CHAPTER 9

Triangle Trigonometry and Applications

Scaling a cliff is difficult for even the most experienced athletes. Success requires physical strength and agility as well as mental focus. Safety-conscious climbers arm themselves with proper gear and with an in-depth knowledge of the surfaces they climb. They ask, "Are the rocks stable?", "Are there a sufficient number of handholds?", "How high is this cliff?"

Using basic trigonometry skills, a climber can estimate the height of a cliff before she begins her ascent. These skills can be used no matter the height of the cliff or the difficulty of the terrain.

SECTION 9.1

Right Triangle Trigonometry

LEARNING OBJECTIVES

- Describe the six trigonometric ratios
- Use the appropriate trigonometric ratio to solve real-world problems involving right triangles
- Describe the relationships among the trigonometric ratios

GETTING STARTED

In September 2006, two amateur California naturalists purportedly discovered the "World's Tallest Tree." (*Source:* Associated Press, September 8, 2006) The duo found three trees that reportedly are taller than the world record holder at that time. Prior to this, the tallest tree as recorded in the Guinness Book of World Records was the Stratosphere Giant, a redwood tree found in Redwood National Park, California, measuring 370 feet tall. Each of the three newly discovered redwood trees, also found in Redwood National Park, are taller than 370 feet. The tallest, named Hyperion, is 378.1 feet tall. The next tallest, the Helios, stands at 376.3 feet; Icarus, the third, reaches 371.2 feet. (*Source:* sfgate.com)

In this section we investigate methods of measuring very tall objects, such as redwood trees. With right triangle trigonometry, we can solve a variety of real-world problems such as this.

Measuring with Shadows and Similar Triangles

One way to measure a very tall object such as a redwood tree is to use the geometry of similar triangles. To make sense of this idea, we will consider a simplified situation. Imagine a shadow cast by the tree along the ground as shown in Figure 9.1. Using a tape measure, we determine the shadow is 504 feet in length. Now imagine a 6-foot-tall man positioning himself so that his shadow, measuring 8 feet, reaches the top of the tree's shadow.

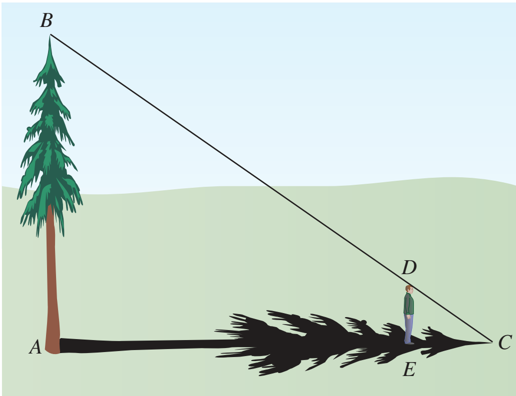


Figure 9.1

The larger triangle made by the tree and its shadow, ΔABC, is **similar** to the smaller triangle made by the man and his shadow, ΔEDC. This means that the lengths of the corresponding sides of the two triangles are proportional and the corresponding angles have equal measure. That is, the ratio of $\overline{AB}$ to $\overline{ED}$ is the same as the ratio of $\overline{AC}$ to $\overline{EC}$ and the ratio of $\overline{BC}$ to $\overline{DC}$. Because these ratios are the same, we say that the three pairs of sides are **proportional**. Angles with equal measure are said to be **congruent**. We write $\angle A \cong \angle E$ and $\angle B \cong \angle D$. The symbol "$\cong$" means "is congruent to."

SIMILAR TRIANGLES

Two triangles, ΔABC and ΔDEF, are similar if

- Corresponding angles are **congruent**.
- Corresponding sides are **proportional**.

Armed with this information, we can now find the height of the tree. We know the following:

Height of person: $\overline{DE} = 6$ feet

Length of person's shadow: $\overline{EC} = 8$ feet

Length of tree's shadow: $\overline{AC} = 504$ feet

Height of tree: $\overline{AB} = H$ feet

Since ΔABC is similar to ΔEDC we know their corresponding sides are proportional. Therefore,

$$\frac{\overline{AB}}{\overline{DE}} = \frac{\overline{AC}}{\overline{EC}}$$

$$\frac{H \text{ feet}}{6 \text{ feet}} = \frac{504 \text{ feet}}{8 \text{ feet}}$$

There are several ways to solve this equation for H, the height of the tree. We choose to cross multiply.

$$\frac{H \text{ feet}}{6 \text{ feet}} = \frac{504 \text{ feet}}{8 \text{ feet}}$$

$$\frac{H}{6} = \frac{504}{8}$$

$$(H)(8) = (6)(504) \qquad \text{Cross multiply.}$$

$$H = \frac{(6)(504)}{8} \qquad \text{Divide by 8.}$$

$$H = 378$$

The height of this hypothetical redwood tree is 378 feet.

■ Making the Connection between Geometry and Trigonometry

The *hypotenuse* of a right triangle is the side opposite the right angle. The *opposite* side for an angle whose measure is θ is the side directly across from the angle. The *adjacent* side for an angle whose measure is θ is the remaining side (see Figure 9.2).

Now let's scale this right triangle so that it fits inside of the unit circle, as shown in Figure 9.3.

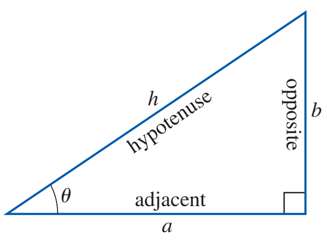


Figure 9.2

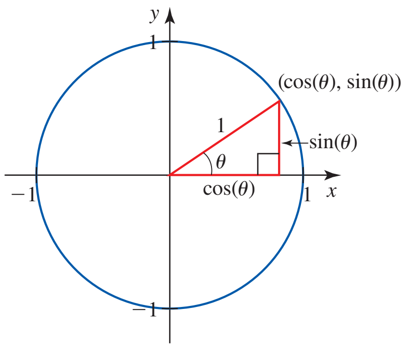


Figure 9.3

Because the original triangle is similar to the resized triangle contained within the unit circle, the side lengths are proportional. We have the following right triangle definitions:

$$\frac{\sin(\theta)}{1} = \frac{b}{h} \qquad \frac{\cos(\theta)}{1} = \frac{a}{h} \qquad \frac{\sin(\theta)}{\cos(\theta)} = \frac{b}{a}$$

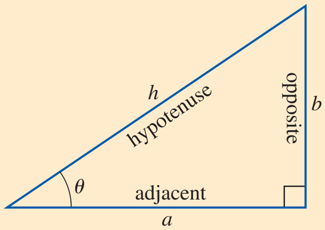
$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

We find the right triangle definitions of the remaining trigonometric functions by taking the reciprocal of these three. We get

$$\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

We summarize the results.

The Six Trigonometric Function Ratios



$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$$

$$\csc(\theta) = \frac{\text{hypotenuse}}{\text{opposite}} \qquad \sec(\theta) = \frac{\text{hypotenuse}}{\text{adjacent}} \qquad \cot(\theta) = \frac{\text{adjacent}}{\text{opposite}}$$

EXAMPLE 1 ■ Using Trigonometric Functions

While standing 200 feet away from the base of the Hyperion redwood tree, a forestry major measures an angle of 62.12° to the top of the tree. What is the height of the tree?

Solution We draw a picture to represent the situation, as shown in Figure 9.4. We then use trigonometry to solve for H.

$$\tan(62.12°) = \frac{H}{200}$$

$$200\tan(62.12°) = H$$

$$200(1.890) \approx H$$

$$H \approx 378$$

The height of the tree is 378 feet.

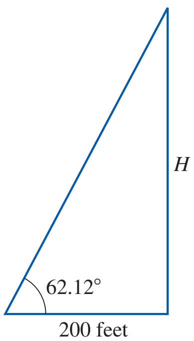


Figure 9.4

EXAMPLE 2 ■ Measuring the Hypotenuse

One of the Seven Wonders of the Ancient World is the Great Pyramid of Giza located near Cairo, Egypt. The pyramid was built by the Egyptian pharaoh Khufu of the Fourth Dynasty around the year 2560 B.C. to serve as a tomb when he died.

(*Source:* ce.eng.usf.edu) The pyramid was 481 feet high when it was built. The angle the side makes with the ground is 51.83°. The square bottom of the pyramid measures roughly 755.3 feet on each side. What is the distance along one of the slanted edges from the ground to the tip of the pyramid?

Solution We begin by drawing a picture of one of the faces of the pyramid and labeling the information we know. See Figure 9.5. The hypotenuse, d, of the right triangle is the length that is unknown; the lengths of the sides opposite and adjacent are known. Therefore, we may use either the cosine or sine ratios to solve for the length, d. To show that either may be used, we will solve the problem both ways.

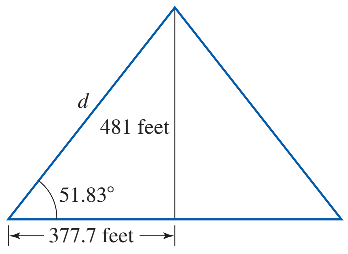


Figure 9.5

$$\cos(51.83°) = \frac{377.7}{d} \qquad \sin(51.83°) = \frac{481}{d}$$

$$d\cos(51.83°) = 377.7 \qquad d\sin(51.83°) = 481$$

$$d = \frac{377.7}{\cos(51.83°)} \qquad d = \frac{481}{\sin(51.83°)}$$

$$d \approx \frac{377.7}{0.6180} \qquad d \approx \frac{481}{0.7862}$$

$$d \approx 611.2 \qquad d \approx 611.8$$

We attribute the slight difference to rounding and inexact measurements.

Angles of Elevation and Depression

Often we are given information about either the **angle of elevation** (Figure 9.6a) or **angle of depression** (Figure 9.6b) when solving problems involving right triangle trigonometry. These angles can be measured using professional tools such as the theodolite or homemade tools such as a transit.

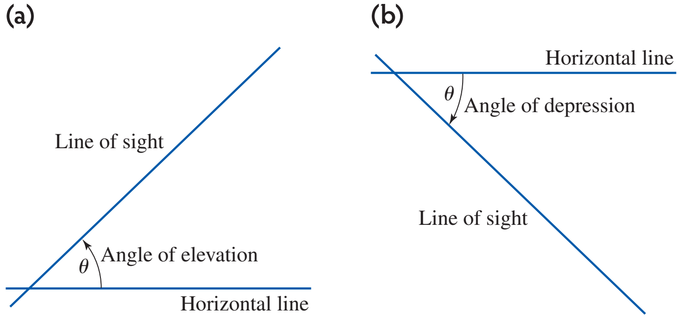


Figure 9.6

Angles of Elevation and Depression

The **angle of elevation** is the angle from an imaginary horizontal line and the observer's line of sight to an object that is *above* the horizontal line.

The **angle of depression** is the angle from an imaginary horizontal line and the observer's line of sight to an object that is *below* the horizontal line.

EXAMPLE 3 ■ Using Angle Measures

Prior to Global Positioning System (GPS) technology, fire lookout towers played a major role in helping people spot forest fires. While some are still in operation, they are being used less and less. Upon spotting a fire, the person in the lookout tower uses a tool called an Osborne Fire Finder (see photo) to determine the distance to the fire. Suppose the fire lookout spotted a fire at an angle of depression of 2.86° and that the height above the ground at the point of spotting the fire was 150 feet. How far from the fire is the base of the fire tower?

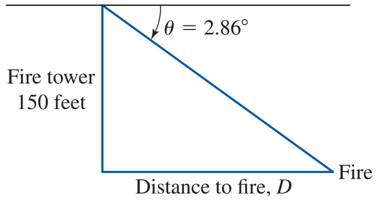


Figure 9.7

Solution We begin with the sketch of the problem situation shown in Figure 9.7. Note that the sketch is not drawn to scale. We consider the right triangle where one of the legs of the triangle is the height of the lookout tower and the other leg is the unknown distance to the fire. The angle of depression, whose measure is $\theta = 2.86°$, is not a part of the triangle that we need to use to solve the problem. However, we use that angle's measure to determine the measure of the angle from the fire tower to the line of sight (hypotenuse): $90° - 2.86° = 87.14°$. Now we use the tangent ratio to solve the problem.

$$\tan(87.14°) = \frac{D}{150}$$

$$150\tan(87.14°) = D$$

$$D = 3002.5$$

The distance from the base of the lookout tower to the fire is about 3002.5 feet.

EXAMPLE 4 ■ Expressing the Six Trigonometric Functions

Given the right triangle shown in Figure 9.8, find the value of each of the following: $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\sec(\theta)$, $\csc(\theta)$, and $\cot(\theta)$.

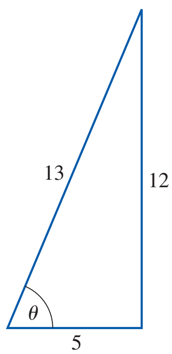


Figure 9.8

Solution We use the definitions of the six trigonometric ratios as follows.

$$\sin(\theta) = \frac{12}{13} \qquad \cos(\theta) = \frac{5}{13} \qquad \tan(\theta) = \frac{12}{5}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{13}{12} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} = \frac{13}{5} \qquad \cot(\theta) = \frac{1}{\tan(\theta)} = \frac{5}{12}$$

■ Determining Angle Measure

Recall that we can use inverse trigonometric functions to determine the measure of an angle when the trigonometric ratio is known. In the next example, we do this to determine the measure of an angle of elevation.

EXAMPLE 5 ■ Calculating an Angle of Elevation

In 1999, two of the authors visited San Francisco for a math conference. While walking along a steep San Francisco street, they wondered what the angle of elevation of the street was. To answer their question, they measured a brick in the foundation of a nearby home, as represented in Figure 9.9. Determine the angle of elevation of the hill.

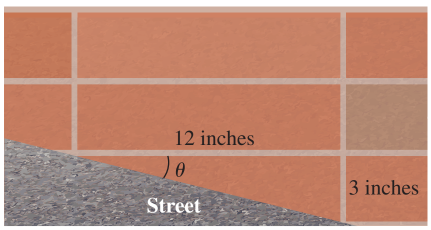


Figure 9.9

Solution We wish to find the measure of the angle the street makes with the hypothetical horizontal line. Since the top and bottom of the brick are parallel lines, this angle has the same measure, θ, as the angle the street makes with the top of the brick. Using this angle as our point of reference, we see that the opposite side length of the right triangle is known (3 inches). Also, the adjacent side length is known (12 inches). Therefore, we use the tangent function to find the required angle measure.

$$\tan(\theta) = \frac{3}{12}$$

$$\theta = \tan^{-1}\left(\frac{3}{12}\right)$$

$$\theta \approx 14.04°$$

The street has an angle of elevation that measures 14.04°.

SUMMARY

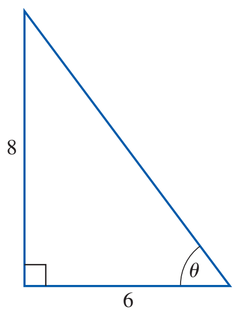
In this section you learned how to solve problems that can be represented using right triangles. You learned the six trigonometric ratios and how to use them to find missing lengths and missing angle measures in right triangles.

9.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–6, evaluate the six trigonometric functions of θ. That is, evaluate $\sin(\theta)$, $\cos(\theta)$, $\tan(\theta)$, $\sec(\theta)$, $\csc(\theta)$, $\cot(\theta)$.

1.



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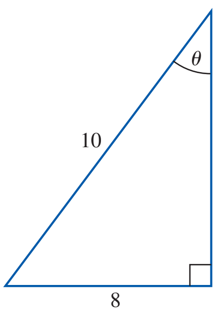

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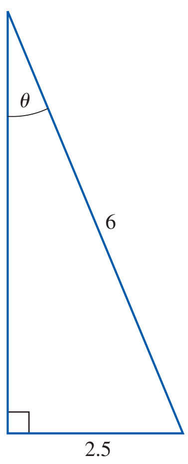

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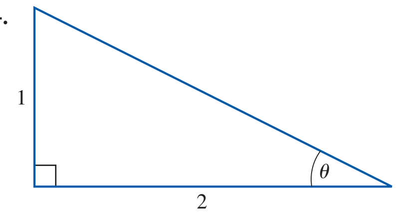

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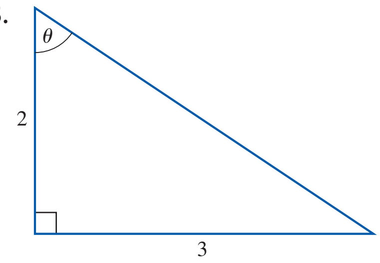

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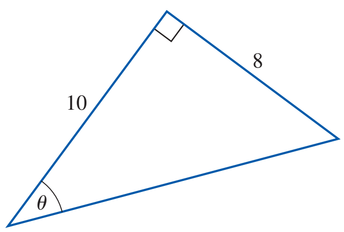
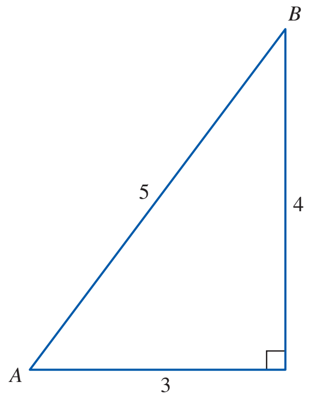


In Exercises 7–11, find each of the values.

a. $\sin(A)$
b. $\cos(B)$
c. $\sin(B)$
d. $\cos(A)$

7.



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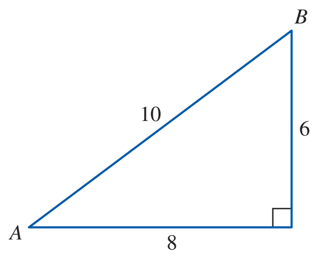

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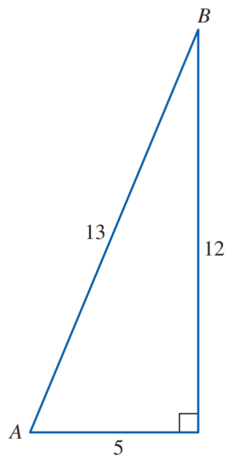

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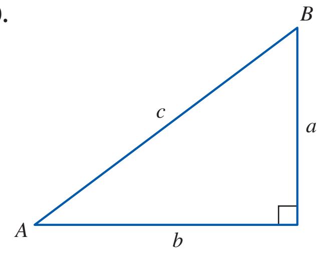
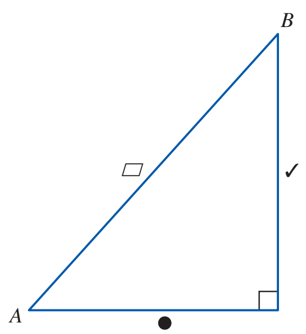


11. (symbols are intentional)



12. Refer back to Exercises 7–11 and express the relationship between the measures of $\angle A$ and $\angle B$. Explain this relationship as clearly as you can.

13. Refer back to Exercises 7–11 and express the relationship between $\sin(A)$ and $\cos(B)$. Explain this relationship as clearly as you can.

14. Refer back to Exercises 7–11 and express the relationship between $\sin(B)$ and $\cos(A)$. Explain this relationship as clearly as you can.

In Exercises 15–23, a trigonometric value is given. Determine if the angle measure, θ, is

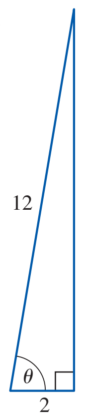
a. Less than 30°.
b. Between 30° and 60°.
c. Between 60° and 90°.

Explain how you know.

15. $\sin(\theta) = 0.15$
16. $\tan(\theta) = 3.8$
17. $\sin(\theta) = 0.65$
18. $\cos(\theta) = 0.95$
19. $\tan(\theta) = 1$
20. $\cos(\theta) = 0.75$
21. $\tan(\theta) = 0.05$
22. $\sin(\theta) = \dfrac{1}{4}$
23. $\cos(\theta) = 0.01$

In Exercises 24–28, determine the indicated angle measure, θ.

24.



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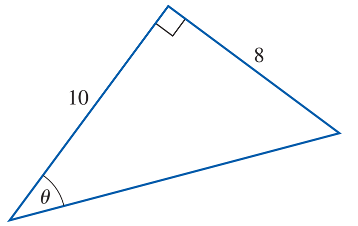

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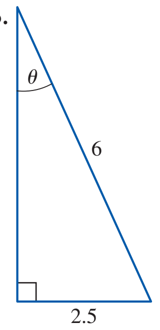

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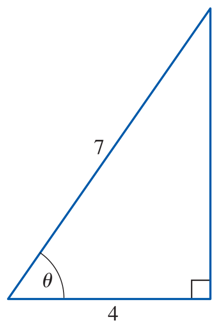

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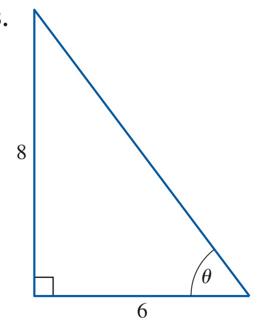


29. Hot Air Balloon To tether a hot air balloon to the ground, long ropes must be used. Suppose that the rope makes an angle of elevation with the ground measuring 30° and is tethered 100 feet from the bottom of the balloon.

a. Assuming the rope is attached to the top of the balloon, what is the height of the balloon?

b. Suppose that when fully inflated, the balloon is 100 feet tall and the angle of elevation of the rope measures 42°. What is the minimum amount of additional rope needed to tether the balloon when fully inflated?

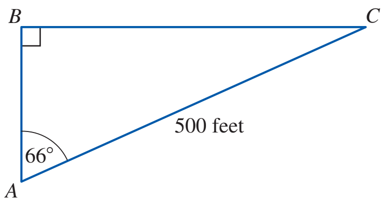
30. Space Shuttle Launch Suppose you are 4000 feet away from the launch pad of the Space Shuttle. You are told that 7 seconds after liftoff, the Space Shuttle is 530 feet above the ground.

a. What is the measure of the angle of elevation from you to the Space Shuttle?

b. What is the average rate of change of the measure of the angle of elevation between liftoff (0 seconds) and 7 seconds?

c. Do you think that the rate of change of the measure of the angle of elevation later in the flight of the shuttle will be faster or slower than the result in part (b)? Explain your answer.

31. Speed Trap A theodolite is used to measure the distance from a particular point (A) to a car traveling down a flat and level road ($\overline{BC}$). Ten seconds later, the device measures the distance to the car and the angle of rotation (C). Using the diagram, determine if the car was speeding. Assume the speed limit is 45 mph.



■ SHOW YOU KNOW

32. You are given a right triangle problem situation to solve. You know the measure of one side and one acute angle. Can you find the measure of every other side and every other angle? If so, explain how. If not, explain why not.

33. You are given a right triangle problem situation to solve. You know the measure of all three sides. Can you find the measure of each of the angles? If so, explain how. If not, explain why not.

34. You are given a right triangle problem situation to solve. You know the measure of all three angles. Can you find the measure of each of the three sides? If so, explain how. If not, explain why not.

35. Is there an angle measure whose sine and cosine are equal? If so, what is it and why are these ratios equal?

36. Which of the three ratios—sin(A), cos(A), or tan(A)—is equal to the slope of a line m? Explain.

■ MAKE IT REAL

37. Pittsburgh Incline Downtown Pittsburgh is located at the intersection of three rivers: the Ohio, the Allegheny, and the Monongahela. In the late 1800s workers came down to the city on a cable car system, or incline, from the residential area located at the top of a ridge. The incline is now a historical site. At the top of the ridge, a sign with some of the facts related to the incline are given. The grade (angle of elevation's measure) is listed as 35.58°. The length of the incline track (the hypotenuse) is 635 feet. What is the vertical distance of the incline track from the bottom to the top of the ridge?

