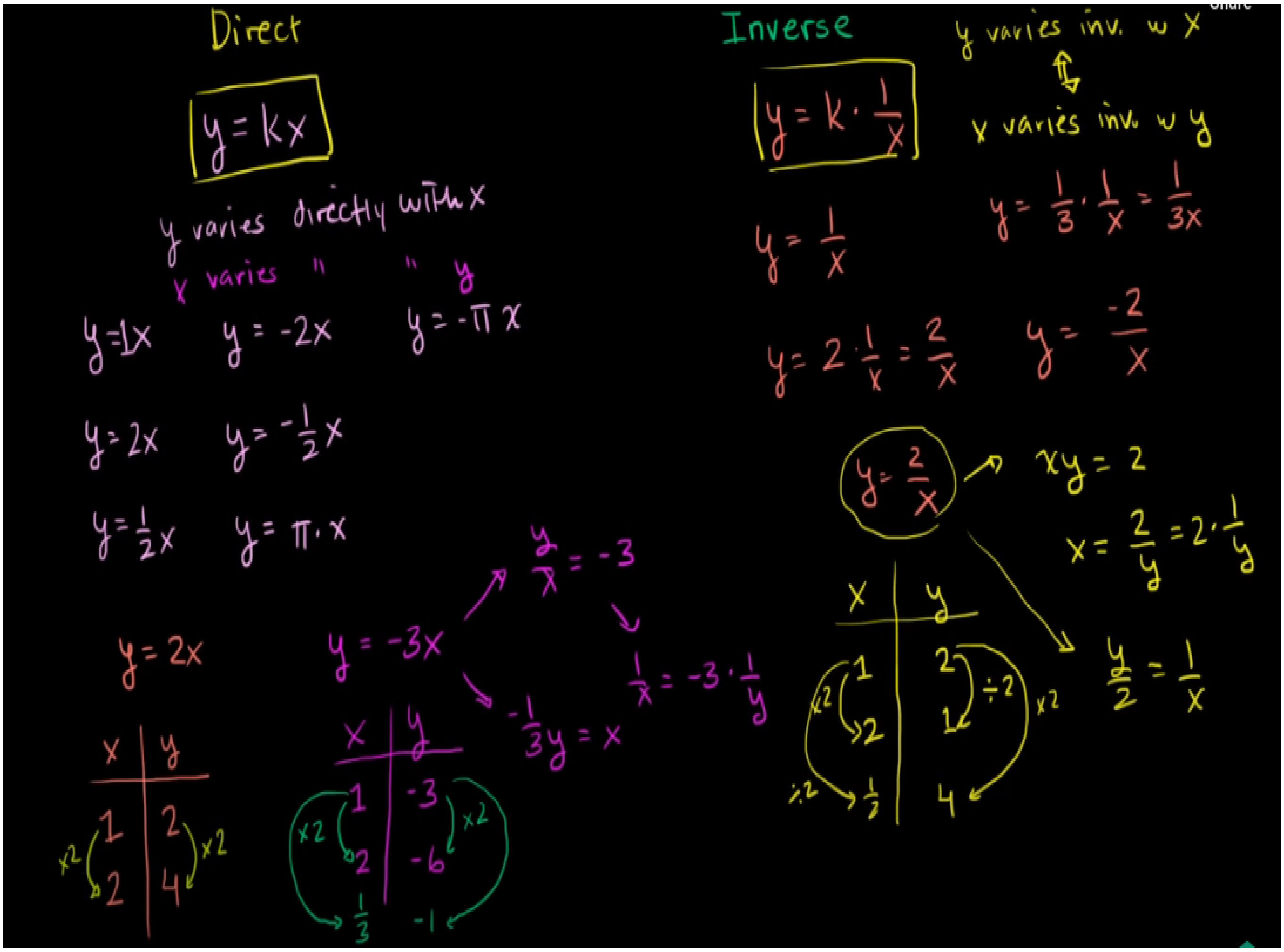
Module 1 - Modeling With Other Types of Functions

