exactly 2 servings of almonds, there is no way to create a 150-gram trail mix that contains exactly 16 grams of fiber.

EXAMPLE 5 ■ Using a System of Equations in a Real-World Context

A chef has been commissioned to create a party mix containing pretzels, bagel chips, and Chex® cereal for a corporate gathering of 180 people. He estimates that on average each person will consume 1/2 cup of the party mix. The Original Chex Party Mix recipe calls for 9 cups of Chex cereal, 1 cup of pretzels, and 1 cup of bagel chips. (*Source:* www.chex.com) However, the chef plans to modify the recipe so that there are half as many bagel chips as pretzels and three times as much cereal as pretzels. How many cups of each ingredient will be needed to make the party mix?

Solution Let p be the number of cups of pretzels, b be the number of cups of bagel chips, and c be the number of cups of Chex cereal. Since each of the 180 people is expected to consume 1/2 cup of party mix, a total of 90 cups of the party mix is needed. Therefore,

$$p + b + c = 90$$

Since there are half as many bagel chips as pretzels, we have

$$b = 0.5p$$
$$2b = p$$
$$-p + 2b = 0$$

Since there is three times as much cereal as pretzels, we obtain

$$c = 3p$$
$$-3p + c = 0$$

No other constraints are given, so the system of equations and corresponding matrix are

$$\begin{aligned} p + b + c &= 90 \\ -p + 2b \quad &= 0 \\ -3p \quad + c &= 0 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 1 & 1 & 90 \\ -1 & 2 & 0 & 0 \\ -3 & 0 & 1 & 0 \end{array}\right]$$

Using technology, we reduce the matrix to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 10 \\ 0 & 0 & 1 & 60 \end{array}\right]$$

This corresponds with the system

$$p = 20$$
$$b = 10$$
$$c = 60$$

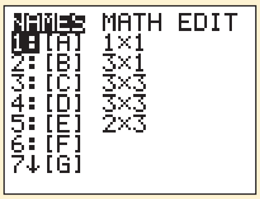
The chef should use 20 cups of pretzels, 10 cups of bagel chips, and 60 cups of cereal.

SUMMARY

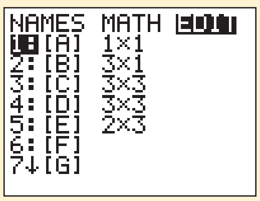
In this section you learned basic matrix notation and saw how to use matrices to represent and solve systems of linear equations. You learned how to formulate linear systems from real-world data and use the solutions to these systems to make decisions. You also learned how to use technology to simplify an augmented matrix to reduced row echelon form.

TECHNOLOGY TIP ■ ENTERING A MATRIX

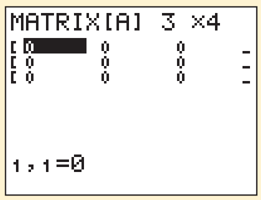
1. Activate the Matrix Menu by pressing [2nd] [x^{-1}]. You may or may not have some matrices displayed in the name list.



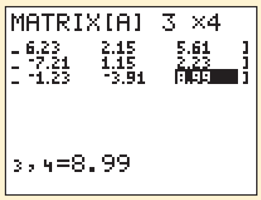
2. Create the augmented matrix A by moving the cursor to **EDIT** and pressing [ENTER].



3. Enter the dimensions of the matrix. Type the number of rows and press [ENTER]. Then type the number of columns and press [ENTER]. The example matrix is a 3 × 4 augmented matrix.



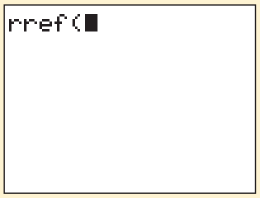
4. Enter the individual values of the matrix. Use the arrow keys to move from one entry to another. When you have entered all of the values, press [2nd] [MODE] to exit the Matrix Editor.



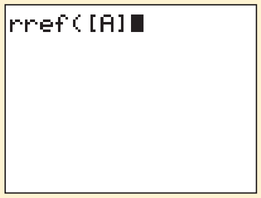
TECHNOLOGY TIP ■ FINDING THE REDUCED ROW ECHELON FORM OF A MATRIX

1. Activate the Matrix Math Menu by pressing 2nd x^{-1} and moving the cursor to the `MATH` menu item.

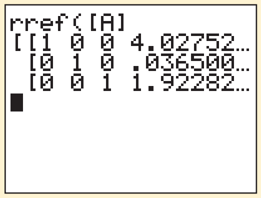
2. Select the `rref` operation by scrolling down the list to item `B:`. Press ENTER to place the operation on the home screen. (The `rref(` function will convert a matrix to reduced row echelon form.)



3. Select the matrix A from the Matrix Names Menu by pressing 2nd x^{-1}, moving the cursor to matrix A, and pressing ENTER. This places the matrix A on the home screen.



4. Calculate the reduced row echelon form of the matrix by pressing ENTER.



8.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, determine the dimensions of the matrix A.

1. $A = \begin{bmatrix} 19 & 5 & 1 \\ 2 & 8 & 18 \end{bmatrix}$

2. $A = \begin{bmatrix} -10 \\ 42 \\ 17 \end{bmatrix}$

3. $A = \begin{bmatrix} 14 & 0 & -5 \end{bmatrix}$

4. $A = \begin{bmatrix} -4 & 5 & 8 & 11 \\ 11 & 5 & 4 & -6 \end{bmatrix}$

5. $A = \begin{bmatrix} 17 \end{bmatrix}$

In Exercises 6–10, write the augmented matrix as a system of linear equations using the variables x, y, and z as appropriate.

6. $\left[\begin{array}{ccc|c} 2 & 1 & 0 & 11 \\ 1 & 0 & 4 & 0 \end{array}\right]$

7. $\left[\begin{array}{cc|c} 1 & 3 & 12 \\ 4 & 1 & 7 \\ -1 & 6 & 9 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & 2 & 19 \\ 6 & 0 & 3 & 22 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 3 & -1 & 9 & 13 \\ -3 & 1 & 0 & -31 \end{array}\right]$

10. $\left[\begin{array}{ccc|c} 5 & 2 & 4 & 11 \\ -1 & 2 & 2 & 3 \\ 5 & 10 & 3 & 18 \end{array}\right]$

In Exercises 11–20,

a. Rewrite the system of equations as an augmented matrix.

b. Use technology to simplify the matrix to reduced row echelon form.

c. Identify the solution(s) to the system of equations. If the system is inconsistent or dependent, so state.

11. $\begin{aligned} 2x + 5y &= 2 \\ 3x - 5y &= 3 \end{aligned}$

12. $\begin{aligned} 6x + 2y &= 10 \\ -x - 2y &= -5 \end{aligned}$

13. $\begin{aligned} -2x + 6y + 4z &= 10 \\ 4x - 12y + 2z &= -20 \\ 3x + 4y - z &= 11 \end{aligned}$

14. $\begin{aligned} 10x - y + z &= -7 \\ 9x - 2y + 4z &= 7 \\ x + 2y - 4z &= -17 \end{aligned}$

15. $\begin{aligned} 3x - 2y + z &= 6 \\ 11x - 20y - z &= 0 \\ y + z &= 3 \end{aligned}$

16. $\begin{aligned} 9x - 6y \quad &= 0 \\ 4x + 5y \quad &= 23 \\ x \quad - z &= 6 \end{aligned}$

17. $\begin{aligned} x - y &= -5 \\ 9x + y &= 25 \\ 29x + y &= 65 \end{aligned}$

18. $\begin{aligned} x - 2y &= -7 \\ 6x + 5y &= 94 \\ 10x + 3y &= 114 \end{aligned}$

19. $$\begin{aligned} 2x - 4y &= 16 \\ 9x + y &= -4 \\ -3x + 6y &= -24 \end{aligned}$$

20. $$\begin{aligned} x - y &= 3 \\ 6x + 7y &= 44 \\ 6x - 7y &= 16 \end{aligned}$$

SHOW YOU KNOW

21. Explain the conceptual meaning of the solution to a system of equations.

22. How are systems of equations and augmented matrices related?

23. What row operations result in an equivalent system of equations?

24. Explain what is meant by an inconsistent system of equations.

25. Suppose a friend missed class and asks you what is meant by "reduced row echelon form." How do you respond?

MAKE IT REAL

In Exercises 26–35, set up and solve the system of linear equations using matrices.

26. Investment Choices The table shows the average annual rate of return of a variety of TIAA-CREF investment accounts over a 10-year period.

CREF Variable Annuity Accounts	10-Year Average
Bond Market	5.86%
Equity Index	7.29%
Global Equities	6.54%
Growth	3.02%
Inflation-Linked Bond	6.33%
Money Market	3.68%
Social Choice	6.97%
Stock	7.55%

Source: www.tiaa-cref.com; as of 6/30/07

An investor chooses to invest \$3000 in the Inflation-Linked Bond, Global Equities, and Stock accounts. He assumes he will be able to get a return equal to the 10-year average and wants the total return on his investment to be 7%. He decides to invest three times as much money in the Inflation-Linked Bond account as in the Global Equities account. How much money should he invest in each account? (For ease of computation, round each percentage to the nearest whole number percent, e.g., 7.55% = 8%.)

27. Investment Choices An investor chooses to invest \$5000 in the Global Equities, Money Market, and Social Choice accounts shown in Exercise 26. She wants to put five times as much money in the Money Market account as in the Global Equities account. She assumes she will be able to get a return equal to the 10-year average and wants the total return on her investment to be 6%.

How much money should she invest in each account? (For ease of computation, round each percentage to the nearest whole number percent.)

28. Resource Allocation: Sandwiches A plain hamburger requires one ground beef patty and a bun. A cheeseburger requires one ground beef patty, one slice of cheese, and a bun. A double cheeseburger requires two ground beef patties, two slices of cheese, and a bun.

Frozen hamburger patties are typically sold in packs of 12; hamburger buns, in packs of 8; and cheese slices, in packs of 24.

A family is in charge of providing burgers for a neighborhood block party. They have purchased 13 packs of buns, 11 packs of hamburger patties, and 3 packs of cheese slices. How many of each type of sandwich should they make if they want to use up all of the buns, patties, and cheese slices?

29. Pet Nutrition: Food Cost PETsMART sold the following varieties of dog food in June 2003. The price shown is for an 8-pound bag.

Pro Plan Adult Chicken & Rice Formula
25% protein, 3% fiber, \$7.99

Pro Plan Adult Lamb & Rice Formula
28% protein, 3% fiber, \$7.99

Pro Plan Adult Turkey & Barley Formula
26% protein, 3% fiber, \$8.49

Source: www.petsmart.com

A dog breeder wants to make 120 pounds of a mix containing 27% protein and 3% fiber. Give two possible answers for how many 8-pound bags of each dog food variety the breeder should buy. For each answer, calculate the total cost of the dog food.

30. Pet Nutrition: Food Cost PETsMART sold the following varieties of dog food in June 2003.

Nature's Recipe Venison Meal & Rice Canine
20% protein, \$21.99 per 20-pound bag

Nutro Max Natural Dog Food
27% protein, \$12.99 per 17.5-pound bag

PETsMART Premier Oven Baked Lamb Recipe
25% protein, \$22.99 per 30-pound bag

Source: www.petsmart.com

A dog breeder wants to make 300 pounds of a mix containing 22% protein. Give two possible answers for how many bags of each dog food variety the breeder should buy. For each answer, calculate the total cost of the dog food. (*Hint:* Note each bag is a different weight. Fractions of bags may not be purchased.)

31. Utilization of Ingredients A custard recipe calls for 3 eggs and 2.5 cups of milk. A vanilla pudding recipe calls for 2 eggs and 2 cups of milk. A bread pudding recipe calls for 2 eggs, 2 cups of milk, and 8 slices of bread.

A stocked kitchen contains 18 eggs, 1 gallon of milk, and 24 slices of bread. How many batches of each recipe should a chef make to use up all of the ingredients?

32. First Aid Kit Supplies Safetymax.com sells first aid and emergency preparedness supplies to businesses. A company that assembles first aid kits for consumers purchases 3500 1-inch by 3-inch plastic adhesive bandages, 1800 alcohol wipes, and 220 tubes of antibiotic ointment from Safetymax.com.

The company assembles compact, standard, and deluxe first aid kits for sale to consumers. A *compact* first aid kit contains 20 plastic adhesive bandages, 8 alcohol wipes, and 1 tube of antibiotic ointment. A *standard* first aid kit contains 40 plastic adhesive bandages, 20 alcohol wipes, and 2 tubes of antibiotic ointment. A *deluxe* first aid kit contains 50 plastic adhesive bandages, 28 alcohol wipes, and 4 tubes of antibiotic ointment.

How many of each type of kit should the company assemble to use up all of the bandages, wipes, and antibiotics ordered?

