Study Guide for

Precalculus: A Make It Real Approach

by Wilson, Adamson, Cox, and O'Bryan

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Preface

The *Study Guide for Precalculus: A Make It Real Approach* is designed to help you master the content of your Precalculus course. The sections in this guide relate to the corresponding sections in the textbook. Each section is organized as described below.

Objectives

These are the tasks you should be able to complete upon completing the section.

Concepts and Definitions

These are the important ideas and mathematical terms that are addressed in the section.

Examples

These illustrate how to apply the section concepts to solve specific problems.

Exercises

These provide you with opportunities to practice problem solving. Complete solutions are also provided so that you can check your work.

Exams

Two practice exams are provided at the end of each chapter. By completing the exams, you can self-assess your mastery of the chapter objectives

As an author and educator, I continually seek to improve my own teaching effectiveness. Please feel free to provide me with candid feedback about what worked well and what needs to be improved in this guide. Contact me directly at scott@makeitreallearning.com.

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Functions and Modeling

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1.1 Functions and Modeling

Objectives

- Identify uses for symbolic, numeric, and graphical forms of mathematical models.
- Analyze mathematical models and use them to create and answer real-world questions.

Concepts and Definitions

- Mathematical Model: A graphical, verbal, numerical, and symbolic representation of a problem situation. The model helps us understand the nature of the problem situation and is often helpful in making predictions or solving the problem.

Examples

- **Example 1: A Decision-Factor Equation**
  Create a decision-factor equation, or mathematical model, that represents the situation where gas mileage, color, and number of miles are the three most important characteristics of the used car you hope to purchase.

  Solution:
  A number of different decision-factor equations could be created. Consider the following decision-factor equation which favors high gas mileage, the color red, and low miles.

  $$Decision\ Factor = 20 \cdot Gas\ Mileage + Color - \frac{Number\ of\ Miles}{100}\ \ where$$

  $$Color = \begin{cases} 50 \text{ if Red} \\ 25 \text{ Blue} \\ 0 \text{ if Other} \end{cases}$$

  Suppose a red car that gets 24 mpg and has 25,000 miles is to be compared to a blue car that gets 30 mpg and has 40,000 miles. We can use the equation as follows.

  Red: $Decision\ Factor = 20(24) + 50 - \frac{25,000}{100} = 280$

  Blue: $Decision\ Factor = 20(30) + 25 - \frac{40,000}{100} = 225$

  In this example, the red car had the higher decision-factor indicating it best met the criteria.

- **Example 2: Using a Decision-Factor Equation**
  Use the decision-factor equation,

$$\textit{Decision Factor} = 20 \cdot \textit{Gas Mileage} + \textit{Color} - \frac{\textit{Number of Miles}}{100} \text{ where}$$

$$\textit{Color} = \begin{cases} 50 \text{ if Red} \\ 25 \text{ Blue} \\ 0 \text{ if Other} \end{cases}$$

to determine which of the following three cars to purchase.

Car1: Blue, 30 MPG, 12,000 miles
Car 2: Red, 15 MPG, 15,000 miles
Car 3: White, 40 MPG, 25,000 miles

Solution:

For Car 1, we use the color scale and assign $\textit{Color} = 25$. *We assign* $\textit{Gas Mileage} = 30$ *and* $\textit{Number of Miles} = 12,000$.

$$\begin{aligned} \textit{Decision Factor} &= 20 \cdot 30 + 25 - \frac{12,000}{100} \\ &= 505 \end{aligned}$$

For Car 2, we use the color scale and assign $\textit{Color} = 50$. *We assign* $\textit{Gas Mileage} = 15$ *and* $\textit{Number of Miles} = 15,000$.

$$\begin{aligned} \textit{Decision Factor} &= 20 \cdot 15 + 50 - \frac{15,000}{100} \\ &= 200 \end{aligned}$$

For Car 3, we use the color scale and assign $\textit{Color} = 0$. *We assign* $\textit{Gas Mileage} = 40$ *and* $\textit{Number of Miles} = 25,000$.

$$\begin{aligned} \textit{Decision Factor} &= 20 \cdot 40 + 0 - \frac{25,000}{100} \\ &= 550 \end{aligned}$$

According to our mathematical model, we should purchase Car 3. Because of the heavy weight we put on fuel economy (gas mileage), it had a higher decision-factor than either of the other two cars.

Exercises

1. Analyze the given mathematical model given in numerical form that shows the weekend gross revenue for the movie Thor (boxofficemojo.com) from 2011. For this model,

   a. describe any trends you notice,

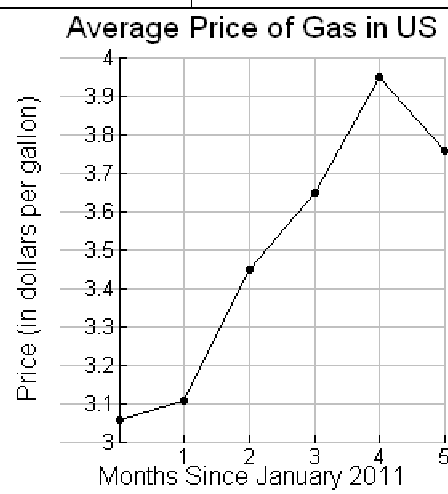
   b. write and answer a question, related to the situation, that is of interest to you.

Weekend	Gross Revenue (millions of dollars)
May 6 – 8	65.7
May 13 – 15	34.7
May 20 – 22	15.5
May 27 – 29	6.5
June 3 – 5	4.3

2. Analyze the given mathematical model given in graphical form that shows the average price of gasoline in the United States in 2011. For the model,

   a. describe any trends you notice,

   b. write and answer a question, related to the situation, that is of interest to you.



Exercise Solutions

1. a. The gross box office revenue for the movie Thor is decreasing each subsequent weekend. Gross revenue dropped by $31 million from the first to second weekend followed by a drop of $19.2 million from the second to the third weekend. The gross revenue then dropped by $9 million from the third to the fourth weekend and by $2.2 million from the fourth to the fifth weekend.

   b. What is the total gross box office revenue for the movie Thor over the given weekends? If we sum up the revenues from the individual weekends, we find that the total revenue is $126.7 million.

2. a. From January until May, the price of gasoline is increasing. The price increases relatively slowly at first, and then increases at a much greater rate before reaching a peak in May. The price of gasoline then deceases from May to June.

   b. If the price of gasoline continues to decrease as it did from May to June, what will the price of gasoline be in September, 2011? We see that the price of gasoline dropped from about $3.95 in May to about $3.76 in June. This is a drop of $0.19. If the price continues to drop $0.19 each month from June to September (3 months), the price will drop $0.19 three more times or $0.57. Subtracting this from $3.76, we predict that the price of gasoline in September 2011 will be $3.19.

1.2 Functions and Function Notation

Objectives

- Write and interpret functions using function notation.
- Explain how a function is a process or a correspondence.
- Solve function equations for a given variable using an equation, table, and graph.

Concepts and Definitions

- **Variable:** A variable is a quantity that changes value.
- **Constant:** A constant is a numeric value that remains the same.
- **Function (Single Variable):** A single-variable function is a process or correspondence relating two quantities in which each input value generates exactly one output value.
- **Vertical Line Test:** If each vertical line drawn on a graph intersects the graph in at most one place, the graph is the graph of a function.
- **Function Notation:** The component parts of function notation include an input, an output, and a function. We use the symbols $f(x) = y$.
- **Evaluating a Function:** The process of finding the output of a function that corresponds with a given input is called evaluating a function.
- **Solving a Function Equation:** The process of finding the input of a function that corresponds with a given output is called solving a function equation.
- **Condensed Function Notation:** When calculating particular function values, it is customary to condense the function notation $D = f(h)$ to $D(h)$. The equation $D(a) = b$ means that when $h = a$, $D = b$. This equation is written as $D = f(a) = b$ in standard function notation.

Examples

- **Example 1: Evaluating a Function Using Function Notation**
  Evaluate the function $s(t) = t^3 + 4t$ at $t = 3$, $t = \Delta$, and $t = a^2$.

  Solution:

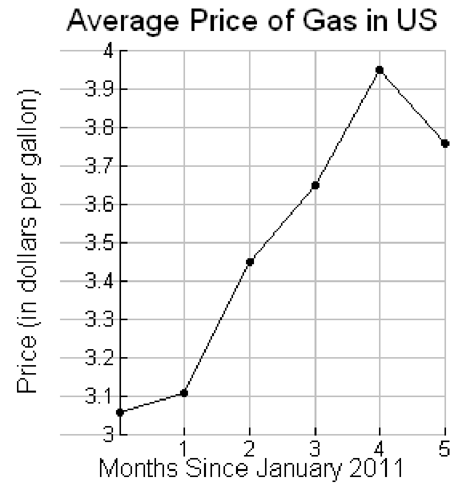
$$\begin{aligned} s(3) &= (3)^3 + 4(3) \\ &= 27 + 12 \\ &= 39 \end{aligned}$$

$$s(\Delta) = (\Delta)^3 + 4(\Delta)$$

$$\begin{aligned} s(a^2) &= (a^2)^3 + 4(a^2) \\ &= a^6 + 4a^2 \end{aligned}$$

- **Example 2: Evaluating and Solving a Function Equation from a Graph**

The price of gasoline in the United States has been fluctuating during 2011. As shown in the following figure, the average cost for gasoline per month in dollars per gallon (C) is a function of the number of months since January 2011 ($t = 0$). We write this in function notation as $f(t) = C$.



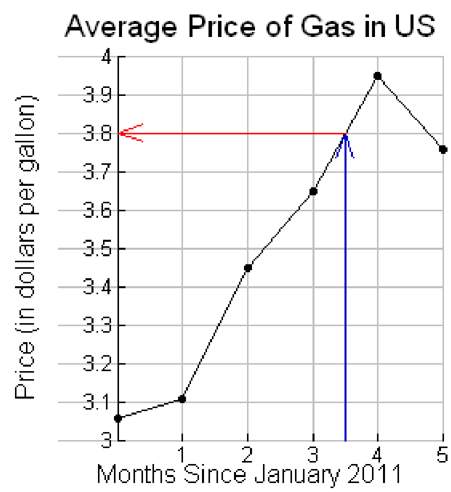
Use the graph to do each of the following:

a. Estimate $f(3.5)$ and explain in a complete sentence what your solution means in its real-world context.

b. Find a value of t such that $f(t) = 3.30$. and explain in a complete sentence what your solution means in its real-world context.

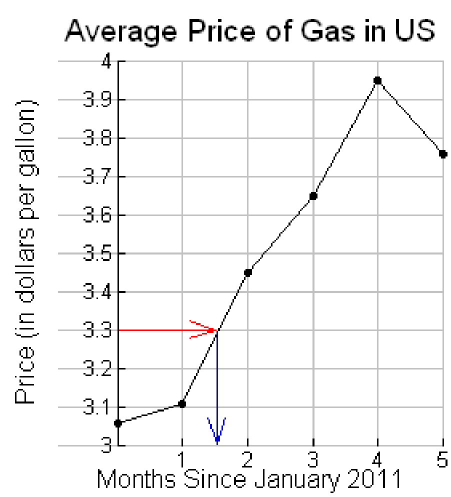
Solution:

a. To estimate $f(3.5)$ means to find the value of the average price of gasoline 3.5 months after January 2011. That is, the middle of April 2011. We find the t value of 3.5 on the horizontal axis, go "up" to the function graph, and then "over" to the vertical axis and find a C value of approximately 3.80.



Therefore, according to our mathematical model in the middle of April 2011 the average cost of gasoline in the United States was $3.80 per gallon.

b. We are to find a value of t such that $f(t) = 3.30$. That is, we are to estimate the month in which the average cost of gasoline in the United States was $3.30 per gallon. We locate 3.30. on the vertical axis, go "over" to the function graph, and then "down" to the horizontal axis to find a t value of approximately 1.5.



Therefore, the price of gasoline was $3.30 per gallon in the middle of February 2011, according to our mathematical model.

Exercises

1. The function $T = f(d) = 0.00151d - 0.062701$ gives the relationship between the time, T (in seconds) that it takes a sniper's bullet to hit its target at a distance, d in yards.

   a. Evaluate $f(900)$ and explain in a complete sentence what your solution means in its real-world context.

   b. Find a value of d such that $f(d) = 1.25$ and explain in a complete sentence what your solution means in its real- world context.

2. Your credit score C may depend on a score for your payment history (p), the amounts owed (a), the length of your credit history (l), the amount of your new credit requested (n), and a numerical score for the types of credit you have in use (t). One model used to calculate a FICO score is weighted in the following manner: 35% payment history (p), 30% amounts owed (a), 15% length of credit history (l), 10% new credit (n), and 10% types of credit used (t). (Source: www.fico.com) Suppose $C = f(p,a,l,n,t) = 0.035p + 0.30a + 0.15l + 0.10n + 0.10t$, evaluate $f(1100, 900, 760, 580, 150)$ and then explain in a complete sentence what your solution means in its real-world context.

Exercise Solutions

1. a. $T = f(900) = 0.00151(900) - 0.062701 = 1.2963$

   When a sniper shoots from a distance of 900 yards, it will take 1.2963 seconds for the bullet to hit its target.

   b.

$$
\begin{aligned}
0.00151(d) - 0.062701 &= 1.25 \\
0.00151(d) &= 1.312701 \\
d &= \frac{1.312701}{0.00151} \approx 869.34
\end{aligned}
$$

   If it takes 1.25 seconds for a sniper's bullet to hit its target, the sniper fired the gun from a distance of about 869.34 yards.

2. $f(1100, 900, 760, 580, 150) = 797$ means that someone with a payment history $p = 1100$, amounts owed $a = 900$, length of credit history $l = 760$, amount of new credit requested $n = 580$, and a numerical score for the types of credit $t = 150$ would have a credit score of 797, a very good score which will likely qualify for some of the lowest interest rates.

1.3 Functions Represented by Tables and Formulas

Objectives

- Determine if a data table represents a function.
- Calculate and interpret the meaning of an average rate of change from a table.
- Create and use basic function formulas to model real-world situations.

Concepts and Definitions

- **Average Rate of Change:** The hypothetical value describing the constant rate of change necessary to move from $(a, f(a))$ to $(b, f(b))$ in a linear fashion. This value is also the slope of the secant line between the two points $(a, f(a))$ to $(b, f(b))$.
- **Average rate of change:** $m = \dfrac{f(b) - f(a)}{b - a}$

Examples

- **Example 1: Determining an Average Rate of Change**
  Using the data in the consumer debt table given below, calculate the average rate of change in consumer debt between 1990 and 1995. Then explain in a complete sentence what the number means in a real-world context. *Source: Federal Reserve Board*

d Year	*C* Total Consumer Debt (in billions of dollars)
1985	593.00
1990	789.10
1995	1095.80
2000	1556.25
2005	2175.25

Solution:

To calculate the average rate of change, we divide the difference of the outputs by the difference in the corresponding inputs.

From the table, we have $(1990, 789.10)$ *and* $(1995, 1095.80)$

$$\frac{debt_2 - debt_1}{year_2 - year_1} = \frac{1095.80 - 789.10}{1995 - 1990} = 61.34$$

This average rate of change tells us that if the total consumer debt was to increase at a constant annual rate from \$593 billion in 1985 to \$2175.25 billion in 2005, that constant rate would have had to have been \$61.34 billion per year. More succinctly, we say that the total consumer debt increased by an average of 61.34 billion dollars per year between 1990 and 1995.

- **Example 2: Using a Multivariable Formula**

  $B(p, r, n, t) = p\,(1 + \frac{r}{n})^{nt}$ is the formula used to calculate the balance, B, in an investment account that is a lump sum invested compounded a fixed number of times a year for a certain number of years. The independent variable p is the amount invested, r is the nominal rate as a decimal, n is the number of times the compounding occurs in a year, and t is the years the money is invested. Compute the value of $B(500, 0.05, 12, 2)$ and explain what this value means in the context of this situation.

  Solution:

  $B(500, 0.05, 12, 2) = 500(1 + \frac{0.05}{12})^{12 \cdot 2} = 552.47$ *If \$500 is invested at a nominal rate of 5% compounded monthly for 2 years, the balance of this investment account at the end of the 2 years is \$552.47.*

Exercises

1. Estimate the number of private trips that were given to U. S. Lawmakers in April 2006.

Private Trips Given to US Lawmakers in 2006		
m **Month**	***N*** **Number of Trips**	**Average Rate of Change**
January	158	
February	61	
March	29	

Source: Political Money Line

2. Sujatha is buying a 2006 True Red MX-5 Mazda Miata with the premium option package advertised on www.edmunds.com for \$27,695 in June, 2006. Taxes and fees amount to an additional \$3,450.00. Sujatha will make a down payment of \$5,000. E-Loan at edmunds.com will loan her the money for five years at 6.15% or for six years at 6.69%.

How much extra would her payments be if she financed the car for five years instead of six? How much money would she save?

Exercise Solutions

1. At the current average rate of change of $-32\frac{\text{trips}}{\text{month}}$, the estimated number of trips in April 2006 is less than 0. We estimate that lawmakers are no longer making private trips in April 2006. In reality, the average rate of change of $-32\frac{\text{trips}}{\text{month}}$ did not continue but the number of private trips certainly is greatly reduced.

Private Trips Given to US Lawmakers in 2006		
m **Month**	***N*** **Number of Trips**	**Average Rate of Change**
January	158	$-97\frac{\text{trips}}{\text{month}}$
February	61	
March	29	$-32\frac{\text{trips}}{\text{month}}$

2. We use the formula $M(p,i,n) = p\left(\frac{i(1+i)^n}{(1+i)^n - 1}\right)$ to compute the monthly payment for each interest rate.

Case 1:

$$M(26145, 0.005125, 60) = 26145\left(\frac{0.005125(1+0.005125)^{60}}{(1+0.005125)^{60} - 1}\right) = \$571.94$$

Case 2:

$$M(26145, 0.0009583, 72) = 26145\left(\frac{0.0009583(1+0.0009583)^{60}}{(1+0.0009583)^{60} - 1}\right) = \$448.61$$

At \$571.94 per month for 5 years, Sujatha will pay a total of $\$571.94 \cdot 60 = \$34316.40$. At \$448.61 per month for 6 years, Sujatha will pay a total of $\$448.61 \cdot 72 = \$32299.92$. If she were to finance the car for 5 years instead of 6, she would pay an extra \$123.33 per month. She would save a total of \$2016.48.

1.4 Functions Represented by Graphs

Objectives

- Graph functions on the rectangular coordinate system.
- Find a function's practical domain and practical range values from its graph.
- Determine the vertical intercept (initial value) and horizontal intercepts of a function from a graph and interpret their real-world meanings.

Concepts and Definitions

- **Domain:** The set of all possible values of the independent variable (input) of a function is called its domain**.** When we limit the domain to values that make sense in the real-world context of a problem, we get the practical domain.
- **Range:** The set of all possible values of the dependent variable (output) of a function is called its range. When we limit the range to values that make sense in the real-world context of a problem, we get the practical range.

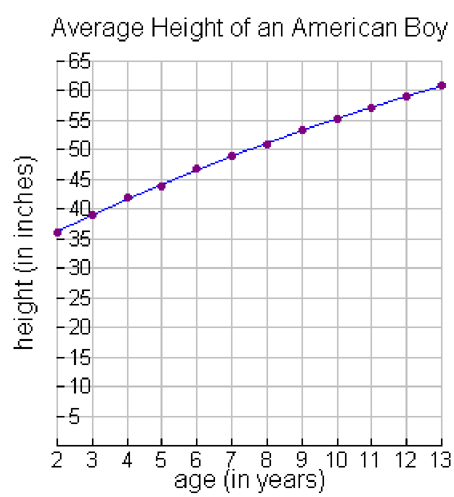
Examples

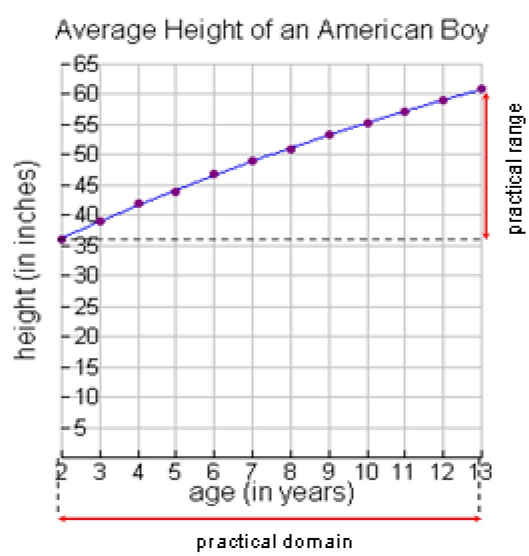
- **Example 1: Determining Practical Domain and Range**
  Use the graph of the average height function, $H(t) = -.05t^2 + 2.99t + 30.44$, for American males ages two to thirteen to determine the practical domain and range.

  Solution:

  The practical domain for the height function is $2 \le t \le 13$ *because the graph goes from ages 2 to 13 in the horizontal direction. It seems reasonable to choose to extend the domain a few years earlier and later than what is provided in the original table of values. If we extend the domain as described, we may conclude the practical domain could possibly be* $1 \le t \le 15$. *However, it couldn't be* $-10 \le t \le 65$ *because no one can be* -10 *years-old and males don't continue to grow in height up to age 65. In order for this model to make sense, we must choose reasonable or practical boundaries to the domain.*

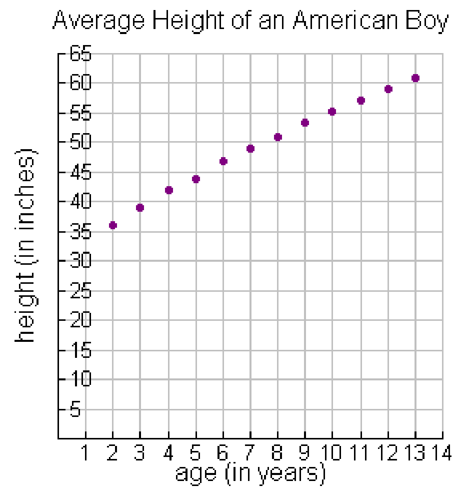
  The practical range for the function is $36 \le H \le 61$ *for the ages 2 to 13 because the height of the children goes from 36 to 61 inches. As with the practical domain, it is*





reasonable to slightly extend the range to smaller and larger heights. See the diagram for a pictorial representation of the practical domain and practical range.

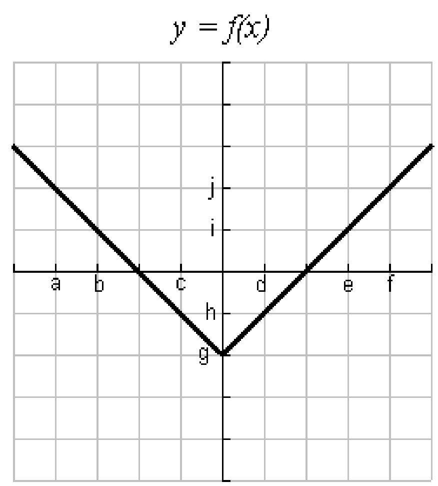
- **Example 2: Interpreting the Real-world Meaning of a Graph**
  The table at the start of the section provided us with input values for children's ages and output values for their associated heights. If we graphically represent these data, we get the graph shown. Interpret the real-world meaning of $(2, 36)$ and $(11, 57.25)$.



Solution:

Remember that each point on the graph represents a single data point from the table. The point $(2, 36)$ *means that the average height for a two-year-old male in America is* 36 *inches. The point* $(11, 57.25)$ *means the average height of an eleven-year old boy is* 57.25 *inches.*

Exercises



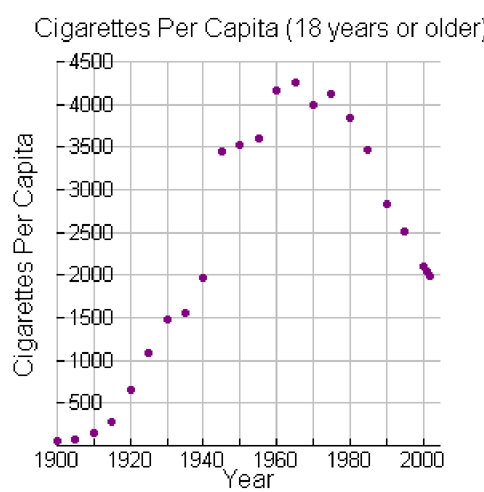
1. Refer to the graph of $f(x)$ with x- values *a, b, c, d, e, f* and y-values *g, h,* and *i*. Determine a possible solution to each equation and a function value for each expression.

   a. $f(a)$

   b. $f(x) = j$

   c. $f(c)$

   d. $f(x) = h$



2. The number of cigarettes consumed by adults over the age of 18 in the United States, c, is a function of the years since 1900, t, and is given in the graph below. (Source: www.infoplease.com)

   a. Determine the practical domain and range for $m(y)$.

   b. Explain information the graph provides its real-world context.

c. What does the vertical intercept mean in its real-world context?

Exercise Solutions

1. a. $f(a) = j$

b. If $f(x) = j$, $x = a, f$

c. $f(c) = h$

d. If $f(x) = h$, $x = c, d$

2. a. Answers may vary. The practical domain is $[0,120]$ which corresponds to the years 1900 to 2020. The practical range is $(0, 4300)$ which corresponds to the number of cigarettes consumed per capita in that year.

b. The per capita consumption of cigarettes increases from 1900 to about 1965. Then the per capita consumption of cigarettes decreases. The trend is that fewer cigarettes are being consumed over recent history.

c. The vertical intercept, $(0,100)$ represents that in 1900 ($t = 0$), about 100 cigarettes are consumed per capita.

1.5 Functions Represented by Words

Objectives

- Convert words representing function relationships into symbolic and graphical representations.
- Translate functions given in equations, tables, and graphs into words.

Concepts and Definitions

- Recognize a function given in words and translate into mathematical symbols.

Examples

- **Example 1: Translating a Function in Words into Symbolic Notation**
  As of June 13, 2011, a forest fire known as the Wallow Fire burned an average of 30,144 acres per day (azcentral.com). Write a function formula showing the total number of acres burned as a function of the number of days elapsed since the fire started.

  Solution:

  With an average of 30,144 acres per day burned, we find the total number of acres burned by multiplying the 30,144 acres per day by the number of days that have elapsed since the start of the fire. We express this as a function by writing $A(d)=30144d$ *where* d *represents the number of days since the start of the fire and* A *represents the number of acres burned.*

- **Example 2: Interpreting in Words the Meaning of a Function Equation**
  Based on data from 2000 – 2004, the amount of revenue generated by cell phone sales may be modeled by

  $$C\left(p\right)=-0.002745p^3+6.173p^2-4633p+1{,}169{,}000$$

  million dollars where p is the amount of revenue generated by pager sales (in millions of dollars). What does $C\left(811\right)=8077$ mean?

  Solution:

  When \$811 million of revenue are generated by pager sales, \$8077 million of revenue is generated by cell phone sales.

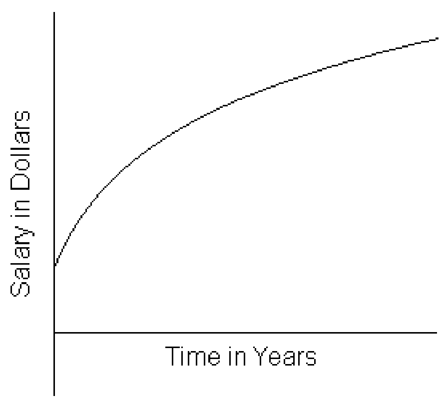
Exercises

1. According to the University of California-Berkeley School of Business (www.haas.berkeley.edu), “The average starting salaries for graduates continue to increase, but at a decreasing rate over years prior.” Sketch a graph to represent fall enrollments as a function of time.

2. Due to inflation, the buying power of the dollar decreases over time. Based on data from 1990 – 2004, the buying power of the current dollar may be modeled by $V(t) = -0.0107t + 0.836$ of a 1980 dollar where t is the number of years after 1990. What does $V(37) = 0.44$ mean?

Exercise Solutions

1.



2. In 2027 (37 years after 1990), the value of a dollar is predicted to be 0.44 of a 1980 dollar. That is, in 2027, the value of the dollar will be 44% of the value of a 1980 dollar.

1.6 Preview of Inverse Functions

Objectives

- Explain the relationship between a function and its inverse.
- Explain and use inverse function notation to solve real-world problems.
- Find the inverse of a function from a table or graph and interpret its practical meaning.

Concepts and Definitions

- A function f^{-1} (read " f inverse") is the function whose inputs are the outputs of f and whose outputs are the inputs of f.

Examples

- **Example 1: Interpreting an Inverse Function Model**
  As of June 13, 2011, a forest fire known as the Wallow Fire burned an average of 30,144 acres per day (azcentral.com). The function formula $A(d) = 30144d$ models the total number of acres, A, burned d days after the start of the fire. The inverse function for this situation is $d^{-1}(A) = \dfrac{A}{30144}$. Evaluate $d^{-1}(500000)$ and explain the practical meaning of the solution.

  Solution:

  The notation $d^{-1}(500000)$ *takes 500000 acres burned as the input and determines how many days have elapsed since the start of the fire it would take before burning that many acres.*

  $$\begin{aligned} d^{-1}(500000) &= \frac{500000}{30144} \\ &\approx 16.6 \end{aligned}$$

  At the average rate of 30,144 acres per day, it will take 16.6 days for the fire to grow to a size of 500,000 acres.

- **Example 2: Making Sense of an Inverse Function**
  For the give situation, do the following:

  a. describe the input and output variables as a function, if possible;

  b. describe what information the inverse relationship to each situation would provide;

  c. determine whether or not the inverse relationship represents a function and justify.

  The length of time it takes for a movie theater to empty at the end of a movie depends on the number of people who attended the showing of the movie.

Solution:

a. The input variable is the number of people attending the movie and the output is the time needed to empty the theater. That is, the time to empty the theater is a function of the number of attendees.

b. The inverse function would determine the number of people in a theater given the total time needed to empty the theater. That is, the number of attendees is a function of the time to empty the theater.

c. On average, the more time needed to empty the theater, the more people that must have attended the movie showing. That is, the function is strictly decreasing meaning that the inverse is indeed a function.

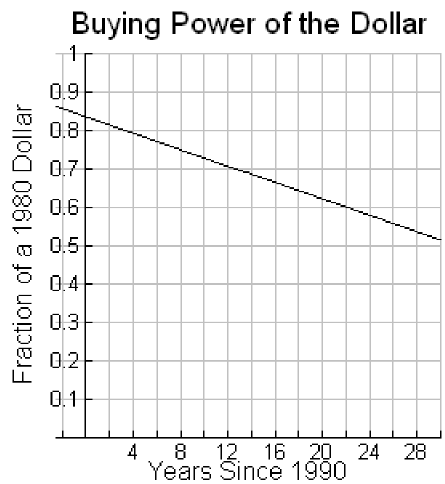
Exercises

1. The function $T = f(d) = 0.00151d - 0.062701$ gives the relationship between the time, T (in seconds) that it takes a sniper's bullet to hit its target at a distance, d in yards.

   a. Find the inverse of this function.

   b. Describe what information the inverse relationship would provide.

2. Due to inflation, the buying power of the dollar decreases over time. Based on data from 1990 – 2004, the buying power of the current dollar may be modeled by $V(t) = -0.0107t + 0.836$ of a 1980 dollar where t is the number of years after 1990. The graph of the function is shown. Use the graph to find the value of $t^{-1}(0.6)$. Describe what information this provides.



Exercise Solutions

1. a. $d = f^{-1}(T) = \dfrac{T + 0.062701}{0.00151}$

   b. The inverse function takes time (in seconds) as the input and outputs the distance from the target. That is, if we know the time it took for a sniper's bullet to hit its target, we can determine the distance from which the bullet was shot.

2. $t^{-1}(0.6) = 22$ which tells us that 22 years after 1990 (2012), the value of a dollar is 0.6 of a 1980 dollar. That is, the value of a 2012 dollar will be 60% of the value of a 1980 dollar.

Chapter 1 Exam A

1. Estimate the function values or solve for x as indicated using the graph of $h(x)$ below.

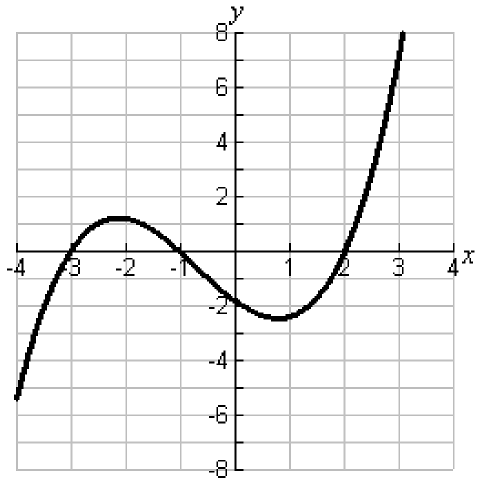
   a. $h(3)$

   b. $h(x) = -5$

   c. $h(x) = 0$

   d. $h(0)$

   e. $h(-2)$



2. The table gives the average number of acres per farm in the United States between 1950 and 2000.

   a. Complete the table.

Average Number of Acres per Farm in the United States		
d **Year (*d* = 0 is 1900)**	***A*** **Average Number of Acres**	**Average Rate of Change**
50	207.78	
60	306.23	
70	379.08	
80	426.33	
90	447.98	
100	444.03	

b. Using words, explain what any one of the rate of change computations means in the context of this situation.

c. Find $A(90)$ and explain what it means.

d. Using the idea of average rate of change, find $A(103)$ and explain what it means.

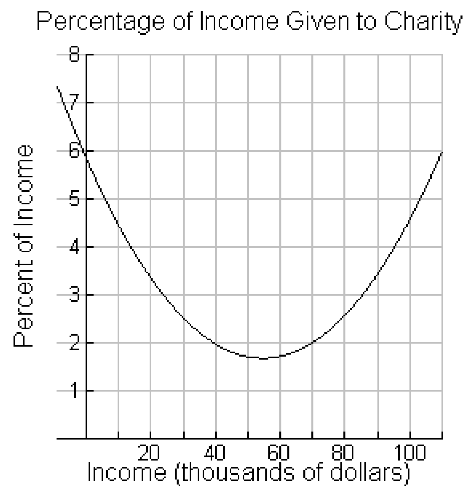
3. According to a survey conducted in 1990, the percentage P of income that Americans give to charities is related to their annual income, I, in thousands of dollars. For families with an annual income of $100,000 or less, the percentage is approximately given by

$$P = f(I) = 0.0014I^2 - 0.1529I + 5.855$$

a. A graph of $P = f(I)$ is provided. Use the graph to solve the equation $4 = f(I)$ and explain what the result means.

b. Use the graph to determine the value of $f(55)$. Explain what the result means.

c. Find the average rate of change between $I = 20$ and $I = 50$ and explain what the value means in the context of this situation.



4. Create a graph to match this story. Graph ***time*** on the horizontal axis and ***height above the ground*** on the vertical axis.

My son Zachary starts off by standing on a trampoline. He begins bouncing and after a few bounces is reaching a maximum height with each bounce. After a few bounces that take him to a maximum height, he stops exerting effort and after some time, returns to his initial position of just standing on the trampoline.

Chapter 1 Exam A Solutions

1. Estimate the function values or solve for x as indicated using the graph of $h(x)$ below.

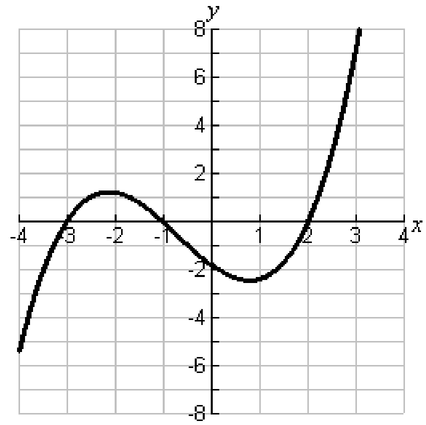
   a. $h(3) = 7$

   b. $h(x) = -5,\ x = -4$

   c. $h(x) = 0,\ x = -3, -1, 2$

   d. $h(0) = -2$

   e. $h(-2) \approx 1.1$



2. The table gives the average number of acres per farm in the United States between 1950 and 2000.

   a. Complete the table.

Average Number of Acres per Farm in the United States		
d **Year** **(*d* = 0 is 1900)**	***A*** **Average Number of Acres**	**Average Rate of Change**
50	207.78	$\frac{306.23 - 207.78}{10} = 9.845$ acres per year
60	306.23	$\frac{379.08 - 306.23}{10} = 7.285$ acres per year
70	379.08	$\frac{426.33 - 379.08}{10} = 4.725$ acres per year
80	426.33	$\frac{447.98 - 426.33}{10} = 2.165$ acres per year
90	447.98	$\frac{444.03 - 447.98}{10} = -0.395$ acres per year
100	444.03	

b. Using words, explain what any one of the rate of change computations means in the context of this situation.

From 1990 to 2000, the average number of acres per farm in the United States is decreasing at a rate of 0.395 acres per year.

c. Find $A(90)$ and explain what it means.

$A(90) = 447.98$

This means that in 1990 (90 years after 1900), the average number of acres per farm in the United States was 447.98 acres.

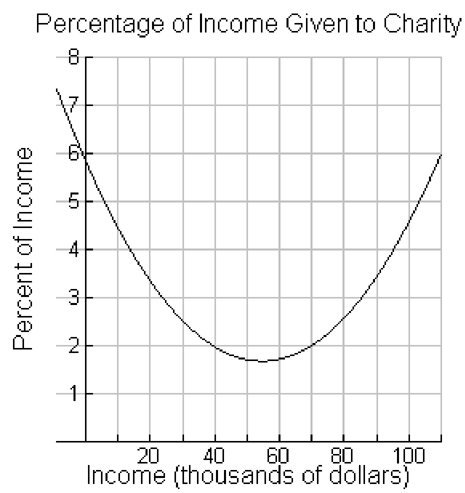
d. Using the idea of average rate of change, find $A(103)$ and explain what it means.

$$\begin{aligned} A(103) &= 444.03 - 0.395(3) \\ &= 442.845 \text{ acres} \end{aligned}$$

The predicted average number of acres per farm in 2003 (103 years after 1900) is 442.845 acres.

3. According to a survey conducted in 1990, the percentage P of income that Americans give to charities is related to their annual income, I, in thousands of dollars. For families with an annual income of \$100,000 or less, the percentage is approximately given by

$$P = f(I) = 0.0014I^2 - 0.1529I + 5.855$$



a. A graph of $P = f(I)$ is provided. Use the graph to solve the equation $4 = f(I)$ and explain what the result means.

$$4 = f(I)$$
$$I \approx \$15,000 \text{ and } \$95,000$$

If the annual income is \$15,000 or \$95,000, the model predicts that these Americans give 4% of their income to charity.

b. Use the graph to determine the value of $f(55)$. Explain what the result means.

$$f(55) \approx 1.8$$

This means that for Americans making \$55,000 in annual income, the model predicts that they give 1.8% of their income to charity.

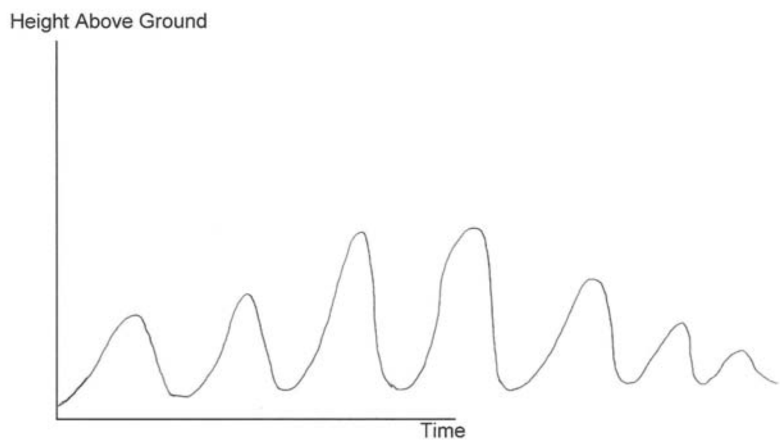
c. Find the average rate of change between $I = 20$ and $I = 50$ and explain what the value means in the context of this situation.

$$\frac{f(50) - f(20)}{50 - 20} = \frac{1.8 - 3.4}{30} = \frac{-1.6}{30} \approx -0.0533$$

On the interval from \$20,000 to \$50,000, for each additional thousand dollars in annual income, Americans give an average of 0.0533 percent less of their income to charity.

4. Create a graph to match this story. Graph ***time*** on the horizontal axis and ***height above the ground*** on the vertical axis.

My son Zachary starts off by standing on a trampoline. He begins bouncing and after a few bounces is reaching a maximum height with each bounce. After a few bounces that take him to a maximum height, he stops exerting effort and after some time, returns to his initial position of just standing on the trampoline.



Chapter 1 Exam B

1. Author Bill James, in his book *Baseball Abstracts*, created a formula to measure the success of a professional baseball player. The level of success was determined by the number of runs that were created by a player (since creating runs is the object of the game). James' formula is:

$$R(H,W,T,B)=\frac{(H+W)\cdot T}{B+W}$$

where R is *Runs Created*, H is number of *Hits*, W is number of *Walks*, T is *Total Bases*, and B is number of *At Bats*.

a. Joe Mauer is a star player for the Minnesota Twins. Using the stats found at www.mlb.com, evaluate the function $R(1019,436,1496,3126)$ and explain what $R(1019,436,1496,3126)$ means.

b. Assuming all other numbers stayed the same, how many hits would Joe Mauer need to have had in order to have a *Runs Created* (R) value of 700?

2. The total money spent by Americans to watch movies at a theater is recorded in the table below (Source: Statistical Abstract of the United States, 2007, Table 1218).

a. Complete the table.

Admissions to Motion Picture Theaters		
y **Year** **(*y* = 0 is 1990)**	***E*** **Expenditure** **(in billion dollars)**	**Average Rate of Change**
0	5.1	
5	5.6	
10	8.6	
14	9.9	

b. Using words, explain what any one of the rate of change computations means in the context of this situation.

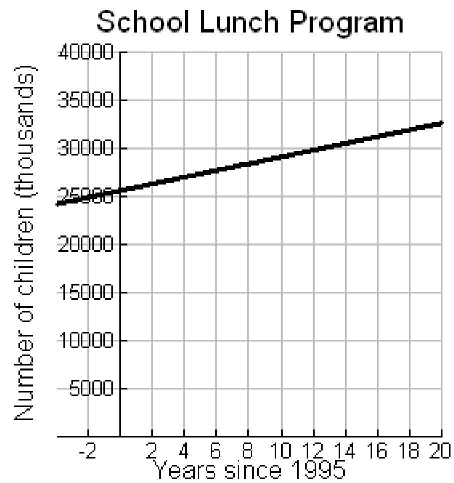
c. Find $A(10)$ and explain what it means.

d. Using the idea of average rate of change, find $A(20)$ and explain what it means.

4. Based on data from 1995-2004, the number of children participating in the federal school lunch program may be modeled by
$L = f(t) = 351.53t + 25587$ thousand children
where t is the number of years after 1995 (Source: USDA - www.nass.usda.gov).

a. A graph of $L = f(t)$ is given provided. Use the graph to solve the equation $30,000 = f(t)$. Explain what the result means.

b. Use the graph to determine the value of $f(20)$. Explain what the result means.



c. Find the average rate of change between t = 0 and t = 12 and explain what the value means in the context of this situation.

4. Sketch a graph to match this story – graph ***time*** on the horizontal axis and ***distance from home*** on the vertical axis.

My son Matthew leaves home to go to school. He drives at a constant average speed of 35 safe miles per hour. After 4 minutes, he realizes that he forgot his homework at home and turns around to retrieve the homework. Once home, it takes him 10 minutes to find the homework. He then drives back toward school at a constant average speed of 42 miles per hour. He arrives at school which is 8 miles from home.

Chapter 1 Exam B Solutions

1. Author Bill James, in his book *Baseball Abstracts*, created a formula to measure the success of a professional baseball player. The level of success was determined by the number of runs that were created by a player (since creating runs is the object of the game). James' formula is:

$$R(H,W,T,B)=\frac{(H+W)\cdot T}{B+W}$$

where R is *Runs Created*, H is number of *Hits*, W is number of *Walks*, T is *Total Bases*, and B is number of *At Bats*.

a. Joe Mauer is a star player for the Minnesota Twins. Using the stats found at www.mlb.com, evaluate the function $R(1019, 436, 1496, 3126)$ and explain what $R(1019, 436, 1496, 3126)$ means.

$$\begin{aligned} R(1019,436,1496,3126) &= \frac{(1019+436)\cdot 1496}{3126+436} \\ &= 611.1 \end{aligned}$$

Joe Mauer has created about 611 runs over his career according to the formula.

b. Assuming all other numbers stayed the same, how many hits would Joe Mauer need to have had in order to have a *Runs Created* (R) value of 700?

$$\begin{aligned} 700 &= \frac{(H+436)\cdot 1496}{3126+436} \\ 700(3126+436) &= (H+436)\cdot 1496 \\ \frac{2493400}{1496} &= H+436 \\ \frac{2493400}{1496}-436 &= H \\ 1231 &\approx H \end{aligned}$$

If Joe Mauer had 1231 total hits, the runs created would have been 700.

2. The total money spent by Americans to watch movies at a theater is recorded in the table below (Source: Statistical Abstract of the United States, 2007, Table 1218).

a. Complete the table.

Admissions to Motion Picture Theaters		
y **Year** **(*y* = 0 is 1990)**	***E*** **Expenditure** **(in billion dollars)**	**Average Rate of Change**
0	5.1	
		$\frac{5.6-5.1}{5}=\frac{0.5}{5}=0.1$ billion dollars per year
5	5.6	
		$\frac{8.6-5.6}{5}=\frac{3}{5}=0.6$ billion dollars per year
10	8.6	
		$\frac{9.9-8.6}{4}=\frac{1.3}{4}=0.325$ billion dollars per year
14	9.9	

b. Using words, explain what any one of the rate of change computations means in the context of this situation.

On the interval from 10 to 14 years after 1990 (2000 – 2004), the expenditures at motion picture theaters is growing at an average rate of $0.325 billion per year.

c. Find $A(10)$ and explain what it means.

$A(10)=8.6$

Ten years after 1990 (2000), expenditures at motion picture theaters were $8.6 billion.

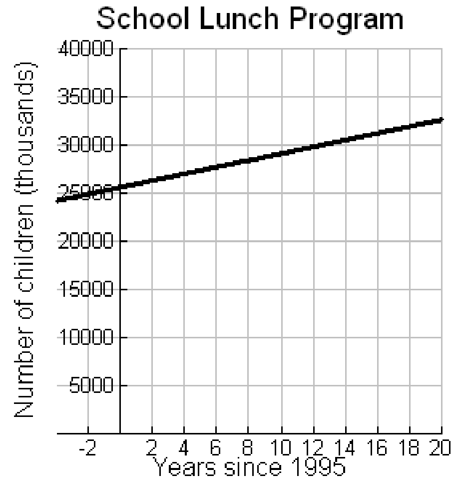
d. Using the idea of average rate of change, find $A(20)$ and explain what it means.

$$\begin{aligned} A(20) &= 9.9+0.325(6) \\ &= \$11.85 \text{ billion} \end{aligned}$$

The predicted expenditure at motion picture theaters in 2010 (20 years after 1990) is $11.85 billion.

3. Based on data from 1995-2004, the number of children participating in the federal school lunch program may be modeled by $L = f(t) = 351.53t + 25587$ thousand children where t is the number of years after 1995 (Source: USDA - www.nass.usda.gov).

a. A graph of $L = f(t)$ is provided. Use the graph to solve the equation $30{,}000 = f(t)$. Explain what the result means.



$30{,}000 = f(t)$

$I \approx 12$ years

When there are 30,000 (thousand) children in the school lunch program, the model predicts that this occurred 12 years after 1995 (2007).

b. Use the graph to determine the value of $f(20)$. Explain what the result means.

$f(20) \approx 32500$

This means that 20 years after 1995 (2015), the model predicts that there will be 32,500 (thousand) or 32,500,000 children participating in the school lunch program.

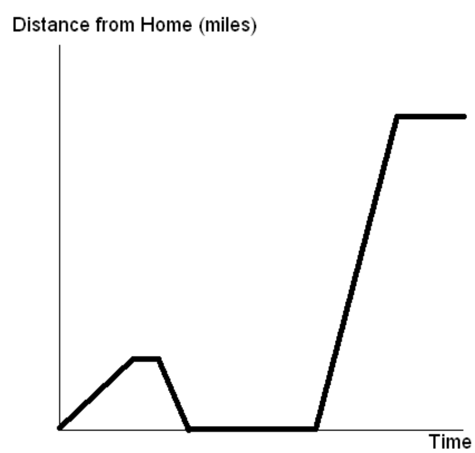
c. Find the average rate of change between *t* = 0 and *t* = 12 and explain what the value means in the context of this situation.

$$\frac{f(12)-f(0)}{12-0} = \frac{30000-25000}{12} = \frac{5000}{12} \approx 416.67$$

On the interval from 0 to 12 (1995 to 2007), for each additional year since 1995, the number of children participating in the school lunch program increases by 416.67 (thousand) children.

4. Sketch a graph to match this story – graph ***time*** on the horizontal axis and ***distance from home*** on the vertical axis.

My son Matthew leaves home to go to school. He drives at a constant average speed of 35 safe miles per hour. After 4 minutes, he realizes that he forgot his homework at home and turns around to retrieve the homework. Once home, it takes him 10 minutes to find the homework. He then drives back toward school at a constant average speed of 42 miles per hour. He arrives at school which is 8 miles from home.



Linear Functions and Matrices

2

2.1 Functions with Constant Rates of Change

Objectives

- Explain the real-world meaning of slope, horizontal intercept, and vertical intercept.
- Write linear functions in slope-intercept form, point-slope, and standard form.
- Graph linear functions.

Concepts and Definitions

- **Constant Rate of Change:** A function has a constant rate of change m if, for any change in the independent variable, the dependent variable always changes by exactly m times as much.
- **Units of a Rate of Change:** The units of a rate of change are the units of the dependent variable divided by the units of the independent variable. We commonly write this as "units of output per unit of input."
- **Linear Function:** Any function with a constant rate of change is called a linear function.
- **Slope of a Linear Function:** The constant rate of change of a linear function is called the slope of the function. The term slope is also used to refer to the steepness of the graph of a linear function.

Formulas

- $slope = m = \dfrac{\text{change in the dependent varible}}{\text{change in the independent varible}} = \dfrac{y_2 - y_1}{x_2 - y_1}$
- A linear function with slope m and vertical intercept $(0, b)$ is written in slope intercept form as $y = mx + b$. The value b is commonly referred to as the initial value.
- A linear function with slope m and a point (x_1, y_1) is written in point-slope form as $y - y_1 = m(x - x_1)$.
- A linear function can be written in standard form as $Ax + By = C$ where A, B, and C are real numbers.

Examples

- **Example 1: Interpreting the Meaning of Slope and Intercepts**
  Many car dealers offer 0% financing as an incentive to attract new car buyers. In January 2007, Ford Motor Company offered 0.0% financing for 60 months on its 2006 F-150 SuperCrew pickup truck (Source: www.ford.com). The Manufacturer's Suggested Retail Price (MSRP) on the Flareside 150" XLT version of the truck was \$31,290 (Source: www.nada.com). Construct a function for the loan balance as a function of the number of months since the truck was purchased, assuming the buyer

paid the MSRP and financed the full amount with the 0.0% loan. As a part of the process, interpret the practical meaning of the slope and intercepts.

Solution:

Since the purchase price of the truck was $31,290, the point $(0, 31290)$ *is on the line. This is the vertical intercept and represents the initial value of the loan. Since the loan will be paid off in 60 months, the point* $(60, 0)$ *is on the line. This is the horizontal intercept and represents how long it will take for the loan balance to reach $0. We calculate the slope using the average rate of change formula and the intercepts.*

$$\text{slope} = \frac{31{,}290 - 0}{0 - 60} \frac{\text{dollars}}{\text{months}} = -521.50 \text{ dollars per month}$$

This indicates that the loan balance is decreasing at a constant rate of 521.50 dollars per month. In other words, the monthly loan payment is $521.50. (If the interest rate wasn't 0.0%, this rate of change would not be constant.) The linear equation for the loan balance as a function of months since the truck was purchased is given by

$B(m) = -521.50m + 31{,}290$ dollars *where* B *is the loan balance and* m *is the number of months since the truck was purchased.*

- **Example 2: Finding a Linear Model from a Verbal Description**

  Based on data from 1980 – 1999, per capita consumption of milk as a beverage has been decreasing by approximately 0.219 gallons per year. In 1997, the per capita consumption of milk as a beverage was 24.0 gallons (Source: Statistical Abstract of the United States, 2001; Table 202, p 129). Find an equation for per capita milk consumption as a function of years since 1980.

  Solution:

  The slope of the line is $m = -0.219$. *The point* $(17, 24.0)$ *lies on the line since 1997 is 17 years after 1980. The point-slope form of the line is* $y - 24.0 = -0.219(t - 17)$*.If preferred, the equation may be rewritten in slope-intercept form*

$$y = -0.219(t - 17) + 24.0$$
$$y = -0.219t + 3.723 + 24.0$$
$$y = -0.219t + 27.723$$

or standard form

$$0.219t + y = 27.723$$

Exercises

1. Determine the slope, the vertical intercept, and the horizontal intercept of the linear function $f(x) = -2x + 5$.

2. Suppose that your car currently has an odometer reading of 26,500 miles. On average, suppose you drive your car 250 miles per week. Write a linear function formula that models the car's odometer reading as a function of the number of weeks elapsed.

Exercise Solutions

1. The slope of the function $f(x) = -2x + 5$ is $m = -2$. The vertical intercept is $(0, 5)$. The horizontal intercept is found by determining the value of x such that $f(x) = -2x + 5 = 0$.

$$\begin{aligned} -2x + 5 &= 0 \\ -2x &= -5 \\ x &= \frac{5}{2} \end{aligned}$$

The horizontal intercept is $(\frac{5}{2}, 0)$.

2. With an initial value of 26,500 miles and a constant rate of change of 250 miles per week, we write a linear function formula to model the car's odometer reading as a function of the number of weeks elapsed. In the function formula, R represents the odometer reading and w represents the number of weeks that have elapsed.

$$R = f(w) = 26500 + 250w$$

2.2 Modeling with Linear Functions

Objectives

- Determine if two quantities are directly proportional.
- Construct linear models of real-world data sets and use them to predict results.
- Find the inverse of a linear function and interpret its meaning in a real-world context.

Concepts and Definitions

- **Direct Proportionality:** Two quantities are directly proportional when one quantity is a constant multiple of the other. That is, $y = kx$ for a constant k. k is called the constant of proportionality.
- **Inverse of a Linear Function:** To find the inverse of a linear function, first write the function in the form $y = mx + b$. Next, solve the function equation for x. Finally, write the inverse function using the notation $x = f^{-1}(y)$.

Examples

- **Example 1: Using Direct Proportionality to Convert Units of Measure**
  Long distance runners may run marathons (26.2 miles), half marathons (13.1 miles), 10K, or 5K (K stands for kilometers) races. To compare the length of these runs, we use the relationship $1 \text{ kilometer} \approx 0.621 \text{ miles}$. In miles, how long is a 10K race? A 5K race?

  Solution:
  To convert from kilometers to miles, we use the function formula $M(K) = 0.621 \cdot K$ *where* K *represents the number of kilometers we want to convert to* M *miles. Therefore, a 5K race is* $M = 0.621 \cdot 5 = 3.105 \text{ miles}$. *A 10K race is* $M = 0.621 \cdot 10 = 6.21 \text{ miles}$.

