CHAPTER 10

Trigonometric Identities

As consumers, we are skilled in quickly determining if two amounts of money are equivalent. For example, we easily recognize that two dimes and a nickel are equivalent to a quarter, or that three dimes do not have the same value as three quarters, even though both groups contain the same number of coins.

The skill of recognizing when two different-looking quantities, such as two trigonometric expressions, are mathematically equivalent helps us become more efficient in solving problems.

SECTION 10.1 Basic Identities

LEARNING OBJECTIVES

- Use the fundamental trigonometric identities
- Use the even–odd function identities
- Use the cofunction identities
- Use the Pythagorean identities

GETTING STARTED

In places where roof tiles are used, the tiles are delivered and distributed on the roof prior to being installed. Objects resting on a sloped surface seem to defy the force of gravity, but they actually maintain their position because of friction. Through the use of trigonometric identities, we can quantify how the tile stays on the roof.

In this section we define trigonometric identities and use these identities to help simplify trigonometric expressions and solve trigonometric equations.

Definition of an Identity

Consider the following expressions.

$$3 + 1 \qquad 2^2 \qquad 6 - 2 \qquad \frac{48}{12} \qquad \frac{1}{2}(8)$$

Each expression is equivalent to 4. Thus for any of these, we can replace the expression with the number 4.

Even though expressions that contain a variable can take on a number of different values, variable expressions can also be equivalent. For example, each of the following variable expressions simplifies to $2x - 6$.

$$2(x - 3) \qquad x + x - 6 \qquad \frac{4x - 12}{2} \qquad -2x - 6 + 4x$$

Since these expressions are equivalent, evaluating them at the same value of x will yield the same result. For example, when $x = 5$ we have

$$\begin{aligned} 2(5 - 3) &= 2(2) \\ &= 4 \end{aligned} \qquad \begin{aligned} 5 + 5 - 6 &= 10 - 6 \\ &= 4 \end{aligned}$$

$$\begin{aligned} \frac{4(5) - 12}{2} &= \frac{20 - 12}{2} \\ &= \frac{8}{2} \\ &= 4 \end{aligned} \qquad \begin{aligned} -2x - 6 + 4x &= -2(5) - 6 + 4(5) \\ &= -10 - 6 + 20 \\ &= 4 \end{aligned}$$

Likewise, evaluating the simplified expression $2x - 6$ at $x = 5$ also yields the same result.

$$\begin{aligned} 2(5) - 6 &= 10 - 6 \\ &= 4 \end{aligned}$$

In general, if two variable expressions are always equivalent, we say they form an **identity**.

Identity

Two variable expressions that are always equivalent are said to form a mathematical **identity**.

EXAMPLE 1 ■ **Determining If an Identity Exists**

Determine if each of the following equations is an identity.

a. $(x - 6)(x + 2) = x^2 - 4x - 12$

b. $4(x - 2) = 2(2x + 6)$

Solution To be an identity, the expressions on either side of the equal sign must be equivalent for all values of x. To verify if the equation is an identity, we simplify each expression as much as possible.

a. We have

$$\begin{aligned}(x - 6)(x + 2) &= x^2 - 4x - 12\\ x^2 + 2x - 6x - 12 &= x^2 - 4x - 12\\ x^2 - 4x - 12 &= x^2 - 4x - 12\end{aligned}$$

Since the simplified expression on the left side of the equal sign is the same as the expression on the right side, the equation $(x - 6)(x + 2) = x^2 - 4x - 12$ is an identity.

b. We have

$$\begin{aligned}4(x - 2) &= 2(2x + 6)\\ 4x - 8 &= 4x + 12\end{aligned}$$

Multiplying a number by 4 and then subtracting 8 is not the same as multiplying a number by 4 and adding 12. Therefore, $4(x - 2) = 2(2x + 6)$ is *not* an identity.

■ Fundamental Trigonometric Identities

In Chapters 8 and 9 we worked with some identities in trigonometry. For example, the definitions of tangent, cotangent, secant, and cosecant are all identities.

Fundamental Trigonometric Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

Trigonometric identities often make it easier to evaluate a trigonometric expression or solve a trigonometric equation.

EXAMPLE 2 ■ **Using Identities to Solve a Trigonometric Equation**

Solve $x = \tan(\theta)\csc(\theta)\cos(\theta)$ for x and write the result in simplified form.

Solution Since x is already isolated, we only need to write the result in simplified form. To do this, we rewrite $\tan(\theta)$ and $\csc(\theta)$ using the fundamental trigonometric identities.

$$\begin{aligned} x &= \tan(\theta)\csc(\theta)\cos(\theta) \\ &= \left[\frac{\sin(\theta)}{\cos(\theta)}\right]\left[\frac{1}{\sin(\theta)}\right][\cos(\theta)] \\ &= \frac{\sin(\theta)\cos(\theta)}{\sin(\theta)\cos(\theta)} \\ &= 1 \end{aligned}$$

Thus $x = \tan(\theta)\csc(\theta)\cos(\theta)$ is equivalent to $x = 1$.

Trigonometric identities allow us to rewrite trigonometric expressions in a simpler form to use when working with expressions or equations.

EXAMPLE 3 ■ **Using Fundamental Identities**

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When an object is at rest, all of the forces acting on it are in balance. The forces acting on a roof tile resting on a sloped roof are related in the formula $w \cdot \sin(\theta) = \mu_s \cdot w \cdot \cos(\theta)$, where w is the weight of the roof tile, θ is the measure of the roof's angle of incline, and μ_s is the *frictional coefficient* for two surfaces. See Figure 10.1. Use a fundamental trigonometric identity to explain how the frictional coefficient is related to the angle of incline for the roof.

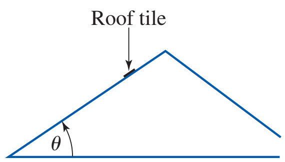


Figure 10.1

Solution We begin by solving the formula $w \cdot \sin(\theta) = \mu_s \cdot w \cdot \cos(\theta)$ for μ_s.

$$\begin{aligned} w \cdot \sin(\theta) &= \mu_s \cdot w \cdot \cos(\theta) \\ \frac{\cancel{w} \cdot \sin(\theta)}{\cancel{w} \cdot \cos(\theta)} &= \frac{\mu_s \cdot \cancel{w \cdot \cos(\theta)}}{\cancel{w \cdot \cos(\theta)}} \\ \frac{\sin(\theta)}{\cos(\theta)} &= \mu_s \end{aligned}$$

We use the fundamental identity $\tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$ to write μ_3 in terms of a single trigonometric function: $\tan(\theta) = \mu_s$.

Thus, since $\tan(\theta)$ always calculates a slope, the frictional coefficient is equivalent to the slope of the roof with an angle of incline measuring θ. If the frictional coefficient is known for two surfaces, such as wood and tile, we can use this relationship to determine the angle at which an object will begin to slide down the slope. We will explore this concept in the exercise set.

■ Even–Odd Function Identities

In Chapter 3, we explained that even functions are symmetric with respect to the vertical axis. That is, their graphs remain unchanged after being reflected horizontally about the vertical axis. Functions with even symmetry have the property that $f(-x) = f(x)$.

As we consider the graph of the cosine function shown in Figure 10.2, we see that reflecting this graph about the vertical axis will result in the same graph. Therefore, $f(\theta) = \cos(\theta)$ is an even function and $\cos(-\theta) = \cos(\theta)$.

Odd functions are symmetric to the origin. This means the function's graph reflected horizontally will be identical to the function's graph reflected vertically. Functions with odd symmetry thus have the property $f(-x) = -f(x)$.

As we consider the graph of the sine function (blue curve) and the negative sine function (red dashed curve) shown in Figure 10.3, we see that reflecting the sine graph about the vertical axis will result in the same negative sine graph. Thus sine is an odd function and $\sin(-\theta) = -\sin(\theta)$.

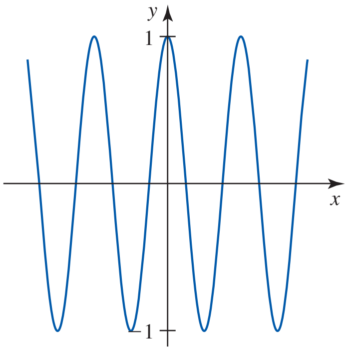


Figure 10.2

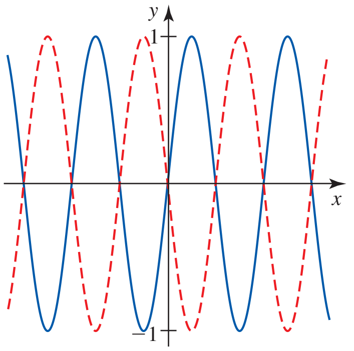


Figure 10.3

We use the fundamental trigonometric identities and our knowledge about the even nature of the cosine function and the odd nature of the sine function to determine which of the four remaining trigonometric functions are even and odd.

$$\begin{aligned}\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)}\\ &= \frac{-\sin(\theta)}{\cos(\theta)} && \text{since } \cos(-\theta) = \cos(\theta) \text{ and } \sin(-\theta) = -\sin(\theta)\\ &= -\tan(\theta)\end{aligned}$$

Since $\tan(-\theta) = -\tan(\theta)$, tangent is an odd function. It follows that the cotangent function is odd since

$$\begin{aligned}\cot(-\theta) &= \frac{1}{\tan(-\theta)}\\ &= \frac{1}{-\tan\theta}\\ &= -\cot(\theta)\end{aligned}$$

Similarly, $\sec(\theta)$ is even and $\csc(\theta)$ is odd because

$$\begin{aligned}\sec(-\theta) &= \frac{1}{\cos(-\theta)} & \csc(-\theta) &= \frac{1}{\sin(-\theta)}\\ &= \frac{1}{\cos(\theta)} & &= \frac{1}{-\sin(\theta)}\\ &= \sec(\theta) & &= -\csc(\theta)\end{aligned}$$

We summarize these **even–odd identities** as follows.

Even–Odd Identities

Cosine and secant functions are **even** functions, giving us the following identities.

$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$

Sine, cosecant, tangent, and cotangent are **odd** functions, giving us the following identities.

$$\sin(-\theta) = -\sin(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$
$$\csc(-\theta) = -\csc(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

EXAMPLE 4 ■ Using Even–Odd Identities to Evaluate Expressions

Evaluate each of the following expressions using even–odd identities.

a. $\cos\left(-\frac{\pi}{4}\right)$

b. $\csc\left(-\frac{7\pi}{3}\right)$

Solution

a. $\cos\left(-\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{4}\right)$

$$= \frac{\sqrt{2}}{2}$$

b. $\csc\left(-\frac{7\pi}{3}\right) = -\csc\left(\frac{7\pi}{3}\right)$

A radian measure of $\frac{7\pi}{3}$ indicates more than one revolution around the circle. Thus, we subtract 2π to find its corresponding measure between 0 and 2π.

$$\frac{7\pi}{3} - 2\pi = \frac{7\pi}{3} - \frac{6\pi}{3}$$
$$= \frac{\pi}{3}$$

We now use $\frac{\pi}{3}$ in place of $\frac{7\pi}{3}$ in the equation.

$$-\csc\left(\frac{7\pi}{3}\right) = -\csc\left(\frac{\pi}{3}\right)$$
$$= -\frac{2\sqrt{3}}{3}$$

EXAMPLE 5 ■ Using Even–Odd Identities to Solve Equations

Find all solutions to each of the following equations over the interval $[0, 2\pi]$.

a. $\tan(-\theta)\csc(\theta) = 1.4$

b. $\frac{3\sin(-\theta)}{\sin(\theta)} = 5\cot(\theta)$

Solution

a.

$$\tan(-\theta)\csc(\theta) = 1.4$$

$$-\tan(\theta)\csc(\theta) = 1.4 \qquad \tan(-\theta) = -\tan(\theta)$$

$$-\frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\sin(\theta)} = 1.4 \qquad \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \text{ and } \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$-\frac{\sin(\theta)}{\cos(\theta)\sin(\theta)} = 1.4$$

$$-\frac{\cancel{\sin(\theta)}}{\cos(\theta)\cancel{\sin(\theta)}} = 1.4$$

$$-\frac{1}{\cos(\theta)} = 1.4$$

$$-1 = 1.4\cos(\theta)$$

$$-\frac{1}{1.4} = \cos(\theta)$$

$$\theta = \cos^{-1}\left(-\frac{1}{1.4}\right) \qquad \text{Write in inverse form.}$$

$$\theta \approx 2.3664 \qquad \text{One solution is between } \frac{\pi}{2} \text{ and } \pi.$$

$$\theta \approx 3.9168 \qquad \text{One solution is between } \pi \text{ and } \frac{3\pi}{2} \text{ (use reference angles).}$$

b.

$$\frac{3\sin(-\theta)}{\sin(\theta)} = 5\cot(\theta)$$

$$\frac{-3\sin(\theta)}{\sin(\theta)} = 5\cot(\theta) \qquad \sin(-\theta) = -\sin(\theta)$$

$$\frac{-3\cancel{\sin(\theta)}}{\cancel{\sin(\theta)}} = 5\cot(\theta)$$

$$-3 = 5\cot(\theta)$$

$$-\frac{3}{5} = \cot(\theta)$$

$$-\frac{3}{5} = \frac{1}{\tan(\theta)} \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$-\frac{5}{3} = \tan(\theta) \qquad \text{Use the reciprocal of each side.}$$

$$\theta = \tan^{-1}\left(-\frac{5}{3}\right) \qquad \text{Write in the inverse form.}$$

$$\theta \approx -1.0304 \qquad \text{Evaluate with a calculator.}$$

$$\theta \approx -1.0304 + \pi \approx 2.1112 \qquad \text{One solution is between } \frac{\pi}{2} \text{ and } \pi.$$

$$\theta \approx -1.0304 + 2\pi \approx 5.2528 \qquad \text{One solution is between } \frac{3\pi}{2} \text{ and } 2\pi.$$

Cofunction Identities

Cofunction identities show relationships between pairs of functions, such as the cosine and sine functions and the tangent and cotangent functions. We develop the cofunction identities for cosine and sine here; the remaining four are left to the exercises.

First, we consider a right triangle with acute angles measuring θ and ϕ (read "fee" or "fy"), shown in Figure 10.4. Using the definitions of cosine and sine in right triangle trigonometry, we find the following relationships.

