CHAPTER 7

Modeling with Other Types of Functions

The Maricopa Community College District typically offers annual salary increases for faculty for the first several years of employment. A *salary increase* is designed to increase employees' standard of living. Additionally, the Faculty Association seeks to negotiate a COLA each year. A COLA, or *cost-of-living allowance*, is designed to keep pace with inflation and thus allows employees to maintain their standard of living. Modeling the projected earnings of a faculty member working in the district requires the use of a combination of functions including linear, exponential, and piecewise as well as a product of functions.

SECTION 7.1 Combinations of Functions

LEARNING OBJECTIVES

- Compute the sum, difference, product, or quotient of functions to model a real-world situation
- Determine the practical and theoretical domain of the combination of functions

GETTING STARTED

Advances in technology and medicine as well as a focus on healthy living have resulted in an increase in the life expectancy of men and women in the United States. The life expectancy of women is greater than that for men, but the gap is closing.

In addition to modeling life expectancy as a function of year, we can model the gap between the life expectancies of men and women. We can then use that model to predict if the life expectancy of men can ever be equal to that of women.

In this section we learn how to combine functions by addition, subtraction, multiplication, and division. This knowledge helps us better understand the relationships between quantities and make predictions such as for life expectancies.

Combining Functions Graphically and Numerically

The graph in Figure 7.1 shows the life expectancy of men and women born between 1980 and 2004. (*Source:* Statistical Abstract of the United States, 2007, Table 98) Looking at the overall trend in life expectancies in Figure 7.2, we see that the difference in life expectancy between women and men (D) decreases as time increases.

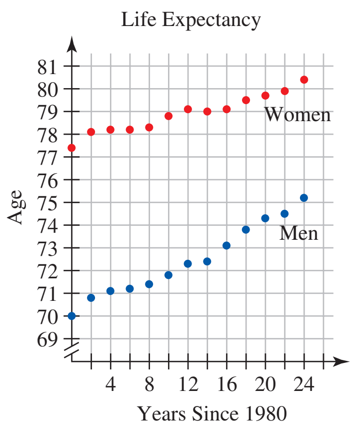


Figure 7.1

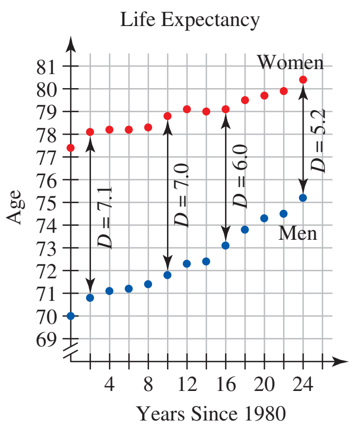


Figure 7.2

To see this more clearly, we calculate the differences throughout the data set, as shown in Table 7.1.

Table 7.1

Birth Year t	Male Life Expectancy (years) M	Female Life Expectancy (years) F	Difference in Life Expectancy $D = F - M$
1980	70.0	77.4	7.4
1982	70.8	78.1	7.3
1984	71.1	78.2	7.1
1986	71.2	78.2	7.0
1988	71.4	78.3	6.9

Table 7.1 (*continued*)

Birth Year t	Male Life Expectancy (years) M	Female Life Expectancy (years) F	Difference in Life Expectancy $D = F - M$
1990	71.8	78.8	7.0
1992	72.3	79.1	6.8
1994	72.4	79.0	6.6
1996	73.1	79.1	6.0
1998	73.8	79.5	5.7
2000	74.3	79.7	5.4
2002	74.5	79.9	5.4
2004	75.2	80.4	5.2

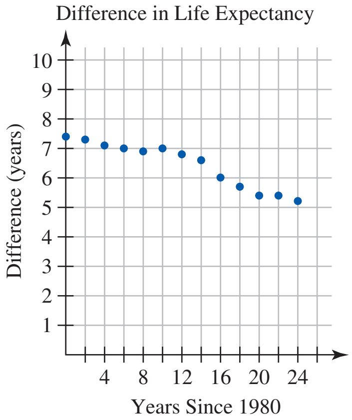


Figure 7.3

The table confirms the overall trend that the life expectancy gap is decreasing as time increases. The difference in life expectancy generally decreases from 7.4 years in 1980 to 5.2 years in 2004. We create a scatter plot of the differences in Figure 7.3 and observe the same trend.

■ Combining Functions Symbolically

We have just seen how to combine functions using graphs and tables. In the next example we show how to combine functions represented symbolically.

EXAMPLE 1 ■ Combining Functions Symbolically

Using linear regression, we determine that the life expectancy for women can be modeled by $F(t) = 0.106t + 77.6$ and the life expectancy for men by $M(t) = 0.204t + 70.0$, where t is the number of years since 1980.

a. Determine a function that models the difference in life expectancy between men and women. Name this function $D(t)$.

b. Interpret the meaning of the slope and vertical intercept of $D(t)$ in the context of this situation.

c. Graph $D(t)$ along with the scatter plot shown in Figure 7.3. Comment on the accuracy of the model.

d. Describe the practical domain and range of $D(t)$.

Solution

a. As we did graphically in Figure 7.2 and numerically in Table 7.1, we determine the model for the differences in life expectancies by subtracting the function $M(t) = 0.204t + 70.0$ from $F(t) = 0.106t + 77.6$. We combine the like terms to simplify the function.

$$\begin{aligned}D(t) &= F(t) - M(t)\\ D(t) &= 0.106t + 77.6 - (0.204t + 70.0)\\ D(t) &= -0.098t + 7.6\end{aligned}$$

b. The slope of -0.098 means the difference in life expectancy between men and women is decreasing by 0.098 years of age per year. The vertical intercept of (0, 7.6) means that the difference in 1980 is 7.6 years. (The actual difference is 7.4 but the model predicts 7.6.)

c. Figure 7.4 shows the graph of $D(t) = -0.098t + 7.6$ along with the scatter plot of the data. We see that this function models the data reasonably well.

d. The practical domain for the function is the years for which the model can reasonably represent the situation. Determining the interval of values over which the model is valid is somewhat subjective. We choose the interval $0 \le t \le 30$, which means we reserve the use of our model for years between 1980 ($t = 0$) and 2010 ($t = 30$).

The practical range for the function is the differences in life expectancy for which the model can reasonably represent the situation. Based on the practical domain, we determine that the interval $4.66 \le D \le 7.60$ is the practical range. This corresponds to the difference in life expectancy in 2010 (4.66) and in 1980 (7.60).

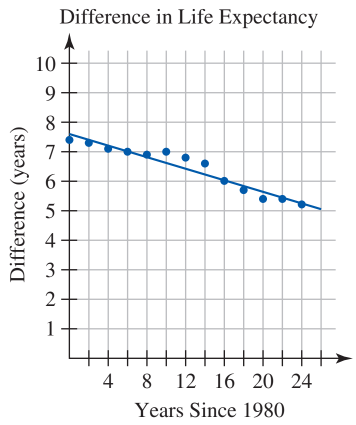


Figure 7.4

EXAMPLE 2 ■ **Making a Prediction Using a Combination of Functions**

Use the function model $D(t) = -0.098t + 7.6$ to predict the difference in life expectancy for men and women born in 2010. Then confirm the result using $M(t) = 0.204t + 70.0$ and $F(t) = 0.106t + 77.6$.

Solution Since 2010 is 30 years since 1980, we evaluate the function at $t = 30$.

$$\begin{aligned} D(30) &= -0.098(30) + 7.6 \\ &= 4.66 \end{aligned}$$

The difference between the life expectancy of men and women born in 2010 is predicted to be 4.66 years.

We confirm this result and determine the life expectancies by evaluating both $M(t) = 0.204t + 70.0$ and $F(t) = 0.106t + 77.6$ at $t = 30$ and subtracting the results.

$$\begin{aligned} M(30) &= 0.204(30) + 70.0 \\ &= 76.12 \end{aligned} \qquad \begin{aligned} F(30) &= 0.106(30) + 77.6 \\ &= 80.78 \end{aligned}$$

Women born in 2010 are expected to live 80.78 years and men born in 2010 are expected to live 76.12 years. The difference between these two values is 4.66 years, which confirms our result.

■ Dividing Functions

The U.S. federal government brings in money (revenue) primarily through taxes paid by its citizens. It spends money (outlays) on military, social programs, and education, among other programs. Currently, the government spends more money than it receives. To make up for this deficit, it borrows money by selling securities like Treasury bills, notes, bonds, and savings bonds to the public. According to the U.S. Treasury Department, the government's national debt is an accumulation of the deficit spending experienced each year. (*Source:* www.treasurydirect.gov)

Based on data from 1940 to 2004, the national debt can be modeled by the exponential function $D(t) = 0.086(1.07^t)$, where t is measured in years since 1940 and D is measured in trillions of dollars. (Modeled from www.whitehouse.gov data) The population of the United States can be modeled by the linear function $P(t) = 2.55t + 127$, where t is years since 1940 and P is millions of people. (*Source:* Modeled from www.census.gov data) We use these models in the next two examples as we combine functions using division.

EXAMPLE 3 ■ Combining Functions Using Division

Suppose every person (including children) in the United States was to pay an equal amount of money to eliminate the national debt. Using the functions D and P just defined, create a function A that would give the amount of money each person in the United States would need to pay. Then evaluate and interpret $A(70)$.

Solution The function D gives the total national debt (in trillions of dollars) for a given year (since 1940). If we divide this value by the population for that same year, we get an average amount of debt per person in the United States.

$$A(t) = \frac{D(t)}{P(t)} = \frac{0.086(1.07^t)}{2.55t + 127} \frac{\text{trillion dollars}}{\text{million people}}$$

Note the units used in this combination of functions is trillion dollars per million people.

Evaluating $A(70)$ will provide us the amount of money required per person to completely pay for the national debt in 2010, since 2010 is 70 years after 1940.

$$\begin{aligned} A(70) &= \frac{0.086(1.07)^{70}}{2.55(70) + 127} \\ &\approx 0.0321 \text{ trillion dollars per million people.} \end{aligned}$$

We express this result as a fraction and simplify.

$$\begin{aligned} \frac{0.0321 \text{ trillion dollars}}{1 \text{ million people}} &= \frac{0.0321(1{,}000{,}000{,}000{,}000)\text{dollars}}{1{,}000{,}000 \text{ people}} \\ &= \frac{32{,}100{,}000{,}000 \text{ dollars}}{1{,}000{,}000 \text{ people}} \\ &= \$32{,}100 \text{ per person} \end{aligned}$$

If everyone paid an equal amount, every man, woman, and child in the United States would have to pay \$32,100 in 2010 in order to pay off the national debt.

EXAMPLE 4 ■ Determining the Domain and Range

The function $A(t) = \dfrac{0.086(1.07^t)}{2.55t + 127}$ gives the amount of money (in trillions of dollars per million people) that each person in the United States would need to pay in year t (where t is the number of years since 1940) to eliminate the national debt.

a. Determine the theoretical domain of this function.

b. Determine the practical domain of this function. Then graph the function.

Solution

a. The theoretical domain refers to the set of input and output values of this function independent of the real-world context. In this case, the theoretical domain is all values of t except the value that makes the denominator zero. To find this value, we set the denominator equal to zero and solve for t.

$$\begin{aligned} 2.55t + 127 &= 0 \\ 2.55t &= -127 \\ t &= \frac{-127}{2.55} \\ t &\approx -49.8 \end{aligned}$$

The theoretical domain is all values of t except $t = -49.8$.

b. The practical domain refers to input values that seem reasonable for the real-world context. We know it is not good practice to extrapolate too far outside of the original data set. Since the original data set included years 1940–2004, we choose domain values that represent the years 1940–2010. Thus we describe the practical domain to be values of t such that $0 \le t \le 70$. (Note: This choice of domain values is subjective and a different choice of values may also be acceptable.) Figure 7.5 shows the graph of

$$A(t) = \frac{0.086 \cdot 1.07^t}{2.55t + 127}$$ using the practical domain.

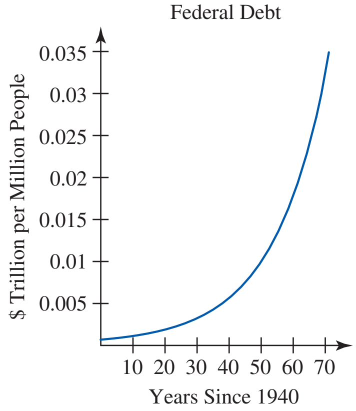


Figure 7.5

■ Multiplying Functions

The National Automobile Dealers Association (NADA) reports statistics on the sales of new vehicles in the United States. The number of new vehicles sold per dealership in the United States is given in Table 7.2 and Figure 7.6. We model these data using the quartic function $V(t) = 0.216t^4 - 4.50t^3 + 27.6t^2 - 28.6t + 653$, as seen in Figure 7.7.

Table 7.2

Number of New Vehicles Sold	
Years Since 1995 t	**New Vehicles Sold per Dealership** V
0	648
1	664
2	668
3	694
4	759
5	783
6	785
7	774
8	769
9	779
10	788

Source: www.autoexcemag.com

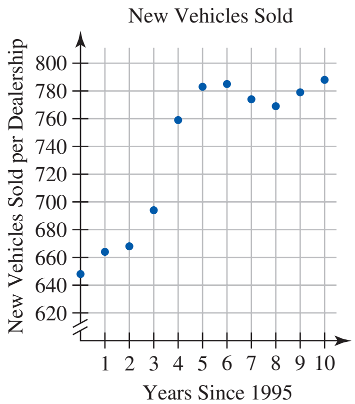


Figure 7.6

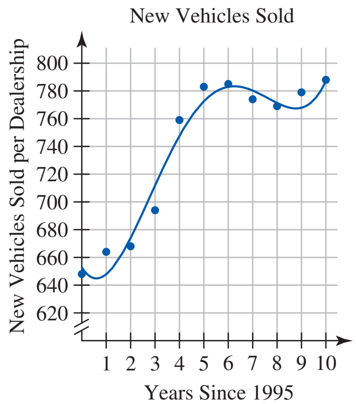


Figure 7.7

The NADA also reports the average retail selling price of these new vehicles. Table 7.3 and Figure 7.8 provide this information. We model these data using the linear function $P(t) = 777t + 21{,}000$, as seen in Figure 7.9.

Table 7.3

New Vehicle Sales Price	
Years Since 1995 t	Average Retail Selling Price (dollars) P
0	20,450
1	21,900
2	22,650
3	23,600
4	24,450
5	24,900
6	25,800
7	26,150
8	27,550
9	28,050
10	28,400

Source: www.autoexcemag.com

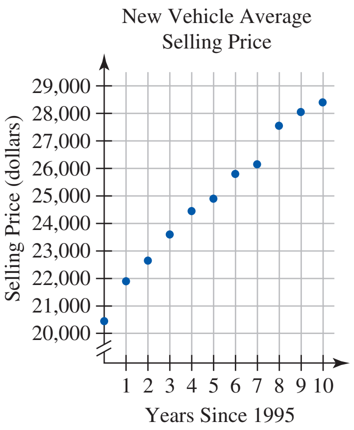


Figure 7.8

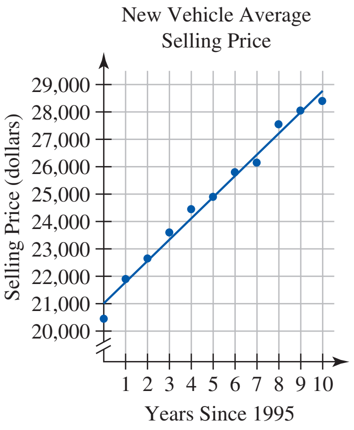


Figure 7.9

In the next example, we use these models to investigate the product of two functions.

EXAMPLE 5 ■ **Interpreting the Product of Two Functions**

The function $V(t) = 0.216t^4 - 4.50t^3 + 27.6t^2 - 28.6t + 653$ models the number of new vehicles sold, where t is measured in years since 1995 and V is the number of new vehicles per dealership in the United States. The function $P(t) = 777t + 21{,}000$ models the average retail selling price, P, of a new vehicle t years after 1995.

a. Find and interpret $f(15)$ given $f(t) = V(t) \cdot P(t)$.

b. Express the function $f(t) = V(t) \cdot P(t)$ symbolically.

c. Interpret the meaning of the function $f(t) = V(t) \cdot P(t)$.

Solution

a. We first evaluate and interpret $V(15)$.

$$\begin{aligned} V(15) &= 0.216(15)^4 - 4.50(15)^3 + 27.6(15)^2 - 28.6(15) + 653 \\ &\approx 2180 \text{ (accurate to 3 significant digits)} \end{aligned}$$

This tells us that 15 years after 1995 (2010), the number of new vehicles sold per dealership will be 2180 vehicles.

Next we evaluate and interpret $P(15)$.

$$\begin{aligned} P(15) &= 777(15) + 21{,}000 \\ &= 32{,}655 \\ &\approx 32{,}700 \text{ (accurate to 3 significant digits)} \end{aligned}$$

This tells us that 15 years after 1995 (2010), the average retail selling price of a new vehicle will be approximately \$32,700.

To evaluate $f(15)$, we calculate $f(15) = V(15) \cdot P(15)$.

$$\begin{aligned} f(15) &= V(15) \cdot P(15) \\ &= 2180 \cdot 32{,}700 \\ &\approx 71{,}300{,}000 \text{ (accurate to 3 significant digits)} \end{aligned}$$

This tells us that 15 years after 1995 (2010), a dealership will generate approximately \$71 million in income from the sales of new vehicles. That is, if 2180 new vehicles are sold at an average cost of \$32,700 each, roughly \$71 million in income is generated.

b. We now express $f(t) = V(t) \cdot P(t)$ symbolically.

$$\begin{aligned} f(t) &= V(t) \cdot P(t) \\ &= (0.216t^4 - 4.50t^3 + 27.6t^2 - 28.6t + 653) \cdot (777t + 21{,}000) \end{aligned}$$

c. Since $V(t)$ is the number of new vehicles sold per dealership and $P(t)$ is the average retail selling price per vehicle (in dollars), we interpret $f(t) = V(t) \cdot P(t)$ to be the total income from new vehicle sales per dealership t years after 1995. That is, if each dealership sold $V(t)$ vehicles at a price of $P(t)$ each, the income for the dealership would be $f(t)$ dollars.

The following box summarizes our work so far in this section and introduces formal notation for the combination of functions.

COMBINATIONS OF FUNCTIONS

Sum of two functions	$(f + g)(x) = f(x) + g(x)$
Difference of two functions	$(f - g)(x) = f(x) - g(x)$
Product of two functions	$(f \cdot g)(x) = f(x) \cdot g(x)$
Quotient of two functions	$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$

EXAMPLE 6 ■ Understanding Combinations of Functions

Now let's consider the conceptual aspects of combining functions when a real-world context is not provided. Given the graphs of two functions f and g shown in Figure 7.10, sketch the graph of the given combination of functions.

a. $(f + g)(x)$

b. $\left(\frac{f}{g}\right)(x)$

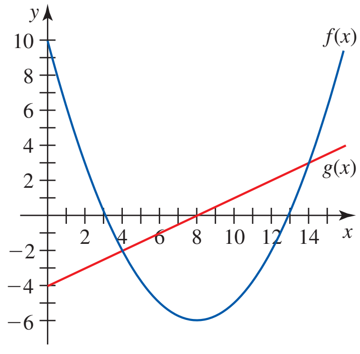


Figure 7.10

Solution

a. We know $(f + g)(x) = f(x) + g(x)$. In terms of the graphs, we are adding together the y-values of each function while leaving the x-values unchanged. Although there are many ways to approach this problem, one of the easiest is to create a table of values such as Table 7.4 for f and g and then sum the function values. To ensure as much accuracy as possible, we select the x-values of the intercepts, intersection points, and extrema for each graph. We use our best judgment to estimate the coordinates.

Then we connect the plotted points of $f(x) + g(x)$ with a smooth curve, as in Figure 7.11.

Table 7.4

x	$f(x)$	$g(x)$	$f(x) + g(x)$
0	10	−4	6
3	0	−2.5	−2.5
4	−2	−2	−4
8	−6	0	−6
13	0	2.5	2.5
14	3	3	6

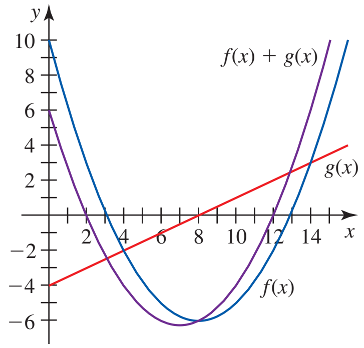


Figure 7.11

b. We use the same points used in part (a) and calculate the quotient of the two function values, as shown in Table 7.5.

A vertical asymptote occurs where $g(x) = 0$ provided $f(x) \neq 0$. We connect the plotted points of $\dfrac{f(x)}{g(x)}$ with a smooth curve, as in Figure 7.12.

Table 7.5

x	$f(x)$	$g(x)$	$\dfrac{f(x)}{g(x)}$
0	10	−4	−2.5
3	0	−2.5	0
4	−2	−2	1
8	−6	0	undefined
13	0	2.5	0
14	3	3	1

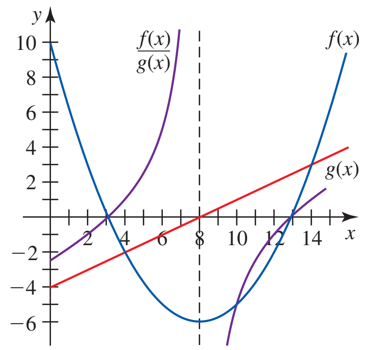


Figure 7.12

In practice, we typically use technology such as a graphing calculator to graph functions. However, graphing functions by hand helps our understanding of the underlying function concepts.

SUMMARY

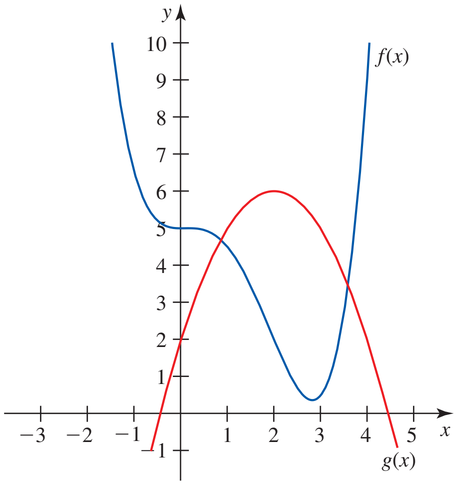
In this section you learned how to combine functions by adding, subtracting, multiplying, and dividing them. You also learned how to interpret combinations of functions in real-world contexts and use the combinations to define additional function relationships and make predictions. Additionally, you examined graphs and tables to better understand the conceptual meanings of combinations of functions.

7.1 EXERCISES

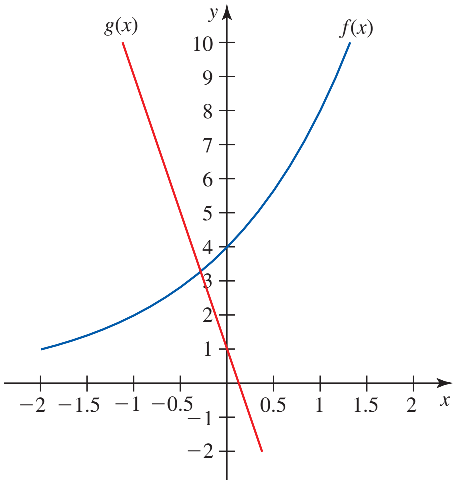
SKILLS AND CONCEPTS

In Exercises 1–5, use the graphs of functions f and g to evaluate each given function. If it is not possible to evaluate the function, explain why not.

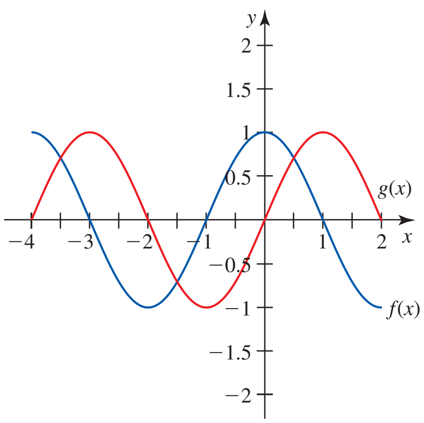
1. **a.** $(f + g)(2)$ **b.** $(f \cdot g)(2)$



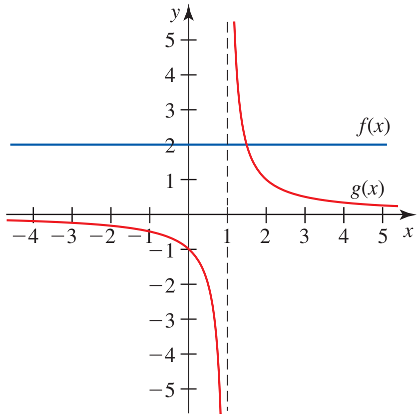
2. **a.** $(g - f)(0)$ **b.** $\left(\frac{g}{f}\right)(0)$



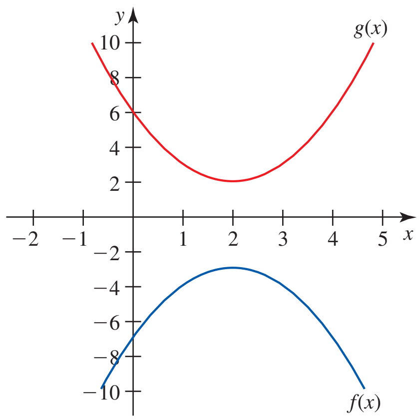
3. **a.** $(g \cdot f)(-2)$ **b.** $\left(\frac{f}{g}\right)(-2)$



4. **a.** $(g + f)(1)$ **b.** $(f \cdot g)(1)$



5. **a.** $(g + f)(1)$ **b.** $(g \cdot f)(2)$



In Exercises 6–10, use the given table to evaluate each given function. If it is not possible to evaluate the function, explain why not.

6.

x	−2	0	2	4	6
$f(x)$	1	5	11	19	29
$g(x)$	5	1	5	17	37

a. $(f + g)(4)$
b. $(f - g)(4)$
c. $\left(\frac{f}{g}\right)(4)$
d. $(f \cdot g)(4)$

7.

x	−2	0	2	4	6
$f(x)$	−1	1	3	5	7
$g(x)$	−6	−4	−2	0	2

a. $(f + g)(4)$
b. $(f - g)(4)$
c. $\left(\frac{f}{g}\right)(4)$
d. $(f \cdot g)(4)$

8.

x	0	1	2	3	4
$f(x)$	0	1	8	27	64
$g(x)$	0	−1	−8	−27	−64

a. $(f+g)(4)$
b. $(f-g)(4)$
c. $\left(\dfrac{f}{g}\right)(4)$
d. $(f \cdot g)(4)$

9.

x	−4	0	4	8	16
$f(x)$	2	4	0	4	8
$g(x)$	−2	−6	−5	−7	−9

a. $(f+g)(4)$
b. $(f-g)(4)$
c. $\left(\dfrac{f}{g}\right)(4)$
d. $(f \cdot g)(4)$

10.

x	4	8	12	16	20
$f(x)$	0	2	4	6	8
$g(x)$	1	1	2	3	5

a. $(f+g)(4)$
b. $(f-g)(4)$
c. $\left(\dfrac{f}{g}\right)(4)$
d. $(f \cdot g)(4)$

In Exercises 11–20, combine the functions f and g symbolically as indicated. Then state the domain of the combination.

11. $f(x) = 3x - 5 \qquad g(x) = -4x + 7$
a. $(f+g)(x)$
b. $(f-g)(x)$

12. $f(x) = -3x + 6 \qquad g(x) = 5x - 10$
a. $(f-g)(x)$
b. $\left(\dfrac{f}{g}\right)(x)$

13. $f(x) = x - 1 \qquad g(x) = x + 3$
a. $(f \cdot g)(x)$
b. $\left(\dfrac{g}{f}\right)(x)$

14. $f(x) = x^2 + 2x - 1 \qquad g(x) = -2x^2 - 5$
a. $(f+g)(x)$
b. $(f \cdot g)(x)$

15. $f(x) = x^2 - 4 \qquad g(x) = 2x + 8$
a. $(g-f)(x)$
b. $\left(\dfrac{f}{g}\right)(x)$

16. $f(x) = \sqrt{x-2} \qquad g(x) = 5$
a. $(f+g)(x)$
b. $(f \cdot g)(x)$

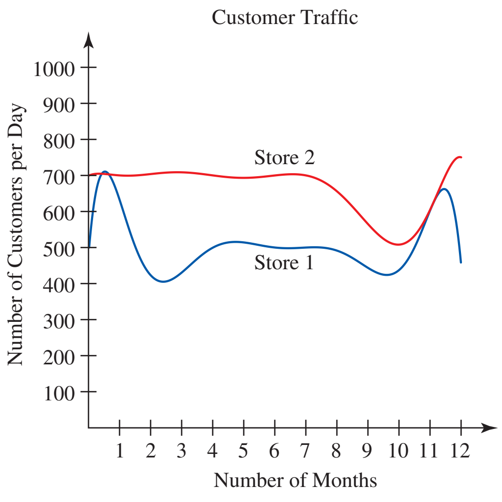
17. $f(x) = 2^x \qquad g(x) = x$
a. $(g \cdot f)(x)$
b. $\left(\dfrac{g}{f}\right)(x)$

18. $f(x) = \ln(x) \qquad g(x) = 4x$
a. $(f+g)(x)$
b. $(f \cdot g)(x)$

19. $f(x) = 2x^3 + x^2 - 4x + 7$
$g(x) = -x^2 + 7x - 2$
a. $(f+g)(x)$
b. $(g \cdot f)(x)$

20. $f(x) = x^3 - x^2 + 3x - 1$
$g(x) = 2x^3 - 2x^2 - 5x + 6$
a. $(g \cdot f)(x)$
b. $(f+g)(x)$

21. Operating a Business Suppose two large warehouse stores are operating in the same medium-sized city. Corporate headquarters has asked them to estimate the number of employees they will need for the upcoming year. Each store manager produced a graph (based on historical data) of the anticipated number of customers per day at that store. Their two graphs are shown on the same coordinate system. The horizontal axis represents the number of months in the upcoming year. The vertical axis represents the number of people expected at a store each day.



a. What does the point (6, 700) represent on Store 2's graph?

b. The function f represents the number of customers per day at Store 1 and g represents the number of customers at Store 2, both as a function of the number of months, m. Interpret the meaning of $(f+g)(m)$.

c. Since the corporate office hires for both stores, it needs an estimate of the stores' *total* number of customers. Sketch a graph of the daily number of customers at both stores.

22. High School Prom

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Students on a prom committee need to set the price for prom tickets. From past experience, they know that if tickets cost \$200 per couple, 100 couples buy tickets for the prom. They also predict from past experience that for every \$5 decrease in price, an additional 2 couples will buy tickets.

a. Create a function representing the price, P, of a prom ticket per couple as a function of the number of \$5 price decreases, n.

b. Create a function representing the number of couples attending prom, A, as a function of the number of \$5 price decreases, n.