- **Example 2: Constructing a Linear Model from a Verbal Description**
  Verizon's "Nationwide Talk 450" cell phone plans cost \$39.99 per month in 2011. The plan included free weekend minutes and 450 whenever minutes. Additional minutes used cost \$0.45 per minute. (Source: www.verizon.com) Write a function that will give the monthly cell phone cost as a function of the number of additional minutes used. (*Verizon* defines *additional minutes* to be the minutes used beyond a subscriber's monthly allotment.) Then calculate the cost of using a total of 600 non-weekend minutes.

  Solution:
  We let c *be the monthly cell phone cost (in dollars) and* m *be the number of additional minutes used. Since the monthly cost depends upon the number of additional minutes used,* c *is the dependent variable and* m *is the independent*

variable. The first indicator that this is a linear function is the constant rate of change: $0.45 per minute. This will be the slope of the linear function. The next indicator is the $39.99 fixed value. Since this value is not dependent upon the number of minutes used, it will be the initial value of our function. We have

$$c(m) = 0.45m + 39.99$$

We are asked to find the cost of using 600 non-weekend minutes. (Since weekend minutes are free, they will not affect the monthly cell phone cost.) Since 450 minutes are included with the plan, 150 additional minutes will be used. We evaluate the function at $m = 150$.

$$\begin{aligned} c(150) &= 0.45(150) + 39.99 \\ &= 67.50 + 39.99 \\ &= 107.49 \end{aligned}$$

It costs $107.49 to use 600 non-weekend minutes. This result is significant. Although we only used 25% more minutes than the plan allowed, the monthly cost nearly tripled!

Exercises

1. Find the inverse of the function $y = f(x) = 6x - 19$. Write your solution in inverse function notation, $f^{-1}(y)$.

2. The consumption of whole milk in the United States has been steadily decreasing for many years as people switch to drinking low fat and no fat milk. In 1980, Americans consumed an average of 20 gallons of whole milk per person each year. In 2000, that number dropped to about 8 gallons per person each year.

   a. Create a linear function formula representing the number of gallons, G, of whole milk consumed as a function of time, t, in years since 1980.

   b. Determine when, according to this linear model, the number of gallons of whole milk consumed by Americans will be 0.

   c. Represent graphically the solution to part b.

Exercise Solutions

1.

$$y = 6x - 19$$
$$\frac{y+19}{6} = x$$
$$x = f^{-1}(y) = \frac{y+19}{6}$$

2. a. From the problem statement, we know that the point $(0, 20)$ is vertical intercept for this linear function. We also know that the line contains the point $(20, 8)$. The constant rate of change (slope) can be computed as follows.

$$m = \frac{8-20}{20-0} = \frac{-12}{20} = \frac{-3}{5}$$

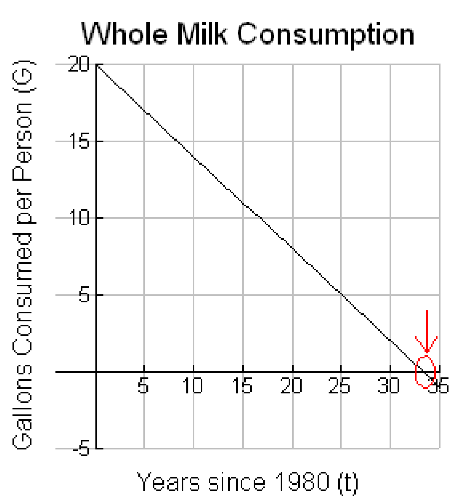
Therefore, the linear function formula is $G(t) = -\frac{3}{5}t + 20$.

b. The number of years elapsed since 1980 when the number of gallons of whole milk consumed per person each year can be found by solving the equation $G(t) = -\frac{3}{5}t + 20 = 0$.

$$-\frac{3}{5}t + 20 = 0$$
$$-\frac{3}{5}t = -20$$
$$t = -20 \cdot \left(-\frac{5}{3}\right) = \frac{100}{3} \approx 33.3$$

According to the linear model, 33.3 years after 1980 (2014), the number of gallons of whole milk consumed per person is zero.

c. The graph confirms our work.



2.3 Linear Regression

Objectives

- Use linear regression to find the equation of the line of best fit.
- Use a linear regression model to make predictions.
- Explain the meaning of the correlation coefficient (r) and the coefficient determination (r^2).

Concepts and Definitions

- **Interpolation:** The process of predicting the output value for an input value that lies between the maximum and minimum input values of the data set.
- **Extrapolation**: The process of predicting the output value for an input value that comes before the minimum input value or after the maximum input value of a data set.
- **Linear Regression Model:** The equation of the line that best fits a data set, as determined by the least value of the total sum of the squares of the residuals, is known as the linear regression model or the least squares regression line.
- **Coefficient of Determination:** The coefficient of determination, r^2, is a value that describes the strength of fit of a linear regression model to a set of data. The closer the value of r^2 is to 1, the stronger the fit.

Examples

- **Example 1: Computing a Linear Regression Model**
  In an episode of the television program *Mythbusters*, data were collected showing the time, t, it takes for a sniper's bullet to hit its target when shot from a distance of d yards. Find the linear regression model for the data and explain what the slope means in the context of this situation.

d **Distance (yards)**	t **Time to Target (in seconds)**
200	0.231
500	0.597
1200	1.791

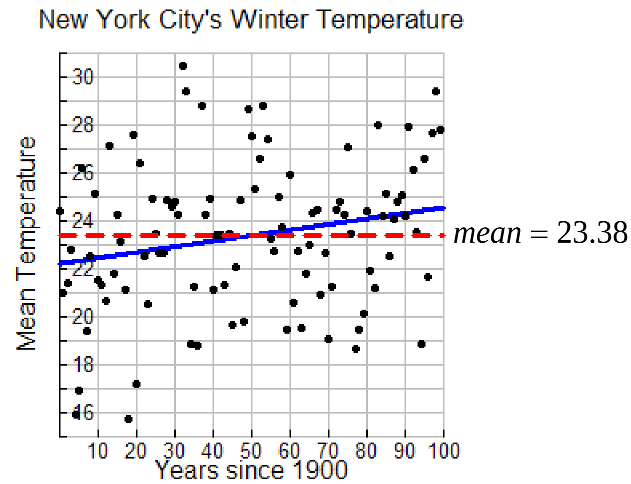
  Solution:

  We use a graphing calculator to find the linear regression model.

  $t(d) = 0.00151d - 0.062701$

  The slope of 0.00151 indicates that for every additional yard added to the distance from the target, it will take 0.00151 more seconds for the sniper's bullet to reach the target. Proportionally, for every 1000 additional yards added to the distance, it will take 1.51 more seconds for the bullet to reach its target.

- **Example 2: Estimating the Coefficient of Determination**
  The data in the graph show the average winter temperature in New York City for years after 1900 (www.ncdc.noaa.gov). The horizontal line in the graph is the mean of the data set. Determine whether the coefficient of determination is closer to 0, closer to 0.5 or closer to 1 and explain how you know.



  Solution:
  It appears as though the coefficient of determination would have a value that is close to 0. We see that the total error (differences between the actual data points and the mean) will be relatively large due to the fact that the data are so spread out around the mean of 23.38^o. However, the explained error (differences between the regression line and the mean) is relatively small. The coefficient of determination value, r^2, *is the ratio* $\dfrac{\text{Total Explained Error}}{\text{Total Error}}$.
  This ratio is close to 0 since the numerator is relatively small and denominator is relatively large.

Exercises

The per capita consumption of cheese in the United States is shown in the table (*Source: Statistical Abstract of the United States, 2011, Table 213*).

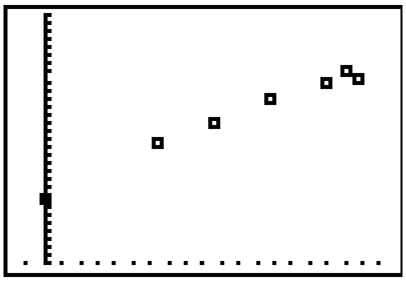
t Years since 1980	c Cheese consumption (pounds per person)
0	17.5
10	24.6
15	26.9
20	29.8
25	31.6
27	33.1
28	32.4

1. a. Use graphing technology to create a scatter plot of these data.
   b. Use graphing technology to create a linear regression model for these data. Use the model to describe the relationship between the cheese consumption by Americans and the number of years sincd 1980.
   c. Use the coefficient of determination, r^2, and the correlation coefficient, r, to describe the strength of fit of the linear regression model to the data. Explain why the values of r^2 and r make sense in this situation.
2. a. Use the linear regression model to predict the amount of cheese consumption per person in 2015.

b. Use the linear regression model to predict the year in which the cheese consumption is 40 pounds per person.

Exercise Solutions

1. a.



b. $c(t) = 0.536t + 16.41$

There is a positive correlation between the number of years since 1980 and the average cheese consumption in pounds per person.

c. With a coefficient of determination (r^2) value of 0.982 and a correlation coefficient value (r) 0.991, we claim that the relationship between t and c is very strong.

2. a. In 2015 (35 years after 1980), the predicted cheese consumption per person is $c(35) = 0.536(35) + 16.41 = 37.18$ pounds of cheese per person.

b. To determine when the per capita cheese consumption is 40 pounds per person, we solve the equation $c(t) = 0.536t + 16.41 = 40$.

$$
\begin{aligned}
0.536t + 16.41 &= 40 \\
0.536t &= 40 - 16.41 \\
0.536t &= 23.59 \\
t &= \frac{23.59}{0.536} \\
t &\approx 44.01
\end{aligned}
$$

The linear regression model predicts that 44 years after 1980 (2024), the per capita consumption of cheese will be 40 pounds.

2.4 Systems of Linear Equations

Objectives

- Determine the solution to a system of equations algebraically, graphically, and using technology and interpret the real-world meaning of the results.
- Use the substitution and elimination methods to solve linear systems that model real-world scenarios.
- Determine if systems of linear equations are dependent or inconsistent and explain the real-world meaning of the results.

Concepts and Definitions

- **System of Linear Equations in Two Variables:** A system of linear equations in two variables is a group of two or more linear equations that use the same variables.
- **Equivalent System of Equations:** A system of equations can be modified to create an equivalent system of equations that has the same solution as the original system.
- **Inconsistent System of Equations:** A system of equations without a solution is said to be inconsistent.
- **Dependent System of Equations:** A system of equations with infinitely many solutions is said to be dependent**.**

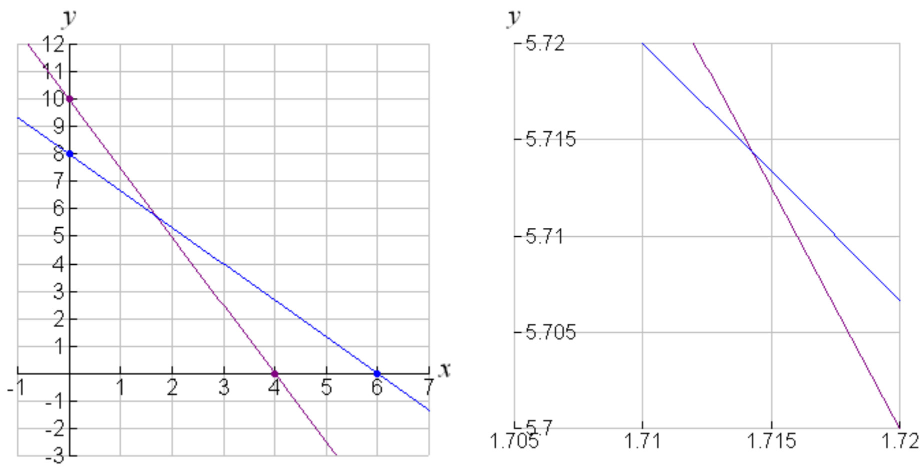
Examples

- **Example 1: Finding the Point of Intersection of Two Lines Graphically**
  Find the solution to the system of linear equations. (That is, find the point of intersection of the graphs of the two lines.) $\begin{aligned} 5x+2y&=20 \\ 4x+3y&=24 \end{aligned}$

  Solution:
  We begin by finding the intercepts for both lines using the technique identified in the box above.

  $5x+2y=20$ h: $(4,0)$ v: $(0,10)$ *and* $4x+3y=24$ h: $(6,0)$ v: $(0,8)$

  We plot the points and connect them with a straight line. It appears from the graph that the lines intersect at or near $(1.7, 5.6)$. *We zoom in to get a better estimate. The lines appear to intersect at about* $(1.714, 5.714)$.



- **Example 2: Using the Substitution Method to Solve a System of Equations**

Solve the system of equations. $\begin{aligned} 5x + 2y &= 12 \\ 4x - \quad y &= 7 \end{aligned}$

Solution:
We solve the second equation for y.

$$\begin{aligned} 4x &= y + 7 \\ y &= 4x - 7 \end{aligned}$$

Substituting this result in for y in the equation $5x + 2y = 12$ *yields*

$$\begin{aligned} 5x + 8x - 14 &= 12 \\ 13x &= 26 \\ x &= 2 \end{aligned}$$

Since $y = 4x - 7$, $y = 4(2) - 7 = 1$. *Therefore, the solution to the system of equations is* $x = 2$ *and* $y = 1$. *We verify the accuracy of this answer by substituting these values into original equations.*

$$\begin{aligned} 5x + 2y &= 12 \\ 5(2) + 2(1) &= 12 \\ 12 &= 12 \end{aligned} \qquad \begin{aligned} 4x - \quad y &= 7 \\ 4(2) - (1) &= 7 \\ 7 &= 7 \end{aligned}$$

Exercises

1. Based on data from 1992 – 2000, the annual value of printing products shipped in the United States may be modeled by $V(t) = 3203.0t + 80732$ million dollars and the annual value of furniture and related products shipped in the United States may be modeled by $V(t) = 3511.7t + 48294$ million dollars where t is the number of years since the end of

1992 (Source: Modeled from Statistical Abstract of the United States, 2001). According to the models, when will the annual value of products shipped in the printing industry and the furniture industry be the same?

2. Solve the system of equations: $\begin{aligned} x+2y&=8 \\ -x+4y&=4 \\ x-y&=2 \end{aligned}$

Exercise Solutions

1. We will replace the V in the second equation with the value of V from the first equation and solve for t.

$$\begin{aligned} 3203.0t+80732&=3511.7t+48294 \\ 32438&=308.7t \\ t&=105.08 \end{aligned}$$

We solve for the value of $V(t)$ by substituting in $t=105.08$ into either equation.

$V(105.08)=3203.0(105.08)+80732=417300$ (accurate to five significant digits)

According to the model, at the end of 2097 ($t=105$) the annual value of products shipped in the printing and furniture industries will be approximately equal ($417,300 million dollars). Although we found a mathematical solution, we are a bit skeptical of the real-life accuracy of the result since it is so far outside of the original data set.

2. We will find the point of intersection of the first two lines and then check to see if the point is a solution to the third equation. The third equation may be written as $y=x-2$. Substituting this value in for y in the first equation and solving for x yields

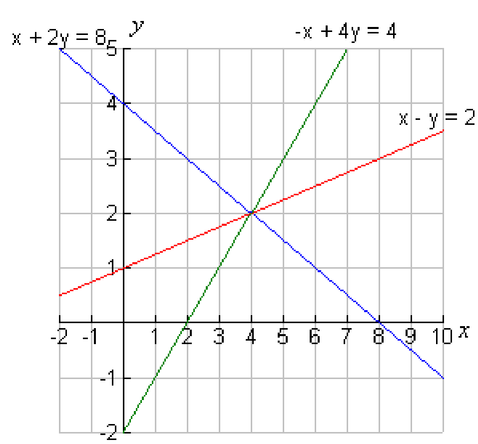
$$\begin{aligned} x+2(x-2)&=8 \\ x+2x-4&=8 \\ 3x&=12 \\ x&=4 \end{aligned}$$

Since $y=x-2$, $y=4-2=2$

The point of intersection of the first two lines is $(4,2)$. We will check to see if this point satisfies the second equation.

$$\begin{aligned} x-y&=2 \\ 4-2&=2 \\ 2&=2 \end{aligned}$$

We graphically confirm the result.



2.5 Using Matrices to Solve Linear Systems

Objectives

- Write systems of linear equations in augmented matrices.
- Use technology to reduce augmented matrices to reduced row echelon form.
- Solve linear systems using matrices and interpret the meaning of the results in a real-world situation.

Concepts and Definitions

- **An $m \times n$ Matrix:** An $m \times n$ matrix A is an array of numbers with m rows and n columns. The plural of matrix is matrices.
- **Row Operations:** For any augmented matrix of a system of equations, the following row operations yield an augmented matrix of an equivalent system of equations:
  - Interchange (change the position of) two rows.
  - Multiply a row by a nonzero number.
  - Add a nonzero multiple of one row to a nonzero multiple of another row and replace either row with the result.
- **Reduced Row Echelon Form:** An augmented matrix is said to be in reduced row echelon form if it satisfies each of the following conditions:
  - The leading entry (first nonzero entry) in each row is a 1.
  - The leading entry of each row is the only nonzero entry in its corresponding column.
  - The leading entry in each row is to the right of the leading entry in the row above it.
  - All rows of zeros are at the bottom of the matrix.

Examples

- **Example 1: Determining the Dimensions and Entry Values of a Matrix**
  Determine the dimensions of the matrix A and the value of the entries a_{12}, a_{21}, and a_{24}.

$$A = \begin{bmatrix} 2 & 3 & 5 & 7 \\ 11 & 13 & 17 & 19 \\ 23 & 29 & 31 & 37 \end{bmatrix}.$$

  Solution:
  The matrix has three rows and four columns so it is a 3×4 matrix. For this matrix $a_{12} = 3$, $a_{21} = 11$, *and* $a_{24} = 19$.

- **Example 2: Solving a System of Equations with Matrices**
  For the system

$$\begin{aligned} 2x + 6y - 4z &= 8 \\ 2x - 12y + 4z &= 20 \\ x + \;4y - \;\;z &= 2 \end{aligned}$$

a. rewrite the system of equations as an augmented matrix .

b. use technology to simplify the matrix to reduced row echelon form.

c. identify the solution(s) to the system of equations. If the system is inconsistent or dependent, so state.

Solution:

a. $\left[\begin{array}{ccc|c} 2 & 3 & -4 & 8 \\ 2 & -12 & 4 & 20 \\ 1 & 4 & -1 & 2 \end{array}\right]$

b. We use a graphing calculator to simplify the matrix to reduced row echelon form. See the technology tip at the end of Chapter 8 in textbook to learn this procedure.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{70}{13} \\ 0 & 1 & 0 & -\frac{14}{13} \\ 0 & 0 & 1 & -\frac{12}{13} \end{array}\right]$$

c. Based on the matrix, we see that the solution to the system of equations is $x = \frac{70}{13}, y = -\frac{14}{13}, z = -\frac{12}{13}$.

Exercises

The following table shows the average annual rate of return of a variety of TIAA-CREF investment accounts over a ten-year period (*Source*: www.tiaa-cref.com).

An investor chooses to invest $10,000 in the Bond Market, Inflation-Linked Bond, and Social Choice accounts. He assumes he will be able to get a return equal to the 10-year average and wants the total return on his investment to be 5.5%. He decides to invest two times as much money in the Bond Market account as in the Social Choice account. How much money should she invest in each account? To answer this question, do each of the following.

As of 6/30/11

CREF Variable Annuity Accounts	**10 year average**
Bond Market	5.29%
Equity Index	3.04%
Global Equities	3.61%
Growth	1.24%
Inflation-Linked Bond	6.48%
Money Market	1.97%
Social Choice	4.51%
Stock	3.91%

1. Set up a system of equations to model this situation. Then, create an augmented matrix that could be used to solve the system of equations.
2. Use technology to simplify the matrix to reduced row echelon form and then identify the solution(s) to the system of equations.

Exercise Solutions

1. We let x represent the amount of money invested in the Bond Market account, y represent the amount of money invested in the Inflation-Linked Bond account, and z represent the amount of money invested in the Social Choice account.

$$\begin{aligned} x+y+z&=10000 \\ x+0y-2z&=0 \\ 0.0529x+0.0648y+0.0451z&=0.055(10000) \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 1 & 0 & -2 & 0 \\ 0.0529 & 0.0648 & 0.0451 & 550 \end{array}\right]$$

2. We use a graphing calculator to simplify the augmented matrix to reduced row echelon form.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 10000 \\ 1 & 0 & -2 & 0 \\ 0.0529 & 0.0648 & 0.0451 & 550 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3333.33 \\ 0 & 1 & 0 & 5000 \\ 0 & 0 & 1 & 1666.67 \end{array}\right]$$

The investor should invest $3,333.33 in the Bond Market, $5,000.00 in the Inflation-Linked Bond, and $1,666.67 in the Social Choice account. One might wonder why the investor wouldn't just invest all the money in the account earning the highest return. Due to risk analysis, investors often invest their money in many accounts to avoid a catastrophe if one investment goes bad.

2.6 Matrix Operation and Applications

Objectives

- Identify properties of matrix addition and scalar multiplication.
- Use matrix addition and scalar multiplication in real-world situations.

Concepts and Definitions

- **Matrix Addition:** An $m \times n$ matrix *A* and an $m \times n$ matrix *B* of the same dimension can be added together to form a new $m \times n$ matrix, *C*. The value of the entry in the *ith* row and *jth* column of *C* is $c_{ij} = a_{ij} + b_{ij}$. If *A* and *B* are not of the same dimension, matrix addition is undefined.
- **Scalar Multiplication:** An $m \times n$ matrix *A* and a real number *k* can be multiplied together to form a new $m \times n$ matrix, *C*. The value of the entry in the *ith* row and *jth* column of *C* is $c_{ij} = k \cdot a_{ij}$.

Examples

- **Example 1: Calculating the Sum of Two Matrices**

  Calculate $C = A + B$ given $A = \begin{bmatrix} 5 & -2 & 7 \\ -3 & 8 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & 7 & -2 \\ -9 & -8 & 3 \end{bmatrix}$.

  Solution:

$$\begin{aligned} C &= A + B \\ &= \begin{bmatrix} 5 & -2 & 7 \\ -3 & 8 & -1 \end{bmatrix} + \begin{bmatrix} 6 & 7 & -2 \\ -9 & -8 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 5+6 & -2+7 & 7+(-2) \\ (-3)+(-9) & 8+(-8) & (-1)+3 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 5 & 5 \\ -12 & 0 & 2 \end{bmatrix} \end{aligned}$$

- **Example 2: Multiplying by a Scalar**

  Calculate $C = -\frac{1}{3}A$ given $A = \begin{bmatrix} 18 & -15 \\ 1 & 0 \\ -6 & 3 \end{bmatrix}$.

Solution:

$$C = -\tfrac{1}{3}A$$
$$= -\tfrac{1}{3}\begin{bmatrix} 18 & -15 \\ 1 & 0 \\ -6 & 3 \end{bmatrix}$$
$$= \begin{bmatrix} -\tfrac{1}{3}\cdot 18 & -\tfrac{1}{3}\cdot(-15) \\ -\tfrac{1}{3}\cdot(1) & -\tfrac{1}{3}\cdot 0 \\ -\tfrac{1}{3}\cdot(-6) & -\tfrac{1}{3}\cdot 3 \end{bmatrix}$$
$$= \begin{bmatrix} -6 & 5 \\ -\tfrac{1}{3} & 0 \\ 2 & -1 \end{bmatrix}$$

Exercises

1. The table shows a partial listing of the base salary schedule for administrators and professional staff (professors) at Harvard University 2011 – 12 (*Source*: www.employment.harvard.edu). Suppose we are now thinking about salaries for the 2012 – 13 school year and want to predict a new salary schedule based on the predicted cost of living allowance (COLA) of 1.2% (*Source*: www.ssa.gov). Create a new salary schedule for the 2012 – 13 school year for Harvard administrators and professional staff.

Grade	Minimum	Maximum
056	51,500	84,263
057	57,900	97,200
058	65,800	113,200
059	76,700	134,300

2. Let $A = \begin{bmatrix} 2 & -4 & 5 \\ -1 & 3 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -7 & 4 \\ 0 & -8 & -5 \end{bmatrix}$, and $O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Calculate $2A-(3B+O)$.

Exercise Solutions

1. We can solve the problem by writing the table as a matrix and multiplying by a scalar. Since salaries are to increase by 1.2 percent, the 2012 – 13 salaries will be 101.2 percent of the 2011 – 12 salaries. We will multiply the matrix by 1.04.

$$S = \begin{bmatrix} 51{,}500 & 84{,}263 \\ 57{,}900 & 97{,}200 \\ 65{,}800 & 113{,}200 \\ 76{,}700 & 134{,}300 \end{bmatrix}$$

$$1.012S = 1.012\begin{bmatrix} 51{,}500 & 84{,}263 \\ 57{,}900 & 97{,}200 \\ 65{,}800 & 113{,}200 \\ 76{,}700 & 134{,}300 \end{bmatrix} \approx \begin{bmatrix} 52{,}118 & 85{,}274 \\ 58{,}595 & 98{,}366 \\ 66{,}590 & 114{,}558 \\ 77{,}620 & 135{,}912 \end{bmatrix}$$

The new matrix represents the predicted 2012 – 13 pay scale for some Harvard employees.

Grade	Minimum	Maximum
056	52,118	85,274
057	58,595	98,366
058	66,590	114,558
059	77,620	135,912

2.

$$2A-(3B+O) = 2\begin{bmatrix} 2 & -4 & 5 \\ -1 & 3 & 6 \end{bmatrix} - \left(3\cdot\begin{bmatrix} 4 & -7 & 4 \\ 0 & -8 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4 & -8 & 10 \\ -2 & 6 & 12 \end{bmatrix} + (-1)\left(\begin{bmatrix} 12 & -21 & 12 \\ 0 & -24 & -15 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)$$

$$= \begin{bmatrix} 4 & -8 & 10 \\ -2 & 6 & 12 \end{bmatrix} + (-1)\begin{bmatrix} 12 & -21 & 12 \\ 0 & -24 & -15 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -8 & 10 \\ -2 & 6 & 12 \end{bmatrix} + \begin{bmatrix} -12 & 21 & -12 \\ 0 & 24 & 15 \end{bmatrix}$$

$$= \begin{bmatrix} -8 & 13 & -2 \\ -2 & 30 & 27 \end{bmatrix}$$

Chapter 2 Exam A

1. Stamped on most urinals in public men's restrooms is the fact that 3.8 liters of water are used per flush.

   a. Write a linear function formula describing the amount of water used, W, as a function of the number of times the urinal is flushed, f.

   b. Solve $W(f) = 638.4$ for f and describe what the result means in the context of this situation.

2. The table of data shown is ***linear***.

x	4		8	11	18
y	11.2	16.6	18.4		

   a. Determine each of the missing table values.

   b. Generate the linear function (where *y* is a function of *x*) that would model the table values. Show all your work to generate the appropriate value for the constant rate of change (slope) and the vertical intercept (*y*-intercept). In other words, you are not allowed to use linear regression on your graphing calculator.

3. China is the world's most populated country and has experienced approximately linear population growth from 1950 to 2010. (Source: www.census.gov).

Year	1950	1960	1970	1980	1990	2000	2010
Population in Millions	562.2	650.7	820.4	984.7	1148.3	1263.6	1330.1

   a. Determine a linear regression model for the population of China as a function of time in years (where 1950 corresponds to year 0).

   b. Interpret the meaning of the slope (constant rate of change) in the context of this situation.

   c. Use your linear model to estimate the population of China in 2012. Show or explain your work on how you were able to make this estimation.

4. The population of the Republic of Maldives can be modeled by the function $M(t) = 0.005t + 0.036$ million people. The population of the United Arab Emirates can be modeled by the function $E(t) = 0.080t - 0.778$ where t is the number of years since 1950. Algebraically determine when the two countries will have the same population (Source: www.census.gov). Explain why your result makes sense and confirm your results graphically

Chapter 2 Exam A Solutions

1. Stamped on most urinals in public men's restrooms is the fact that 3.8 liters of water are used per flush.

   a. Write a linear function formula describing the amount of water used, W, as a function of the number of times the urinal is flushed, f.

   $W(f) = 3.8f$

   b. Solve $W(f) = 638.4$ for f and describe what the result means in the context of this situation.

   $$\begin{aligned} 3.8f &= 638.4 \\ f &= 168 \end{aligned}$$

   When 638.4 liters of water are used, we know that the urinal was flushed 168 times.

2. The table of data shown is ***linear***.

x	4	7	8	11	18
y	11.2	16.6	18.4	23.8	36.4

   a. Determine each of the missing table values.

   Since $m = \dfrac{18.4 - 11.2}{8 - 4} = 1.8$, we can determine the missing table values.

   b. Generate the linear function (where *y* is a function of *x*) that would model the table values. Show all your work to generate the appropriate value for the constant rate of change (slope) and the vertical intercept (*y*-intercept). In other words, you are not allowed to use linear regression on your graphing calculator.

   We know the constant rate of change (slope) is 1.8. To get the y-intercept, we can "back up" 4 units at a rate of 1.8 per unit from 11.2 ($11.2 - 4 \cdot 1.8 = 4.2$) to get $y = 1.8x + 4.2$.

3. China is the world's most populated country and has experienced approximately linear population growth from 1950 to 2010. (Source: www.census.gov).

Year	1950	1960	1970	1980	1990	2000	2010
Population in Millions	562.2	650.7	820.4	984.7	1148.3	1263.6	1330.1

   a. Determine a linear regression model for the population of China as a function of time in years (where 1950 corresponds to year 0).

   $P(t) = 13.776t + 552.42$

b. Interpret the meaning of the slope (constant rate of change) in the context of this situation.

The population of China is increasing at a rate of 13.77 million people per year from 1950 to 2010.

c. Use your linear model to estimate the population of China in 2012. Show or explain your work on how you were able to make this estimation.

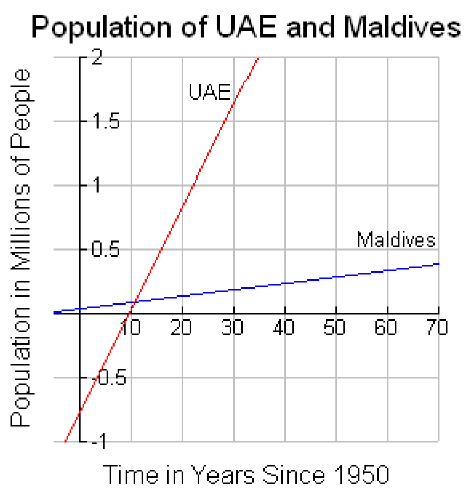
$P(62) = 13.776(62) + 552.42 = 1406.5$

According to the linear model, the population of China in 2012 will be 1,406,500,000.

4. The population of the Republic of Maldives can be modeled by the function $M(t) = 0.005t + 0.036$ million people. The population of the United Arab Emirates can be modeled by the function $E(t) = 0.080t - 0.778$ where t is the number of years since 1950. Algebraically determine when the two countries will have the same population (Source: www.census.gov). Explain why your result makes sense and confirm your results graphically.

$$\begin{aligned} M(t) &= E(t) \\ 0.005t + 0.036 &= 0.080t - 0.778 \\ -0.075t &= -0.814 \\ t &= \frac{-0.814}{-0.075} \approx 10.85 \end{aligned}$$

Approximately 10.85 years after 1950 (into 1961), the population of the United Arab Emirates (UAE) was the same as the population of the Republic of Maldives. This makes sense since the rate of population growth is so much greater for the UAE (0.08 million people per year) than for the Maldives (0.005 million people per year). A graph confirms the result.



Chapter 2 Exam B

1. When eating in a restaurant, many people leave a 20% tip if service is excellent.

   a. Write a linear function representing the amount of the tip, T, as a function of the total bill (amount owed), B.

   b. Solve $T(B) = 17.59$ for B and describe what the result means in the context of this situation.

For question 2, use the data in the table below which give the average U.S. passenger car fuel efficiency in miles per gallon (Source: www.bts.gov).

Year, t	1980	1985	1990	1995	2000	2005
Fuel Efficiency, E (miles per gallon)	16.0	17.5	20.3	21.1	21.9	22.1

2. Calculate a linear regression model for these data where 1980 corresponds to year 0.

   a. Interpret the meaning of the constant rate of change (slope) in the context of this situation.

   b. Interpret the meaning of the vertical intercept (y-intercept) in the context of this situation.

   c. Describe the meaning of the coefficient of determination (r^2) for this linear regression model.

   d. Evaluate $E(35)$ and explain what this value means in the context of the situation.

   e. Solve $E(t) = 40$ for t and explain what the result means in the context of the situation.

3. The average annual return (over the ten year period prior to June 30, 2011) of two mutual funds offered by Harbor Fund is shown in the table (*Source*: Harbor Fund). Suppose you have $12,000 to invest in these two accounts. Assuming the accounts will earn the returns specified in the table over the next year, how much should you invest in each account if you want to earn 6%?

	Average Annual Return
Capital Appreciation	2.78%
Small Cap Growth	7.02%

4. An employee of a construction company earns $35 per hour as a supervisor. She is required to work at least 40 hours weekly. As a side business, she earns $15 per hour tutoring students at the local community college. To maintain her current standard of living, she must earn at least $1500 per week. To maintain her quality of life, she limits her workload to no more 65 hours per week. Use a system of linear inequalities to determine if she is able to meet her income requirement as well as her workload constraint. Explain.

Chapter 2 Exam B Solutions

1. When eating in a restaurant, many people leave a 20% tip if service is excellent.

a. Write a linear function representing the amount of the tip, T, as a function of the total bill (amount owed), B.

$T = 0.20 \cdot B$

b. Solve $T(B) = 17.59$ for B and describe what the result means in the context of this situation.

$$17.59 = 0.20 \cdot B$$
$$\frac{17.59}{0.20} = B$$
$$87.95 = B$$

If we knew that the tip paid was \$17.59, then the total bill at the restaurant was \$87.95.

For question 2, use the data in the table below which give the average U.S. passenger car fuel efficiency in miles per gallon (Source: www.bts.gov).

Year, *t*	1980	1985	1990	1995	2000	2005
Fuel Efficiency, *E* (miles per gallon)	16.0	17.5	20.3	21.1	21.9	22.1

2. Calculate a linear regression model for these data where 1980 corresponds to year 0.

$E(t) = 0.254t + 16.64$

a. Interpret the meaning of the constant rate of change (slope) in the context of this situation.

Fuel efficiency is increasing at a rate of 0.254 miles per gallon for each additional year.

b. Interpret the meaning of the vertical intercept (y-intercept) in the context of this situation.

According to the linear model, in 1980 ($t = 0$), the average fuel efficiency was 16.64 miles per gallon.

c. Describe the meaning of the coefficient of determination (r^2) for this linear regression model.

The value of the coefficient of determination is $r^2 \approx 0.902$. We say that 90.2% of the error is explained by the linear regression model. There is a reasonably good strength of fit between the linear model and the given data.

d. Evaluate $E(35)$ and explain what this value means in the context of the situation.

$E(35) \approx 25.54$ mpg . In 2015 ($t = 35$), the linear model predicts that the average passenger car in the United States will get 35.54 miles per gallon.

e. Solve $E(t) = 40$ for t and explain what the result means in the context of the situation.

$$40 = 0.254t + 16.64$$
$$23.36 = 0.254t$$
$$91.97 \approx t$$

Approximately 91.97 years after 1980 (into 2072), the linear model predicts that the average passenger car in the United States will achieve 40 miles per gallon.

3. The average annual return (over the ten year period prior to June 30, 2011) of two mutual funds offered by Harbor Fund is shown in the table (*Source*: Harbor Fund). Suppose you have $12,000 to invest in these two accounts. Assuming the accounts will earn the returns specified in the table over the next year, how much should you invest in each account if you want to earn 6%?

	Average Annual Return
Capital Appreciation	2.78%
Small Cap Growth	7.02%

We let x represent the amount of money invested in the Capital Appreciation account and y represent the amount of money invested in the Small Cap Growth account.

$$x + y = 12000$$
$$0.0278x + 0.0702y = 0.06(12000)$$

We choose to solve by converting the system of linear equations to an augmented matrix. We will then use a graphing calculator to simplify the augmented matrix to reduced row echelon form.

$$\begin{bmatrix} 1 & 1 & 12000 \\ 0.0278 & 0.0702 & 0.06 \cdot 12000 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2886.79 \\ 0 & 1 & 9113.21 \end{bmatrix}$$

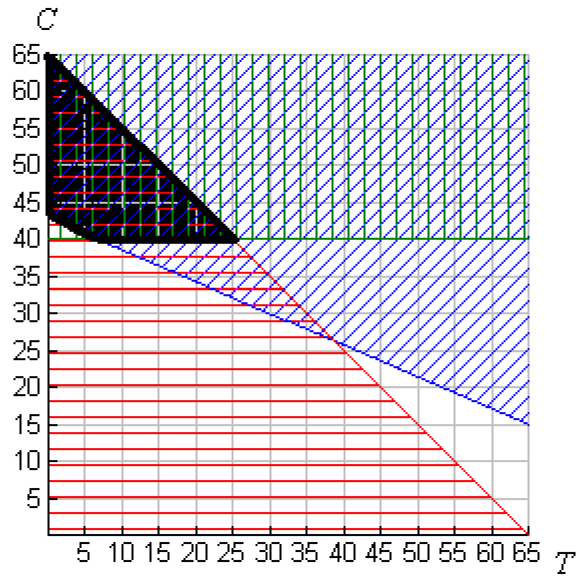
Therefore, the investor should invest $2,886.79 in the Capital Appreciation account and $9,113.21 in the Small Cap Growth account.

4. An employee of a construction company earns $35 per hour as a supervisor. She is required to work at least 40 hours weekly. As a side business, she earns $15 per hour tutoring students at the local community college. To maintain her current standard of living, she must earn at least $1500 per week. To maintain her quality of life, she limits her workload to no more 65 hours per week. Use a system of linear inequalities to

determine if she is able to meet her income requirement as well as her workload constraint. Explain.

We graph the system of inequalities:

$$35C + 15T \geq 1500$$
$$C + T \leq 65$$
$$C \geq 40$$



The overlapping shaded region, in black, represents all of the combinations of the number hours worked as a construction supervisor and the number of hours worked as a tutor so that all of the constraints are met. That is, any value of C and T within the shaded region will provide the needed weekly income while keeping the total number of hours worked at 65 or less and ensuring that at least 40 hours per week are spent in her main job as a construction supervisor.

Transformations of Functions

3

3.1 Horizontal and Vertical Shifts

Objectives

- Identify what change in a function equation results in a vertical shift.
- Identify what change in a function equation results in a horizontal shift.

Concepts and Definitions

- **Vertical Shifts:** The graph of $g(x) = f(x) + k$ is the graph of f shifted vertically by $|k|$ units. If k is positive, the shift is upward. If k is negative, the shift is downward.
- **Horizontal Shifts:** The graph of $g(x) = f(x-h)$ is the graph of f shifted horizontally by $|h|$ units. If h is positive, the shift is to the right. If h is negative, the shift is to the left.

Examples

- **Example 1: Shifting a Graph Vertically**
  Based on graphs generated by wolframalpha.com, the life expectancy of women and men has been increasing at nearly the same constant rate from 1940 to 2010. The life expectancy of women is about 5 years greater than that for men. Using $M(t)$ and $W(t)$ to represent the life expectancy of men, M, and women, W, relative to time, t, write an equation that shows the relationship between M and W. What kind of transformation does this situation represent?

  ***Solution*:**
  Since the values of $M(t)$ are 5 years less than the values of, we say $M(t) = W(t) - 5$. Also, since the values of $W(t)$ are 5 years more than that values of $M(t)$, we say $M(t) + 5 = W(t)$. This is an example of a vertical transformation. The function $W(t)$ is shifted up 5 units compared to $M(t)$. We can also say the function $M(t)$ is vertically shifted down 5 units compared to $W(t)$.

- **Example 2: Shifting a Graph Horizontally** Professional football kicker Adam Vinatieri's career field goal accuracy through 2006 is shown in the table. The length of a field goal attempt is measured from where the holder sets the ball (yard line) to where the goal post is located in the back of the end zone, which is 10 yards in length (*Source:* sports.espn.go.com). Given that the function $P(l)$ gives the percentage of field goals made from a distance of l yards, explain what $P(l-10)$ represents in this context and describe the transformation that the function $P(l)$ undergoes.

Length of Kick (yards) l	Percent of Field Goals Made $P(l)$
18–19	100
20–29	96.36
30–39	83.81
40–49	72.48
50 and beyond	44.44

Solution:

The function $P(l-10)$ *subtracts 10 yards from the length of the field goal as given in the table. This has the effect of taking away the length of the end zone which is 10 yards. Therefore,* $P(l-10)$ *would show the percent of field goals made as a function of the yard line at which the holder sets the ball. This represents a horizontal shift where the function* $P(l-10)$ *is shifted to the left 10 units compared to* $P(l)$.

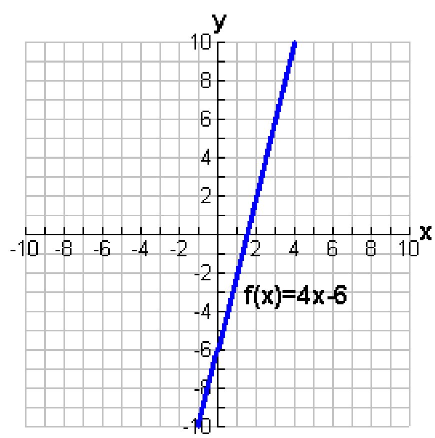
Exercises

1. Graph each of the transformations of f and write the formula for the transformed function in the form $y = mx + b$.

   a. $f(x)+4$

   b. $f(x+5)$

   c. $f(x-2)-4$



2. Use the table below to evaluate the expression or to solve the equation as appropriate.

   a. $f(x)-7.5$ when $x=1$.

   b. $f(x-2)$ when $x=0$.

   c. $f(x-2)=10$

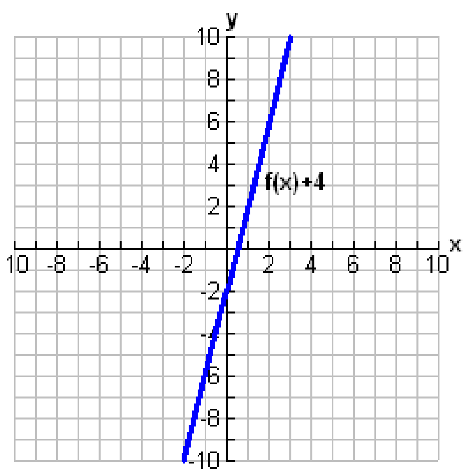
   d. $f(x+3)-2=4$

.

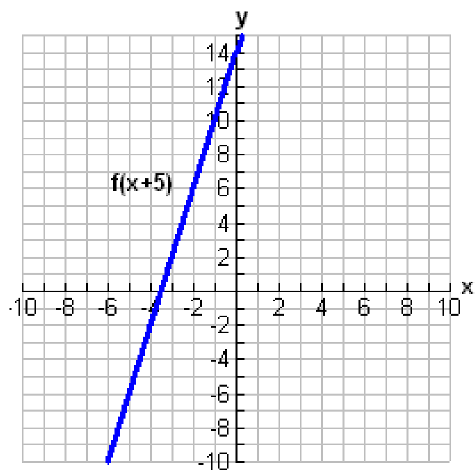
x	$f(x)$
–4	12
–3	10
–2	7
–1	2
0	–1
1	–3
2	4
3	5
4	6

Exercise Solutions

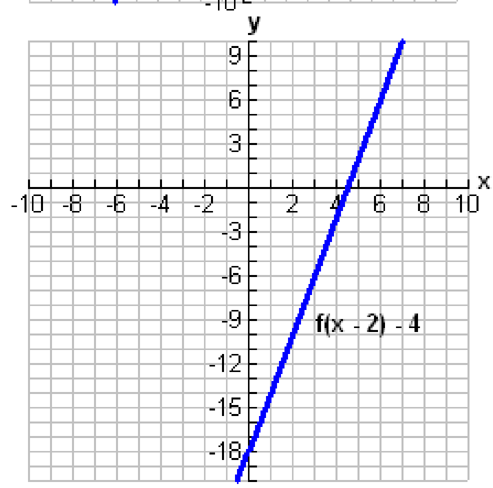
1. a. $f(x)+4=4x-6+4=4x-2$



b. $f(x+5)=4(x+5)-6=4x+14$



c. $f(x-2)-4=4(x-2)-6-4=4x-18$



2. a. $f(x)-7.5$

when $x=1$, $f(1)-7.5=-3-7.5=-10.5$.

b. $f(x-2)$ when $x=0$,

$f(0-2)=f(-2)=7$

c. $f(x-2)=10$, since $f(-3)=10$, $x=-1$

d. $f(x+3)-2=4$ implies that $f(x+3)=6$. Since $f(4)=6$, $x=1$.

3.2 Horizontal and Vertical Reflections

Objectives

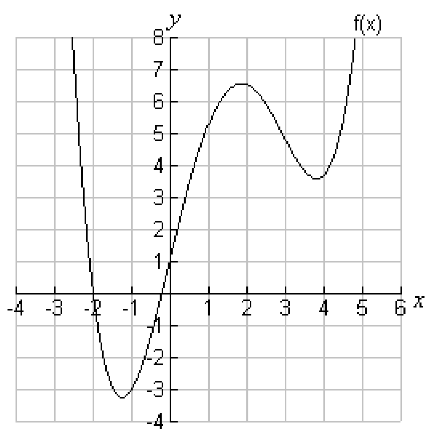
- Identify what change in a function equation results in a horizontal reflection.
- Identify what change in a function equation results in a vertical reflection.
- Understand the concept of symmetry and determine if a function is even, odd, or neither.

Concepts and Definitions

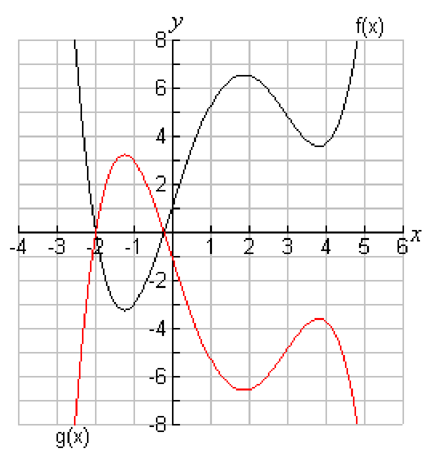
- **Vertical Reflections:** The graph of $g(x) = -f(x)$ is the graph of f reflected vertically about the horizontal axis.
- **Horizontal Reflections:** The graph of $g(x) = f(-x)$ is the graph of f reflected horizontally about the vertical axis.
- **Even Functions:** A function f is an even function if $f(-x) = f(x)$. Even functions are symmetric with respect to the vertical axis.
- **Odd Functions:** A function f is an odd function if $f(-x) = -f(x)$. Odd functions are symmetric with respect to the origin.

Examples

- **Example 1: Vertical Reflections**
  Given the graph of f, graph $g(x) = -f(x)$.



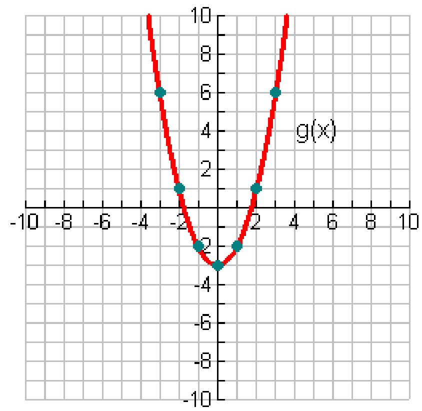
***Solution*:**
As shown in the figure, to obtain the function g, we make all positive values of f negative and all negative values of f positive.



- **Example 2: Observing Even Symmetry**
  For the function $f(x) = x^2 - 3$, create a table of values, graph, and formula to model $g(x)$ if $g(x) = f(-x)$.

  Solution:
  If $g(x) = f(-x)$, then:

x	$f(x)$	$g(x) = f(-x)$
–3	6	$f(-(-3)) = f(3) = 6$
–2	1	$f(-(-2)) = f(2) = 1$
–1	–2	$f(-(-1)) = f(1) = -2$
0	–3	$f(-(0)) = f(0) = -3$
1	–2	$f(-(1)) = f(-1) = -2$
2	1	$f(-(2)) = f(-2) = 1$
3	6	$f(-(3)) = f(-3) = 6$



$f(x) = x^2 - 3$
$g(x) = f(-x) = (-x)^2 - 3$
$g(x) = x^2 - 3$

You should have noticed that something very interesting happened in each of the three representations we used to model $g(x)$. Not only does this function display the same symmetry as $f(x)$, it is the exact same function! The horizontal reflection, which reflected f across its line of symmetry (the vertical line $x = 0$, or the vertical axis), created a function g that is indistinguishable from the original function f.

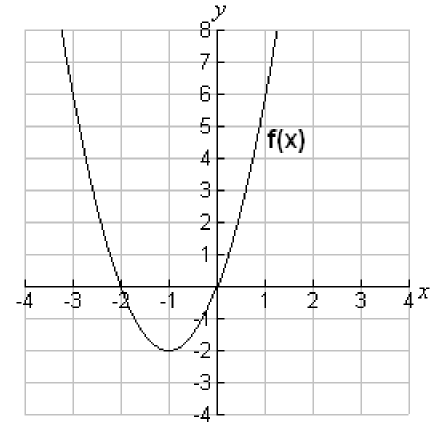
Exercises

1. Determine if the function $f(x) = 2x^2 + 4x$ displays even symmetry, odd symmetry, or neither.

2. Determine if the function $g(x) = 2x^2 - 4$ displays even symmetry, odd symmetry, or neither.

Exercise Solutions

1. We examine the graph of $f(x)$ first:



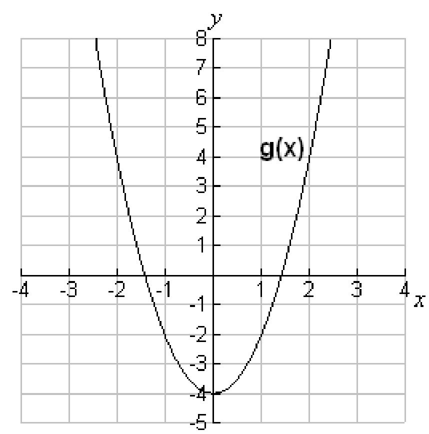
It does not appear that this function will have either even or odd symmetry. The function does not have a line of symmetry at $x = 0$, so it will not have even symmetry. Furthermore, it will not generate the same function when reflected vertically and horizontally. This observation can be verified algebraically as well:

$$\begin{aligned} f(x) &= 2x^2 + 4x \\ f(-x) &= 2(-x)^2 + 4(-x) \\ &= 2(x^2) - 4x \\ &= 2x^2 - 4x \end{aligned}$$

$$\begin{aligned} f(x) &= 2x^2 + 4x \\ -f(x) &= -(2x^2 + 4x) \\ &= -2x^2 - 4x \end{aligned}$$

Since $f(x) \neq f(-x)$, we know that the function does not have even symmetry. Also, since $f(-x) \neq -f(x)$, we can also verify that the function does not have odd symmetry.

2. We examine the graph of $g(x)$ first:



It appears that this function will have even symmetry. The function has a line of symmetry at $x = 0$, so it will have even symmetry. However, it will not generate the same function when reflected vertically and horizontally so it is not an odd function. This observation can be verified algebraically:

$$\begin{aligned} f(x) &= 2x^2 - 4 \\ f(-x) &= 2(-x)^2 - 4 \\ &= 2(x^2) - 4 \\ &= 2x^2 - 4 \end{aligned}$$

$$\begin{aligned} f(x) &= 2x^2 - 4 \\ -f(x) &= -(2x^2 - 4) \\ &= -2x^2 + 4 \end{aligned}$$

Since $f(x) = f(-x)$, we know that the function has even symmetry.
Since $f(-x) \neq -f(x)$, we can verify that the function does not have odd symmetry.

3.3 Vertical Stretches and Compressions

Objectives

- Identify what change in a function equation results in a vertical stretch.
- Identify what change in a function equation results in a vertical compression.

Concepts and Definitions

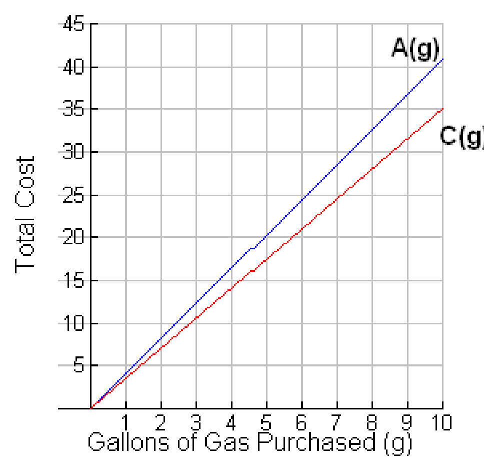
- **Vertical Stretches and Compressions:** The graph of $g(x) = af(x)$ is the graph of f stretched or compressed vertically by a factor of $|a|$ units. If $|a| > 1$, the transformation is a vertical stretch. If $0 < |a| < 1$, the transformation is a vertical compression.

Examples

- **Example 1: Stretching or Compressing a Function Vertically**
  On June 21, 2011, the price per gallon for unleaded gasoline in Fairbanks, Alaska was \$4.08 (Source: www.alaskagasprices.com).

  a. Write a formula for the function $C(g)$, the total cost in dollars of purchasing g gallons.

  b. On the same day, the cost of gas in Amarillo, Texas was \$3.51 per gallon (Source: www.texasgasprices.com). Describe how $A(g)$, the total cost of g gallons of gas in Amarillo, can be found by performing a stretch or a compression (determine which) on C, then use function notation to write A in terms of C. Graph $A(g)$ and $C(g)$ on the same axes to illustrate the stretch or compression (determine which) on C.

  Solution:
  a. $C(g) = 4.08g$

  b. With $A(g) = 3.51g$ *, we see that* $A(g) = \dfrac{3.51}{4.08}C(g) \approx 0.86C(g)$*. Since* $0 < |a| < 1$*, the transformation is a vertical compression.*

- **Example 2: Vertical Stretches and Compressions**
  Use the table of values shown in the table to evaluate each expression.

a. $0.5 f(x+5)$ when $x=-4$.

b. $1.5 f(x)-8$ when $x=2$.

c. $-5 f(x)+3.5$ when $x=-1$.

x	$f(x)$
–4	7
–3	13
–2	29
–1	20
0	6
1	–1
2	–4
3	–13
4	–28

Solution:

a. $0.5 f(-4+5)=0.5 f(1)=0.5(-1)=-0.5$

b. $1.5 f(2)-8=1.5(-4)-8=-6-8=-14$

c. $-5 f(-1)+3.5=-5(20)+3.5=-96.5$

Exercises

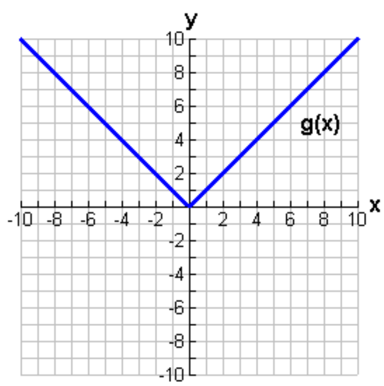
1. Match the graph of the transformation of $g(x)=|x|$ with its formula. Do not use a calculator. Note that some graphs will not be used.