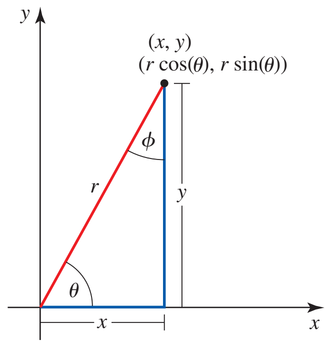


Figure 10.4

$$\cos(\theta) = \frac{x}{r} \qquad \sin(\phi) = \frac{x}{r}$$

$$\cos(\phi) = \frac{y}{r} \qquad \sin(\theta) = \frac{y}{r}$$

This tells us the cosine and sine values of the two acute angles are related as follows.

$$\cos(\theta) = \sin(\phi) \qquad \text{since } \cos(\theta) = \frac{x}{r} \text{ and } \sin(\phi) = \frac{x}{r}$$

$$\sin(\theta) = \cos(\phi) \qquad \text{since } \cos(\phi) = \frac{y}{r} \text{ and } \sin(\theta) = \frac{y}{r}$$

Since the two acute angles in a right triangle add up to 90°, we can rewrite ϕ in terms of θ:

$$\begin{aligned} \theta + \phi &= 90^\circ \\ \phi &= 90^\circ - \theta \end{aligned}$$

Thus

$$\begin{aligned} \cos(\theta) &= \sin(\phi) \\ &= \sin(90^\circ - \theta) \end{aligned} \qquad \begin{aligned} \sin(\theta) &= \cos(\phi) \\ &= \cos(90^\circ - \theta) \end{aligned}$$

The six **cofunction identities** are summarized in the following box.

Cofunction Identities

θ in degrees	**θ in radians**
$\cos(\theta) = \sin(90^\circ - \theta)$	$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$
$\sin(\theta) = \cos(90^\circ - \theta)$	$\sin(\theta) = \cos\left(\frac{\pi}{2} - \theta\right)$
$\tan(\theta) = \cot(90^\circ - \theta)$	$\tan(\theta) = \cot\left(\frac{\pi}{2} - \theta\right)$
$\sec(\theta) = \csc(90^\circ - \theta)$	$\sec(\theta) = \csc\left(\frac{\pi}{2} - \theta\right)$
$\csc(\theta) = \sec(90^\circ - \theta)$	$\csc(\theta) = \sec\left(\frac{\pi}{2} - \theta\right)$
$\cot(\theta) = \tan(90^\circ - \theta)$	$\cot(\theta) = \tan\left(\frac{\pi}{2} - \theta\right)$

EXAMPLE 6 ■ Using Cofunction Identities to Solve an Equation

Find all solutions to the equation $\cos\left(\frac{\pi}{2} - \theta\right) = 0.8$ over the interval $[0, 2\pi]$ using a cofunction identity.

Solution We substitute $\sin(\theta)$ for $\cos\left(\frac{\pi}{2} - \theta\right)$ to simplify the problem.

$$\cos\left(\frac{\pi}{2} - \theta\right) = 0.8$$

$$\sin(\theta) = 0.8 \qquad \cos\left(\frac{\pi}{2} - \theta\right) = \sin(\theta)$$

$$\theta = \sin^{-1}(0.8) \qquad \text{Write in inverse form.}$$

$$\theta \approx 0.9273 \qquad \text{Evaluate with a calculator.}$$

We use reference angles with sine to find the second solution in Quadrant II.

$$\begin{aligned} \theta &\approx \pi - 0.9273 \\ &\approx 2.2143 \end{aligned}$$

Therefore, $\theta \approx 0.9273$ and $\theta \approx 2.2143$ are solutions to $\cos\left(\frac{\pi}{2} - \theta\right) = 0.8$.

EXAMPLE 7 ■ Using Cofunction Identities to Simplify an Expression

Use cofunction identities to write the expression $\cot\left(\frac{\pi}{2} - \theta\right)\sec\left(\frac{\pi}{2} - \theta\right)$ as a single trigonometric function.

Solution

$$\begin{aligned} \cot\left(\frac{\pi}{2} - \theta\right)\sec\left(\frac{\pi}{2} - \theta\right) &= \tan(\theta)\csc(\theta) && \text{since } \cot\left(\frac{\pi}{2} - \theta\right) = \tan(\theta) \text{ and } \sec\left(\frac{\pi}{2} - \theta\right) = \csc(\theta) \\ &= \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{1}{\sin(\theta)} && \text{since } \tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \text{ and } \csc(\theta) = \frac{1}{\sin(\theta)} \\ &= \frac{\cancel{\sin(\theta)}}{\cos(\theta)\cancel{\sin(\theta)}} \\ &= \frac{1}{\cos(\theta)} \\ &= \sec(\theta) \end{aligned}$$

JUST IN TIME ■ EXPONENTS AND TRIGONOMETRIC FUNCTIONS

When a trigonometric function is raised to an exponent, such as $[\cos(\theta)]^2$ and $[\tan(\theta)]^4$, we typically use the notation $\cos^2(\theta)$ and $\tan^4(\theta)$ to avoid confusion with $\cos(\theta)^2$ and $\tan(\theta)^4$, which are equivalent to $\cos(\theta^2)$ and $\tan(\theta^4)$.

Pythagorean Identities

We know that the (x, y) coordinates of the endpoint of an arc on the unit circle are given by $(\cos(\theta), \sin(\theta))$. If we draw a right triangle using the radius of the unit circle as the hypotenuse as shown in Figure 10.5a, we see that the lengths of the legs are also given by $\cos(\theta)$ and $\sin(\theta)$.

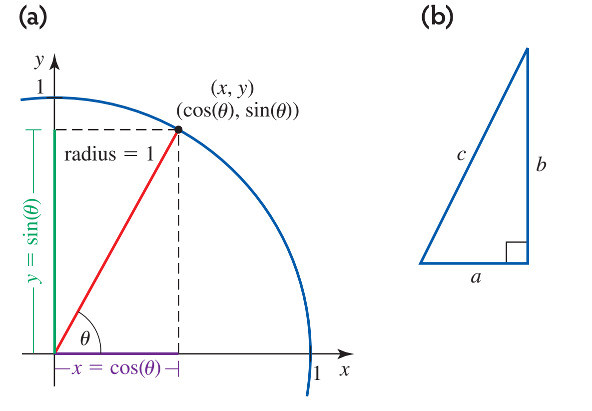


Figure 10.5

Comparing Figures 10.5a and 10.5b and using the Pythagorean theorem, we see the following relationship between cosine and sine.

$$a^2 + b^2 = c^2 \qquad \textbf{Pythagorean theorem}$$

$$\cos^2(\theta) + \sin^2(\theta) = 1 \qquad \boldsymbol{a = \cos(\theta), b = \sin(\theta), \text{ and } c = 1}$$

The equation $\cos^2(\theta) + \sin^2(\theta) = 1$ is called a **Pythagorean identity**.

EXAMPLE 8 ■ Using a Pythagorean Identity to Solve an Equation

Find all solutions to the equation $\cos^2(\theta) + \cos^2(\theta)\tan^2(\theta) = 2.6\cos(\theta)$ over the interval $[0, 2\pi]$.

Solution One problem-solving technique is to write all trigonometric functions in terms of cosine and sine. This will often simplify the problem.

$$\cos^2(\theta) + \cos^2(\theta)\tan^2(\theta) = 2.6\cos(\theta)$$

$$\cos^2(\theta) + \frac{\cos^2(\theta)}{1} \cdot \frac{\sin^2(\theta)}{\cos^2(\theta)} = 2.6\cos(\theta) \qquad \textbf{since } \tan^2(\theta) = \frac{\sin^2(\theta)}{\cos^2(\theta)}$$

$$\cos^2(\theta) + \frac{\cancel{\cos^2(\theta)}\sin^2(\theta)}{\cancel{\cos^2(\theta)}} = 2.6\cos(\theta)$$

$$\cos^2(\theta) + \sin^2(\theta) = 2.6\cos(\theta)$$

$$1 = 2.6\cos(\theta) \qquad \textbf{since } \cos^2(\theta) + \sin^2(\theta) = 1$$

$$0.3846 \approx \cos(\theta)$$

$$\theta \approx \cos^{-1}(0.3846)$$

$$\theta \approx 1.1760 \qquad \textbf{Evaluate using a calculator.}$$

$$\theta \approx 2\pi - 1.1760 \qquad \textbf{Use reference angles.}$$

$$\approx 5.1072$$

Using the basic Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$, we establish the other two Pythagorean identities. First, we see what happens when we divide the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\cos^2(\theta)$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \qquad \text{Divide each side by } \cos^2(\theta).$$

$$\frac{\cancel{\cos^2(\theta)}}{\cancel{\cos^2(\theta)}} + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)} \qquad \text{Separate the fraction.}$$

$$1 + \tan^2(\theta) = \sec^2(\theta) \qquad \frac{\sin^2(\theta)}{\cos^2(\theta)} = \tan^2(\theta) \text{ and } \frac{1}{\cos^2(\theta)} = \sec^2(\theta)$$

This gives us the identity $1 + \tan^2(\theta) = \sec^2(\theta)$. This identity is true for all values of θ except those that make $\cos(\theta) = 0$, in which case both sides of the identity are undefined.

The third Pythagorean identity occurs when we divide the Pythagorean identity $\cos^2(\theta) + \sin^2(\theta) = 1$ by $\sin^2(\theta)$.

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$\frac{\cos^2(\theta) + \sin^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)} \qquad \text{Divide each side by } \sin^2(\theta).$$

$$\frac{\cos^2(\theta)}{\sin^2(\theta)} + \frac{\cancel{\sin^2(\theta)}}{\cancel{\sin^2(\theta)}} = \frac{1}{\sin^2(\theta)} \qquad \text{Separate the fraction.}$$

$$\cot^2(\theta) + 1 = \csc^2(\theta) \qquad \frac{\cos^2(\theta)}{\sin^2(\theta)} = \cot^2(\theta) \text{ and } \frac{1}{\sin^2(\theta)} = \csc^2(\theta)$$

The identity $\cot^2(\theta) + 1 = \csc^2(\theta)$ will be true for all values of θ except those that make $\sin(\theta) = 0$, in which case both sides of the identity are undefined.

We summarize the **Pythagorean identities** in the following box.

PYTHAGOREAN IDENTITIES

$$\cos^2(\theta) + \sin^2(\theta) = 1 \qquad 1 + \tan^2(\theta) = \sec^2(\theta) \qquad \cot^2(\theta) + 1 = \csc^2(\theta)$$

These identities can appear in various forms. For example, $\cos^2(\theta) + \sin^2(\theta) = 1$ may be written as $\sin^2(\theta) = 1 - \cos^2(\theta)$ or $\cos^2(\theta) = 1 - \sin^2(\theta)$, while $\cot^2(\theta) + 1 = \csc^2(\theta)$ may be written as $\cot^2(\theta) = \csc^2(\theta) - 1$ or $1 - \csc^2(\theta) = -\cot^2(\theta)$. Each of these alternative forms is valid.

EXAMPLE 9 ■ Using Pythagorean Identities to Simplify Expressions

Use trigonometric identities to write each of the following as a single trigonometric function or as a whole number.

a. $[1 - \cos^2(\theta)][\cot^2(\theta)]$

b. $[\sec(\theta) + \tan(\theta)][\sec(\theta) - \tan(\theta)]$

Solution

a. We notice $1 - \cos^2(\theta)$ is present in this expression. We also note that solving the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ for $\sin^2(\theta)$ yields $\sin^2(\theta) = 1 - \cos^2(\theta)$. Therefore,

$$[1 - \cos^2(\theta)][\cot^2(\theta)] = \sin^2(\theta)\cot^2(\theta) \qquad \text{since } \sin^2(\theta) = 1 - \cos^2(\theta)$$

$$= \frac{\sin^2(\theta)}{1} \cdot \frac{\cos^2(\theta)}{\sin^2(\theta)} \qquad \text{since } \cot^2(\theta) = \frac{\cos^2(\theta)}{\sin^2(\theta)}$$

$$= \frac{\cancel{\sin^2(\theta)}\cos^2(\theta)}{\cancel{\sin^2(\theta)}}$$

$$= \cos^2(\theta)$$

b.

$$[\sec(\theta) + \tan(\theta)][\sec(\theta) - \tan(\theta)]$$

$$\sec(\theta) \cdot \sec(\theta) - \sec(\theta) \cdot \tan(\theta) + \tan(\theta) \cdot \sec(\theta) - \tan(\theta) \cdot \tan(\theta)$$

$$\sec^2(\theta) - \cancel{\sec(\theta)\tan(\theta)} + \cancel{\sec(\theta)\tan(\theta)} - \tan^2(\theta)$$

$$\sec^2(\theta) - \tan^2(\theta)$$

$$1 \qquad \sec^2(\theta) - \tan^2(\theta) = 1 \text{ since } 1 + \tan^2(\theta) = \sec^2(\theta)$$

So $[\sec(\theta) + \tan(\theta)][\sec(\theta) - \tan(\theta)] = 1$.

SUMMARY

In this section you learned the definition of a mathematical identity. You also learned the fundamental trigonometric identities, cofunction identities, even–odd identities, and Pythagorean identities and saw how to use them to solve trigonometric equations or simplify trigonometric expressions.

10.1 EXERCISES

SKILLS AND CONCEPTS

1. Make a list of all of the trigonometric identities covered in this section. This will be a helpful reference throughout the chapter that you can refer to easily while working on other exercises. You may also want to include alternative versions of the Pythagorean identities. For example, the identity $\cos^2(\theta) + \sin^2(\theta) = 1$ can be written as $\sin^2(\theta) = 1 - \cos^2(\theta)$, $\cos^2(\theta) = 1 - \sin^2(\theta)$, $\sin^2(\theta) - 1 = -\cos^2(\theta)$, and $\cos^2(\theta) - 1 = -\sin^2(\theta)$.

In Exercises 2–7, use fundamental trigonometric identities to simplify each expression by writing it as a single trigonometric function or as a whole number.

2. $\tan(\theta)\cos(\theta)$
3. $\tan(x)\csc(x)\sec(x)$
4. $\dfrac{\sec(x)}{\tan(x)}$
5. $\dfrac{\sec(\beta)}{\csc(\beta)}$
6. $\dfrac{\cos(\theta)}{\tan(\theta)} \cdot \sin(\theta)$
7. $\dfrac{\sin(\theta)\sec(\theta)}{\tan(\theta)}$

In Exercises 8–11, use fundamental trigonometric identities to solve each equation. Find all solutions over the interval [0, 2π]. Verify your solutions by graphing on a graphing calculator.

8. $\cot(\theta)\sin(\theta) = 0.75$
9. $\csc(x)\sin^2(x) = 0.19$
10. $-2\sec(\alpha) = \csc(\alpha)$
11. $\tan^2(\theta) = 0.33\sec^2(\theta)$

In Exercises 12–17, use even–odd identities to simplify each expression by writing it in terms of a single trigonometric function with a positive angle or as a whole number.

12. $\cos(-\theta)$
13. $\dfrac{\cos(-\theta)}{\sin(-\theta)}$
14. $\tan(-\beta)\cos(-\beta)$
15. $\cot(\theta)\csc(-\theta)\sec(-\theta)$
16. $\csc^2(-x)$
17. $\dfrac{\sin^2(-\phi)}{\tan^2(\phi)}$

In Exercises 18–21, use even–odd identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$. Verify your solutions by graphing on a graphing calculator.

18. $\sin(-\theta) = 0.73$

19. $-3.95 = \sec(-x) - 2$

20. $-3\left(\dfrac{\sin(-\theta)}{\cos(-\theta)}\right) = 4.12$

21. $-9.1 = \dfrac{1}{\sin(-\phi)} - 3.5$

In Exercises 22–27, use cofunction identities to simplify each expression by writing it in terms of a single trigonometric function with an angle measuring θ or as a whole number.