Power Functions

General Notes

Video Resources

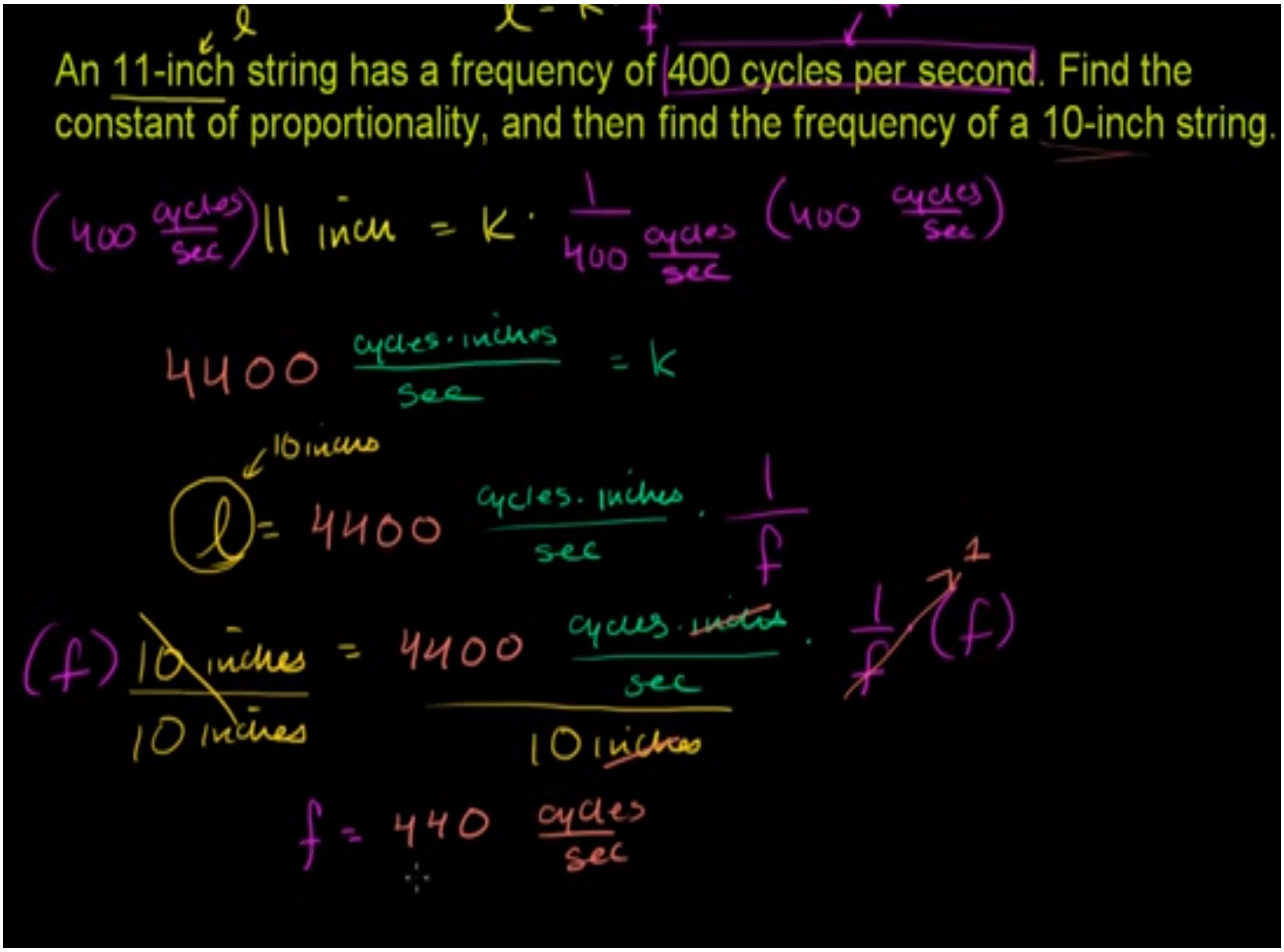
Direct and Inverse Variation



- **Direct Variation:** Scaling up **x** or **y** also scales the opposite by the same factor.
  - *I.e. If **x** is doubled, **y** is doubled.*
- **Inverse Variation:** Scaling up **x** or **y** inversely scales the opposite by the same factor.
  - *I.e. If **x** is doubled, **y** is halved.*
- As seen in the video and image, the equation can be rewritten in various ways to fool the reader. Because of this, always try to simplify the equation to either:
  - **Direct Variation: y = kx**