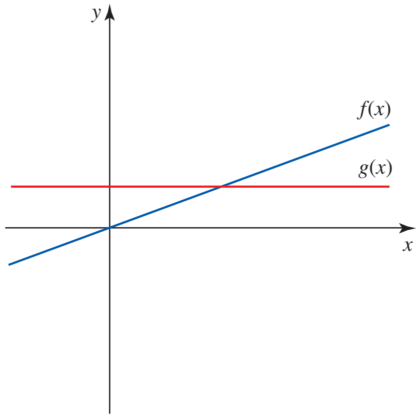
c. Compute and interpret the meaning of the function $(P \cdot A)(n)$.

d. Graph $(P \cdot A)(n)$ and describe the behavior of the function in the context of the situation.

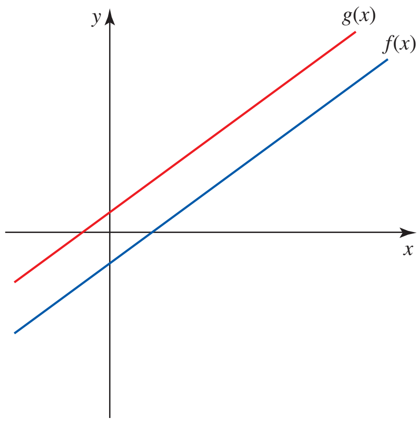
e. Solve $(P \cdot A)(n) = 0$ for n and explain what these values mean in the context of the situation.

In Exercises 23–32, use the given graphs of the functions f and g to sketch a graph of each combination of functions.

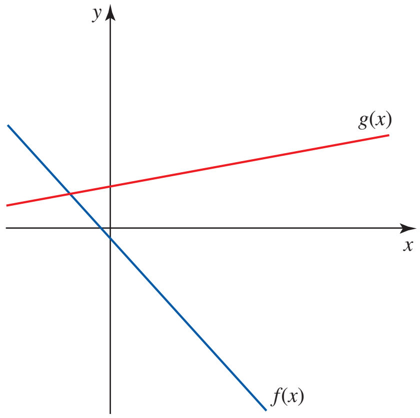
23. Sketch $(f + g)(x)$.



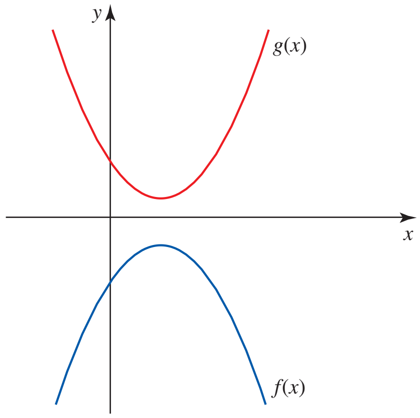
24. Sketch $(f - g)(x)$.



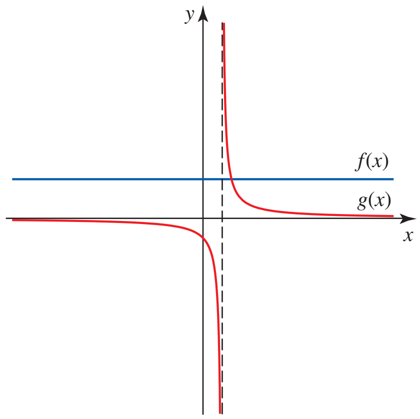
25. Sketch $(f \cdot g)(x)$.



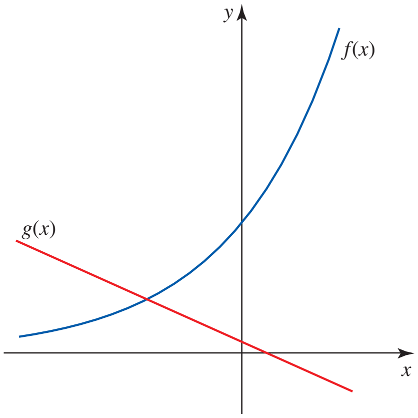
26. Sketch $\left(\frac{f}{g}\right)(x)$.



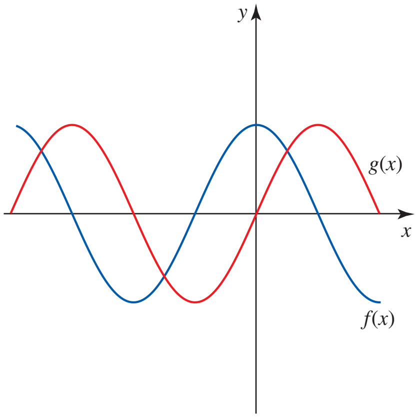
27. Sketch $(f + g)(x)$.



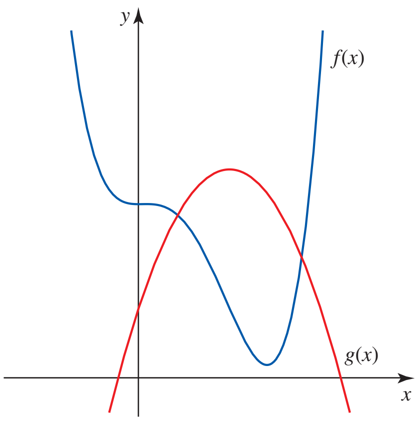
28. Sketch $(f \cdot g)(x)$.



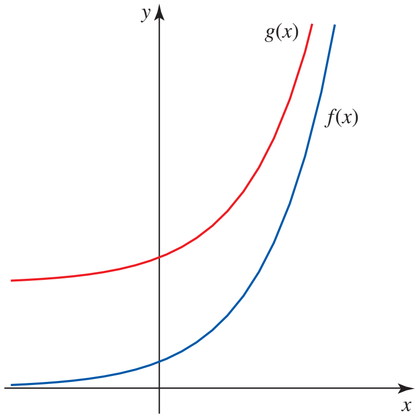
29. Sketch $\left(\frac{f}{g}\right)(x)$.



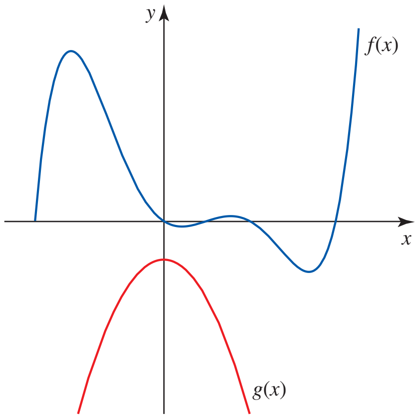
30. Sketch $(f + g)(x)$.



31. Sketch $(f - g)(x)$.



32. Sketch $(f \cdot g)(x)$.



■ SHOW YOU KNOW

33. Suppose the function f has x-intercept $(a, 0)$ and a function g passes through the point (a, b). What are the coordinates of the function $h(x) = f(x) + g(x)$ at $x = a$?

34. A function $h(x) = f(x) + g(x)$ has an x-intercept at $x = a$. What is the relationship between $f(a)$ and $g(a)$?

35. Under what conditions will $h(x) = \dfrac{f(x)}{g(x)}$ have a vertical asymptote?

36. How are the horizontal intercepts of $h(x) = \dfrac{f(x)}{g(x)}$ related to the horizontal intercepts of f?

37. The units of a function s are million dollars and the units of a function q are dollars per hour. What are the units of $r(t) = \dfrac{s(t)}{q(t)}$?

38. The points of intersection of the graphs of a function f and a function g have what graphical significance for the function $j(x) = f(x) - g(x)$?

39. The graph of a function g is a horizontal line and the graph of a function h is a concave up parabola that lies above the graph of g. Describe the graph of $j(x) = g(x) - h(x)$.

40. The graph of a function f is a line with slope m. The graph of a function g is a line with slope $-\frac{1}{m}$. Both functions pass through the origin. Describe the graph of $f(x) = j(x) \cdot h(x)$.

41. If the units of f are dollars per day and the units of g are days per week, what are the units of $h(t) = f(t) \cdot g(t)$?

42. A function f is positive for all values of x and a function g is negative for all values of t. Does $h(t) = f(t) \cdot g(t)$ have any horizontal intercepts? Explain.

MAKE IT REAL

43. Stopping Distance The total stopping distance required to stop a moving car is typically based on reaction distance and braking distance. The reaction distance is the distance the car travels from the moment the driver decides to apply the brake to the moment the brake is applied. Braking distance is the distance traveled from the moment the driver applies the brake to the moment the car comes to a complete stop. Use the data in the table to respond to the following items.

Stephen Aaron Rees/Shutterstock.com

Speed (mph) S	Reaction Distance (feet) R	Braking Distance (feet) B
20	22.0	22.2
25	27.5	34.7
30	33.0	50.0
35	38.5	68.0
40	44.0	88.8
45	49.5	112.4
50	55.0	138.8

Source: Minnesota Driving Manual

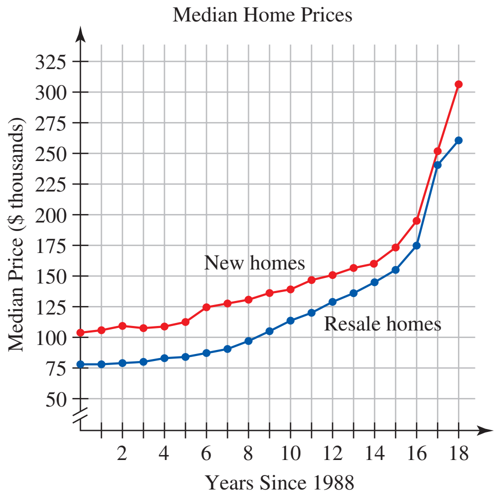
a. Determine a linear regression model representing $R(S)$, the reaction distance as a function of the speed of the car.

b. Determine a quadratic regression model representing $B(S)$, the braking distance as a function of the speed of the car.

c. Compute the function $(R + B)(S)$ and explain what it means in the context of this situation.

d. Evaluate and interpret $(R + B)(75)$.

44. Buying and Selling Homes The graph shows the median price of new and resale homes in the Greater Phoenix, Arizona, area. (*Source:* www.poly.asu.edu)



The median price of resale homes as a function of years since 1988 is given by $R(t)$, in thousands of dollars. The median price of new homes as a function of years since 1988 is given by $N(t)$, in thousands of dollars.

a. Interpret the meaning of $(N - R)(t)$.

b. Estimate and interpret $(N - R)(6)$.

c. Sketch a graph of $(N - R)(t)$.

d. Interpret the meaning of $\left(\frac{N}{R}\right)(t)$.

e. Estimate and interpret $\left(\frac{N}{R}\right)(17)$.

f. Sketch a graph of $\left(\frac{N}{R}\right)(t)$.

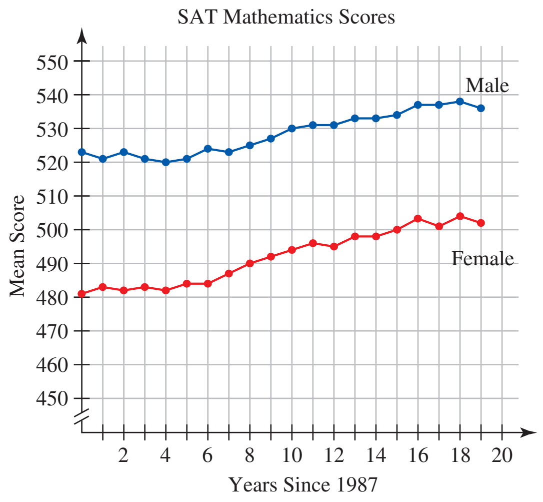
45. Registered Vehicles The number of registered vehicles in the United States, in millions, can be modeled by $V(t) = 3.383t + 154.8$, where t is the number of years since 1980. (*Source: Statistical Abstract of the United States, 2007*, Table 1074)

The population of the United States, in millions, can be modeled by $P(t) = 2.821t + 225.5$, where t is the number of years since 1980. (*Source: Statistical Abstract of the United States, 2006*, Table 2)

a. Compute and interpret $\left(\frac{V}{P}\right)(t)$.

b. Determine the value of t such that $\left(\frac{V}{P}\right)(t) = 1$. Explain what this means in this context.

46. SAT Scores The function M represents the mean SAT score for males relative to the number of years, t, since 1987. The function F represents the mean SAT score for females relative to the number of years, t, since 1987. Use the graph of these functions to respond to the following items.



***Source*: www.collegeboard.com**

a. Explain what information $M(15)$ provides.

b. Explain what information $F(15)$ provides.

c. Interpret the meaning of $(M - F)(t)$.

d. Sketch a graph of $(M - F)(t)$. Describe what information can be gleaned from this graph. That is, what trend do you notice and what does it mean in the context of the situation?

47. Soda Consumption The function $S(t) = -0.22t + 53.18$ represents the per capita consumption of soda pop in the United States, where S is measured in gallons of soda and t represents years since 2000. (***Source: Statistical Abstract of the United States, 2007*, Table 201**) The function $P(t) = 2870.9t + 282{,}426$ represents the population, P, of the United States in thousands as a function of t, the number of years since 2000. (***Source: Statistical Abstract of the United States, 2006*, Table 2**)

a. Compute and interpret $(S \cdot P)(t)$.

b. Graph $(S \cdot P)(t)$ and describe the behavior of this function.

c. Evaluate and interpret $(S \cdot P)(10)$.

48. Violent Crime The second column in the table shows the total number of violent crimes committed in the United States in the given years. The third column shows the number of violent crimes that are classified as murders. Use this table to respond to the following items.

Years Since 1990 t	Total Number of Violent Crimes (thousands) V	Total Number of Murders (thousands) M
0	1820	23
1	1912	25
2	1932	24
3	1926	25
4	1858	23
5	1799	22
6	1689	20
7	1636	18
8	1534	17
9	1426	16
10	1425	16
11	1439	16
12	1424	16
13	1384	17
14	1367	16

(continued)

***Source: Statistical Abstract of the United States, 2007*, Table 295**

a. Interpret the meaning of $\left(\frac{M}{V}\right)(t)$.

b. Create a fourth column in the table showing the values of $\left(\frac{M}{V}\right)(t)$.

c. Use a graphing calculator to create a scatter plot of $\left(\frac{M}{V}\right)$ as a function of t.

d. Determine and create a polynomial regression model for $\left(\frac{M}{V}\right)(t)$. Explain why you decided on the polynomial you used to model these data.

49. Football Game Attendance and Player Salaries Attendance at National Football League games has consistently increased over the past decade. The total attendance, in thousands of people, at all NFL games during the year can be modeled by $A(t) = 323.3t + 17{,}284$, where t is the number of years since 1990. (***Source: Statistical Abstract of the United States, 2007*, Table 1228**) Player salaries have also increased over this same time period. The average salary, in thousands of dollars, of an NFL player can be modeled by

$$S(t) = -0.1812t^4 + 5.758t^3 - 54.05t^2 + 199.6t + 311.5$$

(***Source: Statistical Abstract of the United States, 2007*, Table 1228**)

a. Calculate and interpret the meaning of $A(18)$.

b. Calculate and interpret the meaning of $S(18)$.

c. Determine and interpret the meaning of $\left(\frac{S}{A}\right)(t)$.

d. Determine the practical domain and range of $\left(\frac{S}{A}\right)(t)$.

50. Cheese Consumption The function $C(t) = 0.5161t + 18.89$ represents the per capita consumption of cheese in the United States, where C is measured in pounds of cheese and t represents years since 1980. (***Source: Statistical Abstract of the United States, 2007*, Table 202**) The function $P(t) = 2.943t + 222.7$ represents the population of the

United States, in millions of people, as a function of the number of years since 1980. **(*Source:* Modeled from www.census.gov)**

a. Compute and interpret $(C \cdot P)(t)$.

b. Graph $(C \cdot P)(t)$ and describe the behavior of this function.

c. Evaluate and interpret $(C \cdot P)(30)$.

■ STRETCH YOUR MIND

Exercises 51–55 are intended to challenge your understanding of the combination of functions. For each, suppose f(x) is always increasing and g(x) is always decreasing. Determine if each of the following combinations of functions is always increasing, always decreasing, or is it impossible to tell. Fully explain your rationale for your response.

51. $(f + g)(x)$

52. $(f - g)(x)$

53. $(f \cdot g)(x)$

54. $\left(\frac{f}{g}\right)(x)$

55. $f^{-1}(x) + g^{-1}(x)$

SECTION 7.2 Piecewise Functions

LEARNING OBJECTIVES

- Define piecewise functions using equations, tables, graphs, and words
- Determine function values of piecewise functions from a graph, equation, and table

GETTING STARTED

Many parking facilities assess fees based on the length of time a vehicle is left in the care of the attendants. Often the amount charged increases over set time intervals. When the parking fee is best defined by different equations over distinct time intervals, a piecewise function can be used to describe the fee assessment structure.

In this section we investigate piecewise functions such as this and discuss whether a given piecewise function is continuous or discontinuous. We demonstrate how to create piecewise functions in table, graph, and equation form and use the functions to determine specific values.

■ Piecewise Functions

Throughout this text we have seen that many different types of functions can be used to model real-world situations. We now look at scenarios that are best modeled by a combination of functions over distinct intervals of the domain.

Consider Table 7.6, which shows the fees to park in the East Economy Garage at Sky Harbor International Airport in Phoenix, Arizona for a single day. We see that for any time over 0 minutes through 60 minutes, the fee is \$4.00; for time over 60 through 120 minutes, the fee is \$8.00; and for any time over 120 minutes (for one day), the fee is \$10.00. Graphing this information as in Figure 7.13, we see that $F(m)$ is a discontinuous combination of three linear functions. Note that we use an open circle to denote that a value is not included in the function and an arrow to denote that the function continues beyond the graph.

Table 7.6

Parking Time (minutes) *m*	Parking Fee (dollars) *F*
Over 0 through 60	4.00
Over 60 through 120	8.00
Over 120	10.00

Source: www.phxskyharbor.com

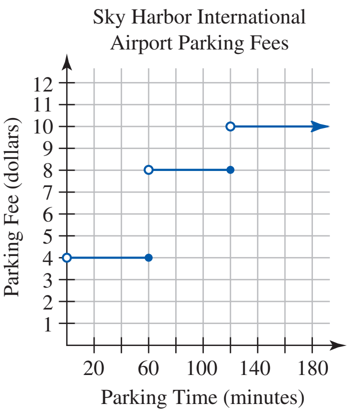


Figure 7.13

A combination of functions such as the parking fee function is called a **piecewise function**.

Piecewise Functions

A **piecewise function** is defined using two or more expressions over given intervals of the domain. Piecewise functions are written in the form

$$f(x) = \begin{cases} \text{Rule 1} & \text{if Condition 1} \\ \text{Rule 2} & \text{if Condition 2} \\ \text{Rule 3} & \text{if Condition 3} \\ \vdots & \vdots \end{cases}$$

The conditions define the input values for which each rule applies. The graphs of piecewise functions may be *continuous* or *discontinuous*. Intuitively, a **discontinuous** function is one with a "break," "hole," or "jump" in its graph and a **continuous** function is one whose graph can be drawn without lifting one's pencil.

Defining a Piecewise Function with an Equation

The rules of a piecewise function may be an algebraic expression such as $2x + 9$ or a verbal expression such as "pays \$260 per credit," as the following examples show.

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \qquad f(\text{age}) = \begin{cases} \text{not eligible for driver's license} & \text{if age} < 16 \\ \text{eligible for driver's license} & \text{if age} \geq 16 \end{cases}$$

$$h(t) = \begin{cases} 5t - 6 & \text{if } t \leq 1 \\ t^2 & \text{if } 1 < t < 5 \\ -3t & \text{if } t \geq 5 \end{cases} \qquad y = \begin{cases} x^2 - 2x + 1 & \text{if } x < 0 \\ \text{undefined} & \text{if } x = 0 \\ -x + 4 & \text{if } x > 0 \end{cases}$$

To model the parking-fee schedule using an algebraic equation, let's look again at the different pieces that define $F(m)$. For example, for any time up to and including 60 minutes, the parking fee is \$4.00. We write

$$F(m) = 4 \quad \text{if } 0 < m \leq 60$$

Combining the separate equations for each level of parking fees, we have

$$F(m) = \begin{cases} 4 & \text{if } \quad 0 < m \leq 60 \\ 8 & \text{if } \quad 60 < m \leq 120 \\ 10 & \text{if } 120 < m \leq 1440 \end{cases} \qquad \text{since there are 1440 minutes in 1 day}$$

It is important to note that $F(m)$ is a single function defined in many *pieces*, not many functions.

EXAMPLE 1 ■ Evaluating Piecewise Functions

Use the parking-fee function, $F(m)$, to evaluate each of the following. Then explain what each result means in its real-world context.

a. $F(40)$

b. $F(76)$

c. $F(480)$

d. $F(1800)$

Solution

a. To evaluate $F(40)$ means to find the fee for parking 40 minutes in the garage. Since 40 is greater than 0 but less than or equal to 60, it satisfies the condition $0 < m \leq 60$. The corresponding "rule" is \$4.00. So $F(40) = 4$, which means the fee for parking 40 minutes is \$4.00.

b. To evaluate $F(76)$ means to find the fee for parking 76 minutes in the garage. Since 76 is greater than 60 but less than or equal to 120, it satisfies the condition $60 < m \leq 120$. The corresponding "rule" is \$8.00. So $F(76) = 8$, which means the fee for parking 76 minutes is \$8.00.

c. To evaluate $F(480)$ means to find the fee for parking 480 minutes in the garage. Since 480 is greater than 120, it satisfies the condition $120 < m \leq 1440$. The corresponding "rule" is \$10.00. So $F(480) = 10$, which means the fee for parking 480 minutes is \$10.00.

d. To evaluate $F(1800)$ means to find the fee for parking 1800 minutes in the garage *on a single day*. Since 1800 is greater than 1440, this value of m does not satisfy any of the conditions for the piecewise function. Therefore, $F(1800)$ is undefined.

EXAMPLE 2 ■ Solving Piecewise Functions

Use the parking-fee function, $F(m)$, to solve each of the following equations for m. Then explain what each result means in its real-world context.

a. $F(m) = 4$

b. $F(m) = 9$

Solution

a. To solve $F(m) = 4$ for m means to determine how many minutes a person can park in the parking garage for \$4.00. From the function equation, we find that \$4.00 is associated with the condition $0 < m \leq 60$. Thus a person who pays the \$4.00 fee may park for more than 0 minutes but no more than 60 minutes.

b. To solve $F(m) = 9$ for m means to determine how many minutes a person can park in the parking garage for \$9.00. From the function equation, we see there is no \$9.00 parking-fee rule. Since there is no parking fee of exactly \$9.00, we are unable to solve the equation. There is no way to spend exactly \$9.00 on parking.

EXAMPLE 3 ■ Creating and Using a Piecewise Function

Simply Fresh Designs is a small business that designs and prints creative greeting cards featuring client photos. The 2006 pricing structure for ordering personalized 4-inch by 6-inch holiday greeting cards from Simply Fresh Designs is shown in Table 7.7.

Table 7.7

Design, Print, & Ship		
4 by 6	\$1.00/card \$0.75/card for orders of 50 or more	white $4\frac{3}{4}$ by $6\frac{1}{2}$ envelope included
(25 minimum order)		
\$6 shipping fee on all orders		

Source: **www.simplyfreshdesigns.com**

a. Create a piecewise function that can be used to calculate the total cost of purchasing n cards.

b. Use the piecewise function to calculate the cost of purchasing 40 cards and 50 cards.

c. How many cards could we buy if we want to spend at most \$45.00?

Solution

a. We let n represent the number of cards and $C(n)$ represent the total cost (in dollars) of purchasing n cards. To determine the function conditions, we note that the card pricing changes from \$1.00 per card to \$0.75 per card when the order size reaches 50 cards. We also note the minimum order size is 25 cards. We have

$$C(n) = \begin{cases} \text{Rule 1} & \text{if } 25 \le n < 50 \\ \text{Rule 2} & \text{if } 50 \le n \end{cases}$$

To determine the function rules, we note that the total cost of the cards is a function of the individual card price plus the shipping cost (\$6.00). If fewer than 50 cards are ordered the cost is $C(n) = 1.00n + 6.00$. If 50 or more cards are ordered, the cost is $C(n) = 0.75n + 6.00$. We now have

$$C(n) = \begin{cases} 1.00n + 6.00 & \text{if } 25 \le n < 50 \\ 0.75n + 6.00 & \text{if } 50 \le n \end{cases}$$

b. We are asked to use the piecewise function to calculate the cost of purchasing 40 cards and 50 cards.

$$\begin{aligned} C(40) &= 1.00(40) + 6.00 \\ &= 46.00 \end{aligned} \qquad\qquad \begin{aligned} C(50) &= 0.75(50) + 6.00 \\ &= 43.50 \end{aligned}$$

It costs \$46.00 to order 40 cards. It costs \$43.50 to order 50 cards.

Due to the volume discount, the cost of 50 cards is actually \$2.50 less than the 40-card cost.

c. To determine how many cards we could buy if we wanted to spend at most \$45.00, we first use the first rule of the function.

$$\begin{aligned} 1.00n + 6.00 &= 45.00 \\ n &= 39 \end{aligned}$$

Since $25 \le 39 < 50$, the solution meets the required condition for the rule. We could order 39 cards for \$45.00.

Now we consider the second rule of the function.

$$\begin{aligned} 0.75n + 6.00 &= 45.00 \\ 0.75n &= 39 \\ n &= 52 \end{aligned}$$

Since $50 \le 52$, the solution meets the required condition for the rule. We could order 52 cards for \$45.00.

For a price of \$45.00, we could either order 39 cards or 52 cards. In this case, we are able to get an additional 13 cards for no additional cost by understanding piecewise functions.

When a scatter plot of a data set appears to have distinct pieces, we can use regression to find a model equation for each piece. The resulting piecewise function is often the best fit to the data.

EXAMPLE 4 ■ Using Regression to Create a Piecewise Function

From 1981 to 1995, the annual number of adult and adolescent AIDS deaths in the United States increased dramatically. However, from 1995 to 2001, the annual death rate plummeted, as shown in Table 7.8.

Table 7.8

Adult and Adolescent AIDS Deaths in the U.S.	
Years Since 1980 *t*	Number of Deaths during Year *D*
1	122
2	453
3	1481
4	3474
5	6877
6	12,016
7	16,194
8	20,922
9	27,680
10	31,436
11	36,708
12	41,424
13	45,187
14	50,071
15	50,876
16	37,646
17	21,630
18	18,028
19	16,648
20	14,433
21	8963

Source: HIV/AIDS Surveillance Report, Dec. 2001; Centers for Disease Control and Prevention, p. 30

a. Draw a scatter plot of the data.

b. Based on the scatter plot, what types of functions will best model each piece? Explain.

c. Use regression to find a piecewise function, $D(t)$, to model the AIDS death data.

Solution

a. The scatter plot is shown in Figure 7.14.

b. From 1981 to 1995, the data is increasing. From 1995 until 2001, the data is decreasing. From 1981 until about 1989, the data appear concave up. A quadratic function may fit this piece well. Between 1989 and 1994, the data appear somewhat linear. A linear function may fit this piece well. Between 1995 and 2001, the scatter plot looks somewhat like a cubic function.

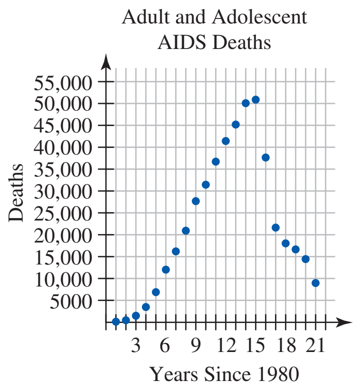


Figure 7.14

c. We use the data for each piece to calculate the corresponding regression models. For the quadratic model, we used data from $t = 1$ through $t = 9$. For the linear model, we used data from $t = 9$ through $t = 14$. For the cubic model, we used data from $t = 15$ through $t = 21$. The resultant function is

$$D(t) = \begin{cases} 417.5t^2 - 682.0t + 101.0 & \text{if } 1 \le t \le 9 \\ 4512.1t - 13{,}138 & \text{if } 9 < t < 15 \\ -381.1t^3 + 21{,}913.3t^2 - 422{,}152.0t + 2{,}739{,}830.5 & \text{if } 15 \le t \le 21 \end{cases}$$

EXAMPLE 5 ■ Graphing a Piecewise Function from an Equation

In 1998, Sammy Sosa of the Chicago Cubs and Mark McGuire of the St. Louis Cardinals were engaged in a race to break the all-time Major League Baseball single-season home run record set in 1962 by Roger Maris of the New York Yankees. As is shown in Figure 7.15, early in the season Sosa was hitting home runs at a relatively slow pace but in June he hit a record number of home runs. In the latter part of the season, his home run pace slowed from his scorching June pace but still remained quite brisk.

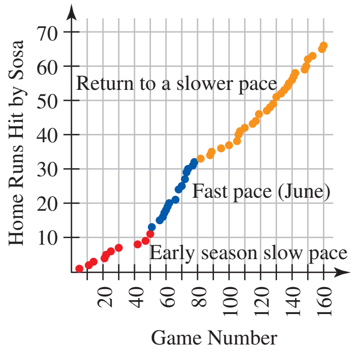


Figure 7.15

Source: www.amstat.org

The number of home runs Sosa hit in 1998 can be modeled by the piecewise function

$$H(g) = \begin{cases} 0.22g - 0.22 & \text{if } 0 \le g < 55 \\ 2.23(1.035^g) & \text{if } 55 \le g < 82 \\ 0.0021g^2 - 0.07g + 23.95 & \text{if } 82 \le g \le 163 \end{cases}$$

where H is the cumulative number of home runs hit and g is the number of games played. (Note: Recall we can choose the "best" function for a data set by considering the type of change in the output values [e.g., first differences, second differences, and percentage change], the shape of the graph, and the r and r^2 values. The difference with piecewise functions is that we consider distinct intervals of the domain that seem to be modeled better by different function types rather than trying to model the entire domain with a single equation.)

a. Sketch a graph of $H(g)$. Be sure to label the axes appropriately.

b. Use the formula of $H(g)$ to evaluate $H(30)$ and $H(60)$. Explain what each solution means in its real-world context.

c. Solve $H(g) = 30$ for g.

d. Use $H(g)$ to predict the number of home runs Sosa hit in 1998. Does the mathematical model show he would break Maris's record of 62 home runs? Explain. (Note: The 1998 Major League Baseball season was 162 games, but the Cubs played 163 games in that season due to a one-game playoff with the San Francisco Giants.)

Solution

a. The graph is shown in Figure 7.16.

b. To evaluate $H(30)$ means to find the cumulative number of home runs Sosa had hit after 30 games. Since $g = 30$ satisfies the condition $0 \le g < 55$, we use the corresponding rule. Therefore,

$$\begin{aligned} H(30) &= 0.22(30) - 0.22 \\ &= 6.38 \\ &\approx 6 \end{aligned}$$

We estimate Sosa had hit 6 home runs after 30 games.

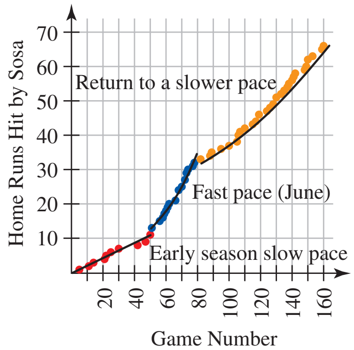


Figure 7.16

To evaluate $H(60)$ means to find the number of home runs Sosa had hit after 60 games. Since $g = 60$ satisfies the condition $55 \le g \le 82$, we use the corresponding rule. Therefore,

$$\begin{aligned} H(60) &= 2.23(1.035^{60}) \\ &= 17.57 \\ &\approx 18 \end{aligned}$$

We estimate Sosa had hit 18 home runs after 60 games.

c. To solve $H(g) = 30$ for g, we need to use the graph in Figure 7.17 to find the game in which Sosa hit his 30th home run. We locate 30 home runs on the vertical axis and then go over to the graph and then down to the horizontal axis to find the game that is associated with 30 home runs. It appears that at about game 75 Sosa hit his 30th home run of the season.