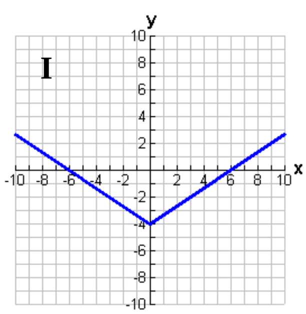
a. $F(x)=\dfrac{2}{3}|x-4|$

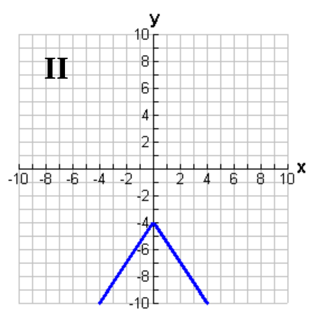
b. $B(x)=-\dfrac{3}{2}|x|-4$

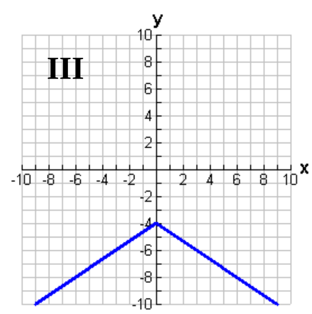
c. $C(x)=-\dfrac{2}{3}|x|-4$

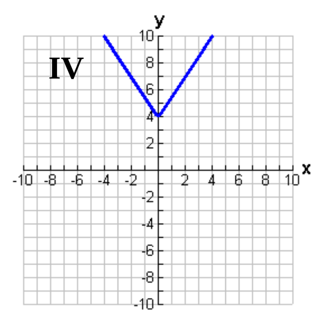
d. $D(x)=\dfrac{3}{2}|x-4|$

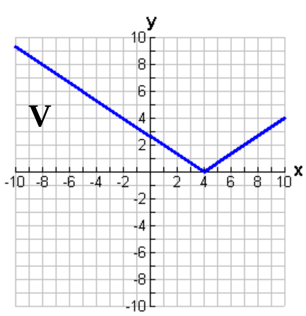


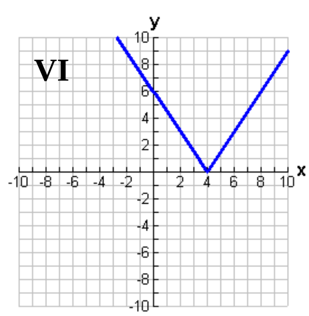




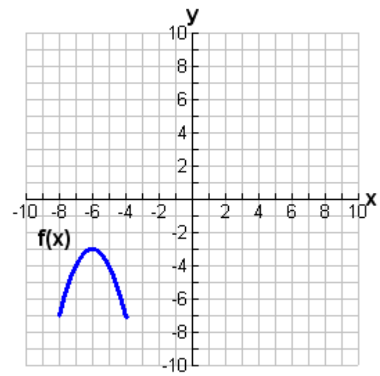








2. List coordinate points that lie on the transformation of f such that $g(x) = \frac{1}{3} f(x)$ and $k(x) = 2f(-x)$. Then describe the transformation.



Exercise Solutions

1. a. V b. II c.III d.VI

2. The table shows three points such that $g(x) = \frac{1}{3} f(x)$ and $k(x) = 2f(-x)$.

$f(x)$	$g(x)$	$k(x)$
$(-6,-3)$	$(-6,-1)$	$(6,-6)$
$(-5,-4)$	$(-5,-\frac{4}{3})$	$(5,-8)$
$(-8,-7)$	$(-8,-\frac{7}{3})$	$(8,-14)$

The function $g(x)$ represents a vertical compression of the function $f(x)$.

The function $k(x)$ represents a horizontal reflection and a vertical stretch of the function $f(x)$.

3.4 Horizontal Stretches and Compressions

Objectives

- Identify what change in a function equation results in a horizontal stretch.
- Identify what change in a function equation results in a horizontal compression.

Concepts and Definitions

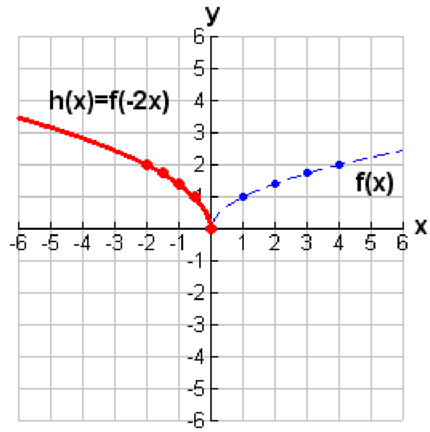
- **Horizontal Stretches:** The graph of $g(x) = f(bx)$ is the graph of *f* stretched horizontally by a factor of $\left|\frac{1}{b}\right|$ units if $\left|\frac{1}{b}\right| > 1$.
- **Horizontal Compressions:** The graph of $g(x) = f(bx)$ is the graph of *f* compressed horizontally by a factor of $\left|\frac{1}{b}\right|$ units if $\left|\frac{1}{b}\right| < 1$.

Examples

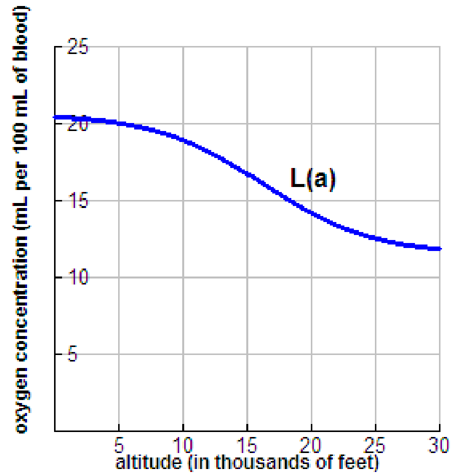
- **Example 1: Stretching a Function Horizontally**
  Consider the function $f(x) = \sqrt{x}$. Create a table, graph, and formula for h if $h(x) = f(-2x)$.

  Solution:
  As we perform the horizontal compression with a negative compression factor, we see a combination of a horizontal compression and horizontal reflection.

x	$f(x) = \sqrt{x}$	$h(x) = f(-2x)$
–2	*undefined*	$f\left(-2(-2)\right) = f(4) = 2$
–1.5	*undefined*	$f\left(-2(-1.5)\right) = f(3) \approx 1.732$
–1	*undefined*	$f\left(-2(-1)\right) = f(2) \approx 1.414$
–0.5	*undefined*	$f\left(-2(-0.5)\right) = f(1) = 1$
0	0	$f\left(-2(0)\right) = f(0) = 0$
1	1	$f\left(-2(1)\right) = f(-2) = \text{undefined}$



- **Example 2: Using Stretches and Compressions to Change Units**
  Based on data from sea level to 30,000 ft, the total level of oxygen in the blood stream in ml O_2/100ml blood may be modeled by $L(a)$ where *a* represents the altitude in thousands of feet ($a = 10$ means 10,000 feet). (Source: Modeled from Supplemental Oxygen for the General Aviation Pilot, 2007, Table 4).



a. Create an estimated table of values for a new function that models the oxygen concentration h feet above sea level.

b. Use function notation to write the new function in terms of L.

Solution:

a. Function O is a horizontal stretch of L because the same output values now occur for input values 1000 times as great. We can also say L is a horizontal compression of O because the same output values occur for input values 0.001 times as great.

Altitude (thousands of feet) a	**Level of Oxygen** $L(a)$	**Altitude (feet)** $h=1000a$	**Level of Oxygen** $O(h)$
0	20.2	0	20.2
10	18.5	10000	18.5
20	14	20000	14
30	12	30000	12

b. Since $h=1000a$ and $a=0.001h$, we can relate $O(h)$ and $L(a)$ as follows:

$$O(h)=L(1000a) \text{ or } L(a)=O(0.001h)$$

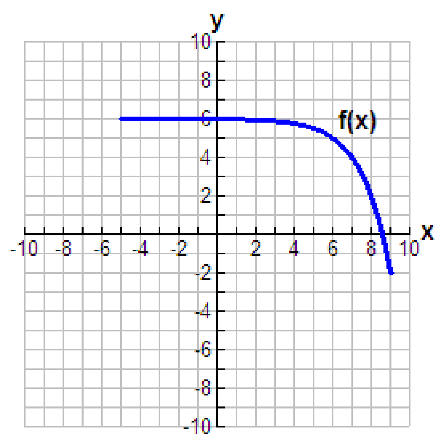
Exercises

1. List three coordinate points that lie on each transformation of the given function.

a. $g(x)=f\left(\frac{1}{3}x\right)$

b. $h(x)=f\left(-5x\right)$

c. $k(x)=2f(0.5x)$



2. Based on data from 1970 to 2004, the U.S. production of crude oil may be modeled by $P(t) = -0.2689t + 20.67$ quadrillion BTU where t is the number of years since 1970. (Source: Modeled from Statistical Abstract of the United States, 2007, Table 895).

a. Function $R(d)$ models the U.S. production of crude oil in quadrillion BTU where d is the number of decades since 1970. Explain the transformation on P required to create R, and then use function notation to write R in terms of P.

b. Rewrite the formula for $R(d)$ as a function of d.

c. Discuss the relationship between the slopes of P and R.

Exercise Solutions

1. The table shows three points such that $g(x) = f\left(\frac{1}{3}x\right)$, $h(x) = f\left(-5x\right)$,and $k(x) = 2f(0.5x)$.

$f(x)$	$g(x)$	$h(x)$	$k(x)$
$(0,6)$	$(0,6)$	$(0,6)$	$(0,-12)$
$(6,5)$	$(2,5)$	$(-30,5)$	$(3,10)$
$(8,2)$	$(\frac{8}{3},2)$	$(-40,2)$	$(4,4)$

2. a. Function R is a horizontal compression of P because the same output values now occur for input values 0.1 times as great. We can also say P is a horizontal stretch of R because the same output values occur for input values 10 times as great. We write $R(d) = P(0.1t)$.

b.

$$P(t) = R(10d) = -0.2689(10d) + 20.67$$
$$R(10d) = -2.689d + 20.67$$

c. The slope of R is 10 times greater than the slope of P.

Chapter 3 Exam A

1. Use the following table to determine the transformations on f required to create the indicated function. Use function notation to write each transformed function in terms of f .

a. $g(x)$

b. $h(x)$

c. $j(x)$

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
–4	0	–3	0	
–3	2	–1	–4	
–2	5	0	–10	0
–1	9	1	–18	2
0	5	7	–10	5
1	–1	11	2	9
2	–2	7	4	5
3	–3	4	6	
4	–5	2	10	

2. Complete the following table of values as much as possible. You will not have enough information to fill in every missing number.

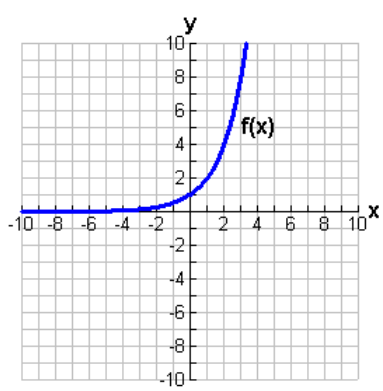
x	$f(x)$	$f(-x)$	$-f(x)$	$-f(x+1)$	$f(-x)-5$
–4	7				
–3	12				
–2	19				
–1	21				
0	15				
1	7				
2	–3				
3	–15				
4	–27				

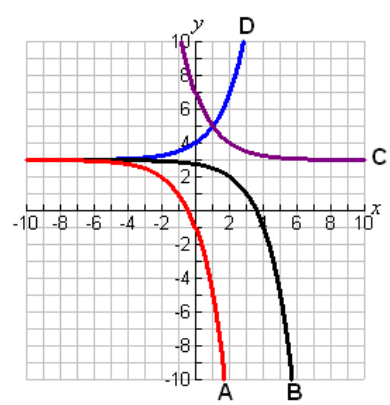
3. Match the formula for the transformation of $f(x)=2^x$ with its graph. One of the graphs will not have a match. Do not use a calculator.

I. $E(x)=-2^{x-2}+3$

II. $G(x)=-2^{x+2}+3$

III. $H(x)=2^{-x+2}+3$

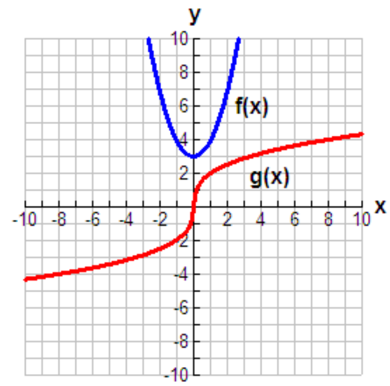




4. Sketch a graph each of the transformations of f or g.

   a. $y = f(3x)$

   b. $y = g(0.5x) - 3$



Chapter 3 Exam A Solutions

1. Use the following table to determine the transformations on f required to create the indicated function. Use function notation to write each transformed function in terms of f .

a. $g(x) = f(-x) + 2$

b. $h(x) = -2f(x)$

c. $j(x) = f(x-2)$

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
–4	0	–3	0	
–3	2	–1	–4	
–2	5	0	–10	0
–1	9	1	–18	2
0	5	7	–10	5
1	–1	11	2	9
2	–2	7	4	5
3	–3	4	6	
4	–5	2	10	

2. Complete the following table of values as much as possible. You will not have enough information to fill in every missing number.

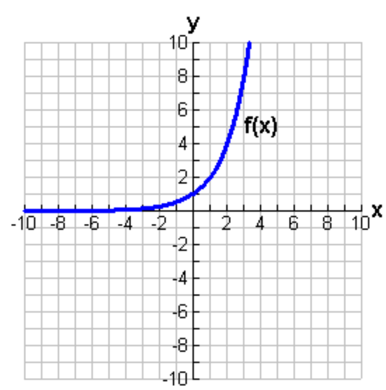
x	$f(x)$	$f(-x)$	$-f(x)$	$-f(x+1)$	$f(-x)-5$
–4	7	–27	–7	–12	–32
–3	12	–15	–12	–19	–20
–2	19	–3	–19	–21	–8
–1	21	7	–21	–15	2
0	15	15	–15	–7	10
1	7	21	–7	3	16
2	–3	19	3	15	14
3	–15	12	15	27	7
4	–27	7	27		2

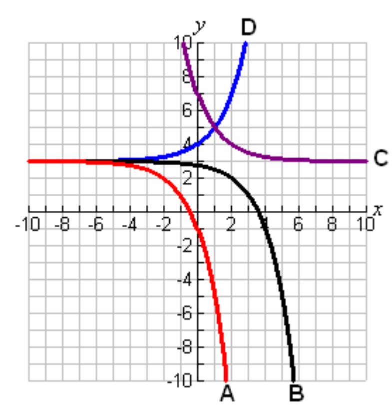
3. Match the formula for the transformation of $f(x) = 2^x$ with its graph. One of the graphs will not have a match. Do not use a calculator.

I. $E(x) = -2^{x-2} + 3$ (B)

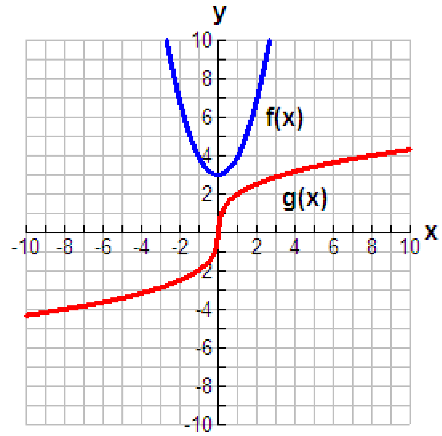
II. $G(x) = -2^{x+2} + 3$ (A)

III. $H(x) = 2^{-x+2} + 3$ (C)



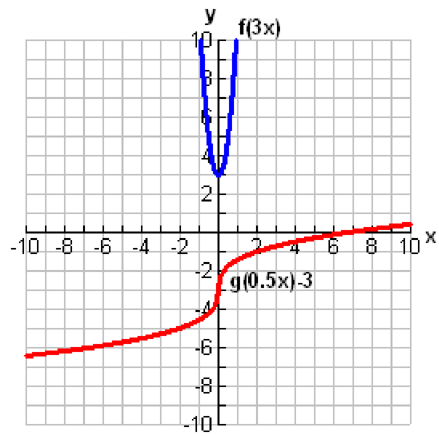


4. Sketch a graph each of the transformations of f or g.



a. $y = f(3x)$

b. $y = g(0.5x) - 3$



Chapter 3 Exam B

1. Use the following table to determine the transformations on f required to create the indicated function, then use function notation to write each transformed function in terms of f.

   a. $g(x)$

   b. $h(x)$

   c. $j(x)$

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
–4	0	1	–5	
–3	2	–1	–3	
–2	5	–4	–2	0
–1	9	–8	–1	5
0	5	–4	0	5
1	–1	2	9	–2
2	–2	3	5	–5
3	–3	4	2	
4	–5	6	0	

2. Determine whether the function $f(x) = x^3 - x$ has odd symmetry, even symmetry, or neither. Support your response algebraically.

The following table shows the number of fighter aircraft produced by four of the major powers in World War II from 1940 to 1944. Use the table to answer questions 3 and 4.

3. Suppose function $g(d)$ models the number of fighter aircraft produced by Germany where the data are aligned so that 1940 represents year 0. Explain the transformation on G required to create g, and then use function notation to write g in terms of G.

Year y	**Germany** $G(y)$	**USSR** $U(y)$	**United Kingdom** $K(y)$
1940	2,746	4,574	4,283
1941	3,744	7,086	7,064
1942	5,515	9,924	9,849
1943	10,898	14,590	10,727
1944	26,326	17,913	10,730

4. Suppose function $k(d)$ models the number of fighter aircraft produced by the United Kingdom where the function values are given in thousands of aircraft. For example, 4,000 would be expressed as 4 (in thousands). Explain the transformation on K required to create k, and then use function notation to write k in terms of K.

Chapter 3 Exam B Solutions

1. Use the following table to determine the transformations on f required to create the indicated function, then use function notation to write each transformed function in terms of f.

a. $g(x) = -f(x)+1$

b. $h(x) = f(-x)$

c. $j(x) = f(2x)$

x	$f(x)$	$g(x)$	$h(x)$	$j(x)$
–4	0	1	–5	
–3	2	–1	–3	
–2	5	–4	–2	0
–1	9	–8	–1	5
0	5	–4	0	5
1	–1	2	9	–2
2	–2	3	5	–5
3	–3	4	2	
4	–5	6	0	

2. Determine whether the function $f(x) = x^3 - x$ has odd symmetry, even symmetry, or neither. Support your response algebraically.

$$\begin{aligned} f(x) &= x^3 - x \\ f(-x) &= (-x)^3 - (-x) \\ &= -x^3 + x \end{aligned} \qquad \begin{aligned} f(x) &= x^3 - x \\ -f(x) &= -(x^3 - x) \\ &= -x^3 + x \end{aligned}$$

Since $f(x) \neq f(-x)$, we know that the function does not have even symmetry. Also, since $f(-x) \neq -f(x)$, we can also verify that the function does not have odd symmetry.

The following table shows the number of fighter aircraft produced by four of the major powers in World War II from 1940 to 1944. Use the table to answer questions 3 and 4.

3. Suppose function $g(d)$ models the number of fighter aircraft produced by Germany where the data are aligned so that 1940 represents year 0. Explain the transformation on G required to create g, and then use function notation to write g in terms of G.

Year y	**Germany** $G(y)$	**USSR** $U(y)$	**United Kingdom** $K(y)$
1940	2,746	4,574	4,283
1941	3,744	7,086	7,064
1942	5,515	9,924	9,849
1943	10,898	14,590	10,727
1944	26,326	17,913	10,730

To align the data so that 1940 represents year 0, we will shift the data horizontally 1940 units to the left. That is, outputs of $g(d)$ will be the same as outputs of $G(y)$ 1940 units earlier. Therefore, we say $g(d) = G(y-1940)$ or $G(y) = g(d+1940)$.

4. Suppose function $k(d)$ models the number of fighter aircraft produced by the United Kingdom where the function values are given in thousands of aircraft. For example, 4,000 would be expressed as 4 (in thousands). Explain the transformation on K required to create k, and then use function notation to write k in terms of K.

Function $k(d)$ is a vertical compression of K because the output values for k are 0.001 times as large as the output values for K for the same input values. We can also say K is a vertical stretch of k because the output values for K are 1000 times as large as the output values for k for the same input values. We say $k(d) = 0.001K(y)$ or $K(y) = 1000k(d)$

Quadratic Functions

4

4.1 Variable Rates of Change

Objectives

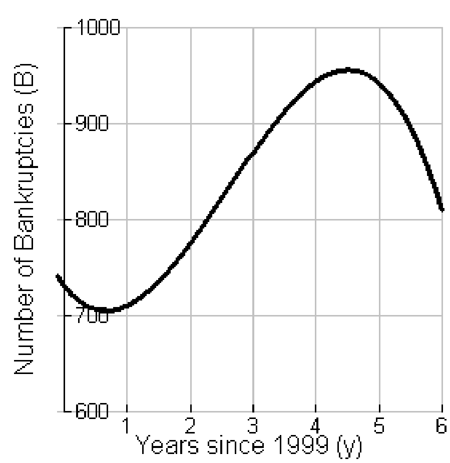
- Understand rates of change in a function model.
- Calculate first and second differences of a table of data.
- Determine the concavity and increasing/decreasing behavior of a function from a table or graph.
- Interpret the meaning of inflection points in real-world contexts.

Concepts and Definitions

- **Increasing Function:** An increasing function is a function whose output values increase as its input values increase.
- **Decreasing Function:** A decreasing function is a function whose output values decrease as its input values increase.
- **Variable Rate of Change:** Any function whose rate of change varies (is not constant) is said to have a variable rate of change. All nonlinear functions have variable rates of change.
- **Inflection Point:** The point on a graph where the function changes concavity is called an inflection point. The inflection point is the point where the instantaneous rate of change is locally maximized or minimized.

Examples

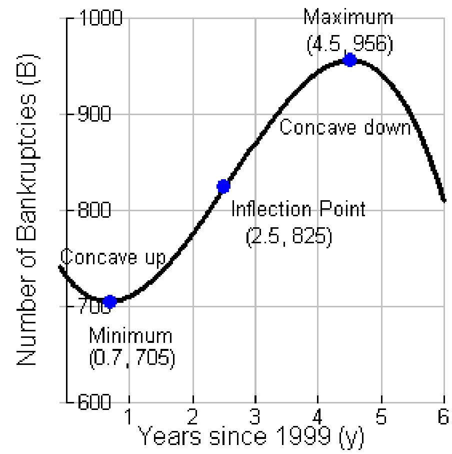
- **Example 1: Interpreting Inflection Points in a Real-World Context**
  The consumer credit industry lobbied Congress for nearly ten years in an effort to pass bankruptcy reform. The industry went to great lengths to paint a picture of consumers using bankruptcy as a means of financial planning, running up huge credit card bills with complete disregard for their ability to repay them and then discharging them in bankruptcy when the well ran dry. On October 17, 2005, the changes the lobbyists had requested were scheduled to take effect. In August 2005, the New York Times reported, “Debtors in Rush to Bankruptcy as Change Nears” (Source: www.nytimes.com). The following graph is a model of the number of Chapter 11 non-business bankruptcies petitions filed, B, in millions of people from 1999 ($y = 0$) to 2005 ($y = 6$). From the graph, do the following:



a. Estimate the intervals over which the function is increasing, decreasing, concave up, and concave down.

b. Determine if there are any inflection points on the graph.

Solution:



a. Estimating from the graph, it appears that the function increases from near $y=1$ *to near* $y=4.5$*. The function decreases from* $y=0$ *to almost* $y=1$ *and then again from* $y=4.5$ *to* $y=6$*. The function appears concave up from* $y=0$ *to about* $y=2.5$ *and the function looks like it is concave down from* $y=2.5$ *to* $y=6$*.*

b. The concavity appears to change from concave up to concave down at $y=2.5$ *so the point* $(2.5, 825)$ *is the approximate inflection point. The minimum value of the function looks like it is near* $(0.7, 705)$ *and the maximum near* $(4.5, 950)$.

- **Example 2: Estimating the Instantaneous Rate of Change from a Formula**
The number of students, S, who took one or more courses in mathematics at Chandler-Gilbert Community College from 1992 to 2003 may be modeled by $S(t)=46.5t^2-188.7t+2180.6$ where $t=0$ is 1992. Estimate the instantaneous rate of change for the year 2002 and explain the meaning of the solution in its real-world context.

Solution:

To estimate the instantaneous rate of change for the year 2002, we find how many students took mathematics classes in both 2002 and a nearby year. We could choose 2001 or 2003 since they are closest to the year 2002. Choosing 2003, we find the number of students by evaluating $S(10)$ *and* $S(11)$.

$$S(10)=46.5(10)^2-188.7(10)+2180.6 \approx 4944 \text{ students}$$

$$S(11)=46.5(11)^2-188.7(11)+2180.6 \approx 5731 \text{ students}$$

We estimate the instantaneous rate of change by finding $\dfrac{\Delta S}{\Delta t}$.

$$\frac{\Delta S}{\Delta t}=\frac{5731-4944}{11-10}\,\frac{\text{students}}{\text{year}}=787 \text{ students per year}$$

We estimate that the number of students taking one or more math classes at Chandler-Gilbert Community College was increasing at a rate of 787 students per year in 2002.

Exercises

1. Determine if each function has a constant or a variable rate of change over the interval provided. Explain or show how you know.

a.

x	y
-4	29
-3	16
-2	7
-1	2
0	1
1	4
2	11
3	22

b.

x	y
-4	13
-3	10
-2	7
-1	4
0	1
1	–2
2	–5
3	–8

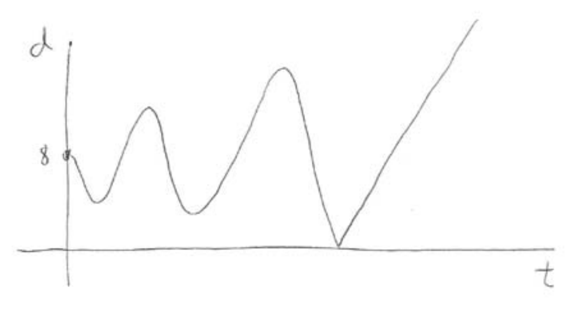
2. A moth starts out eight feet from a light, flies closer to the light, then farther away, then closer than before, then farther away. Finally the moth hits the bulb and flies off. Sketch a possible graph of the distance of the moth from the light as a function of the time. Make sure to label the independent and dependent axes.

Exercise Solutions

1. a. Using the idea of first and second differences, we conclude that the function has a variable rate of change as the $y-$values do not change at a constant rate or equal changes in x.

b. Using the idea of first and second differences, we conclude that the function has a constant rate of change since the $y-$values change at a constant rate of –3 as x increase by 1.

2. Answers may vary. A sample response is given.



4.2 Modeling with Quadratic Functions

Objectives

- Recognize the relationship between a quadratic equation and its graph.
- Use second differences to determine if a quadratic equation represents a data set.
- Construct and use quadratic models to predict unknown results and interpret these findings in a real-world context.

Concepts and Definitions

- **Quadratic Equation in Standard Form:** A function equation of the form $y = ax^2 + bx + c$ with constants a, b, and c and with $a \neq 0$ is called a quadratic equation in standard form. The graph of a quadratic equation is a parabola.
- **The Meaning of the Parameters of a Quadratic Equation:** In the quadratic equation $y = ax^2 + bx + c$, the parameters a, b, and c represent the following:
  - $a =$ one half of the rate of change in the rate of change
  - $b =$ the instantaneous rate of change at $x = 0$ (initial rate)
  - $c =$ the value of y at $x = 0$ (initial value)
- **Difference Properties of Quadratic Functions Given in Tables:** For equally spaced input values, quadratic functions have *linear* first differences and *constant* second differences.

Examples

- **Example 1: Using Quadratic Regression to Extrapolate**

The table shows the per capita personal income for people living in California for selected years (*Source*: Bureau of Economic Analysis). Find a quadratic regression model to represent these data and use the model to predict the average personal income in California in 2011.

t **Years (since 1993)**	***P*** **Personal Income (dollars)**
0	22,833
1	23,348
2	24,339
3	25,373
4	26,521
5	28,240
6	29,772
7	32,149

Solution:

Using the quadratic regression feature of a graphing calculator, we find the model $P(t) = 123.73t^2 + 445.58t + 22847$. *To find the predicted personal income in 2011, we evaluate* $P(18)$ *since 2011 is 18 years since 1993 and so* $t = 18$.
$P(18) = 123.73(18)^2 + 445.58(18) + 22847 \approx 70956$. *According to the model, in 2011 the per capita personal income in California is $70,956.*

- **Example 2: Interpreting the Meaning of the Parameters in a Quadratic Equation**
  Based on data from 1975 – 2004, the number of non-federal hospitals may be modeled by $n(f) = 0.0318f^2 - 9.7454f + 6039$ non-federal hospitals where f is the number of federal hospitals. (Source: Modeled from Health, United States 2006.) Explain the practical meaning of the parameters of the quadratic function model.

  ***Solution*:**
  According to the model, when there are no federal hospitals ($f = 0$), there are 6039 non-federal hospitals. This initial value of non-federal hospitals initially decreases at a rate of 9.7454 non-federal hospitals for every additional federal hospital. This rate of change is increasing at a rate of 0.0636 ($2 \cdot 0.0318$) non-federal hospitals per federal hospital for each additional federal hospital. Since the rate of change is increasing, the graph of the function is concave up.

Exercises

1. Analyze the two tables below. One represents a situation involving a linear function and one represents a situation involving a quadratic function. Determine which is which and explain how you know.

x	y
0	0
1	0.5
2	1
3	1.5
4	2

x	y
0	–1
1	–1
2	1
3	5
4	11

2. The table shows the per capita personal income for people living in Colorado for selected years (*Source*: Bureau of Economic Analysis). Find a quadratic regression model to represent these data and use the model to predict the per capita personal income in Colorado in 2011.

Years (since 1993) ***t***	**Personal Income (dollars)** ***P***
0	22,196
1	23,055
2	24,289
3	25,514
4	27,067
5	28,764
6	30,206
7	32,434

Exercise Solutions

1. The first table represents a linear function. The y-values increase at a constant 0.5 for every increase of 1 in the x-values. The second table represents a quadratic function. While the first differences are not constant, the second differences show that the first differences increase by 2 for every increase of 1 in the x-values.

2. Using the quadratic regression feature of a graphing calculator, we find the model $P(t) = 80.994t^2 + 890.17t + 22158$. To find the predicted personal income in 2011, we evaluate $P(18)$ since 2011 is 18 years since 1993 and so $t = 18$.
 $P(18) = 80.994(18)^2 + 890.17(18) + 22158 \approx 64423$. According to the model, in 2011 the per capita personal income in Colorado is $64,423.

4.3 Quadratic Function Graphs and Forms

Objectives

- Recognize and use the vertex, standard, and factored forms of quadratic functions.
- Determine the vertex, horizontal intercepts, and vertical intercept of a quadratic function from its equation, data table, or graph.
- Use the quadratic formula to solve real-world problems.

Concepts and Definitions

- **Vertex Form of a Quadratic Equation:** The equation of a parabola written in the form $y = a(x-h)^2 + k$ with $a \neq 0$ is said to be in vertex form. The point (h,k) is called the vertex of the parabola.
- The maximum value of a concave down parabola occurs at the vertex. A concave down parabola does not have a minimum value.
- The *minimum* value of a *concave up* parabola occurs at the vertex. A concave up parabola does not have a maximum value.
- **Horizontal Intercepts of a Quadratic Function in Vertex Form:** The horizontal intercepts of a quadratic function $y = a(x-h)^2 + k$ with $a \neq 0$ occur at $x = h \pm \sqrt{-\frac{k}{a}}$ provided $-\frac{k}{a} > 0$. If $-\frac{k}{a} < 0$, the parabola does not have any horizontal intercepts.
- ***x*-Coordinate of the Vertex:** The *x*-coordinate of the vertex of a quadratic function in standard form $y = ax^2 + bx + c$, is $x = -\frac{b}{2a}$.
- **The Quadratic Formula:** The horizontal intercepts of with $y = ax^2 + bx + c$ with $a \neq 0$ occur at $x = \frac{-b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$. If the discriminant $b^2 - 4ac$ is negative, the function does not have any horizontal intercepts.
- **Factored Form of a Quadratic Equation:** The equation of a parabola written in the form $y = a(x-x_1)(x-x_2)$ with $a \neq 0$ is said to be in factored form. The horizontal intercepts of the parabola are $(x_1, 0)$ and $(x_2, 0)$ .The vertex of the parabola lies halfway between the horizontal intercepts at $x = \frac{x_1 + x_2}{2}$. The vertical intercept is $(0, ax_1x_2)$.

Examples

- **Example 1: Analyzing Vertex Form of a Quadratic Equation**
  Determine the coordinates of the vertex and horizontal intercepts of the parabola $y = -2(x+4)^2 - 5$. If no horizontal intercepts exist, so state.

Solution:
The vertex of the parabola is $(-4,5)$. *The horizontal intercepts are found when* $y=0$.

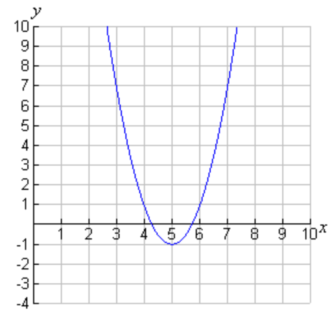
$$\begin{aligned}0&=-2(x+4)^2-5\\5&=-2(x+4)^2\\-\frac{5}{2}&=-(x+4)^2\\\frac{5}{2}&=(x+4)^2\\\pm\sqrt{\frac{5}{2}}&=x+4\\x&=-4\pm\sqrt{\frac{5}{2}}\end{aligned}$$

The horizontal intercepts are located at $(-4+\sqrt{\frac{5}{2}},0)$ and $(-4-\sqrt{\frac{5}{2}},0)$.

- **Example 2: Sketching Graphs of Parabolas**
  Sketch the graph of the quadratic function $y=2(x-5)^2-1$ by hand.

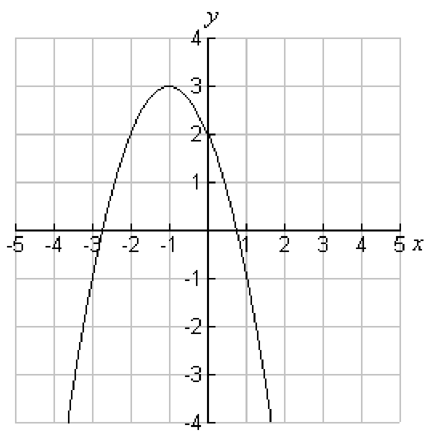
Solution:

We know the vertex of the parabola is $(5,-1)$. *The parabola is concave up and passes through the points* $(4,1)$ *and* $(6,1)$.



Exercises

1. Determine the equation of the quadratic function from its graph.



2. According to www.environmentalgraffiti.com, one of the world's high bungee jumps can be done at Europabrücke Bridge in Austria. The bridge is 630 feet above the ground. How long would it take for a pebble dropped off the bridge to reach the ground? (For the purpose of this model, neglect air resistance.) The height of the pebble as a function of the time elapsed since dropping the pebble is $h(t) = -16t^2 + 630$.

Exercise Solutions

1. We choose to use the vertex form of a quadratic equation since the vertex is readily found to be $(-1, 3)$ so $y = a(x+1)^2 + 3$. We also see that graph has a vertical intercept of $(0, 2)$.

$$\begin{aligned} 2 &= a(0+1)^2 + 3 \\ -1 &= a \end{aligned}$$

Finally, $y = -(x+1)^2 + 3$.

2. We want to know when the pebble will hit the ground so we need to determine the value of t that satisfies the equation $0 = -16t^2 + 630$.

$$\begin{aligned} &0 = -16t^2 + 630 \\ &16t^2 = 630 \\ &t^2 = \frac{630}{16} \\ &t = \pm\sqrt{\frac{630}{16}} \\ &t \approx \pm 6.27 \text{ seconds} \end{aligned}$$

It makes sense to only consider the positive solution. Neglecting air resistance, it takes about 6.27 seconds for a pebble to hit the ground when dropped from a height of 630 feet.

Chapter 4 Exam A

Answer items 1 and 2 using the following situation.

The number of property crimes (burglaries, larceny theft, and motor vehicle theft) increased in the United States from 1984 to 1991. The number decreased from 1991 to 2000 (*Source: Federal Bureau of Investigation*). Using data from the FBI, the following quadratic model was created to represent the number of property crimes (in millions), *C*, as a function of the number of years since 1980 (year 0 corresponds to 1980).

$$C(t) = -0.0357t^2 + 0.80t + 7.62$$

1. Explain the meaning of the parameters in the model in their real-world context.

2. Evaluate $C(20)$ and explain what this value means in the context of the given situation.

3. For each of the following verbal descriptions of a quadratic function model, write a specific model (that you make up) that matches the description. Write the model in general form $y = ax^2 + bx + c$.

   a. The quadratic model has a positive vertical intercept, decreases initially, and then increases at an ever increasing rate.

   b. The quadratic model is concave down, has a maximum value, and has a vertical intercept of (0, 0).

4. A coin is thrown upward from the top of the St. Louis Gateway Arch (630 feet) at an initial velocity of 58 feet per second. Recall the distance off the ground would be found by using the function $h(t) = -16t^2 + v_0 t + h_0$.

   a. At its peak (vertex), what is the height off the ground of this coin?

   b. How long will it take the coin to hit the ground?

   c. How fast will the coin be traveling at impact?

Chapter 4 Exam A Solutions

Answer items 1 and 2 using the following situation.

The number of property crimes (burglaries, larceny theft, and motor vehicle theft) increased in the United States from 1984 to 1991. The number decreased from 1991 to 2000 (*Source: Federal Bureau of Investigation*). Using data from the FBI, the following quadratic model was created to represent the number of property crimes (in millions), *C*, as a function of the number of years since 1980 (year 0 corresponds to 1980).

$$C(t) = -0.0357t^2 + 0.80t + 7.62$$

1. Explain the meaning of the parameters in the model in their real-world context.

In 1980 ($t = 0$), the model predicts the number of property crimes in the United States was 7.62 million. This number initially increases at a rate of 0.80 million property crimes per year. This rate of change is decreasing at a rate of 0.0714 million property crimes per year for each additional year.

2. Evaluate $C(20)$ and explain what this value means in the context of the given situation.

We evaluate the function to get $C(20) = -0.0.357(20)^2 + 0.80(20) + 7.62 = 9.34$. This means that 20 years after 1980 (2000), the model predicts that the number of property crimes is 9.34 million property crimes.

3. For each of the following verbal descriptions of a quadratic function model, write a specific model (that you make up) that matches the description. Write the model in general form - $y = ax^2 + bx + c$.

a. The quadratic model has a positive vertical intercept, decreases initially, and then increases at an ever increasing rate.

Answers can vary but $c > 0$, $b < 0$, and $a > 0$. An example is $y = 3x^2 - 2x + 5$.

b. The quadratic model is concave down, has a maximum value, and has a vertical intercept of (0, 0).

Answers can vary but $c = 0$, $b > 0$, and $a < 0$. An example is $y = -3x^2 + 2x$.

4. A coin is thrown upward from the top of the St. Louis Gateway Arch (630 feet) at an initial velocity of 58 feet per second. Recall the distance off the ground would be found by using the function $h(t) = -16t^2 + v_0 t + h_0$.

a. At its peak (vertex), what is the height off the ground of this coin?

We use $h(t) = -16t^2 + v_0 t + h_0$ with the specific initial values substituted in to obtain $h(t) = -16t^2 + 58t + 630$. The vertex is located at $t = -\frac{58}{2(-16)} = \frac{58}{32} = \frac{29}{16}$. At this time, the height of the coin is $h(\frac{29}{16}) = -16(\frac{29}{16})^2 + 58(\frac{29}{16}) + 630 = 682.5625$ feet. After throwing the coin from the top of the arch, it reaches a maximum height (its peak) of about 682.6 feet after about 1.8 seconds.

b. How long will it take the coin to hit the ground?

We find the time it takes for the coin to hit the ground by solving the equation $h(t) = -16t^2 + 58t + 630 = 0$.

$$h(t) = -16t^2 + 58t + 630 = 0$$
$$t = \frac{-58 \pm \sqrt{58^2 - 4(-16)(630)}}{2(-16)}$$
$$t \approx -4.72, 8.34$$

Disregarding the negative result, we say that it will take about 8.34 seconds to hit the ground.

c. How fast will the coin be traveling at impact?

We know that the coin hits the ground after 8.34 seconds. To estimate the speed of the coin at impact, we find the average rate of change from 8.24 to 8.34 seconds.

$$\text{speed} = \frac{h(8.34) - h(8.24)}{8.34 - 8.24} = \frac{0 - 21.5584}{-0.1} = 215.584 \text{ feet per second}$$

We estimate the speed of the coin at impact to be about 215.6 feet per second.

Chapter 4 Exam B

The data in the table show the cumulative number of homicides due to a romantic triangle (Source: *Crime in the United States 2000*, Uniform Crime Report, FBI). Use the data to answer questions 1 - 4.

Years (Since 1990) ***t***	**Homicides between the Start of 1991 and the End of the Year Shown** ***H***
1	314
2	648
3	1,088
4	1,459
5	1,739
6	1,928
7	2,104
8	2,291
9	2,428
10	2,550

1. Use quadratic regression and a graphing calculator to find the quadratic function that best fits the data set.
2. Use the regression model from #1 to find the vertex of the quadratic model. Interpret the meaning of this vertex in the context of the situation.
3. Find the zeros of the quadratic function and interpret the meaning of these zeros in the context of the situation. Then, write the function in intercept form.
4. Sketch a graph of the function and use the graph to demonstrate your understanding of the idea of constant second differences which is a key characteristic of quadratic functions. For the quadratic function model found in #1, what is this constant second difference and what does it tell us with respect to this situation?

Chapter 4 Exam B Solutions

The data in the table show the cumulative number of homicides due to a romantic triangle (Source: *Crime in the United States 2000*, Uniform Crime Report, FBI). Use the data to answer questions 1 - 4.

Years (Since 1990) ***t***	**Homicides between the Start of 1991 and the End of the Year Shown** ***H***
1	314
2	648
3	1,088
4	1,459
5	1,739
6	1,928
7	2,104
8	2,291
9	2,428
10	2,550

1. Use quadratic regression and a graphing calculator to find the quadratic function that best fits the data set.

$$H(t) = -20.455t^2 + 471.81t - 152.53$$

2. Use the regression model from #1 to find the vertex of the quadratic model. Interpret the meaning of this vertex in the context of the situation.

$$t = -\frac{471.81}{2\cdot(-20.455)} \approx 11.533$$

$$H(11.533) = -20.455(11.533)^2 + 471.81(11.533) - 152.53 \approx 2568.1$$

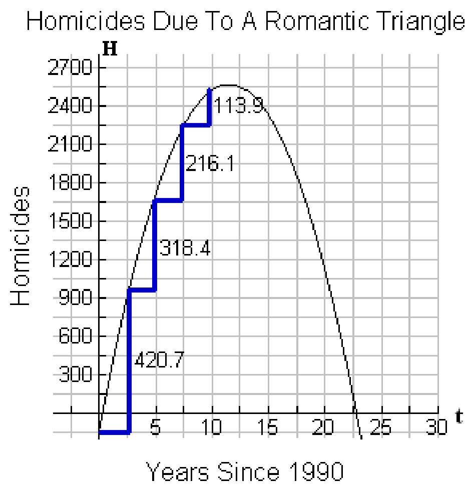
The vertex is $(11.533, 2568.1)$. This vertex tells us that, according to the quadratic model, the maximum number of homicides due to a romantic triangle is about 2568 homicides and that this occurred 11.533 years after 1990 (into 2002).

3. Find the zeros of the quadratic function and interpret the meaning of these zeros in the context of the situation. Then, write the function in intercept form.

$$t = \frac{-471.81 \pm \sqrt{471.81^2 - 4(-20.455)(-152.53)}}{2(-20.455)}$$
$$\approx 0.3279, 22.738$$

This tells us that, as predicted by the quadratic model, there will be no homicides due to a romantic triangle when $t \approx 0.3279$ and 22.738 which correspond to 1991 and 2013 respectively.

4. Sketch a graph of the function and use the graph to demonstrate your understanding of the idea of constant second differences which is a key characteristic of quadratic functions. For the quadratic function model found in #1, what is this constant second difference and what does it tell us with respect to this situation?



Notice that the rate of change is decreasing for each additional 2.5 years since 1990. Consider each rate of change where t changes by 2.5. The average rates of change are 168.28, 127.36, 86.44, and 45.56. Notice that the change in these rates of change is about -40.9. That is, the rate of change is decreasing at a rate of 40.9 homicides per year for each additional year. This constant second difference is approximately twice the value of a in the quadratic model ($2 \cdot 20.455 = 40.91$).

Polynomial, Power, and Rational Functions

5

5.1 Higher-Order Polynomial Function Modeling

Objectives

- Use higher-order polynomial functions to model real-world situations.
- Use the language of rate of change to describe the behavior of a higher-polynomial function.
- Find the inverse of a polynomial function.

Concepts and Definitions

- **Property of Cubic Functions:** Any function with constant third differences is a cubic function.
- **Standard Form of a Cubic Function:** A cubic function has an equation of the form $y = ax^3 + bx^2 + cx + d$ with constants $a, b, c,$ and d and $a \neq 0$.
- **Inflection Point of a Cubic Function:** The point on a graph where the concavity changes is called an inflection point. All cubic functions have exactly one inflection point.
- **Polynomial Function:** For whole number n, a function of the form $y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$ with $a \neq 0$ is called a polynomial function of degree *n*. Each $a_i x^i$ is called a term. The a_i are real-number values called the coefficients of the terms.
- **End Behavior of a Polynomial Function:** For any polynomial function, as x approaches $\pm\infty$, $f(x)$ approaches $\pm\infty$. That is, as the magnitude (absolute value) of x gets larger and larger, the magnitude of the function values will also get larger and larger.

Examples

- **Example 1: Interpreting Inflection Points in a Real-World Context**
  Consider the cubic function, $y = x^3$. Show that the third differences of the cubic function are constant.

  ***Solution*:**
  We will use tables to show that each of the statements is true. We show that the cubic function $y = x^3$ has a constant third difference of 48 for equivalent changes in x.

x	y	Δy	$\Delta(\Delta y)$	$\Delta(\Delta(\Delta y))$
–10	–1000	488	–192	48
–8	–512	296	–144	48
–6	–216	152	–96	48
–4	–64	56	–48	48
–2	–8	8	0	48
0	0	8	48	48
2	8	56	96	48
4	64	152	144	48
6	216	296	192	
8	512	488		
10	1000			

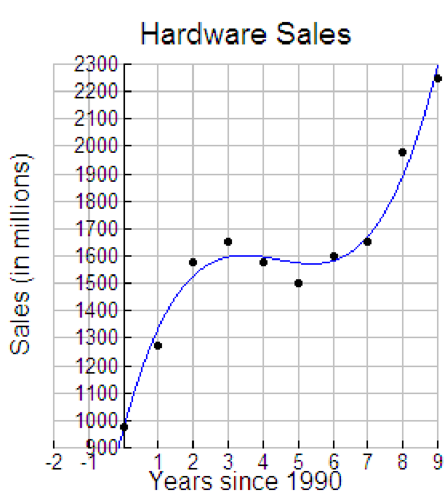
t **Years (Since 1990)**	***S*** **Sales (millions of dollars)**
0	975
1	1,275
2	1,575
3	1,650
4	1,575
5	1,500
6	1,600
7	1,650
8	1,980
9	2,250

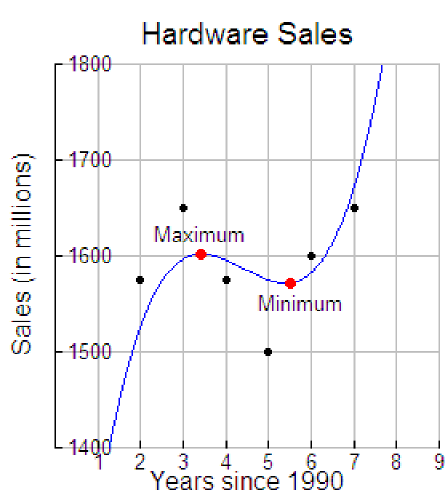
Use this table for Example 2 below.

- **Example 2: Calculating Relative Extremes**
The electronic gaming hardware factory sales industry has experienced ups and downs over the past several years. The data in the table show the sales (in millions of dollars) from 1990 – 1999 (Source: Statistical Abstract of the United States, 2001; Table 1005). Find any extreme values of the model and explain what they mean in the context of the situation.

Solution:

The cubic model is $S(t)=8.451t^3-111.9t^2+470.0t+965.0$*. The function increases at a decreasing rate (concave down) until around* $t=3$ *(1993). The function then decreases until around* $t=5$ *(1995). Beyond 2005, the function increases at an ever increasing rate (concave up). The extreme values occur where the function "turns" from increasing to decreasing (relative maximum) and from decreasing to increasing (relative minimum). Let's zoom in on these relative extremes so that they can be seen more clearly. We use our graphing calculator to find the exact location (in terms of the year) of the relative extremes as well as the output values at these locations as*





predicted by the cubic model. The relative maximum tells us that when $t \approx 3.45$ *(1993–1994), the model estimates about \$1602 million in sales were achieved. The relative minimum tells us that when* $t \approx 5.38$ *(1995–1996), the model estimates about \$1572 million in sales were achieved.*

Exercises

1. Use rates of change to determine if the given point is a relative maximum or a relative minimum: $y = -3x^2 - 3x + 2$; $(-0.5, 2.75)$

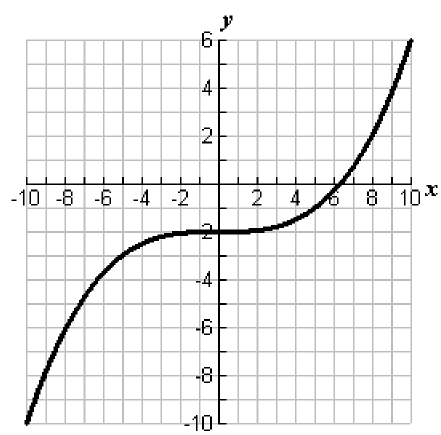
2. Sketch the graph of a function with the given description. Then classify the function by its degree.

   The function is always increasing. It is initially concave down but changes to concave up.

Exercise Solutions

1. Given that the point is either a relative maximum or a relative minimum, we determine which by observing that the given function is a quadratic which is either always concave up or always concave down. We know that the initial rate of change is -3 units of y for each additional unit of x. This rate of change decreases by 6 units of y for each additional unit of x as x increases thus making the function concave down. Therefore, the given point must be a relative minimum.

2. With a single change in concavity (one inflection point), we know that the function is a cubic function. Cubic functions have two observable concavities and one change in concavity. A possible graph of the function is shown.



5.2 Power Functions

Objectives

- Use power functions to model real-world situations.
- Use the language of rate of change to describe the behavior of power functions.
- Determine whether a power function represents direct or inverse variation.

Concepts and Definitions

- **Power Function:** A function with equation of the form $y = ax^b$ where a and b are constants, is called a power function.
- **Direct Variation:** A power functions $y = ax^b$ with $b > 0$ represents direct variation. We say that "y varies directly with x^b" or "y is directly proportional to x^b." a is called the constant of proportionality.
- **Inverse Variation:** A power functions $y = ax^{-c} = \dfrac{a}{x^c}$ with $c > 0$ represents inverse variation. We say that "y varies inversely with x^c" or "y is inversely proportional to x^c." a is called the constant of proportionality.

Examples

- **Example 1: Analyzing Power Functions**
  Using the language of rate of change, describe the power function, $y = a \cdot x^b$, in each case.

  a. $b > 1$ b. $0 < b < 1$ c. $b < 0$

  Solution:
  a. When $b > 1$, we obtain graphs that are increasing and concave up. That is, as x increases, y increases at an ever increasing rate. The greater the value of b, the greater this rate of increase will be. Also, we see that all power functions with $b > 1$ pass through the point $(1, a)$ since $1^b = 1$ for any $b > 1$.

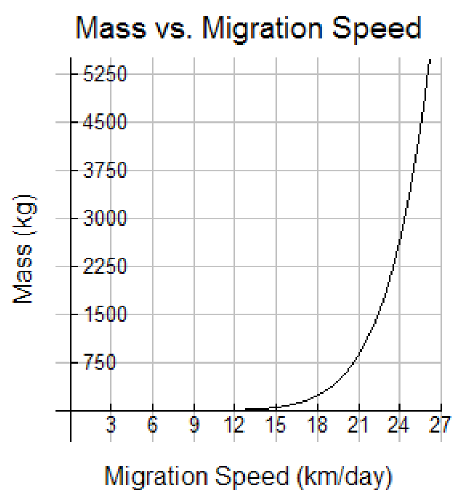
  b. When $0 < b < 1$, we obtain graphs that are increasing and concave down. That is, as x increases, y decreases at an ever decreasing rate. The greater the value of b, the slower this rate of decrease will be. Also, we see that all power functions with $0 < b < 1$ pass through the point $(1, a)$ since $1^b = 1$ for any $0 < b < 1$.

  c. When $b < 0$, we obtain graphs that are decreasing and concave up. That is, as x increases, y decreases at an ever decreasing rate. The more negative the value of b, the greater this rate of decrease will be. Also, we see that all power *functions with* $b < 0$ *pass through the point* $(1, a)$ *since* $1^b = \frac{1}{1^b} = 1$ *for all* $b < 0$.

- **Example 2: Computing and Interpreting the Inverse of a Power Function**
  In Section 5.2 of the text, we see that the migration speed of runners could be modeled using the power function $V = f(m) = 9.31 \cdot m^{0.12}$ kilometers per day where m is the mass of the runner. (The grey wolf, polar bear, and elephant are examples of runners.) Find the function $m = f^{-1}(V)$ and describe its meaning. Confirm your results by graphing the inverse function.

  ***Solution*:**
  The function $V = f(m) = 9.31 \cdot m^{0.12}$ *takes the mass (kg) of an animal,* m*, and gives the migration speed,* V*, in km/day. The inverse,* $m = f^{-1}(V)$*, takes the migration speed,* V*, and gives the mass of that animal. We find the function* $m = f^{-1}(V)$ *by solving* $V = f(m) = 9.31 \cdot m^{0.12}$ *for* m.

$$V = 9.31 \cdot m^{0.12}$$
$$\frac{V}{9.31} = m^{0.12}$$
$$\frac{V}{9.31} = m^{12/100} = m^{3/25}$$
$$\left(\frac{V}{9.31}\right)^{25} = m^3$$
$$\sqrt[3]{\left(\frac{V}{9.31}\right)^{25}} = m$$
$$m = f^{-1}(V) = \sqrt[3]{\left(\frac{V}{9.31}\right)^{25}}$$



Exercises

1. A power function is given in numerical form. Determine whether the power function represents a direct or inverse variation and explain how you know.

x	y
0	0
2	3.793
4	4.579
6	5.113
8	5.529
10	5.875
12	6.173
14	6.437

2. Complete each table so that it accurately represents the verbal description of a power function given: The value of y is inversely proportional to the square of the value of x with a constant of proportionality of 4.

x	1	2	3	4
y				

Exercise Solutions

1. We see that as the value of x increases, the value of y increases. Furthermore, as x increases, y increases at an ever decreasing rate. This is characteristic of a power function where $b > 0$ and thus, this is an example of a power function with a direct variation of the form $y = ax^b$ with $b > 0$.

2. If the value of y is inversely proportional to the square of the value of x with a constant of proportionality of 4, then $y = 4 \cdot \frac{1}{x^2} = \frac{4}{x^2}$. The table is:

x	1	2	3	4
y	$\frac{4}{1^2} = 4$	$\frac{4}{2^2} = 1$	$\frac{4}{3^2} = \frac{4}{9}$	$\frac{4}{4^2} = \frac{4}{16} = \frac{1}{4}$

.

5.3 Rational Functions

Objectives

- Find and interpret the meaning of asymptotes in real-world applications.
- Analyze rational function graphs and identify vertical and horizontal asymptote as well as removable discontinuities.
- Find the domain of rational functions.
- Find the inverse of a rational function.

Concepts and Definitions

- **Rational Function:** A rational function is a function of the for $f(x)=\dfrac{s(x)}{t(x)}$ where $s(x)$ and $t(x)$ are polynomial functions with $t(x)\neq 0$.
- **Vertical Asymptote:** A vertical asymptote of a function $f(x)$ is a vertical line, $x=a$, that the graph of $f(x)$ approaches but does not cross. More formally, as x approaches a, $f(x)$ approaches $\pm\infty$. Symbolically, we write this as $x\to a$, $f(x)\to\pm\infty$.
- **Horizontal Asymptotes:** A horizontal asymptote of a function f is a horizontal line that the graph of f approaches as x approaches positive or negative infinity. More formally, a horizontal asymptote occurs at $y=b$ if and only if the graph of f approaches the line $y=b$ as x approaches either ∞ or $-\infty$.