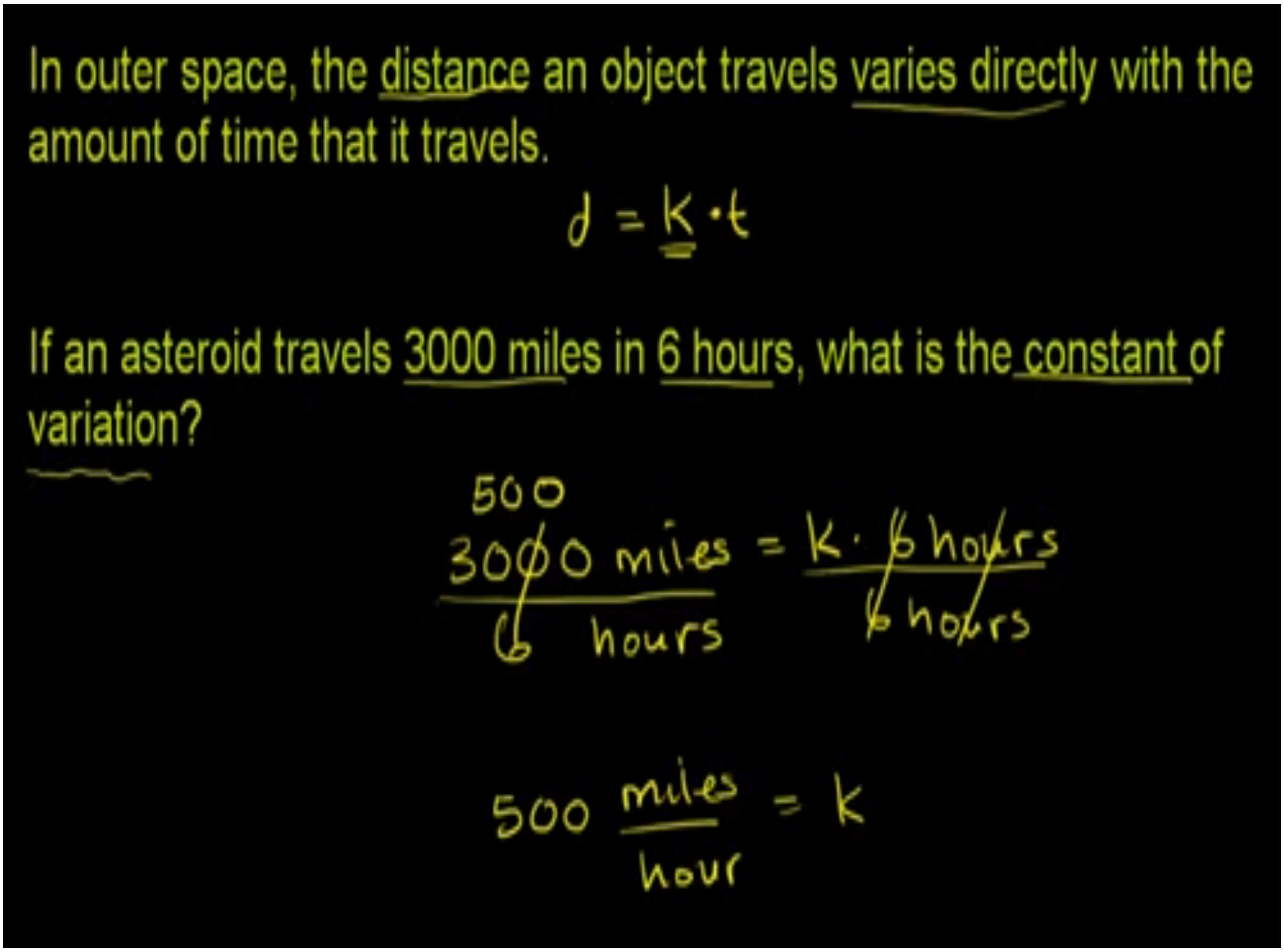
- **Inverse Variation: $y = {}^{k}\!/\!_{x}$**

Inverse Variation Application



> **Full problem:**
>
> On a string instrument, the length of a string varies inversely as the frequency of its vibrations (the vibrations are what give string instruments their sound!).
>
> An 11-inch string has a frequency of 400 cycles per second. Find the constant of proportionality, and then find the frequency of a 10-inch string.