22. $\sin\left(\dfrac{\pi}{2} - \theta\right)$

23. $\csc\left(\dfrac{\pi}{2} - z\right)$

24. $\cos(90° - \theta)\sec(\theta)$

25. $\sin\left(\dfrac{\pi}{2} - \theta\right)\tan(\theta)$

26. $\dfrac{\sin\left(\dfrac{\pi}{2} - \theta\right)}{\cos\left(\dfrac{\pi}{2} - \theta\right)}$

27. $\dfrac{\tan\left(\dfrac{\pi}{2} - m\right)}{\sin\left(\dfrac{\pi}{2} - m\right)}$

In Exercises 28–31, use cofunction identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$. Verify your solutions by graphing on a graphing calculator.

28. $\cos\left(\dfrac{\pi}{2} - \theta\right) = -0.61$

29. $4 = \csc\left(\dfrac{\pi}{2} - \theta\right) + 1$

30. $-1 = \cos(\theta)\cot\left(\dfrac{\pi}{2} - \theta\right)$

31. $-5\cot\left(\dfrac{\pi}{2} - x\right)\sec\left(\dfrac{\pi}{2} - x\right) - 3.5 = -10$

In Exercises 32–35, evaluate each pair of expressions. (Hint: Refer to the Just In Time box in this section.)

32. $\cos^2\left(\dfrac{\pi}{5}\right)$ and $\cos\left(\dfrac{\pi}{5}\right)^2$

33. $\sin\left(\dfrac{11\pi}{7}\right)^2$ and $\sin^2\left(\dfrac{11\pi}{7}\right)$

34. $\tan^3\left(\dfrac{\pi}{3}\right)$ and $\tan\left(\dfrac{\pi}{3}\right)^3$

35. $\cot^4\left(\dfrac{2\pi}{3}\right)$ and $\cot\left(\dfrac{2\pi}{3}\right)^4$

In Exercises 36–41, use Pythagorean identities to simplify each expression.

36. $\sec^2(\theta) - \tan^2(\theta)$

37. $\sec^2(\alpha)[1 - \sin^2(\alpha)]$

38. $\cos^2(\theta) + \tan^2(\theta)\cos^2(\theta)$

39. $\dfrac{\sec^2(\phi) - 1}{\sin^2(\phi)}$

40. $[\csc(\theta) + \cot(\theta)][\csc(\theta) - \cot(\theta)]$

41. $[\sec(\theta) + 1][\sec(\theta) - 1]$

In Exercises 42–45, use Pythagorean identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$. Verify your solutions by graphing on a graphing calculator.

42. $[1 - \sin^2(x)][\sec(x)] = -0.1$

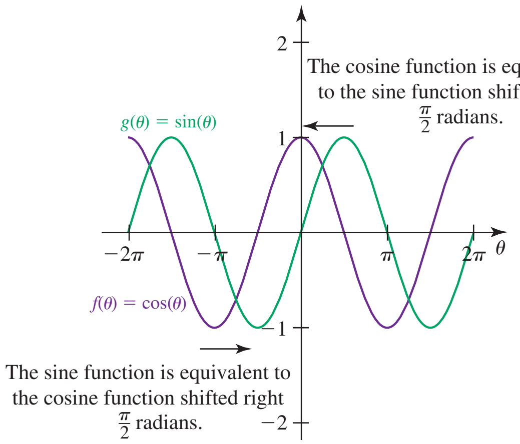
43. $\dfrac{\cos(\theta)}{1 - \sin^2(\theta)} + 2 = -1$

44. $\sin(w) = 2.7[1 - \cos^2(w)]$

45. $1 - \sin^2(\theta)\cot^2(\theta) = 0.8$

In Exercises 46–48, complete the indicated tasks and answer any given questions.

46. The figure shows the graphs of cosine and sine.



Use the information in the figure to write two cofunction identities that relate the cosine and sine functions.

47. Use graphs and/or diagrams to verify that the secant function has even symmetry while the cosecant and cotangent functions have odd symmetry.

48. In calculus, Pythagorean identities are very useful for rewriting certain expressions to make calculations easier. For example, consider the expressions $\sqrt{1 + x^2}$, $\sqrt{1 - x^2}$, and $\sqrt{x^2 - 1}$. A *trigonometric substitution* is when the x-value is replaced by $\sin(\theta)$, $\tan(\theta)$, or $\sec(\theta)$, and the expression is then simplified.

a. Which of the three trigonometric values should be substituted for x in each of the three expressions $\sqrt{1 + x^2}$, $\sqrt{1 - x^2}$, and $\sqrt{x^2 - 1}$ to simplify them? Make the substitutions, then simplify each expression.

b. What would you substitute for x in the expression $\sqrt{16 - x^2}$? Perform the substitution and simplify the expression.

In Exercises 49–52, solve each trigonometric equation by factoring. Find all solutions over the interval $[0, 2\pi]$. (Hint: Start by performing a substitution, such as letting $x = \cos(\theta)$ or $x = \sin(\theta)$.) Verify your solutions by graphing on a graphing calculator.

49. $\cos^2(\theta) + 4\cos(\theta) - 5 = 0$

50. $2\sin^2(\theta) - \sin(\theta) = 1$

51. $3\sin^2(\theta) - 10\sin(\theta) = 6\sin(\theta) - 12$

52. $3\sin^2(\theta) - 2\cos(\theta) = 3\cos(\theta) + 1$

SHOW YOU KNOW

53. What are the differences between an expression, an equation, and an identity?

54. How do you determine if an equation is an identity?

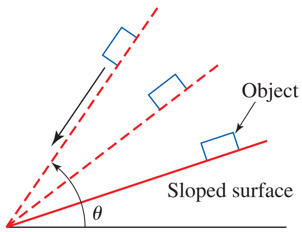
55. Why are trigonometric identities useful in solving equations?

56. Explain why the cofunction identities $\tan\left(\frac{\pi}{2}-\theta\right)=\cot(\theta)$ and $\cot\left(\frac{\pi}{2}-\theta\right)=\tan(\theta)$ are true. Use graphs and/or diagrams in your explanation.

57. Explain why the cofunction identities $\sec\left(\frac{\pi}{2}-\theta\right)=\csc(\theta)$ and $\csc\left(\frac{\pi}{2}-\theta\right)=\sec(\theta)$ are true. Use graphs and/or diagrams in your explanation.

MAKE IT REAL

In Exercises 58–61, refer to Example 3.



58. Friction A brick resting on wood has a frictional coefficient of 0.6. What is the minimum angle measure at which the brick will slide down a wooden surface? **(*Source:* www.roymech.co.uk)**

59. Friction A copper object resting on a steel surface has a frictional coefficient of 0.53. What is the minimum angle measure at which the object will slide down a steel surface? **(*Source:* frictioncenter.siu.edu)**

60. Friction A rubber object resting on concrete has a frictional coefficient that varies from 1.0 to 4.0 depending on conditions such as moisture. **(*Source:* frictioncenter.siu.edu)**

a. Find the range of angles for which a rubber object would begin to slide down a concrete slope.

b. Based on your results to part (a), explain at least one reason that rubber is used for the soles of running shoes.

61. Friction Following are the frictional coefficients for a number of materials and surfaces. Which of these objects would remain stationary on a 20° incline? **(*Source:* www.roymech.co.uk)**

A. A Teflon-coated object on a steel surface: $\mu_s = 0.04$

B. A leather object resting on an oak surface: $\mu_s = 0.61$

C. A Plexiglas object resting on a steel surface: $\mu_s = 0.4$

D. A steel object on a polystyrene surface: $\mu_s = 0.35$

STRETCH YOUR MIND

Exercises 62–66 are intended to challenge your understanding of simplifying trigonometric expressions and solving trigonometric equations.

62. Show that $\dfrac{1}{\sin(\theta)+1}+\dfrac{1}{\csc(\theta)+1}$ simplifies to a whole number.

63. Rewrite the expression $\dfrac{\sin(\pi-\theta)}{\cos(\theta-\pi)}$ using a single trigonometric function.

64. Rewrite the expression $\dfrac{1-2\sin^2(\theta)}{\sin(\theta)\cos(\theta)}+\tan(\theta)$ using a single trigonometric function.

65. Find all solutions to the equation $\cos^4(\theta)+2\cos^2(\theta)\sin^2(\theta)+\sin^4(\theta)=1$ over the interval $[-2\pi, 2\pi]$.

66. Find all solutions to the equation $\dfrac{1+\sin(\theta)}{\cos(\theta)}+\dfrac{\cos(\theta)}{1+\sin(\theta)}=-4$ over the interval $[-4\pi, 0]$.

SECTION 10.2 Verifying Identities

LEARNING OBJECTIVES

- Explain how trigonometric identities are determined
- Use algebraic manipulation to verify the equivalency of two identities
- Use trigonometric identities to describe equivalent relationships

GETTING STARTED

In Chapter 8 we defined cosine and sine as "lengths" associated with the unit circle. In Chapter 9 we discovered how cosine and sine are related to right triangles. We used this knowledge to solve for missing angles and side lengths in triangles. In addition, we learned about four other trigonometric functions: tangent, secant, cosecant, and cotangent.

In this section we use the identities introduced in Section 10.1 to determine if two trigonometric expressions are equivalent. We also see that all trigonometric functions may be represented as lengths associated with a unit circle, as shown in Figure 10.6, and that these relationships can be proved using trigonometric identities.

The Six Trigonometric Functions

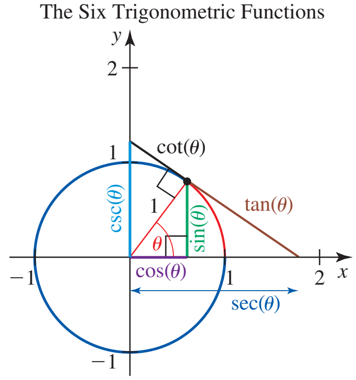


Figure 10.6

Verifying Identities

The process of verifying the equivalence of two expressions is different from the process of solving equations. When solving equations, we move quantities back and forth across the equal sign using mathematical operations. When verifying identities, we seek to make one side of the equation look like the other side of the equation, but we may not move any quantities across the equal sign because we do not *know* the two sides are equivalent—this is what we want to show. Instead, we rewrite the expression on one side of the equal sign using known identities until we succeed in making it look like the other side of the equal sign. This process is difficult for many people because a certain path to the solution is often unclear. The key to success is to recognize that if a chosen strategy begins to make an expression more complex instead of simpler, a restart using a different strategy may be needed. In fact, it is common to restart the simplification process multiple times in a single problem. This should not be viewed as a failure. It is part of the learning process. With practice, anyone can become skilled in verifying identities.

EXAMPLE 1 ■ Verifying Identities

Verify each identity.

a. $\cot(\theta)\tan(\theta) = 1$

b. $\csc(\theta)\tan(\theta) = \cos(\theta)\sec^2(\theta)$

Solution

a. To verify an identity means to show that the expressions on either side of the equal sign are equivalent. Thus, we want to show that, for *any* input for which the expressions are defined, the equation will always be true.

$$\cot(\theta)\tan(\theta) \stackrel{?}{=} 1$$

$$\frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)} \stackrel{?}{=} 1 \qquad \text{Rewrite using cosine and sine.}$$

$$\frac{\cancel{\cos(\theta) \cdot \sin(\theta)}}{\cancel{\cos(\theta) \cdot \sin(\theta)}} \stackrel{?}{=} 1$$

$$1 = 1$$

The expression $\cot(\theta)\tan(\theta)$ will always be 1 for any angle where it is defined. Thus $\cot(\theta)\tan(\theta) = 1$ is an identity.

b. We use identities from Section 10.1 to simplify the right side of $\csc(\theta)\tan(\theta) = \cos(\theta)\sec^2(\theta)$.

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \cos(\theta)\sec^2(\theta)$$

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \cos(\theta) \cdot \frac{1}{\cos^2(\theta)} \qquad \text{since } \sec^2(\theta) = \frac{1}{\cos^2(\theta)}$$

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \frac{\cos(\theta)}{\cos^2(\theta)}$$

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \frac{\cancel{\cos(\theta)}}{\cos(\theta)\cancel{\cos(\theta)}} \qquad \text{since } \cos^2(\theta) = \cos(\theta)\cos(\theta)$$

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \frac{1}{\cos(\theta)}$$

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \sec(\theta) \qquad \text{since } \sec(\theta) = \frac{1}{\cos(\theta)}$$

We have simplified the right side of the potential identity by writing it in terms of a single trigonometric function, but we have yet to show that the two sides are equivalent. We now simplify the left side.

$$\csc(\theta)\tan(\theta) \stackrel{?}{=} \sec(\theta)$$

$$\frac{1}{\sin(\theta)} \cdot \frac{\sin(\theta)}{\cos(\theta)} \stackrel{?}{=} \sec(\theta) \quad \text{Use fundamental trigonometric identities.}$$

$$\frac{\cancel{\sin(\theta)}}{\cancel{\sin(\theta)} \cdot \cos(\theta)} \stackrel{?}{=} \sec(\theta)$$

$$\frac{1}{\cos(\theta)} \stackrel{?}{=} \sec(\theta)$$

$$\sec(\theta) = \sec(\theta) \quad \text{since } \sec(\theta) = \frac{1}{\cos(\theta)}$$

Thus $\csc(\theta)\tan(\theta) = \cos(\theta)\sec^2(\theta)$ is an identity.

Notice we did not move any quantities across the equal sign as we verified the identity. Instead, we simplified each side as far as possible. There are no definitive steps to take each time we are verifying an identity. However, some general guidelines will help us be efficient.

HOW TO: ■ VERIFY TRIGONOMETRIC IDENTITIES

To verify trigonometric identities,

1. Simplify one side at a time. It is often best to work with the side that appears more complex. Always stay on one side of the equation at a time.
2. Combine fractions on the same side using addition, subtraction, multiplication, or division as necessary.
3. If any of the trigonometric functions are being squared, you may be able to use a Pythagorean identity to simplify the expression. If there are no exponents, try writing all trigonometric functions in terms of cosine and sine.
4. Keep your goal in mind. Sometimes looking at the less complex side of the potential identity will give you a clue about what identities will help you get there.
5. If one approach does not work, try another. Often, it only takes one substitution or one key step to unlock the entire identity.

EXAMPLE 2 ■ Verifying an Identity

Verify the identity $\sec(\theta) - \tan(\theta)\sin(\theta) = \cos(\theta)$.

