CHAPTER 8

Trigonometric Functions

On August 13, 2003, the largest electrical blackout in North American history struck Connecticut, New Jersey, New York, Massachusetts, Vermont, Michigan, Ohio, Pennsylvania, and the Canadian province of Ontario. The outage impacted the lives of about 50 million people, leaving some without power for days. Financial losses due to the outage were estimated to be between \$4 billion and \$10 billion. (*Source:* U.S.-Canada Power System Outage Task Force, reports.energy.gov) The photo shows the New York skyline during the outage.

To reduce the frequency of such outages, energy organizations monitor and attempt to predict power usage each day throughout the year. The cyclical nature of daily power consumption can be modeled by periodic functions, making power usage predictions remarkably accurate.

8.1 Periodic Functions
8.2 Angle Measure
8.3 Unit Circle and Trigonometric Functions
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SECTION 8.1 Periodic Functions

LEARNING OBJECTIVES

- Determine if a real-world data table or graph is periodic
- Determine the period, midline, amplitude, and frequency of a periodic function
- Extrapolate function values for a periodic function given in a graph or table

GETTING STARTED

Many people commute to work or school and must plan for travel time to arrive at their destination on time. In addition, delivery truck drivers and others whose work efficiency is affected by freeway travel times can benefit from a detailed analysis of average travel times throughout the day. For example, by scheduling deliveries before and after peak travel times, drivers can reduce the amount of time spent in traffic and increase the number of deliveries that can be made in a day. Typical commute times on a given segment of a road can be modeled by a periodic function.

In this section we explore periodic functions and see how to apply these functions to real-world situations such as analyzing commute times. We define certain aspects of these functions—namely, the midline, amplitude, and frequency. We also show how to recognize the periodicity of a function given a graph or a table.

■ Periodic Functions

A **periodic function** is a function whose output values repeat at regular intervals, or **periods**. Such a function is said to have **periodicity**.

PERIODIC FUNCTIONS

A function f is **periodic** if $f(x + p) = f(x)$ for all x and for a positive constant p. The value p is called the **period** of the function.

From our knowledge of transformations, we know the graph of $f(x + p)$ is the graph of $f(x)$ shifted left p units. A periodic function has the property that when its graph is shifted left or right p units (where p is the period of the function), the resultant graph is the same as the original graph.

Let's explore the concepts related to periodic functions in the context of commuter travel times. The Washington Department of Transportation regularly monitors traffic flows and reports average and current travel times to commuters via its website (www.wsdot.wa.gov). The data is updated every five minutes throughout the day so commuters can estimate their travel times on key segments of freeway. A sample report is shown in Figure 8.1. The Washington State Transportation Department graph in Figure 8.2 shows the average travel time from Bellevue to Seattle on I-90 on Tuesdays in 2003 at various times throughout the day.

If we assume that average travel times for Wednesdays and Thursdays are identical to those for Tuesdays, we can say the travel-time function has a period of 24 hours. That is, every 24 hours, the travel-time pattern will repeat. (Note: The actual weekly data show that travel times for Fridays through Mondays vary significantly from the Tuesday travel times; therefore, we limit the periodic model to Tuesdays through Thursdays.)

When working with periodic functions, we are often interested in the **midline** of the function.

Travel times as of 10:10 A.M. Wednesday, June 13, 2007

State Route/ Interstate	Route Description	Distance (miles)	Average Travel Time (minutes)	Current Travel Time (minutes)	Via HOV (min.)
167	Auburn to Renton	9.8	17	12	10
405	Bellevue to Bothell	9.7	11	15	10
405 5	Bellevue to Everett	23.2	25	31	24
405 5	Bellevue to Federal Way	24.9	29	30	26
405 90	Bellevue to Issaquah	9.8	10	11	11
405 520	Bellevue to Redmond	6.8	8	10	8
405 90 5	Bellevue to Seattle	10.7	14	13	12
	Via Westbound Express Lanes	10.7	15	13	12

Figure 8.1

Source: www.wsdot.wa.gov

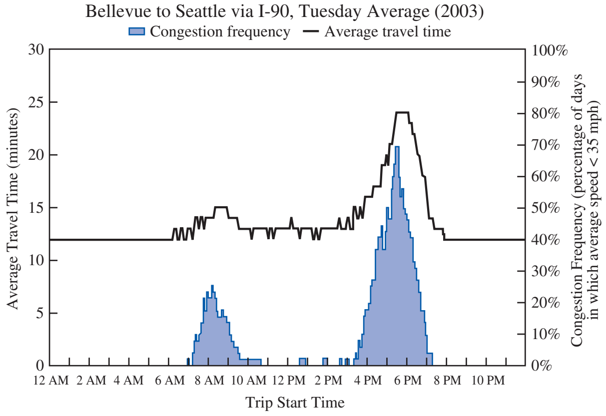


Figure 8.2

Source: www.wsdot.wa.gov

MIDLINE (CENTERLINE)

The **midline** or **centerline** of a periodic function f is the horizontal line that lies halfway in between the maximum and minimum output values of the function. In other words, the midline is the horizontal line with equation

$$y = \frac{\text{max of } f + \text{min of } f}{2}$$

For the travel-time function, it appears the maximum travel time is about 24 minutes and the minimum travel time is about 12 minutes. The midline is

$$y = \frac{24 + 12}{2}$$
$$= 18 \text{ minutes}$$

We draw a red dashed line on the graph, as shown in Figure 8.3, to indicate the location of the midline.

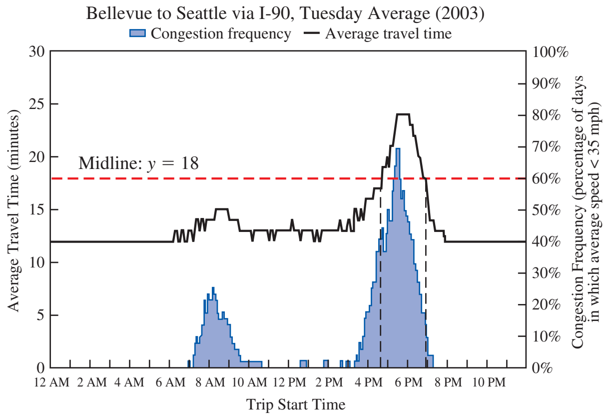


Figure 8.3

Source: www.wsdot.wa.gov

If the graph is above the midline, this means the travel time exceeds the average of the maximum and minimum travel times. We see this occurs between about 4:45 P.M. and 7:00 P.M. (indicated by black dashed lines). At all other times, the travel time is less than the average of the maximum and minimum travel times.

Another important characteristic of periodic functions is the **amplitude** of the function.

Amplitude

The **amplitude** of a periodic function f is the distance from the midline to the maximum or minimum output value of the function. In symbolic terms,

$$\text{amplitude} = \text{max of } f - \text{midline}$$

Equivalently, the amplitude is half of the distance between the maximum and minimum output values of the function. That is,

$$\text{amplitude} = \frac{\text{max of } f - \text{min of } f}{2}$$

For the travel-time function, we have

$$\begin{aligned}\text{amplitude} &= \frac{\text{max of } f - \text{min of } f}{2} \\ &= \frac{24 - 12}{2} \\ &= 6 \text{ minutes}\end{aligned}$$

This tells us the *maximum* commute time is 6 minutes *more* than the average of the maximum and minimum travel times. Similarly, the *minimum* commute time is 6 minutes *less* than the average of the maximum and minimum travel times. We indicate the amplitude with additional red dashed lines on the graph in Figure 8.4.

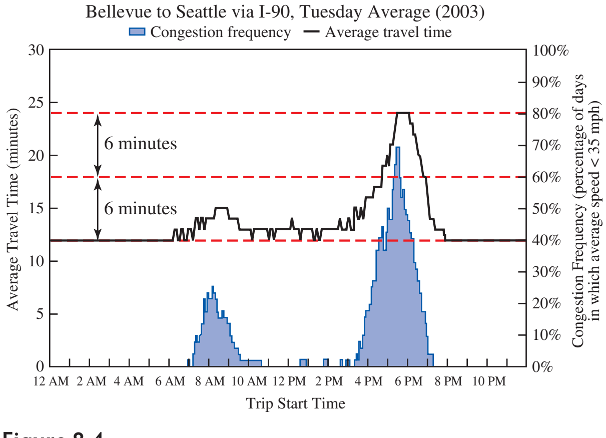


Figure 8.4

Source: www.wsdot.gov

A third characteristic of periodic functions is **frequency**. The frequency of a periodic function is closely related to its period.

Frequency

The **frequency** of a periodic function is the reciprocal of its period. That is,

$$\text{frequency} = \frac{1}{\text{period}}$$

For the travel-time function, the period is 24 hours so the frequency is $\frac{1}{24}$. Because the units of the frequency are not immediately obvious, let's investigate. Recall that there are 24 hours per 1 period. We can represent this as

$$\frac{24 \text{ hours}}{1 \text{ period}}$$

Now we take the reciprocal of this expression to find the units of the frequency.

$$\frac{1 \text{ period}}{24 \text{ hours}}$$

Thus the frequency is $\frac{1}{24}$ period per hour. That is, for each hour that passes, the travel-time function moves through $\frac{1}{24}$ of its period.

A periodic function's frequency is generally measured in the number of cycles (periods) per unit of time such as minutes, seconds, and so on. Figure 8.5 shows periodic functions with different frequencies, where the green graph has the highest frequency and the black graph has the lowest frequency.

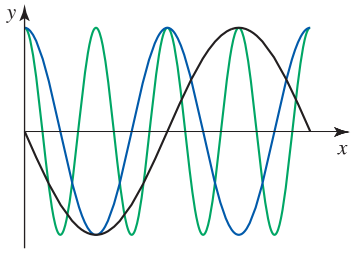


Figure 8.5

EXAMPLE 1 ■ Determining the Frequency of a Periodic Function

Two siblings (a boy and a girl) begin to play on the swing set. At rest, both swings are at approximately 2 feet off the ground but, as the children pump their legs, each swing's height off the ground changes. The graphs in Figure 8.6 display a 14-second interval of time (from 16 seconds to 30 seconds) and the heights of each swing seat (in feet). Use the graphs to determine the midline, amplitude, period, and frequency of each graph and explain what they mean in terms of each child's swinging action.

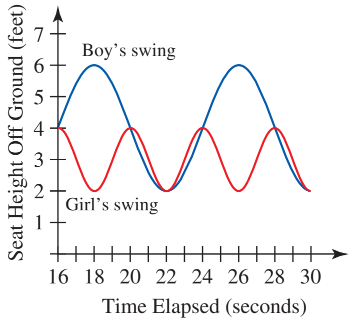


Figure 8.6

Solution We find the midline of each graph by calculating the average between the maximum and minimum swing seat heights for each child and display the midlines with dashed lines in Figure 8.7.

$$\text{boy's: } \frac{6+2}{2} = 4 \text{ feet} \qquad \text{girl's: } \frac{4+2}{2} = 3 \text{ feet}$$

We calculate the amplitude of each graph by calculating half the distance between the maximum and minimum swing seat heights.

$$\text{boy's: } \frac{6-2}{2} = 2 \text{ feet} \qquad \text{girl's: } \frac{4-2}{2} = 1 \text{ foot}$$

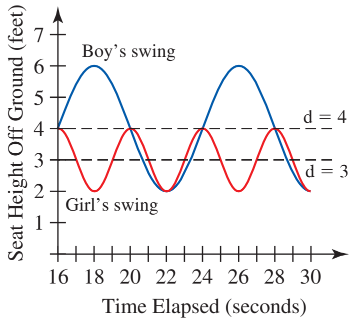


Figure 8.7

The amplitude of the boy's graph is 2 feet and the amplitude of the girl's graph is 1 foot. This means the boy's swing seat height varies from 2 feet above and 2 feet below the midline and the girl's from 1 foot below to 1 foot above the midline.

We determine the period for each of the graphs by finding out how long it takes each child's seat to return to its starting position. At 16 seconds, the boy's seat height is 4 feet. Eight seconds later, the seat returns to the same position. Therefore, the boy's period is 8 seconds. At 16 seconds, the girl's seat height is 4 feet. Four seconds later, her seat is at the same position. Therefore, the girl's period is 4 seconds.

Since frequency is the reciprocal of period, we have

$$\text{boy's frequency: } \frac{1 \text{ cycle}}{8 \text{ seconds}} = 0.125 \text{ cycles per second}$$

and

$$\text{girl's frequency: } \frac{1 \text{ cycle}}{4 \text{ seconds}} = 0.25 \text{ cycles per second}$$

Since the girl's frequency (0.25) is twice as large as the boy's frequency (0.125), the girl is traveling twice as fast as the boy.

Let's now turn our attention to how periodic functions may be transformed vertically or horizontally and what this means in a real-world context.

EXAMPLE 2 ■ Transforming Periodic Functions

The California Independent System Operator (ISO) is a not-for-profit electric transmission organization that predicts and monitors California's energy usage. Through close monitoring, the organization seeks to avoid the crippling electrical blackouts that can shut down a region. To help electricity consumers do their part in conserving energy, the organization publishes and regularly updates a daily power usage forecast on its website (www.caiso.com).

Figure 8.8 shows California ISO's electricity demand forecast for June 12 and June 13, 2007.

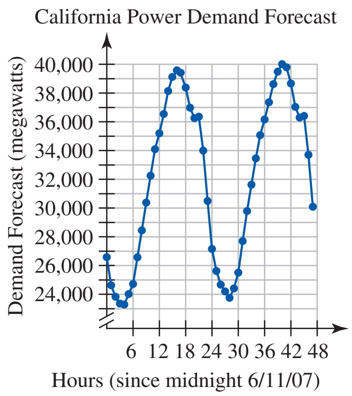


Figure 8.8

Source: www.caiso.com

a. Explain in terms of the rate of change why you think the data change as they do. Also, give a reasonable rationale for why the graph appears to be periodic in nature.

b. Estimate the midline and amplitude and explain what each value means in terms of power usage.

c. Suppose that through an aggressive energy conservation campaign, consumers were persuaded to reduce their collective electricity usage by 2000 megawatts each hour. How would the graph of the actual usage compare to the graph of the usage forecast?

d. Let $E(h)$ represent the predicted power usage function with E representing megawatts and h representing the number of hours since midnight June 11, 2007. Use function notation to symbolize the transformed function that approximates the actual amount of electricity used each hour if consumers reduce their collective electricity usage by 2000 megawatts each hour.

e. Over the two-day interval from June 12–June 13, 2007, the data appear to be periodic. Do you think this graphical model would be accurate in predicting usage for January 12–January 13, 2008? Explain.

Solution

a. Between midnight and 5 A.M., power usage decreases with a rate of change that is becoming less negative. Between 5 A.M. and around 4 to 5 P.M., hourly power usage increases rapidly from roughly 23,500 megawatts to about 39,500 megawatts. Between 5 P.M. and midnight, energy usage tends to decline, becoming more and more negative. The units of the hourly power usage rate of change are megawatts per hour.

Since most people sleep at night, it seems reasonable that energy usage at night will be less than that in the day. Furthermore, since air conditioners consume a lot of electricity during hot summer days, we expect the demand in the day to be greater than that during the cooler summer nights. The highest predicted usage on both dates occurred between noon and 8 P.M. It seems reasonable that the daily pattern of power usage should be about the same for most weekdays. The data appears to be periodic, at least in the short term. However, variable temperatures may increase or decrease the demand so using the model to extrapolate values too far away from June 12, 2007, will likely yield unreliable results.

b. Although this data set is not perfectly periodic, the maximum power usage appears to be around 39,750 megawatts and the minimum power usage appears to be around 23,500 megawatts. The midline is given by

$$\begin{aligned} y &= \frac{39{,}750 + 23{,}500}{2} \\ &= 31{,}625 \end{aligned}$$

The usage value midway between the maximum and minimum usage values is 31,625 megawatts.

The amplitude is given by

$$\begin{aligned} \text{amplitude} &= 39{,}750 - 31{,}625 \\ &= 8125 \end{aligned}$$

The maximum usage is 8125 megawatts above the midline and the minimum usage is 8125 megawatts below the midline. We graph both the midline and the amplitude on Figure 8.9.

c. If all of the usage forecast values are decreased by 2000 megawatts, the graph will be shifted downward by 2000, as shown in Figure 8.10.

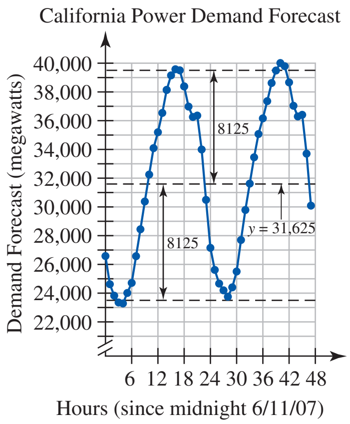


Figure 8.9

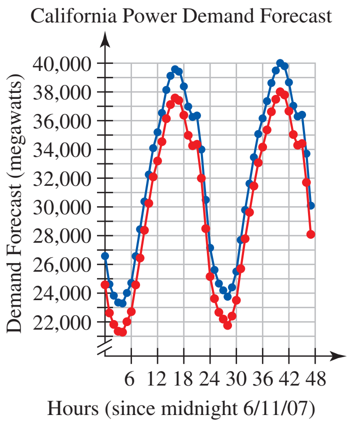


Figure 8.10

d. Since $E(h)$ gives the original energy forecast, the function $N(h) = E(h) - 2000$ gives the energy usage if the consumers collectively use 2000 fewer megawatts of electricity each hour.

e. Although the function may be effective in predicting energy usage for days on or around June 12, 2007, it is unlikely to be accurate in forecasting electricity usage for January 12–13, 2008. There is a substantial difference in daytime temperatures between June 12 and January 12; heaters (many powered by natural gas), not air conditioners, will be more widely used on January 12 and are more likely to be running at night. ■

EXAMPLE 3 ■ Determining Periodic Behavior from Tabular Data

Table 8.1 shows the number of new houses sold each month for two consecutive years in the United States. Is the data set periodic from year to year? Explain.

Table 8.1

Month (1 = January) *m*	New Houses Sold in 2005 (in thousands) *H*	New Houses Sold in 2006 (in thousands) *N*
1	1203	1185
2	1319	1084
3	1328	1126
4	1260	1097
5	1286	1087
6	1274	1073
7	1389	969
8	1255	1000
9	1244	1004
10	1336	952
11	1214	987
12	1239	1019

Source: www.census.gov

Solution To be perfectly periodic, the number of new homes sold each month in 2006 must be the same as the number sold in 2005 in the same month. Even if the data values do not match up perfectly, we may be able to model the data by a periodic function if the data demonstrates the same increasing and decreasing behavior each year.

We immediately conclude that this data set is not perfectly periodic since the data values differ from year to year. Over the first three months of the year, home sales increased in 2005. However, in 2006, home sales first decreased then increased. In 2006, home sales decreased monthly between March and July. In contrast, in 2005 home sales oscillated between increasing and decreasing over the same time period. These results convince us home sales in these years were not periodic by year.

We now look at how to extrapolate periodic data given in both graphical and tabular forms.

EXAMPLE 4 ■ Extrapolating Periodic Graphical Data

The graph in Figure 8.11 models the K–8 enrollment in American public schools from 1970 to 2000. Assuming that the pattern established here is periodic, sketch a graph of the function from 1970 to 2030. Also, provide an estimate for how many K–8 students there will be in the years 2010, 2020, and 2030.

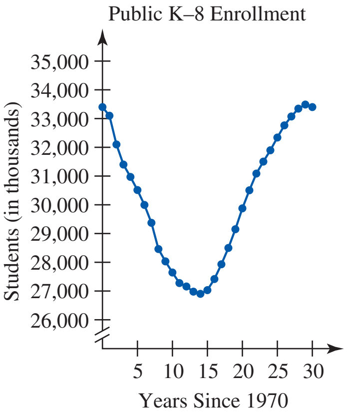


Figure 8.11

Source: Statistical Abstract of the United States, 2001, Table 232

Solution We have been told to assume the data is periodic so we may use the current graph to generate the data for the subsequent 30-year period. We do so in Figure 8.12, where the graph extends the pattern to the year 60.

We find the output for each year on the graph, as indicated on Figure 8.13, and predict the public school K–8 enrollment for the years 2010, 2020, and 2030 to be 27,900, 29,900, and 33,400, respectively.

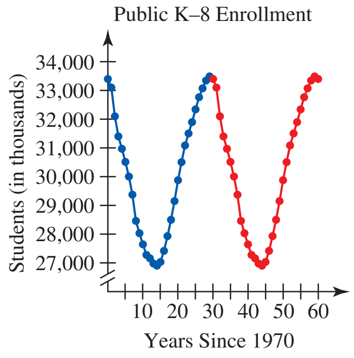


Figure 8.12

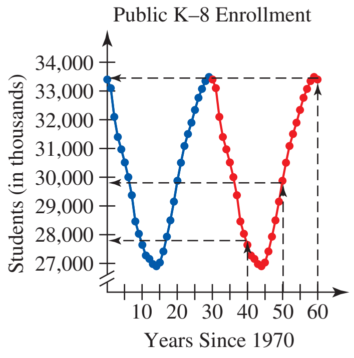


Figure 8.13

EXAMPLE 5 ■ Extrapolating Periodic Data from a Table

Table 8.2 shows the numbers of hours of daylight in Minneapolis, Minnesota, on the 21st day of the month for 2007 and part of 2008. Determine if the data appears to be periodic. If so, predict the number of hours of daylight for the remaining days of 2008.

Table 8.2

Day	2007 Hours of Daylight	2008 Hours of Daylight
Jan 21	9:22	9:21
Feb 21	10:44	10:44
Mar 21	12:12	12:14
Apr 21	13:47	13:49
May 21	15:03	15:05
Jun 21	15:37	15:37
Jul 21	15:06	
Aug 21	13:48	
Sep 21	12:14	
Oct 21	10:42	
Nov 21	9:21	
Dec 21	8:46	

Source: aa.usno.navy.mil

Solution Since the hours of daylight for 2008 are very close to the hours of daylight for 2007, it appears the hours of daylight are periodic with a period of 12 months. We estimate the remaining values for 2008 simply by copying the 2007 values, as shown in Table 8.3.

Table 8.3

Day	2007 Hours of Daylight	2008 Hours of Daylight
Jul 21	15:06	15:06
Aug 21	13:48	13:48
Sep 21	12:14	12:14
Oct 21	10:42	10:42
Nov 21	9:21	9:21
Dec 21	8:46	8:46

SUMMARY

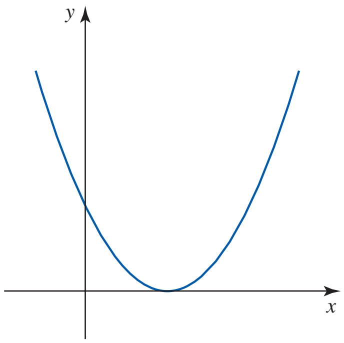
In this section you learned to recognize the periodicity of a function given a graph or table. You also learned about the midline and amplitude of periodic functions and learned to distinguish between the period and frequency.

8.1 EXERCISES

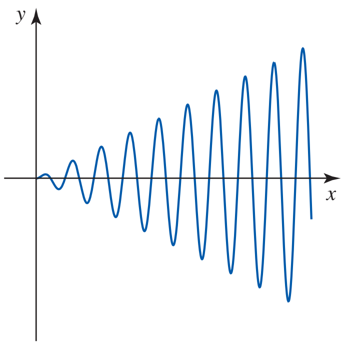
SKILLS AND CONCEPTS

In Exercises 1–8, determine if the given function appears to be periodic in nature.

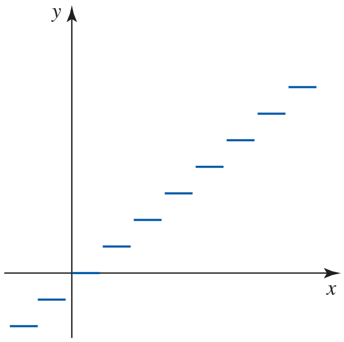
1.



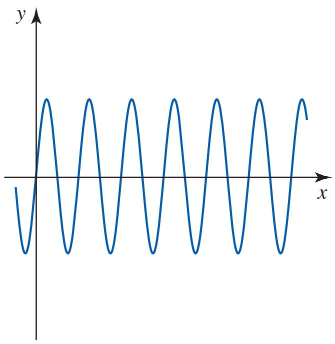
2.



3.



4.



5.

x	y
0	2
1	10
2	14
3	2
4	10
5	14
6	2
7	10
8	14
9	2
10	10

6.

x	y
−5	9
−4	15
−3	13
−2	12
−1	6
0	7
1	98
2	1
3	−9
4	6
5	4

7.

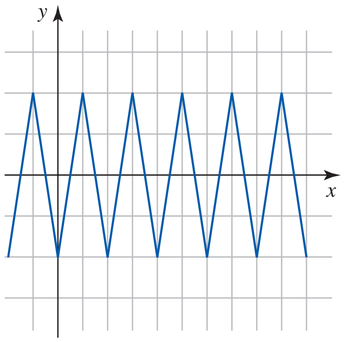
x	y
−1	7
0	8
1	9
2	8
3	7
4	8
5	9
6	8
7	7
8	8
9	9

8.

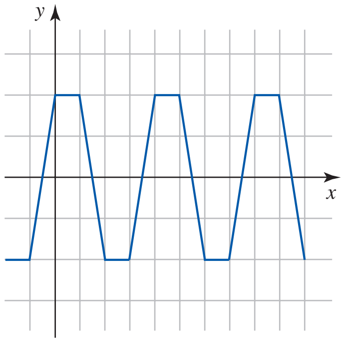
x	y
60	a
70	b
80	c
90	f
100	r
110	f
120	c
130	b
140	a
150	b
160	c

In Exercises 9–16, estimate the period of each function. (Note: Assume each horizontal line of the grid on the graphs is one unit.)

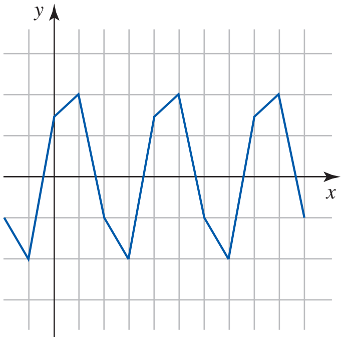
9.



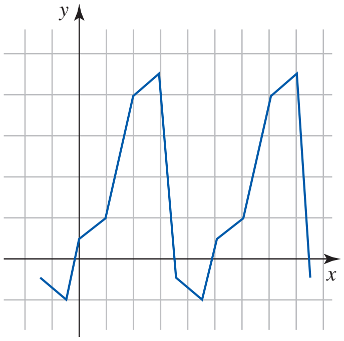
10.



11.



12.



13.

x	y
0	2
1	10
2	14
3	2
4	10
5	14
6	2
7	10

14.

x	y
12	3
24	6
36	9
48	8
60	3
72	6
84	9
96	8

15.

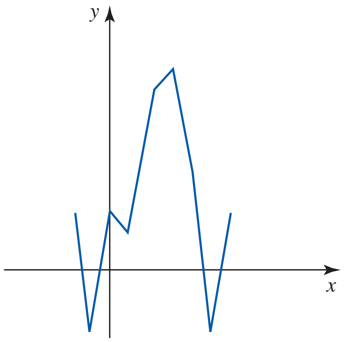
x	y
−38	2
−37	1
−36	2
−35	1
−34	2
−33	1
−32	2

16. Assume the x-values are equally spaced.

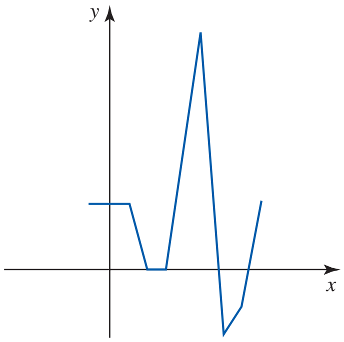
x	y
a	1
b	2
c	3
d	4
e	5
f	4
g	3
h	2
i	1
j	2
k	3

In Exercises 17–20, assume the functions are periodic and that one period is shown. Using the concept of periodicity, complete one more cycle before and after the values given in each graph and table.

17.



18.



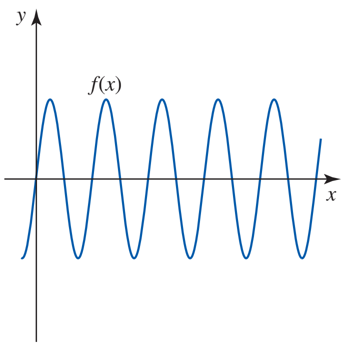
19.

x	y
0	
1	
2	
3	
4	10
5	14
6	2
7	8
8	
9	
10	
11	

20.

x	y
¶ − 4	
¶ − 3	
¶ − 2	
¶ −1	
¶	∂
¶ + 1	Σ
¶ + 2	©
¶ + 3	®
¶ + 4	∂
¶ + 5	
¶ + 6	
¶ + 7	
¶ + 8	

In Exercises 21–24, two periodic functions, f and g, are given. Transform the function as described.



x	$g(x)$
0	3
1	4
2	5
3	3
4	4
5	5
6	3

21. Sketch $f(x) + 3$.

22. Sketch $f(x - 1)$.

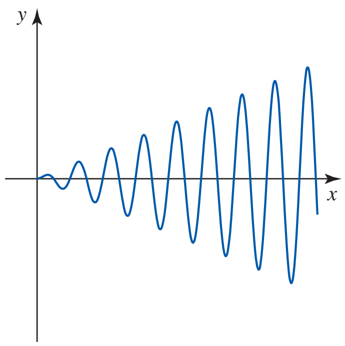
23. Add a third column in the table of $g(x)$ for values of $g(x + 2)$.

24. Add a third column in the table of $g(x)$ for values of $2g(x - 1)$.

In Exercises 25–28, indicate which of the graphs A–D may possibly model the statement. The independent and dependent variables are defined for each exercise. (Note: There may be more than one correct answer.) Justify your conclusion.

25. Ferris Wheel A man takes a ride on a Ferris wheel. The independent variable is elapsed time since the man got on the Ferris wheel and the dependent variable is the man's height above the ground.

A.



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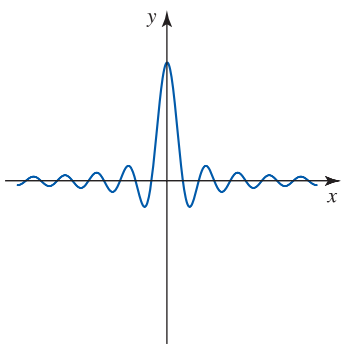
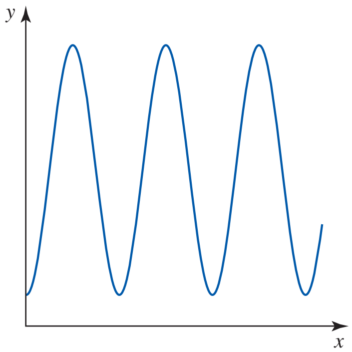
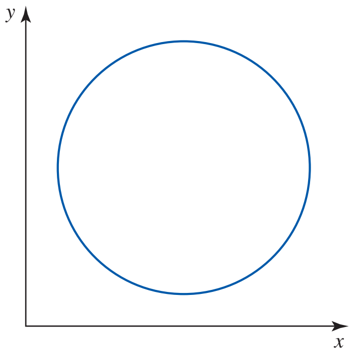
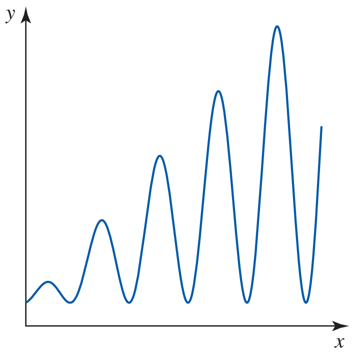
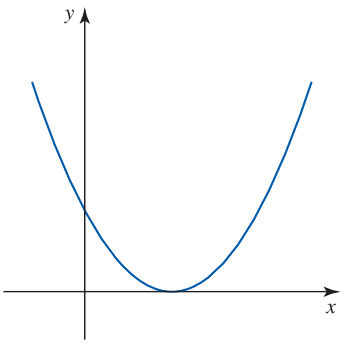


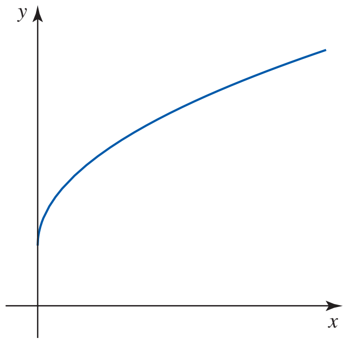
C. 

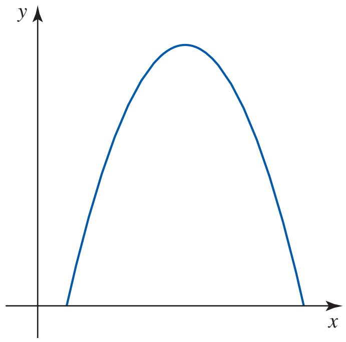
D. 

26. A Child Swinging A child swings back and forth on a swing. The independent variable is elapsed time from when he got on the swing and the dependent variable is the child's height above the ground.

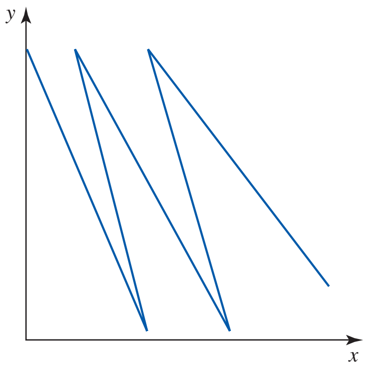
A. 

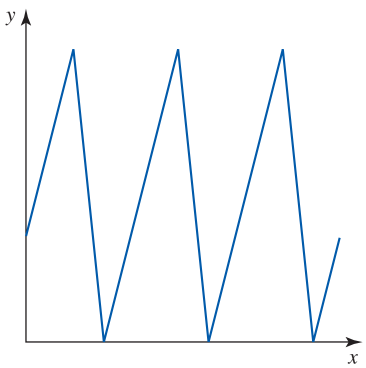
B. 

C. 

D. 

27. Family Vacation A family's motor home uses a large quantity of gasoline during an extended family trip. The independent variable is elapsed time since the family began their trip and the dependent variable is the amount of gasoline remaining in their tank. On a particular day, the family does not stop the vehicle other than to gas up the motor home.

A. 

B. 

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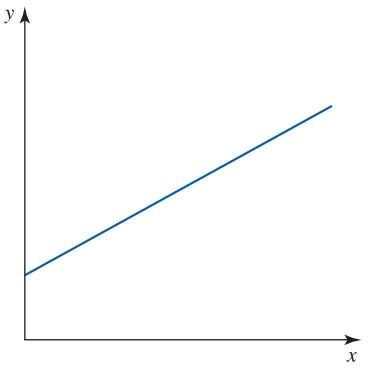

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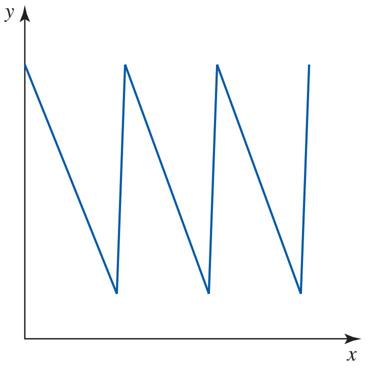
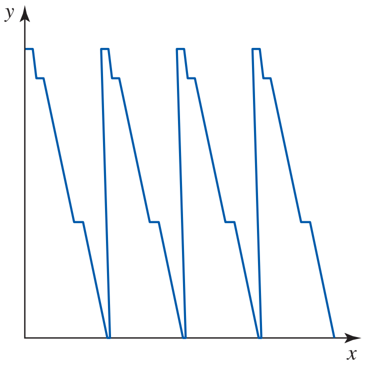
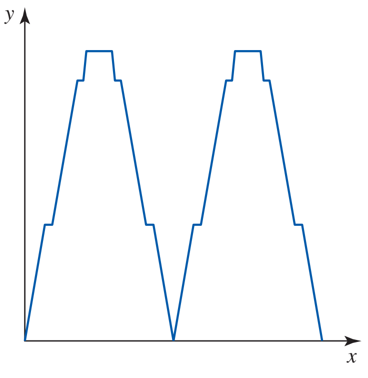


28. Subway Commute A subway makes repeated trips transporting passengers back and forth between two destination points within the city. The independent variable is elapsed time since the subway began transporting passengers and the dependent variable is the distance from its original starting point.

A.



B.



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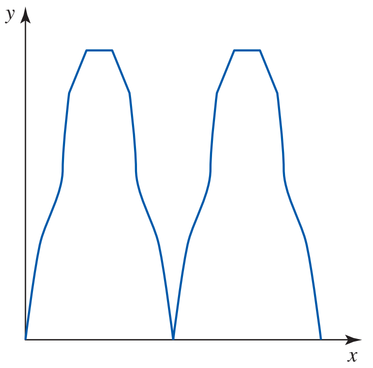

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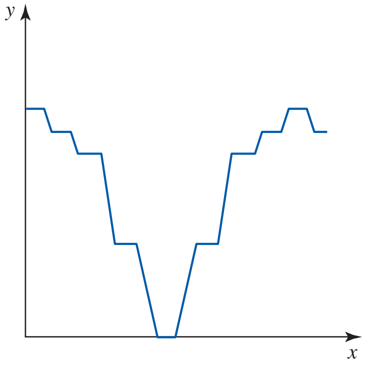


29. Daily Sales The table depicts the hypothetical number of customers that are served each hour at a 24-hour fast-food restaurant on a typical weekday.

Time of Day (0 = 12:00 A.M.) *h*	Number of Customers *c*
0	1
1	0
2	0
3	1
4	3
5	9
6	15
7	23
8	55
9	50
10	45
11	75
12	115
13	114
14	64
15	35
16	36
17	98
18	111
19	89
20	60
21	44
22	31
23	9

a. Estimate the average rate of change between 6 A.M. and 11 A.M. Interpret your answer in terms of the context.

b. Estimate the instantaneous rate of change at 12:00 P.M. (noon). Interpret your answer.

c. Estimate where the highest values are in the morning, mid-day, and evening, and explain what happens before and after each in terms of rate of change. Why would it make sense that these three highest values occur when they do?

d. Do you think this table of data would be periodic over a week of time? Explain.

■ SHOW YOU KNOW

30. In your own words, explain what it means for a function to be "periodic."

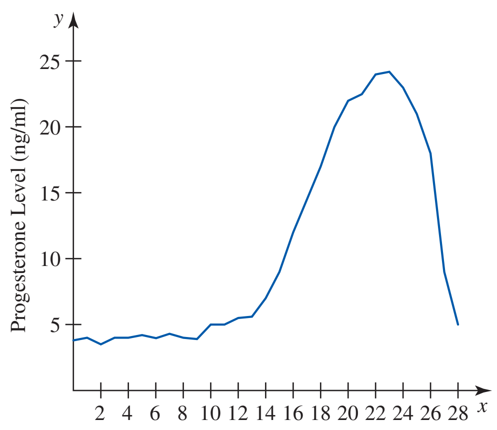
31. Explain the difference between the period and the frequency of a periodic function. Be sure to talk about their relative meanings—do not simply state formulas or algorithms for calculating their values.

32. What is the relationship between the maximum value, minimum value, midline, and amplitude of a periodic function?

33. In this section we have seen many examples of periodic functions, including hours of daylight during the year and traffic commute times during the week. Describe a real-world situation not found in this section that you think can be represented by a periodic function and explain why this situation is periodic.

■ MAKE IT REAL

34. Female Hormones The cycle of hormone production in the female human body regulates many important bodily functions. Progesterone is one such hormone that in the average woman fluctuates on approximately a 28-day cycle, as shown in the graph.



Source: **www.early-pregnancy-tests.com**

Describe what the graph tells about women's monthly cycle of progesterone production.

35. Alternating Current The electricity used in American households is in the form of alternating current (AC). Alternating current is typically 120 volts and 60 hertz in the United States. This means that the voltage cycles from −120 volts to +120 volts and back to −120 volts and 60 cycles occur each second. In answering the following questions, assume that at $t = 0$ there is no voltage at a given outlet.

a. Sketch $v(t)$, the voltage as a function of time, for the first 0.1 second.

b. State the period, the amplitude, and the midline of the graph you made in part (a). Explain the real-world meaning of each.

36. Pneumonia and Influenza Mortality According to the Centers for Disease Control and Prevention, the percentage of all deaths caused by pneumonia and influenza over the years 2003 to 2006 in 122 American cities is shown in the following graph together with a *seasonal baseline* and an *epidemic threshold*. The input values (which repeat) are measured in weeks since January 1 of the corresponding year.

Source: **www.cdc.gov/flu/weekly/**

a. Explain the purpose of the periodic functions representing the *seasonal baseline* and the *epidemic threshold*.

b. Estimate the period, the amplitude, and the midline for the *seasonal baseline* and the *epidemic threshold*. Explain the real-world meaning of each.

c. Determine from the graph the weeks and years in which there was an epidemic level of deaths from pneumonia and influenza.

37. Utilities Budget A utility bill for electricity usage may comprise a large part of a family's yearly budget. To predict the amount of money required for electricity, a family may look at a prior year's costs. The table shows the monthly utility costs for the year 2006 for a single-family residence.

Month (1 = January) m	Cost (dollars) c
1	124.19
2	111.16
3	116.30
4	133.71
5	270.07
6	320.11
7	250.23
8	391.19
9	345.91
10	256.44
11	124.87
12	126.98

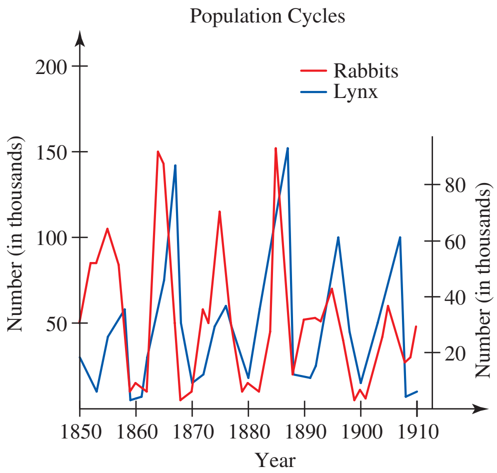
Source: Author's data

a. Assume the charge for electricity remained the same for 2007, and create a graph of $c(m)$ that will predict the cost each month in 2007.

b. Now suppose the cost of electricity increased by 4% for 2008 and remained at that level through 2009. Create a graph showing the function that will predict the 2008 and 2009 monthly electric bills.

c. If you chose to add an electrically heated swimming pool to your home at the start of 2007, how would this change affect the graph of this function?

38. Population Cycles Populations of some species go through periods of repeated growth (boom) and decline (bust). The graph shows the 10-year cyclical fluctuations in the population of the snowshoe rabbit and its primary predator, the lynx, in the Hudson Bay area of Canada from 1850 to 1910. What claims can one make based on this data, if any? Justify your response.



Source: users.rcn.com

■ STRETCH YOUR MIND

Exercises 39–40 are intended to challenge your understanding of periodic functions.

39. Pendulum Swings The period of a pendulum is defined as the time taken for two swings of the pendulum (left to right and back again). The formula for the period, T (in seconds), is the multivariable function, $T(l, g) = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum measured in meters from the pivot point to the bob's center of gravity, and g is the gravitational field strength (or acceleration due to gravity).

a. Sketch the graph of $T(l, 9.8)$ from $0 < l \le 5$.

b. Explain what information the graph of this function provides regarding the period of a pendulum and its length.

c. Evaluate $T(0.5, 9.8)$, $T(0.7, 9.8)$, and $T(1.0, 9.8)$.

40. Sunspots In the mid-1800s, Samuel Heinrich Schwabe (an amateur astronomer) declared that the number of sunspots varied cyclically, reaching a peak about every 10 years. He based his claim on 17 years of daily observations. Advancing Schwabe's work, Swiss astronomer Rudolf Wolf compiled the history of sunspot variations as far back as 1745 and determined that the period for sunspot activity would be more accurately estimated to be 11 years rather than 10. A minimum number of sunspots occurred in 1760 and a maximum occurred about 5.5 years later. (*Source:* www-spof.gsfc.nasa.gov)

Use this information and the 11-year cycle of sunspots to sketch a possible graph of the sunspot activity for the years 2000 to 2050. Then estimate when the minimum and maximum number of sunspots will occur over this 50-year timespan.

SECTION 8.2 Angle Measure

LEARNING OBJECTIVES

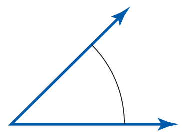
- Explain the useful definition of angle measure
- Measure angles in degrees and radians
- Convert angle measure between units of degree and radian

GETTING STARTED

Many of us have an intuitive understanding of *angle* measure. Angles play a big role in architecture, construction, surveying, geometry, backyard pool design, highway construction, and even some sporting activities. For example, to be successful in the game of billiards, a player must bank the balls off of the cushions at specific angles. In construction, surveyors measure distances and angles to map out locations for the construction project. Surveyors use a tool called a theodolite, which is a telescope mounted on a tripod that allows the surveyor to measure angles with great precision: $\frac{1}{60}$ of 1 degree.

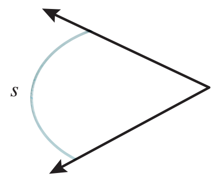
In this section we define the term *angle* and look at the different ways that angles are measured. We also provide a way to think about angle measure that will be useful in our later study of trigonometric functions.

■ Angles



From a geometric point of view, an **angle** is the figure formed by two rays (or segments) sharing a common endpoint. We could think of the angle as the amount of opening between the two rays. While this is a fairly common interpretation of an angle measure, we will formalize this notion to create a method of measuring angles that will be very useful in subsequent sections.

To measure an angle, we imagine drawing an arc, centered at the vertex of the angle, from one ray to the other. The length of this arc, s, is a portion of the circumference of an entire circle of radius, r. That is, the arc length s is directly proportional to the circumference C of the circle. So for some constant k, $s = kC$. Recall the circumference of a circle is $C = 2\pi r$. The constant of proportionality k tells what proportion of the circumference is included in the arc length.



Since $s = kC$ and $C = 2\pi r$, we have

$$\begin{aligned} s &= kC \\ &= k(2\pi r) \\ &= (2\pi k)r \end{aligned}$$

This result shows us the arc length, s, is directly proportional to the radius, r, of the circle with constant of proportionality $2\pi k$.

For example, if the arc length is equal to half of the circumference of the corresponding circle, $k = \dfrac{1}{2}$ and

$$\begin{aligned} s &= (2\pi k)r \\ &= (2\pi \tfrac{1}{2})r \\ &= \pi r \end{aligned}$$

When the arc length is half of the circumference of the corresponding circle, the arc length is equal to π times the radius.