33. Concert Ticket Sales On the weekend of July 23–25, 2004, the House of Blues Sunset Strip in Hollywood, California, hosted three concerts: Saves the Day, Jet, and BoDeans. Saves the Day tickets cost \$15, Jet tickets cost \$20, and BoDeans tickets cost \$22. (***Source:*** **www.ticketmaster.com**)

If concert planners expected that a total of 1200 tickets would be sold over the weekend, how many tickets needed to be sold for each concert to bring in \$23,500? Find three different solutions.

34. Concert Ticket Sales On July 7, 2004, Shania Twain was scheduled to perform at the TD Waterhouse Centre in Orlando, Florida. The center offered 18,039 seats for the concert in three seating classifications: floor, lower, and upper. Based upon their location, tickets were offered at three different prices: \$80, \$65, and \$45. (***Source:*** **www.ticketmaster.com**) Suppose the average price of a floor ticket was \$80, the average price of a lower ticket was \$65, and the average price of an upper ticket was \$45. If concert planners expected a total of 12,000 tickets would be sold, including 5 times as many tickets on the lower level as on the floor, how many of each type of ticket would have to be sold to reach \$675,000 in ticket revenue?

35. Concert Ticket Sales On June 24, 2004, Madonna was scheduled to perform at Madison Square Gardens in New York, New York. Tickets were offered at four different prices: \$49.50, \$94.50, \$154.50, and \$304.50. (***Source:*** **www.ticketmaster.com**) If concert planners expected a total of 25,000 tickets would be sold, including 10 times as many of the least expensive ticket as of the most expensive ticket and one-third as many \$154.50 tickets as of \$94.50 tickets, how many of each type of ticket would need to be sold to earn \$2,130,000 in revenue?

■ STRETCH YOUR MIND

Exercises 36–40 are intended to challenge your understanding of linear system applications.

36. Amusement Park Rides Rides at an amusement park require 3, 4, 5, or 6 tickets. A family purchases 75 tickets and receives 2 tickets from a park guest who had leftover tickets. The family wants to use all their tickets and wants the sum of 5- and 6-ticket rides to be twice as much as the sum of 3- and 4-ticket rides. What is the largest possible number of 6-ticket rides that the family can go on subject to these constraints?

37. Assortment of Coins A coin purse contains 64 coins including pennies, nickels, dimes, and quarters. The total value of the coins is \$4.94. There is the same number of nickels as dimes. How many of each type of coin is in the bag?

38. Investments The following table shows the average annual rate of return of a variety of TIAA-CREF investment accounts over a 10-year period.

CREF Variable Annuity Accounts	10-Year Average
Bond Market	5.86%
Equity Index	7.29%
Global Equities	6.54%
Growth	3.02%
Inflation-Linked Bond	6.33%
Money Market	3.68%
Social Choice	6.97%
Stock	7.55%

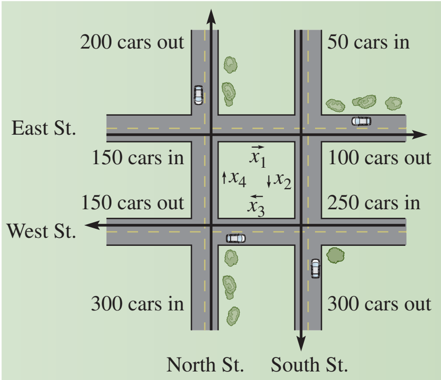
Source: **www.tiaa-cref.com; as of 6/30/07**

An investor wants to earn a 7% annual return on a \$68,500 investment. The investor expects the annual return on each account will be equal to the 10-year rate, rounded to the nearest whole number percentage.

The investor wants to invest the same amount of money in the Bond Market account as in the Global Equities account and twice as much in the Social Choice account as in the Growth account. How much money should the investor place in each account?

39. Traffic Flow

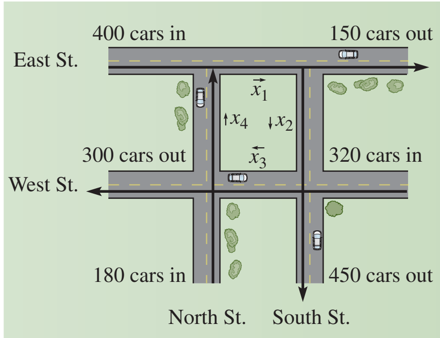
The figure shows the flow of traffic at four city intersections.



For each intersection, the number of cars entering the intersection must equal the number of cars leaving the intersection. For example, the number of cars entering the intersection of North and West is $x_3 + 300$. The number of cars leaving the intersection is $x_4 + 150$. Therefore, $x_3 + 300 = x_4 + 150$.

Find two separate sets of values x_1, x_2, x_3, and x_4 that work in the traffic-flow system.

40. Traffic Flow Repeat Exercise 39 for the following traffic-flow diagram. Then explain who could benefit from this type of traffic-flow analysis.



SECTION 8.2 Matrix Operations and Applications

LEARNING OBJECTIVES

- Identify properties of matrix addition and scalar multiplication
- Use matrix addition and scalar multiplication in real-world situations

GETTING STARTED

Many schools use a salary schedule to determine how much to pay their teachers. Typically, as teachers gain experience and participate in professional development activities, they are advanced on the salary schedule. However, due to inflation (rising prices), a teacher's buying power may be reduced even though his or her salary is increased. A cost of living allowance (COLA) is designed to offset the effects of inflation on a person's salary.

In this section we further develop the concept of matrices by demonstrating matrix addition and scalar multiplication. We also demonstrate how to use these techniques to analyze real-world financial situations such as COLAs for teachers.

■ Matrix Addition

We add two matrices of the same dimension by adding their corresponding entries. For example, to add the 2×2 matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$ to get matrix C, we have

$$\begin{aligned} C &= A + B \\ &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1+4 & 2+(-2) \\ 3+(-3) & 4+1 \end{bmatrix} && \textbf{Add corresponding entries.} \\ &= \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \end{aligned}$$

Matrix Addition

An $m \times n$ matrix A and an $m \times n$ matrix B can be added together to form a new $m \times n$ matrix, C. The value of the entry in the ith row and jth column of C is $c_{ij} = a_{ij} + b_{ij}$.

If A and B are not of the same dimension, matrix addition is undefined.

EXAMPLE 1 ■ **Calculating the Sum of Two Matrices**

Calculate $C = A + B$ given $A = \begin{bmatrix} 2 & 0 \\ -1 & 6 \\ 4 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 9 & 2 \\ -7 & -6 \\ 8 & -1 \end{bmatrix}$.

Solution

$$\begin{aligned} C &= A + B \\ &= \begin{bmatrix} 2 & 0 \\ -1 & 6 \\ 4 & 5 \end{bmatrix} + \begin{bmatrix} 9 & 2 \\ -7 & -6 \\ 8 & -1 \end{bmatrix} \\ &= \begin{bmatrix} 2+9 & 0+2 \\ -1+(-7) & 6+(-6) \\ 4+8 & 5+(-1) \end{bmatrix} \\ &= \begin{bmatrix} 11 & 2 \\ -8 & 0 \\ 12 & 4 \end{bmatrix} \end{aligned}$$

EXAMPLE 2 ■ **Using Matrix Addition in a Real-World Context**

According to the package labeling, a 0.5-ounce serving of M&Ms Mini Baking Bits contains 3.5 grams of fat, 9 grams of carbohydrates, and 1 gram of protein. A 1-ounce serving of Walmart Party Size Party Peanuts contains 14 grams of fat, 5 grams of carbohydrates, and 8 grams of protein. A 1/2-cup serving of Diamond Sliced Almonds contains 15 grams of fat, 6 grams of carbohydrates, and 6 grams of protein. Create a matrix for each ingredient showing the fat, carbohydrate, and protein content of the product. Then add the three matrices together and interpret the meaning of the result.

Solution The matrix for each ingredient will be of the form $B = \begin{bmatrix} \textit{fat} \\ \textit{carbohydrates} \\ \textit{protein} \end{bmatrix}$. We have

$$\text{M\&Ms: } M = \begin{bmatrix} 3.5 \\ 9 \\ 1 \end{bmatrix}$$

$$\text{Peanuts: } P = \begin{bmatrix} 14 \\ 5 \\ 8 \end{bmatrix}$$

$$\text{Almonds: } A = \begin{bmatrix} 15 \\ 6 \\ 6 \end{bmatrix}$$

We then add the three matrices.

$$M + P + A = \begin{bmatrix} 3.5 \\ 9 \\ 1 \end{bmatrix} + \begin{bmatrix} 14 \\ 5 \\ 6 \end{bmatrix} + \begin{bmatrix} 15 \\ 6 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 3.5 + 14 + 15 \\ 9 + 5 + 6 \\ 1 + 6 + 6 \end{bmatrix}$$

$$= \begin{bmatrix} 32.5 \\ 20 \\ 13 \end{bmatrix}$$

This means that if we create a mix from one serving of each ingredient, the mix will contain 32.5 grams of fat, 20 grams of carbohydrates, and 13 grams of protein.

Scalar Multiplication

There are two types of multiplication used with matrices: **scalar multiplication** and **matrix multiplication**. We will address scalar multiplication here and matrix multiplication in the next section. The term **scalar** means *constant* or *number*. Scalar multiplication "scales" the entries of a matrix by making them larger or smaller by a given factor.

For example, if $A = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 4 & 6 \\ 5 & 7 & 9 \end{bmatrix}$ then

$$5A = 5\begin{bmatrix} 1 & 3 & 0 \\ -2 & 4 & 6 \\ 5 & 7 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \cdot 1 & 5 \cdot 3 & 5 \cdot 0 \\ 5 \cdot (-2) & 5 \cdot 4 & 5 \cdot 6 \\ 5 \cdot 5 & 5 \cdot 7 & 5 \cdot 9 \end{bmatrix}$$

Multiply each entry by the scalar 5.

$$= \begin{bmatrix} 5 & 15 & 0 \\ -10 & 20 & 30 \\ 25 & 35 & 45 \end{bmatrix}$$

In this example, the scalar 5 increased the magnitude of each entry by a factor of 5.

Scalar Multiplication

An $m \times n$ matrix A and a real number k can be multiplied together to form a new $m \times n$ matrix, C. The value of the entry in the ith row and jth column of C is $c_{ij} = k \cdot a_{ij}$.

EXAMPLE 3 ■ Multiplying by a Scalar

Calculate $C = -\frac{1}{2}A$ given $A = \begin{bmatrix} 2 & 0 \\ -4 & 6 \\ 1 & 8 \end{bmatrix}$.

Solution

$$\begin{aligned} C &= -\tfrac{1}{2}A \\ &= -\tfrac{1}{2}\begin{bmatrix} 2 & 0 \\ -4 & 6 \\ 1 & 8 \end{bmatrix} \\ &= \begin{bmatrix} -\frac{1}{2}\cdot 2 & -\frac{1}{2}\cdot 0 \\ -\frac{1}{2}\cdot(-4) & -\frac{1}{2}\cdot 6 \\ -\frac{1}{2}\cdot 1 & -\frac{1}{2}\cdot 8 \end{bmatrix} \\ &= \begin{bmatrix} -1 & 0 \\ 2 & -3 \\ -\frac{1}{2} & -4 \end{bmatrix} \end{aligned}$$

EXAMPLE 4 ■ Using Scalar Multiplication in a Real-World Context

Table 8.4 shows the base salary schedule for educators at University of Nevada–Las Vegas (UNLV) for 2005–2006.

Table 8.4

Rank	Minimum	Maximum
IV: Professor	59,683	123,958
III: Associate	43,681	90,722
II: Assistant	37,103	77,061
I: Instructor	30,584	63,520

Source: hr.unlv.edu

For the 2006–2007 school year, faculty received a 4% cost of living adjustment. Create a new salary schedule for the 2006–2007 school year for UNLV faculty.

Solution We can solve the problem by writing the table as a matrix and multiplying by a scalar. Since salaries are to increase by 4%, the 2006–2007 salaries will be 104% of the 2005–2006 salaries.

$$\begin{aligned} 104\% &= \frac{104}{100} \\ &= 1.04 \end{aligned}$$

Thus we will multiply the matrix by 1.04 to increase each entry by 4%.

$$S = \begin{bmatrix} 59{,}683 & 123{,}958 \\ 43{,}681 & 90{,}722 \\ 37{,}103 & 77{,}061 \\ 30{,}584 & 63{,}520 \end{bmatrix}$$

$$\begin{aligned} 1.04S &= 1.04\begin{bmatrix} 59{,}683 & 123{,}958 \\ 43{,}681 & 90{,}722 \\ 37{,}103 & 77{,}061 \\ 30{,}584 & 63{,}520 \end{bmatrix} \\ &\approx \begin{bmatrix} 62{,}070 & 128{,}916 \\ 45{,}428 & 94{,}351 \\ 38{,}587 & 80{,}143 \\ 31{,}807 & 66{,}061 \end{bmatrix} \end{aligned}$$

The new matrix represents the calculated 2006–2007 pay scale for UNLV faculty, as shown in Table 8.5.