Examples

- **Example 1: Analyzing Rational Functions**

  Let $f(x)=\dfrac{1}{x-4}$.

  a. Complete a table for x-values close to 4. What happens to the values of $f(x)$ as $x\to 4^{+}$ and $x\to 4^{-}$?

  b. Use a table of values to determine what happens to the values of $f(x)$ as $x\to\infty$? As $x\to-\infty$?

  c. Without a calculator, graph $f(x)$. Give equations for the vertical and horizontal asymptotes.

  Solution:

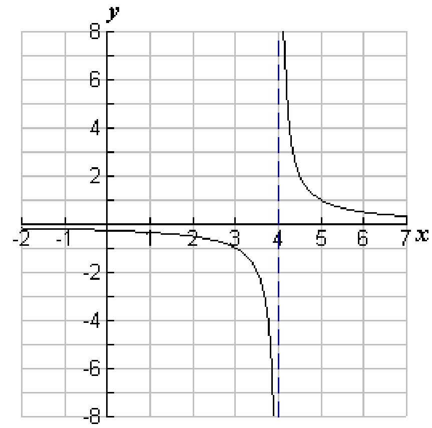
  *a. As $x\to 4^{+}$ **and** $x\to 4^{-}$, $f(x)$ goes off to ∞ and $-\infty$ respectively.*

x	3	3.9	3.99	4	4.01	4.1	5
$f(x)$	-1	-10	-100	undefined	100	10	1

b. As $x \to \pm\infty$, $f(x) \to 0$

x	5	50	500	5,000	50,000
$f(x)$	1	$\frac{1}{46}$	$\frac{1}{496}$	$\frac{1}{4996}$	$\frac{1}{49996}$

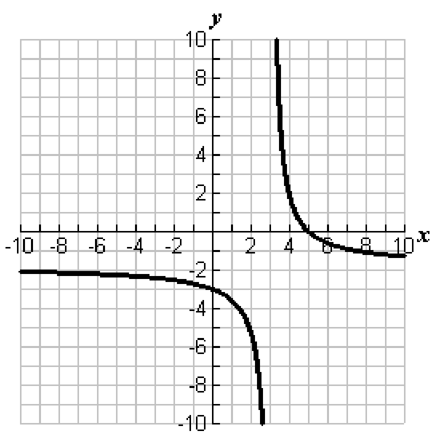
x	–5	–50	–500	–5,000	–50,000
$f(x)$	$\frac{1}{9}$	$-\frac{1}{54}$	$-\frac{1}{504}$	$-\frac{1}{5004}$	$-\frac{1}{50004}$



c. V.A.: $x = 4$ *and H.A.:* $y = 0$

- **Example 2: Sketching Graphs of Rational Functions**
  Draw a possible sketch of a graph for a function $f(x)$ that has one vertical asymptote at $x = 3$, and a horizontal asymptote at $y = -2$. The graph of $f(x)$ has a horizontal intercept at $(5, 0)$ and a vertical intercept at $(0, -4)$. The graph of $f(x)$ is concave down for $-\infty < x < 3$ and concave up for $3 < x < \infty$.

  Solution:



Exercises

1. Write the formula for a rational function that satisfies the function, $g(x)$, that has two vertical asymptotes, one of which must be the line $x = -4$ and the other must be the line $x = 3$.

2. The function $A = f(g) = \dfrac{800,000,000 + 10g}{g}$ gives the average cost per gram, A, of producing g grams of a new drug.

a. What does the function $g = f^{-1}(A)$ represent in this situation?

b. Find the inverse function $g = f^{-1}(A)$.

Exercise Solutions

1. We know that the function $g(x)$ has two vertical asymptotes one at $x = -4$ and the other at $x = 3$. Therefore, the factors $(x-3)$ and $(x+4)$ must be in the denominator of this rational function. We will say that $g(x) = \dfrac{2}{(x-3)(x+4)}$.

2. a. The function $A = f(g) = \dfrac{800,000,000 + 10g}{g}$ gives the average cost, A, of producing g grams of new drug. Therefore, the function $g = f^{-1}(A)$ gives the number of grams of the drug produced, g, for a given average cost, A.

b. To find the inverse function, $g = f^{-1}(A)$, we solve the function $A = f(g) = \dfrac{800,000,000 + 10g}{g}$ for g.

$$
\begin{aligned}
&A = \frac{800,000,000 + 10g}{g} \\
&gA = 800,000,000 + 10g \\
&gA - 10g = 800,000,000 \\
&g\left(A - 10\right) = 800,000,000 \\
&g = \frac{800,000,000}{A - 10} \\
&g = f^{-1}(A) = \frac{800,000,000}{A - 10}
\end{aligned}
$$

Chapter 5 Exam A

1. Three numerical representations of functions are shown in a table. The functions are either linear, quadratic, cubic, or none of these. Use successive differences as appropriate to determine what type of function is represented by each data table.

x	$f(x)$	$g(x)$	$h(x)$
0.0	2.50	0.00	8.00
1.5	4.38	43.88	16.44
3.0	6.25	54.00	23.75
4.5	8.13	50.63	29.94
6.0	10.00	54.00	35.00
7.5	11.88	84.38	38.94
9.0	13.75	162.00	41.75
10.5	15.63	307.13	43.44
12.0	17.50	540.00	44.00
13.5	19.38	880.88	43.44

2. The data in the table show the percentage of 10th graders who admit to using inhalants as a drug (www.ojp.usdoj.gov).

   a. Use regression to determine a mathematical model representing these data.

   c. Calculate any relative extreme values and explain what they mean in the context of the situation.

t **Years (Since 1990)**	*D* **Drug Use (percent)**
1	15.7
2	16.6
3	17.5
4	18.0
5	19.0
6	19.3
7	18.3
8	18.3
9	17.0
10	16.6
11	15.2
12	13.5
13	12.7

3. Write a power function representing the verbal statement: The quality, Q, of men's sperm sample is inversely proportional to the amount of time spent on a cell phone, t (Source: www.clevelandclinic.org).

4. Find the horizontal and vertical asymptotes, if there are any, for the function $f(x) = \dfrac{3x}{4x-1}$.

Chapter 5 Exam A Solutions

1. Three numerical representations of functions are shown in a table. The functions are either linear, quadratic, cubic, or none of these. Use successive differences as appropriate to determine what type of function is represented by each data table.

x	$f(x)$	$g(x)$	$h(x)$
0.0	2.50	0.00	8.00
1.5	4.38	43.88	16.44
3.0	6.25	54.00	23.75
4.5	8.13	50.63	29.94
6.0	10.00	54.00	35.00
7.5	11.88	84.38	38.94
9.0	13.75	162.00	41.75
10.5	15.63	307.13	43.44
12.0	17.50	540.00	44.00
13.5	19.38	880.88	43.44

The function $f(x)$ is a linear function with a constant rate of change of approximately 1.88 for every increase in x of 0.5. The function $g(x)$ is a cubic function with a constant third difference of about 20.25 for every increase in x of 0.5. The function $h(x)$ is quadratic with a constant second difference of about 1.25 for every increase in x of 0.5.

2. The data in the table show the percentage of 10th graders who admit to using inhalants as a drug (www.ojp.usdoj.gov).

t **Years (Since 1990)**	***D*** **Drug Use (percent)**
1	15.7
2	16.6
3	17.5
4	18.0
5	19.0
6	19.3
7	18.3
8	18.3
9	17.0
10	16.6
11	15.2
12	13.5
13	12.7

a. Use regression to determine a mathematical model representing these data.

$$D(t) = 0.0029t^3 - 0.1894t^2 + 1.8654t + 13.765$$

c. Calculate any relative extreme values and explain what they mean in the context of the situation.

There is a relative maximum at $(5.66, 18.78)$. This tells us that 5.66 years after 1990 (into 1996), the percentage of 10th graders who admit to using inhalants as a drug was at an all-time high of 18.78%.

3. Write a power function representing the verbal statement: The quality, Q, of men's sperm sample is inversely proportional to the amount of time spent on a cell phone, t (Source: www.clevelandclinic.org).

$$Q(t) = \frac{a}{t}$$

4. Find the horizontal and vertical asymptotes, if there are any, for the function $f(x) = \dfrac{3x}{4x-1}$.

This function has a vertical asymptote at $x = \dfrac{1}{4}$ since $4x - 1 = 0$ when $x = \dfrac{1}{4}$. This function has a horizontal asymptote at $y = \dfrac{3}{4}$ since $f(x) \to \dfrac{3}{4}$ as $x \to \pm\infty$.

Chapter 5 Exam B

1. For the function $y = -2x^4 + 3x^3 + 2x^2 - x + 2$,

   a. Describe the end behavior of the given polynomial function.

   b. Make a table of values that confirms the end behavior you described. Create your table in such a way that it shows what happens to function values as $x \to \infty$ and $x \to -\infty$.

2. Based on data from 1990 to 2005, the U.S. exports of frozen and canned vegetables are provided in the table (Source: Statistical Abstract of the United States: 2007, Table 819).

   a. Use regression to find the quartic model for these data and explain why a quartic polynomial function best models the data.

   b. Describe what the concave up and concave down portions of the graph tell us about the relationship between the quantities.

t **Years (since 1990)**	***V*** **Exports of vegetables (in thousand metric tons)**
0	529
5	892
10	1112
12	1065
13	1010
14	1048
15	1083

3. Suppose that the cost, C (in millions of dollars), for the federal government to seize p percent of an illegal drug as it enters the country is $C = \dfrac{528p}{100 - p} \quad 0 \le p < 100$.

   a. Draw an accurate graph of this rational function. Clearly label your graph showing the labels on the x– and y–axes. That is, label the x-min, x-max, y-min, and y-max.

   b. According to this model, would it be possible to seize 100% of the drug? Explain. Use the idea of asymptotes in your explanation.

4. When leasing a new car from a dealer, people are usually allowed to drive an average of 12,000 per year during the lease. Let M represent the fuel efficiency of a car in miles per gallon.

   a. Explain what the power function $G(M) = \dfrac{12,000}{M}$ represents. Clearly describe the units involved to help explain your response.

   b. Solve $G(M) = 875$ for M and explain what the result means.

Chapter 5 Exam B Solutions

1. For the function $y = -2x^4 + 3x^3 + 2x^2 - x + 2$,

   a. Describe the end behavior of the given polynomial function.

   The function $y = -2x^4 + 3x^3 + 2x^2 - x + 2$ is a polynomial function of degree 4. This is also known as a quartic function. With a leading coefficient that is negative, we know that as $x \to \pm\infty$, $y \to -\infty$.

   b. Make a table of values that confirms the end behavior you described. Create your table in such a way that it shows what happens to function values as $x \to \infty$ and $x \to -\infty$.

x	–5,000	–500	–50	–5	5	50	500	5,000
$f(x)$	-1×10^{15}	-1×10^{11}	-1.3×10^{7}	–1633	–863	-1.2×10^{7}	-1×10^{11}	-1×10^{15}

2. Based on data from 1990 to 2005, the U.S. exports of frozen and canned vegetables are provided in the table (Source: Statistical Abstract of the United States: 2007, Table 819).

t **Years (since 1990)**	***V*** **Exports of vegetables (in thousand metric tons)**
0	529
5	892
10	1112
12	1065
13	1010
14	1048
15	1083

   a. Use regression to find the quartic model for these data and explain why a quartic polynomial function best models the data.

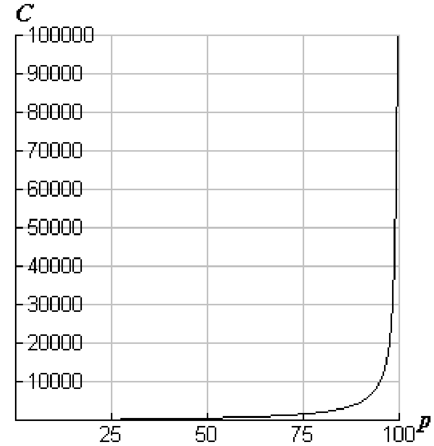
   We can somewhat see the two inflection points and the two changes in concavity thus the three distinct concavities in the graph. The coefficient of determination is quite high compared to other regression possibilities. Therefore, the quartic regression model is: $V(t) = 0.1657t^4 - 5.1119t^3 + 44.837t^2 - 44.493t + 528.99$.

   b. Describe what the concave up and concave down portions of the graph tell us about the relationship between the quantities.

   Initially, the amount of vegetable exports is increasing at an increasing rate (concave up). By about 1995, the amount of vegetable exports, while still increasing, is increasing at a decreasing rate (concave down) and by about 1999, has turned and begins to decrease (still concave down). In about 2001, we see the amount of vegetable exports to begin to decrease but at a slower and slower rate (concave up). After 2003, the amount of vegetable exports begins to increase at an ever increasing rate (still concave up).

3. Suppose that the cost, C (in millions of dollars), for the federal government to seize p percent of an illegal drug as it enters the country is

$$C = \frac{528p}{100 - p} \qquad 0 \le p < 100.$$



a. Draw an accurate graph of this rational function. Clearly label your graph showing the labels on the x – and y – axes. That is, label the x -min, x -max, y -min, and y -max.

b. According to this model, would it be possible to seize 100% of the drug? Explain. Use the idea of asymptotes in your explanation.

No, it would not be possible to seize 100% of the drug. We see a vertical asymptote at $p = 100$ which means that as $p \to 100$, $C \to \infty$.

4. When leasing a new car from a dealer, people are usually allowed to drive an average of 12,000 per year during the lease. Let M represent the fuel efficiency of a car in miles per gallon.

a. Explain what the power function $G(M) = \dfrac{12,000}{M}$ represents. Clearly describe the units involved to help explain your response.

Since 12,000 is measure in miles and M in miles per gallon, the function $G(M) = \dfrac{12,000}{M}$ would out put the total number of gallons of gasoline to drive 12,000 miles with a car getting M miles per gallon.

b. Solve $G(M) = 875$ for M and explain what the result means.

$$G(M) = \frac{12,000}{M} = 875$$
$$12,000 = 875M$$
$$M = \frac{12,000}{875} \approx 13.71 \text{ miles per gallon}$$

A car using 875 gallons of gasoline over the course of a year would have a fuel efficiency of 13.71 miles per gallon.

Exponential and Logarithmic Functions

6

6.1 Percentage Change

Objectives

- Calculate change factors from tables and graphs.
- Calculate percentage rates of change from tables, graphs, and change factors.
- Recognize that functions with a constant percentage change are exponential functions.

Concepts and Definitions

- **Exponential Function:** A function of the form $y = ab^x$ with $a \neq 0$, $b > 0$, and $b \neq 1$ is called an exponential function.
  - a is called the initial value of the function.
  - b is called the growth factor if $b > 1$ and the decay factor if $b < 1$.
- **Change Factors and the Percentage Rate of Change:** The change factor, b, of an exponential function is given by $b = 1 + r$, where r is the percentage change rate (as a decimal). If $r > 0$, b is called a growth factor and r is called the percentage growth rate. If $r < 0$, b is called a decay factor and r is called the percentage decay rate.
- **Exponential Growth and Decay:** Any function that increases at a constant percentage rate is said to demonstrate exponential growth. This growth may be very rapid (e.g., 44% per year) or very slow (e.g., 0.01% per year). Any function that decreases at a constant percentage rate is said to demonstrate exponential decay. This decay may be very rapid (e.g., 250% per year) or very slow (e.g., 20.02% per year).

Examples

- **Example 1: Writing the Formula for an Exponential Function**
  The growth factor for enrollment in higher education is 1.02. In 2007, it is reported that 18.2 million students were enrolled in degree granting institutions of higher education (*Source*: nces.ed.gov). Write the equation of the exponential function that represents this situation.

  Solution:
  Let t represent the number of years since 2007 and E be the enrollment in degree granting institutions of higher education. Since 2007 is 0 years after 2007, the initial value of the enrollment function is 18.2 million. We have $E(t) = 18.2(b)^t$. Since the growth factor $b = 1.02$, the exponential function equation is $E(t) = 18.2(1.02)^t$.
- **Example 2: Finding Change Factors**
  In 1976 the percentage of students enrolled in degree granting institutions of higher education was 3.1%. This percentage increased to 11.4% in 2007. Assuming that these

data have a common growth factor, determine the 31-year growth factor and the annual growth factor.

Solution:

To find the 31-year growth factor, we divide the percentage of Hispanic students for the year 2007 by the percentage of Hispanic students for the year 1976.

$$\begin{aligned} \textit{growth factor} &= \frac{11.4}{3.1} \\ &\approx 3.68 \end{aligned}$$

The five-year growth factor is approximately 3.68.

The annual growth factor is given by

$$\begin{aligned} b &= \left(\frac{11.4}{3.1}\right)^{1/(31-0)} \\ &\approx 3.68^{1/31} \\ &\approx 1.043 \end{aligned}$$

The annual growth factor is approximately 1.043.

Exercises

1. The average annual inflation rate (or cost of living) was 4.13% from 1957 to 2007 (*Source*: www.inflationdata.com). Taking into account the inflation rate, what is cost of buying something in 2007 that cost $1.00 in 1957?

2. The number of cell phones in the United States from 1990 to 2000 increased nearly 40% each year (*Source*: hypertextbook.com). There were 2 million cell phones in 1990. What is the yearly growth factor and what formula models the growth in the number of cell phones from 1990 to 2000? If the number of cell phones has continued to grow at this rate, what is the predicted number of cell phones in 2011?

Exercise Solutions

1. With an annual inflation rate of 4.13%, the growth factor is 1.0413. With an initial value of $1, we can write the exponential function as $B(t) = 1 \cdot 1.0413^t$. We are interested in a 50 year change in time so $t = 50$.

$$B(50) = 1 \cdot 1.0413^{50} \approx 7.56$$

We can now claim that what $1 purchase in 1957 will take $7.56 in 2007.

2. With a 40% increase in the number of cell phones each year, the growth factor is $b = 1.40$. With an initial value of 2 million cell phones in 1990, we write the function formula for the number of cell phones, C, as a function of the number of years since 1990.

$$C(t) = 2(1.40)^t$$

In 2011, 21 years after 1990, the model predicts that the number cell phones in the United States is $C(21) = 2(1.40)^{21} \approx 2342.7 \text{ million cell phones}$. This is about 2.3 billion cell phones. It may be that the 40% rate has not kept pace after 2000 and the number of cell phones is actually much less.

6.2 Exponential Function Modeling and Graphs

Objectives

- Construct exponential models algebraically from tables or words.
- Use exponential regression to model real-world data sets.
- Graph exponential functions given in equations, tables, or words.

Concepts and Definitions

- **The Graphical Significance of the Change Factor:** The change factor, b, controls the steepness and increasing/decreasing behavior of the exponential function $y = a \cdot b^x$. For positive a,
  - if $b > 1$, the graph is increasing and increasing the value of b will make the graph increase more rapidly.
  - if $0 < b < 1$, the graph is decreasing and decreasing the value of b will make the graph decrease more rapidly.
- **Graphical Meaning of the Initial Value:** The exponential function $y = a \cdot b^x$ has vertical intercept $(0, a)$, the initial value of the function.

Examples

- **Example 1: Determining a Percentage Rate of Decay from a Half-Life**
  Colbalt-60 has a half-life of 5.26 years. What percentage of the substance decays each year?

  Solution:
  Since the substance is decaying exponentially, the amount remaining may be modeled by $y = ab^t$. *Since half of the initial value remains after 5.26 years, we know*

$$\tfrac{1}{2}a = ab^{5.26}$$
$$\tfrac{1}{2} = b^{5.26}$$
$$\left(\tfrac{1}{2}\right)^{\frac{1}{5.26}} = \left(b^{5.26}\right)^{\frac{1}{5.26}}$$
$$b \approx 0.8765$$

  Since $b = 1 + r$, $r = -0.1235$. *The amount of Cobalt-60 remaining is decreasing at a rate of 12.35% per day.*

- **Example 2: Using Exponential Regression to Model a Data Set**
  Find the equation of the exponential function that best fits the data set. Then forecast the per capita bottled water consumption in 2012.

Years (since 1980) *t*	**Per capita Bottled Water Consumption (gallons)** *w*
0	2.4
5	4.5
10	8.0
14	10.7
15	11.6
16	12.5
17	13.1
18	16.0
19	18.1

***Solution*:**
Using exponential regression, we obtain the model of best fit. Per capita bottled water consumption may be modeled by $w(t) = 2.593(1.106)^t$ *gallons where* t *is the number of years since the end of 1980.*

To forecast consumption in 2012, we need to evaluate the function at $t = 32$.

$$w(32) = 2.593(1.106)^{32} \approx 65.16 \text{ gallons}$$

We estimate that per capita bottled water consumption in 2012 is 65.16 gallons.

Exercises

1. The median household income in the United States was \$50,221 in 2009. At what annual percentage rate would median salaries have to increase in order for the median income to double by March 2025?

2. A population growth model, based on the World Health Organization's 2006 World Health Statistics, for China is $P(t) = 1,323.345\left(1.007\right)^t$. The population is in millions t years after 2006.

   a. Find the annual growth/decay rate of the population.

   b. State whether the population is increasing or decreasing and explain how you know.

Exercise Solutions

1. Since we are assuming a constant percentage growth, an exponential model may be used to model the median household income.

$$\begin{aligned}2(50,221) &= 50,221(1+r)^{16}\\ 2 &= (1+r)^{16}\\ (2)^{\frac{1}{16}} &= \left((1+r)^{16}\right)^{\frac{1}{16}}\\ 1.044 &\approx 1+r\\ r &\approx 0.044\end{aligned}$$

A doubling time of 16 years corresponds with an annual percentage rate of about 4.4%.

2. a. The population model for China was given to be $P(t)=1,323.345(1.007)^t$. We see a growth factor of 1.007. This means that the growth rate is $1.007-1=0.007=0.7\%$.

b. Since the growth factor is greater than 1, we know that the population is increasing. In fact, we know that the population is increasing at a rate of 0.7% per year.

6.3 Compound Interest and Continuous Growth

Objectives

- Use the compound interest formula to calculate the future value of an investment.
- Construct and use continuous growth models.
- Use exponential models to predict and interpret unknown results.

Concepts and Definitions

- **Periodic Rate:** The periodic rate is calculated as
  $\text{periodic rate} = \dfrac{\text{nominal interest rate}}{\text{compounding frequency}}$.
- **Compound Interest:** The future value A of an initial investment P is given by
  $A = P\left(1+\dfrac{r}{n}\right)^{nt}$ where n is the compounding frequency (the number of times interest is paid per year), t is the number of years the money is invested, and r is the nominal interest rate in decimal form $\left(r = \dfrac{\text{nominal rate}}{100\%}\right)$.
- **Annual Percentage Yield:** The annual percentage yield of an investment earning an nominal interest rate r compounded n times per year is given by
  $$\begin{aligned} APY &= \left(1+\frac{r}{n}\right)^{n} - 1 \\ &= \text{annual growth factor} - 1 \end{aligned}$$
- **The Number *e*:** e is the irrational number $2.718281828\ \ldots$
- **Continuous Compound Interest:** The future value A of an initial investment P earning a continuous compound interest rate r is given by $A = Pe^{rt}$ where t is the number of years after the initial investment is made. The annual percentage yield is $APY = e^{r} - 1$.
- **General Form of an Exponential Function Using *e*:** Exponential functions of the form $f(x) = ab^{x}$ may be written in the form $f(x) = ae^{kx}$ where a is the initial value and k is the continuous growth rate.
- **Continuous Growth (decay) Rate:** The continuous growth (decay) rate of an exponential function is the value of k that makes the following relationship true: $e^{k} = b$ where b is the growth (decay) factor of an exponential function. If $k > 0$, then $b > 1$ and the function is growing. If $k < 0$, then $0 < b < 1$ and the function is decaying.

Examples

- **Example 1: Modeling Compound Interest with Exponential Functions**
  An investor makes a $12,000 initial deposit in a CD with a 3.25% interest rate compounded quarterly. Do the following:

a. Use the compound interest formula to find an exponential function to model the future value of the investment.

b. Determine the annual percentage yield for the investment.

Solution:
a.

$$\begin{aligned} A &= 12,000\left(1+\frac{0.032}{4}\right)^{4t} \\ &= 12,000(1+0.008)^{4t} \\ &= 12,000\left[(1.008)^4\right]^t \\ &= 12,000(1.0324)^t \end{aligned}$$

So, $A(t) = 12,000(1.0324)^t$ *models the future value of the CD.*

b. From the exponential model, we can see that the annual growth factor is 1.0324 and the corresponding annual percentage yield is 3.24% (1.0324 − 1).

- **Example 2: Continuous Compound Interest**
  Find the future value after 20 years of $8,000 invested in an account with a 4.75% nominal interest rate compounded continuously.

  Solution:
  Using the formula for continuous compound interest,

$$\begin{aligned} A &= Pe^{rt} \\ &= 8,000e^{0.0475(20)} \\ &= 8,000e^{0.95} \\ &= 8,000(2.5857) \\ &= \$20,685.60 \end{aligned}$$

  *After four years, the account will be worth $20,685.*60.

Exercises

1. Rewrite the function $f(x) = 5,000e^{0.12x}$ in the form $f(x) = ab^x$.

2. Which CD will have a higher annual percentage yield (APY)? Quoted rates are from www.bankrate.com and are accurate as of June 2011. First Internet Bank of Indiana offers

a nominal rate of 2.37% compounded monthly and Aurora Bank offers a nominal rate of 2.36% compounded daily.

Exercise Solutions

1. Recall that $b = e^k$, so we can find the growth factor b by evaluating $e^{0.12}$.

$$\begin{aligned} f(x) &= 5,000e^{0.12x} \\ &= 5,000\left(e^{0.12}\right)^x \\ &= 5,000(1.1275)^x \end{aligned}$$

The value of the function grows 12.75% each time x changes by 1.

2. We begin by writing the balance, F, for First Internet Bank of Indiana and A for Aurora Bank as a function of time in years, t, since the account was opened.

$$F(t) = P(1+\frac{0.024}{12})^{12t} \text{ and } A(t) = P(1+\frac{0.0239}{365})^{365t}$$

The annual percentage yield (APY) is found using

$$\begin{aligned} APY &= \left(1 + \frac{r}{n}\right)^n - 1 \\ &= \text{ annual growth factor } - 1 \end{aligned}$$

The APY for the First Interstate Bank of Indiana is $(1+\frac{0.0237}{12})^{12} - 1 = 0.02396$.

The APY for the Aurora Bank is $(1+\frac{0.0236}{365})^{365} - 1 = 0.02388$.

The APY is greater for the First Interstate Bank of Indiana at 2.396%. The bank will report an APY of 2.4%. For the Aurora Bank, the APY is 2.388%. The bank will report 2.39%.

6.4 Solving Exponential and Logarithmic Equations

Objectives

- State and use the rules of logarithms.
- Solve exponential equations using logarithms and interpret the real-world meaning of the results.
- Solve logarithmic equations using exponentiation and interpret the real-world meaning of the results.

Concepts and Definitions

- **Logarithmic Functions:** Let b and x be real numbers with $b > 0$, $b \neq 1$, and $x > 0$. The function $y = \log_b(x)$ is called a logarithmic function. The value b is called the base of the logarithmic function. We read the expression $\log_b(x)$ as "log base b of x."
- **Inverse Relationship between Logarithmic and Exponential Functions:** $y = \log_b(x)$ is equivalent to $b^y = x$.
- **Logarithm Rule 1:** $\log_b(b^m) = m$
- **Logarithm Rule 2:** $b^{\log_b(m)} = m$
- **Logarithm Rule 3:** $\log_b(m \cdot n) = \log_b(m) + \log_b(n)$
- **Logarithm Rule 4:** $\log_b\left(\frac{m}{n}\right) = \log_b(m) - \log_b(n)$
- **Logarithm Rule 5**: $\log_b(m^n) = n \log_b(m)$
- **Common and Natural Logarithms**: The common logarithm, $y = \log_{10}(x)$, is typically written as $y = \log(x)$. This is equivalent to $x = 10^y$. The natural logarithm, $y = \log_e(x)$, is typically written as $y = \ln(x)$. This is equivalent to $x = e^y$.
- **Change of Base Formula:** For all $x > 0$, $y = \log_b(x)$ may be written as $y = \frac{\log(x)}{\log(b)}$ or $y = \frac{\ln(x)}{\ln(b)}$. In either of these forms, the logarithm may be evaluated with a calculator.

Examples

- **Example 1: Evaluating a Logarithm**
  a. Find the value of y given that $y = \log_5(125)$.

  b. Estimate the value of y given that $y = \log_7(40)$.

Solution:

a. $y = \log_5(125)$ *answers the question: "What exponent do we place on 5 in order to get 125?" That is, what value of* y *makes the equation* $5^y = 125$ *true? Since* $5^3 = 125$, $y = 3$. *Symbolically, we write* $\log_5(125) = 3$ *and say the "logarithm base 5 of 125 is 3."*

b. $y = \log_7(40)$ *answers the question: "What exponent do we place on 7 in order to get 40?" That is, what value of* y *makes the equation* $7^y = 40$ *true? The answer to this question is not a whole number. Since* $7^1 = 7$ *and* $7^2 = 49$, *we know that* y *is a number between 1 and 2.*

- **Example 2: Solving an Exponential Equation Using Logarithms**
  A Population growth model, based on the World Health Organization's 2006 World Health Statistics, for China is $P(t) = 1{,}323.345(1.007)^t$. The population is in millions t years after 2006. Use the model to predict when the population of China will be 2,000 million people (2 billion people).

  Solution:

$$P(t) = 1{,}323.345(1.007)^t$$
$$2000 = 1{,}323.345(1.007)^t$$
$$1.511 = (1.007)^t$$
$$\ln(1.511) = \ln(1.007)^t$$
$$\ln(1.511) = t\ \ln(1.007)$$
$$\frac{\ln(1.511)}{\ln(1.007)} = t$$
$$59.17 \approx t$$

  According to the model, the population of China will reach 2000 million (2 billion) about 59 years after 2006 (2065).

Exercises

1. State the two integer values between which the expression $\log_5\left(\frac{1}{30}\right)$ falls. Then use the change of base formula to evaluate the expression exactly.

2. Celestial objects, such as stars, moons, and planets, all have different *apparent magnitudes*, which is a measure of how bright the object appears to people on Earth with a lower magnitude indicating a brighter object. The function $B(d) = (2.512)^d$ describes how

many times brighter one object is than another if the difference in their apparent magnitudes is d (Source: liftoff.msfc.nasa.gov). Sirius in the Canis Major constellation is about 5.445 times brighter than Procyon in the Canis Minor constellation (Source: www.windows.ucar.edu). What is the difference in their apparent magnitudes?

Exercise Solutions

1. We can estimate $\log_5\left(\frac{1}{30}\right)$ by thinking about the exponent we place on 5 to get $\frac{1}{30}$. We know that $5^{-2} = \frac{1}{25}$ and $5^{-3} = \frac{1}{125}$. Therefore, the value of $\log_5\left(\frac{1}{30}\right)$ lies between -2 and -3. We can be more specific using the change of base formula.

$$\log_5\left(\frac{1}{30}\right) = \frac{\log\frac{1}{30}}{\log 5} = \frac{\ln\frac{1}{30}}{\ln 5} \approx -2.11$$

2. We use the function $B(d) = \left(2.512\right)^d$ and the given information to set up the following equation. Then, we use logarithms to solve the equation and interpret the result.

$$\begin{aligned} 5.445 &= \left(2.512\right)^d \\ d &= \log_{2.512} 5.455 \\ d &= \frac{\log 5.455}{\log 2.512} \\ d &\approx 1.84 \end{aligned}$$

The difference in the apparent magnitudes of Sirius and Procyon.

6.5 Logarithmic Function Modeling

Objectives

- Graph logarithmic functions from equations and tables.
- Use logarithmic regression to model real-world data sets.
- Use logarithms to linearize exponential data to find an exponential model.

Concepts and Definitions

- **Inverse of a Logarithmic Function:** The inverse of the exponential function $p = ab^t$ is the logarithmic function $t = \log_b\left(\frac{p}{a}\right) = \frac{\ln p - \ln a}{\ln b} = \frac{\ln p}{\ln b} - \frac{\ln a}{\ln b}$.
- **Find an Exponential Model Using Logarithms:** To find an exponential model for t as a function of p, do the following:
    - Calculate $\log p$ for each value of p.
    - Use linear regression on the data set with input value t and output value $\log p$.
    - Write the linear regression equation in the form $\log p = mt + b$.
    - Rewrite the logarithmic equation in exponential form:
      $p = 10^{mt+b} = \left(10^{mt}\right)\left(10^{b}\right) = \left(10^{b}\right)\left(10^{m}\right)^{t}$.

Examples

- **Example 1: Finding a Logarithmic Function Model Algebraically** The table of data shows the total school expenditures and expenditures per pupil between the 1990–91 school and the 2002 – 2003 school year.

  Source: National Center for Education Statistics, Table 34

School Year t	Expenditure per pupil (in dollars) E	Total School Expenditure (in billion dollars) T
1990 – 91	4902	202.0
1991 – 92	5023	211.2
1992 – 93	5160	220.9
1993 – 94	5327	231.5
1994 – 95	5529	243.9
1995 – 96	5689	255.1
1996 – 97	5923	270.2
1997 – 98	6189	285.5
1998 – 99	6508	302.9
1999 – 2000	6912	323.9
2000 – 01	7380	348.4
2001 – 02	7727	368.4
2002 – 03	8044	387.6

  Do the following:

  a. Find a logarithmic function model for the total school expenditure as a function of expenditure per pupil. (Use the first and last points in the data set to find the equation of the model algebraically.)

c. Verify the accuracy of your results by graphing a scatter plot of the data together with the model.

Solution:
We substitute the first and last points from the data table into the equation to determine the values of a *and* b.

$$202.0 = \log_b\left(\frac{4902}{a}\right) \qquad 387.6 = \log_b\left(\frac{8044}{a}\right)$$
$$b^{202.0} = \frac{4902}{a} \qquad b^{387.6} = \frac{8044}{a}$$
$$a = \frac{4902}{b^{202.0}} \qquad a = \frac{8044}{b^{387.6}}$$

Setting the two equations equal to each other, we solve for b.

$$\begin{aligned}\frac{4902}{b^{202.0}} &= \frac{8044}{b^{387.6}}\\ b^{185.6} &= \frac{8044}{4902}\\ &\approx 1.002672\end{aligned}$$

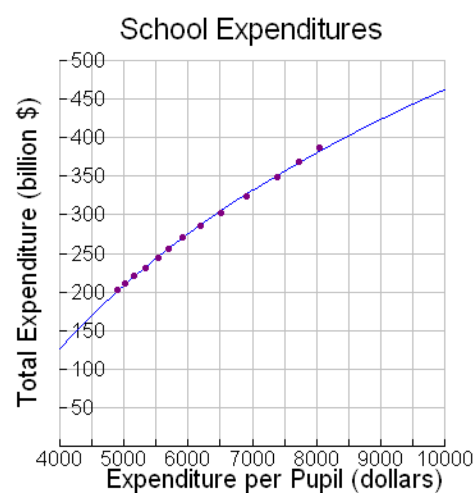
We substitute this result into $a = \dfrac{4902}{b^{202.0}}$ *and solve for* a.

$$a = \frac{4902}{(1.002672)^{202.0}} \approx 2859$$

The resultant logarithmic model is $T = \log_{1.002672}\left(\dfrac{E}{2859}\right)$. *This model may be simplified as shown below.*

$$\begin{aligned}T &= \log_{1.002672}\left(\frac{E}{2859}\right)\\ &= \frac{\ln E - \ln 2859}{\ln(1.002672)} = 374.7\ln E - 2982\end{aligned}$$

Based on our analysis of the situation and the graph, the results seem reasonable



- **Example 2: Linearizing Data**
  Determine algebraically the equation of the linear function that passes through points of the form $(t, \log p)$.

t	p
–4	2048
–1	32
0.5	4
1	2

Solution:
We first calculate $\log p$ *and then use linear regression to find the linear regression function for these data:* $\log y = -0.6021t + 0.9031$.

t	$\log p$
–4	3.3113
–1	1.5051
0.5	0.6021
1	0.3010

Exercises

The data in the table shows the average salary of basketball players in the NBA (*Source*: Statistical Abstract of the United States, 2001; Table 1324). Use the data to answer exercises 1 and 2.

Years (Since 1980) t	**NBA Average Salary (\$1000s)** S
0	170
5	325
10	750
15	1,900
16	2,000
17	2,200
18	2,600

1. Use linear regression to find a model of the form $\log S = mt + b$.

2. Solve the equation in part (b) for S and rewrite the result in the standard form of an exponential function.

Exercise Solutions

1. $\log(S) = 0.0679t + 2.2080$

2.

$$\log(S) = 0.0679t + 2.2080$$
$$S = 10^{0.0679t+2.2080} = 10^{0.0679t}10^{2.2080}$$
$$= 1.1692^t \cdot 161.44$$
$$= 161.44 \cdot 1.1692^t$$

Chapter 6 Exam A

1. Assume that a scientist sampled data that she believes is growing exponentially. A laboratory assistant has recorded data over the last six hours and entered the following data. One of the points has been entered in error. Which is it? Explain your reasoning.

Time *h* **(hours)**	**Number** *n*
0	10.65
1	18.75
2	33.00
3	47.65
4	102.20
5	179.85
6	316.55

2. The data to the right show how the postal rates have increased for first-class letters weighing up to 1 ounce.

   a. Find an exponential model to fit the data.

   b. Discuss how well the model fits the data.

   c. Use your model to predict the postage rate in the year 2012.

Year of rate increase *y*	**Postal Rate** *p* **(cents)**
1919	2
1932	3
1958	4
1963	5
1968	6
1971	8
1974	10
1975	13
1978	15
1981	20
1985	22
1988	25
1991	29
1995	32
1999	33
2001	34
2002	37
2006	39

3. Based on data from August 1, 1952, through May 31, 2007, the average annual return on an investment in the CREF Stock account was 10.72%. (*Source*: www.tiaa-cref.com). According to the model, how much would a $1000 investment made in the account on August 1, 1952, be worth on August 1, 2007?

4. Solve for x: $2.1^{5.7x} + 4 = 25.8$

Chapter 6 Exam A Solutions

1. Assume that a scientist sampled data that she believes is growing exponentially. A laboratory assistant has recorded data over the last six hours and entered the following data. One of the points has been entered in error. Which is it? Explain your reasoning.

Time h **(hours)**	**Number** n
0	10.65
1	18.75
2	33.00
3	47.65
4	102.20
5	179.85
6	316.55

Knowing that the data is exponential, we should see a common ratio between data points. This common ratio should be approximately 1.76 since this is the ratio for the majority of subsequent values of n. However, the ratios $\frac{47.65}{33.00}$ and $\frac{102.2}{47.65}$ do not fit this trend. The data point at $t = 3$ and $n = 47.65$ is the questionable data point. In all other cases, the common ratio is 1.76. To make this true at $t = 3$, we would have to see $n = 58.08$.

2. The data to the right show how the postal rates have increased for first-class letters weighing up to 1 ounce.

Year of rate increase y	**Postal Rate** p **(cents)**
1919	2
1932	3
1958	4
1963	5
1968	6
1971	8
1974	10
1975	13
1978	15
1981	20
1985	22
1988	25
1991	29
1995	32
1999	33
2001	34
2002	37
2006	39

a. Find an exponential model to fit the data.

We take y to be the number of years since 1900.

$$p(y) = 0.6655 \cdot 1.0397^{y}$$

b. Discuss how well the model fits the data.

The model fits the data adequately. With an $r^2 \approx 0.92$, the strength of the fit is not exceptional.

c. Use your model to predict the postage rate in the year 2012.

$$p(112) = 0.6655 \cdot 1.0397^{112} \approx 52.3$$

In 2012, we predict that postage rate is 52.3 cents per stamp.

3. Based on data from August 1, 1952, through May 31, 2007, the average annual return on an investment in the CREF Stock account was 10.72%. (*Source*: www.tiaa-cref.com). According to the model, how much would a $25,000 investment made in the account on August 1, 1952, be worth on August 1, 2020?

$$A(t) = 25000(1+0.1072)^{t}$$

$$\begin{aligned} A(68) &= 25000(1+0.1072)^{68} \\ &= 25,428,061.45 \end{aligned}$$

In 2020, the account will have a value of $25,428,061.45.

4. Solve for x: $2.1^{5.7x} + 4 = 25.8$

$$\begin{aligned}
2.1^{5.7x} + 4 &= 25.8 \\
2.1^{5.7x} &= 21.8 \\
5.7x &= \log_{2.1} 21.8 \\
5.7x &= \frac{\log 21.8}{\log 2.1} \\
x &= \frac{1}{5.7} \cdot \frac{\log 21.8}{\log 2.1} \approx 0.7287
\end{aligned}$$

Chapter 6 Exam B

1. You are given models for the population in millions of different countries t years after 2005. For each exercise, determine the year in which the models predict the populations will be equal (*Source*: World Health Organization's 2006 statistics).

$$\text{U.S.: } U(t) = 298.2\left(1.009\right)^t \text{ and Indonesia: } I(t) = 222.8\left(1.012\right)^t$$

2. Since air pressure decreases as altitude increases, many airplanes have pressure altimeters that measure the atmospheric pressure to determine the altitude of the airplane. While specific weather conditions in a location can require calibration for accuracy, the basic formula used by these devices is $a(p) = -26,216 \ln\left(\dfrac{p}{101.304}\right)$ where a is the altitude in feet and p is the pressure in kilopascals (1 kilopascal is about 0.145 psi in pressure). (*Source*: www.ozarkaerospace.com) Find the inverse of $a(p)$ and explain what the inverse function models.

3. Use the given information to determine which CD will have a higher annual percentage yield (APY). Quoted rates are from www.bankrate.com and are accurate as of April 20, 2009.

   Bank #1 – GMAC Bank offers a nominal rate of 3.44% <u>compounded continuously</u> with an initial investment of $100,000.

   Bank #2 – State Bank of India offers a nominal rate of 3.6% <u>compounded quarterly</u> with an initial investment of $100,000.

4. Solve for x: $\log\left(\dfrac{x}{4}\right) + \log\left(16\right) = -2$

Chapter 6 Exam B Solutions

1. You are given models for the population in millions of different countries t years after 2005. For each exercise, determine the year in which the models predict the populations will be equal (*Source*: World Health Organization's 2006 statistics).

U.S.: $U(t) = 298.2(1.009)^t$ and Indonesia: $I(t) = 222.8(1.012)^t$

$$298.2(1.009)^t = 222.8(1.012)^t$$
$$\frac{(1.009)^t}{(1.012)^t} = \frac{222.8}{298.2}$$
$$\left(\frac{1.009}{1.012}\right)^t = \frac{222.8}{298.2}$$
$$0.9970^t = 0.7471$$
$$t = \log_{0.9970} 0.7471 = \frac{\ln 0.7471}{\ln 0.9970} \approx 97.04$$

The two countries are predicted to have the same population about 97 years after 2005 (2102).

2. Since air pressure decreases as altitude increases, many airplanes have pressure altimeters that measure the atmospheric pressure to determine the altitude of the airplane. While specific weather conditions in a location can require calibration for accuracy, the basic formula used by these devices is $a(p) = -26,216 \ln\left(\frac{p}{101.304}\right)$ where a is the altitude in feet and p is the pressure in kilopascals (1 kilopascal is about 0.145 psi in pressure). (*Source*: www.ozarkaerospace.com) Find the inverse of $a(p)$ and explain what the inverse function models.

$$a = -26,216 \ln\left(\frac{p}{101.304}\right)$$
$$\frac{a}{-26,216} = \ln\left(\frac{p}{101.304}\right)$$
$$e^{-\frac{a}{26,216}} = \frac{p}{101.304}$$
$$101.304e^{-\frac{a}{26,216}} = p$$
$$p^{-1}(a) = 101.304e^{-\frac{a}{26,216}}$$

Given the altitude, a, the function $p^{-1}(a)$ outputs the pressure in kilopascals.

3. Use the given information to determine which CD will have a higher annual percentage yield (APY). Quoted rates are from www.bankrate.com and are accurate as of April 20, 2009.

Bank #1 – GMAC Bank offers a nominal rate of 3.44% compounded continuously with an initial investment of $100,000.

$APY = e^{0.0344} - 1 \approx 0.035$

Bank #2 – State Bank of India offers a nominal rate of 3.6% compounded quarterly with an initial investment of $100,000.

$APY = (1 + \frac{0.036}{4})^4 - 1 \approx 0.036$

Bank #2 has a higher APY of 3.6% versus 3.5% for Bank #1.

4. Solve for x: $\log\left(\frac{x}{4}\right) + \log(16) = -2$

$$\begin{aligned}
\log(\frac{x}{4}) + \log(16) &= -2 \\
\log(\frac{x}{4} \cdot 16) &= -2 \\
\log(4x) &= -2 \\
10^{-2} &= 4x \\
\frac{1}{100} &= 4x \\
x &= \frac{1}{400}
\end{aligned}$$

Modeling with Other Types of Functions

7

7.1 Combinations of Functions

Objectives

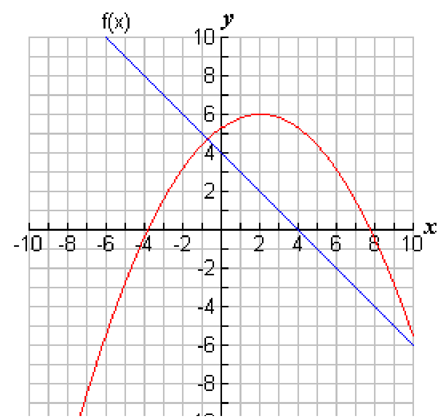
- Compute the sum, difference, product, or difference of functions to model a real-world situation.
- Determine the practical and theoretical domain of the combinations of functions.

Concepts and Definitions

- **Combinations of Functions:**
  - Sum of two functions: $(f+g)(x) = f(x)+g(x)$
  - Difference of Two Functions: $(f-g)(x) = f(x)-g(x)$
  - Product of Two Functions: $(f \cdot g)(x) = f(x)\cdot g(x)$
  - Quotient of Two Functions: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)},\ g(x) \neq 0$

Examples

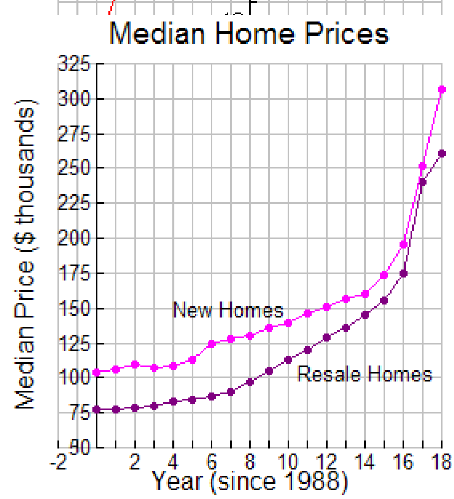
- **Example 1: Combining Functions Graphically**
  Use the graphs of functions $f(x)$ and $g(x)$ to evaluate the given functions. If it is not possible to evaluate the function, explain why not.



  a. $(f+g)(2)$ b. $(f \cdot g)(2)$

  Solution:
  a. Examining the graph, we find that $f(2)=2$ *and* $f(2)=6$. *Therefore,* $(f+g)(2) = f(2)+g(2) = 2+6 = 8$.
  b. $(f \cdot g)(2) = f(2)\cdot g(2) = 2\cdot 6 = 12$

- **Example 2: Interpreting the Combination of Functions**
  The graph shows the median price of new and resale homes in the Greater Phoenix, Arizona area (Source: www.poly.asu.edu).



  The median price of resale homes as a function of years since 1988 is given by $R(t)$, in thousands of dollars. The median price of new homes as a function of years since 1988 is given by $N(t)$, in thousands of dollars.

a. Interpret the meaning of the function $(N-R)(t)$.

b. Estimate and interpret $(N-R)(6)$.

c. Interpret the meaning of the function $\left(\frac{N}{R}\right)(t)$.

d. Estimate and interpret $\left(\frac{N}{R}\right)(17)$.

Solution:

a. $(N-R)(t)$ *is the difference in the median price of resale homes and the median price of new homes.*

b. $(N-R)(6)=N(6)-R(6)\approx 125-88=37$ *There is a $37,000 difference between the median price of resale homes and the median price of new homes.*

c. $\left(\frac{N}{R}\right)(t)$ *is the ratio comparing the median price of new homes with the median price of resale homes.*

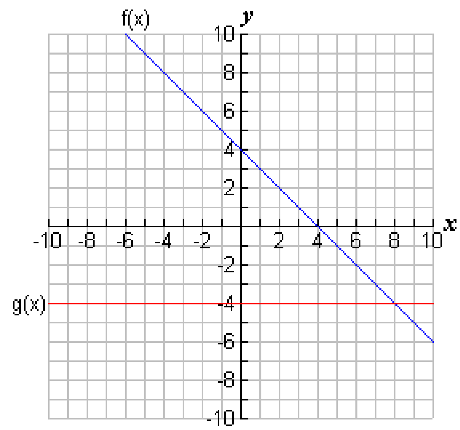
d. $\left(\frac{N}{R}\right)(17)=\frac{N(17)}{R(17)}\approx\frac{250}{240}\approx 1.04$ *The ratio comparing the median price of new homes with the median price of resale homes is just slightly greater than 1. This tells us that the median prices are very similar in 2005.*

Exercises

1. Combine the functions $f(x)=3x-5$ and $g(x)=-4x+7$ symbolically as indicated. Then state the domain of this combination.

   a. $(f+g)(x)$

   b. $(f-g)(x)$



2. Graphs of the functions $f(x)$ and $g(x)$ are given to the right. Using the graph, sketch a graph of the combination of functions $(f+g)(x)$.

Exercise Solutions

1. a.

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ &=(3x-5)+(-4x+7)\\ &=-x+2\end{aligned}$$

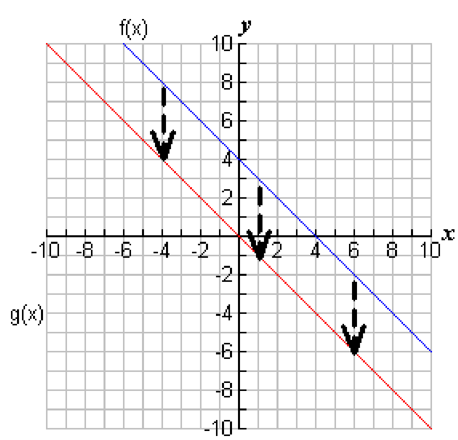
The domain of the function $(f+g)(x)$ is the set of all real numbers.

b.

$$\begin{aligned}(f-g)(x) &= f(x)-g(x)\\ &=(3x-5)-(-4x+7)\\ &=7x-12\end{aligned}$$

The domain of $(f-g)(x)$ is the set of all real numbers.

2. When analyzing the graph, we see that $g(x)$ is the constant function $g(x)=-4$. When adding this function to $f(x)$ to obtain a graph of $(f+g)(x)$, we would add the constant -4 to all the output values of $f(x)$. This would have the effect of shifting the graph of $f(x)$ down 4 units. Note that the graph shows some examples of the downward shift of 4 units.



7.2 Piecewise Functions

Objectives

- Define piecewise functions using equations, tables, graphs and words.
- Determine function values of piecewise functions from a graph, equation, and table.

Concepts and Definitions

- **Piecewise Function:** A piecewise function is a function which is defined using two or more formulas over given intervals of the domain. Piecewise functions are written in the form

$$f(x)=\begin{cases}\text{Rule 1} & \text{if Condition 1}\\ \text{Rule 2} & \text{if Condition 2}\\ \text{Rule 3} & \text{if Condition 3}\\ \vdots & \vdots\end{cases}$$

The conditions define the domain values for which each rule applies.

Examples

- **Example 1: Graphing a Piecewise Function**

Given that $f(x)=\begin{cases}x^2 & \text{if } x\le -1\\ 0.6x-6 & \text{if } x>3\end{cases}$, do the following:

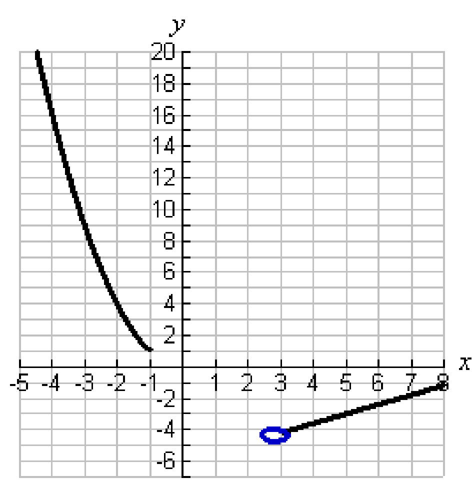
a. Describe what each piece of the graph looks like using what you have previously learned.

b. Sketch the graph of the piecewise function.

Solution:

a. For values of x less than or equal to –1, the shape will be a parabola. For x values that are greater than 3, the graph will be linear with the vertical intercept –6 and slope of 0.6. It is important to note that there are no function values defined for $-1<x\le 3$.

b. Our description from part (a) is helpful as we graph the function. The function has two pieces one which is parabolic and the other



which is linear. Note that for values of $f(x)$ that are greater than 1 and less than or equal to 3, this function is not defined.

- **Example 2: Evaluating a Piecewise Function**

Given that $f(x)=\begin{cases} x^2 & \text{if } x\le -1 \\ 0.6x-6 & \text{if } x>3 \end{cases}$, do the following:

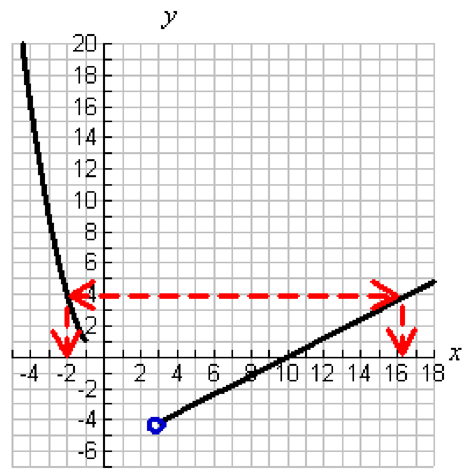
a. Evaluate: $f(-4)$, $f(3)$, $f(5)$

b. Solve $f(x)=4$ for x.

***Solution*:**

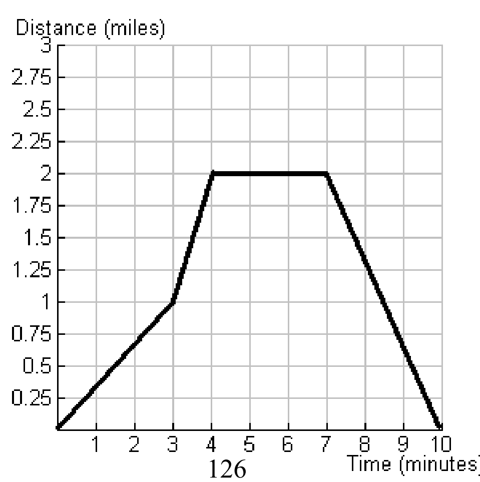
a. To evaluate $f(-4)$ means to find the value of the function when $x=-4$. We know from $f(x)$ that when $x\le -1$ we are to evaluate the function using $f(x)=x^2$. Therefore, $f(-4)=(-4)^2=16$. To evaluate $f(3)$ means to find the value of the function when $x=3$. We can see from $f(x)$ above that the function is not defined when $x=3$. Therefore, $f(3)$ is undefined. To evaluate $f(5)$ means to find the value of the function when $x=5$. We know from $f(x)$ that when $x>3$ we are to evaluate the function using $f(x)=0.6x-6$. Therefore, $f(5)=0.6(5)^2-6=9$.

b. To solve $f(x)=4$ for x using the graph we need to find 4 on the y axis and then locate where the graph has an output value of 4. Next we find the x value that is associated with the y value of 4. We find that there are two answers for this problem, -2 and $16\frac{2}{3}$.