Solution We begin by rewriting the trigonometric functions on the left side using cosine and sine.

$$\sec(\theta) - \tan(\theta)\sin(\theta) \stackrel{?}{=} \cos(\theta)$$

$$\frac{1}{\cos(\theta)} - \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{\sin(\theta)}{1} \stackrel{?}{=} \cos(\theta) \qquad \textbf{Rewrite using cosine and sine.}$$

$$\frac{1}{\cos(\theta)} - \frac{\sin^2(\theta)}{\cos(\theta)} \stackrel{?}{=} \cos(\theta) \qquad \textbf{Multiply } \frac{\sin(\theta)}{\cos(\theta)} \cdot \frac{\sin(\theta)}{1}.$$

$$\frac{1 - \sin^2(\theta)}{\cos(\theta)} \stackrel{?}{=} \cos(\theta) \qquad \textbf{Combine fractions.}$$

$$\frac{\cos^2(\theta)}{\cos(\theta)} \stackrel{?}{=} \cos(\theta) \qquad \cos^2(\theta) = 1 - \sin^2(\theta), \text{ since } \cos^2(\theta) + \sin^2(\theta) = 1$$

$$\frac{\cancel{\cos(\theta)}\cos(\theta)}{\cancel{\cos(\theta)}} \stackrel{?}{=} \cos(\theta) \qquad \text{since } \cos^2(\theta) = \cos(\theta)\cos(\theta)$$

$$\cos(\theta) = \cos(\theta)$$

We may also use graphing to help verify possible identities. However, as previously mentioned, graphs can sometimes be misleading. The only way to verify identities without any doubt is to use algebra.

EXAMPLE 3 ■ Verifying an Identity

Use a graph to determine if it is possible for each of the following to be an identity. If it is possible, verify the identity. If it is not, provide a counterexample to show the statement is not true.

a. $\dfrac{\sec(-\theta)}{\tan(-\theta)} = -\csc(\theta)$

b. $\cot^2(\theta)\cos^2(\theta) = \sin^2(\theta)$

Solution

a. We begin by graphing $y = \dfrac{\sec(-\theta)}{\tan(-\theta)}$ and $y = -\csc(\theta)$, as shown in Figure 10.7. It appears the two graphs coincide.

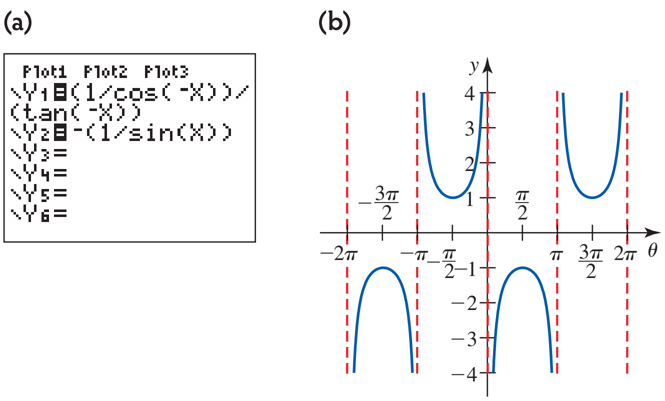


Figure 10.7

Thus, this may be an identity. We verify the identity using algebra, beginning with the more complex left side.

$$\frac{\sec(-\theta)}{\tan(-\theta)} \stackrel{?}{=} -\csc(\theta)$$

$$\frac{\sec(\theta)}{-\tan(\theta)} \stackrel{?}{=} -\csc(\theta) \qquad \textbf{sec}(-\theta) = \textbf{sec}(\theta) \textbf{ and } \textbf{tan}(-\theta) = -\textbf{tan}(\theta)$$

$$\frac{\sec(\theta)}{-\tan(\theta)} \stackrel{?}{=} -\csc(\theta)$$

$$\frac{\frac{1}{\cos(\theta)}}{-\frac{\sin(\theta)}{\cos(\theta)}} \stackrel{?}{=} -\csc(\theta) \qquad \text{Rewrite the left side using cosine and sine.}$$

$$\frac{1}{\cos(\theta)} \cdot -\frac{\cos(\theta)}{\sin(\theta)} \stackrel{?}{=} -\csc(\theta) \qquad \text{To divide, multiply by the reciprocal.}$$

$$-\frac{\cancel{\cos(\theta)}}{\cancel{\cos(\theta)}\sin(\theta)} \stackrel{?}{=} -\csc(\theta)$$

$$-\frac{1}{\sin(\theta)} \stackrel{?}{=} -\csc(\theta)$$

$$-\csc(\theta) = -\csc(\theta)$$

b. We begin by graphing $y = \cot^2(\theta)\cos^2(\theta)$ and $y = \sin^2(\theta)$, as shown in Figure 10.8.

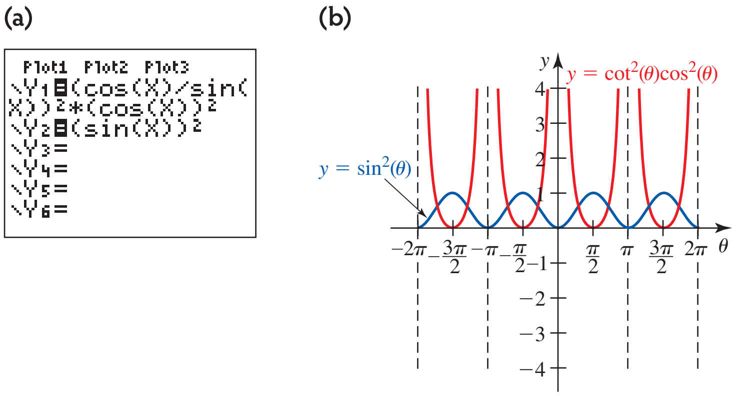


Figure 10.8

It is clear the expressions are not equivalent. We will provide a counterexample to prove the point. Let $\theta = \frac{\pi}{2}$.

$$\cot^2(\theta)\cos^2(\theta) \stackrel{?}{=} \sin^2(\theta)$$

$$\cot^2\left(\frac{\pi}{2}\right)\cos^2\left(\frac{\pi}{2}\right) \stackrel{?}{=} \sin^2\left(\frac{\pi}{2}\right)$$

$$(0)^2(0)^2 \stackrel{?}{=} (1)^2$$

$$0 \neq 1$$

EXAMPLE 4 ■ **Verifying an Identity**

Verify the identity $\cot^2(x) = \csc^2(x) - \cot(x)\tan(x)$.

Solution We begin with the right side since it is more complex. We will use a Pythagorean identity since we have some squared terms.

$$\cot^2(x) \stackrel{?}{=} \csc^2(x) - \cot(x)\tan(x)$$

$$\cot^2(x) \stackrel{?}{=} \csc^2(x) - \frac{\cos(x)}{\sin(x)} \cdot \frac{\sin(x)}{\cos(x)} \qquad \text{Rewrite using cosine and sine.}$$

$$\cot^2(x) \stackrel{?}{=} \csc^2(x) - \frac{\cancel{\cos(x) \cdot \sin(x)}}{\cancel{\cos(x) \cdot \sin(x)}}$$

$$\cot^2(x) \stackrel{?}{=} \csc^2(x) - 1$$

$$\cot^2(x) = \cot^2(x) \qquad \csc^2(x) - 1 = \cot^2(x) \text{ since } \cot^2(x) + 1 = \csc^2(x)$$

EXAMPLE 5 ■ **Verifying an Identity**

Verify the identity $\dfrac{\tan\left(\frac{\pi}{2} - \theta\right)}{\csc(\theta)} = \cos(\theta)$.

Solution We notice that the input of one of the trigonometric functions is $\frac{\pi}{2} - \theta$, indicating that we may use a cofunction identity. In this case, we use the identity $\tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$ to rewrite the expression on the left side.

$$\frac{\tan\left(\frac{\pi}{2} - \theta\right)}{\csc(\theta)} \stackrel{?}{=} \cos(\theta)$$

$$\frac{\cot(\theta)}{\csc(\theta)} \stackrel{?}{=} \cos(\theta) \qquad \tan\left(\frac{\pi}{2} - \theta\right) = \cot(\theta)$$

$$\frac{\frac{\cos(\theta)}{\sin(\theta)}}{\frac{1}{\sin(\theta)}} \stackrel{?}{=} \cos(\theta) \qquad \text{Rewrite in terms of cosine and sine.}$$

$$\frac{\cos(\theta)}{\sin(\theta)} \cdot \frac{\sin(\theta)}{1} \stackrel{?}{=} \cos(\theta) \qquad \text{To divide, multiply by the reciprocal.}$$

$$\frac{\cos(\theta)\cancel{\sin(\theta)}}{\cancel{\sin(\theta)}} \stackrel{?}{=} \cos(\theta)$$

$$\cos(\theta) = \cos(\theta)$$

EXAMPLE 6 ■ **Verifying an Identity**

Verify the identity $\csc(\beta) = \dfrac{\cot(\beta) + 1}{\cos(\beta) + \sin(\beta)}$.

Solution Our first step is to write the numerator in terms of cosine and sine.

$$\csc(\beta) \stackrel{?}{=} \frac{\cot(\beta) + 1}{\cos(\beta) + \sin(\beta)}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\dfrac{\cos(\beta)}{\sin(\beta)} + 1}{\cos(\beta) + \sin(\beta)} \qquad \text{Write in terms of cosine and sine.}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\dfrac{\cos(\beta)}{\sin(\beta)} + 1}{\cos(\beta) + \sin(\beta)}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\dfrac{\cos(\beta)}{\sin(\beta)} + \dfrac{\sin(\beta)}{\sin(\beta)}}{\cos(\beta) + \sin(\beta)} \qquad \text{Get common denominators in the numerator.}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\dfrac{\cos(\beta) + \sin(\beta)}{\sin(\beta)}}{\cos(\beta) + \sin(\beta)} \qquad \text{Combine the fractions in the numerator.}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\cos(\beta) + \sin(\beta)}{\sin(\beta)} \cdot \frac{1}{\cos(\beta) + \sin(\beta)} \qquad \text{To divide, multiply by the reciprocal.}$$

$$\csc(\beta) \stackrel{?}{=} \frac{\cancel{\cos(\beta) + \sin(\beta)}}{\sin(\beta)(\cancel{\cos(\beta) + \sin(\beta)})}$$

$$\csc(\beta) \stackrel{?}{=} \frac{1}{\sin(\beta)}$$

$$\csc(\beta) = \csc(\beta)$$

EXAMPLE 7 ■ **Verifying an Identity**

Verify the identity $\dfrac{1 + \tan(\theta)}{1 + \cot(\theta)} = \tan(\theta)$.

Solution We will begin simplifying the left side since it is more complex than the right side. Since we want the right side to simplify to tangent, we will try to write the left side of the identity using tangent.

$$\frac{1 + \tan(\theta)}{1 + \cot(\theta)} \stackrel{?}{=} \tan(\theta)$$

$$\frac{1 + \tan(\theta)}{1 + \dfrac{1}{\tan(\theta)}} \stackrel{?}{=} \tan(\theta) \qquad \cot(\theta) = \frac{1}{\tan(\theta)}$$

$$\frac{1 + \tan(\theta)}{\dfrac{\tan(\theta)}{\tan(\theta)} + \dfrac{1}{\tan(\theta)}} \stackrel{?}{=} \tan(\theta) \qquad \text{Use common denominators to combine terms.}$$

$$\frac{1 + \tan(\theta)}{\dfrac{\tan(\theta) + 1}{\tan(\theta)}} \stackrel{?}{=} \tan(\theta)$$

$$\frac{1+\tan(\theta)}{1}\cdot\frac{\tan(\theta)}{\tan(\theta)+1}\stackrel{?}{=}\tan(\theta) \qquad \textbf{To divide, multiply by the reciprocal.}$$

$$\frac{\tan(\theta)\cancel{(1+\tan(\theta))}}{\cancel{(\tan(\theta)+1)}}\stackrel{?}{=}\tan(\theta) \qquad \mathbf{1+\tan(\theta)=\tan(\theta)+1}$$

$$\tan(\theta)=\tan(\theta)$$

There are other methods for verifying this identity, such as rewriting the trigonometric functions in terms of cosine and sine. We will return to this example in Exercise 59 at the end of this section and verify the identity using different approaches.

It is sometimes necessary to multiply by an expression equal to the number 1 to help simplify identities. Look for expressions such as $1-\sin(\theta)$ or $\cos(\theta)+1$ in a fraction on either side of the potential identity. Our next example will show how we can use a Pythagorean identity to help verify identities of this form.

EXAMPLE 8 ■ Verifying an Identity

Verify the identity $\dfrac{\sin(\theta)}{1-\cos(\theta)}=\dfrac{1+\cos(\theta)}{\sin(\theta)}$.

Solution We start on the left side. We recall that the product of two binomials of the form $(a+b)(a-b)$ will expand to become a^2-b^2. Also, if $a=1$ and $b=\cos(\theta)$ or $b=\sin(\theta)$, we will get the expression $1-\cos^2(\theta)$ or $1-\sin^2(\theta)$, which are components of a Pythagorean identity.

$$\frac{\sin(\theta)}{1-\cos(\theta)}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)}$$

$$\frac{\sin(\theta)}{1-\cos(\theta)}\cdot\frac{1+\cos(\theta)}{1+\cos(\theta)}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)} \qquad \textbf{Multiply the left side by } \frac{\mathbf{1+\cos(\theta)}}{\mathbf{1+\cos(\theta)}}.$$

$$\frac{[\sin(\theta)][1+\cos(\theta)]}{[1-\cos(\theta)][1+\cos(\theta)]}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)}$$

$$\frac{[\sin(\theta)][1+\cos(\theta)]}{1-\cos^2(\theta)}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)} \qquad \mathbf{[1-\cos(\theta)][1+\cos(\theta)]=1-\cos^2(\theta)}$$

$$\frac{[\sin(\theta)][1+\cos(\theta)]}{\sin^2(\theta)}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)} \qquad \mathbf{\sin^2(\theta)=1-\cos^2(\theta)\text{ since }\cos^2(\theta)+\sin^2(\theta)=1}$$

$$\frac{\cancel{[\sin(\theta)]}[1+\cos(\theta)]}{\cancel{\sin(\theta)}\cdot\sin(\theta)}\stackrel{?}{=}\frac{1+\cos(\theta)}{\sin(\theta)} \qquad \mathbf{\text{since }\sin^2(\theta)=\sin(\theta)\sin(\theta)}$$

$$\frac{1+\cos(\theta)}{\sin(\theta)}=\frac{1+\cos(\theta)}{\sin(\theta)}$$

■ A Construction Approach to Analyzing the Six Trigonometric Functions

One approach to understanding the values of the six trigonometric functions for a given angle measure is to use the unit circle. When we first discussed cosine and sine, we located the endpoint of the arc on the unit circle associated with an angle measuring θ and then determined their values by determining the "length" of the horizontal and vertical components of each coordinate. Similarly, the values of all the trigonometric functions can be represented as lengths associated with the unit circle according to Figure 10.9.

The proofs to verify that these lengths are accurate representations for the value of each trigonometric function rely on Pythagorean identities. The three right triangles shown in Figure 10.10 match up exactly with the three Pythagorean identities.

In the exercise set, we will use Pythagorean identities to verify other relationships given in Figure 10.9.