If the arc length is equal to one-sixth of the circumference, $k = \frac{1}{6}$ and

$$\begin{aligned} s &= (2\pi k)r \\ &= (2\pi\tfrac{1}{6})r \\ &= \frac{\pi}{3}r \end{aligned}$$

When the arc length is one-sixth of the circumference of the corresponding circle, the arc length is equal to $\frac{\pi}{3}$ times the radius.

For simplicity, we represent the constant of proportionality in the equation $s = (2\pi k)r$ with K. That is, $K = 2\pi k$. The simplified equation is $s = Kr$, where $K = 2\pi k$.

JUST IN TIME ■ PROPORTIONALITY

A quantity y is said to be **directly proportional** to a quantity x if $y = kx$ for some constant value k, called the **constant of proportionality**. Consider the following examples.

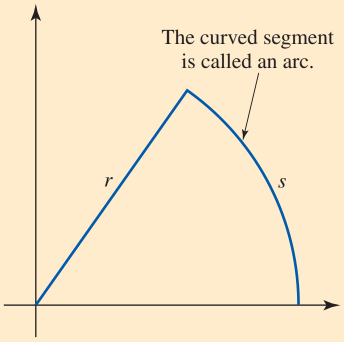
1. The circumference of a circle is directly proportional to its radius with a constant of proportionality 2π. That is, $C = 2\pi r$.
2. The total cost of filling a gas tank is directly proportional to the number of gallons put in the tank. The constant of proportionality is the gasoline price per gallon. At a price of \$3.489 per gallon, the cost is $C = 3.489x$.

Relationship between Arc Length and Radius

The length, s, of an arc is directly proportional to the radius of the arc, r, with constant of proportionality K. That is,

$$s = Kr$$

If k is the proportion of the circumference of a circle of radius r that coincides with the arc, then $K = 2\pi k$.



EXAMPLE 1 ■ Relating Radius and Arc Length

An arc's length is equal to $\frac{2}{3}$ of the circumference of the corresponding circle. Write the equation for the arc length as a function of the radius.

Solution Since $K = 2\pi k$,

$$\begin{aligned} K &= 2\pi\left(\frac{2}{3}\right) \qquad \text{since } k = \frac{2}{3} \\ &= \frac{4\pi}{3} \end{aligned}$$

So $s = \frac{4\pi}{3}r$ is the equation for arc length as a function of radius for an arc that is equal to $\frac{2}{3}$ of the circumference of the corresponding circle.

EXAMPLE 2 ■ **Determining When Arc Length Equals Radius**

Under what conditions is the arc length equal to the radius?

Solution When the arc length equals the radius, the equation $s = Kr$ becomes $r = Kr$. Solving this equation for K yields $K = 1$. We have

$$\begin{aligned} K &= 2\pi k \\ 1 &= 2\pi k \\ \frac{1}{2\pi} &= k \\ 0.159 &\approx k \end{aligned}$$

So when the arc length is about 15.9% of the circumference, the arc length and the corresponding radius are equal.

■ Measuring Angles Using Radians

When measuring quantities, we may use any number of different measurement systems. For example, although the numbers 212, 100, and 373.15 are different from each other, they each represent the boiling temperature for water when the appropriate units are used: 212 degrees Fahrenheit = 100 degrees Celsius = 373.15 kelvin. Likewise, we may assign any numeric value we want to a particular angle as long as we have clearly defined the measurement system. For angles, two different measurement systems are commonly used: degrees and radians. Due to its close connection with arc length, we first look at radian measure.

We begin by considering the **unit circle**, a circle of radius 1 centered at the origin. Imagine the radius of the circle is a piece of string. Take the piece of string and lay it on the outer edge of the circle, starting at the point (1, 0). Draw a line from the center of the circle to the uppermost end of the string as indicated in Figure 8.14. The angle formed by this line and the horizontal axis is defined to be 1 **radian**. That is, the measure of the angle θ that corresponds with an arc length of 1 radius is 1 radian.

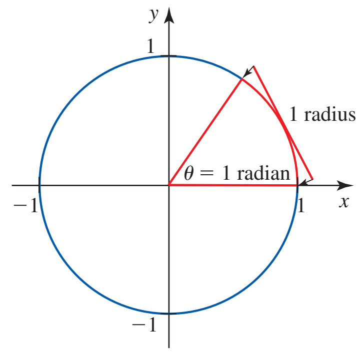


Figure 8.14

Radian Measure

The measure of an angle θ that corresponds with an arc length of 1 radius in the counterclockwise direction is 1 **radian**.

EXAMPLE 3 ■ **Determining the Number of Radians in One Revolution of the Unit Circle**

Using the idea that 1 radian is the measure of the angle that corresponds with an arc length of 1 radius, determine how many radians are in one complete revolution of a circle.

Solution We continue to imagine that the radius of the unit circle is a piece of string and that we want to use the piece of string to measure the circumference of the whole circle. We imagine laying the string on the outer edge of the circle, with one end beginning at the point (1, 0), and measuring the number of 1-radius lengths required to span the entire circle. In Figure 8.15, each change in color represents the length of 1 radius.

It takes six complete radius lengths to span the circumference of the whole circle plus a little bit more. In fact, the "little bit more" is approximately 0.28 of one radius. It takes approximately 6.28 radius lengths to span the circumference of the circle. The exact number of lengths is 2π ($2\pi \approx 6.2832$).

Since each radian corresponds with an arc length of 1 radius on the unit circle, it takes 2π radians to generate an arc length equal to the circumference of the circle. This is where we get the formula for the circumference of a circle, $C = 2\pi r$.

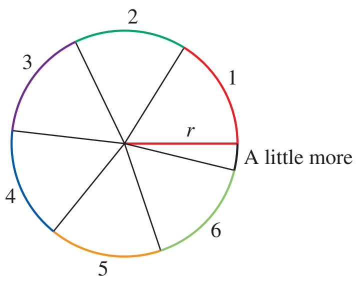


Figure 8.15

Linking the idea of angle measure with the idea of arc length has significant consequences. One radian yields an arc length of 1 radius, 2 radians yields an arc length of 2 radii, and so on. In general, θ radians yields an arc length of θ radii. Symbolically, we write $s = \theta r$, where s is the arc length and r is the length of the radius.

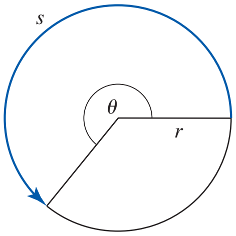


Figure 8.16

Arc Length as a Function of Angle Measure in Radians

The length, s, of the arc corresponding with an angle measure of θ radians is

$$s = \theta r$$

where r is the length of the radius. See Figure 8.16. For the unit circle, $s = \theta$.

We showed earlier that the length, s, of an arc is directly proportional to the radius of the arc, r, with constant of proportionality K. That is, $s = Kr$. When measuring angles in radians, the constant of proportionality equals the angle measure. That is, $K = \theta$.

EXAMPLE 4 ■ Determining the Radian Measure of an Angle

What is the radian measure of the angle that corresponds with $\frac{1}{4}$ of the circumference of a circle of radius 3?

Solution The circumference of a circle corresponds with an angle of 2π. Therefore, $\frac{1}{4}$ of the circumference of a circle corresponds with $\frac{1}{4}$ of 2π.

$$\begin{aligned}\theta &= \frac{1}{4(2\pi)}\\ &= \frac{2\pi}{4}\\ &= \frac{\pi}{2}\end{aligned}$$

Observe that the radius of the circle was not used in the calculation of the angle measure. For any circle (regardless of the length of the radius), $\frac{1}{4}$ of the circumference corresponds with an angle measure of $\frac{\pi}{2}$. The measure of the angle is $\frac{\pi}{2}$ radians.

EXAMPLE 5 ■ Determining the Proportion of a Circumference That Corresponds with an Angle

What proportion of a circle's circumference corresponds with an angle of $\frac{4\pi}{3}$ radians?

Solution The arc length is

$$s = \theta r$$
$$s = \frac{4\pi}{3} r$$

We calculate the proportion by dividing the arc length by the circumference.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{\frac{4\pi}{3} r}{2\pi r}$$
$$= \left(\frac{4\pi r}{3}\right)\frac{1}{2\pi r}$$
$$= \left(\frac{4}{3}\right)\left(\frac{1}{2}\right)$$
$$= \frac{4}{6}$$
$$= \frac{2}{3}$$

An angle of $\frac{4\pi}{3}$ radians corresponds with $\frac{2}{3}$ of the circumference of the circle.

■ Sketching Angles

An angle is represented graphically by an *initial side* and a *terminal side*, as shown in Figure 8.17. The positive horizontal axis forms the *initial side* of any angle in standard position.

To sketch a *positive* angle, we begin on the positive horizontal axis and sweep out an arc in the *counterclockwise* direction. To sketch a *negative* angle, we begin on the positive horizontal axis and sweep out an arc in the *clockwise* direction. The magnitude of the angle determines the position of the terminal side, as is demonstrated in Example 6.

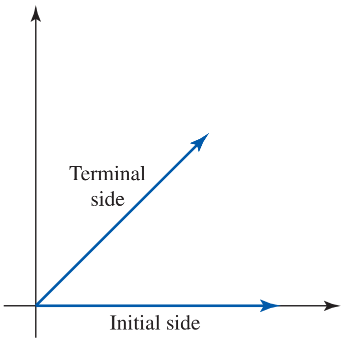


Figure 8.17

EXAMPLE 6 ■ Sketching Angles Measured in Radians

Sketch each of the following angles measured in radians.

a. $\theta = \pi$

b. $\theta = -\pi$

c. $\theta = \frac{\pi}{4}$

d. $\theta = -\frac{2\pi}{3}$

e. $\theta = 4$

Solution

a. We know an angle of 2π represents an arc length that corresponds with the circumference of an entire circle. Therefore, to sketch the angle $\theta = \pi$, we represent an arc length that traverses one-half the circumference of a circle in the counterclockwise direction, as shown in Figure 8.18.

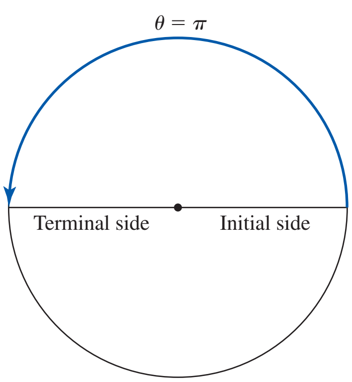


Figure 8.18

b. The magnitude of the angle is π; however, since the angle is negative, the arc length traverses the circumference in the clockwise direction to represent $\theta = -\pi$. See Figure 8.19.

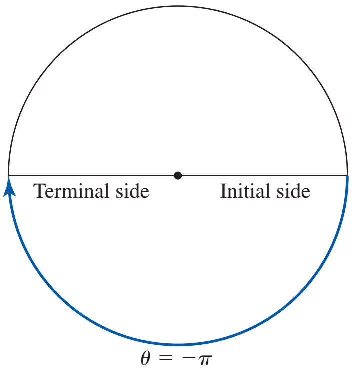


Figure 8.19

c. The angle $\theta = \dfrac{\pi}{4}$ is equivalent to $\theta = \dfrac{1}{4}\pi$.

This angle is one-quarter of the angle $\theta = \pi$,

as shown in Figure 8.20.

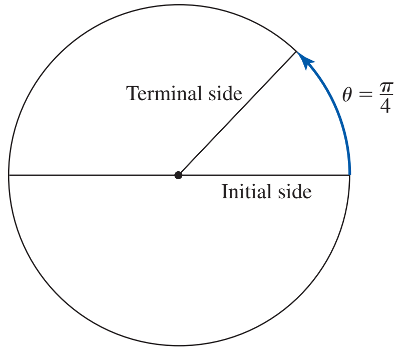


Figure 8.20

d. The angle $\theta = -\dfrac{2\pi}{3}$ is equivalent to $\theta = -\dfrac{2}{3}\pi$.

We then see this angle is two-thirds of the angle

$\theta = \pi$ and, since it is negative, the arc length traverses the circle in the clockwise direction. See Figure 8.21.

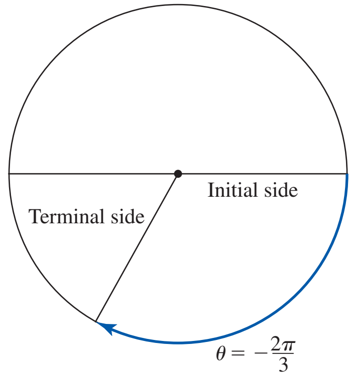


Figure 8.21

e. Recall for the unit circle, the numeric value of the angle is equivalent to the arc length. In this case, the angle and arc length are 4. We first determine the proportion of the circumference that the arc length represents.

$$\frac{\text{arc length}}{\text{circumference}} = \frac{4}{2\pi}$$
$$\approx 0.637$$

The terminal side of the angle will occur close to $\dfrac{2}{3}$ (66.7%) of the way around the circle in the counterclockwise direction, as shown in Figure 8.22.

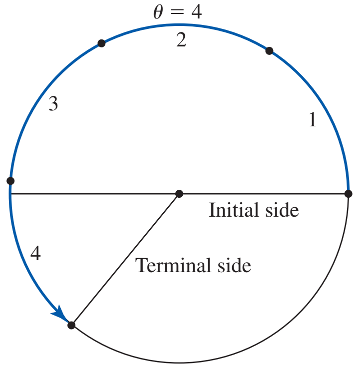


Figure 8.22

Angles That Measure Greater Than 2π Radians

We can measure and sketch angles with $\theta > 2\pi$. Since each measure of 2π corresponds with one complete revolution around the circle, an angle greater than 2π indicates that the angle represents more than one revolution around the circle, as indicated in Figure 8.23. For example, we may think of $\theta = 2.5\pi$ as

$$\begin{aligned}\theta &= 2\pi + 0.5\pi \\ &= 2\pi + \tfrac{1}{2}\pi \\ &= 2\pi + \tfrac{1}{4}(2\pi) \\ &= 1 \text{ revolution} + \tfrac{1}{4} \text{ revolution}\end{aligned}$$

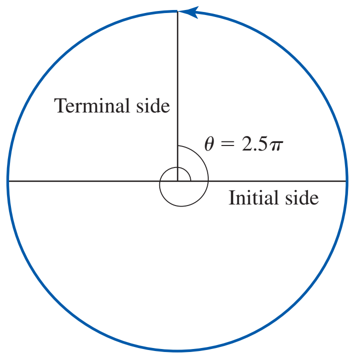


Figure 8.23

The terminal side of $\theta = 2.5\pi$ is in the same position as the angle whose arc length is one-quarter of the circumference of the circle, in this case $\frac{\pi}{2}$.

In Section 8.3, we will discuss *reference angles*, which will further help us identify angles such as $\theta = 2.5\pi$ and $\theta = \frac{\pi}{2}$ that share a terminal side.

Measuring Angles Using Degrees

Another common way to measure angles is in *degrees*. Because of historical convention established by the ancient Babylonians, an angle whose arc length is $\frac{1}{360}$ of the circumference of a circle is known as 1 **degree** (notated 1°). The number 360 was a good choice because 2, 3, 4, 5, 6, 8, 9, 10 and so on are factors of 360. Consequently, determining the angle measure for $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \frac{1}{8}, \frac{1}{9}, \frac{1}{10}$ of a circle's circumference results in whole-number angles. For example, the measure of the angle that corresponds with $\frac{1}{10}$ of the circumference of the circle is $\frac{1}{10}(360°) = 36°$.

PEER INTO THE PAST

WHY ARE THERE 360° IN A CIRCLE?

The reason that we divide a circle into 360° is based on historical convention. The ancient Babylonians (approximately 3000 B.C.), who lived in Mesopotamia (southern Iraq), used a base-60 number system. Therefore, it made sense to them to use 360° as opposed to 100°, which might make more sense to us today. Back then, people believed that the Sun circled Earth and that it took about 360 days for the Sun to complete one circuit. It is believed that the ancient Babylonians invented the 360° sundial as a way to keep track of time.

We should note that the use of 360 is very convenient since 360 is divisible by 2, 3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, and so on. This makes it very easy to divide a circle up into smaller parts using nice, "neat" fractions.

Sources: www.wonderquest.com and mathforum.org

Degree Measure

The measure of an angle θ that corresponds with an arc length of $\frac{1}{360}$ of the circumference of a circle is 1 degree.

EXAMPLE 7 ■ Sketching Angles Measured in Degrees

Sketch each of the following angles measured in degrees.

a. $\theta = 180°$
b. $\theta = -180°$
c. $\theta = 120°$
d. $\theta = -225°$
e. $\theta = 72°$

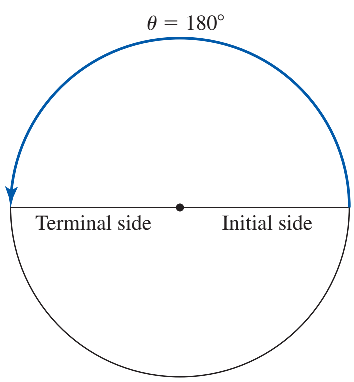


Figure 8.24

Solution

a. $\theta = 180°$ is the angle whose arc length is $\frac{180}{360}$ or $\frac{1}{2}$ of the full circumference of the circle, as shown in Figure 8.24.

b. $\theta = -180°$ is the same as $\theta = 180°$ except the arc length traverses the circumference of the circle in the clockwise direction, as indicated in Figure 8.25.

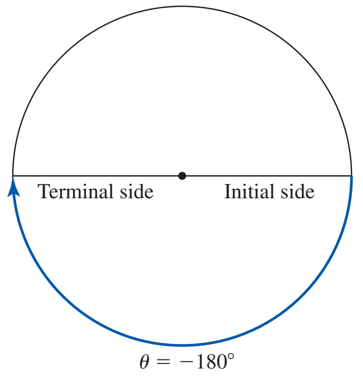


Figure 8.25

c. $\theta = 120°$ is the angle whose arc length is $\frac{120}{360}$ or $\frac{1}{3}$ of the full circumference of the circle. See Figure 8.26.

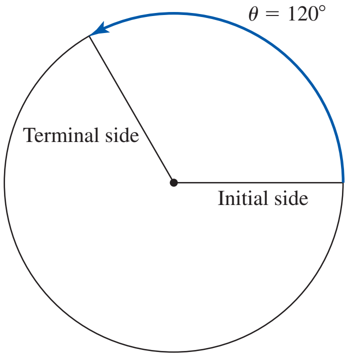


Figure 8.26

d. $\theta = -225°$ is the angle whose arc length is $\frac{225}{360}$ or $\frac{5}{8}$ of the full circumference of the circle traversed in the clockwise direction, as shown in Figure 8.27.

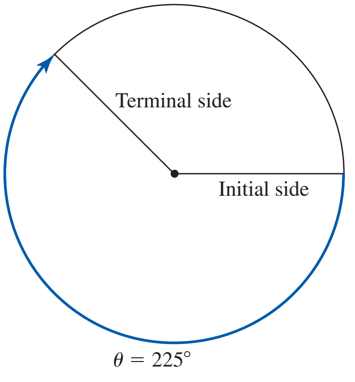


Figure 8.27

e. $\theta = 72°$ is the angle whose arc length is $\frac{72}{360}$ or $\frac{1}{5}$ of the full circumference of the circle. See Figure 8.28.

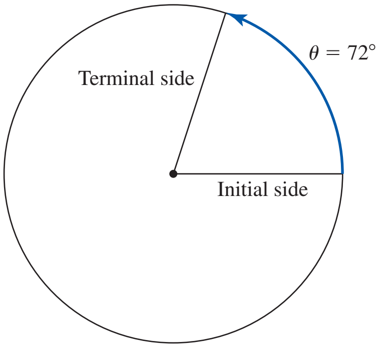


Figure 8.28

Earlier in this section, we established that an arc length, s, is related to its radius, r, by the equation $s = Kr$. Using our knowledge of the circumference of a circle, we showed that $K = 2\pi k$, where k represented the proportion of the circumference included in the arc length. We revisit these equations now to determine the relationship between the arc length and the degree measure of the corresponding angle. We have

$$\begin{aligned} s &= Kr \\ &= (2\pi k)r && \textbf{since } K = 2\pi k \end{aligned}$$

If θ is an angle in degrees, then $k = \frac{\theta}{360}$ gives us the proportion of the circumference included in the arc length. Substituting this result, we have

$$\begin{aligned} s &= (2\pi k)r \\ &= \left(2\pi\left(\frac{\theta}{360}\right)\right)r \\ &= \left(\frac{\pi}{180}\theta\right)r \end{aligned}$$

Arc Length as a Function of Angle Measure in Degrees

The length, s, of the arc corresponding with θ degrees is

$$s = \left(\frac{\pi}{180}\theta\right)r$$

where r is the length of the radius. For the unit circle, $s = \frac{\pi}{180}\theta$.

■ Converting from Degree Measure to Radian Measure

Since 360 degrees is equivalent to 2π radians, we can easily convert from degrees to radians. We first observe that

$$\begin{aligned} \frac{2\pi \text{ radians}}{360 \text{ degrees}} &= \frac{\pi \text{ radians}}{180 \text{ degrees}} \\ &= 1 && \textbf{since the quantities in the numerator and denominator are equal} \end{aligned}$$

We will use this fact as we convert from degrees to radians. For example, to convert 30° to radians we have

$$\begin{aligned} 30 \text{ degrees} &= 30 \text{ degrees} \cdot \frac{\pi \text{ radians}}{180 \text{ degrees}} && \textbf{since } \frac{\pi \textbf{ radians}}{\textbf{180 degrees}} = 1 \\ &= \frac{30 \text{ degrees}}{180 \text{ degrees}} \cdot \pi \text{ radians} \\ &= \frac{1}{6}\pi \text{ radians} \\ &= \frac{\pi}{6} \text{ radians} \end{aligned}$$

So $30° = \frac{\pi}{6}$ radians.

HOW TO: ■ CONVERT AN ANGLE MEASURE FROM DEGREES TO RADIANS

To convert degree measure to radian measure, multiply θ degrees by $\frac{\pi \text{ radians}}{180 \text{ degrees}}$.

EXAMPLE 8 ■ Converting Degree Measure to Radian Measure

Convert each angle from degree to radian measure. Explain why your result makes sense.

a. $90°$ **b.** $150°$ **c.** $-45°$ **d.** $1°$

Solution

a. To convert from degree to radian measure, we multiply the degree measure by $\dfrac{\pi \text{ radians}}{180°}$.

$$\begin{aligned} 90° \cdot \frac{\pi \text{ radians}}{180°} &= \frac{90° \cdot \pi}{180°} \text{ radians} \\ &= \frac{\pi}{2} \text{ radians} \end{aligned}$$

This makes sense because $90°$ is $\frac{1}{4}$ of $360°$ and $\dfrac{\pi}{2}$ radians is $\frac{1}{4}$ of 2π radians.

b. To convert from degree to radian measure, we multiply the degree measure by $\dfrac{\pi \text{ radians}}{180°}$.

$$\begin{aligned} 150° \cdot \frac{\pi \text{ radians}}{180°} &= \frac{150° \cdot \pi}{180°} \text{ radians} \\ &= \frac{5\pi}{6} \text{ radians} \end{aligned}$$

We see $150° = \dfrac{5\pi}{6}$ radians.

To check our work, we verify that each angle represents the same proportion of the circumference included in the associated arc length.

$$\frac{150°}{360°} = \frac{5}{12} \qquad \begin{aligned} \frac{\frac{5\pi}{6} \text{ radians}}{2\pi \text{ radians}} &= \frac{5\pi}{6} \cdot \frac{1}{2\pi} \\ &= \frac{5}{12} \end{aligned}$$

Both angles correspond with arc lengths that are $\dfrac{5}{12}$ of the circumference of the circle.

c.

$$\begin{aligned} -45° \cdot \frac{\pi \text{ radians}}{180°} &= \frac{-45° \cdot \pi}{180°} \text{ radians} \\ &= -\frac{\pi}{4} \text{ radian} \end{aligned}$$

This makes sense because both angles correspond with arc lengths that are $\dfrac{1}{8}$ of the circumference of the circle traversed in the clockwise direction $\left(\dfrac{45}{360} = \dfrac{\pi/4}{2\pi} = \dfrac{1}{8}\right)$.

d.

$$\begin{aligned} 1° \cdot \frac{\pi \text{ radians}}{180°} &= \frac{1° \cdot \pi}{180°} \text{ radians} \\ &= \frac{\pi}{180} \text{ radian} \end{aligned}$$

This makes sense because both angles correspond with arc lengths that are $\dfrac{1}{360}$ of the circumference of the circle $\left(\dfrac{1}{360} = \dfrac{\pi/180}{2\pi}\right)$.

Converting from Radian Measure to Degree Measure

Since 360 degrees is equivalent to 2π radians, we can easily convert from radians to degrees. We first observe that

$$\begin{aligned}\frac{360 \text{ degrees}}{2\pi \text{ radians}} &= \frac{180 \text{ degrees}}{\pi \text{ radians}} \\ &= 1 \qquad \text{since the quantities in the numerator and denominator are equal}\end{aligned}$$

We will use this fact as we convert from radians to degrees. For example, to convert $\frac{7\pi}{4}$ radians to degrees we have

$$\begin{aligned}\frac{7\pi}{4} \text{ radians} &= \frac{7\pi}{4} \text{ radians} \cdot \frac{180 \text{ degrees}}{\pi \text{ radians}} \qquad \text{since } \frac{180 \text{ degrees}}{\pi \text{ radians}} = 1 \\ &= \frac{1260\pi \text{ radians}}{4\pi \text{ radians}} \text{ degrees} \\ &= 315 \text{ degrees}\end{aligned}$$

So $\frac{7\pi}{4}$ radians is equal to 315°.

HOW TO: ■ CONVERT AN ANGLE MEASURE FROM RADIANS TO DEGREES

To convert radian measure to degree measure, multiply θ radians by $\frac{180 \text{ degrees}}{\pi \text{ radians}}$.

EXAMPLE 9 ■ Converting Radian Measure to Degree Measure

Convert each angle from radian to degree measure. Then explain why the result makes sense.

a. $\frac{\pi}{6}$ radian

b. $\frac{3\pi}{2}$ radians

c. 3.5 radians

d. $-\frac{3\pi}{4}$ radians

e. 1 radian

Solution

a. To convert from radian to degree measure, we multiply the radian measure by $\frac{180°}{\pi}$.

$$\begin{aligned}\frac{\pi}{6} \cdot \frac{180°}{\pi} &= \frac{180°}{6} \\ &= 30°\end{aligned}$$

This result makes sense because $\frac{\pi}{6}$ represents $\frac{1}{12}$ of the full circumference of a circle and 30° is $\frac{1}{12}$ of 360°.

b.
$$\begin{aligned}\frac{3\pi}{2}\cdot\frac{180°}{\pi} &= \frac{540°}{2}\\ &= 270°\end{aligned}$$

This result makes sense because $\frac{3\pi}{2}$ represents $\frac{3}{4}$ of the full circumference of a circle and 270° is $\frac{3}{4}$ of 360°.

c.
$$\begin{aligned}3.5\cdot\frac{180°}{\pi} &= \frac{630°}{\pi}\\ &\approx 200.54°\end{aligned}$$

This result makes sense because 3.5 radians is between π radians and $\frac{3\pi}{2}$ and 200.54° is between 180° and 270°.

d.
$$\begin{aligned}-\frac{3\pi}{4}\cdot\frac{180°}{\pi} &= -\frac{540°}{4}\\ &= -135°\end{aligned}$$

This result makes sense because $-\frac{3\pi}{4}$ represents $\frac{3}{8}$ of the full circumference of a circle traversed in a clockwise direction and 135° is $\frac{3}{8}$ of 360°.

e.
$$\begin{aligned}1\cdot\frac{180°}{\pi} &= \frac{180°}{4}\\ &\approx 57.3°\end{aligned}$$

This result makes sense because 1 radian is the arc length spanned when 1 radius length is wrapped around the circumference of the circle. We have seen in this section that this always corresponds with a length along the circumference that is in the first quadrant and less than 90°.

Ideally, all angles would be appropriately labeled with units (e.g., degrees, radians) all of the time; however, mathematicians frequently drop the unit "radian" off of angles in radian measure. Therefore, we will assume any unitless angle measure is measured in radians. For example, $\theta = 4$ is interpreted as $\theta = 4$ radians.

SUMMARY

In this section you learned that the length of an arc is directly proportional to the radius of the corresponding circle. You discovered that when the angle is measured in radians, the angle measure is the constant of proportionality. You also learned that angles are typically measured using radians or degrees. Additionally, you discovered how to convert from one angle measure to another.

8.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–8, describe what fraction of the circumference of a full circle is spanned by an angle with the given measure.

1. $\theta = \frac{\pi}{4}$ radian

2. $\theta = 2.5$ radians

3. $\theta = \frac{\pi}{10}$ radian

4. $\theta = \frac{3\pi}{5}$ radians

5. $\theta = \frac{11\pi}{6}$ radians

6. $\theta = \frac{4\pi}{3}$ radians

7. $\theta = \frac{7\pi}{8}$ radians

8. $\theta = 8$ radians

In Exercises 9–16, draw a sketch of an angle with the given measure. Clearly indicate what you are looking at when you view the angle you have sketched.

9. $\theta = \frac{1}{2}$ radian

10. $\theta = 210°$

11. $\theta = 6\pi$ radians

12. $\theta = -330°$

13. $\theta = -\frac{4\pi}{5}$ radians

14. $\theta = 175°$

15. $\theta = 45$ radians

16. $\theta = 12\pi°$

In Exercises 17–21, convert each angle measure from degrees to radians.

17. $\theta = 15°$

18. $\theta = -60°$

19. $\theta = 335°$

20. $\theta = 120°$

21. $\theta = 410°$

In Exercises 22–26, convert each angle measure from radians to degrees.

22. $\theta = \frac{\pi}{7}$

23. $\theta = \frac{\pi}{18}$

24. $\theta = \frac{4\pi}{3}$

25. $\theta = 8.5$

26. $\theta = -10$

27. What is the length of the arc that corresponds with an angle whose measure is 150° in a circle of radius 2 feet?

28. What is the length of the arc that corresponds with an angle whose measure is $\frac{2\pi}{3}$ radian in a circle of radius 3 meters?

29. What is the length of the arc that corresponds with an angle whose measure is 315° in a circle of radius 4 yards?

30. What is the length of the arc that corresponds with an angle whose measure is 4 radians in a circle of radius 6 centimeters?

31. What is the length of the arc that corresponds with an angle whose measure is 10° in a circle of radius 3 meters?

32. Without using a calculator, rank the following angle measures in order from most negative to most positive.

$$\frac{3\pi}{4}, \frac{\pi}{2}, -\frac{\pi}{5}, \frac{5\pi}{6}, -\frac{\pi}{9}$$

33. What angle measure, in radians, corresponds to 2.25 rotations around a unit circle?

34. What angle measure, in radians, corresponds to -0.6 rotations around a unit circle?

35. Consider a circle of radius 2π. Find the arc length that corresponds with an angle of 60°.

36. Consider a circle of radius 2π. Find the arc length that corresponds with an angle of $-270°$.

37. Consider a circle of radius 2π. Find the arc length that corresponds with an angle of $A°$.

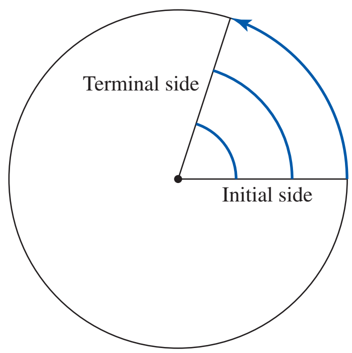
SHOW YOU KNOW

38. Pretend you are explaining the idea of radian measure of angles to a classmate who missed class. Write out an explanation or describe how you would help your classmate understand this idea.

39. Pretend you are explaining to another student the procedure for converting the measure of an angle from degrees to radians. She asks, "Why do you multiply by $\frac{\pi}{180}$?" How do you respond?

40. Explain why, without using computations, an angle measuring 180° is equivalent to one measuring π radians.

41. Explain why the measure of the angle shown is the same regardless of the location in which the angle is measured. Use the definition $\theta = k\frac{s}{r}$ in your explanation.



MAKE IT REAL

In Exercises 42–46, match each angle measured in radians with the picture where that angle measure is seen, approximately. Each angle measure matches with one picture.

42. $\frac{5\pi}{6}$

43. $\frac{\pi}{3}$

44. 0.25

45. π

46. 2π

A.

B.

C.

D.

E.

In Exercises 47–51, describe a real-world object, picture, or situation where you would see approximately each of the following angle measures. Draw a sketch of the angle to accompany your description.

47. $90°$

48. $\frac{\pi}{4}$ radian

49. 2π radians

50. $180°$

51. 1 radian

STRETCH YOUR MIND

Exercises 52–54 are intended to challenge your understanding of angle measure. Another unit of angle measure is the gradian. Just as there are 2π radians and $360°$ in a circle, there are 400 gradians in a circle.

52. What gradian measure represents the arc length spanning one-quarter of a circle? Half circle? One and one-half circle?

53. Convert 1 radian to gradians.

54. Recall the definition of angle uses $\theta = k\frac{s}{r}$. What is the constant of proportionality, k, when using gradians as the angle measure?

SECTION 8.3

LEARNING OBJECTIVES

- Describe the relationship between the cosine and sine functions and the length of the arc swept out by the angle
- Describe the relationship between the coordinates of a point on the unit circle and the cosine and sine functions
- Use reference angles to determine cosine and sine values

Unit Circle and Trigonometric Functions

GETTING STARTED

The Ferris wheel is named after George Ferris, a bridge engineer who designed the attraction for the 1893 World's Columbian Exposition in Chicago, Illinois. Ferris's wheel had a diameter of 250 feet and was an engineering marvel that astonished the world (*Source:* www.hydeparkhistory.org) It is now common to find Ferris wheels all over the world.

In this section we learn about the properties of circles as well as how to describe the coordinates of a point on the circle. These insights will help us understand ideas such as a person's height above the ground while riding a Ferris wheel.

■ Defining the Circle

A circle is defined as the set of points that are all located a fixed distance (the radius) away from another point called the center of the circle (Figure 8.29a).

The *unit circle* is the basic example of a circle from which we will develop our understanding of circle-related ideas (Figure 8.29b).

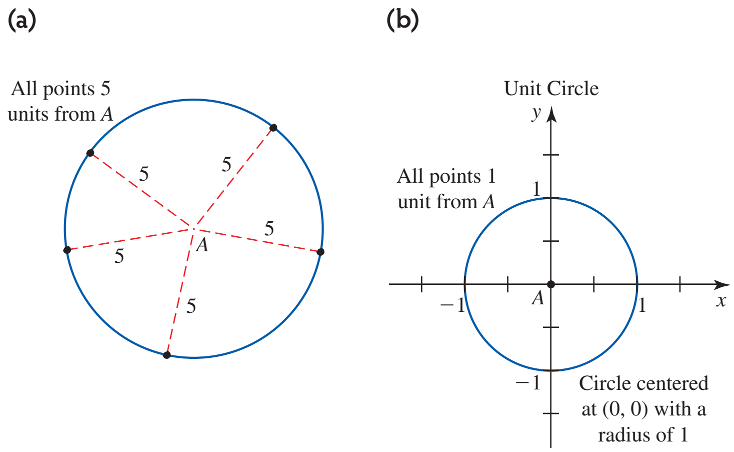


Figure 8.29

UNIT CIRCLE

The **unit circle** is a circle with its center at (0, 0) and a radius of 1 unit.

We can describe points on the unit circle in a number of ways. One is to describe the angle that corresponds with where a point is located on the arc of the circle. For example, Figure 8.30a shows that the line through the center of the circle and point B creates a 30° angle $\left(\frac{\pi}{6} \text{ radians}\right)$. Similarly, Figure 8.30b shows that the line through the center of the circle and point C creates a 255° angle $\left(\frac{17\pi}{12} \text{ radians}\right)$.

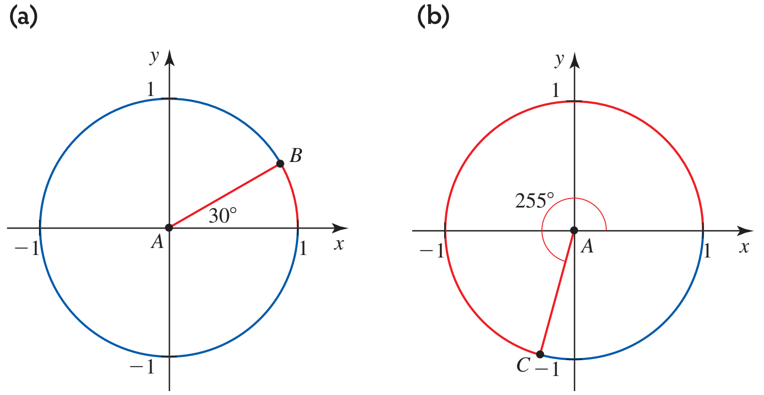


Figure 8.30

When talking about angle measures throughout this section, we assume the angles are in standard position. That is, the initial side of the angle coincides with the positive x-axis. We place the circle on a coordinate plane so that we may define points on the circle using familiar (x, y) coordinates.

EXAMPLE 1 ■ Estimating $x(\theta)$ and $y(\theta)$

As the arc length increases along a unit circle and the corresponding angle changes from 0° to 90° in increments of 10°, estimate the x- and y-values of the endpoint of the arc.

Solution We start by estimating the coordinates of the endpoint of the arc corresponding with 10°, comparing its position to the x- and y-axes. See Figure 8.31.

It appears this point is located at approximately (0.98, 0.17). Similarly, we can find the coordinates for points where the arcs correspond with 20° and 30°, as shown in Figures 8.32a and 8.32b.

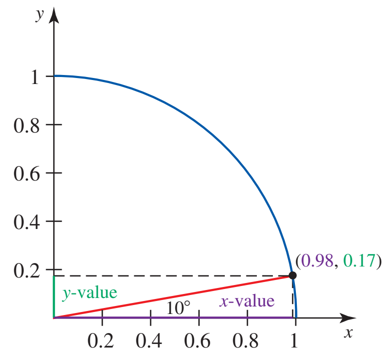


Figure 8.31

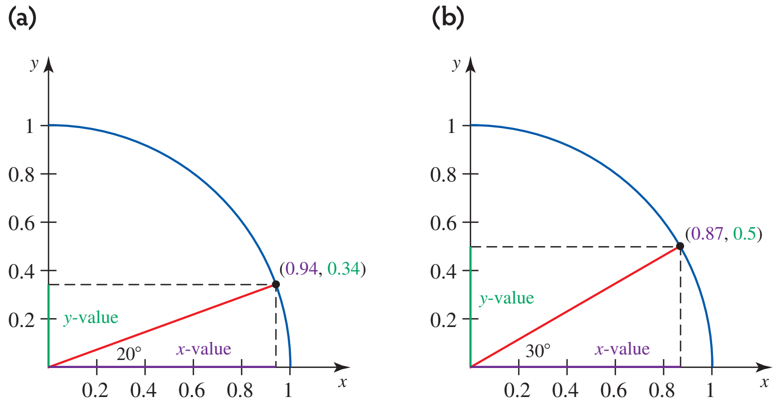


Figure 8.32

We can fill in a table of values for the remaining coordinates around the circle in the first quadrant in the same way (see Table 8.4 and Figure 8.33). We have also estimated the endpoint of the arc corresponding with 45° due to its unique characteristic of being equidistant from both axes.

Table 8.4

Angle (degrees) θ	Angle (radians) θ	Horizontal Position $x(\theta)$	Vertical Position $y(\theta)$	Coordinate Point (x, y)
0°	0	1	0	(1, 0)
10°	$\pi/18$	0.98	0.17	(0.98, 0.17)
20°	$\pi/9$	0.94	0.34	(0.94, 0.34)
30°	$\pi/6$	0.87	0.5	(0.87, 0.5)
40°	$2\pi/9$	0.77	0.64	(0.77, 0.64)
45°	$\pi/4$	0.71	0.71	(0.71, 0.71)
50°	$5\pi/18$	0.64	0.77	(0.64, 0.77)
60°	$\pi/3$	0.5	0.87	(0.5, 0.87)
70°	$7\pi/18$	0.34	0.94	(0.34, 0.94)
80°	$4\pi/9$	0.17	0.98	(0.17, 0.98)
90°	$\pi/2$	0	1	(0, 1)

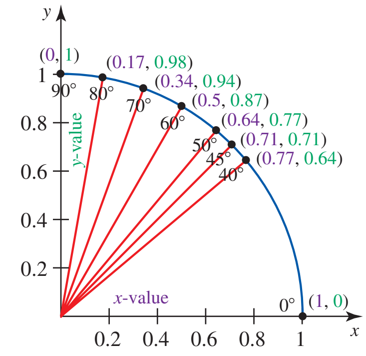


Figure 8.33

Unlike many of the functions we have worked with up to this point, functions describing these coordinates are not functions with x as the input and y as the output. Rather, both the horizontal position (x) and the vertical position (y) are functions of the angle (θ).

The ideas seen in Example 1 may be applied to arcs of any measure. Note that the x- and y-values may become negative depending on the quadrant in which the endpoint of the arc is located. See Figures 8.34a and 8.34b.

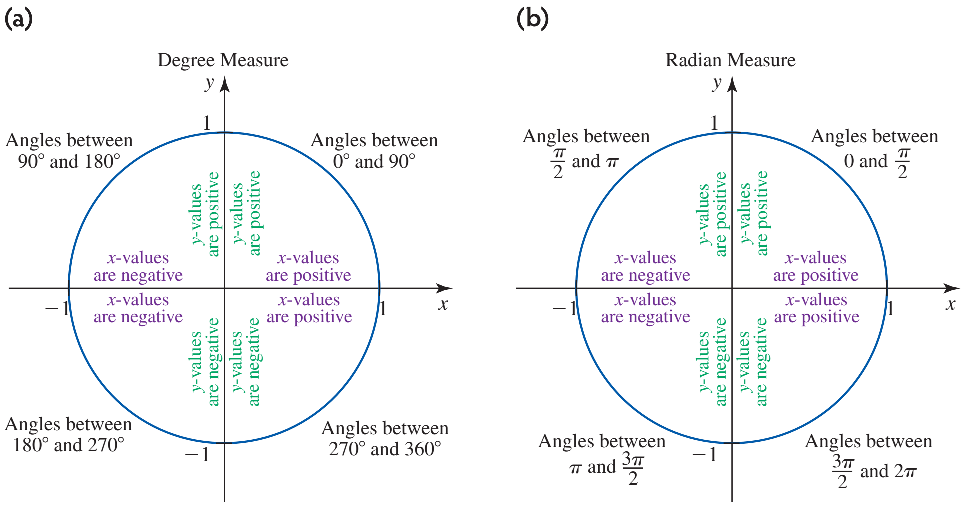


Figure 8.34

EXAMPLE 2 ■ Estimating $x(\theta)$ and $y(\theta)$

Estimate $x(\theta)$ and $y(\theta)$ for the endpoint of the arc corresponding with the given angle.

a. $\theta = 325°\left(\dfrac{65\pi}{36} \text{ radians}\right)$ **b.** $\theta = 110°\left(\dfrac{11\pi}{18} \text{ radians}\right)$

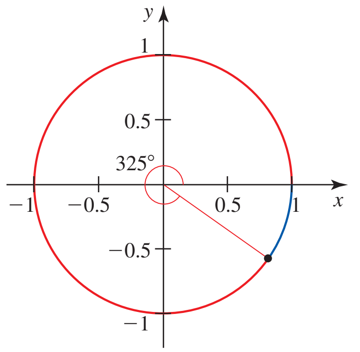


Figure 8.35

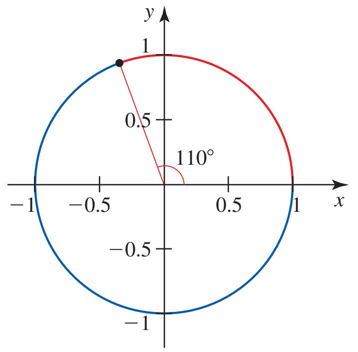


Figure 8.36

Solution

a. By estimation, $x(325°) \approx 0.82$ and $y(325°) \approx -0.57$, so we estimate that the endpoint of the arc corresponding with a 325° angle $\left(\dfrac{65\pi}{36} \text{ radians}\right)$ is located at $(0.82, -0.57)$. See Figure 8.37.

b. By estimation, $x(110°) \approx -0.34$ and $y(110°) \approx 0.94$, so we estimate that the endpoint of the arc corresponding with 110° $\left(\dfrac{11\pi}{18} \text{ radians}\right)$ is at $(-0.34, 0.94)$. See Figure 8.38.

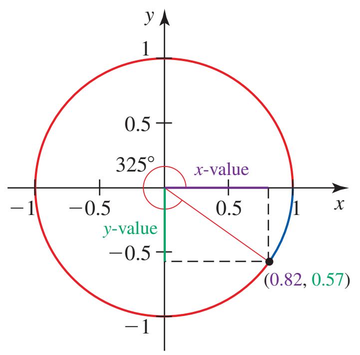


Figure 8.37

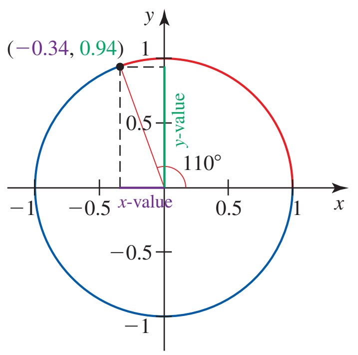


Figure 8.38

■ Circles with Radii Other Than 1

When we estimated the horizontal and vertical components of the points on the unit circle (Examples 1 and 2), each value was between −1 and 1 since the endpoint positions may not exceed the radius of the circle. However, if the circle is larger, the horizontal and vertical components will increase in magnitude. For example, if the radius of the circle is 5, the coordinates of the endpoint of an arc will range from −5 to 5, as shown in Figures 8.39a and 8.39b.

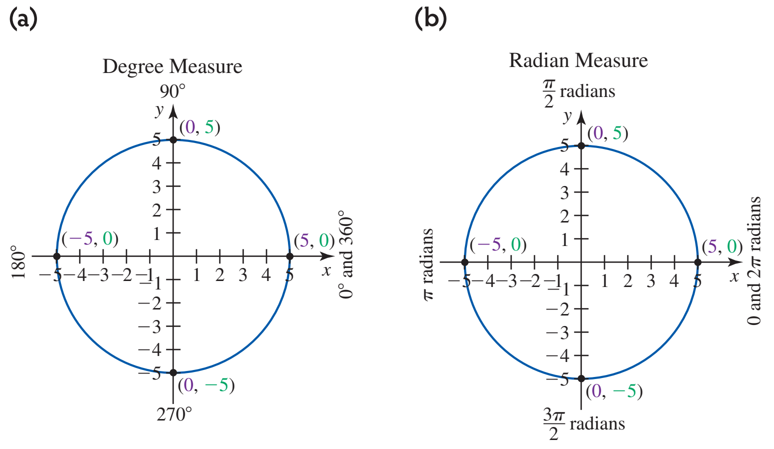


Figure 8.39

Instead of repeating the point estimation process to create a new table, we look at this as a transformation of the original situation. The unit circle has grown by a factor of 5, so the x- and y-values of the coordinates will all change by a factor of 5, as shown in Table 8.5.