Direct Variation Application



Power Function Behavior - Rate of Change

Rate of change can describe power functions in the following ways:

1. **Identifying Increases or Decreases**
    - The rate of change can tell us whether a power function is increasing or decreasing.
    - A **positive** rate of change means that the function is **increasing**.
    - A **negative** rate of change indicates that it is **decreasing**.
2. **Describing Speed of Change**
    - The magnitude *(or absolute value)* of the rate of change describes how quickly the function is increasing or decreasing.
    - A larger rate of change means that the function is changing more rapidly.
3. **Derivative of Power Functions**
    - The rate of change of a power function is represented by its derivative.

- For a power function of the form $y = ax^n$, the derivative is $y' = nax^{(n-1)}$. This gives the rate of change of the function at any point.

4. **Understanding the Shape**
   - The second derivative (rate of change of the rate of change) helps us understand the shape of a power function graph.
   - If the second derivative is constantly **positive**, the graph of the function is **concave up**.
   - If the second derivative is constantly **negative**, the graph is **concave down**.
5. **Analyzing Turning Points**
   - In power functions, turning points occur where the derivative (rate of change) is zero.
6. **Behavior at Infinity**
   - The behavior of a power function as it approaches infinity or negative infinity depends on the power. For example, if the power is positive, the function tends towards infinity as x approaches infinity. If the power is negative, the function tends towards zero as x approaches infinity.
7. **Predicting Future Values**
   - The derivative at a point can give us information about the function's behavior near that point. However, it's not generally accurate to say that it allows us to predict future values of the function, especially for non-linear functions like power functions.
8. **Rate of Change in Real World Contexts**
   - In real world applications, the rate of change of power functions can represent a variety of phenomena, such as the growth or decay of a population, the spread of a disease, or the change in velocity of an object. However, the specific interpretation of the rate of change depends on the context and the specific power function being considered.

Power Functions Definition

A function with the equation of the form:

> $y = ax^b$

where **a** and **b** are constants, is called a **power function**.

- The main difference between a power function and a polynomial function is that in a power function the exponent, **b**, can be any real number rather than just a positive integer.
- A power function is a single-term function, whereas a polynomial function may have multiple terms.

Solving Power Functions

Method 1 - Graphing

1. Input the table of values into Desmos and then use power regression to find the
   equation of the function.
   - The formula is: $\mathbf{y = ax^b}$
2. **a** is the slope of the line and **b** is the exponent.
3. Example: $\mathbf{y = 9.30693x^{0.12}}$

Method 2 - Algebraically

To solve a power function equation $c = ax^b$ for x, apply the following steps.

General Procedure for $c = ax^b$

1. Divide each side by a.

$$\frac{c}{a} = x^b$$

2. Raise each side to $\frac{1}{b}$.

$$\left(\frac{c}{a}\right)^{1/b} = (x^b)^{1/b}$$

3. Simplify.

$$x = \left(\frac{c}{a}\right)^{1/b}$$

Specific Procedure for $58 = 27x^{0.75}$

1. $\frac{58}{27} = x^{0.75}$

2. $\left(\frac{58}{27}\right)^{1/0.75} = (x^{0.75})^{1/0.75}$

3. $x = \left(\frac{58}{27}\right)^{1/0.75} \approx 2.77$

When solving power function equations using the streamlined method shown here, it is important to remember the meaning of the rational exponent. For example, since $0.75 = \frac{3}{4}$, we have

$$\begin{aligned} x &= \left(\frac{58}{27}\right)^{1/0.75} \\ &= \left(\frac{58}{27}\right)^{\frac{1}{3/4}} \\ &= \left(\frac{58}{27}\right)^{4/3} \\ &= \sqrt[3]{\left(\frac{58}{27}\right)^4} \end{aligned}$$