Exercises

1. The following graph illustrates a trip a family took from home in four parts. Create a piecewise function formula for the *speed* traveled as a function of time in minutes.



2. According to www.city-data.com, the average cost of electricity in a Phoenix apartment (one story, no pool) depends on the size of the apartment.

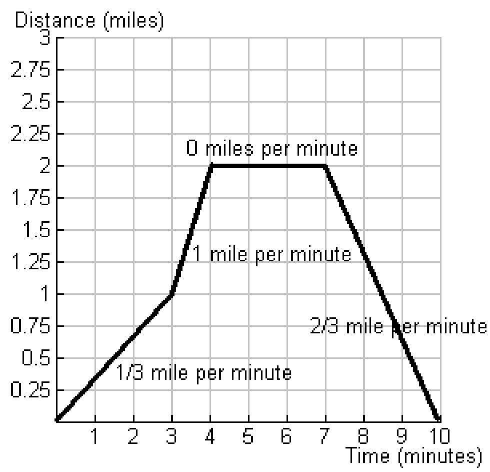
Size of Apartment S **(square feet)**	**Average Monthly Cost of Electricity** C
$S < 1000$	\$111
$1001 < S \le 1500$	\$124
$1501 < S \le 2000$	\$138
$2001 < S \le 2500$	\$157

a. Express the information in the chart as the piecewise function $C(S)$.

b. Evaluate $C(1750)$ and interpret its meaning in the context of this situation.

Exercise Solutions

1. We are asked to write a function for the speed traveled during the trip. The graph shows the speeds over each of the four sections of the trip. Recall, speed can be found by calculating distance (in miles) divided by time (minutes).



$$S(t) = \begin{cases} \dfrac{1}{3} & \text{if } 0 \le t < 1 \\ 1 & \text{if } 1 \le t < 4 \\ 0 & \text{if } 4 \le t < 7 \\ \dfrac{2}{3} & \text{if } 7 \le t \le 10 \end{cases}$$

Note that t is measured in minutes and S is the speed in miles per minute.

2. a. The piecewise function where S is the size of the apartment in square feet and C is the average cost of electricity is $C(S) = \begin{cases} 111 & \text{if } S < 1000 \\ 124 & \text{if } 1001 < S \le 1500 \\ 138 & \text{if } 1501 < S \le 2000 \\ 157 & \text{if } 2001 < S \le 2500 \end{cases}$.

b. We evaluate the function $C(1750) = 138$. This tells us that an apartment that is 1750 square feet will have average electricity costs of \$138.

7.3 Composition of Functions

Objectives

- Compose two or more functions using tables, equations, or graphs.
- Create, use, and interpret function composition notation in a real-world context.

Concepts and Definitions

- **Composition of Functions:** The function $h(x) = f(g(x))$ is the composition of the function f with the function g. The function h is called a composite function.

Examples

- **Example 1: Composing Functions Defined by Equations**
  Based on data from 1989 – 1998, restaurant sales in the United States may be modeled by $S(r) = 0.9291r - 160.0$ billion dollars where *r* is the number of restaurants (in thousands).The number of restaurants may be modeled by $r(t) = 0.3405t^2 + 6.910t + 330.9$ thousand restaurants where *t* is the number of years since 1989 (*Source*: Modeled from Statistical Abstract of the United States, 2001; Table 1268). Calculate $S(r(6))$ and interpret its meaning in the context of this situation.

  Solution:

  $r(6) = 0.3405(6)^2 + 6.910(6) + 330.9 = 384.618$

  $S(r(6)) = s(384.618) = 0.9291(384.618) - 160.0 \approx 197.35$

  $S(r(6)) \approx 197.35$

  This means that 6 years after 1989 (1995), restaurant sales are $197.35 billion.

- **Example 2: Decomposing Functions**
  Decompose the composition function $h(x) = f(g(x))$ into $f(g)$ and $g(x)$ where $h(x) = \left|3x^2 - 9\right|$. (Note: There may be more than one possible correct answer.)

  Solution:

  $f(g) = |g|$ *and* $g(x) = 3x^2 - 9$

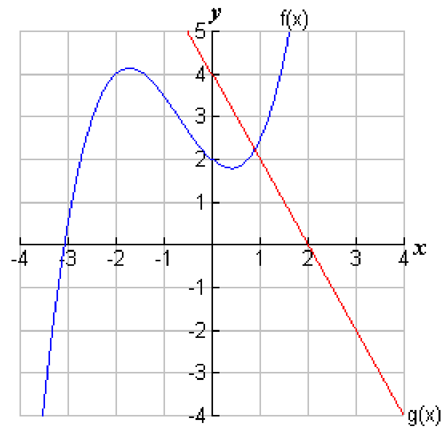
Exercises

1. Using the graph, estimate the following:

   a. $f(g(2))$

   b. $g(f(-2))$

   c. $g(g(0))$



2. A coupon for \$10 off a purchase of \$100 or more to an electronics store arrives in your mailbox.

   a. Write a function $C(x)$ for the final cost of your purchase if you choose to use it. Be sure to define your variables.

   b. What is the domain and range of the function $C(x)$ assuming that you will want to use your coupon?

   c. You read in the newspaper that this weekend only there is a sale that offers 5% off any purchase at the store so you decide to wait and use your coupon until this weekend to save even more money. Write a function $T(p)$ for the final cost of a customer's purchase with this sale (don't include the coupon yet). Be sure to define your variables.

   d. Write the function $T(C(x))$. Explain what this function represents in the context of the situation (Hint: Which discount is considered first?)

Exercise Solutions

1. a. $f(g(2)) = 2$

   b. $g(f(-2)) = -4$

   c. $g(g(0)) = -4$.

2. a. $C(x) = x - 10$, where x is the total purchase, $x \geq 100$, and C is the final cost to the customer.

   b. Domain: $x > 100$ and Range: $C \geq 90$.

   c. $T(p) = 0.95p$, where p is the purchase price and T is the total cost to the customer.

   d. $T(C(x)) = T(x-10) = 0.95(x-10)$. In this case, we first use the \$10 off coupon to reduce the purchase price by \$10. Then, we multiply by 0.95 to take an additional 5% off of the purchase price. The function $T(C(x))$ gives the total cost to the customer after applying both the $10 off coupon and the 5% off sale in the order described.

7.4 Logistic Functions

Objectives

- Graph logistic functions from equations and tables.
- Use logistic models to predict and interpret unknown results.

Concepts and Definitions

- **Logistic Functions:** A logistic function is a function of the form $N(t) = \dfrac{L}{1+Ae^{-Bt}}$ The number L is the upper limiting value of the function $N(t)$. The lower limiting value is $y=0$, the horizontal axis.

Examples

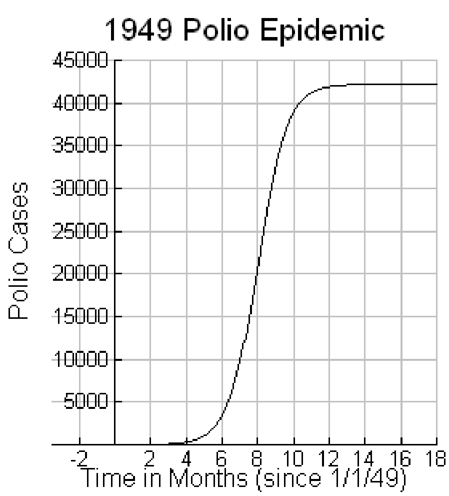
- **Example 1: Exploring Logistic Functions in a Real-World Context**
  The second worst polio epidemic in United States history occurred in 1949. The cumulative number of cases of polio that year may be modeled by the logistic function $P(t) = \dfrac{42183.911}{1+21,484.253e^{-1.248911t}}$ cases where t is the number of months since January 1, 1949. (Source: Modeled from Twelfth Annual Report, The National Foundation for Infantile Paralysis, 1949)

  a. Graph the function over the practical domain.

  b. Discuss the concavity of $P(t)$ from January through August by explaining what information this gives us about the spread of polio.

  c. Discuss the concavity of $P(t)$ from August through December by explaining what information this gives us about the spread of polio.

  Solution:
  a.



b. The function is concave up from January to August. This tells us that cumulative number of cases of polio is increasing at an ever increasing rate.

c. The function is concave down from August to December. This tells us that cumulative number of cases of polio is still increasing but at a decreasing rate.

- **Example 2: Using Logistic Regression to Model Data**

Use logistic regression to find the logistic model for the data showing the cost of a 30-second advertising spot during the Super Bowl. Forecast the cost of a 30-second television advertisement in 2013.

Year (Since 1987) *t*	**Cost of 30-second TV advertising spot for Super Bowl (in millions of dollars)** *A*
0	0.55
5	0.85
10	1.22
14	2.10
15	2.31
16	2.19
17	2.31
18	2.42
19	2.50

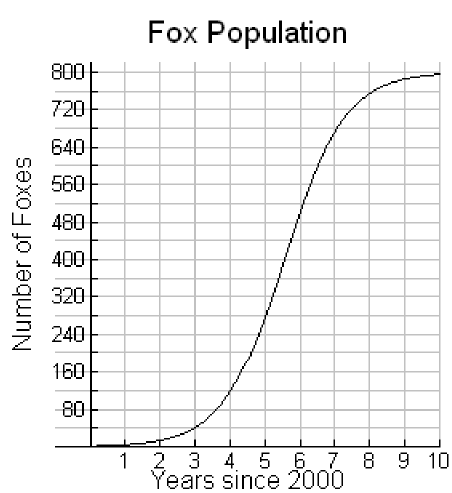
Solution:

Logistic regression for the cost of a 30-second advertising spot for the Super Bowl gives the function $A(t) = \dfrac{3.331}{1+6.483e^{-0.161t}}$. *We forecast the cost of a 30-second advertisement in 2013 by evaluating* $A(27) = \dfrac{3.331}{1+6.483e^{-0.161(27)}} \approx 3.073$. *In 2013, we predict that it will cost 3.073 million dollars for a 30-second Super Bowl advertisement.*

Exercises

1. If $f(x) = \dfrac{32}{1+18e^{-0.65x}}$ is a logistic function, what is the limiting value? Is this logistic function an increasing or decreasing logistic function? How can you tell?

2. A logistic growth curve is sketched below. Estimate the limiting value and the inflection point and explain what each means in the real-world context.



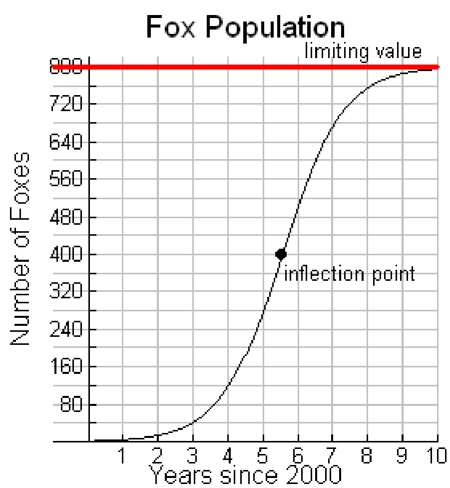
Exercise Solutions

1. The limiting value is 32. We know this because the limiting value is determined by thinking about the function value as the value of x gets larger and larger. As x gets larger and larger, $18e^{-0.65x}$ gets smaller and smaller, approaching 0.

$$f(x) = \frac{32}{1+0}$$

Therefore, the denominator of the function approaches 1 and the entire function must then approach 32. This is an increasing logistic function because of the exponent $-0.65x$. Since the value of B in the general logistic model, $N(t) = \frac{L}{1+Ae^{-Bt}}$, is positive, we have a situation demonstrating logistic growth.

2. We estimate the limiting value to be 800 foxes since the graph seems to be "leveling off" near the 800 fox value. The inflection point appears to occur at $(5.5, 400)$. This means that into 2006, the number of foxes is 400 and the population is increasing at the greatest rate. After this, the fox population continues to increase but at a decreasing rate. The inflection point is where the function changes from concave up (prior to $t = 5.5$) to concave down (after $t = 5.5$).



7.5 Choosing a Mathematical Model

Objectives

- Select the "best" function to model a given real-world data set in equation, graph, table, or word form.
- Use the appropriate function to model real-world data sets.
- Use appropriate models to predict and interpret unknown results.

Concepts and Definitions

- **Choosing a Mathematical Model:**

Function Type	Model Equation	Key Graphical Features
Linear	$y = mx + b$	Line
Quadratic	$y = ax^2 + bx + c$	Concave up everywhere or concave down everywhere
Cubic	$y = ax^3 + bx^2 + cx + d$	Changes concavity exactly once, no horizontal asymptotes
Quartic	$y = ax^4 + bx^3 + cx^2 + dx + f$	Changes concavity zero or two times, no horizontal asymptotes
Exponential	$y = ab^x$ with $a > 0$	Concave up, horizontal asymptote at $y = 0$
Logarithmic	$y = a + b\ln(x)$ with $b > 0$	Concave down, vertical asymptote at $x = 0$
Logistic	$y = \dfrac{c}{1 + ae^{-bx}} + k$	Changes concavity exactly once, horizontal asymptotes at $y = k$ and $y = c + k$

- **Choosing Between Multiple Best-Fit Models:** When two or more models fit a data set equally well and we have no additional information which lead us to believe one model would be better than the other, we choose the simplest function as the model.

Examples

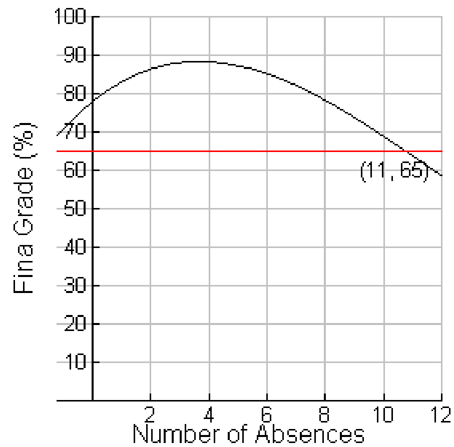
- **Example 1: Choosing a Model from a Verbal Description**
  A car company has found that there is a relationship between the amount of money it spends on advertising and the number of cars it sells. When it spends \$50,000 on advertising, it sells 500 cars. For each additional \$5000 spent, it sells 20 more cars.

  a. Explain why a linear model may be used to model this situation.

  b. Let a be the amount of money they spend on advertising. Find a formula for the number of cars c sold as a function of a.

  Solution:

  a. In the verbal description, we read, "for each additional \$5000 spent, it sells 20 more cars." This language suggests a constant rate of change (20 cars per \$5000



or \$5000 per 20 cars). A situation involving a constant rate of change is best modeled with a linear function.

b. We know that the constant rate of change is 20 cars per \$5000 or $\frac{20}{5000} = \frac{1}{250}$*. In simplified form, we interpret the rate of change to mean that one additional car is sold for every \$250 spent on advertising. To find the vertical intercept, we use the fact that 500 cars are spent when \$50,000 in advertising is spent.*

There are 200 sets of \$250 in \$50,000 ($\frac{\$50,000}{\$250} = 200$*). For each of the sets, 1 car is sold. Or, for each \$250 less spent on advertising, 1 fewer car is sold. So, if no money is spent on advertising, 300 cars (*$500-200$*) are sold. Therefore,* $c(a) = \frac{1}{250}a + 300$.

- **Example 2: Choosing a Model from a Table of Values**

Construct a scatterplot for the data obtained in a hypothetical study on the number of absences and the Final Grades of seven randomly selected students from an Intermediate Algebra class. The data are shown below.

Number of Absences	Final Grade (%)
6	82
2	86
15	43
9	74
12	58
5	90
8	78

a. Choose a function to model these data.

b. Find $G(10)$ and interpret its meaning in the real-world context.

c. Solve $G(x) = 65$ and interpret its meaning in the real-world context.

Solution:

a. We choose to model the data with a cubic polynomial function. The scatterplot suggests a concave down, decreasing function. When comparing the quadratic model to the cubic model, we get a coefficient of determination value much closer to 1 with the cubic polynomial model. Thus, $G(A) = 0.029A^3 - 0.992A^2 + 6.126A + 77.80$.

b. We evaluate $G(10) = 0.029(10)^3 - 0.992(10)^2 + 6.126(10) + 77.80 \approx 68.86$.

This means that if a student has 10 absences, the expected final grade is 68.86%.

c. We solve $G(x) = 85$ *graphically. Using a graphing calculator, we the point of intersection to be* $(10.76, 65)$*. Contextually, we say that a student with 11 absences can expect to earn a final grade of 65%.*

Exercises

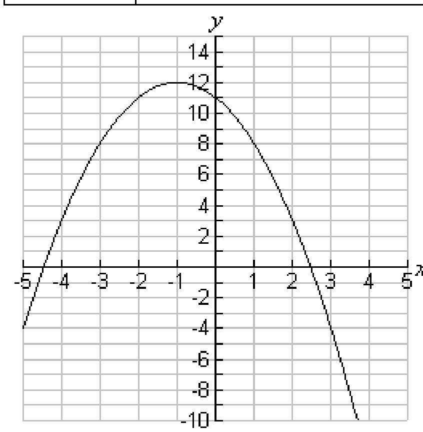
1. Use a graph of the function $y=-x^2-2x+11$ to determine over what interval(s) the function is increasing or decreasing. Determine over what interval(s) the function is concave up or concave down. Identify the coordinate of the y-intercept.

2. Find the equation of the mathematical model (if possible) which you believe will most accurately forecast the indicated result. Justify your conclusions. Forecast the average consumer's Christmas spending in 2012 (*Source*: National Retail Federation).

Year (Since 2002) t	**Average consumer spending on Christmas gifts (in dollars)** S
0	649
1	672
2	702
3	738
4	791

Exercise Solutions

1. The quadratic function has a vertical intercept at $(0,11)$. From this vertical intercept, the function initially decreases (based upon the $-2x$ term). This rate of change becomes more and more negative as x increases (based upon the $-x^2$ term). Thus, the function increases on $(-\infty,-2)$ (note that the vertex is $x=-\frac{b}{2a}=-\frac{-2}{2(-1)}=-2$). The function decreases on $(-2,\infty)$. The quadratic function is concave down.



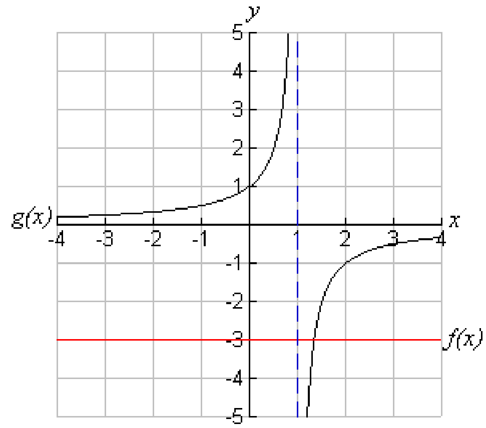
2. We choose the quadratic regression model $S(t)=4.714t^2+16.14t+649.8$. We base this upon two strategies. First, we observe that average consumer spending, S, increases but at an increasing rate (Note the ΔS column**)**. Second, when comparing the coefficient of determinations of the linear and quadratic functions, it is much closer to 1 with the quadratic model. We forecast the average consumer's Christmas spending in 2012 by evaluating the function $S(10)=4.714(10)^2+16.14(10)+649.8=1282.6$.

Year (Since 2002) t	**Average consumer spending on Christmas gifts (in dollars)** S	ΔS
0	649	23
1	672	30
2	702	36
3	738	53
4	791	

In 2012, we predict that the average consumer will spend $1282.60 on Christmas gifts.

Chapter 7 Exam A

1. Use the graphs of functions $f(x)$ and $g(x)$ to evaluate the given functions. If it is not possible to evaluate the function, explain why not.



a. $(g+f)(2)$

b. $(f\cdot g)(0)$

c. $(f-g)(1)$

d. $(\frac{g}{f})(2)$

2. A car rental company charges \$15.32 to rent a compact car for one day from St. Louis International Airport plus \$2.50 for every 100 miles (or portion of 100 miles) that is driven over 200 miles (*Source*: www.avis.com).

a. Graph the function $R(m)$ where R is the cost to rent the car if driven m miles.

b. Evaluate $R(200)$ and $R(201)$. Explain why there is such a large discrepancy for driving one additional mile.

c. Solve $R(m)=20.32$ for m. Interpret the meaning of these value(s) for m.

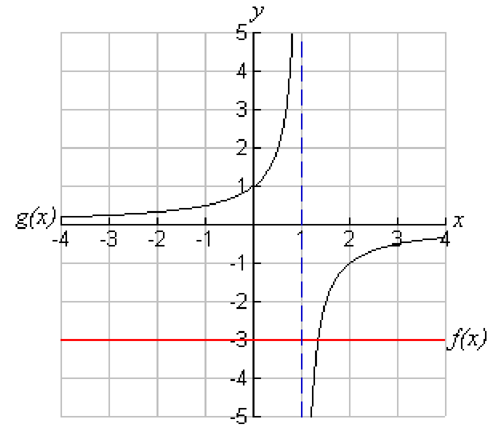
3. The average price of a DVD player may be modeled by $P(d)=1024-90.98\cdot\ln d$ dollars, where d is the number of DVD players sold since 1997 (in thousands). The number of DVD players sold may be modeled by $d(t)=-178.88t^3+1854.05t^2-1353t+542.70$ thousand DVD players where t is the number of years since 1997. (*Source*: www.ce.org). Write a formula for $P(d(t))$. Then evaluate and interpret $P(d(9))$.

4. Explain why the table of values below may be modeled by a decreasing logistic function.

x	y
0	15
1	14
2	12
3	8
4	2
5	1
6	0.5
7	0.3
8	0.1

Chapter 7 Exam A Solutions

1. Use the graphs of functions $f(x)$ and $g(x)$ to evaluate the given functions. If it is not possible to evaluate the function, explain why not.



a. $(g+f)(2) = g(2) + f(2) = -1 + (-3) = -4$

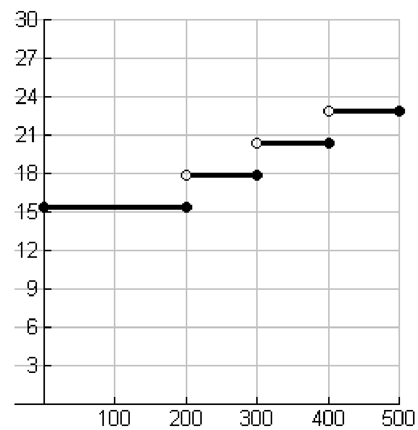
b. $(f \cdot g)(0) = f(0) \cdot g(0) = (-3)(1) = -3$

c. $(f-g)(1) = f(1) - g(1) = undefined$

d. $(\frac{g}{f})(2) = \frac{g(2)}{f(2)} = \frac{-1}{-3} = \frac{1}{3}$

2. A car rental company charges \$15.32 to rent a compact car for one day from St. Louis International Airport plus \$2.50 for every 100 miles (or portion of 100 miles) that is driven over 200 miles (*Source*: www.avis.com).

a. Graph the function $R(m)$ where R is the cost to rent the car if driven m miles.



b. Evaluate $R(200)$ and $R(201)$. Explain why there is such a large discrepancy for driving one additional mile.

$R(200) = 15.32$ and $R(201) = 17.82$

The cost for driving 200 miles exactly is \$15.32 but the cost increases by \$2.50 to \$17.82 if the car is driven an extra mile and then the additional mileage charge is implemented.

c. Solve $R(m) = 20.32$ for m. Interpret the meaning of these value(s) for m.

If $R(m) = 20.32$, then $300 < m \le 400$. The car can be driven between 300 (exclusive) and 400 (inclusive) miles for a fee of \$20.32.

3. The average price of a DVD player may be modeled by $P(d) = 1024 - 90.98 \cdot \ln d$ dollars, where d is the number of DVD players sold since 1997 (in thousands). The number of DVD players sold may be modeled by $d(t) = -178.88t^3 + 1854.05t^2 - 1353t + 542.70$ thousand DVD players where t is the number of years since 1997. (*Source*: www.ce.org). Write a formula for $P(d(t))$. Then evaluate and interpret $P(d(9))$.

$$\begin{aligned} P(d(t)) &= P(-178.88t^3 + 1854.05t^2 - 1353t + 542.70) \\ &= 1024 - 90.98\ln(-178.88t^3 + 1854.05t^2 - 1353t + 542.70) \end{aligned}$$

$$\begin{aligned} P(d(9)) &= 1024 - 90.98\ln(-178.88(9)^3 + 1854.05(9)^2 - 1353(9) + 542.70) \\ &= 1024 - 90.98\ln(8140.23) \\ &\approx 204.76 \end{aligned}$$

Nine years after 1997 (2006), the average price of a DVD player is predicted to be $204.76 according to the models.

4. Explain why the table of values below may be modeled by a decreasing logistic function.

x	y
0	15
1	14
2	12
3	8
4	2
5	1
6	0.5
7	0.3
8	0.1

Examine the values of y. Initially as x increases, y decreases slowly. As x continues to increase, y decreases by more and more. Then, as x continues to increase more, y continues to decrease but by less and less. Graphically, we would see a decreasing function that is concave down followed by concave up. This is characteristic of a decreasing logistic function.

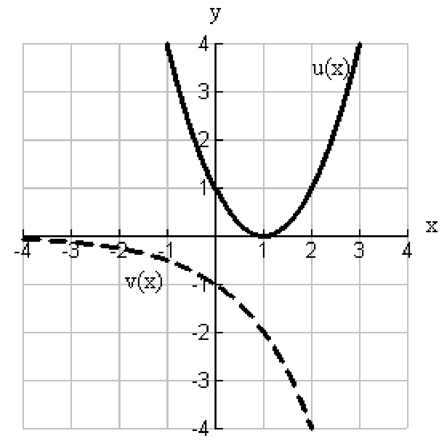
Chapter 7 Exam B

1. Let $u(x)$ and $v(x)$ be two functions defined by the two graphs below. Estimate:

   a. $v(u(1))$

   b. $u(v(-4))$

   c. $v(u(0))+v(u(-1))$



2. A long-distance telephone carrier offered the following rates for calls outside the state of residence made using its rechargeable prepaid calling card: \$0.05 per minute for the first 10 minutes, \$0.03 per minute for each minute thereafter (*Source*: MCI.com). Suppose that to compete with cell phones that have free long distance, the company has decided to charge for the exact duration of the call instead of rounding up to the next full minute for any fraction of a minute.

   a. Write the piecewise equation for the function $T(m)$ where T is the total cost for a call lasting m minutes.

   b. Evaluate $T(7.3)$ and explain what the numerical value means in the real-world context.

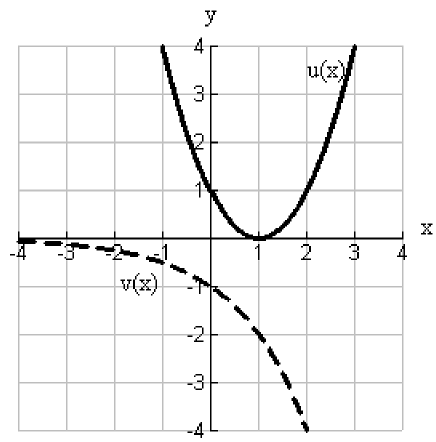
   c. Sketch the graph of $T(m)$ over the domain from 0 to 60 minutes.

3. Cell phone sales may be modeled by $C(p)=-0.0027p^3+6.173p^2-4633p+1{,}169{,}000$ millions dollars where p is the amount of sales of pagers sold in the same year in millions of dollars. Pager sales may be modeled by $p(t)=-20.64t^2+61.47t+751.71$ millions dollars where t is the number of years since 2000. (*Source*: Statistical Abstract of the United States 2006, Table 1003). Write a formula for $C(p(t))$ and then evaluate and interpret $C(p(10))$.

4. Charles' Law tells us that the volume of an inflated balloon increases as the air temperature rises. The following tables display the data from experimental measurements for a particular balloon. Find a function, $v(f)$, that models the relationship shown in the table. Justify your conclusions.

Temperature (Fahrenheit) f	**Volume (cubic inches)** v
30	33.2
40	37.8
43	38.9
46	39.0
52	39.6
60	41.2
64	42.2
70	44.1

Chapter 7 Exam B Solutions



1. Let $u(x)$ and $v(x)$ be two functions defined by the two graphs below. Estimate:

   a. $v(u(1)) = -1$

   b. $u(v(-4)) = 1$

   c. $v(u(0)) + v(u(-0.5)) = -2 + -4 = -6$

2. A long-distance telephone carrier offered the following rates for calls outside the state of residence made using its rechargeable prepaid calling card: \$0.05 per minute for the first 10 minutes, \$0.03 per minute for each minute thereafter (*Source*: MCI.com). Suppose that to compete with cell phones that have free long distance, the company has decided to charge for the exact duration of the call instead of rounding up to the next full minute for any fraction of a minute.

   a. Write the piecewise equation for the function $T(m)$ where T is the total cost for a call lasting m minutes.

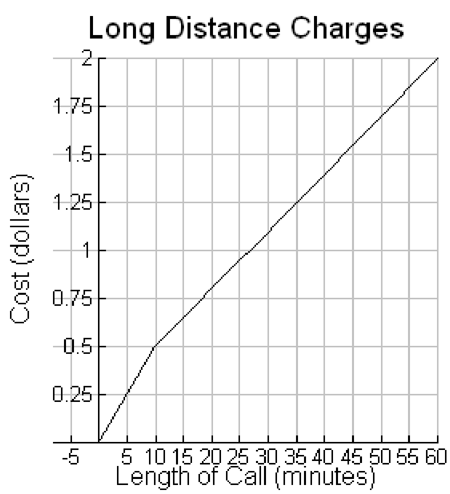
$$T(m) = \begin{cases} 0.05m & \text{if } 0 \le m \le 10 \\ 0.5 + 0.03(m-10) & \text{if } m > 10 \end{cases}$$

   b. Evaluate $T(7.3)$ and explain what the numerical value means in the real-world context.

$$T(7.3) = 0.05(7.3) = 0.365$$

   This means that a call lasting 7.3 minutes will cost \$0.365.

   c. Sketch the graph of $T(m)$ over the domain from 0 to 60 minutes.



3. Cell phone sales may be modeled by $C(p) = -0.0027p^3 + 6.173p^2 - 4633p + 1{,}169{,}000$ millions dollars where p is the amount of sales of pagers sold in the same year in millions of dollars. Pager sales may be modeled by $p(t) = -20.64t^2 + 61.47t + 751.71$ millions dollars where t is the number of years since 2000. (*Source*: Statistical Abstract of the United States 2006, Table 1003). Write a formula for $C(p(t))$ then evaluate and interpret $C(p(10))$.

$$C(p(t)) = C(-20.64t^2 + 61.47t + 751.71)$$
$$= -0.0027(-20.64t^2 + 61.47t + 751.71)^3 + 6.173(-20.64t^2 + 61.47t + 751.71)^2 - 4633(-20.64t^2 + 61.47t + 751.71) + 1{,}169{,}000$$
$$C(p(10)) =$$
$$-0.0027(-20.64(10)^2 + 61.47(10) + 751.71)^3 + 6.173(-20.64(10)^2 + 61.47(10) + 751.71)^2 - 4633(-20.64(10)^2 + 61.47(10) + 751.71) + 1{,}169{,}000$$
$$= 5313195.851$$

Ten years after 2000 (2010), the models predict that about 5,313,196 cell phones will be sold.

4. Charles' Law tells us that the volume of an inflated balloon increases as the air temperature rises. The following tables display the data from experimental measurements for a particular balloon. Find a function, $v(f)$, that models the relationship shown in the table. Justify your conclusions.

Temperature (Fahrenheit) f	**Volume (cubic inches)** v
30	33.2
40	37.8
43	38.9
46	39.0
52	39.6
60	41.2
64	42.2
70	44.1

We choose to use the cubic regression model $v(f) = 3.675f^3 - 0.058f^2 + 3.127f - 18.68$.
In examining the data and scatterplot, we see that the volume does not increase at a constant rate as the temperature increases. Rather, we see the volume increasing concave down at first and then the volume continues to increase but concave up. This is characteristic of a cubic model. Furthermore, the coefficient of determination is much closer to 1 with the cubic regression model than it is with a linear or quadratic model.

Trigonometric Functions

8

8.1 Periodic Functions

Objectives

- Recognize the periodicity of a function given in a graph or table set in a real-world context.
- Generate additional periodic behavior of a function from a graph or table.

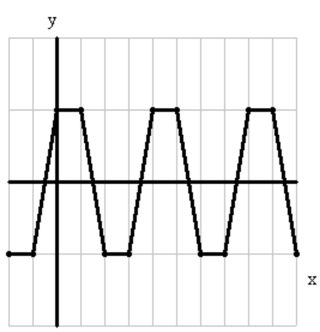
Concepts and Definitions

- **Periodic Functions:** A function f is periodic if $f(x+p)=f(x)$ for all x and for a positive constant p. The value p is called the period of the function.
- **Midline (Centerline):** The midline or centerline of a periodic function f is the horizontal line that lies halfway in between the maximum and minimum output values of the function. In other words, the midline is the horizontal line with equation $y=\dfrac{\text{max of } f + \text{ min of } f}{2}$.
- **Amplitude:** The amplitude of a periodic function f is the distance from the midline to the maximum or minimum output value of the function. In symbolic terms, $\text{amplitude} = \text{max of } f - \text{midline}$. Equivalently, the amplitude is half of the distance between the maximum and minimum output values of the function. That is, $\text{amplitude} = \dfrac{\text{max of } f - \text{min of } f}{2}$.
- **Frequency:** The frequency of a periodic function is the reciprocal of its period. That is $\text{frequency} = \dfrac{1}{\text{period}}$.

Examples

- **Example 1: Estimating the Period of a Function**
  Estimate the period of each function. Note: Assume that each horizontal tick mark on the graphs is one unit.

a.



b.

x	y
0	2
1	10
2	14
3	2
4	10
5	14
6	2
7	10

Solution:

a. The period of this function is $p = 4$. *We observe that* $f(x+p) = f(x)$ *when* $p = 4$.

b. The period of this function is $p = 3$. *We observe that* $f(x+p) = f(x)$ *when* $p = 3$.

- **Example 2: Graphing Periodic Behavior**
The following description is based on one person's experiences with freeway traffic during a typical weekday in a metropolitan region. At 4:00 am, traffic is relatively light. As time goes on, traffic flow increases more and more quickly reaching a peak at about 7:00 am. Traffic flow then decreases until it reaches a minimum traffic flow at 3:00. Traffic flow again increases at an ever increasing rate reaching a peak traffic flow at 5:00 pm. Again, traffic flow decreases to a minimum at 9:00 am. Traffic flow remains constant until about 4:00 am where the pattern repeats.

a. Sketch a graph of $F(t)$, the traffic flow (in cars per minute passing a particular point on road) F as a function of time, t, in hours since midnight. Note: You do not need to express a particular scale on the vertical axis. Rather, communicate the overall idea expressed in the description.

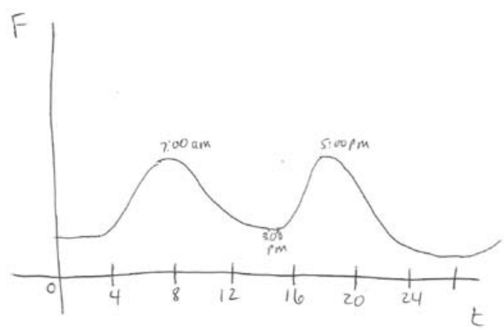
b. At what time, approximately, is the instantaneous rate of change of $F(t)$ zero? What does this mean in the context of the situation?

c. At what time, approximately, is the instantaneous rate of change of $F(t)$ the most positive? What does this mean in the context of the situation?

d. At what time, approximately, is the instantaneous rate of change of $F(t)$ the most negative? What does this mean in the context of the situation?

Solution:

a. An example of a graph is shown.



b. The instantaneous rate of change of $F(t)$ *is zero at about 7:00 am, 3:00 pm, and 5:00 pm. This is where the function is not increasing or decreasing and has reached a maximum or minimum value.*

c. The instantaneous rate of change of $F(t)$ *is most positive at about 6:00 am and 4:30 pm. This is where the function is increasing at the greatest rate (is steepest). Traffic flow is increasing at the* *greatest rate at these times as rush hour is busiest in the morning and in the evening.*

d. *The instantaneous rate of change of $F(t)$ is most negative at about 10:00 am and 7:00 pm. This is where the function is decreasing at the greatest rate (is steepest). Traffic flow is decreasing at the greatest rate at these times as rush hour is slowing down in the morning and in the evening.*

Exercises

1. Complete one more cycle before and after the values given in the table using the concept of periodicity.

x	y
10	
11	
12	
13	–5
14	4
15	3
16	0
17	–5
18	
19	
20	
21	

2. With respect to dental hygiene, it is common practice to visit the dentist twice per year or every six months. One of these is expressed using the meaning of period and one is expressed with the meaning of frequency. Explain which is which and why.

Exercise Solutions

1. See the table to the right. Based on the values given, we conclude the that period is $p = 4$. The pattern established can be continued using this period.

x	y
10	4
11	3
12	0
13	–5
14	4
15	3
16	0
17	–5
18	4
19	3
20	0
21	–5

2. The period is the length of time between visits which is six months. The frequency is how many times something we visit the dentist which is twice per year. Note that we could express the period as $\frac{1}{2}$ since we visit the dentist every half-year. Frequency is defined as $\frac{1}{\text{period}} = \frac{1}{\frac{1}{2}} = 2$.

8.2 Angle Measure

Objectives

- Explain the useful definition of angle measure.
- Measure angles in degrees and radians.
- Convert angle measure between units of degree and radian.

Concepts and Definitions

- **Relationship between Arc Length and Radius:** The length, s, of an arc is directly proportional to radius of the arc, r, with constant of proportionality K. That is, $s = Kr$. If k is the proportion of the circumference of a circle of radius r that coincides with the arc, then $K = 2\pi k$.
- **Radian Measure:** The measure of an angle, θ, that corresponds with an arc length of 1 radius in the counter-clockwise direction is 1 radian.
- **Arc Length as a Function of Angle Measure in Radians:** The length, s, of the arc corresponding with an angle measure of θ radians is $s = \theta r$ where r is the length of the radius. For the unit circle, $s = \theta$.
- **Degree Measure:** The measure of an angle, θ, that corresponds with an arc length of $\frac{1}{360}$ of the circumference of a circle is 1 degree.
- **Arc Length as a Function of Angle Measure in Degrees:** The length, s, of the arc corresponding with θ degrees is $s = (\frac{\pi}{180}\theta)r$ where r is the length of the radius. For the unit circle, $s = \frac{\pi}{180}\theta$.
- **Converting an Angle from Degrees to Radians:** To convert to radian measure, multiply θ degrees by $\frac{\pi \text{ radians}}{180 \text{ degrees}}$.
- **Converting an Angle from Radians to Degrees:** To convert to degree measure, multiply θ radians by $\frac{180 \text{ degrees}}{\pi \text{ radians}}$.

Examples

- **Example 1: Determining the Proportion of a Circumference that Corresponds with an Angle**
  Describe what fraction of the circumference of a full circle is spanned by the angle $\theta = \frac{5\pi}{6}$.

Solution:

We can rewrite $\theta = \frac{5\pi}{6} = \frac{5}{6}\pi$ *. The angle* $\theta = \frac{5\pi}{6}$ *spans* $\frac{5}{6}$ *of half the circumference of the circle.*

$$C = 2\pi r$$

$$\frac{1}{2}C = \pi r$$

$$\frac{5}{6}(\frac{1}{2}C) = \frac{5}{6}\pi r$$

$$\frac{5}{12}C = \frac{5}{6}\pi r$$

The angle $\theta = \frac{5\pi}{6}$ *spans* $\frac{5}{12}$ *of the circumference of a full circle.*

- **Example 2: Converting Angle Measures**

  a. Convert the angle $\theta = 600^\circ$ to radian measure.

  b. Convert the angle $\theta = 4$ to degree measure.

  Solution:

  a.

$$\theta = 600^\circ = 600^\circ \cdot \frac{\pi \text{ radians}}{180^\circ}$$
$$= \frac{600\pi}{180} = \frac{10\pi}{3}$$

  b.

$$\theta = 4 \text{ radians} = 4 \cdot \frac{180^\circ}{\pi \text{ radians}}$$
$$= \frac{4 \cdot 180^\circ}{\pi} \approx 229.2^\circ$$

Exercises

1. What is the length of the arc which corresponds to an angle of 170° in a circle of radius 3.5 feet?

2. a. Convert the angle $\theta = 220^\circ$ to radians.

   b. Convert the angle $\theta = \frac{11\pi}{3}$ to degrees.

Exercise Solutions

1.

$$\begin{aligned} s &= (\frac{\pi}{180}\theta)r \\ &= (\frac{\pi}{180}\cdot 170^\circ)3.5 \\ &= \frac{17\pi}{18}\cdot 3.5 \\ &= \frac{59.5\pi}{18} \approx 10.38 \text{ feet} \end{aligned}$$

2. a.

$$\begin{aligned} \theta = 220^\circ &= 220^\circ \cdot \frac{\pi \text{ radians}}{180^\circ} \\ &= \frac{220\pi}{180} = \frac{11\pi}{9} \end{aligned}$$

b.

$$\begin{aligned} \theta &= \frac{11\pi}{3} \text{ radians} = \frac{11\pi}{3} \text{ radians} \cdot \frac{180^\circ}{\pi \text{ radians}} \\ &= \frac{11\cdot 180^\circ}{3} \approx 660^\circ \end{aligned}$$

8.3 Unit Circle and Trigonometric Functions

Objectives

- Describe the relationship between the cosine and sine functions and the length of the arc swept out by the angle.
- Describe the relationship between the coordinates of a point on the unit circle and the cosine and sine functions.
- Use reference angles to determine cosine and sine values.

Concepts and Definitions

- **The Unit Circle:** The unit circle is a circle with its center at $(0,0)$ and a radius of 1 unit.
- **Cosine Function:** The cosine of an angle whose measure is θ, denoted $\cos(\theta)$, is the horizontal position of the endpoint of the corresponding arc on the unit circle.
- **Horizontal Position of a Point on a Circle:** The horizontal position of a point on the arc of a circle of radius r is given by $x(\theta) = r \cdot \cos(\theta)$, where θ is the measure of the corresponding angle.
- **The Sine Function:** The sine of an angle whose measure is θ, denoted $\sin(\theta)$, is the vertical position of the endpoint of the corresponding arc on the unit circle.
- **Vertical Position of a Point on a Circle:** The vertical position of a point on the arc of a circle of radius r is given by $y(\theta) = r \cdot \sin(\theta)$, where θ is the measure of the corresponding angle.
- **Reference Angle:** A reference angle is the smallest positive angle formed by the terminal side of the angle θ and the x-axis.

Examples

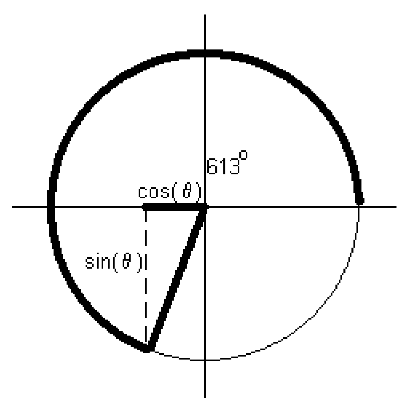
- **Example 1: Using and Interpreting Cosine and Sine**
  Estimate the value of each expression. Then explain what the value means in the context of the unit circle. Use a calculator to confirm your estimate.

  a. $\cos(613^\circ)$

  b. $\sin(613^\circ)$

  Solution:

  a. *The* $\cos(613^\circ)$ *represents the horizontal* (x) *position of the endpoint of the arc on the unit circle corresponding with a* 613° *angle. We estimate* $\cos(613^\circ) \approx -0.3$. *Using a calculator, we find* $\cos(613^\circ) \approx -0.2924$.



b. *The* $\sin(613^\circ)$ *represents the vertical (* y *) position of the endpoint of the arc on the unit circle corresponding with a* 613° *angle. We estimate* $\sin(613^\circ) \approx -0.9$. *Using a calculator, we find* $\sin(613^\circ) \approx -0.9563$.

- **Example 2: Using the Sine Function**
  The original Ferris wheel designed by George Ferris had a radius of 125 feet. If a person boards the Ferris wheel from the bottom (ground level), how high off the ground is the person after they have traveled $\frac{5}{6}$ of the way around the wheel?

  Solution:
  Since a full revolution of the wheel corresponds to an angle measuring $360°$, *traveling* $\frac{5}{6}$ *of the way around the Ferris wheel corresponds to an angle of 300°. However, in this case, the angle is not being measured from its standard position on the positive* x*-axis. Instead, it is starting from the position corresponding with* $-90°$.
  By subtracting 90° from 300°, we may use the sine function to determine the height. Since $125\sin(300^\circ - 90^\circ) = 125\sin(210^\circ) = -62.5$, *the seat is 62.5 feet below the horizontal line passing through the center of the Ferris wheel (the* x*-axis). To find the total height, we will need to add the length of the radius, 125 feet.*

$$\begin{aligned}\text{height above the ground} &= 125 + 125\sin(210^\circ)\\ &= 125 + (-62.5) = 62.5\end{aligned}$$

  A person who has traveled $\frac{5}{6}$ *around the Ferris wheel is 62.5 feet above the ground.*

Exercises

1. Use a calculator to evaluate each expression. Then explain what the value means in the context of the unit circle. The angles are all measured in radians

   a. $\cos(-\frac{12\pi}{11})$

   b. $\sin(\frac{68\pi}{19})$

2. You are given the following cosine value for the endpoint of the arc corresponding with an angle measuring θ degrees: $\cos(\theta) = 0.43$. For this value of cosine, do the following.

   a. Draw a unit circle and indicate the two possible positions on this circle where the given value of cosine would be true

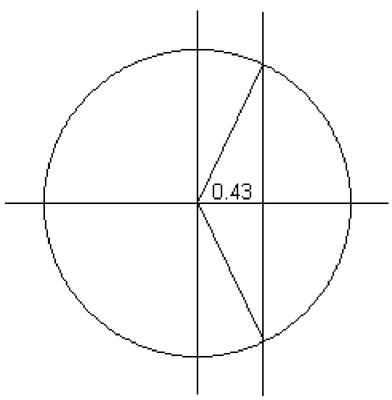
b. Estimate the value of $\sin(\theta)$ for each of these points.

Exercise Solutions

1. a. The $\cos(-\frac{12\pi}{11})$ represents the horizontal (x) position of the endpoint of the arc on the unit circle corresponding with a $-\frac{12\pi}{11}$ angle. Using a calculator we find $\cos(-\frac{12\pi}{11}) \approx -0.9595$.

b. The $\sin(\frac{68\pi}{19})$ represents the vertical (y) position of the endpoint of the arc on the unit circle corresponding with a $\frac{68\pi}{19}$ angle. Using a calculator we find $\sin(\frac{68\pi}{19}) \approx -0.9694$.

2. a.



b. When $\cos(\theta) = 0.43$, we estimate $\sin(\theta) \approx 0.95$ and $\sin(\theta) \approx -0.95$.

8.4 Graphing Sine and Cosine

Objectives

- Use the unit circle to construct the graphs of the cosine and sine functions.
- Graph cosine and sine including transformations of each.
- Solve trigonometric equations graphically.

Concepts and Definitions

- **Transformed Cosine and Sine Functions**: For functions of the form $x(\theta) = A\cos(B(\theta - C)) + D$ or $y(\theta) = A\sin(B(\theta - C)) + D$, we generalize the graphical meaning of their parameters as follows.
    - $|A|$, the vertical stretch factor, is the amplitude of the graph. If $A < 0$, then the function is also reflected vertically.
    - $|B|$, is the angular frequency, which is the number of periods that will be completed in the interval $0^\circ \leq \theta \leq 360^\circ$ or $0 \leq \theta \leq 2\pi$ for θ. If $B < 0$, then the function is also reflected horizontally.
    - $\dfrac{360^\circ}{|B|}$ or $\dfrac{2\pi}{|B|}$ radians is the period of the function, the length of the interval required to complete one complete cycle.
    - $\dfrac{|B|}{360^\circ}$ or $\dfrac{|B|}{2\pi}$ radians is the frequency of the function, the fraction of a period completed within a one-unit interval.
    - $|C|$ is the horizontal shift of the function. If C is positive, the shift is to the right. If C is negative, the shift is to the left.
    - $|D|$ is the vertical shift of the function. If D is positive, the shift is upward. If D is negative, the shift is downward. The midline is located at $x = D$ or $y = D$.
- **Phase Shifts:** The phase shift is the portion of one period by which the function is shifted horizontally instead of the amount it is shifted.
- **Determining Phase Shift:** For functions of the form $x(\theta) = A\cos(B(\theta - C)) + D$ or $y(\theta) = A\sin(B(\theta - C)) + D$, the **phase shift** is $|BC|$. The value $\dfrac{|BC|}{360^\circ}$ or $\dfrac{|BC|}{2\pi}$ tells us the portion of one period that the graph has been shifted horizontally.

Examples

- **Example 1: Multiple Transformations of Cosine and Sine**
  Answer all of the following questions for the function $f(\theta) = 7\sin(\theta + \pi) + 3$
  $f(\theta) = 7\sin(\theta + \pi) + 3$, θ is in radians.

a. What is the period, the amplitude, frequency, and the equation of the midline?

b. Graph the function without using a calculator. Graph at least one complete period.

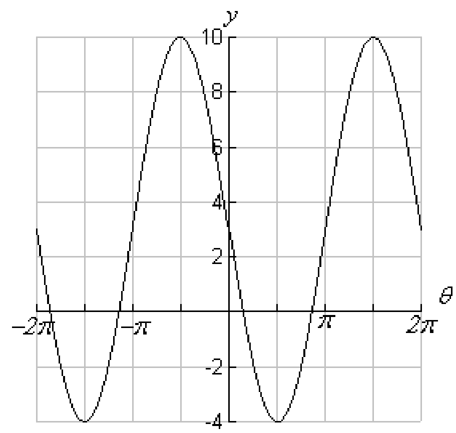
c. Describe the transformation of the function as related to the graph of $f(\theta)=\sin(\theta)$.

Solution:

a. The period is $\frac{2\pi}{|1|}=2\pi$. The amplitude is $A=7$. The equation of the midline is $y=3$.

b. The graph is shown.

c. The function $f(\theta)=7\sin(\theta+\pi)+3$, *when related to* $f(\theta)=\sin(\theta)$, *has been shifted upward 3 units, stretched vertically by a factor of 7, and shifted to the left* π *units.*



- **Example 2: Using a Cosine Model in a Real-World Context**

The function $A(m)=-18,000\cos(\frac{\pi}{6}(m-1))+59,000$ models the number of workers in the agricultural industry m months after January 2004 in Clatsop County.

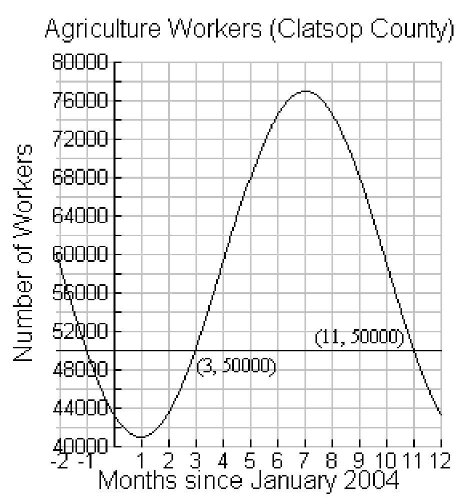
a. Graph $A(m)$ over the interval $0\le m\le 12$.

b. Graphically solve $50,000=-18,000\cos(\frac{\pi}{6}(m-1))+59,000$ and explain what the result means in the context of this situation.

Solution:

a. The graph is shown.

b. We graph $y=50,000$ *along with* $A(m)=-18,000\cos(\frac{\pi}{6}(m-1))+59,000$ *to see where the two graphs intersect. Therefore, the solutions to the equation on* $0\le m\le 12$ *are* $(3,50000)$ *and* $(11,50000)$. *Three months after January (April 2004) and 11 months after January (December 2004), the number of workers in the agricultural industry hit 50,000 workers.*



Exercises

New York City is located north of the equator. Osorno, Chile is located the same distance from the equator but is south of the equator. The number of hours of daylight in New York City d days after March 21, 2010 may be modeled by $N(d) = 2.925\sin(\frac{2\pi}{365}d) + 12.18$. The number of hours of daylight in Osorno, Chile d days after March 21, 2010 may be modeled by $S(d) = -2.908\sin(\frac{2\pi}{365}d) + 12.18$. Be sure to use radian mode. (*Source*: Modeled from data at aa.usno.navy.mil)

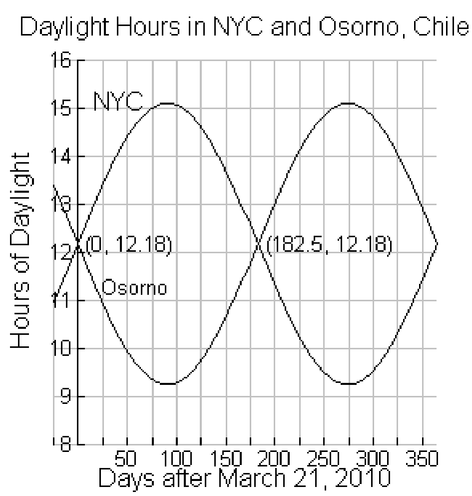
1. Without graphing, describe any similarities and differences between functions N and S.

2. Graphically, find the intersection points of N and S and explain what these values represent.

Exercise Solutions

1. The two functions are vertical reflections relative to one another. The have the same amplitude ($|A| = 2.908$), the same vertical shift ($|D| = 12.18$), and the same period ($\text{period} = 365 \text{ days}$). Because they are the same distance from the equator but one city is north and one city is south, they will have opposite values (number of hours of daylight) on the same day. For example, on about June 21 (first day of summer in NYC, winter in Osorno), the number of daylight hours in New York is about 3 hours more than the midline ($12.18 + 3 = 15.18$) and the number of daylight hours in Osorno is about 3 hours less than the midline ($12.18 - 3 = 9.18$).

2. The functions N and S intersect at spring equinox (March 21) and fall equinox (September 21). See the graph.



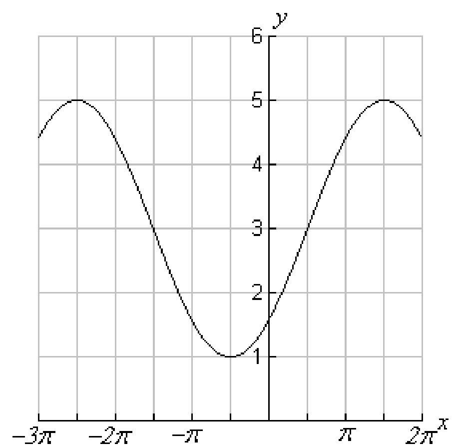
8.5 Sinusoidal Modeling

Objectives

- Generate equations for sine and cosine functions from tables and graphs.
- Graph sine and cosine functions from equations and tables.
- Use sine and cosine functions to model real-world data sets.

Examples

- **Example 1: Generating an Equation from the Graph of a Sinusoid**
  Find a formula for the sinusoidal function. You may use either sine or cosine.



Solution:
To find the sine formula for the wave, we need to determine what A, B, C, and D are in the standard sine formula $y(\theta) = A\sin(B(\theta - C)) + D$. The amplitude is $|A|$ and on the graph it is 2 because that is the height of the wave above the midline. Therefore, $A = 2$. The frequency, B, is the constant that determines the period. We know that the period for a sine wave (in degrees) is 2π radians. This wave's period is 4π so $B = \frac{2\pi}{4\pi} = \frac{1}{2}$. The horizontal shift, C, is $\frac{\pi}{2}$. Finally, D is the vertical shift which is 3 units in this case. The formula for the function is $y(\theta) = 2\sin(\frac{1}{2}(\theta - \frac{\pi}{2})) + 3$.