Table 8.5

Angle (degrees) θ	Angle (radians) θ	Horizontal Position on the Unit Circle	Vertical Position on the Unit Circle	Horizontal Position on a Circle with Radius 5 $x(\theta)$	Vertical Position on a Circle with Radius 5 $y(\theta)$	Coordinate Point on a Circle of Radius 5 (x, y)
0°	0	1	0	$5(1) = 5$	$5(0) = 0$	(5, 0)
30°	$\pi/6$	0.87	0.5	$5(0.87) \approx 4.4$	$5(0.50) = 2.5$	(4.4, 2.5)
60°	$\pi/3$	0.5	0.87	$5(0.50) = 2.5$	$5(0.87) \approx 4.4$	(2.5, 4.4)
90°	$\pi/2$	0	1	$5(0) = 0$	$5(1) = 5$	(0, 5)

A similar conclusion can be made about the points when $\theta > 90°$.

This demonstrates the importance of the unit circle and the estimations of $x(\theta)$ and $y(\theta)$ made earlier. For example, $x(60°) = 0.5$ for the unit circle. This means that the x-coordinate of the endpoint of the arc corresponding with a 60° angle on the unit circle is 0.5, which is half the length of the radius. This relationship is the same for a circle of *any* size: If the radius of the circle is 10, then $x(60°)$ is half the radius, or 5; if the radius is 4, then $x(60°)$ is half the radius, or 2.

Thus, if we have a circle with a radius of 12 and want to know the coordinates of the endpoint of the arc corresponding with a 70° angle on the circle, we may look to the unit circle to find the solution. Using Table 8.6, we see the horizontal position is 34% of one radius length and the vertical position is 94% of one radius length.

Table 8.6

Angle (degrees) θ	Angle (radians) θ	Horizontal Position on the Unit Circle $x(\theta)$	Percentage of the Length of One Radius for Any Circle	Vertical Position on the Unit Circle $y(\theta)$	Percentage of the Length of One Radius for Any Circle
0°	0	1	100%	0	0%
10°	$\pi/18$	0.98	98%	0.17	17%
20°	$\pi/9$	0.94	94%	0.34	34%
30°	$\pi/6$	0.87	87%	0.5	50%
40°	$2\pi/9$	0.77	77%	0.64	64%
50°	$5\pi/18$	0.64	64%	0.77	77%
60°	$\pi/3$	0.5	50%	0.87	87%
70°	$7\pi/18$	0.34	34%	0.94	94%
80°	$4\pi/9$	0.17	17%	0.98	98%
90°	$\pi/2$	0	0%	1	100%

This gives us a coordinate point at (4.1, 11) since

$$x(70°) = 0.34(12) \approx 4.1 \qquad y(70°) = 0.94(12) \approx 11$$

■ Cosine and Sine Functions

The ideas we just examined are the basis for the trigonometric functions *cosine* and *sine*. The **cosine** function (abbreviated as **cos**) is defined as the horizontal position of the endpoint of an arc on the unit circle. This is exactly what the function $x(\theta)$ represents when applied to the unit circle. Thus, when dealing with a unit circle,

$$x(\theta) = \cos(\theta)$$

Cosine Function

The **cosine** of an angle θ, denoted $\cos(\theta)$, is the horizontal position of the endpoint of the corresponding arc on the unit circle.

We have just discussed how to find the horizontal position of the endpoint of an arc on any circle by multiplying $\cos(\theta)$ by the radius. Therefore, for a circle of radius r, the horizontal position—denoted by $x(\theta)$—is found by

$$x(\theta) = r \cdot \cos(\theta)$$

Horizontal Position of a Point on a Circle

The horizontal position of a point on the arc of a circle of radius r is given by $x(\theta) = r \cdot \cos(\theta)$, where θ is the corresponding angle.

PEER INTO THE PAST

WHY DO WE USE THE TERMS *COSINE* AND *SINE*?

The first time the sine value as a function of an angle appears in writing is in a sixth-century Hindu document called the *Aryabhatiya*. In this work, the sine value is identified as half of a chord within the circle, and thus it was called *ardha-jya*, which means "half-chord." This term was eventually shortened to *jiva*. Later, this important document was translated to Arabic, but the term *jiva* was left untranslated. When it was read, however, its sound was very similar to another Arabic word, *jaib*, which means "fold" or "bay" (as in "Hudson Bay"). Thus, when the term was finally translated into Latin—the language of European scholars—it was written as *sinus*, which is the Latin word for "bay" or "curve." *Sinus* was eventually shortened to *sine*.

Cosine earned its name through its connection with the sine values of angles. For angles between 0° and 90°, the cosine of the angle measure is equivalent to the sine of the complementary angle (complementary angles are angles with measures that sum to 90°). Thus, it is possible to find the sine and cosine values of angles from 0° to 45° and have enough information to fill in the values of sine and cosine for the remaining angles up to 90°.

Source: Eli Maor, *Trigonometric Delights*, Princeton University Press, 1998.

Similarly, the **sine** function (abbreviated as **sin**) is defined as the vertical position of the endpoint of an arc on the unit circle. This is what the function $y(\theta)$ represents when applied to the unit circle. Hence, when working with the unit circle,

$$y(\theta) = \sin(\theta)$$

SINE FUNCTION

The **sine** of an angle θ, denoted $\sin(\theta)$, is the vertical position of the endpoint of the corresponding arc on the unit circle.

And for a circle with radius r, the position is

$$y(\theta) = r \cdot \sin(\theta)$$

VERTICAL POSITION OF A POINT ON A CIRCLE

The vertical position of a point on the arc of a circle of radius r is given by $y(\theta) = r \cdot \sin(\theta)$, where θ is the corresponding angle.

Cosine and sine are often written without units, but it can be useful to think of their unit as "radius length" since the numbers they return tell us the horizontal and vertical positions of a point as portions of one radius length. For example, we can think of $\cos(60°) = 0.5$ as "The horizontal position of the endpoint of the arc corresponding with a 60° angle is equal to half of one radius length."

EXAMPLE 3 ■ Using and Interpreting Cosine and Sine

Evaluate each of the following and explain what the answer represents in the context of arcs of circles.

a. $\cos(20°)$

b. $4\sin(10°)$

c. $(\cos(40°), \sin(40°))$

Solution

a. $\cos(20°)$ represents the horizontal (x) position of the endpoint of the arc on the unit circle corresponding with a 20° angle. From Figure 8.40, we estimate $\cos(20°) \approx 0.94$.

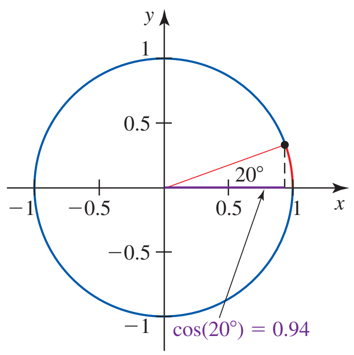


Figure 8.40

b. When a constant is multiplied by either a cosine or sine value, then the circle we are working with has a radius other than 1. In this case, the radius of the circle is 4. From Table 8.6, our estimate of the vertical position of the endpoint of an arc corresponding to a 10° angle was 0.17, or 17% of the radius of the circle (see Figure 8.41a). Thus, the vertical position of the endpoint of the arc on a circle of radius 4 is 17% of 4 (see Figure 8.41b):

$$\begin{aligned} 4\sin(10°) &\approx 4(0.17) \\ &\approx 0.69 \end{aligned}$$

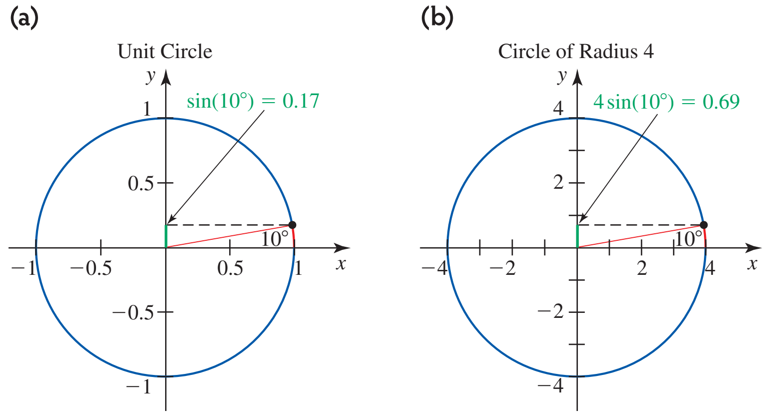


Figure 8.41

c. (cos(40°), sin(40°)) is the (x, y) coordinate point at the endpoint of the arc on the unit circle corresponding with a 40° angle, as shown in Figure 8.42. The coordinate is (0.77, 0.64) since

$$\cos(40°) \approx 0.77 \qquad \sin(40°) \approx 0.64$$

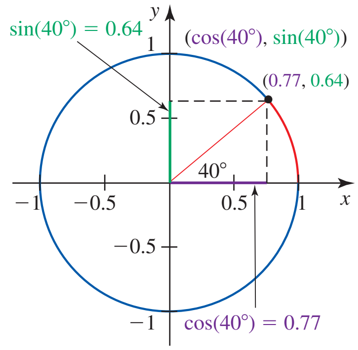


Figure 8.42

■ Using a Calculator to Evaluate Cosine and Sine

The visual estimates of the coordinate points on a circle we have used so far are limiting because the numbers are not exact values. Furthermore, it is difficult to accurately estimate the difference between positions of the endpoint of an arc when the corresponding angles differ by small amounts, such as between cos(13°) and cos(13.1°). Fortunately, calculators evaluate the cosine and sine values with much greater precision and for any angle value. (See the Technology Tip at the end of the section for details.)

EXAMPLE 4 ■ Evaluating Cosine and Sine Using Technology

Evaluate cos(213.68°) and sin(213.68°) and interpret their meaning for the unit circle and for a circle with a radius other than 1.

Solution A calculator gives the screens shown in Figure 8.43.

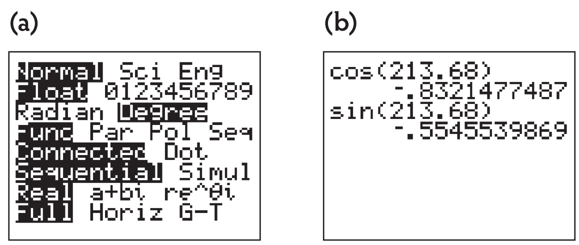


Figure 8.43

Thus the endpoint of the arc on the unit circle corresponding to an angle of 213.68° is located at about (−0.8321, −0.5546).

If the radius of the circle is not 1, then the x-coordinate is 83.21% of one radius length in the negative horizontal direction and the y-coordinate is 55.46% of one radius length in the negative vertical direction. For example, if the radius is 7, then the coordinates of the endpoint of the arc are $(-5.8247, -3.8822)$ since

$$x\text{-coordinate} = (-0.8321)(7) \qquad y\text{-coordinate} = (-0.5546)(7)$$
$$= -5.8247 \qquad\qquad\qquad\qquad = -3.8822$$

$$(-5.8247, -3.8822)$$

We have thus far used $x(\theta)$ and $y(\theta)$ to name the functions finding horizontal and vertical positions, respectively. However, the use of cosine and sine now make this distinction clear, so we can now use function notation such as $f(\theta) = \cos(\theta)$.

EXAMPLE 5 ■ Using the Sine Function

The original Ferris wheel designed by George Ferris had a radius of 125 feet. If a person boards the Ferris wheel from the bottom (ground level, ignoring seat height), how high off the ground is the person after they have traveled $\frac{2}{3}$ of the way around the wheel?

Solution Since a full revolution of the wheel corresponds to an angle measuring 360°, traveling $\frac{2}{3}$ of the way around the Ferris wheel corresponds to an angle of 240° $\left(\frac{2}{3}(360°) = 240°\right)$. However, in this case, the angle is not being measured from its standard position on the positive x-axis. Instead, it is starting from the position corresponding with $-90°$. See Figure 8.44. By subtracting 90° from 240°, we can use the sine function to determine the height. Since

$$\begin{aligned} 125\sin(240° - 90°) &= 125\sin(150°) \\ &= 62.5 \text{ feet} \end{aligned}$$

The person is 62.5 feet above the horizontal line passing through the center of the Ferris wheel (the x-axis). To find the total height, we will need to add the length of the radius, 125 feet. See Figure 8.45 and the following calculation.

$$\begin{aligned} \text{height above ground} &= 125 + 125\sin(150°) \\ &= 125 + 62.5 \\ &= 187.5 \text{ feet} \end{aligned}$$

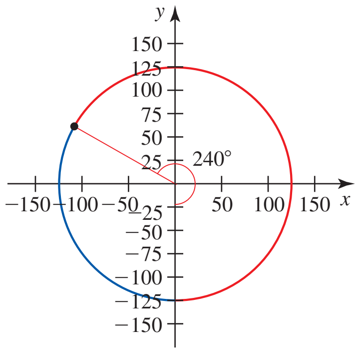


Figure 8.44

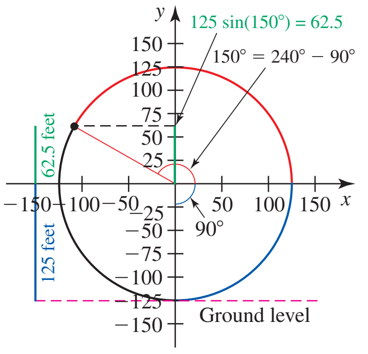


Figure 8.45

A person who has traveled $\frac{2}{3}$ around the Ferris wheel is 187.5 feet above the ground.

Domain and Range of the Cosine and Sine Functions

Earlier in the chapter, we discussed what it meant for angles to be negative (measured in the clockwise direction). We also indicated that when an angle surpasses 360° (2π radians), it is making more than one revolution around the circle. As the endpoint of the arc moves around the circle more than one time in a clockwise or counterclockwise direction, it will repeat all of the same values for $\cos(\theta)$ and $\sin(\theta)$ for every 360° (or 2π radians) that the angle changes. This occurs because no matter which direction we move on the circle or how many revolutions we make, we are still on the same circle and still passing through the same points. Thus, we may find the cosine or sine of *any* angle measure. Therefore, the domain of $\cos(\theta)$ and $\sin(\theta)$ includes all real numbers.

Since $\cos(\theta)$ and $\sin(\theta)$ represent horizontal and vertical positions of a point on the unit circle, and since their units are portions of a radius length, the values for these functions will always be between -1 and 1. It is impossible for the x- or y-value of a coordinate on the unit circle to be farther from the origin than the one radius. Thus, the ranges for both $x(\theta) = \cos(\theta)$ and $y(\theta) = \sin(\theta)$ are identical and include all values between -1 and 1.

EXAMPLE 6 ■ Finding Values of Cosine and Sine

Find the coordinates at the endpoints of the arc corresponding with a 405° angle and a $\frac{19\pi}{6}$ radian angle on the unit circle. Compare these to the coordinates of the endpoints of arcs corresponding with angles between 0° and 360° or 0 and 2π radians on the unit circle.

Solution

$$\cos(405°) \approx 0.7071 \qquad \cos\left(\frac{19\pi}{6}\right) \approx -0.8660$$

$$\sin(405°) \approx 0.7071 \qquad \sin\left(\frac{19\pi}{6}\right) \approx -0.5$$

$$(0.7071, 0.7071) \qquad (-0.8660, -0.5)$$

Since $405° = 360° + 45°$, the terminal side of a 405° angle is in the same position as the terminal side of a 45° angle. The endpoint of the corresponding arc is approximately (0.7071, 0.7071), as shown in Figure 8.46.

Since $\frac{7\pi}{6} + 2\pi = \frac{19\pi}{6}$, the terminal side of a $\frac{19\pi}{6}$ radian angle is in the same position as the terminal side of a $\frac{7\pi}{6}$ radian angle. See Figure 8.47.

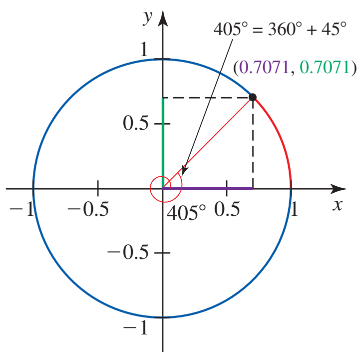


Figure 8.46

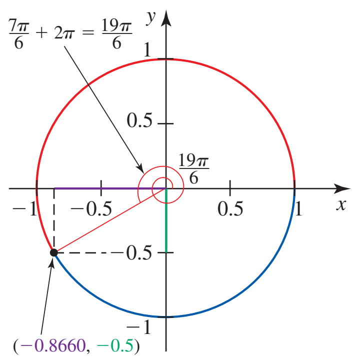


Figure 8.47

As Example 6 shows, we determine the endpoints of arcs corresponding with angles that are greater than 360° (or 2π radians) by looking at corresponding angles between 0° and 360° (or 0 and 2π radians). If the arc completes between one and two revolutions around the circle, we can find the corresponding angle by subtracting 360° or 2π once. If the arc makes between two and three complete revolutions around the circle, we can find the corresponding angle by subtracting 360° or 2π twice. This pattern will continue for arcs that make more than three complete revolutions. For arcs marked off in the clockwise direction (negative arcs), we instead *add* 360° or 2π radians to the angle the appropriate number of times until we get an angle that is between 0° and 360° or 0 and 2π radians.

Reference Angles

You may have noticed that some coordinates for the endpoints of arcs outside of the first quadrant are very similar to those corresponding to angles between 0° and 90° (or 0 and $\frac{\pi}{2}$ radians). For example, the endpoint of an arc corresponding with 225° on the unit circle is located at $(-0.71, -0.71)$ while the endpoint of an arc corresponding with a 45° arc on the unit circle is located at $(0.71, 0.71)$. This is not a coincidence. We observe that circles are symmetric about both the x- and y-axes, meaning that every point (x, y) on the circle also has corresponding points at $(-x, y)$, $(x, -y)$, and $(-x, -y)$ on the circle, as illustrated in Figure 8.48. These points have the same *reference angle*, as shown in Figures 8.49a–8.49d (on the next page).

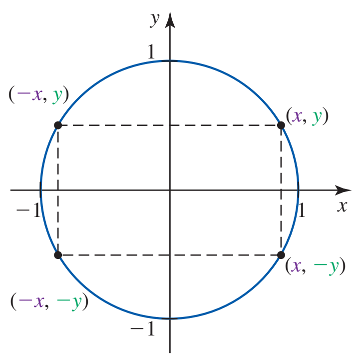


Figure 8.48

> **REFERENCE ANGLE**
>
> A **reference angle** is the smallest positive angle formed by the terminal side of the angle θ and the x-axis.

These graphs in Figure 8.49 demonstrate two important points. First, the reference angle will always measure between 0° and 90° (or 0 and $\frac{\pi}{2}$ radians). Second, when the reference angles are equivalent, the x- and y-values of the points on the circle will have the same magnitudes, although the signs will differ depending on the quadrant in which the point is located.

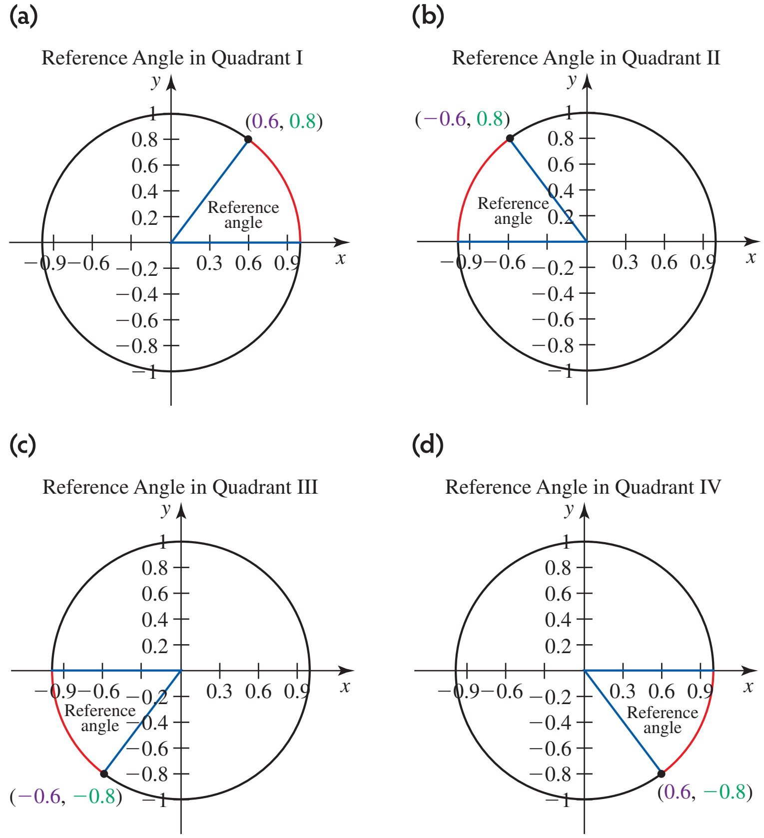


Figure 8.49

EXAMPLE 7 ■ Using Reference Angles

a. The coordinate (0.8480, 0.5299) lies on the endpoint of the arc on the unit circle corresponding with a 32° angle. Find three angles between 0° and 360° with the same reference angle and give the coordinates of the endpoint of each corresponding arc.

b. The coordinate (0.3827, 0.9239) lies on the endpoint of an arc on the unit circle corresponding with $\frac{3\pi}{8}$ radians. Find three angles between 0 and 2π with the same reference angle and give the coordinates of the endpoint of each corresponding arc.

Solution

a. The three remaining arcs will be in the second, third, and fourth quadrants and will all have a 32° reference angle. Thus, we may find these angle measures by adding or subtracting 32° from 180° and 360°. The coordinates of the endpoint of each arc will match the coordinates for the endpoint of the arc corresponding with a 32° angle, but the signs will vary according to the quadrant. See Figure 8.50.

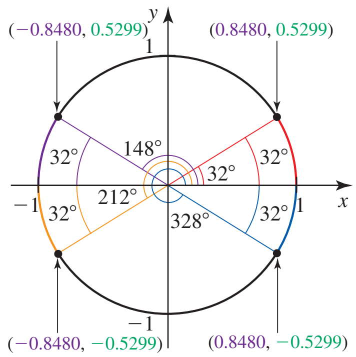


Figure 8.50

Quadrant II	*Quadrant III*	*Quadrant IV*
$180° - 32° = 148°$	$180° + 32° = 212°$	$360° - 32° = 328°$
$(-0.8480, 0.5299)$	$(-0.8480, -0.5299)$	$(0.8480, -0.5299)$

b. Each angle will have a reference angle measuring $\frac{3\pi}{8}$ radians. Thus, we may find these angle measures by adding or subtracting $\frac{3\pi}{8}$ radians from π radians and 2π radians. The coordinates of the endpoint of each arc will match the coordinates for the endpoint of the arc corresponding with an angle measuring $\frac{3\pi}{8}$ radians, but the signs will vary according to the quadrant. See Figure 8.51.

Quadrant II	*Quadrant III*	*Quadrant IV*
$\pi - \frac{3\pi}{8} = \frac{5\pi}{8}$	$\pi + \frac{3\pi}{8} = \frac{11\pi}{8}$	$2\pi - \frac{3\pi}{8} = \frac{13\pi}{8}$
$(-0.3827, 0.9239)$	$(-0.3827, -0.9239)$	$(0.3827, -0.9239)$

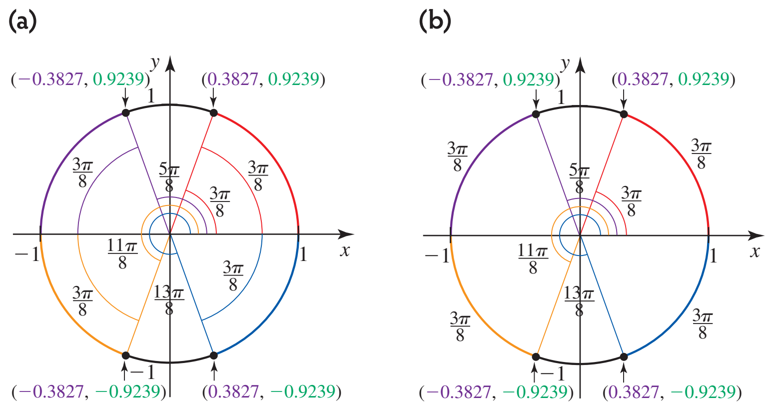


Figure 8.51

■ Exact Values of Cosine and Sine on the Unit Circle

The values we have been using for cosine and sine are rounded versions of the exact values of these functions. Certain angles appear often enough that it is beneficial to find the *exact* cosine and sine values for these angles. We may find these values using basic concepts from geometry. We begin with an equilateral triangle.

Consider an equilateral triangle arranged inside of a unit circle as shown in Figure 8.52a. Since two of the three sides are radii of the circle, and since an equilateral triangle has three sides of the same length, we know all three sides are 1 unit long. In addition, an equilateral triangle has three angles of the same measure. Since the angles in any triangle have a sum of 180°, each of the angles of the equilateral triangle must measure 60°. By using an **altitude** of the triangle (a line drawn from a vertex perpendicular to the opposite side), we can bisect this equilateral triangle, creating the right triangle shown in Figure 8.52b.

PEER INTO THE PAST

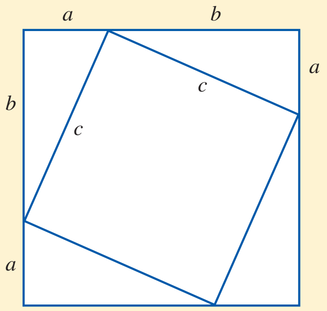
PYTHAGORAS AND HIS THEOREM

Pythagoras of Greece was born between 580 and 572 B.C. and died between 500 and 490 B.C. He is respected as a great mathematician and scientist. Because folklore and legend cloud his work, little is known about his life and teachings. He taught his followers that everything is related to mathematics and that numbers are the ultimate reality. Pythagoras believed that through mathematics everything could be predicted and measured through the rhythmic cycles and patterns found in nature.

Pythagoras is best known for the theorem that bears his name—by tradition he is credited with its discovery and proof.

The Pythagorean theorem states that in any right triangle the square of the length of the hypotenuse (side opposite the right angle) is equal to the sum of the squares of the lengths of the legs of the triangle. The theorem can be written symbolically $a^2 + b^2 = c^2$, where c represents the length of the hypotenuse, and a and b represent the lengths of the other two sides.

One of the simplest proofs of the theorem is shown here. (*Source:* www-history.mcs.st-and.ac.uk)



$$c^2 + 4\left(\frac{1}{2}ab\right) = (a + b)^2$$

$$c^2 + 2ab = a^2 + 2ab + b^2$$

$$c^2 = a^2 + b^2$$

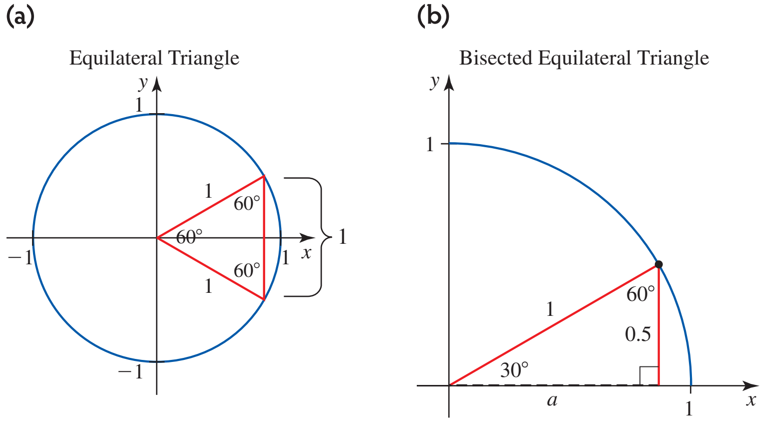


Figure 8.52

The length of the hypotenuse of the triangle is 1 and the length of the shortest leg is 0.5 or $\frac{1}{2}$. To find the length of the other leg (marked a in the figure), we use the Pythagorean theorem.

$$a^2 + b^2 = c^2$$
$$a^2 + \left(\frac{1}{2}\right)^2 = (1)^2$$
$$a^2 + \frac{1}{4} = 1$$
$$a^2 = \frac{3}{4}$$
$$a = \sqrt{\frac{3}{4}}$$
$$a = \frac{\sqrt{3}}{\sqrt{4}}$$
$$a = \frac{\sqrt{3}}{2}$$

The horizontal position of the vertex lying on the circle is $a = \frac{\sqrt{3}}{2}$. The vertical position is $\frac{1}{2}$. Thus $\cos(30°) = \frac{\sqrt{3}}{2}$ and $\sin(30°) = \frac{1}{2}$, as shown in Figure 8.53a. We may

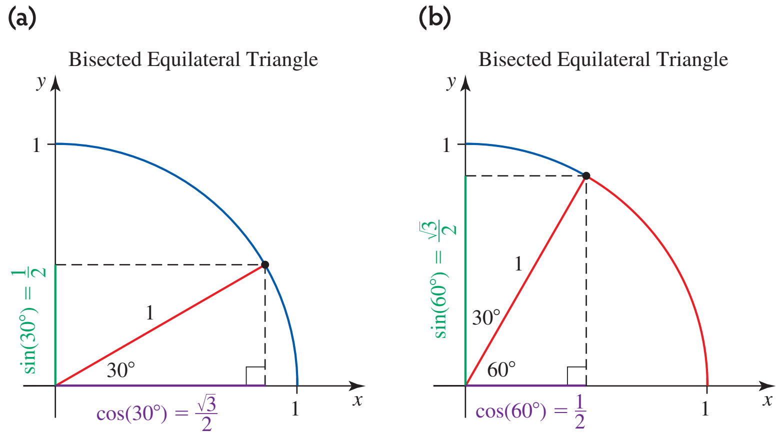


Figure 8.53

find the exact values for cos(60°) and sin(60°) by drawing an equilateral triangle bisected by the y-axis and following the same process. The results are shown in Figure 8.53b, giving us $\cos(60°) = \frac{1}{2}$ and $\sin(60°) = \frac{\sqrt{3}}{2}$.

In radians, $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$, $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$, and $\sin\left(\frac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$. Since the values of cosine and sine are reversed for these angles, it is easy to get them confused. However, by remembering that $\frac{\sqrt{3}}{2} \approx 0.8660$ and $\frac{1}{2} = 0.5$, we simply need to ask "to which axis is the coordinate at the endpoint of the arc closer, the x-axis or the y-axis?" If it is closer to the x-axis, the x-coordinate will be the larger value $\left(\frac{\sqrt{3}}{2}\right)$. If it is closer to the y-axis, the y-coordinate will be the larger value $\left(\frac{\sqrt{3}}{2}\right)$.

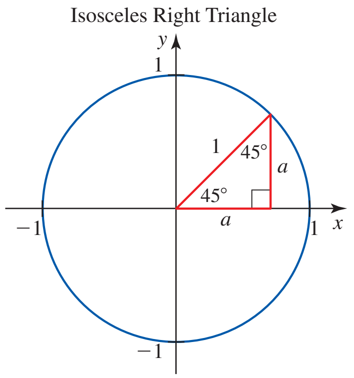


Figure 8.54

In addition to 30° and 60°, 45° is considered to be one of the *standard* angles in the first quadrant. This result is derived from an *isosceles right triangle*. An **isosceles right triangle** is a right triangle whose legs are of equal length. In Figure 8.54, we arrange a right triangle in the unit circle, with a hypotenuse of length 1 and each leg of length a. Since in triangles sides of equal length have equal angles opposite the sides, the two non-90° angles must be equal. The sum of the two non-right angles is 90°, so each angle must be $\frac{90°}{2} = 45°$.

The length of the legs of the triangle represent cos(45°) and sin(45°). We again utilize the Pythagorean theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + a^2 &= (1)^2 \\ 2a^2 &= 1 \\ a^2 &= \frac{1}{2} \\ a &= \sqrt{\frac{1}{2}} \\ a &= \frac{\sqrt{1}}{\sqrt{2}} \\ a &= \frac{1}{\sqrt{2}} \end{aligned}$$

Therefore $\cos(45°) = \frac{1}{\sqrt{2}}$ and $\sin(45°) = \frac{1}{\sqrt{2}}$. However, when writing fractions that include radicals, such as $\sqrt{2}$, it is customary to **rationalize the denominator**, which means to write the fraction so that the denominator does not contain a radical.

$$\begin{aligned} a &= \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} \\ &= \frac{1 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} \\ &= \frac{\sqrt{2}}{(\sqrt{2})^2} \\ &= \frac{\sqrt{2}}{2} \end{aligned}$$

So $\cos(45°) = \dfrac{\sqrt{2}}{2}$ and $\sin(45°) = \dfrac{\sqrt{2}}{2}$. In radians, $\cos\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$ and $\sin\left(\dfrac{\pi}{4}\right) = \dfrac{\sqrt{2}}{2}$. See Figure 8.55.

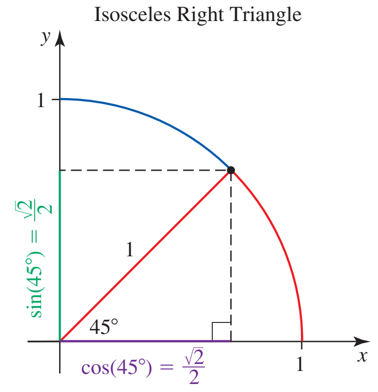


Figure 8.55

JUST IN TIME ■ RATIONALIZING DENOMINATORS

Historically, it has been a common practice to rationalize denominators for a number of reasons. First, rationalizing denominators gives us a standard way to write numbers, which is helpful when comparing two values. For example, it is not clear that $\dfrac{1}{\sqrt{2}}$ and $\dfrac{\sqrt{2}}{2}$ have the same value, but it becomes apparent after rationalizing. Thus, a student who gets $\dfrac{1}{\sqrt{2}}$ as an answer might be very frustrated when their teacher, peers, and textbook say the answer is $\dfrac{\sqrt{2}}{2}$ unless she understands that the two are equivalent. Furthermore, when doing calculations by hand, such as dividing, it is easier to use long division with $\dfrac{\sqrt{2}}{2}$ than with $\dfrac{1}{\sqrt{2}}$.

The importance of rationalizing denominators has decreased in recent years as calculators and computers now perform most calculations. For example, it only takes a few seconds to verify that $\dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$ using a calculator. On the other hand, many textbooks use $\dfrac{\sqrt{2}}{2}$ as the value for cos(45°) and sin(45°) instead of $\dfrac{1}{\sqrt{2}}$. Thus, learners should be aware of the practice of rationalizing denominators and know how to recognize equivalent fractions in rationalized form.

We combine the preceding results into Table 8.7 to show the exact values of cosine and sine for major angles in the first quadrant and on the adjacent axes. Then, based on what we know about reference angles, we extend this table to include angles outside of the first quadrant, as shown in Figure 8.56.

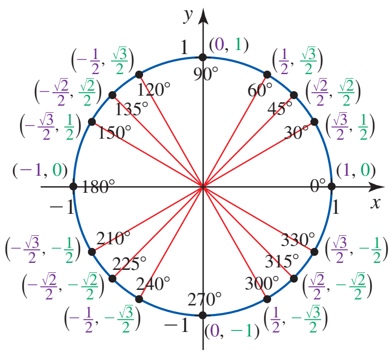


Figure 8.56

Table 8.7

Angle Measure (degrees) θ	Angle Measure (radians) θ	Reference Angle's Measure	$\cos(\theta)$	$\sin(\theta)$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$30°$ or $\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$45°$	$\frac{\pi}{4}$	$45°$ or $\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$60°$	$\frac{\pi}{3}$	$60°$ or $\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$90°$	$\frac{\pi}{2}$	$90°$ or $\frac{\pi}{2}$	0	1
$120°$	$\frac{2\pi}{3}$	$60°$ or $\frac{\pi}{3}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$
$135°$	$\frac{3\pi}{4}$	$45°$ or $\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$
$150°$	$\frac{5\pi}{6}$	$30°$ or $\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$
$180°$	π	0	-1	0
$210°$	$\frac{7\pi}{6}$	$30°$ or $\frac{\pi}{6}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$225°$	$\frac{5\pi}{4}$	$45°$ or $\frac{\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$240°$	$\frac{4\pi}{3}$	$60°$ or $\frac{\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$270°$	$\frac{3\pi}{2}$	$90°$ or $\frac{\pi}{2}$	0	-1
$300°$	$\frac{5\pi}{3}$	$60°$ or $\frac{\pi}{3}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$
$315°$	$\frac{7\pi}{4}$	$45°$ or $\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$
$330°$	$\frac{11\pi}{6}$	$30°$ or $\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$
$360°$	2π	0	1	0

EXAMPLE 8 ■ Finding Exact Values for Cosine and Sine

Find the exact value of each of the following expressions.

a. $\sin\left(\frac{4\pi}{3}\right)$

b. $\cos(855°)$

c. $\cos\left(-\frac{11\pi}{6}\right)$

Solution

a. Consulting the table, we find $\sin\left(\frac{4\pi}{3}\right) = -\frac{\sqrt{3}}{2}$.

b. First, we need to find the corresponding angle between 0° and 360°.

$$\begin{aligned} 855^\circ - 2(360^\circ) &= 135^\circ \\ \cos(855^\circ) &= \cos(135^\circ) \\ &= -\frac{\sqrt{2}}{2} \end{aligned}$$

c. We first find the corresponding angle between 0 and 2π radians.

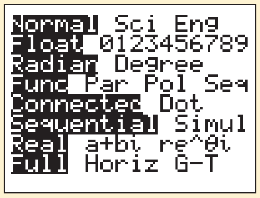
$$\begin{aligned} -\frac{11\pi}{6} + 2\pi &= \frac{\pi}{6} \\ \cos\left(-\frac{11\pi}{6}\right) &= \cos\left(\frac{\pi}{6}\right) \\ &= \frac{\sqrt{3}}{2} \end{aligned}$$

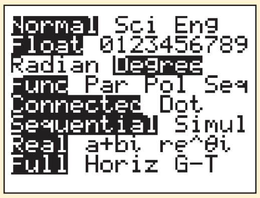
SUMMARY

In this section you learned about the unit circle and saw how the coordinate points on a circle are found using cosine and sine. You discovered how to use reference angles to find cosine and sine values of any angle. You also learned about the domain and range of these functions and how to express the exact values of cosine or sine for standard angles.

TECHNOLOGY TIP ■ CALCULATING COSINE AND SINE

1. Press MODE to verify that your calculator is in the proper units (Radian or Degree) according to the angle given.

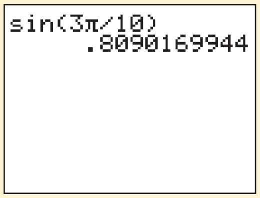


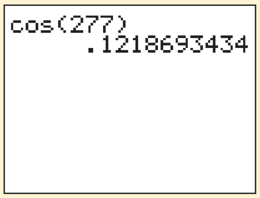


2. Press COS or SIN on your calculator, then input the angle value and close the parentheses. Rounding will often be necessary when using the calculator. It is customary to round the values of trigonometric functions to 4 decimal places.

$$\sin\left(\frac{3\pi}{10}\text{ radians}\right) \approx 0.8090$$

$$\cos(277^\circ) \approx 0.1219$$

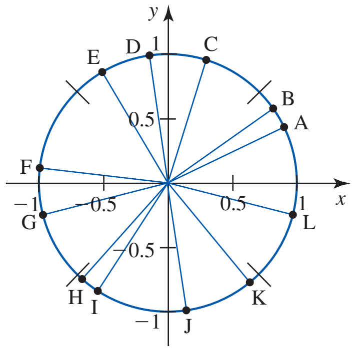




8.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, use the following figure.



For the indicated point,

a. Estimate θ, the angle between 0° and 360° where the point lies.

b. Estimate $\cos(\theta)$ and $\sin(\theta)$.

1. A
2. E
3. G
4. K

In Exercises 5–8, use the preceding figure. For the indicated point,

a. Estimate θ, the angle between 0 and 2π radians where the point lies.

b. Estimate $\cos(\theta)$ and $\sin(\theta)$.

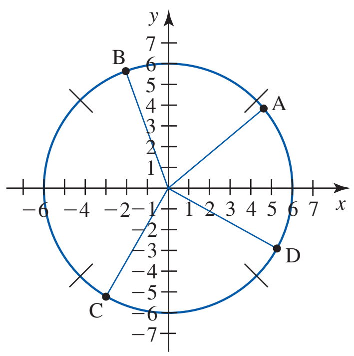
5. B
6. D
7. H
8. L

In Exercises 9–12, use the following figure. When we are working with a circle whose radius is not 1, we use $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to find the coordinates of the endpoint of an arc. We can rewrite these formulas as $\frac{x}{r} = \cos(\theta)$ and $\frac{y}{r} = \sin(\theta)$ by dividing by r.

a. Estimate the (x, y) coordinates for the indicated point.

b. Use your answer in part (a) and the fact that the circle has a radius of 6 to find $\cos(\theta)$ and $\sin(\theta)$.

c. Explain what your answer in part (b) represents.



9. A
10. B
11. C
12. D

In Exercises 13–16, use a calculator to evaluate each expression. Then explain what the value means in the context of circles.

13. $\cos(13.5°)$
14. $\sin(119°)$
15. $0.5\sin(537°)$
16. $(\cos(-203.6°), \sin(-203.6°))$

In Exercises 17–20, use a calculator to evaluate each expression. Then explain what the value means in the context of circles. The angles are all measured in radians.

17. $\cos\left(\frac{4\pi}{7}\right)$
18. $\sin\left(-\frac{\pi}{9}\right)$
19. $4.2\sin\left(-\frac{7\pi}{6}\right)$
20. $\left(8\cos\left(\frac{11\pi}{3}\right), 8\sin\left(\frac{11\pi}{3}\right)\right)$

In Exercises 21–26, find the exact value of each expression, then draw a picture to represent the meaning behind the value you found.

21. $\cos(30°)$
22. $\cos(-315°)$
23. $2.5\sin(90°)$
24. $18\cos(\pi)$
25. $\sin\left(\frac{2\pi}{3}\right)$
26. $-\sin\left(\frac{5\pi}{4}\right)$

In Exercises 27–30, you are given the cosine value for the endpoint of the arc corresponding with an angle measuring θ°. For each value of cosine,

a. Draw a unit circle.

b. Indicate the two possible positions on this circle where the given value of cosine would be true.

c. Estimate the value of $\sin(\theta)$ for each of these points.

27. $\cos(\theta) = 0.43$
28. $\cos(\theta) = 0.76$
29. $\cos(\theta) = -0.6$
30. $\cos(\theta) = -0.24$
31. Based on your responses to Exercises 27–30, how many positions on a circle generally have the same value for cosine? What is true about the sine values at each of these positions?

In Exercises 32–34, you are given the sine value for the endpoint of the arc corresponding with an angle of $\theta°$. For each value of sine,

a. Draw a unit circle.

b. Indicate the two possible positions on this circle where the corresponding angle will have the given value of sine.

c. Estimate the value of $\cos(\theta)$ for each of these points.

32. $\sin(\theta) = 0.16$

33. $\sin(\theta) = -0.52$

34. $\sin(\theta) = -0.1$

35. Based on your responses to Exercises 32–34, how many positions on a circle generally have the same value for sine? What is true about the cosine values at each of these positions?

36. For each of the indicated intervals, describe whether the cosine and sine of θ are positive or negative and determine whether they are increasing or decreasing.

a. $0 < \theta < \frac{\pi}{2}$ **b.** $\frac{\pi}{2} < \theta < \pi$

c. $\pi < \theta < \frac{3\pi}{2}$ **d.** $\frac{3\pi}{2} < \theta < 2\pi$

In Exercises 37–40, you are given the cosine and sine value for the endpoint of an arc.

a. Find three angles between 0° and 360° with the same reference angles.

b. Find the cosine and sine values for each of the angles found in part (a).

c. Draw a diagram that shows all four angles and their cosine and sine values.

37. $\cos(53°) = 0.6018$ and $\sin(53°) = 0.7986$

38. $\cos(76.2°) = 0.2385$ and $\sin(76.2°) = 0.9711$

39. $\cos(148°) = -0.8480$ and $\sin(148°) = 0.5299$

40. $\cos(262.4°) = -0.1323$ and $\sin(262.4°) = -0.9912$

In Exercises 41–44, you are given the cosine and sine value for the endpoint of an arc.

a. Find three angles between 0 and 2π radians with the same reference angles.

b. Find the cosine and sine values for each of the angles found in part (a).

c. Draw a diagram that shows all four angles and their cosine and sine values.

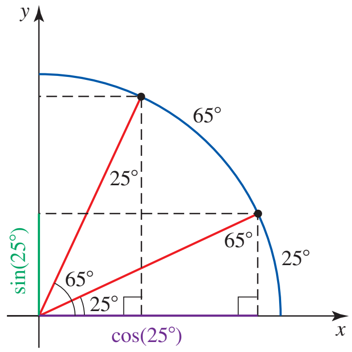
41. $\cos\left(\frac{\pi}{5}\right) = 0.8090$ and $\sin\left(\frac{\pi}{5}\right) = 0.5878$

42. $\cos\left(\frac{2\pi}{9}\right) = 0.7660$ and $\sin\left(\frac{2\pi}{9}\right) = 0.6428$

43. $\cos\left(\frac{9\pi}{8}\right) = -0.9239$ and $\sin\left(\frac{9\pi}{8}\right) = -0.3827$

44. $\cos\left(\frac{12\pi}{7}\right) = 0.6235$ and $\sin\left(\frac{12\pi}{7}\right) = -0.7818$

45. Throughout this section you may have noticed that the cosine value of one angle may be the same sine value as another angle. In fact, there is a pattern to when $\cos(\theta) = \sin(\phi)$. In the first quadrant, the rule is that the sine of an angle is the same as the cosine of its complement (complementary angles have a sum of 90° or $\frac{\pi}{2}$ radians). Use the following diagram to explain why this is true.



46. As the length of an arc increases, the cosine and sine values of its endpoint pass through all of the points in their range in a specific order. In fact, they pass through the same values *in the same order* as each other. The only difference is that they do not begin at the same value.

a. Starting with $\theta = 90°$, find $\sin(\theta)$ in 15° intervals until $\theta = 180°$.

b. Starting with $\theta = 0°$, find $\cos(\theta)$ in 15° intervals until $\theta = 90°$.

c. Compare your results in parts (a) and (b) and explain what you observe.

