# 8.2 Angle Measure, Arc Length, and Area of a Sector

## Getting Started

Many of us have an intuitive understanding of *angle* measure. Angles play a big role in architecture, construction, surveying, geometry, backyard pool design, highway construction, and even some sporting activities. For example, to be successful in the game of billiards, a player must bank the balls off of the cushions at specific angles.

In construction, surveyors play an important role in measuring distances and angles in order to map out locations for the construction project. Surveyors use a tool called a theodolite for measuring angles. The theodolite is a telescope mounted on a tripod that allows the surveyor to measure angles with great precision: $\frac{1}{60}$ of 1 degree!

In this section, we look at the different ways that angles are measured. We also provide a way to think about angle measure that will be useful in our later study of trigonometric functions.

## Angles and Angle Measure

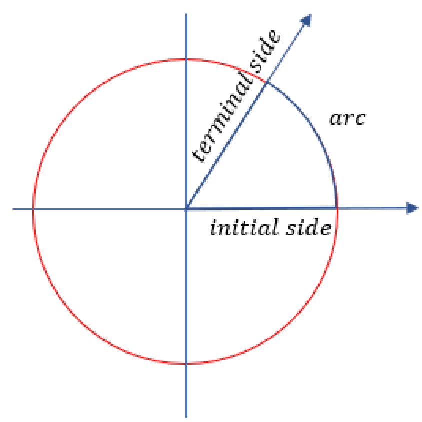
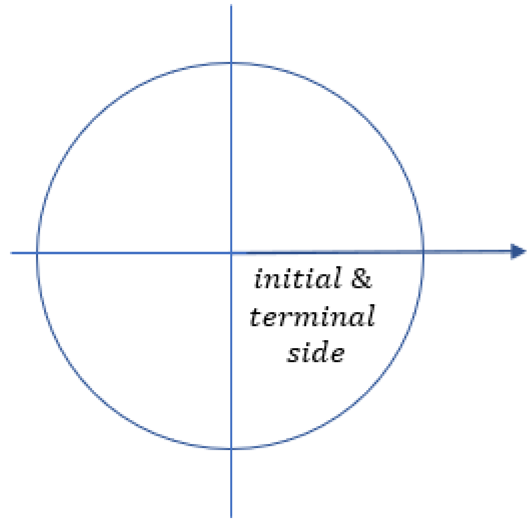


Figure 8.14

From a geometric point of view, an **angle** is the figure formed by two rays (or segments) sharing a common endpoint. If we position the common endpoint at the origin and rotate the angle so that one of the rays forming the angle lies along the horizontal axis, we can relate the mathematical concept of angle with the mathematical concept of a circle. We refer to the ray aligned with the horizontal axis as the *initial side* of the angle and the other ray as the *terminal side* of the angle. We draw a circular arc between the initial and terminal sides (Figure 8.14). Imagine rotating the terminal side of the angle counterclockwise until the terminal side lies on top of the initial side forming a circle (Figure 8.15). The corresponding angle is referred to as a *full angle* or *complete angle*. Notice that the arc length of a full angle equals the length of the circumference of the circle. Recall that the length of the circumference of the circle, $C$, is given by $C = 2\pi r$ where $r$ is the length of the radius of the circle. Consequently, any angle less than a full angle will have an arc length less than the length of the circumference.

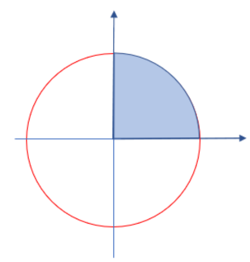



To allow us to compare different angles, it is useful to create a way to quantify each angle. A variety of systems of angle measure have been developed over the years to quantify the relative size of an angle. One of the most intuitive ways to quantify an angle is to identify the fraction of the circumference of the circle that is associated with the angle.

A common approach used to develop an angle measure system is to divide a full angle into a number of equally-sized smaller angles. In the degree system, the full angle is divided into 360 smaller angles. Each of these smaller angles measures 1 degree and has a corresponding arc length equal to $\frac{1}{360}$ of the length of the circumference. In the gradian system, the full angle is divided in to 400 smaller angles. Each of the smaller angles measures 1 gradian and has a corresponding arc length equal to $\frac{1}{400}$ of the length of the circumference. In the quadrant system (used by the mathematician Euclid), the full angle is divided into four smaller angles. Each of the smaller angles measures 1 quadrant and has a corresponding arc length equal to $\frac{1}{4}$ of the length of the circumference. In the sextant system (used by the Babylonians), the full angle is divided into six smaller angles. Each of the smaller angles measures 1 sextant and has a corresponding arc length equal to $\frac{1}{6}$ of the length of the circumference.