Likewise, to find a cosine formula we need to know what the parameters A, B, C, and D are in the standard cosine formula $y(\theta) = A\sin(B(\theta - C)) + D$. The sine and cosine functions will have the same amplitude, period, and vertical shift. We must only determine the horizontal shift. To do this, we observe there is a maximum value for the function at $\frac{3\pi}{2}$ instead of at $0°$. Therefore, we shift the graph left $\frac{3\pi}{2}$ from the vertical axis. This gives us the formula $y(\theta) = 2\cos(\frac{1}{2}(\theta - \frac{3\pi}{2})) + 3$.

- **Example 2: Generating a Sinusoidal Model from a Table of Data**
  The table below displays the average high and low temperatures over the course of a year since January (*Source*: weather.com). Find a formula for the average high

temperature in International Falls $T(m)$ as a function of the month of the year. Graph the data along with the formula you find.

Solution:

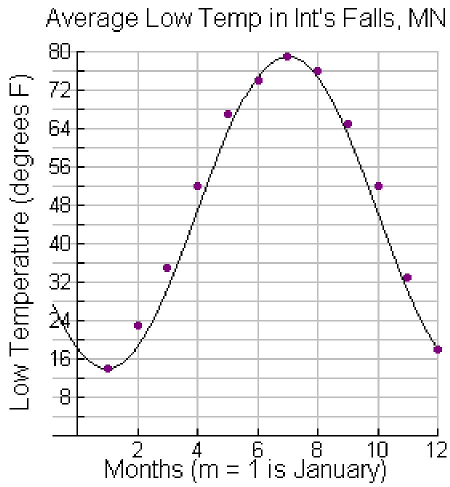
The maximum average low temperature value is 79 and the minimum low temperature value is 14.

Month m **(**$m=1$ **is January)**	**International Falls, MN Average Low Temperature (degrees Fahrenheit)** T
1	14
2	23
3	35
4	52
5	67
6	74
7	79
8	76
9	65
10	52
11	33
12	18

$$\text{amplitude} = \frac{79-14}{2} = \frac{65}{2} = 32.5$$

$$\text{midline} = \frac{79+14}{2} = \frac{93}{2} = 46.5$$

The amplitude is 32.5 and the midline is $y = 46.5$. This means the temperature varies 32.5 degrees above and below the midline of 46.5 degrees. From the data table, we observe that the period is 12 months. The frequency is the reciprocal of the period and equals $\frac{1}{12}$. Since the function values start at the midline, increase to the maximum value, decrease to the minimum value, and return to the midline, a sine function model can be used for the data. The standard form for a sine function is $T(m) = A\sin(B(m-C)) + D$ and we know $A = 32.5$, $C = 4$, and $D = 46.5$. To find the value of B, we use $B = \frac{2\pi}{\text{period}}$ and have $B = \frac{2\pi}{12} = \frac{\pi}{6}$. The equation that models these data is



$$T(m) = 32.5\sin\left(\frac{\pi}{6}(m-4)\right) + 46.5.$$

Exercises

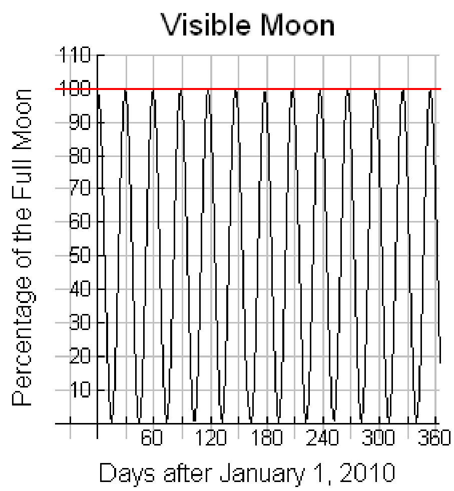
1. State the period, amplitude, phase shift and midline for the function $y = 6.25\sin(2(t-4)) - 5$.

2. The function $M(t) = 50\cos(\frac{2\pi}{29.53}t) + 50$ models the percentage M of the full moon visible t days after January 1, 2010. Use this model to determine all days in 2010 where a full moon was visible.

Exercise Solutions

1. The amplitude of the function $y = 6.25\sin(2(t-4)) - 5$ is $A = 6.25$. The midline is $y = -5$.The period is $B = 29.53$. Compared to a standard sine function, this function is shifted 4 units to the right.

2. We graphically solve the equation $100 = 50\cos(\frac{2\pi}{29.53}t) + 50$. We see that $M = 100$ (100% of the moon is visible) each time the graph reaches a peak.



Using the "intersect" feature on the graphing calculator, we find that 100% of the moon is visible when
$d = 29.53, 59.06, 88.59, 118.12, 147.65, 177.18, 206.71, 236.24, 265.77, 295.3, 324.83, 354.36$.

These values of d correspond January 30 (20.53), March 2 (59.06), March 31 (88.59), April 30 (118.12), May 29 (147.65), June 28 (177.18), July 27 (206.71), August 26 (236.24), September 24 (265.77), October 24 (295.3), November 22 (324.83), December 22 (354.36).

8.6 Other Trigonometric Functions

Objectives

- Define tangent, cotangent, cosecant, and secant in terms of sine and cosine.
- Use the language of rate of change to describe the behavior of trigonometric functions.

Concepts and Definitions

- **Tangent Function:** The tangent function gives the slope of the terminal side of an angle in standard position measured in θ degrees or radians. That is, $m = \tan(\theta) = \dfrac{\sin(\theta)}{\cos(\theta)}$.
- **Cotangent Function:** The cotangent of an angle, denoted $\cot(\theta)$, is the reciprocal of the tangent function and represents the reciprocal of the slope of the terminal side of an angle measuring θ degrees or radians. That is $\cot(\theta) = \dfrac{1}{\tan(\theta)} = \dfrac{\cos(\theta)}{\sin(\theta)}$.
- **Secant Function:** The secant of an angle θ, denoted $\sec(\theta)$, is the reciprocal of the cosine value at θ. That is, $\sec(\theta) = \dfrac{1}{\cos(\theta)}$.
- **Cosecant Function:** The cosecant of an angle θ, denoted $\csc(\theta)$, is the reciprocal of the sine value at θ. That is, $\csc(\theta) = \dfrac{1}{\sin(\theta)}$.

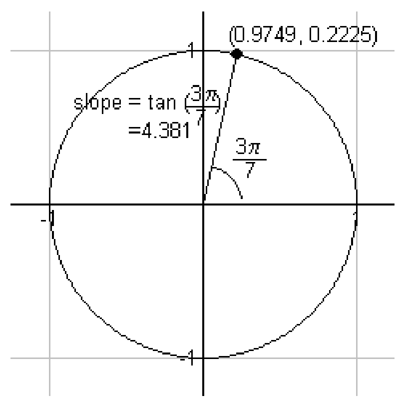
Examples

- **Example 1: Using the Tangent Function**

  Evaluate the expression $\tan(\frac{3\pi}{7})$ and explain what this value represents in the context of angles.

  Solution:

  Using technology, we find $\tan(\frac{3\pi}{7}) \approx 4.381$. *This tells us that the slope of the terminal side of a* $\frac{3\pi}{7}$ *angle in standard position is 4.381.*



- **Example 2: Evaluating Trigonometric Expressions**

  Find the exact value of each expression without using a calculator.

  a. $\tan(60^\circ)$

b. $\cot(45^\circ)$

c. $\sec(150^\circ)$

d. $\csc(-\frac{\pi}{6})$

Solution:

$$a.\ \tan(60^\circ) = \frac{\sin(60^\circ)}{\cos(60^\circ)} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

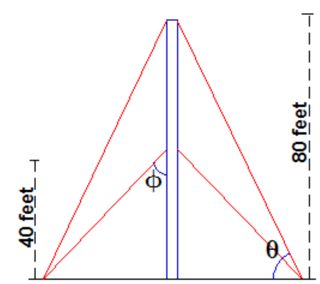
$$b.\ \cot(45^\circ) = \frac{\cos(45^\circ)}{\sin(45^\circ)} = \frac{\sqrt{2}/2}{\sqrt{2}/2} = 1$$

$$c.\ \sec(150^\circ) = \frac{1}{\cos(150^\circ)} = \frac{1}{-\cos(30^\circ)} = \frac{1}{-\sqrt{3}/2} = -\frac{2}{\sqrt{3}}$$

$$d.\ \csc(-\frac{\pi}{6}) = \frac{\cos(-\frac{\pi}{6})}{\sin(-\frac{\pi}{6})} = \frac{\sqrt{3}/2}{-1/2} = -\sqrt{3}$$

Exercises

In Exercises 1 – 2, use the following information. Guy wires are wires attached to tall towers, poles, or structures to provide stability and support. A 80-foot microwave tower is constructed with guy wires attached at the top of the tower and halfway to the top of the tower as shown in the diagram. The length of each guy wire depends on the angle it is constructed with the ground.



1. The function $D(\theta) = 80\cot(\theta)$ models the distance from the tower the guy wire must be attached to the ground given the wire makes an angle of θ with the ground.

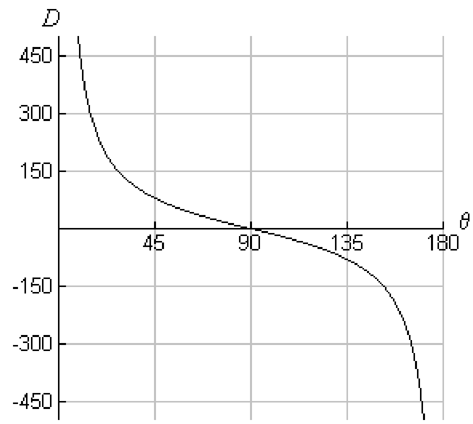
   a. Graph $D(\theta) = 80\cot(\theta)$ and determine the practical domain for this function? What is the practical range?

   b. Find $D(54^\circ)$ and explain what this value represents in the context of the problem.

2. The function $G(\phi) = 40\sec(\phi)$ models the length of the guy wire attached halfway up the tower when it makes an angle of ϕ with the tower. How long will the guy wire need to be if it makes an angle of 42° with the tower?

Exercise Solutions

1. a. The mathematical domain such that the length of the guy wire is positive is $0 < \theta < 180^\circ$. However, as $\theta \to 0$, $D \to \infty$. We will have to decide upon a more reasonable angle so that the length is not unreasonably large. Similarly, as $\theta \to 90^\circ$, $D \to 0$. We make the practical domain $20^\circ \le \theta \le 75^\circ$. With this practical domain, the practical range is $80\cot(75^\circ) \le D \le 80\cot(20^\circ)$ or $21.4 \le D \le 219.8$.



b.

$$
\begin{aligned}
D(\theta) &= 80\cot(\theta) \\
D(54^\circ) &= 80\cot(54^\circ) \\
&= 80 \cdot \frac{\cos(54^\circ)}{\sin(54^\circ)} = \frac{1}{\tan(54^\circ)} \cdot \\
&= 58.1234
\end{aligned}
$$

If the guy wire is attached to the ground such that it makes an angle of 54° then the guy wire will need to be 58.1234 feet in length.

2.

$$
\begin{aligned}
G(\phi) &= 40\sec(\phi) \\
G(42^\circ) &= 40\sec(42^\circ) \\
&= 40 \cdot \frac{1}{\cos(42^\circ)} \\
&= 53.83
\end{aligned}
$$

If the guy wire is attached to the tower such that it makes an angle of 42° then the guy wire will need to be 53.53 feet in length.

8.7 Inverse Trigonometric Functions

Objectives

- Use inverse trigonometric functions to solve trigonometric equations algebraically.
- Use trigonometric models of real-world data sets for questions whose answers require inverse trigonometric functions.

Concepts and Definitions

- **Inverse Cosine Function:** The function $f^{-1}(x) = \cos^{-1}(x)$ is the inverse cosine function. The inverse cosine function takes the x-coordinate of the endpoint of an arc as its input and outputs an angle measure that corresponds with an arc whose endpoint has this x-coordinate.
- **Inverse Sine Function:** The function $f^{-1}(y) = \sin^{-1}(y)$ is the inverse sine function. The inverse sine function takes the y-coordinate of the endpoint of an arc as its input and outputs an angle measure that corresponds to an arc whose endpoint has this y-coordinate.
- **Inverse Tangent Function:** The function $f^{-1}(x) = \tan^{-1}(x)$ is the inverse tangent function. The inverse tangent function takes the slope of the terminal side of an angle as its input and outputs one of the angle measures whose terminal side has this slope.
- **Domain and Range for Inverse Trigonometric Functions:**
    - The inverse cosine function $f^{-1}(x) = \cos^{-1}(x)$ has a domain of $-1 \le x \le 1$ and a range of $0 \le f^{-1}(x) \le 180^\circ$ or $0 \le f^{-1}(x) \le \pi$.
    - The inverse sine function $f^{-1}(y) = \sin^{-1}(y)$ has a domain of $-1 \le y \le 1$ and a range of $-90^\circ \le f^{-1}(y) \le 90^\circ$ or $-\frac{\pi}{2} \le f^{-1}(y) \le \frac{\pi}{2}$.
    - The inverse tangent function $f^{-1}(x) = \tan^{-1}(x)$ as a domain of all real numbers and a range of $-90^\circ \le f^{-1}(x) \le 90^\circ$ or $-\frac{\pi}{2} \le f^{-1}(x) \le \frac{\pi}{2}$.

Examples

- **Example 1: Using the Inverse Cosine Function**
  The angle whose vertex is a point on earth and whose rays extend from this point to the sun and the horizon, respectively, is known as the altitude of the sun. This angle changes as the sun rises and sets each day. Negative angles indicate that the sun is below the horizon and the location marked by the point is dark. The altitude of the sun in Raleigh, North Carolina, on June 26, 2012 may be modeled by the function $A(h) = 63.75\cos(\frac{\pi}{12}(h-6.5)) + 13.45$ where h is the number of hours since 6:00 am (*Source*: Modeled from data at aa.usno.navy.mil). Algebraically, solve

$22 = 63.75\cos(\frac{\pi}{12}(h-6.5))+13.45$ over the interval $[-2, 14]$. Then interpret the practical meaning of the result.

Solution:

$$22 = 63.75\cos(\frac{\pi}{12}(h-6.5))+13.45$$

$$8.55 = 63.75\cos(\frac{\pi}{12}(h-6.5))$$

$$0.1341 = \cos(\frac{\pi}{12}(h-6.5))$$

$$\cos^{-1}(0.1341) = \frac{\pi}{12}(h-6.5)$$

$$\frac{12}{\pi}\cos^{-1}(0.1341) = h-6.5$$

$$\frac{12}{\pi}\cos^{-1}(0.1341)+6.5 = h$$

$$h \approx 12$$

The altitude of the sun is 22° *about 12 hours after 6:00 am or at 6:00 pm.*

- **Example 2: Evaluating Inverse Trigonometric Expressions**
  Evaluate each of the expressions exactly. All angles are in radians in the first quadrant.

a. $\cos(\sin^{-1}(\frac{\sqrt{3}}{2}))$

b. $\sec(\csc^{-1}(\frac{2\sqrt{3}}{3}))$

Solution:

a. First, we evaluate $\sin^{-1}(\frac{\sqrt{3}}{2})$. *This expression states that there is an angle whose sine is* $\frac{\sqrt{3}}{2}$. *We recall that* $\sin(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$. *Therefore, we now have* $\cos(\sin^{-1}(\frac{\sqrt{3}}{2})) = \cos(\frac{\pi}{3}) = \frac{1}{2}$.

b. First, we evaluate $\csc^{-1}(\frac{2\sqrt{3}}{3})$. *This expression states that there is an angle whose cosecant is* $\frac{2\sqrt{3}}{3}$. *We recall that* $\csc(\frac{\pi}{3}) = \frac{1}{\sin(\frac{\pi}{3})} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$. *Therefore, we now have* $\sec(\csc^{-1}(\frac{2\sqrt{3}}{3})) = \sec(\frac{\pi}{3}) = \frac{1}{\cos(\frac{\pi}{3})} = \frac{1}{\frac{1}{2}} = 2$.

Exercises

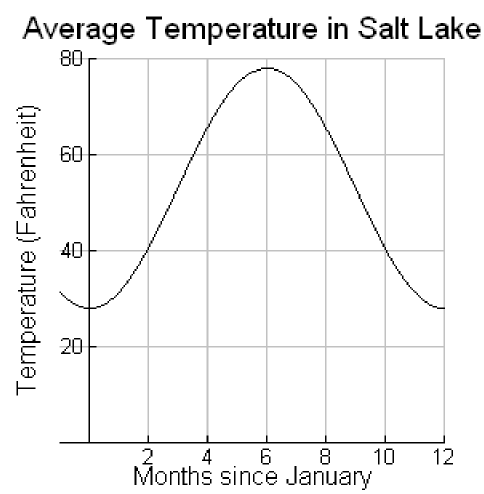
The average temperature in Salt Lake City, Utah, may be modeled by the function $T(m) = 25\sin(\frac{\pi}{6}(m-3)) + 53$ degrees Fahrenheit where m is the number of months since January.

1. Graph the function, and then identify an appropriate restricted domain such that its inverse will also be a function.

2. Find the inverse function and state its domain and range.

Exercise Solutions

1. After graphing the function, we choose to restrict the domain to $0 \le m \le 6$. This domain ensures that we have all of the output values of T represented once.



2.

$$T = 25\sin(\frac{\pi}{6}(m-3)) + 53$$

$$T - 53 = 25\sin(\frac{\pi}{6}(m-3))$$

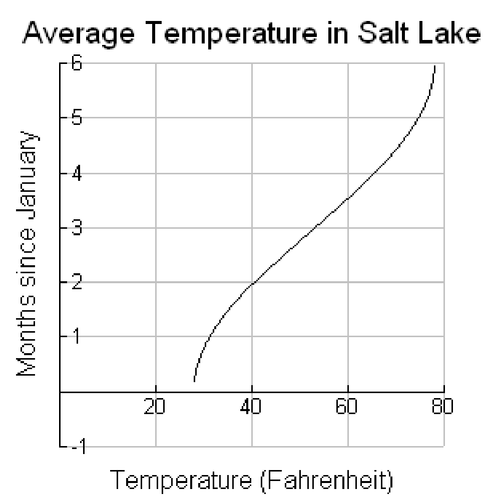
$$\frac{T-53}{25} = \sin(\frac{\pi}{6}(m-3))$$

$$\sin^{-1}(\frac{T-53}{25}) = (\frac{\pi}{6}(m-3))$$

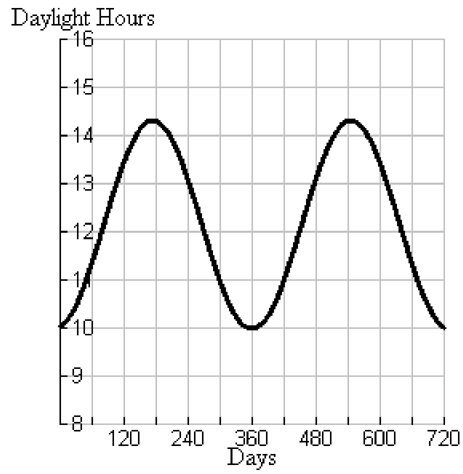
$$\frac{6}{\pi}\sin^{-1}(\frac{T-53}{25}) = m-3$$

$$\frac{6}{\pi}\sin^{-1}(\frac{T-53}{25}) + 3 = m^{-1}(T)$$

The domain for this inverse function is $28 \le T \le 78$. The range is $0 \le m \le 6$.



Chapter 8 Exam A



1. The graphs show the number of hours of daylight at different times of the year for Birmingham, AL (*Source*: Modeled from data at aa.usno.navy.mil). The data was recorded every ten days for two years beginning January 1, 2007. Find the amplitude, period, and phase shift from January 1st for the graph and then provide a formula for the sinusoidal function.

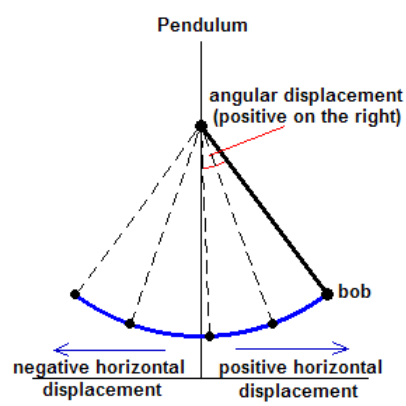
2. Use a calculator to evaluate each expression. Then explain what the value means in the context of the unit circle.

   a. $\sin(119^{\circ})$

   b. $\cos(\frac{2\pi}{9})$

   c. $\tan(\frac{10\pi}{7})$

3. A pendulum consists of a bob (a weight) attached at the end of a rod or string of fixed length. Pendulums are commonly found in grandfather clocks. When pulled back a certain distance, the pendulum will begin to swing back and forth along an arc. As it moves along this arc, it changes its horizontal displacement relative to its position at rest.



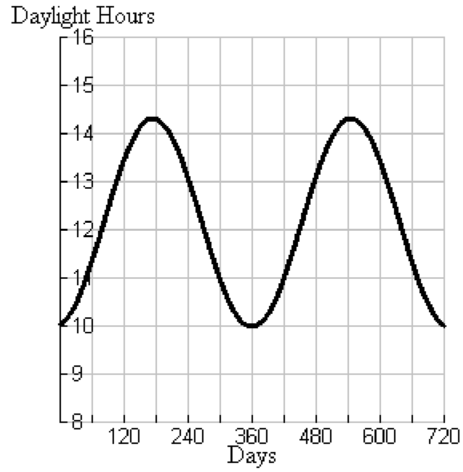
   a. The angular displacement is the angle created between the pendulum and the vertical line representing its position at rest. Explain why trigonometric functions are useful for finding the horizontal displacement of the pendulum's bob when you know the angular displacement.

   b. If a pendulum has a length of 3 feet and swings such that its angular displacement θ has domain of $-26° \le \theta \le 26°$, find the maximum horizontal displacement.

4. The voltage V in an electrical circuit is given by the formula $V(t) = 5\cos(120\pi t)$ where t is the time measured in seconds. Solve $5\cos(120\pi t) = -4.2$. Explain what your answer represents.

Chapter 8 Exam A Solutions

1. The graphs show the number of hours of daylight at different times of the year for Birmingham, AL (*Source*: Modeled from data at aa.usno.navy.mil). The data was recorded every ten days for two years beginning January 1, 2007. Estimate the amplitude, period, and phase shift from January 1st for the graph and then provide a formula for the sinusoidal function.



The amplitude (distance above and below the midline of $H = 12.2$) is approximately $A = 2$. The period is 365 days. The graph is shifted about 80 days to the right from January 1st. Therefore, we can create the following sinusoidal function model.

$$H(t) = 2\sin(\frac{2\pi}{365}(t-80)) + 12.2$$

2. Use a calculator to evaluate each expression. Then explain what the value means in the context of the unit circle.

a. $\sin(119^\circ) \approx 0.875$

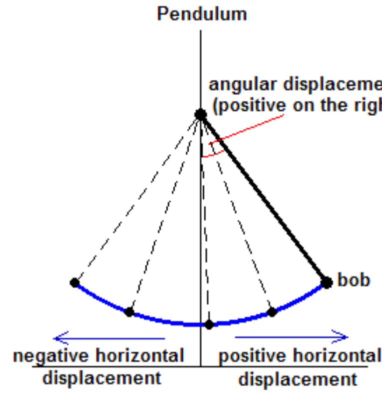
The $\sin(119^\circ)$ represents the vertical (y) position of the endpoint of the arc on the unit circle corresponding with a 119° angle.

b. $\cos(\frac{2\pi}{9}) \approx 0.7660$

The $\cos(\frac{2\pi}{9})$ represents the horizontal (x) position of the endpoint of the arc on the unit circle corresponding with a $\frac{2\pi}{9}$ radian angle.

c. $\tan(\frac{10\pi}{7}) \approx 4.381$

The slope of the terminal side of a $\frac{10\pi}{7}$ angle in standard position is 4.381.



3. A pendulum consists of a bob (a weight) attached at the end of a rod or string of fixed length. Pendulums are commonly found in grandfather clocks. When pulled back a certain distance, the pendulum will begin to swing back and forth along an arc. As it moves along this arc it changes its horizontal displacement relative to its position at rest.

   a. The angular displacement is the angle created between the pendulum and the vertical line representing its position at rest. Explain why trigonometric functions are useful for finding the horizontal displacement of the pendulum's bob when you know the angular displacement.

   As the pendulum swings back and forth, the horizontal displacement varies as the angular displacement changes. This is best represented by the periodic behavior modeled by trigonometric functions.

   b. If a pendulum has a length of 3 feet and swings such that its angular displacement θ has domain of $-26° \le \theta \le 26°$, find the maximum horizontal displacement.

$$\cos(26^\circ) = \frac{h}{3}$$
$$h = 3\cos(26^\circ)$$
$$h \approx 2.696$$

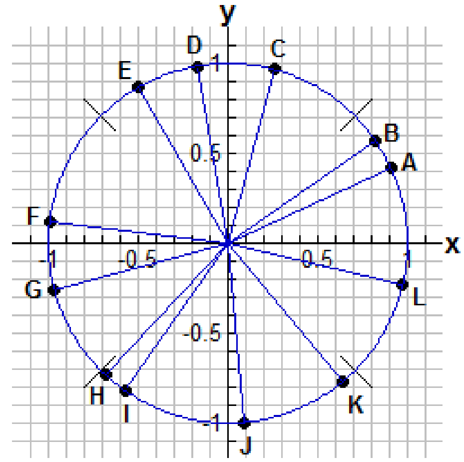
4. The voltage V in an electrical circuit is given by the formula $V(t) = 5\cos(120\pi t)$ where t is the time measured in seconds. Solve $5\cos(120\pi t) = -4.2$. Explain what your answer represents.

$$5\cos(120\pi t) = -4.2$$
$$\cos(120\pi t) = \frac{-4.2}{5}$$
$$120\pi t = \cos^{-1}\left(\frac{-4.2}{5}\right)$$
$$t = \frac{\cos^{-1}\left(\frac{-4.2}{5}\right)}{120\pi}$$
$$t \approx 0.0068$$

At time $t \approx 0.0068$, the voltage in an electric circuit is –4.2 volts.

Chapter 8 Exam B

1. Use the figure to do the following.



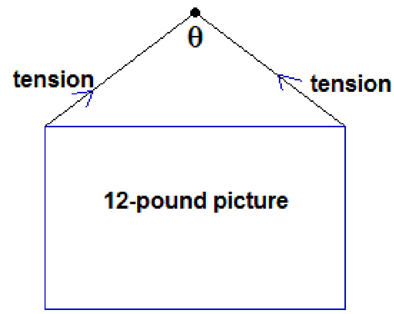
a. Estimate θ, the angle between 0 and 2π radians, where the point lies.

b. Estimate $\cos(\theta)$ and $\sin(\theta)$.

i. B ii. D iii. F

iv. H v. J vi. L

2. When a 12-pound picture is hung as shown, the tension T in pounds on each side of the wire is given by the function $T\theta) = 12\sec(\frac{\theta}{2})$.

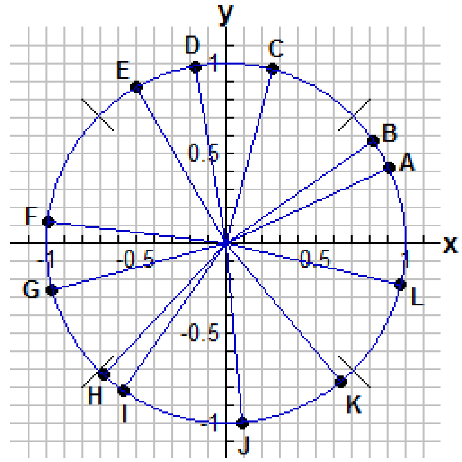


What angle will create a tension of 19 pounds on one side of the wire?

3. Graph one period of function $f(\theta) = 6\cos(3\theta) + 1$ then state where the rate of change is positive and where the concavity is positive. θ is in degrees. Find all intersection points of f and $g(\theta) = -4$ over the period of f you graphed.

4. The function $H(d) = 3.78\sin(\frac{2\pi}{365}d) + 12.2$ models the number of hours of daylight in Seattle d days after March 21, 2010. Solve $15.5 = 3.78\sin(\frac{2\pi}{365}d) + 12.2$ over the interval $[-79, 285]$. (*Source*: Modeled using data from aa.usno.navy.mil).

Chapter 8 Exam B Solutions



1. Use the figure to do the following.

   a. Estimate θ, the angle between 0 and 2π radians, where the point lies.

   b. Estimate $\cos(\theta)$ and $\sin(\theta)$.

i. B $(\frac{1}{5}\pi)$ ii. D $(\frac{9}{16}\pi)$ iii. F $(\frac{15}{16}\pi)$

$\cos(\theta) \approx 0.8$ $\cos(\theta) \approx -0.2$ $\cos(\theta) \approx -0.95$

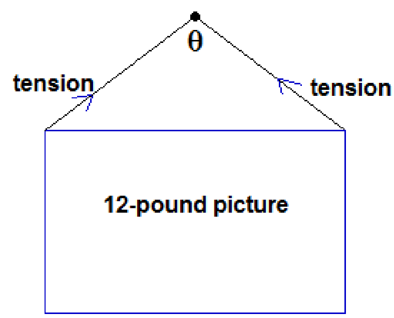
$\sin(\theta) \approx 0.6$ $\sin(\theta) \approx 0.95$ $\sin(\theta) \approx 0.1$

iv. H $(\frac{5}{4}\pi)$ v. J $(\frac{25}{16}\pi)$ vi. L $(\frac{11}{6}\pi)$

$\cos(\theta) \approx -0.7$ $\cos(\theta) \approx 0.1$ $\cos(\theta) \approx 0.95$

$\sin(\theta) \approx -0.75$ $\sin(\theta) \approx -0.95$ $\sin(\theta) \approx -0.2$

2. When a 12-pound picture is hung as shown, the tension T in pounds on each side of the wire is given by the function $T(\theta) = 12\sec(\frac{\theta}{2})$.



   What angle will create a tension of 19 pounds on one side of the wire?

$$T(\theta) = 12\sec(\frac{\theta}{2})$$

$$19 = 12\sec(\frac{\theta}{2})$$

$$\frac{19}{12} = \sec(\frac{\theta}{2})$$

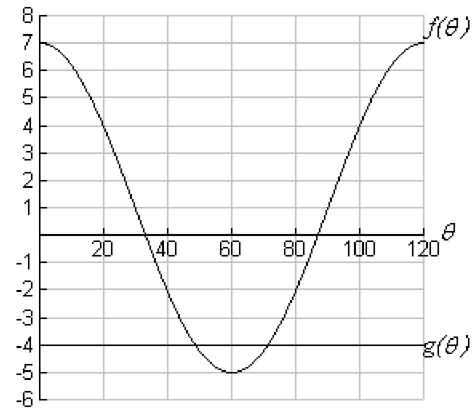
$$\sec^{-1}(\frac{19}{12}) = \frac{\theta}{2}$$

$$\cos^{-1}(\frac{12}{19}) = \frac{\theta}{2}$$

$$2\cos^{-1}(\frac{12}{19}) = \theta$$

$$\theta \approx 101.67^\circ$$

3. Graph one period of function $f(\theta) = 6\cos(3\theta) + 1$ then state where the rate of change is positive and where the concavity is positive. θ is in degrees. Find all intersection points of f and $g(\theta) = -4$ over the period of f you graphed.

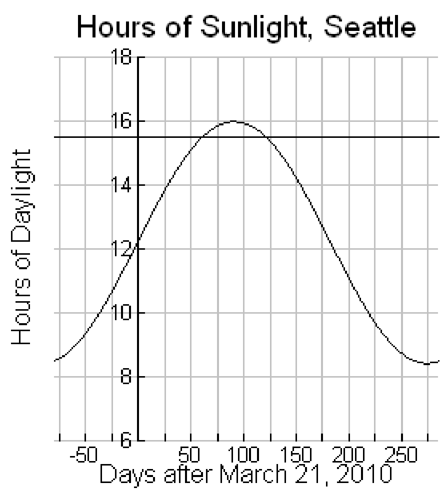


The rate of change is positive when $60^\circ < \theta < 120^\circ$. The function is concave up approximately when $25^\circ < \theta < 95^\circ$.

$6\cos(3\theta) + 1 = g(\theta) = -4$ when $\theta \approx 48.8^\circ$ and $\theta \approx 71.2^\circ$.

4. The function $H(d) = 3.78\sin(\frac{2\pi}{365}d) + 12.2$ models the number of hours of daylight in Seattle d days after March 21, 2010. Solve $15.5 = 3.78\sin(\frac{2\pi}{365}d) + 12.2$ over the interval $[-79,\ 285]$. (*Source*: Modeled using data from aa.usno.navy.mil).

$$15.5 = 3.78\sin(\frac{2\pi}{365}d) + 12.2$$
$$3.3 = 3.78\sin(\frac{2\pi}{365}d)$$
$$\frac{3.3}{3.78} = \sin(\frac{2\pi}{365}d)$$
$$\sin^{-1}(\frac{3.3}{3.78}) = \frac{2\pi}{365}d$$
$$\frac{365}{2\pi}\sin^{-1}(\frac{3.3}{3.78}) = d$$
$$d \approx \frac{365}{2\pi}\cdot 1.061 \text{ and } \frac{365}{2\pi}\cdot 2.080$$
$$d \approx 61.6 \text{ and } 120.8$$



Seattle will experience 15.5 hours of daylight about 62 days after March 21, 2010 and 120.8 days after March 21, 2010.

Triangle Trigonometry and Applications

9

9.1 Right Triangle Trigonometry

Objectives

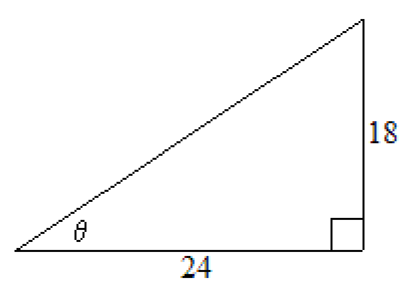
- Describe the six trigonometric ratios.
- Use the appropriate trigonometric ratio to solve real-world problems involving right triangles.
- Describe the relationships among the trigonometric ratios.

Concepts and Definitions

- **Similar Triangles:** Two triangles, $\triangle ABC$ and $\triangle DEF$ are similar if corresponding angles are congruent and corresponding sides are proportional.
- **Six Trigonometric Functions:**
  - $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$
  - $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$
  - $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$
  - $\csc\theta = \dfrac{\text{hypotenuse}}{\text{opposite}}$
  - $\sec\theta = \dfrac{\text{hypotenuse}}{\text{adjacent}}$
  - $\cot\theta = \dfrac{\text{adjacent}}{\text{opposite}}$
- **Angles of Elevation and Depression**: The angle of elevation is the angle from an imaginary horizontal line and the observer's line of sight to an object that is above the horizontal line. The angle of depression is the angle from an imaginary horizontal line and the observer's line of sight to an object that is below the horizontal line.

Examples

- **Example 1: Expressing the Six Trigonometric Functions**
  Evaluate the six trigonometric functions of θ. That is, evaluate $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$.



Solution:
We use the Pythagorean Theorem to find the length of the hypotenuse of the right triangle.

$$24^2 + 18^2 = h^2$$
$$900 = h^2$$
$$h = \sqrt{900} = 30$$

We use the definitions of the six trigonometric ratios as follows.

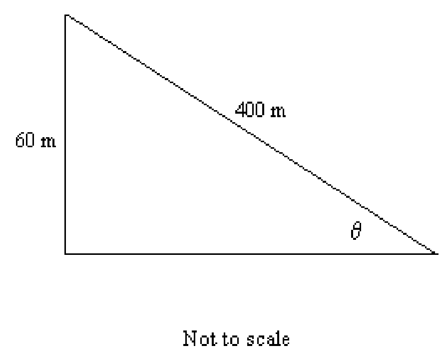
$$\cos\theta = \frac{24}{30} = \frac{4}{5} \qquad \sin\theta = \frac{18}{30} = \frac{3}{5} \qquad \tan\theta = \frac{18}{24} = \frac{3}{4}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{4} \qquad \csc\theta = \frac{1}{\sin\theta} = \frac{5}{3} \qquad \cot\theta = \frac{1}{\tan\theta} = \frac{4}{3}$$

- **Example 2: Calculating an Angle of Elevation**
  Ski Dubai is an indoor skiing facility in the Mall of the Emirates in Dubai, United Arab Emirates. The longest run is 400 meters with a drop of 60 meters (www.skidxb.com). What angle does this ski run make with the horizontal?

  Solution:

  We wish to find the slope that the ski run makes with the hypothetical horizontal line. Consider the diagram and see why the sine function is used to find the required angle.



$$\sin\theta = \frac{60}{400}$$
$$\theta = \sin^{-1}\left(\frac{60}{400}\right)$$
$$\theta \approx 8.63^\circ$$

The ski slope has an angle of elevation of 8.63°.

Exercises

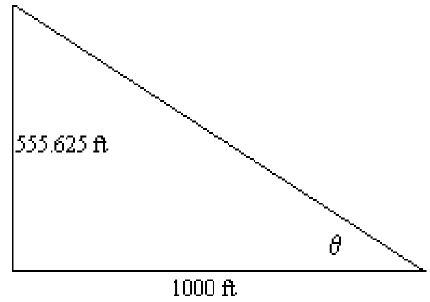
1. Suppose you are standing 1000 feet away from the base of the Washington Monument. We know that the monument is 555 feet $\frac{5}{8}$ inches in height. What is the angle of elevation from your position to the top of the monument?

2. Suppose that the angle of depression from the top of the Eiffel Tower to a point F feet from the base of the tower is $26°$. The Eiffel Tower is 986 feet tall. Find the value of F.

Exercise Solutions

1. Consider the diagram and see why the tangent function is used to find the required angle.

$$\tan\theta = \frac{555.625}{1000}$$
$$\theta = \tan^{-1}\left(\frac{555.625}{1000}\right)$$
$$\theta \approx 29.06^\circ$$

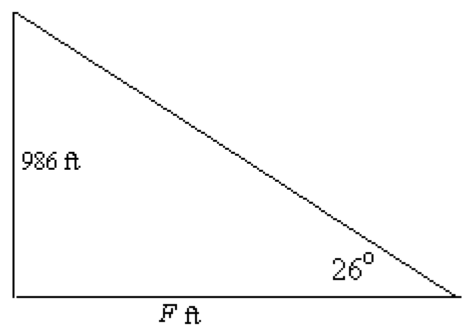


The angle of elevation is 29.06°.

Not to scale

2. Consider the diagram and see why the tangent function is used to find the required angle.

$$\tan 26^\circ = \frac{986}{F}$$
$$F\tan 26^\circ = 986$$
$$F = \frac{986}{\tan 26^\circ}$$
$$F \approx 2021.6$$



The value of F is approximately 2021.6 feet.

Not to scale

9.2 Law of Cosines

Objectives

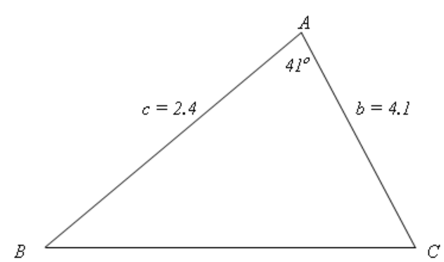
- Develop the formula for the Law of Cosines.
- Demonstrate the relationship between the Pythagorean Theorem and the Law of Cosines.
- Apply the Law of Cosines to real-world scenarios.

Concepts and Definitions

- **Law of Cosines:** For $\triangle ABC$, $c^2 = a^2 + b^2 - 2ab\cos C$ where c is the side opposite angle C. Similarly, $a^2 = b^2 + c^2 - 2bc\cos A$ and $b^2 = a^2 + c^2 - 2ac\cos B$.

Examples

- **Example 1: Solving a Triangle**
  Solve each triangle using any correct technique.



Solution:

We are provided with the lengths of 2 sides of the triangle and the measure of the included angle. We have a SAS situation and the Law of Cosines is applicable.

$$a^2 = 2.4^2 + 4.1^2 - 2(2.4)(4.1)\cos 41^\circ$$
$$a^2 \approx 7.717$$
$$a \approx 2.778$$

We use the Law of Cosines again to find the measure of angle C.

$$2.4^2 = 2.778^2 + 4.1^2 - 2(2.778)(4.1)\cos C$$
$$\cos C \approx 0.8239$$
$$C \approx 34.5^\circ$$

We know that the sum of the measures of any triangle is 180°. *The measure of angle B is* $180^\circ - 41^\circ - 34.5^\circ = 104.5^\circ$.

Example 2: Applying the Law of Cosines
Two airplanes leave Reagan National Airport in Washington D.C. at noon. The air traffic controller notes that the two are traveling away from one another at an

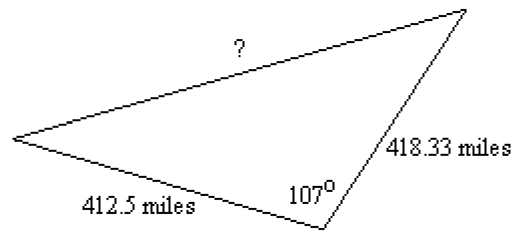
angle of $107°$. One of the airplanes is traveling at 495 miles per hour and the other 502 miles per hour. How far apart are the two planes at 12:50 p.m.?

Solution:

We first find the distance that each plane has traveled in the 50 minutes that has elapsed since leaving the airport.

$$d_1 = 495 \text{ mph} \cdot \frac{50}{60} \text{ hours} = 412.5 \text{ miles}$$

$$d_2 = 502 \text{ mph} \cdot \frac{50}{60} \text{ hours} = 418\frac{1}{3} \text{ miles}$$



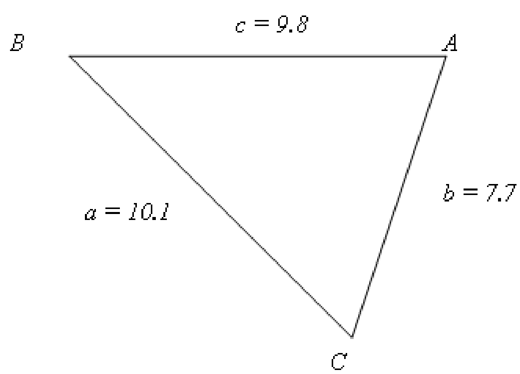
Use the diagram to see how to set up the Law of Cosines and compute the desired distance between the planes.

$$d^2 = 412.5^2 + 418.33^2 - 2(412.5)(418.33)\cos 107°$$
$$d^2 = 33.9889$$
$$a = 5.83$$

The two airplanes are 5.83 miles apart at 12:50 p.m.

Exercises

1. Solve each triangle using any correct technique.



2. Two points, P and O, on opposite sides of a pond are staked in order to determine its width. A third point, N, is located on land 250 feet from P and 180 feet from O and angle $N = 78°$. What is the pond's width?

Exercise Solutions

1. We are provided with the lengths of all 3 sides of the triangle. We have a SSS situation and the Law of Cosines is applicable. We first find the measure of angle *C*.

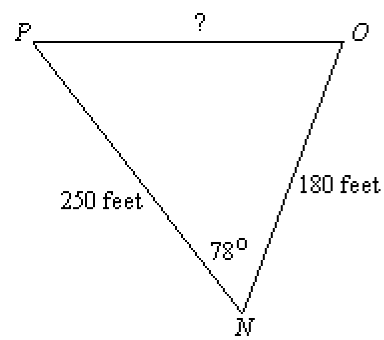
$$9.8^2 = 10.1^2 + 7.7^2 - 2(10.1)(7.7)\cos C$$
$$\cos C \approx 0.4196$$
$$C \approx \cos^{-1}(0.4196)$$
$$C \approx 65.19^\circ$$

We use the Law of Cosines again to find the measure of angle *B*.

$$7.7^2 = 10.1^2 + 9.8^2 - 2(10.1)(9.8)\cos B$$
$$\cos B \approx 0.7009$$
$$C \approx \cos^{-1}(0.7009)$$
$$C \approx 45.5^\circ$$

We know that the sum of the measures of any triangle is 180°. The measure of angle A is $180^\circ - 65.19^\circ - 45.5^\circ = 69.31^\circ$.

2. We begin by sketching a picture of the situation.



Not to scale

We know the lengths of two sides and the measure of the included angle (SAS) and can apply the Law of Cosines.

$$n^2 = 250^2 + 180^2 - 2(250)(180)\cos 78^\circ$$
$$n^2 \approx 76187.95$$
$$n \approx 276.02$$

The width of the pond is about 276 feet.

9.3 Law of Sines

Objectives

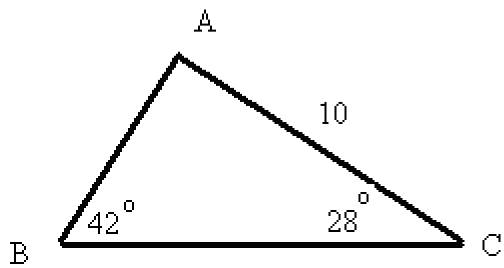
- Develop the formula for the Law of Sines
- Apply the Law of Sines to real-world scenarios

Concepts and Definitions

- **Law of Sines:** In any $\triangle ABC$, the ratio of the sine of an angle to the length of the side opposite the angle is a constant. That is, $\frac{\sin B}{b} = \frac{\sin C}{c} = \frac{\sin A}{a}$ and $\frac{b}{\sin B} = \frac{c}{\sin C} = \frac{a}{\sin A}$, for angles A, B, and C and sides a, b, and c.

Examples

- **Example 1: Using the Law of Sines to solve an AAS Triangle**
  Find the measure of angle A, the length of AB, and the length of BC.



Solution:
We find the measure of angle A.

$$A = 180^o - 42^o - 28^o = 110^o$$

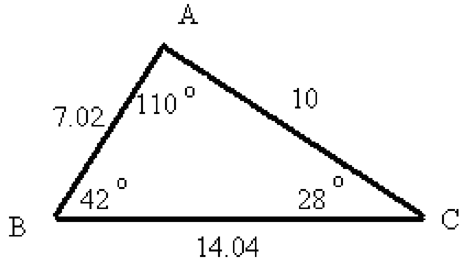
We know to use the Law of Sines to find the missing sides because we are given two angles and a side.

$$\frac{\sin 28^o}{c} = \frac{\sin 42^o}{10} \qquad\qquad \frac{\sin 110^o}{a} = \frac{\sin 42^o}{10}$$

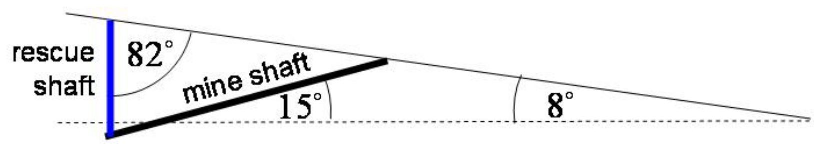
$$c(\sin 42^o) = (10)(\sin 28^o) \qquad\qquad a(\sin 42^o) = (10)(\sin 110^o)$$

$$c = \frac{(10)(\sin 28^o)}{\sin 42^0} \approx 7.02 \qquad\qquad a = \frac{(10)(\sin 110^o)}{\sin 42^0} \approx 14.04$$

Therefore, the triangle has the following measures.



- **Example 2: Using the Law of Sines**
  While playing in an open mine shaft, a child becomes trapped in the tunnel when the entrance caves in. Rescuers, concerned about the possibility of a total mine collapse, decide not to try to reopen the entrance as a part of the rescue. Instead, they move up the mountain a distance and drill a vertical opening to the bottom of the mine shaft, from which they will rescue the child (See figure). The mine shaft is 45 feet long and runs at an angle of $15°$ down from horizontal. In addition, the hillside slopes at an angle of $8°$ up from horizontal. Using the diagram, determine how far uphill from the mine shaft rescuers have to go to dig and how far down they will have to dig to reach the bottom of the mine shaft.



Solution:
First, we find the obtuse angle that the mine shaft makes with the hill. We do this knowing that the sum of the measures of the angles of any triangle is $180°$.

$$180^\circ - 15^\circ - 8^\circ = 157^\circ$$

Therefore, the acute angle that the mine shaft makes with the hill is $23°$ *and the angle between the mine shaft and the rescue shaft is* $180^\circ - 82^\circ - 23^\circ = 75^\circ$.

Using the Law of Sines, we can determine the distance, d *, uphill from the mine shaft rescuers have to dig the rescue shaft.*

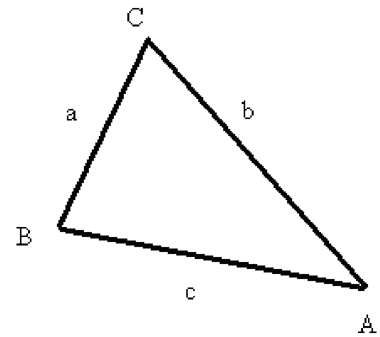
$$\frac{45}{\sin 82^\circ} = \frac{d}{\sin 75^\circ}$$
$$d \sin 82^\circ = 45 \sin 75^\circ$$
$$d = \frac{45 \sin 75^\circ}{\sin 82^\circ} \approx 43.89$$

The rescuers should begin digging the rescue shaft 43.89 feet uphill from the mine shaft. Again using the Law of Sines, we can determine the depth of the rescue shaft.

$$\frac{45}{\sin 82^\circ} = \frac{r}{\sin 23^\circ}$$
$$r \sin 82^\circ = 45 \sin 23^\circ$$
$$r = \frac{45 \sin 23^\circ}{\sin 82^\circ} \approx 17.76$$

The rescue shaft will be 17.76 feet deep.

Exercises



1. Use triangle *ABC* to find the missing sides and angles (if possible) if $a = 21,\ \mathrm{A} = 152°,\ b = 33$. If there are two solutions, find both. If there is no solution, write "no solution".

2. The Earth (E) and Mars (M) are approximately 9.3 million and 14.2 million miles from the Sun (S), respectively. (*Source*: www.enchantedlearning.com) Find the distance between the two planets when $\angle SEM$ is $21°$.

Exercise Solutions

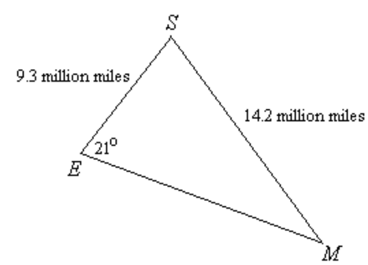
1. Since the sum of the measures of the angles of a triangle is 180° and since $152^\circ + 47.54^\circ = 199.54^\circ$, the triangle with the given measurements cannot exist. There is no solution.

$$\frac{21}{\sin 152^\circ} = \frac{33}{\sin B}$$

$$\sin B = \frac{33 \sin 152^\circ}{21}$$

$$B = \sin^{-1}\left(\frac{33 \sin 152^\circ}{21}\right) \approx 47.54^\circ$$

2. We begin by drawing a diagram to represent the situation. We use the Law of Sines to compute the measure of angle *M*. We then find the measure of angle *S* and the required distance.



$$\frac{14.2}{\sin 21^\circ} = \frac{9.3}{\sin M}$$

$$\sin M = \frac{9.3 \sin 21^\circ}{14.2}$$

$$M = \sin^{-1}\left(\frac{9.3 \sin 21^\circ}{14.2}\right) \approx 13.57^\circ$$

$$\frac{14.2}{\sin 21^\circ} = \frac{d}{\sin 145.43^\circ}$$

$$d = \frac{14.2 \sin 145.43^\circ}{\sin 21^\circ}$$

$$d \approx 22.48$$

Note that the measure of angle *S* is $180^\circ - 21^\circ - 13.57^\circ = 145.43^\circ$

The distance between Earth and Mars is about 22.48 million miles.

9.4 Polar Coordinates

Objectives

- Graph and classify equations in polar coordinates.
- Convert between polar and rectangular coordinates.
- Describe the relationship between Cartesian (rectangular) and polar coordinates using trigonometry.

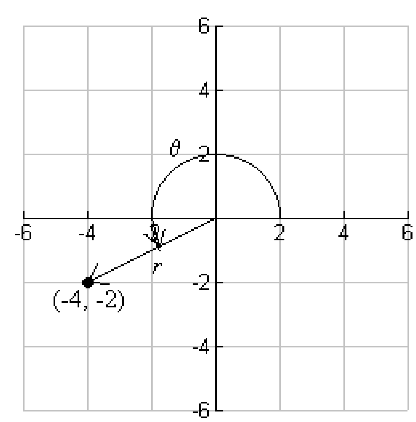
Concepts and Definitions

- **Polar Coordinates:** The polar coordinates (r,θ) represent a point in the polar plane where r is the distance from the pole to the point and θ is the angle formed with the horizontal.
- **Relationships between Polar and Cartesian Coordinates:** The polar coordinates (r,θ) and the corresponding Cartesian coordinates (x,y) are related as follows.
    - $x = r\cos\theta$
    - $y = r\sin\theta$
    - $r = \sqrt{x^2+y^2}$
    - $\theta = \tan^{-1}\left(\frac{y}{x}\right)$ for $x \neq 0$.

Examples

- **Example 1: Converting from Cartesian to Polar Coordinates**
  Convert the coordinates $(x, y) = (-4,-2)$ to polar coordinates.

  Solution:
  We begin by graphing the point in the Cartesian plane. We know $r = \sqrt{x^2+y^2}$.

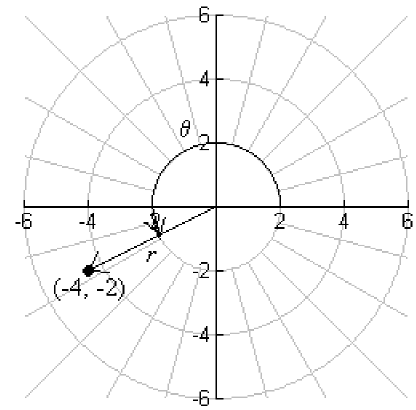


$$r = \sqrt{(-4)^2+(-2)^2}$$
$$r = \sqrt{20}$$
$$r \approx 4.472$$

We also know $\theta = \tan^{-1}\left(\frac{y}{x}\right),\ x \neq 0$.

$$\theta = \tan^{-1}\left(\frac{-2}{-4}\right)$$
$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$
$$\theta \approx 0.4636$$

The inverse tangent function returns angles in Quadrants I and IV only. Since $\theta \approx 0.4636$ *is in Quadrant I, we need to add* π *to get to Quadrant III.*



$$\theta \approx 0.4636 + \pi$$
$$\theta \approx 3.605$$

The point $(x, y) = (-4, -2)$ *is equivalent to* $(r, \theta) = (4.472, 3.605)$.

- **Example 2: Graphing Polar Equations**
  Graph the polar equation $r = \sin(2\theta) + 2$ using a graphing calculator. Choose a viewing window so that you obtain a "good" result.

  Solution:

  We use the following window to get a "good" result.

