CHAPTER 5

Polynomial, Power, and Rational Functions

On March 24, 1989, the Exxon Valdez oil tanker spilled more than 10.8 million gallons of oil, polluting more than 1300 miles of Alaskan shoreline. The cleanup effort cost approximately \$2.1 billion. (*Source:* www.evostc.state.ak.us)

People overseeing cleanup efforts must determine when the expected gains from spending additional money on cleanup are not worth the additional cost. Cost-benefit decisions such as this can sometimes be modeled with rational functions.

SECTION 5.1

Higher-Order Polynomial Functions

LEARNING OBJECTIVES

- Use higher-order polynomial functions to model real-world situations
- Use the language of rate of change to describe the behavior of a higher-polynomial function
- Find the inverse of a polynomial function

GETTING STARTED

The U.S. Postal Service limits the size of the packages it will send. To determine if a package qualifies to be sent Parcel Post®, USPS employees add the length of the package to the distance around the thickest part (girth). The sum of the length and girth cannot exceed 130 inches. For a rectangular package with equal height and width, the relationship between the length of the package and its volume can be described by a cubic polynomial function.

In this section we look at higher-order *polynomial* functions, beginning with cubic functions. We investigate the rates of change of these functions and discuss concavity, inflection points, and end behavior of their graphs. We also see how polynomial functions are applied to a variety of real-world contexts, including the size of shipping boxes.

Cubic Functions

Figure 5.1 is a representation of six boxes whose length plus girth equal 130 inches. The length, width, height, and volume of each box are recorded in Table 5.1. Recall that volume is the product of length, width, and height. That is, $V = lwh$.

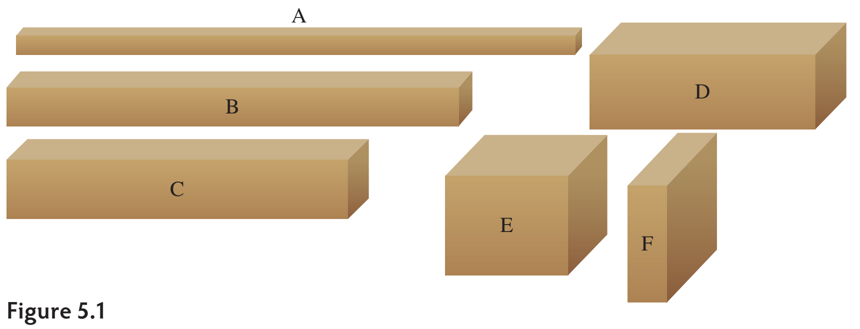


Figure 5.1

Table 5.1

Box	Length (inches)	Width (inches)	Height (inches)	Volume (cubic inches)
A	110	5	5	2750
B	90	10	10	9000
C	70	15	15	15,750
D	50	20	20	20,000
E	30	25	25	18,750
F	10	30	30	9000

To determine the relationship between the width and volume of the box we calculate successive differences. That is, we find the first differences (ΔV), second differences ($\Delta(\Delta V)$), and third differences ($\Delta(\Delta(\Delta V))$), as shown in Table 5.2. This strategy requires that each of the widths in the table be equally spaced. In this case, the widths are equally spaced 5 inches apart.

Table 5.2

Box	Width (inches)	Volume (cubic inches)	First Difference ΔV	Second Difference $\Delta(\Delta V)$	Third Difference $\Delta(\Delta(\Delta V))$
A	5	2750	6250	500	−3000
B	10	9000	6750	−2500	−3000
C	15	15,750	4250	−5500	−3000
D	20	20,000	−1250	−8500	
E	25	18,750	−9750		
F	30	9000			

We know from Chapter 4 that functions with constant first differences are linear and functions with constant second differences are quadratic. Functions with constant third differences, as in Table 5.2, are **cubic functions**.

Property of Cubic Functions

Any function with constant third differences is a **cubic function**.

EXAMPLE 1 ■ Finding the Equation of a Cubic Function Algebraically

A rectangular package has a square end (see Figure 5.2). The sum of the length and girth of the package is equal to 130 inches. Find an equation that relates the width of the package to the volume of the package.

Figure 5.2

Solution The girth of the package is $4w$ because the distance across the top of the package, the distance down the front of the package, the distance across the bottom of the package, and the distance up the back of the package are each w inches. Since the sum of the length and girth is 130 inches, we have

$$l + 4w = 130$$
$$l = 130 - 4w$$

The volume of a rectangular box is the product of its length, width, and height. Thus,

$$V = lwh$$
$$V = lw^2 \qquad \text{since } h = w$$
$$V = (130 - 4w)w^2 \qquad \text{since } l = 130 - 4w$$
$$V = -4w^3 + 130w^2$$

The equation that relates the volume of the package to its width is $V = -4w^3 + 130w^2$.

Standard Form of a Cubic Function

A cubic function has an equation of the form

$$y = ax^3 + bx^2 + cx + d$$

with constants a, b, c, and d and $a \neq 0$.

Graphs of Cubic Functions

In Example 1 we saw that the width and the volume of the package were related by $V = -4w^3 + 130w^2$. We can verify the accuracy of the equation by graphing it together with a scatter plot of the data presented in Table 5.1, as shown in Figure 5.3.

Observe that the graph is initially concave up but changes to concave down around $w = 10$. The point where the function changes concavity is called an **inflection point**.