Table 8.5

Rank	Minimum	Maximum
IV: Professor	62,070	128,916
III: Associate	45,428	94,351
II: Assistant	38,587	80,143
I: Instructor	31,807	66,061

To aid in the application of the concepts we have discussed in this section, we provide a summary of key properties for the processes.

Matrix Addition and Scalar Multiplication Properties

Let A, B, and C be $m \times n$ matrices and let c and k be real numbers. Let O be the $m \times n$ **zero matrix** (a matrix with entries of all zeros). The following properties hold.

Matrix Addition Properties

1. Additive Associative $\quad A + (B + C) = (A + B) + C$
2. Additive Commutative $\quad A + B = B + A$
3. Additive Identity $\quad A + O = O + A = A$
4. Additive Inverse $\quad (-A) + A = A + (-A) = 0$

Scalar Multiplication Properties

5. Distributive $\quad c(A + B) = cA + cB$
6. Distributive $\quad (c + k)A = cA + kA$
7. Multiplicative Associative $\quad c(kA) = (ck)A$
8. Scalar Unit $\quad 1A = A$
9. Scalar Zero $\quad 0A = O$

No matrix subtraction properties are listed because we view matrix subtraction as a combination of scalar multiplication and matrix addition. In other words, the matrix expression $A - B$ is equivalent to $A + (-1)B$. As shown in Example 5, in practice we typically simplify the matrix expression $A - B$ by subtracting the entries of B from the corresponding entries of A.

EXAMPLE 5 ■ Calculating the Difference of Two Matrices

Let $A = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -5 & 0 \\ 9 & 11 & 20 \end{bmatrix}$ and $B = \begin{bmatrix} -4 & 7 & 3 \\ 10 & -9 & 10 \\ -2 & 4 & 8 \end{bmatrix}$. Show $A - B$ is equivalent to $A + (-1)B$.

Solution

$$A - B = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -5 & 0 \\ 9 & 11 & 20 \end{bmatrix} - \begin{bmatrix} -4 & 7 & 3 \\ 10 & -9 & 10 \\ -2 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4-(-4) & -1-7 & 2-3 \\ 3-10 & -5-(-9) & 0-10 \\ 9-(-2) & 11-4 & 20-8 \end{bmatrix}$$

Subtract the entries of B from the corresponding entries of A.

$$= \begin{bmatrix} 8 & -8 & -1 \\ -7 & 4 & -10 \\ 11 & 7 & 12 \end{bmatrix}$$

$$A + (-1)B = \begin{bmatrix} 4 & -1 & 2 \\ 3 & -5 & 0 \\ 9 & 11 & 20 \end{bmatrix} + (-1)\begin{bmatrix} -4 & 7 & 3 \\ 10 & -9 & 10 \\ -2 & 4 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 & 2 \\ 3 & -5 & 0 \\ 9 & 11 & 20 \end{bmatrix} + \begin{bmatrix} (-1)(-4) & (-1)7 & (-1)3 \\ (-1)10 & (-1)(-9) & (-1)10 \\ (-1)(-2) & (-1)4 & (-1)8 \end{bmatrix}$$

Multiply the entries of B by the scalar -1.

$$= \begin{bmatrix} 4 & -1 & 2 \\ 3 & -5 & 0 \\ 9 & 11 & 20 \end{bmatrix} + \begin{bmatrix} 4 & -7 & -3 \\ -10 & 9 & -10 \\ 2 & -4 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -8 & -1 \\ -7 & 4 & -10 \\ 11 & 7 & 12 \end{bmatrix}$$

Matrix expressions may combine one or more matrix operations. Example 6 includes matrix addition and scalar multiplication with three different matrices including the zero matrix.

EXAMPLE 6 ■ Solving a Matrix Algebra Problem Involving the Zero Matrix

Let $A = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -5 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 7 & 0 \\ 3 & -2 & -4 \end{bmatrix}$, and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Calculate $A - (2B + O)$.

Solution

$$A - (2B + O) = \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -5 \end{bmatrix} - \left(2 \cdot \begin{bmatrix} 5 & 7 & 0 \\ 3 & -2 & -4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -5 \end{bmatrix} + (-1)\left(\begin{bmatrix} 10 & 14 & 0 \\ 6 & -4 & -8 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

Multiply B by 2. Rewrite "−" as "+(−1)."

$$= \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -5 \end{bmatrix} + (-1)\begin{bmatrix} 10 & 14 & 0 \\ 6 & -4 & -8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 4 \\ -1 & 2 & -5 \end{bmatrix} + \begin{bmatrix} -10 & -14 & 0 \\ -6 & 4 & 8 \end{bmatrix}$$

Multiply $2B$ by -1.

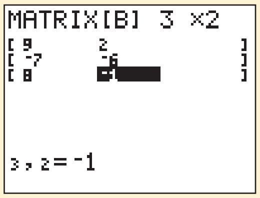
$$= \begin{bmatrix} -9 & -14 & 4 \\ -7 & 6 & 3 \end{bmatrix}$$

SUMMARY

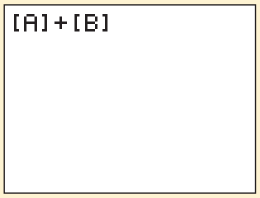
In this section you learned how to do matrix addition and scalar multiplication. You saw how these techniques could be used in life and business to analyze real-world situations.

TECHNOLOGY TIP ■ ADDING TWO MATRICES

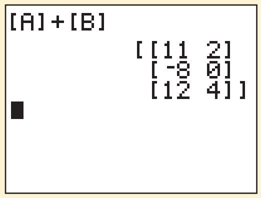
1. Enter matrix A and matrix B into the calculator using the Matrix Menu.



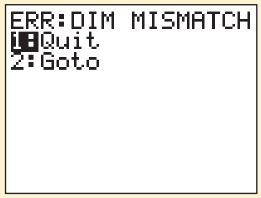
2. Use the Matrix Names Menu to place matrix A on the home screen. Press the [+] key. Then place matrix B on the home screen.



3. Press [ENTER] to display the sum of the two matrices.

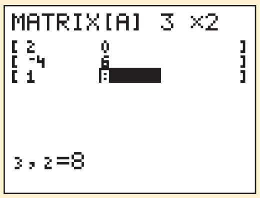


Error Alert:
If the **DIM MISMATCH** error message appears, double-check that you have entered the matrices correctly. If you have, this message tells you that the matrix addition is not possible because the two matrices are of different dimensions.

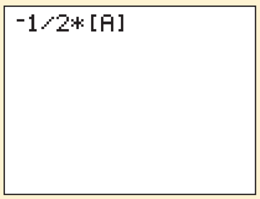


TECHNOLOGY TIP ■ SCALAR MULTIPLICATION

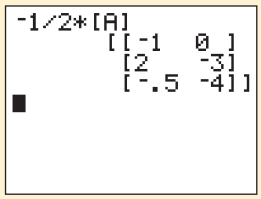
1. Enter matrix A into the calculator using the Matrix Menu.



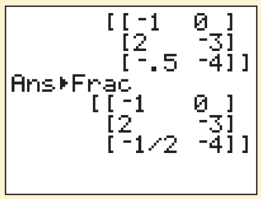
2. Type in the scalar on the home screen, press the [×] key, and then use the Matrix Names Menu to place matrix A on the home screen.



3. Press [ENTER] to display the product of the scalar and the matrix.



4. If you would like to convert decimal entries to fractions, press [MATH], select **FRAC**, then press [ENTER].



8.2 EXERCISES

SKILLS AND CONCEPTS

In Exercise 1–10, perform the indicated matrix operation, if possible, given the following matrices. Solve these problems without technology.

$$A = \begin{bmatrix} 1 & 6 \\ 3 & 4 \\ 5 & 2 \end{bmatrix}, B = \begin{bmatrix} -5 & 0 \\ 7 & 8 \\ -9 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} 2 & 3 & 0 \\ -1 & 4 & 1 \\ 5 & -2 & 0 \end{bmatrix}, D = \begin{bmatrix} 1 & -3 & 4 \\ 0 & 5 & 7 \\ 9 & 8 & 2 \end{bmatrix}$$

1. $A + B$
2. $A - B$
3. $B + C$
4. $C - D$
5. $D + C$
6. $2A$
7. $3B$
8. $2A + 3B$
9. $-2C + 2D$
10. $2A + 2C$

In Exercises 11–20, use technology to simplify the matrix expressions given the following matrices.

$$A = \begin{bmatrix} 1.2 & 6.3 & 0.4 \\ -9.1 & 4.2 & 1.7 \\ 0.9 & -2.0 & 0.3 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.4 & -0.3 & 0.4 \\ -2.8 & 5.5 & 7.1 \\ 9.2 & 8.6 & 2.0 \end{bmatrix}$$

11. $1.2A$
12. $-2.3B$
13. $1.2A - 2.3B$
14. $4.1A + 0.1B$
15. $-1.1A + 2.9B$
16. $2.9A + 0.1B$
17. $-8.7A + 8.7B$
18. $-A - 9.2B$
19. $7.8A + 9.9B$
20. $-1.7A - 2.1B$

21. Personal Debt A couple is planning to get married. He has the following debts: a \$2700 consumer loan at Loan Shark Larry's, a \$26,500 car loan, and an \$8200 credit card balance at Risky Bank, plus a \$2700 consumer loan at Mastercraft Jewelers (for the ring). She has the following debts: a \$1200 car loan, an \$82,500 home mortgage, and a \$200 credit card balance at Risky Bank, plus a \$250 consumer loan at Mastercraft Jewelers. Determine their combined debts of each type at the various financial institutions.

22. Food Supply Bed and breakfasts attract visitors with their intimate ambiance and delicious cuisine. Many establishments serve wedding brunches in addition to providing lodging services.

A bed and breakfast hostess is planning an upcoming wedding brunch. She plans to make 4 dozen muffins (2 dozen apple and 2 dozen blueberry) and 6 fruit crisps (3 apple and 3 blueberry). The amount of flour, sugar, and fruit required to make a single fruit crisp is shown in Table A and the amount required to make a dozen muffins is shown in Table B.

Table A

Crisp	Flour	Sugar	Fruit
apple	1/2 cup	3/4 cup	3 cups
blueberry	1/2 cup	1 cup	4 cups

Table B

Muffins	Flour	Sugar	Fruit
apple	2 cups	1/4 cup	3/4 cup
blueberry	2 cups	1/3 cup	1 cup

Use matrices to create a table that shows how much flour, sugar, and fruit will be required for the apple desserts and the blueberry desserts.

SHOW YOU KNOW

23. What is scalar multiplication?

24. Explain why two matrices must have the same dimension for them to be added together.

25. Describe the conditions under which the sum of two matrices results in a matrix consisting only of zeros.

26. Create an argument to justify the commutative property, $A + B = B + A$, for the sum of two matrices.

27. Suppose the difference, $A - B$, of two matrices is calculated. Describe the relationship between $A - B$ and the difference $B - A$.

MAKE IT REAL

In Exercises 28–41, use matrix addition and/or scalar multiplication to find the solution.

28. Faculty Salaries The 2001–2002 pay scale for the Green River Community College faculty is shown in the table. For the 2002–2003 academic year, the faculty received a 3.432% cost of living increase. Determine the 2002–2003 pay scale.

	240 Credits (dollars)	300 Credits (dollars)	360 Credits (dollars)
Level 1	35,131	38,742	42,354
Level 2	37,016	40,627	44,238
Level 3	38,900	42,511	46,122
Level 4	40,784	44,395	48,006

Source: **Green River Community College**

29. Faculty Salaries If the Green River Community College faculty in Exercise 28 gets a 3.432% raise annually, determine the 2005–2006 pay scale.

30. **Condiment Prices** On July 20, 2002, Albertsons.com advertised the following items at the indicated prices.

	Albertsons	Hunt's	Kraft
24-ounce ketchup	\$0.69	\$1.89	
18-ounce barbecue sauce		\$0.99	\$0.99
32-ounce mayonnaise	\$2.19		\$3.19

If the prices are only affected by inflation and the annual rate of inflation is 3%, determine the price of each of the items on July 20, 2006.

31. **Soda Prices** On July 20, 2002, Albertsons.com advertised the following items at the indicated prices.

	Albertsons	A&W	Henry Weinhard's
6-pack root beer	\$1.59	\$1.67	\$4.50
12-pack cream soda	\$3.18	\$3.34	\$9.00
2-liter club soda	\$0.99		

If the prices are only affected by inflation and the annual rate of inflation is 3%, determine the price of each of the items on July 20, 2006.

32. **Auto Prices** The average trade-in values of a Volkswagen New Beetle and Volkswagen Golf in July 2002 are shown in Table C. The average retail values of the two vehicles are shown in Table D.