Rational Exponents and Radicals

Rational exponents can be expressed using radical notation. For example,

$$x^{1/2} = \sqrt{x}$$
$$x^{1/3} = \sqrt[3]{x}$$
$$x^{1/4} = \sqrt[4]{x}$$

If the exponent is not a unit fraction like these, we use properties of exponents to first rewrite the expression.

$$x^{2/3} = (x^2)^{1/3} = \sqrt[3]{x^2}$$
$$x^{0.35} = x^{35/100} = x^{7/20} = (x^7)^{1/20} = \sqrt[20]{x^7}$$

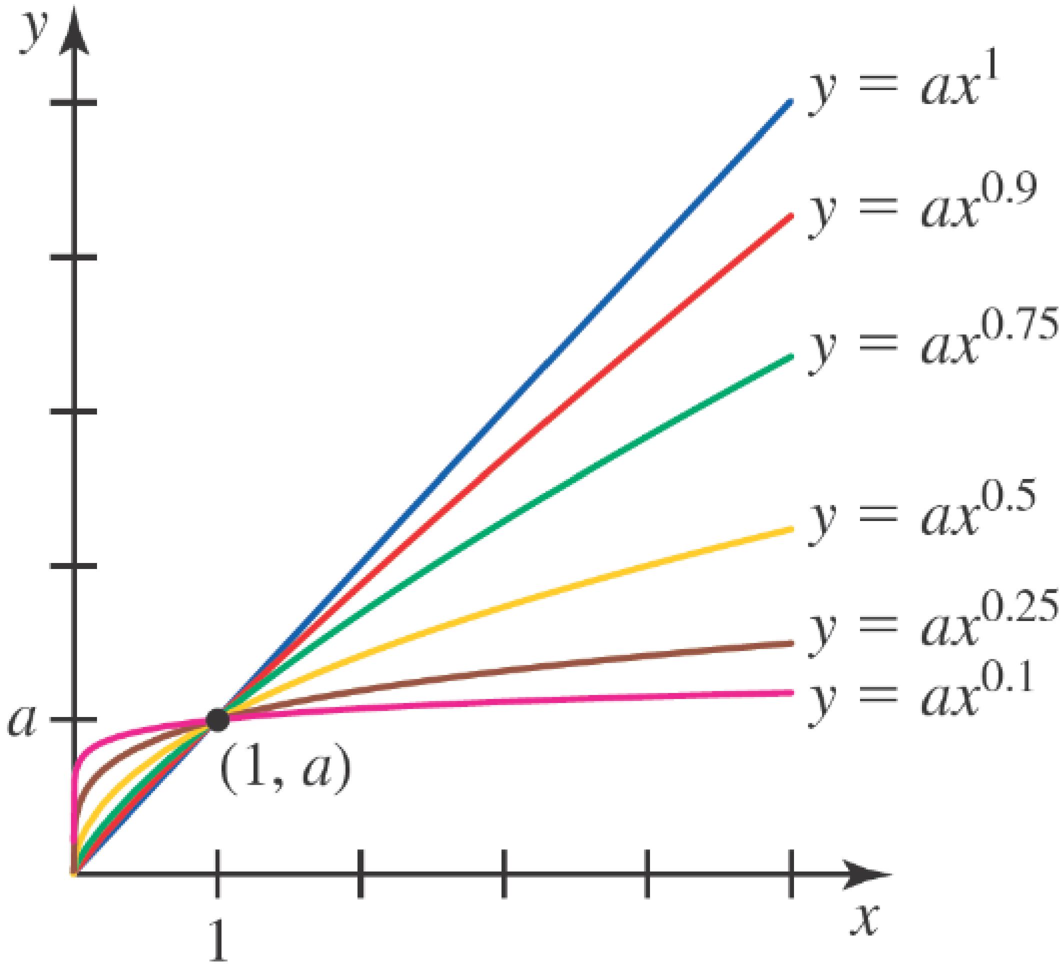
These radical expressions can also be written as