  $\theta \min: 0$

  $\theta \max: 2\pi$

  θ step: 0.1

  X min: -4

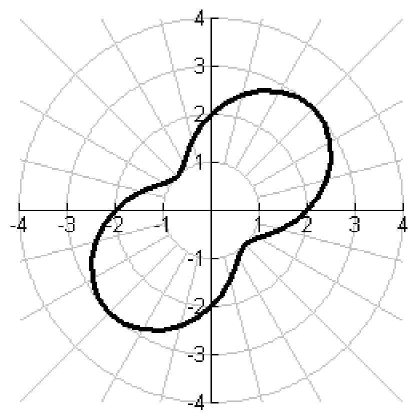
  X max: 4

  X step: 1

  Y min: -4

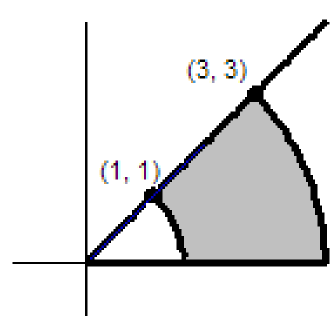
  Y max: 4

  Y step: 1

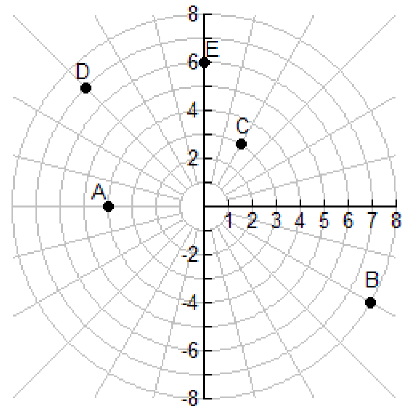


Exercises

1. Consider the diagram. Use the diagram given to express the shaded region using inequalities in polar coordinates.



2. Consider each of the points shown in the polar plane.
   Label each point in two different ways.



Exercise Solutions

1. We see that $1 \le r \le 3$.

   The angle, θ, can be determined using the inverse tangent function.

$$\tan^{-1}\frac{1}{1} = \tan^{-1}\frac{3}{3} = \tan^{-1}1 = \frac{\pi}{4} \text{ or } 45^\circ$$

   We describe the shaded region using inequalities in polar coordinates:

$$1 \le r \le 3 \text{ and } 0 \le \theta \le \frac{\pi}{4} \text{ in radians or } 0 \le \theta \le 45^\circ.$$

2. Answers may vary. Two possible correct answers are given.

   $A(4, \pi)$ or $A(-4, 0)$

   $B(8, -\frac{\pi}{6})$ or $B(-8, \frac{5\pi}{6})$

   $C(3, \frac{\pi}{3})$ or $C(-3, \frac{4\pi}{3})$

   $D(7, \frac{3\pi}{4})$ or $D(-7, -\frac{\pi}{4})$

   $E(6, \frac{\pi}{2})$ or $E(-6, -\frac{\pi}{2})$

9.5 Vectors

Objectives

- Use vectors to model and solve real-world situations
- Determine the magnitude and direction of a vector
- Resolve a vector into components
- Add, subtract, and scale vectors graphically and algebraically.

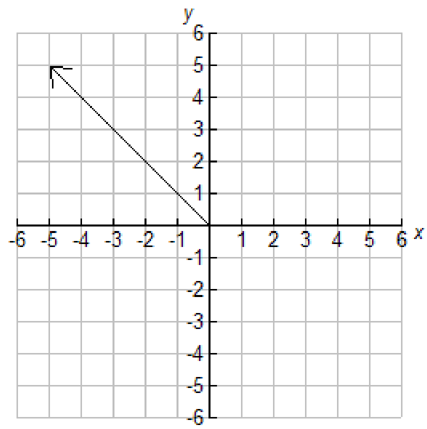
Concepts and Definitions

- **Vector:** A vector is any quantity with both magnitude and direction.
- **Displacement Vector in Component Form:** A displacement vector from one point to another is represented by an arrow with the tail located at the first point and the tip is located at the second point. If the tail is located at point (a,b) and the tip is located at the point (c,d), then the displacement vector, $\vec{V}$, is written in component form as $\vec{V}=\langle c-a,d-b\rangle$.
- **Vector Sum:** Given two vectors, $\vec{A}=\langle a,b\rangle$ and $\vec{B}=\langle c,d\rangle$, the sum $\vec{A}+\vec{B}$ is found by adding corresponding components: $\vec{A}+\vec{B}=\langle a+c,b+d\rangle$.
- **Computing Magnitude and Direction of a Given Vector:** The magnitude of a vector is given by $\vec{A}=\langle a,b\rangle$ is $|\vec{A}|=\sqrt{a^2+b^2}$ **.** The direction is determined by:
    - $\theta=\tan^{-1}\left(\frac{b}{a}\right)$ for a vector with a tip in the first or fourth quadrants.
    - $\theta=\tan^{-1}\left(\frac{b}{a}\right)+180^\circ$ for a vector with a tip in the second or third quadrants.
- **Scalar Multiplication:** Given a vector $\vec{A}=\langle a,b\rangle$ and a scalar, k, the product $k\cdot\vec{A}$ is the vector $k\cdot\vec{A}=k\cdot\langle a,b\rangle=\langle k\cdot a,k\cdot b\rangle$.
- **Vector Subtraction:** Given two vectors $\vec{A}=\langle a,b\rangle$ and $\vec{B}=\langle c,d\rangle$, the difference $\vec{A}-\vec{B}$ is given by $\vec{A}+\left(-\vec{B}\right)$ where $-\vec{B}=\langle -c,-d\rangle$. That is, $\vec{A}-\vec{B}=\langle a-c,b-d\rangle$.

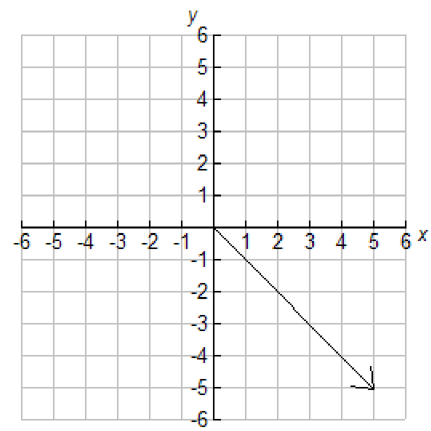
Examples

- **Example 1: Expressing Vectors in Component Form**
  Write each vector in component form.

a.



b.



Solution:

a. The vector has a horizontal component of -5 *and a vertical component of* 5 *so the vector, in component form, is expressed* $\langle -5,5\rangle = -5\vec{i}+5\vec{j}$.

b. The vector has a horizontal component of 5 *and a vertical component of* -5 *so the vector, in component form, is expressed* $\langle 5,-5\rangle = 5\vec{i}-5\vec{j}$.

- **Example 2: Computing with Vectors**
  Perform the indicated operation.

$\vec{v} = 6\vec{i}+2\vec{j}$ $\qquad \vec{w} = -5\vec{i}-3\vec{j}$ $\qquad \vec{r} = -4\vec{j}$ $\qquad \vec{u} = 7\vec{i}$

a. $\vec{v}+2\vec{w}$ b. $\frac{1}{2}\vec{v}-\vec{w}$ c. $3\vec{v}-2\vec{r}+\vec{w}$ d. $-\vec{w}+\vec{r}-\vec{u}$

Solution:

a. $\vec{v}+2\vec{w} = 6\vec{i}+2\vec{j}+2(-5\vec{i}-3\vec{j}) = -4\vec{i}-4\vec{j}$

b. $\frac{1}{2}\vec{v}-\vec{w} = \frac{1}{2}(6\vec{i}+2\vec{j})-(-5\vec{i}-3\vec{j}) = 8\vec{i}+4\vec{j}$

c. $3\vec{v}-2\vec{r}+\vec{w} = 3(6\vec{i}+2\vec{j})-2(-4\vec{j})+(-5\vec{i}-3\vec{j}) = 13\vec{i}+11\vec{j}$

d. $-\vec{w}+\vec{r}-\vec{u} = -(-5\vec{i}-3\vec{j})+(-4\vec{j})-7\vec{i} = -2\vec{i}-\vec{j}$

Exercises

1. Find each of the following.

a. Find another vector, $\vec{B}$, that is in the opposite direction as $\vec{W} = -6\vec{i}+3\vec{j}$ and is only one unit in length.

b. Find another vector, $\overrightarrow{P}$, that is twice the length of the vector $\overrightarrow{K} = \langle 2, -1 \rangle$ but in the opposite direction.

2. To travel to New York City, New York, from Boston, Massachusetts, a pilot needs to fly at heading of E 52.4° N (*Source*: www.aeroplanner.com). Suppose an airplane is flying the route at a speed of 540 miles per hour at a fixed altitude with no wind factor. As the plane reaches a certain point, it encounters wind with a velocity of 35 miles per hour in the direction E 10° N.

a. Express the velocity of the wind, $\overrightarrow{W}$, and the velocity of the airplane, $\overrightarrow{A}$, as a vector.

b. What is the resultant velocity of the plane? Round to the nearest tenth.

c. What is the resultant speed of the plane? Round to the nearest tenth and include units.

d. What is the resultant direction of the plane? That is, what is the measure of the angle that the resultant vector forms with the positive x-axis?

Exercise Solutions

1. a. $\overrightarrow{B} = \frac{6}{\sqrt{45}}\vec{i} - \frac{3}{\sqrt{45}}\vec{j} = \frac{6}{3\sqrt{5}}\vec{i} - \frac{3}{3\sqrt{5}}\vec{j}$

b. $\overrightarrow{P} = -2\langle 2, -1 \rangle = \langle -4, 2 \rangle$

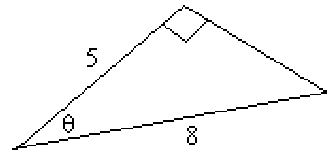
2. a. $\overrightarrow{W} = \langle 540\cos 37.6^\circ, 540\sin 37.6^\circ \rangle$ and $\overrightarrow{A} = \langle 35\cos 80^\circ, 35\sin 80^\circ \rangle$

b. $\overrightarrow{W} + \overrightarrow{A} = \langle 540\cos 37.6^\circ, 540\sin 37.6^\circ \rangle + \langle 35\cos 80^\circ, 35\sin 80^\circ \rangle \approx \langle 433.9, 363.9 \rangle$

c. $\left|\overrightarrow{W} + \overrightarrow{A}\right| = \sqrt{(433.9)^2 + (363.9)^2} \approx 566.3$ MPH

d. $\tan^{-1}\frac{363.9}{433.9} \approx 40^\circ$ or E 50° N

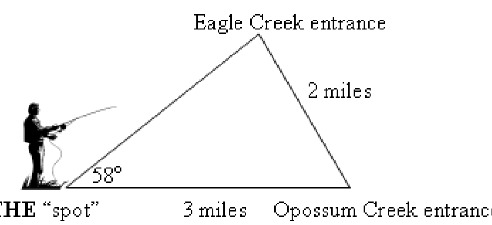
Chapter 9 Exam A



1. Evaluate the six trigonometric functions of θ. That is, evaluate $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$.

2. Two lighthouses mark a marine pass between two piers in Le Palais, Belle Isle, France. Assume the two lighthouse beacons are 200 feet apart. If the angle to the lighthouses from the ship is $10°$ and the ship is midway between the two lighthouses, find the distance between the ship and either lighthouse.

3. Joe has a favorite fishing spot where he always seems to have great luck catching the "big one". Since he is your close friend, Joe decided to leave you a map to the spot using familiar landmarks to help you find it.



   a. Explain why Joe must have been telling you a "fish story" (a lie).

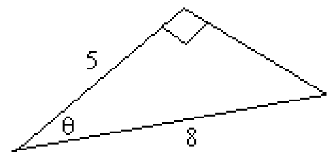
   b. Find the two possible distances between "the spot" and the Eagle Creek entrance using the correct angle, 28^o instead of the angle Joe told you.

4. a. Convert the Cartesian coordinates $(2, -\sqrt{3})$ to polar coordinates.

   b. Convert the polar coordinates $(6, \frac{2\pi}{3})$ to Cartesian coordinates.

Chapter 9 Exam A Solutions

1. Evaluate the six trigonometric functions of θ. That is, evaluate $\sin\theta$, $\cos\theta$, $\tan\theta$, $\sec\theta$, $\csc\theta$, $\cot\theta$.



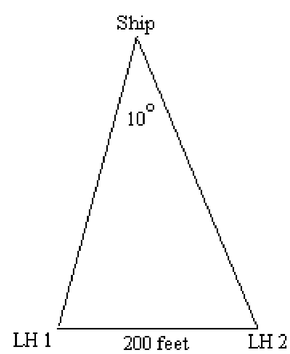
Using the Pythagorean Theorem, we find the length of the third side of the triangle.

$$w^2 = 64 - 25 = 39$$
$$w = \sqrt{39}$$

$$\sin\theta = \frac{\sqrt{39}}{8} \qquad \cos\theta = \frac{5}{8} \qquad \tan\theta = \frac{\sqrt{39}}{5}$$

$$\csc\theta = \frac{1}{\frac{\sqrt{39}}{8}} = \frac{8}{\sqrt{39}} = \frac{8\sqrt{39}}{39} \qquad \sec\theta = \frac{1}{\frac{5}{8}} = \frac{8}{5} \qquad \cot\theta = \frac{5}{\sqrt{39}} = \frac{5\sqrt{39}}{39}$$

2. Two lighthouses mark a marine pass between two piers in Le Palais, Belle Isle, France. Assume the two lighthouse beacons are 200 feet apart. If the angle to the lighthouses from a ship coming into shore is $10°$ and the ship is midway between the two lighthouses, find the distance between the ship and either lighthouse.



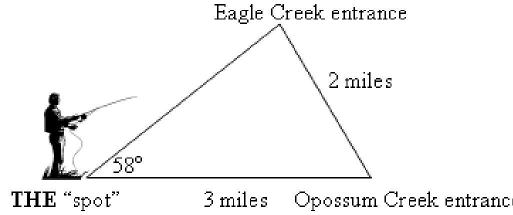
Not to scale

We begin by sketching a picture to represent the situation. We are given the fact that the triangle is isosceles. Therefore, we know each base angle is 85°. Using the Law of Sines, we find the required distance.

$$\frac{200}{\sin 10^\circ} = \frac{d}{\sin 85^\circ}$$
$$d = \frac{200 \sin 85^\circ}{\sin 10^\circ}$$
$$d \approx 1147.4$$

The distance between the ship and either lighthouse is 1147.4 feet.

3. Joe has a favorite fishing spot where he always seems to have great luck catching the "big one." Since he is your close friend, Joe decided to leave you a map to the spot using familiar landmarks to help you find it.



a. Explain why Joe must have been telling you a "fish story" (a lie).

We use the Law of Sines to show that it is impossible for the given triangle to exist.

$$\frac{2}{\sin 58^\circ} = \frac{3}{\sin E}$$

$$\sin E = \frac{3\sin 58^\circ}{2}$$

$$E = \sin^{-1}\left(\frac{3\sin 58^\circ}{2}\right) \approx \sin^{-1}(1.27)$$

It is impossible to compute $\sin^{-1}(1.27)$ so we know that Joe must be telling a "fish story."

b. Find the two possible distances between "the spot" and the Eagle Creek entrance using the correct angle, 28°, instead of the angle Joe told you.

$$\frac{2}{\sin 28^\circ} = \frac{3}{\sin E}$$

$$\sin E = \frac{3\sin 28^\circ}{2}$$

$$E = \sin^{-1}\left(\frac{3\sin 28^\circ}{2}\right)$$

$$\approx 44.8^\circ \text{ or } 135.2^\circ$$

The angle at Opossum Creek entrance is either 107.2° or 16.8°.

$$\frac{2}{\sin 28^\circ} = \frac{d}{\sin 107.2^\circ} \qquad d = \frac{2\sin 107.2^\circ}{\sin 28^\circ} \approx 4.07$$

or

$$\frac{2}{\sin 28^\circ} = \frac{d}{\sin 16.8^\circ} \qquad d = \frac{2\sin 16.8^\circ}{\sin 28^\circ} \approx 1.23$$

The two possible distances are 4.07 miles and 1.23 miles.

4. a. Convert the Cartesian coordinates $(2,-\sqrt{3})$ to polar coordinates.

$$(2,-\sqrt{3}) \Rightarrow (2^2 + (-\sqrt{3})^2, \tan^{-1}\frac{-\sqrt{3}}{2}) \Rightarrow (7,-0.714) \text{ or } (7,-40.89^\circ)$$

b. Convert the polar coordinates $(6,\frac{2\pi}{3})$ to Cartesian coordinates.

$$(6,\frac{2\pi}{3}) \Rightarrow (6\cos\frac{2\pi}{3}, 6\sin\frac{2\pi}{3}) \Rightarrow (-3,3\sqrt{3})$$

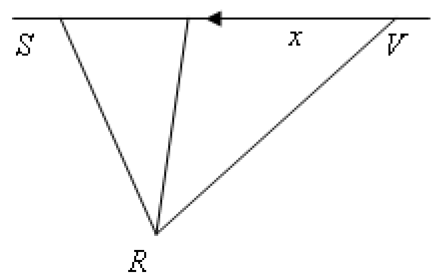
Chapter 9 Exam B

1. The Burj Dubai, currently under construction in the city of Dubai in the United Arab Emirates, will be the tallest building in the world when it is completed. While its ultimate height is being kept secret, it is projected to be approximately 800 meters in height upon completion. (www.newscientist.com). On June 24, 2007, the height of the building reached 493 meters (www.burjdubaiskyscraper.com). Suppose you are standing 500 meters from the base of the tower.

   a. Compute the angle of elevation to the top of the tower (493 meters).

   b. Suppose that after the tower is completed, you went back to the same position, 500 meters from the base of the tower. What will the angle of elevation to the top of the tower be at that time? (Assume the final height will be 800 meters.)

2. The figure shows a vehicle moving due west and approaching a 30-watt transmitter on a 30 meter tall radio tower. The effective range of the transmitter is 15 kilometers. To the nearest kilometer, for what distance x will the vehicle be within range of the transmitter at R if, when the car is at V, the measure of $\angle SVR = 43°$, and the distance from V to R is 21 kilometers.



3. Graph the polar equation $r = \cos(5\theta) + 5$ using a graphing calculator. Choose a viewing window so that you obtain a "good" result.

4. To travel to New York City, New York, from Boston, Massachusetts, a pilot needs to fly at heading of E 52.4° N (*Source*: www.aeroplanner.com). Suppose an airplane is flying the route at a speed of 540 miles per hour at a fixed altitude with no wind factor.

   a. The velocity of the plane may be considered a vector. Explain why it is a vector and describe the two important parts of the vector.

   b. Express the velocity of the plane, $\vec{A}$, as a vector in component form.

   c. Find the magnitude of $\vec{A}$ and describe its significance in the context of the problem situation. Show all work.

Chapter 9 Exam B Solutions

1. The Burj Dubai, currently under construction in the city of Dubai in the United Arab Emirates, will be the tallest building in the world when it is completed. While its ultimate height is being kept secret, it is projected to be approximately 800 meters in height upon completion. (www.newscientist.com). On June 24, 2007, the height of the building reached 493 meters (www.burjdubaiskyscraper.com). Suppose you are standing 500 meters from the base of the tower.

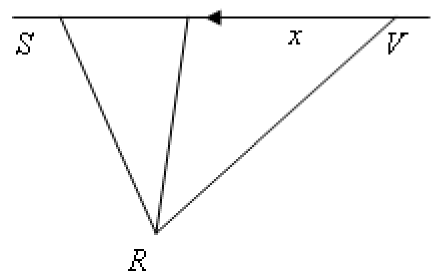
   a. Compute the angle of elevation to the top of the tower (493 meters).

$$\theta = \tan^{-1}\frac{493}{500} \approx 44.6^\circ$$

   b. Suppose that after the tower is completed, you went back to the same position, 500 meters from the base of the tower. What will the angle of elevation to the top of the tower be at that time? (Assume the final height will be 800 meters.)

$$\theta = \tan^{-1}\frac{800}{500} \approx 58^\circ$$

2. The figure shows a vehicle moving due west and approaching a 30-watt transmitter on a 30 meter tall radio tower. The effective range of the transmitter is 15 kilometers. To the nearest kilometer, for what distance x will the vehicle be within range of the transmitter at R if, when the car is at V, the measure of $\angle SVR = 43°$, and the distance from V to R is 21 kilometers.



We need to determine the value of x when the distance from R to the vehicle is 15 miles. We do this using the Law of Cosines.

$$\begin{aligned}
15^2 &= x^2 + 21^2 - 21x\cos(43^\circ) \\
0 &= x^2 + 21^2 - 21x\cos(43^\circ) - 15^2 \\
0 &= x^2 - 21\cos(43^\circ)x - 216 \\
0 &= x^2 - 15.358x - 216 \\
x &= \frac{15.358 \pm \sqrt{15.358^2 - 4(1)(-216)}}{2(1)} \\
x &\approx 24.26, -8.903
\end{aligned}$$

Using the practical result, the vehicle will be in range of the transmitter when $x \approx 24.26$ kilometers.

3. Graph the polar equation $r = \cos(5\theta) + 5$ using a graphing calculator. Choose a viewing window so that you obtain a "good" result.

We use the following window to get a "good" result.

$\theta \min : 0$

$\theta \max : 2\pi$

θ step : 0.1

X min : -6

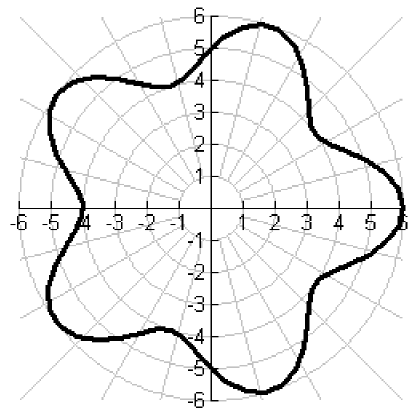
X max : 6

X step: 1

Y min : -6

Y max : 6

Y step: 1



4. To travel to New York City, New York, from Boston, Massachusetts, a pilot needs to fly at heading of E 52.4° N (*Source*: www.aeroplanner.com). Suppose an airplane is flying the route at a speed of 540 miles per hour at a fixed altitude with no wind factor.

a. The velocity of the plane may be considered a vector. Explain why it is a vector and describe the two important parts of the vector.

A vector quantity has both magnitude and direction. In this case, the velocity of the plane has magnitude reported as a speed and a direction given as the heading of the plane.

b. Express the velocity of the plane, $\vec{A}$, as a vector in component form.

$$\vec{A} = \left\langle 540\cos 37.6^{\circ}, 540\sin 37.6^{\circ} \right\rangle \approx \left\langle 427.84, 329.48 \right\rangle$$

c. Find the magnitude of $\vec{A}$ and describe its significance in the context of the problem situation. Show all work.

$$\left|\vec{A}\right| = \sqrt{(540\cos 37.6^{\circ})^2 + (540\sin 37.6^{\circ})^2} \approx \sqrt{427.84^2 + 329.48^2}$$
$$\approx 540$$

The magnitude of $\vec{A}$ corresponds to the speed of the airplane which is 540 miles per hour. The computations show this to be true.

Trigonometric Identities

10

10.1 Basic Identities

Objectives

- Use the fundamental trig identities.
- Use the even-odd function identities.
- Use the cofunction identities.
- Use the Pythagorean identities.

Concepts and Definitions

- **Identity:** Two variable expressions that are always equivalent are said to form a mathematical identity.
- **Fundamental Trigonometric Identities:** $\tan(\theta)=\dfrac{\sin(\theta)}{\cos(\theta)}$, $\cot(\theta)=\dfrac{\cos(\theta)}{\sin(\theta)}$,

  $\sec(\theta)=\dfrac{1}{\cos(\theta)}$, $\csc(\theta)=\dfrac{1}{\sin(\theta)}$
- **Even-Odd Identities:** Cosine and secant functions are even functions giving us the following identities: $\cos(-\theta)=\cos(\theta)$ and $\sec(-\theta)=\sec(\theta)$. Sine, tangent, cotangent, and cosecant are odd functions, giving us the following identities:
  - $\sin(-\theta)=-\sin(\theta)$ $\quad\tan(-\theta)=-\tan(\theta)$
    $\csc(-\theta)=-\csc(\theta)$ $\quad\cot(-\theta)=-\cot(\theta)$
- **Cofunction Identities:**

θ in degrees	θ in radians
$\cos(\theta)=\sin(90°-\theta)$	$\cos(\theta)=\sin\left(\dfrac{\pi}{2}-\theta\right)$
$\sin(\theta)=\cos(90°-\theta)$	$\sin(\theta)=\cos\left(\dfrac{\pi}{2}-\theta\right)$
$\tan(\theta)=\cot(90°-\theta)$	$\tan(\theta)=\cot\left(\dfrac{\pi}{2}-\theta\right)$
$\sec(\theta)=\csc(90°-\theta)$	$\sec(\theta)=\csc\left(\dfrac{\pi}{2}-\theta\right)$
$\csc(\theta)=\sec(90°-\theta)$	$\csc(\theta)=\sec\left(\dfrac{\pi}{2}-\theta\right)$
$\cot(\theta)=\tan(90°-\theta)$	$\cot(\theta)=\tan\left(\dfrac{\pi}{2}-\theta\right)$

- **Pythagorean Identities:**
  $\cos^2(\theta)+\sin^2(\theta)=1,\ 1+\tan^2(\theta)=\sec^2(\theta),\ \cot^2(\theta)+1=\csc^2(\theta)$

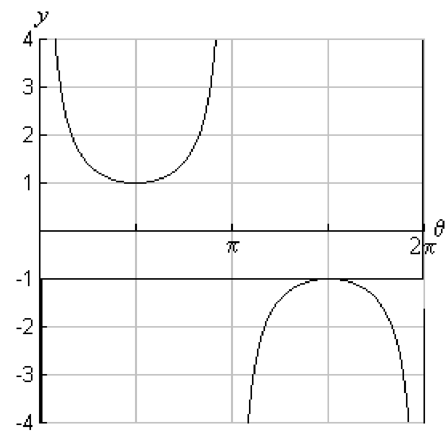
Examples

- **Example 1: Using Fundamental Identities**
  Use fundamental trigonometric identities to simplify the expression $\sec x \cdot \cot x$ by writing it as a single trigonometric function or as a whole number.

  Solution:
  We apply the fundamental identities to simplify the expression.
  $$\sec x \cdot \cot x = \frac{1}{\cos x} \cdot \frac{\cos x}{\sin x} = \frac{1}{\sin x} = \csc x$$

- **Example 2: Using Identities to Solve Equations**
  Use trigonometric identities to solve the equation
  $\tan(\frac{\pi}{2} - \theta)\cot(\theta) - \csc^2(\theta) = \csc(\theta)$. Find all solutions over the interval $[0,\ 2\pi]$.
  Verify your solutions by graphing.

  Solution:
  $$\tan(\frac{\pi}{2} - \theta)\cot(\theta) - \csc^2(\theta) = \csc(\theta)$$
  $$\cot(\theta)\cot(\theta) - \csc^2(\theta) = \csc(\theta)$$
  $$\cot^2(\theta) - \csc^2(\theta) = \csc(\theta)$$
  $$-1 = \csc(\theta)$$
  $$-1 = \frac{1}{\sin(\theta)}$$
  $$\sin(\theta) = -1$$
  $$\theta = \frac{3\pi}{2}$$



  As we see in the solution process, the left side of the equation is equivalent to $y = -1$. *The function* $y = \csc(\theta)$ *intersects* $y = -1$ *when* $\theta = \frac{3\pi}{2}$.

Exercises

1. Use trigonometric identities to simplify the expression $\left(\sin(\theta) + \cos(\theta)\right)^2 + \left(\sin(\theta) - \cos(\theta)\right)^2$ by writing it in terms of a single trigonometric function or as a whole number.

2. Use trigonometric identities to solve the equation $\sin(\theta)\cos(\theta)\tan(\theta) + \cos^2(\theta) = -2\tan(\theta)$. Find all solutions over the interval $[0,\ 2\pi]$. Verify your solutions by graphing.

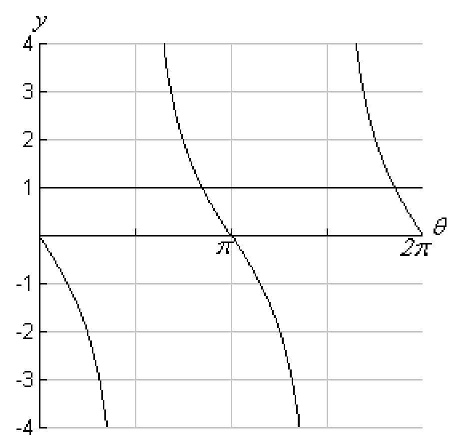
Exercise Solutions

1.

$$
\begin{aligned}
&\left(\sin(\theta)+\cos(\theta)\right)^2+\left(\sin(\theta)-\cos(\theta)\right)^2\\
&=\sin^2(\theta)+2\sin(\theta)\cos(\theta)+\cos^2(\theta)+\sin^2(\theta)-2\sin(\theta)\cos(\theta)+\cos^2(\theta)\\
&=\sin^2(\theta)+\cos^2(\theta)+\sin^2(\theta)+\cos^2(\theta)\\
&=1+1\\
&=2
\end{aligned}
$$

2.

$$
\begin{aligned}
\sin(\theta)\cos(\theta)\tan(\theta)+\cos^2(\theta)&=-2\tan(\theta)\\
\sin(\theta)\cos(\theta)\frac{\sin(\theta)}{\cos(\theta)}+\cos^2(\theta)&=-2\tan(\theta)\\
\sin^2(\theta)+\cos^2(\theta)&=-2\tan(\theta)\\
1&=-2\tan(\theta)\\
\tan(\theta)&=-\frac{1}{2}\\
\theta&=\tan^{-1}(-\frac{1}{2})\\
\theta&\approx 2.68,\ 5.82\text{ radians}
\end{aligned}
$$



As we see in the solution process, the left side of the equation is equivalent to $y=1$. The function $y=-2\tan(\theta)$ intersects $y=1$ when $\theta\approx 2.68$ and 5.82.

10.2 Verifying Identities

Objectives

- Explain how trigonometric identities are determined.
- Use algebraic manipulation to verify the equivalency of two identities.
- Use trigonometric identities to describe equivalent relationships.

Concepts and Definitions

- **Verifying Trigonometric Identities:**
    - Simplify one side at a time. It is often best to work with the side that appears more complex. Always stay on one side of the equation at a time.
    - Combine fractions on the same side using addition, subtraction, multiplication, or division as necessary.
    - If any of the trigonometric functions are being squared, you may be able to use a Pythagorean identity to simplify the expression. If there are no exponents, try writing all trigonometric functions in terms of cosine and sine.
    - Keep your goal in mind. Sometimes looking at the less complex side of the potential identity will give you a clue about what identities will help you get there.
    - If one approach does not work, try another. Oftentimes, it only takes one substitution or one key step to unlock the entire identity.

Examples

- **Example 1: Verifying Identities**

Verify that $\dfrac{1+\tan(\theta)}{\sin(\theta)+\cos(\theta)} = \sec(\theta)$ is an identity.

Solution:

$$\frac{1+\tan(\theta)}{\sin(\theta)+\cos(\theta)} = \sec(\theta)$$
$$1+\tan(\theta) = \sec(\theta)(\sin(\theta)+\cos(\theta))$$
$$1+\tan(\theta) = \sec(\theta)\sin(\theta)+\sec(\theta)\cos(\theta)$$
$$1+\tan(\theta) = \frac{1}{\cos(\theta)}\sin(\theta)+\frac{1}{\cos(\theta)}\cos(\theta)$$
$$1+\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}+1$$
$$1+\tan(\theta) = 1+\tan(\theta)$$

Example 2: Verifying Identities

Verify that $\dfrac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\sin(\theta)-\cos(\theta)$ is an identity.

Solution:

$$\frac{\sin^2(-\theta)-\cos^2(-\theta)}{\sin(-\theta)-\cos(-\theta)}=\sin(\theta)-\cos(\theta)$$
$$\frac{(\sin(-\theta)-\cos(-\theta))(\sin(-\theta)+\cos(-\theta))}{\sin(-\theta)-\cos(-\theta)}=\sin(\theta)-\cos(\theta)$$
$$\sin(-\theta)+\cos(-\theta)=\cos(\theta)-\sin(\theta)$$
$$-\sin(\theta)+\cos(\theta)=\cos(\theta)-\sin(\theta)$$
$$\cos(\theta)-\sin(\theta)=\cos(\theta)-\sin(\theta)$$

Exercises

1. Verify that $\sin(\phi)=\dfrac{\cos\left(\frac{\pi}{2}-\phi\right)+\sec(\phi)\sin(\phi)}{\sec(\phi)+1}$ is an identity.

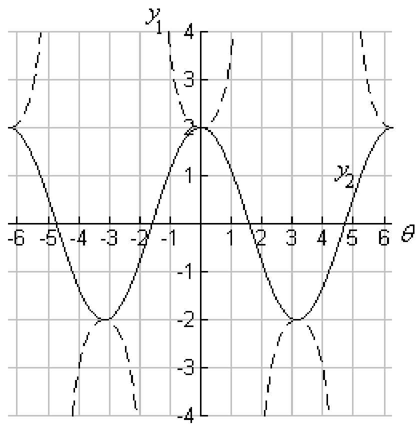
2. Use a graph to determine if it is likely that the trigonometric equation $\csc(\theta)\tan(\theta)+\sec(\theta)=2\cos(\theta)$ is an identity. If it appears to be an identity, use algebra to verify. If it does not appear to be an identity, provide a counterexample to demonstrate that the two expressions are not equivalent.

Exercise Solutions

1.

$$\sin(\phi)=\frac{\cos\left(\frac{\pi}{2}-\phi\right)+\sec(\phi)\sin(\phi)}{\sec(\phi)+1}$$
$$\sin(\phi)=\frac{\sin(\phi)+\sec(\phi)\sin(\phi)}{\sec(\phi)+1}$$
$$\sin(\phi)=\frac{\sin(\phi)(1+\sec(\phi))}{\sec(\phi)+1}$$
$$\sin(\phi)=\sin(\phi)$$

2. We begin by graphing $y_1=\csc(\theta)\tan(\theta)+\sec(\theta)$ and $y_2=2\cos(\theta)$.



The graph shows that the functions $y_1 = \csc(\theta)\tan(\theta) + \sec(\theta)$ and $y_2 = 2\cos(\theta)$ are not equivalent and therefore, $\csc(\theta)\tan(\theta) + \sec(\theta) = 2\cos(\theta)$ is not an identity.

We can choose a value for θ to create a counterexample. If $\theta = 1$ radian , then $y_1 = \csc(1)\tan(1) + \sec(1) \approx 3.70$ and $y_2 = 2\cos(1) \approx 1.08$. This again shows that trigonometric equation $\csc(\theta)\tan(\theta) + \sec(\theta) = 2\cos(\theta)$ is not an identity.

10.3 Other Identities

Objectives

- Use the sum and difference identities.
- Use sum to product and product-to-sum identities.
- Use double and half-angle identities.

Concepts and Definitions

- **Sum and Difference Identities:**
  - $\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$ and $\cos(\theta-\phi)=\cos(\theta)\cos(\phi)+\sin(\theta)\sin(\phi)$
  - $\sin(\theta+\phi)=\sin(\theta)\cos(\phi)+\cos(\theta)\sin(\phi)$ and $\sin(\theta-\phi)=\sin(\theta)\cos(\phi)-\cos(\theta)\sin(\phi)$
  - $\tan(\theta+\phi)=\dfrac{\tan(\theta)+\tan(\phi)}{1-\tan(\theta)\tan(\phi)}$ and $\tan(\theta-\phi)=\dfrac{\tan(\theta)-\tan(\phi)}{1+\tan(\theta)\tan(\phi)}$
- **Sum-to-Product Identities:**
  - $\cos(\theta)+\cos(\phi)=2\cos\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$ and $\cos(\theta)-\cos(\phi)=-2\sin\left(\dfrac{\theta+\phi}{2}\right)\sin\left(\dfrac{\theta-\phi}{2}\right)$
  - $\sin(\theta)+\sin(\phi)=2\sin\left(\dfrac{\theta+\phi}{2}\right)\cos\left(\dfrac{\theta-\phi}{2}\right)$ and $\sin(\theta)-\sin(\phi)=2\sin\left(\dfrac{\theta-\phi}{2}\right)\cos\left(\dfrac{\theta+\phi}{2}\right)$
- **Product-to-Sum Identities:**
  - $\cos(\theta)\cos(\phi)=\dfrac{\cos(\theta+\phi)+\cos(\theta-\phi)}{2}$
  - $\sin(\theta)\sin(\phi)=\dfrac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}$
  - $\sin(\theta)\cos(\phi)=\dfrac{\sin(\theta+\phi)+\sin(\theta-\phi)}{2}$
- **Double Angle Identities**:
  - $\cos(2\theta)=\cos^2(\theta)-\sin^2(\theta)=2\cos^2\theta-1=1-2\sin^2\theta$
  - $\sin(2\theta)=2\sin(\theta)\cos(\theta)$
  - $\tan(2\theta)=\dfrac{2\tan(\theta)}{1-\tan^2(\theta)}$
- **Half Angle Identities:**
  - $\cos\left(\dfrac{\theta}{2}\right)=\pm\sqrt{\dfrac{1+\cos(\theta)}{2}}$ and $\sin\left(\dfrac{\theta}{2}\right)=\pm\sqrt{\dfrac{1-\cos(\theta)}{2}}$
  - $\tan\left(\dfrac{\theta}{2}\right)=\dfrac{1-\cos(\theta)}{\sin(\theta)}$ or $\tan\left(\dfrac{\theta}{2}\right)=\dfrac{\sin(\theta)}{1+\cos(\theta)}$

- The quadrant where $\frac{\theta}{2}$ falls determines whether the positive or negative square root is used.

Examples

- **Example 1: Using the Sum or Difference Identities**
  Find the exact value of the expression $\cos(165°)$ using sum and difference identities.

  Solution:
  The measure 165° does not appear on our table of exact values for trigonometric function. In order to find $\cos(165°)$*, we can to write 165° as the sum of two angles whose exact trigonometric values we know. In this case, we use* $45°+120°$.

$$\cos(\theta+\phi)=\cos(\theta)\cos(\phi)-\sin(\theta)\sin(\phi)$$
$$\cos(45°+120°)=\cos(45^\circ)\cos(120°)-\sin(45^\circ)\sin(120°)$$
$$\cos(165°)=\frac{\sqrt{2}}{2}\cdot(-\frac{1}{2})-\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{3}}{2}$$
$$\cos(165°)=-\frac{\sqrt{2}}{4}-\frac{\sqrt{6}}{4}$$
$$\cos(15°)=\frac{-\sqrt{2}-\sqrt{6}}{4}$$

  We may check our work. Using a calculator, $\frac{-\sqrt{2}-\sqrt{6}}{4}\approx-0.9659$ *and* $\cos(165°)\approx-0.9659$.

- **Example 2: Using the Sum-to-Product Identities**
  Use the sum-to-product identities to find the exact value of the expression $\sin(255°)-\sin(15°)$.

  Solution:

$$\sin(\theta)-\sin(\phi)=2\sin\left(\frac{\theta-\phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right)$$
$$\sin(225^\circ)-\sin(15^\circ)=2\sin\left(\frac{225^\circ-15^\circ}{2}\right)\cos\left(\frac{225^\circ+15^\circ}{2}\right)$$
$$=2\sin(105^\circ)\cos(120^\circ)$$

$$\begin{aligned}\sin(255^\circ)-\sin(15^\circ) &= 2\sin\left(105^\circ\right)\cos\left(120^\circ\right)\\ &= 2\sin\left(60^\circ+45^\circ\right)\cos\left(120^\circ\right)\\ &= 2(\sin(60^\circ)\cos(45^\circ)+\cos(60^\circ)\sin(45^\circ))\cos(120^\circ)\\ &= 2((\frac{\sqrt{3}}{2}\cdot\frac{\sqrt{2}}{2}+\frac{1}{2}\frac{\sqrt{2}}{2}))(-\frac{1}{2})\\ &= -\frac{\sqrt{6}+\sqrt{2}}{4}\end{aligned}$$

Exercises

1. Use double angle identities to solve the equation $-6\sin(\theta)\cos(\theta)=2$. Find all solutions over the interval $\left[0,\ 2\pi\right]$.

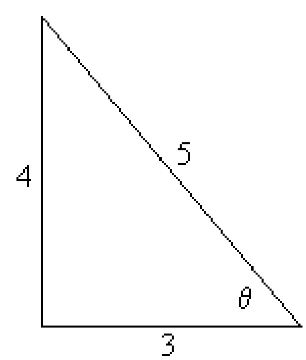
2. Given that $0<\theta<\frac{\pi}{2}$ and $\sin\left(\theta\right)=\frac{4}{5}$, find the exact value for expressions $\tan(\frac{\theta}{2})$.

Exercise Solutions

1.

$$\begin{aligned}-6\sin(\theta)\cos(\theta) &= 2\\ -3(2\sin(\theta)\cos(\theta)) &= 2\\ -3\sin(2\theta) &= 2\\ \sin(2\theta) &= -\frac{2}{3}\\ 2\theta &= \sin^{-1}(-\frac{2}{3})\\ \theta &= \frac{\sin^{-1}(-\frac{2}{3})}{2}\approx 1.936, 2.777, 5.077, 5.918\end{aligned}$$

2. Since $\sin\left(\theta\right)=\frac{4}{5}$, we can sketch a triangle with this characteristic and find $\cos\left(\theta\right)=\frac{3}{5}$ by using the Pythagorean Theorem.



$$\tan(\frac{\theta}{2})=\frac{1-\cos\left(\theta\right)}{\sin\left(\theta\right)}=\frac{1-\frac{3}{5}}{\frac{4}{5}}=\frac{\frac{2}{5}}{\frac{4}{5}}=\frac{2}{4}=\frac{1}{2}$$

Chapter 10 Exam A

1. Use fundamental trigonometric identities to simplify each expression by writing it as a single trigonometric function or as a whole number.

   a. $\tan(x)\csc(x)\sec(x)$

   b. $\dfrac{\sec(x)}{\tan(x)}$

2. Use trigonometric identities to solve each equation. Find all solutions over the interval $[0, 2\pi]$.

   a. $\tan(y)\cos(y) = \sin^2(y)$

   b. $-1 = \cos(\theta)\cot\left(\dfrac{\pi}{2} - \theta\right)$

3. Verify the trigonometric identity $\sin\left(\dfrac{\pi}{2} - \theta\right)\tan(-\theta) = -\sin(\theta)$.

4. Use a graph to determine if it is likely that the trigonometric equation $\sin^2(\theta)\sec^2(\theta) - \sin^2(\theta) = \sin^4(\theta)\csc^2(\theta)$ is an identity. If it appears to be an identity, use algebra to verify. If it does not appear to be an identity, provide a counterexample to demonstrate that the two expressions are not equivalent.

Chapter 10 Exam A Solutions

1. Use fundamental trigonometric identities to simplify each expression by writing it as a single trigonometric function or as a whole number.

a. $\tan(x)\csc(x)\sec(x)$

$$\frac{\sin(x)}{\cos(x)}\cdot\frac{1}{\sin(x)}\cdot\frac{1}{\cos(x)}=\frac{1}{\cos^2(x)}=\sec^2(x)$$

b. $\dfrac{\sec(x)}{\tan(x)}$

$$\frac{\dfrac{1}{\cos(x)}}{\dfrac{\sin(x)}{\cos(x)}}=\frac{1}{\cos(x)}\cdot\frac{\cos(x)}{\sin(x)}=\frac{1}{\sin(x)}=\csc(x)$$

2. Use trigonometric identities to solve each equation. Find all solutions over the interval $[0,\ 2\pi]$.

a. $\tan(y)\cos(y)=\sin^2(y)$

$$\frac{\sin(y)}{\cos(y)}\cdot\cos(y)=\sin^2(y)$$

$$\sin^2(y)-\sin(y)=0$$

$$\sin(y)(\sin(y)-1)=0$$

$$\sin(y)=0 \text{ or } \sin(y)=1$$

$$y=0,\pi,2\pi,\frac{\pi}{2}$$

b. $-1=\cos(\theta)\cot\left(\dfrac{\pi}{2}-\theta\right)$

$$-1=\cos(\theta)\tan(\theta)$$

$$-1=\cos(\theta)\cdot\frac{\sin(\theta)}{\cos(\theta)}$$

$$-1=\sin(\theta)$$

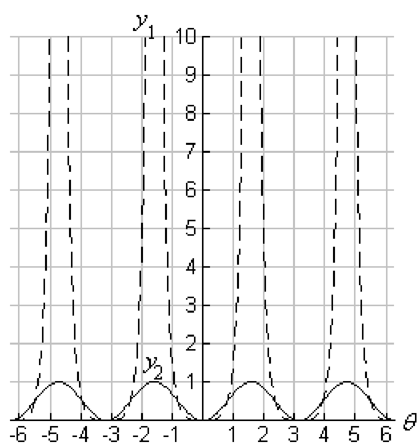
$$\theta=\frac{3\pi}{2}$$

3. Verify the trigonometric identity $\sin\left(\frac{\pi}{2}-\theta\right)\tan(-\theta)=-\sin(\theta)$.

$$\cos(\theta)(-\tan(\theta))=-\sin(\theta)$$
$$-\cos(\theta)\cdot\frac{\sin(\theta)}{\cos(\theta)}=-\sin(\theta)$$
$$-\sin(\theta)=-\sin(\theta)$$

4. Use a graph to determine if it is likely that the trigonometric equation $\sin^2(\theta)\sec^2(\theta)-\sin^2(\theta)=\sin^4(\theta)\csc^2(\theta)$ is an identity. If it appears to be an identity, use algebra to verify. If it does not appear to be an identity, provide a counterexample to demonstrate that the two expressions are not equivalent.

We begin by graphing $y_1=\sin^2(\theta)\sec^2(\theta)-\sin^2(\theta)$ and $y_2=\sin^4(\theta)\csc^2(\theta)$.



The graph shows that the functions $y_1=\sin^2(\theta)\sec^2(\theta)-\sin^2(\theta)$ and $y_2=\sin^4(\theta)\csc^2(\theta)$ are not equivalent and therefore, $\sin^2(\theta)\sec^2(\theta)-\sin^2(\theta)=\sin^4(\theta)\csc^2(\theta)$ is not an identity.

We can choose a value for θ to create a counterexample. If $\theta=1$ radian, then $y_1=\sin^2(1)\sec^2(1)-\sin^2(1)\approx 1.717$ and $y_2=\sin^4(1)\csc^2(1)\approx 0.708$. This again shows that trigonometric equation $\sin^2(\theta)\sec^2(\theta)-\sin^2(\theta)=\sin^4(\theta)\csc^2(\theta)$ is not an identity.

Chapter 10 Exam B

1. Use trigonometric identities to solve each equation. Find all solutions over the interval $\left[0,\ 2\pi\right]$.

   a. $1.4 = \dfrac{\cos(-\alpha)}{2} + 1$

   b. $\cos\left(\dfrac{\pi}{2} - \theta\right) = -0.61$

2. Find the exact value of each expression using sum and difference identities.

   a. $\tan\left(\dfrac{\pi}{12}\right)$

   b. $\sin\left(\dfrac{9\pi}{8}\right)$

3. Verify each identity using the trigonometric identities from this chapter.

   a. $\dfrac{\cos(8x) - \cos(2x)}{\cos(8x) + \cos(2x)} = -\tan(5x)\tan(3x)$

   b. $2 - \cos(\theta) = \cos^2\left(\dfrac{\theta}{2}\right) + 3\sin^2\left(\dfrac{\theta}{2}\right)$

4. Find the exact value of the trigonometric expression $\csc\left(\dfrac{\theta}{2}\right)$ using the given information.

$$\frac{3\pi}{2} < \theta < 2\pi \text{ and } \cos(\theta) = \frac{\sqrt{3}}{4}$$

Chapter 10 Exam B Solutions

1. Use trigonometric identities to solve each equation. Find all solutions over the interval $\left[0,\ 2\pi\right]$.

a. $1.4 = \dfrac{\cos\left(-\alpha\right)}{2} + 1$

$$\begin{aligned} 0.4 &= \frac{\cos(\alpha)}{2} \\ 0.8 &= \cos(\alpha) \\ \alpha &= \cos^{-1}(0.8) \\ \alpha &\approx 0.644, 5.640 \end{aligned}$$

b. $\cos\left(\dfrac{\pi}{2} - \theta\right) = -0.61$

$$\begin{aligned} \sin(\theta) &= -0.61 \\ \theta &= \sin^{-1}(-0.61) \\ \theta &\approx 5.627, 3.798 \end{aligned}$$

2. Find the exact value of each expression using sum and difference identities.

a. $\tan(\dfrac{\pi}{12})$

$$\tan(\frac{\pi}{12}) = \tan(\frac{1}{2}\cdot\frac{\pi}{6}) = \frac{1-\cos(\frac{\pi}{6})}{\sin(\frac{\pi}{6})} = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \frac{\frac{2-\sqrt{3}}{2}}{\frac{1}{2}} = 2-\sqrt{3}$$

b. $\sin(\dfrac{9\pi}{8})$

$$\sin(\frac{9\pi}{8}) = \sin(\frac{1}{2}\cdot\frac{9\pi}{4}) = \pm\sqrt{\frac{1-\cos(\frac{9\pi}{4})}{2}} = \pm\sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \pm\sqrt{\frac{2-\sqrt{2}}{4}} = \pm\frac{\sqrt{2-\sqrt{2}}}{2}$$

Because the angle $\dfrac{9\pi}{8}$ is located in quadrant III, the value of the sine function is negative. Therefore, $\sin(\dfrac{9\pi}{8}) = -\dfrac{\sqrt{2-\sqrt{2}}}{2}$.

3. Verify each identity using the trigonometric identities from this chapter.

a. $\dfrac{\cos(8x)-\cos(2x)}{\cos(8x)+\cos(2x)}=-\tan(5x)\tan(3x)$

$$\frac{-2\sin\left(\frac{8x+2x}{2}\right)\sin\left(\frac{8x-2x}{2}\right)}{2\cos\left(\frac{8x+2x}{2}\right)\cos\left(\frac{8x-2x}{2}\right)}=-\tan(5x)\tan(3x)$$

$$-\frac{\sin(5x)\sin(3x)}{\cos(5x)\cos(3x)}=-\tan(5x)\tan(3x)$$

$$-\tan(5x)\tan(3x)=-\tan(5x)\tan(3x)$$

b. $2-\cos(\theta)=\cos^2\left(\dfrac{\theta}{2}\right)+3\sin^2\left(\dfrac{\theta}{2}\right)$

$$2-\cos(\theta)=\frac{1+\cos(\theta)}{2}+3\cdot\frac{1-\cos(\theta)}{2}$$

$$2-\cos(\theta)=\frac{1+\cos(\theta)+3-3\cos(\theta)}{2}$$

$$2-\cos(\theta)=\frac{4-2\cos(\theta)}{2}$$

$$2-\cos(\theta)=\frac{2(2-\cos(\theta))}{2}$$

$$2-\cos(\theta)=2-\cos(\theta)$$

4. Find the exact value of the trigonometric expression $\csc\left(\dfrac{\theta}{2}\right)$ using the given information.

$$\frac{3\pi}{2}<\theta<2\pi \text{ and } \cos(\theta)=\frac{\sqrt{3}}{4}$$

We use the half-angle formula to get started. We are given that the angle is located in quadrant IV so:

$$\sin\left(\frac{\theta}{2}\right)=-\sqrt{\frac{1-\cos(\theta)}{2}}=-\sqrt{\frac{1-\frac{\sqrt{3}}{4}}{2}}=-\sqrt{\frac{\frac{4-\sqrt{3}}{4}}{2}}=-\sqrt{\frac{4-\sqrt{3}}{8}}$$

$$\frac{1}{\sin\left(\frac{\theta}{2}\right)}=\csc\left(\frac{\theta}{2}\right)=\sqrt{\frac{8}{4-\sqrt{3}}}$$

Conic Sections 11

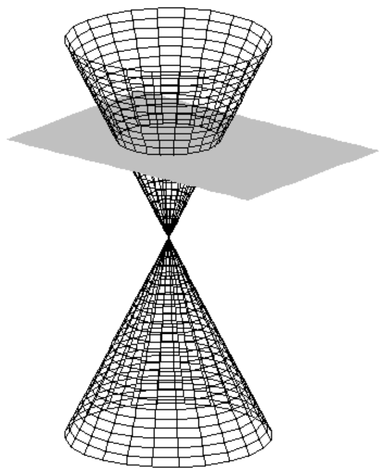
11.1 Slices of the Cone

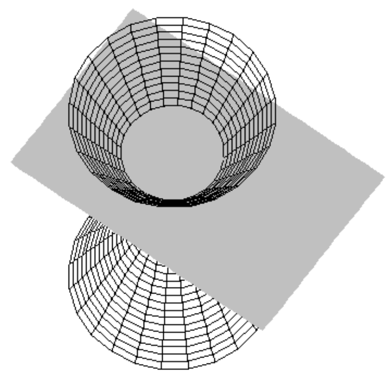
Objectives

- Explain how slices of a cone form the different conic sections: circle, ellipse, parabola, and hyperbola.

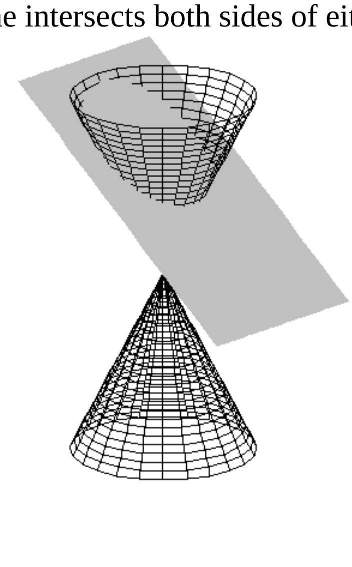
Concepts and Definitions

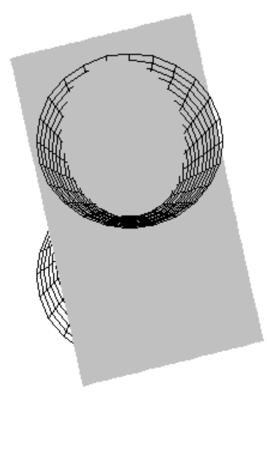
- **The Circle as a Conic Section:** A circle is formed when a horizontal plane intersects either of the cones (not at the vertex). The radius of the circle depends on the site of the intersection.



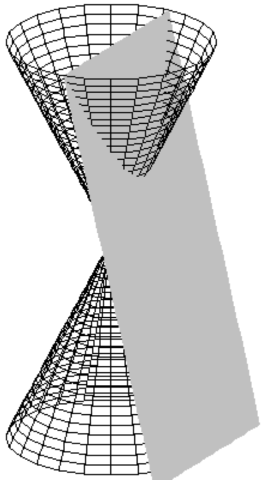
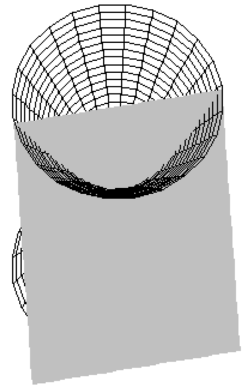


- **The Ellipse as a Conic Section:** An ellipse is formed when a non-horizontal plane intersects both sides of either cone.

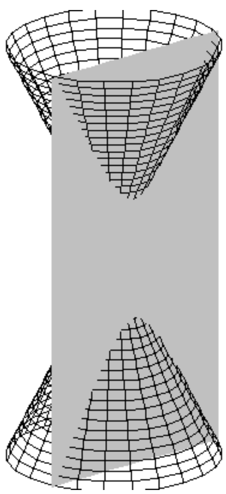
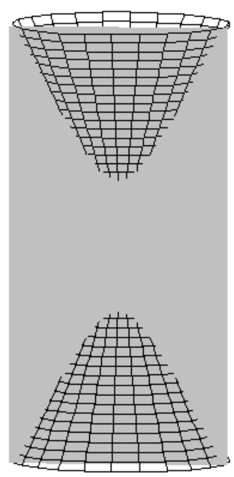




- **The Parabola as a Conic Section:** A parabola is formed when a plane intersects the top (or bottom) of a cone and a corresponding side.

- **The Hyperbola as a Conic Section:** A hyperbola, which consists of two parts, is formed when a vertical plane intersects both cones.