38. Ladder Safety A report describing a firefighter training accident stated that a 24-foot extension ladder was placed against a wall at a 75° angle with the ground. At the time of the accident, the top of the ladder was at a height of 20 feet 1 inch. (***Source:*** **www.cdc.gov**)

a. Was the ladder at its full 24-foot length? If so, explain how you know. If not, how long was the ladder?

b. The report states the ladder "was positioned at an angle of about 75° (4 to 1 ratio)." Use trigonometry to confirm or refute the claim that there was a 4 to 1 ratio. What do you think was meant by this?

39. Largest Escalator The CNN Center in Atlanta, Georgia, has the largest freestanding escalator in the country. It ascends 8 stories (85 feet) and is approximately 200 feet in length.

a. What is the measure of the angle of elevation of the escalator?

b. If it takes 2 minutes to get to the top of the escalator, how fast is the escalator moving?

40. Leaning Tower of Pisa The Leaning Tower of Pisa is leaning because of unstable ground upon which it was built. According to the following statement: *"The inclination is c. $5\frac{1}{2}$ degrees towards the south; this means that the seventh cornice protrudes about 4.5m over the first cornice."* (***Source:*** **terre.duomo.pisa.it**)

The horizontal protrusions at each level are referred to as cornices. What is the vertical distance between the first and the seventh cornice?

41. Steep Grade Road Sign In mountainous regions, it is common to see signs warning drivers of a steep grade. If a sign warns of a 14% grade, that means that for every 100 feet traveled horizontally, the elevation will decrease by 14 feet. Determine the angle of elevation on a portion of roadway with a 14% grade.

42. Evel Knievel's Rocket Launching On September 8, 1974, daredevil Evel Knievel attempted to jump the Snake River Canyon in southern Idaho. A special rocket was built for the stunt. It was launched off a ramp shown in the picture. The ramp was 300 feet in length and rose vertically 200 feet. (***Source:*** **www.canosoarus.com**)

a. Calculate the measure of the angle of elevation that the ramp makes with the ground.

b. The horizontal crossbeam is located halfway to the top of the ramp. Calculate the length of this crossbeam.

43. Parasailor A parasailor is tethered to a boat with a 110-foot cable and is 74 feet above the point of attachment.

a. What is the measure of the angle of depression from the parasailor to the boat?

b. How far, horizontally, is the parasailor from the boat?

44. Hoover Dam Two trigonometry students wrote the following in a report they created about Hoover Dam.

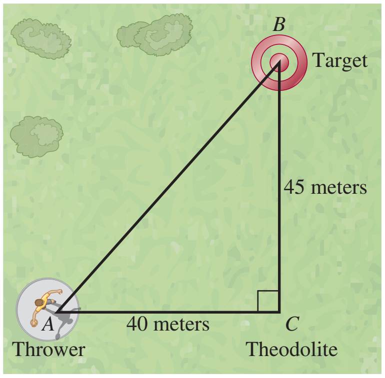
> My friend stood at the top of the dam and used a theodolite to measure the angle of depression to my position, 1558 feet from the base of the dam. The measure of the angle of depression was 25°. Using right triangle trigonometry, we calculated the height of Hoover Dam to be 726 feet.

Discuss the accuracy of the report. Are the values reported by the students reasonably accurate? Why or why not? (*Note:* The true height of Hoover Dam is 726.4 feet.) (***Source:*** **www.usbr.gov**)

45. Trigonometry in Sports During the Olympic Games, judges use a theodolite to measure angles and distances in track and field events such as the discus, hammer throw, and shot put. (***Source:*** **www.leica-geosystems.com**)

In the discus event, a referee sticks a target into the ground at the place the discus hits the turf. The theodolite measures the distance to the target using lasers.

Suppose a discus thrower (A) tosses the discus and the discus lands at point (B). A theodolite is set up 40 meters from the throwing circle (C). The theodolite calculates the distance to the target (B) is 45 meters. What is the official distance of the throw? What is the measure of angle A?



■ STRETCH YOUR MIND

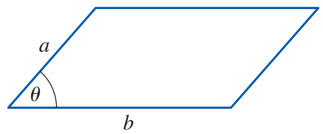
Exercises 46–52 are intended to challenge your understanding of right triangle trigonometry.

46. Sketch three right triangles, ΔABC, such that $\sin(A) = \frac{2}{3}$. In each, label all side and angle measures.

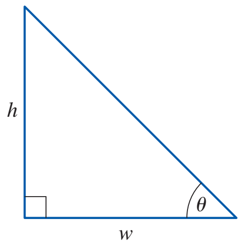
47. Sketch three right triangles, ΔABC, such that $\cot(A) = 2$. In each, label all side and angle measures.

48. Sketch three right triangles, ΔABC, such that $\sec(A) = \frac{5}{3}$. In each, label all side and angle measures.

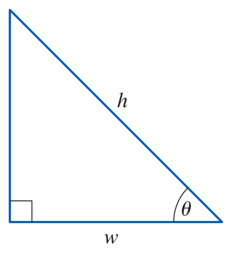
49. Describe how right triangle trigonometry can be used to find the area of a parallelogram.



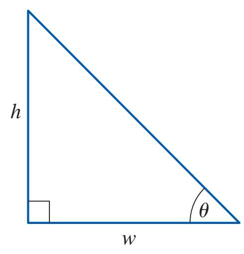
50. In the diagram, suppose the length w is fixed. Write two equations that relate the length h and the angle measure θ.



51. In the diagram, suppose the length w is fixed. Write two equations that relate the length h and the angle measure θ.



52. In the diagram, suppose the length w is fixed while h is free to vary. Express the function $\theta(h)$ and explain what information it provides.



SECTION 9.2 Law of Cosines

LEARNING OBJECTIVES

- Develop the formula for the Law of Cosines
- Demonstrate the relationship between the Pythagorean theorem and the Law of Cosines
- Apply the Law of Cosines to real-world scenarios

GETTING STARTED

The usefulness of trigonometry in surveying, navigation, and astronomy has been seen throughout history. The Egyptians used concepts from trigonometry in land surveying and in building the pyramids. The Greeks and Babylonians, pioneers in the field of astronomy, used trigonometry to develop early theories for planetary position and motion. Today, trigonometry is also used in physics to further our understanding of space. In 2004, NASA's Gravity Probe B satellite was launched into orbit to test a key prediction of Albert Einstein's general theory of relativity: that the fabric that makes up space and time is curved and dragged by the gravity of celestial bodies such as Earth. (*Source:* www.einstein.stanford.edu) To test this hypothesis, scientists employed a law of trigonometry known as the Law of Cosines.

In this section we develop the formula for the Law of Cosines, demonstrate the relationship between the Pythagorean theorem and the Law of Cosines, and apply the law to real-world scenarios.

■ Types of Triangles

If none of the angles in a triangle is a right angle, the triangle is called **oblique**. Each angle of the triangle is either **acute** (greater than 0° but less than 90°) or **obtuse** (greater than 90° but less than 180°). An oblique triangle with three acute angles is called an **acute triangle** (Figure 9.10a). An oblique triangle with two acute angles and one obtuse angle is called an **obtuse triangle** (Figure 9.10b).

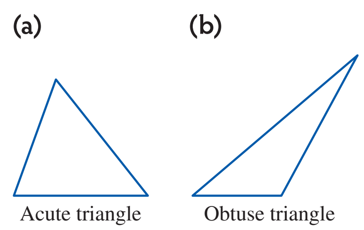


Figure 9.10

As before, we label angles with capital letters such as A, B, and C and the sides directly across from these angles with their corresponding lowercase letters such as a, b, and c. See Figure 9.11.

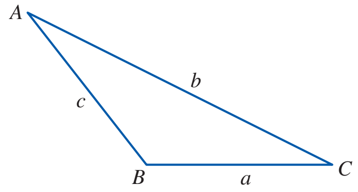


Figure 9.11

The phrase **solve a triangle** means to find the lengths of its sides and the measurements of its angles. To solve any triangle, we only need to know the length of one side and two other pieces of information: two more sides, a side and an angle, or two angles. Letting S represent a side length and A represent an angle measure, the three pieces of information may be arranged as shown in Table 9.1 and Figure 9.12.

Table 9.1

Case	Meaning	Description
SSS	Side–Side–Side	Three sides
SAS	Side–Angle–Side	Two sides and the angle between them
ASA	Angle–Side–Angle	Two angles and the side shared by the angles
SAA	Side–Angle–Angle	Two angles and a side opposite one angle
SSA	Side–Side–Angle	Two sides and an angle opposite one side

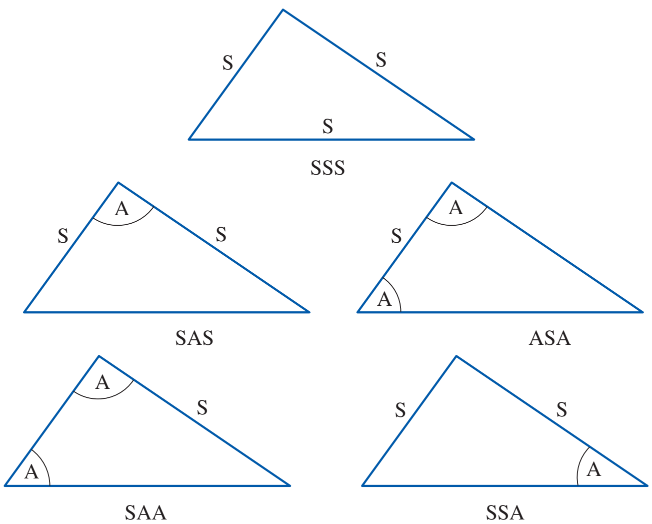


Figure 9.12

■ Law of Cosines

The Law of Cosines is used to solve oblique triangles with all three sides known (SSS) and triangles with two sides and the included angle (SAS) given. (We will investigate the remaining cases (ASA, SAA, SSA) when we investigate the Law of Sines in the next section.)

EXAMPLE 1 ■ **Solving an Oblique Triangle Using Right Triangle Trigonometry**

Solve the oblique triangle shown in Figure 9.13.

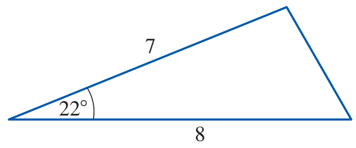


Figure 9.13

Solution All of our trigonometric tools up to this point rely on right triangles, and this is not a right triangle. However, we can turn this oblique triangle into two right triangles by drawing an **altitude**, a line segment drawn from one vertex perpendicular to the opposite side. We also add labels to the unknown sides and angles. See Figure 9.14. Notice how the altitude breaks up the 8-unit side into two pieces. We do not know the exact length of each piece, so we use x and $8 - x$ to represent their lengths, noting that their sum must be 8 units. As shown in Figure 9.15, we begin solving with the right triangle on the left, using a cosine and sine ratio to find the values of x and h, respectively.

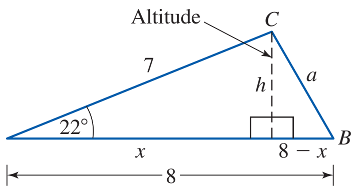


Figure 9.14

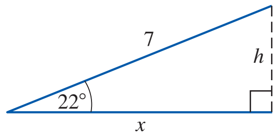


Figure 9.15

$$\cos(22°) = \frac{x}{7} \qquad \sin(22°) = \frac{h}{7}$$
$$x = 7\cos(22°) \qquad h = 7\sin(22°)$$
$$x \approx 6.49 \qquad h \approx 2.62$$

Now we solve the triangle on the right. Since $x \approx 6.49$, then $8 - x \approx 1.51$, and since $h \approx 2.62$, then we get the triangle shown in Figure 9.16. The known sides allow us to set up a tangent ratio to solve for the measure of angle B.

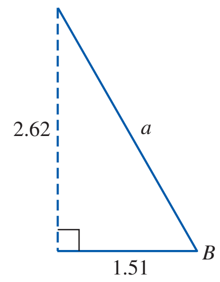


Figure 9.16

$$\tan(B) \approx \frac{2.62}{1.51}$$
$$B \approx \tan^{-1}\left(\frac{2.62}{1.51}\right)$$
$$B \approx 60.0°$$

We can solve for the length of a using either a trigonometric ratio or the Pythagorean theorem with the smaller right triangle. We use the Pythagorean theorem.

$$1.51^2 + 2.62^2 \approx a^2$$
$$a^2 \approx 9.14$$
$$a \approx \sqrt{9.14}$$
$$a \approx 3.02$$

Since the sum of the angle measures of a triangle is 180°, we complete the solution by finding the measure of angle C, as follows. The solved triangle is illustrated in Figure 9.17.

$$180° - 22° - 60.0° = 98.0°$$

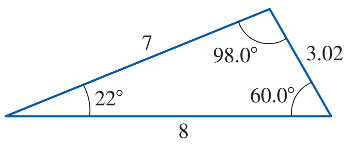


Figure 9.17

Example 1 demonstrates it is possible to solve oblique triangles using basic trigonometric ratios. However, the process can be tedious and time consuming. The Law of Cosines is a shorter process that will achieve the same goal.

Developing the Law of Cosines

To develop the Law of Cosines, we begin with oblique triangle ABC. As in Example 1, we begin by drawing an altitude, as in Figure 9.18.

If we apply the Pythagorean theorem to the triangle on the right, we get

$$(a - x)^2 + h^2 = c^2$$
$$a^2 - 2ax + x^2 + h^2 = c^2$$

Likewise, if we apply the Pythagorean theorem to the triangle on the left, we get

$$x^2 + h^2 = b^2$$

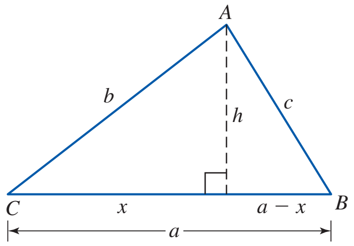


Figure 9.18

Since $x^2 + h^2 = b^2$, we may substitute this result into the first equation.

$$a^2 - 2ax + (x^2 + h^2) = c^2$$
$$a^2 - 2ax + b^2 = c^2 \quad \text{Substitute } b^2 \text{ for } x^2 + h^2.$$
$$a^2 + b^2 - 2ax = c^2 \quad \text{Rearrange terms.}$$

Observe from the triangle that $\cos(C) = \dfrac{x}{b}$. Therefore, $x = b\cos(C)$. We use this fact to eliminate the x in the equation.

$$a^2 + b^2 - 2a(x) = c^2$$
$$a^2 + b^2 - 2a(b\cos(C)) = c^2$$

This result is called the **Law of Cosines**.

Law of Cosines

For ΔABC, $c^2 = a^2 + b^2 - 2ab\cos(C)$, where c is the length of the side opposite angle C. Similarly, $a^2 = b^2 + c^2 - 2bc\cos(A)$ and $b^2 = a^2 + c^2 - 2ac\cos(B)$.

EXAMPLE 2 ■ Solving an SSS Triangle

Solve the triangle ABC shown in Figure 9.19. The measures of the three sides are $a = 4$, $b = 3$, and $c = 6$.

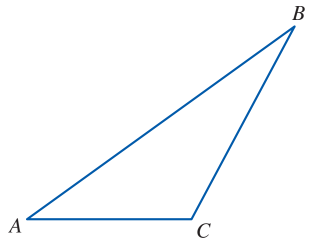


Figure 9.19

Solution We need to find the measurements of its angles since we know the length of all three sides. We use the Law of Cosines $c^2 = a^2 + b^2 - 2ab\cos(C)$ with $a = 4$, $b = 3$, and $c = 6$, as shown in Figure 9.20. To find angle C, we have

$$6^2 = 4^2 + 3^2 - 2(4)(3)\cos(C)$$
$$36 = 16 + 9 - 24\cos(C)$$
$$11 = -24\cos(C)$$
$$\frac{11}{-24} = \cos(C)$$
$$-0.4583 = \cos(C)$$
$$\cos^{-1}(-0.4583) = C$$
$$117.28° = C$$

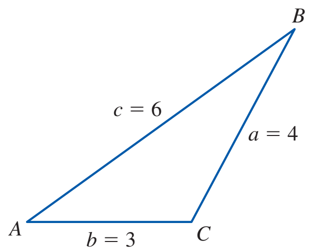


Figure 9.20

We can now find angle B by using a variation of the Law of Cosines.

$$\begin{aligned}
b^2 &= a^2 + c^2 - 2ac\cos(B) \\
3^2 &= 4^2 + 6^2 - 2(4)(6)\cos(B) \\
9 &= 16 + 36 - 48\cos(B) \\
-43 &= -48\cos(B) \\
\frac{-43}{-48} &= \cos(B) \\
0.8958 &= \cos(B) \\
\cos^{-1}(0.8958) &= B \\
26.39° &= B
\end{aligned}$$

Since the sum of the angle measures of a triangle is 180°, the measure of angle A is given by

$$180° - 117.28° - 26.39° = 36.33°$$

Thus $A = 36.33°$, $B = 26.39°$, and $C = 117.28°$.

The Law of Cosines and the Pythagorean Theorem

The Law of Cosines and the Pythagorean theorem are related. Observe the Law of Cosines ($c^2 = a^2 + b^2 - 2ab\cos(C)$) and the Pythagorean theorem ($c^2 = a^2 + b^2$) differ only by the term $-2ab\cos(C)$. Whenever $\cos(C) = 0$, the two equations are equal. Since $\cos^{-1}(0) = 90°$, the Law of Cosines and Pythagorean theorem are equivalent whenever $C = 90°$. Thus the Pythagorean theorem is a special case of the Law of Cosines that works specifically with right triangles.

EXAMPLE 3 ■ Applying the Law of Cosines

In a follow-up study to the 2004 Gravity Probe B investigation mentioned earlier, a physics experiment on a grand scale has been initiated. At NASA's Jet Propulsion Laboratory, scientists are testing Einstein's theory of relativity, which states that the Sun's gravity causes starlight to bend, shifting the apparent position of stars in the sky. Using the Solar System as a giant laboratory, the scientists plan to test two seemingly competing theories regarding the nature and behavior of space and time: Einstein's theory of relativity and the more recent idea of quantum mechanics. Einstein's theory states that gravity from celestial bodies such as the Sun causes light to bend. Quantum mechanics says that gravity bends light because it causes space to warp, as though space and time were a sheet of flat flexible material that depresses under the weight of celestial objects. The depressions in the sheet cause light to bend or curve slightly as it passes by the large objects. (See illustration at left.) (For more information go to science.nasa.gov/headlines/y2004/26mar_einstein.htm.)

The experiment, known as the Laser Astrometric Test Of Relativity (LATOR), will look at how the sun's gravity deflects beams of laser light emitted by two small satellites. The two satellites will orbit on opposite sides of the Sun at about the same distance from the Sun as Earth. This will form an oblique triangle between Earth and the satellites, with laser beams as its sides, as shown in Figure 9.21. Note that one beam passes very close to the Sun. As is shown in the figure, the scientists

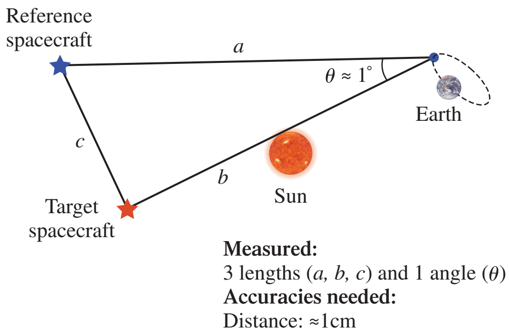


Figure 9.21
Source: NASA

anticipate that the angle between the two satellites as viewed from Earth will be approximately 1°. We want to verify that prediction.

Using Figure 9.21 and assuming the distance from Earth to the target spacecraft is 297 million kilometers, determine the measure of angle θ. Also find the other two angles formed by the spacecrafts and Earth.

Solution We use the Law of Cosines because we know all three side lengths of the triangle (SSS) and do not know any of the angle measures. We denote the sides as $a = 300$ million kilometers, $b = 297$ million kilometers, and $c = 5$ million kilometers. We want to find θ, which we denote as C in our formula.

$$
\begin{aligned}
c^2 &= a^2 + b^2 - 2ab\cos(C) \\
5^2 &= 300^2 + 297^2 - 2(300)(297)\cos(C) && \textit{c = 5, a = 300, b = 297} \\
25 &= 90{,}000 + 88{,}209 - 178{,}200\cos(C) && \text{Solve for angle } C. \\
-178{,}184 &= -178{,}200\cos(C) \\
\frac{-178{,}184}{-178{,}200} &= \cos(C) \\
0.9999102 &= \cos(C) \\
\cos^{-1}(0.9999102) &= C \\
0.768^\circ &= C \\
C &\approx 1^\circ
\end{aligned}
$$

We see angle C rounds to 1°, as the scientists predict.

To find the next angle, we use an equivalent form of the Law of Cosines.

$$
\begin{aligned}
a^2 &= b^2 + c^2 - 2bc\cos(A) \\
300^2 &= 297^2 + 5^2 - 2(297)(5)\cos(A) && \textit{c = 5, a = 300, b = 297} \\
90{,}000 &= 88{,}209 + 25 - 2970\cos(A) && \text{Solve for angle } A. \\
1766 &= -2970\cos(A) \\
\frac{1766}{-2970} &= \cos(A) \\
-0.5946 &= \cos(A) \\
\cos^{-1}(-0.5946) &= A \\
126.48^\circ &= A
\end{aligned}
$$

To find the measure of angle B we use the fact that the sum of the angles of a triangle is 180°.

$$
\begin{aligned}
B &= 180^\circ - 0.768^\circ - 126.48^\circ \\
B &= 52.75^\circ
\end{aligned}
$$

The solved triangle is shown in Figure 9.22.

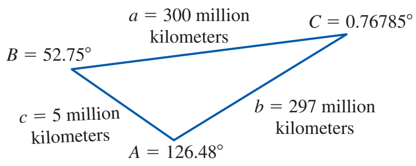


Figure 9.22

EXAMPLE 4 ■ Applying the Law of Cosines

At the 2000 Summer Olympics in Sydney, Australia, surveying equipment was used to save valuable television transmission time and to avoid long, unnecessary delays. A *total station* (an advanced, electronic theodolite) was set up near all discus, shot-put, and javelin events. The total station measured angles and distances quickly and recorded the thrown distances to millimeter accuracy. (*Source:* www.leica-geosystems.com)

Figure 9.23 displays the discus event. The total station is set up prior to each event a known distance from the thrower. After a contestant throws the discus, a referee sticks a target in the ground at the impact location. The surveyor then points the telescope of the total station to the target and reads the distance using a laser aimed at the landing location. Using the figure, find the distance d the discus was thrown.

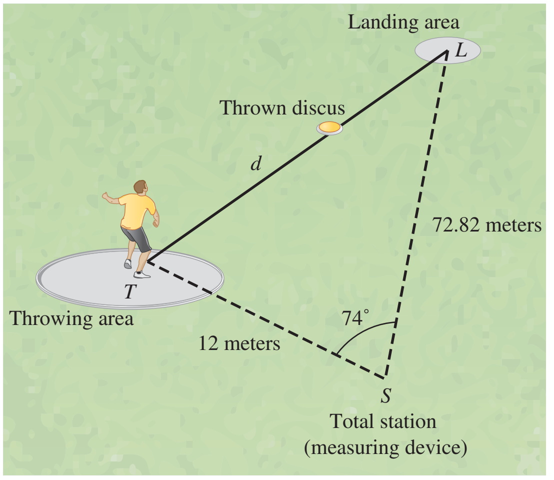


Figure 9.23

Solution We need to find the measure of d, the distance the discus was thrown. Since we are given two sides and the included angle (SAS), we can use the Law of Cosines to find d.

$$d^2 = 72.82^2 + 12^2 - 2(72.82)(12)\cos(74°)$$
$$d^2 = 5302.75 + 144 - 1747.68\cos(74°)$$
$$d^2 = 5446.75 - 1747.68\cos(74°)$$
$$d^2 = 5446.75 - 481.73$$
$$d^2 = 4965.02$$
$$d = \sqrt{4965.02}$$
$$d = 70.46$$

The discus was thrown 70.46 meters.