Table C

Average Trade-In		
	Golf	New Beetle
2000 Model	\$11,000	\$11,850
2001 Model	\$11,875	\$13,175

Source: **www.nada.com**

Table D

Average Retail		
	Golf	New Beetle
2000 Model	\$13,050	\$14,000
2001 Model	\$14,025	\$15,475

Source: **www.nada.com**

Use matrices to create a table that shows the average dealer markup for each of the vehicles.

33. **Auto Prices** The average trade-in values of a Honda Civic and Honda Accord in July 2002 are shown in Table E. The average retail values of the two vehicles are shown in Table F.

Table E

Average Trade-In		
	Accord	Civic
2000 Model	\$14,800	\$8,925
2001 Model	\$16,575	\$9,850

Source: **www.nada.com**

Table F

Average Retail		
	Accord	Civic
2000 Model	\$17,100	\$10,800
2001 Model	\$18,975	\$11,825

Source: **www.nada.com**

Use matrices to create a table that shows the average dealer markup for each of the vehicles.

34. **Energy Usage** The amount of natural gas and coal energy produced and consumed in the United States is shown in the tables. Use matrix operations to create a table that shows the difference between energy production and consumption.

Energy Production		
Years Since 1960	Natural Gas (quadrillion BTUs)	Coal (quadrillion BTUs)
0	12.66	10.82
10	21.67	14.61
20	19.91	18.60
30	18.36	22.46
40	19.74	22.66

Energy Consumption		
Years Since 1960	Natural Gas (quadrillion BTUs)	Coal (quadrillion BTUs)
0	12.39	9.84
10	21.80	12.27
20	20.39	15.42
30	19.30	19.25
40	23.33	22.41

***Source: Statistical Abstract of the United States, 2001*, Table 891**

35. **Energy Policy** If you were a lobbyist for the natural gas industry, how would you use the results of Exercise 34 to persuade legislators to support further natural gas exploration?

36. **Energy Usage** The amount of nuclear electric power and coal energy produced in the United States is shown in the tables. Use matrix operations to create a table that shows the difference between energy production and consumption.

Energy Production		
Years Since 1960	Nuclear Electric Power (quadrillion BTUs)	Coal (quadrillion BTUs)
0	0.01	10.82
10	0.24	14.61
20	2.74	18.60
30	6.16	22.46
40	8.01	22.66

Energy Consumption		
Years Since 1960	Nuclear Electric Power (quadrillion BTUs)	Coal (quadrillion BTUs)
0	0.01	9.84
10	0.24	12.27
20	2.74	15.42
30	6.16	19.25
40	8.01	22.41

Source: Statistical Abstract of the United States, 2001, Table 891

37. Energy Policy An environmentalist believes the United States should only produce as much energy as it consumes. Based on the results of Exercise 36, which energy technology (coal or nuclear electric power) seems to best support the environmentalist's position? Defend your conclusion.

38. Renewable Energy Consumption The amount of renewable energy consumed in the United States in various years is shown in the tables. Use matrices to make a table showing the total amount of energy consumed between the beginning of 1997 and the end of 1999.

1997 Energy Consumption	
Renewable Energy Type	Energy Consumed (quadrillion BTUs)
conventional hydroelectric power	3.94
geothermal energy	0.33
biomass	2.98
solar energy	0.07
wind energy	0.03

1998 Energy Consumption	
Renewable Energy Type	Energy Consumed (quadrillion BTUs)
conventional hydroelectric power	3.55
geothermal energy	0.34
biomass	2.99
solar energy	0.07
wind energy	0.03

1999 Energy Consumption	
Renewable Energy Type	Energy Consumed (quadrillion BTUs)
conventional hydroelectric power	3.42
geothermal energy	0.33
biomass	3.51
solar energy	0.08
wind energy	0.04

Source: Statistical Abstract of the United States, 2001, Table 896

39. Average Energy Consumption Using the results from Exercise 38, use matrix operations to construct a table showing the average annual amount of energy consumed between the start of 1997 and the end of 1999.

40. Energy Usage Trends Using the data tables from Exercise 38 and the results from Exercise 39, construct a table showing the difference between the 1999 energy consumption and the average annual energy consumption. What conclusions can you draw from your result?

41. Organized Physical Activity The tables show the percentage of high school students involved in physical education classes. Based on the information given, can you determine the percentage of ninth-graders who exercise 20 or more minutes per class? Justify your answer.

Males Enrolled in a Physical Education Class		
Grade	Enrolled in a P.E. Class (percent)	Exercised 20 Minutes or More per Class (percent)
9	60.7	82.1
10	82.3	84.4
11	65.3	79.4
12	44.6	82.0

Females Enrolled in a Physical Education Class		
Grade	Enrolled in a P.E. Class (percent)	Exercised 20 Minutes or More per Class (percent)
9	51.5	69.6
10	75.6	72.5
11	56.6	70.2
12	36.8	68.0

Source: Statistical Abstract of the United States, 2001, Table 1246

■ STRETCH YOUR MIND

Exercises 42–45 are intended to challenge your ability to apply matrix addition and scalar multiplication properties. For each exercise, use

$$A = \begin{bmatrix} 2 & 15 & 0 \\ 6 & -4 & -9 \\ -8 & \frac{1}{2} & \frac{2}{3} \end{bmatrix}, C = \begin{bmatrix} 12 & 1 & 4 \\ 3 & 0 & 5 \\ -2 & \frac{3}{2} & -\frac{1}{3} \end{bmatrix}$$

42. Solve the matrix equation for B.

$$5A - B = C$$

43. Solve the matrix equation for B.

$$-2A + 3B = C$$

44. Solve the matrix equation for B.

$$\tfrac{1}{2}A + \tfrac{2}{3}B = \tfrac{1}{6}C$$

45. Solve the matrix equation for B.

$$-100A - 200B = 330C$$

SECTION 8.3 Matrix Multiplication and Inverse Matrices

LEARNING OBJECTIVES

- Use matrix multiplication
- Find inverse matrices using technology
- Use inverse matrices to solve matrix equations

GETTING STARTED

In 2011, Harkins Theaters charged \$9.50 per adult, \$5.50 per child, and \$6.50 per senior. (*Source:* www.movietickets.com) For theater-going families, determining the total cost for an evening at the movies is more complicated than multiplying the total number of tickets purchased by a fixed price since the prices per ticket vary. Fortunately, the mathematical concept of matrix multiplication can be used to address this issue.

In this section we demonstrate how to do matrix multiplication, and show how to find the inverse of a 2×2 matrix. We will also look at real-world applications of matrix multiplication such as the cost of taking a family to the movies.

■ Matrix Multiplication

Recall that a $1 \times n$ row matrix consists of a single row and n columns; an $n \times 1$ column matrix consists of n rows and a single column; and an $n \times n$ square matrix consists of n rows and n columns. We will use each of these types of matrices as we discuss matrix multiplication.

In the last section, we introduced the concept of scalar multiplication—the multiplication of a number and a matrix. Now we consider matrix multiplication—the multiplication of two or more matrices. Since matrix multiplication is a somewhat strange process, we will introduce the concept with a simple example before giving a formal definition.

EXAMPLE 1 ■ Using Matrix Multiplication in a Real-World Context

A theater charges \$9.50 per adult, \$5.50 per child, and \$6.50 per senior. What will be the total admission cost for a group of 12 adults, 16 children, and 4 seniors?

Solution We first calculate the total cost as

$$\begin{aligned}\text{total cost} &= \underset{\text{adults}}{9.50(12)} + \underset{\text{children}}{5.50(16)} + \underset{\text{seniors}}{6.50(4)}\\ &= 114 + 88 + 26\\ &= 228\end{aligned}$$

The total admission cost for the group is \$228.

We can obtain the same result by representing the individual admission cost as a row matrix and the number of guests as a column matrix.

$$C = [9.50 \quad 5.50 \quad 6.50] \qquad \text{individual admission cost matrix}$$

$$N = \begin{bmatrix} 12 \\ 16 \\ 4 \end{bmatrix} \qquad \text{number of guests matrix}$$

Now we can write the product of the 1×3 matrix C and the 3×1 matrix N as

$$CN = \begin{bmatrix} 9.50 & 5.50 & 6.50 \end{bmatrix} \begin{bmatrix} 12 \\ 16 \\ 4 \end{bmatrix}$$

From our initial solution, we know the total cost is given by the expression

$$9.50(12) + 5.50(16) + 6.50(4)$$

To combine the elements of each matrix to end up with the desired expression, we multiply each entry of the row matrix by the corresponding entry of the column matrix. In other words, we multiply the first entry in the row matrix by the first entry in the column matrix, and so on. We then obtain the final result by summing each of the individual terms. Therefore,

$$\begin{aligned} CN &= \begin{bmatrix} 9.50 & 5.50 & 6.50 \end{bmatrix} \begin{bmatrix} 12 \\ 16 \\ 4 \end{bmatrix} \\ &= [9.50(12) + 5.50(16) + 6.50(4)] \\ &= [114 + 88 + 26] \\ &= [228] \end{aligned}$$

The total ticket cost is \$228.

To determine the units of the 1×1 matrix, we note that the units of C were dollars per person and the units of N were persons. The units of the product of the matrices is the product of the units of the matrices, in the same order. Therefore,

$$\begin{aligned} \text{units of } CN &= \left(\frac{\text{dollars}}{\text{person}}\right)(\text{persons}) \\ &= \left(\frac{\text{dollars}}{\cancel{\text{person}}}\right)(\cancel{\text{persons}}) \\ &= \text{dollars} \end{aligned}$$

The process for multiplying a row matrix and a column matrix is summarized as follows.

The Product of a Row Matrix and a Column Matrix

The product of a $1 \times n$ row matrix A and an $n \times 1$ column matrix B is the 1×1 square matrix given by

$$\begin{aligned} AB &= \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} \\ &= [a_1 b_1 + a_2 b_2 + \cdots + a_n b_n] \end{aligned}$$

EXAMPLE 2 ■ Determining the Product of a Row Matrix and a Column Matrix

Calculate AB given $A = [1 \quad 2]$ and $B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

Solution

$$\begin{aligned} AB &= [1 \quad 2]\begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= [1(3) + 2(4)] \\ &= [3 + 8] \\ &= [11] \end{aligned}$$

The solution is the 1×1 matrix [11]. Note this is a matrix with a single entry (11) not the number 11.

EXAMPLE 3 ■ Determining the Product of a Row Matrix and a Column Matrix

Calculate AB given $A = [2 \quad 0 \quad -1 \quad 8]$ and $B = \begin{bmatrix} 5 \\ 7 \\ 4 \\ -2 \end{bmatrix}$.

Solution

$$\begin{aligned} AB &= [2 \quad 0 \quad -1 \quad 8]\begin{bmatrix} 5 \\ 7 \\ 4 \\ -2 \end{bmatrix} \\ &= [2(5) + 0(7) + (-1)(4) + 8(-2)] \\ &= [10 + 0 - 4 - 16] \\ &= [-10] \end{aligned}$$

The method we used to calculate the product of a $1 \times n$ row matrix and an $n \times 1$ column matrix can be used to calculate the product of any two matrices (provided the product exists). Consider the following $m \times n$ matrix A and $n \times p$ matrix B, where we have boxed the ith row of A and the jth column of B.

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \boxed{a_{i1}} & \boxed{a_{i2}} & \boxed{\cdots} & \boxed{a_{in}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \cdot \begin{bmatrix} b_{11} & b_{12} & \cdots & \boxed{b_{1j}} & \cdots & b_{1p} \\ b_{21} & b_{22} & \cdots & \boxed{b_{2j}} & \cdots & b_{2p} \\ \vdots & \vdots & \ddots & \boxed{\vdots} & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & \boxed{b_{nj}} & \cdots & b_{np} \end{bmatrix} \end{aligned}$$

We determine the entry c_{ij} of matrix C by calculating

$$\begin{aligned} [c_{ij}] &= [a_{i1} \quad a_{i2} \quad \cdots \quad a_{in}]\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix} \\ &= [a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj}] \end{aligned}$$

Matrix Multiplication

An $m \times n$ matrix A and an $n \times p$ matrix B can be multiplied together to form a new $m \times p$ matrix, C. The value of the entry in the ith row and jth column of C is the product of the ith row of A and the jth column of B.

EXAMPLE 4 ■ Determining the Product of Two Matrices

Find $C = AB$ given $A = \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Solution A is a 3×2 matrix and B is a 2×1 matrix. Since the number of columns of A matches the number of rows of B, matrix multiplication is possible. The dimensions of $C = AB$ are 3×1. The 3 comes from the number of rows of A and the 1 comes from the number of columns of B.