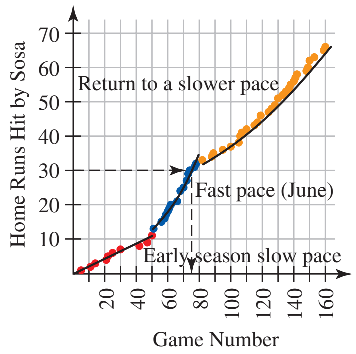


Figure 7.17

d. Since the Cubs played 163 games in 1998, we need to evaluate $H(163)$. Since $g = 163$ satisfies the condition $82 \le g \le 163$, we apply the corresponding rule.

$$\begin{aligned} H(163) &= 0.0021(163)^2 - 0.07(163) + 23.95 \\ &= 68.33 \\ &\approx 68 \end{aligned}$$

According to our model, Sosa hit 68 home runs in 1998 and broke Maris's record of 62. (Actually, he hit 66 and broke the record set by Maris.)

EXAMPLE 6 ■ Creating a Table and Graph for the Absolute Value Function

The absolute value function, $f(x) = |x|$ is formally defined as the piecewise function

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ x & \text{if } x \ge 0 \end{cases}$$

Create a table of values and graph of $f(x) = |x|$ for $-4 \le x \le 4$. Then find $f(-2.5)$.

Solution We create Table 7.9 by using selected x-values between -4 and 4. To create the graph, we plot these points to see the pattern and then connect the points. The graph of the absolute value function, shown in Figure 7.18, includes the points in the table as well as the value of the function for all values $-4 \le x \le 4$. Note that this is a piecewise continuous function.

Table 7.9

x	y
-4	$-(-4) = 4$
-3	$-(-3) = 3$
-2	$-(-2) = 2$
-1	$-(-1) = 1$
0	0
1	1
2	2
3	3
4	4

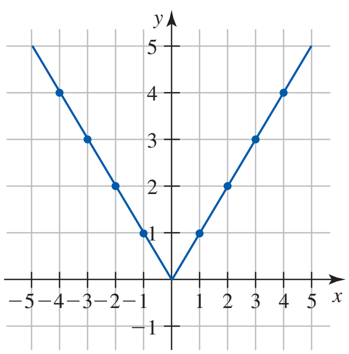


Figure 7.18

Since $x = -2.5$ satisfies the condition $x < 0$, we apply the corresponding rule, $f(x) = -x$, to find $f(-2.5)$.

$$\begin{aligned} f(-2.5) &= -(-2.5) \\ &= 2.5 \end{aligned}$$

We can easily verify our result by looking at the graph.

SUMMARY

In this section you learned that the mathematical modeling of data can be represented by piecewise functions and that piecewise functions can be represented by tables, equations, and graphs. You also learned some piecewise functions are discontinuous, characterized by breaks, holes, and jumps. Additionally, you discovered that piecewise functions lend flexibility to mathematical modeling by using a variety of functions over the domain of the function.

7.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, graph each piecewise function.

1. $f(x) = \begin{cases} 5 & \text{if } -3 \le x < 1 \\ -4 & \text{if } x \ge 1 \end{cases}$

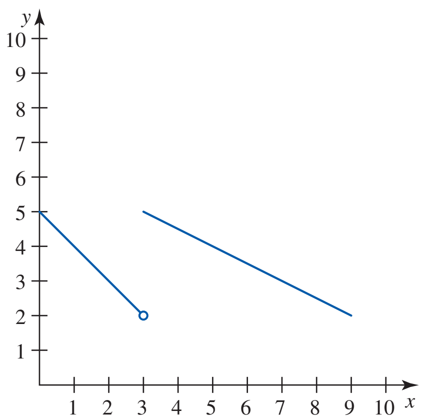
2. $f(x) = \begin{cases} -2x & \text{if } x < 0 \\ -3x^2 & \text{if } x \ge 0 \end{cases}$

3. $f(x) = \begin{cases} -x^2 & \text{if } x \le 2 \\ \sqrt{x} & \text{if } 2 < x < 5 \\ 7 + 15x & \text{if } x \ge 5 \end{cases}$

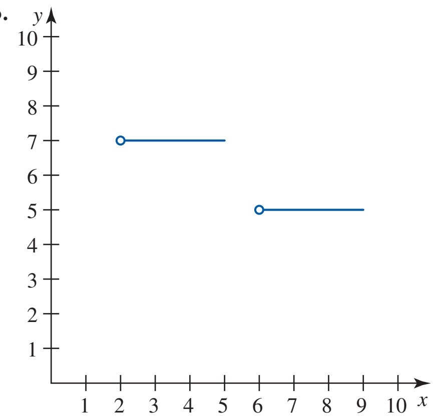
4. $f(x) = \begin{cases} x - 1 & \text{if } -2 \le x < 0 \\ 2^x & \text{if } 1 < x < 6 \\ 10x & \text{if } x \ge 6 \end{cases}$

In Exercises 5–8, write the equation for each piecewise function, $f(x)$.

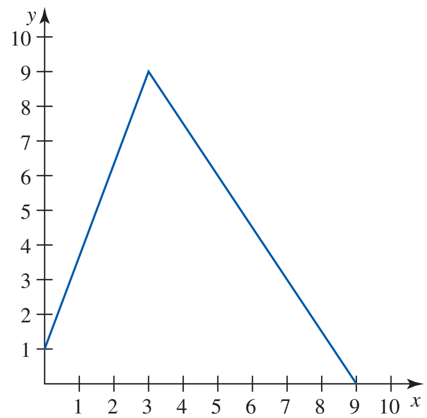
5.



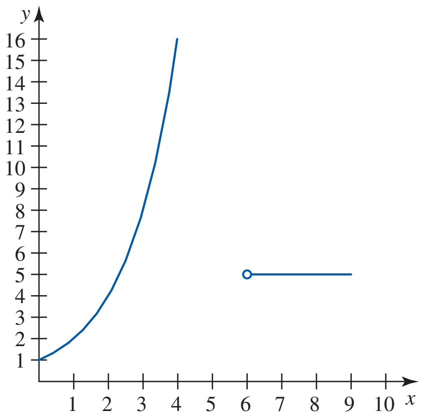
6.



7.



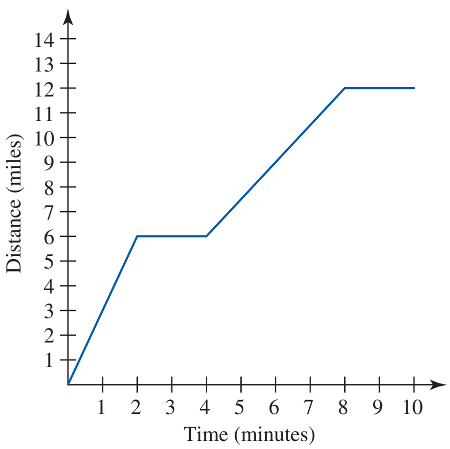
8.



9. The following data table is best defined by a piecewise function, $f(x)$, that can be modeled by an exponential, quadratic, and linear function over three distinct intervals of the domain. Specify over which interval the data is exponential, quadratic, and linear. Then find a formula for $f(x)$.

x	y		x	y		x	y
1	2		6	18		11	14
2	4		7	19		12	12
3	8		8	20		13	11
4	16		9	21		14	11
5	17		10	17			

10. The following graph illustrates the four parts of a trip a family took from home. Use the graph to answer the questions.



a. When did the family take a break?

b. When was their speed the slowest?

c. When was their speed the fastest?

11. **Author Royalties** An author is about to publish her first book, which will be sold for \$35. The author has negotiated to be paid a set amount of \$30,000 for the first 20,000 copies sold, royalties of 13% on the next 7500 copies, and 22% on any additional copies.

a. Write a piecewise function that specifies the total amount the author will be paid, P, based on the number of copies of the book, b, sold.

b. Sketch a graph of the function over a practical domain.

c. Evaluate $P(42{,}556)$ and explain what the numerical value of the solution means in the real-world context of this problem.

d. Solve $P(b) = 42{,}556$ for b and explain what the numerical value of the solution means in the real-world context of this problem.

SHOW YOU KNOW

12. What is a piecewise function?

13. Explain how to create the equation for a piecewise function from data in tabular or graphical form.

14. Explain how to evaluate a piecewise function represented in graphical, tabular, or symbolic form.

15. What does it mean for a function to be continuous or discontinuous?

16. What are the different types of discontinuities?

17. What are the advantages and disadvantages of choosing to model data with a piecewise function?

18. Why are some data sets best modeled by piecewise functions?

MAKE IT REAL

19. **Postage** According to the U.S. Postal Service the single-piece rate to send first-class mail (letters, cards, flats, and parcels) in 2007 is displayed in the table below.

Weight Not Over (ounces) w	Single-Piece Charge (dollars) c
1	\$0.39
2	\$0.63
3	\$0.87
4	\$1.11
5	\$1.35
6	\$1.59
7	\$1.83
8	\$2.07
9	\$2.31
10	\$2.55
11	\$2.79
12	\$3.03
13	\$3.27
for each additional ounce over 13	+ \$0.24

Source: U.S. Postal Service

a. Write a function, $c(w)$, where w is the weight of a first-class letter in ounces and c is the required postage fee in dollars.

b. Evaluate $c(7)$ and explain what the numerical value means in the real-world context.

c. Solve $c(w) = 3.99$ for w and explain what the numerical value means in the real-world context.

20. Ironman Competition A 2.4-mile swim, 112-mile bike ride, and a 26.2-mile run make up an Ironman Triathlon competition. A certain triathlete averages a swimming speed of 2.6 mph, a cycling speed of 18 mph, and a running speed of 10 mph. Assume there is no transitioning time from one segment of the race to another. (*Source:* www.ironmanarizona.com) (*Hint:* Recall that distance = rate · time.)

a. Develop a piecewise function for the speed, S, of the participant as a function of his time, t, in minutes.

b. Sketch a graph of the piecewise function $S(t)$.

c. Describe how each piece of the function $S(t)$ models the three segments of the race.

21. Cost of Electricity Most utility companies charge a higher rate when customers use more than a certain amount of energy. In warm climates, they also charge more during summer months when electric use is higher. As an example, the table shows the residential electric rates for Austin, Texas in 2003. (Note: These figures include a fuel charge of 2.265¢ per kWh.)

Energy Used (kilowatts) k	Energy Charge (cents per kilowatt hour (kWh)) E
$0 < k \leq 500$	\$0.058
$k > 500$ May–Oct.	\$0.10
$k > 500$ Nov.–Apr.	\$0.083

Source: www.michaelbluejay.com

a. Derive a formula for the piecewise function, $E(k)$, that models the energy cost per kilowatt hour used from May to October.

b. Evaluate $E(450)$ for a customer in the month of August.

c. Derive a formula for the piecewise function, $E(k)$, that models the energy cost per kilowatt hour used from November to April.

d. Evaluate $E(450)$ for a customer in the month of December.

22. Windchill Factor The windchill factor, W, is a measure of how cold it feels when taking into account both the temperature (degrees Fahrenheit), t, and the wind velocity (miles per hour), v. In the fall of 2001 the National Weather Service began using a revised windchill formula based on new research on how wind and cold air affects people. (*Source:* www.weather.gov) The formula the National Weather Service uses is the multi-variable piecewise function:

$$W(t, v) = \begin{cases} t & \text{if } 0 \leq v < 4 \\ 35.74 + 0.6215t - 35.75v^{0.16} \\ \quad + 0.4275t(v^{0.16}) & \text{if } 4 \leq v \leq 45 \\ 1.60t - 55 & \text{if } v > 45 \end{cases}$$

a. Evaluate $W(32, 25)$ and interpret the meaning of the solution in the real-world context.

b. Sketch a graph of $W(20, v)$ and, using the concept of rate of change, explain what information the shape of the graph gives regarding the windchill factor and the temperature.

c. How fast must the wind blow for the temperature of 20° to feel like −5°?

d. At what wind speeds do the windchill factor and the actual temperature feel the same?

23. Federal Income Tax Rates Federal income tax rates depend on the amount of taxable income received. The following tax rate schedule shows how the tax rate was determined for a single filer for 2006.

2006 Federal Tax Rate Schedules		
Note: These tax rate schedules are provided so that you can compute your federal estimated income tax for 2006. To compute your actual income tax, please see the instructions for 2006 Form 1040, 1040A, or 1040EZ as appropriate when they are available.		
Schedule X Single		
If taxable income is over	**But not over**	**The tax is:**
\$0	\$7,550	10% of the amount over \$0
\$7,550	\$30,650	\$755 plus 15% of the amount over 7,550
\$30,650	\$74,200	\$4,220.00 plus 25% of the amount over 30,650
\$74,200	\$154,800	\$15,107.50 plus 28% of the amount over 74,200
\$154,800	\$336,550	\$37,675.50 plus 33% of the amount over 154,800
\$336,550	no limit	\$97,653.00 plus 35% of the amount over 336,550

Source: www.irs.gov

a. Write a piecewise function for income tax, T, as a function of taxable income, i, for single filers.

b. Use the function from part (a) to calculate the tax for the following taxable income levels: \$2400; \$29,700; \$59,300; \$129,000; and \$345,000.

c. Use the function to estimate your income tax for 2006.

24. Music Museum Experience Music Project, an interactive music museum in Seattle, charged adult visitors a \$19.95 admission fee in 2005. Groups of 15 or more could enter for \$14.50 per person. (*Source:* www.emplive.com)

a. Create a table for group cost as a function of group size. (Show the cost for groups of 1 to 20 people.)

b. Write the total cost of admission, C, as a function of the number of people in a group, n. Confirm that the function and the table from part (a) are in agreement.

c. Calculate the cost of admitting a group of 14 people and the cost of admitting a group of 15 people.

d. Determine the largest group you could bring in and still remain below the cost of a 14-person group.

25. Buffet Dinner The following sign shows the admission price for Amazing Jake's entertainment center in 2005.

Source: www.amazingjakes.com

a. Write an equation for the weekend buffet price, W, as a function of a person's age, a.

b. Use $W(a)$ to calculate the buffet price for a 3-year-old, 6-year-old, 9-year-old, and 24-year-old person.

c. Explain why it is impossible to evaluate $W(2)$ using the table provided.

d. If a family comprised of a husband and wife (under age 65) and children ages infant, 3, 5, 8, and 12 years go to Amazing Jakes on the weekend, what will it cost them to enter?

26. Disneyland The following sign shows the cost of Disneyland theme park tickets for 2007.

Ticket Prices

	Ages 3-9	Ages 10+
5-Day Park Hopper® Bonus Ticket Save up to $40 per person!	~~$199.00~~ $159.00	~~$229.00~~ $189.00
4-Day Park Hopper® Bonus Ticket Save up to $30 per person!	~~$179.00~~ $149.00	~~$209.00~~ $179.00
3-Day Park Hopper® Bonus Ticket Save up to $20 per person!	~~$149.00~~ $129.00	~~$179.00~~ $159.00
2-Day Park Hopper® Ticket	$102.00	$122.00
1-Day Park Hopper® Ticket	$73.00	$83.00
Single-Day Theme Park Ticket	$53.00	$63.00

Source: www.disneyland.com

a. Write an equation for the 3-Day Park Hopper admission price, H, as a function of a person's age, a.

b. What is the domain and range of the function from part (a)?

c. Write an equation for the adult admission price for a Park Hopper ticket, A, as a function of the number of days, d, to be spent in the park.

d. What is the domain and range of $A(d)$?

e. A husband and wife want to take their children to Disneyland on a 3-Day Park Hopper pass next summer. If the children's ages at the time of the trip will be 1, 3, 6, 9, and 13, what will it cost the family to enter?

27. Taxi Fare In 2006 in New York City, one company's initial charge for a taxi ride was $2.50 for the first $\frac{1}{5}$ of a mile, and $0.40 for each additional $\frac{1}{5}$ of a mile. (*Source*: www.schallerconsult.com)

a. Create a table showing the cost of a trip as a function of its length. Your table should start at 0 and go up to 1 mile in $\frac{1}{5}$ of a mile intervals.

b. What is the cost for a 10-mile taxi ride?

c. How far can a person ride for $3.00?

d. Graph the taxi cab fare function in part (a).

28. Museum Admission The Museum of Science and Industry in Chicago has special rates for school and tour groups with a four-week advance registration. The museum's 2007 prices for tour groups of 20 or more people are shown in the table.

Attendee's Age a	Admission Price P
$a \geq 65$	$8.00
$12 \leq a < 65$	$9.50
$3 \leq a \leq 11$	$6.00
$a < 3$	Free

Source: www.msichicago.org

a. Determine the cost of attending the museum with fifteen 11-year-olds, seven 12-year-olds, two 29-year-olds, and one 42-year-old.

b. Can the table be used to determine how many 16-year-old students can go to the museum for $100.00? Explain.

c. Graph the museum admission price function $P(a)$.

d. Create a piecewise function for the admission price table for groups of 20 or more people. Write the price as a function of a person's age.

29. Durango Ski Resort The ski lift rates for 2006–2007 in Durango, CO, are shown in the table, based on the age of the skier.

Skier's Age (years) *a*	Charge (dollars) *c*
12 and under	31
13 to 18	43
19 to 61	59
62 and over	43

Source: **www.onthesnow.com**

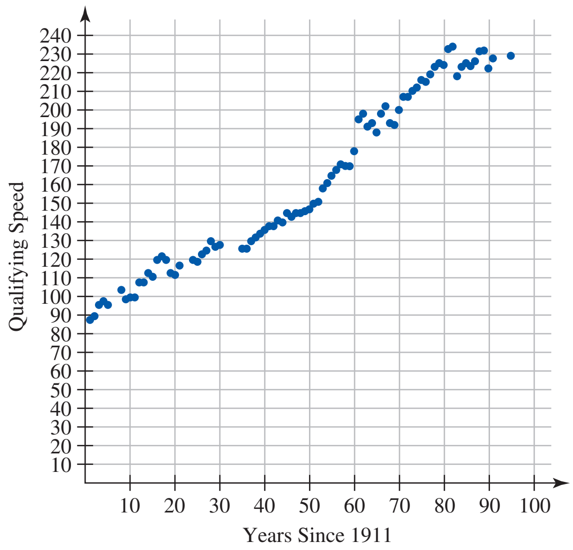
a. A family holding their reunion in Durango decides to go skiing. The following table displays the number of people from each age group that want to ski. How much will it cost for them all to go?

Skier's Age (years) *a*	Number Who Want to Ski
12 and under	11
13 to 18	7
19 to 61	25
62 and over	1

b. Sketch a graph of the function $c(a)$ that models the cost to ski as a function of the age of the skier.

c. Create a piecewise function for $c(a)$ that models the cost to ski as a function of the age of the skier.

30. The Indianapolis 500 A scatter plot of the qualifying speeds for the Indianapolis 500 from 1911 to 2007 is shown in the figure.



Source: **www.indy500.com**

a. The function $s(y) = 1.707y + 77.679$ is the line of best fit for these data, where s is the speed and y is the years since 1911. Evaluate $s(100)$ and explain the meaning of the value in this real-world context.

b. Why do you suppose the qualifying speed for 1911 was 0 mph for the Indianapolis 500? Explain what you think would happen to the parameters (slope and y-intercept) of the line of best fit if this point is removed from the scatter plot and a new linear regression is done.

c. Examine the scatter plot of the qualifying speeds from 1911 to 2007. Give a reasonable explanation for the horizontal and vertical gaps in the data that occur for the following years:

- 1916–1919
- 1941–1946
- 1971–1972
- 1996–1997

d. Would a piecewise function be a better mathematical model for the Indianapolis 500 qualifying speeds than one linear function? Why or why not?

31. Cost of Natural Gas In many parts of the country, the natural gas bills for customers are calculated on gas meter readings that are taken every month. A *therm* is a unit that measures the amount of natural gas needed to produce 100,000 BTUs of heat. Each month, a customer's gas bill states how many therms were used. For single-family residential customers of Southwest Gas Corporation in Southern Nevada during the 2007 Summer Season (May–October), the gas delivery charge for natural gas was billed in the following manner.

- \$8.50 basic service charge + \$1.20 per therm for 0 to 15 therms (including all tariffs and adjustments)
- \$8.50 basic service charge + \$1.02 per therm (including all tariffs and adjustments) for greater than 15 therms

(*Source:* www.swgas.com)

a. Construct a table for this data, using t to represent the total number of therms used and C to represent the total monthly charge for natural gas delivery. Use therms from 0 to 50 in increments of 5 therms.

b. Write a formula for the piecewise function, $C(t)$, that represents the monthly delivery charge to a residential customer based on the number of therms, t, used.

c. Sketch the graph of the function $C(t)$.

d. What is the delivery charge if a customer used 76 therms during the month of June?

STRETCH YOUR MIND

Exercises 32–39 are intended to challenge your understanding of piecewise functions.

Many companies that have mail order catalogs charge their customers shipping and handling fees for purchased products. The fees that JCPenney charged in 2007 for their catalog purchases are shown in the table.

Total Product Cost (dollars) t	Shipping and Handling Charge (dollars) S
Up to \$25.00	\$5.95
\$25.01 to \$40.00	\$7.50
\$40.01 to \$50.00	\$8.50
\$50.01 to \$75.00	\$11.50
\$75.01 to \$100.00	\$14.95
\$100.01 to \$150.00	\$18.95
\$150.01 to \$200.00	\$22.95
\$200.01 to \$300.00	\$25.95
\$300.01 to \$500.00	\$29.95
Over \$500.00	\$39.95

Source: www4.jcpenney.com

32. What are the shipping and handling charges a customer incurs if the total product cost is \$50.00? \$50.01?

33. Notice that there is a \$3.00 increase in shipping and handling charges for product costs that differ by only \$0.01 from \$50.00 to \$50.01. Give a possible reason for this increase.

34. What are the shipping and handling charges a customer incurs if the total product cost is \$562.00? \$1000.00?

35. Sketch a graph of the shipping and handling cost, S, as a function of the total product cost, t, in dollars. Be sure to label and scale the axes appropriately.

36. Write a formula for the shipping and handling cost, S, as a function of the total cost, t, in dollars.

37. Sketch a graph of the total cost of a purchase including the shipping and handling fee, B, as a function of the total product cost, t, in dollars. Be sure to label and scale the axes appropriately.

38. Write a formula for the total cost of a purchase including the shipping and handling fee, B, as a function of the total of the product cost, t, in dollars.

39. Compare and contrast the graphs of $S(t)$ and $B(t)$. Explain in terms of the real-world context the differences that exist in the graphs.

SECTION 7.3 Composition of Functions

LEARNING OBJECTIVES

- Compose two or more functions using tables, equations, or graphs
- Create, use, and interpret function composition notation in a real-world context

GETTING STARTED

In many fields, an employee with an advanced educational degree can move into positions of responsibility and higher pay more rapidly than less-educated coworkers. This may be due in part to the belief that an employee with an advanced degree has a deeper knowledge base to draw from and is more capable of handling responsibilities. In effect, the time a student spends earning a degree can in turn produce a higher salary.

In many real-world situations such as this, it is common for the output of one function to be used as the input of another function, resulting in a composition of functions. In this section we investigate the concepts behind function composition and learn how to compose and decompose functions using tables, graphs, and equations.

■ Understanding Composition of Functions

Consider Table 7.10, which shows the 2004 median annual salaries of American employees, the associated educational degree acquired, and the accumulated full-time academic years typically spent to attain such degrees.

Table 7.10

Accumulated Time in College (academic years) t	Degree Attained d	Median Annual Salary (dollars) s
0	High School Diploma	20,733
2	Associate's	30,026
4	Bachelor's	38,880
6	Master's	50,693
8	Doctorate	72,073

Source: U.S. Census Bureau, www.census.gov/population/www/socdemo/education/cps2005.html

EXAMPLE 1 ■ Understanding Composition of Functions

Use Table 7.10 to answer the following questions.

a. If an employee spent 4 years in college, what degree did she earn?

b. What is the median annual salary of an employee with a Doctorate?

c. If an employee attended school for 6 years, what degree did he earn? What is the median annual salary of an employee with a Master's degree?

d. What is the median annual salary of an employee who spent 6 years in college?

Solution

a. According to Table 7.10, a person who attends college for 4 years earns a Bachelor's degree. Notice *academic years* is the input of the function and *degree* is the output. The function used does not reveal anything about the person's median annual salary.

b. According to the table, an employee with a Doctorate degree earns a median annual salary of $72,073. Observe *degree attained* is the input of the function and *median annual salary* is the output. This function does not reveal anything about the academic years spent in college.

c. According to the table, a person who attends college for 6 years earns a Master's degree. The median annual salary of a person with a Master's degree is $50,693. For the first question, *academic years* is the input and *degree attained* is the output. However, for the second question, the *degree attained* is the input and the *median annual salary* is the output of the function.

d. According to the table, an employee with 6 years of college earns a median annual salary of $50,693. (This assumes the employee earned a Master's degree.) We are using a function that has *academic years* as the input and *median annual salary* as the output. In effect, we skipped the step of finding the employee's degree and calculated the salary directly.

In parts (c) and (d) of Example 1 we arrived at the same answer for the employee's median annual salary but took two different paths to get there. Table 7.11 compares the processes involved.

Table 7.11

The two-step method used in part (c)		
Input →	Output/Input →	Output
Accumulated Time in College (academic years)	Degree Attained	Median Annual Salary (dollars)
6 →	Master's →	50,693
The function composition method used in part (d)		
Input →		Output
Accumulated Time in College (academic years)		Median Annual Salary (dollars)
6 →		50,693

The process outlined in part (d) demonstrates function composition, which occurs when the output of one function is used as an input to another function. In this case, the output of the *degree attained* function was the input to the *median annual salary* function.

Composition Notation

Continuing with the salary example, we define the variables as follows:

- t is the accumulated time in college (academic years).
- d is the degree attained.
- s is the median annual salary.

It is important to understand the relationship between these variables.

- d and t

  We know d is a function of t because the degree attained by an employee depends on the accumulated number of academic years. We write the relationship as $d(t)$.

- s and d

  We know s is a function of d because the median annual salary of an employee depends on the degree attained. We write the relationship as $s(d)$.

- s, d, and t

  We have shown that s is a function of d and that d is a function of t. That is, salary (s) depends on the degree earned (d), which depends on the accumulated number of years spent in college (t). However, we have also seen these two functions can be combined in a **composition of functions**: s is a function of d and d is a function of t. This relationship is sometimes represented with the notation $(s \circ d)(t)$; however, we will use the more intuitive notation, $s(d(t))$.

Composition of Functions

The function $h(x) = f(g(x))$ is the **composition** of the function f with the function g. Function h is called a *composite* function.

Creating Composite Functions

In the next few examples, we'll look at how to create and evaluate composite functions defined by tables, equations, and graphs.

EXAMPLE 2 ■ Creating a Composite Function Defined by Tables

More than 500 new energy drinks were launched worldwide in 2006, each one with promises of weight loss, increased endurance, and legal highs. Newer products joined top-sellers Red Bull and Monster to make up a $3.4 billion a year industry that grew by 80% in 2005. (*Source:* www.cnn.com) Most of these beverages are caffeine-laden, and nutritionists warn that the drinks can hook people, especially teenagers, on an unhealthy jolt-and-crash cycle. Although energy drinks such as Red Bull, Rock Star, Rush, Monster, and Hype contain some nutritional vitamins and supplements, many contain other ingredients that are not necessarily good for you. One particular brand of energy drink has 80 milligrams of caffeine in an 8.3-ounce can.

The effects of caffeine in the bloodstream on the heart rate of a typical 175-pound male adult are displayed in Table 7.12. It is also known that the amount of caffeine in a person's bloodstream dissipates over time. The amount of caffeine in a 175-pound adult male's bloodstream would typically dissipate as shown in Table 7.13, where time, t, is the number of hours after the caffeine is ingested.

Table 7.12

Caffeine Level (milligrams) c	Heart Rate (beats per minute) r
0	80
50	82
131	84
150	85
198	86
261	88
300	89

Source: Dr. Brent Alvar, Chandler-Gilbert Community College Wellness and Fitness Director

Table 7.13

Time (hours) t	Caffeine Level (milligrams) c
0	300
1	261
2	227
3	198
4	172
5	150
6	131

Source: Dr. Brent Alvar, Chandler-Gilbert Community College Wellness and Fitness Director

Using Tables 7.12 and 7.13, create a table that shows the heart rate, r, as a function of the time, t.

Solution We begin by writing the two tables adjacent to each other, as shown in Tables 7.14 and 7.15.

Table 7.14

Time (hours) t	Caffeine Level (milligrams) c
0	300
1	261
2	227
3	198
4	172
5	150
6	131

Table 7.15

Caffeine Level (milligrams) c	Heart Rate (beats per minute) r
0	80
50	82
131	84
150	85
198	86
261	88
300	89

We see that the values for the caffeine levels appear to be the same in many cases. We create a three-column table (Table 7.16) to capture what we know. We now eliminate the rows of the table with blank spaces and delete the middle column to get Table 7.17, which relates time, t, to heart rate, r.

Table 7.16

Time (hours) t	Caffeine Level (milligrams) c	Heart Rate (beats per minute) r
0	300	89
1	261	88
2	227	
3	198	86
4	172	
5	150	85
6	131	84
	50	82
	0	80

Table 7.17

Time (hours) t	Heart Rate (beats per minute) r
0	89
1	88
3	86
5	85
6	84

EXAMPLE 3 ■ Evaluating a Composite Function from a Table

Use the tables and the solution in Example 2 to answer the following.

a. Evaluate $r(c(3))$ first by using $c(t)$ and $r(c)$ and then by using the composite function $r(c(t))$. Explain the meaning of the answer in its real-world context.

b. Explain the difference between the two ways to arrive at the answer in part (a).

Solution

a. One way to evaluate $r(c(3))$ is to first evaluate $c(3)$. We find from Table 7.14 that $c(3) = 198$. Now we evaluate $r(198)$ by looking at Table 7.15 and find $r(198) = 86$. The second way is to skip the intermediate step and evaluate $r(c(3))$ using Table 7.17. We see that $r(c(3)) = 86$, which means that 3 hours after the caffeine is ingested, the heart rate is 86 beats per minute.

b. The difference is that we can find the answer in three steps by using the two functions $r(c)$ and $c(t)$ separately, or we can skip the middle step by using the composite function $r(c(t))$.

EXAMPLE 4 ■ Creating a Composition of Functions Defined by Equations

The effect of caffeine, c, in milligrams on a person's heart rate, r, in beats per minute can be modeled by the linear function $r(c) = 80 + 0.03c$. The dissipation of caffeine, c, in milligrams from the bloodstream over time since ingestion, t, in hours can be modeled by the exponential function $c(t) = 300(0.87)^t$.

a. Write the equation for the composite function $r(c(t))$. Explain the input and output variables in terms of the real-world context.

b. Evaluate $r(c(3))$ and explain the meaning of the solution in terms of the real-world context.

c. Find the practical domain of $r(c)$, $c(t)$, and $r(c(t))$. Refer to the tables in Example 2 as needed.