$$\sqrt[3]{x^2} = (\sqrt[3]{x})^2$$
$$\sqrt[20]{x^7} = (\sqrt[20]{x})^7$$

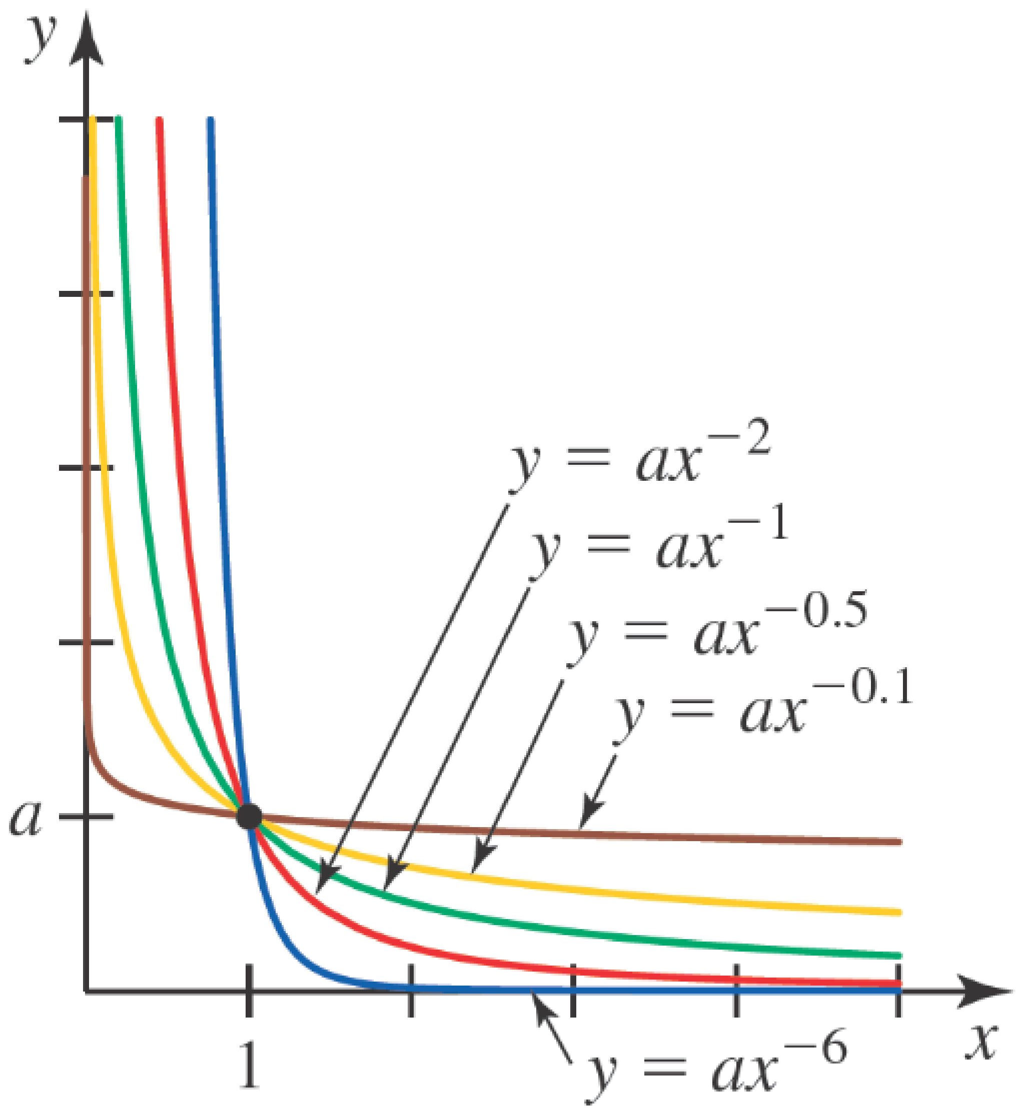
In general,

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

Power Function Graph - **x ≥ 0** and **0 < b < 1**



Power Function Graph - **x > 0** and **b < 0**



Direct and Inverse Variation

Given a power function of the form $\mathbf{y = ax^b}$ with **a > 0**.