EXAMPLE 5 ■ Choosing between Techniques

Given the triangles in Figures 9.24–9.26, state whether you can use the Pythagorean theorem, a trigonometric function, the Law of Cosines, or a combination of these to solve each triangle.

a.

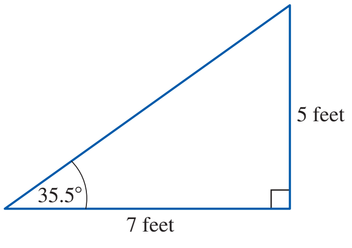


Figure 9.24

b.

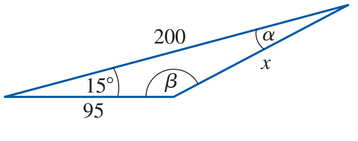


Figure 9.25

c.

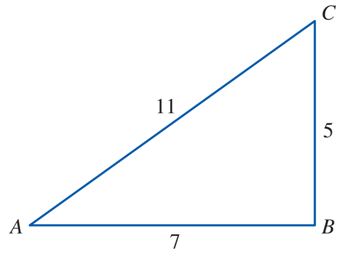


Figure 9.26

Solution

a. The triangle is a right triangle and we know the lengths of two sides. Therefore, we can use the Pythagorean theorem to find the length of the third side. However, we could use the trigonometric functions cosine or secant as well, since we are seeking the length of the hypotenuse. To find the measure of the third angle, we can use the fact that the sum of the angle measures of a triangle is 180°.

b. Since we are given SAS and do not know the measure of the remaining two angles, we would use the Law of Cosines to solve the triangle.

c. Even though the triangle looks like a right triangle, we do not have enough information to determine if it is. Therefore, since we are given the length of all three sides (SSS), we would use the Law of Cosines to solve the triangle.

SUMMARY

In this section you learned the Law of Cosines and its relationship to the Pythagorean theorem. You also learned to apply the Law of Cosines to real-world scenarios.

9.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10, find the length of the side opposite the given angle.

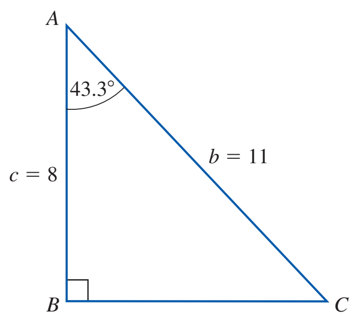
1. In ΔABC, $b = 4$, $c = 5$, and $A = 53°$
2. In ΔABC, $a = 7$, $c = 10$, and $B = 37°$
3. In ΔPQR, $p = 3$, $q = 2$, and $R = 131°$
4. In ΔJKL, $j = 8$, $k = 5$, and $L = 175°$
5. In ΔEFG, $e = 36.5$, $g = 46.3$, and $F = 71°$
6. In ΔTRE, $r = 2.875$, $e = 6.013$, and $T = 116°$
7. In ΔMNO, $m = 6$, $n = 2$, and $O = 31°$
8. In ΔABC, $a = 4$, $b = 2$, and $C = 35°$
9. In ΔABC, $a = 1$, $b = 3$, and $C = 28°$
10. In ΔABC, $a = 7$, $b = 8$, and $C = 10°$

In Exercises 11–20, find the measure of the specified angle.

11. $\angle A$ in ΔABC; if $a = 2$, $b = 3$, and $c = 4$
12. $\angle C$ in ΔABC; if $a = 6$, $b = 7$, and $c = 9$
13. $\angle T$ in ΔPAT; if $p = 6$, $a = 7$, and $t = 12$
14. $\angle E$ in ΔMEG; if $m = 13$, $e = 21$, and $g = 17$
15. $\angle Y$ in ΔXYZ; if $x = 7$, $y = 5$, and $z = 11$
16. $\angle O$ in ΔNOT; if $n = 1604$, $o = 2143$, and $t = 1448$
17. $\angle Q$ in ΔSQR; if $s = 1643$, $q = 2455$, and $r = 1998$
18. $\angle A$ in ΔABC; if $a = 3$, $b = 5$, and $c = 6$
19. $\angle C$ in ΔABC; if $a = 5$, $b = 6$, and $c = 8$
20. $\angle C$ in ΔABC; if $a = 5$, $b = 5$, and $c = 9$

In Exercises 21–26, solve each triangle using any correct technique.

21.



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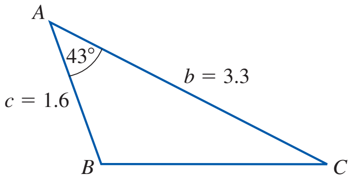

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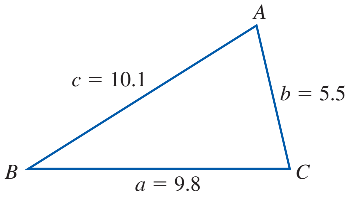

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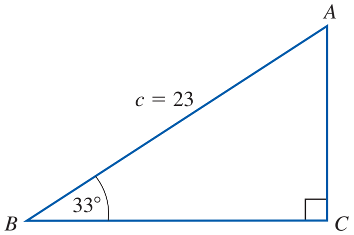
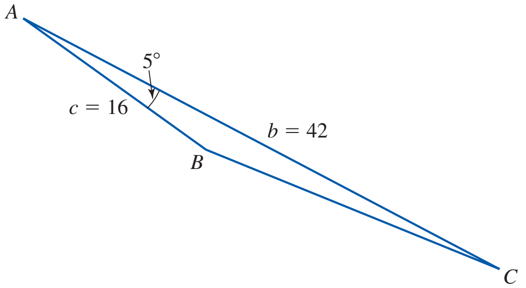
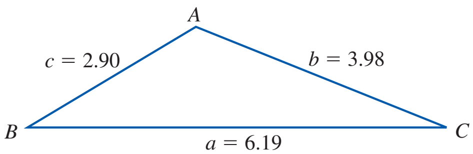


25.

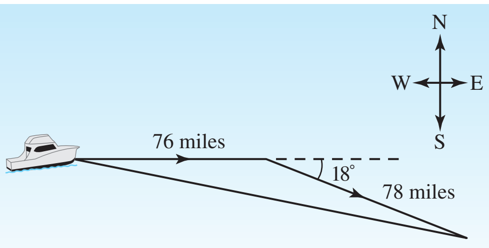


26.



27. Height of a Flagpole A student views the flagpole outside her school and wonders how tall it is. The flagpole has a shadow 66 feet, 5 inches long. She estimates the measure of the angle of elevation to the top of the flagpole to be 36°. How tall is the pole?

28. Ship at Sea A ship travels 76 miles due east, then adjusts its course 18° southward. After traveling 78 miles, how far is the ship from where it began?



SHOW YOU KNOW

29. Explain how the Law of Cosines and the Pythagorean theorem are related. Use diagrams, tables, and/or graphs in your explanation.

30. Describe how you know when the Law of Cosines can be used to solve a triangle.

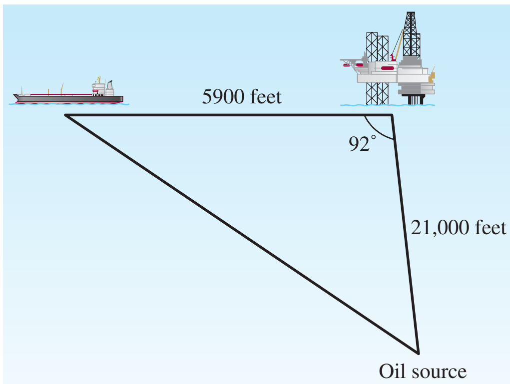
31. Is there such a triangle that allows you to find the third side with either the Pythagorean theorem or the Law of Cosines? Justify your answer.

32. As C changes, describe what happens to the term $-2ab\cos(C)$ in the Law of Cosines. Make sure to explain what happens when $0° < C < 90°$, when $C = 90°$, and when $90° < C < 180°$.

MAKE IT REAL

In Exercises 33–40, draw a diagram when one is not supplied. Then apply the Law of Cosines, the Pythagorean theorem, or a trigonometric function to find the length of the specified side or measure of the specified angle.

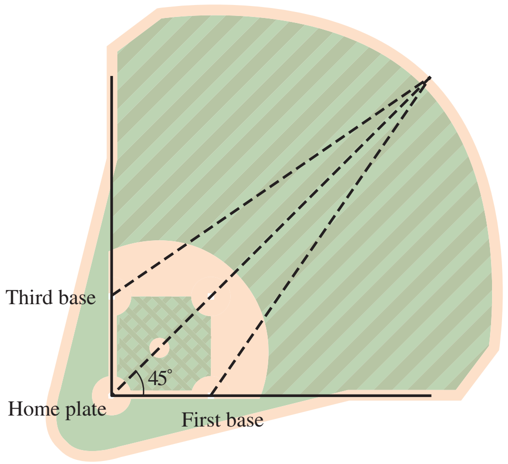
33. Depth of Oil With oil prices high and demand outstripping supply, oil companies are venturing into rugged areas that were previously ignored. Geologists work with engineers to locate oil-rich areas offshore. Recent advances in technology make it possible to access oil reserves as deep as 21,000 feet below the water's surface. Computers monitor the treacherous currents swirling around floating oil rigs and adjust drills to make sure they remain on course. (***Source:*** **www.msnbc.msn.com**) Use the diagram to answer the following questions.



a. How far is the oil tanker from the oil source?

b. At what angle measure is the tanker from the oil source in relation to the oil rig?

34. Yankee Stadium Yankee Stadium, home of the New York Yankees, has a distance of 408 feet from home plate to dead center field. (*Source:* **www.ballparks.com**) The distance from one base to the next base is 90 feet for all major league fields. How far is it from dead center to first base? Third base?

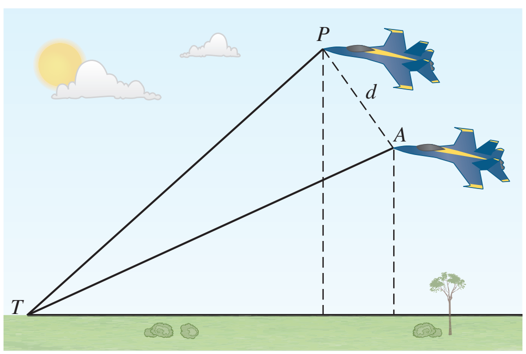


35. Submarine Races The nuclear-powered USS Triton was the first submarine to go completely around the world without surfacing, a feat she accomplished in 1960. Her underwater speed was estimated to be around 21 knots. In contrast, one of the United States' most recent submarines, the USS Virginia, has an underwater speed of 25 knots. (*Source:* **www.navy.mil**)

Although built decades apart, suppose these two submarines engaged in an underwater race. The submarines start at the same point and travel for 3 hours in different directions. Given that the angle between their paths measures 28°, how far apart are they at the end of the 3 hours? (*Hint:* 1 knot = 1.15 miles per hour.)

36. Airplane Distances The Blue Angels, the Navy's elite flying squadron, have performed for more than 427 million fans since their inception in 1946. (*Source:* **www.blueangels.navy.mil**)

Suppose the two Blue Angel jets at points P and A in the diagram have elevations of 1575 feet and 1000 feet, respectively. Both are flying east toward the viewing crowd at T. From T, the measure of the angle of elevation of jet P is 5°, and the measure of the angle of elevation of jet A is 2.8°.



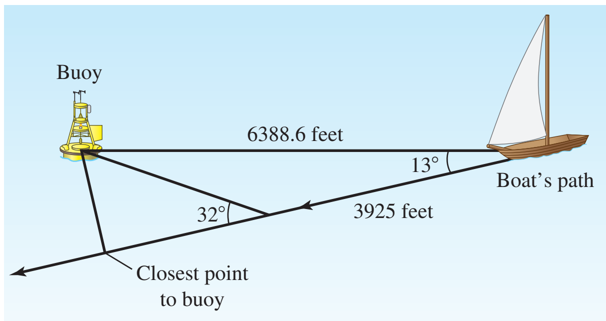
a. How far from the crowd is each of the jets?

b. How far in front of jet A is the nose tip of jet P?

c. What is the ground distance from the nose tip of each jet from the crowd?

37. Sailboats and Sea Lions The National Marine Fisheries Service has issued mandatory regulations requiring people stay at least 100 yards from certain marine mammals in the wild, including sea lions. (*Source:* **http://www.nmfs.noaa.gov**)

As a sailboat sails through a harbor, the captain sights some sea lions lounging on a buoy at an angle measuring 13° to the path of the boat. The sailboat then goes another 3925 feet and records that the buoy now makes an angle measuring 32°.



a. How far is the sailboat from the sea lions at the second sighting if at the first sighting the distance to the buoy was 6388.6 feet?

b. The children onboard want a close-up view of the sea lions. What is the closest they will come to the sea lions if the boat stays on it current course? At the closest point, will the boat be a legal distance from the mammals?

c. The ship continues past the buoy on the same path. After going 20,000 feet past the buoy, one child takes one last look at the sea lions. At what angle is the child's line of sight back to the buoy?

38. Airport Positions The distance between the Baltimore, Maryland, airport (BWI) and the Albuquerque, New Mexico, airport (ABQ) is 1448.1 nautical miles. The distance between BWI and the Fresno, California, airport (FAT) is 2023.2 nautical miles. The distance between ABQ and FAT is 645.2 nautical miles. (*Source:* **www.cgsnetwork.com**) With the three airports being the vertices of a triangle, what is the measure of the angle at BWI?

39. Aircraft Landing The process of landing an aircraft is one of the most difficult challenges a pilot encounters. In the late 1920s airports used rotating lights to mark landing fields after dark. Approach lighting was added in the early 1930s to indicate to pilots the correct angle of descent. (*Source:* **www.centennialofflight.gov**)

Suppose an airplane flying at an altitude of 8 miles has an angle of depression measuring 7.78° to a landing strip.

a. What is the plane's ground distance from the landing strip?

b. If the plane descends on a path with a constant angle of depression measuring 7.78°, how far will it travel in air distance before it touches down?

40. **Mountain Height** Radhanath Sikdar, a mathematician and surveyor from India, calculated the height of Mount Qomolangma (Mount Everest) in 1852. While standing a distance of 240 kilometers from the peak, he determined that the height of the mountain was 8839 meters. (***Source:*** **www.wikipedia.com**)

a. What was the measure of the angle of elevation from his position to the peak?

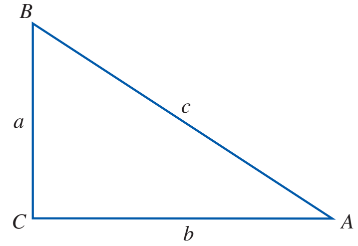
b. If he would have made his calculation from 100 kilometers closer to the peak, what would have been the measure of the angle of elevation?

STRETCH YOUR MIND

Exercises 41–44 are intended to challenge your understanding of the Law of Cosines.

41. For a certain triangle with side lengths a and b of a given length, demonstrate the effect the term $-2ab\cos(C)$ from the Law of Cosines has on the measure of the third side c as the measure of angle C grows larger than 90° and approaches 180°. Use diagrams as a part of your explanation.

42. From the Pythagorean theorem, we know that the square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the other two side lengths ($c^2 = a^2 + b^2$) with a, b, and c as shown below in ΔABC.



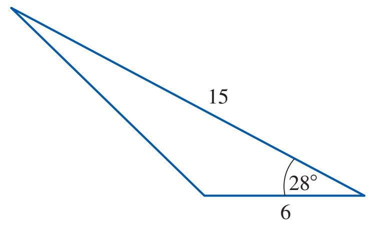
a. What is the effect on the relationship of $c^2 = a^2 + b^2$ if $\angle C$ becomes acute while a and b remain constant?

b. What is the effect on the relationship of $c^2 = a^2 + b^2$ if $\angle C$ becomes obtuse while a and b remain constant?

c. Is there an $\angle C$ that maintains the relationship $c^2 = a^2 + b^2$? Explain.

43. In ΔXYZ, $x = 3$, $y = 7$, and $z = 11$. Explain why you cannot find the measure of angle Z.

44. Solve the following oblique triangle using right triangle trigonometry and not the Law of Cosines.



SECTION 9.3 Law of Sines

LEARNING OBJECTIVES

- Develop the formula for the Law of Sines
- Apply the Law of Sines to real-world scenarios

GETTING STARTED

After years of lobbying the federal government and private fundraising, city leaders from St. Louis, Missouri, secured the approval and resources necessary to build a monument to revitalize the city. The Jefferson National Expansion Memorial (commonly called the Gateway Arch) on the shore of the Mississippi River would commemorate the westward expansion of the early American pioneers who faced and overcame daunting odds. In 1948, after the biggest nationwide architectural design competition in American history, a committee chose the plan of a little-known engineer, Eero Saarinen, for its deceptively simple, yet elegant stainless steel arch structure. At 630 feet tall and 630 feet wide, the arch would be more than twice as tall as the Statue of Liberty.

During construction from 1963 to 1965, engineers made numerous important mathematical calculations to ensure the "Gateway to the West" would be the masterpiece they had all envisioned. (*Source:* www.nps.gov/jeff) Specifically, they used the Law of Sines to keep each leg centered correctly. In this section, we will develop the formula for the Law of Sines, apply it to real-world scenarios, and use it in solving triangles.

Law of Sines

When solving oblique triangles for missing angle measures and side lengths, we use the Law of Sines to compute the remaining side lengths of a triangle if two angle measures and a side length are known (AAS or ASA) or when two side lengths and one of the nonincluded angle measures are known (SSA). Before we develop the Law of Sines, let's analyze these types of triangles using basic trigonometric ratios.

EXAMPLE 1 ■ **Solving an Oblique Triangle Using Right Triangle Trigonometry**

Solve the oblique triangle shown in Figure 9.27.

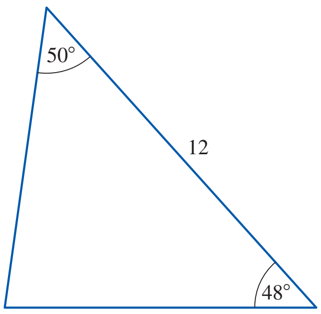


Figure 9.27

Solution A logical first step is to find the measure of the missing angle. Since the sum of the angle measures in a triangle is 180°, we have

$$180° - 48° - 50° = 82°$$

We draw an altitude to create two right triangles, then label the diagram as in Figure 9.28. Starting with the triangle on the right (Figure 9.29), we use a sine and cosine ratio to find the lengths of h and x, respectively.

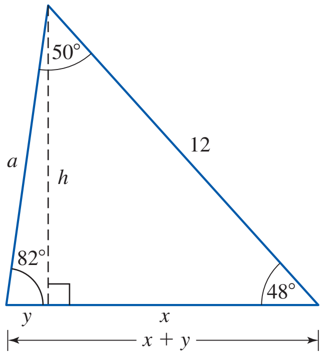


Figure 9.28

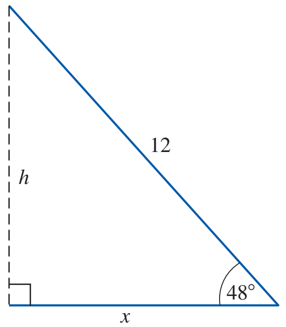


Figure 9.29

$$\sin(48°) = \frac{h}{12} \qquad \cos(48°) = \frac{x}{12}$$

$$h = 12\sin(48°) \qquad x = 12\cos(48°)$$

$$h \approx 8.92 \qquad x \approx 8.03$$

Now that we know the value of h, we use the triangle on the left (Figure 9.30) and a tangent and sine ratio to find the lengths of y and a, respectively.

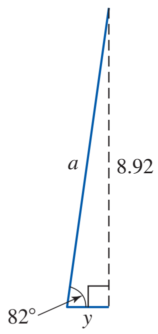


Figure 9.30

$$\tan(82°) = \frac{h}{y} \qquad \sin(82°) = \frac{h}{a}$$

$$y\tan(82°) = h \qquad a\sin(82°) = h$$

$$y = \frac{h}{\tan(82°)} \qquad a = \frac{h}{\sin(82°)}$$

$$y \approx \frac{8.92}{\tan(82°)} \qquad a \approx \frac{8.92}{\sin(82°)}$$

$$y \approx 1.25 \qquad a \approx 9.01$$

We add the lengths of x and y to get approximately 9.28. The solved triangle is shown in Figure 9.31.

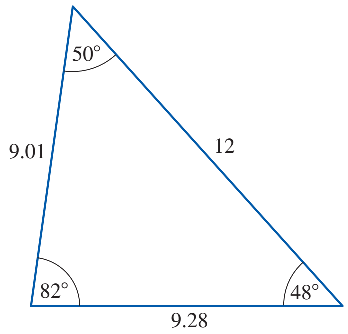


Figure 9.31

Example 1 shows that we can solve an oblique triangle using basic trigonometric ratios. However, with the Law of Sines we can solve them more quickly.

Developing the Law of Sines

To develop the formula for the Law of Sines, let's consider the triangle ABC shown in Figure 9.32, with h as the length of the altitude to the side with length a and k as the length of the altitude to the side with length b. We will determine the sine of various angles and compare their ratios. (Recall that the sine function is defined as $\sin(\theta) = \dfrac{\text{opposite}}{\text{hypotenuse}}$.)

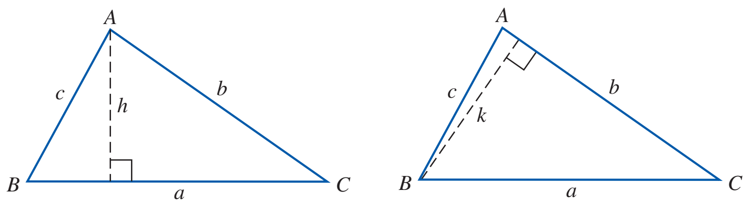


Figure 9.32

We have

$$\sin(B) = \frac{h}{c} \qquad \sin(C) = \frac{h}{b} \qquad \sin(C) = \frac{k}{a} \qquad \sin(A) = \frac{k}{c}$$

$$h = c\sin(B) \qquad h = b\sin(C) \qquad k = a\sin(C) \qquad k = c\sin(A)$$

We now set the two pairs of ratios equal to each other.

$$h = c\sin(B) \quad \text{and} \quad h = b\sin(C) \qquad\qquad k = a\sin(C) \quad \text{and} \quad k = c\sin(A)$$

$$c\sin(B) = b\sin(C) \quad \textbf{Substitute for } \boldsymbol{h}. \qquad\qquad a\sin(C) = c\sin(A) \quad \textbf{Substitute for } \boldsymbol{k}.$$

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} \quad \textbf{Divide by sine terms.} \qquad\qquad \frac{c}{\sin(C)} = \frac{a}{\sin(A)} \quad \textbf{Divide by sine terms.}$$

Thus we conclude that in triangle ABC,

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

By taking reciprocals of each of these terms, we additionally discover that

$$\frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a}$$

This relationship between sides and angles is known as the **Law of Sines**.

Law of Sines

In any ΔABC, the ratio of the sine of an angle to the length of the side opposite the angle is a constant. That is,

$$\frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a} \quad \text{and} \quad \frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

for angles A, B, and C and side lengths a, b, and c.

■ Triangulation

We now return to the Gateway Arch example to see how the Law of Sines can be applied in a real-world situation. Engineers working on the arch knew it was critical to make precise measurements as the two arch legs were being erected simultaneously. Even a slight deviation in any direction would result in the legs failing to meet at the top. Throughout the entire process, engineers had to maintain accuracy to $\frac{1}{64}$ of an inch. (*Source:* The History Channel, "*The St. Louis Arch*," 2001) To achieve the extreme accuracy that was required, *triangulation* was used. **Triangulation** is the process of using two known points to find the coordinates of an unknown point by constructing a triangle with the three points as vertices and using the Law of Sines to determine the unknown point.