47. Will the trend observed in Exercise 46 continue all the way around the circle? Defend your reasoning using diagrams to strengthen your argument.

48. Fill in the blanks to formalize the observations you made in Exercises 46 and 47.

a. $\cos(\theta) = \sin($_________$)$

b. $\sin(\theta) = \cos($_________$)$

■ SHOW YOU KNOW

49. When you evaluate $\cos(342°)$ with your calculator and get the value 0.9511, does this value represent a length? Justify your answer.

50. Explain why it is impossible for $\cos(\theta) = 1.5$ or $\sin(\theta) = -3$.

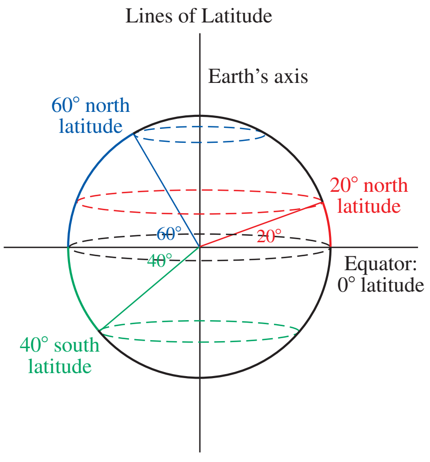
51. Cosine and sine functions are periodic, meaning that their values repeat at regular intervals. Explain, with reference to the unit circle, why these functions are periodic and state the period of each function.

52. Explain how the cosine and sine values of angles on the unit circle can be used to find coordinates of points on *any* circle, and explain why this works.

53. A fellow classmate mistakenly thinks that $\cos(45°) = 0.5$ and $\sin(45°) = 0.5$. Explain why he might think this and why it is not possible. Use diagrams as necessary.

MAKE IT REAL

In Exercises 54–56, use the following information. The circumference of a circle is $C = 2\pi r$. Positions on Earth's surface are described according to lines of latitude and longitude. Lines of latitude are drawn around the Earth parallel to the equator at regular intervals determined by the arc drawn from the equator (see the figure). Each line of latitude forms a circle around the Earth.



54. Earth's Circumference What is the circumference of Earth at the equator, assuming the Earth's radius is 3963 miles?

55. Earth's Circumference

a. What happens to the circumference of the circles formed by the lines of latitude as you move farther away from the equator?

b. What is the largest possible degree for a line of latitude and what is the circumference of the circle at that line? Where is this located?

56. Earth's Circumference The formula $C(\theta) = 2\pi(3963\cos(\theta))$ models the circumference of the circle formed by a line of latitude θ degrees from the equator.

a. Explain why $3963\cos(\theta)$ models the radius of the circle formed by a line of latitude θ degrees from the equator.

b. Find the radius of the circle formed by the Tropic of Cancer (23.5° north latitude) and the Tropic of Capricorn (23.5° south latitude).

c. Explain why it is not required to use negative values for θ to get the correct circumference in the Southern Hemisphere.

57. Ferris Wheels The London Eye is a Ferris wheel constructed on the banks of the River Thames in London. The London Eye has a radius of about 221 feet and is boarded from the bottom. (*Source:* www.aboutbritain.com)

Determine the height of a person from the bottom of the London Eye after traveling each of the following portions of a revolution.

a. 1/3 of the way around

b. 5/12 of the way around

c. 9/10 of the way around

d. after completing 6/5 revolutions

58. Ferris Wheels The Singapore Flyer is a Ferris wheel constructed on top of a three-story building. The wheel itself has a diameter of 492 feet, but the bottom of the wheel sits nearly 50 feet off the ground. Determine the height from the ground of a person after boarding from the bottom and traveling each of the following portions of a revolution. (*Source:* www.singaporeflyer.com.sg)

a. halfway around

b. 1/10 of the way around

c. 4/5 of the way around

d. 4.65 full revolutions

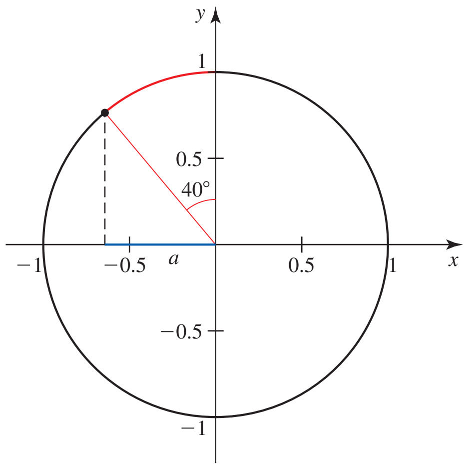
STRETCH YOUR MIND

Exercises 59–63 are intended to challenge your understanding of cosine and sine functions.

59. Use the Pythagorean theorem to write an equation involving $\cos(\theta)$, $\sin(\theta)$, and the radius in a unit circle. (*Hint:* Draw a picture of their relationship first.)

60. Explain how you can use your answer to Exercise 59 to check if it is possible for $\cos(\theta) = 0.7642$ and $\sin(\theta) = 0.4391$.

61. The following diagram shows an angle of 40° beginning at the positive y-axis. Explain why the expression $-\sin(40°)$ may be used to find the horizontal position a.



62. Consider the equation $\sin(-\theta) = -\sin(\theta)$. Is this equation true for all values of θ? Defend your answer using an explanation and diagrams.

63. Given that $\cos(\theta) = x$ and $\sin(\theta) = y$, evaluate each expression.

a. $\cos(-\theta)$

b. $\sin(\theta + 2\pi n)$

c. $\sin(\pi - \theta)$

d. $\cos(\theta + \pi)$

SECTION 8.4 Graphing Cosine and Sine Functions

LEARNING OBJECTIVES

- Use the unit circle to construct the graphs of the cosine and sine functions
- Graph cosine and sine including transformations of each
- Solve trigonometric equations graphically

GETTING STARTED

The number of hours of daylight varies throughout the year, with more hours of sunlight in the summer and less in the winter. Depending on where you live, the change in the number of hours of daylight is more or less dramatic. For example, there are up to 7.5 more hours of sunlight in the summer than in the winter in Seattle, Washington, while the difference is only 4.5 hours in Los Angeles, California. (*Source:* http://aa.usno.navy.mil)

In this section we explain how to graph cosine and sine functions. We also show how to transform them and what these transformations represent with respect to the unit circle. We also apply our understanding of cosine and sine functions to real-world periodic phenomena, such as the number of hours of daylight throughout the year.

■ Graphing $x(\theta) = \cos(\theta)$ and $y(\theta) = \sin(\theta)$

To graph $x(\theta) = \cos(\theta)$ and $y(\theta) = \sin(\theta)$, we need to change from the xy-coordinate system to a system with the angle measure as the independent variable and the trigonometric function value of the angle as the dependent variable. In these systems, the angle measure is the input value and the trigonometric function value of the angle is the output.

The graphs in Figure 8.57 show the transition from the unit circle to the graph of $x(\theta) = \cos(\theta)$. In Figure 8.57a, we see the points on the unit circle labeled with their angle measure (in degrees) and the corresponding horizontal (x) position of the endpoint for the corresponding arcs. However, these points are still on the xy-plane. We take these points and graph them as θx-coordinates in Figure 8.57b. Note the x-value appears on the vertical axis in this situation because it is the output of the function.

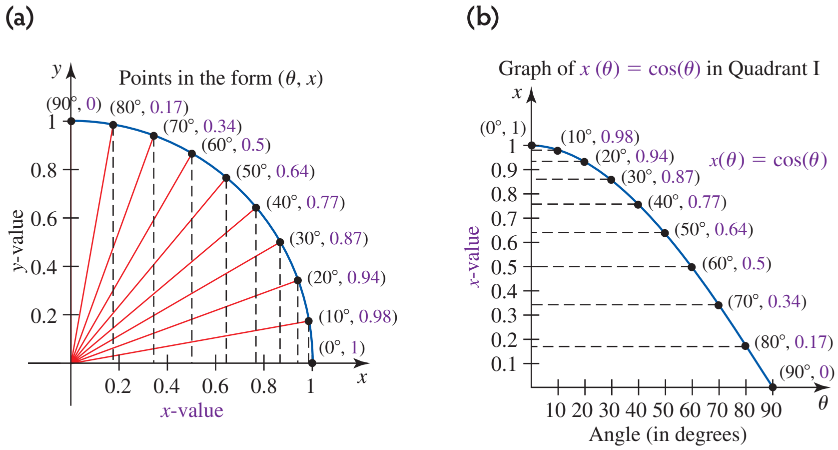


Figure 8.57

Looking closely at the function, we see that as θ changes, the x-value is not changing as much for angles close to 0° as it is for angles closer to 90°. To see why, let's examine what happens as we move around the circle using 10° intervals of θ in the first quadrant, as illustrated in Figure 8.58.

We see that the same change in the angle measurement yields a greater change in the horizontal component of each coordinate point as the angle measure approaches 90° and a greater change in the vertical component of each coordinate point when the angle is closer to 0°.

We follow the same process to graph $y(\theta) = \sin(\theta)$ from the first quadrant of the unit circle. See Figures 8.59a and 8.59b.

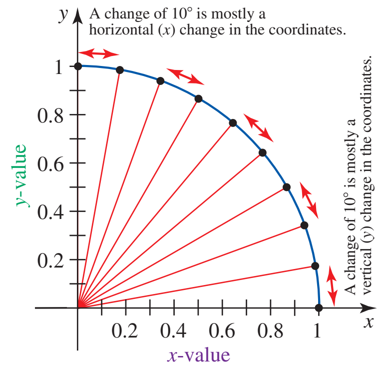


Figure 8.58

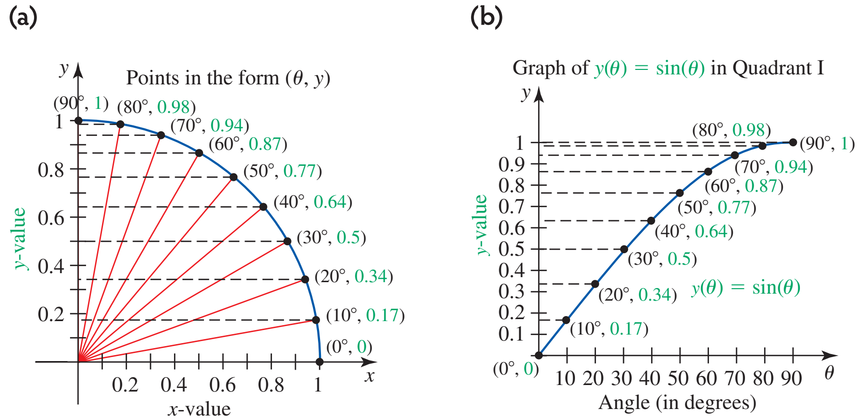


Figure 8.59

To complete the graphs of $x(\theta) = \cos(\theta)$ and $y(\theta) = \sin(\theta)$, we apply the same reasoning as we continue around the unit circle. See Figures 8.60a–8.60d.

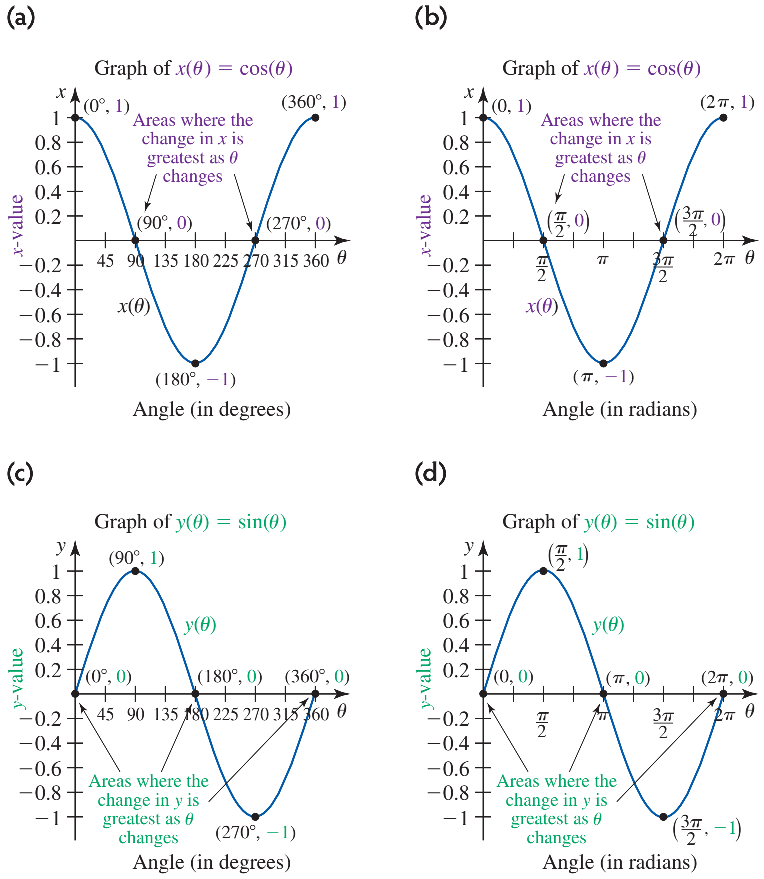


Figure 8.60

JUST IN TIME ■ INTERVAL NOTATION

The inequality of $a \le x \le b$ is represented as $[a, b]$ and the inequality $a < x < b$ is represented as (a, b) in interval notation. Similarly, $a \le x < b$ is represented as $[a, b)$ and $a < x \le b$ is represented as $(a, b]$. The interval $[a, b]$ is called a **closed** interval because it includes the endpoints. The interval (a, b) is called an **open** interval because it excludes the endpoints. Since interval notation is less cumbersome than inequality notation, it is frequently used.

The graphs of $x(\theta) = \cos(\theta)$ and $y(\theta) = \sin(\theta)$ in Figure 8.60 show the points corresponding with the interval [0°, 360°] and [0, 2π radians], respectively. But we know we may calculate cosine and sine for *any* angle, although all of the possible range values are shown in the graphs of Figure 8.60. Cosine and sine are *periodic* functions and these graphs show only one period for each. As the angle continues to increase past 360° or 2π radians, the values shown will repeat themselves every 360° or 2π radians. As θ decreases to be less than 0° or 0 radians, the values will also repeat, but in the opposite order as θ continues to decrease (moving left on the graph). This allows us to expand the graphs to include more periods, as shown in Figures 8.61a–8.61d.

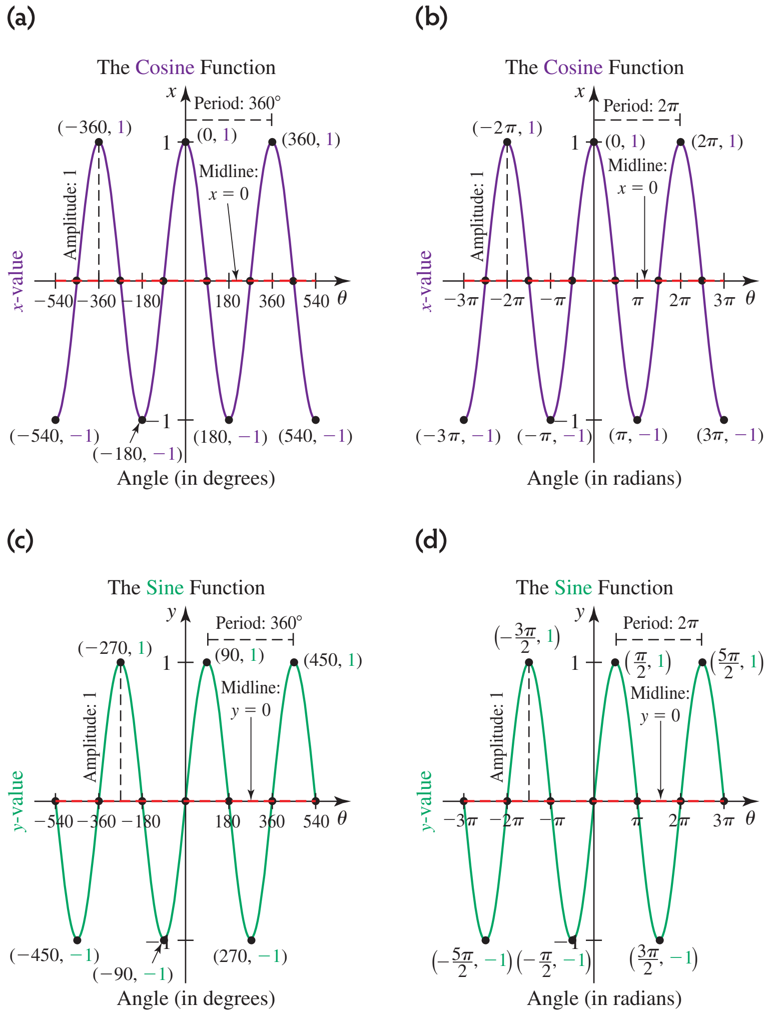


Figure 8.61

■ Characteristics of Cosine and Sine Functions

The characteristics of $\cos(\theta)$ and $\sin(\theta)$ are described in Table 8.8 and Figure 8.62.

Table 8.8

Function	Amplitude	Period	Frequency	Midline	Vertical Intercept	Horizontal Intercepts
$x(\theta) = \cos(\theta)$	1	$360°$ or 2π	$\frac{1}{360}$ or $\frac{1}{2\pi}$	$x = 0$	$x(0) = 1$	$\theta = 90° + 180°n$ or $\theta = \frac{\pi}{2} + \pi n$
$y(\theta) = \sin(\theta)$	1	$360°$ or 2π	$\frac{1}{360}$ or $\frac{1}{2\pi}$	$y = 0$	$y(0) = 0$	$\theta = 180°n$ or $\theta = \pi n$

A careful look at the graphs of cosine and sine shows they have identical patterns. The two functions are separated by a horizontal shift of one-quarter of a period. We will explore this idea in greater depth in the exercises at the end of this section.

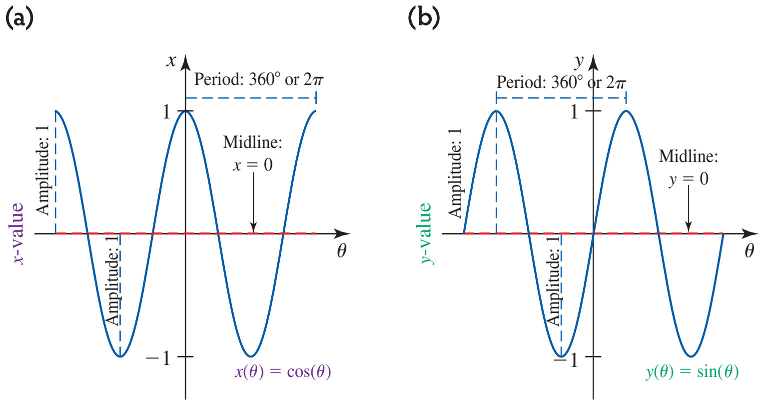


Figure 8.62

■ Transformations of Cosine and Sine Functions

The transformations of cosine and sine work just like the transformations we studied in Chapter 3.

EXAMPLE 1 ■ Shifting Cosine and Sine Functions

Graph each of the following functions.

a. $x(\theta) = \cos(\theta) + 2$; θ is in degrees.

b. $y(\theta) = \sin\left(\theta + \dfrac{2\pi}{3}\right)$; θ is in radians.

Solution

a. As shown in Figure 8.63, the graph of $x(\theta) = \cos(\theta) + 2$ is the graph of $x(\theta) = \cos(\theta)$ shifted upward 2 units. A vertical shift of the function changes the vertical position of the function but otherwise has no effect on the shape of the graph.

b. As shown in Figure 8.64, the graph of $y(\theta) = \sin\left(\theta + \dfrac{2\pi}{3}\right)$ is the graph of $y(\theta) = \sin(\theta)$ shifted left $\dfrac{2\pi}{3}$ radians.

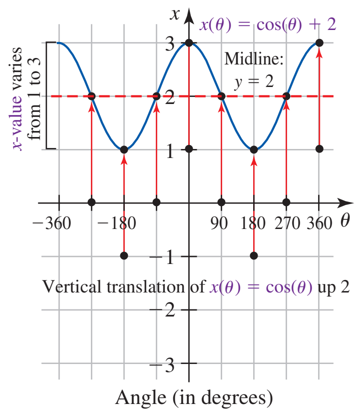


Figure 8.63

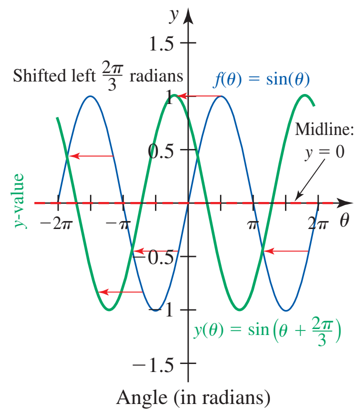


Figure 8.64

A horizontal shift of the function changes the horizontal position of the function but otherwise has no effect on the shape of the graph.

EXAMPLE 2 ■ Reflecting the Cosine and Sine Functions

Graph $f(\theta) = \sin(-\theta)$ and $g(\theta) = -\sin(\theta)$, then compare the graphs. Repeat for $h(\theta) = \cos(-\theta)$ and $j(\theta) = -\cos(\theta)$.

Solution Figure 8.65a shows that the graph of $f(\theta) = \sin(-\theta)$ is a horizontal reflection of the sine function. Figure 8.65b shows that the graph of $g(\theta) = -\sin(\theta)$ is a vertical reflection of the sine function.

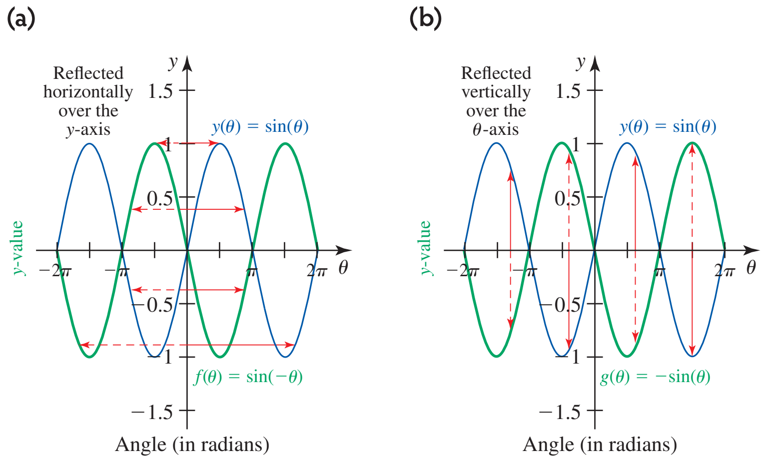


Figure 8.65

The graphs of $f(\theta) = \sin(-\theta)$ and $g(\theta) = -\sin(\theta)$ are identical, meaning that the sine function possesses *odd symmetry* (see Chapter 3).

The graph of $h(\theta) = \cos(-\theta)$ is a horizontal reflection of the cosine function, and the graph of $j(\theta) = -\cos(\theta)$ is a vertical reflection of the cosine function. However, since the cosine function is symmetric about the vertical axis, a horizontal reflection will not change the graph's appearance. Thus, the cosine function possesses *even symmetry* (see Chapter 3). A vertical reflection does change the appearance of the cosine function, as shown in Figure 8.66.

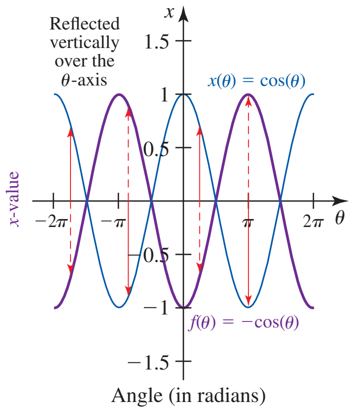


Figure 8.66

EXAMPLE 3 ■ Vertically Stretching and Compressing Cosine and Sine Functions

Graph each of the following functions. The angle θ is in degrees.

a. $x(\theta) = 5\cos(\theta)$

b. $y(\theta) = \dfrac{2}{5}\sin(\theta)$

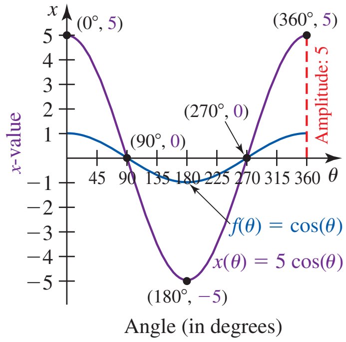


Figure 8.67

Solution

a. Figure 8.67 shows that the graph of $x(\theta) = 5\cos(\theta)$ is the cosine function vertically stretched by a factor of 5.

b. Figure 8.68 shows that the graph of $y(\theta) = \frac{2}{5}\sin(\theta)$ is the sine function vertically compressed by a factor of $\frac{2}{5}$.

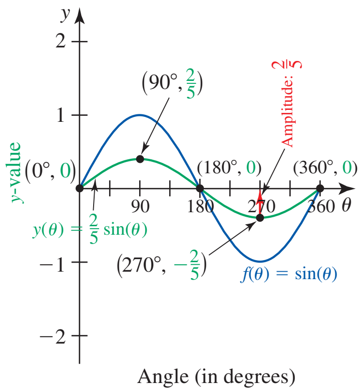


Figure 8.68

Therefore, vertical compressions and stretches affect the amplitude of a trigonometric function. The new amplitude of a vertically stretched cosine or sine function will be identical to the vertical stretch factor.

EXAMPLE 4 ■ Horizontally Stretching and Compressing Cosine and Sine Functions

Graph each of the following functions and interpret them with reference to the unit circle.

a. $x(\theta) = \cos(4\theta)$

b. $y(\theta) = \sin\left(\frac{1}{2}\theta\right)$

Solution

a. The graphs (in degrees and in radians) are shown in Figures 8.69a and 8.69b. Since 4θ changes 4 times as much when compared to θ, $x(\theta) = \cos(4\theta)$ will have a period one-fourth of 360°. Thus, θ must only change from 0° to 90° for 4θ to change from 0° to 360°, making the period of $x(\theta) = \cos(4\theta)$ 90°, or $\frac{360°}{4}$. In radians, the period is $\frac{\pi}{2}\left(\text{or } \frac{2\pi}{4}\right)$. The "4" in $x(\theta) = \cos(4\theta)$ is called the **angular frequency**—the number of periods the function will complete as θ changes by 360° or 2π radians.

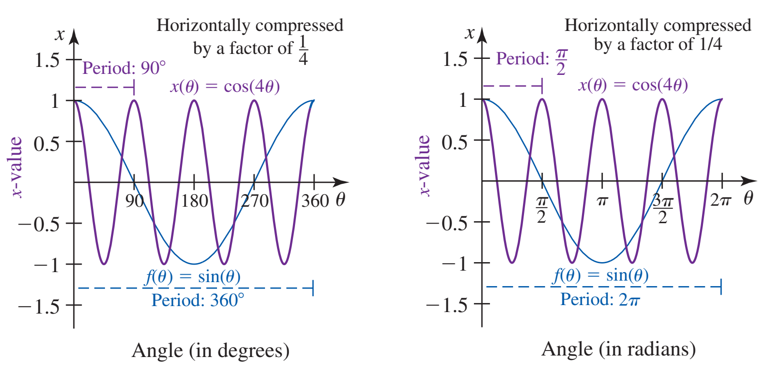


Figure 8.69

b. The graphs (in degrees and in radians) are shown in Figures 8.70a and 8.70b. The measure $\frac{1}{2}\theta$ changes only half as much as θ, meaning that θ will have to change twice as much for $y(\theta) = \sin\left(\frac{1}{2}\theta\right)$ to complete one period of sine than for $f(\theta) = \sin(\theta)$. Specifically, θ must change from 0° to 720° for $\frac{1}{2}\theta$ to change from 0° to 360°, making the period of $y(\theta) = \sin\left(\frac{1}{2}\theta\right)$ equal 720° or $\frac{360°}{1/2}$. In radians, the period is $4\pi\left(\text{or } \frac{2\pi}{1/2}\right)$. Furthermore, the angular frequency is $\frac{1}{2}$, telling us that the function will complete half of a period as θ changes by 360° or 2π radians.

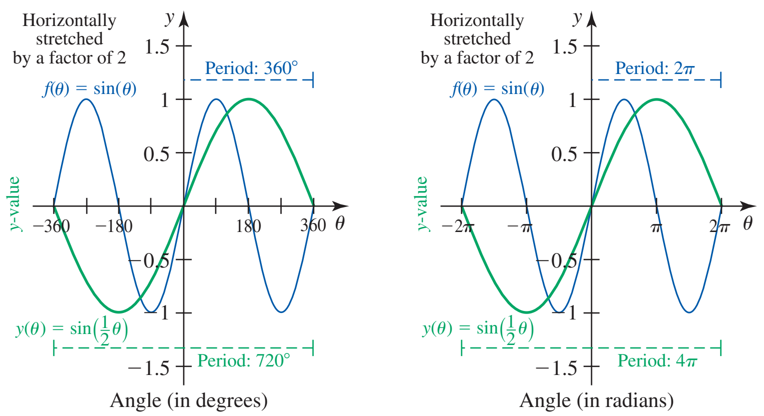


Figure 8.70

Example 4 demonstrates that horizontal stretches and compressions impact the period of a trigonometric function.

We summarize all of the above observations as follows.

Transformed Cosine and Sine Functions

For functions $x(\theta) = A\cos(B(\theta - C)) + D$ or $y(\theta) = A\sin(B(\theta - C)) + D$, we generalize the graphical meaning of their parameters as follows.

- $|A|$, the vertical stretch factor, is the *amplitude* of the graph. If $A < 0$, then the function is also reflected vertically.
- $|B|$ is the *angular frequency*, which is the number of periods that will be completed in the interval [0°, 360°] or [0, 2π radians] for θ. If $B < 0$, then the function is also reflected horizontally.
- $\frac{360°}{|B|}$ or $\frac{2\pi}{|B|}$ radians is the *period* of the function, the length of the interval required to complete one complete cycle.
- $\frac{|B|}{360°}$ or $\frac{|B|}{2\pi}$ radians is the *frequency* of the function, the fraction of a period completed within a one-unit interval.
- $|C|$ is the *horizontal shift* of the function. If C is positive, the shift is to the right. If C is negative, the shift is to the left.
- $|D|$ is the *vertical shift* of the function. If D is positive, the shift is upward. If D is negative, the shift is downward. The midline is located at $x = D$ or $y = D$.

EXAMPLE 5 ■ Multiple Transformations of Cosine and Sine Functions

Graph each of the following functions.

a. $y(\theta) = -\sin\left(2\left(\theta + \frac{\pi}{6}\right)\right) - 5$; θ is measured in radians.

b. $x(\theta) = \frac{4}{3}\cos\left(\frac{1}{4}(\theta - 180°)\right)$; θ is measured in degrees.

Solution

a. We create the graph for $y(\theta) = \overset{3}{-}\sin\left(\overset{1}{2}\left(\theta \overset{2}{+} \frac{\pi}{6}\right)\right) \overset{4}{-} 5$ by performing four transformations on $f(\theta) = \sin(\theta)$ in the following order: (1) a horizontal compression by a factor of $\frac{1}{2}$, (2) a horizontal shift left $\frac{\pi}{6}$, (3) a vertical reflection, and (4) a shift downward 5 units. See Figures 8.71a–8.71d.

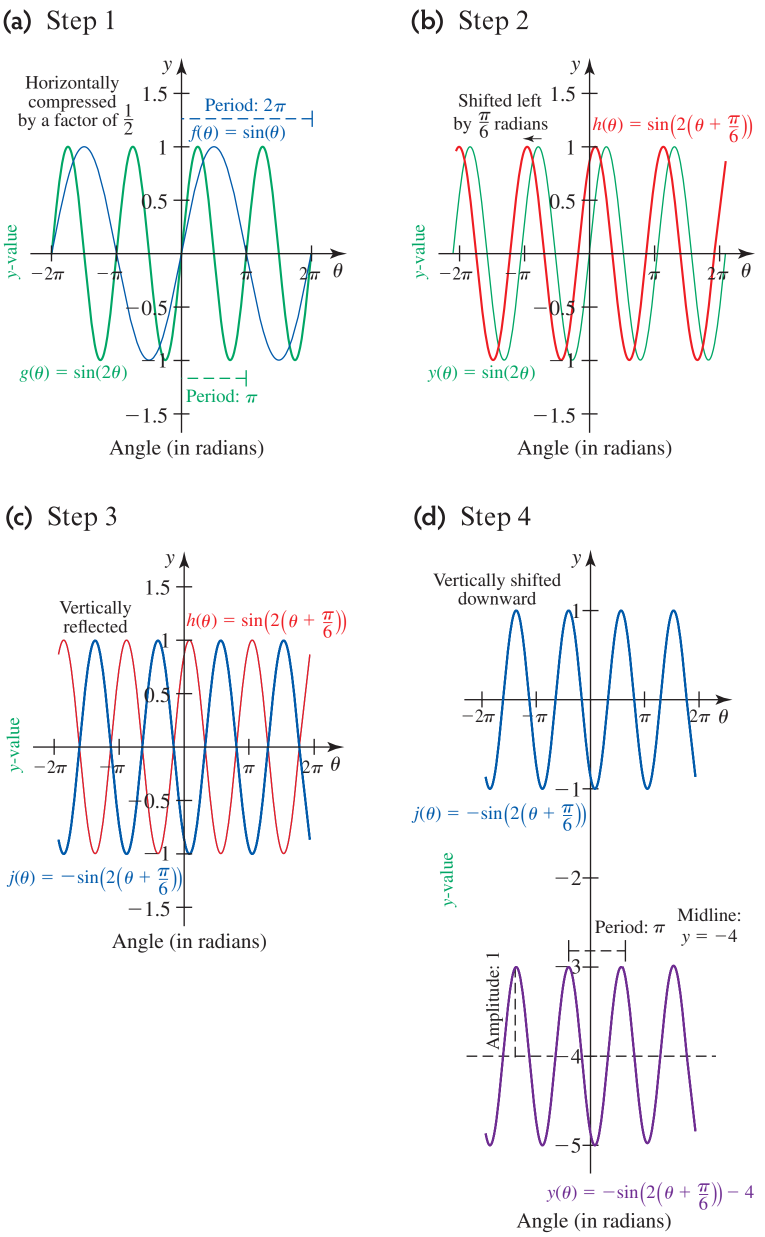


Figure 8.71

b. We create the graph for $x(\theta) = \overset{3}{\frac{4}{3}} \cos\left(\overset{1}{\frac{1}{4}}(\theta - \overset{2}{180^\circ})\right)$ by performing three transformations on $f(\theta) = \cos(\theta)$ in the following order: (1) a horizontal stretch by a factor of 4, (2) a horizontal shift to the right 180°, and (3) a vertical stretch by a factor of $\frac{4}{3}$. See Figures 8.72a–8.72d.

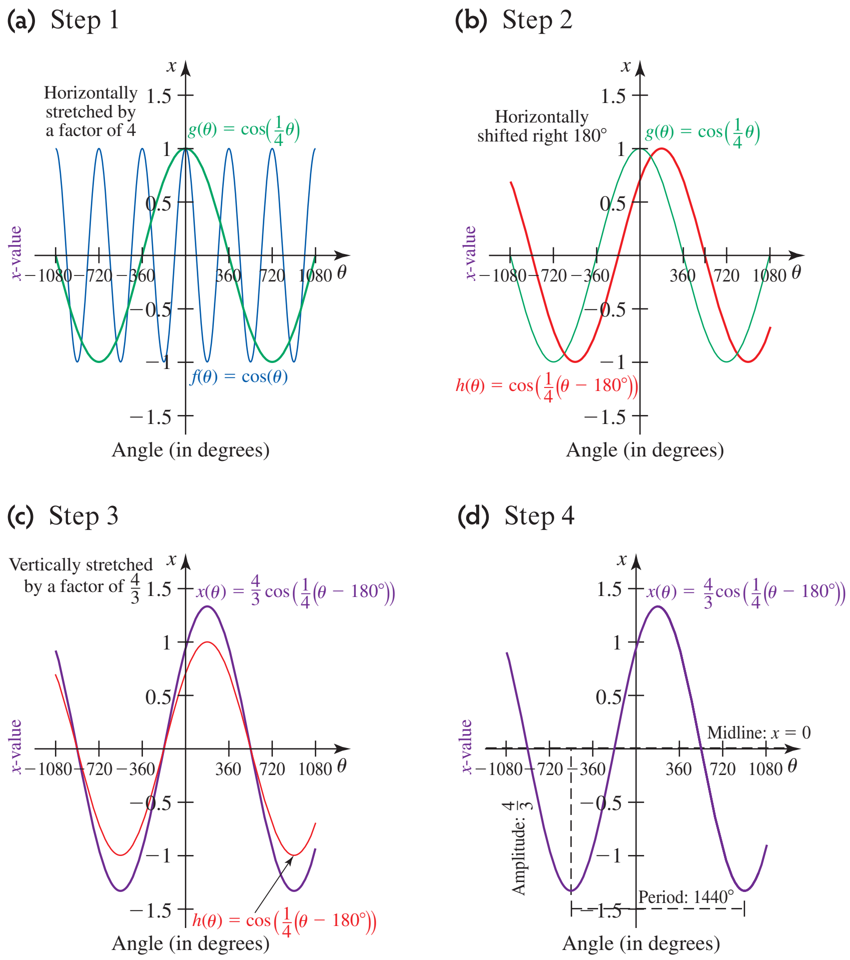


Figure 8.72

■ Phase Shifts

In Example 5, we saw that the horizontal shifts of a function are apparent when the input to the cosine or sine function is written in factored form, such as $2\left(\theta + \frac{\pi}{6}\right)$ in $y(\theta) = -\sin\left(2\left(\theta + \frac{\pi}{6}\right)\right) - 5$ or $\frac{1}{4}(\theta - 180)$ in $x(\theta) = \frac{4}{3}\cos\left(\frac{1}{4}(\theta - 180)\right)$. Writing these inputs in expanded form "hides" the horizontal shift, but it gives us another piece of information: the **phase shift**.

> **PHASE SHIFT**
>
> The **phase shift** is the *portion of one period* by which the function is shifted horizontally instead of the *amount* it is shifted.

Consider $x(\theta) = \frac{4}{3}\cos\left(\frac{1}{4}(\theta - 180)\right)$. Distributing the $\frac{1}{4}$ yields $x(\theta) = \frac{4}{3}\cos\left(\frac{1}{4}(\theta - 45)\right)$. The phase shift is 45°, which means that shifting the graph right 180° is equivalent to shifting the graph right $\frac{45}{360}$ of one period. Reducing the fraction tells us that this is a translation of $\frac{1}{8}$ of one period. See Figure 8.73.

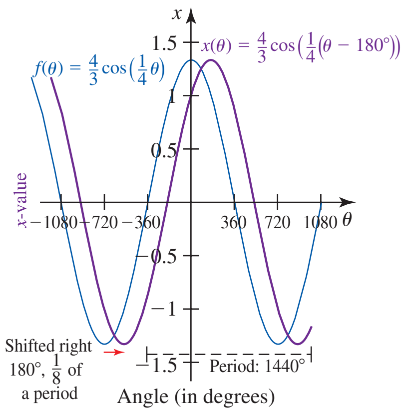


Figure 8.73

With $y(\theta) = -\sin\left(2\left(\theta + \frac{\pi}{6}\right)\right) - 5$ in radians the same reasoning applies. Distributing the 2 inside of sine yields $y(\theta) = -\sin\left(2\theta + \frac{\pi}{3}\right) - 5$. The phase shift is $\frac{\pi}{3}$. This tells us that we need to shift the graph horizontally a portion of one period equal to $\frac{\pi/3}{2\pi}$.

$$\begin{aligned}\frac{\pi/3}{2\pi} &= \frac{\pi}{3}\cdot\frac{1}{2\pi}\\ &= \frac{\pi}{6\pi}\\ &= \frac{1}{6}\end{aligned}$$

Thus, shifting the graph left by $\frac{\pi}{6}$ means that we are shifting it left $\frac{1}{6}$ of one period.

> **DETERMINING PHASE SHIFT**
>
> For the functions $x(\theta) = A\cos(B(\theta - C)) + D$ or $y(\theta) = A\sin(B(\theta - C)) + D$, the **phase shift** is $|BC|$. The value $\frac{|BC|}{360°}$ or $\frac{|BC|}{2\pi}$ tells us the portion of one period that the graph has been shifted horizontally.

■ Solving Trigonometric Equations by Graphing

So far we have worked with calculating cosine and sine values when we are given the angle. With the ability to graph these functions, we may now find the angle given the values of cosine and sine.

EXAMPLE 6 ■ Solving Trigonometric Equations by Graphing

Solve each of the following equations for θ values over the interval $[0, 2\pi]$ by graphing. Then interpret the solution.

a. $\cos(\theta) = 0.33$

b. $3\sin(2\theta) = 2.15$

Solution

a. Using the Technology Tip at the end of this section, we first verify that the calculator is in Radian mode. To solve this equation by graphing, we need to graph two functions, $y = \cos(x)$ and $y = 0.33$, and find their intersection point(s). Since we only want angles between 0 and 2π, we adjust our viewing window accordingly to get the results shown in Figure 8.74.

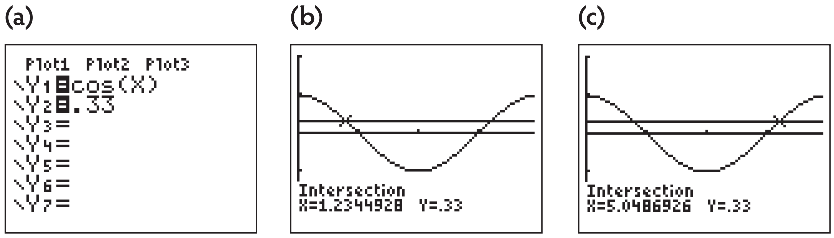


Figure 8.74

Angles measuring 1.2345 radians and 5.0487 radians have cosine values equal to 0.33.

b. We graph the equations $y = 3\sin(2x)$ and $y = 2.15$ and find their intersection point(s) as shown in Figure 8.75. Notice, however, that the sine function is being horizontally compressed by $\frac{1}{2}$. This means that as θ changes from 0 to 2π we will have two periods of sine, giving us four solutions to this equation on the interval $[0, 2\pi]$.

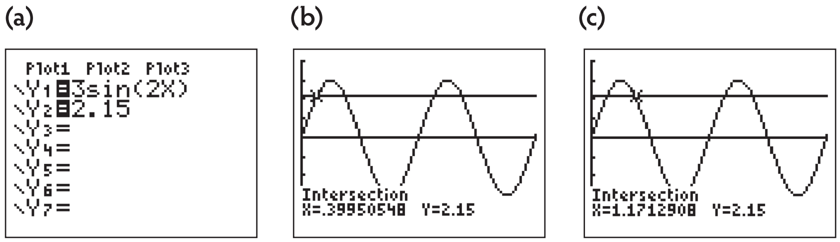


Figure 8.75

We can follow the same process to find the remaining two solutions, or we can simply add π to the first two solutions since the remaining solutions will be in the same positions in the second period. Our results tell us that a vertical position of 2.15 occurs when θ is 0.3995, 1.1713, 3.5411, and 4.3129 radians.

Modeling with Cosine and Sine Functions

As we saw earlier in this chapter, there are many situations in the real world that display periodic behavior. Cosine and sine functions can be used to model many of these situations.

EXAMPLE 7 ■ Using a Sine Model in a Real-World Context

The function $H(d) = 3.78\sin\left(\frac{2\pi}{365}d\right) + 12.2$ models the number of hours of daylight in Seattle d days after March 21, 2010. Use this information to answer the following questions. (*Hint:* Be sure to use radian mode on your calculator.) (*Source:* Modeled from data at aa.usno.navy.mil)

a. What is the period, the amplitude, and the equation of the midline? What do these represent in the real-world context?

b. How many hours of daylight are there on March 21? March 1?

c. Graph H over the interval $-80 \le t \le 285$ (the data for all of 2010). Describe the rate of change and concavity of the function and explain what this information tells you about this situation.

d. What day(s) of the year will have 15 hours of daylight?

Solution

a. This is a sine function being stretched horizontally by $\frac{365}{2\pi}$. So, while the original period was 2π, the new period will be $\frac{365}{2\pi}$ times as much, or 365 days $\left(\frac{365}{2\pi} \cdot 2\pi = 365\right)$. Alternatively, we could use the fact that $\frac{2\pi}{|B|}$ will calculate the period.

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{2\pi/365} \\ &= 2\pi \cdot \frac{365}{2\pi} \\ &= 365\end{aligned}$$

This tells us that the cycle will repeat every 365 days.

The equation of the midline is $H = 12.2$ and the amplitude is 3.78 hours. We use these to find the greatest and least number of hours of daylight.

$$\begin{array}{cc} 12.2 + 3.78 & 12.2 - 3.78 \\ 15.98 & 8.42 \end{array}$$

Seattle has at most 15.98 hours and at least 8.42 hours of daylight in a day.

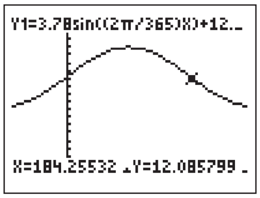
b. On March 21, $d = 0$, and on March 1, $d = -20$.

$$\begin{aligned} H(0) &= 3.78 \sin\left(\frac{2\pi}{365}(0)\right) + 12.2 \\ &= 3.78 \sin(0) + 12.2 \\ &= 0 + 12.2 \\ &= 12.2 \text{ hours} \end{aligned}$$

$$\begin{aligned} H(-20) &= 3.78 \sin\left(\frac{2\pi}{365}(-20)\right) + 12.2 \\ &= 3.78 \sin(-0.3443) + 12.2 \\ &= -1.276 + 12 \\ &\approx 10.9 \text{ hours} \end{aligned}$$

c. The graph and viewing window specifications are shown in Figure 8.76. From $t = -80$ to $t = 0$ the graph appears to be concave up and increasing, meaning that the number of daylight hours is increasing, and increasing by more and more for each day that passes. After $t = 0$ the function becomes concave down. The number of daylight hours is still increasing, but now by smaller and smaller increments each day. After $t = 92$ or 93 (around June 21), the number of daylight hours begins to decrease, and it does so by larger and larger amounts each day until about $t = 176$ (around September 21) when the graph becomes concave up, meaning that the number of daylight hours decreases by smaller amounts each day.

(a)



(b)

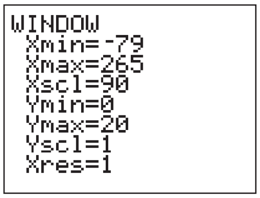


Figure 8.76

d. As shown in Figure 8.77, we graph the equation $y = 15$ and find the intersection of this graph and function H. We should expect to find two solutions since we have graphed one period of the function and we know that for one period of cosine or sine there are two domain values that give us the same cosine or sine value.