If we wanted to, we could make up our own system of angle measure. For example, we could divide a full angle into 10 equally-size smaller angles and refer to the measure of the smaller angle as 1 *decel*, an angle whose corresponding arc length would equal $\frac{1}{10}$ of the length of the circumference. The measure of the full angle would equal 10 decels. No matter the system of measure, it is useful to think of the measure of an angle as it relates to the circumference of the corresponding circle.

**Example 1: Determining the Measure of an Angle in Different Systems of Angle Measure**



A full angle is divided into four equally-sized smaller angles. What is the measure of the smaller angle in degrees, gradians, quadrants, sextants, and decels?

**Solution**

Because the smaller angle is ¼ of a full angle, its arc length with be ¼ of the length of the circumference and its angle measure will be ¼ of the measure of a full angle.

degrees: $\frac{1}{4}(360 \text{ degrees}) = 90 \text{ degrees}$

gradians: $\frac{1}{4}(400 \text{ gradians}) = 100 \text{ gradians}$

quadrants: $\frac{1}{4}(4 \text{ quadrants}) = 1 \text{ quadrant}$

sextants: $\frac{1}{4}(6 \text{ sextants}) = 1.5 \text{ sextants}$

decels: $\frac{1}{4}(10 \text{ decels}) = 2.5 \text{ decels}$

So 90 degrees, 100 gradians, 1 quadrant, 1.5 sextants and 2.5 decels are equivalent angle measures for ¼ of a full angle.

The proportional thinking illustrated in Example 1 can be used to convert an angle measure from one system to another as illustrated in Example 2.

**Example 2: Converting from One System of Angle Measure to Another**

How many degrees are equivalent to 300 gradians?

**Solution**

A full angle measures 400 gradians so 300 gradians is $\frac{300}{400}$ of a full angle. We know that a full angle in the degree system has measure 360 degrees. Multiplying 360 degrees by $\frac{300}{400}$ gives us the degree measure equivalent to 300 gradians.

$$\frac{300}{400}\,(360 \text{ degrees}) = 270 \text{ degrees}$$

So 300 gradians is equivalent to 270 degrees. This angle has an arc length that is $\frac{300}{400}$ of the length of the circumference.

**CONVERTING BETWEEN DIFFERENT SYSTEMS OF ANGLE MEASURE**

The process for converting the measure of an angle from one system to another is as follows:

$$new\ system\ angle\ measure = \frac{original\ system\ angle\ measure}{original\ system\ full\ angle\ measure} \cdot new\ system\ full\ angle\ measure$$

**Example 3: Comparing Angle Measures from Different Systems**

Which of the angles has the greatest angle measure: 120 gradians, 4 sextants, 2 quadrants, or 100 degrees?

**Solution**

One of the most efficient ways to answer the question is to determine what fraction of a full angle is represented each angle.

$$\frac{120 \text{ gradians}}{400 \text{ gradians}} = 0.30 \qquad \frac{4 \text{ sextants}}{6 \text{ sextants}} \approx 0.67 \qquad \frac{2 \text{ quadrants}}{4 \text{ quadrants}} = 0.50 \qquad \frac{100 \text{ degrees}}{360 \text{ degrees}} \approx 0.28$$

Observe that 120 gradians is 30% of the measure of a full angle, 4 sextants is about 67% of the measure of a full angle, 2 quadrants is 50% of the measure of a full angle, and 100 degrees is about 28% of the measure a full angle. Because 4 sextants has the greatest percentage of the measure of a full angle, it is the angle with the greatest angle measure.

**Example 4: Determining Angle Measure from an Arc Length**

An angle has an arc length the equals $\frac{1}{3}$ of the circumference of a circle. What is the measure of the angle in degrees, radians, and sextants?

**Solution:**

Because the arc length is $\frac{1}{3}$ of the circumference, we know the angle measure will be $\frac{1}{3}$ of the angle measure of a full angle.