EXAMPLE 2 ■ Using the Law of Sines for an ASA Triangle

During the process of constructing the legs of the Gateway Arch, engineers used four monuments (posts) that formed a rectangle 300 feet by 720 feet centered on the arch legs (see Figure 9.33). The four posts were set, two behind each leg of the arch, and the distance between those two monuments was measured. Surveyors used a theodolite to measure angles. Engineers used the two north monuments for triangulation on the south leg and the two south monuments for triangulation on the north leg. By applying the Law of Sines, the engineers precisely determined any given point along the arch legs. (*Source:* The History Channel, "*The St. Louis Arch*," 2001) Figure 9.33 represents one of the triangles formed by the surveyors as they were aligning the south leg.

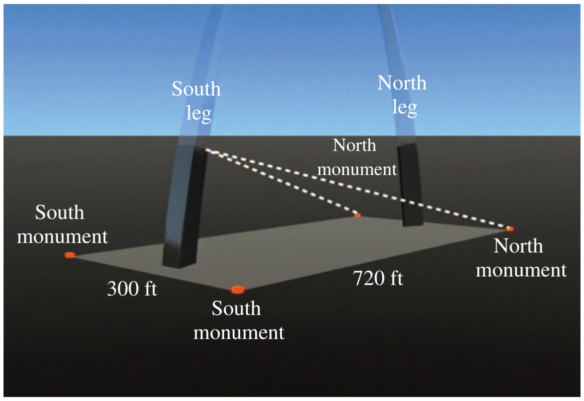


Figure 9.33

The triangle in Figure 9.34 provides the measurements for one such triangle as the section was being moved into place. Determine the values of the missing angle measure A, and the two side lengths b and c.

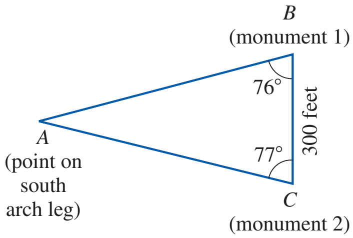


Figure 9.34

Solution In Figure 9.34, the unknown angle A measures $180° - B - C$, since the sum of the three angle measures of a triangle is 180°. Therefore, the measure of angle A is

$$180° - 76° - 77° = 27°$$

To determine the lengths of the remaining two sides of the triangle, we use the Law of Sines since we are given the measure of two angles (B and C) and the included side (a). See Figure 9.35. Using the Law of Sines to find c, we have

$$\frac{\sin(77°)}{c} = \frac{\sin(27°)}{300 \text{ feet}}$$
$$c(\sin(27°)) = (300 \text{ feet})(\sin(77°))$$
$$c = \frac{(300 \text{ feet})(\sin(77°))}{\sin(27°)}$$
$$c \approx 643.8 \text{ feet}$$

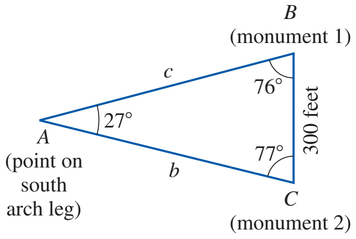


Figure 9.35

Using the Law of Sines to find b gives

$$\frac{\sin(76°)}{b} = \frac{\sin(27°)}{300 \text{ feet}}$$
$$b(\sin(27°)) = (300 \text{ feet})(\sin(76°))$$
$$b = \frac{(300 \text{ feet})(\sin(76°))}{\sin(27°)}$$
$$b \approx 641.2 \text{ feet}$$

Therefore, the solved triangle ABC has the measures shown in Figure 9.36.

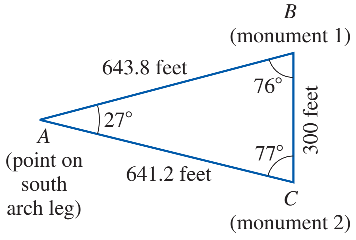


Figure 9.36

EXAMPLE 3 ■ Using the Law of Sines to Solve an ASA Triangle

In triangle ABC, $A = 35°$, $B = 15°$, and $c = 5$. Find a, b, and angle C.

Solution We first draw a diagram for the triangle described, as shown in Figure 9.37. The measure of angle C is

$$C = 180° - 35° - 15°$$
$$C = 130°$$

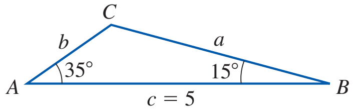


Figure 9.37

We use the Law of Sines to find the missing sides because we are given two angle measures and their included side. Although we may use either form of the Law of Sines, $\frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$ or $\frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a}$, it is most efficient to have the variable in the numerator. We'll use $\frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$ since we are solving for a side. (If we were solving for an angle, we would have used the other form.)

$$\frac{a}{\sin(35°)} = \frac{5}{\sin(130°)}$$
$$a = \frac{5\sin(35°)}{\sin(130°)}$$
$$a \approx 3.74$$

$$\frac{b}{\sin(15°)} = \frac{5}{\sin(130°)}$$
$$b = \frac{5\sin(15°)}{\sin(130°)}$$
$$b \approx 1.69$$

Therefore, the solved triangle has the measures shown in Figure 9.38.

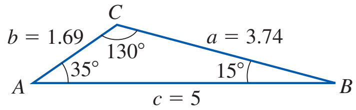


Figure 9.38

EXAMPLE 4 ■ Using the Law of Sines to Solve an AAS Triangle

The diagram in Figure 9.39, which is drawn to scale, shows the relative positions of Chicago, New York City, and Washington, DC. The angle at Chicago measures 14.91° and the angle at New York City measures 47.18°. The distance between New York City and Washington, DC, is 209 miles. How far is Chicago from New York City and Washington, DC?

Figure 9.39

Solution We relabel the triangle as in Figure 9.40 for computational ease. We are told that $A = 14.91°$, $B = 47.18°$, and $a = 209$. We need to find b (the distance from Chicago to Washington, DC) and c (the distance from Chicago to New York City). We use the Law of Sines because we are given two angles and a side.

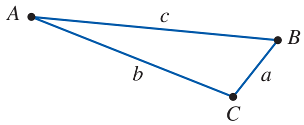


Figure 9.40

$$\begin{aligned} \frac{209}{\sin(14.91°)} &= \frac{b}{\sin(47.18°)} \\ b &= \frac{209\sin(47.18°)}{\sin(14.91°)} \\ b &\approx 596 \end{aligned}$$

The distance between Chicago and Washington, DC, is about 596 miles.

To set up the next ratio, we first need to know C.

$$\begin{aligned} C &= 180° - A - B \\ C &= 180° - 14.91° - 47.18° \\ C &= 117.91° \end{aligned}$$

Using the Law of Sines, we find c.

$$\begin{aligned} \frac{209}{\sin(14.91°)} &= \frac{c}{\sin(117.91°)} \\ c &= \frac{209\sin(117.91°)}{\sin(14.91°)} \\ &\approx 718 \end{aligned}$$

The distance between Chicago and New York City is about 718 miles.

Using the Law of Sines with an SSA triangle often results in what is referred to as the *ambiguous case* because more than one triangle may be generated. In fact, the known information may result in one triangle, two triangles, or no triangle at all. We look at this next.

EXAMPLE 5 ■ Exploring the SSA Triangle

Both of the triangles shown in Figure 9.41 have sides of length 7 and 9 and contain a 43° angle. However, angle A for the first triangle is acute whereas the corresponding angle of the second triangle, B, is obtuse. Find the measure of angles A and B in the triangles.

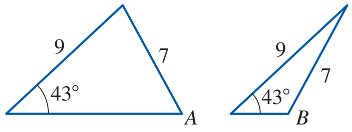


Figure 9.41

Solution As shown in Figure 9.42, we begin with the first triangle. To find A, we use the Law of Sines since we have two sides and a nonincluded angle.

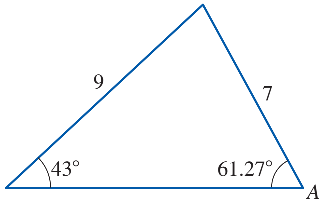


Figure 9.42

$$\frac{\sin(A)}{9} = \frac{\sin(43°)}{7}$$
$$7\sin(A) = 9\sin(43°)$$
$$\sin(A) = \frac{9\sin(43°)}{7}$$
$$\sin(A) = 0.8769$$
$$\sin^{-1}(\sin(A)) = \sin^{-1}(0.8769)$$
$$A = \sin^{-1}(0.8769)$$
$$A = 61.27°$$

Now, as shown in Figure 9.43, we move to the second triangle. To find B, we recall that sine is positive in both the first and second quadrants. Therefore, there are two angles such that $\sin(\theta) = 0.8769$. $A = 61.27°$ is one of the angles. To find the measure of the other angle, B, we find the angle in the second quadrant with a reference angle of 61.27°.

$$B = 180° - 61.27° = 118.73°$$

Thus there are two triangles with sides 7 and 9 and nonincluded angle measuring 43°.

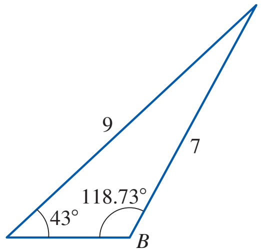


Figure 9.43

EXAMPLE 6 ■ Using the Law of Sines for an SSA Triangle

The design for a new playground calls for a children's slide that is 12.5 feet long inclined 30° with the horizontal. (This is the measure of the angle the slide would create with the ground if it did not bend at the bottom.) To reach the top of the slide the children will climb a ladder from the other side that is 6.4 feet long. For slides to be safe for children to use, the angle that the ladder makes with the horizontal should measure 80° or less. Determine whether or not the design of the playground slide is in compliance with the standard. Draw a sketch that shows the slide, ladder, and ground. (*Note:* There are two correct answers. In your solution show them both and describe why only one will make sense in the real-world context.

Solution We first draw a diagram for the triangle described. See Figure 9.44. To show that the measure of angle B is less than or equal to 80°, we employ the Law of Sines.

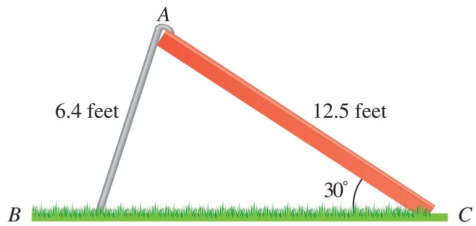


Figure 9.44

$$\frac{\sin(B)}{12.5} = \frac{\sin(30°)}{6.4}$$
$$6.4\sin(B) = 12.5\sin(30°)$$
$$\sin(B) = \frac{12.5\sin(30°)}{6.4}$$
$$\sin(B) = 0.9766$$
$$B = \sin^{-1}(0.9766)$$

Because sine is positive in the first and second quadrants, there are two answers for the measure of angle B.

$$B = 77.6° \quad \text{and} \quad \begin{aligned} B &= 180° - 77.6° \\ B &= 102.4° \end{aligned}$$

We show the first angle, $B = 77.6°$, in Figure 9.45. This slide is in compliance since $77.6° < 80°$. Angle $B = 102.4°$ is not in compliance since $102.4° > 80°$. Besides, as shown in Figure 9.46, an angle measuring 102.4° is unreasonable since a ladder sloped backward would be impossible for young children (and most adults) to climb.

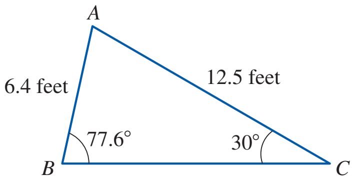


Figure 9.45

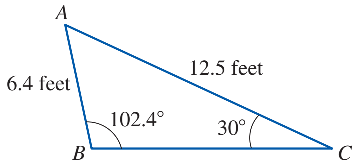


Figure 9.46

Summary of Triangle Solving Techniques

To solve any triangle with one known side length and two other pieces of known information (two side lengths, a side length and an angle measure, or two angle measures), use the following techniques.

Right Triangles

- Use the Pythagorean theorem to determine side lengths.
- Use the six trigonometric functions to find side lengths or angle measures.

Oblique Triangles

- Use the Law of Cosines for triangles with three known side lengths or two known side lengths and the measure of the angle between them (SAS or SSS).
- Use the Law of Sines for triangles with two known angle measures and one known side length (ASA, AAS) and for triangles with two known side lengths and the measure of an angle opposite a known side (SSA).

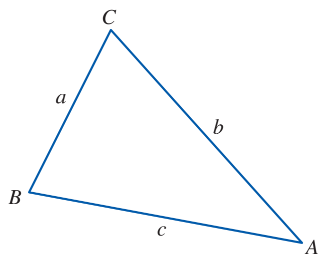
SUMMARY

In this section you learned where the Law of Sines comes from and how to apply it to real-world scenarios. You also learned how to use the Law of Sines to solve triangles.

9.3 EXERCISES

SKILLS AND CONCEPTS

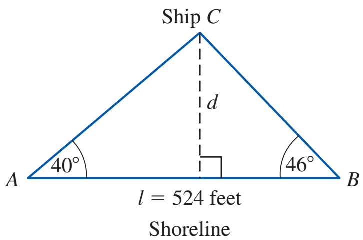
In Exercises 1–20, use triangle ABC to find the missing side lengths and angle measures (if possible). If there are two solutions, find both. If there is no solution, so state.



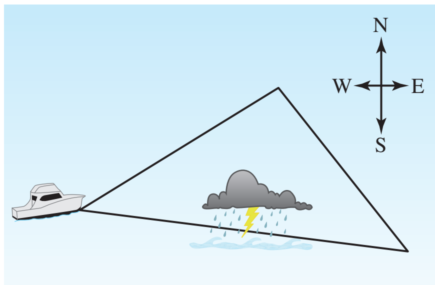
1. $a = 6$, $B = 77°$, $C = 14°$
2. $a = 8$, $B = 19°$, $C = 75°$
3. $a = 15$, $B = 98°$, $C = 15°$
4. $a = 4$, $B = 48°$, $C = 114°$
5. $A = 16°$, $B = 37°$, $c = 3$
6. $A = 96°$, $B = 23°$, $c = 6$
7. $A = 106°$, $B = 8°$, $c = 16$
8. $A = 76°$, $B = 48°$, $c = 10$
9. $A = 19°$, $C = 82°$, $b = 24$
10. $A = 10°$, $C = 160°$, $b = 4$
11. $A = 94°$, $C = 31°$, $b = 8$
12. $A = 4°$, $C = 7°$, $b = 4$
13. $A = 19°$, $B = 82°$, $b = 24$
14. $A = 34°$, $B = 80°$, $b = 12$
15. $A = 28°$, $B = 67°$, $b = 23$
16. $A = 4°$, $B = 111°$, $b = 19$
17. $a = 34$, $A = 80°$, $b = 12$
18. $a = 26$, $A = 140°$, $b = 43$
19. $a = 5$, $A = 64°$, $b = 15$
20. $a = 13$, $A = 136°$, $b = 3$

In Exercises 21–23, apply the Law of Sines, the Pythagorean theorem, or a trigonometric function to the given situation.

21. **Distance to a Ship** An observer at A measures angle A between the shore and the ship, and an observer at B does likewise for angle B. If the length l is 524 feet, find the distance d from shore to ship.



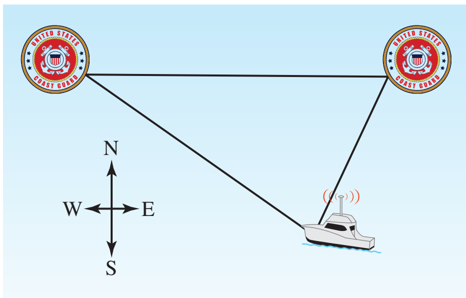
22. **Storm at Sea** A ship receives notification of a storm to its east. To avoid the storm, it proceeds at a heading of 43° north of east for 167 nautical miles. It then changes to a heading of 59° south of east.



   a. How far does the ship travel on a heading of 59° south of east until it intersects its original course?

   b. How far out of its way did the ship go to avoid the storm?

23. **U.S. Coast Guard** One U.S. Coast Guard station is located 105 miles due west of a second station. A ship in distress sends a signal received by each station. The location of the ship from the first station is 34° west of south and the location of the ship from the second station is 41° east of south.



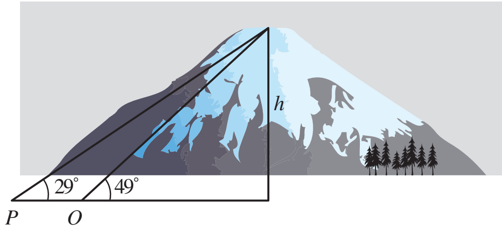
   a. How far is each station from the ship?

   b. If a search and rescue helicopter capable of flying 175 miles per hour is dispatched from the nearest station to the ship, how long will it take to reach the ship?

In Exercises 24–26, solve each problem for the missing values. Use the Law of Sines or the Law of Cosines as appropriate.

24. **Mountain Height** Two observers view the same mountain peak from two points on level ground and 3 miles apart. The angle of elevation at P to the peak is 29°. For

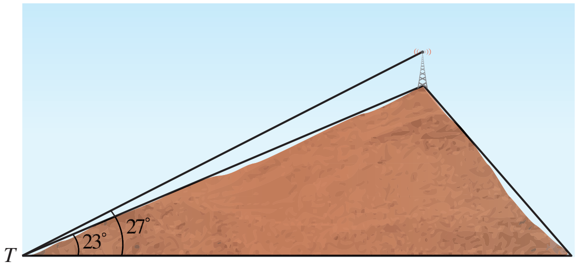
the other observer, the angle of elevation at O to the peak measures 49°.



a. Find the distance from P to the summit.

b. Find the height of the mountain h.

25. **Radio Tower** A radio tower is located on top of a hill, as shown in the figure. The distance from point T on the ground to the top of the hill is 1964 feet. The angle of elevation from T to the base of the tower measures 23° and the angle of elevation from T to the top of the tower measures 27°.



a. Find the height of the hill.

b. Find the height of the tower.

26. **Height of a Kite** Three children are flying a kite and are located as displayed in the diagram.

a. Estimate the angle of elevation from each child to the kite by looking at the figure.

b. If child C_1 is approximately 3 feet, child C_2 is approximately 5 feet, and child C_3 is approximately 11 feet from being directly under the kite, estimate the distance from each child's feet to the kite using the angle of elevation from part (a).

In Exercises 27–32, two sides and an angle are given. Determine whether the given information results in one triangle, two triangles, or no triangle at all. Find the missing values of all sides and angles for the triangles that result.

27. $a = 3, b = 2, A = 50°$

28. $b = 5, c = 3, B = 100°$

29. $b = 4, c = 6, B = 20°$

30. $a = 4, b = 5, A = 60°$

31. $a = 2, c = 1, C = 25°$

32. $a = 2, c = 1, C = 120°$

■ SHOW YOU KNOW

33. Describe under what conditions it is appropriate to use the Law of Sines.

34. Explain how triangulation is used to solve real-world problems.

35. Explain what is meant by the ambiguous case as it relates to the Law of Sines.

36. Suppose a classmate who missed the development of the Law of Sines is wondering why there are three parts to the Law of Sines formula. That is, how would you explain that

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

37. The Law of Sines states that

$$\frac{b}{\sin(B)} = \frac{c}{\sin(C)} = \frac{a}{\sin(A)}$$

and

$$\frac{\sin(B)}{b} = \frac{\sin(C)}{c} = \frac{\sin(A)}{a}$$

Explain why the Law of Sines may be expressed either way.

■ MAKE IT REAL

38. **Height of a Tree** Trees located on a sloping hillside turn to grow vertically by responding to gravity and light rays. (***Source:*** **www.madsci.org**) To measure the height of a tree on a slope it is helpful to know the angle between the tree and the sloped ground.

If a tree is growing vertically at an angle of 76° with respect to the sloped ground and the angle to the top of the tree from a location uphill 110 feet from the base of the tree is 48°, how tall is the tree?

39. **Building Height** As of 2006, officials in Bellevue, Washington, limited the heights of community business buildings in the city to 45 feet. (***Source:*** **www.ci.bellevue.wa.us**) One of the city officials wants to know if an existing building meets the height limits. At a distance of 100 feet away from the base of the building, the official measures the angle of elevation to the top of the building to be 26°.

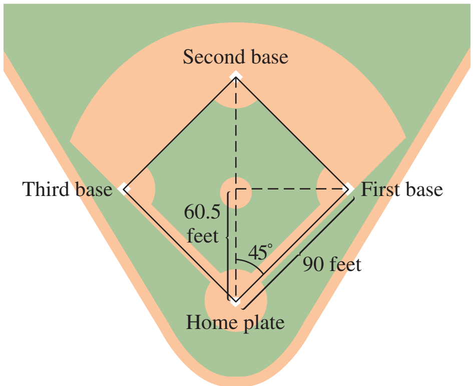
a. Does the building meet the city code for height? Explain.

b. What is the maximum angle of elevation from a distance of 100 feet for which a building will meet the city code for height?

40. Forest Fire Observation In the Adirondack Mountains near Tupper Lake, New York, forest fire observers used fire observation towers for over 70 years. The people who staffed the towers reported forest fires and smoke sightings. As many as 57 towers were used by the observers; however, in the early 1980s, the State of New York determined that the towers were no longer needed since aerial observations had become commonplace. (*Source:* www.tupperlake.net)

Suppose a forest ranger in one of the observation towers in the Adirondacks sights a fire 33° east of north while a ranger in a tower 15 miles due east of the first ranger sights the same fire at 46° west of north. How far is the fire from each ranger?

41. Baseball Diamond A major league baseball diamond is a square with 90 feet on a side. The pitching rubber is located 60.5 feet from home plate on the diagonal from home plate to second base.

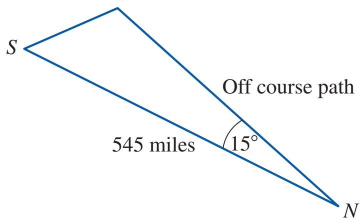


a. How far is a pitcher's throw to first base from the pitching rubber?

b. How far is a pitcher's throw to second base from the pitching rubber?

c. If a pitcher faces home plate, through what angle does he need to turn to face third base?

42. Airplane Distance While flying from New Orleans, Louisiana, *N*, to San Antonio, Texas, *S*, a distance of approximately 545 miles, a pilot had to go off course by 15° to avoid turbulence.

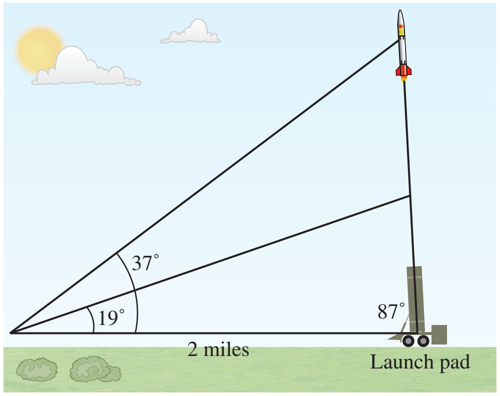


a. The pilot corrects his course after 10 minutes and maintains an average speed of 245 miles per hour. Through what angle should the pilot now turn to head toward San Antonio?

b. For the trip to last a total of 2.5 hours, what new average speed should the pilot achieve?

43. Patriot Missile Launch A Patriot missile may be launched at a variety of angles. After release, its guidance system turns the missile and adjusts its speed to reach its desired destination. (*Source:* www.army-technology.com)

Suppose a missile is turned to an angle of 87° after its initial launch. An observer 2 miles from the launch pad sees the missile after launch at an angle of elevation measuring 19°. Five seconds later, the missile is observed at an angle of elevation of 37°.