Solution

a. To form the composition $r(c(t))$, we begin by placing the equation for $c(t)$ into function $r(c(t))$ as the input.

$$\begin{aligned} r(c) &= 80 + 0.03c \\ r(c(t)) &= r(300(0.87^t)) && \text{since } c(t) = 300(0.87)^t \\ &= 80 + 0.03(300(0.87^t)) \\ &= 80 + 9(0.87^t) \end{aligned}$$

Therefore, $r(c(t)) = 80 + 9(0.87^t)$. The input to the function $r(c(t))$ is time in hours and the output is heart rate in beats per minute.

b. To evaluate $r(c(3))$, we input 3 hours into the composite function $r(c(t))$. $r(c(3)) = 80 + 9(0.87^3) = 85.9$. This means that 3 hours after the caffeine is ingested the person's heart rate is about 86 beats per minute.

c. Referring to Table 7.15, we see that the practical domain of $r(c)$ is all values from 0 milligrams to 300 milligrams. That is, $0 \le c \le 300$. Referring to Table 7.14, we see that $c(t)$ has a practical domain of all values from 0 to 6 hours. That is, $0 \le t \le 6$. (Note that the domain should realistically include more hours to allow for the caffeine to be totally removed from the person's system, which could take 24 hours or more.) Finally, the practical domain for the composite function, $r(c(t))$, is again $0 \le t \le 6$ because this function is defined in this context for only the first 6 hours after the person ingested the caffeine.

EXAMPLE 5 ■ Creating a Composition of Functions Defined by Graphs

Figure 7.19 shows the functions f and g. Draw the graph of $f(g(x))$.

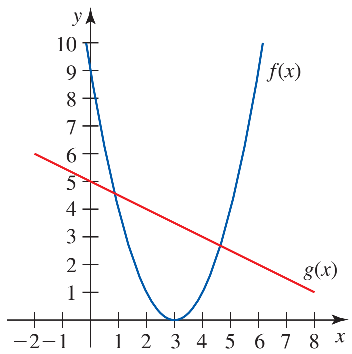


Figure 7.19

Solution We begin by creating a table of values for $g(x)$. Then, using the values of $g(x)$ as inputs, we use the graph of $f(x)$ to find $f(g(x))$. See Table 7.18. For example, we see $g(0) = 5$ and $f(5) = 4$. We plot the points using the first and third columns of the table and connect the points with a smooth curve, as shown in Figure 7.20.

Table 7.18

x	$g(x)$	$f(g(x))$
−2	6	9
0	5	4
2	4	1
4	3	0
6	2	1
8	1	4

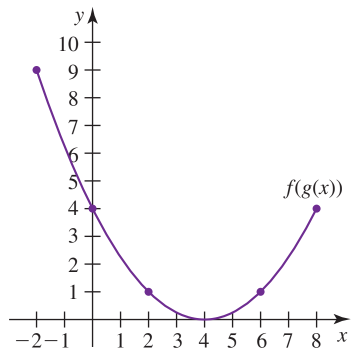


Figure 7.20

EXAMPLE 6 ■ **Creating Composite Functions Symbolically**

Given $f(x) = \sqrt{x + 3}$ and $g(x) = \dfrac{2}{x + 1}$, find $f(g(x))$ and $g(f(x))$.

Solution We first find $f(g(x))$.

$$f(g(x)) = f\left(\frac{2}{x + 1}\right) \qquad \text{The function } g(x) \text{ is the input.}$$

$$= \sqrt{\left(\frac{2}{x + 1}\right) + 3} \qquad \text{Replace the } x \text{ in } f(x) \text{ with } \frac{2}{x + 1}.$$

We next find $g(f(x))$.

$$g(f(x)) = g(\sqrt{x + 3}) \qquad \text{since } f(x) = \sqrt{x + 3}$$

$$= \frac{2}{\sqrt{x + 3} + 1}$$

EXAMPLE 7 ■ **Decomposing Functions Symbolically**

Each given function is a composite function of the form $f(g(x))$. Decompose the functions into two functions $f(x)$ and $g(x)$.

a. $f(g(x)) = |3x - 2|$

b. $f(g(x)) = \dfrac{2}{x^2 - 1}$

c. $f(g(x)) = 2^{6x-1}$

d. $f(g(x)) = (x - 2)^4 - (x - 2)^3 + (x - 2)^2$

Solution

a. The function $f(g(x))$ takes the absolute value of $3x - 2$. Two functions that can be used for the composition are $f(x) = |x|$ and $g(x) = 3x - 2$. This is not a unique solution; $f(x) = |x - 2|$ and $g(x) = 3x$ also work.

b. The function $f(g(x))$ consists of a rational function and a quadratic function. Two functions that can be used for the composition are $f(x) = \dfrac{2}{x}$ and $g(x) = x^2 - 1$. This is not a unique solution; $f(x) = \dfrac{1}{x - 1}$ and $g(x) = x^2$ also work.

c. The function $f(g(x))$ raises 2 to the power $6x - 1$. Two functions that can be used for the composition are $f(x) = 2^x$ and $g(x) = 6x - 1$. Additional solutions exist.

d. The function $f(g(x))$ raises $x - 2$ to the fourth, third, and second power. Two functions that can be used for the composition are $f(x) = x^4 - x^3 + x^2$ and $g(x) = x - 2$. Additional solutions exist.

SUMMARY

In this section you investigated the concepts that underlie function composition. You discovered that in many real-world situations it is common for the output of one function to be the input of another function. You also learned how to create composite functions and to evaluate them using tables, graphs, and equations and then how to decompose such functions symbolically.

7.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–10, rewrite each pair of functions, $f(g)$ and $g(t)$, as one composite function, $f(g(t))$, if possible. Then evaluate $f(g(2))$.

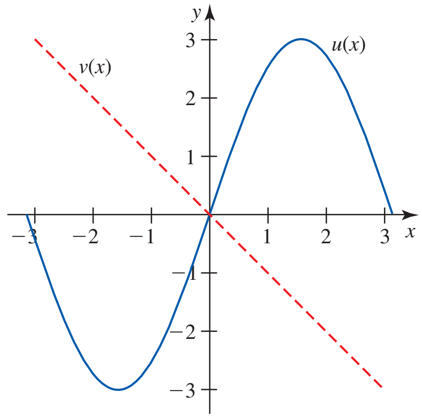
1. $f(g) = 3g^2 - g + 4 \quad g(t) = 5 - 4t$
2. $f(g) = 3e^g \quad g(t) = 2t^2$
3. $f(g) = \dfrac{5}{g} \quad g(t) = 1 + 4(2^{-0.4t})$
4. $f(g) = \sqrt{8g^2 + 8g - 1} \quad g(t) = |3t^3| - t$
5. $f(g) = 5g - 4 \quad g(t) = t^2 - 3t + 6$
6. $f(g) = \sqrt[3]{g} \quad g(t) = t - 5$
7. $f(g) = 6g + 3 \quad g(t) = t^2 - 2t - 6$
8. $f(g) = \dfrac{4}{1 - 5g} \quad g(t) = \dfrac{1}{t}$
9. $f(g) = g^3 - 4g^2 + 2g - 3 \quad g(t) = t + 1$
10. $f(g) = \dfrac{1 - g}{g} \quad g(t) = \dfrac{2t - 2}{4t + 1}$

In Exercises 11–16, decompose the composite function $h(x) = f(g(x))$ into $f(g)$ and $g(x)$. (Note: There may be more than one possible correct answer.)

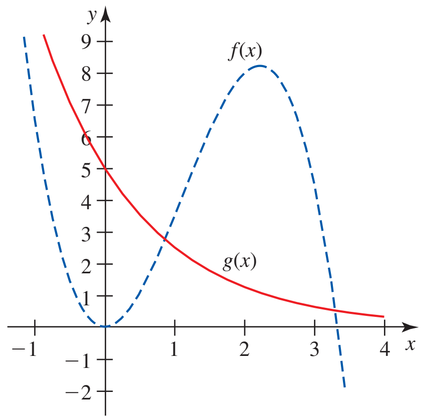
11. $h(x) = (2x - 2)^5$
12. $h(x) = \sqrt[3]{x^2 - 7}$
13. $h(x) = \dfrac{1}{(x - 2)^6}$
14. $h(x) = \left(\dfrac{2 - x^3}{2 + x^3}\right)^2$
15. $h(x) = (\sqrt{x} - 3)^4$
16. $h(x) = e^{(3x - \pi)}$

In Exercises 17–20, decompose the function into two functions. The unfamiliar operational symbols are used to focus on the idea of decomposition rather than the operations involved.

17. $h(x) = (x - 1)^{\heartsuit}$
18. $h(x) = \sqrt[\Uparrow]{x^{\oplus} - x}$
19. $h(x) = \left(\dfrac{4}{(x \diamond 5)}\right)^{©}$
20. $h(x) = |4x \int 9|$
21. Let $u(x)$ and $v(x)$ be two functions defined by the two graphs in the figure. Estimate the following.
    - **a.** $v(u(-2))$
    - **b.** $u(v(1))$
    - **c.** $v(u(0)) + v(u(3))$
    - **d.** $v^{-1}(u(2))$



22. Using the two graphs in the figure, estimate the following.
    - **a.** $f(g(2))$
    - **b.** $g(f(1))$
    - **c.** $f(f(-0.5))$
    - **d.** $g(g(3))$
    - **e.** $g^{-1}(f(1.5))$
    - **f.** $g^{-1}(f(3))$



23. Use the first three columns in the table to fill in the values for $k(n(x))$. (Note: Some function values may be undefined.)

x	$n(x)$	$k(x)$	$k(n(x))$
0	2	6	
1	1	3	
2	6	4	
3	3	2	
4	4	5	
5	5	9	

24. Let $k(x)$ and $n(x)$ be the functions from Exercise 23. Construct a table of values for $n(k(x))$. (Note: some function values may be undefined.)

25. Find the equations for the functions in parts (a)–(f). Let $f(x) = x^2 - 2$, $g(x) = \dfrac{2}{x + 3}$, and $h(x) = \sqrt{x}$.

a. $f(g(x))$

b. $g(f(x))$

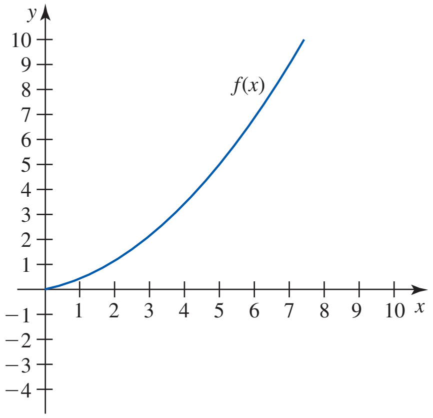
c. $f(h(x))$

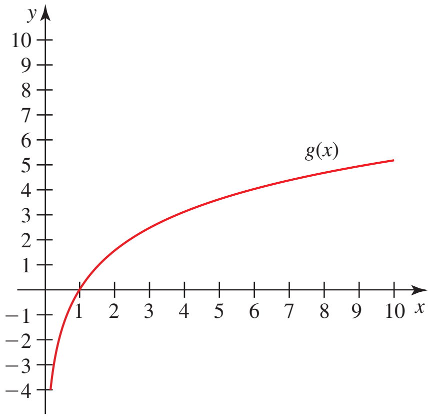
d. $h(f(x))$

e. $h(h(x))$

f. $g(f(h(x)))$

26. Use the following graphs of $f(x)$ and $g(x)$ to sketch a graph of functions $f(g(x))$ and $g(f(x))$. Do the two composite functions have the same graphs? Why or why not?





27. Find $f(f(3))$ for

$$f(x) = \begin{cases} 4 & \text{if } x \leq 0 \\ 2x + 1 & \text{if } 0 < x < 4 \\ x^2 - 4 & \text{if } x \geq 4 \end{cases}$$

In Exercises 28–29, select the correct answer from those offered and explain why you chose as you did.

28. Ripple Spread A pebble is dropped into a lake, creating a circular ripple that travels outward at a speed of 4 centimeters per second. Which of the following functions express the area, A, of the circle as a function of the number of seconds, s, that have passed since the pebble hits the lake?

A. $A = 16\pi s$

B. $A = 16\pi r^2$

C. $A = 16\pi s^2$

D. $A = 4\pi s^2$

E. $A = \pi r^2$

29. Which of the following functions defines the area, A, of a square as a function of its perimeter, p?

A. $A = \dfrac{p^2}{16}$

B. $A = s^2$

C. $A = \dfrac{p^2}{4}$

D. $A = 16s^2$

E. $p = 4\sqrt{A}$

30. Quality Control A baker suspects his oven may be malfunctioning so he begins inspecting muffins at his bakery. He notes the first two batches of muffins he cooks each day are not acceptable because they are not cooked enough in the middle. After those first two batches, the oven generally is warmed up enough and works fine. Since the muffins will be packaged for shipping, the muffins are left to cool in a climate-controlled environment after cooking. The muffins are then properly packaged by a machine in cellophane 95% of the time. Let m be the number of muffin batches cooked in a day.

a. Write an equation for $n(m)$ that represents the number of muffin batches that are properly cooked.

b. Write an equation for $p(n)$ that represents the number of muffin batches that are properly packaged.

c. Now find $p(n(52))$, and interpret its meaning in this real-world context.

31. Seaside Restaurant A world-renowned seaside restaurant's motto is "shellfish directly from the sea." The restaurant prides itself in its delectable scallop dish. Management hires local fishermen to harvest scallops in the nearby area, and customers can enjoy their meals while watching the fishermen arrive at the dock with that day's catch. Unfortunately, the number of shellfish in the area has been decreasing due to the increased amount of pollutants that have been found to be in the ocean.

There are two related functions involved in this scenario. One is the number of scallops per acre, f, as a function of the amount of pollutants found in the ocean, w, in grams per liter $\left(\dfrac{\text{g}}{\text{liter}}\right)$, modeled by $f(w) = 23 - 4w$.

The second is the price (in dollars) of a scallop dinner, p, as a function of the number of scallops per acre, f, modeled by $p(f) = 39.95 - 1.10f$.

a. Determine the composite function $p(f(w))$ and explain what it defines.

b. How much did a scallop dinner cost before there were any measurable pollutants in the ocean? Solve this problem in two different ways—one using composition and one not.

c. According to the model, what will be the price of a dinner when pollution has killed all of the scallops? (Note: This situation will require scallops to be imported.)

d. Solve $p(f(w)) = 21.50$ and interpret the practical meaning of the results.

32. College Student Workload Many college students work to pay tuition. The number of hours students work may affect the number of credits they can take. In turn, study time in preparation for class is dependent on the number of credits taken. The tables show examples of these relationships.

Hours Worked per Week h	Credits Taken c
$0 \le h < 4$	18
$4 \le h < 8$	17
$8 \le h < 12$	16
$12 \le h < 16$	15
$16 \le h < 20$	14
$20 \le h < 24$	13
$24 \le h < 28$	12

Credits Taken c	Study Hours per Week s
12	12
13	14
14	16
15	20
16	25
17	31
18	39

a. Construct a table showing the relationship between the number of hours worked per week and the number of hours of study per week.

b. Sketch a graph of the new relationship of the number of hours worked per week and the resulting number of hours of study per week.

c. Describe how the new relationship between the number of hours worked per week and the resulting number of hours of study per week is an example of a composition of functions.

d. Considering the information given in the tables, give an example of a situation in which a student reduces the hours worked and thus increases the number of credits and makes more time available for leisure activities.

SHOW YOU KNOW

33. Explain what is meant by function composition.

34. Make up your own situation using a composition of functions and three interrelated variables.

In Exercises 35–39, determine whether it is reasonable for the pairs of functions to be combined by function composition. If so, give function notation for the new function and explain what the input and output values would be.

35. $p(l)$ is the revenue in pesos from the sale of l laptop computers. $D(p)$ is the dollar value of p pesos.

36. $l(s)$ is the profit generated by the sale of s servings of lemonade. $s(h)$ is the number of servings of lemonade a child can make in h hours.

37. $r(y)$ is the home loan interest rate as a function of the year. $h(r)$ is the number of new homes that are built in a city as a function of the home loan interest rate.

38. $c(s)$ is the number of calories consumed as a function of the number of sodas consumed. $s(w)$ is the number of sodas consumed after w weeks.

39. $a(t)$ is the number of accidents caused by drunk drivers in the year t. $c(t)$ is the number of automobiles that have passed by an intersection in t hours.

MAKE IT REAL

40. Table A shows temperatures measured in Celsius, C, and their corresponding temperatures in Fahrenheit, F. Table B shows temperatures in Fahrenheit, F, and their corresponding temperatures in Celsius, C.

Table A

Temperature (°F)	Temperature (°C)
−4	−20
5	−15
14	−10
23	−5
32	0
41	5
50	10
59	15
68	20
77	25
86	30

Table B

Temperature (°C)	Temperature (°F)
−20	−4
−15	5
−10	14
−5	23
0	32
5	41
10	50
15	59
20	68
25	77
30	86

Use the tables to evaluate the following expressions.

a. $C(F(5))$

b. $F(C(23))$

c. $F(0)$

41. Temperature Conversion In chemistry it is sometimes necessary to convert temperatures from one unit of measure to another. Specifically, there are three measures of temperature typically used: Fahrenheit, Celsius, and kelvin. The formula $C(F) = \frac{5}{9}(F - 32)$ is used to convert Fahrenheit to Celsius and the formula $K(C) = C + 273$ is used to convert Celsius to kelvin.

a. Find the formula for the composite function $K(C(F))$.

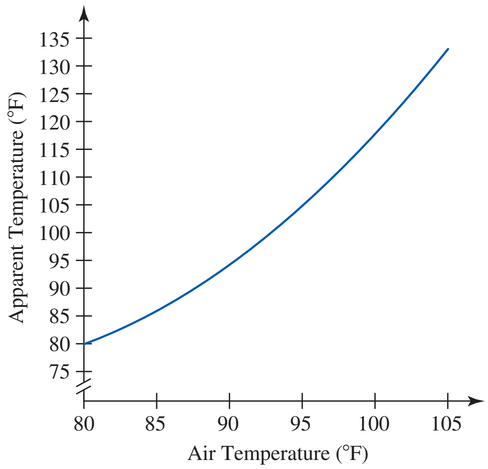
b. Explain the input and output for $K(C(F))$ in terms of the real-world context.

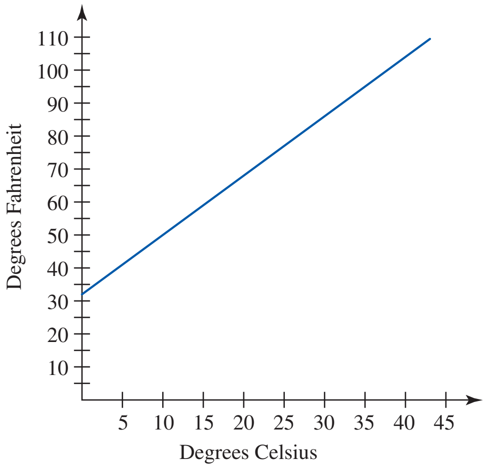
c. Evaluate $K(C(81))$ and explain the meaning of the solution in the real-world context.

d. Solve $K(C(F)) = 572$ for F and explain the meaning of the solution in the real-world context.

42. Heat Index The heat index is used to describe the *apparent temperature*, how hot it feels when both temperature and humidity are taken into consideration. For mild temperatures, humidity has little effect on the apparent temperature. However, for high temperatures, humidity dramatically affects the apparent temperature. When the humidity is 50%, the apparent temperature can be modeled by $a(F) = 0.04643F^2 - 6.464F + 299.9$ degrees Fahrenheit, where F is the air temperature in degrees Fahrenheit. This model is valid for air temperatures between 80 and 105 degrees Fahrenheit. (*Source:* **www.crh.noaa.gov**) Outside of the United States, Celsius is the common unit of measure for temperature, and U.S. citizens traveling abroad sometimes have difficulty translating Celsius temperatures into the more familiar Fahrenheit temperatures. The function $F(C) = 1.8C + 32$ converts degrees Celsius, C, into degrees Fahrenheit, F.

Apparent Temperature at 50% Humidity





Using the graphs of $F(C)$ and $a(F)$,

a. Graph the composite function $a(F(C))$ using axes that are labeled and scaled appropriately.

b. Describe the graphs of functions $F(C)$ and $a(F)$ and compare them with the graph of the composite function $a(F(C))$.

c. Use both $F(C)$ and $a(F)$ as well as $a(f(c))$ to determine what the apparent temperature is when the air temperature is 35 degrees Celsius.

43. Gasoline Prices In the following table, $P(y)$ is the purchasing power of a dollar (how much the dollar is "worth") as measured by the consumer price index in year y, using 2000 as the base year for the value of \$1.00. $G(P)$ is the average price of a gallon of regular unleaded gasoline when the dollar purchasing power is P dollars. Using function notation, write a composition of functions showing the price of gasoline in year-2000 dollars.

Years Since 2000 y	Dollar Purchasing Power P	Average U.S. Gasoline Price (per gallon) G
0	1.00	1.39
1	1.03	1.66
2	1.04	1.36
3	1.07	1.44
4	1.10	1.81
5	1.13	2.19
6	1.17	2.87

Sources: **www.eia.doe.gov and www.measuringworth.com**

44. Take-Home Pay A student earns \$8.75 per hour working as a tutor in her college's learning center. Her weekly take-home pay (after withholdings) is 92.35% of her gross weekly earnings.

a. Write a function for her gross weekly earnings as a function of the number of hours she works.

b. Write a function for her weekly take home pay as a function of her gross weekly earnings.

c. Create a composite function that will calculate her weekly take-home pay as a function of the number of hours she works. Then calculate her take-home pay when she works 16 hours.

45. **Dress Sizes** A dress that is size J in Japan is size A in Australia, where $J(A) = A - 1$. A dress that is size A in Australia is size K in the United Kingdom, where $A(K) = K + 2$. A dress that is size K in the United Kingdom is size E in Europe, where $K(E) = E - 28$. A dress that is size E in Europe is size U in the United States, where $E(U) = U + 30$. (*Source:* www.asknumbers.com)

a. Find an equation for the composite function $J(A(K(E(U))))$ and explain what the input and output values to this function represent in the real-world context.

b. Evaluate $J(A(K(E(10))))$ and explain what the numerical solution means in the real-world context.

c. Solve $J(A(K(E(U)))) = 17$ for U and explain the solution in its real-world context.

46. **Money Conversion** In North America, there are three forms of currency: the Mexican peso, the Canadian dollar, and the American dollar. Although exchange rates vary, the conversion rates for these monetary systems can be modeled mathematically.

In May 2007 the function $A(C) = 0.911078C$ converted Canadian dollars, C, to American dollars, A. The function $A(M) = 0.0925834M$ converted Mexican pesos, M, to American dollars, A. (*Source:* www.x-rates.com)

a. Using the information that the two functions $A(C)$ and $A(M)$ provide, find a function to convert Mexican pesos to Canadian dollars.

b. Sketch graphs for $A(C)$, $A(M)$, and $C(A(M))$. Explain what the independent and dependent variables are for each function.

c. What would the composite function $M(A(C))$ have as the input and output values? Create a table of values for $M(A(C))$ for five monetary values of your choice.

47. **Height and Weight** The height and weight of healthy children generally increase as they get older. The function

$$a(h) = 38.05 - \sqrt{-24.39h + 2137}$$

models the age a of a 2- to 18-year-old boy with height h (in inches). The function

$$w(a) = -0.034a^3 + 1.245a^2 - 5.077a + 34.346$$

models the average weight of a 2- to 18-year-old boy, where w is the weight (in pounds) when the boy is a years old. (*Source:* Modeled from www.cdc.gov data)

a. Find the function composition $w(a(h))$.

b. Determine the independent and dependent variables of $w(a(h))$ and explain their meaning in this real-world context.

c. Calculate $w(a(60))$ and interpret its meaning in this real-world context.

48. **Lumber and Wood Products Industry** Based on data from 1995 to 1999, the average annual earnings of an employee in the lumber and wood products industry can be modeled by $s(n) = -0.7685n^2 + 1295n - 516{,}596$ dollars where n is the number of employees in thousands. (*Source: Statistical Abstract of the United States, 2001*, Table 979)

Based on data from 1995 to 1999, the number of employees in the lumber and wood products industry can be modeled by

$$n(t) = 3.143t^2 + 5.029t + 772.5 \text{ thousand employees}$$

where t is the number of years since 1995. (*Source: Statistical Abstract of the United States, 2001*, Table 979)

a. Calculate $s(n(4))$.

b. Interpret the meaning of the answer in part (a).

c. According to the model, what was the average annual earnings in 1998?

49. **Insurance Premiums** Among other things, car insurance premiums depend on the value of the vehicle, which depends on the age (year) of the vehicle. Suppose you are shopping for a used Cadillac Escalade with approximately 40,000 miles and need to determine how much the insurance premium will be for any particular car. Table C gives the 2007 clean retail value of the Escalade with 40,000 miles for different years.

Table C

Model Year of Vehicle t	2007 Clean Retail Value of Vehicle (with 40,000 miles) v
2000	\$17,250
2001	\$22,350
2002	\$27,225
2003	\$28,950
2004	\$32,900
2005	\$36,525
2006	\$40,450

Source: www.nada.com

Table D gives estimated insurance premiums for the Escalade from multiple car insurance companies.

Table D

Vehicle Value v	Insurance Premium (every 6 months) p
\$10,000–\$20,000	\$200.00
\$20,001–\$25,000	\$500.00
\$25,001–\$30,000	\$800.00
\$30,001–\$40,000	\$1000.00
\$40,001–\$50,000	\$1200.00

a. Create a linear model to represent the 2007 value, v, of the vehicle as a function of the model year, t. Let $t = 0$

correspond to the year 2000. Write several sentences explaining exactly what this model represents in this context. Finally, use the model to predict the 2007 value of a 2007 Cadillac Escalade.

b. Create a graph of the insurance premium paid every 6 months, p, as a function of the value of the vehicle, v. Label and scale the axes appropriately.

c. Explain how the situation of determining the insurance premium paid every 6 months, p, given the model year of the vehicle, t, is an example of the composition of two functions.

d. What is the 2007 insurance premium charged for a 2003 Cadillac Escalade? Express your answer using function notation.

e. Determine which vehicle you could buy (by indicating the model year) if you could afford insurance premiums of \$800 every 6 months. Express your results using function notation.

f. Use $v(t)$, the 2007 value of the Escalade as a function of the model year, to predict how much the insurance premiums would be for a 2005 Escalade.

STRETCH YOUR MIND

Exercises 50–52 are intended to challenge your understanding of composite functions.

50. Craftsmanship At a factory in Louisville, Kentucky, workers assemble custom chairs using traditional techniques that take more time than modern assembly techniques. However, a worker assembling chairs at the factory slows down throughout the day as he gets tired and bored. The time it takes him to assemble one chair can be modeled by $c(t) = 3 + 0.1t$, where c is the number of hours it takes to assemble a chair and t is the number of hours he has been working that day. (Note: Since it takes him a range of time to complete the chair, t is assumed to be the number of hours he has been working when he begins making the chair.) Assume the worker is paid \$15.50 per hour.

a. Find $c(6)$ and explain what this means in the context of the situation.

b. Explain what the rate of change is for the function $c(t)$ and what it represents in the context of the situation. Be as descriptive as possible.

c. What does the "3" represent in the function equation? Be as descriptive as possible.

d. Write a function $w(t)$ representing the worker's total wages after working for t hours.

e. What are the labor costs for the factory to have a worker make a chair when that worker starts on the chair 3.5 hours into his shift? Explain how you determined your answer.

f. Write the function $w(c(t))$. What are the input and output variables of this function?

g. Write the function $c(w(t))$. Does this function make sense in the context of the problem? Explain your reasoning.

h. The factory currently has workers completing 12-hour shifts (meaning the workers work for 12 hours a day instead of the standard 8 hours, but the workers work fewer days per week). Should the factory continue to use this schedule or go to a traditional 8-hour shift? Explain your reasoning.

i. If a worker does not complete a chair during his shift, he leaves the chair for a worker on the next shift to complete. How many hours of work is put into a chair if one worker (working a 12-hour shift) starts on the chair 11 hours into his shift and leaves the unfinished chair for the next worker to complete? What are the labor costs to the factory for making this chair?

j. Each chair requires approximately \$27 worth of materials to manufacture. Write a function that will give the total costs to the factory for making a single chair.

k. Based on your responses to the preceding questions, what is a reasonable price to charge per chair to ensure that the company earns a profit on each chair? Write a function that shows the factory's profit on a single chair.

51. Solar Energy Solar panels convert energy from the sun into usable electricity to power homes and businesses. The larger the solar panel, the more energy from the sun it can absorb. Figure A shows the relationship, $s(A)$, between the size of a solar panel (in square meters) and the amount of energy it can absorb from the sun (in watts) in San Antonio, Texas. However, solar panels are not 100% efficient, meaning they cannot convert 100% of the energy they absorb into usable electricity. Figure B shows the relationship, $p(s)$, between the amount of energy absorbed and the amount of usable electricity created (both in watts).

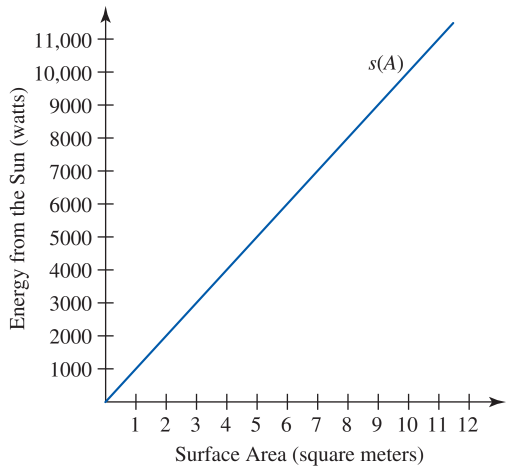


Figure A

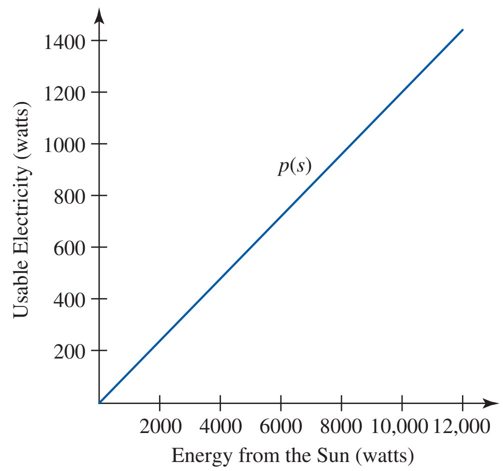


Figure B

a. How much usable electricity can be created from a solar panel that has a surface area of 8 square meters?

b. Your friend in San Antonio is bragging about how his house is hooked up to a solar panel, which saves a bunch of money on his electric bill. He claims he gets 950 watts of usable electricity from his solar panel. About how large is his panel?

c. Create a table of values for the function $p(s(A))$ using at least five coordinate points, then make a graph of this function making sure to label your axes with the appropriate variables.

52. Life Insurance Many employers offer supplemental life insurance to their employees to help offset the costs should an employee suddenly die. The cost of the insurance depends on the age of the employee. The following table shows the monthly insurance rates for $20,000 of supplemental insurance.