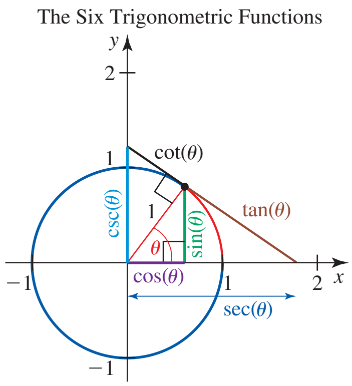


Figure 10.9

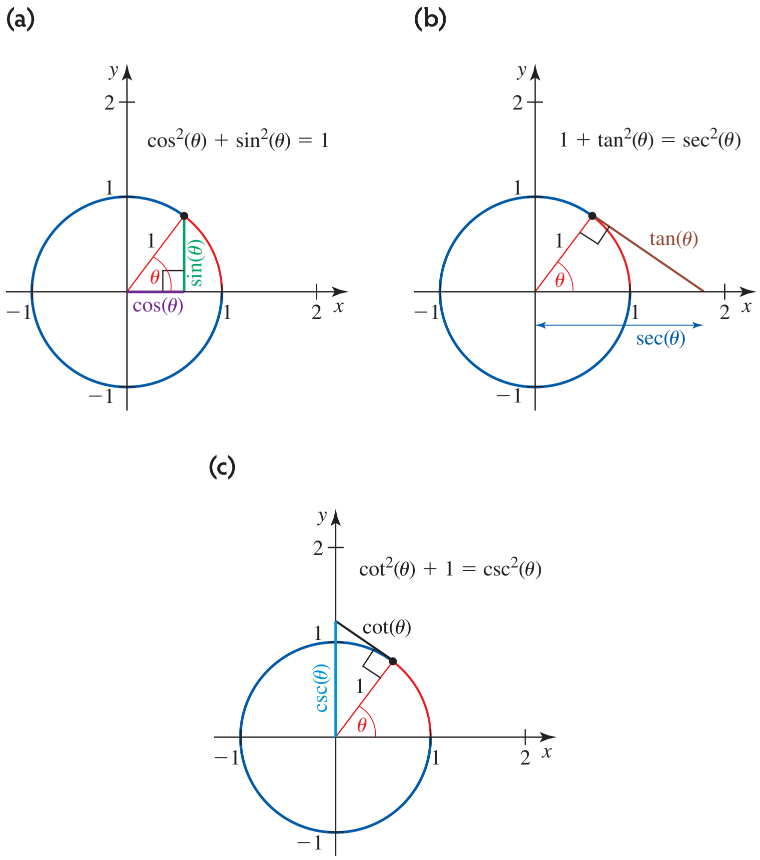


Figure 10.10

SUMMARY

In this section you learned how to verify that two trigonometric expressions are equivalent. You also learned how the six trigonometric functions can be represented as lengths associated with the unit circle.

10.2 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–7, use fundamental identities to verify each identity. (See Examples 1, 6, and 7.)

1. $\cot(\theta)\cos(\theta) = \sin(\theta)$

2. $\cos(\theta)\sec(\theta) = 1$

3. $\tan^2(\theta) = \sin^2(\theta)\sec^2(\theta)$

4. $\cos(\beta) + [\cot(\beta)][1 - \sin(\beta)] = \cot(\beta)$

5. $\dfrac{\cot(\theta)}{\cos(\theta)} = \csc(\theta)$

6. $\dfrac{\sin(\theta) + \cos(\theta)}{\sin(\theta)\cos(\theta)} = \csc(\theta) + \sec(\theta)$

7. $\dfrac{\sin(\theta)}{\sin(\theta) - \cos(\theta)} = \dfrac{1}{1 - \cot(\theta)}$

In Exercises 8–13, use Pythagorean identities to verify each identity. (See Examples 2 and 4.)

8. $[\sin^2(x)][1 + \cot^2(x)] = 1$

9. $\cot(\beta)\tan(\beta) - \cos^2(\beta) = \sin^2(\beta)$

10. $\cos^2(\theta) = \dfrac{\cos^2(\theta) + \sin^2(\theta)}{1 + \tan^2(\theta)}$

11. $[\cos(x) + \sin(x)]^2 = 2\cos(x)\sin(x) + 1$

12. $2\cos^2(\theta) + \sin^2(\theta) = 1 + \cos^2(\theta)$

13. $[1 + \tan^2(x)][\cos^2(x) - 1] = -\tan^2(x)$

In Exercises 14–19, use even–odd identities to verify each identity. (See Example 3.)

14. $-\csc(\theta) = \dfrac{1}{\sin(-\theta)}$

15. $\dfrac{\csc(-x)}{\sec(-x)} = -\cot(x)$

16. $\sec^2(\theta) = 1 + \tan^2(-\theta)$

17. $\cos(-\theta)\cos(\theta) - \sin(\theta)\sin(-\theta) = 1$

18. $1 = \csc^2(-y)\sin^2(-y)$

19. $\dfrac{\sin^2(-\theta)}{\tan^2(\theta)} = \cos^2(\theta)$

In Exercises 20–25, use cofunction identities to verify each identity. (See Example 5.)

20. $\sin\left(\dfrac{\pi}{2} - \theta\right)\sec(\theta) = 1$

21. $\cot(x) = \dfrac{\sin\left(\dfrac{\pi}{2} - x\right)}{\cos\left(\dfrac{\pi}{2} - x\right)}$

22. $\sin(\beta) = \cos(\beta)\cot\left(\dfrac{\pi}{2} - \beta\right)$

23. $\cos^2\left(\dfrac{\pi}{2} - \alpha\right) + \cos^2(-\alpha) = 1$

24. $-\cot^2(x) = \dfrac{\cot(x)\cos(x)}{\sin\left(\dfrac{\pi}{2} - x\right)\tan(-x)}$

25. $\dfrac{\sec(\theta)\tan\left(\dfrac{\pi}{2} - \theta\right)}{1 - \csc^2(\theta)} = -\sec(\theta)\tan(\theta)$

In Exercises 26–29, use a graph to determine if it is likely that each trigonometric equation is an identity. If it appears to be an identity, use algebra to verify. If it does not appear to be an identity, provide a counterexample to demonstrate that the two expressions are not equivalent. (Hint: To input $\tan^2(\theta)$ into your calculator, use $(\tan(\theta))^2$.)

26. $\dfrac{\sin(\theta)}{\tan(\theta)} = \sec(\theta)$

27. $\cos(\theta)\sec(\theta) + \tan^2(\theta) = \sec^2(\theta)$

28. $[\cos(\theta)][\tan(\theta) + \cot(\theta)] = \csc(\theta)$

29. $\csc(\theta)\tan(\theta) + \sec(\theta) = 2\cos(\theta)$

In Exercises 30–39, verify each identity.

30. $\cot(\theta)\tan(-\theta) = -1$

31. $1 = \tan(\phi)\tan\left(\dfrac{\pi}{2} - \phi\right)$

32. $\sec^2\left(\dfrac{\pi}{2} - \theta\right) - 1 = \cot^2(\theta)$

33. $\tan^2(\theta)\cos^2(\theta) + \cot^2(\theta)\sin^2(\theta) = 1$

34. $\sec(\theta)\cos(-\theta) - \sin^2(\theta) = \cos^2(\theta)$

35. $1 = \sec(\alpha) \cdot \dfrac{\sin(\alpha)}{\tan(\alpha)}$

36. $\sin(\theta)\tan(\theta) = \dfrac{\cos^2\left(\dfrac{\pi}{2} - \theta\right)}{\cos(\theta)}$

37. $\dfrac{\sec(\theta)}{\cos(\theta)} = \sec^2(\theta)$

38. $\tan^2(y) - \sin^2(y)\tan^2(y) = \sin^2(y)$

39. $\tan(\theta) - \dfrac{\sec^2(\theta)}{\tan(\theta)} = -\cot(\theta)$

In Exercises 40 and 41,

a. Trace the given figure.

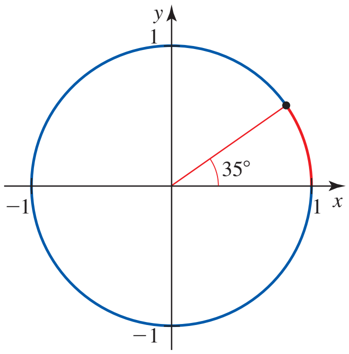
b. Draw the construction showing the six trigonometric functions as lengths.

c. Using a ruler, measure the length of the radius and each of the six lengths that represent the values of the trigonometric functions for the given angle measure.

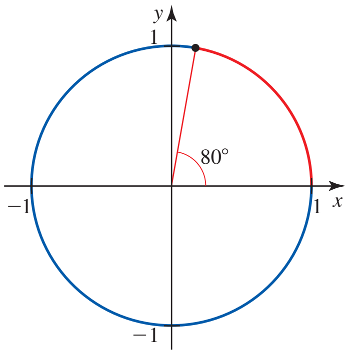
d. Divide each of the six lengths for the trigonometric values you measured by the length of the radius. These are the values for the six functions of the given angle measure on the unit circle.

e. Using a calculator, find the values of the six trigonometric functions for the given angle measure and compare these values to the values you found in part (d).

40.



41.

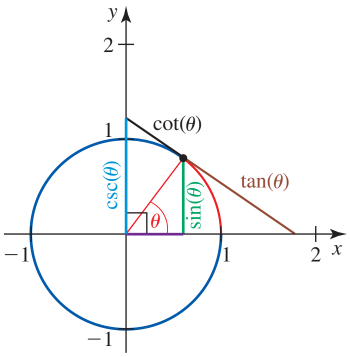


42. Explain how the values of each of the six trigonometric functions change as the angle measure increases in the first quadrant. Compare this behavior to the graphs of each of the six functions on the interval (0°, 90°).

In Exercises 43–46, you are shown identities implied by the diagram of the six trigonometric functions as they relate to the unit circle. Verify each identity.

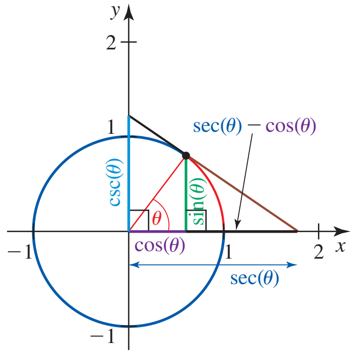
43. Use similar triangles.

$$\frac{\sin(\theta)}{\csc(\theta)} = \frac{\tan(\theta)}{\tan(\theta) + \cot(\theta)}$$



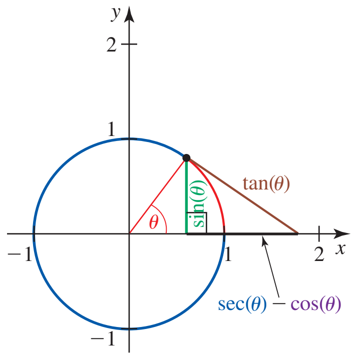
44. Use similar triangles.

$$\frac{\sec(\theta) - \cos(\theta)}{\sec(\theta)} = \frac{\sin(\theta)}{\csc(\theta)}$$



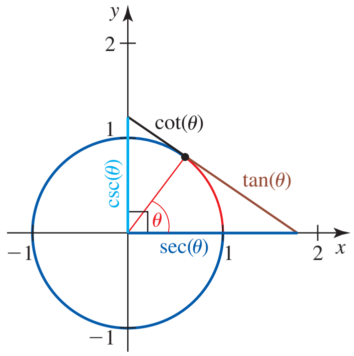
45. Use the Pythagorean theorem.

$$\sin^2(\theta) + [\sec(\theta) - \cos(\theta)]^2 = \tan^2(\theta)$$



46. Use the Pythagorean theorem.

$$\sec^2(\theta) + \csc^2(\theta) = [\cot(\theta) + \tan(\theta)]^2$$



In Exercises 47–50, verify each identity. In each case, you will be multiplying binomials of the form $(a + b)(a - b) = a^2 - b^2$. This skill will be necessary to complete Exercises 51–58.

47. $[1 + \sin(\theta)][1 - \sin(\theta)] = \cos^2(\theta)$

48. $[1 + \cos(\theta)][1 - \cos(\theta)] = \sin^2(\theta)$

49. $[\csc(\theta) + \cot(\theta)][\csc(\theta) - \cot(\theta)] = 1$

50. $\dfrac{1}{1 + \cos(\theta)} + \dfrac{1}{1 - \cos(\theta)} = 2\csc^2(\theta)$

In Exercises 51–54, verify each identity. (See Example 8.)

51. $\dfrac{\sec(\theta) - 1}{1 - \cos(\theta)} = \sec(\theta)$

52. $\sec(\theta) - \tan(\theta) = \dfrac{\cos(\theta)}{1 + \sin(\theta)}$

53. $\dfrac{\sec(x) + 1}{\tan(x)} = \dfrac{\tan(x)}{\sec(x) - 1}$

54. $\dfrac{\cos(-\theta)}{1 + \sin(-\theta)} = \sec(\theta) + \tan(\theta)$

In Exercises 55–58, use factoring to verify each identity. (See Example 9.)

55. $\dfrac{\sin^2(\theta)}{1 - \cos(\theta)} = 1 + \cos(\theta)$

56. $1 - \sin(x) = \dfrac{\cos^2(x)}{1 + \sin(x)}$

57. $\dfrac{\csc^2(\phi) - 1}{\csc(\phi) - 1} = \csc(\phi) + 1$

58. $\dfrac{\sin^2(\beta) - 9}{\sin(\beta) - 3} = \sin(\beta) + 3$

59. In Example 7, we verified the identity $\dfrac{1 + \tan(\theta)}{1 + \cot(\theta)} = \tan(\theta)$ using only one of several different options available. Verify this identity using two methods different from the one used in Example 7.

■ SHOW YOU KNOW

60. This section provided you with some tips on verifying identities. Provide a list of suggestions (in your own words) on how to verify identities and share it with a classmate who is struggling.

61. Explain why rewriting trigonometric functions using cosine and sine is a good technique to use when verifying an identity.

62. A classmate says $\sin(\theta)\cot(\theta) = 1 - \sin(\theta)$ is an identity. His justification is that if $\theta = \frac{\pi}{2}$, then

$$\sin\left(\frac{\pi}{2}\right)\cot\left(\frac{\pi}{2}\right) = 1 - \sin\left(\frac{\pi}{2}\right)$$
$$1 \cdot 0 = 1 - 1$$
$$0 = 0$$

Is your classmate correct? Explain.

63. A classmate says since $\cos^2(\theta) + \sin^2(\theta) = 1$ is an identity, then solving for cosine yields $\cos(\theta) = \sqrt{1 - \sin^2(\theta)}$, which must also be an identity. Do you agree the expressions $\cos(\theta)$ and $\sqrt{1 - \sin^2(\theta)}$ are equivalent? Explain.

■ STRETCH YOUR MIND

Exercises 64–69 are intended to challenge your abilities to verify identities. For each, verify the identity.