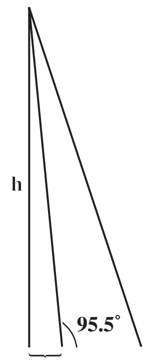


a. How far did the missile travel during the 5-second interval?

b. What was the missile's average speed during this 5-second interval?

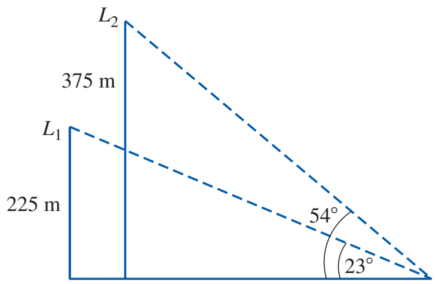
c. If the missile continues on the same path (87° from vertical), what will its angle of elevation be 17 seconds after the first sighting?

44. Height of the Tower of Pisa Using the photograph, estimate the height, *h*, of the Leaning Tower of Pisa. Also, estimate how far from vertical (90°) the tower is leaning.



45. The "Phoenix Lights" Something bizarre appeared in the Arizona sky on the night of March 13, 1997. It was witnessed by thousands of people. Neither researchers nor witnesses have yet figured out what Arizonans saw in the event now dubbed "the Phoenix Lights." (*Source:* www.azcentral.com) That has not stopped many from trying to figure it out, however.

The images above include a photograph of the lights and an artist's interpretation of the source of the lights. Assume the lights were first sighted by a family at a point L_1 due east from them at an angle of elevation of 23° from the ground and at an altitude of 225 meters. Then suppose the lights were next sighted at a point L_2 due east at an angle of 54° and an altitude of 375 meters. What is the distance that the lights traveled from L_1 to L_2?



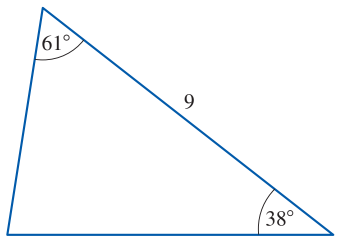
■ STRETCH YOUR MIND

Exercises 46–47 are intended to challenge your understanding of triangle trigonometry.

46. Width of the Columbia River While Lewis and Clark were exploring the American west in 1805, William Clark used triangulation to measure the width of the Columbia River near its junction with the Snake River. Lewis placed two markers 148 poles apart on the side of the river he was on. (*Note:* a pole was a 5.5-yard measuring device used by surveyors at the time.)

Using the reproduced photograph of the region from 1945, show how Lewis found the width of the Columbia to be 152 poles (836 yards). (*Hint:* Rotate your textbook so that North is at the top of the photo.) (*Note:* The notation N 84° W means 84° west of north and S $28\frac{1}{2}$°E means $28\frac{1}{2}$° east of south.)

47. Solve the following oblique triangle using right triangle trigonometry and not the Law of Sines.



SECTION 9.4 Polar Coordinates

LEARNING OBJECTIVES

- Graph and classify equations in polar coordinates
- Convert between polar and rectangular coordinates
- Describe the relationship between Cartesian (rectangular) and polar coordinates using trigonometry

GETTING STARTED

There are over 8000 artificial objects orbiting Earth. Of those, nearly 2500 are satellites, some operative and some inoperative. (*Source:* liftoff.msfc.nasa.gov) The rest of the objects are space junk—nose cone shrouds, hatch covers, rocket bodies, payloads that have disintegrated or exploded, and even objects that have escaped from manned spacecraft during operations.

Satellites may orbit Earth in several ways. The three most common ways are geostationary orbits, polar orbits, and inclined orbits. According to NASA (*source:* asd-www.larc.nasa.gov), a geostationary orbit is one in which the satellite stays in the same relative position over Earth by orbiting in the same plane as Earth's rotation (equatorial plane). A polar orbit is one where the plane in which the satellite orbits is perpendicular to Earth's equatorial plane. An inclined orbit is one where the orbit of the satellite is somewhere in between a geostationary orbit (angle of inclination is 0 degrees) and a polar orbit (angle of inclination is 90 degrees).

In this section we investigate the polar coordinate system, which allows us to locate an object that moves in a circular path around a fixed center. We also discuss the relationship between the more well-known Cartesian, or rectangular, coordinate system and the polar coordinate system.

The Polar Coordinate System

Let's consider a satellite orbiting Earth in an inclined orbit. Iridium 19, a communications satellite launched in 1997, orbits Earth every 100 minutes at an inclination of 86.4°. (*Source:* www.n2yo.com) If we visualize this orbit in a two-dimensional plane (as shown in Figure 9.47), we see it traces out a circular path around the globe.

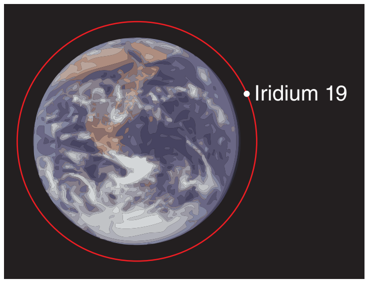


Figure 9.47

To locate the Iridium 19 satellite in this plane, we superimpose a Cartesian coordinate system on this diagram and locate the satellite by indicating the horizontal and vertical distances from the origin, as shown in Figure 9.48.

Because of the circular path of the satellite, we can also describe its location in the plane using **polar coordinates**. We locate the satellite by its distance, r, from the origin (called the **pole**) and the angle, θ, this radius makes with the horizontal axis. We name this point (r, θ), as shown in Figure 9.49.

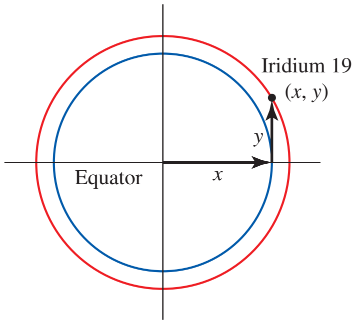


Figure 9.48

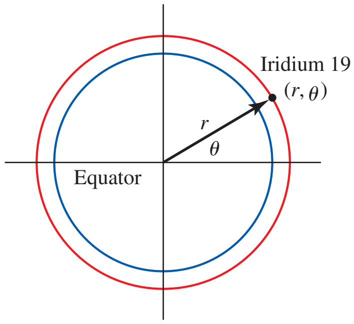


Figure 9.49

PEER INTO THE PAST

THE CARTESIAN COORDINATE SYSTEM

The Cartesian, or rectangular, coordinate system is named after the famous mathematician and philosopher René Descartes of France. Legend has it that Descartes was lying in bed and noticed a fly on the ceiling. This inspired him to think of a way to describe the location of the fly. Choosing a beginning location called the origin, (0, 0), he counted horizontally a certain number of tiles and then vertically as a way to describe the fly's location. The development of this system of graphing led to what we now know as analytic geometry, which bridges algebra and geometry.

POLAR COORDINATES

The **polar coordinates** (r, θ) represent a point in the polar plane where r is the distance from the pole to the point and θ is the measure of the angle formed with the horizontal.

Rather than plotting polar coordinates in the Cartesian coordinate plane, we use the polar coordinate plane. Several concentric circles representing the distances r along with the terminal sides of several different angle measures are shown in the polar coordinate plane in Figure 9.50.

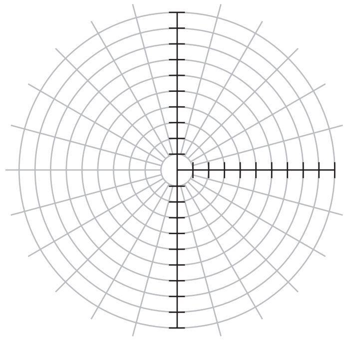


Figure 9.50

EXAMPLE 1 ■ Locating a Point in the Polar Coordinate Plane

The radius of Earth is approximately 6378 kilometers. The Iridium 19 satellite orbits Earth at an altitude of 778 kilometers. At a particular time, the satellite creates an angle measuring 0.5 radian with the horizontal axis. What is its location in terms of polar coordinates? Plot this point using a polar coordinate plane.

Solution If we take the center of Earth to be the pole, then the distance to the satellite is the radius of Earth added to the altitude of the satellite:

$$6378 \text{ km} + 778 \text{ km} = 7156 \text{ km}$$

With a reported angle of 0.5 radian, we write the location of the satellite in polar coordinates as $(r, \theta) = (7156, 0.5)$. To plot this point in the polar coordinate plane, we begin on the positive horizontal axis and measure 0.5 radian or approximately 28.6°. Along the terminal side of that angle, we count out 7156 kilometers, where the scale is 1000 kilometers for each of the concentric circles. See Figure 9.51.

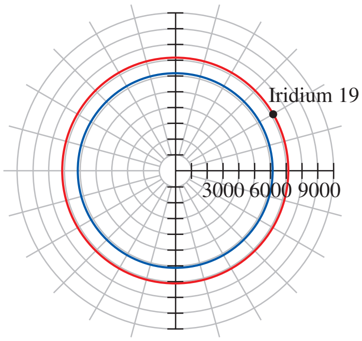


Figure 9.51

EXAMPLE 2 ■ Determining the Coordinates of a Point in the Polar Plane

Give three possible coordinates for the polar point given in Figure 9.52. Give the angle measure in radians.

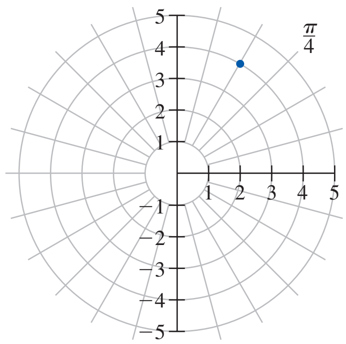


Figure 9.52

Solution We first determine the value of θ. Using the positive horizontal axis as the initial side, we see the angle measuring θ has a terminal side of $\frac{4\pi}{12} = \frac{\pi}{3}$. Along the terminal side of this angle, the point is located 4 units from the pole. We label this point $\left(4, \frac{\pi}{3}\right)$, as shown in Figure 9.53.

We could also say the terminal side of the angle is $\frac{\pi}{3}$ plus one complete revolution. That is, $\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$. Using this angle, we label the point $\left(4, \frac{7\pi}{3}\right)$, as in Figure 9.54. Extending this idea, we can infer there are infinitely many labels for this point.

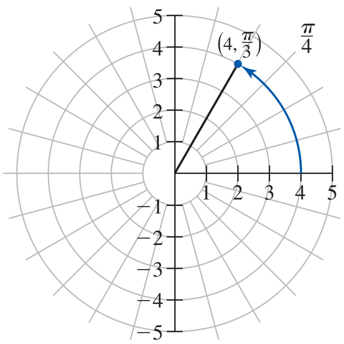


Figure 9.53

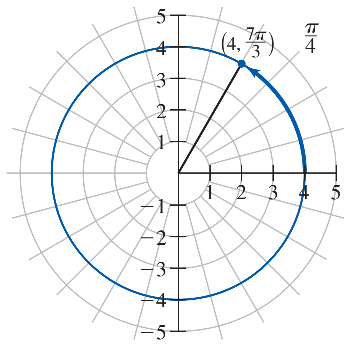


Figure 9.54

Finally, we can express the location of the point using a negative angle measure and negative value for r. Instead of counting out r units along the terminal side of the angle, we count out r units in the opposite direction, labeling the value of r as a negative in $\left(-4, -\frac{2\pi}{3}\right)$. See Figure 9.55.

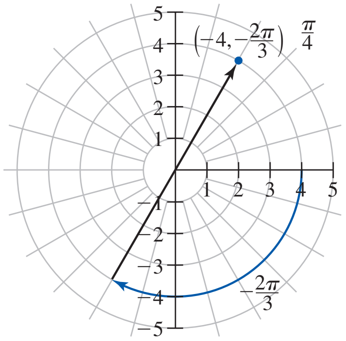


Figure 9.55

■ Using Polar Coordinates to Describe Relationships

Polar coordinates are sometimes preferred for symbolically describing real-world phenomena such as satellite locations. In the two-dimensional plane we examined previously, we saw that a satellite follows a circular path around Earth. To describe this path in polar coordinates, we use equations of the form $r =$ *something*. We can also use graphs to express relationships using polar coordinates. The Technology Tip at the end of the section shows how to use a graphing calculator to produce graphs of polar equations.

EXAMPLE 3 ■ Using Polar Coordinates

The DirecTV-4S satellite was launched in November 2001 to provide over 300 television channels to 41 different communities. (*Source:* science.nasa.gov) It is in a geostationary orbit at an altitude of 35,800 kilometers, as depicted in Figure 9.56. Using the center of Earth as the pole, express the geostationary orbit of this satellite using an equation in polar coordinates. (*Hint:* Earth's radius is about 6378 kilometers.) Graph this equation.

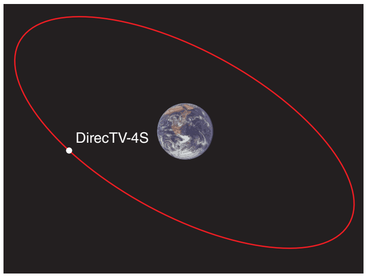


Figure 9.56

Solution We can describe the path of the satellite by recognizing that the value of the radius, r, remains constant. That is, for any value of θ, the polar coordinates (r, θ) will have a constant value for r. Using the center of Earth as the pole, we find $r = 6378 + 35{,}800 = 42{,}178$ kilometers. We write the equation $r = 42{,}178$ to represent the circular orbit of the satellite. The graph is shown in Figure 9.57.

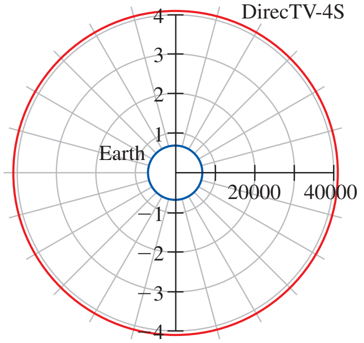


Figure 9.57

■ Natural Applications of Polar Coordinates

The chambered nautilus is a sea creature found in the tropical Pacific Ocean. A newly hatched nautilus wears a shell divided into four small chambers. As a nautilus grows, it gains more living space by building new chambers connected to the old ones. Adult shells have 30 chambers. (*Source:* www.mbayaq.org)

The pattern of the chambered nautilus shell was first described by the Greek mathematician Archimedes and is now known as the "Archimedes spiral." As we follow the curve of the shell from the very center as it curls around to the tip of its shell, we see that as the angle increases, the distance, r, from the pole also increases. We can describe this pattern using polar coordinates as in Example 4.

EXAMPLE 4 ■ Graphing in Polar Coordinates

Consider the equation $r = \theta$. Complete a table of values for the equation and use it to describe the polar curve formed by the equation. Then graph the equation.

Solution We begin by making Table 9.2 to help understand the nature of the polar curve. We use radians for the angle measure and write the values of r as decimals.

Table 9.2

θ	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π	$\frac{5\pi}{4}$	$\frac{3\pi}{2}$	$\frac{7\pi}{4}$	2π
r	0	0.785	1.57	2.36	3.14	3.93	4.71	5.50	6.28

We see that as θ increases, r increases as well. That is, the points move farther and farther from the pole as θ increases. We plot these points on the polar plane as in Figure 9.58.

We now plot the equation $r = \theta$ (Figure 9.59a) and see that it creates the Archimedes spiral (Figure 9.59b).

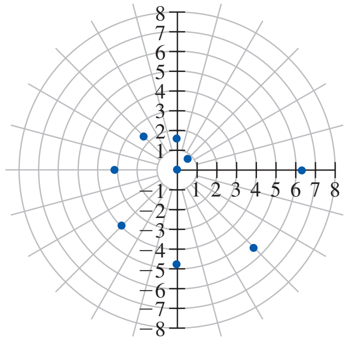


Figure 9.58

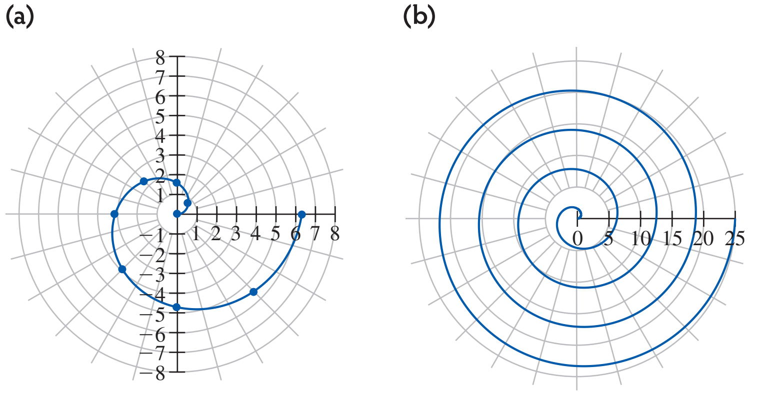


Figure 9.59

Polar equations often create very interesting and beautiful graphs. In the next example, we will look at equations of the form $r = a\cos(n\theta)$.

EXAMPLE 5 ■ Graphing Polar Equations

Graph the equation $r = 2\cos(3\theta)$ for $0 \le \theta \le 2\pi$. Write a justification for the behavior of the graph. Then, explore several other equations of the form $r = 2\cos(n\theta)$ and describe the graph and its connection to the value of n.

Solution We begin by graphing $r = 2\cos(3\theta)$ as in Figure 9.60. The graph resembles a flower with three petals. The greatest value of r is $r = 2$ because $-1 \le \cos(3\theta) \le 1$ and it follows that $-2 \le 2\cos(3\theta) \le 2$. Since $r = 2\cos(3\theta)$, $-2 \le r \le 2$. To see why the graph reaches $r = 2$ three times, note that the interval $0 \le \theta \le 2\pi$ is equivalent to $0 \le 3\theta \le 6\pi$. We know $\cos(3\theta) = \pm 1$ when $3\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi$—that is, when $\theta = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}, 2\pi$. At these angle values, $2\cos(3\theta) = \pm 2$. The reason we only see three petals is that the period of this function is $\frac{2\pi}{3}$. After $\frac{2\pi}{3}$, the graph begins repeating itself.

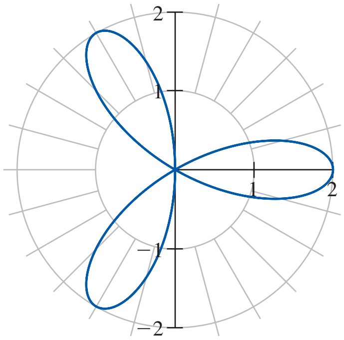


Figure 9.60

Now let's explore other values of n. Consider the graph of $r = 2\cos(5\theta)$ shown in Figure 9.61, a 5-petal flower. In Figure 9.62, we see $r = 2\cos(6\theta)$ produces a 12-petal flower. The interval $0 \le \theta \le 2\pi$ is equivalent to $0 \le 6\theta \le 12\pi$. Also, $\cos(6\theta) = \pm 1$ when $6\theta = 0, \pi, 2\pi, 3\pi, 4\pi, 5\pi, 6\pi, 7\pi, 8\pi, 9\pi, 10\pi, 11\pi, 12\pi$—that is, when $\theta = 0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, 2\pi$. Therefore $r = \pm 2$ for each of these 12 values of θ. In general, $r = a\cos(n\theta)$ produces a n-petal rose when n is odd and a $2n$-petal rose when n is even.

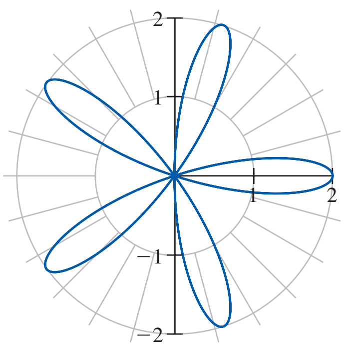


Figure 9.61

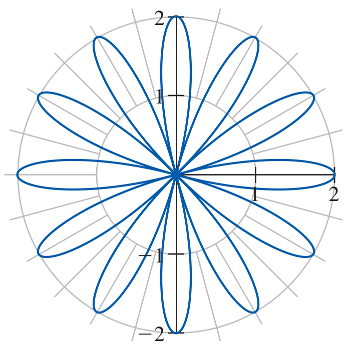
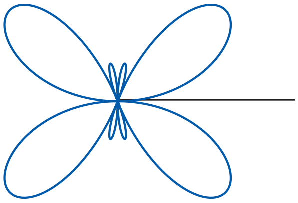


Figure 9.62

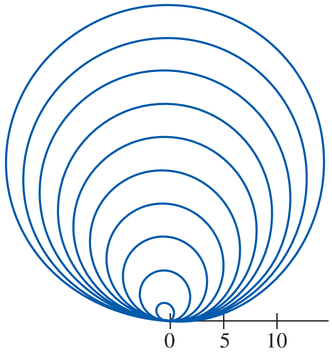
■ Other Polar Graphs

Polar equations can be used to create a variety of beautiful graphs. We encourage you to experiment! We provide a few graphs here together with their equations to pique your interest. The possibilities are endless. For each, challenge yourself to describe why the graphs appear as they do.

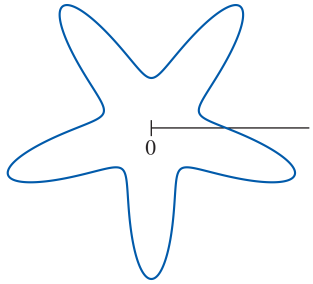
$r(\theta) = \cos(\theta)\sin(3\theta)$



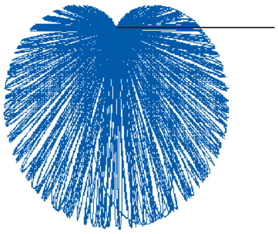
$r(\theta) = \theta \sin(\theta)$



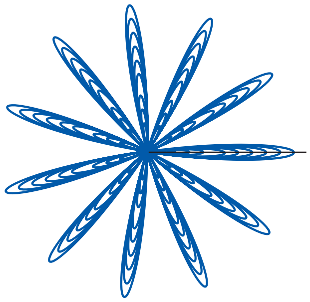
$r(\theta) = \dfrac{3}{2 + \sin(5\theta)}$



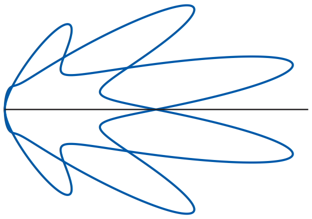
$r(\theta) = \cos(\theta^2) - \sin(\theta)$



$r(\theta) = \theta \cos(11\theta)$



$r(\theta) = \dfrac{\cos(\theta)}{2 + \sin(7\theta)}$



EXAMPLE 6 ■ Investigating Polar Graphs

Graph the polar equation $r = \cos(5\theta) + 10$ using a graphing calculator. Choose a viewing window so you obtain a good result. Then write an explanation for why the graph makes sense. For example, explain why the maximum value of r makes sense, why the behavior of the graph makes sense, and so on.

Solution Using the Technology Tip at the end of this chapter, we begin by entering the equation $r = \cos(5\theta) + 10$ into the calculator and producing a graph in the standard viewing window. See Figure 9.63.

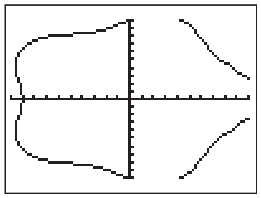


Figure 9.63

We do not consider this a good viewing window since part of the graph is not visible. We know $\cos(5\theta)$ always produces values between -1 and 1, inclusive. Adding 10 will produce values for r such that $9 \le r \le 11$. We change the viewing window to include these values for r and obtain the graph shown in Figure 9.64.

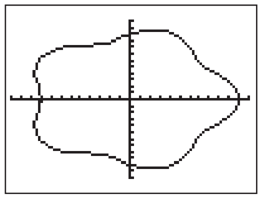


Figure 9.64

The interval $0 \le \theta \le 2\pi$ is equivalent to $0 \le 5\theta \le 10\pi$. Also, $\cos(5\theta) = 1$ when $5\theta = 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi$—that is, when $\theta = 0, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}, \frac{10\pi}{5}$. Therefore, $\cos(5\theta) + 10 = 11$ at each of these values of θ.