Rates for $20,000 Supplemental Insurance		
Age	Mid-Interval A	Monthly Rate (dollars) R
25–27	26	0.72
28–30	29	0.80
31–33	32	1.00
34–36	35	1.20
37–39	38	1.44
40–42	41	2.16
43–45	44	3.44
46–48	47	4.16
49–51	50	5.76
52–54	53	7.40
55–57	56	10.28
58–60	59	12.80
61–63	62	17.64
64–66	65	25.40
67–69	68	35.48
70–72	71	54.72
73–75	74	78.32

Source: www.bussvc.wisc.edu

a. Find the *percentage* change in monthly insurance rates between each mid-interval age. Then find the average of these percentage changes and explain what this value means in the context of this situation.

b. We use the mid-interval age to represent each age interval so that we may create a function for this situation. (Note the ages change by 3 years). Write an exponential function that models the insurance rate, R, as a function of the mid-interval age, A.

c. Describe the relationship between the quantities R and A as provided by the function. In other words, verbally describe what information the function provides.

d. Use logarithms to find the value of A if $R(A) = 36$ and explain what the value of A means in the real-world context.

SECTION 7.4 Logistic Functions

LEARNING OBJECTIVES

- Graph logistic functions from equations and tables
- Use logistic models to predict and interpret unknown results

GETTING STARTED

In North America, the influenza season typically starts around the month of October. The "flu" is a contagious respiratory illness caused by influenza viruses that can cause mild to severe illness sometimes leading to death. The number of people infected with the flu grows slowly at first, then more quickly, then slowly again, eventually leveling off. Such behavior (in growth or decay) can be modeled with a logistic function.

In this section we show how to discriminate between exponential and logistic models, illustrate how to graph logistic functions from tables and equations, and use logistic regression to model real-world data and make predictions. We also use the language of rate of change to describe the behavior of logistic growth and decline.

Exponential Growth with Constraints

An important fact about exponential growth functions is that even though growth may seem slow over the short run, it ultimately becomes very rapid. For instance, a population growth rate of 2% per year may seem small; but it means that the population will double every 35 years. A population of 10,000 today that is growing at 2% annually will double three times over the next 105 years, yielding a population of 80,000. However, it is unrealistic to expect exponential growth to continue forever. In most situations, there are factors that ultimately limit the growth (such as available land, natural resources, or food supplies).

Logistic Growth

Flu viruses mainly spread from person to person through the coughing or sneezing of people with the virus. Sometimes people become infected by touching something with flu viruses on it and then touching their mouth or nose. Most healthy adults are able to infect others beginning one day before symptoms develop and up to five days after becoming sick. (*Source:* www.cdc.gov) As more and more people catch the flu they, in turn, also begin to infect others. Thus the total number of people who have contracted the flu begins to grow exponentially, increasing at an increasing rate, as in Figure 7.21.

However, this trend cannot continue indefinitely. People with the flu will begin to come in contact with people who already have been infected, have built up immunity, or have been vaccinated. As Figure 7.22 shows, this will cause the rate of increase in the number of people who have had the flu to begin to decline—that is, we see the number of people who have had the flu increasing at a decreasing rate.

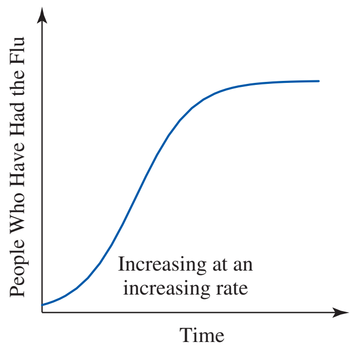


Figure 7.21

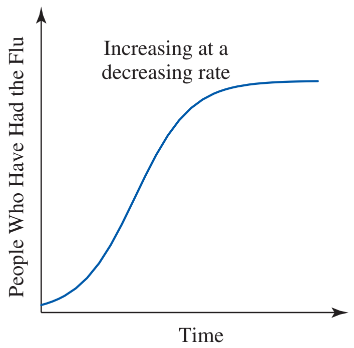


Figure 7.22

Eventually, the total number of flu cases will level off because there is only a limited number of people who will be able to contract the flu. Therefore, Figure 7.23 represents the total number of people who have had the flu as a combination of very slow exponential growth (labeled *a*), rapid exponential growth (labeled *b*), followed by a slower increase (labeled *c*), and then a leveling off (labeled *d*). At the inflection point the rate at which the flu is spreading is the greatest. The horizontal asymptote (red dashed horizontal line) represents the **limiting value** for the number of people who will contract the flu.

The mathematical model for such behavior is called a **logistic function**.

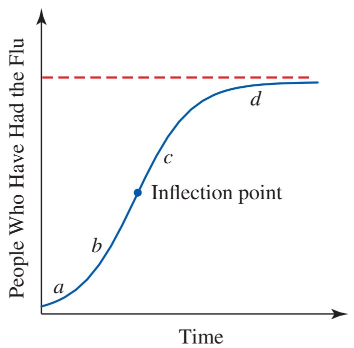


Figure 7.23

PEER INTO THE PAST

LOGISTIC FUNCTIONS

The English economist Thomas Robert Malthus (1766–1834) was a pioneer in population study. He believed that poverty and starvation were unavoidable because the human population tends to grow exponentially while food supplies tend to grow linearly. Malthusian theory laid the groundwork for others to investigate the notion of carrying capacities (limiting values) for populations.

P. F. Verhulst followed up Malthus by introducing the logistic model in 1838 to better describe the growth of human populations. According to the PNAS (Proceedings of the National Academy of Sciences of the United States of America), R. Pearl and L. J. Reed derived the same model in 1920 to describe the growth of the population of the United States since 1790 and to attempt predictions of that population at future times.

It is known that the logistic model for population growth, like the exponential model, has limitations. According to Pearl and Reed, in 1920 the population was to stabilize at about 197 million people. Of course, this level has been far surpassed. At the time of this text's writing there are over 300 million residents of the United States.

Source: www.pubmedcentral.nih.gov

LOGISTIC FUNCTIONS

A **logistic function** is a function of the form

$$N(t) = \frac{L}{1 + Ae^{-Bt}}$$

The number L is the **upper limiting value** of the function $N(t)$. The **lower limiting value** is $y = 0$, the horizontal axis.

EXAMPLE 1 ■ Exploring Logistic Functions in a Real-World Context

The Centers for Disease Control monitor flu infections annually, paying particular attention to flu-like symptoms in children from birth to 4 years old. The cumulative number of children 0–4 years old who visited the CDC's sentinel providers with flu-like symptoms during the 2006–2007 flu season are displayed in Figure 7.24, together with a logistic model. (A sentinel provider is a medical provider who has agreed to report flu data to a national surveillance network from early October through mid-May.) (*Source:* www.cdc.gov)

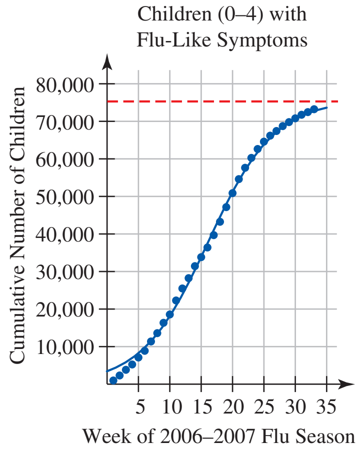


Figure 7.24

a. Explain why a logistic function may better model the 2006–2007 flu infection rate than an exponential function.

b. Using the language of rate of change, describe the behavior of the graph and relate this to what it tells about the real-world context.

c. The formula for the logistic function that models the cumulative number of reported children with flu-like symptoms is $H(w) = \dfrac{75{,}700}{1 + 21.31e^{-0.1897w}}$, where w is the week of the flu season. What does the limiting value mean in the real-world context? How is the limiting value represented in the formula for the function $H(w)$ and its graph?

d. Estimate the coordinates for the point of inflection and explain what each coordinate means in the real-world context.

Solution

a. A logistic function models the growth in the cumulative number of children with flu-like symptoms better than an exponential function because a logistic function grows slowly at first, then more quickly, and finally levels off at a limiting value. An exponential function, on the other hand, may grow slowly at first but then will increase at an ever increasing rate, ultimately exceeding the available number of children who could conceivably become infected with flu-like illnesses.

b. The graph of the function model is increasing throughout the interval; however, the rate at which the graph is increasing varies. Initially, the graph is concave up, indicating that the cumulative number of reported cases is increasing at an ever-increasing rate. Around week 17, the graph changes to concave down, indicating that the cumulative number of reported cases are increasing at a lesser and lesser rate. Around week 32, the graph is increasing at such a slow rate that it appears to level off. This indicates that the number of newly reported cases in weeks 32 and beyond is so small that it has a negligible effect on the cumulative number of cases.

c. The limiting value is approximately 75,700 children. It appears that no more than 75,700 children were reported to have flu-like symptoms during the season. This value is represented in the formula for $H(w)$ by the value 75,700 found in the numerator and on the graph by the horizontal asymptote at 75,700.

d. From the graph of $H(w)$, an estimate for the coordinates of the inflection point is approximately (17, 40,000). This means at week 17 of the 2006–2007 flu season (late January), the number of children with flu-like symptoms was increasing most rapidly.

Logistic Decay

Many real-world data sets are modeled with decreasing rather than increasing logistic functions. As was the case with logistic growth functions, logistic decay functions have an upper limiting value, L, and a lower limiting value of $y = 0$.

EXAMPLE 2 ■ Recognizing a Logistic Decay Function

As shown in Table 7.19, the infant mortality rate in the United States has been falling since 1950. (An *infant* is a child under 1 year of age.)

a. Using Table 7.19, show how the rate of change records the decline in the infant death mortality rate and discuss why this suggests that a logistic model may fit the data.

b. From the table, predict the future lower limiting value and explain what the numerical value means in the real-world context. Does this value seem reasonable?

c. Create a scatter plot of the data and estimate the upper and lower limiting values.

Table 7.19

Years Since 1950 y	Infant Mortality Rate (deaths per 1000 live births) M
0	29.2
10	26.0
20	20.0
30	12.6
35	10.6
40	9.2
45	7.6
50	6.9
59	6.4

Source: www.cdc.gov

Solution

a. If we calculate the decrease in the infant mortality rate (ΔM) and the yearly rate of change of the infant mortality rate over each interval of time given in Table 7.20, we can determine the behavior of the function $M(y)$. (Note: We must exercise caution because the intervals between the years provided are not the same.)

Table 7.20

Years Since 1950 y	Infant Mortality Rate (deaths per 1000 live births) M	Change in Infant Mortality Rate (deaths per 1000 live births) ΔM	Rate of Change in Infant Mortality Rate per Year $\frac{\Delta M}{\Delta y}$
0	29.2		
10	26.0	−3.2	−0.32
20	20.0	−6.0	−0.60
30	12.6	−7.4	−0.74
35	10.6	−2.0	−0.40
40	9.2	−1.4	−0.28
45	7.6	−1.6	−0.32
50	6.9	−0.7	−0.14
59	6.4	−0.5	−0.056

From the rate of change, we can see that the decline in the infant mortality rate tends to drop slowly at first, then more dramatically, and then levels off. The change in the infant mortality rate suggests a logistic model.

b. We estimate the lower limiting value to be approximately 6.0 deaths per 1000 live births because we expect the values to level off near 6.0.

c. The scatter plot is shown in Figure 7.25. From the scatter plot, we predict the upper limiting value will be around 35 and the lower limiting value will be around 6.

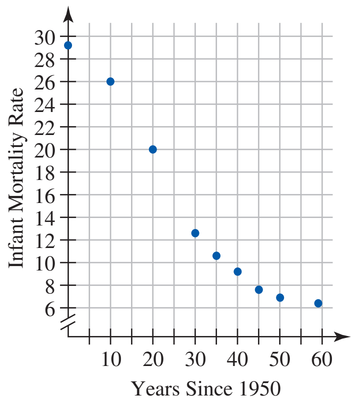


Figure 7.25

We said earlier that all logistic functions have a lower limiting value at $y = 0$. We can use logistic regression to model data sets with this lower limiting value. But when a data set appears to be logistic but has a different lower limiting value, as in Example 2, we need to align the data before using logistic regression to model the function. We demonstrate in Example 3.

EXAMPLE 3 ■ Using Logistic Regression to Model Data

Refer to Table 7.20 and Figure 7.25 in Example 2.

a. Use logistic regression to find the logistic function, $M(y)$, for the data. Comment on the model's fit to the data.

b. Align the data by subtracting the estimated lower limiting value from each output value in the data set. Then redo the logistic regression for the aligned data.

c. Create the aligned logistic model by adding back the lower limiting value to the model equation in part (b). Describe how well this new function fits the data set.

Solution

a. Using the Technology Tip for logistic regression at the end of this section, we find

$$M(y) = \frac{46.96}{1 + 0.5616e^{0.0481y}}, \text{ graphed in Figure 7.26.}$$

Although the original data looked logistic, this logistic model does not appear to fit the data as well as we might have expected.

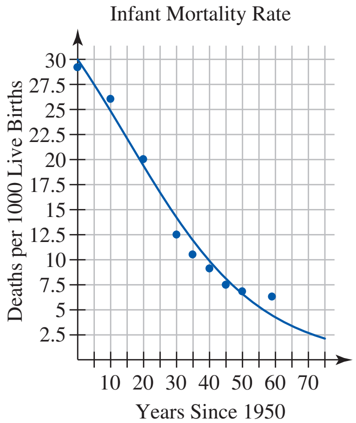


Figure 7.26

b. We align the data by subtracting the estimated lower limiting value of 6.0 from the outputs as shown in Table 7.21. (See also the second Technology Tip at the end of this section.)

Table 7.21

Years Since 1950 y	Infant Mortality Rate (deaths per 1000 live births) M	Aligned Output Data $M - 6.0$
0	29.2	23.2
10	26.0	20.0
20	20.0	14.0
30	12.6	6.6
35	10.6	4.6
40	9.2	3.2
45	7.6	1.6
50	6.9	0.9
59	6.4	0.4

After data alignment, we recalculate the logistic regression to get $M(y) = \dfrac{25.18}{1 + 0.08327e^{0.1144y}}$, graphed in Figure 7.27.

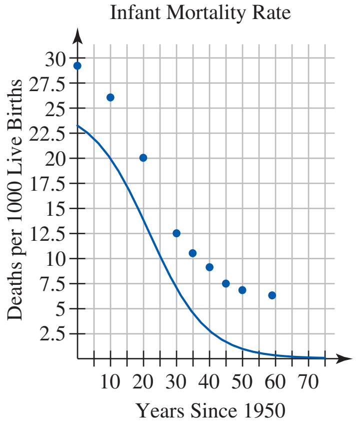


Figure 7.27

c. We notice the logistic regression model for the aligned data lies below the scatter plot. Therefore, we add the limiting value of 6.0 to the equation to shift the dataupward. This process gives the function $N(y) = \dfrac{25.18}{1 + 0.08327e^{0.1144y}} + 6.0$. The original scatter plot and the graph of the function N are shown in Figure 7.28. The aligned model fits the data much better than the original logistic model.

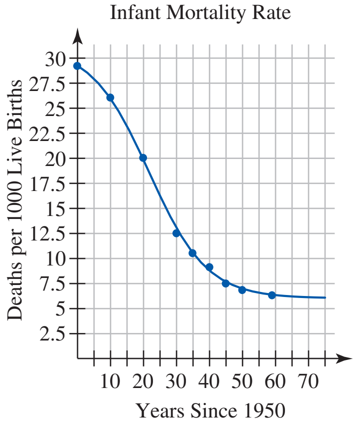


Figure 7.28

EXAMPLE 4 ■ Extrapolating Exponential and Logistic Growth

The digital video disc (DVD) player was introduced to the market in the first quarter of 1997. At the end of 2006, Nielsen Media Research reported more U.S. households own DVD players (81.2% of all households) than VCRs (79.2% of households). (*Source:* www.nielsenmedia.com) The sales of DVD hardware from 1997 to 2006 are given in Table 7.22.

Table 7.22

Years Since 1997 y	DVD Hardware Sales ($ millions) D
0	0.305
1	0.946
2	3.550
3	9.877
4	16.662
5	25.113
6	33.734
7	37.125
8	36.737
9	32.660

a. Use logistic regression to determine the function, $D(y)$, of the form $N(t) = \dfrac{L}{1 + Ae^{-Bt}}$ that best models the growth in DVD sales from 1997 to 2004. Determine the value of L and explain what this number means in the real-world context.

b. Based on the data from 2005 and 2006, does it appear the model can be used to accurately forecast future sales? Explain why or why not.

Solution

a. Using the Technology Tip and logistic regression on the data from 1997 to 2004, we find $L(y) = \dfrac{40.69}{(1 + 57.17e^{-0.92y})}$ million dollars is the best-fit logistic function for DVD sales (see Figure 7.29). The limiting value, L, is 40.69, which means DVD sales will level off at \$40,690,000 per year.

b. For the years 2005 and 2006, we evaluate the function at $t = 8$ and $t = 9$.

$$L(8) = \frac{40.69}{(1 + 57.17e^{-0.92(8)})} = 39.3 \qquad L(9) = \frac{40.69}{(1 + 57.17e^{-0.92(9)})} = 40.1$$

The model forecasts 39.3 million dollars in sales in 2005 and 40.1 million dollars in sales in 2006. In actuality, there were 36.7 and 32.7 million dollars in sales, respectively. The model predicts a leveling off of sales whereas the actual data shows a decline in sales in 2005 and 2006 (see Figure 7.30). Consequently, the model does not appear to accurately model future sales.

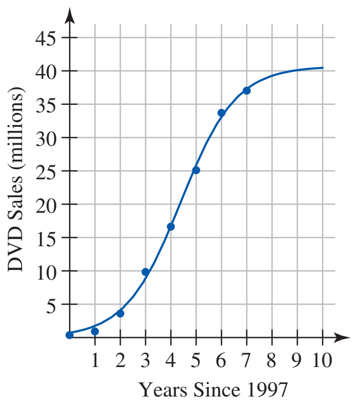


Figure 7.29

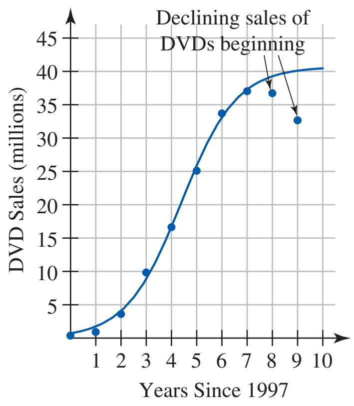


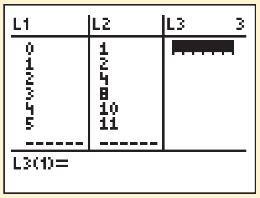
Figure 7.30

SUMMARY

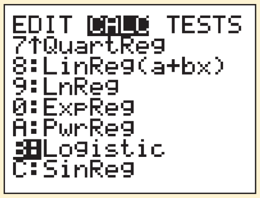
In this section you learned to graph logistic functions from tables and equations, and use logistic regression to model and make predictions from real-world data. You also learned to use the language of rate of change to describe the behavior of logistic growth and decay.

TECHNOLOGY TIP ■ LOGISTIC REGRESSION

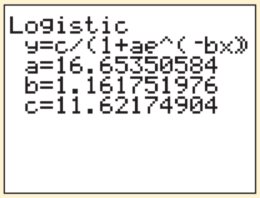
1. Enter the data using the Statistics Menu List Editor.



2. Bring up the Statistics Menu Calculate feature and select item `B:Logistic` and press [ENTER].

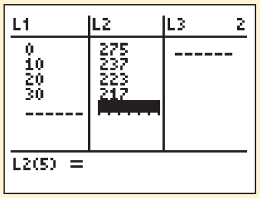


3. If you want to automatically paste the regression equation into the Y = Editor so that you can easily graph the model, press the key sequence [VARS]; `Y-VARS`; `Function`; `Y1` and press [ENTER]. Otherwise press [ENTER].

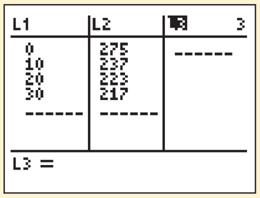


TECHNOLOGY TIP ■ ALIGNING A DATA SET

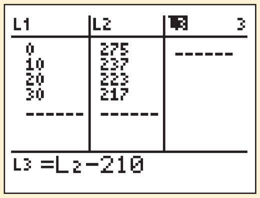
1. Enter the data using the Statistics Menu List Editor.



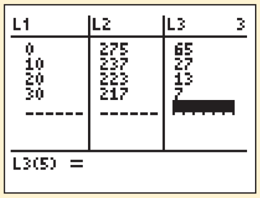
2. Move the cursor to the top of `L3`. We want the entries in `L3` to equal the entries in `L2` minus the amount of the vertical shift (in this case, 210). To do this, we must enter the equation **L3=L2−210** as in Step 3.



3. Press [2nd], then type **2** to place `L2` on the equation line at the bottom of the viewing window. Then type **−** and **210** to subtract 210.



4. Press [ENTER] to display the list of aligned values in `L3`.



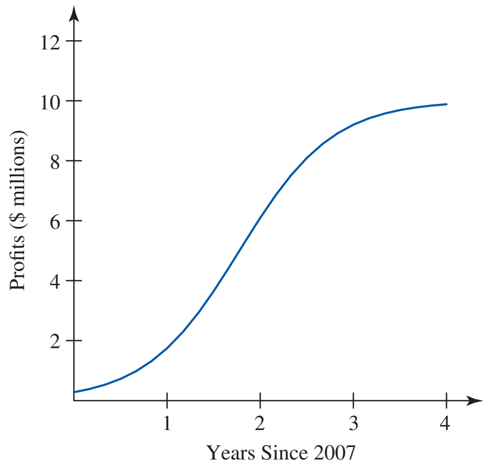
7.4 EXERCISES

■ SKILLS AND CONCEPTS

1. Create a table of values for a logistic function that is increasing.
2. Create a table of values for a logistic function that is decreasing.
3. A logistic growth curve is sketched in the figure. Estimate the limiting value and the inflection point, and explain what each means in the real-world context. Also provide the year you would recommend that the com-

pany release the upgraded product so their profits do not begin to fall.

Product Sales



4. Explain why the following table of values can be modeled by a decreasing logistic function.

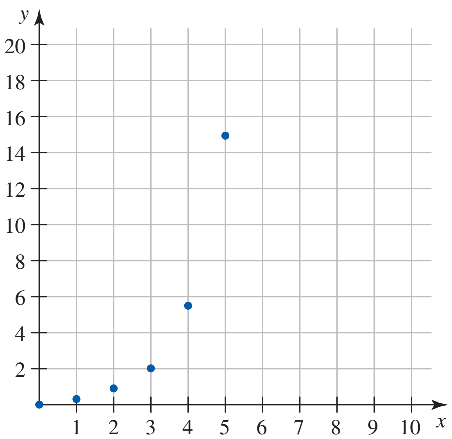
x	y
0	15
1	14
2	12
3	8
4	2
5	1
6	0.5
7	0.3
8	0.1

5. If $f(x) = \dfrac{12}{1 + 16e^{-0.8x}}$ is a logistic function, what is the limiting value?

6. If $f(x) = \dfrac{113}{1 + 46e^{-4x}}$ is a logistic function, what is the limiting value?

In Exercises 7–12, identify the scatter plots as linear, exponential, logistic, or none of these. If you identify the scatter plot as none of these, explain why.

7.



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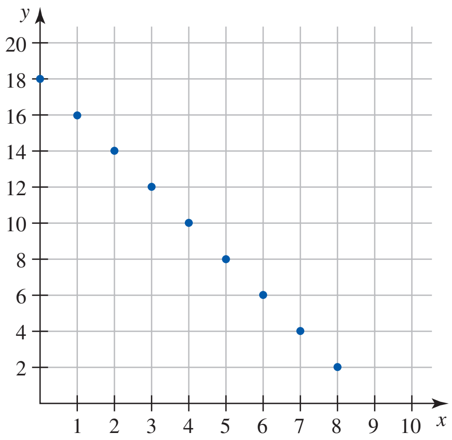

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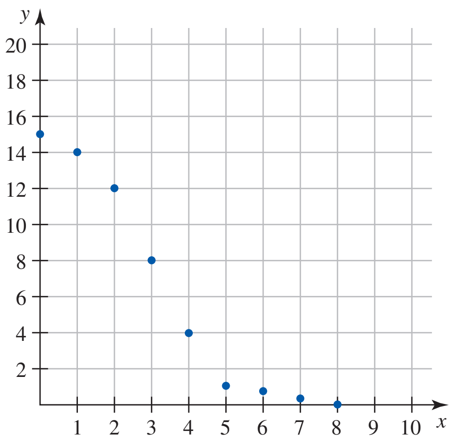

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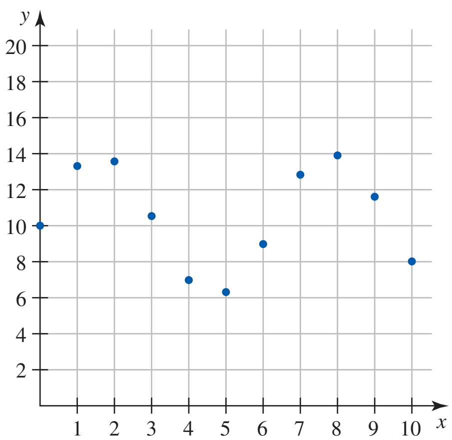
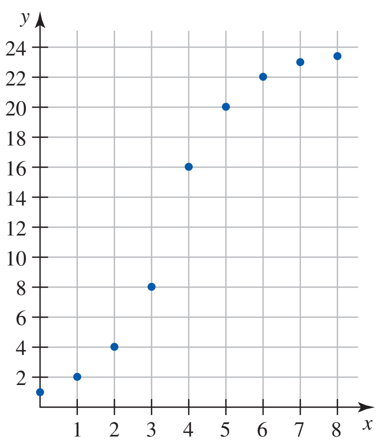
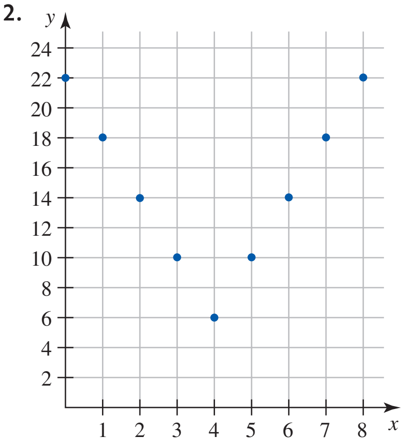


11.



12.



13. Agriculture Agriculture plays a significant role in the economy of the midwestern United States. One of the most important crops produced is corn. Much research is done to produce the most hearty, disease- and drought-resistant plants possible. The growth rate of the corn crop is important to farmers as well. With the relatively short growing seasons in some parts of the Midwest, adapting varieties of corn to fit a region's growing season is an important science. (***Source:*** **www.age.uiuc.edu**)

Suppose a new corn crop is planted and measured each day after it first breaks through the soil. The height of the plant, $c(t)$, can be modeled by the logistic function $c(t) = \dfrac{96}{1 + 18(0.955)^t}$ inches, where t is the number of days since the corn plant first emerged from the soil.

a. Use a graphing calculator to graph the height over 130 days.

b. What is the practical domain of $c(t)$? What is the range? What does this tell about the height of the corn plant?

c. Estimate the coordinates of the inflection point and explain what each value means in the real-world context.

d. Estimate the limiting value and explain what this value means in the real-world context.

14. Value of Stocks Suppose the value of a share of one of your stocks increases at an increasing rate from 2004 through 2008 and then increases at a decreasing rate from 2008 to 2012. Fill in the table below with reasonable values that reflect these value changes.

Year	Value of Stock (dollars)		Year	Value of Stock (dollars)
2004	22.00		2009	
2005			2010	
2006			2011	
2007			2012	
2008	50.00			

15. Pyramid Schemes A pyramid scheme is a nonsustainable business model that involves the exchange of money primarily for enrolling other people into the scheme, usually without a product or service being rendered. Most pyramid schemes take advantage of the confusion that exists between genuine business models and convincing scams. The essential idea behind each scam is that the individual makes only one payment, but is promised to receive exponential benefits from other people as a reward.

Suppose the scheme initiator promises huge financial gains for investors who join his Profit Club for a \$100 fee. He recruits two investors, who in turn each recruit two more investors, and so on. Each new club member pays the \$100 joining fee. Half of the fee goes to the person who recruited the club member and the other half is paid to the person who recruited the recruiter as a "bonus." Each recruiter must pay half of all bonuses received to the person who recruited him as a "bonus." This process is to continue indefinitely. Describe the flaw in this business model.

SHOW YOU KNOW

16. How are exponential and logistic functions similar and how are they different?

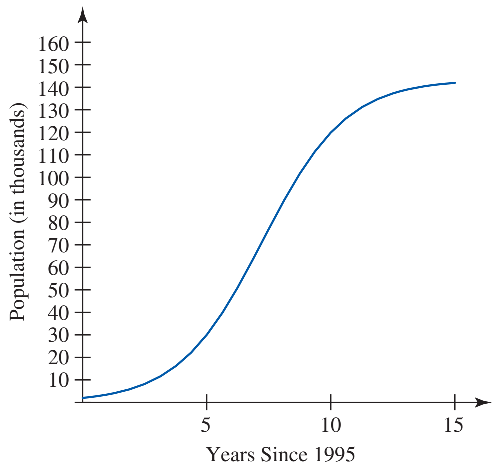
17. Explain how to graph logistic functions represented by a table or an equation.

18. Describe a logistic function in terms of a rate of change.

19. What is important to keep in mind when modeling data sets with a logistic function?

MAKE IT REAL

20. Environmental Carrying Capacity The graph approximates the hypothetical population of roadrunners, R, in thousands of birds within the confines of a designated area within the Tonto National Forest in Arizona as a function of the year y.



Ecologists refer to growth of the type shown here as *logistic population growth*. Coyotes are one of the roadrunners' natural predators and affect the number of roadrunners within the national forest.

a. Explain in general terms how the roadrunner population changes over time.

b. When did the population reach 60,000?

c. Evaluate $R(6)$ and explain what the numerical value means in this context.

d. During which period of years is the graph concave up? Explain what this means about the population growth during this period.

e. During which period is the graph concave down? Explain what this means about the population growth during this period.

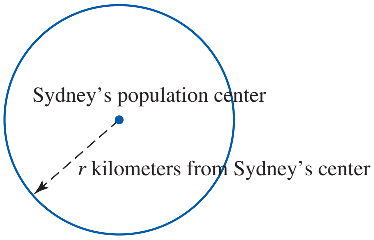
f. When does a point of inflection appear to occur on the graph? Explain how this point may be interpreted in terms of the growth rate.

g. What is the *environmental carrying capacity* of this forest for the roadrunner? That is, what is the maximum number of roadrunners that the environment seems to be able to support (the limiting value)?

21. **Population Density** The population of Sydney, Australia, is distributed in suburbs around the city's center. Let S be the number of people (in millions) living within r kilometers of the city's center. Then $S(r)$ can be approximated by

$$S(r) = \frac{1.796}{(1 + 3.05e^{(-0.21r)})}$$

(***Source:*** **www.environment.gov.au**)



a. Use the graph of $S(r)$ to approximate the number of people living in Sydney including the suburbs.

b. Use the graph of $S(r)$ to estimate the distance in kilometers from Sydney's center to the edge of town where not very many people live.

c. Estimate the distance from Sydney's center to the most densely populated neighborhoods.

22. **Computer Transistors** In 1965 Moore's Law predicted that the number of transistors on a circuit would double every 2 years. In Chapter 6, we modeled the growth of the number of transistors with the exponential function $T(y) = 2300(2)^y$, where y is the number of years since 1971. The following table shows the actual number of transistors for different Intel computers. Do you believe the mathematical model $T(y)$ can be used to extrapolate the future growth in the number of transistors or would a logistic function be a better choice? Justify your answer.