64. $\sin^4(\theta) = 1 - 2\cos^2(\theta) + \cos^4(\theta)$

65. $\dfrac{\cos^4(\theta) + \sin^2(\theta)\cos^2(\theta) + \sin^2(\theta)}{\cos^2(\theta)} = \sec^2(\theta)$

66. $\dfrac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)} = \dfrac{\cot(\theta) + \cot(\phi)}{\cot(\theta)\cot(\phi) - 1}$

67. $\dfrac{\sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)}{\cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)} = \dfrac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)}$

68. $\sec^4(\theta) - \tan^4(\theta) = \dfrac{1 + \sin^2(\theta)}{\cos^2(\theta)}$

69. $x^2 + y^2 = [x\sin(\theta) + y\cos(\theta)]^2 + [x\cos(\theta) - y\sin(\theta)]^2$

SECTION 10.3 Other Trigonometric Identities

LEARNING OBJECTIVES

- Use the sum and difference identities
- Use the sum-to-product and product-to-sum identities
- Use double-angle and half-angle identities

GETTING STARTED

When two or more musical notes of different frequencies are played together, the combination of their intensities and frequencies create acoustic *beats*. The manner in which the sounds combine can either be *consonant* or *dissonant*. Consonant sounds, such as common chords used by most musicians, are pleasant. Dissonant sounds sound as if they conflict. To tune an instrument, a person compares the notes the instrument plays with a standard note played by an electronic tuning machine or a tuning fork. The person adjusts the instrument until the two notes, when played together, produce no beat. Trigonometric identities can help us work with the formulas used to model the combination of musical notes.

In this section we discuss trigonometric identities involving the sum or difference of angles, the sum or difference of two trigonometric functions, the product of two trigonometric functions, and trigonometric functions involving double angles and half angles. We focus on stating and using common identities rather than verifying them.

Sum and Difference Identities

In Chapter 8, we demonstrated how to find the exact values of trigonometric functions with reference angles measuring 0, $\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}$, or $\frac{\pi}{2}$. For all other measures, we used a calculator to generate rounded estimates of the function values. With the sum and difference identities, we can begin to find exact values of trigonometric functions for many more angles. Specifically, the sine, cosine, or tangent of the sum of two angle measures can be determined using the **sum and difference identities**.

Sum and Difference Identities

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$
$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$
$$\sin(\theta - \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)$$

$$\tan(\theta + \phi) = \frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)}$$

$$\tan(\theta - \phi) = \frac{\tan(\theta) - \tan(\phi)}{1 + \tan(\theta)\tan(\phi)}$$

EXAMPLE 1 ■ Using the Difference Identity for Cosine

Find the exact value for $\cos(15°)$.

Solution To find $\cos(15°)$, we need to write 15° as the difference of two angle measures whose exact trigonometric values we know. In this case, we use $45° - 30°$.

$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$$

$$\cos(45° - 30°) = \cos(45°)\cos(30°) + \sin(45°)\sin(30°) \qquad \theta = 45° \text{ and } \phi = 30°$$

$$\cos(15°) = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \cdot \frac{1}{2} \qquad \text{Use exact values.}$$

$$\cos(15°) = \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$$

$$\cos(15°) = \frac{\sqrt{6} + \sqrt{2}}{4}$$

We use a calculator to check our work: $\frac{\sqrt{6} + \sqrt{2}}{4} \approx 0.9659$ and $\cos(15°) \approx 0.9659$.

EXAMPLE 2 ■ Using Sum and Difference Identities

Find the exact value for each of the following.

a. $\sin\left(-\frac{\pi}{12}\right)$

b. $\sin(285°)$

Solution

a. Our first step is to find two angle measures whose sum or difference is $-\frac{\pi}{12}$.

$$\frac{2\pi}{12} - \frac{3\pi}{12} = -\frac{\pi}{12}$$

$$\frac{\pi}{6} - \frac{\pi}{4} = -\frac{\pi}{12}$$

We find $\frac{\pi}{6} - \frac{\pi}{4}$ meets these requirements. We now use the difference identity for sine.

$$\sin(\theta - \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)$$

$$\sin\left(\frac{\pi}{6} - \frac{\pi}{4}\right) = \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{4}\right) \qquad \theta = \frac{\pi}{6} \text{ and } \phi = \frac{\pi}{4}$$

$$\sin\left(-\frac{\pi}{12}\right) = \frac{1}{2}\cdot\frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2} \qquad \text{Use exact values.}$$

$$= \frac{\sqrt{2}}{4} - \frac{\sqrt{6}}{4} \qquad \sqrt{a}\cdot\sqrt{b} = \sqrt{a\cdot b}$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

We check this answer with a calculator: $\frac{\sqrt{2} - \sqrt{6}}{4} \approx -0.2588$ and $\sin\left(-\frac{\pi}{12}\right) \approx -0.2588$. We also note $-\frac{\pi}{12}$ is an arc whose endpoint lies in Quadrant IV, and that sine values are negative in Quadrant IV since the endpoint lies below the x-axis.

b. We begin by finding two angles whose measures have a sum or difference of 285° and for which we know the exact values of the trigonometric functions. We use 225° + 60°. We now use the sum identity for sine. We expect the final value to be negative since 285° terminates in Quadrant IV, where the sine values are negative.

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$

$$\sin(225° + 60°) = \sin(225°)\cos(60°) + \cos(225°)\sin(60°) \qquad \theta = 225° \text{ and } \phi = 60°$$

$$\sin(285°) = \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{1}{2} + \left(-\frac{\sqrt{2}}{2}\right)\cdot\frac{\sqrt{3}}{2} \qquad \text{Use exact values.}$$

$$= -\frac{\sqrt{2}}{4} + \left(-\frac{\sqrt{6}}{4}\right)$$

$$= \frac{-\sqrt{2} - \sqrt{6}}{4}$$

Again, we check our answer using a calculator: $\frac{-\sqrt{2} - \sqrt{6}}{4} \approx -0.9659$ and $\sin(285°) \approx -0.9659$.

EXAMPLE 3 ■ Using a Sum or Difference Identity

The Ferris wheel at Odaiba in Tokyo, Japan, has a diameter of 100 meters. Riders reach a maximum height of 115 meters off the ground during a 16-minute revolution. Rider A rides the Odaiba Ferris wheel and her height above the ground in meters can be modeled by the formula $H_A(t) = -50\cos\left(\frac{\pi}{8}t\right) + 65$, where t is in minutes since boarding. After Rider A travels for 2 minutes, Rider B boards the Ferris wheel. Rider B's height above the ground in meters can be modeled by the formula $H_B(t) = -50\cos\left[\frac{\pi}{8}(t-2)\right] + 65$, where t is in minutes since Rider A boarded. When was the first time both riders were the same height above the ground? (*Source:* Tokyo-i .tripod.com)

Solution The riders will be the same height above the ground when $H_A(t) = H_B(t)$. Thus, we set their formulas equal to each other and simplify.

$$-50\cos\left(\frac{\pi}{8}t\right) + 65 = -50\cos\left[\frac{\pi}{8}(t-2)\right] + 65$$

$$-50\cos\left(\frac{\pi}{8}t\right) = -50\cos\left[\frac{\pi}{8}(t-2)\right]$$

$$\cos\left(\frac{\pi}{8}t\right) = \cos\left[\frac{\pi}{8}(t-2)\right]$$

$$\cos\left(\frac{\pi}{8}t\right) = \cos\left(\frac{\pi}{8}t - \frac{\pi}{4}\right)$$

At this point, we can use the difference identity for cosine on the right side.

$$\cos\left(\frac{\pi}{8}t\right) = \cos\left(\frac{\pi}{8}t\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{8}t\right)\sin\left(\frac{\pi}{4}\right) \qquad \text{Use the difference identity.}$$

$$\cos\left(\frac{\pi}{8}t\right) = \cos\left(\frac{\pi}{8}t\right)\cdot\frac{\sqrt{2}}{2} + \sin\left(\frac{\pi}{8}t\right)\cdot\frac{\sqrt{2}}{2} \qquad \text{Use exact values.}$$

$$\frac{\cancel{\cos\left(\frac{\pi}{8}t\right)}}{\cancel{\cos\left(\frac{\pi}{8}t\right)}} = \frac{\cancel{\cos\left(\frac{\pi}{8}t\right)}\cdot\frac{\sqrt{2}}{2}}{\cancel{\cos\left(\frac{\pi}{8}t\right)}} + \frac{\sin\left(\frac{\pi}{8}t\right)\cdot\frac{\sqrt{2}}{2}}{\cos\left(\frac{\pi}{8}t\right)}$$

$$1 = \frac{\sqrt{2}}{2} + \tan\left(\frac{\pi}{8}t\right)\cdot\frac{\sqrt{2}}{2} \qquad \text{since } \frac{\sin\left(\frac{\pi}{8}t\right)}{\cos\left(\frac{\pi}{8}t\right)} = \tan\left(\frac{\pi}{8}t\right)$$

$$1 = \frac{\sqrt{2}}{2}\left[1 + \tan\left(\frac{\pi}{8}t\right)\right] \qquad \text{Factor out } \frac{\sqrt{2}}{2} \text{ from the right side.}$$

$$1\cdot\frac{2}{\sqrt{2}} = 1 + \tan\left(\frac{\pi}{8}t\right) \qquad \text{Multiply both sides by } \frac{2}{\sqrt{2}}.$$

$$\frac{2}{\sqrt{2}} - 1 = \tan\left(\frac{\pi}{8}t\right)$$

$$\frac{\pi}{8}t = \tan^{-1}\left(\frac{2}{\sqrt{2}} - 1\right) \qquad \text{Write in inverse form.}$$

We evaluate $\tan^{-1}\left(\frac{2}{\sqrt{2}} - 1\right)$ using a calculator to get about 0.3927 radians. The other angle with the same tangent value during one rotation will be $0.3927 + \pi \approx 3.5343$ radians. We therefore set up two equations and solve.

$$\frac{\pi}{8}t \approx 0.3927 \qquad \frac{\pi}{8}t \approx 3.5343$$

$$t \approx \frac{8}{\pi} \cdot 0.3927 \qquad t \approx \frac{8}{\pi} \cdot 3.5343$$

$$t = 1 \qquad t = 9$$

We find they will be at the same height 1 minute and 9 minutes after Rider A boards the Ferris wheel. However, only the answer $t = 9$ makes sense in this context because Rider B has not yet boarded the Ferris wheel at $t = 1$.

EXAMPLE 4 ■ Using Sum and Difference Identities

Find the exact value for csc(195°).

Solution We begin by rewriting the expression using sine.

$$\csc(195°) = \frac{1}{\sin(195°)}$$

We rewrite 195° as 240° − 45° and use the difference identity for sine. (Note: There are other correct choices, such as using 135° + 60°.)

$$\begin{aligned}
\csc(195°) &= \frac{1}{\sin(195°)} \\
&= \frac{1}{\sin(240° - 45°)} \\
&= \frac{1}{\sin(240°)\cos(45°) - \cos(240°)\sin(45°)} && \text{Use the difference identity.} \\
&= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\frac{\sqrt{2}}{2} - \left(-\frac{1}{2}\right)\frac{\sqrt{2}}{2}} && \text{Use exact values.} \\
&= \frac{1}{-\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}} \\
&= \frac{1}{\frac{-\sqrt{6} + \sqrt{2}}{4}} \\
&= \frac{4}{-\sqrt{6} + \sqrt{2}}
\end{aligned}$$

If we rationalize the denominator by multiplying by $\frac{-\sqrt{6} - \sqrt{2}}{-\sqrt{6} - \sqrt{2}}$, we get the simplified solution $-\sqrt{6} - \sqrt{2}$.

Common Student Errors

It is important to avoid common errors when working with sum and difference identities. The following box shows some of these common errors.

Common Errors

$$\cos(\theta + \phi) \neq \cos(\theta) + \cos(\phi) \qquad \cos(\theta - \phi) \neq \cos(\theta) - \cos(\phi)$$
$$\sin(\theta + \phi) \neq \sin(\theta) + \sin(\phi) \qquad \sin(\theta - \phi) \neq \sin(\theta) - \sin(\phi)$$
$$\tan(\theta + \phi) \neq \tan(\theta) + \tan(\phi) \qquad \tan(\theta - \phi) \neq \tan(\theta) - \tan(\phi)$$

As these errors indicate, we must be careful to use the appropriate identities and not try to apply the distributive property to these expressions. The parentheses in, say, the expression $\cos(\theta + \phi)$ do not indicate multiplication because "cos" does not represent a number we can use to multiply. Rather "cos" is an operation itself indicating that we are to find the x-coordinate of the endpoint of an arc with angle measuring $\theta + \phi$.

■ Sum-to-Product Identities

When calculating the sum or difference of the sine or cosine of two angles, it is useful to rewrite the sum or difference as a product. The **sum-to-product identities** are used for this purpose.

Sum-to-Product Identities

$$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$

$$\sin(\theta) + \sin(\phi) = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$

$$\sin(\theta) - \sin(\phi) = 2\sin\left(\frac{\theta - \phi}{2}\right)\cos\left(\frac{\theta + \phi}{2}\right)$$

EXAMPLE 5 ■ **Using Sum-to-Product Identities to Find Exact Values**

Find the exact value for $\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{17\pi}{12}\right)$.

Solution

$$\sin(\theta) - \sin(\phi) = 2\sin\left(\frac{\theta - \phi}{2}\right)\cos\left(\frac{\theta + \phi}{2}\right)$$

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{17\pi}{12}\right) &= 2\sin\left(\frac{\frac{\pi}{12} - \frac{17\pi}{12}}{2}\right)\cos\left(\frac{\frac{\pi}{12} + \frac{17\pi}{12}}{2}\right) \\
&= 2\sin\left(\frac{-\frac{16\pi}{12}}{2}\right)\cos\left(\frac{\frac{18\pi}{12}}{2}\right) \\
&= 2\sin\left(-\frac{16\pi}{24}\right)\cos\left(\frac{18\pi}{24}\right)
\end{aligned}$$

$$\begin{aligned}
\sin\left(\frac{\pi}{12}\right) - \sin\left(\frac{17\pi}{12}\right) &= 2\sin\left(-\frac{2\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) \\
&= -2\sin\left(\frac{2\pi}{3}\right)\cos\left(\frac{3\pi}{4}\right) && \textbf{since } \sin(-\theta) = -\sin(\theta) \\
&= -2\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}}{2}
\end{aligned}$$

Sinusoidal functions can be used to model the intensity of sounds, such as those produced when a note is played on a piano. The function $I(t) = \cos(2\pi f t)$ models the intensity, I, of sound, where f is the frequency of the note played in Hz (cycles per second) and t is the time in seconds. When two sound waves interact, they produce either *constructive interference* (the sounds combine to become louder), *destructive interference* (the sounds combine to become softer), or some combination of the two that creates *beats* in the noise. A beat makes the combined sound seem to "vibrate" as it alternates between sounding louder and sounding softer. (*Note:* In a graph of the model, louder sounds have intensities farther from the horizontal axis and softer sounds have intensities closer to the horizontal axis.)