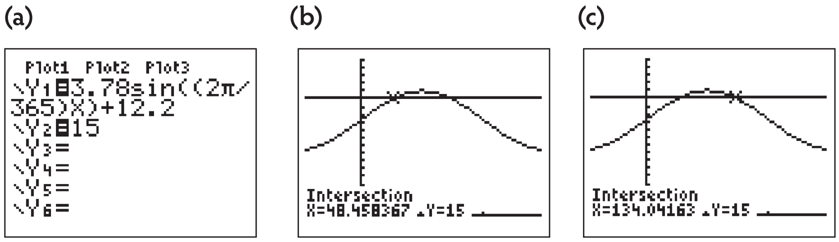


Figure 8.77

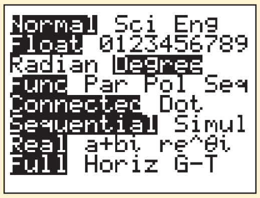
There are about 15 hours of daylight on two days during the year: $t \approx 48$ (May 8) and $t \approx 134$ (August 2). ■

SUMMARY

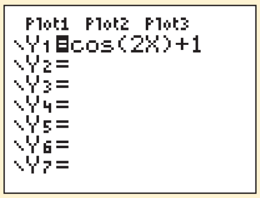
In this section you learned how to graph the cosine and sine functions and transformations of these functions. You also learned how to solve trigonometric equations using a graph. Additionally, you discovered that cosine and sine functions can be used to model real-world situations.

TECHNOLOGY TIP ■ GRAPHING COSINE AND SINE

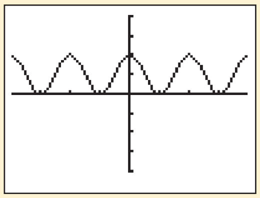
1. Press MODE to verify that your calculator is in the proper units (Radian or Degree) according to the angle given.



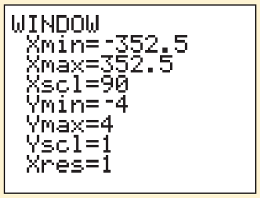
2. Use Y = to enter the sine or cosine function you want to graph using the COS or SIN button, as appropriate. The calculator will open a set of parentheses for you after pressing COS or SIN. Be sure to close the set of parentheses at the appropriate time.



3. Under the ZOOM menu, select 7:ZTrig to create a graph using a window size that fits most basic trigonometric functions.



4. Press WINDOW and adjust the maximum and minimum values to see more or less of the graph.



8.4 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–7, answer all of the following questions for each function.

- **a.** What is the period, the amplitude, the frequency, and the equation of the midline?
- **b.** What are the maximum and minimum values of f?
- **c.** Graph the function without using a calculator. Make sure to graph at least one complete period.
- **d.** Where are the horizontal intercepts (if they exist) located?
- **e.** Where is the vertical intercept located?
- **f.** Describe the transformation of the function as related to the graph of $f(\theta) = \sin(\theta)$ or $f(\theta) = \cos(\theta)$, as appropriate.

1. $f(\theta) = 4\cos(\theta)$; θ is in degrees.

2. $f(\theta) = \cos(2\theta)$; θ is in radians.

3. $f(\theta) = \sin\left(\frac{1}{4}\theta\right)$; θ is in degrees.

4. $f(\theta) = \cos(\theta - 120°)$; θ is in degrees.

5. $f(\theta) = 3\sin\left(\theta + \frac{\pi}{3}\right)$; θ is in radians.

6. $f(\theta) = -0.75\cos\left(\frac{1}{3}(\theta + 30°)\right)$; θ is in degrees.

7. $f(\theta) = -\sin\left(4\left(\theta - \frac{\pi}{6}\right)\right)$; θ is in radians.

8. Determine the phase shift for each function in Exercises 1–7, then interpret what the phase shift tells you about that function.

In Exercises 9–10,

- **a.** Determine the number of periods the function will complete on the interval $0 \le \theta \le 360°$.
- **b.** How many values of θ between 0° and 360° will make the equation $f(\theta) = 0.5$ true?
- **c.** Graph the function using a calculator. (*Hint:* Set your calculator to Degree mode and your window range to Xmin=0 and Xmax=360.) Find the solutions described in part (b).

9. $f(\theta) = \cos(2\theta)$

10. $f(\theta) = 2\sin(4(\theta + 200°)) - 1$

In Exercises 11–12,

- **a.** Determine the number of periods the function will complete on the interval $0 \le \theta < 2\pi$.
- **b.** How many values of θ between 0 and 2π radians will make the equation $f(\theta) = 0.25$ true?
- **c.** Graph the function using a calculator. (*Hint:* Set your calculator to Radian mode and your window range to Xmin=0 and Xmax=2π.) Find the solutions described in part (b).

11. $f(\theta) = \sin(3\theta)$

12. $f(\theta) = 4\cos(2(\theta + \pi))$

In Exercises 13–18, state whether or not each of the following equations is true. Justify your conclusion.

13. $-\cos(\theta) = \cos(-\theta)$

14. $-\sin(\theta) = \sin(-\theta)$

15. $\cos(\theta) = \cos(-\theta)$

16. $\sin(\theta) = \sin(-\theta)$

17. $\cos(\theta - \pi) = \sin\left(\theta + \frac{3\pi}{2}\right)$

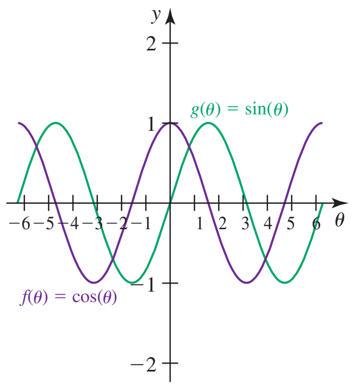
18. $\cos(\theta) = -\sin(\theta - 90°)$

In Exercises 19–20, calculate the average rate of change of the function from $\theta = \frac{\pi}{8}$ to $\theta = \frac{\pi}{6}$.

19. $f(\theta) = 2\sin(\theta) + 1$

20. $f(\theta) = 3\sin(3\theta)$

In Exercises 21–24, use the graph of $f(\theta) = \cos(\theta)$ and $g(\theta) = \sin(\theta)$ to sketch graphs of the each of the following functions.



21. $f(\theta) + g(\theta)$

22. $f(\theta) - g(\theta)$

23. $(f \circ g)(\theta)$

24. $(g \circ f)(\theta)$

25. Which of the functions in Exercises 21–24 are periodic?

26. We know $f(\theta) = \cos(\theta)$ has even symmetry and that $g(\theta) = \sin(\theta)$ has odd symmetry. Do any of the functions in Exercises 21–24 possess even or odd symmetry?

27. Consider the function $f(\theta) = \cos(\theta) + \theta$, where θ is in radians.

- **a.** Create a table and graph of f.
- **b.** Does this function have a midline or amplitude?
- **c.** Is this function periodic? Justify your conclusion.

SHOW YOU KNOW

28. Explain the similarities and differences between the graphs of the cosine and sine functions.

29. Explain why horizontally or vertically shifting the cosine or sine function does not change the period.

30. Explain why changes to the period of a cosine or sine function will never affect the amplitude.

31. A classmate tells you that it is never necessary to use the cosine function—that *every* cosine function may instead be written as a shifted sine function. Do you agree or disagree? Explain.

MAKE IT REAL

In Exercises 32–34, use the following information. Seasonal employment is a major part of the economy in many places. The function

$$L(m) = -550\cos\left(\frac{\pi}{6}(m-1)\right) + 3300$$

models the number of people employed in leisure and hospitality fields in Clatsop County, Oregon, m months after January 2004. (Hint: Be sure to use Radian mode on your calculator.) (***Source:*** **Modeled using estimates from a graph at www.qualityinfo.org**)

32. Seasonal Employment Find the amplitude, period, equation of the midline, and phase shift for L. Then explain what each tells about this situation.

33. Seasonal Employment

a. During what month are the fewest number of people employed in these industries in Clatsop County?

b. During what month are the most people employed?

c. What are the fewest and most employees working in this field during the year?

d. Give a possible explanation for why employment in these industries follows this pattern.

34. Seasonal Employment Graph L over the interval $0 \le m \le 12$. Describe its rate of change and concavity throughout this interval and explain what this tells about employment in these industries throughout the year.

In Exercises 35–40, use the table showing the average temperature in Indianapolis, Indiana. Average temperatures are calculated by taking the average temperature for each day over a 24-hour period, and then finding the average of the daily temperatures for each month. (Hint: Be sure to use Radian mode on your calculator.)

Month of the Year (1 = January) m	Average Temperature (°F) $T(m)$
1	29.8°
2	33.0°
3	41.9°
4	53.3°
5	62.8°
6	71.3°
7	75.3°
8	74.4°
9	66.9°
10	55.1°
11	43.6°
12	32.8°

(*continued*)

Source: **www.engr.udayton.edu**

35. Average Temperatures

a. Plot the data for Indianapolis's average monthly temperature.

b. Explain why a cosine or sine function will better model the data than a polynomial function.

c. Determine the period, amplitude, and equation of the midline from the table or graph.

Peter Igel/Shutterstock.com

36. Average Temperatures

a. Graph the function

$$T(m) = -22.75\cos\left(\frac{\pi}{6}(m-1)\right) + 52.55$$

together with the points in the table on the same set of axes.

b. Is this model a good fit for the data?

c. State the period, amplitude, and equation of the midline of the function. Then compare these with your answers from Exercise 35, part (c).

d. Describe the rate of change and concavity of the function, then explain what this tells about the average temperatures in Indianapolis throughout the year.

e. Explain why a negative cosine function was used to model this situation.

37. Average Temperatures Using the function $T(m)$ given in Exercise 36,

a. Evaluate $T(10)$ and explain the meaning of the value in this context.

b. During what month is the average temperature the greatest? What is the greatest average temperature?

c. During what month is the average temperature the lowest? What is the lowest average temperature?

d. What does the value $T(2.5)$ tell you? Find this value.

38. Average Temperatures The function

$$H(m) = -18.20\cos\left(\frac{\pi}{6}(m-1)\right) + 60.46$$

models the average temperatures in degrees Fahrenheit in Huntsville, Alabama, during month m of the year. Using the function $T(m)$ given in Exercise 37, graph H and T together using a calculator. Describe the similarities and differences between the two functions.

39. Average Temperatures The function

$$S(m) = 9.40\cos\left(\frac{\pi}{6}(m-1)\right) + 57.8$$

models the average temperatures in degrees Fahrenheit in Sydney, Australia, during month m of the year. (***Source:*** **Modeled from data at www.engr.udayton.edu)**

a. Graph H (from Exercise 38) and S together using a calculator.

b. Describe the similarities and differences between these two functions.

c. Why was a positive cosine function used to model S?

d. What accounts for the differences between the average temperatures in these locations?

e. What is the phase shift in S? Interpret this value.

f. Suppose we wanted to use a negative cosine function to model S instead. Fill in the blank to create a function that models the average temperatures in Sydney.

$$S(m) = -9.40\cos\left(\frac{\pi}{6}(\_\_\_\_\_)\right) + 57.8$$

40. Average Temperatures Solve the following equations by graphing using the appropriate functions from Exercises 35–39. Then interpret the solutions.

a. $T(m) = 45$

b. $H(m) = 75$

c. $S(m) = 58$

d. $S(m) = 80$

In Exercises 41–47, use the following information. The function

$$M(t) = 50\cos\left(\frac{2\pi}{29.53}(t)\right) + 50$$

models the percentage M of the full moon visible t days after January 1, 2010. (Hint: Be sure to use Radian mode on your calculator. (***Source:*** **Modeled from data at http://aa.usno.navy.mil)**

41. Phases of the Moon What are the smallest and largest percentages of the full moon visible? Explain how these values are represented in the formula.

42. Phases of the Moon What percentage of the full moon was visible on each of the following dates?

a. January 5

b. January 16

c. February 5

43. Phases of the Moon Function M is created by transforming the function $f(t) = \cos(t)$. However, it was not necessary to horizontally shift the cosine function to create M. What does this tell you?

44. Phases of the Moon What is the period of M? Interpret this value in the context of the situation.

45. Phases of the Moon How many periods will be completed over the interval $0 \le t \le 119$ (the first four months of the year)? How many days during this interval will 75% of the full moon be visible? Explain.

46. Phases of the Moon

a. Graph M over the interval $0 \le t \le 119$.

b. Find the days during this interval where 75% of the full moon will be visible.

c. Describe the rate of change and concavity over the interval $0 \le t \le 15$ and explain what this information means in the context of the situation.

47. Phases of the Moon In the year 2020, the first full moon will occur on January 10. Write the function $P(d)$, the percentage of the full moon visible d days after January 1, 2020 and explain how this function is a transformation of M.

In Exercises 48–49, use the following information. If you have traveled to a foreign country, you may have discovered that different countries use different standards for the voltage and frequency of their alternating current (AC) power supply. In an AC system, the current changes direction, creating positive and negative voltages according to the sine formula

$$V(t) = P\sin(2\pi f t)$$

where V is the voltage, P is the peak voltage, f is the frequency, and t is time in seconds.

48. Alternating Current The United States uses a peak voltage of 120 volts with a frequency of 60 cycles per second.

a. Find the function for the voltage of the alternating current in the United States.

b. What is the period of this function? What does this tell you?

c. How many times per second will the voltage be 0.5? Explain how you found this answer.

d. Graph one period of this function, then use your graph to find the time(s) when the voltage will be 50.

49. Alternating Current The United Kingdom uses a peak voltage of 240 volts with a frequency of 50 cycles per second.

a. Explain how you could create the function for the United Kingdom's voltage by performing transformations on the function created in Exercise 48. Then write the function.

b. What is the period of the United Kingdom's voltage function? What does this tell you?

c. How many times per second will the voltage be 0.5? Explain how you found this answer.

d. Graph one period of this function, then use your graph to find the time(s) when the voltage will be 50.

STRETCH YOUR MIND

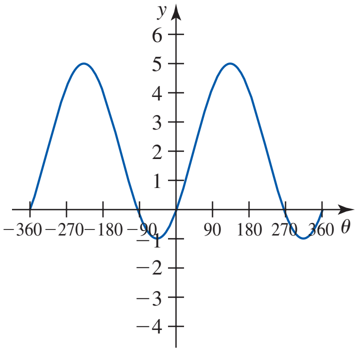
Exercises 50–55 are intended to challenge your understanding of graphing cosine and sine functions.

50. It can be said that $f(\theta) = \sin(\theta)$ will complete one full cycle (period) whenever θ changes by 360° or 2π radians. Using this idea, answer the following questions.

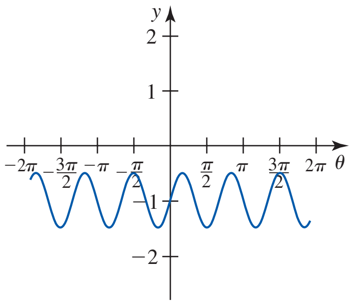
a. Describe the behavior of $h(x) = \sin(x^2)$ for $x > 0$. Then make a rough sketch of the graph of h without using a calculator. Is this function periodic? Explain.

b. Describe the behavior of $j(x) = \sin\left(\frac{1}{x}\right)$ for $x > 0$. Then sketch a rough graph of j without using a calculator. Is this function periodic? Explain.

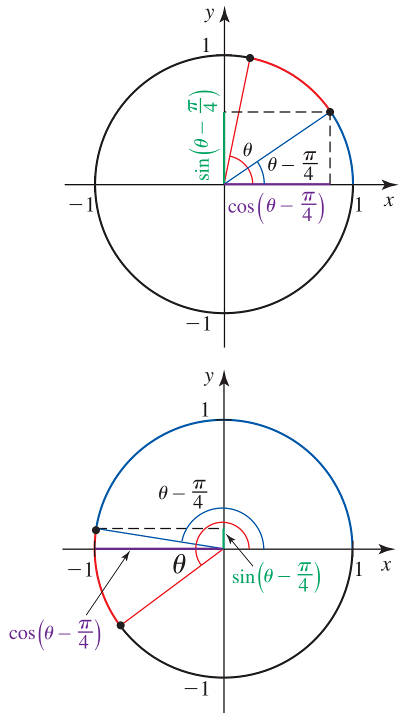
51. Create two functions—one using cosine and the other using sine—that have the following graph if θ is in degrees.



52. Create two functions—one using cosine and the other using sine—that have the following graph if θ is in radians.

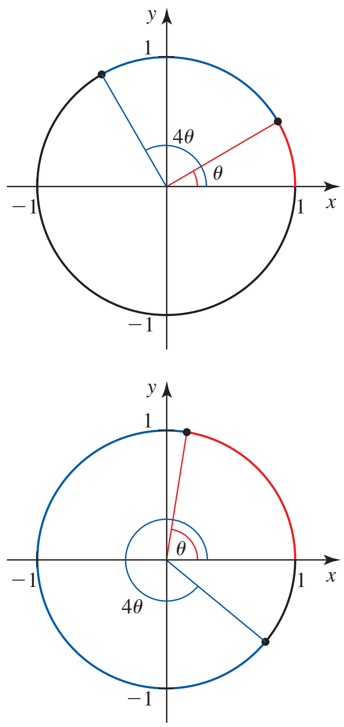


53. The following graphs show how to find $\cos\left(\theta - \frac{\pi}{4}\right)$ and $\sin\left(\theta - \frac{\pi}{4}\right)$ using the unit circle.



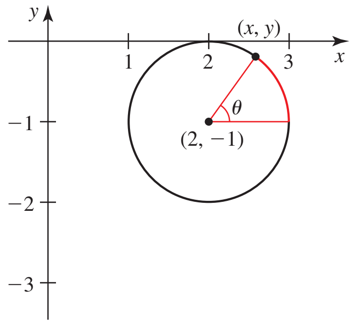
Explain, using these figures, why the graphs of $x(\theta) = \cos\left(\theta - \frac{\pi}{4}\right)$ and $y(\theta) = \sin\left(\theta - \frac{\pi}{4}\right)$ will be shifts of the original cosine and sine functions and what the new graphs represent with respect to the unit circle.

54. The following figures show the relationship between θ and 4θ.



Explain, using these figures, why the graphs of $x(\theta) = \cos(4\theta)$ and $y(\theta) = \sin(4\theta)$ will be horizontal compressions of the original cosine and sine functions and what the new graphs represent with respect to the unit circle.

55. The following diagram shows a circle of radius 1 centered at $(2, -1)$.



Draw the graphs of the x- and y-coordinates of the endpoints of arcs on this circle as functions of the angle θ. Explain what transformation(s) of the original cosine and sine functions are necessary to create these graphs.

SECTION 8.5 Modeling with Trigonometric Functions

LEARNING OBJECTIVES

- Generate equations for sine and cosine functions from tables and graphs
- Graph sine and cosine functions from equations and tables
- Use sine and cosine functions to model real-world data sets

GETTING STARTED

The Japanese word *tsunami* translates as "harbor wave" and is used internationally to refer to a series of waves (known as a *wave train*) traveling at hundreds of miles per hour across the ocean with extremely long wavelengths. While in the deep ocean, wave trains may have hundreds of miles between wave crests. As the waves approach shore, their speed decreases due to friction as they begin to "feel" the bottom. As the waves strike shore, they may flood low-lying coastal areas and cause massive destruction and significant loss of life, such as in Japan in March 2011. The behavior of tsunamis can be modeled and analyzed with sinusoidal functions (cosine and sine).

In this section we generate graphs and formulas for cosine and sine functions from verbal descriptions and tables of data. We also show how to use sinusoidal functions to model real-world situations to make predictions. In addition, we show how to solve sinusoidal equations using tables and graphs.

■ Sinusoidal Models

Tsunamis are generally the result of a sudden rise or fall of a section of the earth's crust under or near the ocean (in other words, an earthquake). This type of seismic disturbance can displace the water column, creating a rise or fall in the level of the ocean above. This rise or fall in sea level is the initial formation of a tsunami wave. (*Source:* www.tsunami.org)

On April 1, 1946, an earthquake with a reported magnitude of 7.8 on the Richter scale occurred in the Aleutian Islands off of the coast of Alaska. Almost 5 hours later, the largest and most destructive tsunami waves ever to strike the Hawaiian Islands penetrated more than half a mile inland in some areas. Between wave crests, the drawdown

Laupahoehoe Point Beach Park, today

(pull back of the ocean from the shoreline) exposed some areas of the sea floor 500 feet in the seaward direction.

No warning was given or possible for this tsunami. As a consequence, a total of 159 tsunami-related fatalities resulted from this single destructive event. In quick response to the 1946 tsunami tragedy, scientists developed a warning system that has detected nearly every Pacific-wide tsunami since. (*Source:* www.tsunami.org)

1946 tsunami hitting Hawaiian Islands

EXAMPLE 1 ■ Generating a Sinusoidal Model from a Verbal Description

Tsunami waves generated by an earthquake of magnitude 7.8 on the Richter scale occur 10 to 20 minutes apart and travel at speeds greater than 500 miles per hour. The amount of time between successive wave crests, known as the wave period, varies from only a few minutes to more than an hour. When the April 1946 tsunami hit Laupahoehoe Point, Hawaii, it had a 27.3-foot amplitude (from normal level) and a wave period of 12 minutes. (*Source:* www.wcatwc.gov)

a. Assuming the height of the water surface above the sea floor at the base of the pier can be modeled with a sinusoidal function, generate a graph for the sine function model, given that the height was 12 feet before the tsunami hit Laupahoehoe Point. For the purposes of the graphical model, assume the water level at the pier first went down from its normal level, then rose an equal distance above its normal level, and finally returned to its normal level. Label the dependent and independent axes.

b. Write a function, $d(t)$, that models the situation.

Solution

a. We know that the height of the water surface above the sea floor at the pier is 12 feet initially. We represent this on the graph in Figure 8.78 by making 12 feet the vertical intercept at time 0 minutes. We also know the amplitude of the waves is 27.3 feet. We use this figure to find the maximum and minimum values that the water surface height reaches.

$= 12 + 27.3$ $\qquad\qquad$ $= 12 - 27.3$

$= 39.3$ feet **maximum water surface height** $\qquad$ $= -15.3$ feet **minimum water surface height**

We were told the water level went down before returning to its normal level and then rose an equal amount. This tells us the graph decreases first 27.3 feet before increasing to 27.3 feet above normal level.

We also know the wave period is 12 minutes. This is shown on the graph by making the graph return to 12 feet (the initial water surface height) after 12 minutes. Finally, because it is stated that the wave is sinusoidal, we draw a negative sine curve.

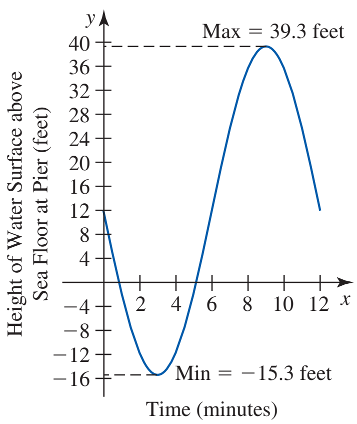


Figure 8.78

b. In the prior section, we learned about transforming the sine functions using the standard equation $y(\theta) = A\sin(B(\theta - C)) + D$. The value of A is -27.3 feet because the amplitude of the tsunami is 27.3 feet. We make it negative because the water receded first from the pier.

We know the tsunami had a wave period of 12 minutes. We use this information to find B.

$$\frac{2\pi}{|B|} = \text{period}$$

$$\frac{2\pi}{|B|} = 12$$

$$\frac{2\pi}{12} = |B|$$

$$|B| = \frac{\pi}{6}$$

So $B = \pm\frac{\pi}{6}$. We choose $B = \frac{\pi}{6}$ since we only need a horizontal stretch, not a horizontal stretch and horizontal reflection. We do not need to shift the graph horizontally, so $C = 0$. The vertical shift is $D = 12$ because the normal water surface height at the pier is 12 feet. The sine function that models this tsunami is

$$\begin{aligned} d(t) &= -27.3\sin\left(\frac{2\pi}{12}(t-0)\right) + 12 \\ &= -27.3\sin\left(\frac{\pi}{6}(t)\right) + 12 \end{aligned}$$

EXAMPLE 2 ■ **Using a Sinusoidal Graph to Model Real-World Data**

The graph in Figure 8.79 models the water surface height above the ocean floor at the base of the pier in Laupahoehoe Point, Hawaii, during the April 1946 tsunami.

a. Evaluate $d(4)$ and explain what this means in the real-world context of this exercise.

b. Solve the equation $d(t) = 25$ for t. Find all values of t that occur between time 0 minutes and time 12 minutes.

c. According to the graph of $d(t)$, what will be the minimum water surface height? How do you interpret this answer in terms of what happened in the real world?

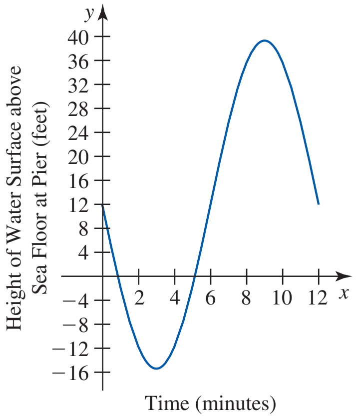


Figure 8.79

Solution

a. Using the graph in Figure 8.79 to evaluate $d(4)$, we find the input of 4 minutes on the horizontal axis and then locate the output value on the vertical axis. It appears that $d(4) \approx -12$. This means that at 4 minutes, the water surface height is 12 feet below the level of the ocean floor at the pier. Since the ocean floor slants away from the pier, the ocean floor is exposed between the pier and the ocean for some distance.

b. To solve the equation $d(t) = 25$, we draw the horizontal line at $y = 25$, as shown in Figure 8.80. This line intersects the graph in two places on the interval from 0 minutes to 12 minutes. We move vertically from the points of intersection to the horizontal axis and determine $t \approx 7$ and $t \approx 11$.

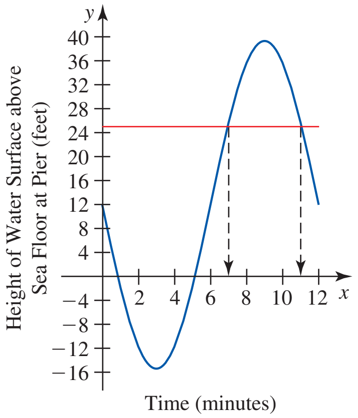


Figure 8.80

The water surface height is at 25 feet at approximately 7 minutes and 11 minutes into the first period.

c. From the graph of $d(t)$, we estimate the minimum water surface height is about -15 feet. This means the base of the pier is 15 feet *above* the water surface height. As a result, the ocean floor at the pier and some distance away from the pier is exposed. How much of the ocean floor between the pier and the ocean depends on the slope of the ocean floor, which is unknown to us.

EXAMPLE 3 ■ Generating a Sinusoidal Model from a Table of Data

The Willis Tower (formerly the Sears Tower) in Chicago was designed to be flexible enough to bend rather than break in strong winds, but rigid enough so people are comfortable working and living in it without feeling the movements of the building. (*Source:* www.911research.wtc7.net)

On a windy Chicago day (35-mph winds), the distance, s, that the top of the Willis Tower sways varies sinusoidally with the time, t, in seconds. For such days, engineers have designed the tower to sway a maximum distance of 12 inches from vertical in each direction. (*Source:* www.dupont.com) Table 8.9 gives hypothetical sway distances for specific times based on a sinusoidal function.

View from top of Willis Tower

Table 8.9

Time (seconds) t	Sway Distance from Vertical (inches) d
0	0.0
6	6.0
12	10.4
18	12.0
24	10.4
30	6.0
36	0.0
42	−6.0
48	−10.4
54	−12.0
60	−10.4
66	−6.0
72	0.0

a. Using the table of values for $d(t)$, give the midline, amplitude, period, and frequency. Explain what each means in the real-world context.

b. Write a sinusoidal equation expressing the distance that the building sways.

c. Using $d(t)$, predict the sway after 9 seconds.

Solution

a. The maximum sway value is 12 and the minimum sway value is -12.

$$\text{amplitude} = \frac{\max - \min}{2} = \frac{12 - (-12)}{2} = 12 \qquad \text{midline} = \frac{\max + \min}{2} = \frac{12 + (-12)}{2} = 0$$

The amplitude is 12 and the midline is $y = 0$. This means the building sways 12 inches in either direction from the vertical position.

From the data table, we observe the function values begin at 0 inches, increase to 12 inches, decrease to -12 inches, and then return to 0 inches. Since this cycle takes 72 seconds, the period is 72.

The frequency is the reciprocal of the period and equals $\frac{1}{72}$. The portion of a period that occurs over a 1-second interval is $\frac{1}{72}$.

b. Since the function values start at the midline, increase to the maximum value, decrease to the minimum value, and return to the midline, a sine function model should be used for the data. The standard form for a sine function is $y(\theta) = A\sin(B(\theta - C)) + D$. From the description in part (a), we know $A = 12$, $C = 0$, and $D = 0$. To find the value of B, we use the formula $B = \frac{2\pi}{\text{period}}$ to get $B = \frac{2\pi}{72} = \frac{\pi}{36}$. Therefore, the equation for the distance d in inches that the Willis Tower sways in t seconds is $d(t) - 12\sin\left(\frac{\pi}{36}t\right)$.

c. We evaluate $y(9)$ to find the sway after 9 seconds.

$$\begin{aligned} d(9) &= 12\sin\left(\frac{\pi}{36}(9)\right) \\ &\approx 8.5 \end{aligned}$$

The sway is 8.5 inches after 9 seconds.

EXAMPLE 4 ■ Generating an Equation from the Graph of a Sinusoid

Pure sounds produce single sine waves on an oscilloscope. Find both the sine and cosine formula for the wave in the graph of a pure sound shown in Figure 8.81. On the vertical scale, the voltage is measured with each increment representing 0.5 volts, and on the horizontal scale each increment represents 30 milliseconds.

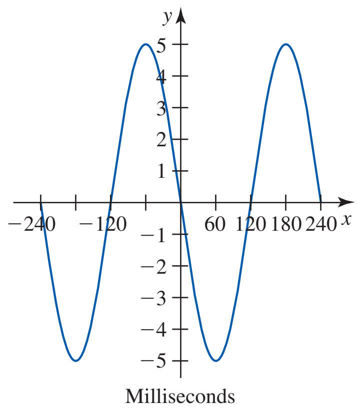


Figure 8.81

Solution To find the sine formula for the wave, we need to determine what A, B, C, and D are in the standard sine function $y(\theta) = A\sin(B(\theta - C)) + D$. The amplitude is $|A|$ and on the graph it is 5 because that is the height of the wave above the midline. Also, we note the graph is the reflection of the standard sine wave because it decreases after 0 milliseconds instead of increasing. Therefore, $A = -5$. The frequency, B, is the constant that determines the period. We know that the period for a sine wave is 360 milliseconds. This wave's period is 240 milliseconds so $B = \frac{360}{240} = 1.5$. The graph of the wave is not shifted horizontally from the vertical axis so C, the horizontal shift, is 0. Neither is the graph of this wave shifted vertically, so D, the vertical shift, is also 0. The equation for this function is therefore $y(\theta) = -5\sin(1.5\theta)$.

Likewise, to find a cosine equation we need to know what the parameters A, B, C, and D are in the standard cosine function $y(\theta) = A\cos(B(\theta - C)) + D$. The sine and cosine functions will have the same amplitude, period, and vertical shift. We must only determine the horizontal shift. To do this, we observe there is a maximum value for the function at -60 milliseconds instead of at 0 milliseconds. Therefore, we shift the graph left 60 milliseconds from the vertical axis. This gives us the equation $y(\theta) = 5\cos(1.5(\theta + 60))$.

SUMMARY

In this section you learned to generate graphs and equations for sine and cosine functions from verbal descriptions, graphs, and tables of data. In addition, you saw how to use sinusoidal functions to model real-world situations and to make predictions and interpret results.

8.5 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, state the period, amplitude, and midline.

1. $y = 5\sin(4(t - 2)) + 3$

2. $y = 7\cos(0.5(t + 4)) - 2$

3. $y = 8\cos(2t)$

4. $y = 3\sin(\pi(t - 2)) - 0.8$

In Exercises 5–8, determine the vertical and horizontal shifts from a basic sine or cosine function.

5. $y = 3\sin(7(t - 1)) + 2$

6. $y = -2\cos(t - 2) - 3.5$

7. $y = 6\cos\left(3\left(t + \dfrac{\pi}{2}\right)\right) - 9$

8. $y = \sin(\pi t) - 1$

In Exercises 9–12, sketch one period of the function without using a calculator.

9. $y = 4\cos(3t)$

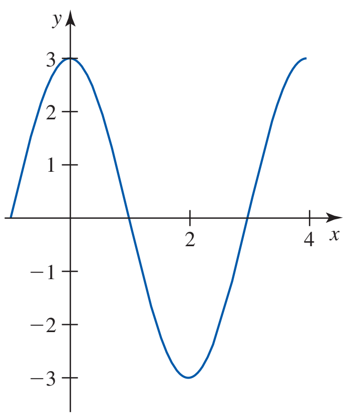
10. $y = 4\sin\left(t + \dfrac{\pi}{4}\right)$

11. $y = 4\sin(\pi(t + \pi)) - 3$

12. $y = 4\cos\left(2\left(t - \dfrac{\pi}{2}\right)\right) + 4$

In Exercises 13–20, find an equation for the sinusoidal function shown in the graph. You may use either sine or cosine.

13.



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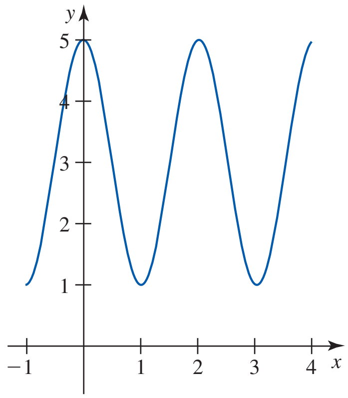

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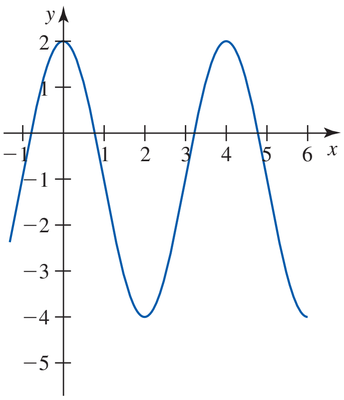
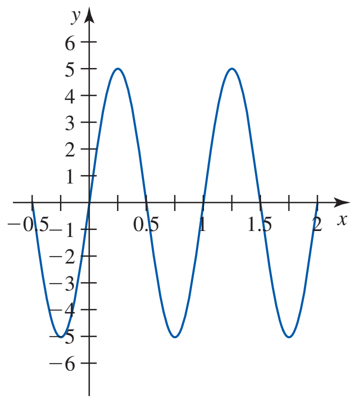


16.



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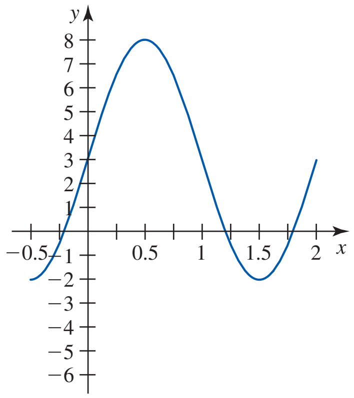

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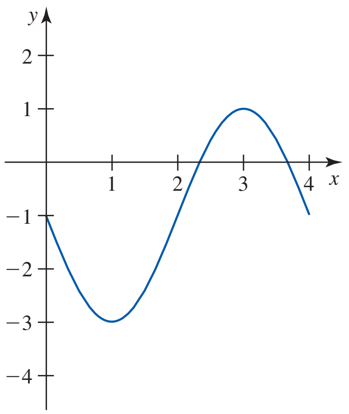

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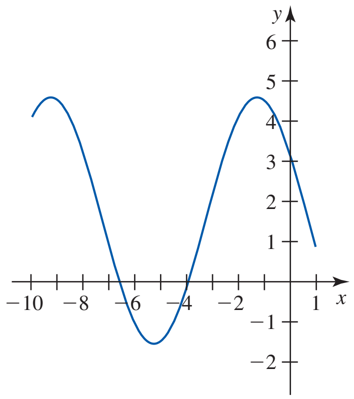

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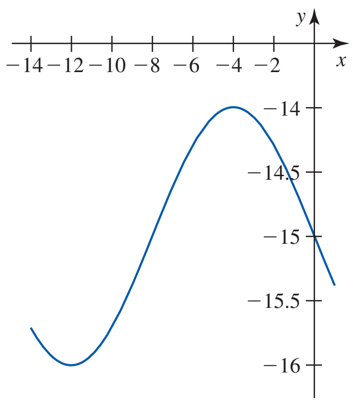


In Exercises 21–25, find a possible equation for the trigonometric function whose values are shown in the table.

21.

x	y
0	2
0.1	3.2
0.2	3.9
0.3	3.9
0.4	3.2
0.5	2
0.6	0.8
0.7	0.1
0.8	0.1
0.9	0.8
1.0	2

22.

x	y
−5	4
−4	5.2
−3	6
−2	6
−1	5.2
0	4
1	2.8
2	2
3	2
4	2.8
5	4

23.

x	y
0	1
5	1.3
10	1.5
15	1.5
20	1.3
25	1
30	0.7
35	0.5
40	0.5
45	0.7
50	1

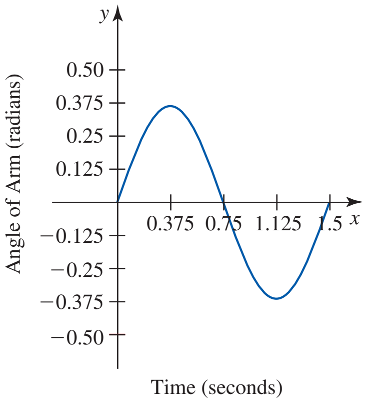
24.

x	y
−2	−4
−1	−5.2
0	−6
1	−6
2	−5.2
3	−4
4	−2.8
5	−2
6	−2
7	−2.8
8	−4

25.

x	y
1	0.5
3	0.7
5	0.5
7	0.3
9	0.5
11	0.7
13	0.5
15	0.3
17	0.5
19	0.7
21	0.5

26. Forearm The photo shows a schematic diagram of a rhythmically moving arm. The upper arm rotates back and forth from the point A; the position of the arm is measured by the angle a (in radians) between the actual position and the downward vertical position as shown in the graph.



a. Find an equation of the form $y(\theta) = A\sin(B(\theta - C)) + D$ for the graph $a(t)$.

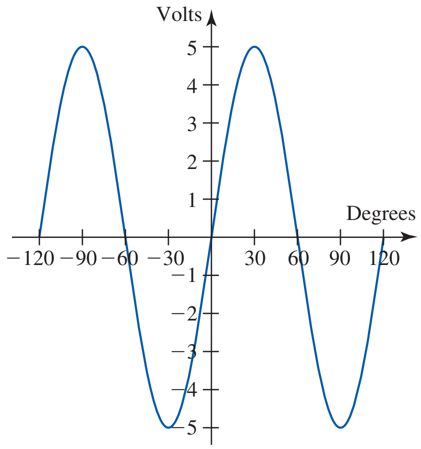
b. How long does it take for a complete movement (swing from vertical then back-and-forth back to vertical) of the upper arm?

c. At what times during one cycle is the upper arm in the vertical position?

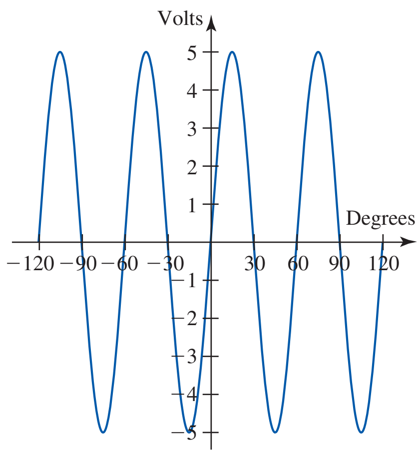
d. What is the fifth time that the arm is at -0.1 radians?

In Exercises 27–28, find a sine and cosine function for the wave of a pure sound shown in the graph. On the vertical scale, each increment represents 1 volt, and on the horizontal scale each increment represents 30°.

27.



28.



29. Bouncing Spring When a weight is attached to the end of a spring, pulled down to extend the spring, and then released, the weight begins to bounce up and down. When this occurs, the distance the weight is from the floor, d, in centimeters varies sinusoidally over a short period of time, t, in seconds. Assume a stopwatch reads 0.5 seconds when the weight reaches its first high point 42 centimeters above the floor. The next low point, 11 centimeters above the floor, occurs at 1.2 seconds.

a. Sketch a graph of this sinusoidal function.

b. Write an equation expressing the distance the weight is from the floor in terms of the number of seconds the stopwatch reads.

c. Predict the distance from the floor when the stopwatch reads 3.4 seconds.

d. What was the distance from the floor when timing started?

e. Predict the first positive value of time at which the weight is 38 centimeters above the floor.

f. Can the height the weight is off the floor be accurately modeled by the function $d(t)$ indefinitely? Explain.

30. Product Demand Cyclical behavior is common in the business world as the seasonal fluctuations in the demand for certain equipment are largely dependent on the weather. Common examples would include sales of surfing equipment, swimwear, and snow shovels. Assume the amount of money spent on snow skis, S, in the Colorado Springs, Colorado, vicinity can be modeled by

$$S(m) = 480.02\sin(0.49m + 1.76) + 1026.1$$

thousand dollars where m is the number of months since January 1 ($m = 0$).

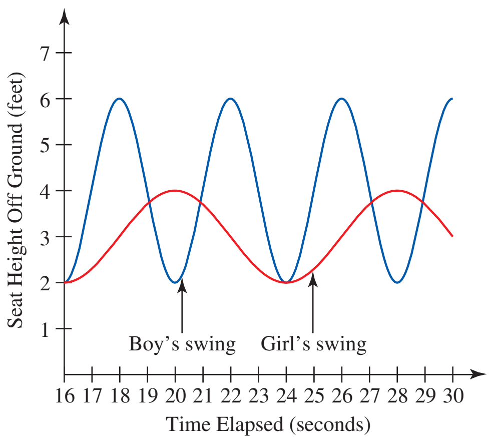
a. Calculate the amplitude and explain what the numerical value means in this context.

b. Determine the vertical shift and explain what the numerical value means in this context.

c. Determine the phase shift and explain what the numerical value means in this context.

d. Determine the period and explain what the numerical value means in this context.

31. Swinging Two siblings (a boy and a girl) begin to play on the swing set. The graphs in the figure display a 14-second interval of time and the heights of each child in feet. Use the graphs to determine the sinusoidal equations for each child's distance from the ground as a function of the time.



32. Paddlewheel Boat As a paddlewheel turns to drive a boat downriver, each paddle blade moves in such a way that its distance, d, from the water's surface is a sinusoidal function of time. When a stopwatch reads 5 seconds, one of the paddle blades was at its highest, 18 feet above the water's surface. Assume the wheel's diameter is 20 feet, and it completed a revolution every 9 seconds.

a. Sketch a graph of this sinusoidal function.

b. Write a function, $d(t)$, expressing the distance above the river the blade is in terms of t.

c. Find each value and interpret its meaning in the real-world context:

i. $d(6)$ ii. $d(2.9)$

iii. $d(-3.5)$ iv. $d(t) = 9.7$

v. $d(t) = d(6)$ vi. $d(t) = -8$

d. Where was the paddlewheel blade when the stopwatch was started?

e. Find when the paddle entered the surface of the water. Give all answers for the first complete turn of a wheel.

f. Find when the paddle is 7 feet above the water during the second turn of the wheel.

SHOW YOU KNOW

33. How can you tell if a sinusoidal function can model a data set well?

34. Are sinusoidal functions appropriate for modeling all periodic functions? Defend your conclusion.

35. One of your classmates was absent and doesn't know how to construct a sinusoidal model when given periodic data in a table. Write an explanation to help her understand the process.

36. Explain why the period of a function and the horizontal stretch/compression factor, B, are related by the formula $\text{period} = \dfrac{360°}{|B|}$ or $\text{period} = \dfrac{2\pi}{|B|}$.

MAKE IT REAL

37. The Bay of Fundy In Canada, the Bay of Fundy tides are in constant motion and are the highest in the world. The tides may rise and fall at a rate between 6 and 8 feet per hour and have a difference between low and high water levels of nearly 50 feet. In addition, the period of time between low and high tides is 6 hours and 13 minutes. At a particular location the depth of the water, f, in feet is given as a function of time, t, in hours since midnight by $f(t) = A\cos(B(t - C)) + D$.

a. What is the physical meaning of D?

b. What is the value of A?

c. What is the value of B?

d. What is the physical meaning of C?

38. Utilities Budget The table shows the monthly electricity costs for the year 2006 of a single-family residence. Explain why a sinusoidal function should not be used to model the electricity cost as a function of month.

Month (1 = January) m	Cost (dollars) c
1	124.19
2	111.16
3	116.30
4	133.71
5	270.07
6	320.11
7	250.23
8	391.19
9	345.91
10	256.44
11	124.87
12	126.98

39. Voltage The voltage V in an electrical circuit is given by the formula $V(t) = 5\cos(120\pi t)$, where t is the time measured in seconds.

a. Find the amplitude and the period.

b. Determine the frequency of the function and describe what this means in the real-world context.

c. Evaluate $V(0)$, $V(0.03)$, $V(0.06)$, $V(0.09)$, and $V(0.12)$.

d. Graph $V(t)$ for $0 \le t \le \dfrac{1}{30}$.

40. Voltage The voltage V in an electrical circuit is given by the formula $V(t) = 3.8\cos(40\pi t)$, where t is the time measured in seconds.

a. Find the amplitude and the period.

b. Determine the frequency of the function and describe what this means in the real-world context.

c. Evaluate $V(0.02)$, $V(0.04)$, $V(0.08)$, $V(0.12)$, and $V(0.14)$.

d. Graph one period of $V(t)$.

41. Pneumonia and Influenza Mortality According to the Centers for Disease Control and Prevention, the epidemic threshold and the seasonal baseline over the years 2003 to 2006 for pneumonia and influenza in 122 American cities can be modeled as shown in the graph in weeks since January 1 of each year. The epidemic threshold means that when the number of cases exceeds this number it is considered an epidemic or very serious problem. The seasonal baseline is the expected minimum number of cases. (*Source:* www.cdc.gov/flu/weekly)

a. Use the graph of the epidemic threshold to determine the sinusoidal equation for the percent of all deaths, p, as a function of the week of the year, w.

b. Explain what the period, the amplitude, and the midline mean in terms of the real-world context for the epidemic threshold graph.

c. Evaluate $p(30)$ for the year 2005 and explain what this number means.

d. Estimate in what weeks and what years the maximum epidemic threshold for the percentage of deaths due to pneumonia and influenza occur from 2003 to 2006.

e. Estimate in what weeks and what years the minimum in the epidemic threshold occurs from 2003 to 2006.

f. Give the intervals over which the function $p(w)$ is increasing and decreasing for 2005. What information does each interval give us?

g. What does the gap between the epidemic threshold and seasonal baseline graphs mean?