Intel Product Name	Years Since 1971 y	Number of Transistors on Integrated Circuit T
4004 Microprocessor	0	2300
8008 Microprocessor	1	3500
8080 Microprocessor	3	6000
8086 Microprocessor	7	29,000
286 Microprocessor	11	134,000
386 Microprocessor	14	275,000
486 Microprocessor	18	1,200,000
Pentium	22	3,100,000
Pentium II	26	7,500,000
Pentium III	28	9,500,000
Pentium IV	29	42,000,000
Itanium 2	31	220,000,000
Dual Core Itanium 2	35	1,700,000,000

Source: **www.intel.com**

23. **Social Diffusion** Sociologists use the phrase "social diffusion" to describe the way information spreads through a population. (***Source:*** **www.jstor.org**) The information might be a rumor, a cultural fad, or news about a technical innovation.

Assume the number of people, P, who have received some type of information as a function of the time, t, in days is modeled with the logistic function,

$$P(t) = \frac{145}{1 + 150e^{-0.3t}}$$

a. Based on the graph of $P(t)$, describe what the rate of change of the function tells about how the news of an event spreads over time.

b. Estimate the coordinates of the point of inflection and explain what each value means in this context.

c. Find the average rate of change from $t = 5$ to $t = 25$ for the function and explain the meaning of this rate of change.

d. Is there a limiting value for the function? If so, estimate the value and explain what it means in the context of the spread of information through a population.

24. **Stay-at-Home Moms** In recent years, the number of married mothers who choose to stay at home to care for their

families and bypass an outside career has increased. The function

$$M(t) = \frac{948.2}{1 + 27.23e^{-1.110t}} + 4700$$

models the number of married couple families with stay-at-home mothers (in thousands) as a function of the number of years since 1999, t. (*Source: Statistical Abstract of the United States, 2006*, Table 59)

a. Describe what the rate of change of the function tells about how the number of married, stay-at-home mothers is changing over time.

b. Estimate the coordinates of the inflection point and explain what each value means in this context.

c. Find the average rate of change from $t = 3$ to $t = 7$ for the function and explain the meaning of this rate of change.

d. What are the limiting values for the function and what do they mean in the context of the stay-at-home mothers?

25. Internet Access Based on data from 1994 to 2000, the percentage of public school classrooms with Internet access can be modeled by

$$P(y) = \frac{85.88}{1 + 31.40e^{-0.9250y}}$$

where P is the percentage of public schools with Internet access and y is the number of years since the end of 1994.

a. Graph the function over a practical domain.

b. Does it seem reasonable that the function would have a limiting value? Explain.

c. Estimate the rate at which the percentage of public schools with Internet access was changing at the end of 2000.

26. Women with Ph.D.s The number of women in the United States who have earned a Ph.D., N, has dramatically increased in the years since 1900, t. The table displays the data for selected years from 1900 to 1999.

Years Since 1900 t	Number of Ph.D.s Awarded to Women N
0	23
10	44
20	88
30	313
40	429
50	620
60	1042
70	3971
80	9408
90	13,106
95	16,414
99	17,493

Source: Modeled from U.S. Doctorates in the 20th Century, National Science Foundation, 2006, Figure 3-2

a. Explain in general terms how the number of women earning Ph.D.s changes with time.

b. Estimate the year in which the number of women earning Ph.D.s first surpassed 1000.

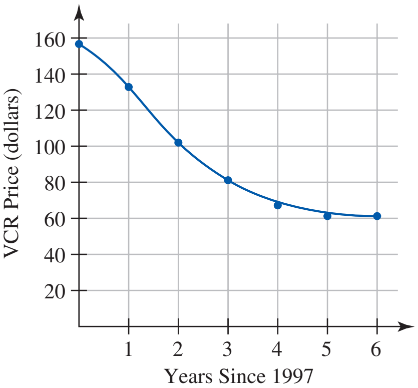
c. Looking at the average rate of change between successive years, does it appear there is a limiting value for the number of Ph.D.s earned by women? If so, estimate what this number will be.

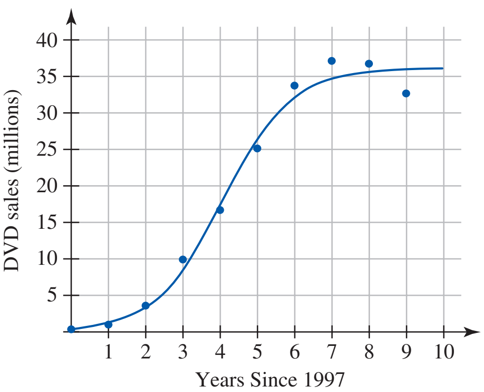
d. Create a scatter plot of the data and determine during which period the data is increasing and concave up. Explain what this means about the growth in Ph.D.s for women.

e. During which period is the data increasing and concave down? Explain what this means about the growth in Ph.D.s for women.

f. Estimate where the point of inflection occurs from the graph. What historical events may have been occurring at this time to affect the number of Ph.D.s being earned by women?

27. VCR Prices vs. DVD Sales The price of a VCR since 1997 (the year DVD players entered the market) and the sales of DVD players since 1997 in the United States are shown in the following two graphs.





(*continued*)

a. Using the graphs, explain how the two may be interrelated. Make sure to include in your explanation why the price of a VCR is declining while the sales of DVD players is increasing.

b. Explain why each graph behaves in such a way as to follow the pattern of a logistic function.

28. **Telephone Usage** The following table gives the percentages of American households with telephones.

Years Since 1935 t	Percentage of American Households with a Telephone T
0	31.8
5	36.9
10	46.2
15	61.8
20	71.5
25	78.3
30	84.6
35	90.5
45	93.0
50	93.0
55	93.3
60	93.9

Source: Statistical Abstract of the United States, 2007, Table 1111

a. Use the table to estimate the point of inflection. (Be sure to show how the table was used.) What is the significance of the inflection point in this situation?

b. From the table, estimate the instantaneous rate of change in 1970.

c. What is the meaning of the instantaneous rate of change you estimated in part (b)? What are its units?

d. Do you think there should be a limiting value for this data? If so, explain what you think it is. Justify your conclusion.

29. **Cable TV Subscribers** The following table gives the number of basic cable television subscribers since 1975 (in thousands).

Years Since 1975 t	Number of Basic Cable Subscribers (thousands) C
0	9,800
5	17,500
10	35,440
15	50,520
20	60,550
25	66,250
26	66,732
27	66,472
28	66,050
29	65,727
30	65,337

Source: Statistical Abstract of the United States, 2007, Table 1126

a. Use the table to estimate the point of inflection. (Be sure to show how the table was used.) What is the significance of the inflection point in this situation?

b. From the table, estimate the instantaneous rate of change in 2002.

c. What is the meaning of the value you found in part (b)? What are its units?

30. **Pediatric AIDS** Based on data from 1992 to 2001, the estimated number of new pediatric AIDS cases in the United States in year t can be modeled by

$$P(t) = \frac{919.9}{1 + 0.0533e^{0.07758t}} + 90 \text{ cases}$$

where t is the number of years since the end of 1992. (**Source: Centers for Disease Control and Prevention data**)

a. How many new pediatric AIDS cases are estimated to have occurred in 2000?

b. At what rate was the estimated number of new pediatric AIDS cases changing at the end of 2000?

c. Based on the results of part (b) and the graph of $P(t)$, does it look like efforts to reduce pediatric AIDS in the United States are generating positive results? Explain.

31. **Adult Internet Usage** The percentage of adults using the Internet at home, school, or work to access websites or to send and receive email has grown dramatically since 1995. Use the table to suggest whether an exponential or logistic model will be the best choice to extrapolate the percentage of adult Internet users past 2005.

Years Since 1995 t	Percentage of Adult Internet Users P
0	14
5	53
9	59
10	69

32. **Hotel/Motel Occupancy** Based on data from 1990 to 1999, the average hotel/motel room rate can be modeled by

$$R(t) = \frac{27.80}{(1 + 47.23e^{-0.6425t})} + 57 \text{ dollars per day}$$

where t is the number of years since the end of 1990. (***Source:*** ***Statistical Abstract of the United States, 2007***, **Table 1264**)

Approximate the rate at which the hotel/motel room rate was changing at the end of 1997 and at the end of 1999.

33. **Information Technology** Based on data from 1990 to 2000, the percentage of the economy attributed to the information technology sector may be modeled by

$$P(t) = \frac{2.956}{1 + 30.15e^{-0.5219t}} + 7.9$$

where P is the percentage of the economy and t is the number of years since the end of 1990. (***Source: Statistical Abstract of the United States, 2001***, **Table 1122**) Approximate the rate at which the information technology percentage of the economy was changing at the end of 1995 and at the end of 2000.

34. Chicken Pox

Years Since 1985 T	Reported Cases of Chicken Pox (thousands) C
0	178.2
5	173.1
10	120.6
14	46
15	27.4
16	22.5
17	22.8
18	20.9

***Source: Statistical Abstract of the United States, 2006*, Table 177**

A chicken pox vaccine became widely available in the United States in 1995. According to the model, does the vaccine appear to be having an effect? Does the model project that chicken pox will be eradicated? Explain.

35. Video Game Sales Blizzard Entertainment released the engaging real-time strategy game "Warcraft III: Reign of Chaos" on July 2, 2002. By July 22, 2002, more than a million copies of the game had been sold. **(*Source:* www.pcgameworld.com)**

Assuming a total of 8 million copies of the game will be sold, sketch a graph of a logistic growth model for "Warcraft III" game sales.

In Exercises 36–40, use logistic regression to find the logistic model for the data. Then answer the given questions.

36. Deadly Fights over Money

Years Since 1990 t	Homicides H
1	520
2	483
3	445
4	387
5	338
6	328
7	287
8	241
9	213
10	206

***Source: Crime in the United States 1995, 2000*, Uniform Crime Report, FBI**

Use the logistic model $H(t)$ to estimate the point of inflection and the rate that money-related homicides were decreasing in the year 2001.

37. Cassette Tape Sales

Years Since 1993 t	Percent of Music Market (percentage points) P
0	38.0
1	32.1
2	25.1
3	19.3
4	18.2
5	14.8
6	8.0
7	4.9
8	3.4
9	2.4

***Source:* Recording Industry of America**

According to the model, estimate the year that the cassette market share will drop below 1%. Approximately at what rate will the cassette tape market share be decreasing at that time?

38. *My Big Fat Greek Wedding* Box Office Sales

Weekend	Week Number	Cumulative Gross Box Office Sales (dollars)
Apr. 19–21 (2002)	1	597,362
June 28–30	11	19,340,988
Sept. 6–8	21	95,824,732
Nov. 15–17	31	199,574,370
Jan. 24–26 (2003)	41	236,448,697
Apr. 4–6	51	241,437,427

***Source:* www.boxofficeguru.com**

a. According to the table, were cumulative box office sales for *My Big Fat Greek Wedding* increasing at a higher rate in week 1 or in week 51?

b. IFC Films was the distributor for *My Big Fat Greek Wedding.* If you were a marketing consultant to IFC Films, what would you tell the company about forecasted box office sales beyond week 51?

39. Value of Fabricated Metals Shipments

Years Since 1992 t	Shipment Value ($ millions) V
0	170,403
1	177,967
2	194,113
3	212,444
4	222,995
5	242,812

(*continued*)

Years Since 1992 t	Shipment Value ($ millions) V
6	253,720
7	256,900
8	258,960

Source: Statistical Abstract of the United States, 2001, Table 982

(*Hint:* Before creating the model, align the data by subtracting 170,000 from each value of V. After doing logistic regression, add back the 170,000 to the resultant model equation.)

Explain what the point of inflection and limiting value mean in the real-world context.

40. Calculator Technology The rapid advancement in calculator technology over the past 30 years has led to a decrease in the price of a basic, four-function calculator.

In August 1972, the four-function Sinclair Executive calculator was introduced for around $150. By the end of the decade, comparable calculators were selling for under $10.

Suppose the table gives the average price of a four-function calculator from 1972 to 1980.

Years Since 1972 t	Calculator Cost (dollars) C
0	150
1	132
2	110
3	85
4	62
5	42
6	27
7	17
8	10

a. Create a scatter plot of the data. Is this scatter plot increasing or decreasing? Does there appear to be an inflection point? What does the inflection point tell about four-function calculator prices?

b. Do you think there is a limiting value as t increases? If so, what is it and what does it represent?

■ STRETCH YOUR MIND

Exercises 41–42 are intended to challenge your understanding of logistic functions.

41. AIDS Deaths in the United States From 1981 to 1995, the number of adult and adolescent AIDS deaths in the United States increased dramatically. However, from 1995 to 2001 the annual death rate plummeted, as shown in the table.

Years Since 1981 t	Number of Deaths during Year A
0	122
1	453
2	1481
3	3474
4	6877
5	12,016
6	16,194
7	20,922
8	27,680
9	31,436
10	36,708
11	41,424
12	45,187
13	50,071
14	50,876
15	37,646
16	21,630
17	18,028
18	16,648
19	14,433
20	8963

Source: HIV/AIDS Surveillance Report, Dec. 2001; Centers for Disease Control and Prevention, p. 30

a. Create a scatter plot of the data.

b. Find a best-fit function, $d(t)$, for the number of deaths from 1981 to 1995.

c. Find a best-fit function, $c(t)$, for the number of deaths from 1995 to 2001.

d. Explain the trend in the number of AIDS deaths from 1981 to 1995.

e. Explain the trend in the number of AIDS deaths from 1995 to 2001.

f. Write the piecewise function $A(t)$ that models the number of deaths due to AIDS from 1981 to 2001.

g. Evaluate $A(32)$ and $A(35)$. Do these values seem reasonable? Explain.

42. Canadian Population The population of Canada has grown at different rates from 1861 to 2001 and is displayed in the table.

Years Since 1861 t	Canadian Population (millions) C
0	3.230
10	3.689
20	4.325
30	4.833
40	5.371
50	7.207
60	8.788
70	10.377
80	11.507
90	13.648
100	18.238
110	21.568
120	24.820
130	28.031
140	31.050

(*continued*)

Source: www.sustreport.org

The Sustainability Reporting Program estimates the population of Canada to be 33,369,000 in the year 2011; 35,393,000 in the year 2021; and 42,311,000 in the year 2050. Create a scatter plot of the data in the table and use it to find the mathematical model that those who wrote the Sustainability Report could have used to arrive at their estimates.

SECTION 7.5 Choosing a Mathematical Model

LEARNING OBJECTIVES

- Select the "best" function to model a real-world data set given in equation, graph, table, or word form
- Use the appropriate function to model real-world data sets
- Use appropriate models to predict and interpret unknown results

GETTING STARTED

The U.S. Congress periodically debates the pros and cons of increasing the federal minimum wage. For instance, lawmakers must consider how the increase in the wage will impact business owners' ability to hire employees as well as how the increase in pay will positively impact employees' lives. The process of determining a fair and reasonable minimum wage includes investigating previous wage levels and inflation rates as well as analyzing economic trends. Knowing how to choose a mathematical model is an important skill in situations such as this.

In this section we revisit what we have learned throughout the text by studying how interpolation, extrapolation, function end behavior, and a function's rate of change are used when choosing the "best" mathematical model. We also consider the reasonableness of predictions and determine the practical domain of the models we choose.

■ Choosing a Mathematical Model from a Table of Values

Recall that a *mathematical model* is a graphical, verbal, numerical, and symbolic representation of a problem situation. The model helps us understand the nature of the problem situation and make predictions. Mathematical models are used frequently to represent real-world situations so it is important to master the process of choosing an appropriate model.

EXAMPLE 1 ■ Choosing a Model from a Table of Values

The *Fair Labor Standards Act* was passed by Congress in 1938 to set a minimum hourly wage for American workers. Table 7.23 displays the wages set for the year 1938 and each year in which the wage was raised.

Table 7.23

Years Since 1938 t	Minimum Wage (dollars) M		Years Since 1938 t	Minimum Wage (dollars) M
0	0.25		37	2.10
1	0.30		38	2.30
7	0.40		40	2.65
12	0.75		41	2.90
18	1.00		42	3.10
23	1.15		43	3.35
25	1.25		52	3.80
29	1.40		53	4.25
30	1.60		58	4.75
36	2.00		59	5.15

Source: usgovinfo.about.com

a. Determine whether a linear, a quadratic, or an exponential function is the most appropriate mathematical model for these data. Explain as you go along why you did not choose the two types of functions you reject.

b. Use your model to predict the minimum wage in 2008.

Solution

a. To determine if these data are best modeled by a linear function or a nonlinear function (quadratic or exponential), we first calculate the average rate of change over each time interval, as shown in Table 7.24. (Note that the time intervals are not equal.)

Table 7.24

Years Since 1938 t	Minimum Wage (dollars) M	Average Rate of Change $\frac{\Delta M}{\Delta t}$
0	0.25	
1	0.30	0.05
7	0.40	0.02
12	0.75	0.07
18	1.00	0.04
23	1.15	0.03
25	1.25	0.05
29	1.40	0.04
30	1.60	0.20
36	2.00	0.07
37	2.10	0.10
38	2.30	0.20
40	2.65	0.18
41	2.90	0.25
42	3.10	0.20
43	3.35	0.25
52	3.80	0.05
53	4.25	0.45
58	4.75	0.10
59	5.15	0.40

We know that to be effectively modeled with a linear function, the data must have a relatively constant rate of change. This data set has neither a constant nor nearly constant rate of change so we rule out the linear model. To confirm our conclusion, we draw the scatter plot in Figure 7.31. The overall trend seems to show the data increasing at an increasing rate and, therefore, it is nonlinear.

We next check to see if the data may be best modeled by a quadratic function. Using quadratic regression, we find the quadratic model of best fit, $Q(t) = 0.0012t^2 + 0.01t + 0.29$, and graph it in Figure 7.32.

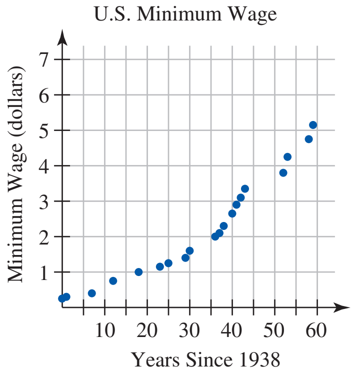


Figure 7.31

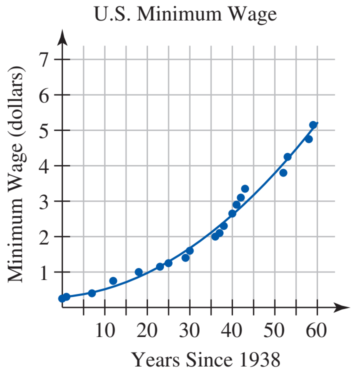


Figure 7.32

The graph of the quadratic function appears to be a good fit, but exponential functions can also model data that are increasing at an increasing rate. To test the exponential model, we calculate the annual growth factors to see if they are relatively constant, as shown in Table 7.25. Recall that to find the annual growth factor over a time span greater than 1 year, we must find the nth root of the quotient between two consecutive years. For example, for the growth over the years from 1939 and 1945, we find the 6-year growth factor of $\frac{0.40}{0.30} \approx 1.33$ and the annual growth factor of $\sqrt[6]{1.33} \approx 1.05$.

Table 7.25

Years Since 1938 t	Minimum Wage (dollars) M	Annual Growth Factor
0	0.25	$\frac{0.30}{0.25} \approx 1.20$
1	0.30	$\frac{0.40}{0.30} \approx 1.33$ $\sqrt[6]{1.33} \approx 1.05$
7	0.40	$\frac{0.75}{0.40} \approx 1.88$ $\sqrt[5]{1.88} \approx 1.13$
12	0.75	1.05
18	1.00	1.03
23	1.15	1.04
25	1.25	1.03
29	1.40	1.14
30	1.60	1.04

Table 7.25 (*continued*)

Years Since 1938 t	Minimum Wage (dollars) M	Annual Growth Factor
36	2.00	1.05
37	2.10	1.10
38	2.30	1.07
40	2.65	1.09
41	2.90	1.07
42	3.10	1.08
43	3.35	1.01
52	3.80	1.12
53	4.25	1.02
58	4.75	1.08
59	5.15	

Since the growth factors are nearly constant—ranging between 1.01 and 1.20—we use exponential regression to determine the exponential model of best fit: $E(t) = 0.33(1.05^t)$. Its graph is shown (in red) in Figure 7.33, along with the quadratic model (in blue). Like the quadratic model, the exponential model appears to fit the trend in the data relatively well over the domain from 1938 to 1998.

To determine which of the two functions is the best choice, we need to consider which one will give us the most reasonable estimate for the minimum wage in years after 1998. To find out, we expand the graphs to year 75 (or 2013) as shown in Figure 7.34.

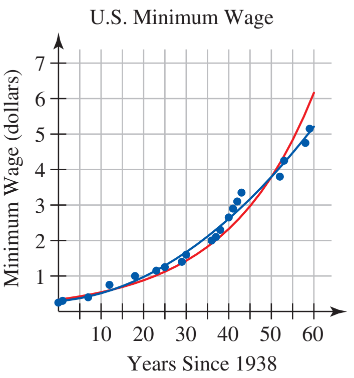


Figure 7.33

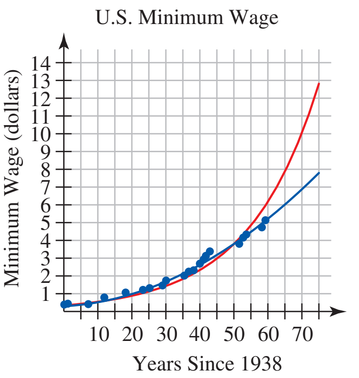


Figure 7.34

Now we can see that the exponential function departs from the trend in the data quite markedly while the quadratic function continues to model the trend in the data reasonably well. Therefore, we choose the quadratic function, $Q(t) = 0.0012t^2 + 0.01t + 0.29$ as the most appropriate model.

b. To predict the minimum wage in year 70 (or 2008), we use the model from part (a) to evaluate $Q(70)$.

$$\begin{aligned} Q(70) &= 0.0012(70)^2 + 0.01(70) + 0.29 \\ &= 6.87 \end{aligned}$$

We estimate the minimum wage in 2008 (year 70) will be \$6.87.

The following strategies are useful when you are given a table of data to model.

Model Selection Strategies to Use When Given a Table of Data

1. Consider the first and second differences for linear and quadratic functions, respectively. Also consider the change factor between outputs for exponential functions.
2. Create a scatter plot of the data.
3. Determine if the scatter plot looks like one or more of the standard mathematical functions: linear, quadratic, cubic, quartic, exponential, logarithmic, logistic.
4. Find a mathematical model for each function type selected in Step 3.
5. Use all available information to anticipate the expected behavior of the data outside of the data set. Eliminate models that do not exhibit the expected behavior. (Sometimes it is expedient to switch the order of Steps 4 and 5.)
6. Choose the simplest model from among the models that meet the criteria.

In accomplishing Step 3, it is helpful to recognize key graphical features exhibited by the data set, especially the concavity of the scatter plot. Table 7.26 summarizes these features.

Table 7.26

Function Type	Model Equation	Key Graphical Features
Linear	$y = mx + b$	Line
Quadratic	$y = ax^2 + bx + c$	Concave up everywhere or concave down everywhere
Cubic	$y = ax^3 + bx^2 + cx + d$	Changes concavity exactly once, no horizontal asymptotes
Quartic	$y = ax^4 + bx^3 + cx^2 + dx + f$	Changes concavity zero or two times, no horizontal asymptotes
Exponential	$y = ab^x$ with $a > 0$	Concave up, horizontal asymptote at $y = 0$
Logarithmic	$y = a + b\ln(x)$ with $b > 0$	Concave down, vertical asymptote at $x = 0$
Logistic	$y = \dfrac{c}{1 + ae^{-bx}} + k$	Changes concavity exactly once, horizontal asymptotes at $y = k$ and $y = c + k$

EXAMPLE 2 ■ Choosing a Model from a Table of Values

As shown in Table 7.27, the average hotel/motel room rate increased between 1990 and 1999.

Table 7.27

Years Since 1990 t	Room Rate (dollars) R		Years Since 1990 t	Room Rate (dollars) R
0	57.96		5	66.65
1	58.08		6	70.93
2	58.91		7	75.31
3	60.53		8	78.62
4	62.86		9	81.33

Source: Statistical Abstract of the United States, 2007, Table 1264

a. Choose a mathematical model for the data.

b. Forecast the average hotel/motel room rate for 2004.

c. Compare your estimated room rate for 2004 with the actual amount of \$86.24. What does this tell you about the model you chose?

Solution

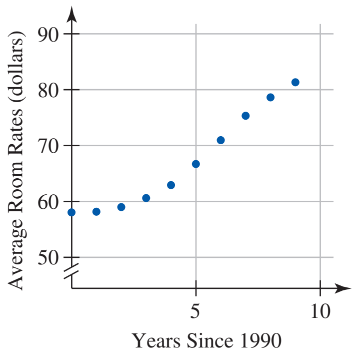
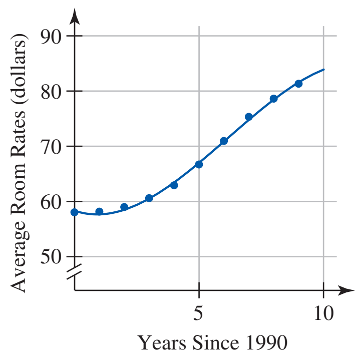
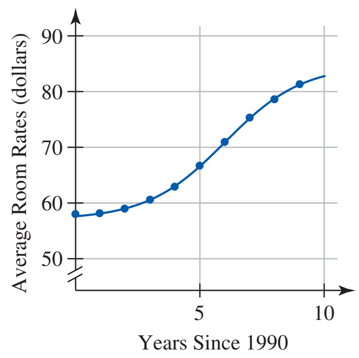
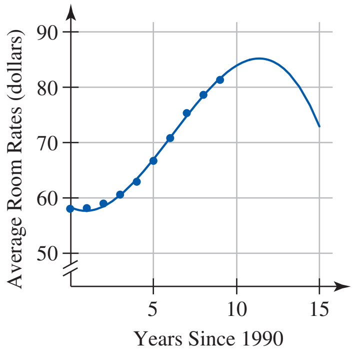
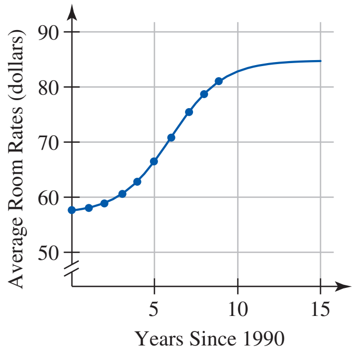


Figure 7.35

a. We first draw the scatter plot shown in Figure 7.35. The graph is initially concave up but changes concavity near $t = 6$. Since the graph changes concavity exactly once, a cubic or a logistic model may fit the data.

<table>
<tr><th>R as a Cubic Function of t</th><th>R as a Logistic Function of t</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$R(t) = -0.04815t^3 + 0.8868t^2 - 1.495t + 58.32$</td><td>$R(t) = \dfrac{27.80}{(1 + 47.23e^{-0.6425t})} + 57$</td></tr>
</table>

Both the cubic and logistic models appear to fit the data extremely well. To determine which model will be best for forecasting R in 2004 ($t = 14$), we extend the domain for each graph to include the interval $-1 \leq t \leq 15$.

R as a Cubic Function of *t*	*R* as a Logistic Function of *t*
	

Now we see that the cubic model decreases at about 2002 ($t = 12$) and continues to decrease. Since room rates have increased every year since 1990, it is unlikely the room rate will drop at any point in the future. On the other hand, the logistic model levels off, which seems to better model room rates.

b. Evaluating the logistic model at $R(14)$, we forecast the 2004 room rate to be \$84.64.

c. When we compare the projected average room rate of \$84.64 to the actual cost in 2004 of \$86.24, we see the logistic model is quite accurate and our choice was correct. However, we need to exercise caution in extrapolating much farther because it is doubtful the average room rate will remain constant.

It is important to note there is always a certain degree of uncertainty when choosing a model from several that seem to fit the data. Two different people may select two different models as their "best" model. For this reason, it is important to always explain the reasoning behind a particular selection.

EXAMPLE 3 ■ Choosing a Model from a Table of Values

As shown in Table 7.28, the average price of a movie ticket increased between 1975 and 2005. Find a mathematical model for the data and forecast the average price of a movie ticket in 2010.

Table 7.28

Years Since 1975 t	Movie Ticket Price (dollars) P
0	2.05
5	2.69
10	3.55
15	4.23
20	4.35
25	5.39
30	6.40

Source: www.boxofficemojo.com

Solution We first draw the scatter plot shown in Figure 7.36. The first four data points appear to be somewhat linear so our initial impression is that a linear model may fit the data well. However, the fifth data point is not aligned with the first four, so we know the data set is not perfectly linear. However, a linear model may still fit the data fairly well.

We could also conclude that the scatter plot is concave down on the interval $0 \le t \le 15$ and concave up on $15 \le t \le 30$. Since the scatter plot appears to change concavity once and does not have any horizontal asymptotes, a cubic model may work well.

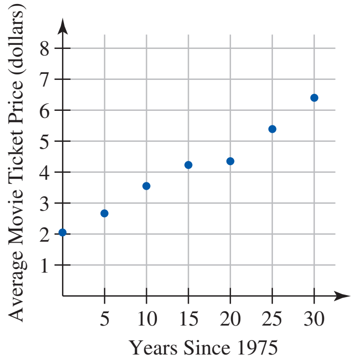
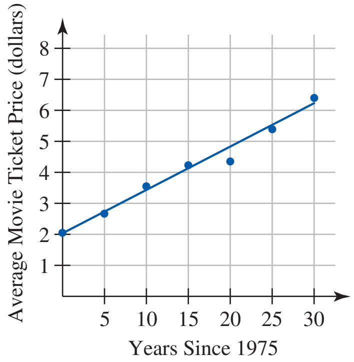
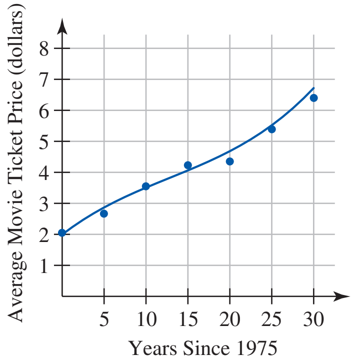


Figure 7.36

P* as a Function of *t	***P* as a Function of *t***
	
$P(t) = 0.14t + 2.03$	$P(t) = 0.0002t^3 - 0.0077t^2 + 0.2087t + 1.9871$

In 2010, $t = 35$. Evaluating each function at $t = 35$ yields

$$P(35) = 0.14(35) + 2.03$$
$$\approx \$6.93$$
$$P(35) = 0.0002(35)^3 - 0.0077(35)^2 + 0.2087(35) + 1.9871$$
$$\approx \$8.43$$

The linear model predicts a ticket price of \$6.93, and the cubic model predicts a price of \$8.43.

It is difficult to know which of these estimates of the average price of a movie ticket is the most accurate since both models seem to fit the data equally well. Additionally, we have no other information that would lead us to believe one model would be better than the other. Because both models seem to fit the data equally well, we choose the simplest one, which is $P(t) = 0.14t + 2.03$.