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} \\ &= \begin{bmatrix} 1(5) + 0(6) \\ 3(5) + 7(6) \\ 4(5) + (-2)(6) \end{bmatrix} && \begin{matrix} \text{Row 1 of } A \text{ times Column 1 of } B \\ \text{Row 2 of } A \text{ times Column 1 of } B \\ \text{Row 3 of } A \text{ times Column 1 of } B \end{matrix} \\ &= \begin{bmatrix} 5 + 0 \\ 15 + 42 \\ 20 - 12 \end{bmatrix} \\ &= \begin{bmatrix} 5 \\ 57 \\ 8 \end{bmatrix} \end{aligned}$$

Notice the entries of C are the product of the rows of A and the column of B. For example, c_{21} is the product of the second row of A, $[3 \quad 7]$, and the first column of B, $\begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Note that matrix multiplication can be performed only if the number of columns of the first matrix is equal to the number of rows of the second matrix.

EXAMPLE 5 ■ Determining the Product of Two Matrices

Find $C = DA$ given $D = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}$.

Solution D is a 2×2 matrix and A is a 3×2 matrix. The matrix $C = DA$ cannot be computed since the number of columns of D (2) is not equal to the number of rows of A (3). Let's try to do the multiplication anyway just to see what happens. To calculate the entry, c_{21}, we need to multiply the second row of D by the first column of A.

$$\begin{aligned} [c_{21}] &= [7 \quad 6]\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \\ &= [7(1) + 6(3) + ?(4)] \end{aligned}$$

We see that we cannot perform the calculation.

Using the matrices from Example 5, we next reverse the order in which the matrices are to be multiplied.

EXAMPLE 6 ■ **Determining the Product of Two Matrices**

Find $C = AD$ given $A = \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix}$.

Solution A is a 3×2 matrix and D is a 2×2 matrix so the number of columns in A (2) now equals the number of rows in D (2). The matrix $C = AD$ is a 3×2 matrix.

$$\begin{aligned} C &= AD \\ &= \begin{bmatrix} 1 & 0 \\ 3 & 7 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 7 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 1(2) + 0(7) & 1(5) + 0(6) \\ 3(2) + 7(7) & 3(5) + 7(6) \\ 4(2) + (-2)(7) & 4(5) + (-2)(6) \end{bmatrix} \\ &= \begin{bmatrix} 2 & 5 \\ 55 & 57 \\ -6 & 8 \end{bmatrix} \end{aligned}$$

From Examples 5 and 6, we see matrix multiplication is not commutative. That is, in general, $AB \neq BA$.

EXAMPLE 7 ■ **Using Technology to Determine the Product of Two Matrices**

Find $C = AB$ given $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 4 & 5 & 6 & 7 \\ 5 & 6 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} -1.1 & 1.8 \\ 0.4 & 4.2 \\ -0.2 & 1.0 \\ -3.9 & 1.1 \end{bmatrix}$.

Solution A is a 4×4 matrix and B is a 4×2 matrix so AB will be a 4×2 matrix. Due to the size of the matrices and the complexity of the entries in matrix B, we will use technology to calculate the product as demonstrated in the first Technology Tip at the end of the section. The result is

$$\begin{aligned} C &= AB \\ &= \begin{bmatrix} -16.5 & 17.6 \\ -11.7 & 9.5 \\ -30.9 & 41.9 \\ -4.5 & 41.2 \end{bmatrix} \end{aligned}$$

Before we continue to another real-world application of matrix multiplication, we summarize relevant properties of the process.

Matrix Multiplication Properties

Let A, B, and C be matrices. Let I be an identity matrix (a square matrix with 1s on the diagonal and 0s elsewhere) and let O be a zero matrix. Given that the dimensions of the matrices allow each of the operations to be performed, the following properties hold.

1. Multiplicative Associative $\quad A(BC) = (AB)C$
2. Multiplicative Identity $\quad AI = IA = A$
3. Distributive $\quad A(B + C) = AB + AC$
4. Distributive $\quad (A + B)C = AC + BC$
5. Multiplication by a Zero Matrix $\quad OA = AO = O$
6. Not Commutative $\quad AB \neq BA$

EXAMPLE 8 ■ **Using Matrix Multiplication in a Real-World Context**

Breakfast cereal connoisseurs often enjoy mixing cereals to create a new breakfast taste. A connoisseur working at a bed and breakfast wants to report the nutritional content of various mixtures of Honey Nut Cheerios®, Rice Crunch-Ems!, and Corn Crunch-Ems! to his health-conscious guests. From the package labeling he determines the nutritional content of each cereal and records it in Table 8.6.

Table 8.6

	Honey Nut Cheerios®	Rice Crunch-Ems!	Corn Crunch-Ems!
Protein	3 grams/cup	1.6 grams/cup	2 grams/cup
Carbohydrates	24 grams/cup	20.8 grams/cup	27 grams/cup
Fat	1.5 grams/cup	0 grams/cup	0 grams/cup

Source: Health Valley Rice and Corn Crunch-Ems! labels and General Mills Honey Nut Cheerios® label

His first mixture will contain 1 cup of Honey Nut Cheerios, 2 cups of Rice Crunch-Ems!, and 1 cup of Corn Crunch-Ems!. His second mixture will contain 2 cups of Honey Nut Cheerios®, 4 cups of Rice Crunch-Ems!, and 3 cups of Corn Crunch-Ems!. Determine the amount of protein, carbohydrates, and fat in each 1-cup serving of the mixtures.

Solution We represent the nutrition content table with the matrix

$$N = \begin{array}{c} \begin{array}{ccc} \text{Honey} & \text{Rice} & \text{Corn} \end{array} \\ \begin{bmatrix} 3.0 & 1.6 & 2.0 \\ 24.0 & 20.8 & 27.0 \\ 1.5 & 0.0 & 0.0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Protein} \\ \text{Carbs} \\ \text{Fat} \end{array}$$

We represent mixture ingredients by Table 8.7 and the corresponding matrix.

Table 8.7

	Mixture 1	Mixture 2
Honey	1.0 cup	2.0 cups
Rice	2.0 cups	4.0 cups
Corn	1.0 cup	3.0 cups

$$M = \begin{array}{c} \begin{array}{cc} \text{Mix 1} & \text{Mix 2} \end{array} \\ \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 3 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Honey} \\ \text{Rice} \\ \text{Corn} \end{array}$$

Notice the columns of N and the rows of M represent the same cereal (Honey Nut Cheerios, Rice Crunch-Ems!, and Corn Crunch-Ems!). The matrix NM will be a 3×2 matrix with rows representing protein, carbohydrates, and fat and columns representing Mixture 1 and Mixture 2.

$$NM = \begin{bmatrix} 3.0 & 1.6 & 2.0 \\ 24.0 & 20.8 & 27.0 \\ 1.5 & 0.0 & 0.0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 1 & 3 \end{bmatrix}$$

$$= \begin{array}{c} \begin{array}{cc} \text{Mix 1} & \text{Mix 2} \end{array} \\ \begin{bmatrix} 8.2 & 18.4 \\ 92.6 & 212.2 \\ 1.5 & 3.0 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Protein} \\ \text{Carbs} \\ \text{Fat} \end{array}$$

The units of N are *grams/cup* and the units of M are *cups*, so the unit of their product is *grams*.

Converting the matrix back to a table we construct Table 8.8, which shows the total amount of protein, carbohydrates, and fat in each mixture.

Table 8.8

	Mixture 1	**Mixture 2**
Protein	8.2 grams	18.4 grams
Carbohydrates	92.6 grams	212.2 grams
Fat	1.5 grams	3.0 grams

The first mixture contains 4 cups of cereal and the second contains 9 cups of cereal.

Dividing the terms in the first column by 4 cups and the entries in the second column by 9 cups, we get the values shown in Table 8.9.

Table 8.9

	Mixture 1	**Mixture 2**
Protein	2.05 grams/cup	2.04 grams/cup
Carbohydrates	23.15 grams/cup	23.58 grams/cup
Fat	0.375 grams/cup	0.33 grams/cup

Since the original data was accurate to 1 decimal place, we will round the table entries to 1 decimal place. We obtain the values in Table 8.10, which show the nutritional content of each of the mixtures.

Table 8.10

	Mixture 1	**Mixture 2**
Protein	2.1 grams/cup	2.0 grams/cup
Carbohydrates	23.2 grams/cup	23.6 grams/cup
Fat	0.4 grams/cup	0.3 grams/cup

■ Inverse Matrices

As we previously mentioned, the **identity matrix**, I_n, is the $n \times n$ square matrix with 1s along the main diagonal of the matrix and 0s elsewhere. For example, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $I_1 = [1]$ are all identity matrices. (The subscript on the I is often omitted but the dimensions of I are typically implied by the context of the problem.) We are often interested in matrices that have the property that $AB = BA = I$ when we solve systems of linear equations. In general, $AB \neq BA$, since matrix multiplication is not commutative. However, if two $n \times n$ matrices are inverses of each other, then $AB = BA = I$.

Inverse Matrices

An $n \times n$ matrix A and an $n \times n$ matrix B are inverses of each other if and only if $AB = BA = I_n$. We say $B = A^{-1}$ (read "A inverse").

A matrix with an inverse is said to be **invertible**. A matrix without an inverse is said to be **singular**.

EXAMPLE 9 ■ Determining If One Matrix Is the Inverse of Another

Let $A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}$. Determine if matrix B is the inverse of matrix A.

Solution

$$\begin{aligned} AB &= \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 6-5 & -15+15 \\ 2-2 & -5+6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

$$\begin{aligned} BA &= \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 6-5 & 10-10 \\ -3+3 & -5+6 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I_2 \end{aligned}$$

Since $AB = BA = I_2$, $B = A^{-1}$.

You may have noticed that the matrices in Example 9 looked remarkably similar to each other. In fact, for a 2×2 invertible matrix, it is fairly simple to calculate the inverse.

Inverse of a 2 × 2 Matrix

The inverse of an invertible 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$A^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

The quantity $ad - bc$ is called the **determinant** of the matrix A and is often written $\det(A)$. If $\det(A) = 0$, then $\dfrac{1}{ad - bc}$ is undefined and A is singular.

This method works only for 2×2 matrices. For larger matrices, we use technology.

EXAMPLE 10 ■ **Finding the Inverse of a 2 ×2 Matrix**

Find the inverse of matrix $A = \begin{bmatrix} 4 & 2 \\ 5 & 3 \end{bmatrix}$, if it exists.

Solution

$$\det(A) = 4(3) - 2(5)$$
$$\det(A) = 2$$

Since the determinant of A is not zero, the inverse of A exists.

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ -5 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1.5 & -1 \\ -2.5 & 2 \end{bmatrix}$$

EXAMPLE 11 ■ **Using the Determinant to Determine If a Matrix Is Singular**

Find the inverse of matrix $A = \begin{bmatrix} 2 & 6 \\ 1 & 3 \end{bmatrix}$, if it exists.

Solution Since $\det(A) = 2(3) - 6(1) = 0$, A is singular. That is, A does not have an inverse.

Notice that Row 1 is equal to twice Row 2. Whenever one row of a matrix is a multiple of another row, the determinant of the matrix will be zero.

We can algebraically determine the inverse of an invertible matrix of any size by augmenting the matrix with the identity matrix and then using row reduction. For example, if $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, we add on the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. We then use row operations to find the inverse matrix.

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{array}\right] \quad \textbf{Augment } \boldsymbol{A} \textbf{ with } \boldsymbol{I}.$$

$$\left[\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 2 & 3 & -1 \end{array}\right] \quad \textbf{3 times Row 1} - \textbf{Row 2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 2 & 3 & -1 \end{array}\right] \quad \textbf{Row 1} - \textbf{Row 2}$$

$$\left[\begin{array}{cc|cc} 1 & 0 & -2 & 1 \\ 0 & 1 & \frac{3}{2} & -\frac{1}{2} \end{array}\right] \quad \textbf{one-half of Row 2}$$

With the left-hand side of the augmented matrix reduced to the identity matrix, the matrix on the right-hand side of the augmented matrix is A^{-1}. We verify the result by multiplying A by A^{-1}.

$$\begin{aligned} AA^{-1} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix} \\ &= \begin{bmatrix} 1(-2) + 2(\frac{3}{2}) & 1(1) + 2(-\frac{1}{2}) \\ 3(-2) + 4(\frac{3}{2}) & 3(1) + 4(-\frac{1}{2}) \end{bmatrix} \\ &= \begin{bmatrix} -2 + 3 & 1 + (-1) \\ -6 + 6 & 3 + (-2) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= I \end{aligned}$$

Multiplying A^{-1} by A will yield the same result.