42. Hours of Daylight The table displays the number of hours of daylight, H, in Seattle, Washington, d days after March 21, 2010. Use the table of values to answer the following questions. (*Hint:* Be sure to use Radian mode on your calculator.)

Days after March 21, 2010 d	Number of Daylight Hours H
0	12.20
30	14.07
60	15.45
90	15.98
120	15.53
150	14.21
180	12.36
210	10.48
240	9.04
270	8.43
300	8.80
330	10.06
360	11.88

Source: aa.usno.navy.mil

a. Find an equation for the sinusoidal function, $H(d)$, to model the number of daylight hours in Seattle.

b. Use $H(d)$ to estimate when the number of daylight hours will be 13 hours.

c. Find the average rate of change of $H(d)$ from $d = 30$ to $d = 120$. What does this rate mean?

43. Full Moon The table displays the percentage M of the full moon visible t days after January 1, 2010.

Days after January 1, 2010 t	Percentage of Moon Visible M
0	100
3	91
6	63
9	31
12	8
15	0
18	8
21	29
24	59
27	87
30	100

Source: aa.usno.navy.mila

a. Find an equation for the sinusoidal function, $M(t)$, to model the percentage of the full moon visible t days after January 1, 2010.

b. Use $M(t)$ to estimate when the moon will be 33% and 50% visible.

c. Find the average rate of change of $M(t)$ from $t = 10$ to $t = 25$. What does this rate mean?

44. New York Temperature The table displays the average high and low temperatures over the course of a year since January.

Month (1 = January) m	New York Average High (degrees Fahrenheit) H	New York Average Low (degrees Fahrenheit) L
1	39	26
2	42	29
3	50	35
4	60	45
5	71	55
6	79	64
7	85	70
8	83	69
9	76	61
10	65	50
11	54	41
12	44	32

Source: weather.com

a. Find an equation for the average high temperature in New York, $H(m)$, as a function of the month of the year.

b. Find an equation for the average low temperature in New York, $L(m)$, as a function of the month of the year.

c. Compare the two functions $H(m)$ and $L(m)$ in terms of the amplitude, period, and horizontal and vertical shifts. Explain any discrepancies in terms of the context of the real world.

d. Use a function transformation to write $L(m)$ in terms of $H(m)$.

45. Death Valley Temperature Air temperature is a cyclic phenomenon with a cycle of 12 months. We derive the function $T(t)$ to estimate the temperature T in degrees at time t in months based on the average taken over several years. Note this function is not an accurate indicator of temperature at a specific time of day since it is normally cooler in the morning, warmer during the day, and cooler again at night. Also, for any particular year and at different times of day the temperature will vary from the average. However, it will allow us to approximate the average temperature for certain dates.

Death Valley in California is in the Mojave Desert and even during the coldest times of the year the average low temperature is quite high. Based on the average temperature data, we make the following assumptions:

- The minimum monthly low temperature of 37°F occurs at the end of January ($t = 1$).
- The maximum monthly low temperature of 86°F occurs at the end of July ($t = 7$).
- Temperature is periodic (cyclic) on an annual basis.

a. Write a function $T(t)$ to describe the average daily low temperature at time t using a sinusoidal function.

b. Graph one cycle of the function $T(t)$ and label the axes appropriately.

c. Label the minimum and maximum values on the graph of $T(t)$ noting both the input and output values.

d. What is the period?

e. Determine the amplitude.

f. Determine the horizontal shift.

g. Determine the midline.

h. What is the temperature at the end of April? In what month does the temperature again reach that level?

i. When will the average low temperature be 82°F? Give the month and approximate day of the month.

j. Indicate on the graph when $T(t) = 78$ and then find the solutions for t over the interval $0 \le t \le 12$.

■ STRETCH YOUR MIND

Exercises 46–50 are intended to challenge your understanding of sinusoidal models.

46. Space Shuttle When the Space Shuttle is fired into orbit from a site such as Cape Canaveral, Florida, which is not on the equator, it goes into an orbit that takes it alternatively north and south of the equator. Its distance from the equator is approximately sinusoidal over time. (***Source:*** **www.cse.ssl.berkeley.edu**)

Suppose the Space Shuttle is fired into orbit from Cape Canaveral and 12 minutes later it reaches its farthest distance north of the equator, 2485 miles. Half a cycle later it reaches its farthest distance south of the equator on the other side of Earth, also 2485 miles. Assume the Space Shuttle completes one orbit every 82 minutes.

a. Sketch one complete cycle of the number of miles, m, the Space Shuttle is from the equator at time t minutes since liftoff. (When the Shuttle is north of the equator consider the distance positive and when it is south of the equator consider the distance negative.)

b. Find an equation expressing $m(t)$.

c. Evaluate $m(30)$, $m(55)$, and $m(180)$ and explain what the values mean in this context.

d. What is the smallest positive time t for when the Space Shuttle is 1000 miles south of the equator?

e. Estimate how far Cape Canaveral is from the equator.

47. Satellite Communications Satellites orbit Earth in an elliptical path. The apogee of the orbit is the highest point the satellite is from Earth and the perigee is the lowest point. Suppose on a Low Earth Orbit (ranging from 0 to 1240 miles) a satellite is at its apogee of $d = 900$ miles at $t = 0$ hours and at its perigee of $d = 56$ miles at $t = 1$ hour.

a. Write an equation for the distance, d, the satellite is above the surface of Earth as a function of time, t.

b. Predict the first four positive values of t for which the satellite is 280 miles from the surface.

c. For how many consecutive minutes will the satellite be below 300 miles?

48. Power Utilization The first table displays the estimated power usage for the California ISO service area for a day in 2007.

Time Since Midnight (hours) t	California Power Usage (megawatts) C
0	14,235
1	12,565
2	13,637
3	13,750
4	14,432
5	13,797
6	14,013
7	12,968
8	12,159
9	11,661
10	10,661
11	8928
12	7277
13	5809

(continued)

Time Since Midnight (hours) t	California Power Usage (megawatts) C
14	3880
15	2681
16	2253
17	2904
18	4056
19	5353
20	7693
21	7432
22	7563
23	9992

Source: www.caiso.com

The second table displays the amount of power used by Arizona Power Service (APS) customers per hour for a day in the winter of 2007.

Time Since Midnight (hours) t	Arizona Power Usage (megawatts) A
0	3830
1	3813
2	3859
3	3887
4	4010
5	4273
6	4630
7	4825
8	4731
9	4553
10	4322
11	4096
12	3897
13	3698
14	3547
15	3472
16	3510
17	3822
18	4305
19	4367
20	4318
21	4213
22	3921
23	3665

Source: Tom Glock, Manager, Power Operations for Arizona Public Service

a. After analyzing a scatter plot for both the power usage of California, $C(t)$, and Arizona, $A(t)$, explain if it makes sense to use a sinusoidal function to model each.

b. Describe the power requirements of each state in words.

c. Find $P(t) = C(t) + A(t)$ and explain what the input and output values of $P(t)$ are.

d. What is the total power that the two power companies must prepare to provide and why is this important to know? Keep in mind the overall maximum power needed at any particular hour.

49. Octaves Musical notes are sounds with specific frequencies that are spaced apart according to a mathematical rule. In most Western music, the notes are organized into a 12-tone scale (see the table). A set of 12 consecutive notes is called an *octave*.

Note	Frequency (Hz)
C (middle C)	262
C sharp	277
D	294
D sharp	311
E	330
F	349
F sharp	370
G	392
G sharp	415
A	440
A sharp	466
B	494
C	523

Source: www.vibrationdata.com

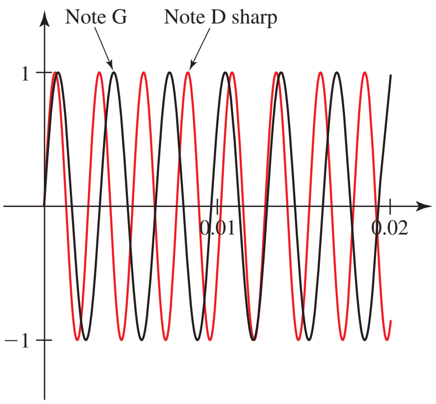
The mathematical models for the frequencies of the notes D sharp and G are

$$F_{\text{D sharp}}(t) = k\sin(2\pi \cdot 311t)$$

and

$$F_{\text{G}}(t) = k\sin(2\pi \cdot 392t)$$

where k is the loudness of the notes at the instant each note is played, and t is the time in seconds. If we assume that the loudness k is the same for both notes, we can graph each note as shown in the figure. (Note the very short time interval graphed: from 0 to 0.02 seconds.)



a. Graph the sum of these two notes.

b. At what times do the maximum and minimum points occur on the interval 0 to 0.02 seconds?

c. What is the frequency of the musical chord formed by playing both D sharp and G?

50. Damped Oscillation When a weight attached to a spring is released and begins to bob up and down, it does not continue to rise and drop the same distance for very long. As the weight moves toward equilibrium, its up-and-down movements become smaller. Suppose that we raise the weight 7 centimeters above its rest position and release it at time $t = 0$, where t is in seconds. A function that can model the behavior of the weight is $w(t) = 7(0.5^t)\cos(2\pi t)$. Graph the function $w(t)$ and determine how long it takes for the weight to come to a rest. (Assume the object is at rest when $w(t) < 0.005$.)

SECTION 8.6 Other Trigonometric Functions

LEARNING OBJECTIVES

- Define tangent, cotangent, secant, and cosecant in terms of sine and cosine
- Use the language of rate of change to describe the behavior of trigonometric functions

GETTING STARTED

Steep downhill portions of roads are very dangerous for large trucks carrying heavy loads. A driver must use a combination of vehicle brakes and engine braking (using a low gear) to maintain a safe speed and control of the vehicle. If the brakes fail from overheating or if the driver does not maintain an engaged transmission, there is potential for a deadly accident. (*Source:* www.usroads.com) What exactly does it mean when a road sign warns of an 8% downhill grade, such as this sign near Moab, Utah?

In this section we discuss the trigonometric functions tangent, cotangent, secant, and cosecant. We show how each of these functions are connected to cosine and sine, how each behaves, and how to graph each of them. We also see how these trigonometric functions are used to model real-world situations such as understanding road grades.

■ The Tangent Function

The **tangent function** (abbreviated **tan**) has an angle measure as its input, and its output gives the slope of the terminal side of an angle in standard position. Recall that the terminal side of an angle θ passes through the points (0, 0) and $(\cos(\theta), \sin(\theta))$. We define tangent using the basic equation for slope.

$$\tan(\theta) = m = \frac{\text{change in } y}{\text{change in } x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{\sin(\theta) - 0}{\cos(\theta) - 0} = \frac{\sin(\theta)}{\cos(\theta)}$$

Using the unit circle, we can see how the tangent function uses this equation and how it relates to cosine and sine.

EXAMPLE 1 ■ Finding the Slope of the Terminal Side of an Angle

Find the slope of the terminal side of a 35° angle and explain how cosine and sine are involved in this calculation.

Solution We first locate the endpoint of the arc corresponding with a 35° angle on the unit circle shown in Figure 8.82. Recall that we can express the coordinates of any point on the unit circle in terms of the angle measure. In this case, the coordinate is (cos(35°), sin(35°)). We need to find the slope of the line going through this point and through the origin.

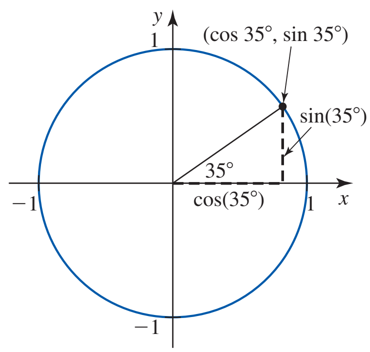


Figure 8.82

$$\tan(35°) = \frac{\sin(35°) - 0}{\cos(35°) - 0}$$
$$= \frac{\sin(35°)}{\cos(35°)}$$
$$\approx 0.7002$$

Thus we see that the tangent function is the ratio of the sine and cosine functions.

Tangent Function

The **tangent function** gives the slope of the terminal side of an angle in standard position measured in θ degrees or radians. That is,

$$m = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

The Tangent Function and Technology

We can calculate tangent values with a calculator by using the $\boxed{\text{TAN}}$ key or by finding the ratio $\dfrac{\sin(\theta)}{\cos(\theta)}$. For example, the calculations for the slope of the terminal side of an angle measuring $\dfrac{\pi}{8}$ radians is shown in Figure 8.83 using both methods.

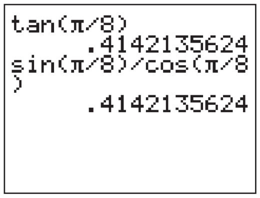


Figure 8.83

EXAMPLE 2 ■ Using the Tangent Function

Evaluate the expression $\tan\left(\dfrac{8\pi}{7}\right)$ and explain what this value represents in the context of angles.

Solution Using technology, we find $\tan\left(\dfrac{8\pi}{7}\right) \approx 0.4816$. This tells us that the slope of the terminal side of an $\dfrac{8\pi}{7}$ angle in standard position is 0.4816, as shown in Figure 8.84.

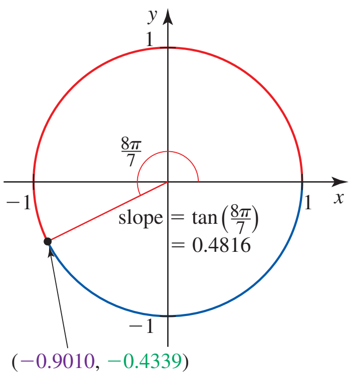


Figure 8.84

Graph of the Tangent Function

To understand what the graph of the tangent function will look like, let's examine Figure 8.85 to see how the rates of change of the terminal sides of angles change as we move around the circle.

In the first quadrant, the slope starts at 0 (horizontal) and increases until it becomes infinitely large near the positive y-axis. When we reach the positive y-axis, the slope is undefined because the line segment is vertical. As we continue into the second quadrant, we see that the slope is very steep in the negative direction and increases toward 0 as we approach the negative x-axis. Thereafter the slope continues to increase until it is once again infinitely large near the

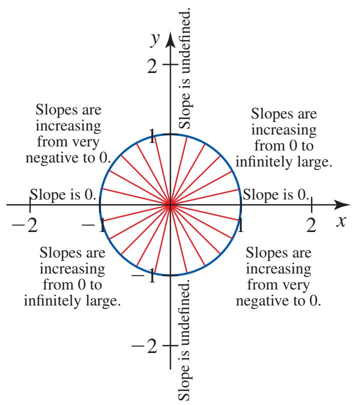


Figure 8.85

negative y-axis. The slope becomes undefined as the line segment becomes vertical, then in the fourth quadrant the slope again starts very steep in the negative direction and increases to 0.

The graph of $f(\theta) = \tan(\theta)$, in degrees and in radians, is shown in Figure 8.86. The tangent function has horizontal intercepts at $(0°, 0)$, $(180°, 0)$, and $(360°, 0)$ and the function is undefined when $\theta = 90°$ and $\theta = 270°$.

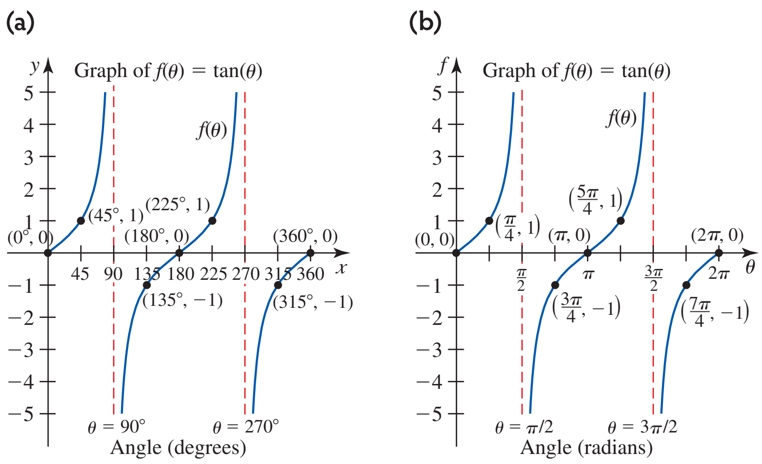


Figure 8.86

Looking closely at the graph of $f(\theta) = \tan(\theta)$, we see there are two angles between 0° and 360° (0 and 2π radians) with the same tangent value, and we observe these occur for θ-values that are separated by 180° (π radians). The tangent function, like cosine and sine, is a periodic function but has a period of 180° (π radians).

Due to the tangent function's periodic nature, its horizontal intercepts and vertical asymptotes occur at regular intervals. Since $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$, the horizontal intercept occurs when $\frac{\sin(\theta)}{\cos(\theta)} = 0$, which happens each time $\sin(\theta) = 0$. Similarly, the vertical asymptotes occur when $\frac{\sin(\theta)}{\cos(\theta)}$ is undefined, which happens each time $\cos(\theta) = 0$. See Figure 8.87.

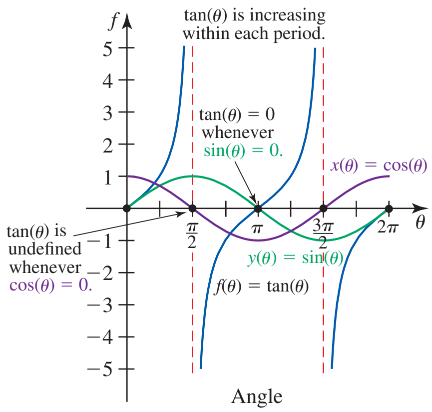


Figure 8.87

Table 8.10

	Period	Vertical Intercept	Horizontal Intercepts Occur When $\sin(\theta) = 0$	Vertical Asymptotes Occur When $\cos(\theta) = 0$
$f(\theta) = \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$	180° or π radians	$f(0) = 0$	$\theta = 180°n$ or $\theta = \pi n$ radians for integer n	$\theta = 90° + 180°n$ or $\theta = \frac{\pi}{2} + \pi n$ radians for integer n

The domain of the tangent function includes all real numbers except where the function is undefined, which happens when $\cos(\theta) = 0$ (at $\theta = 90° + 180°n$ or $\theta = \frac{\pi}{2} + \pi n$ radians for all integer values of n). The range includes all real numbers since the slope of the terminal side of an angle can vary between being infinitely steep in the negative and positive directions.

EXAMPLE 3 ■ Using the Tangent Function

Roads have varying degrees of steepness. A road's steepness is reported as a percent and is calculated by dividing the change in elevation by the change in horizontal distance. Explain what it means to have an 8% downhill grade on a road. Then find the angle of descent for the road in degrees.

Solution An 8% downhill grade is a road that declines in elevation $\dfrac{8 \text{ vertical feet}}{100 \text{ horizontal feet}}$. In other words, for every 100-foot change in horizontal position, the road elevation decreases 8 feet (or, for every 1-foot horizontal change the elevation decreases by 0.08 feet).

If we denote the angle of descent as θ, then the 8% downhill grade tells us the following.

$$\begin{aligned}\tan(\theta) &= \text{slope of the road} \\ &= \frac{\text{vertical change}}{\text{horizontal change}} \\ &= -\frac{8}{100}\end{aligned}$$

Since we know the road's slope but not the angle whose terminal side gives us this slope, we solve the equation $\tan(\theta) = -\dfrac{8}{100}$ by graphing, as shown in Figure 8.88.

(a) (b) (c)

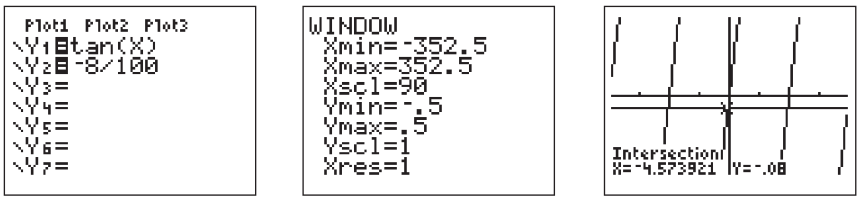


Figure 8.88

There are multiple solutions to this equation, but the incline or decline of roads will not vary too far from 0°. Keep in mind that a road that is 0° is completely level and a road that is 90° is completely straight up. The angle of descent for this road is about 4.57° below the horizontal.

■ Reciprocal Trigonometric Functions

Cosine, sine, and tangent are the three trigonometric functions typically given the most attention. The remaining three trigonometric functions are created by finding the *reciprocals* of the three main functions. Recall that the **reciprocal** of a quantity a is $1/a$. The product of a number and its reciprocal is 1. For example, the values in the following number pairs are reciprocals of each other: 2 and $\frac{1}{2}$, $\frac{5}{3}$ and $\frac{3}{5}$, or -5 and $-\frac{1}{5}$.

Although the reciprocal trigonometric functions are not used as often as cosine, sine, and tangent, they are still widely used for simplifying and solving trigonometric equations.

Cotangent Function

Similar to the tangent function, the **cotangent function**, abbreviated **cot**, is a way to describe the slope of a line. The cotangent of an angle is defined to be the reciprocal of the tangent function.

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Substituting $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ for $\tan(\theta)$ will give us a clearer picture of how the cotangent value relates to cosine and sine.

$$\begin{aligned}\cot(\theta) &= \frac{1}{\tan(\theta)}\\ &= \frac{1}{\frac{\sin(\theta)}{\cos(\theta)}} && \text{Substitute } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.\\ &= 1\cdot\frac{\cos(\theta)}{\sin(\theta)}\\ &= \frac{\cos(\theta)}{\sin(\theta)}\end{aligned}$$

This shows us that the cotangent function is a ratio of the change in x to the change in y of the terminal side of an angle in standard position, which is the reciprocal of the slope equation.

Cotangent Function

The **cotangent function** of an angle, denoted **cot(θ)**, is the reciprocal of the tangent function and represents the reciprocal of the slope of the terminal side of an angle measuring θ degrees or radians.

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$

Also,

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

EXAMPLE 4 ■ Finding and Interpreting the Cotangent Value

Some of the most recognizable structures on Earth are the pyramids of Egypt. While building these structures, the Egyptians utilized the cotangent relationship (which they called the *seked*) by keeping careful measurements of the lengths of each horizontal level as they increased the pyramid's height. (*Source:* Eli Maor, *Trigonometric Delights*, Princeton University Press, 1998)

The Great Pyramid at Giza, built for Khufu between 2589 and 2566 B.C. has sides that make a 51.843° angle with the ground. Find cot(51.843°) and explain what this value represents. (*Source:* www.pbs.org)

Solution To find cot(51.843°), we use a calculator to evaluate $\dfrac{1}{\tan(51.843°)}$ and find $\cot(51.843°) \approx 0.7857$, or about $\dfrac{78.57}{100}$. This tells us the distance from the horizontal center of the pyramid to the edge gains 0.7857 foot each time the height increases by 1 foot.

Since the width is twice the distance from the horizontal center to the edge, a 1-foot increase in height corresponds with a 1.5714-foot increase in width.

Since the cotangent function is so closely connected to the tangent function, we see many similarities between their graphs. Because they are reciprocals, as $\tan(\theta)$ increases, the fraction $\dfrac{1}{\tan(\theta)}$ decreases. Much like the tangent function, the cotangent function has horizontal intercepts and vertical asymptotes at regular intervals. We find a horizontal intercept whenever $\cos(\theta) = 0$ since this makes the ratio $\dfrac{\cos(\theta)}{\sin(\theta)} = 0$, and a vertical asymptote whenever $\sin(\theta) = 0$, which makes the ratio $\dfrac{\cos(\theta)}{\sin(\theta)}$ undefined. See Figure 8.89 and Table 8.11.

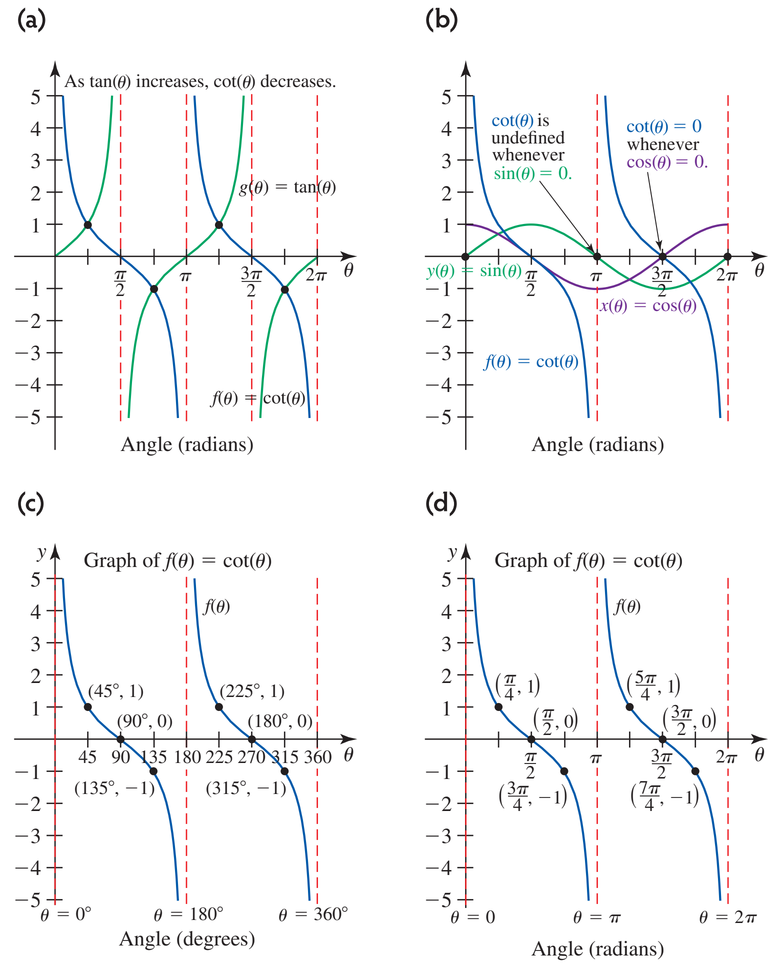


Figure 8.89

Table 8.11

	Period	Vertical Intercept	Horizontal Intercepts Occur When $\cos(\theta) = 0$	Vertical Asymptotes Occur When $\sin(\theta) = 0$
$f(\theta) = \cot(\theta) = \dfrac{\cos(\theta)}{\sin(\theta)}$	$180°$ or π radians	none	$\theta = 90° + 180°n$ or $\theta = \dfrac{\pi}{2} + \pi n$ radians for integer n	$\theta = 180°n$ or $\theta = \pi n$ radians for integer n

The cotangent function has a domain of all real numbers except where it is undefined, which occurs when $\sin(\theta) = 0$ (at $\theta = 180°n$ or $\theta = \pi n$ radians for all integer values of n). The range of the cotangent function includes all real numbers.

EXAMPLE 5 ■ Using the Graph of Cotangent

Pharoah Khafre, believed to be Khufu's son, constructed his pyramid next to his father's in Giza between 2558 and 2532 B.C. His pyramid was 471 feet tall with a square base measuring 704 feet on each side. Use the cotangent function to find the angle of incline for the side of Khafre's pyramid. (*Source:* www.pbs.org)

Solution We begin the problem with a diagram representing the situation, shown in Figure 8.90.

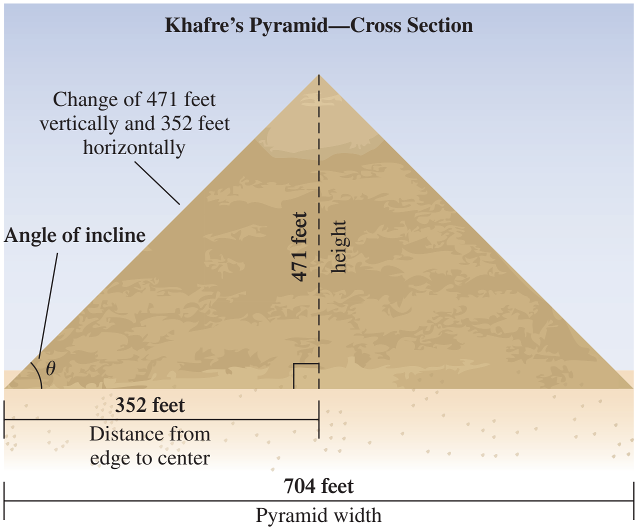


Figure 8.90

This diagram shows that the distance from the center to the edge of the pyramid changes by 352 feet as the height increases by 471 feet, giving a cotangent value of $\frac{352}{471}$, or about 0.7473 (the horizontal changes 0.7473 foot as the height increases by 1 foot). To find the angle, we solve the equation $\cot(\theta) = \frac{352}{471}$ by graphing, as shown in Figure 8.91. See the Technology Tip at the end of the section for details.

Since the function is periodic, there are many solutions. However, only an angle between 0° and 90° makes sense in this situation because 0° is flat and 90° is vertical. Also, we expect the angle of the pyramid's incline to be closer to 45°, which is halfway between flat (0°) and vertical (90°). The side of the pyramid has a 53.223° incline.

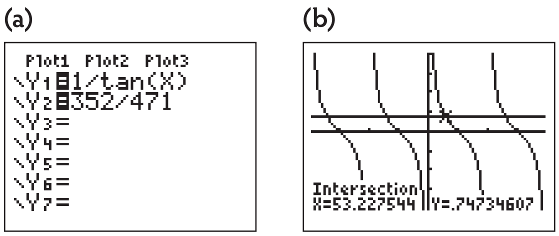


Figure 8.91

Secant Function

The **secant function** is the reciprocal of the cosine function.

Secant Function

The **secant** of an angle θ, denoted **sec(θ)**, is the reciprocal of the cosine value at θ.

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

The graph of the secant function, in degrees and in radians, is shown in Figure 8.92. The value of the secant function is the value of the fraction $\frac{1}{\cos(\theta)}$, so the secant varies as the cosine varies. As $\cos(\theta)$ increases, $\sec(\theta)$ decreases and as $\cos(\theta)$ decreases, $\sec(\theta)$ increases. Furthermore, the value of secant is undefined at regular intervals, whenever $\cos(\theta) = 0$. It also has the same period as the cosine function since $\frac{1}{\cos(\theta)}$ repeats its values as $\cos(\theta)$ repeats its values. (See Table 8.12.)

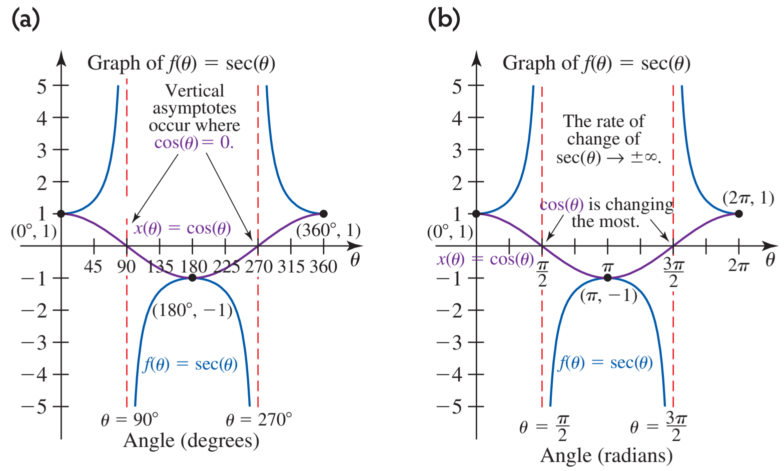


Figure 8.92

Table 8.12

	Period	Vertical Intercept	Horizontal Intercepts	Vertical Asymptotes Occur When $\cos(\theta) = 0$
$f(\theta) = \sec(\theta) = \frac{1}{\cos(\theta)}$	360° or 2π radians	$f(0) = 1$	none	$\theta = 90° + 180°n$ or $\theta = \frac{\pi}{2} + \pi n$ for integer n

To further explore the connectedness of $\cos(\theta)$ and $\sec(\theta)$, consider how the rate of change of $\cos(\theta)$ determines the rate of change of $\sec(\theta)$. As indicated in Figure 8.92b, when the rate of change of $\cos(\theta)$ is close to zero (near its extrema), the rate of change for $\sec(\theta)$ is also close to zero. As the rate of change of $\cos(\theta)$ moves away from 0, the rate of change of $\sec(\theta)$ also moves away from 0.

The value of $\sec(\theta)$ will never fall between -1 and 1 because $|\cos(\theta)|$ is never greater than 1. Instead, $|\cos(\theta)| < 1$ gives us secant values such as $\frac{1}{0.5} = 2$ or $\frac{1}{0.1} = 10$. Therefore, the range of the secant function does not include all real numbers; there is a gap between -1 and 1 that the secant function graph does not cross.

The domain includes all real numbers except where $\cos(\theta) = 0$ (at $\theta = 90° + 180°n$ or $\theta = \frac{\pi}{2} + \pi n$ radians for all integer values of n).

Cosecant Function

The **cosecant function** is the reciprocal of the sine function.

Cosecant Function

The **cosecant function** of an angle θ, denoted **csc(θ)**, is the reciprocal of the sine value at θ.

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

The graph of the cosecant function, in degrees and in radians, is shown in Figure 8.93. The summary of its characteristics is shown in Table 8.13. All our observations regarding the secant function apply to the cosecant function as well, with one small difference—the cosecant function is associated with the sine function instead of the cosine function.

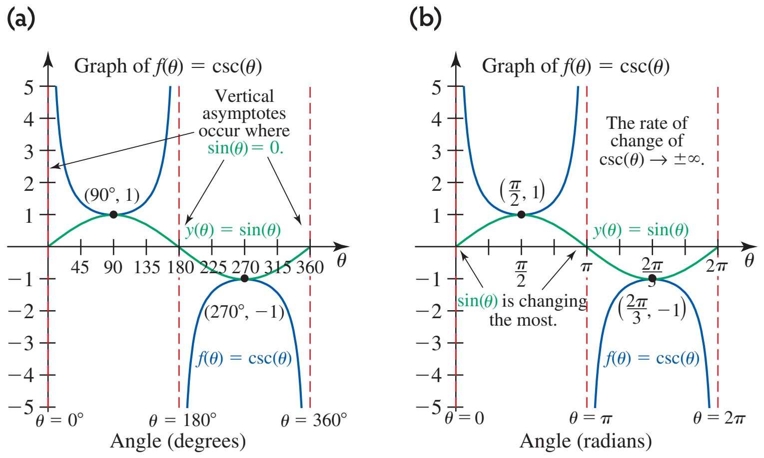


Figure 8.93

Table 8.13

	Period	Vertical Intercept	Horizontal Intercepts	Vertical Asymptotes Occur When $\sin(\theta) = 0$
$f(\theta) = \csc(\theta) = \frac{1}{\sin(\theta)}$	360° or 2π radians	none	none	$\theta = 180°n$ or $\theta = \pi n$ for integer n

Like the secant function, the range includes all real numbers except those between -1 and 1. The domain includes all real numbers except where $\sin(\theta) = 0$ (at $\theta = 180°n$ or $\theta = \pi n$ radians for all integer values of n), which makes the function undefined.

Exact Values with Other Trigonometric Functions

We have already determined the exact cosine and sine values at key angles, and this information will allow us to determine the exact values of the remaining trigonometric functions at the same angles. We demonstrate a few examples of how to find these exact values and then provide a complete table. In the examples, some of these values contain radicals in the denominators. All of the values in Table 8.14 have rationalized denominators.

$$\tan(60°) = \frac{\sin(60°)}{\cos(60°)} = \frac{\sqrt{3}/2}{1/2} = \frac{\sqrt{3}}{2}\cdot\frac{2}{1} = \sqrt{3}$$

$$\sec\left(\frac{\pi}{2}\right) = \frac{1}{\cos(\pi/2)} = \frac{1}{0} \quad \text{undefined}$$

$$\csc(30°) = \frac{1}{\sin(30°)} = \frac{1}{1/2} = 2$$

$$\cot\left(\frac{\pi}{4}\right) = \left(\frac{\cos(\pi/4)}{\sin(\pi/4)}\right) = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

Table 8.14

Exact Values of the Six Trigonometric Functions for Major Reference Angles							
Angle (degrees) θ	Angle (radians) θ	Cosine $\cos(\theta)$	Sine $\sin(\theta)$	Tangent $\tan(\theta)$	Cotangent $\cot(\theta)$	Secant $\sec(\theta)$	Cosecant $\csc(\theta)$
0°	0	1	0	0	undefined	1	undefined
30°	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
90°	$\frac{\pi}{2}$	0	1	undefined	0	undefined	1

Other exact values can be determined using reference angles while making necessary adjustments to the sign of each value based on the quadrant. The diagrams in Figure 8.94 show the signs of the six trigonometric functions in each of the four quadrants.

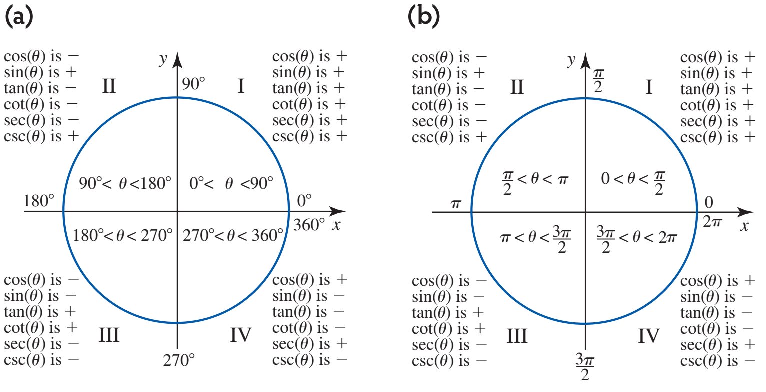


Figure 8.94

EXAMPLE 6 ■ Evaluating Secant and Cosecant

Evaluate each of the following expressions.

a. $\sec\left(\frac{3\pi}{5}\right)$ using a calculator

b. $\csc\left(\frac{8\pi}{3}\right)$ without using a calculator

Solution

a.

$$\sec\left(\frac{3\pi}{5}\right) = \frac{1}{\cos\left(\frac{3\pi}{5}\right)}$$
$$\approx -3.2361$$

b. First, we recognize $\frac{8\pi}{3}$ indicates an angle greater than 2π. We find the corresponding angle between 0 and 2π.

$$\frac{8\pi}{3} - 2\pi = \frac{8\pi}{3} - \frac{6\pi}{3}$$
$$= \frac{2\pi}{3}$$

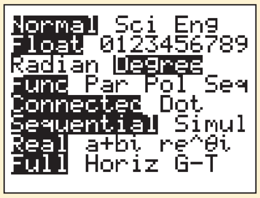
So $\csc\left(\frac{8\pi}{3}\right) = \csc\left(\frac{2\pi}{3}\right)$. The angle $\frac{2\pi}{3}$ has a reference angle measuring $\frac{\pi}{3}$, and $\csc\left(\frac{\pi}{3}\right) = \frac{2\sqrt{3}}{3}$ according to Table 8.14. However, $\frac{2\pi}{3}$ is in the second quadrant, so we need to double-check the sign value for cosecant in this quadrant. In this case, cosecant is positive in Quadrant II, so $\csc\left(\frac{2\pi}{3}\right) = \frac{2\sqrt{3}}{3}$, telling us that $\csc\left(\frac{8\pi}{3}\right) = \frac{2\sqrt{3}}{3}$.

SUMMARY

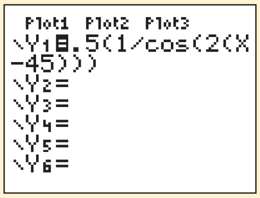
In this section you learned to define, graph, and use the tangent function. You also learned about the three reciprocal trigonometric functions: cotangent, secant, and cosecant. Additionally, you learned how to graph these four trigonometric functions and how to describe their behaviors in relationship to the cosine and sine functions.

TECHNOLOGY TIP ■ GRAPHING OTHER TRIGONOMETRIC FUNCTIONS

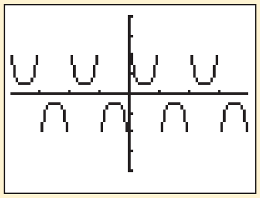
1. Press MODE to verify that your calculator is in the proper units (Radian or Degree) according to the situation.



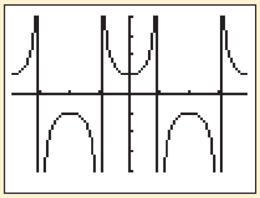
2. Use Y= to enter the trigonometric function you wish to graph. Note that only the tangent function is directly programmed into the calculator. For all other functions, you must express them using cosine, sine, or tangent according to the relationships $\cot(x) = \dfrac{1}{\tan(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, and $\csc(x) = \dfrac{1}{\sin(x)}$. For example, the correct way to enter $y = 0.5\sec(2(x - 45))$ is shown.

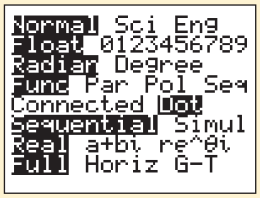


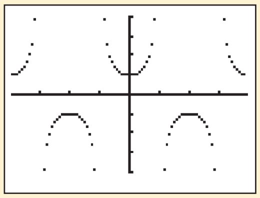
3. Under the `ZOOM` menu, select `7:ZTrig` to create a graph using a window size that fits most basic trigonometric functions. Notice that some technology does not show the entire range of values due to the limitations of pixel density on the screen. Use the WINDOW feature to further adjust the viewing window if necessary.



4. In some cases your calculator may create extra lines that are not part of your actual graph because it connects points that should not be connected, as shown. To remove these lines you may change your calculator from `Connected` mode to `Dot` mode after pressing the MODE button.







8.6 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, use your calculator to evaluate each expression, then explain what your solution means in the context of angles.

1. $\tan(175°)$

2. $\tan\left(-\dfrac{5\pi}{7}\right)$

3. $\cot(80°)$

4. $\cot(8)$

In Exercises 5–10, find the exact value of each expression without using a calculator, then explain what your solution means in the context of angles.

5. $\tan(30°)$

6. $\cot\left(\dfrac{3\pi}{2}\right)$

7. $\cot(810°)$

8. $\tan(-\pi)$

9. $\tan(135°)$

10. $\cot\left(-\dfrac{2\pi}{3}\right)$

In Exercises 11–14, determine whether the expression is equivalent to cos(θ), sin(θ), tan(θ), or cot(θ).

11. $\dfrac{\sec(\theta)}{\csc(\theta)}$

12. $\dfrac{\csc(\theta)}{\sec(\theta)}$

13. $\dfrac{\tan(\theta)}{\sec(\theta)}$

14. $\dfrac{\cot(\theta)}{\csc(\theta)}$

In Exercises 15–20, find the exact value of each expression.

15. $\sec(0)$

16. $\csc(180°)$

17. $\sec(-30°)$

18. $\csc\left(\dfrac{5\pi}{6}\right)$

19. $\csc(-855°)$

20. $\sec\left(\dfrac{8\pi}{3}\right)$

In Exercises 21–22, determine the exact values of the indicated trigonometric functions without a calculator.

21. Use the fact that $\cos(60°) = \frac{1}{2}$ to determine sec(60°).

22. Use the facts that $\cos(210°) = -\frac{\sqrt{3}}{2}$ and $\sin(210°) = -\frac{1}{2}$ to find cot(210°).

In Exercises 23–26, draw a circle and an angle θ such that the given information is true.

23. $\tan(\theta)$ is positive and $\cos(\theta)$ is negative.

24. $\sin(\theta)$ is positive and $\cot(\theta)$ is negative.

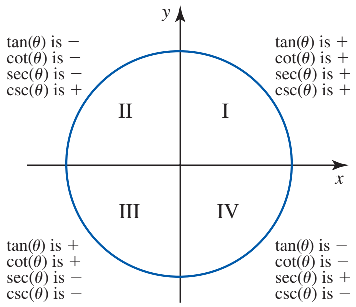
25. $\sec(\theta)$ is positive and $\csc(\theta)$ is positive.

26. $\csc(\theta)$ is negative and $\tan(\theta)$ is negative.

27. Given that $\sec(\theta) = 2$, draw a circle and show the two possible locations for the endpoint of an arc corresponding with an angle θ. Repeat for ϕ given that $\csc(\phi) = -3$.

SHOW YOU KNOW

28. The figure shows the signs of tangent, cotangent, secant, and cosecant values in each of the four quadrants. Explain why each of the functions has the indicated sign in each quadrant.



29. Let θ and ϕ be two angles such that $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, and $\theta \ne \phi$. What must be true about these angles if $\tan(\theta) = \tan(\phi)$? Explain your reasoning, then explain whether $\cot(\theta) = \cot(\phi)$ will also be true for that relationship.

30. Explain why it is possible for tangent, cotangent, secant, and cosecant values to be greater than 1 even though it is not possible for cosine or sine values to be greater than 1.

31. Explain, in your own words, why the slope of the terminal side of an angle in standard position is always found using $\tan(\theta)$ instead of $r\tan(\theta)$ even if the angle is measured on a circle with a radius other than 1.

32. Fill in the blanks with either <, >, or = to correctly complete each statement, then explain why the statement must be true. Do not use a calculator.

a. $\cos\left(\frac{\pi}{8}\right)$ ______ $\sin\left(\frac{\pi}{8}\right)$

b. $\tan\left(\frac{\pi}{8}\right)$ ______ $\cot\left(\frac{\pi}{8}\right)$

c. $\sec\left(\frac{\pi}{8}\right)$ ______ $\csc\left(\frac{\pi}{8}\right)$

d. $\cos\left(\frac{\pi}{8}\right)$ ______ $\sec\left(\frac{\pi}{8}\right)$

e. $\sin\left(\frac{\pi}{8}\right)$ ______ $\csc\left(\frac{\pi}{8}\right)$

MAKE IT REAL

33. Road Grades The Clinton-Pavilion Road through the Pavilion Mountains in British Columbia has an 8.53° uphill climb at its steepest point. Find tan(8.53°) and cot(8.53°), then explain what these values represent in this situation. **(*Source:* www.pleadwalk.org)**

34. Road Grades The westbound U.S. Highway 74 truck route through North Carolina near the Tuckaseegee River has a downhill grade of 5%. **(*Source:* www.southeastroads.com)**

a. Explain what a 5% downhill grade means, then find the angle of descent for the road using the graph of the tangent function.

b. Explain how you could have used the cotangent function to have found the answer to part (a).

c. What is the elevation change over 2.5 miles on a road with a 5% downhill grade?