Relationship between Cartesian and Polar Coordinates

Using right triangle trigonometry and the Pythagorean theorem, we can determine the relationships between Cartesian and polar coordinates. That is, if we know the coordinates of a point in the Cartesian system, (x, y), we can determine the same location in polar coordinates, (r, θ). Or, if we know the coordinates of a point in the polar system, (r, θ), we can determine the same location in Cartesian coordinates.

We examine the generic graph in Figure 9.65 to determine these relationships. Suppose we know the values of r and θ and wish to determine the corresponding values of x and y.

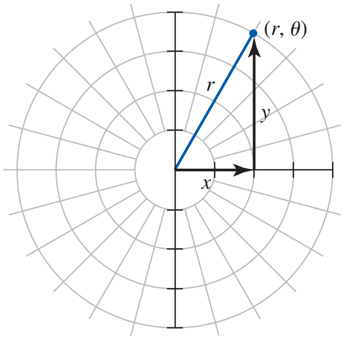


Figure 9.65

We have the following relationships.

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\theta) = \frac{y}{r}$$

$$x = r\cos(\theta) \qquad y = r\sin(\theta)$$

That is, if we know (r, θ), we can compute (x, y).

We can also determine (r, θ) if we know (x, y). Consider the graph in Figure 9.66. Using the Pythagorean theorem and the definition of tangent, we have

$$x^2 + y^2 = r^2 \qquad \tan(\theta) = \frac{y}{x}$$

$$r = \sqrt{x^2 + y^2} \qquad \theta = \tan^{-1}\left(\frac{y}{x}\right),\ x \neq 0$$

We take θ to be the reference angle and will determine the quadrant in which it lies through the context of the situation.

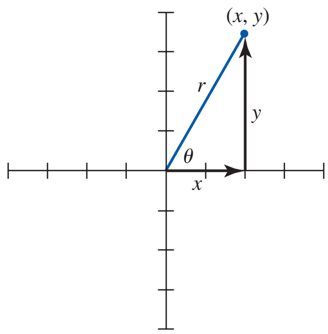


Figure 9.66

Relationships between Polar and Cartesian Coordinates

The polar coordinates (r, θ) and the corresponding Cartesian coordinates (x, y) are related as follows.

$$x = r\cos(\theta) \quad \text{and} \quad y = r\sin(\theta)$$

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \text{ for } x \neq 0$$

EXAMPLE 7 ■ Converting from Polar to Cartesian Coordinates

Using the polar coordinates for the Iridium 19 satellite, $(r, \theta) = (7156, 0.5)$, determine the equivalent Cartesian coordinates, (x, y).

Solution Since we know the polar coordinates, we find the x- and y-coordinates using $x = r\cos(\theta)$ and $y = r\sin(\theta)$. (Remember that the angle is in radians.)

$$x = 7156\cos(0.5)$$
$$x = 6279.98$$

$$y = 7156\sin(0.5)$$
$$y = 3430.77$$

The polar coordinates $(r, \theta) = (7156, 0.5)$ are equivalent to the Cartesian coordinates $(x, y) = (6279.98, 3430.77)$, as shown in Figure 9.67.

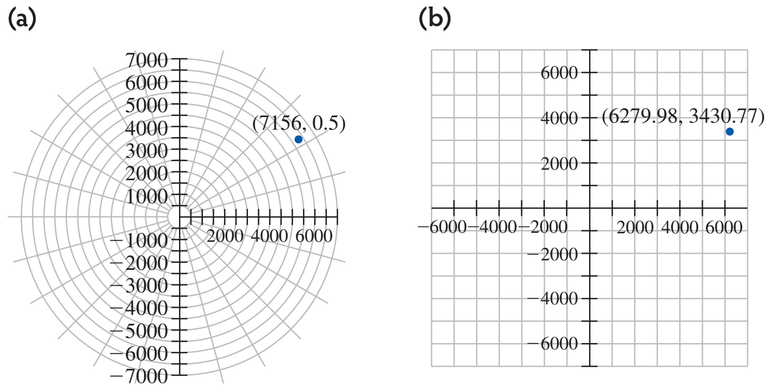


Figure 9.67

EXAMPLE 8 ■ Converting from Cartesian to Polar Coordinates

Convert the Cartesian coordinates $(x, y) = (-4, 6)$ to polar coordinates.

Solution We begin by graphing the point in the Cartesian plane, as shown in Figure 9.68. We know $r = \sqrt{x^2 + y^2}$. Thus,

$$r = \sqrt{(-4)^2 + (6)^2}$$
$$r = \sqrt{52}$$
$$r \approx 7.211$$

We also know $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x \neq 0$. Therefore,

$$\theta = \tan^{-1}\left(\frac{6}{-4}\right)$$
$$\theta = \tan^{-1}\left(-\frac{3}{2}\right)$$
$$\theta \approx -0.9828$$

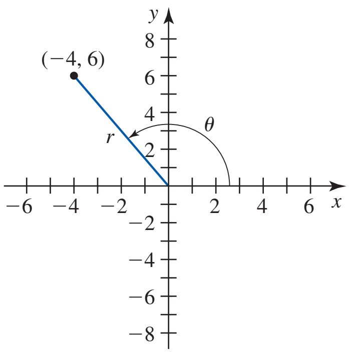


Figure 9.68

The inverse tangent function returns angles in Quadrants I and IV only. Since $\theta \approx -0.9828$ is in Quadrant IV, we need to add π to get to Quadrant II.

$$\theta \approx -0.9828 + \pi$$
$$\theta \approx 2.159$$

The point $(x, y) = (-4, 6)$ is equivalent to $(r, \theta) = (7.211, 2.159)$. See Figure 9.69.

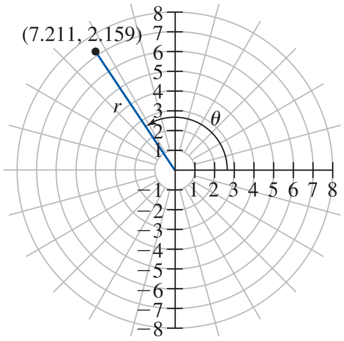


Figure 9.69

■ Describing Graphs Using Polar Coordinates

In calculus, it is sometimes necessary to describe a region in a plane using polar coordinates. We practice this skill in the next example.

EXAMPLE 9 ■ Describing a Region with Polar Coordinates

Farmers sometimes allow their fields to be shaped so that beautiful geometric designs can be seen from the air. A relatively simple crop circle is shown in the photo on the left. Use polar coordinates to describe the circular, ring region shown in the inset.

Solution Since we do not know the scale used for this ring, we will impose a coordinate system over the picture and generically use "units" as the unit of length. See Figure 9.70.

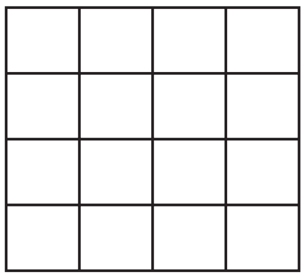


Figure 9.70

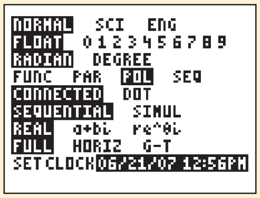
We know that polar coordinates describe the value of r, the distance from the origin, and the angle θ measured from the positive x-axis. In this crop circle, with the coordinate system shown, the value of r varies from approximately $r = 1.6$ units at the inner circle to approximately $r = 1.9$ units at the outer circle. We express this region using the inequality $1.6 \le r \le 1.9$. Since the crop circle ring in question completes one revolution, θ varies from 0 radians to 2π radians. We express this as $0 \le \theta \le 2\pi$.

SUMMARY

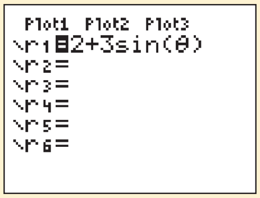
In this section you learned how to locate a point in a plane using polar coordinates. You also learned how to model real-world phenomena using graphs of polar equations as well as strategies for making sense of the behavior of the polar graph. Additionally, you learned how to convert from Cartesian coordinates to polar coordinates and vice versa.

TECHNOLOGY TIP ■ POLAR GRAPHS

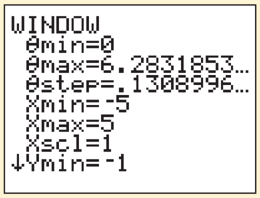
1. Press [MODE], scroll to **POL** and press [ENTER]. This will change the calculator to polar mode.



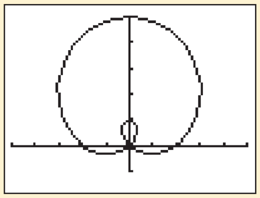
2. Press [Y=] and enter the equation of a polar graph, $r(\theta)$. Press [X,T,θ,n] to access the variable θ. In this example, we graph the polar equation $r = 2 + 3\sin(\theta)$.



3. Press **WINDOW**. Enter appropriate values for **θmin** and **θmax**. In radians, consider trying **θmin=0** and **θmax=2π**. Enter a value for **θstep**. This determines the frequency at which pixels will be plotted. The smaller the **θstep**, the more accurate the graph (but it will plot more slowly). The rest of the window settings can be made as usual.



4. Press [GRAPH]. You may need to adjust the window to get a good view of the polar graph.

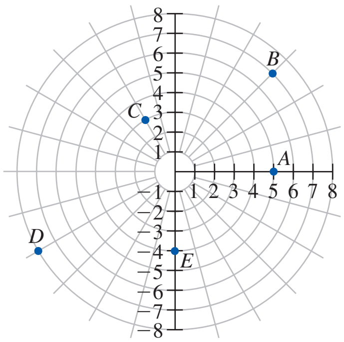


9.4 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–5, consider each of the points shown in the polar plane. Label each point in two different ways.

1. Point A
2. Point B
3. Point C
4. Point D
5. Point E



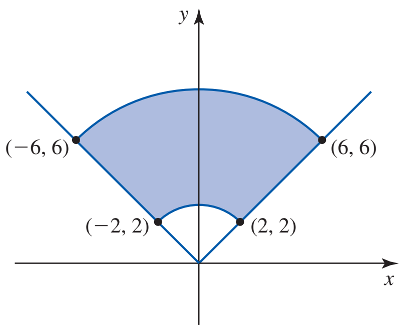
In Exercises 6–15, convert the Cartesian coordinates to polar coordinates.

6. (2, 2)
7. (3, 0)
8. (0, −5)
9. (−4, 0)
10. $(\sqrt{3}, 2)$
11. $(1, -\sqrt{3})$
12. (5, 2)
13. (−1, −5)
14. (4, −3)
15. (−6, 5)

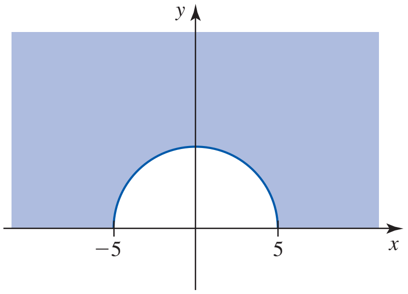
In Exercises 16–25, convert the polar coordinates to Cartesian coordinates.

16. (2, 0)
17. $\left(3, \frac{\pi}{2}\right)$
18. $(2, \pi)$
19. $\left(-3, \frac{\pi}{6}\right)$
20. $\left(5, \frac{5\pi}{6}\right)$
21. $\left(2, -\frac{\pi}{3}\right)$
22. (0.5, 1)
23. (−2, 4)
24. (−4, 0)
25. (−5, −2)

26. Give inequalities for r and θ to describe the given region in polar coordinates.



27. Give inequalities for r and θ to describe the given region in polar coordinates. Note the shaded region extends indefinitely in both the x- and y-directions.



In Exercises 28–32, graph the polar equation using a graphing calculator. Choose a viewing window so that you obtain a good result. Then, write an explanation for why the graph makes sense.

28. $r = 4\cos(5\theta)$

29. $r = 4\sin(2\theta)$

30. $r = 4\sin(3\theta)$

31. $r = 4\sin(5\theta)$

32. $r = 1 + 2\sin(\theta)$

In Exercises 33–42, graph the polar equation using a graphing calculator. Choose a viewing window so that you obtain a good result.

33. $r = 1 + 3\sin(\theta)$

34. $r = 1 + 4\sin(\theta)$

35. $r = 1 + 2\cos(\theta)$

36. $r = 1 + 3\cos(\theta)$

37. $r = 1 + 4\cos(\theta)$

38. $r = 5\sin(\theta)\cos^2(\theta)$

39. $r = 5\cos(\theta)\sin^2(\theta)$

40. $r = \dfrac{2}{\sqrt{\theta}}$

41. $r = \dfrac{4\sin(\theta)}{\theta}$

42. $r = \dfrac{10\sin(\theta)}{\theta}$

SHOW YOU KNOW

43. A classmate missed class and asks you to explain the purpose of polar coordinates. How do you respond?

44. Write a detailed explanation on how to convert from Cartesian to polar coordinates. Then write an explanation for why the formulas work in this process.

45. Write a detailed explanation on how to convert from polar to Cartesian coordinates. Then write an explanation for why the formulas work in this process.

46. Write a detailed explanation of the behavior of the polar graph of $r = C$, where C is a constant.

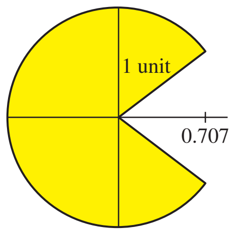
47. Write a detailed explanation of the behavior of the polar graph of $r = \theta$.

MAKE IT REAL

48. Compact Disc An audio compact disc (CD) has a diameter of 12 centimeters. The diameter of the hole in the center of the CD is 1.5 centimeters. Write a description of the CD using polar coordinates. That is, use inequalities to describe the range of values of r and θ that would describe the CD.

49. Pizza Slice A 16-inch-diameter pizza is cut into 8 equal pieces. Use inequalities to describe one slice of pizza in polar coordinates.

50. Pac-Man The video game Pac-Man was introduced to the world in 1980 by game designer Toru Iwatani. One day during lunch, he took a slice of pizza from the whole pie and was inspired to create Pac-Man. (***Source:*** **www.arcade-history.com**) Use the diagram given to express Pac-Man using inequalities in polar coordinates.



STRETCH YOUR MIND

Exercises 51–55 are designed to challenge your understanding of polar graphs.

51. Graph the equation $r = a + n\sin(\theta)$ for different values of a while holding n constant. Describe the relationship between the behavior of the graph and the value of a.

52. Graph the equation $r = a + n\sin(\theta)$ for different values of n while holding a constant. Describe the relationship between the behavior of the graph and the value of n.

53. Graph the equation $r = a + n\cos(\theta)$ for different values of a while holding n constant. Describe the relationship between the behavior of the graph and the value of a.

54. Graph the equation $r = a + n\cos(\theta)$ for different values of n while holding a constant. Describe the relationship between the behavior of the graph and the value of n.

55. Repeat Exercises 51–54 but for the polar equations $r = a - n\sin(\theta)$ and $r = a - n\cos(\theta)$.

SECTION 9.5 Vectors

LEARNING OBJECTIVES

- Use vectors to model and solve real-world situations
- Determine the magnitude and direction of a vector
- Resolve a vector into components
- Add, subtract, and scale vectors graphically and algebraically

GETTING STARTED

On the ground, the wind has a negligible effect on an airplane's speed, but in the air, airplanes may be significantly impacted by wind. In fact, when pilots calculate the time needed to complete a journey, they distinguish between ground speed and airspeed. *Ground speed* is the speed of the airplane relative to the ground and *airspeed* is the difference between the ground speed and the wind speed. If there is no wind speed, the airspeed is equal to the ground speed. If there is a tailwind (wind blowing in the direction of the airplane's travel), the airspeed will be greater than the ground speed and if there is a headwind (wind blowing in the opposite direction of the airplane's travel), the airspeed will be less than the ground speed. In addition to having an influence on the speed of the plane, wind may influence the direction the plane is traveling. If the wind is blowing against the side of the plane, the airspeed is not impacted, but the wind may alter the direction the plane is traveling.

In this section we study vector quantities like wind speed where both magnitude (e.g., speed, force, mass, length) and direction are relevant. We learn to use vectors to model and solve real-world situations.

■ Geometric Vectors

The Dubai International Airport in Dubai, United Arab Emirates, is one of the largest and busiest airports in the world. As with most airports, it has a moving walkway system to quickly move people from one part of the airport terminal to another. We may think of the velocity of the walkway as a vector quantity. **Vector** quantities have both a *magnitude* (the speed of the walkway in this case) and *direction* (the direction the walkway is moving). (Scalar quantities have only magnitude.)

> **VECTOR**
>
> A **vector** is any quantity with both magnitude and direction.

Suppose the walkway is moving at a speed of 2 miles per hour and we designate the direction to be in the positive vertical direction on the Cartesian plane. To represent the vector quantity geometrically, we use an arrow connecting two points as in Figure 9.71a, where we connect the points (0, 0) and (0, 2). The length, or **magnitude**, of the vector corresponds to the speed of the walkway. This magnitude, together with the direction of 90°, makes this a vector quantity. The beginning point is known as the **tail** of the vector and the end point is known as the **tip** of the vector. There is no special reason why the tail is located at (0, 0). In fact, as shown in Figure 9.71b, we could place this 2-unit-long vertical vector anywhere on the Cartesian coordinate system. When the tail of the vector is located at the origin, we refer to the vector as a **position vector**.

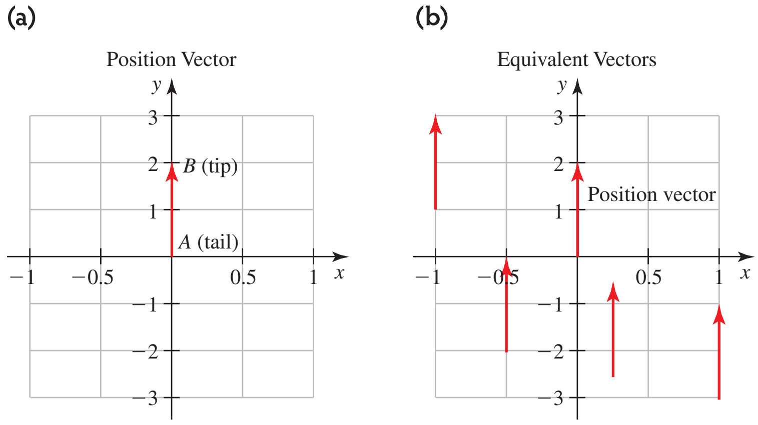


Figure 9.71

Symbolically, we express the position vector by determining how much distance is displaced horizontally and vertically from tail to tip. In this case, the horizontal displacement is 0 units and the vertical displacement is 2 units. Therefore, we express the vector as $\overrightarrow{AB} = \langle 0, 2\rangle$. We call this the **displacement vector in component form**.

Displacement Vector in Component Form

A **displacement vector** from one point to another is represented by an arrow with the tail located at the first point and the tip located at the second point. If the tail is located at point (a, b) and the tip is located at the point (c, d), then the displacement vector, $\vec{V}$, is written in component form as

$$\vec{V} = \langle c - a, d - b\rangle$$

Suppose we see a person walking on the moving walkway at a pace of 3 miles per hour. Because the walkway is moving at 2 miles per hour, the combined speed of the person is 5 miles per hour. We express this sum geometrically by placing the vectors (the moving walkway and the person walking) tail to tip in the Cartesian plane and connecting the tail of the first vector with the tip of the second vector. See Figure 9.72a.

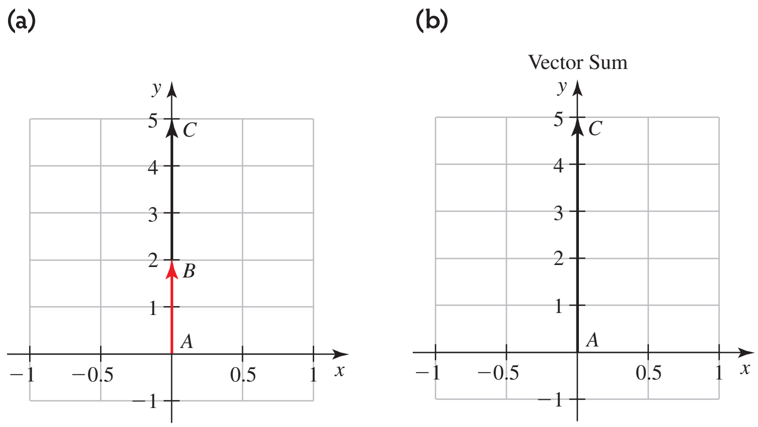


Figure 9.72

We symbolically say $\vec{AB} + \vec{BC} = \vec{AC}$. We can also compute the **vector sum** (Figure 9.72b) by adding the horizontal and then the vertical components of the vectors.

$$\vec{AB} + \vec{BC} = \vec{AC}$$
$$\langle 0, 2\rangle + \langle 0, 3\rangle = \langle 0 + 0, 2 + 3\rangle$$
$$\vec{AC} = \langle 0, 5\rangle$$

Vector Sum

Given two vectors $\vec{A} = \langle a, b\rangle$ and $\vec{B} = \langle c, d\rangle$, the sum $\vec{A} + \vec{B}$ is found by adding corresponding components.

$$\vec{A} + \vec{B} = \langle a + c, b + d\rangle$$

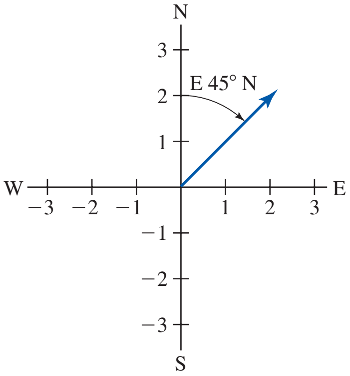


Figure 9.73

Measuring Angles in an Aviation Context

In aviation, angles are measured from due north. That is, due north is 0° and angles are described as being east or west of north. For example, 45° east of north (E 45° N) is an angle in the first quadrant, as shown in Figure 9.73.

EXAMPLE 1 ■ **Using Vectors**

Suppose an airplane pilot wishes to file a flight plan from one airport to another. The destination airport is 200 miles due south. The pilot anticipates an average airspeed of 80 miles per hour. Indicate the heading as a vector quantity in component form and then calculate the flight time.

Solution Since the location of the destination airport is due south of the starting point, the airplane should head E 180° N (due south) at 80 miles per hour. Having both magnitude (speed of 80 miles per hour) and direction (due south), this is a vector quantity. We write this velocity vector as $\vec{V} = \langle 0, -80\rangle$ and graph it as shown in Figure 9.74.

At the average speed of 80 miles per hour, it will take 2.5 hours to travel the 200 miles to the destination airport since $\frac{200 \text{ miles}}{80 \text{ miles per hour}} = 2.5$ hours.