We can determine the inverse matrix for all invertible 2×2 matrices by augmenting a generic matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with the identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and row reducing.

$$\left[\begin{array}{cc|cc} a & b & 1 & 0 \\ c & d & 0 & 1 \end{array}\right] \quad \textbf{Augment } \boldsymbol{A} \textbf{ with the identity matrix } \boldsymbol{I}.$$

$$= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & ad - bc & -c & a \end{array}\right] \quad -cR_1 + aR_2 \to R_2$$

$$= \left[\begin{array}{cc|cc} a & b & 1 & 0 \\ 0 & 1 & \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right] \quad \frac{1}{ad - bc}R_2 \to R_2$$

$$= \left[\begin{array}{cc|cc} a & 0 & 1 - b\left(\dfrac{-c}{ad - bc}\right) & -b\left(\dfrac{a}{ad - bc}\right) \\ 0 & 1 & \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right] \quad R_1 - bR_2 \to R_1$$

$$= \left[\begin{array}{cc|cc} a & 0 & 1 + \dfrac{bc}{ad - bc} & \dfrac{-ab}{ad - bc} \\ 0 & 1 & \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right] \qquad \text{Simplify.}$$

$$= \left[\begin{array}{cc|cc} 1 & 0 & \dfrac{1}{a}\left(1 + \dfrac{bc}{ad - bc}\right) & \dfrac{1}{a}\left(\dfrac{-ab}{ad - bc}\right) \\ 0 & 1 & \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{array}\right]$$

Although the left-hand side of the augmented matrix is I, the right-hand side does not yet look like $\dfrac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. However, we can show the matrix is equivalent to the general 2×2 inverse matrix with a few additional steps.

$$A^{-1} = \begin{bmatrix} \dfrac{1}{a}\left(1 + \dfrac{bc}{ad - bc}\right) & \dfrac{1}{a}\left(\dfrac{-ab}{ad - bc}\right) \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{1}{a}\left(\dfrac{ad - bc}{ad - bc} + \dfrac{bc}{ad - bc}\right) & \dfrac{-ab}{a(ad - bc)} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} \dfrac{ad - bc + bc}{a} & \dfrac{-ab}{a} \\ -c & a \end{bmatrix}$$

$$= \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

As mentioned earlier, this process of augmenting a matrix with the identity matrix and row reducing can be used for any size square matrix. However, in practice, we typically use a graphing calculator to find the inverse of a matrix. The second Technology Tip at the end of the section shows how to do this.

EXAMPLE 12 ■ Finding the Inverse of a Square Matrix

Use technology to find the inverse of the matrix $A = \begin{bmatrix} 6 & 14 & 16 \\ 4 & 1 & 14 \\ 1 & 4 & 6 \end{bmatrix}$.

Solution Using the Technology Tip at the end of this section, we determine

$$A^{-1} = \begin{bmatrix} 0.25 & 0.1 & -0.9 \\ 0.05 & -0.1 & 0.1 \\ -0.075 & 0.05 & 0.25 \end{bmatrix}$$

We can also write the inverse matrix as

$$A^{-1} = \begin{bmatrix} \frac{1}{4} & \frac{1}{10} & -\frac{9}{10} \\ \frac{1}{20} & -\frac{1}{10} & \frac{1}{10} \\ -\frac{3}{40} & \frac{1}{20} & \frac{1}{4} \end{bmatrix} \quad \text{or} \quad A^{-1} = \frac{1}{40}\begin{bmatrix} 10 & 4 & -36 \\ 2 & -4 & 4 \\ -3 & 2 & 10 \end{bmatrix}$$

Solving Systems of Linear Equations Using Matrix Equations

We will now return to the process of solving systems of linear equations with the added capability of matrix algebra. Consider the system of linear equations

$$\begin{aligned} x - 4y - z &= -5 \\ 3y + z &= 7 \\ 2x + y &= 10 \end{aligned}$$

Let's define A to be the coefficient matrix of the system. That is, the entries of A are the coefficients of the variables of the system.

$$A = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

Let's define a column matrix X to be the variable matrix of the system. That is, the entries of X are the variables of the system.

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Finally, let's define the column matrix B to be the constant matrix of the system. That is, the entries of B are the constants from the right-hand side of the equal sign of the system

$$B = \begin{bmatrix} -5 \\ 7 \\ 10 \end{bmatrix}$$

Now let's consider the matrix product AX.

$$\begin{aligned} AX &= \begin{bmatrix} 1 & -4 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \\ &= \begin{bmatrix} 1x - 4y - 1z \\ 0x + 3y + 1z \\ 2x + 1y + 0z \end{bmatrix} \end{aligned}$$

But from the system of equations we know

$$\begin{aligned} 1x - 4y - 1z &= -5 \\ 0x + 3y + 1z &= 7 \\ 2x + 1y + 0z &= 10 \end{aligned}$$

So

$$\begin{aligned} AX &= \begin{bmatrix} -5 \\ 7 \\ 10 \end{bmatrix} \\ &= B \end{aligned}$$

Therefore, a system of linear equations may be represented by the matrix equation $AX = B$ and the solution to the system of equations is given by the matrix X.

To solve the matrix equation for X, we left-multiply both sides by the matrix A^{-1}. (Since matrix multiplication is not commutative, we use the terms *left-multiply* and *right-multiply* to designate on which side to place the matrix being inserted into the equation.)

$$\begin{aligned} AX &= B \\ A^{-1}(AX) &= A^{-1}B && \text{Left-multiply by } A^{-1}. \\ (A^{-1}A)X &= A^{-1}B && \text{Associative Property} \\ IX &= A^{-1}B && A^{-1}A = I \\ X &= A^{-1}B && IX = X \end{aligned}$$

So the product of the inverse of the coefficient matrix and the constant matrix is the solution matrix. Using the algebraic or technological methods previously introduced, it may be shown that the inverse of $A = \begin{bmatrix} 1 & -4 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 2 & 3 & -1 \end{bmatrix}$.

Thus the solution to the matrix equation $\begin{bmatrix} 1 & -4 & -1 \\ 0 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \\ 10 \end{bmatrix}$ is the product of A^{-1} and B.

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ 2 & 3 & -1 \end{bmatrix}\begin{bmatrix} -5 \\ 7 \\ 10 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} \end{aligned}$$

So $x = 4$, $y = 2$, and $z = 1$.

This method of solving systems of linear equations is extremely efficient and works well for large systems of linear equations with unique solutions. Unfortunately, if a system is dependent (has multiple solutions), we cannot use this method because the coefficient matrix will not be invertible.

EXAMPLE 13 ■ Solving a System of Equations Using Matrix Algebra

Write the system of equations as a matrix equation and solve.

$$\begin{aligned} x + y + z &= 7 \\ 3x - y + z &= 21 \\ -x + 2y + 2z &= 2 \end{aligned}$$

Solution The system of equations is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 2 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 21 \\ 2 \end{bmatrix}$$

The solution to the system is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 2 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 7 \\ 21 \\ 2 \end{bmatrix}$$

Using technology we determine that

$$\begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{6} & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{5}{6} & \frac{1}{2} & \frac{2}{3} \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \\ -1 & 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 21 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & 0 & -\frac{1}{3} \\ \frac{7}{6} & -\frac{1}{2} & -\frac{1}{3} \\ -\frac{5}{6} & \frac{1}{2} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} 7 \\ 21 \\ 2 \end{bmatrix}$$

We use technology to multiply the resultant matrices and get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 6 \end{bmatrix}$$

So $x = 4$, $y = -3$, and $z = 6$ is the solution to the system of equations.

■ Analyzing Real-World Data with Matrices

With a well-developed set of matrix algebra skills, we can set up matrix equations to represent real-world data with relative ease. Since the entries of the matrices in many real-world situations are decimal numbers, we will often use technology to determine the desired solution.

EXAMPLE 14 ■ Using Matrix Algebra in a Real-World Context

Financial advisors often counsel their clients to diversify their investments into a variety of accounts of varying levels of performance and risk. The average annual return (over the 10-year period prior to June 30, 2002) of two mutual funds offered by Harbor Fund is shown in Table 8.11.

Table 8.11

	Average Annual Return
Capital Appreciation Fund	12.69%
Bond Fund	7.97%

Source: Harbor Fund account statement

High-performance accounts typically have greater volatility than lower-performance accounts. For example, although the Capital Appreciation Fund has the higher average annual reurn over the 10-year period, it earned −23.42% in a recent year while the Bond Fund earned 10.86% in the same year.

An investor has $1000 to invest in the two accounts. Assuming the accounts will earn the returns specified in the table over the next year, how much should she invest in each account if she wants to earn 8%, 10%, or 12%?

Solution Let x be the amount invested in the Capital Appreciation Fund and y be the amount invested in the Bond Fund. Since the sum of the individual investments is $1000, we have

$$x + y = 1000$$

The annual return on each account is the product of the rate of return and the amount of money invested in the account. For the Capital Appreciation Fund, the annual return is given by $0.1269x$. For the Bond Fund, the annual return is given by $0.0797y$. The combined return of the two accounts is then $0.1269x + 0.0797y$. If we let r be the desired rate of return on the \$1000 investment, then $1000r$ is the dollar amount of the return. Since these two expressions must be equal, we have

$$0.1269x + 0.0797y = 1000r$$

For the 8% return, $r = 0.08$. Therefore,

$$\begin{aligned} 0.1269x + 0.0797y &= 1000(0.08) \\ 0.1269x + 0.0797y &= 80 \end{aligned}$$

Consequently, we have the system of equations

$$\begin{aligned} x + y &= 1000 && \textbf{total investment} \\ 0.1269x + 0.0797y &= 80 && \textbf{total return on investment} \end{aligned}$$

and the corresponding matrix equation

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 0.1269 & 0.0797 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1000 \\ 80 \end{bmatrix}}$$

Similarly, for the 10% return, we have the system of equations

$$\begin{aligned} x + y &= 1000 && \textbf{total investment} \\ 0.1269x + 0.0797y &= 100 && \textbf{total return on investment} \end{aligned}$$

and the corresponding matrix equation

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 0.1269 & 0.0797 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1000 \\ 100 \end{bmatrix}}$$

Likewise, for the 12% return, we have the system of equations

$$\begin{aligned} x + y &= 1000 && \textbf{total investment} \\ 0.1269x + 0.0797y &= 120 && \textbf{total return on investment} \end{aligned}$$

and the corresponding matrix equation

$$\overset{A}{\begin{bmatrix} 1 & 1 \\ 0.1269 & 0.0797 \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} 1000 \\ 120 \end{bmatrix}}$$

Notice that although the constant matrices of each matrix equation differ, each of the three matrix equations has the same coefficient matrix, A. We know the solution to the matrix equation $AX = B$ is $X = A^{-1}B$. The inverse of A is given by

$$A^{-1} \approx \begin{bmatrix} -1.689 & 21.19 \\ 2.689 & -21.19 \end{bmatrix}$$

Therefore, for the 8% return, we have

$$\begin{aligned} X &= A^{-1}B \\ &= \begin{bmatrix} -1.689 & 21.19 \\ 2.689 & -21.19 \end{bmatrix} \begin{bmatrix} 1000 \\ 80 \end{bmatrix} \\ &= \begin{bmatrix} 6.36 \\ 993.64 \end{bmatrix} \end{aligned}$$

She should invest \$6.36 in the Capital Appreciation Fund and \$993.64 in the Bond Fund to earn an 8% return.

For the 10% return, we have

$$X = A^{-1}B$$

$$= \begin{bmatrix} -1.689 & 21.19 \\ 2.689 & -21.19 \end{bmatrix} \begin{bmatrix} 1000 \\ 100 \end{bmatrix}$$

$$= \begin{bmatrix} 430.08 \\ 569.92 \end{bmatrix}$$

She should invest \$430.08 in the Capital Appreciation Fund and \$569.92 in the Bond Fund to earn a 10% return.

For the 12% return, we have

$$X = \begin{bmatrix} -1.689 & 21.19 \\ 2.689 & -21.19 \end{bmatrix} \begin{bmatrix} 1000 \\ 120 \end{bmatrix}$$

$$= \begin{bmatrix} 853.81 \\ 146.19 \end{bmatrix}$$

She should invest \$853.81 in the Capital Appreciation Fund and \$146.19 in the Bond Fund to earn a 12% return.

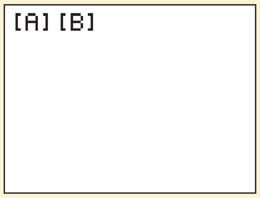
Recall we *assumed* she would earn the 10-year average annual return on all of her investments. Since none of the returns are guaranteed, which blend of investments she decides to choose will depend on her risk tolerance.

SUMMARY

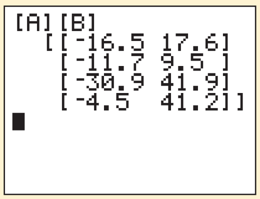
In this section you learned how to do matrix multiplication and use it in real-world scenarios. You also learned how to find the inverse of a matrix and use it to solve systems of equations.

TECHNOLOGY TIP ■ MATRIX MULTIPLICATION

1. Enter matrix A and matrix B into the calculator using the Matrix Menu. Use the Matrix Names Menu to place matrix A on the home screen. Then place matrix B on the home screen.