Degrees: $\frac{1}{3}(360 \text{ degrees}) = 120 \text{ degrees}$

Radians: $\frac{1}{3}(2\pi \text{ radians}) = \frac{2\pi}{3} \text{ radians}$

Sextants: $\frac{1}{3}(6 \text{ sextants}) = 2 \text{ sextants}$

**Example 5: Determining Arc Length**

An angle has measure 12 degrees. If the corresponding circle has a radius of 20 feet, what is the arc length corresponding with 12 degrees?

**Solution**

We know that 12 degrees is $\frac{12}{360}$ of a full angle. Consequently, the arc length will be $\frac{12}{360}$ of the length of the circumference of the corresponding circle.

$$arc\ length = \frac{12}{360} \cdot 2\pi(20)$$

$$arc\ length = \frac{4}{3}\pi$$

$$arc\ length \approx 4.19 \text{ feet}$$

The arc length of a 12-degree angle corresponding with a circle with radius 20 feet is about 4.19 feet.

**DETERMINING ARC LENGTH**

The arc length of an angle corresponding with a circle of radius $r$ units is given by:

$$arc\ length = \frac{angle\ measure}{full\ angle\ measure} \cdot circumference$$

or, equivalently,

$$arc\ length = \frac{angle\ measure}{full\ angle\ measure} \cdot 2\pi r$$

**Example 6: Finding the Angle Measure Corresponding to a Given Arc Length**

For a given angle and corresponding circle, the length of the radius and the arc length are equal.

What is the measure of the angle in degrees?

**Solution:**

$$arc\ length = \frac{angle\ measure}{full\ angle\ measure} \cdot 2\pi r$$

$$r = \frac{angle\ measure}{360\text{ degrees}} \cdot 2\pi(r)$$

$$\frac{360}{2\pi}\text{degrees} = angle\ measure$$

$$angle\ measure \approx 57.3\text{ degrees}$$

If an angle has the property that the arc length equals the radius of the corresponding circle, the angle has measure of approximately 57.3 degrees.

## Radian Measure

Relating the length of an arc to the radius of the corresponding circle is a powerful mathematical idea that is essential to calculus. In fact, the radian system of angle measure is based entirely on this relationship.

**RADIAN MEASURE**

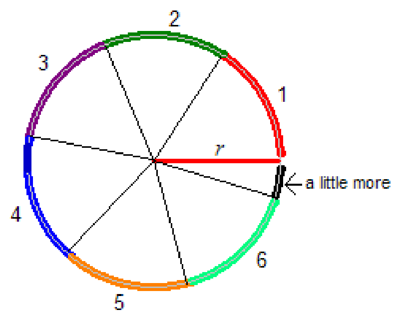
The measure of an angle $\theta$ corresponding with an arc length equal to the length of the radius of the corresponding circle is 1 **radian**.

**Example 6: Determining the Number of Radians in a Full Angle**

Using the idea that one radian is the measure of the angle that corresponds with an arc length equal to the length of one radius, determine the angle measure of a full angle in radians.

**Solution:**

We continue to imagine that the radius of the unit circle is a piece of string and that we want to use the piece of string to measure the circumference of the whole circle. We imagine laying the string on the outer edge of the circle, with one end beginning at the point $(1, 0)$, and measuring the number of 1 radius lengths required to span the entire circle. In the diagram below, each change in color represents the length of 1 radius.




It takes six complete radii lengths to span the circumference of the whole circle plus a little bit more. In fact, the "little bit more" is approximately 0.28 of one radius. It takes approximately 6.28 radii to span the circumference of the circle. The exact number of lengths is $2\pi$ ($2\pi \approx 6.2832$). Since each radian corresponds with an arc length of one radius on the unit circle, it takes $2\pi$ radians to generate an arc length equal to the circumference of the circle. This is where the formula for the circumference of a circle, $C = 2\pi r$, comes from.

**Example 7: Converting from Degrees to Radians**

What angle measure (in radians) is equal to 45 degrees?

**Solution**

$$\begin{aligned} radian\ measure &= \frac{45 \text{ degrees}}{360 \text{ degrees}} \cdot 2\pi \text{ radians} \\ &= \frac{1}{8} \cdot 2\pi \text{ radians} \\ &= \frac{\pi}{4} \text{radians} \end{aligned}$$

An angle of 45 degrees has the same measure as an angle of $\frac{\pi}{4}$ radians.

**Example 8: Converting from Radians to Degrees**

What angle measure (in degrees) is equal to 3 radians?