It is important to note that in this type of problem we are not trying to "hit" each data point with our model. Rather, we are attempting to capture the overall trend.

Choosing between Multiple Best-Fit Models

When two or more models fit a data set equally well and we have no additional information (such as more current data) that would lead us to believe one model would be better than the other, we choose the simplest function as the model.

Choosing a Mathematical Model from a Verbal Description

Certain verbal descriptions often hint at a particular mathematical model to use. By watching for key phrases, we can narrow the model selection process. Table 7.29 presents some typical phrases, their interpretation, and possible models.

Table 7.29

Choosing a Mathematical Model from a Verbal Description		
Phrase	**Interpretation**	**Possible Model**
Salaries are projected to increase by \$800 per year for the next several years.	The graph of the salary function will have a constant rate of change, $m = 800$.	Linear $S(t) = 800t + b$
The company's revenue has decreased by 5% annually for the past 6 years.	Since the annual decrease is a *percentage* of the previous year's revenue, the dollar amount by which revenue decreases annually is decreasing. The revenue graph is decreasing and concave up.	Exponential $R(t) = a(1 - 0.05)^t = a(0.95)^t$
Product sales were initially slow when the product was introduced but sales increased rapidly as the popularity of the product increased. Sales are continuing to increase but not as quickly as before.	The graph of the sales function may have a horizontal asymptote at or near $y = 0$ and a horizontal asymptote slightly above the maximum projected sales amount.	Logistic
Enrollments have been dropping for years. Each year we lose more students than we did the year before.	The graph of the enrollment function is decreasing since enrollments are dropping. It will also be concave down since the rate at which students are dropping continues to increase in magnitude.	Quadratic
Company profits increased rapidly in the early 1990s but leveled off in the late 1990s. In the early 2000s, profits again increased rapidly.	The graph of the profit function increases rapidly, then levels off, then increases rapidly again.	Cubic

EXAMPLE 4 ■ Choosing a Model from a Verbal Description

In its 2001 Annual Report, the Coca-Cola Company reported:

> "Our worldwide unit case volume increased 4 percent in 2001, on top of a 4 percent increase in 2000. The increase in unit case volume reflects consistent performance across certain key operations despite difficult global economic conditions. Our business system sold 17.8 billion unit cases in 2001." (*Source:* Coca-Cola Company 2001 Annual Report, p. 46)

Find a mathematical model for the unit case volume of the Coca-Cola Company.

Solution Since the unit case volume is increasing at a constant percentage rate (4%), an exponential model should fit the data well. We have

$V(t) = ab^t$, where t is the number of years since the end of 2001

The report says that the initial number of unit cases sold was 17.8 billion, so

$$V(t) = 17.8(1 + 0.04)^t \text{ billion unit cases}$$
$$= 17.8(1.04)^t \text{ billion unit cases}$$

The mathematical model for the unit case volume is $V(t) = 17.8(1.04)^t$.

Sometimes a data set cannot be effectively modeled by any of the aforementioned functions. In these cases, we look to see if we can model the data with a piecewise function.

EXAMPLE 5 ■ Choosing a Piecewise Function Model

From 1981 to 1995, the annual number of adult and adolescent AIDS deaths in the United States increased dramatically after an initial outbreak of the disease. However, from 1995 to 2001 the annual death rate plummeted as shown in Table 7.30.

Table 7.30

Years Since 1981 t	Number of Deaths during Year D		Years Since 1981 t	Number of Deaths during Year D
0	122		11	41,424
1	453		12	45,187
2	1481		13	50,071
3	3474		14	50,876
4	6877		15	37,646
5	12,016		16	21,630
6	16,194		17	18,028
7	20,922		18	16,648
8	27,680		19	14,433
9	31,436		20	8963
10	36,708			

Source: HIV/AIDS Surveillance Report; Dec. 2001; Centers for Disease Control and Prevention, p. 30

a. Find the mathematical model that best models the data.

b. Use the model to forecast the number of adult and adolescent AIDS deaths in 2004.

c. Then identify factors you think may have led to the dramatic decline in AIDS deaths after 1995.

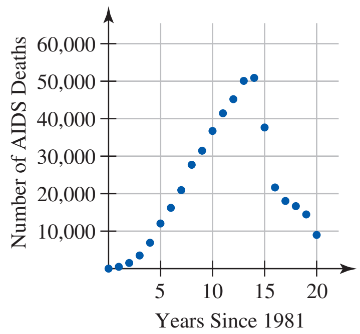


Figure 7.37

Solution

a. We first draw the scatter plot shown in Figure 7.37. The data set appears to exhibit logistic behavior up until 1995. After 1995, the graph appears to display cubic behavior. We will use a piecewise function to model the data set. We use a graphing calculator to determine each of the model pieces by using logistic and cubic regression. (The data point associated with $t = 14$ was used in both pieces of the model.) We obtain the following function and the graph in Figure 7.38.

$$P(t) = \begin{cases} \dfrac{53{,}955}{1 + 38.834e^{-0.45127t}} & \text{if } 0 \le t \le 14 \\ -381.06t^3 + 20{,}770t^2 - 379{,}469t + 2{,}339{,}211 & \text{if } t > 14 \end{cases}$$

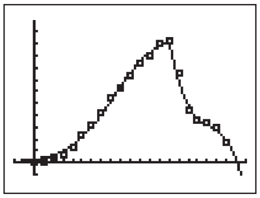


Figure 7.38

Our piecewise model appears to fit the data set very well.

b. To forecast the number of AIDS deaths in 2004 ($t = 23$), we use the second part of the piecewise function since $23 > 14$.

$$\begin{aligned} P(23) &= -381.06(23)^3 + 20{,}770(23)^2 - 379{,}469(23) + 2{,}339{,}211 \\ &= -37{,}603 \text{ adult and adolescent AIDS deaths} \end{aligned}$$

Despite the fact the model fit the data well, using the model to forecast the 2004 mortality rate yields an unreasonable result: It is impossible to have a negative number of deaths. Even from the data set, we can only estimate that the number of AIDS deaths in years beyond 2001 will be somewhere between 0 and 8963 (the 2001 figure).

c. Public education into the causes of the disease and advances in medical interventions may have been factors that led to the dramatic decline in AIDS deaths.

Some data sets cannot be effectively modeled by any of the standard mathematical functions or even a piecewise function. We look at this in the next example.

EXAMPLE 6 ■ Analyzing Data Not Easily Modeled with a Common Function

The number of firearms detected during airport passenger screening is shown in Table 7.31. After the terrorist attacks on September 11, 2001, airline screening became much more thorough. The increased security initially had a deterrent effect. According to the Bureau of Transportation Statistics, there were 1071 firearms detected in 2001 and 650 firearms detected in 2004. Nevertheless, in 2005 there were 2217 firearms detected, indicating the deterrent effect may have worn off or that detection methods may have improved. (*Source:* www.bts.gov)

Estimate the number of firearms detected by airport screeners in 2008.

Table 7.31

Years Since 1980 t	Firearms Detected F		Years Since 1980 t	Firearms Detected F
0	1914		14	2994
5	2913		15	2390
10	2549		16	2155
11	1644		17	2067
12	2608		18	1515
13	2798		19	1552

Source: Statistical Abstract of the United States, 2001, Table 1062

Solution The scatter plot in Figure 7.39 does not resemble any of the standard mathematical functions. It also does not appear that a piecewise function will fit the data well. Based on the table data and additional data, the number of firearms detected in a given year appears to be somewhat random, ranging from about 650 firearms to roughly 3000 firearms. Using this additional information, we can estimate that the number of firearms detected in 2008 was somewhere between 650 and 3000.

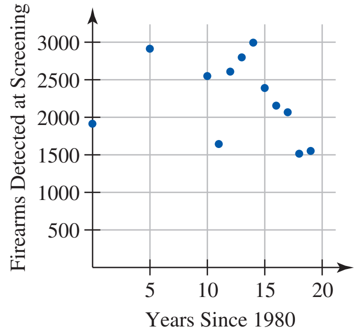


Figure 7.39

SUMMARY

In this section you learned strategies and techniques for selecting mathematical models. You learned to select the "best" function to model based on the rate of change of the data, the scatter plot, and the reasonableness of using the data to extrapolate. You also discovered that sometimes more than one function can be used to model the same data set, and sometimes no function works.

7.5 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–8, use a graph of the function to determine over what interval(s) the function is increasing or decreasing. Determine over what interval(s) the function is concave up or concave down. Identify the coordinate of the y-intercept, if there is one.

1. $y = 2(0.75)^x$

2. $y = 5 - 15x$

3. $y = 3x^2 - 2x + 5$

4. $y = \sqrt{x} + 7$

5. $y = \dfrac{45}{1 + 7e^{.05x}}$

6. $y = \dfrac{4}{1 - 5x}$

7. $y = -0.3(2.8)^x$

8. $y = -0.5x^2 - 2x$

9. Candy Bar Prices Candy bars currently cost \$0.60 each. The price of a candy bar is expected to increase by 3% per year in the future. Find a mathematical model for the price of a candy bar.

10. Calculator Prices A calculator currently costs \$87. The price of the calculator is expected to increase by \$3 per year. Find a mathematical model for the price of the calculator.

11. Mortality Rates There are presently 95 members of a high school graduating class who are still living. The number of surviving class members is decreasing at a rate of 4% per year. Find a mathematical model for the number of surviving class members.

12. Product Sales Growth An electronics company is introducing a new product next year. The company anticipates sales will be initially slow but will increase rapidly once people become aware of the product. It anticipates that monthly sales will start to level off in 18 months at about \$200,000. It predicts sales for the first two months will be \$12,000 and \$19,000, respectively. Develop a mathematical model to forecast monthly product sales.

13. Club Membership A business club is concerned about its decreasing number of members. Two years ago, it had 200 members. Last year it had 165 members and this year it has 110 members. If nothing changes, the club expects to lose even more members next year than it lost this year. Develop a mathematical model for the club membership.

14. Baseball Card Values A baseball card increases in value according to the function, $b(t) = \dfrac{5}{2}t + 100$, where b is the value of the card (in dollars) and t is the time (in years) since the card was purchased (that is, $t = 1$ represents one year after the card was purchased). Which of the following describe(s) what $\dfrac{5}{2}$ conveys about the situation?

I. The card's value increases \$5 every 2 years.

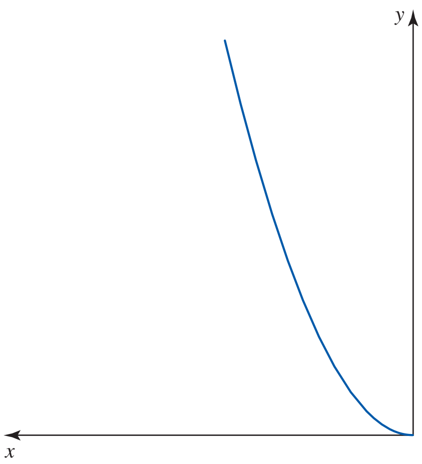
II. The card's value increases by a factor of $\dfrac{5}{2}$ every year.

III. The card's value increases by $\frac{5}{2}$ dollars per year.

A. I only **B.** II only
C. III only **D.** I and III only
E. I, II, and III

15. Create four data sets with at least eight data points in each: the first linear, the second quadratic, the third cubic, and the fourth exponential. Show why each data set represents the type of function it does by using successive differences or consecutive ratios.

16. Below is a portion of the graph of f. Is $f(x)$ exponential, quadratic, cubic, or is there not enough information to tell? Explain.



SHOW YOU KNOW

17. What makes a function the "best" to model a data set?

18. How do you use the "best" function to extrapolate effectively?

19. If a cubic model and a linear model both fit a data set with a coefficient of determination of $r^2 = 0.999$, which model should be used to interpolate? Justify your conclusion.

MAKE IT REAL

20. Population Growth The town of Queen Creek, Arizona, was founded in 1989. In 1990, there were 2667 people living in the town. The town grew rapidly in the 1990s, due in large part to new home construction in the area. There were 4316 people living in the town in 2000 and 4940 people in 2001. The Arizona Department of Commerce estimated the 2002 population of Queen Creek at 5555 people.

Find an exponential function to model the population of Queen Creek. Using this model, what is the maximum projected population of the city? (*Hint:* First align the data.)

The Arizona Department of Commerce estimated the 2005 population of Queen Creek at 15,890. How accurately did the model predict the 2005 population? Explain.

21. Housing Prices On March 12, 2004, a builder priced a new home in Queen Creek, Arizona, at \$143.9 thousand. The builder's sales rep told the author that the price for that home style would increase on March 16, March 30, and April 13, to \$148.9 thousand, \$151.9 thousand, and \$153.9 thousand, respectively. (***Source:*** **Fulton Homes**) Find a mathematical model for the price of the new home style.

22. Federal Tax Rates Federal income tax rates depend on the amount of taxable income received. In 2003, federal income taxes were calculated as follows. For single filers, the first \$7000 earned was taxed at 10%. The next \$21,400 earned was taxed at 15%. The next \$40,400 earned was taxed at 25%.

For example, the tax of a single filer who earned \$25,000 would be calculated as follows:

10% tax on the first \$7000:
$\$7000 \times 0.10 = \$700$

Amount to be taxed at a higher rate:
$\$25{,}000 - \$7000 = \$18{,}000$

15% tax on the next \$18,000:
$\$18{,}000 \times 0.15 = \$2700$

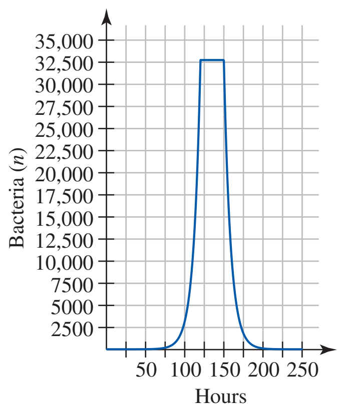
The filer's total tax is

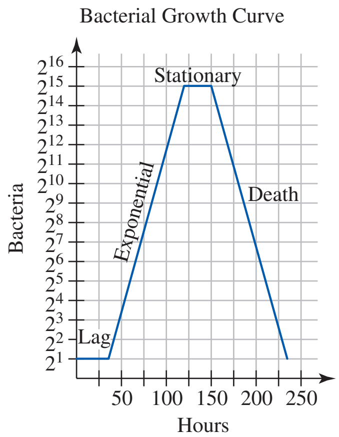
$$\$700 + \$2700 = \$3400$$

Find a mathematical model for income tax as a function of taxable income for single filers.

23. Measurement of Bacterial Growth The study of how bacteria grow requires bacterial enumeration (cell counting). If bacteria are held in a closed system like a test tube, after a period of time bacterial growth can occur. Initially this happens when a wall is created that divides the one bacterium into two "daughter cells."

Two graphs are shown. The first graph shows the number of bacteria in the population over time. Microbiologists typically represent the bacterial growth shown in the first graph with the second graph, which is called the *bacterial growth curve*. On the bacterial growth curve, the vertical axis uses a logarithmic scale. Notice that each tick mark on the vertical axis represents a power of 2. Moving from one tick mark to the next represents a doubling of the population.



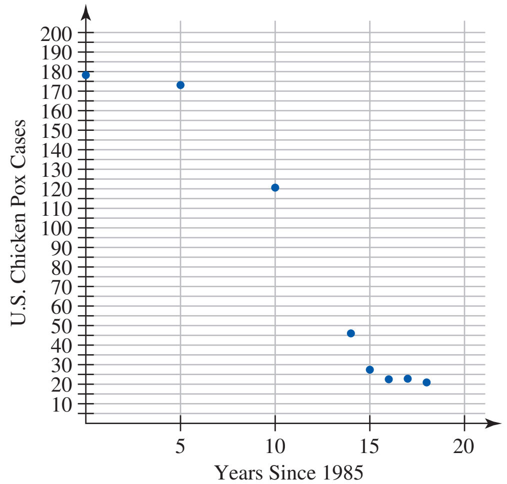


Describe each of the four phases of bacterial growth labeled on the bacterial growth curve (lag phase, exponential phase, stationary phase, and death phase) in terms of the number of viable (living) bacteria and rate of change.

In Exercises 24–25, choose an appropriate model for the graphs. Then explain what the rate of change tells about the relationship between the independent and dependent variables.

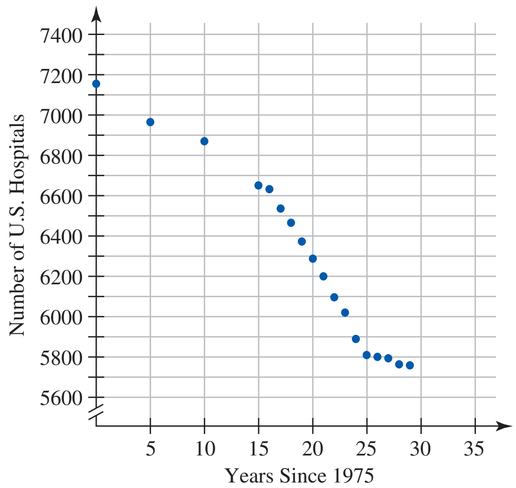
24. Confirmed Cases of Chicken Pox

NatUlrich/Shutterstock.com



***Source: Statistical Abstract of the United States, 2006*, Table 177**

25. Number of Hospitals



***Source: Health, United States, 2006*, p. 364**

In Exercises 26–36, find the equation of the mathematical model that will most accurately forecast the indicated result (if possible). Then use the model to find the value. Justify your conclusions.

26. Health Care Spending

Years Since 1990 t	Per Capita Spending on Physician and Clinical Services (dollars) p
0	619
5	813
8	914
9	954
10	1010
11	1085
12	1162
13	1249

***Source: Statistical Abstract of the United States, 2006*, Table 121**

Forecast spending on physician and clinical services for 2008.

27. Chapter 11 Bankruptcies

Years Since 1998 t	Nonbusiness Chapter 11 Bankruptcies B
0	981
1	731
2	722
3	745
4	894
5	966
6	935

***Source: Statistical Abstract of the United States, 2006*, Table 749**

Forecast the number of nonbusiness Chapter 11 bankruptcies in 2010.

28. Community College Education Costs

Years Since 1997–1998 t	Maricopa Community College Resident Cost per Credit (dollars) C
0	37
1	38
2	40
3	41
4	43
5	46
6	51
7	55
8	60
9	65
10	65

***Source:* www.dist.maricopa.edu**

Forecast the cost per credit for county resident students attending college in the district in 2010–2011.

29. DVD Players Sold

Average Price of DVD (dollars) *p*	DVD Players Sold (thousands) *D*
489.97	349
390.18	1079
270.00	4072
201.55	8499
165.00	12,707
142.00	17,090
123.00	21,994
108.60	19,990

Source: Consumer Electronics Association, www.ce.org

Forecast the DVD players sold when the average price of a DVD player is $100.

30. Prescription Drug Prices

Years Since 1995 *t*	Average Brand-Name Drug Price (dollars) *D*
0	40.22
2	49.55
3	53.51
4	60.66
5	65.29
6	69.75
7	77.49
8	85.57
9	95.86

Source: Statistical Abstract of the United States, 2006, Table 126

Forecast the average prescription drug price in 2010.

31. Community College Education Costs

Years Since 1984–1985 *t*	Washington State Community Colleges Tuition and Fees (dollars) *F*
0	581
1	699
2	699
3	759
4	780
5	822
6	867
7	945
8	999
9	1125
10	1296
11	1350
12	1401
13	1458
14	1515
15	1584
16	1641
17	1743

(continued)

Source: Washington State Higher Education Coordinating Board, Higher Education Statistics, September 2001

Forecast the annual tuition and fees at a Washington state community college in 2008.

32. Golf Course Facilities

Years Since 1980 *t*	Golf Facilities *G*	Years Since 1980 *t*	Golf Facilities *G*
0	12,005	14	13,683
5	12,346	15	14,074
6	12,384	16	14,341
7	12,407	17	14,602
8	12,582	18	14,900
9	12,658	19	15,195
10	12,846	20	15,489
11	13,004	21	15,689
12	13,210	22	15,827
13	13,439	23	15,899

Source: Statistical Abstract of the United States, 2004–2005, Table 1240

Forecast the number of golf facilities in the United States in 2008.

33. Per Capita Personal Income

Years Since 1993 *t*	Per Capita Personal Income—Florida (dollars) *F*
0	21,320
1	21,905
2	22,942
3	23,909
4	24,869
5	26,161
6	26,593
7	27,764

Source: Bureau of Economic Analysis, www.bea.gov

Forecast the per capita personal income in Florida in 2004.

34. Air Carrier Accidents

Year t	Air Carrier Accidents A		Year t	Air Carrier Accidents A
1992	18		1997	49
1993	23		1998	50
1994	23		1999	52
1995	36		2000	54
1996	37			

***Source:* *Statistical Abstract of the United States, 2001*, Table 1063**

Forecast the number of air carrier accidents in 2002.

35. Chipotle Mexican Grill Revenue

Franchise Royalties and Fees ($1000s) f	Restaurant Sales ($1000s) s
267	131,331
753	203,892
1493	314,027
2142	468,579
2618	625,077

***Source:* Chipotle Mexican Grill, Inc, 2005 Annual Report**

Forecast the amount of restaurant sales (in thousands) when the franchise royalties and fees reach $3,000,000.

36. Yogurt Production

Years Since 1997 t	Yogurt (million pounds) y		Years Since 1997 t	Yogurt (million pounds) y
0	1574		5	2311
1	1639		6	2507
2	1717		7	2707
3	1837		8	2990
4	2003			

***Source:* *Statistical Abstract of the United States, 2007*, Table 846**

Forecast the amount of yogurt produced in 2010.

In Exercises 37–43, find the model that best fits the data. Then use the model to answer the given questions.

37. Magazine Advertising

Years Since 1990 t	Magazine Advertising Expenditures ($ millions) M
0	6803
1	6524
2	7000
3	7357
4	7916
5	8580
6	9010

(continued)

Years Since 1990 t	Magazine Advertising Expenditures ($ millions) M
7	9821
8	10,518
9	11,433
10	12,348

***Source:* *Statistical Abstract of the United States, 2001*, Table 1272**

According to the model, how much money was spent on magazine advertising in 2002?

38. Cable TV Advertising

Years Since 1990 t	Cable TV Advertising Expenditures ($ millions)
0	2457
1	2728
2	3201
3	3678
4	4302
5	5108
6	6438
7	7237
8	8301
9	10,429
10	12,364

***Source:* *Statistical Abstract of the United States, 2001*, Table 1272**

According to the model, how much money was spent on cable television advertising in 2002?

39. Advertising Using the models from Exercises 37 and 38, determine in what year cable television advertising expenditures are expected to exceed magazine advertising expenditures.

40. Radio Advertising

Years Since 1990 t	Radio Advertising Expenditures ($ millions) R
0	8726
1	8476
2	8654
3	9457
4	10,529
5	11,338
6	12,269
7	13,491
8	15,073
9	17,215
10	19,585

***Source:* *Statistical Abstract of the United States, 2001*, Table 1272**

According to the model, when will radio advertising exceed $25 billion?

41. Yellow Pages Advertising

Years Since 1990 t	Yellow Pages Advertising Expenditures ($ millions) Y
0	8926
1	9182
2	9320
3	9517
4	9825
5	10,236
6	10,849
7	11,423
8	11,990
9	12,652
10	13,367

Source: Statistical Abstract of the United States, 2001, **Table 1272**

According to the model, when will yellow page advertising exceed $15 billion?

42. Average Hourly Earnings in Manufacturing Industries

Years Since 1980 t	Michigan Average Hourly Earnings (dollars per hour) M
0	9.52
1	10.53
2	11.18
3	11.62
4	12.18
5	12.64
6	12.80
7	12.97
8	13.31
9	13.51
10	13.86
11	14.52
12	14.81
13	15.36
14	16.13
15	16.31
16	16.67
17	17.18
18	17.61
19	18.38
20	19.20

Source: Statistical Abstract of the United States, 2001, **Table 978**

According to the model, what will the average hourly wage be in Michigan manufacturing industries in 2003?

43. Average Hourly Earnings in Manufacturing Industries

Years Since 1980 t	Florida Average Hourly Earnings (dollars per hour) F
0	5.98
1	6.53
2	7.02
3	7.33
4	7.62
5	7.86
6	8.02
7	8.16
8	8.39
9	8.67
10	8.98
11	9.30
12	9.59
13	9.76
14	9.97
15	10.18
16	10.55
17	10.95
18	11.43
19	11.83
20	12.28

Source: Statistical Abstract of the United States, 2001, **Table 978**

According to the model, what will the average hourly wage be in Florida manufacturing industries in 2003?

44. Hourly Earnings Based on the wage data in Exercises 42 and 43, do you think it would be better to start up a manufacturing business in Florida or Michigan? Justify your answer and explain what other issues might impact your decision.

45. Engine Torque Torque is the power a car engine generates. The horsepower reflects the amount of work that the engine is doing based on the gearing and revolutions per minute. Therefore, the torque of an engine depends on the speed of the engine. The table gives the hypothetical torque for different V8 engine speeds (in thousands of revolutions per minute).

Engine Speed (rpm in 1000s) s	Engine Torque (ft-lb) T
1	180
2	390
3	510
4	545

(*continued*)

Engine Speed (rpm in 1000s) s	Engine Torque (ft-lb) T
5	517
6	410
7	360

Source: craig.backfire.ca

a. Draw a scatter plot of the data.

b. Is a linear or quadratic function more appropriate to model this data? Explain.

c. Use regression to find the linear or quadratic model that best fits the data.

d. Use the model in part (c) to find $T(5.5)$. Explain what the solution means in its real-world context.

e. Use the model in part (c) to estimate $T(s) = 100$. Explain what the solution means in its real-world context.

f. Find the coordinates of the vertex (turning point) of the graph of the quadratic function in part (c). Explain what this point means in this context.

g. Find the average rate of change from $s = 2$ to $s = 4$. Explain what the solution means in this context.

h. Find the rate of change at $s = 4$. Explain what the solution means in this context.

■ STRETCH YOUR MIND

Exercises 46–54 are intended to challenge your understanding of mathematical modeling.

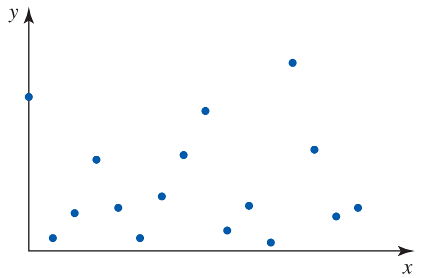
46. The graph of a mathematical model passes through all of the points of a data set. A classmate claims that the model is a perfect forecaster of future results. How would you respond?

47. A scatter plot is concave up and increasing on the interval $0 \le x \le 5$, concave down and increasing on the interval $5 \le x \le 8$, and decreasing at a constant rate on the interval $8 \le x \le 15$. Describe two different mathematical models that may fit the data set.

48. Describe how a business owner can benefit from mathematical modeling, despite the imprecision of a model's results.

49. Daily fluctuations in the stock market make the share price of a stock very difficult to model. What approach would you take if you wanted to model the long-term performance of a particular stock?

50. You are asked by your boss to model the data shown in the following scatter plot. How would you respond?



51. Men's Health Medical researchers have found that there is a relationship between a person's blood pressure and weight. In males 35 years of age, for every 5-pound increase in the person's weight there is generally an increase in the systolic blood pressure of 2 millimeters of mercury (mmHg). Moreover, for a male 35 years of age and 190 pounds the preferred systolic blood pressure is 125. (*Source:* Chandler-Gilbert Community College Wellness program, www.cgc.edu)

a. Construct a formula for a 35-year-old male's systolic blood pressure as a function of weight.

b. What is the slope of the equation? What is the meaning of the slope in terms of the real-world context?

c. What is the vertical intercept of the equation? What is the meaning of the vertical intercept in this context?

52. Cigarette Consumption The data for the per capita consumption of cigarettes over the last century is shown in the table.

Years Since 1900 y	Per Capita Cigarette Consumption c
0	54
10	151
20	665
30	1485
40	1976
50	3552
60	4171
70	3985
80	3851
90	2827
100	2092

Source: Tobacco Outlook Report, Economic Research Service, U.S. Dept. of Agriculture

a. Determine what type of mathematical model may best fit the data. Choose from linear, quadratic, or exponential. Explain why you think the function you chose is the *best* choice. Be sure to justify your choice by showing differences and ratios, the scatter plot, and what would seem most reasonable for the future. Defend your choice and refute the other two possibilities.

b. Use regression to find the function that best fits the data and name the function $c(y)$.

c. Evaluate and interpret the real-world meaning of $c(109)$.

d. Solve and interpret the real-world meaning of $c(y) = 100$.

e. Using the function $c(y)$ from part (b), approximate the year in which per capita cigarette consumption will reach its maximum.

f. Find the average rate of change from 1940 to 1950 and explain what your answer means in the context of this problem. Considering this decade, why would your answer seem reasonable?

g. Find the average rate of change from 1980 to 1990 and explain what your answer means in the context of this problem. Considering this decade, why would your answer seem reasonable?

h. Estimate the instantaneous rate of change for the year 1995 using the model you chose. Interpret the meaning of your answer in the context of this problem.

53. Population Explosion Phoenix, Arizona is located in Maricopa County. Maricopa County's population is expected to nearly double in less than four decades, which could stretch the limits of currently projected water supplies and all other municipal services. The table gives the population of Maricopa County every 10 years from 1900 through 2000.

Years Since 1900 t	Population of Maricopa County P
0	20,457
10	34,488
20	89,567
30	150,970
40	186,193
50	331,770
60	663,510
70	971,228
80	1,509,260
90	2,122,100
100	3,000,000

Source: Maricopa Association of Governments and the U.S. Bureau of Census

a. What kind of function best models the data? Choose from linear, quadratic, or exponential. Explain why the function you chose is the best choice. (Note: None of these three functions fits really well, but you still need to choose the "best.") Be sure to justify your choice using differences and ratios, a scatter plot, and what would seem most reasonable for the future. Defend your choice and refute the other two possibilities.

b. Use regression to find the function that best fits the data and name the function $m(y)$.

c. Evaluate and interpret the real-world meaning of $m(150)$.

d. Solve and interpret the real-world meaning of $m(y) = 4{,}000{,}000$.

e. Evaluate $\dfrac{m(88) - m(52)}{88 - 52}$ and interpret what it means in this context.

f. Estimate the rate of change for the year 1995 using the model you chose. Interpret the meaning of your answer in the context of this problem.

g. How far into the future do you feel the model will be effective in predicting the future population? Give the reasons you chose the year you did.

54. Major League Baseball Players' Strike During the 2001 Major League Baseball season while the players were contemplating going on strike, one of the authors wrote the following letter to the editor of the Arizona Republic newspaper. Read the letter and then answer the question.

As a mathematics professor at a community college in the Valley, I couldn't help but do some mathematical analysis of the Major League Baseball Players' salaries to determine for myself what the "predicament" is that the players find themselves in that may "force" them to strike. I found the players' ***average*** *salaries for the last 20 years on the USA Today website. If the trend continues as it has over this time period predicted* ***average*** *players' salaries for future years are:*

2003	*$2,883,458*
2004	*$3,408,631*
2005	*$4,055,009*
2010	*$9,854,997*
2015	*$22,307,696*

Oh, now I know what to tell my 11-year-old son who asked me why the players want to strike. They really do need more money! I would be worried about my income too with low wages like these. Maybe my son's school can start a penny drive for our beloved Diamondbacks!