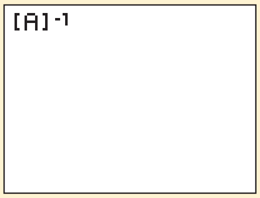
2. Press ENTER to display the product of the two matrices.



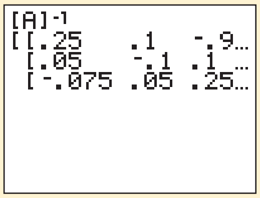
Error Alert:
If this error message appears, double-check that you have entered the matrices correctly. If you have, this message tells you that the matrix multiplication is not possible because the number of columns of the first matrix is not the same as the number of rows of the second matrix.

TECHNOLOGY TIP ■ FINDING THE INVERSE OF A MATRIX

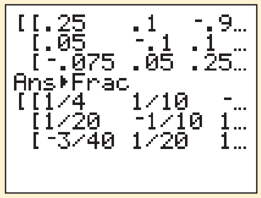
1. Enter matrix A into the calculator using the Matrix Menu. Use the Matrix Names Menu to place matrix A on the home screen. Then press the $\boxed{x^{-1}}$ button.



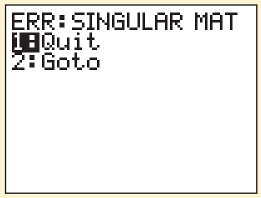
2. Press $\boxed{\text{ENTER}}$ to display the inverse of the matrix.



3. To convert the entries to fractions, press $\boxed{\text{MATH}}$ then **FRAC**. Press $\boxed{\text{ENTER}}$.



Error Alert:
If this error message appears, double-check that you have entered the matrix correctly. If you have, this message tells you that the matrix is singular (not invertible).



8.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10, perform the indicated operation. As appropriate, use the matrices

$$A = \begin{bmatrix} 3 & 4 \\ -1 & -2 \end{bmatrix}, B = \begin{bmatrix} 2 & -4 \\ 1 & 2 \end{bmatrix},$$

$$C = \begin{bmatrix} 8 & 4 \\ 6 & 1 \\ 7 & 3 \end{bmatrix}, D = \begin{bmatrix} -2 & 0 & 3 \\ 0 & 4 & 3 \\ 1 & 7 & 0 \end{bmatrix}$$

If the specified operation is undefined, so state.

1. $\begin{bmatrix} 3 & 9 & -2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 12 \end{bmatrix}$

2. $\begin{bmatrix} -2 & 9 & 0.5 \end{bmatrix}\begin{bmatrix} 1.5 \\ 4 \\ -6 \end{bmatrix}$

3. AB

4. BA

5. CD

6. DC

7. CA

8. AC

9. CB

10. $A^{-1}A$

In Exercises 11–15, use the determinant formula to determine if the matrix is invertible or singular. If the matrix is invertible, find its inverse.

11. $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

12. $B = \begin{bmatrix} -2 & 4 \\ 1 & 2 \end{bmatrix}$

13. $C = \begin{bmatrix} 9 & 6 \\ 3 & 2 \end{bmatrix}$

14. $D = \begin{bmatrix} 0 & -1 \\ 4 & 4 \end{bmatrix}$

15. $E = \begin{bmatrix} -0.5 & -0.7 \\ 4.0 & 3.4 \end{bmatrix}$

In Exercises 16–25, write the system of equations as a matrix equation, $AX = B$, and solve using an inverse matrix and technology.

16. $\begin{aligned} x + y + z &= 6 \\ 2x - y + z &= 3 \\ 4x - 2y + 3z &= 9 \end{aligned}$

17. $\begin{aligned} 5x + y + z &= 1 \\ 4x - 2y + 5z &= -2 \\ x - 7y + 6z &= -7 \end{aligned}$

18. $\begin{aligned} 3x + 2y + 3z &= 6 \\ 2x - 5y + z &= -11 \\ 4x + 2y + 3z &= 3 \end{aligned}$

19. $\begin{aligned} x + y + z &= 1 \\ 2x - y + z &= 4 \\ 4x - 2y + 3z &= 9 \end{aligned}$

20. $\begin{aligned} x + y + z &= 3 \\ 2x - y + z &= 13 \\ 4x - 2y + 3z &= 28 \end{aligned}$

21. $\begin{aligned} 3x - y + 2z &= 17 \\ 4x + 3y + 5z &= 12 \\ 6x + 8y + 4z &= 4 \end{aligned}$

22. $\begin{aligned} 3x - y + 2z &= 12 \\ 4x + 3y + 5z &= 45 \\ 6x + 8y + 4z &= 64 \end{aligned}$

23. $2x + 5y + 3z = 10$
$x + 4y + 2z = 7$
$0x + 2y + 4z = 6$

24. $2x + 5y + 3z = 0$
$x + 4y + 2z = -1$
$0x + 2y + 4z = 2$

25. $x + y + z = 1$
$2x - y + z = 1$
$4x - 2y + 3z = 3$

26. Grade Point Average Students at Green River Community College must earn 90 credits to obtain an Associate of Arts degree. Three students with an existing 2.9 GPA hope to increase their cumulative GPA to 3.5. The first student has earned 30 credits; the second, 45 credits; and the third, 55 credits. Is it possible for all three students to increase their cumulative GPA to 3.5 by the time they obtain 90 credits?

27. Executive Bonus Plan A small company rewards its upper management by offering annual bonuses. Each executive receives a percentage of the profits that remain after the bonuses of all of the executives have been deducted from the company's profits. The CEO receives 5%; the CFO, 4%; and the Vice President, 3%. What will be the bonus amount of each executive if the company's annual profit is \$500,000? \$800,000? \$1,000,000?

SHOW YOU KNOW

28. How do inverse matrices make solving matrix equations easier?

29. Explain why the number of columns in the first matrix must be the same as the number of rows in the second matrix when two matrices are multiplied together.

30. Suppose row matrix P represents the prices of 10 items sold at a concession stand. Suppose column matrix N represents the number of items sold for each of the items in matrix P. Explain what the product of matrix P and matrix N represents.

31. A friend missed class and asks you to explain how to use the inverse matrix to solve a system of linear equations. How do you respond?

32. Describe the difference between an invertible matrix and a singular matrix.

MAKE IT REAL

In Exercises 33–37, use matrix multiplication to find the solution.

33. Lumber Manufacturing Payroll The number of employees working in the lumber and wood products manufacturing industry is shown in the first table.

Manufacturing Employees: Lumber and Wood Products	
Years Since 1995 *t*	**Full-Time Employees (thousands)** *E*
0	772
1	782
2	794
3	816
4	843

Source: Statistical Abstract of the United States, 2001, Table 979

The average annual wage/salary of a full-time employee working in the lumber and wood products manufacturing industry is shown in the next table.

Manufacturing Salaries: Lumber and Wood Products	
Years Since 1995 *t*	**Average Annual Wage/Salary (dollars)** *S*
0	25,110
1	26,148
2	27,382
3	28,278
4	29,040

Source: Statistical Abstract of the United States, 2001, Table 979

From 1995 through 1999, what was the total amount of money spent on employee wages and salaries?

34. Rubber and Plastics Payroll The number of employees working in the rubber and plastics manufacturing industry is shown in the first table.

Manufacturing Employees: Rubber and Plastics Industry	
Years Since 1995 *t*	**Full-Time Employees (thousands)** *N*
0	963
1	965
2	984
3	998
4	994

Source: Statistical Abstract of the United States, 2001, Table 979

The average annual wage/salary of a full-time employee working in the rubber and plastics manufacturing industry is shown in the next table.

Manufacturing Average Earnings: Rubber and Plastics Industry	
Years Since 1995 *t*	Earnings (thousands) *E*
0	29,867
1	30,898
2	32,237
3	33,574
4	34,508

***Source: Statistical Abstract of the United States, 2001*, Table 979**

From 1995 through 1999, what was the total amount of money spent on employee earnings?

35. Fruit Smoothie Nutritional Content The Vita-Mix Super 5000 is a powerful, blender-like kitchen appliance with a 2+ horsepower motor. Vita-Mix blade tips move at up to 240 miles per hour, easily converting whole fruits into luscious smoothies or grinding wheat kernels into fine flour. (***Source:*** **www.vita-mix.com**)

The author uses his Vita-Mix regularly to make breakfast beverages and was curious about the nutritional content of two different types of whole-fruit smoothies. The ingredients of the first smoothie include 1 large apple, 1 large orange, 1 large banana, and 1 cup of water. The ingredients of the second smoothie include 2 large oranges, l large banana, and 1 cup of water.

Assuming each fruit yields a 1-cup serving, we have the following nutrition information. A large orange contains 53.2 milligrams of vitamin C (ascorbic acid), 40 milligrams of calcium, and 2.4 grams of fiber. A large banana contains 9.1 milligrams of vitamin C, 6 milligrams of calcium, and 2.4 grams of fiber. A large apple contains 5.7 milligrams of vitamin C, 7 milligrams of calcium, and 2.7 grams of fiber. (***Source:*** **USDA**) How much vitamin C, calcium, and fiber is in each smoothie?

36. Fruit Smoothie Nutritional Content One cup of flax seed contains 1.3 milligrams of vitamin C, 199 milligrams of calcium, and 27.9 grams of fiber. If 1 tablespoon of flax seed is added to each of the smoothies in Exercise 35, how much vitamin C, calcium, and fiber is in each smoothie? (*Hint:* There are 16 tablespoons in a cup.)

37. Fruit Smoothie Nutritional Content A tropical fruit smoothie is made from 2 cups of chopped mango, 1 cup of pineapple, 1 cup of coconut, and 2 cups of ice. A piña colada smoothie is made from 2 cups of pineapple, 1 cup of coconut, and 2 cups of ice.

A cup of fresh mango contains 10 milligrams of calcium, 27.7 milligrams of vitamin C, and 1.8 grams of fiber. A cup of fresh pineapple contains 7 milligrams of calcium, 15.4 milligrams of vitamin C, and 1.2 grams of fiber. A cup of raw coconut contains 14 milligrams of calcium, 3.3 milligrams of vitamin C, and 9 grams of fiber. (***Source:*** **USDA**) How much vitamin C, calcium, and fiber is in each smoothie?

38. Alcohol-Related Homicides The table shows the annual number of homicides resulting from an alcohol-related brawl.

Homicides Resulting from an Alcohol-Related Brawl	
Years Since 1990 *t*	Homicides *H*
1	500
2	429
3	383
4	316
5	254
6	256
7	239
8	213
9	203
10	181

Source: **Federal Bureau of Investigation**

Write the number of homicides as a 10×1 column matrix H. Let

$$A = [1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1]$$

Calculate and interpret the meaning of AH.

39. Military Personnel The table shows the number of military personnel in 1996 and 1998.

Year *T*	All Military Personnel (thousands) *m*	Army Personnel (thousands) *a*
1996	1056	491
1998	1004	484

***Source: Statistical Abstract of the United States, 2001*, Tables 499–500**

We can create a 2×2 matrix P with rows representing 1996 and 1998 and columns representing the number of military personnel.

$$P = \begin{array}{c} \begin{array}{cc} m & a \end{array} \\ \begin{bmatrix} 1056 & 491 \\ 1004 & 484 \end{bmatrix} \end{array} \begin{array}{l} \\ \text{Year 1996} \\ \text{Year 1998} \end{array}$$

Let $R = [-1 \quad 1]$ and $C = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Calculate and interpret the meaning of RP and CP.

40. Movie Theater Revenue The table shows the average movie theater ticket price and total theater attendance from 1995 to 1999.

Years Since 1995	Price per Person (dollars)	Attendance (in millions)
0	4.35	1263
1	4.42	1339
2	4.59	1388
3	4.69	1481
4	5.08	1465

Source: Statistical Abstract of the United States, 2001, Table 1244

Use matrices to calculate the accumulated movie theater revenue from ticket sales from the start of 1995 through 1999.

41. **Company Profit** The table shows the net sales and cost of goods sold for the Kellogg Company from 1999 to 2001.

Year	Net Sales ($ millions)	Cost of Goods Sold ($ millions)
1999	6984.2	3325.1
2000	6954.7	3327.0
2001	8853.3	4128.5

Source: Kellogg Company 2001 Annual Report

Use matrices to calculate the accumulated net sales, accumulated cost of goods sold, and accumulated profit for the time period 1999 to 2001.

42. **Company Profit** The table shows the net sales and cost of goods sold for the Coca-Cola Company from 1999 to 2001.

Year	Net Sales ($ millions)	Cost of Goods Sold ($ millions)
1999	19,284	6009
2000	19,889	6204
2001	20,092	6044

Source: Coca-Cola Company 2001 Annual Report

Use matrices to calculate the accumulated net sales, accumulated cost of goods sold, and accumulated profit for the time period 1999 to 2001.

In Exercises 43–46, determine the solution by setting up and solving the matrix equation.