- If b > 0, then it is an **increasing** function and $\mathbf{x^b}$ and **y** vary **directly**.
- If **b < 0**, then it is a **decreasing** function and $\mathbf{x^b}$ and **y** vary **inversely**.

When determining if a power function should be used to model a set of data, look for signs of direct or inverse variation power functions.

Direct Variation

Given a power function of the form $\mathbf{y = ax^b}$ with **a > 0**.

It is Said as either:

- *"**y** varies directly with **x^b**"*
- "***y** is directly proportional to **x^b***"

a is called the **constant of proportionality**.

Inverse Variation

The power function with **$b < 0$** should be rewritten from**
****$y = ax^b$** to **$y = ax^{-c}$**,
where **c** is a positive number equal to **|b|**.

- Alternate form useful for interpreting inverse variation.

Given a power function of the form (with **$c > 0$**):

$$y = ax^{-c} = \frac{a}{x^c},$$

It is Said as either:

- *"**y** varies inversely with **x^c**"*
- "***y** is inversely proportional to **x^c***"

a is called the **constant of proportionality**.

Real-world examples

Related Quantities	Formula	In Words
Cost of a fuel purchase and the amount of fuel bought	$C = kg$	The cost of a fuel purchase is directly proportional to the amount of fuel bought (purchased in **gallons**). **k** is the constant of proportionality and represents the fuel price per gallon.
Area of a circle and its radius	$A = \pi r^2$	The area of a circle is directly proportional to the square of its **radius**. π is the constant of proportionality.
Blood flow in an artery and the radius of the artery	$F = kr^4$	The rate at which blood flows in an artery (in **mL per minute**) is directly proportional to the fourth power of the **radius** of the artery. **k** is the constant of proportionality.
Average earnings per hour when paid a fixed amount of money to complete a task and hours worked	$A = k/x$	The average earnings per hour, **k**, is inversely proportional to the amount of money to complete a task and hours worked. **x** is the constant of proportionality and represents the fixed amount of money paid for the job.
Length of a 4-cubic-foot box with equal height and width and box width	$L = 4/w^2$	The length of a box, **L**, with equal height and width, **w**, is inversely proportional to the square of the width with a constant of proportionality, **4**.

Negative Exponents

Recall the following property of negative exponents:

$$x^{-p} = \frac{1}{x^p}$$

We can make sense of this rule by investigating patterns.

$$x^4 = x \cdot x \cdot x \cdot x$$

$$x^3 = \frac{x \cdot x \cdot x \cdot x}{x} = x \cdot x \cdot x$$

$$x^2 = \frac{x \cdot x \cdot x}{x} = x \cdot x$$

$$x^1 = \frac{x \cdot x}{x} = x$$

For each decrease in 1 of the exponent, we remove one factor of x.

$$x^0 = \frac{x}{x} = 1$$

Continuing to decrease the exponent by 1 and continuing to divide by x produces the following pattern.

$$x^{-1} = \frac{1}{x}$$

$$x^{-2} = \frac{\frac{1}{x}}{x} = \frac{1}{x^2}$$

$$x^{-3} = \frac{\frac{1}{x^2}}{x} = \frac{1}{x^3}$$

Inverses of Power Functions

Power functions that are strictly increasing or decreasing will have an inverse function.

Given: $\boldsymbol{f(x) = ax^n}$

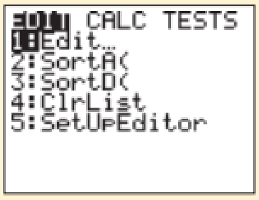
$$
\begin{aligned}
y &= ax^n \\
\frac{y}{a} &= x^n \\
\left(\frac{y}{a}\right)^{1/n} &= (x^n)^{1/n} \\
x &= \left(\frac{y}{a}\right)^{1/n} \\
f^{-1}(y) &= \left(\frac{y}{a}\right)^{1/n}
\end{aligned}
$$

- *inverse variation* and *inverse function* are not the same thing.

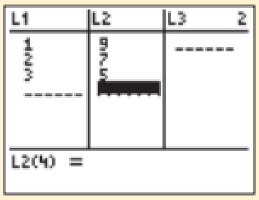
Power Function Regression on a Calculator

1. Press 2nd then 0, scroll to **DiagnosticOn** and press ENTER twice. This will ensure that the correlation coefficient r and the coefficient of determination r^2 will appear.