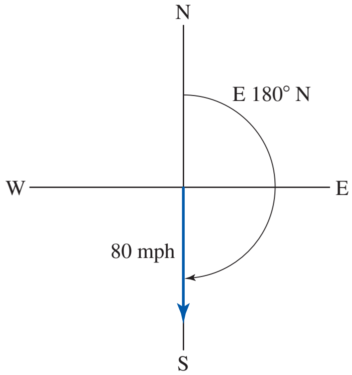


Figure 9.74

Computing the Magnitude and Direction of a Vector

Suppose we need to find the magnitude of the vector $\vec{A} = \langle 2, 5\rangle$. As a position vector, this vector has a horizontal component of 2 and a vertical component of 5. Sketching this as in Figure 9.75, we see that we can use the Pythagorean theorem to compute the magnitude, or length, of the vector (dashed line). We use the symbol $|\vec{A}|$ to represent the magnitude.

$$|\vec{A}|^2 = 2^2 + 5^2$$
$$|\vec{A}|^2 = 4 + 25$$
$$|\vec{A}|^2 = 29$$
$$|\vec{A}| = \sqrt{29}$$

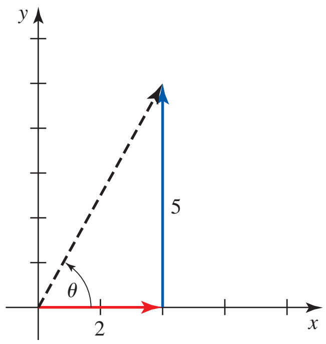


Figure 9.75

The magnitude of the vector is $|\vec{A}| = \sqrt{29}$ or $|\vec{A}| \approx 5.39$ units.

Using trigonometric functions, we can also determine the direction of the vector.

$$\tan(\theta) = \frac{5}{2}$$
$$\theta = \tan^{-1}(2.5)$$
$$\approx 68.2°$$

The vector is headed in the direction of 68.2°.

Computing Magnitude and Direction of a Given Vector

The magnitude of a vector $\vec{A} = \langle a, b \rangle$ is given by

$$|\vec{A}| = \sqrt{a^2 + b^2}$$

The direction is determined by

- $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ for a vector with a tip in the first or fourth quadrant.
- $\theta = \tan^{-1}\left(\frac{b}{a}\right) + 180°$ for a vector with a tip in the second or third quadrant.

Adding Vectors Geometrically

It is sometimes helpful to see the sum of two vectors graphically. Suppose we wish to visualize the sum of the vectors $\vec{W} = \langle -2, 5 \rangle$ and $\vec{Z} = \langle 3, 1 \rangle$. We begin by sketching these vectors in the Cartesian plane as position vectors. (See Figure 9.76.) Recall we may move vectors around the plane as long as we maintain their magnitude (length) and direction. To visualize the sum $\vec{W} + \vec{Z}$, we move vector $\vec{Z}$ (red line) and connect it tail to tip with vector $\vec{W}$ (blue line). Then we connect the tail of vector $\vec{W}$ to the tip of the repositioned vector $\vec{Z}$ (light red line). This vector, $\vec{W} + \vec{Z}$, is known as the **resultant vector** (dashed black line). We see that $\vec{W} + \vec{Z} = \langle -2, 5 \rangle + \langle 3, 1 \rangle = \langle 1, 6 \rangle$.

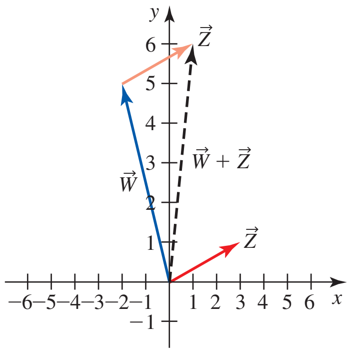


Figure 9.76

EXAMPLE 2 ■ Using Vectors

The airplane in Example 1 was traveling 80 miles per hour due south. Suppose the airplane encounters a 25-mile-per-hour wind blowing from west to east throughout the entire trip. If the pilot keeps the vector heading of due south, how far from the destination will the pilot end up? Express the solution of this problem using geometric vectors. What is the magnitude of the resultant vector?

Solution We first make a diagram of the problem situation, drawing both the vector for the airplane and the wind. See Figure 9.77a. Recall that as long as magnitude and direction are maintained, we can move a vector around the Cartesian plane. Figure 9.77b shows the repositioned wind vector connected tail to tip with the airplane vector.

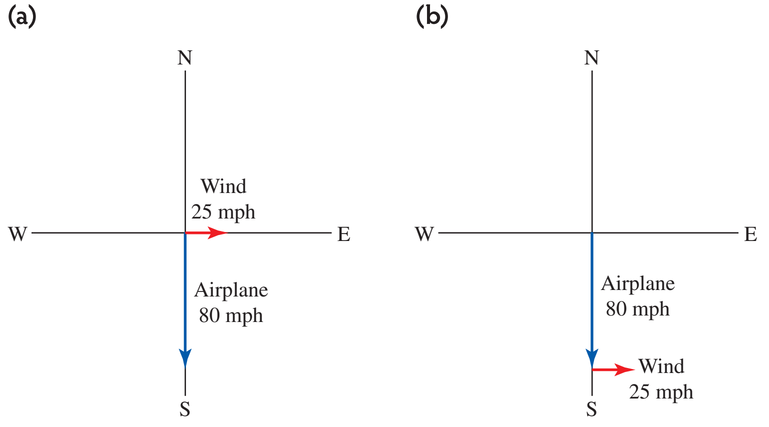


Figure 9.77

The vector sum of the airplane vector and the wind vector is shown in Figure 9.78. As we might expect, the effect of the wind causes the airplane to miss its target by pushing it to the east of the destination airport. Since the airplane takes 2.5 hours to travel the 200-mile distance to its destination, the wind will cause the airplane to travel 62.5 miles east of the airport.

$$25\,\frac{\text{miles}}{\text{hour}} \cdot 2.5 \text{ hours} = 62.5 \text{ miles}$$

The resultant velocity vector is the sum of the velocity of the airplane, $\vec{A} = \langle 0, -80 \rangle$, and the velocity of the wind, $\vec{W} = \langle 25, 0 \rangle$.

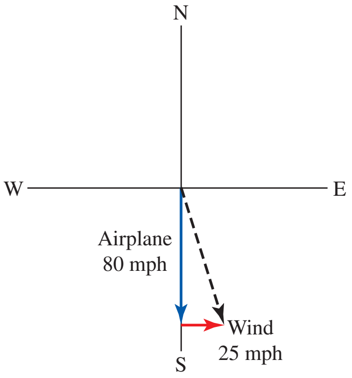


Figure 9.78

$$\vec{R} = \vec{A} + \vec{W}$$
$$\vec{R} = \langle 0, -80 \rangle + \langle 25, 0 \rangle$$
$$\vec{R} = \langle 0 + 25, -80 + 0 \rangle$$
$$\vec{R} = \langle 25, -80 \rangle$$

The magnitude of the resultant vector is the airspeed of the plane as a result of the effect of the wind.

$$|\vec{R}| = \sqrt{(25)^2 + (-80)^2}$$
$$|\vec{R}| = \sqrt{625 + 6400}$$
$$|\vec{R}| = \sqrt{7025}$$
$$|\vec{R}| \approx 83.82$$

The airspeed of the airplane is 83.82 miles per hour.

EXAMPLE 3 ■ **Using Vectors**

Determine the heading necessary so that the airplane in Example 2 will reach the destination airport. The average ground speed is still 80 miles per hour. Sketch a diagram showing the corrected heading and write the heading as a vector in component form.

Solution The pilot will need to set the heading so that the airplane points into the wind in such a way that the horizontal component of its velocity vector compensates for the 25-mile-per-hour wind. We sketch the problem situation in Figure 9.79. We see the hypotenuse of the right triangle is represented by the 80-mile-per-hour average speed and the side opposite the angle is the 25-mile-per-hour wind speed. We can determine the angle needed to correct the heading using trigonometry.

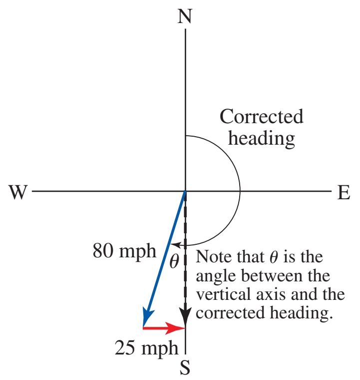


Figure 9.79

$$\sin(\theta) = \frac{25}{80}$$

$$\theta = \sin^{-1}\left(\frac{25}{80}\right)$$

$$\theta = 18.21°$$

The pilot needs to add 18.21° to the heading to correct for the wind. Therefore, the new heading is 180° + 18.21° = 198.21° E of N at 80 miles per hour. We can also refer to this angle as 161.79° W of N. This vector has a horizontal component of −25 to counteract the effects of the wind. We determine the vertical component using the Pythagorean theorem.

$$80^2 = (-25)^2 + y^2$$
$$6400 = 625 + y^2$$
$$y^2 = 6400 - 625$$
$$y^2 = 5775$$
$$y = \sqrt{5775}$$
$$y \approx 75.99$$

The desired heading will be determined by the velocity vector $\vec{V} = \langle -25, -75.99 \rangle$.

■ Scalar Multiplication with Vectors

When we multiply a vector quantity by a scalar, only the magnitude of the vector changes. For example, suppose the velocity of a sprinter in the 100-meter sprint is given by the vector $\vec{S} = \langle 3, 6 \rangle$ measured in meters per second. Suppose the wind is blowing directly at the sprinter's back at a velocity that is $\frac{1}{18}$ the speed of the sprinter. We can compute the velocity of the wind, $\vec{W}$, as a vector quantity.

$$\vec{W} = \frac{1}{18}\vec{S}$$

$$\vec{W} = \frac{1}{18}\langle 3, 6 \rangle$$

$$\vec{W} = \left\langle \frac{1}{18} \cdot 3, \frac{1}{18} \cdot 6 \right\rangle$$

$$\vec{W} = \left\langle \frac{1}{6}, \frac{1}{3} \right\rangle$$

Scalar Multiplication

Given a vector $\vec{A} = \langle a, b \rangle$ and a scalar, k, the product $k\vec{A}$ is the vector

$$\begin{aligned} k \cdot \vec{A} &= k \cdot \langle a, b \rangle \\ &= \langle k \cdot a, k \cdot b \rangle \end{aligned}$$

EXAMPLE 4 ■ Using Scalar Multiplication

Suppose the average velocity vector of a sprinter in the 100-meter sprint is given by $\vec{S} = \langle 3, 6 \rangle$ and that the wind is blowing directly at the sprinter's back at a speed that is $\frac{1}{18}$ as great as the velocity of the sprinter. With the effect of the wind taken into consideration, how long will it take the sprinter to finish the race? According to USA Track and Field rules, if the average speed of the wind blowing in the direction of the runner is greater than 2 meters per second, then a record cannot be accepted. Could a record be accepted in this case?

Solution We saw, using scalar multiplication, that the velocity of the wind is given by the vector $\vec{W} = \left\langle \frac{1}{6}, \frac{1}{3} \right\rangle$. We know the wind velocity vector is in the same direction as the sprinter velocity vector, just smaller in magnitude by $\frac{1}{18}$. We will find the sum of these vectors and use the resultant vector to determine the speed of the sprinter with the effects of the wind considered.

$$\vec{S} + \vec{W} = \langle 3, 6 \rangle + \left\langle \frac{1}{6}, \frac{1}{3} \right\rangle$$

$$\vec{S} + \vec{W} = \left\langle 3 + \frac{1}{6}, 6 + \frac{1}{3} \right\rangle$$

$$\vec{S} + \vec{W} = \left\langle \frac{19}{6}, \frac{19}{3} \right\rangle$$

The magnitude of this resultant vector is the speed of the runner.

$$|\vec{S} + \vec{W}| = \sqrt{\left(\frac{19}{6}\right)^2 + \left(\frac{19}{3}\right)^2}$$

$$|\vec{S} + \vec{W}| = \sqrt{\frac{1805}{36}}$$

$$|\vec{S} + \vec{W}| \approx 7.08 \text{ meters per second}$$

Using this average speed of 7.08 meters per second, we determine the time it will take the sprinter to complete the 100-meter sprint as

$$\frac{100 \text{ meters}}{7.08 \dfrac{\text{meters}}{\text{second}}} = 14.12 \text{ seconds}$$

While this is not a record time, if the wind speed is less than 2 meters per second, a record time would be accepted.

$$|\vec{W}| = \sqrt{\left(\frac{1}{6}\right)^2 + \left(\frac{1}{3}\right)^2}$$

$$|\vec{W}| = \sqrt{\frac{1}{36} + \frac{1}{9}}$$

$$|\vec{W}| = \sqrt{\frac{5}{36}}$$

$$|\vec{W}| \approx 0.37 \text{ meters per second}$$

The wind speed was 0.37 meters per second, so a record time could be accepted.

■ Vector Subtraction

We subtract the vectors through the process of adding the opposite. That is, $\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$. In other words, we first multiply the second vector by the scalar -1 and then add corresponding components of the vectors.

VECTOR SUBTRACTION

Given two vectors $\vec{A} = \langle a, b \rangle$ and $\vec{B} = \langle c, d \rangle$, the difference $\vec{A} - \vec{B}$ is given by $\vec{A} + (-\vec{B})$, where $-\vec{B} = \langle -c, -d \rangle$. That is,

$$\vec{A} - \vec{B} = \langle a - c, b - d \rangle$$

■ Unit Vectors

Unit vectors are vectors that have a length of 1 unit. We can use two special unit vectors to express displacement vectors in a different form. We define these unit vectors as follows.

$$\vec{i} = \langle 1, 0 \rangle$$

$$\vec{j} = \langle 0, 1 \rangle$$

To show how we can use the unit vectors $\vec{i}$ and $\vec{j}$ to express displacement vectors, consider the following.

$$\vec{V} = \langle 4, 5 \rangle$$

$$\vec{V} = \langle 4, 0 \rangle + \langle 0, 5 \rangle$$

$$\vec{V} = 4 \cdot \langle 1, 0 \rangle + 5 \cdot \langle 0, 1 \rangle$$

$$\vec{V} = 4\vec{i} + 5\vec{j}$$

These equivalent forms of expressing vector quantities are conventionally used in mathematics and science. We will use both forms throughout the section. Within a particular problem situation, it is typical to stay with one particular format.

Vectors Used to Measure Force

Another way that vectors may be used is to represent forces. Simply put, a force is a push or a pull on an object. Using pounds as a unit of measure of force, we might say "20 pounds of force were required to move the box across the room" or "we applied 80 pounds of force to the rope to keep the wagon from rolling down the hill." In the next example, we will express forces as vectors in component form and use these vectors to solve a problem.

EXAMPLE 5 ■ Using Vectors

Suppose a 400-pound force is required to stabilize a hot air balloon. Two ropes, held at the angles shown in Figure 9.80, are used as tether lines to hold the balloon in place. Determine the force needed in each of the tether lines to create the 400-pound force. Express each of these forces as vectors in component form.

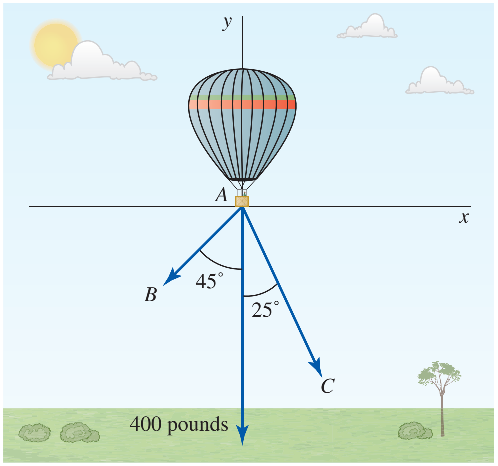


Figure 9.80

Solution We determine the measure of the angle between each vector and the x-axis. Then we draw the horizontal and vertical components of each vector. See Figure 9.81.

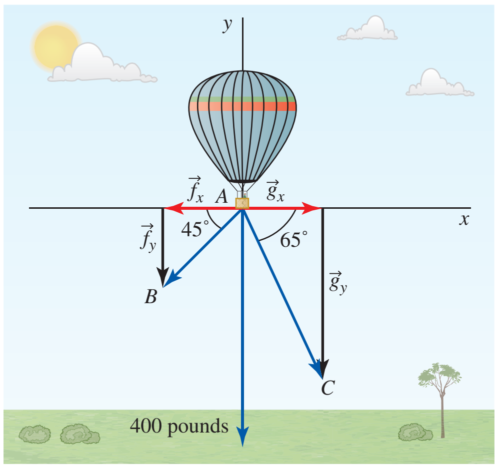


Figure 9.81

We will first determine the vector $\overrightarrow{AB}$. We know $\vec{f}_x + \vec{f}_y = \overrightarrow{AB}$, so we determine the magnitudes of $\vec{f}_x$ and $\vec{f}_y$ by using the cosine and sine functions.

$$\cos(45°) = \frac{|\vec{f}_x|}{|\vec{AB}|} \qquad \sin(45°) = \frac{|\vec{f}_y|}{|\vec{AB}|}$$

$$|\vec{f}_x| = |\vec{AB}|\cos(45°) \qquad |\vec{f}_y| = |\vec{AB}|\sin(45°)$$

The vector $\vec{AB} = \langle -|\vec{AB}|\cos(45°), -|\vec{AB}|\sin(45°)\rangle$. (We add the negatives to both components because the tip of the vector $\vec{AB}$ in standard position lies in the third quadrant.) We represent the vector in component form as

$$\vec{AB} = -|\vec{AB}|\cos(45(°)\vec{i} - |\vec{AB}|\sin(45°)\vec{j}$$

We determine the component form of vector $\vec{AC}$ in a similar way. We know $\vec{g}_x + \vec{g}_y = \vec{AC}$, so we determine the magnitudes of $\vec{g}_x$ and $\vec{g}_y$ by using the cosine and sine functions.

$$\cos(65°) = \frac{|\vec{g}_x|}{|\vec{AC}|} \qquad \sin(65°) = \frac{|\vec{g}_y|}{|\vec{AC}|}$$

$$|\vec{g}_x| = |\vec{AC}|\cos(65°) \qquad |\vec{g}_y| = |\vec{AC}|\sin(65°)$$

The vector $\vec{AC} = \langle |\vec{AC}|\cos(65°), -|\vec{AC}|\sin(65°)\rangle$. (We add the negative to the vertical component because the tip of the vector $\vec{AC}$ in standard position lies in the fourth quadrant.) We represent the vector in component form as

$$\vec{AC} = |\vec{AC}|\cos(65°)\vec{i} - |\vec{AC}|\sin(65°)\vec{j}$$

Next, we apply ideas concerning the balloon situation. For the balloon to be stabilized, $\vec{AB} + \vec{AC} = 0\vec{i} - 400\vec{j}$. That is, the horizontal components should add to 0 since the balloon is not to move horizontally. The vertical components should add to be -400 to overcome the force of the balloon in the upward direction. Thus,

$$-|\vec{AB}|\cos(45°) + |\vec{AC}|\cos(65°) = 0$$

$$-|\vec{AB}|\sin(45°) - |\vec{AC}|\sin(65°) = -400$$

We have a system of two equations with two unknowns, $|\vec{AB}|$ and $|\vec{AC}|$. We will solve the first equation for $|\vec{AC}|$ and then substitute into the second equation.

$$|\vec{AC}|\cos(65°) = |\vec{AB}|\cos(45°)$$

$$|\vec{AC}| = \frac{|\vec{AB}|\cos(45°)}{\cos(65°)}$$

Next, we substitute and solve for $|\vec{AB}|$.

$$-|\vec{AB}|\sin(45°) - |\vec{AC}|\sin(65°) = -400$$

$$-|\vec{AB}|\sin(45°) - \left[\frac{|\vec{AB}|\cos(45°)}{\cos(65°)}\right]\sin(65°) = -400$$

$$|\vec{AB}|\sin(45°) + \left[\frac{|\vec{AB}|\cos(45°)}{\cos(65°)}\right]\sin(65°) = 400$$

$$0.707|\vec{AB}| + 1.52|\vec{AB}| = 400$$

$$2.22|\vec{AB}| = 400$$

$$|\vec{AB}| = \frac{400}{2.22}$$

$$|\vec{AB}| \approx 179.9 \text{ pounds}$$

We now substitute this result into $|\vec{AC}| = \dfrac{|\vec{AB}|\cos(45°)}{\cos(65°)}$ to find the magnitude of $\vec{AC}$.

$$|\vec{AC}| = \frac{179.9\cos(45°)}{\cos(65°)}$$

$$|\vec{AC}| \approx 301 \text{ pounds}$$

The component forms of the vectors are

$$\vec{AB} = -179.9\cos(45°)\vec{i} - 179.9\sin(45°)\vec{j}$$
$$\vec{AB} = -127.2\vec{i} - 127.2\vec{j}$$

$$\vec{AC} = 301\cos(65°)\vec{i} - 301\sin(65°)\vec{j}$$
$$\vec{AC} = 127.2\vec{i} - 272.8\vec{j}$$

SUMMARY

In this section you learned how to use vectors to model and solve real-world situations. You learned to compute the magnitude of a vector and to write a vector in component form. You learned how to add and subtract vectors as well as calculate the scalar multiple of a vector.

9.5 EXERCISES

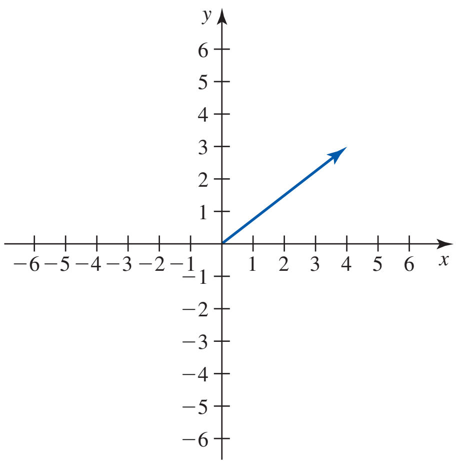
SKILLS AND CONCEPTS

In Exercises 1–10, determine whether the given quantity is a vector or scalar.

1. Velocity of wind
2. Price of gasoline
3. Air temperature
4. Force needed to lift a box
5. Number of students in a class
6. Tension in a suspension bridge cable
7. Water current in river
8. Velocity of a car on the highway
9. Amount of water in swimming pool
10. Force needed to open a door

In Exercises 11–15, write each vector in component form.

11.



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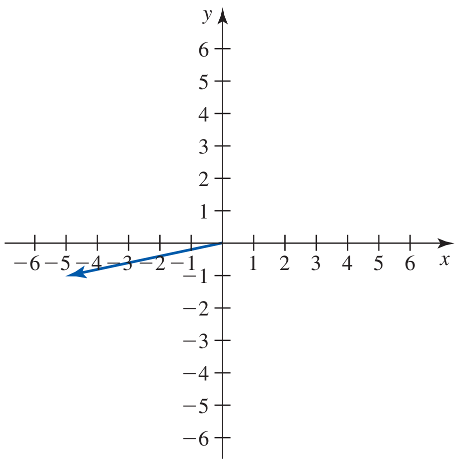

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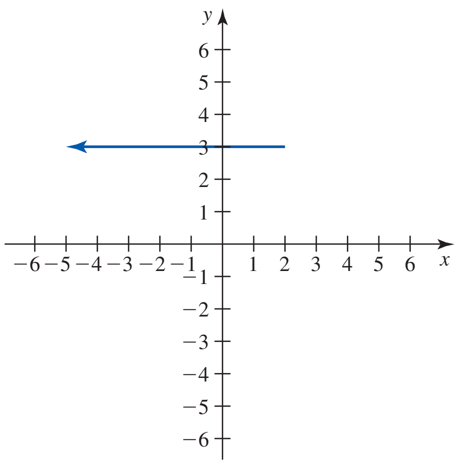

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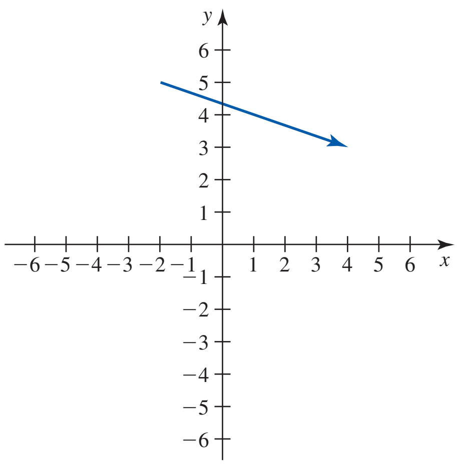

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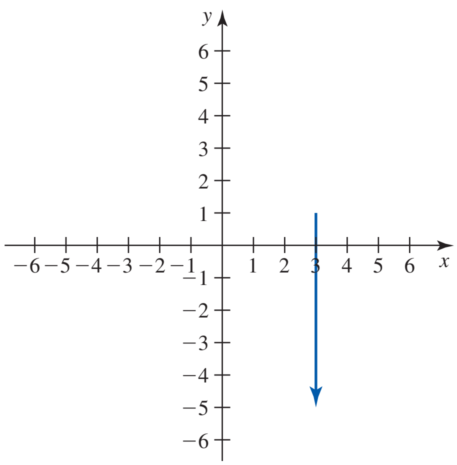


In Exercises 16–25, perform the indicated operation. Then sketch the given vectors and the resultant of the operation.

$$\vec{a} = 3\vec{i} + 5\vec{j} \qquad \vec{b} = -4\vec{i} - \vec{j}$$
$$\vec{c} = 6\vec{j} \qquad \vec{d} = -8i$$

16. $\vec{a} + \vec{b}$

17. $\vec{a} + \vec{c}$

18. $3\vec{a}$

19. $-\vec{b}$

20. $\vec{b} - \vec{a}$

21. $2\vec{c} - \vec{b}$

22. $3\vec{b} - 2\vec{d}$

23. $2\vec{c} - 2\vec{c}$

24. $2\vec{a} + \frac{1}{2}\vec{d}$

25. $2\vec{a} + 3\vec{b}$

In Exercises 26–30, find each of the displacement vectors from one city to another. For these exercises, the origin is the intersection of the equator and the prime meridian. The horizontal and vertical distances are approximated to be the latitude and longitude measurements.