43. **Nutritional Content** A nut distributor wants to determine the nutritional content of various mixtures of pecans (oil roasted, salted), cashews (dry roasted, salted), and almonds (honey roasted, unblanched). Her supplier has provided the following nutrition information.

	Almonds	Cashews	Pecans
Protein	26.2 grams/cup	21.0 grams/cup	10.1 grams/cup
Carbo-hydrates	40.2 grams/cup	44.8 grams/cup	14.3 grams/cup
Fat	71.9 grams/cup	63.5 grams/cup	82.8 grams/cup

Source: www.Nutri-facts.com

Her first mixture, Protein Blend, contains 6 cups of almonds, 3 cups of cashews, and 1 cup of pecans. Her second mixture, Low Fat Mix, contains 3 cups of almonds, 6 cups of cashews, and 1 cup of pecans. Her final mixture, Low Carb Mix, contains 3 cups of almonds, 1 cup of cashews, and 6 cups of pecans. Determine the amount of protein, carbohydrates, and fat in each 1-cup serving of the mixtures.

44. **Floral Costs** A florist purchases her flowers from online flower wholesaler. White Daisies are $3.38 per bunch, Football Mums are $7.40 per bunch, Super Blue Purple Statice is $4.25 per bunch, and Misty Blue Limonium is $5.25 per bunch. (*Source:* FlowerSales.com)

From these flowers she will make three types of bouquets:

	Type 1	Type 2	Type 3
Daisies	1 bunch	1 bunch	2 bunches
Mums	1 bunch	1 bunch	none
Statice	1/2 bunch	none	1/2 bunch
Limonium	none	1/2 bunch	none

What is her flower cost for each type of bouquet? How much should she charge for each bouquet if her markup is 50% of her flower cost?

45. **Return on Investments** The average annual returns (over the 10-year period prior to June 30, 2002) of two mutual funds offered by Harbor Fund are shown in the table.

	Average Annual Return
Money Market	4.50%
Large Cap Value	11.01%

Source: Harbor Fund

Suppose you have $2000 to invest in these two accounts. Assuming the accounts will earn the returns specified in the table over the next year, how much should you invest in each account if you want to earn 6%, 8%, or 10%?

46. **Nutritional Content** A nut distributor wants to determine the nutritional content of various mixtures of pecans (oil roasted, salted), cashews (dry roasted, salted), and almonds (honey roasted, unblanched). Her supplier has provided the following nutrition information.

	Almonds	Cashews	Pecans
Protein	26.2 grams/cup	21.0 grams/cup	10.1 grams/cup
Sugars	20.5 grams/cup	40.7 grams/cup	3.9 grams/cup
Fiber	19.7 grams/cup	4.1 grams/cup	10.4 grams/cup

Source: www.Nutri-facts.com

Her first mixture, Protein Blend, contains 6 cups of almonds, 3 cups of cashews, and 1 cup of pecans. Her second mixture, Low Sugar Mix, contains 2 cups of almonds, 1 cups of cashews, and 7 cups of pecans. Her final mixture, High Fiber Mix, contains 5 cups of almonds, 1 cup of cashews, and 4 cups of pecans. Determine the amount of protein, sugar, and fiber in each 1-cup serving of the mixtures.

STRETCH YOUR MIND

Exercises 47–55 are intended to challenge your understanding of matrix multiplication and matrix inverses.

47. For real numbers a and b, $(a + b)^2 = a^2 + 2ab + b^2$. For $n \times n$ matrices A and B, does $(A + B)^2 = A^2 + 2AB + B^2$? Explain.

48. A **diagonal matrix** A is a matrix with the property that $a_{ij} = 0$ when $i \neq j$. Show that if A is invertible, the inverse of the diagonal matrix

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \text{ is } A^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}.$$

Under what conditions is A singular?

49. Find three 3×3 diagonal matrices that are their own inverses.

50. A **symmetric matrix** A is a matrix with the property that $a_{ij} = a_{ji}$. Show that if A is a 3×3 symmetric matrix, A^2 is a 3×3 symmetric matrix. (*Hint:* $A^2 = A \cdot A$)

51. Show that if A is invertible, A^2 is invertible. (*Hint:* $(AB)^{-1} = B^{-1}A^{-1}$)

52. Show that if A and B are invertible and AB is defined, AB is invertible. (*Hint:* $(AB)^{-1} = B^{-1}A^{-1}$)

53. Find three different 2×2 matrices, A, with the property that $A^2 = I$.

54. Find two different 3×3 matrices that are their own inverses. That is $A \cdot A = I$.

55. Given $A^2 = I$, write A^{-1} in terms of A.

CHAPTER 8 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 8.1

1. How are systems of equations and augmented matrices related?
2. What row operations result in an equivalent system of equations?

SECTIONS 8.2 AND 8.3

3. What is the difference between scalar multiplication and matrix multiplication?

SECTION 8.3

4. How do inverse matrices make solving matrix equations easier?

REVIEW EXERCISES

SECTION 8.1

In Exercises 1–5, write the system of equations as an augmented matrix then simplify the matrix to reduced row echelon form. Identify the solution(s) to the system of equations.

1. $2.1x - y = -8$
$3.4x + y = 8$

2. $5.2x - 1.3y = 12$
$-10.4x + 2.6y = 24$

3. $4x - 8y = -16$
$x + y = 5$

4. $2.9x - 8.1y = 4$
$3.7x + 16.2y = 1.5$

5. $x + y + z = 6$
$2x - y + z = 3$
$3x + 2z = 9$

In Exercises 6–9, use technology to write the matrix in reduced row echelon form.

6. $\left[\begin{array}{ccc|c} 8.1 & 4.3 & 6.2 & 3.1 \\ 2.7 & -2.8 & 7.5 & 7.2 \\ 8.2 & -5.3 & -1.1 & 5.9 \end{array}\right]$

7. $\left[\begin{array}{ccc|c} 2.6 & 3.0 & 0.2 & 9.6 \\ 6.1 & 4.0 & 6.5 & 0.8 \\ 9.1 & -5.0 & -0.8 & 9.9 \end{array}\right]$

8. $\left[\begin{array}{ccc|c} 1 & 5 & -1 & -11 \\ 0 & 0 & 3 & 12 \\ 2 & 4 & -2 & 8 \end{array}\right]$

9. $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$

SECTION 8.2

In Exercises 10–14, perform the indicated matrix operation, if possible, given the following matrices. Solve these problems without technology.

$$A = \begin{bmatrix} 2 & 3 \\ -3 & -4 \\ 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 5 & 6 \\ 3 & -4 \\ 9 & 2 \end{bmatrix}$$

10. $A + B$ **11.** $A - B$
12. $2A$ **13.** $3B$
14. $2A + 3B$

In Exercises 15–19 use technology to simplify the matrix expressions given

$$A = \begin{bmatrix} 1.2 & 6.1 & -0.4 \\ -9.1 & 4.2 & 1.7 \\ 0.9 & 3.0 & 3.3 \end{bmatrix}, B = \begin{bmatrix} 3.4 & -0.3 & -0.4 \\ -3.8 & 5.6 & 7.2 \\ 2.2 & 2.6 & 2.0 \end{bmatrix}$$

15. $3.2A$ **16.** $5.3B$
17. $3.2A - 5.3B$ **18.** $4.1A + 0.1B$
19. $-1.1A + 2.9B$

20. Auto Prices The average trade-in values of a Toyota Celica and Toyota MR2 Spyder in July 2002 are shown in Table G. The average retail values of the two vehicles are shown in Table H.

Table G

Average Trade-In		
	Celica	**MR2 Spyder**
2000 Model	\$13,025	\$18,750
2001 Model	\$14,850	\$20,100

Source: www.nada.com

Table H

Average Retail		
	Celica	**MR2 Spyder**
2000 Model	\$15,300	\$21,300
2001 Model	\$17,250	\$22,725

Source: www.nada.com

Use matrices to create a table that shows the average dealer markup for each of the vehicles.

SECTION 8.3

In Exercises 21–26, perform the indicated operation. As appropriate, use the following matrices.

$$A = \begin{bmatrix} 5 & 8 \\ 1 & 2 \end{bmatrix}, B = \begin{bmatrix} 4 & 5 \\ 2 & 2 \end{bmatrix}, C = \begin{bmatrix} 2 & 1 \\ 5 & 4 \\ 8 & 7 \end{bmatrix}, D = \begin{bmatrix} -1 & 0 & 4 \\ -2 & 2 & -2 \\ -6 & 4 & 1 \end{bmatrix}$$

If the specified operation is undefined, so state.

21. AB **22.** BA
23. CD **24.** A^{-1}
25. B^{-1} **26.** D^{-1}

In Exercises 27–28, use the determinant formula to determine if the matrix is invertible or singular.

27. $A = \begin{bmatrix} 9 & 6 \\ -3 & 2 \end{bmatrix}$ **28.** $B = \begin{bmatrix} -8 & -4 \\ 5 & 3 \end{bmatrix}$

In Exercises 29–30, use technology to find the inverse of the matrix, if it exists. If the matrix is singular, so state.

29. $B = \begin{bmatrix} 2.0 & 6.2 & -0.8 \\ 0 & 3.2 & 5.4 \\ 0 & 0 & -0.5 \end{bmatrix}$

30. $B = \begin{bmatrix} 2.0 & 6.2 & -0.8 \\ 4.2 & 3.2 & 5.4 \\ 6.2 & 9.4 & 4.6 \end{bmatrix}$

In Exercises 31–33, find the inverse of the matrix A using technology. If A is singular, so state.

31. $A = \begin{bmatrix} 6 & -2 & 1 \\ 3 & -1 & 1 \\ 0 & 2 & 4 \end{bmatrix}$

32. $A = \begin{bmatrix} 9 & 7 & 8 \\ 3 & 3 & 3 \\ 6 & 10 & 11 \end{bmatrix}$

33. $A = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ -2 & 0 & 1 \end{bmatrix}$

In Exercises 34–36, write the system of equations as a matrix equation, $AX = B$, and solve.

34. $$\begin{aligned} x + y + z &= 2 \\ 2x - y + z &= -6 \\ 4x - 2y + 3z &= -17 \end{aligned}$$

35. $$\begin{aligned} x + y + z &= 14 \\ x - y + z &= 26 \\ x - y - z &= 2 \end{aligned}$$

36. $$\begin{aligned} 3x + 2y + 3z &= 0 \\ 2x - 5y + z &= 0 \\ 4x + 2y + 3z &= 0 \end{aligned}$$

In Exercises 37–39, determine the solution by setting up and solving the matrix equation.

37. Return on Investment The average annual return (over the 10-year period prior to June 30, 2002) of two mutual funds offered by Harbor Fund is shown in the table.

	Average Annual Return
Growth	5.84%
Capital Appreciation	12.69%

Source: **Harbor Fund**

Suppose you have \$2000 to invest in these two accounts. Assuming the accounts will earn the returns specified in the table over the next year, how much should you invest in each account if you want to earn 7%, 9%, or 11%?

38. Grade Point Average Students at Green River Community College must earn 90 credits to obtain an Associate of Arts degree. A student with an existing 3.1 GPA hopes to increase her cumulative GPA to 3.4. If she has 36 credits now and anticipates that she will be able to earn a 3.7 GPA on her remaining coursework, is it possible for her to increase her cumulative GPA to 3.4 by the time she obtains 90 credits?

39. Floral Costs A florist purchases her flowers from an online flower wholesaler. White Daisies are \$3.38 per bunch, Football Mums are \$7.40 per bunch, and Super Blue Purple Statice is \$4.25 per bunch. (***Source:*** **FlowerSales.com**) From these flowers she will make three types of jumbo bouquets.

Albert Michael Cutri/ Shutterstock.com

	Type 1	Type 2	Type 3
Daisies	2 bunches	3 bunches	2 bunches
Mums	1 bunch	2 bunches	2 bunches
Statice	1 bunch	none	1 bunch

What is her flower cost for each type of bouquet? How much should she charge for each bouquet if her markup is 50% of her flower cost?

Make It Real Project

What to Do

1. Find out your current cumulative grade point average and your total number of graded credits.
2. Determine how many credits are required for your degree program.
3. Find two or three scholarships that require a minimum GPA.
4. If your cumulative GPA is below the required minimum for the scholarship, complete Step 5. Otherwise, complete Step 6.
5. Set up and solve a system of equations to determine how many credits of "A" grade (4.0) a student with your cumulative GPA must earn to meet the GPA requirement. To do this, let x be the number of credits already earned and y be the number of credits left to be earned. Proceed to Step 7.
6. Set up and solve a system of equations to determine how many credits of "C" grade (2.0) a student with your cumulative GPA can earn and still meet the GPA requirement. To do this, let x be the number of credits already earned and y be the number of credits left to be earned.
7. Suppose one of your friends has the same GPA as you but has earned 12 credits fewer than you. Based on your previous calculations, determine if it is mathematically possible for both you and your friend to meet the grade point average requirement of each scholarship by the time you have earned the required number of credits.