35. Pyramids Menkaure ruled Egypt from 2490 to 2472 B.C. and was the third pharaoh to build a pyramid at Giza. His pyramid has an incline of 51.34°. **(*Source:* www.pbs.org)**

a. Find tan(51.34°) and cot(51.34°), then explain what these values mean in this context.

b. Menkaure's pyramid was 216 feet tall when constructed. How wide is his pyramid?

c. Write a function $w(\theta)$ to model the width of a 216-foot-tall pyramid that has an incline of θ degrees.

36. Pyramids The Luxor Casino in Las Vegas is built in the shape of a pyramid to match its Egyptian theme. The casino stands 350 feet tall and has a square base with sides that are 600 feet long. **(*Source:* Luxor Hotel Fact Sheet)**

a. Find the angle of incline for the Luxor Hotel by graphing using the cotangent function.

b. Repeat part (a) using the tangent function and compare your results.

37. Wheelchair Ramps The federal government has established guidelines for the construction of wheelchair ramps. One of these guidelines says that no single ramp may rise more than 30 inches in height over any horizontal distance. **(*Source:* www.usdoj.gov)**

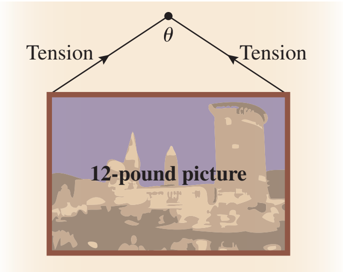
The function $L(\theta) = 30\cot(\theta)$ models the horizontal length of a ramp in inches that rises 30 inches off the ground and makes an angle of θ with the ground.

a. Only changes in height of 1 inch over every 20 inches or less are considered "ramps" by the federal government, and no ramp may increase its vertical distance more than 1 inch for every change of 12 inches of horizontal distance. What is the domain for L? What is its range?

b. Graph L over its domain, then describe the rate of change and concavity and what this information tells you about this situation.

c. The government suggests that ramps increase 1 inch for every 16 to 20 inches of horizontal distance. What is the domain of L for "ideal" ramps?

38. Wire Tension When a 12-pound picture is hung as shown, the tension T in pounds on each side of the wire is given by the function $T(\theta) = 12\sec\left(\frac{\theta}{2}\right)$.



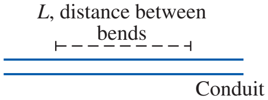
a. What is the practical domain for T? What is the practical range?

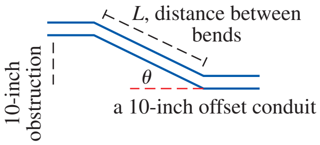
b. Graph T over the practical domain. Describe the rate of change and concavity of T, then explain what this information tells you about this situation.

c. How much tension is on each side of the wire when $\theta = \frac{2\pi}{3}$?

d. If the tension on each side of the wire is 18.6 pounds, what is the value of θ?

39. Electrical Conduits Electrical wiring in homes and businesses is run through conduits to help protect and route the wiring throughout a structure. When this conduit must be bent, such as to move around an obstacle or change levels, electricians use a rule involving the cosecant function to know where to bend the conduit to create the right amount of offset.





The function $L(\theta) = 10\csc(\theta)$ determines the length of conduit between the bends of an offset of 10 inches at an angle of θ degrees. (*Source:* www.porcupinepress.com)

a. What is the reasonable domain of this function? What is the reasonable range?

b. Graph the function over its reasonable domain. Describe the rate of change and concavity of the function, then explain what this information tells you about this situation.

c. If a contractor created a 10-inch offset with a 40° angle, what is the distance between the bends in the offset?

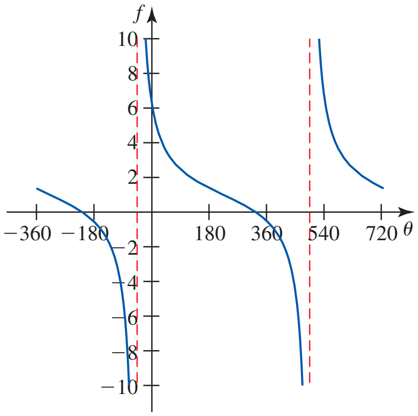
d. At what angle must a contractor bend a conduit to avoid a 10-inch obstruction using 18 inches between the bends in the offset?

e. Electricians try to use a 30° angle for offsets whenever practical. Explain some possible reasons for this.

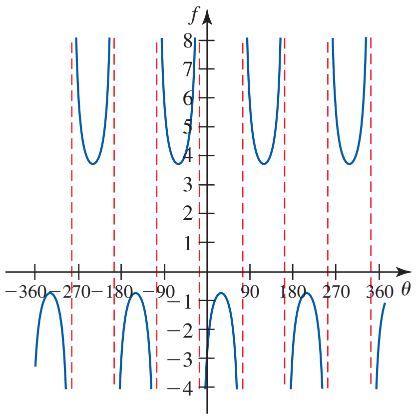
STRETCH YOUR MIND

Exercises 40–44 are intended to challenge your understanding of trigonometric functions.

40. Write a possible formula for the graph of $f(\theta)$.



41. Write a possible formula for the graph of $f(\theta)$.



42. a. Create a table of values for $f(\theta) = \tan(\theta)$ over the domain $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then create the graph for f^{-1}.

b. Discuss why the domain had to be restricted to $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ to talk about f^{-1}.

c. What does the function f^{-1} do?

43. Using a calculator, graph the function $f(\theta) = (\tan(\theta))^2$. (*Hint:* It may be helpful to look at a table of values in addition to the graph.)

a. Compare this function to the function $g(\theta) = \tan(\theta)$. Discuss any similarities and differences and the reasons for each.

b. Graph the function $h(\theta) = (\sec(\theta))^2$ on the same screen. Discuss the relationship between f and h.

c. Write a formula involving $\tan(\theta)$ and $\sec(\theta)$ based on your observations in part (b). How might this formula be useful when working with trigonometric functions?

44. Using a calculator, graph the function $f(\theta) = (\cot(\theta))^2$. (*Hint:* It may be helpful to look at a table of values in addition to the graph.)

a. Compare this function to the function $g(\theta) = \cot(\theta)$. Discuss any similarities and differences and the reasons for each.

b. Graph the function $h(\theta) = (\csc(\theta))^2$ on the same screen. Discuss the relationship between f and h.

c. Write a formula involving $\cot(\theta)$ and $\csc(\theta)$ based on your observations in part (b). How might this formula be useful when working with trigonometric functions?

SECTION 8.7 Inverse Trigonometric Functions

LEARNING OBJECTIVES

- Use inverse trigonometric functions to solve trigonometric equations algebraically
- Use trigonometric models of real-world data sets for questions whose answers require inverse trigonometric functions

GETTING STARTED

The changing tides are an example of periodic phenomena in our world. Many tidal patterns may be modeled using cosine and sine functions. Monitoring tide heights is important for many reasons including safely docking ships, military coordination during times of war, and making adjustments to satellite elevation readings. Novice sailors who fail to consider tidal patterns may find their boats beached on the shore. (*Source:* www.oceanservice.noaa.gov)

In this section we discuss inverse trigonometric functions and their graphs. We look at how to use inverse trigonometric concepts to solve trigonometric equations, such as finding the time of day when the tide will be at a certain height.

■ Solving for an Angle Measure

Suppose we have three angles whose measures are θ, ϕ, and α, and that $\cos(\theta) = 0.8$, $\cos(\phi) = 0.3$, and $\cos(\alpha) = -0.5$. These three equations show us the x-coordinate at the endpoint of the arcs associated with each angle, but in each case we do not know the actual angle measure. Figure 8.95a shows these equations as they relate to the unit circle.

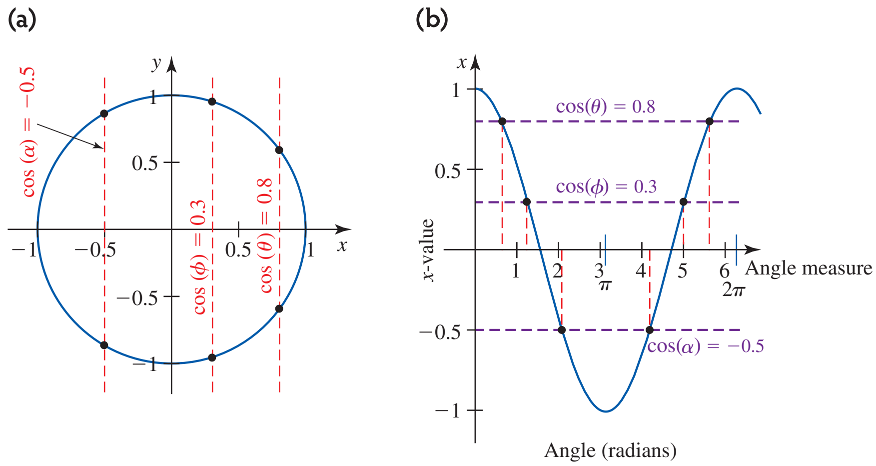


Figure 8.95

We can see there are two points on the circle that make each of the equations true, meaning there are two arcs whose endpoints have each given x-coordinate in one period of the cosine function. We can also draw the cosine function on the interval $[0, 2\pi]$ and solve these equations graphically as we learned earlier in this chapter (see Figure 8.95b.) Using a graphing calculator, we determine the solutions to the equations by finding the points of intersection. The solutions are

$\cos(\theta) = 0.8$	$\cos(\phi) = 0.3$	$\cos(\alpha) = -0.5$
$\theta \approx 0.6435$	$\phi \approx 1.2661$	$\alpha \approx 2.0944$
and	and	and
$\theta \approx 5.6397$	$\phi \approx 5.0171$	$\alpha \approx 4.1888$

However, recall that the cosine function is periodic, and angles with measures less than 0 and greater than 2π are possible. This means we can find these same x-coordinates for an *infinite* number of angles, as shown in Figure 8.96.

We can find each of the remaining solutions by adding or subtracting 2π from the initial solutions a certain number of times. We write the solutions to the equation $\cos(\theta) = 0.8$ as follows for all integer values of n.

$$\theta \approx 0.6435 + 2\pi n$$

and

$$\theta \approx 5.6397 + 2\pi n$$

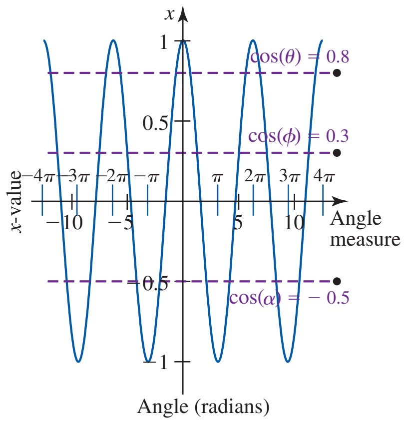


Figure 8.96

For example, let $n = 1$. This gives us angles that have completed one period plus a portion of another period of the cosine graph.

$$\theta \approx 0.6435 + 2\pi(1) \approx 6.9267 \qquad \theta \approx 5.6397 + 2\pi(1) \approx 11.9229$$

We can repeat this process indefinitely, including using negative values for n, to find any of the infinite solutions to these equations.

The process of solving for the input value when the output value is given is a common mathematical procedure. We have learned in previous chapters how to create *inverse functions* to make this process easier. We now demonstrate how to do this with trigonometric functions.

■ Inverse Cosine Function

Consider the equation $\cos(\theta) = 0.8$. This may also be written as $\cos^{-1}(0.8) = \theta$. The notation $\mathbf{cos^{-1}}$ refers to the **inverse cosine function**, which is the function that shows the same relationship as the cosine function, but whose inputs and outputs are reversed. The inverse cosine function takes the x-coordinate of the endpoint of the arc as its input and outputs the angle measure that corresponds to an arc whose endpoint has this x-coordinate.

cosine function

$$\underbrace{\cos(\overbrace{\theta}^{\text{angle measure}})}_{x\text{-coordinate of the endpoint of the arc}} = \underbrace{0.8}_{x\text{-coordinate of the endpoint of the arc}}$$

inverse cosine function

$$\overbrace{\cos^{-1}(\underbrace{0.8}_{x\text{-coordinate of the endpoint of the arc}})}^{\text{angle measure}} = \overbrace{\theta}^{\text{angle measure}}$$

If we define the function $f(\theta) = \cos(\theta)$, then the outputs of f are x-values, meaning $f(\theta) = x$ or $x = \cos(\theta)$. In the inverse form, $\cos^{-1}(x) = \theta$, the x-value is now the input and the θ-value is the output. We write this function as $f^{-1}(x) = \cos^{-1}(x)$.

Inverse Cosine Function

The function $f^{-1}(x) = \cos^{-1}(x)$ is the **inverse cosine function**. The inverse cosine function takes the x-coordinate of the endpoint of an arc as its input and outputs an angle measure that corresponds with an arc whose endpoint has this x-coordinate.

Using Technology to Evaluate the Inverse Cosine Function

Once the relationship $\cos(\theta) = 0.8$ is written in its inverse form $\cos^{-1}(0.8) = \theta$, we can use technology to solve the equation without having to graph the function and determine intersection points. See the Technology Tip at the end of the section for details.

Figure 8.97 shows that the endpoint of an arc that corresponds to an angle of 0.6435 radians has an x-coordinate of 0.8, which is the same angle measure we determined previously by graphing. However, the calculator only shows one of the infinite number of solutions to this equation. We find the remaining solutions using reference angles, as indicated in Figure 8.98. For example, the other solution between 0 and 2π is approximately 5.6397 radians, and is found using a reference angle of 0.6435 radians in Quadrant IV.

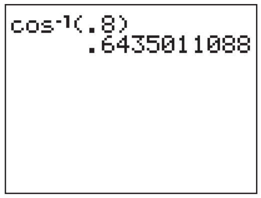


Figure 8.97

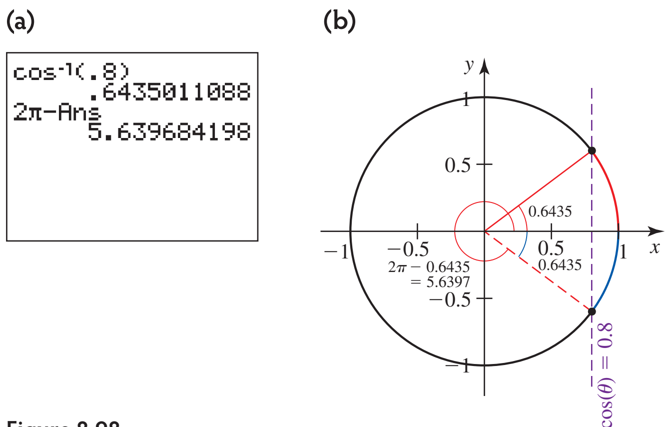


Figure 8.98

We can perform the same process with a calculator in Degree mode if θ is in degrees.

EXAMPLE 1 ■ Using the Inverse Cosine Function

Solve each of the following equations for θ over the interval $0 \le \theta \le 2\pi$, then explain what the solution represents with respect to circles.

a. $\cos(\theta) = -0.25$

b. $6\cos(\theta) = 1.04$

Solution

a. We first write the equation $\cos(\theta) = -0.25$ in its inverse form, $\cos^{-1}(-0.25) = \theta$. Using a calculator, we find $\cos^{-1}(-0.25) \approx 1.8235$, telling us that an angle measuring 1.8235 radians corresponds with an arc on the unit circle whose endpoint has an x-coordinate of -0.25. Using reference angles as shown in Figure 8.99, we find another solution in the third quadrant (the other quadrant where negative x-coordinates exist) at approximately 4.4597 radians.

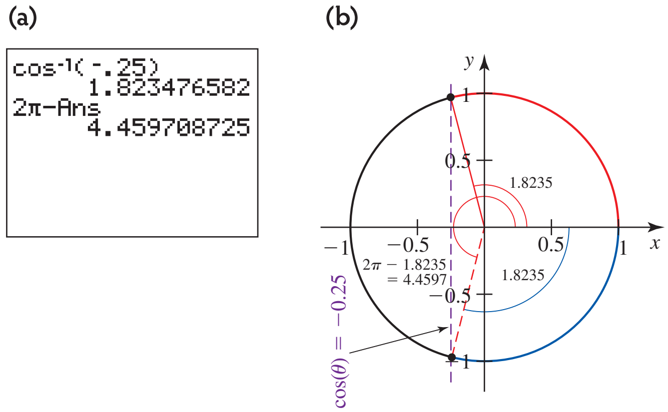


Figure 8.99

b. We recall 6 cos(θ) represents the x-coordinate at the endpoint of an arc on a circle with radius 6. We first divide the equation 6 cos(θ) = 1.04 by 6 to get $\cos(\theta) = \dfrac{1.04}{6} \approx 0.1733$. This tells us that, on the unit circle, an arc whose endpoint has an x-coordinate of 0.1733 will correspond with an angle θ. We now use the inverse cosine function and reference angles to find our solutions at $\theta \approx 1.397$ and $\theta \approx 4.887$. (See Figure 8.100.)

$$\cos(\theta) \approx 0.1733$$

$$\cos^{-1}(0.1733) \approx \theta$$

$$1.397 \approx \theta$$

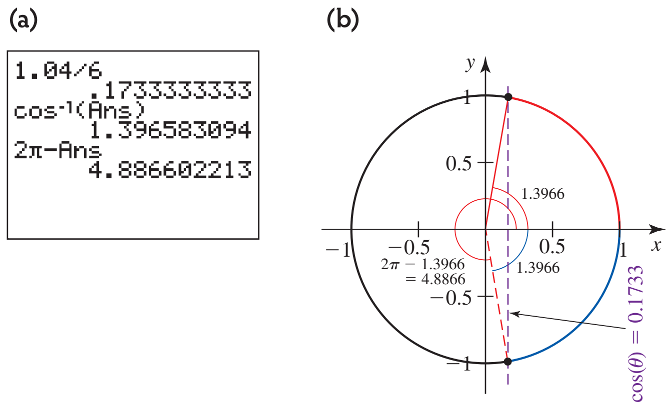


Figure 8.100

This also tells us that the arcs on a circle of radius 6 that correspond with angles of 1.397 and 4.887 radians will have endpoints whose x-coordinates are 1.04.

In the preceding examples we note that the calculator only displays one solution and that this solution is the angle measure closest to 0. We will explore this important observation in greater depth later in this section.

Finding Exact Values Using Inverse Cosine

Recall there are some angles whose exact trigonometric values we use whenever possible. Table 8.15, which we developed in Sections 8.3 and 8.6, can help us find the exact value solutions to inverse trigonometric functions as well.

Table 8.15

Exact Values of the Six Trigonometric Functions for Major Reference Angles							
Angle (degrees) θ	**Angle (radians) θ**	**Cosine $\cos(\theta)$**	**Sine $\sin(\theta)$**	**Tangent $\tan(\theta)$**	**Cotangent $\cot(\theta)$**	**Secant $\sec(\theta)$**	**Cosecant $\csc(\theta)$**
$0°$	0	1	0	0	undefined	1	undefined
$30°$	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$60°$	$\frac{\pi}{3}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$90°$	$\frac{\pi}{2}$	0	1	undefined	0	undefined	1

EXAMPLE 2 ■ **Finding Exact Values Using Inverse Cosine**

Solve each equation for θ using exact values over the interval $0 \le \theta \le 2\pi$, then explain what the solution represents with regard to the unit circle.

a. $\cos(\theta) = \frac{\sqrt{3}}{2}$

b. $9\cos(\theta) + 3 = -1.5$

Solution

a. The cosine value is positive in Quadrants I and IV, and is $\frac{\sqrt{3}}{2}$ for a reference angle of $\frac{\pi}{6}$. For $\cos(\theta) = \frac{\sqrt{3}}{2}$, we write this in inverse form $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \theta$ and say $\cos^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{6}$ since $\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$. The remaining solution will be an angle measuring $2\pi - \frac{\pi}{6}$, or $\frac{11\pi}{6}$ as shown in Figure 8.101. The solutions are $\theta = \frac{\pi}{6}$ and $\frac{11\pi}{6}$

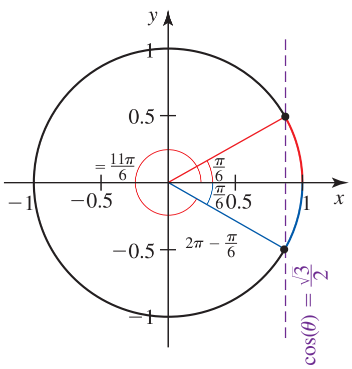


Figure 8.101

b. We begin by solving the equation for $\cos(\theta)$ and then rewrite in inverse form.

$$\begin{aligned} 9\cos(\theta) + 3 &= -1.5 \\ 9\cos(\theta) &= -4.5 \\ \cos(\theta) &= -0.5 \\ \cos^{-1}(-0.5) &= \theta \end{aligned}$$

We are looking for two angle measures that correspond to arcs on the unit circle whose endpoints have an x-coordinate of -0.5. This occurs first in Quadrant II at a reference angle of $\frac{\pi}{3}$, at $\theta = \frac{2\pi}{3}$. Thus, $\cos^{-1}(-0.5) = \frac{2\pi}{3}$ since $\cos\left(\frac{2\pi}{3}\right) = -0.5$. The other solution occurs in Quadrant III (the other quadrant where negative x-coordinates exist) at $\pi + \frac{\pi}{3}$, or $\frac{4\pi}{3}$ (see Figure 8.102). The solutions are $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$.

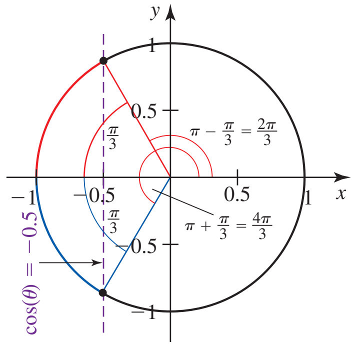


Figure 8.102

EXAMPLE 3 ■ Using the Inverse Cosine Function to Solve Equations

Tide heights are measured in feet above the mean lower low water (MLLW) height, the average of the lowest height during low tide for a given day over a 19-year period. The function $H(t) = 5\cos\left(\frac{2\pi}{13}(t + 3)\right) + 7$ models the height of the tide in feet above MLLW at Cutler Naval Base in Maine t hours after midnight on May 29, 2007. (*Source:* Modeled from a graph at www.tidesonline.noaa.gov)

a. What is the period of H and what does it tell about this situation?

b. Explain what the solution(s) to the equation $5\cos\left(\frac{2\pi}{13}(t + 3)\right) + 7 = 7.25$ will represent, then determine all of the solutions over one period of the function beginning at midnight on May 29, 2007.

Solution

a. We find the period by using $\frac{2\pi}{|B|}$ for the given function.

$$\begin{aligned}\frac{2\pi}{|B|} &= \frac{2\pi}{2\pi/13} \qquad B = \frac{2\pi}{13}\\ &= 2\pi\left(\frac{13}{2\pi}\right)\\ &= 13\end{aligned}$$

The tides at Cutler Naval Base at this time during the year have a period of 13 hours, meaning they go from high tide to low tide and back to high tide once every 13 hours.

b. The solution(s) to $5\cos\left(\frac{2\pi}{13}(t + 3)\right) + 7 = 7.25$ give the number of hours after midnight on May 29, 2007 when the tide height was 7.25 feet above MLLW.

$$\begin{aligned}5\cos\left(\frac{2\pi}{13}(t + 3)\right) + 7 &= 7.25\\ 5\cos\left(\frac{2\pi}{13}(t + 3)\right) &= 0.25\\ \cos\left(\frac{2\pi}{13}(t + 3)\right) &= 0.05\\ \cos^{-1}(0.05) &= \frac{2\pi}{13}(t + 3)\end{aligned}$$

At this step we use a calculator to determine $\cos^{-1}(0.05)$. Remember that for one period of cosine, there will be two values that make $\cos^{-1}(0.05)$ true. We use reference angles to find the second solution.

$$\cos^{-1}(0.05) \approx 1.5208 \qquad \text{One solution is between 0 and } \frac{\pi}{2}.$$

$$2\pi - 1.5208 \approx 4.7624 \qquad \text{The other solution is between } \frac{3\pi}{2} \text{ and } 2\pi.$$

Since $\cos^{-1}(0.05) = \frac{2\pi}{13}(t + 3)$, these solutions tell us the two values that $\frac{2\pi}{13}(t + 3)$ may equal over one period of cosine. We set up and solve two equations.

$$\frac{2\pi}{13}(t + 3) \approx 1.5208 \qquad \cos^{-1}(0.05) \approx 1.5208$$

$$t + 3 \approx 3.1465 \qquad \text{Multiply by } \frac{13}{2\pi}.$$

$$t \approx 0.1465$$

$$\frac{2\pi}{13}(t + 3) \approx 4.7624 \qquad \cos^{-1}(0.05) \approx 4.7624$$

$$t + 3 \approx 9.8535 \qquad \text{Multiply by } \frac{13}{2\pi}.$$

$$t \approx 6.8535$$

Approximately 0.1465 hours after midnight (about 12:09 A.M.) and 6.8535 hours after midnight (about 6:51 A.M.) the tide height will be 7.5 feet above MLLW.

■ Inverse Sine and Inverse Tangent Functions

The sine and tangent functions can also be written in inverse form. We can use these inverse functions to solve equations involving sine and tangent.

EXAMPLE 4 ■ Using the Inverse Sine Function

Given the equation $0.5 \sin(\theta) = 0.35$,

a. Explain what the solution to the equation will represent on the unit circle.

b. Solve the equation for θ on the interval $[0, 2\pi]$ by graphing.

c. Solve the equation for θ on the interval $[0, 2\pi]$ using the inverse sine function.

Solution

a. This equation is used to find angles that correspond with arcs whose endpoints on a circle with a radius of 0.5 have y-coordinates of 0.35. If we multiply each side of the equation by 2, we get $\sin(\theta) = 0.7$. Thus, on the unit circle, the same angle measures will correspond with arcs whose endpoints have a y-coordinate of 0.7. See Figure 8.103.

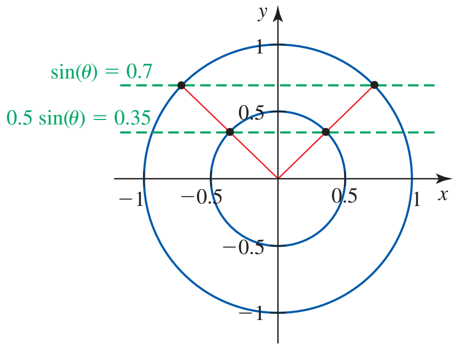


Figure 8.103

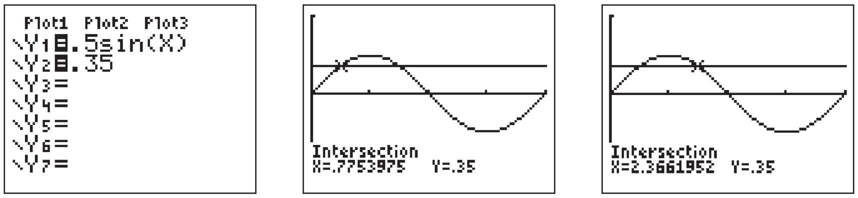
b. 

Figure 8.104

Figure 8.104 shows that angles measuring about 0.7754 and 2.3662 radians will correspond with arcs whose endpoints have a y-coordinate of 0.35 on a circle with a radius of 0.5.

c. This equation will have two solutions over the interval $[0, 2\pi]$. We use the inverse sine function to find the first solution and reference angles (Figure 8.105) to find the second.

$$0.5\sin(\theta) = 0.35$$
$$\sin(\theta) = 0.7$$
$$\sin^{-1}(0.7) = \theta$$
$$0.7754 \approx \theta \qquad \text{One solution is between 0 and } \frac{\pi}{2}.$$
$$\pi - 0.7754 \approx \theta \qquad \text{The other solution is between } \frac{\pi}{2} \text{ and } \pi.$$
$$2.3662 \approx \theta$$

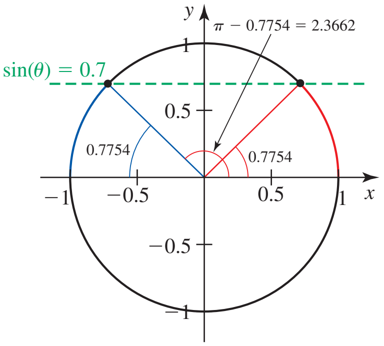


Figure 8.105

EXAMPLE 5 ■ Using the Inverse Tangent Function

Given the equation $\tan(\theta) = -1$,

a. Explain what the solution to the equation will represent.

b. Solve the equation for θ over the interval [0°, 360°] by graphing.

c. Solve the equation for θ over the interval [0°, 360°] using the inverse tangent function.

Solution

a. The equation $\tan(\theta) = -1$ is used to find an angle measure such that the terminal side of the angle has a slope of -1.

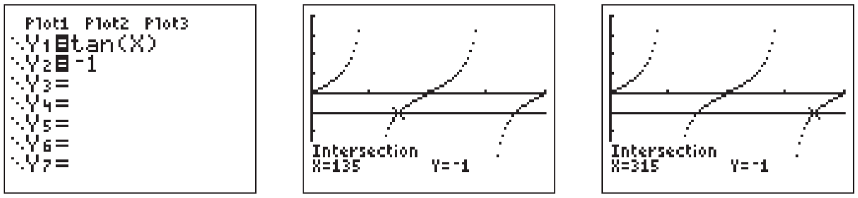
b. 

Figure 8.106

Figure 8.106 shows that angles in standard position measuring 135° and 315° have terminal sides with a slope of –1. We note that 135° and 315° are separated by 180°, a property consistent with our observations in Section 8.6 about angles with the same tangent values.

c. We begin by writing the equation $\tan(\theta) = -1$ in inverse form $\tan^{-1}(-1) = \theta$. From Table 8.15, we see that the tangent function has a magnitude of 1 when the reference angle is 45°, and it will be −1 in Quadrants II and IV.

$$180° - 45° = 135° \quad \text{One solution will be between 90° and 180°.}$$
$$135° + 180° = 315° \quad \text{The other solution will be 180° away.}$$

Another approach is to use a calculator to evaluate $\tan^{-1}(-1)$. When we do this, we get $\tan^{-1}(-1) = -45°$. In this case, a calculator returned a negative angle, which is not in the interval [0°, 360°]. To find the two solutions in the given interval, we need to add 180° (to find the next angle with the same tangent value) and 360° (to find the corresponding angle to −45° on the circle). See Figure 8.107.

$$-45° + 180° = 135°$$
$$-45° + 360° = 315°$$

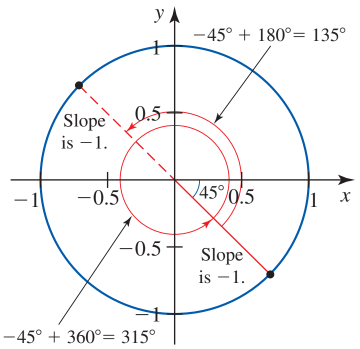


Figure 8.107

Inverse Sine Function

The function $f^{-1}(y) = \sin^{-1}(y)$ is the **inverse sine function**. The inverse sine function takes the y-coordinate of the endpoint of an arc as its input and outputs an angle measure that corresponds with an arc whose endpoint has this y-coordinate.

Inverse Tangent Function

The function $f^{-1}(x) = \tan^{-1}(x)$ is the **inverse tangent function**. The inverse tangent function takes the slope of the terminal side of an angle as its input and outputs one of the angle measures whose terminal side has this slope.

EXAMPLE 6 ■ Using Inverse Trigonometric Functions

Given $\cos(\theta) = 0.6410$ for some θ in the interval $\left[0, \frac{\pi}{2}\right]$, use your knowledge of inverse trigonometric functions to find $\sin(\theta)$.

Solution The equation $\cos(\theta) = 0.6410$ tells us the endpoint of an arc on the unit circle has an x-coordinate of 0.6410. We write this in the inverse form:

$\cos^{-1}(0.6410) = \theta$ for a value of θ between 0 and $\frac{\pi}{2}$. Using a calculator, we get

$$\begin{aligned}\theta &= \cos^{-1}(0.6410)\\ &\approx 0.8750\end{aligned}$$

So $\theta \approx 0.8750$ (which falls between 0 and $\frac{\pi}{2}$). Thus, we find $\sin(0.8750) \approx 0.7675$. An arc that corresponds with a 0.8750 radian angle has its endpoint at (0.6410, 0.7675).

EXAMPLE 7 ■ Using Inverse Trigonometric Functions with a Horizontal Compression

Find all of the solutions for the equation $\sin(3\theta) = 0.4690$ over the interval $0 \le \theta \le 360°$ without graphing.

Solution Over the interval $0 \le \theta \le 360°$ we expect there to be six solutions to the equation $\sin(3\theta) = 0.4690$. As θ varies from 0° to 360°, 3θ varies from 0° to 1080° and the sine function completes three full periods, yielding six different angles that correspond with arcs whose endpoints have 0.4690 as their y-coordinate. We begin by finding the six angle measures between 0° to 1080° that meet this criterion.

$$\sin^{-1}(0.4690) \approx 27.97°$$

The first angle that meets the criterion measures about 27.97°. The next angle is in Quadrant II and can be found using reference angles.

$$180° - 27.97° = 152.03°$$

There are four remaining angles to find over the interval [0°, 1080°], and these will correspond to the first two angles as we make a second and third revolution. See Figure 8.108.

$$27.97° + 360° = 387.97° \qquad 27.97° + 2(360°) = 747.97°$$
$$152.03° + 360° = 512.03° \qquad 152.03° + 2(360°) = 872.03°$$

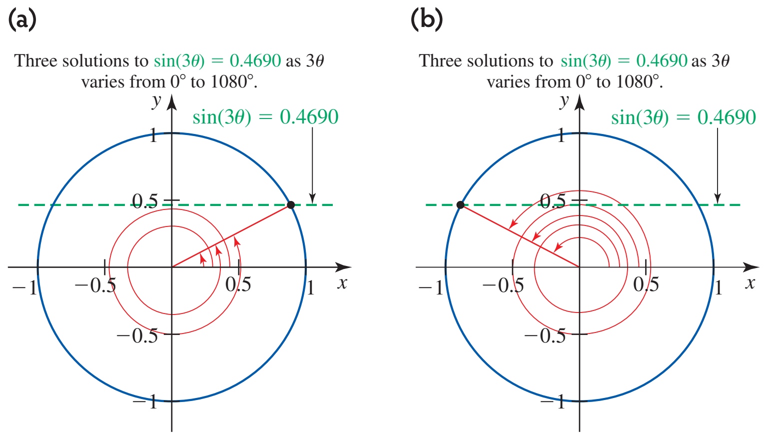


Figure 8.108

As 3θ varies from 0° to 1080°, the angles 27.97°, 152.03°, 387.97°, 512.03°, 747.97°, and 872.03° correspond with arcs whose endpoints have a y-coordinate of 0.4690. We therefore solve the following equations to find the six values of θ over the interval [0°, 360°] that make this true.

$$3\theta \approx 27.97° \qquad 3\theta \approx 152.03° \qquad 3\theta \approx 387.97°$$
$$\theta \approx 9.32° \qquad \theta \approx 50.68° \qquad \theta \approx 129.32°$$

$$3\theta \approx 512.03° \qquad 3\theta \approx 747.97° \qquad 3\theta \approx 872.03°$$
$$\theta \approx 170.68° \qquad \theta \approx 249.32° \qquad \theta \approx 290.68°$$

EXAMPLE 8 ■ Solving Equations with Other Trigonometric Functions

Solve the following equations over the interval [0°, 360°] using inverse trigonometric functions.

a. $\csc(\theta) = -1.4532$

b. $\cot(\theta) = 4.2308$

Solution

a. Since calculators are not able to work with the cosecant, we need to convert this into an equation involving sine.

$$\csc(\theta) = -1.4532$$

$$\frac{1}{\sin(\theta)} = \frac{-1.4532}{1} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\frac{\sin(\theta)}{1} = \frac{1}{-1.4532} \qquad \text{Use the reciprocal of both sides.}$$

$$\sin(\theta) \approx -0.6881 \qquad \text{Evaluate } \frac{1}{-1.4532}.$$

$$\sin^{-1}(-0.6881) \approx \theta \qquad \text{Write in inverse form.}$$

$$-43.48° \approx \theta \qquad \text{Evaluate with a calculator.}$$

The solution given by a calculator is not in the interval [0°, 360°], but it does have a reference angle of 43.48°. The solutions will have this reference angle and will be located in Quadrants III and IV.

$$180° + 43.48° = 223.48° \qquad \text{One solution will be between 180° and 270°.}$$

$$360° - 43.48° = 316.52° \qquad \text{One solution will be between 270° and 360°.}$$

Thus, $\csc(\theta) = -1.4532$ is true for $\theta \approx 223.48°$ and $\theta \approx 316.52°$ over the interval $0° \geq \theta \geq 360°$.

b. We first transform this equation to use tangent instead of cotangent.

$$\cot(\theta) = 4.2308$$

$$\frac{1}{\tan(\theta)} = \frac{4.2308}{1} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\tan(\theta) = \frac{1}{4.2308} \qquad \text{Use the reciprocal of both sides.}$$

$$\tan(\theta) \approx 0.2364 \qquad \text{Evaluate } \frac{1}{4.2308}.$$

$$\tan^{-1}(0.2364) \approx \theta \qquad \text{Write in inverse form.}$$

$$13.30° \approx \theta \qquad \text{Evaluate with a calculator.}$$

The terminal sides of angles measuring 13.30° and 193.30° (180° greater than 13.30°) have a slope of 0.2364. Therefore, $\cot(\theta) = 4.2308$ is true on the interval $0° \geq \theta \geq 360°$ for $\theta \approx 13.30°$ and $\theta \approx 193.30°$.

■ Graphing Inverse Trigonometric Functions

We have noticed that a calculator does not automatically give us all of the solutions to an equation when we use inverse trigonometric functions. In fact, sometimes we get positive angle measures and sometimes we get negative measures. Why? There is a very important reason for this involving their definition as *functions*. Recall that a function has only one output for any given input. To create an inverse trigonometric function, we must first restrict the domain of the trigonometric function so that the function is strictly increasing or decreasing. We can then define the corresponding inverse trigonometric function.

Graphing the Inverse Cosine Function

We use some of the exact values for the cosine function to create a partial table of values that has the cosine of an angle as the input and the angle itself as the output. See Tables 8.16 and 8.17.

Table 8.16

Angle (degrees) θ	Cosine (x-coordinate) $\cos(\theta) = x$
0	1
$\pi/4$	$\sqrt{2}/2$
$\pi/2$	0
$3\pi/4$	$-\sqrt{2}/2$
π	-1
$5\pi/4$	$-\sqrt{2}/2$
$3\pi/2$	0
$7\pi/4$	$\sqrt{2}/2$
2π	1

Table 8.17

Cosine (x-coordinate) x	Angle (degrees) $\cos^{-1}(x) = \theta$
1	0
$\sqrt{2}/2$	$\pi/4$
0	$\pi/2$
$-\sqrt{2}/2$	$3\pi/4$
-1	π
$-\sqrt{2}/2$	$5\pi/4$
0	$3\pi/2$
$\sqrt{2}/2$	$7\pi/4$
1	2π

When we graph the cosine function and the data from Table 8.16, we see it is a function (Figure 8.109a), but when we investigate the graph of Table 8.17, we see it is not a function (Figure 8.109b).

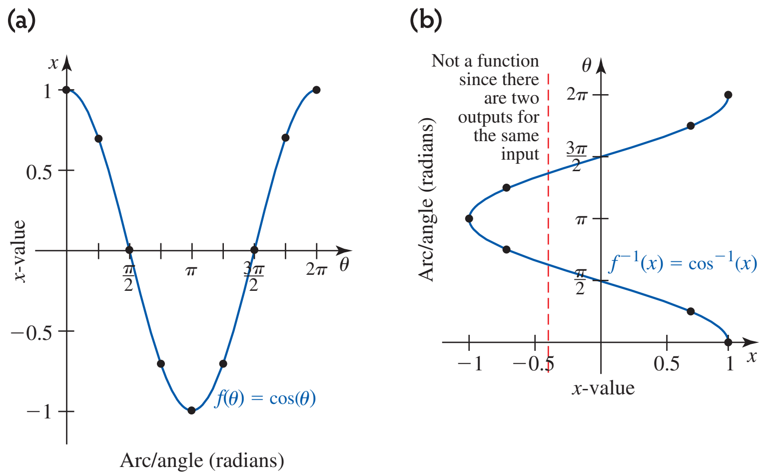


Figure 8.109

In fact, when we graph functions such as $f^{-1}(x) = \cos^{-1}(x)$ using a calculator as shown in Figure 8.110, we see that a calculator graphs the inverse based on a restricted domain for the cosine function.

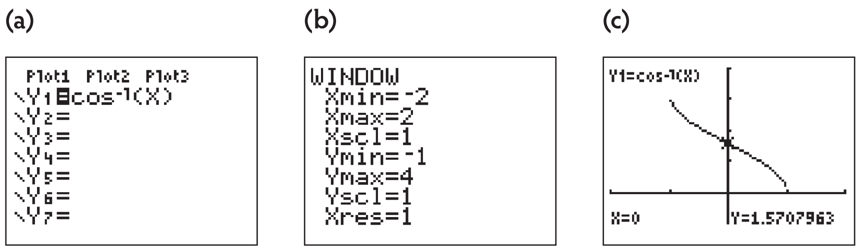


Figure 8.110

It appears that a calculator only shows inputs (x-values) between -1 and 1 and outputs angles between 0 and π. Let's examine the reasons for this domain and range.

As we saw in Figure 8.95, repeated here in Figure 8.111, there were two solutions to each of the equations $\cos(\theta) = 0.8$, $\cos(\phi) = 0.3$, and $\cos(\alpha) = -0.5$ on the unit circle.

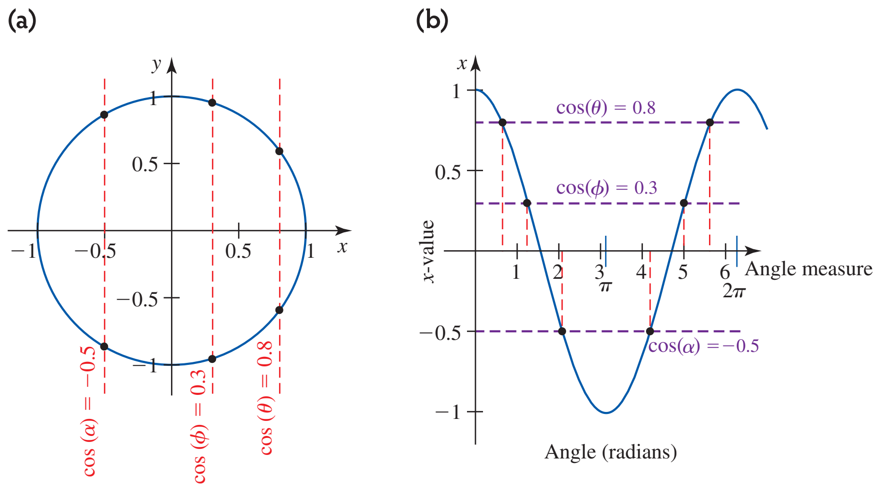


Figure 8.111

If there are two different input values with the same output value, then when we create the inverse relationship we have two outputs for the same input value, which is not a function. Thus, we need to restrict the original domain of the cosine function such that there will be one output associated with a given input.

We notice all of the possible x-coordinates for the endpoint of an arc occur in Quadrants I and II (the top half of the unit circle). That is, for angles between 0° and 180° or 0 and π radians, all of the possible x-coordinates are generated. All of the x-coordinates in Quadrants III and IV match those found in Quadrants I and II. Thus, by restricting the domain of the cosine function to the interval $0° \leq \theta \leq 180°$ or $0 \leq \theta \leq \pi$, we will still be able to represent all of the possible values of the cosine function (values between -1 and 1), but each of these values will only be associated with one angle.

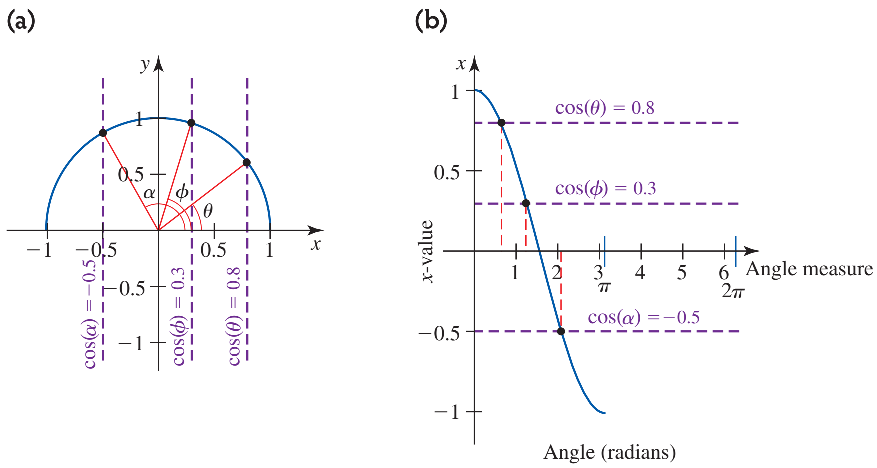


Figure 8.112

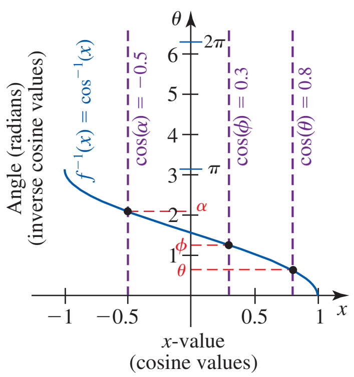


Figure 8.113

When the inverse of the function in Figure 8.112 is graphed (shown in Figure 8.113), we now see the same graph generated by a calculator, showing us that a calculator considers the domain of the inverse cosine function to be $[-1, 1]$ and its range to be $[0°, 180°]$ or $[0, \pi]$. This is why a calculator only returns one angle measure when calculating inverse cosine values. Other angles with the same trigonometric values can always be found using reference angles.

Graphing the Inverse Sine and Inverse Tangent Functions

As with the inverse cosine function, the inverse sine and inverse tangent relations will not be functions unless we restrict the domain of the original sine and tangent functions.

Let's consider the equation $\sin(\theta) = -0.25$. We can examine the solutions to this equation on the unit circle (Figure 8.114a) and the graph of the sine function (Figure 8.114b).