26. From Phoenix (33, −112) to Minneapolis (45, −93)

27. From Minneapolis (45, −93) to Amsterdam (52, 5)

28. From Amsterdam (52, 5) to Dubai (25, 55)

29. From Dubai (25, 55) to Malé (4, 73)

30. From Dubai (25, 55) to Phoenix (33, −112)

31. Find another vector, $\vec{S}$, that is in the same direction as $\vec{T} = 5\vec{i} - 3\vec{j}$ but is only one unit in length.

32. Find another vector, $\vec{P}$, that is twice the length of $\vec{Q} = \langle 2, 6 \rangle$ but is in the opposite direction of $\vec{Q}$.

33. Suppose a displacement vector from (2, 3) to (c, n) is $\vec{W} = \langle 3, -4 \rangle$. Determine the value of c and n.

34. Suppose the sum of three vectors is the vector $\vec{T} = 6\vec{i} + 10\vec{j}$. One of the vectors is $\vec{Z} = \vec{i} - 2\vec{j}$. Determine components of the other two vectors that make the equation true.

35. Determine the point that is the location of the tip of the vector whose tail is located at (4, 5) and is displaced two units in the direction of $\theta = 60°$.

36. Suppose a delivery truck is traveling north at 10 miles per hour and the driver tosses a newspaper out of the window at a speed of 2 miles per hour toward a house located directly to the west relative to the truck.

a. Find the velocity vector of the paper relative to the ground when the driver releases it.

b. Find the speed of the paper at that time.

SHOW YOU KNOW

37. Many people think *velocity* and *speed* to mean the same thing. Explain why these words are not synonymous.

38. Read the FoxTrot comic strip. Write an explanation, including graphs or diagrams, confirming that running "10 yards out, then 10 yards to the right" puts you in the same place as going "$10\sqrt{2}$ yards at a 45-degree angle."

39. Suppose a boat is traveling on a river at velocity $\vec{B}$ and speed of 20 miles per hour. Further suppose that the river has a current of $\vec{C}$ and a speed of 5 miles per hour. Under what conditions will the speed of the boat be 25 miles per hour relative to the land alongside the river? Create a geometric argument for a situation where the relative speed of the boat is not 25 miles per hour.

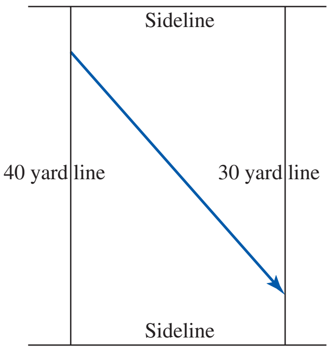
40. The acceleration due to gravity is approximately 9.8 meters per second per second. This means the velocity of a falling object, due to gravity, increases by 9.8 meters per second for each second that the object falls. The force of gravity is a vector that may be expressed as $\vec{G} = -9.8\vec{j}$. Explain why.

41. Vectors express quantities that have magnitude and direction. It is possible to describe the "zero vector." Write an explanation of the zero vector and explain how it is expressed symbolically.

42. Using a geometric representation of vectors, show that a vector sum is commutative. That is, show that $\vec{A} + \vec{B} = \vec{B} + \vec{A}$.

■ MAKE IT REAL

43. Marching Band Drill Marching band members follow detailed instructions to create formations on the field. For example, one move might instruct a marcher to begin 8 steps (5 yards) in front of one sideline on the 40 yard line and move to the position 8 steps (5 yards) in front of the other sideline on the 30 yard line. The sidelines are 50 yards apart. Using the diagram shown, create a displacement vector to represent this marching move. Answers may vary depending on how you describe the points.



44. Hanging a Frame As seen in the photo, a frame is hung on a wall with the center of the hanging cable on the hook. The angle at the hook is 100° and the frame weighs 8 pounds. Determine the tension in each side of the hanging cable and express the tension as a vector in component form.

45. **Aircraft Heading** The heading from Minneapolis, Minnesota, to Los Angeles, California, is E 249° N. (*Source:* www.aeroplanner.com) Suppose an airplane is flying from Minneapolis to Los Angeles at a speed of 605 miles per hour at a fixed altitude with no wind factor.

a. The velocity of the plane may be considered a vector. Explain why it is a vector and describe the two important parts of the vector.

b. Express the velocity of the plane, $\vec{A}$, as a vector in component form.

c. Find the magnitude of $\vec{A}$ and describe its significance in the context of the problem situation. Show all work.

46. **Wind Effect** The heading from Minneapolis, Minnesota, to Los Angeles, California, is E 249° N. (*Source:* www.aeroplanner.com) Suppose an airplane is flying from Minneapolis to Los Angeles at a speed of 605 miles per hour at a fixed altitude with no wind factor. As the plane reaches a certain point, it encounters wind with a velocity of 50 miles per hour in the direction E 30° N.

a. Express the velocity of the wind, $\vec{W}$, and the velocity of the airplane, $\vec{A}$, as a vector.

b. Sketch the resultant vector produced by the sum of $\vec{A}$ and $\vec{W}$. What is the resultant velocity of the plane? Round to the nearest tenth.

c. What is the resultant speed of the plane? Round to the nearest tenth and include units.

d. What is the resultant direction of the plane? That is, what is the measure of the angle that the resultant vector forms with the positive x-axis?

47. **Seed Spitting Contest** The U.S. Watermelon Seed Spitting Contest is held in Pardeeville, Wisconsin, each September. For men, the current seed spitting record is 61 feet 3 inches. (*Source:* www.uncommondays.com)

Suppose in one particular year, the seed spitting contest was being performed in the direction of $\vec{S} = 4\vec{i} + \vec{j}$. During the contest, a steady wind picked up with a velocity of $\vec{V} = \vec{i} + 0.25\vec{j}$ miles per hour.

Suppose further that the rulebook states that a legal wind in the direction of the seed spit must not exceed 2.5 miles per hour. If a contestant spits a seed 63 feet, will the record stand?

48. **The Wright Brothers** On December 17, 1903, Orville Wright flew the world's first aircraft. Later that day he sent the following telegram.

> "SUCCESS FOUR FLIGHTS THURSDAY MORNING ALL AGAINST TWENTY ONE MILE WIND STARTED FROM LEVEL WITH ENGINE POWER ALONE AVERAGE SPEED THROUGH AIR THIRTY ONE MILES LONGEST 57 SECONDS INFORM PRESS HOME CHRISTMAS. OREVELLE [*sic*] WRIGHT" (*Source:* www.first-to-fly.com)

According to Wright's journal, the wind was blowing from the north.

a. Express the velocity of the airplane as a vector in component form.

b. Express the velocity of the wind as a vector in component form.

c. Find the vector sum of the velocity of the airplane and the velocity of the wind.

d. What is the magnitude of the vector sum found in part (c)? Do you think that Orville's brother Wilbur was able to run alongside the airplane? Why or why not?

■ STRETCH YOUR MIND

Exercises 49–53 are intended to challenge your understanding of vectors. Your task is to make an argument for each of the following facts. Your argument may be graphical and/or symbolic.

49. $(c + k)\overrightarrow{AB} = c\overrightarrow{AB} + k\overrightarrow{AB}$

50. $\overrightarrow{AB} + \vec{0} = \overrightarrow{AB}$

51. $k(\vec{A} + \vec{B}) = k\vec{A} + k\vec{B}$

52. $\overrightarrow{AB} + (-\overrightarrow{AB}) = 0$

53. $(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C})$

CHAPTER 9 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 9.1

1. What are the six trigonometric ratios and how can you use them to solve problems?
2. How are the six trigonometric ratios connected to the unit circle?
3. What is an angle of elevation? What is an angle of depression?

SECTION 9.2

4. What is the Law of Cosines?
5. Under what conditions is it appropriate to use the Law of Cosines?
6. How is the Law of Cosines connected to the Pythagorean theorem?

SECTION 9.3

7. What is the Law of Sines?
8. Under what conditions is it appropriate to use the Law of Sines?
9. How is triangulation used to solve real-world problems?

SECTION 9.4

10. How are coordinates described in the polar coordinate system?
11. How can you convert coordinates from the Cartesian system to the polar system? How can you convert from polar coordinates to Cartesian coordinates?
12. How can you describe a geometric region using inequalities in polar coordinates?
13. How can you describe the graph of an equation in polar coordinates?
14. In what kinds of real-world situations is it appropriate to use polar coordinates?

SECTION 9.5

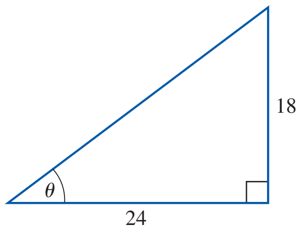
15. What is a vector quantity and how is it different from a scalar quantity?
16. How are vectors represented geometrically?
17. How are vectors added and subtracted, and how is a sum or difference represented geometrically?
18. How can you find the magnitude of a vector? What might this magnitude represent?
19. What is a position vector?
20. How are displacement vectors determined?
21. How can vectors be used to represent real-world situations?

REVIEW EXERCISES

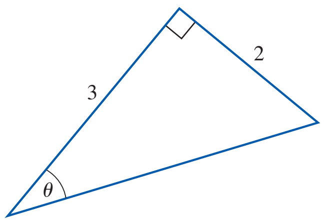
SECTION 9.1

In Exercises 1–2, evaluate the six trigonometric functions of θ. That is, evaluate sin(θ), cos(θ), tan(θ), sec(θ), csc(θ), cot(θ).

1.



2.



In Exercises 3–4, a trigonometric value is given. Determine if the angle measure, θ, is

a. Less than 30°.

b. Between 30° and 60°.

c. Between 60° and 90°.

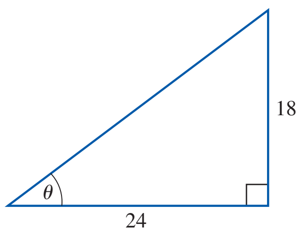
Explain how you know.

3. $\sin(\theta) = 0.98$

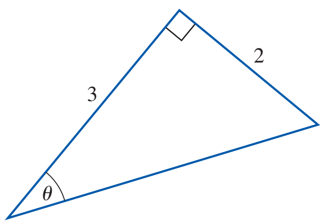
4. $\cos(\theta) = 0.95$

In Exercises 5–6, determine the angle measure of θ.

5.



6.



7. Washington Monument Suppose you are standing 1000 feet away from the base of the Washington Monument, which is 555 feet $\frac{5}{8}$ inches in height. What is the measure of the angle of elevation from your position to the top of the monument?

8. The Burj Dubai The Burj Dubai, currently under construction in the city of Dubai in the United Arab Emirates, will be the tallest building in the world when it is completed. While its ultimate height is being kept secret, it is projected to be approximately 800 meters in height upon completion. (***Source:*** **www.newscientist.com**) On June 24, 2007, the height of the building reached 493 meters. (***Source:*** **www.burjdubaiskyscraper.com**)

Suppose you are standing 500 meters from the base of the tower.

a. Compute the measure of the angle of elevation to the top of the tower (493 meters).

b. Suppose after the tower is completed, you went back to the same position, 500 meters from the base of the tower. What will be the measure of the angle of elevation to the top of the tower at that time? (Assume the final height will be 800 meters.)

SECTION 9.2

In Exercises 9–12, find the length of the side opposite the given angle.

9. In ΔABC, $b = 15$, $c = 19$, and $A = 24°$

10. In ΔABC, $a = 71$, $b = 80$, and $C = 7°$

11. In ΔABC, $b = 5$, $c = 6$, and $A = 59°$

12. In ΔABC, $a = 5$, $c = 8$, and $B = 31°$

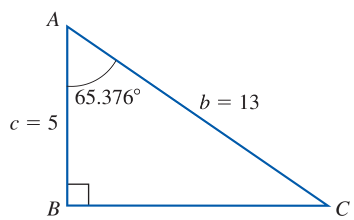
In Exercises 13–14, find the measure of the specified angle.

13. A in ΔABC; if $a = 4$, $b = 5$, and $c = 6$

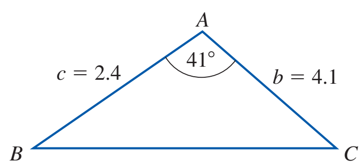
14. C in ΔABC; if $a = 15$, $b = 16$, and $c = 18$

In Exercises 15–16, solve each triangle using any correct technique.

15.



16.



In Exercises 17–18, draw a diagram when one is not supplied. Then apply the Law of Cosines, the Pythagorean theorem, or a trigonometric function to find the length of the specified side or measure of the specified angle.

17. Distance between Airplanes Two airplanes leave Reagan National Airport in Washington DC at noon. The air traffic controller notes the two are traveling away from one another at an angle of 107°. One of the airplanes is traveling at 495 miles per hour and the other 502 miles per hour. How far apart are the two planes at 12:50 P.M.?

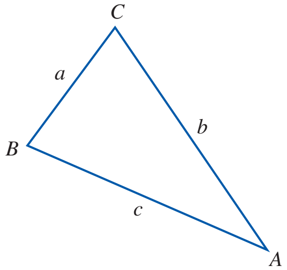
18. Distance between Cities Attempting to fly from Cincinnati, Ohio, to Tallahassee, Florida—a distance of 738 miles—a small plane veers off course by 5°.

a. If the plane has a constant speed of 210 miles per hour and the pilot notices his error after 30 minutes, how far off course is the plane at that time? In other words, at what angle should he turn to get back on track?

b. How fast should the pilot fly the plane to get to Tallahassee in four hours?

SECTION 9.3

In Exercises 19–22, use triangle ABC to find the missing sides and angles (if possible). If there are two solutions, find both. If there is no solution, so state.



19. $a = 8$, $B = 87°$, $C = 24°$

20. $a = 4$, $B = 15°$, $C = 55°$

21. $A = 26°$, $B = 47°$, $c = 13$

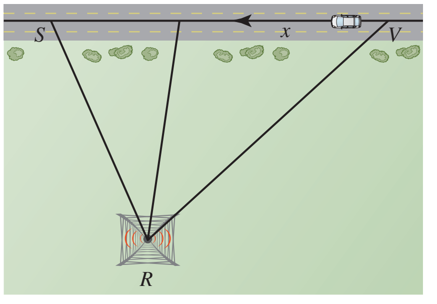
22. $A = 86°$, $B = 13°$, $c = 4.5$

In Exercises 23–26, draw a diagram when one is not supplied. Then apply the Law of Sines, Law of Cosines, the Pythagorean theorem, or a trigonometric function to the situation.

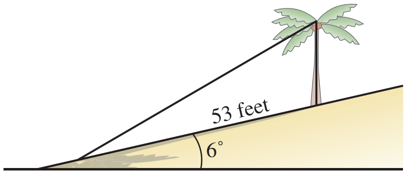
23. Cloud Height Two observers wonder how high the cloud is above their heads. The first person can see the cloud at an angle of elevation measuring 35° and the other at 43°. If the two people are 2500 feet apart, how high is the cloud?

24. Interplanetary Distance Earth (E) and Mars (M) are approximately 9.3 million and 14.2 million miles from the Sun (S), respectively. (***Source:*** **www.enchantedlearning.com**) Find the distance between the two planets when $\angle SEM$ measures 21°.

25. Radio Transmission The figure shows a vehicle moving due west and approaching a 30-watt transmitter on a 30-meter-tall radio tower. The effective range of the transmitter is 15 kilometers. To the nearest kilometer, for what distance x will the vehicle be within range of the transmitter at R if, when the car is at V, the measure of $\angle SVR = 43°$ and the distance from V to R is 21 kilometers?



26. Tree Height A tree grows vertically on the side of a hill that slopes upward from the horizontal by 6°. The tree casts a shadow 53 feet long down the hill when the angle of elevation of the sun measures 15°. How tall is the tree?



SECTION 9.4

In Exercises 27–31, consider each of the points shown in the polar plane. Label each point in two different ways.

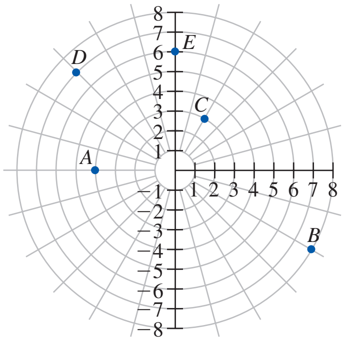
27. Point A

28. Point B

29. Point C

30. Point D

31. Point E



In Exercises 32–34, convert the Cartesian coordinates to polar coordinates.

32. $(3, -4)$

33. $(-1, -1)$

34. $(0, 6)$

In Exercises 35–40, convert the polar coordinates to Cartesian coordinates.

35. $(4, 2\pi)$

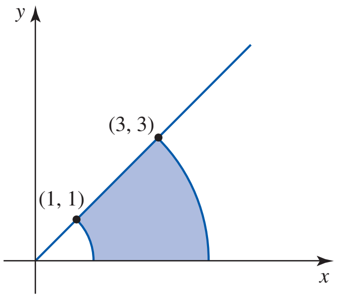
36. $\left(1, \frac{3\pi}{2}\right)$

37. $\left(-2, \frac{\pi}{3}\right)$

38. $\left(-1, -\frac{5\pi}{6}\right)$

39. $\left(6, \frac{2\pi}{3}\right)$

40. Give inequalities for r and θ to describe the shaded region in polar coordinates.



In Exercises 41–42, graph the polar equation using a graphing calculator. Choose a viewing window so you obtain a good result.

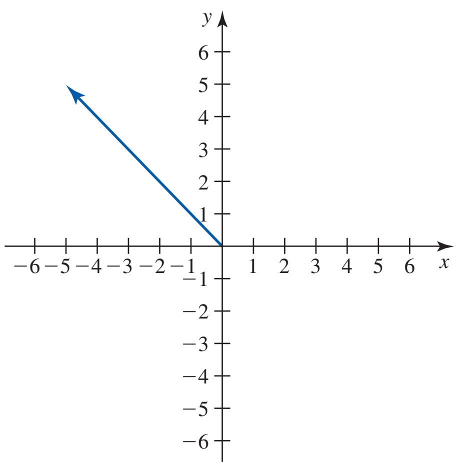
41. $r = \cos(5\theta) + 2$

42. $r = \cos(5\theta) + 5$

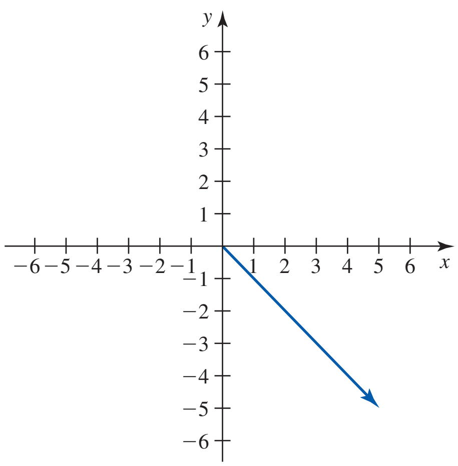
SECTION 9.5

In Exercises 43–44, write each vector in component form.

43.



44.



In Exercises 45–46, perform the indicated operation. Then sketch the given vectors and the resultant of the operation.

$$\vec{v} = 6\vec{i} + 2\vec{j} \qquad \vec{w} = -5\vec{i} - 3\vec{j}$$
$$\vec{r} = -4\vec{j} \qquad \vec{u} = 7i$$

45. $\vec{v} + 2\vec{w}$

46. $-\vec{r} + \vec{u}$

47. Find another vector, $\vec{B}$, that is in the opposite direction as $\vec{W} = -6\vec{i} + 3\vec{j}$ and is only one unit in length.

48. Find another vector, $\vec{P}$, that is twice the length of the vector $\vec{K} = \langle 2, -1\rangle$ but in the opposite direction.

49. Find the value of c such that the magnitude of vector $\vec{Z} = 2\vec{i} + c\vec{j}$ is $|\vec{Z}| = 5$.

50. Flight Heading To travel from New York, New York, to Boston, Massachusetts, a pilot needs to fly at a heading of E 52.4° N. (*Source:* www.aeroplaner.com) Suppose an airplane is flying the route at a speed of 540 miles per hour at a fixed altitude with no wind factor.

a. The velocity of the plane may be considered a vector. Explain why it is a vector and describe the two important parts of the vector.

b. Express the velocity of the plane, $\vec{A}$, as a vector in component form.

c. Find the magnitude of $\vec{A}$ and describe its significance in the context of the problem situation. Show all work.

51. Wind Effects To travel from New York, New York, to Boston, Massachusetts, a pilot needs to fly at a heading of E 52.4° N. (*Source:* www.aeroplaner.com) Suppose an airplane is flying the route at a speed of 540 miles per hour at a fixed altitude with no wind factor. As the plane reaches a certain point, it encounters wind with a velocity of 35 miles per hour in the direction E 10° N.

a. Express the velocity of the wind, $\vec{W}$, and the velocity of the airplane, $\vec{A}$, as a vector.

b. Sketch the resultant vector produced by the sum of $\vec{A}$ and $\vec{W}$. What is the resultant velocity of the plane? Round to the nearest tenth.

c. What is the resultant speed of the plane? Round to the nearest tenth and include units.

d. What is the resultant direction of the plane? That is, what is the measure of the angle that the resultant vector forms with the positive x-axis?

Make It Real Project

What to Do: Part I

1. Identify an object that is at least three times your height.
2. Draw a diagram showing the distances and angles you will need to calculate the object's height.
3. Measure the needed angles and distances and use trigonometry to calculate the height of the object.
4. Write a report describing the situation, the measurements, your trigonometric computations, and your results.

What to Do: Part II

1. A polar graph with equation $r = \pm\sqrt{a^2\theta}$ is called Fermat's spiral. Graph several polar graphs for different values of a. Describe the effect that a has on these graphs.
2. A polar graph with equation $r = \frac{a}{\theta}$ is called a hyperbolic spiral. Graph several polar graphs for different values of a. Describe the effect that a has on these graphs.
3. A polar graph with equation $r = \frac{a\sin(\theta)}{\theta}$ is called a cochleoid. Graph several polar graphs for different values of a. Describe the effect that a has on these graphs.