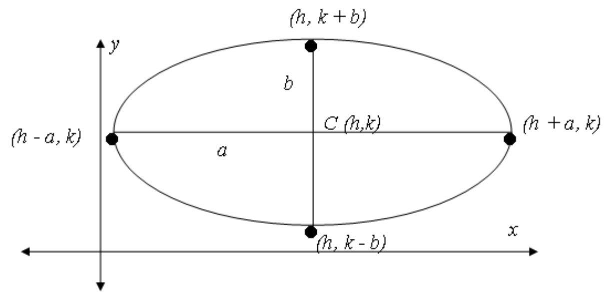
11.2 Ellipses and Circles

Objectives

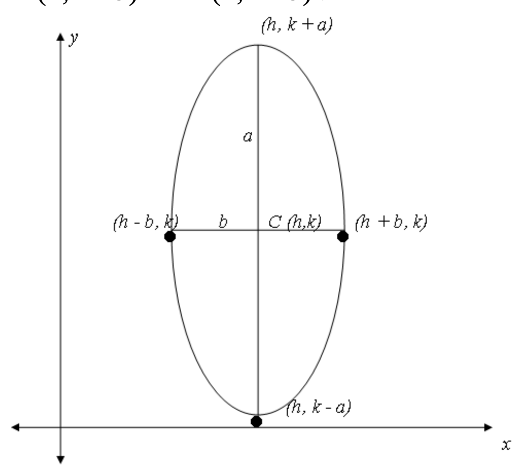
- Generate equations for ellipses and circles given a graph, a set of points, or words
- Apply ellipses and circles to real–world scenarios.

Concepts and Definitions

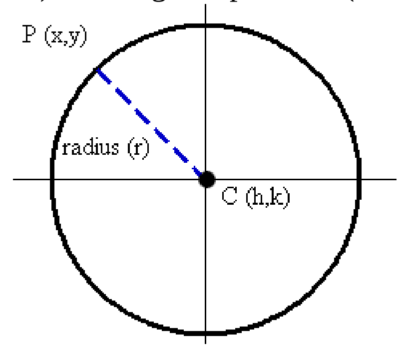
- **Ellipse:** An ellipse is the set of all points in a plane, the sum of whose distances d_1 and d_2 from two fixed points in the plane (the foci) is constant.
- **Ellipse: Major Axis Horizontal, Center at (*h*,*k*):** The standard equation of an ellipse with center at (h,k) and major axis horizontal
$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$
where $a>b>0$ and $b^2=a^2-c^2$.
    - The vertices of the ellipse are $(h+a,k)$ and $(h-a,k)$.
    - The endpoints of the minor axis are $(h,k+b)$ and $(h,k-b)$.



- **The Ellipse: Major Axis Vertical, Center at (*h*,*k*):** The standard equation of an ellipse with center at (h,k) and major axis vertical is
$$\frac{(x-h)^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$
where $a>b>0$ and $b^2=a^2-c^2$.
    - The vertices of the ellipse are $(h,k+a)$ and $(h,k-a)$.
    - The endpoints of the minor axis are $(h+b,k)$ and $(h-b,k)$.



- **Circle:** A circle is the set of all points in a plane at a fixed distance r (called the radius) from a given point C (called the center).
- **Standard Equation of the Circle:** The standard equation of a circle with center at (h,k) and radius r is
$(x-h)^2+(y-k)^2=r^2$.



Examples

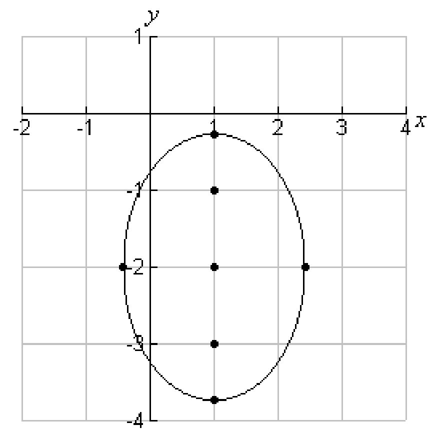
- **Example 1: Describing an Ellipse from its Equation**
  Determine the center, foci, vertices, and the endpoints of the minor axis of the ellipse $3x^2+2y^2-6x+8y+5=0$. Then graph the ellipse.

  Solution:
  We need to complete the square in order to get the equation in the standard form.

$$
\begin{aligned}
3x^2+2y^2-6x+8y+5&=0\\
3(x^2-2x)+2(y^2+4y)&=-5\\
3(x^2-2x+1)+2(y^2+4y+4)&=-5+3+8\\
3(x-1)^2+2(y+2)^2&=6\\
\frac{3(x-1)^2}{6}+\frac{2(y+2)^2}{6}&=\frac{6}{6}\\
\frac{(x-1)^2}{2}+\frac{(y+2)^2}{3}&=1
\end{aligned}
$$

  We know the center is $(1,-2)$, $a^2=3$, *and* $b^2=2$. *Therefore,* $a=\sqrt{3}$ *and* $b=\sqrt{2}$. *We find* c.

$$
\begin{aligned}
b^2&=a^2-c^2\\
2&=3-c^2\\
1&=c^2\\
1&=c
\end{aligned}
$$



  Therefore the foci are $(1,-1)$ *and* $(1,-3)$ *and the vertices are* $(1,-2+\sqrt{3})$ *and* $(1,-2-\sqrt{3})$. *The endpoints of the minor axis are* $(1+\sqrt{2},-2)$ *and* $(1-\sqrt{2},-2)$.

- ***Example 2:*** **Finding the Equation of a Circle by Completing the Square**
  Find the center and radius of the circle with the equation $x^2+y^2-6x+8y-7=0$.

  Solution:
  We begin by arranging the terms of the equation: $(x^2-6x)+(y^2+8y)=7$. *We complete the square to write the equation in standard form.*

$$
\begin{aligned}
(x^2-6x+9)+(y^2+8y+16)&=7+9+16\\
(x-3)^2+(y+4)^2&=32
\end{aligned}
$$

  The circle has the center $(3,-4)$ *and radius* $\sqrt{32}$.

Exercises

1. The Oval Office, the official office of the President of the United States, is located in the West Wing of the White House and is an ellipse. The major axis of the room is 35 feet 10 inches and the minor axis is 29 feet (*Source*: www.whitehouse.gov).

   a. Use these dimensions to determine the equation of the ellipse that models the office and determine the coordinates of the foci and vertices.

   b. Graph the ellipse.

2. A severe weather warning siren can be heard in a circular range represented by the equation $x^2 + y^2 = 42.25$. If a woman is 3.8 miles west and 4 miles south of the siren, can she hear the sound?

Exercise Solutions

1. We position the graph so that the origin is at $(0,0)$. We compute

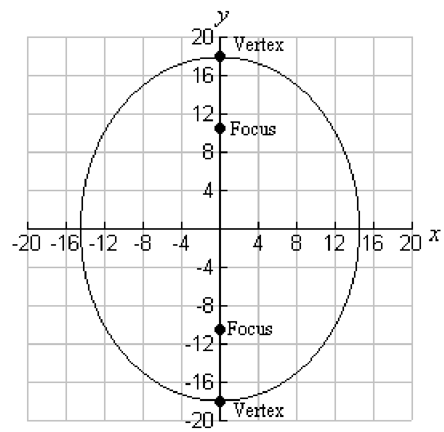
$a = \dfrac{39\frac{5}{6}}{2} = 17\dfrac{11}{12} \approx 17.92$. We compute

$b = \dfrac{29}{2} = 14\dfrac{1}{2} = 14.5$. Therefore,

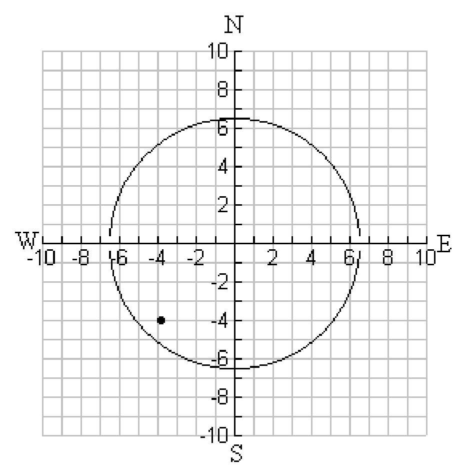
$\dfrac{x^2}{14.5^2} + \dfrac{y^2}{17.92^2} = 1$. We compute c.

$$\begin{aligned} b^2 &= a^2 - c^2 \\ 14.5^2 &= 17.92^2 - c^2 \\ 10.53 &\approx c \end{aligned}$$

The foci are $(0,10.53)$ and $(0,-10.53)$. The vertices are $(0,17.92)$ and $(0,-17.92)$.



2. If the woman can hear the siren, she would have to be on or inside the circular range represented by $x^2 + y^2 = 42.25$. A graph helps us to determine this. Note that $(-3.8)^2 + (-4)^2 = 30.44 < 42.25$, so the woman must be inside the circular range.



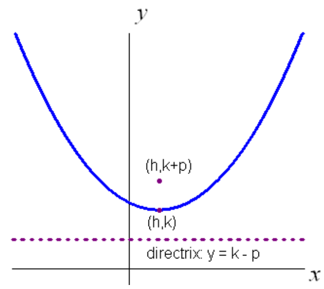
11.3 Parabolas

Objectives

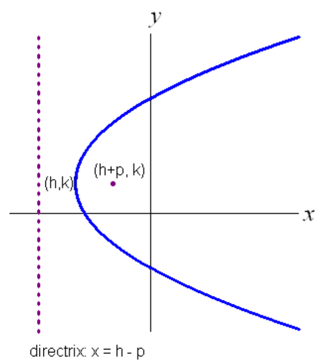
- Generate equations for parabolas given a graph, a set of points, or words.
- Apply parabolas to real-world scenarios.

Concepts and Definitions

- **The Parabola: Axis of Symmetry Vertical, Vertex at (h,k):** The standard equation of a parabola with vertex at (h,k) and axis of symmetry *vertical* is $(x-h)^2 = 4p(y-k)$.



- **The Parabola: Axis of Symmetry Horizontal, Vertex at (h,k):** The standard equation of a parabola with vertex at (h,k) and axis of symmetry *horizontal* is $(y-k)^2 = 4p(x-h)$.



Examples

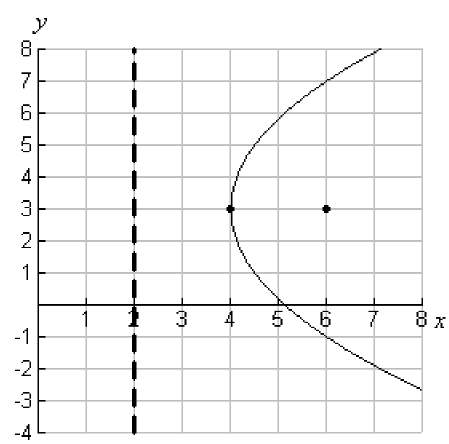
- **Example 1: Finding the Equation of a Parabola**
  Find the equation of the parabola whose directrix is at $x = 2$ and whose vertex is at $(4,3)$. Then graph the parabola.

  Solution:
  The directrix is vertical, so the axis of symmetry will be horizontal. Furthermore, since the vertex is not the origin, the parabola will have an equation of the form $(y-k)^2 = 4p(x-h)$. *Since the vertex is* $(4,3)$, *we have*

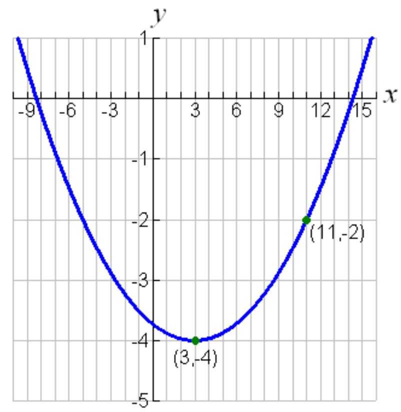
$$(y-3)^2 = 4p(x-4)$$

  Since the vertex is two units away from the directrix, $p = 2$. *Thus*

$$(y-3)^2 = 4(2)(x-4)$$
$$(y-3)^2 = 8(x-4).$$



- **Example 2: Calculating the Equation of a Parabola From a Graph**
  Write the equation for the parabola.



Solution:
The parabola passes through $(11,-2)$. *The vertex appears to be at* $(3,-4)$. *We know the equation will be of the form* $(x-h)^2 = 4p(y-k)$ *since the graph is concave up. We substitute the vertex into the equation.*

$$(x-3)^2 = 4p(y+4)$$

We determine the value of p *by substituting in the point* $(11,-2)$.

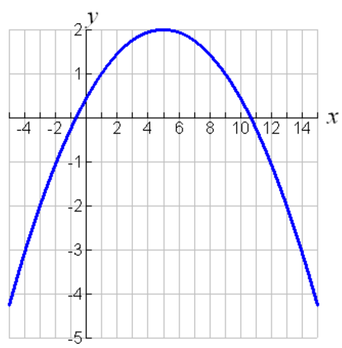
$$\begin{aligned} (x-3)^2 &= 4p(y+4) \\ (11-3)^2 &= 4p(-2+4) \\ 64 &= 8p \\ p &= \frac{64}{8} = 8 \end{aligned}$$

The equation of the parabola is $(x-3)^2 = 32(y+4)$.

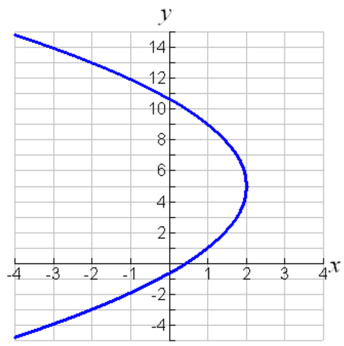
Exercises

1. The graphs of parabolas are given. Match each graph with its equation.

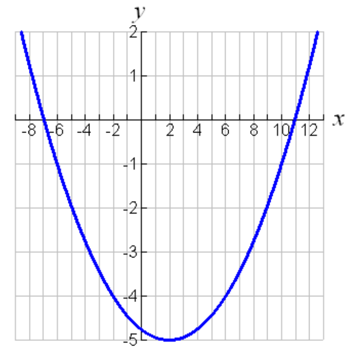
A.



B.



C.



I. $(y-5)^2 = -16(x-2)$ II. $(x-5)^2 = -16(y-2)$ III. $(x-2)^2 = 16(y+5)$

2. Write the equation of the parabola with vertex $(-1,3)$ and focus $(5,3)$. Then graph the equation.

Exercise Solutions

1. B – I, A – II, C – III.

2. Since the focus is 6 units to the right of the vertex, we know that we have a horizontal parabola of the form $(y-k)^2 = 4p(x-h)$. We begin by substituting the coordinates of the vertex for k and h.

$$(y-3)^2 = 4p(x-(-1))$$
$$(y-3)^2 = 4p(x+1)$$

The vertex is 6 units to the left of the focus so $p = 6$.

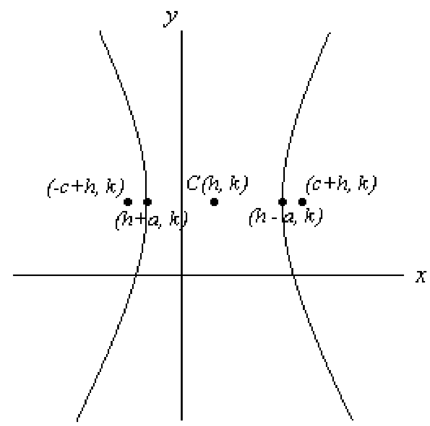
$$(y-3)^2 = 4(6)(x+1)$$
$$(y-3)^2 = 24(x+1)$$

11.4 Hyperbolas

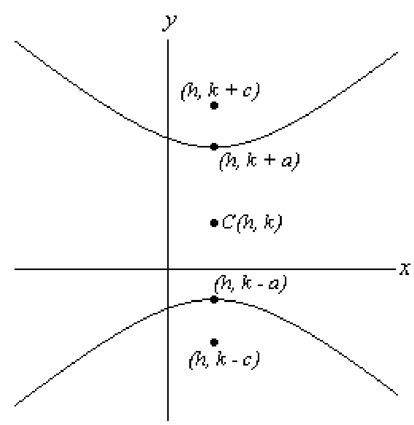
Objectives

- Generate equations for hyperbolas given a graph, a set of points, or words
- Apply hyperbolas to real-world scenarios.

Concepts and Definitions



- **The Hyperbola: Center at (h,k), Horizontal Transverse Axis:** The standard equation of a hyperbola with center at (h,k) and with a horizontal transverse axis is $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$ where $a^2+b^2=c^2$. The vertices of the hyperbola are $(h+a,k)$ and $(h-a,k)$.
- **The Hyperbola: Center at (h,k), Vertical Transverse Axis:** The standard equation of a hyperbola with center at (h,k) and with a vertical transverse axis is $\frac{(y-k)^2}{a^2}-\frac{(x-h)^2}{b^2}=1$, where $a^2+b^2=c^2$. The vertices of the hyperbola are $(h,k+a)$ and $(h,k-a)$.



- **Asymptotes of a Hyperbola:** A hyperbola with a horizontal transverse axis, centered at the origin, has asymptotes with equations $y=\frac{b}{a}x$ and $y=-\frac{b}{a}x$. A hyperbola with a vertical transverse axis, centered at the origin, has asymptotes with equations $y=\frac{a}{b}x$ and $y=-\frac{a}{b}x$.

Examples

- **Example 1: Finding the Equation of a Hyperbola**
  Find an equation and graph the hyperbola centered at the origin and with vertex $(4,0)$ and one focus at $(\sqrt{19},0)$.

Solution:

With a vertex of $(4,0)$ *and focus of* $(\sqrt{19},0)$*, we know the transverse axis is horizontal. The hyperbola is of the standard form* $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$*. The center is* $(0,0)$ *so we know the equation will be* $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ *since* $h=0$ *and* $k=0$*. We know the hyperbola has a vertex at* $(4,0)$*, which means the other vertex is at* $(-4,0)$*. This means* $a=4$*. Finally, since one focus is at* $(\sqrt{19},0)$*, we know* $c=\sqrt{19}$*. Using the relationship* $a^2+b^2=c^2$*, we determine the value of* b*.*

$$\begin{aligned}(4)^2+b^2&=\left(\sqrt{19}\right)^2\\ 16+b^2&=19\\ b^2&=3\end{aligned}$$

We write the equation of the hyperbola.

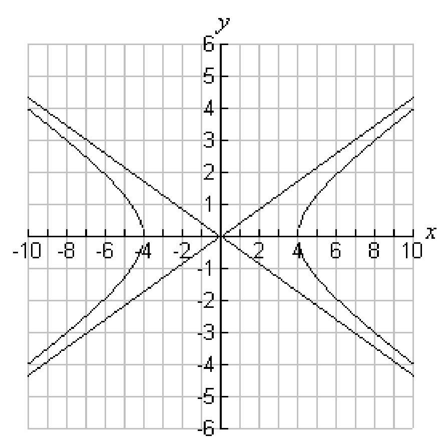
$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2}&=1\\ \frac{x^2}{16}-\frac{y^2}{3}&=1\end{aligned}$$

- **Example 2: Calculating Equations of Asympototes**

Find the equations of the asymptotes for the hyperbola $\frac{x^2}{16}-\frac{y^2}{3}=1$. Then graph the hyperbola along with the asymptotes.

Solution:

From the given equation $\frac{x^2}{16}-\frac{y^2}{3}=1$*, we know* $a=4$ *and* $b=\sqrt{3}$*. We substitute these values in the general equation* $y=\pm\frac{b}{a}x$ *to get* $y=\pm\frac{\sqrt{3}}{4}x$*. The equations of the asymptotes are* $y=\frac{\sqrt{3}}{4}x$ *and* $y=-\frac{\sqrt{3}}{4}x$*. We graph all three equations to confirm our work.*



Exercises

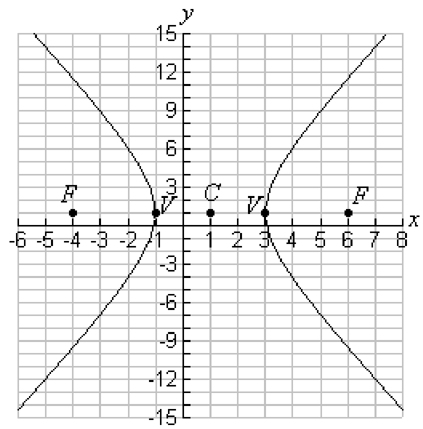
1. Write the equation of the hyperbola in standard form with center at $(1,1)$, vertex at $(3,1)$ and focus at $(6,1)$. Then graph the equation.

2. Find the vertices, foci, and asymptotes of the hyperbola $3x^2+12x-y^2+2y-5=0$. Then indicate whether the hyperbola opens up and down or left and right.

Exercise Solutions

1.

$$a^2+b^2=c^2$$
$$2^2+b^2=5^2$$
$$4+b^2=25$$
$$b^2=21$$
$$\frac{(x-1)^2}{4}-\frac{(y-1)^2}{21}=1$$



2. Find the vertices, foci, and asymptotes of the hyperbola $3x^2+12x-y^2+2y-5=0$. Then indicate whether the hyperbola opens up and down or left and right.

$$3x^2+12x-y^2+2y-5=0$$
$$3(x^2+4x)-1(y^2-2y)=5$$
$$3(x^2+4x+4)-1(y^2-2y+1)=5+12-1$$
$$3(x+2)^2-(y-1)^2=16$$
$$3(x+2)^2-(y-1)^2=16$$
$$\frac{3(x+2)^2}{16}-\frac{(y-1)^2}{16}=1$$
$$\frac{(x+2)^2}{16/3}-\frac{(y-1)^2}{16}=1$$

$$a^2+b^2=c^2$$
$$\frac{16}{3}+16=c^2$$
$$\sqrt{\frac{64}{3}}=c$$
$$\frac{8}{\sqrt{3}}=c$$

$C:(-2,1)$

$V:\ (-2+\frac{16}{3},1),(-2-\frac{16}{3},1)$

$F:\ (-2+\frac{8}{\sqrt{3}},1),(-2-\frac{8}{\sqrt{3}},1)$

opens left and right

Chapter 11 Exam A

1. Write the equation of the ellipse with vertices at $(2,5)$ and $(2,-1)$ and $c=2$. Then graph the equation.

2. In the Museum of Science and Industry in Chicago, Illinois, there is a special room designed to carry the quietest message from the lips of a person speaking to another's ear. The elliptical shape of the room has the following dimensions:

   Width of the ellipse: 13 feet 6 inches

   Length of the ellipse: 47 feet 4 inches

   Distance between foci: 40 feet 7 inches.

   (*Source*: www.msichicago.org/exhibit/whispering/index.html).

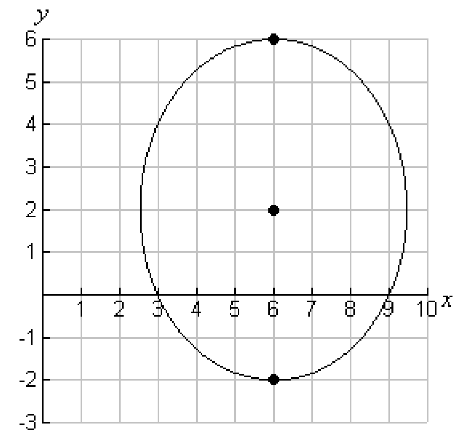
   Give the equation for the ellipse of this whispering gallery.

3. Find the directrix, focus, vertex, and axis of symmetry for the parabola $(y-2)^2=-4(x+3)$. Then graph the equation.

4. Find the vertices, foci, and asymptotes of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$. Then indicate whether the hyperbola opens up and down or left and right.

Chapter 11 Exam A Solutions

1. Write the equation of the ellipse with vertices at $(6,6)$ and $(6,-2)$ and $c=2$. Then graph the equation.



The center of the ellipse is $(6,2)$. Therefore, $a^2=16$. We calculate b^2.

$$\begin{aligned} b^2 &= a^2-c^2 \\ b^2 &= 16-4 \\ b^2 &= 12 \end{aligned}$$

The equation of the ellipse is

$$\frac{(x-6)^2}{12}+\frac{(y-2)^2}{16}=1.$$

2. In the Museum of Science and Industry in Chicago, Illinois, there is a special room designed to carry the quietest message from the lips of a person speaking to another's ear. The elliptical shape of the room has the following dimensions:

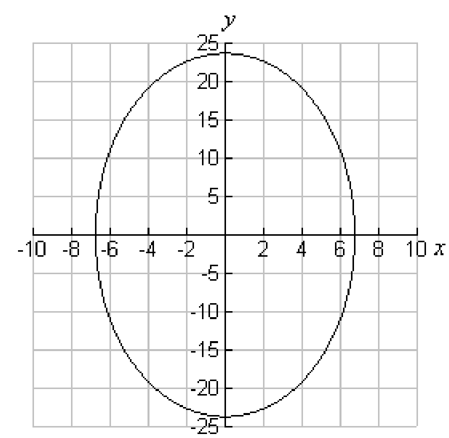
Width of the ellipse: 13 feet 6 inches

Length of the ellipse: 47 feet 4 inches

Distance between foci: 45 feet 4 inches.

(*Source*: www.msichicago.org/exhibit/whispering/index.html).

Give the equation for the ellipse of this whispering gallery.



We start by drawing a graph of the ellipse. We choose to position the ellipse so its center is at $(0,0)$.

We see that $a=23\frac{2}{3}$ and $b=6\frac{3}{4}$. We write the equation of the ellipse.

$$\frac{x^2}{(6.75)^2}+\frac{y^2}{(23.\overline{6})^2}=1$$

3. Find the directrix, focus, vertex, and axis of symmetry for the parabola $\left(y-2\right)^2 = -4\left(x+3\right)$. Then graph the equation.

The parabola is of the form $\left(y-k\right)^2 = 4p\left(x-h\right)$. The directrix is $x=-1$. The focus is $(-4,2)$. The vertex is $(-3,2)$. The axis of symmetry is $y=2$. The parabola opens up to the left.

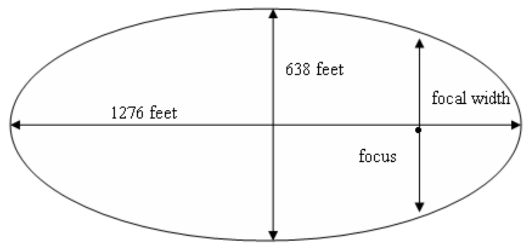
4. Find the vertices, foci, and asymptotes of the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$. Then indicate whether the hyperbola opens up and down or left and right.

$$a^2+b^2=c^2$$
$$9+4=c^2$$
$$\sqrt{13}=c$$
$$C:(0,0)$$
$$V:\ (3,0),(-3,0)$$
$$F:\ (\sqrt{13},0),(-\sqrt{13},0)$$
opens left and right

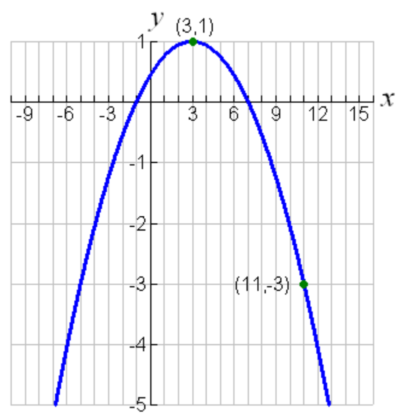
Chapter 11 Exam B

1. The Rose Bowl in Pasadena, California, is in the shape of an ellipse. Use the dimensions of the field to find an equation for the ellipse. Calculate the eccentricity. Rose Bowl Facts: The stadium measures 880 feet from north to south rims and 695 feet from east to west rims. (*Source*: www.rosebowlstadium.com/RoseBowl_general-info.htm)

2. Many race tracks around the world are either elliptical or have portions of the track that are elliptical (*Source*: www.oval.race-cars.com). The Dover International Speedway in Dover, Delaware, is a racetrack designed roughly in the shape of an ellipse that is approximately 1276 feet long and 638 feet wide (*Source*: www.nascar.com). Write the equation of the ellipse.



3. Write the equation for the parabola whose graph is shown.

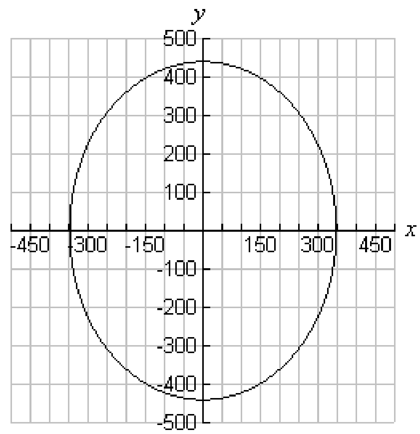


4. Find the vertices, foci, and asymptotes of the hyperbola $x^2 + 8x - y^2 - 6y - 9 = 0$. Then indicate whether the hyperbola opens up and down or left and right.

Chapter 11 Exam B Solutions

1. The Rose Bowl in Pasadena, California, is in the shape of an ellipse. Use the dimensions of the field to find an equation for the ellipse. Calculate the eccentricity. Rose Bowl Facts: The stadium measures 880 feet from north to south rims and 695 feet from east to west rims. (*Source*: www.rosebowlstadium.com/RoseBowl_general-info.htm)

We start by drawing a graph of the ellipse. We choose to position the ellipse so its center is at $(0,0)$. We see that $a = 440$ and $b = 347.5$. We write the equation of the ellipse.



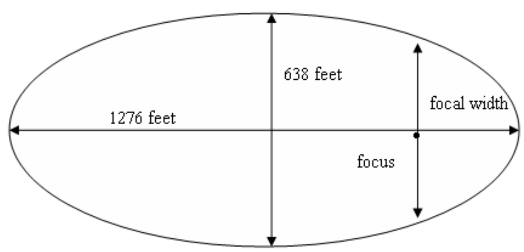
$$\frac{x^2}{(347.5)^2} + \frac{y^2}{(440)^2} = 1$$

To compute the eccentricity, we need to first compute the value of c.

$$b^2 = a^2 - c^2$$
$$347.5^2 = 440^2 - c^2$$
$$c^2 \approx 72843.75$$
$$c \approx 269.9$$

$$e = \frac{c}{a} = \frac{269.9}{440} \approx 0.613$$

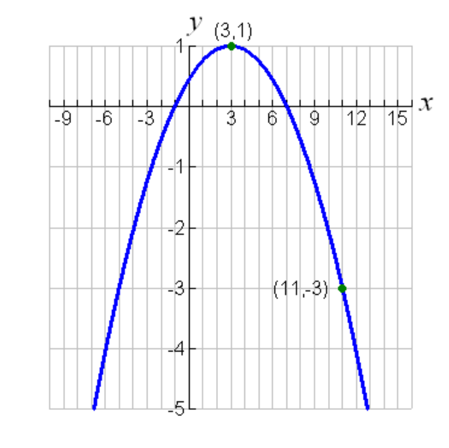
2. Many race tracks around the world are either elliptical or have portions of the track that are elliptical (*Source*: www.oval.race-cars.com). The Dover International Speedway in Dover, Delaware, is a racetrack designed roughly in the shape of an ellipse that is approximately 1276 feet long and 638 feet wide (*Source*: www.nascar.com). Write the equation of the ellipse.



We choose to position the ellipse so its center is at $(0,0)$. From the picture, we see that $a = 1276$ and $b = 638$. We write the equation of the ellipse.

$$\frac{x^2}{(1276)^2} + \frac{y^2}{(638)^2} = 1$$

3. Write the equation for the parabola whose graph is shown.



$$(x-h)^2 = -4p(y-k)$$
$$(x-3)^2 = -4(p)(y-1)$$
$$(11-3)^2 = -4(p)(-3-1)$$
$$64 = -4p(-4)$$
$$64 = 16p$$
$$4 = p$$
$$(x-3)^2 = -16(y-1)$$

4. Find the vertices, foci, and asymptotes of the hyperbola $x^2 + 8x - y^2 - 6y - 9 = 0$. Then indicate whether the hyperbola opens up and down or left and right.

$$x^2 + 8x - y^2 - 6y - 9 = 0$$
$$x^2 + 8x - 1(y^2 + 6y) = 9$$
$$x^2 + 8x + 16 - 1(y^2 + 6y + 9) = 9 + 16 - 9$$
$$(x+4)^2 - (y+3)^2 = 16$$
$$\frac{(x+4)^2}{16} - \frac{(y+3)^2}{16} = 1$$

$$a^2 + b^2 = c^2$$
$$16 + 16 = c^2$$
$$\sqrt{32} = c$$
$$C: (-4,-3)$$
$$V:\ (-4+4,-3),(-4-4,-3) = (0,-3),(-8,-3)$$
$$F:\ (-4+\sqrt{32},-3),(-4-\sqrt{32},-3)$$

opens left and right

Sequences and Series

12

12.1 Sequences

Objectives

- Determine whether a given sequence converges or diverges using graphs, tables, and symbols.
- Find explicit and recursive formulas for sequences.
- Use sequences to analyze real-world situations.

Concepts and Definitions

- **Sequence:** A sequence $\{a_n\}$ is a function whose domain is the set of natural numbers (1, 2, 3, …). The function values are called terms of the sequence. The terms are represented by $a_1, a_{2,} ..., a_n, ...$.

Examples

- **Example 1: Analyzing a Sequence**
  For the sequence $a_n = 2 + 3(n-1)$,

  a. list the first five terms of the given sequence.

  b. predict whether the sequence will converge or diverge.

  c. check the accuracy of your prediction by graphing the first ten terms of the sequence.

  Solution:

  a. $a_1 = 2 + 3(1-1) = 2$

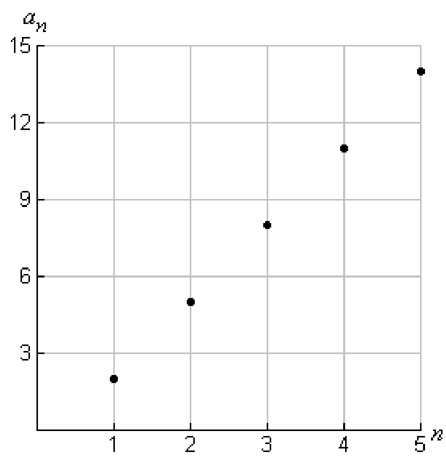
  $a_2 = 2 + 3(2-1) = 5$

  $a_3 = 2 + 3(3-1) = 8$

  $a_4 = 2 + 3(4-1) = 11$

  $a_5 = 2 + 3(5-1) = 14$

  The first five terms of the sequence are $2, 5, 8, 11, 14, \ldots$

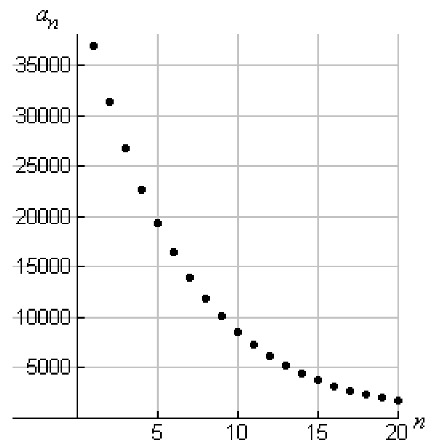


  b. The sequence appears to be growing without bound so is diverging.

  c. The graph shows that the sequence is diverging.

- ***Example 2:*** **Identifying a Convergent Sequence**
  As a rule of thumb, a car depreciates in value by 15% every year (www.bankrate.com). The MSRP for one 2011 Dodge Charger was \$36,945 on July 13, 2011 (www.nada.com). If the car is purchased and kept indefinitely, does the value of the car converge or diverge? If it converges, to what value does it converge?

  Solution:
  We can express the value of the car in subsequent years as the sequence $a_n = 36945(0.85)^{n-1}$. *Note that since the value depreciates by 15% each year, it retains 85% of its value. We graph the sequence to get a sense of the whether the value converges or diverges over time. As time increases, the value continues to decrease but at a decreasing rate. We say that the sequence converges to \$0.*



Exercises

1. Find a formula for the general term of the sequence $6,\ 2,\ \frac{2}{3},\ \frac{2}{9},\ldots$ and state whether the sequence converges. Assume the pattern in the given terms continues.

2. Vicodin is taken by people in need of pain relief. It contains oxycodone and acetaminophen and can be addictive. Dosing is dependent upon the severity of the pain that a person is experiencing. Vicodin has a half-life of about 4 hours (*Source*: www.rxlist.com). If 325-mg of Vicodin is prescribed to be taken every 4 hours, what will the maintenance level be in this case? Use graphs, formulas, and tables to justify your work.

Exercise Solutions

1. We notice that each subsequent term is found by multiplying the previous term by $\frac{1}{3}$.

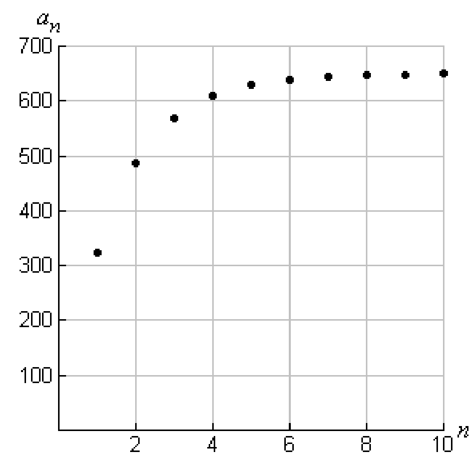
   Starting with the first term, we multiply by $\frac{1}{3}$ to get the second term. We multiply by $\frac{1}{3}$ again (a total of two times) to get the third term, etc. Therefore, the formula for the general term of the sequence is $a_n = 6(\frac{1}{3})^{n-1}$. Notice that the exponent, which keeps track of the number of times we multiply by $\frac{1}{3}$, must be $n-1$ so that the first term comes out to be 6 since $a_1 = 6(\frac{1}{3})^{1-1} = 6(\frac{1}{3})^0 = 6$. The sequence converges since as $n \to \infty$, $a_n \to 0$.

2. Since the half-life is 4 hours and each new dose will be taken every 4 hours, we can write a formula for the nth term of the sequence.

$$a_n = 0.5a_{n-1} + 325$$

That is, each subsequent term of the sequence, representing the amount of Vicodin in the bloodstream, is found by multiplying the previous term by 0.5 and then adding the new dose of 325-mg.

We graph the sequence to explore the maintenance level. It appears as though the maintenance level is about 650-mg. This result makes sense since the sequence shows us that taking half of the previous amount (half of 650 is 325) and adding the new dose (325-mg), would keep us at 650-mg indefinitely.



12.2 Arithmetic and Geometric Sequences

Objectives

- Determine whether a sequence is arithmetic or geometric.
- Use arithmetic and geometric sequences to analyze real-world problems.

Concepts and Definitions

- **Arithmetic Sequence:** An arithmetic sequence, $\{a_n\}$, is a sequence in which each subsequent term of the sequence is found by adding a common difference, d, to the previous term. The formula for the n^{th} term of an arithmetic sequence is $a_n = a_1 + (n-1)\cdot d$.
- **Geometric Sequence:** A geometric sequence, $\{a_n\}$, is a sequence in which each subsequent term of the sequence is found by multiplying the previous term by a common ratio, r. The formula for the n^{th} term of a geometric sequence is $a_n = a_1 \cdot r^{n-1}$.

Examples

- **Example 1: Creating an Arithmetic Sequence**
  Decide whether an arithmetic or geometric sequence will best model the situation. Explain how you know, then write an explicit formula for the sequence.

  For every mile driven on the freeway at a constant speed, 0.05 gallons of gasoline is consumed.

  Solution:
  Since the amount of consumed increases by a constant amount of 0.05 gallons per mile driven, this situation represents an arithmetic sequence. There is a constant difference of 0.05. Assuming no gasoline was consumed at the beginning of the trip, we create the general formula.

$$a_n = a_1 + (n-1)\cdot d$$
$$a_n = 0 + (n-1)\cdot 0.05$$
$$a_n = 0.05n - 0.05$$

- **Example 2: Creating a Geometric Sequence**
  Decide whether an arithmetic or geometric sequence will best model the situation. Explain how you know, then write an explicit formula for the sequence.

  The number of students enrolled at the local college has been increasing at 5.8% each year. Initially, the college enrollment was 4500 students.

Solution:
Since enrollment at the college is increasing by a constant factor of 5.8%, the situation represents a geometric sequence. The first term is 4500 and the ratio of consecutive terms is a constant 1.058.

$$a_n = a_1 \cdot r^{n-1}$$
$$a_n = 4500 \cdot 1.058^{n-1}.$$

Exercises

1. a. Two terms of an arithmetic sequence are $a_1 = 6$ and $a_5 = 20$. Write an explicit formula for the sequence and find a_{10}.

   b. Two terms of a geometric sequence are $a_1 = 3$ and $a_3 = 6.75$. Write an explicit formula for the sequence and find a_6.

2. Tuition at Itasca Community College in Grand Rapids, MN in 2011 was $170.62 per credit for residents of Minnesota (*Source:* www.itascacc.edu).

   a. Explain why this situation may be represented by an arithmetic sequence. What information do the terms of the sequence provide?

   b. Write an explicit formula for the n^{th} term of the sequence.

   c. Compute a_{15} and explain what it means.

Exercise Solutions

1. a. We first find the common difference.

$$d = \frac{20-6}{5-1} = \frac{14}{4} = \frac{7}{2} = 3.5$$

Next, we find the explicit formula for the sequence.

$$a_n = a_1 + (n-1) \cdot d$$
$$a_n = 6 + (n-1) \cdot 3.5$$
$$a_n = 6 + 3.5n - 3.5$$
$$a_n = 3.5n + 2.5$$

Finally, we find a_{10}.

$$a_{10} = 3.5 \cdot 10 + 2.5 = 37.5$$

b. We first find the common ratio.

$$r = \sqrt{\frac{6.75}{3}} = 1.5$$

Next, we find the explicit formula for the sequence.

$$a_n = a_1 \cdot r^{n-1}$$
$$a_n = 6.75 \cdot 1.5^{n-1}$$

Finally, we find a_6.

$$a_6 = 6.75 \cdot 1.5^{6-1} = 6.75 \cdot 1.5^5 \approx 51.258$$

2. a. Since each credit hour costs an additional \$170.62, this situation represents an arithmetic sequence. The common difference is the \$170.62 per credit hour. Each term in the sequence represents the cost for taking n credit hours at Itasca Community College.

b.

$$a_n = a_1 + (n-1) \cdot d$$
$$a_n = 170.62 + (n-1) \cdot 170.62$$
$$a_n = 170.62 + 170.62n - 170.62$$
$$a_n = 170.62n$$

c.

$$a_{15} = 170.62 \cdot 15$$
$$= 2559.3$$

The cost of tuition when enrolling in 15 credit hours at Itasca Community College is \$2,559.30.

12.3 Series

Objectives

- Determine whether a series is arithmetic or geometric.
- Determine whether a series converges or diverges using graphs, tables, and symbols.
- Compute the partial sums of arithmetic and geometric series.
- Use series to solve real-world problems.

Concepts and Definitions

- **Sum of a Finite Arithmetic Sequence:** The sum of the first n terms of a finite arithmetic sequence with first term a_1 and common difference d is given by $S_n = \frac{n}{2}(a_1 + a_n)$.
- **Finite Geometric Series:** A finite geometric series with n terms can be expressed as $a + ar + ar^2 + ar^3 + ... + ar^{n-1}$ where a is the first term of the series and r is the common ratio or constant multiple.
- **Sum of a Finite Geometric Sequence:** The sum of the first n terms of a finite geometric sequence with first term a and common ratio r is $S_n = \frac{a(1-r^n)}{1-r}$ provided that $r \neq 1$.
- **Sum of an Infinite Geometric Series:** The sum of the first n terms of an infinite geometric series $S_n = a + ar + ar^2 + ... + ar^{n-1}$ converges to $S = \frac{a}{1-r}$ as $n \to \infty$ provided that $-1 < r < 1$. Otherwise, the series diverges.

Examples

- **Example 1: Computing the Sum of an Arithmetic Sequence**
  The tallest champagne fountain in the world was built by Luuk Broos in December 1999 (*Source*: timesofindia.indiatimes.com). It was a 56-story tower of champagne glasses that was on display at the Steigenberger Kurhaus Hotel, Scheveningen, Netherlands. The tower was quite complicated with many sections, but for one section of the tower, there was one champagne glass at the top of the section with one additional champagne glass added to each subsequent row. If there were 125 rows, how many champagne glasses were used?

  Solution:
  This is an example of a finite arithmetic series with $d = 1$. *We can find the sum knowing that* $n = 125$, $a_1 = 1$, *and* $a_{125} = 125$.

$$S_{125} = \frac{125}{2}(1+125) = 7875$$

In this section of the tower, 7,875 champagne glasses were needed.

- **Example 2: Calculating the Future Value of an Investment**
  A person deposits \$2000 every year into an account that earns 8% compounded annually. Using sigma notation, express the amount of money that will be in the account after eight years. Then determine the amount of money that will be in the account after eight years.

  Solution:
  Let A_n represent the amount of money in the account immediately after the \$2000 is deposited into the account for the nth time.

$$\begin{aligned} A_1 &= \$2000 \\ A_2 &= \$2000 + \$2000 \cdot 1.08 = \$4160 \end{aligned}$$

Notice A_2, the amount of money in the account after the second year's deposit is made, is found by adding the new \$2000 deposit to the original \$2000 and the interest it earned. We continue this pattern. After the third year's deposit, we will have \$2000 plus the previous years' deposits and the interest they've earned.

$$\begin{aligned} A_3 &= \$2000 + \$4160 \cdot 1.08 \\ &= \$2000 + (\$2000 + \$2000 \cdot 1.08) \cdot 1.08 \\ &= \$2000 + \$2000 \cdot 1.08 + \$2000 \cdot 1.08^2 \\ &= \$6492.80 \end{aligned}$$

We investigate what happens after the deposit is made for year 4 and then look for a pattern to help us write the sum in sigma notation.

$$\begin{aligned} A_4 &= \$2000 + \$6492.80 \cdot 1.08 \\ &= \$2000 + (\$2000 + \$2000 \cdot 1.08 + \$2000 \cdot 1.08^2) \cdot 1.08 \\ &= \$2000 + \$2000 \cdot 1.08 + \$2000 \cdot 1.08^2 + \$2000 \cdot 1.08^3 \\ &= \$9012.22 \end{aligned}$$

We see we have a geometric series that begins with \$2000. Each subsequent term is multiplied by the common multiple 1.08. We write the sum using sigma notation.

$$\begin{aligned} &\sum_{n=1}^{8} 2000 \cdot 1.08^{n-1} \\ &= \$2000 + \$2000 \cdot 1.08 + \$2000 \cdot 1.08^2 + \$2000 \cdot 1.08^3 + ... + \$2000 \cdot 1.08^7 \\ &= \$21,273.26 \end{aligned}$$

After the 8th deposit, the account will have \$21,273.26.

Exercises

1. Attracting large events to a city, such as the Major League Baseball All-Star Game, can boost the local economy. The boost comes in the form of money spent by visitors who pay for lodging, food, transportation, souvenirs, and so on. It is difficult to determine precisely the impact on the economy in terms of dollars, but we estimate about 55% of every dollar introduced into the local economy is spent again locally. Then, 55% of that money is spent again locally. The process continues with 55% of the money spent in the community being spent again locally. It is estimated that during the 2011 All-Star Game in Phoenix, visitors contributed $67 million in direct spending to the region while they were there (*Source*: www.abc15.com). What was the total impact, in dollars, on the local Phoenix economy considering how the money was re-spent locally?

2. Determine if the series is arithmetic, geometric, or neither. Explain how you know.

   a. $\sum_{n=1}^{10} 800 \cdot \left(\frac{1}{4}\right)^n$

   b. $\sum_{n=1}^{10} 800 + \frac{1}{4} \cdot n$

Exercise Solutions

1. We begin with the $67 million that was spent by visitors to Phoenix. Of that amount, 55% was spent again locally. This pattern continues indefinitely. We can express the economic impact as the infinite geometric series, S.

$$S = 67 + 67 \cdot 0.55 + 67 \cdot 0.55^2 + 67 \cdot 0.55^3 + ... = \sum_{n=1}^{\infty} 67 \cdot 0.55^{n-1}$$

We have an infinite geometric series where $a = 67$ and $r = 0.55$.

$$S = \frac{67}{1 - 0.55} \approx 148.89$$

We estimate the total economic impact of the All-Star Game to be $148,890,000.

2. a. We see a common ratio of $\frac{1}{4}$ so the series is a geometric series.

   b. We see a common difference of $\frac{1}{4}$ so the series is an arithmetic series.

Chapter 12 Exam A

1. Find a formula for the general term of the sequence, $8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$, and state whether the sequence converges. Assume the pattern in the given terms continues.

2. The median starting salary of a newly hired teacher in Arizona in 2011 was \$30,408 (*Source*: teacherportal.com). This represents an average 1.44% increase per year over the past 10 years.

   a. Assuming that the median starting salary for a teacher in Arizona will continue to increase at 1.44% per year, explain why this situation can be represented by a geometric sequence. What information would the terms of the sequence represent?

   b. Write an explicit formula for the n^{th} term of the sequence.

   c. Compute a_5 and explain what it means.

3. Express the repeating decimal 0.121212… as a rational number.

4. Find the sum of the first 20 terms of the arithmetic series $1200+1213+1226+1239+\ldots+(13n+1187)+\ldots$.

Chapter 12 Exam A Solutions

1. Find a formula for the general term of the sequence, $8, 2, \frac{1}{2}, \frac{1}{8}, \frac{1}{32}, \ldots$, and state whether the sequence converges. Assume the pattern in the given terms continues.

   We see that each subsequent term of the sequence is found by multiplying the previous term by the common ratio $\frac{1}{4}$. The formula for the general term is $a_n = 8(\frac{1}{4})^{n-1}$. As we continue to multiply previous terms by $\frac{1}{4}$ to get the next term, the terms will become smaller and smaller. Therefore, we can state that the sequence converges.

2. The median starting salary of a newly hired teacher in Arizona in 2011 was $30,408 (*Source*: teacherportal.com). This represents an average 1.44% increase per year over the past 10 years.

   a. Assuming that the median starting salary for a teacher in Arizona will continue to increase at 1.44% per year, explain why this situation can be represented by a geometric sequence. What information would the terms of the sequence represent?

      Since we have a percentage increase, each subsequent term would be found by multiplying the previous term by 1.0144. This is characteristic of a geometric sequence where each term in the sequence represents the median starting salary for a teacher in Arizona in future years.

   b. Write an explicit formula for the n^{th} term of the sequence.

      $a_n = 30408(1.0144)^{n-1}$

   c. Compute a_5 and explain what it means.

      $a_5 = 30408(1.0144)^{5-1} \approx 32197.70$

      This result tells us that in 2015, the median starting salary for a teacher in Arizona will be $32197.70.

3. Express the repeating decimal 0.121212… as a rational number.

   $0.121212\ldots = 0.12 + 0.0012 + 0.000012 + \ldots$

   We see an infinite geometric series where $a = 0.12$ and $r = 0.01$. We find the sum of this infinite geometric series as follows.

$$\begin{aligned} S_n &= \frac{a}{1-r} \\ &= \frac{0.12}{1-0.01} \\ &= \frac{0.12}{0.99} \\ &= \frac{12}{99} \end{aligned}$$

4. Find the sum of the first 20 terms of the arithmetic series $1200+1213+1226+1239+...+(13n+1187)+...$.

We note that $a_1 = 1200$ and $d = 13$. We compute $a_{20} = 13(20)+1187 = 1447$.

$$\begin{aligned} S_n &= \frac{n}{2}(a_1 + a_n) \\ S_{20} &= \frac{20}{2}(1200+1447) \\ &= 10(2647) \\ &= 26470 \end{aligned}$$

Chapter 12 Exam B

1. Adderall is a medicine often used to treat patients with narcolepsy or attention deficit disorder. With a half-life of 10 hours, about 75% of the medicine is eliminated from the bloodstream every 24 hours. A typical prescription calls for the patient to take one 10-mg tablet every 24 hours (*Source*: www.drugs.com).

   a. Generate the first 5 terms of the sequence that would represent the amount of medicine in the bloodstream every 24 hours.

   b. What is the maintenance level in this situation?

   c. How long does it take to reach the maintenance level? Confirm your answer graphically.

2. Two terms of a geometric sequence are $a_1 = 5$ and $a_3 = 2.5$. Write an explicit formula for the sequence and find a_6.

3. Find the sum of the first four terms of the geometric series $\frac{1}{6^0}+\frac{1}{6^1}+\frac{1}{6^2}+\frac{1}{6^3}+...+\frac{1}{6^{n-1}}+...$.

   Determine whether the infinite series converges or diverges. If the series converges, state its sum.

4. Determine if the series $\sum_{n=1}^{5}\frac{1}{2}n^2+1$ is arithmetic, geometric, or neither. Justify your answer.

Chapter 12 Exam B Solutions

1. Adderall is a medicine often used to treat patients with narcolepsy or attention deficit disorder. With a half-life of 10 hours, about 75% of the medicine is eliminated from the bloodstream every 24 hours. A typical prescription calls for the patient to take one 10-mg tablet every 24 hours (*Source*: www.drugs.com).

   a. Generate the first 5 terms of the sequence that would represent the amount of medicine in the bloodstream every 24 hours.

   $a_1 = 10$

   $a_2 = 0.25 \cdot 10 + 10 = 12.5$

   $a_3 = 0.25 \cdot 12.5 + 10 = 13.125$

   $a_4 = 0.25 \cdot 13.125 + 10 = 13.28125$

   $a_5 = 0.25 \cdot 13.28125 + 10 = 13.3203125$

   b. What is the maintenance level in this situation?

   The maintenance level appears to be approximately 13.3-mg of medicine. We confirm this algebraically.

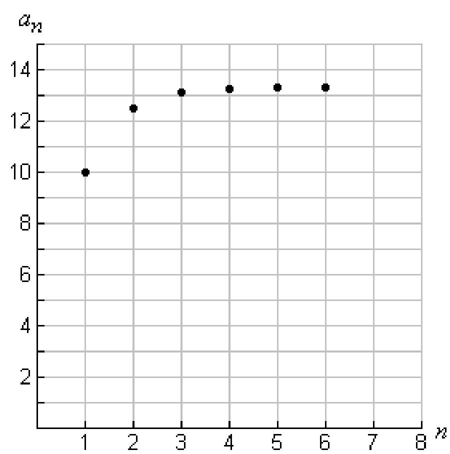
$$\begin{aligned} 0.25k + 10 &= k \\ 10 &= 0.75k \\ k &= \frac{10}{0.75} \approx 13.33 \end{aligned}$$

   c. How long does it take to reach the maintenance level? Confirm your answer graphically.

   We see in part a) that we are very close to reaching maintenance level after the 5th dose. We will consider one more dose.

$$a_6 = 0.25 \cdot 13.3203125 + 10 \approx 13.33$$

   Certainly, after the 6th dose, the maintenance level is reached.



2. Two terms of a geometric sequence are $a_1 = 5$ and $a_3 = 2.5$. Write an explicit formula for the sequence and find a_6.

We first compute the common ratio r. $r = \sqrt{\frac{2.5}{5}} \approx 0.707$. We can then write the explicit formula for the sequence.

$$a_n = 5(0.707)^{n-1}$$

We compute $a_6 = 5(0.707)^{6-1} \approx 0.8832$

3. Find the sum of the first four terms of the geometric series $\frac{1}{6^0} + \frac{1}{6^1} + \frac{1}{6^2} + \frac{1}{6^3} + ... + \frac{1}{6^{n-1}} + ...$.

Determine whether the infinite series converges or diverges. If the series converges, state its sum.

We first find the sum of the first 4 terms.

$$S_n = \frac{a - ar^n}{1-r}$$

$$S_4 = \frac{1 - 1\cdot(\frac{1}{6})^4}{1 - \frac{1}{6}} \approx 1.199$$

With $r = \frac{1}{6}$, the infinite geometric series will converge.

$$S_n = \frac{a}{1-r}$$

$$S_n = \frac{1}{1-\frac{1}{6}} = \frac{5}{6}$$

4. Determine if the series $\sum_{n=1}^{5}(\frac{1}{2}n^2 + 1)$ is arithmetic, geometric, or neither. Justify your answer.

This series has no common difference and no common ratio. Therefore, this series is neither arithmetic nor geometric.