**Solution**

$$degree\ measure = \frac{3 \text{ radians}}{2\pi \text{ radians}} \cdot 360 \text{ degrees}$$
$$\approx 171.9 \text{ degrees}$$

An angle of 3 radians has the same measure of an angle of about 171.9 degrees.

## Area of a Sector

The term *sector* refers to the region bounded by the initial and terminal sides of an angle and the corresponding arc (Figure 8.17). Just as arc length was a fraction of the circumference of a circle, the area of a sector is a fraction of the area of a circle. The area of a circle of radius $r$ units is $A = \pi r^2$ square units.

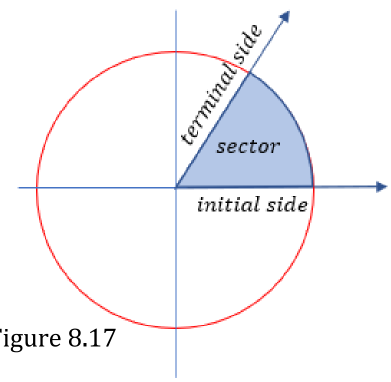


Figure 8.17

**Example 9: Determining the Area of a Sector**

What is the area of the sector corresponding with a circle of radius 10 feet and angle 60 degrees?

**Solution**

The area of the circle is $100\pi$ square feet. Because the angle 60 degrees is $\frac{60}{360}$ of a full angle, the area of the sector will be $\frac{60}{360}$ of the area of the circle.

$$area\ of\ sector = \frac{60}{360} \cdot 100\pi \text{ square feet}$$
$$\approx 52.36 \text{ square feet}$$

The area of the sector is about 52.36 square feet.

**DETERMINING THE AREA OF A SECTOR**

The area of a sector corresponding with an angle and a circle of radius $r$ units is given by:

$$area\ of\ sector = \frac{angle\ measure}{full\ angle\ measure} \cdot circle\ area$$

or, equivalently,

$$area\ of\ sector = \frac{angle\ measure}{full\ angle\ measure} \cdot \pi r^2$$

**Example 10: Determining the Angle Measure for a Sector with Known Area**

A sector corresponding with a circle of radius 10 feet contains 200 square feet of area. What is the measure of the corresponding angle in radians?

**Solution**

$$area\ of\ sector = \frac{angle\ measure}{full\ angle\ measure} \cdot \pi r^2$$

$$200 = \frac{angle\ measure}{2\pi} \cdot \pi (10)^2$$

$$200 = angle\ measure \cdot 50$$

$$angle\ measure = 4 \text{ radians}$$

The measure of the angle is 4 radians.

**Summary**

In this section you learned that different systems of angle measure (including degrees and radians) may be used to quantify angles. You learned how to convert between different systems of angle measure and how to determine the length of an arc and the area of a sector.

# 8.2 Exercises

*Exercises highlighted in blue are new. The remaining exercises are from the 1st Edition. Answers to the new odd exercises are provided at the end of this file.*

## Skills and Concepts

*In Exercises 1 – 8, a full angle is divided into $n$ equally-sized, smaller angles. Determine the measure of the smaller angle in degrees, gradians, sextants, and quadrants.*

1. $n = 2$
2. $n = 24$
3. $n = 3$
4. $n = 40$
5. $n = 6$
6. $n = 60$
7. $n = 20$
8. $n = 90$

*In Exercises 9 – 16, determine which of the listed angles has the smallest measure.*

9. 3 radians, 160 degrees, 2.5 sextants

10. 4.2 radians, 1.5 quadrants, 225 degrees

11. 190 degrees, 200 gradians, 3.5 radians

12. 3 quadrants, 5 sextants, 300 degrees

13. $\frac{\pi}{4}$ radians, 43 degrees, 0.5 quadrants

14. 6 radians, 350 degrees, 5.2 sextants

15. 280 degrees, 292 gradians, 5.1 radians

16. $\frac{3\pi}{4}$ radians, 120 degrees, 1.2 quadrants

For Exercises 17 – 21, convert each angle measured in degrees to radian measure.

17. $\theta = 15°$

18. $\theta = -60°$

19. $\theta = 335°$

20. $\theta = 120°$

21. $\theta = 410°$

For Exercises 22 – 26, convert each angle measured in radians to degree measure.