Signed,
A citizen concerned about our precious baseball players

Year	t	Average Players' Salary
1982	0	241,497
1983	1	289,194
1984	2	329,408
1985	3	371,571
1986	4	412,520
1987	5	412,454
1988	6	438,729
1989	7	497,254
1990	8	597,537
1991	9	851,492
1992	10	1,028,667
1993	11	1,076,089
1994	12	1,168,263
1995 (player's strike year)	13	1,110,766
1996	14	1,119,981
1997	15	1,336,609
1998	16	1,398,831
1999	17	1,611,166
2000	18	1,895,630
2001	19	2,264,403
2002	20	2,383,235

Determine what mathematical model the author used to estimate the average player's salary in the years 2010 and 2015.

CHAPTER 7 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 7.1

1. How do you compute the sum, difference, product, and quotient of functions?
2. How do you determine the sum or difference of functions from a graph or table?
3. What is the practical and theoretical domain for a combination of functions?

SECTION 7.2

4. What is a piecewise function?
5. How do you create the formula for a piecewise function given data in tabular or graphical form?
6. How do you evaluate a piecewise function represented in graphical, tabular, or symbolic form?
7. What does it mean for a function to be continuous or discontinuous?
8. What are the different types of discontinuities?
9. What are the advantages and disadvantages of choosing to model data with a piecewise function?
10. Why are some data sets best modeled by piecewise functions?

SECTION 7.3

11. What is a composition of functions?
12. How do you create composite functions represented in tabular, graphical, and formula form?
13. What is function composition notation and how is it interpreted?
14. How do you evaluate a function composition represented in graphical, tabular, or symbolic form?
15. How do you decompose function compositions?

SECTION 7.4

16. How are exponential and logistic functions similar and how are they different?
17. How do you graph logistic functions represented by a table or formula?
18. How can a logistic function be described in terms of a rate of change?
19. What is important to keep in mind when modeling data sets with a logistic function?

SECTION 7.5

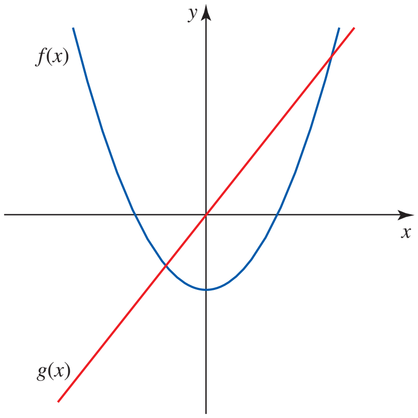
20. What makes a function the "best" to model a data set?
21. How do you use the "best" function to extrapolate effectively?

REVIEW EXERCISES

■ SECTION 7.1 ■

In Exercises 1–2, use the given graph or table of functions f and g to evaluate each given function. If it is not possible to evaluate the function, explain why not.

1.



a. $(f + g)(0)$

b. $(f - g)(0)$

c. $\left(\frac{f}{g}\right)(0)$

d. $(f \cdot g)(0)$

2.

x	−8	−4	0	4	8
$f(x)$	−4	0	4	8	12
$g(x)$	−4	−2	0	2	4

a. $(f + g)(0)$

b. $(f - g)(0)$

c. $\left(\frac{f}{g}\right)(0)$

d. $(f \cdot g)(0)$

In Exercises 3–6, combine the functions f(x) and g(x) symbolically as indicated. Then state the domain and range of the combination.

3. $f(x) = -2x + 3 \qquad g(x) = x - 4$

a. $(f + g)(x)$

b. $(f - g)(x)$

4. $f(x) = x^2 - 1 \qquad g(x) = x - 3$

a. $(f - g)(x)$

b. $\left(\frac{f}{g}\right)(x)$

5. $f(x) = e^x - 1 \qquad g(x) = 2x$

a. $(f \cdot g)(x)$

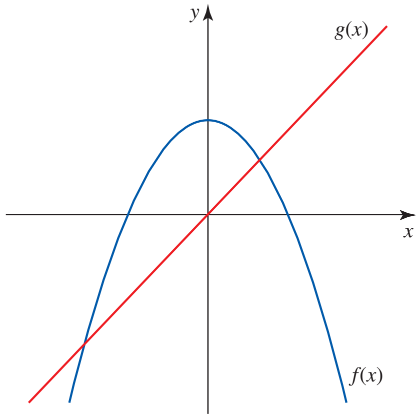
b. $\left(\frac{g}{f}\right)(x)$

6. $f(x) = \ln(x) \qquad g(x) = 2x - 4$

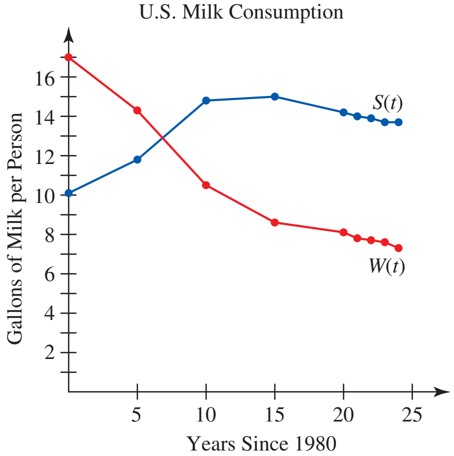
a. $(f + g)(x)$

b. $(f \cdot g)(x)$

7. Use the given graphs of the functions f and g to sketch a graph of each combination of functions: $(f + g)(x)$, $(f - g)(x)$, $(f \cdot g)(x)$, and $\left(\frac{f}{g}\right)(x)$.



8. **Milk Consumption** The graph shows the amount of milk consumed per person in the United States in selected years.



***Source: Statistical Abstract of the United States, 2007,* Table 201**

The per person consumption of whole milk as a function of years since 1980 is given by $W(t)$, in gallons. The per person consumption of reduced fat, light, and skim milk as a function of years since 1980 is given by $S(t)$, in gallons.

a. Interpret the meaning of $(W - S)(t)$.

b. Estimate and interpret $(W - S)(6)$.

c. Sketch a graph of $(W - S)(t)$.

d. Interpret the meaning of $(W + S)(t)$.

e. Estimate and interpret $(W + S)(6)$.

f. Sketch a graph of $(W + S)(t)$.

9. Cheddar Cheese Consumption The amount of cheddar cheese consumed in the United States, in pounds per person, can be modeled by

$$C(t) = 0.034t^2 - 0.036t + 3.76$$

(*Source: Statistical Abstract of the United States, 2007*, Table 202) The cost per pound of cheddar cheese in the United States can be modeled by

$$P(t) = 0.117t^3 - 0.600t^2 + 0.683t + 9.7$$

(*Source: Statistical Abstract of the United States, 2007*, Table 712) In both models, t is the number of years since 2000.

a. Interpret the meaning of $(C \cdot P)(t)$.

b. Evaluate and interpret $(C \cdot P)(10)$.

c. Solve $(C \cdot P)(t) = 50$ and explain what the result means in the context of this situation.

10. Presidential Elections The total number of votes cast in presidential elections, in millions, and the total number of people in the United States of voter age, in millions, are presented in the table, where t is the number of years since 1980. (*Source:* Statistical Abstract of the United States, 2007, Tables 385 and 405)

Years Since 1980 t	Votes Cast (millions) $C(t)$	Voter-Age Population (millions) $V(t)$
0	86.5	157.1
4	92.7	170.0
8	91.6	178.1
12	104.6	185.7
16	96.4	193.7
20	105.6	202.6
24	122.3	215.7

a. Interpret the meaning of $\left(\frac{C}{V}\right)(t)$.

b. Evaluate and interpret $\left(\frac{C}{V}\right)(24)$.

c. Create a scatter plot of $\left(\frac{C}{V}\right)(t)$. Is there an apparent trend? Explain.

■ SECTION 7.2 ■

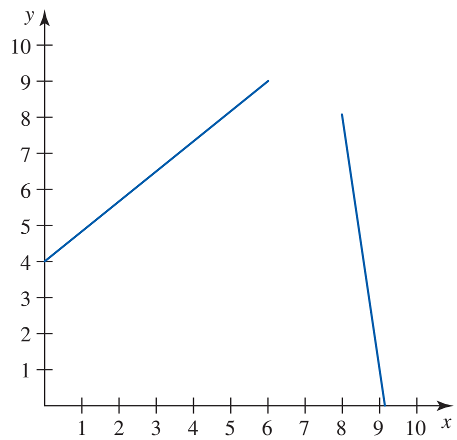
In Exercises 11–12, graph each piecewise function.

11. $f(x) = \begin{cases} -x & \text{if } x \le 4 \\ x^2 & \text{if } 4 < x < 11 \\ 3\ln(x) & \text{if } x \ge 11 \end{cases}$

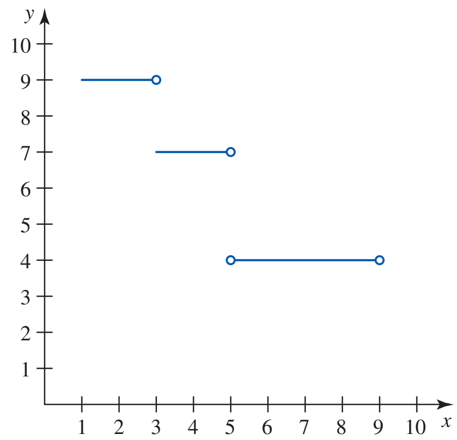
12. $f(x) = \begin{cases} \dfrac{2}{x-4} & \text{if } -3 \le x < 2 \\ 2 & \text{if } 2 < x < 4 \\ 2^x & \text{if } x \ge 6 \end{cases}$

In Exercises 13–14, write the equation for each piecewise function.

13.



14.



15. Cesarean Births The percentages of babies delivered by Cesarean birth are listed in the table for selected years from 1970 to 2005.

Years Since 1970 y	Cesarean Births (percent) C
0	5
5	11
10	17
15	23
20	23
25	21
30	23
35	30

Source: www.census.gov

a. Create a scatter plot of these data.

b. Assuming the data from 1970 to 1995 was quadratic and the data from 1995 to present is exponential, write the equation for the piecewise function $C(y)$ that models the data.

c. Calculate the average rate of change from $y = 0$ to $y = 15$ and explain what this value means in this context.

d. Estimate $C(17)$ and interpret its real-world meaning.

e. Estimate the instantaneous rate of change at $y = 17$ and interpret its real-world meaning.

f. Predict $C(35)$ assuming the trend in the data continues.

16. Filling a Swimming Pool The function,

$$V(t) = \begin{cases} 100t & \text{if } 0 \le t < 20 \\ 2000 & \text{if } 20 \le t < 23 \\ 1500t - 28{,}000 & \text{if } 23 \le t < 38 \\ 32{,}000 & \text{if } 40 \le t \le 48 \end{cases}$$

models the volume of water, V, (in gallons) in a swimming pool in t hours as it is filled.

a. Graph the function $V(t)$ and explain what information each piece of the function gives about how the pool is filled.

b. Evaluate $V(24)$ and explain what the numerical value represents in the real-world context.

17. Grading Scales At Chandler-Gilbert Community College, a professor may assign a student who completes a course a grade of A, B, C, D, or F. An A is worth 4.0 grade points per credit, a B is worth 3.0 grade points per credit, a C is worth 2.0 grade points per credit, a D is worth 1.0 grade point per credit, and an F is worth 0.0 grade points per credit. The table shows a typical grading scale in terms of grade points earned per credit.

Final Course Percentage p	Grade Points per Credit G
$p < 60$	0.0
$60 \le p < 70$	1.0
$70 \le p < 80$	2.0
$80 \le p < 90$	3.0
$90 \le p < 100$	4.0

a. Sketch a graph of the function $G(p)$ where G is the grade points per credit and p is the final course percentage.

b. Solve $G(p) = 3.0$ for p and explain what the numerical answer(s) mean in the real-world context.

18. Credit Card Bill One of the authors received a credit card offer from Capital One® Visa® that offered the following minimum payment terms:

> If the minimum balance is less than \$10, the full balance must be paid. If the balance is \$10 or more, the minimum payment is 3% or \$10, whichever is more.
> (*Source:* Capital One)

a. Create a table of values for the minimum amount due, p, as a function of the unpaid balance, b. Start the table with a \$5 unpaid balance and continue to \$1000 in increments of \$100.

b. Write a piecewise function $p(b)$ for the minimum payment due.

c. Sketch a graph of the function $p(b)$ over the domain $0 \le b \le 1000$.

19. Legoland Ride Policy The following picture was taken at Legoland in California in 2007 while one of the authors was on vacation.

© Frank C. Wilson

At the time of the visit, the author was accompanied by children of the following ages and heights.

Child Number	Age (years)	Height (inches)
1	1	33
2	4	42
3	7	49
4	10	59
5	14	65

a. Determine which of the five children could go on the ride and under what conditions.

b. When reading the sign, the author found the phrase "between 34 inches tall and 6 years of age and 48 inches tall" confusing. Describe the ambiguities that are created by this verbiage.

c. Represent the ride restrictions with a piecewise function. (*Hint:* There are two input variables: age and height.)

d. Referring to the piecewise function in part (c), rewrite the language on the sign to be more clear.

■ SECTION 7.3 ■

In Exercises 20–23, rewrite each pair of functions, $f(g)$ and $g(t)$, as one composite function, $f(g(t))$, if possible. Then evaluate $f(g(2))$.

20. $f(g) = 2g + 3 \qquad g(t) = 3t$

21. $f(g) = 5e^g \qquad g(t) = t^2 + 4$

22. $f(g) = \dfrac{3}{g - 1} \qquad g(t) = \dfrac{2}{t}$

23. $f(g) = \sqrt{g} \qquad g(t) = 2t + 3$

In Exercises 24–29, decompose the composite function $h(x) = f(g(x))$ into $f(g)$ and $g(x)$. (Note: There may be more than one possible correct answer.)

24. $h(x) = \sqrt{\ln(x^2 - 2)}$

25. $h(x) = e^{x^2 - 2}$

26. $h(x) = \dfrac{9}{(x+3)^4}$

27. $h(x) = |-x^3 + 4|$

28. $h(x) = (2x+1)^2 - 5$

29. $h(x) = \dfrac{2}{1+\sqrt{x}}$

30. Use the first three columns in the table to fill in the values for $f(g(x))$. (Note: Some function values may be undefined.)

x	$f(x)$	$g(x)$	$f(g(x))$
0	3	1	
1	5	4	
2	4	0	
3	1	2	
4	0	5	
5	2	3	

31. Find $g(g(4))$ for

$$g(x) = \begin{cases} 4 & \text{if } x \le 2 \\ 2x+1 & \text{if } 2 < x < 6 \\ x^2 - 4 & \text{if } x \ge 8 \end{cases}$$

32. Cigarettes and Heart Disease The function $D(p) = 14.08p - 53.87$ models the death rate due to heart disease from 1974 to 2003 in deaths per 100,000 people as a function of the percentage of people, p, who smoke. The function $p(t) = -0.52t + 35.7$ models the percentage of people who smoke as a function of the years, t, since 1974. (*Source: Statistical Abstract of the United States, 2006*, Table 106) Write a formula for $D(p(t))$ and then evaluate and interpret $D(p(36))$.

33. Motion Picture Screens and Movie Attendance The number of motion picture screens can be modeled by

$$S(m) = -0.000917m^3 + 4.21m^2 - 6440m + 3{,}320{,}000$$

screens where m is the movie attendance, in millions of people. Movie attendance may be modeled by

$$m(t) = -30.36t^2 + 153.12t + 1407.29$$

million people where t is the number of years since 2000. (*Source: Statistical Abstract of the United States, 2006*, Table 1234) Write a formula for $S(m(t))$ then evaluate and interpret $S(m(9))$.

34. Gross Pay vs. Net Pay An employee working as a lifeguard earns \$8.40 per hour.

a. Write a function that gives the employee's gross pay as a function of the number of hours worked. Be sure to identify the meaning of the variables you use.

b. Gross pay is reduced by Social Security (6.2%) and Medicare (1.45%). Additionally, income tax may be withheld and unemployment insurance may be required. Assuming the deductions total 12% of the employee's gross pay, write the employee's net pay as a function of the gross pay. Be sure to identify the meaning of the variables you use.

c. Based on the results of part (a) and (b), write an equation for net pay as a function of the number of hours worked.

d. Starting with the equation in part (c), write an equation for number of hours worked as a function of the net pay. This function is the inverse of the function in part (c). Explain what the input and output variables represent.

35. Cell Phone Billing In November 2006, Cingular Wireless advertised the following rate plan on their website.

Nation 1350 w/Rollover Rate Plan Details	
Monthly Cost (for 1350 Anytime minutes)	\$79.99
Night & Weekend minutes	Unlimited
Mobile to Mobile minutes	Unlimited
Long Distance	\$0.00
Roaming Charges	\$0.00
Additional minutes	\$0.35/minute

a. Write a function that will allow you to calculate the monthly cell phone cost as a function of weekday daytime minutes used. Indicate the meaning of the variables you use.

b. Use the function from part (a) to complete the table. Show all steps leading to your answers.

Minutes Used	Monthly Cost
0	
1000	
1500	
	\$364.19

c. A subscriber to the Nation 1350 plan has observed that when he travels his weekday daytime cell phone usage increases dramatically. When he is at home, he uses roughly 1300 weekday daytime minutes monthly; however, his cell phone usage increases by 100 weekday daytime minutes per day that he travels. (For example, if he travels 2 days in the month, he uses $1300 + 2(100) = 1500$ weekday daytime minutes in the month.)

Write a function that gives the number of weekday daytime minutes he uses in a month as a function of the number of days he travels in the month. Be sure to state the meaning of the variables you use.

d. Assuming the Nation 1350 plan subscriber from part (c) travels at least one day a month, write a function that gives his monthly cell phone bill as a function of the number of days he travels in the month. (*Hint:* Refer to your results for parts (a) and (c).)

e. What will be his monthly cell phone bill if he travels five days in the month?

f. Explain how this situation of determining the monthly cost of this cell phone plan (refer back to parts (d) and (e)) is an example of the composition of two functions.

SECTION 7.4

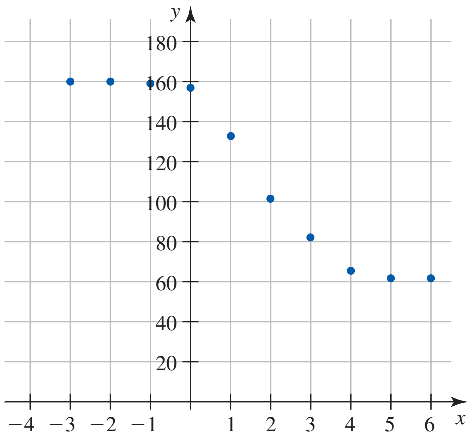
36. Create a table of values for a logistic function that is increasing and has a limiting value at 13.

37. Create a table of values for a logistic function that is decreasing and has a limiting value at 38.

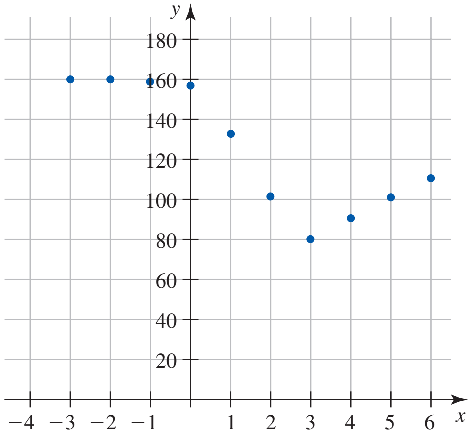
38. If $f(x) = \dfrac{15}{1 + 23e^{-1.8x}}$ is a logistic function, what is the limiting value?

In Exercises 39–40, identify the scatter plots as linear, exponential, logistic, or none of these. If you identify the scatter plot as none of these, explain why.

39.



40.



41. Lawn Mowing At the beginning of winter, I do not have to mow the yard. As spring approaches, I have to mow it some. In spring, the number of times I have to mow grows dramatically as the spring rains come. In the summer, this number declines because it is so hot. In the fall I do not have to mow very much as winter approaches. Sketch a graph that models the cumulative number of times I mow the lawn during a year starting with January 1.

42. Audio Cassette Sales The sales of audio cassettes in 1990 were high before the advent of CDs. As CDs began to catch on, the sales of audio cassettes began to drop but not dramatically at first because CD players were not widespread. In the mid-1990s the sales of audio cassettes dropped quickly and then leveled off in the late 1990s. Sketch a graph that models the annual sales of audio cassettes as a function of the years since 1990.

43. Amusement Park Attendance The cumulative number of visitors to an amusement park that is open year round is given in the table.

Month m	Cumulative Number of Visitors by the End of the Month (thousands) V
January	32
February	55
March	122
April	255
May	530
June	910
July	1465
August	1954
September	2187
October	2401
November	2411
December	2415

a. Find a logistic model to fit the data.

b. Estimate the limiting value and explain what it means in the real-world context.

c. In what month does it appear the point of inflection occurs? What does this mean in this context? Does this seem reasonable? Explain.

44. College Graduates According to the U.S. Department of Education, the number of college graduates increased significantly during the 20th century. The following table gives the number of college graduates (in thousands) from 1900 to 2000.

Years Since 1900 y	Number of College Graduates (thousands) g
0	30
10	54
20	73
30	127
40	223
50	432
60	530
70	878
80	935
90	1017
100	1180

Source: U.S. Department of Education

a. Create a scatter plot of these data.

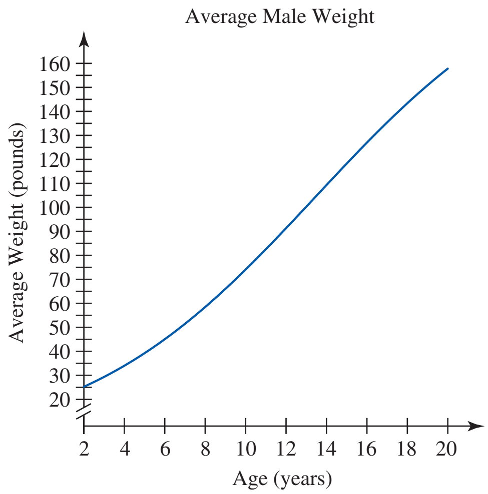
b. Find the equation for the logistic function $g(y)$ that best models the data.

c. Describe how the rate of change of the function $g(y)$ details the growth in the number of college graduates in the 20th century.

d. Estimate the limiting value for the logistic function $g(y)$.

e. Do you think the limiting value you found in part (d) will accurately project the number of college graduates in the beginning years of the 21st century? Why or why not?

45. Male Growth The weight W (in pounds) of the average boy is a function of his age y (in years). Below is a graph of $W(y)$ for males from 2 to 20 years of age.



Source: **www.cdc.gov**

a. Evaluate $W(16)$ and explain what the value means in the real-world context.

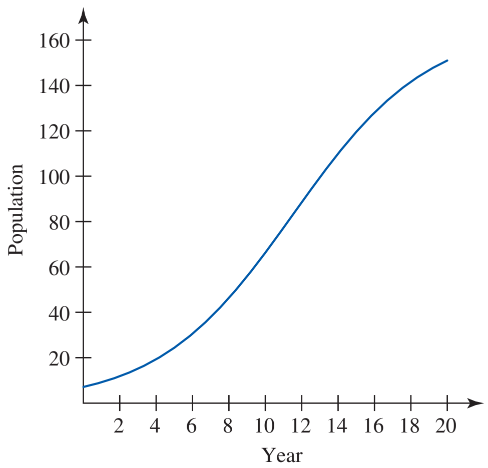
b. Solve $W(t) = 140$ and then explain what the numerical value means in the context of the problem.

c. Over what time period is the graph concave down? Explain what information this provides about the average weight of males.

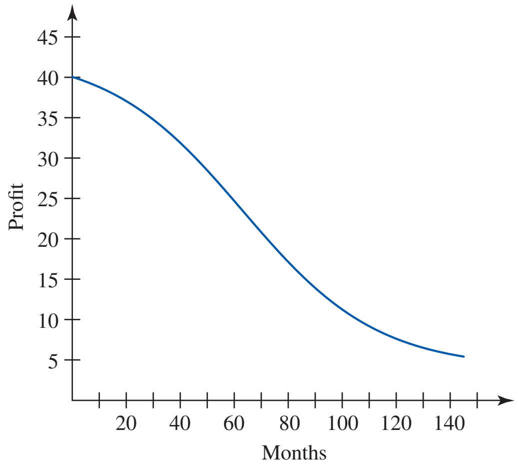
d. Estimate the point of inflection on the graph and explain what the coordinates of this point mean in terms of the average male.

In Exercises 46–47, imagine you are writing the headline for a particular situation and that you are analyzing the following graphs for your story. Write a headline that accurately represents each of the graphs. You may create your own context.

46.



47.



SECTION 7.5

48. Create four data sets with at least eight data points each. The first set should be linear; the second, quadratic; the third, exponential; and the fourth, logistic. Explain why each data set represents the type of function it does.

In Exercises 49–60, find the equation of the mathematical model that will most accurately forecast the indicated result (if possible). Then use the model to find the value. Justify your conclusions.

49. Average National Football League Salaries

Years Since 1976 t	Average NFL Player's Earnings (dollars) E
0	47,500
10	244,700
20	787,500
30	1,700,000

***Source: Sports Illustrated*, July 31, 2006; "Average Joe"**

Forecast the average NFL player's salary in 2010.

50. Average Major League Baseball Salaries

Years Since 1976 t	Average MLB Player's Earnings (dollars) E
0	52,802
10	402,579
20	1,101,455
30	2,857,932

***Source: Sports Illustrated*, July 31, 2006; "Average Joe"**

Forecast the average MLB player's salary in 2010.

51. Average National Basketball Association Salaries

Years Since 1976 t	Average NBA Player's Earnings (dollars) E
0	130,000
10	382,000
20	2,000,000
30	5,000,000

Source: Sports Illustrated, July 31, 2006; "Average Joe"

Forecast the average NBA player's salary in 2010.

52. Average National Hockey Association Salaries

Years Since 1976 t	Average NHL Player's Earnings (dollars) E
0	85,000
10	159,000
20	892,000
30	1,500,000

Source: Sports Illustrated, July 31, 2006; "Average Joe"

Forecast the average NHL player's salary in 2010.

53. Average Professional Golf Association Salaries

Years Since 1976 t	Average PGA Golfer's Earnings (dollars) E
0	25,814
10	97,610
20	257,840
30	973,495

Source: Sports Illustrated, July 31, 2006; "Average Joe"

Forecast the average PGA golfer's salary in 2010.

54. Average Ladies' Professional Golf Association Salaries

Years Since 1976 t	Average LPGA Golfer's Earnings (dollars) E
0	7,185
10	37,850
20	98,363
30	162,043

Source: Sports Illustrated, July 31, 2006; "Average Jane"

Forecast the average LPGA golfer's salary in 2010.

55. Average Association of Tennis Professionals Salaries

Years Since 1976 t	Average ATP Player's Earnings (dollars) E
0	18,034
10	62,892
20	231,827
30	260,000

Source: Sports Illustrated, July 31, 2006; "Average Joe"

Forecast the average ATP player's salary in 2010.

56. Average Women's Tennis Association Salaries

Years Since 1976 t	Average WTA Player's Earnings (dollars) E
10	55,965
20	165,477
30	345,000

Source: Sports Illustrated, July 31, 2006; "Average Jane"

Forecast the average WTA player's salary in 2010.

57. Average Engineer Salaries

Years Since 1976 t	Average Engineer's Earnings (dollars) E
0	20,749
10	42,667
20	60,684
30	80,122

Source: American Federation of Teachers

Forecast the average engineer's salary in 2010.

58. Average Attorney Salaries

Years Since 1976 t	Average Attorney's Earnings (dollars) E
0	24,205
10	50,119
20	66,560
30	100,852

Source: American Federation of Teachers

Forecast the average attorney's salary in 2010.

59. Average Accountant Salaries

Years Since 1976 t	Average Accountant's Earnings (dollars) E
0	24,205
10	50,119
20	66,560
30	71,200

Source: American Federation of Teachers

Forecast the average accountant's salary in 2010.

60. Average K–12 Teacher Salaries

Years Since 1976 t	Average K–12 Teacher's Earnings (dollars) E
0	12,591
10	25,260
20	37,594
30	47,602

Source: American Federation of Teachers

Forecast the average teacher's salary in 2010.

Make It Real Project

What to Do

1. Find a salary schedule for one of the local school districts in your area or use the one provided on the next page for the Maricopa Community College District.
2. For each of the vertical salary "lanes," determine how much each *vertical* step on the pay scale increases a teacher's salary.
3. Teachers who participate in professional development activities move *horizontally* on the pay scale. For each professional development credit hour earned, determine by how much the annual pay is increased.
4. From the salary schedule, pick a realistic salary for a newly hired teacher in the district. Write a function that models her salary as a function of years worked given that she expects to move one step vertically and earn six professional development credits each year.
5. As inflation increases prices, the buying power of the dollar decreases. Consequently, many employers offer a *cost of living allowance* (COLA) to their employees. Assuming a COLA of 3% is applied to the salary schedule annually, revise the function in part (4) to address this fact.
6. Use the model in part (5) to forecast the future salary of a teacher 5 years and 8 years into the future.

Make It Real Project continued

Maricopa Community College District
Residential Faculty Salary Schedule
2007–2008
Effective 7/1/2007

Base Salary
Credit Hour .33% $136.04
Vertical Increment 7% $2886

Step	IP	IP+12	IP+20	IP+24	IP+36	IP+40	IP+48	IP+60	IP+75	Ph.D.
1	$41,225	$42,857	$43,946	$44,490	$46,122	$46,667	$47,755	$49,387	$51,428	$53,469
2	$44,111	$45,743	$46,832	$47,376	$49,008	$49,553	$50,641	$52,273	$54,314	$56,355
3	$46,997	$48,629	$49,718	$50,262	$51,894	$52,439	$53,527	$55,159	$57,200	$59,241
4	$49,883	$51,515	$52,604	$53,148	$54,780	$55,325	$56,413	$58,045	$60,086	$62,127
5	$52,769	$54,401	$55,490	$56,034	$57,666	$58,211	$59,299	$60,931	$62,972	$65,013
6	$55,655	$57,287	$58,376	$58,920	$60,552	$61,097	$62,185	$63,817	$65,858	$67,899
7	$58,541	$60,173	$61,262	$61,806	$63,438	$63,983	$65,071	$66,703	$68,744	$70,785
8	$61,427	$63,059	$64,148	$64,692	$66,324	$66,869	$67,957	$69,589	$71,630	$73,671
9	$64,313	$65,945	$67,034	$67,578	$69,210	$69,755	$70,843	$72,475	$74,516	$76,557
10	$67,199	$68,831	$69,920	$70,464	$72,096	$72,641	$73,729	$75,361	$77,402	$79,443
11	$70,085	$71,717	$72,806	$73,350	$74,982	$75,527	$76,615	$78,247	$80,288	$82,329
12	$72,971	$74,603	$75,692	$76,236	$77,868	$78,413	$79,501	$81,133	$83,174	$85,215
13	$75,857	$77,489	$78,578	$79,122	$80,754	$81,299	$82,387	$84,019	$86,060	$88,101
14				$82,008	$83,640	$84,185	$85,273	$86,905	$88,946	$90,987

Initial Placement (IP) indicates initial placement on the salary schedule for any faculty member with an associates, bachelors, or masters degree. Credit hours are paid for each hour earned.

Wage & Salary for MCCD