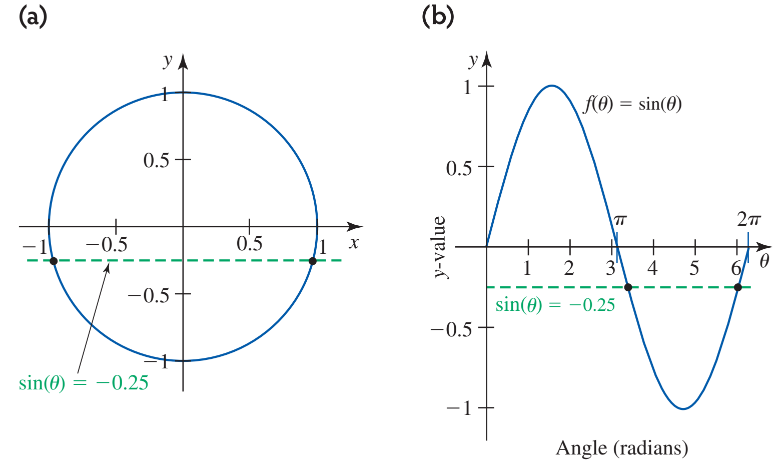


Figure 8.114

As with cosine, we find two solutions to this equation on the circle. We notice each time we get two solutions to an equation such as $\sin(\theta) = -0.25$, one of the solutions is in Quadrant I or IV and the other is in Quadrant II or III. In fact, all of the values of the sine function (y-coordinates from -1 to 1) will occur on either the right side or the left side of the circle (as opposed to either the top or the bottom of the circle with cosine). To create the inverse sine relation and have it be a function, we need to restrict the domain such that we include all of the values of sine and have an interval closest to the origin. We choose the interval $[-90°, 90°]$ or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ shown in Figure 8.115a. (Note: There are other intervals that we could have chosen, but the common convention is to choose the intervals closest to the origin.) With this restricted domain for the sine function, the inverse sine relation will now be a function, shown in Figure 8.115b.

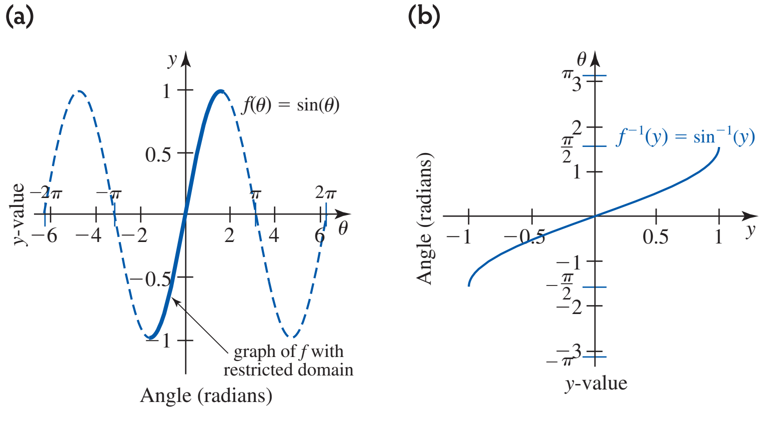


Figure 8.115

The graph shown in Figure 8.115b is identical to what a calculator will display when you graph the inverse sine function. All of the angle measures the calculator returns when we use the inverse sine function will be in the interval [−90°, 90°] or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, showing that this is how a calculator defines the range of the inverse sine function.

To restrict the domain of the tangent function, we simply choose the complete period nearest to the origin since the tangent function outputs all of its slope values exactly once over each of its periods. Thus, if we restrict the domain of tangent to the interval [−90°, 90°] or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (Figure 8.116a), its inverse will be a function (Figure 8.116b).

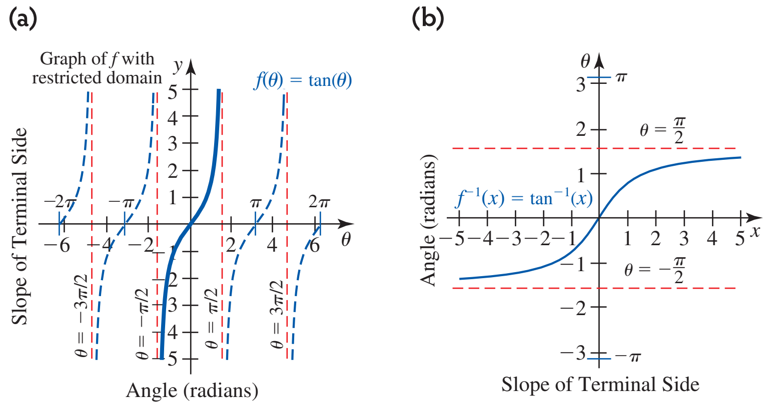


Figure 8.116

The graph we see in Figure 8.116b is identical to the one a calculator creates when we graph the inverse tangent function. Likewise, the calculator restricts the range of the inverse tangent function to the interval [−90°, 90°] or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, which is why we only get angle measures within this interval when we use a calculator to find inverse tangent values.

Domain and Range for Inverse Trigonometric Functions

- The inverse cosine function $f^{-1}(x) = \cos^{-1}(x)$ has a domain of [−1, 1] and a range of [0°, 180°] or $[0, \pi]$.
- The inverse sine function $f^{-1}(y) = \sin^{-1}(y)$ has a domain of [−1, 1] and a range of [−90°, 90°] or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.
- The inverse tangent function $f^{-1}(y) = \tan^{-1}(x)$ has a domain of all real numbers and a range of [−90°, 90°] or $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

EXAMPLE 9 ■ Graphing Inverse Trigonometric Functions and Solving Equations

The sport of rodeo (including the bull riding event) is popular in Cheyenne, Wyoming. The function

$$T(m) = -20.6\cos\left(\frac{\pi}{6}(m-1)\right) + 50$$

models the average temperature in Cheyenne in degrees Fahrenheit during month m of the year. (For this model, m is in radians.) (*Source:* Modeled using data from aa.usno .navy.mil)

a. Graph the function and determine a restricted domain for T such that the inverse will also be a function.

b. Find the formula for the inverse function of T. Explain what the inputs and outputs of this function are.

c. Graph the inverse function you found in part (b).

d. According to your function in part (b), when is the average temperature 35°F in Cheyenne?

Solution

a. After graphing the function as shown in Figure 8.117, we choose to restrict the domain to [1, 7]. This domain ensures that we have all of the output values of T represented once.

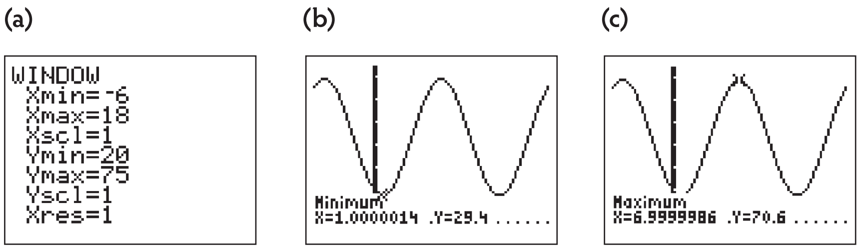


Figure 8.117

b. From part (a) or from using the amplitude and midline of the function we know that the inverse function will have inputs that are average temperatures (T) between 29.4°F and 70.6°F and outputs that are months of the year (m) between 1 and 7 (beginning of January to the beginning of July). We now solve for the input variable in our original function to create the inverse function.

$$-20.6\cos\left(\frac{\pi}{6}(m-1)\right)+50 = T$$

$$-20.6\cos\left(\frac{\pi}{6}(m-1)\right) = T-50$$

$$\cos\left(\frac{\pi}{6}(m-1)\right) = -\frac{T-50}{20.6}$$

$$\frac{\pi}{6}(m-1) = \cos^{-1}\left(-\frac{T-50}{20.6}\right) \qquad \text{Write in inverse form.}$$

$$m-1 = \frac{6}{\pi}\cos^{-1}\left(-\frac{T-50}{20.6}\right)$$

$$m = \frac{6}{\pi}\cos^{-1}\left(-\frac{T-50}{20.6}\right)+1$$

The inverse function is $m(T) = \dfrac{6}{\pi}\cos^{-1}\left(-\dfrac{T-50}{20.6}\right)+1$.

c. The graph is shown in Figure 8.118.

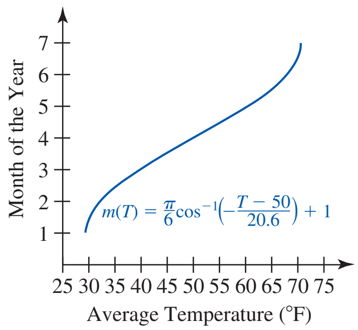


Figure 8.118

d.

$$\begin{aligned} m(35) &= \frac{6}{\pi}\cos^{-1}\left(-\frac{35-50}{20.6}\right)+1 \\ &= \frac{6}{\pi}\cos^{-1}\left(-\frac{15}{20.6}\right)+1 \\ &\approx \frac{6}{\pi}\cos^{-1}(-0.7282)+1 \\ &\approx 2.44 \qquad \text{Evaluate with a calculator.} \end{aligned}$$

The average temperature will be 35°F in about the middle of February. We note that, due to the restricted domain and range of function m, we only have one solution to this equation. We verify this solution using the graph from part (c).

■ Alternative Notation for Inverse Trigonometric Functions

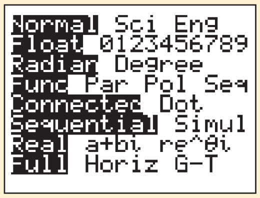
Since the outputs of the inverse trigonometric functions are angle measures that correspond with endpoints of arcs on the unit circle, they are sometimes called arc functions, as in **arc cosine (arccos)**, **arc sine (arcsin)**, and **arc tangent (arctan)**. We use this notation interchangeably with inverse trigonometric notation: for example, $f^{-1}(x) = \cos^{-1}(x)$ can be written $f^{-1}(x) = \arccos(x)$ without changing its meaning. Arc function notation is sometimes preferred because it eliminates the confusion between an exponent of -1 and the -1 that is part of inverse trigonometric function notation.

SUMMARY

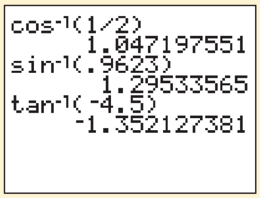
In this section you learned about inverse trigonometric functions and how they are used to solve equations where the value of a trigonometric function is known but not the angle measure. You also learned how to find and graph inverse trigonometric functions and how to restrict the domain of a trigonometric function so that its inverse is also a function.

TECHNOLOGY TIP ■ USING COS^{-1}, SIN^{-1}, AND TAN^{-1}

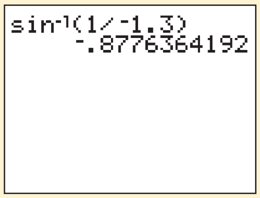
1. Press MODE to verify that your calculator is in the proper units (Radian or Degree) according to the situation.



2. Press 2nd, followed by COS, SIN, or TAN to use `cos`$^{-1}$, `sin`$^{-1}$, or `tan`$^{-1}$, respectively. This will give you one angle that has the given value for the trigonometric function you selected. You will need to use reference angles to find additional angles with the given trigonometric value if this is not the desired solution.



3. To find the values of the inverse secant, cosecant, or cotangent functions, you will need to use the relationships $\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$, $\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right)$, and $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right)$. The method for evaluating $\csc^{-1}(-1.3)$ is shown.



8.7 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, find at least one value for the given expression over the interval [0, 2π], then explain what your answer represents.

1. $\sin^{-1}(0.68)$

2. $\tan^{-1}(0.3267)$

3. $\arccos(-0.4682)$

4. $\arctan(8.4)$

In Exercises 5–8, evaluate each expression exactly, finding at least one value over the interval [0, 2π] without using a calculator. Explain what your answer represents.

5. $\cos^{-1}\left(\frac{1}{2}\right)$

6. $\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

7. $\sin^{-1}\left(-\frac{\sqrt{2}}{2}\right)$

8. $\csc^{-1}(-1)$

In Exercises 9–12, find all solutions to each equation over the interval [0°, 360°] using inverse trigonometric functions. Check your solutions by graphing.

9. $\cos(\theta) = 0.7767$

10. $\tan(\theta) = -11.2960$

11. $6 = 8\cos(\theta)$

12. $-9\cot(\theta) = 3\sqrt{3}$

In Exercises 13–16, find all solutions to each equation over the interval [0, 2π] using inverse trigonometric functions. Check your solutions by graphing.

13. $\sin(\theta) = -0.4390$

14. $3\sin(\theta) = 2.6402$

15. $\cos\left(\frac{1}{2}\theta\right) = -\frac{\sqrt{3}}{2}$

16. $1.4650 = \sec(3\theta)$

17. a. Explain how the tangent and inverse tangent functions can be used to solve the equation $\frac{\sin(\theta)}{\cos(\theta)} = 0.8361$, then find at least one solution.

b. Using the reasoning you gave in part (a), find at least one solution to each of the following equations.

i. $\sin(\theta) = 2.5\cos(\theta)$

ii. $1.32\cos(\theta) = -4\sin(\theta)$

iii. $\cos(\theta) - \sin(\theta) = 0$

18. a. Graph the cotangent, secant, and cosecant functions over restricted domains such that their inverses will be a function.

b. Graph the inverse cotangent function, inverse secant function, and inverse cosecant function based on the restricted domains in part (a).

c. Identify the domain and range for these three inverse functions.

d. Explain how to use your calculator to graph these inverse functions, then graph each using your calculator and compare these graphs to your graphs in part (b).

In Exercises 19–22, find all solutions to the given equation over the interval [0°, 360°] using inverse trigonometric functions.

19. $\cos(\theta + 15°) = 0.8265$

20. $1.5\tan(\theta + 45°) = 3$

21. $-\frac{1}{4}\sin(\theta - 60°) = 1$

22. $3\sec(\theta - 40°) = 8.6$

In Exercises 23–30, evaluate each of the expressions exactly. All angles are in radians in the first quadrant.

23. $\tan\left(\cos^{-1}\left(\frac{1}{2}\right)\right)$

24. $\sin(\tan^{-1}(1))$

25. $\sec(\cot^{-1}(\sqrt{3}))$

26. $\cot(\sec^{-1}(1))$

27. $\tan^{-1}\left(\cot\left(\frac{\pi}{3}\right)\right)$

28. $\cot^{-1}(\sin(0))$

29. $\csc^{-1}\left(\sec\left(\frac{\pi}{6}\right)\right)$

30. $\sin(\csc^{-1}(\sqrt{2}))$

In Exercises 31–34, evaluate each expression using a calculator. All angles are in radians in the first quadrant.

31. $\tan(\sin^{-1}(0.7053))$

32. $\cos(\sin^{-1}(0.2879))$

33. $\sec(\tan^{-1}(1.4602))$

34. $\csc(\sec^{-1}(2.3760))$

In Exercises 35–40, evaluate the indicated trigonometric expression using the given information.

35. If $\cos(\theta) = 0.0532$ over the interval [0°, 90°], find $\sin(\theta)$.

36. If $\tan(\theta) = 1.4269$ over the interval [0°, 90°], find $\cos(\theta)$.

37. If $\sin(\theta) = -0.4850$ over the interval $\left[\pi, \frac{3\pi}{2}\right]$, find $\tan(\theta)$.

38. If $\tan(\theta) = -4.8035$ over the interval $\left[\frac{3\pi}{2}, 2\pi\right]$, find $\csc(\theta)$.

39. If $\sec(\theta) = -3.1333$ over the interval [90°, 180°], find $\cot(\theta)$.

40. If $\csc(\theta) = -4.0023$ over the interval [270°, 360°], find $\sec(\theta)$.

SHOW YOU KNOW

41. One of your classmates says that the equation $\sin\left(\frac{1}{2}\theta\right) = 0.5$ will only have one solution over the interval $0° \le \theta \le 360°$ and gives the following rationale. “The equation $\sin(\theta) = 0.5$ has two solutions over the interval $0° \leq \theta \leq 360°$, but $\sin\left(\frac{1}{2}\theta\right)$ means that the sine function only completes half of one period as θ goes from 0° to 360°. Thus, there will only be one solution to the equation $\sin\left(\frac{1}{2}\theta\right) = 0.5$."

a. Explain what part of his conclusion is correct.

b. Explain what part of his conclusion is incorrect and why.

42. One of your classmates says that the equation $\sec(\theta) = 3$ has no solution. She justifies this claim by plugging $\frac{1}{\cos^{-1}(3)}$ into the calculator and showing you that the calculator outputs a domain error. Is she correct? Explain.

43. Compare and contrast the expressions $\cos(1)$ and $\cos^{-1}(1)$. Assume all angles are in radians.

44. Explain why the six basic inverse trigonometric functions will never have a vertical asymptote despite the fact that four of the original trigonometric functions have vertical asymptotes.

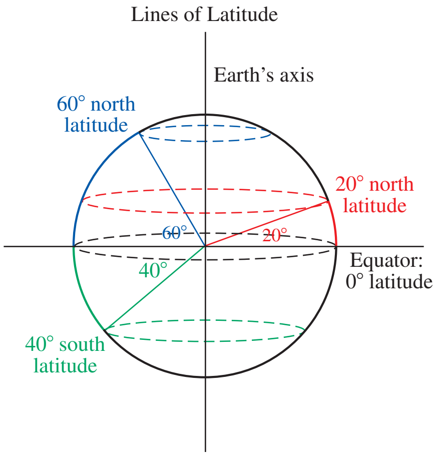
MAKE IT REAL

In Exercises 45–46,

a. Using inverse trigonometric functions, find a solution to the given equation that is reasonable in the context of the problem.

b. Explain what your solution means, and explain why it is a reasonable solution.

45. Lines of Latitude The formula $C(\theta) = 2\pi(3963\cos(\theta))$ models the circumference of the circle in miles formed by a line of latitude θ degrees from the equator.



Solve $2\pi(3963\cos(\theta)) = 11{,}880$. Explain what your answer represents.

46. Wheelchair Ramps The federal government has established guidelines for the construction of wheelchair ramps, one of which says that a single ramp may not increase its height more than 30 inches over any horizontal distance. The function $L(\theta) = 30\cot(\theta)$ models the horizontal length of a ramp in inches that rises 30 inches off the ground and makes an angle of θ with the ground. Solve $30\cot(\theta) = 490$. Explain what your answer represents. (*Source:* www.usdoj.gov)

In Exercises 47–50,

a. Using inverse trigonometric functions, find all solutions to the given equation over the interval provided.

b. Explain what your solution(s) mean in the given context.

c. Use a calculator to verify your solutions.

47. Seasonal Employment The function

$$L(m) = -550\cos\left(\frac{\pi}{6}(m-1)\right) + 3300$$

models the number of people employed in leisure and hospitality fields in Clatsop County, Oregon, m months after January 2004. Solve the equation

$$-550\cos\left(\frac{\pi}{6}(m-1)\right) + 3300 = 4200$$

over the interval [1, 13]. (*Source:* Modeled from a graph at www.qualityinfo.org)

48. Average Temperatures The function

$$S(m) = 9.40\cos\left(\frac{\pi}{6}(m-1)\right) + 57.8$$

models the average monthly temperatures in degrees Fahrenheit in Sydney, Australia during the mth month of the year. Solve

$$53.1 = 9.40\cos\left(\frac{\pi}{6}(m-1)\right) + 57.8$$

on the interval [1, 12]. (*Source:* Modeled using data from www.engr.udayton.edu)

49. Hours of Daylight The number of hours of daylight in New York City d days after March 21, 2010 can be modeled by

$$N(d) = 2.925\sin\left(\frac{2\pi}{365}d\right) + 12.18$$

Solve

$$2.925\sin\left(\frac{2\pi}{365}d\right) + 12.18 = 11.5$$

over the interval [−79, 285]. (*Source:* Modeled using data from aa.usno.navy.mil)

50. Ferris Wheels The function

$$H(\theta) = 246\sin(\theta - 90°) + 296$$

models a person's height in feet on the Singapore Flyer Ferris wheel after boarding at the bottom and traveling θ degrees around the wheel. Solve

$$246\sin(\theta - 90°) + 296 = 110$$

over the interval [0°, 720°]. (*Source:* www.singaporeflyer.com.sg)

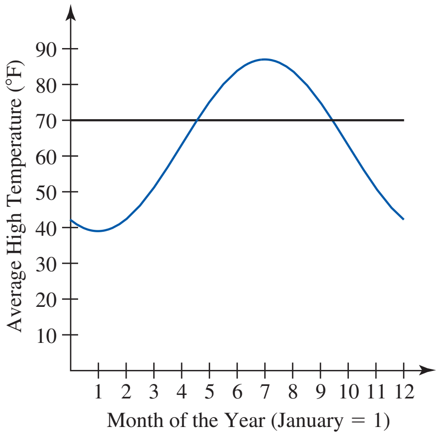
In Exercises 51–54,

a. Find a possible formula for the curve.

b. Find all of the intersection points of the curve and the line shown using inverse trigonometric functions.

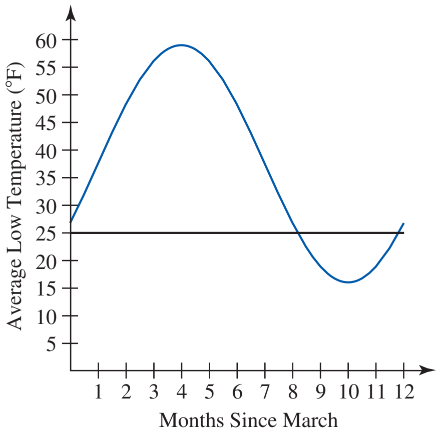
c. Explain what your solutions represent.

51. Average High Temperature



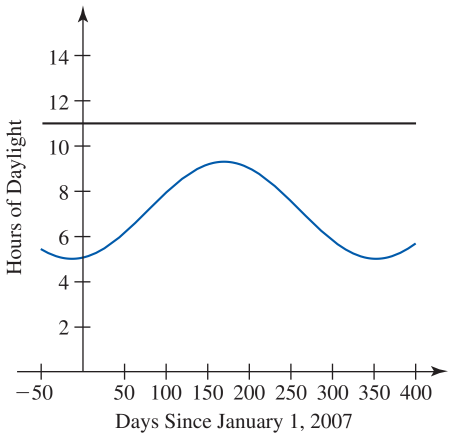
Source: **www.weatherbase.com**

52. Average Low Temperature



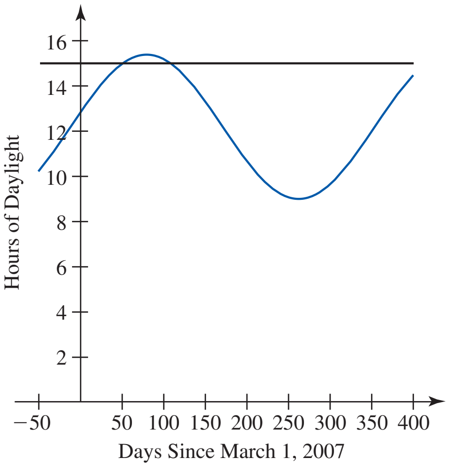
Source: **www.weatherbase.com**

53. Hours of Daylight



Source: **aa.usno.navy.mil**

54. Hours of Daylight



Source: **aa.usno.navy.mil**

In Exercises 55–56,

a. Graph the function, then identify an appropriate restricted domain such that its inverse will also be a function.

b. Find the inverse function and state its domain and range.

c. Graph the inverse function over the domain and range found in part (b).

d. Use the inverse function found in part (b) to answer the given question.

55. Ferris Wheels The function

$$H(t) = 221 \sin\left(\frac{\pi}{15}t - \frac{\pi}{2}\right) + 221$$

models a person's height in feet from the bottom of the London Eye Ferris wheel after boarding at the bottom and traveling around the wheel for t minutes. When is a rider 155 feet higher than where the rider boarded? (***Source:*** **www.aboutbritain.com**)

56. Seasonal Employment The function

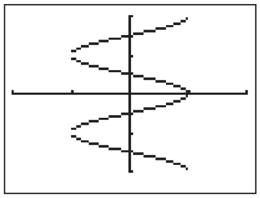
$$A(m) = -18{,}000 \cos\left(\frac{\pi}{6}(m - 1)\right) + 59{,}000$$

models the number of workers in the agricultural industry m months after January 2004 in Clatsop County, Oregon. (***Source:*** **Modeled from a graph at www.qualityinfo.org**) When will the number of workers be 41,000?

STRETCH YOUR MIND

Exercises 57–58 are intended to challenge your understanding of inverse trigonometric functions.

57. The graph is the inverse cosine relation for two periods of the cosine function. Write a series of formulas necessary to duplicate this graph on your calculator.



58. Consider the functions $f(x) = \sin^{-1}(\sin(x))$ and $g(x) = \sin(\sin^{-1}(x))$. Assume all angles are in radians.

a. What are the domain and range of each function?

b. Graph each function separately and explain why they appear the way they do.

c. One of your classmates says that both of these functions are identical, and that they can be simplified to $f(x) = x$ and $g(x) = x$, so $f(x) = g(x)$. Explain why he might have come to this conclusion, but why this is not a correct assumption.

In Exercises 59–60, evaluate each expression without using a calculator. All angles are in radians in the first quadrant.

59. $\cos(\tan^{-1}(\cot(\csc^{-1}(2))))$

60. $\csc^{-1}\left(\cot\left(\sec^{-1}\left(2\sin\left(\frac{\pi}{4}\right)\right)\right)\right)$

CHAPTER 8 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 8.1

1. What does it mean for something to be "periodic"?
2. What is the relationship between maximum value, minimum value, midline, and amplitude?

SECTION 8.2

3. How are arc length and radius related?
4. What is a radian?

SECTION 8.3

5. What is the relationship between the sine and cosine functions and the unit circle?
6. Why will the values of $f(\theta) = r\sin(\theta)$ and $g(\theta) = r\cos(\theta)$ never exceed r?

SECTION 8.4

7. How do you solve a trigonometric equation graphically?
8. How do you solve a trigonometric equation with a data table?

SECTION 8.5

9. How can you tell if a sinusoidal function will model a data set well?
10. How do you construct a sinusoidal model?

SECTION 8.6

11. How are tangent, cotangent, secant, and cosecant related to sine and cosine?
12. Why do tangent, cotangent, cosecant, and secant have restricted domains?

SECTION 8.7

13. What is the relationship between a trigonometric function and its inverse?
14. What are the benefits of using inverse trigonometric functions in solving trigonometric equations?
15. What are the drawbacks of using inverse trigonometric functions in solving trigonometric equations?

REVIEW EXERCISES

SECTION 8.1

1. **Merry-Go-Round** A child begins a ride on a merry-go-round. The independent variable is the elapsed time from when he began to ride and the dependent variable is the child's position in relation to where he started. Sketch a graph to model this scenario. (*Note:* There may be more than one correct answer.)

2. **Solar Irradiance** *Solar irradiance* is radiant energy emitted by the sun resulting from a nuclear fusion reaction that creates electromagnetic energy. The graph shows the pattern that solar irradiance measured in watts per square meter (W/m^2) has been found to follow over time.

   a. Estimate the period, the amplitude, and the midline for the solar irradiance. Explain the real-world meaning of each.

   b. Determine from the graph the years in which solar irradiance was at its lowest.

3. **Surface Water Temperature** A scientific study of the Jack Bay in the lower Patuxent River in Maryland involved measuring the surface temperature of the water. Data were collected from 1998 to 1999.

Time in Months Since Jan. 1, 1998 t	Surface Temperature (degrees Celsius) C
1	5.90
2	6.10
3	6.95
4	15.00
5	19.90
6	23.70
7	27.45
8	28.05
9	25.95
10	19.10
11	11.70
12	12.90
13	5.30
14	6.90
15	6.25
16	13.45
17	19.20
18	23.70
19	26.25
20	28.00
21	23.60
22	17.00
23	12.90
24	8.30

(continued)

Source: **www.dnr.state.md.us**

   a. Estimate the average rate of change between months 8 and 12. Interpret your answer in terms of the context.

   b. Estimate the instantaneous rate of change at month 20. Interpret your answer.

   c. Estimate where the highest and lowest temperatures are in each year and explain what happens before and after each in terms of rate of change.

   d. Does this table of data appear to be periodic? Explain.

4. **Earthquakes** The number of worldwide earthquakes E of different intensities that occurred each year for 2005 and 2006 is displayed in the table.

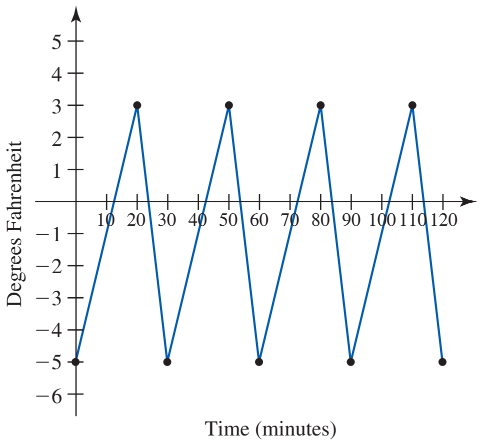
Magnitude (Richter scale value) m	Number of Earthquakes in 2005 E	Number of Earthquakes in 2006 N
8.0 to 9.9	1	1
7.0 to 7.9	10	10
6.0 to 6.9	141	132
5.0 to 5.9	1697	1483
4.0 to 4.9	13,918	13,069
3.0 to 3.9	9189	9953
2.0 to 2.9	4636	4016
1.0 to 1.9	26	19
0.1 to 0.9	0	2

Source: **neic.usgs.gov/neis/eqlists/eqstats.html**

   a. Do these data appear to be periodic? Explain.

   b. Using these data, predict how many earthquakes there will be of magnitude 3.5 on the Richter scale in 2010.

5. Freezer Temperature The graph models the fluctuation in temperature in degrees Fahrenheit inside of a freezer with its door closed.

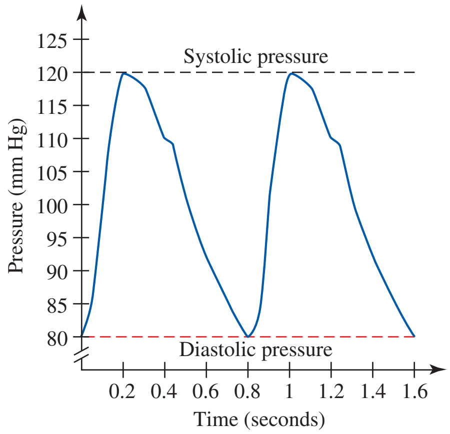


a. Estimate the temperature in the freezer after 50 and 70 minutes.

b. Does this graph represent a periodic function? Explain.

c. Suppose the freezer door is opened after being closed for 120 minutes and is left open for 30 minutes in a 70°F room. Sketch a graph representing the temperature inside the freezer for the 30-minute period that the door is open after the 120-minute period represented in the graph.

6. Blood Pressure The graph gives the variation in blood pressure for a typical person. Systolic and diastolic pressures are the upper and lower limits of the periodic changes in pressure that produce the pulse. The length of time between peaks is called the period of the pulse.



Source: **www.vaughns-1-pagers.com/medicine/blood-pressure.htm**

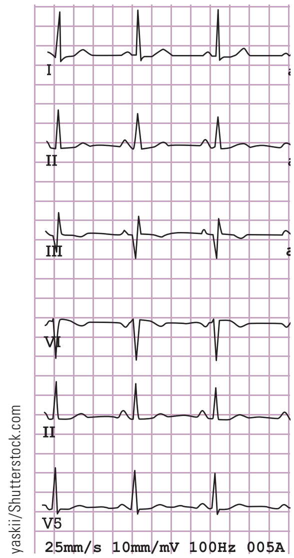
a. Find the amplitude, midline, and period.

b. Find the frequency (also known as the pulse rate—the number of pulse beats in one minute) for the typical person.

c. When a person's blood pressure reaches 160 (systolic pressure) over 100 (diastolic pressure) many doctors require their patients to receive treatment. What aspect(s) of the graph would be affected with these blood pressure readings? Estimate the numerical value(s) of the parameter(s) that has to change.

7. Pilot-Induced Oscillation *Pilot-induced oscillation* is caused when a pilot makes a sequence of corrections in opposing directions. For example, a pilot aiming for a 450-foot-per-minute descent may find herself descending too rapidly at 475 feet per minute. She may then try to correct this by slowing the vertical speed to 450 feet per minute. However, the vertical speed indicator generally lags the actual vertical speed. Consequently, the vertical speed may actually drop below 450 feet per minute before the indicator shows a speed of 450. When the indicator is updated with this slower speed she may then choose to increase her vertical speed causing her to travel at 475 feet per minute again. Given that each cycle of corrections is 5 minutes in length, sketch two cycles of the pilot trying to correct her vertical speed. **(*Source:* www.answers.com/topic/pilot-induced-oscillation)**

8. Electrocardiogram An electrocardiogram is a record of the electrical activity of the heart as a function of time in seconds. Typically doctors analyze the various waves on the printout and use the information as part of a diagnostic process to determine the health of a patient. An electrocardiograph machine outputs paper at a constant rate of 25 millimeters per second and the grid lines on the graph paper are 5 millimeters apart. Estimate the period and frequency of the graph. Also estimate how many beats per minute this person's heart is beating.



Note: **Image is not drawn to scale.**

■ SECTION 8.2 ■

For Exercises 9–10, describe what fraction of the circumference of a full circle is spanned by an angle with the given measure.

9. $\theta = \dfrac{5\pi}{6}$ radians

10. $\theta = 3$ radians

For Exercises 11–12, convert each angle measure in degrees to radian measure.

11. $\theta = 35°$

12. $\theta = -120°$

For Exercises 13–14, convert each angle measure from radians to degrees.

13. $\theta = \dfrac{\pi}{6}$

14. $\theta = -\dfrac{2\pi}{3}$

15. What is the length of the arc that corresponds with an angle with a measure of 75° in a circle of radius 1.5 centimeters?

16. What is the length of the arc that corresponds with an angle of 1.75 radians in a circle of radius 5 inches?

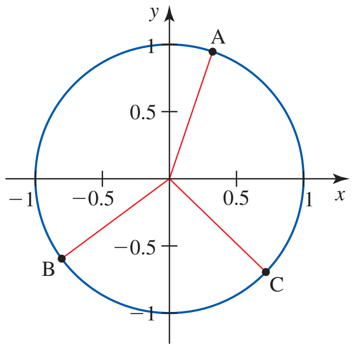
17. In a circle of radius 10 meters, what is the measure of the angle that corresponds with an arc of length 30 meters?

18. List two foundational characteristics of the idea of angle. Explain why you think they are foundational to your understanding of the idea of angle.

19. Boarding For boarders, an "inverted 540" is "an inverted aerial where the athlete performs a 540 degree rotational flip. In other words, the athlete approaches the half pipe wall riding forward, becomes airborne, rotates 540 degrees in a backside direction while performing a front flip, and lands riding forward. Invented by skateboarder Mike McGill (www.snowboarding.com)." Describe what is meant by "rotates 540 degrees."

SECTION 8.3

In Exercises 20–22, use the following graph, which shows the terminal side of three angles (A, B, and C) associated with an arc in standard position.



If the angle measures θ, estimate $\cos(\theta)$ and $\sin(\theta)$.

20. A **21.** B **22.** C

In Exercises 23–24, use a calculator to evaluate each expression, then draw a picture to represent the meaning behind the value you found.

23. $\cos(125°)$

24. $4\sin(213.5°)$

In Exercises 25–26, use a calculator to evaluate each expression, then draw a picture to represent the meaning behind the value you found. The angles are all measured in radians.

25. $\cos\left(\dfrac{\pi}{7}\right)$

26. $\left(\cos\left(\dfrac{9\pi}{5}\right), \sin\left(\dfrac{9\pi}{5}\right)\right)$

In Exercises 27–28, find the exact values of each expression, then draw a picture to represent the meaning behind the value you found.

27. $\cos(45°)$

28. $\sin(240°)$

29. Ferris Wheels The London Eye is a Ferris wheel constructed on the banks of the River Thames in London. The London Eye has a radius of about 221 feet and is boarded from the bottom. **(*Source:* www.aboutbritain.com)**

Determine the height of a person from the bottom of the London Eye after traveling each of the following portions of a revolution.

a. 3/4 of the way around

b. 1/8 of the way around

c. 7/10 of the way around

30. Ferris Wheels The Singapore Flyer is a Ferris wheel constructed on top of a three-story building. The wheel itself has a diameter of 492 feet, but the bottom of the wheel sits nearly 50 feet off the ground. Determine the height from the ground of a person after boarding from the bottom and traveling each of the following portions of a revolution. **(*Source:* www.singaporeflyer.com.sg)**

a. 1/5 of the way around

b. 5/6 of the way around

c. after 8/5 revolutions

SECTION 8.4

In Exercises 31–36, answer all of the following questions for each function.

a. What are the period, amplitude, frequency, and equation of the midline?

b. What are the maximum and minimum values of f?

c. Graph the function without using a calculator. Make sure to graph at least one complete period.

d. Where are the horizontal intercepts located?

e. Where is the vertical intercept located?

f. Describe the transformation of the function as related to the graph of $f(\theta) = \sin(\theta)$ or $f(\theta) = \cos(\theta)$, as appropriate.

31. $f(\theta) = 3\sin(\theta)$; θ is in degrees.

32. $f(\theta) = \sin\left(\dfrac{1}{2}\theta\right)$; θ is in radians.

33. $f(\theta) = \dfrac{2}{3}\cos\left(\theta - \dfrac{\pi}{4}\right)$; θ is in radians.

34. $f(\theta) = \sin\left(1.5\left(\theta + \dfrac{\pi}{3}\right)\right) + 2$; θ is in radians.

35. $f(\theta) = -2\cos\left(\dfrac{1}{5}(\theta + 90°)\right)$; θ is in degrees.

36. $f(\theta) = -4\cos\left(3\left(\theta - \dfrac{\pi}{6}\right)\right) + 1$; θ is in radians.

37. Explain the difference between a phase shift and a horizontal shift.

38. State which transformations have the ability to alter the given characteristic of a trigonometric function.

a. period

b. amplitude

c. midline

d. frequency

e. horizontal intercepts

f. vertical intercept

39. Seasonal Employment The function

$$C(m) = -5000\cos\left(\frac{\pi}{6}(m-3)\right) + 77{,}000$$

models the number of people employed in construction in Oregon m months after January 2002. (*Source:* Modeled using estimates from a graph at www.qualityinfo.org)

a. Find the amplitude, period, equation of the midline, and phase shift for L, then explain what each tells you about this situation.

b. During what month are the fewest number of people employed in construction in Oregon? The most people?

c. Graph C over the interval $0 \le m \le 12$. Describe its rate of change and concavity throughout this interval and explain what this tells you about construction employment in Oregon.

40. Sunrise The function

$$S(d) = 1.325\cos\left(\frac{2\pi}{365}(d+2)\right) + 5.775$$

models the time of sunrise in Washington, Indiana, in Eastern Standard Time (EST) on day d of the year. (*Note:* Daylight savings time is not taken into consideration in this model.) (*Source:* Modeled from data at aa.usno.navy.mil)

a. Find the period, the amplitude, and the equation of the midline for S. Explain what these values tell you about the time that the sun rises in Washington, Indiana.

b. Graph S over the interval [0, 365]. Describe the rate of change and concavity of the function, then explain what this information tells you about this situation.

c. Solve the equation

$$5 = 1.325\cos\left(\frac{2\pi}{365}(d+2)\right) + 5.775$$

over the same interval by graphing, then explain what your solution(s) mean.

41. Average Temperatures The function

$$T(m) = 24.8\sin\left(\frac{\pi}{6}(m-4)\right) + 63.3$$

models the average high temperature in Washington, Indiana, in degrees Fahrenheit during month m of the year. (*Source:* www.weatherbase.com)

a. Find the period, the amplitude, and the equation of the midline for T. Explain what these values tell you about the average high temperature in Washington, Indiana.

b. Solve the equation

$$24.8\sin\left(\frac{\pi}{6}(m-4)\right) + 63.3 = 75$$

over the interval [1, 12] by graphing, then explain what your solution(s) mean.

SECTION 8.5

In Exercises 42–45, state the period, amplitude, and midline.

42. $y = 4\sin(2(t+1)) + 6$

43. $y = 9\cos(3.1(t+2)) - 4$

44. $y = 6\cos(0.5t)$

45. $y = \pi\sin(3t) - 0.04$

In Exercises 46–49, determine the vertical and horizontal shifts.

46. $y = 4\sin(2(t+1)) + 6$

47. $y = 9\cos(3.1(t+2)) - 4$

48. $y = 6\cos(0.5t)$

49. $y = \pi\sin(3t) - 0.04$

In Exercises 50–53, sketch one period of the function without using a calculator.

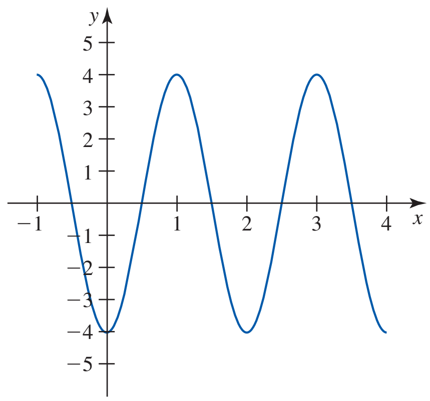
50. $y = 2\cos(5t)$

51. $y = 11\sin\left(t + \frac{3\pi}{4}\right)$

52. $y = 3\sin(\pi(t-\pi)) + 1$

53. $y = \cos\left(t - \frac{5\pi}{2}\right) + 8$

54. Find an equation for the sinusoidal function shown in the graph. You may use either sine or cosine.



SECTION 8.6

In Exercises 55–56, use your calculator to evaluate each expression, then explain what your solution means in the context of angles.

55. tan(113°)

56. cot(230°)

In Exercises 57–58, find the exact value of each expression without using a calculator, then explain what your solution means in the context of angles.

57. $\cot\left(\frac{2\pi}{3}\right)$

58. $\tan\left(\frac{7\pi}{4}\right)$

59. Road Grades The U.S. Department of Agriculture recommends building forest access roads on which water drains away from the road so that the roads may still be used under wet conditions. One method of doing this is to *outslope* the road, or give it a slight slope so that water runs down the road instead of collecting on it. Roads with an outslope have an angle of incline of 2.3°. (***Source:*** **www.na.fs.fed.us**)

a. What is the minimum road grade for an outsloped road?

b. What does cot(2.3°) represent in this situation?

c. What is the elevation change over 0.25 miles on a road with a 2.3° incline?

60. Pyramids The ancient Romans built a pyramid in the Egyptian style called the Pyramid of Cestius. However, the Romans built this pyramid much steeper than the famous pyramids at Giza. (Their use of concrete made it easier to do so.) The sides of the Pyramid of Cestius have an incline of about 67.8°. (***Source:*** **www.romeguide.it**)

a. Find tan(67.8°) and cot(67.8°), then explain what these values mean in this context.

b. The Pyramid of Cestius is 27 meters tall. How wide is it?

In Exercises 61–62, use a calculator to evaluate each expression, then use your answer to find the cosine or sine value for the same arc or angle.

61. sec(81°)

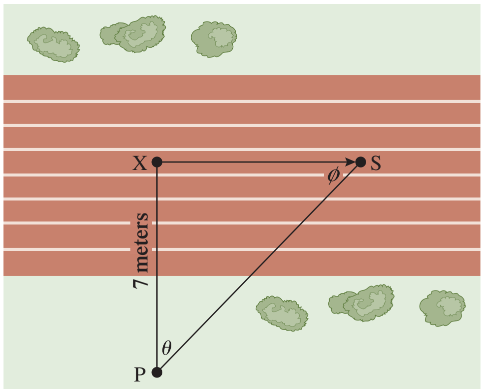
62. csc(163°)

In Exercises 63–64, find the exact value of each expression. Angles are in radians.

63. $\sec\left(\frac{\pi}{6}\right)$

64. $\csc\left(\frac{7\pi}{6}\right)$

For Exercises 65–66, use the following information. A person observing a 100-meter sprinter race at a track meet is located at point P. When the sprinter reaches point X, the sprinter and observer are 7 meters apart.



65. The function $d(\theta) = 7\sec(\theta)$ models the distance in meters between the sprinter and the observer (the distance from P to S) as a function of the angle θ. What is the distance between the sprinter and observer when $\theta = 32°$?

66. The function $s(\phi) = 7\csc(\phi)$ will also model the distance in meters between the sprinter and observer (the distance from P to S), but will be a function of ϕ instead of θ. Find a reasonable solution to the equation $20 = 7\csc(\phi)$ using graphing, then explain what your solution means.

SECTION 8.7

In Exercises 67–68, find at least one value for the given expression over the interval [0, 2π], then explain what your answer represents.

67. $\cos^{-1}(0.82)$

68. $\sin^{-1}(-0.34)$

In Exercises 69–70, evaluate each expression exactly, finding at least one value over the interval [0°, 360°] without using a calculator.

69. $\csc^{-1}(\sqrt{2})$

70. $\tan^{-1}(-\sqrt{3})$

In Exercises 71–73, find all solutions to each equation over the interval [0°, 360°] using inverse trigonometric functions. Check your solutions by graphing.

71. $\cos(\theta) = -0.3926$

72. $3.5013 = 2.1\tan(\theta)$

73. $6\csc(\theta - 20°) = -13$

In Exercises 74–76, find all solutions to each equation over the interval [0, 2π] using inverse trigonometric functions. Check your solutions by graphing.

74. $\sin(\theta) = 0.0184$

75. $\sec\left(\theta + \frac{\pi}{15}\right) = -3.5$

76. $16.8 = 7.6\cot\left(\theta - \frac{\pi}{6}\right)$

In Exercises 77–78,

a. Using inverse trigonometric functions, find a solution to the given equation that is reasonable in the context of the problem.

b. Explain what your solution means, and explain why it is a reasonable solution.

77. Road Grades The function $g(\theta) = 100\tan(\theta)$ models the percent road grade measure as a function of θ, the angle of ascent or descent. Solve $9 = 100\tan(\theta)$.

78. Lines of Latitude The function $C(\theta) = 2\pi(3963\cos(\theta))$ models the circumference of the circle (in miles) formed by a line of latitude θ degrees from the equator. Solve $2\pi(3963\cos(\theta)) = 5650$.

Make It Real Project

What to Do

1. Go to www.weatherbase.com and select "United States" as the region. Select the state and city where you live to get weather data for your city.
2. Use the data provided to create a formula and graph for the average high temperature in your city over the course of one year.
3. Go to http://aa.usno.navy.mil/, then select "Data Services," followed by "Duration of Daylight/Darkness Table for One Year." Enter your state and city information to get the number of hours of daylight per day in your city throughout the year.
4. Use the data provided to create a formula and graph for the number of hours of daylight in a day throughout the year. (*Hint:* To turn 8 hours 14 minutes into a number of hours in decimal form, find $8 + \frac{14}{60}$.)
5. Compare and contrast the graphs you created. Explain the possible reasons for any similarities in these graphs.