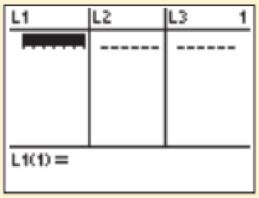
2. Bring up the Statistics Menu by pressing the STAT button.



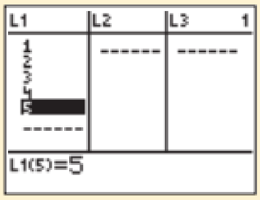
3. Bring up the List Editor by selecting **EDIT** and pressing ENTER.



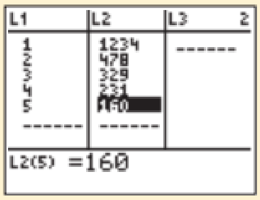
4. If there are data in the lists, clear the lists. Use the arrows to move the cursor to the list heading, L1, then press the CLEAR button and press ENTER. This clears all of the list data. Repeat for each list with data. (Warning: Be sure to use CLEAR instead of DELETE. DELETE removes the entire column.)



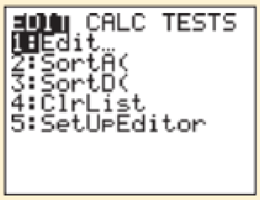
5. Enter the numeric values of the *inputs* in list L1, pressing ENTER after each entry.



6. Enter the numeric values of the *outputs* in list L2, pressing ENTER after each entry.

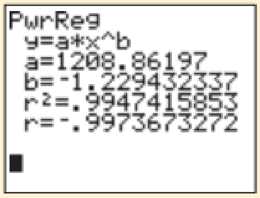


7. Return to the Statistics Menu by pressing the STAT button.



8. Bring up the Calculate Menu by using the arrows to select **CALC**. Use arrows to move down to **A:PwrReg**.

9. Calculate the power equation of the model by selecting **A:PwrReg** and pressing ENTER twice. The power regression model is $y = 1208.86x^{-1.22943}$ and has correlation coefficient $r = -0.9974$.



Math Examples

- If you are asked to fill in a table based on the representation of a power function, fill in fractions rather than decimals.

Math Example 1

- k is the constant of proportionality.

A quantity **W** is inversely proportional to the square root of a quantity **n.**

1. Write out the base equation → **W(n) = k√n**
2. Rewrite the radical as a rational exponent → **W(n) = kn$^{1/2}$**

3. Because it is looking for the inverse the equation goes from $\mathbf{y = ax^b} \rightarrow \mathbf{y = ax^{-c}}$.
    - Where **c** is a positive number equal to **|b|**.
4. Final answer → $\mathbf{W(n) = kn^{-1/2}}$

Math Example 2

Solve the given equation: $\mathbf{4x^{3/2} = 20}$

1. Divide both sides by **4** → $\mathbf{x^{3/2} = 5}$
2. Rewrite the equation as either:
    1. $\mathbf{(x^3)^{1/2} = 5}$
    2. $\mathbf{\sqrt{x^3} = 5}$
3. Square both sides → $\mathbf{x^3 = 25}$
4. Take the cube root of both sides → $\mathbf{x = \sqrt[3]{5}}$
5. Final answer → $\mathbf{x = 2.924017738212866 \approx 2.924}$

Math Example 3

Solve the given equation: $\mathbf{10x^{-2.5} = 60}$

1. Divide both sides by **10** → $\mathbf{x^{-2.5} = 6}$
2. Rewrite the equation as → $\mathbf{1/x^{2.5} = 6}$
3. Multiply both sides by the reciprocal of $\mathbf{1/x^{2.5}}$ → $\mathbf{1 = 6x^{2.5}}$
4. Divide both sides by **6** → $\mathbf{1/6 = x^{2.5}}$
5. Rewrite the equation as → $\mathbf{\sqrt[2.5]{(1/6)} = x}$
6. Final answer → $\mathbf{x = 0.4883593419 \approx 0.488}$

Power Functions