22. $\theta = \dfrac{\pi}{7}$

23. $\theta = \dfrac{\pi}{18}$

24. $\theta = \dfrac{4\pi}{3}$

25. $\theta = 8.5$

26. $\theta = -10$

*In Exercises 27 – 32, determine the arc length corresponding with the given angle and circle radius.*

27. 30 degrees, radius of 5 feet

28. 60 degrees, radius of 0.3 mile

29. 95 degrees, radius of 2 meters

30. 190 degrees, radius of 4 cm

31. 210 degrees, radius of 4 yards

32. 330 degrees, radius of 15 inches

*In Exercises 33 – 37, determine the area of the sector corresponding with the given angle and circle radius.*

33. 30 degrees, radius of 5 feet

34. 95 degrees, radius of 2 meters

35. 190 degrees, radius of 4 cm

36. 210 degrees, radius of 4 yards

37. 330 degrees, radius of 15 inches

## Show You Know

38. Pretend you are explaining the idea of radian measure of angles to a classmate who missed class. Write out an explanation or describe how you would help your classmate understand this idea.

39. A classmate claims that existing systems of angle measure are confusing and hard to remember. She has created a new unit of angle measure: the *centagree.* The are 100 centagrees in a full angle. Explain how you would convert degrees into centagrees.

40. Explain why, without using computations, an angle measuring $180°$ is equivalent to measuring $\pi$ radians.

41. Describe the relationship between the area of a sector and the corresponding arc length for a circle with radius $r$ units.

## Make It Real

For Exercises 42 – 46, match each angle measured in degree with the picture where that angle measure is seen, approximately. Each angle measure matches with one picture.

42. 120°

43. 36°

44. 46°

45. 360°

46. 90°

A. Cupboard door

B. Glass cup

C. Playset

D. Sidewalk and decking

E. Table top

## Stretch Your Mind

Exercises 47 – 50 are intended to challenge your understanding of angle measure. A unit of angular measure used by the mathematician Eratosthenes is the hexacontade. There are 60 hexacontades in a full angle.

47. Describe the relationship between hexacontades and degrees.

48. What procedure should be used to convert hexacontades to radians?

49. What is the arc length and sector area for an angle of 22 hexacontades with a corresponding circle of radius 8 meters?

50. The area of a sector corresponding with a circle of radius 12 inches is 144 square inches. What is the corresponding angle measured in hexacontades?

# 8.2 Answers to New Odd Exercises

1. 180 degrees, 200 gradians, 3 sextants, 2 quadrants

3. 120 degrees, $\frac{400}{3}$ gradians, 2 sextants, $\frac{4}{3}$ quadrants

5. 60 degrees, $\frac{400}{6}$ gradians, 1 sextant, $\frac{2}{3}$ quadrants

7. 18 degrees, 20 gradians, 0.3 sextant, 0.2 quadrant

9. 2.5 radians

11. 200 gradians

13. 43 degrees

15. 292 gradians

27. $\approx 2.62$ feet

29. $\approx 3.32$ meters

31. $\approx 14.66$ yards

33. $\approx 6.54$ square feet

35. $\approx 26.53$ square centimeters

37. $\approx 647.95$ square inches

39. Because a full angle is 100 centagrees and 360 degrees, the two quantities are equal.

    $100\ centagrees = 360\ degrees$

    Thus $\frac{100}{360}\ centagrees = 1\ degree$. Because $n$ degrees is equal to $n \cdot (1\ degree)$, $n$ degrees equals $n \cdot \left(\frac{100}{360}\ centagrees\right)$. So to convert from degrees to centagrees multiply by $\frac{100}{360}$.

41. Suppose that the arc length is equal to $n\%$ of the circumference. Then the area of the corresponding sector is $n\%$ of the area of the circle. That is, the arc length is

    $s = \frac{n}{100}(2\pi r)$ and the sector area is

    $a = \frac{n}{100}(\pi r^2)$. Observe $\frac{n\pi r}{100} = \frac{s}{2}$ and

    $\frac{n\pi r}{100} = \frac{a}{r}$. So $\frac{s}{2} = \frac{a}{r}$ which simplifies to

    $s = \frac{2a}{r}$. The arc length equals twice the area of the sector divided by the length of the radius.

43. E, Table top

45. B, Glass cup

47. Because 60 hexacontades is equivalent to 360 degrees,1 hexacontade is equivalent to 6 degrees.

49. $sector\ area \approx 73.72$ square feet

    $arc\ length \approx 18.43$ feet