EXAMPLE 6 ■ Using a Sum-to-Product Identity

A piano tuner strikes a tuning fork that produces a sound with a frequency of 440 Hz (the frequency of key A4 on the piano. This is called "Concert A" and is the A above Middle C that is used to tune instruments for concert performances). At the same time, the note A4 is played on a perfectly tuned piano.

a. Model the intensity of both the tuning fork and the note A4.

b. Add the equations for the intensities of these two sounds, then rewrite the result as a single trigonometric function.

c. Graph the equations in part (a) and the result from part (b) and explain what the graph shows. Use the interval [0, 0.01] for your graph.

d. Suppose the note A4 is played on a piano that is out of tune such that the key associated with A4 produces a sound with a 420 Hz frequency instead of 440 Hz. Repeat parts (a) and (b), then graph the equation for the combined intensity of the two notes on the interval [0, 0.05]. Explain how the piano tuner knows the piano is out of tune.

Solution

a. We use the function $I(t) = \cos(2\pi f t)$ for two notes with the frequency 440 Hz, denoting them I_T for the tuning fork and I_A for the key A4 on the piano.

$$I_T(t) = \cos(2\pi 440 t) = \cos(880\pi t) \qquad I_A(t) = \cos(880\pi t)$$

b.

$$I_T(t) + I_A(t) = \cos(880\pi t) + \cos(880\pi t)$$
$$I(t) = 2\cos(880\pi t)$$

c. We graph $I(t) = \cos(880\pi t)$ and $I(t) = 2\cos(880\pi t)$ on the same axes, as shown in Figure 10.11. (*Note:* We do not need to graph $I_A(t)$ since it is identical to $I_T(t)$.)

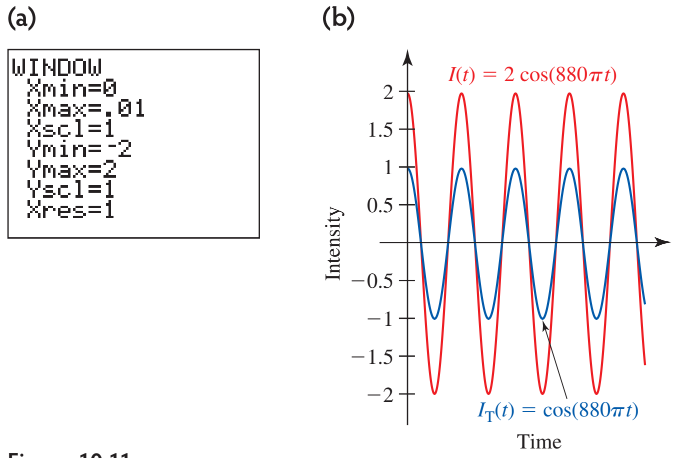


Figure 10.11

The graph shows us that the tuning fork and note played on the piano are creating constructive interference. At all points in time the intensity of the sound is exactly double what it was previously, which causes the combined sound to be perceived as being louder. (This same phenomenon happens with more complex sounds, such as music from a stereo, when two or more properly synchronized speakers are used.)

d. With the piano out of tune, $I_A(t)$ will be altered to reflect a frequency of 420 Hz.

$$I_T(t) = \cos(880\pi t) \qquad I_A(t) = \cos(2\pi \cdot 420 t) = \cos(840\pi t)$$

We now add the intensities of the notes to create a function for their combined intensities.

$$I_T(t) + I_A(t) = \cos(880\pi t) + \cos(840\pi t)$$

$$I(t) = 2\cos\left(\frac{880\pi t + 840\pi t}{2}\right)\cos\left(\frac{880\pi t - 840\pi t}{2}\right) \qquad \text{Use a sum-to-product identity.}$$

$$= 2\cos\left(\frac{1720\pi t}{2}\right)\cos\left(\frac{40\pi t}{2}\right)$$

$$= 2\cos(860\pi t)\cos(20\pi t)$$

(a)

```
WINDOW
 Xmin=0
 Xmax=.05
 Xscl=1
 Ymin=-2
 Ymax=2
 Yscl=1
 Xres=1
```

(b)

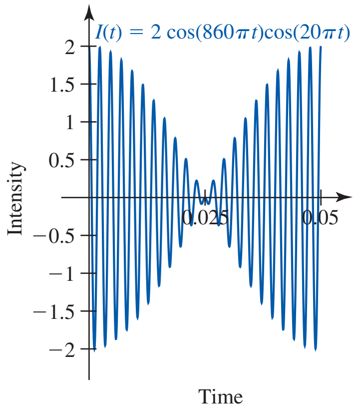


Figure 10.12

The graph of $I(t)$ in Figure 10.12 shows that two notes with frequencies of 440 Hz and 420 Hz create an interesting pattern that is repeated at regular intervals. This pattern shows intervals where the intensity of the sound is greater than either of the original notes and intervals where the intensity of the sound is less than either of the original notes. These alternating periods of high and low intensity that make the combined sound seem to vibrate are what tell the tuner the piano is out of tune. The tuner then tightens or loosens the piano strings until the notes combine perfectly.

Product-to-Sum Identities

Just as we can use identities to convert sums to products, we can use identities to convert products to sums through the use of the **product-to-sum identities**.

Product-to-Sum Identities

$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta + \phi) + \cos(\theta - \phi)}{2}$$

$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2}$$

$$\sin(\theta)\cos(\phi) = \frac{\sin(\theta + \phi) + \sin(\theta - \phi)}{2}$$

EXAMPLE 7 ■ **Using a Product-to-Sum Identity**

Find the exact value of $\cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{12}\right)$.

Solution

$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta + \phi) + \cos(\theta - \phi)}{2}$$

$$\cos\left(\frac{5\pi}{4}\right)\cos\left(\frac{5\pi}{12}\right) = \frac{\cos\left(\frac{5\pi}{4} + \frac{5\pi}{12}\right) + \cos\left(\frac{5\pi}{4} - \frac{5\pi}{12}\right)}{2}$$

$$= \frac{1}{2}\left[\cos\left(\frac{15\pi}{12} + \frac{5\pi}{12}\right) + \cos\left(\frac{15\pi}{12} - \frac{5\pi}{12}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{20\pi}{12}\right) + \cos\left(\frac{10\pi}{12}\right)\right]$$

$$= \frac{1}{2}\left[\cos\left(\frac{5\pi}{3}\right) + \cos\left(\frac{5\pi}{6}\right)\right] \qquad \textbf{Reduce fractions.}$$

$$= \frac{1}{2}\left[\frac{1}{2} + \left(-\frac{\sqrt{3}}{2}\right)\right] \qquad \textbf{Use exact values.}$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{4}$$

■ Double-Angle Identities

The **double-angle identities** are developed by applying the sum and difference identities using $\phi = \theta$.

DOUBLE-ANGLE IDENTITIES

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$= 2\cos^2\theta - 1$$
$$= 1 - 2\sin^2\theta$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

EXAMPLE 8 ■ Using Double-Angle Identities to Solve Equations

Find all of the solutions to the equation $\sin(2\theta) + \cos(\theta) = 0$ over the interval $[0, 2\pi]$.

Solution We use the double-angle identity for sine and then factor to solve the equation.

$$\sin(2\theta) + \cos(\theta) = 0$$

$$2\sin(\theta)\cos(\theta) + \cos(\theta) = 0 \qquad \text{since } \sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\cos(\theta)(2\sin(\theta) + 1) = 0 \qquad \text{Factor out } \cos(\theta).$$

$$\cos(\theta) = 0 \quad \text{or} \quad 2\sin(\theta) + 1 = 0$$

The solutions will occur when $\cos(\theta) = 0$ and when $2\sin(\theta) + 1 = 0$.

$$\cos(\theta) = 0 \qquad\qquad 2\sin(\theta) + 1 = 0$$

$$\theta = \cos^{-1}(0) \qquad\qquad 2\sin(\theta) = -1$$

$$\theta = \frac{\pi}{2} \text{ and } \theta = \frac{3\pi}{2} \qquad\qquad \sin(\theta) = -\frac{1}{2}$$

$$\theta = \sin^{-1}\left(-\frac{1}{2}\right)$$

$$\theta = \frac{7\pi}{6} \text{ and } \theta = \frac{11\pi}{6}$$

There are four solutions: $\theta = \frac{\pi}{2}$, $\theta = \frac{7\pi}{6}$, $\theta = \frac{3\pi}{2}$, and $\theta = \frac{11\pi}{6}$.

■ Half-Angle Identities

To find the sine, cosine, or tangent of half of an angle, we use the **half-angle identities**.

HALF-ANGLE IDENTITIES

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}} \qquad \tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{\sin(\theta)}$$

or

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}} \qquad \tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)}$$

The quadrant where $\left(\frac{\theta}{2}\right)$ falls determines whether the positive or negative square root is used.

EXAMPLE 9 ■ **Using Half-Angle Identities**

Find the exact value of sin(15°) using a half-angle identity.

Solution We use a half-angle identity since 15° is half of 30°, and we know the exact values for trigonometric functions at 30°.

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

$$\sin\left(\frac{30°}{2}\right) = \pm\sqrt{\frac{1-\cos(30°)}{2}}$$

$$\sin(15°) = \pm\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}} \qquad \cos(30°) = \frac{\sqrt{3}}{2}$$

$$= \pm\sqrt{\frac{1}{2}-\frac{\sqrt{3}}{4}} \qquad \text{Dividing by 2 is the same as multiplying by } \frac{1}{2}.$$

$$= \pm\sqrt{\frac{2-\sqrt{3}}{4}} \qquad \text{since } \frac{1}{2} = \frac{2}{4}$$

$$= \frac{\sqrt{2-\sqrt{3}}}{2} \qquad 15° \text{ is in Quadrant I, so sine is positive.}$$

SUMMARY

In this section you learned many additional identities involving trigonometric functions. You also learned how to use these identities to rewrite trigonometric expressions, solve equations, and find the exact values of trigonometric expressions. The identities covered in this section are summarized on the next page.

Sum and Difference Identities

$$\cos(\theta + \phi) = \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)$$
$$\cos(\theta - \phi) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi)$$

$$\sin(\theta + \phi) = \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)$$
$$\sin(\theta - \phi) = \sin(\theta)\cos(\phi) - \cos(\theta)\sin(\phi)$$

$$\tan(\theta + \phi) = \frac{\tan(\theta) + \tan(\phi)}{1 - \tan(\theta)\tan(\phi)}$$
$$\tan(\theta - \phi) = \frac{\tan(\theta) - \tan(\phi)}{1 + \tan(\theta)\tan(\phi)}$$

Sum-to-Product Identities

$$\cos(\theta) + \cos(\phi) = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$
$$\cos(\theta) - \cos(\phi) = -2\sin\left(\frac{\theta + \phi}{2}\right)\sin\left(\frac{\theta - \phi}{2}\right)$$
$$\sin(\theta) + \sin(\phi) = 2\sin\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right)$$
$$\sin(\theta) - \sin(\phi) = 2\sin\left(\frac{\theta - \phi}{2}\right)\cos\left(\frac{\theta + \phi}{2}\right)$$

Product-to-Sum Identities

$$\cos(\theta)\cos(\phi) = \frac{\cos(\theta + \phi) + \cos(\theta - \phi)}{2}$$
$$\sin(\theta)\sin(\phi) = \frac{\cos(\theta - \phi) - \cos(\theta + \phi)}{2}$$
$$\sin(\theta)\cos(\phi) = \frac{\sin(\theta + \phi) + \sin(\theta - \phi)}{2}$$

Double-Angle Identities

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta)\end{aligned}$$

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$
$$\tan(2\theta) = \frac{2\tan(\theta)}{1 - \tan^2(\theta)}$$

Half-Angle Identities

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 + \cos(\theta)}{2}}$$
$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1 - \cos(\theta)}{2}}$$

$$\tan\left(\frac{\theta}{2}\right) = \frac{1 - \cos(\theta)}{\sin(\theta)}$$

or

$$\tan\left(\frac{\theta}{2}\right) = \frac{\sin(\theta)}{1 + \cos(\theta)}$$

10.3 EXERCISES

SKILLS AND CONCEPTS

1. In Section 10.1 you made a list of the basic trigonometric identities for easy reference. Add the identities from this section to that reference sheet.

In Exercises 2–5,

a. For the given angle measure, find a pair of angle measures that meets the following criteria: They have a sum or difference equivalent to the given angle measure and both angles have reference angles measuring 0°, 30°, 45°, 60°, or 90°.

b. Find the exact value of each expression.

2. $\cos(15°)$ **3.** $\sin(-165°)$

4. $\tan(75°)$ **5.** $\sec(285°)$

In Exercises 6–9,

a. For the given angle measure, find a pair of angle measures that meets the following criteria: They have a sum or difference equivalent to the given angle measure and both angles have reference angles measuring 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, or $\frac{\pi}{2}$. For example, the angle measure $\frac{\pi}{12}$ can be written as $\frac{3\pi}{12} - \frac{2\pi}{12}$ $\left(\text{simplified to } \frac{\pi}{4} - \frac{\pi}{6}\right)$.

b. Find the exact value of each expression.

6. $\tan\left(-\frac{7\pi}{12}\right)$ **7.** $\cos\left(\frac{11\pi}{12}\right)$

8. $\sin\left(\frac{17\pi}{12}\right)$ **9.** $\tan\left(\frac{61\pi}{12}\right)$

In Exercises 10–15, find the exact value of each expression.

10. $\sin(43°)\cos(17°) + \cos(43°)\sin(17°)$

11. $\dfrac{\tan(302°) - \tan(167°)}{1 + \tan(302°)\tan(167°)}$

12. $\dfrac{1}{\cos(216°)\cos(546°) + \sin(216°)\sin(546°)}$

13. $\cos\left(\frac{13\pi}{15}\right)\cos\left(\frac{23\pi}{15}\right) + \sin\left(\frac{13\pi}{15}\right)\sin\left(\frac{23\pi}{15}\right)$

14. $\dfrac{\tan\left(\frac{8\pi}{9}\right) + \tan\left(\frac{7\pi}{9}\right)}{1 - \tan\left(\frac{8\pi}{9}\right)\tan\left(\frac{7\pi}{9}\right)}$

15. $\dfrac{1}{\sin\left(\frac{11\pi}{6}\right)\cos\left(\frac{7\pi}{12}\right) - \cos\left(\frac{11\pi}{6}\right)\sin\left(\frac{7\pi}{12}\right)}$

16. Show that using $\cos(315° - 60°)$ or $\cos(225° + 30°)$ give the same result for $\cos(255°)$.

In Exercises 17–22, use the product-to-sum identities to find the exact value of each expression.

17. $\sin(45°)\sin(15°)$

18. $\cos(75°)\cos(165°)$

19. $\sin(15°)\cos(75°)$

20. $\sin\left(\frac{\pi}{12}\right)\cos\left(\frac{\pi}{12}\right)$

21. $\sin\left(\frac{3\pi}{4}\right)\sin\left(\frac{11\pi}{12}\right)$

22. $\cos\left(\frac{13\pi}{24}\right)\sin\left(\frac{31\pi}{24}\right)$

In Exercises 23–30,

a. Rewrite the given angle measure to express it as half of the measure of one of the special angles for which we know the exact trigonometric values. For example, 15° may be rewritten as $\frac{1}{2} \cdot 30°$ or $\frac{30°}{2}$ since we know the exact trigonometric values at 30°. Also, $\frac{5\pi}{8}$ may be written as $\dfrac{\frac{5\pi}{4}}{2}$ since $\frac{5\pi}{8} = \frac{1}{2} \cdot \frac{5\pi}{4}$ and since we know the exact trigonometric values at $\frac{5\pi}{4}$.

b. Find the exact value of the expression.

23. $\sin(75°)$ **24.** $\cos(15°)$

25. $\tan(-22.5°)$ **26.** $\sin(-105°)$

27. $\tan\left(\frac{\pi}{8}\right)$ **28.** $\sin\left(\frac{\pi}{12}\right)$

29. $\tan\left(\frac{7\pi}{12}\right)$ **30.** $\cos\left(\frac{13\pi}{12}\right)$

In Exercises 31–36, use the sum-to-product identities to find the exact value of each expression.

31. $\sin(75°) + \sin(15°)$

32. $\cos(75°) - \cos(345°)$

33. $\sin(-15°) - \sin(285°)$

34. $\cos\left(\frac{7\pi}{12}\right) + \cos\left(\frac{\pi}{12}\right)$

35. $\sin\left(\frac{11\pi}{12}\right) - \sin\left(-\frac{5\pi}{12}\right)$

36. $\cos\left(\frac{\pi}{12}\right) + \cos\left(\frac{19\pi}{12}\right)$

In Exercises 37–42, use double-angle identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$.

37. $\sin(2\theta) + \sin(\theta) = 0$

38. $\sin(2\theta) - \cos(\theta) = 0$

39. $2\sin(x)\cos(x) = 0.7$

40. $-6\sin(\theta)\cos(\theta) = 0.5$

41. $2\cos^2(\theta) - 1 = -\dfrac{2}{3}$

42. $4 - \sin^2(x) = 3.5$

SHOW YOU KNOW

43. Discuss the relationship between θ and ϕ if $\sin(\theta + \phi) = 0$.

44. Explain the relationship between θ and ϕ if $\cos(\theta) + \cos(\phi) = 0$.

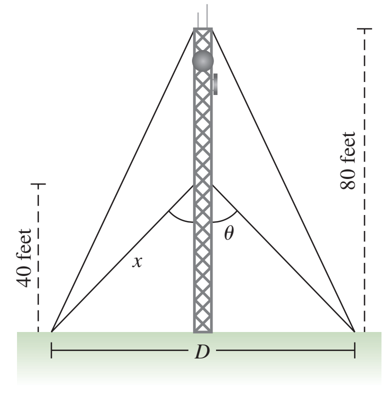
45. State the range for the function $f(\theta) = \sin(2\theta)$. Use your answer to explain why the following *cannot* be an identity: $\sin(2\theta) = 2\sin(\theta)$.

46. Choose a value of θ, then use a diagram of the unit circle to explain what $\cos(2\theta)$ and $\tan\left(\dfrac{\theta}{2}\right)$ represent.

47. The identity formulas for $\cos\left(\dfrac{\theta}{2}\right)$ and $\sin\left(\dfrac{\theta}{2}\right)$ contain the symbol $\pm$. Explain how you know whether to use the positive or negative value obtained using the formula.

MAKE IT REAL

48. Guy Wires Guy wires are wires attached to tall towers to provide stability and support. An 80-foot microwave tower is constructed with guy wires attached at the top of the tower and halfway to the top of the tower s as shown in the diagram.



The distance between where guy wires are attached to the ground on opposite sides of the tower is given by the equation $D = 2x\sin\left(\dfrac{\theta}{2}\right)$. Use a trigonometric identity to rewrite this equation using trigonometric functions of θ instead of $\dfrac{\theta}{2}$.

49. Range of a Launched Object If we neglect air resistance and other environmental factors, the total horizontal distance (in feet) an object will travel when it is launched at an angle of θ with an initial velocity of v_0 feet per second is given by the equation

$$R = \frac{1}{16}v_0^2\sin(\theta)\cos(\theta).$$

a. Use an identity to rewrite this formula using only one trigonometric function.

b. The U.S. Golf Association regulates the manufacture of golf balls. Under very specific testing conditions, the balls to be sold cannot exceed an initial velocity of 250 feet per second when hit. Suppose a golfer hits a ball at a 19° angle (considered optimal) with an initial velocity of 250 feet per second. According to our formula, how far will the ball travel? (***Source:*** **www.golfdigest.com**)

c. Graph R with an initial velocity of 250 feet per second and a variable trajectory angle. What angle allows a golf ball to travel the farthest?

d. Explain some possible reasons for the difference between your answer in part (c) and the 19° optimal angle described in part (b). Research the Internet to better address this question.

In Exercises 50–51, answer the questions that relate to sound. The function $I(t) = \cos(2\pi ft)$ models the intensity of a sound where f is the frequency in Hz (cycles per second) and t is the time elapsed in seconds. For the model, use radian mode on the calculator.

50. Musical Notes A piano player strikes the note A4 (frequency of 440 Hz) and the white key to its left, G4 (frequency of 392 Hz), simultaneously on a piano.

a. Write an equation to model the intensity of each note.

b. Add the intensities of the notes to create a function for the combined intensity of the notes. Use a sum-to-product identity to write your final answer as the product of trigonometric functions.

c. Graph the result to part (b) over the interval $[0, 0.05]$ using a calculator. Explain how the combination of these two notes will sound, especially to a trained ear.

d. Graph the formulas from part (a) together over the interval $[0, 0.02]$. Use this graph to explain the behavior of the function graphed in part (c).

51. Noise Cancellation Noise cancellation technology, such as that being used in high-end headphones and car mufflers, utilize sounds with the same frequency to cancel out unwanted noise. Sounds with the same frequency that are half of a period out of phase with each other will cancel each other with total destructive interference.

a. Compare the graphs of the functions modeling the intensity of two different sounds, A and B, given by $I_A(t) = \cos(120\pi t)$ and $I_B(t) = \cos(120\pi t - \pi)$.

b. Find the sum of the intensities of the sounds from part (a). Use a sum-to-product identity to simplify the result. Explain what you found in terms of noise cancellation technology.

52. Ferris Wheels The function $H_A(t) = -67.5\cos\left(\frac{\pi}{15}t\right) + 67.5$ models the height from the bottom of the London Eye Ferris wheel, in meters, for Rider A t minutes after she boards.

a. Write a formula to model the height from the bottom of the Ferris wheel for Rider B—who boards the London Eye 5 minutes after Rider A—in terms of t, the number of minutes since Rider A boarded.

b. Each rider only makes one revolution on the London Eye before getting off. When were both riders at the same height?

■ STRETCH YOUR MIND

Exercises 53–59 are intended to challenge your understanding of using the trigonometric identities from this section.

53. Find an identity for $\cos(\theta + \phi + \beta)$. (*Hint:* you will need to group angles.)

54. Verify the identity.

$$\sin(6\theta) - \sin(2\theta) + \sin(12\theta) - \sin(8\theta) = 4\sin(2\theta)\cos(7\theta)\cos(3\theta)$$

55. Verify the identity.

$$\cos(5\theta) + \cos(3\theta) + \cos(8\theta) + \cos(10\theta) = 4\cos(\theta)\cos(6.5\theta)\cos(2.5\theta)$$

56. Find the exact value of $\cos(7.5°)$.

57. Find the exact value of $\tan\left(\frac{\pi}{16}\right)$.

58. Rewrite $\sin(3\theta)$ as an expression in terms of $\cos(\theta)$ and $\sin(\theta)$.

59. Rewrite $\cos(3\theta)$ as an expression in terms of $\cos(\theta)$ and $\sin(\theta)$.

CHAPTER 10 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 10.1

1. What are the differences between an expression, an equation, and an identity?
2. How do you determine if an equation is an identity?

SECTION 10.2

3. What are some guidelines to follow when attempting to verify trigonometric identities?
4. How can graphing be used to help in verifying identities?

SECTIONS 10.1 AND 10.3

5. Why are trigonometric identities useful in solving equations?

REVIEW EXERCISES

SECTION 10.1

In Exercises 1–8, use trigonometric identities to simplify each expression by writing it in terms of a single trigonometric function or as a whole number.

1. $\cos^2(\alpha)\sec(\alpha)$
2. $[\cos^2(\theta)][1 + \tan^2(\theta)]$
3. $\cos\left(\frac{\pi}{2} - \theta\right)\sec(\theta)$
4. $\frac{\cos(-\theta)}{\sin(-\theta)}$
5. $\frac{\sec(\theta) - \cos(\theta)}{\tan(\theta)}$
6. $\sec^2(\theta) - \cot^2\left(\frac{\pi}{2} - \theta\right)$
7. $\cos^2(\theta) - \sin^2(\theta)$
8. $[\sin(\theta) + \cos(\theta)]^2 + [\sin(\theta) - \cos(\theta)]^2$

In Exercises 9–16, use trigonometric identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$. Verify your solutions by graphing on a graphing calculator.

9. $-3.1 = \frac{\csc(\theta)}{\cot(\theta)}$
10. $\cos(-\theta)\tan(\theta) = 0.98$
11. $\cos\left(\frac{\pi}{2} - \theta\right)\cot(\theta) = 0.43$
12. $4\sin(\theta)\csc\left(\frac{\pi}{2} - \theta\right) = 4.35$
13. $\frac{3\sin(\theta)}{1 - \cos^2(\theta)} = 7$
14. $\frac{1 + \cot^2(-\theta)}{\csc(\theta)} = 2.6$
15. $\sin(\theta)\cos(\theta)\tan(\theta) + \cos^2(\theta) = -2\tan(\theta)$
16. $\frac{1}{\cot^2(\theta) + 1} = 0.15$

SECTION 10.2

In Exercises 17–25, verify each identity.

17. $\sin(\theta)\csc(\theta) - \sin^2(\theta) = \cos^2(\theta)$
18. $[\sec(x)][\sec(x) - \cos(x)] = \tan^2(x)$
19. $\cos^2(\beta) + \cos^2\left(\frac{\pi}{2} - \beta\right) = 1$
20. $\csc^2(\theta) = \cot^2(\theta) + \sin^2(\theta) + \cos^2(-\theta)$
21. $\sin^2(y) = \frac{1 - \sin^2(y)}{\csc^2(y) - 1}$
22. $\tan(x) = \frac{\cos\left(\frac{\pi}{2} - x\right)}{\sin\left(\frac{\pi}{2} - x\right)}$
23. $\frac{1}{\tan(\theta) + \sec(\theta)} = \frac{\cos(\theta)}{\sin(\theta) + 1}$
24. $\sec(\theta) - \cos(\theta) = \tan(\theta)\sin(\theta)$
25. $\frac{\cos^2(\theta)}{1 + \sin(\theta)} = 1 - \sin(\theta)$

In Exercises 26–29, use a graph to determine if it is likely each trigonometric equation is an identity. If it appears to be an identity, use algebra to verify. If it does not appear to be an identity, provide a counterexample to demonstrate that the two expressions are not equivalent.

26. $\sec^2(x) = \frac{\sec^2(x) - 1}{\sin^2(x)}$
27. $[\csc(x) - 1][\csc(x) + 1] = \tan^2(x)$
28. $[\csc(x)]\left[\frac{\sin(x)}{\tan(x)}\right] = 1$
29. $\sec^2(\theta) + 2 = \tan^2(\theta) + 3$

SECTION 10.3

In Exercises 30–35, find the exact value of each expression using sum and difference identities.

30. $\tan(15^\circ)$
31. $\cos(165^\circ)$
32. $\frac{\tan(140^\circ) - \tan(5^\circ)}{1 + \tan(140^\circ)\tan(5^\circ)}$
33. $\sin\left(\frac{23\pi}{12}\right)$
34. $\cos\left(-\frac{13\pi}{12}\right)$
35. $\cos\left(\frac{13\pi}{8}\right)\cos\left(\frac{3\pi}{8}\right) + \sin\left(\frac{13\pi}{8}\right)\sin\left(\frac{3\pi}{8}\right)$

In Exercises 36–37, use the sum-to-product identities to find the exact value of each expression.

36. $\sin(255^\circ) - \sin(15^\circ)$
37. $\cos\left(\frac{5\pi}{12}\right) + \cos\left(\frac{13\pi}{12}\right)$

In Exercises 38–39, use the product-to-sum identities to find the exact value of each expression.

38. $\cos(165^\circ)\cos(135^\circ)$
39. $\cos\left(-\frac{\pi}{4}\right)\sin\left(\frac{\pi}{12}\right)$

In Exercises 40–41, use double-angle identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$.

40. $\sin(2\theta) - \sin(\theta) = 0$

41. $\cos^2(\theta) - \sin^2(\theta) = 0.75$

In Exercises 42–45, use half-angle identities to find the exact value of each expression.

42. $\sin(67.5°)$

43. $\tan(-105°)$

44. $\cos\left(\frac{\pi}{12}\right)$

45. $\sin\left(-\frac{7\pi}{8}\right)$

In Exercises 46–55, verify each identity using the trigonometric identities from this chapter.

46. $\cos(\pi - \theta) = -\cos(\theta)$

47. $\cos\left(\frac{3\pi}{2} + \theta\right) = \sin(\theta)$

48. $\cot(\pi - \theta) = -\cot(\theta)$

49. $\dfrac{\sin(4\theta)}{\cos(2\theta) + \cos^2(2\theta)} = 2\tan(\theta)$

50. $\dfrac{\cos(8x) - \cos(2x)}{\cos(8x) + \cos(2x)} = \tan(5x)\tan(3x)$

51. $10\sin(\theta)\cos(\theta) = 5\sin(2\theta)$

52. $3\cos(2\theta) = 3\cos^2(\theta) - 3\sin^2(\theta)$

53. $\csc(2\theta) = \dfrac{\csc(\theta)}{2\cos(\theta)}$

54. $\cos(\theta) + 2\sin^2\left(\frac{\theta}{2}\right) = 1$

55. $2 - \cos(\theta) = \cos^2\left(\frac{\theta}{2}\right) + 3\sin^2\left(\frac{\theta}{2}\right)$

Make It Real Project

What to Do

1. Search the Internet to find a list of the frequencies for different notes on a musical instrument (such as a piano). Choose two different notes, and write down their names and frequencies. Also, give the name of the website where you found the information.
2. Use the equation $I(t) = \cos(2\pi ft)$ to model the intensity of each note, where f is the frequency in Hz (cycles per second).
3. If the two notes are played simultaneously their intensities will combine. Add the equations for the intensities of both notes, then use a trigonometric identity to write the combined equation as the product of trigonometric functions.
4. Graph the combined intensity equation found in Question 3. You may need to adjust your window range to see the graph properly. A domain from 0 to 0.05 is a good starting point.
5. Describe the pattern of the combined intensities of the two notes, then explain how the combined notes will sound.
6. Using the Internet, research the term *longitudinal wave*. Write a few paragraphs explaining how sound travels and what sinusoidal curves are modeling when they represent longitudinal waves.