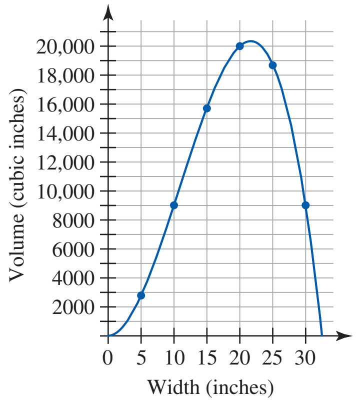


Figure 5.3

Inflection Point of a Cubic Function

The point on a graph where the concavity changes is called an **inflection point**. All cubic functions have exactly one inflection point.

Figure 5.4 shows the graphs with inflection points marked for several cubic functions.

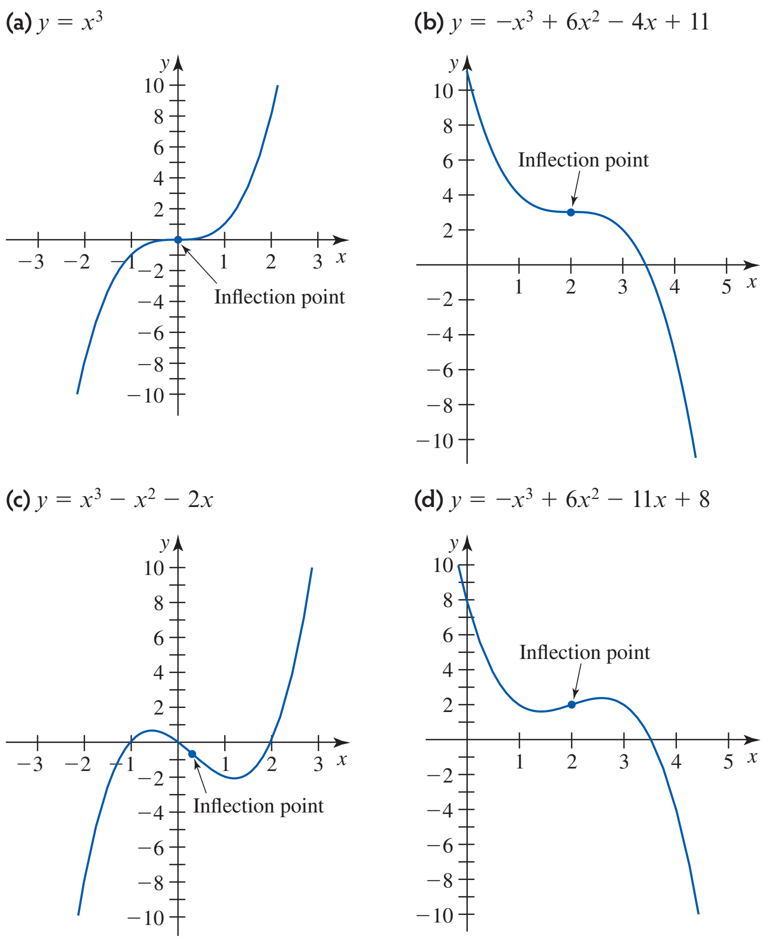


Figure 5.4

Relationship between Inflection Points and Rates of Change

The concavity of the graph and the rates of change of the corresponding function are closely related. When the graph is concave up, the rates of change (and first differences) are increasing. When the graph is concave down, the rates of change (and first differences) are decreasing. Thus inflection points, which occur where the concavity changes, also indicate where the rates of change switch from increasing to decreasing or vice versa.

For example, the graph in Figure 5.5 shows the rates of change of a cubic function at various points. Observe that when the graph is concave down, the rates of change are decreasing (12.9, 6.7, 2.1, 0, −1.9, −3.0). When the graph is concave up, the rates of change are increasing (−3.0, −2.5, 0, 2.1, 4.7, 9.0). At the inflection point (3, 6), the rates of change stop decreasing and begin to increase.

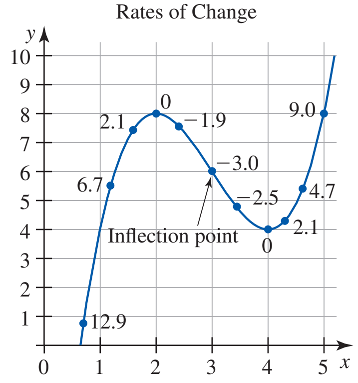


Figure 5.5

■ Modeling with Cubic Functions

Scatter plots that appear to change concavity exactly one time can be modeled by cubic functions. The resultant model will have constant third differences and either increasing or decreasing second differences.

EXAMPLE 2 ■ Using a Cubic Function in a Real-World Context

The per capita consumption of breakfast cereals (ready to eat and ready to cook) since 1980 is given in Table 5.3.

Table 5.3

Years Since 1980 t	Consumption (pounds) C	Years Since 1980 t	Consumption (pounds) C
0	12	10	15.4
1	12	11	16.1
2	11.9	12	16.6
3	12.2	13	17.3
4	12.5	14	17.4
5	12.8	15	17.1
6	13.1	16	16.6
7	13.3	17	16.3
8	14.2	18	15.6
9	14.9	19	15.5

Source: Statistical Abstract of the United States, 2001, Table 202

a. Create a scatter plot of these data and explain what type of function might best model the data.

b. Find the cubic regression model for the situation and graph the model together with the scatter plot.

c. Use the model from part (b) to predict the per capita consumption of breakfast cereal in 2000.

Solution

a. The scatter plot is shown in Figure 5.6. It appears that the per capita consumption (after a brief decline early on) increases at an increasing rate (concave up) until about 1988. Then, the per capita consumption increases at a decreasing rate (concave down) until 1994, where it begins to decrease but remains concave down. Because the graph appears to change concavity once, a cubic model may be appropriate. We see a possible inflection point (where the per capita consumption of cereal is increasing at the greatest rate) at approximately 1988 ($t = 8$).

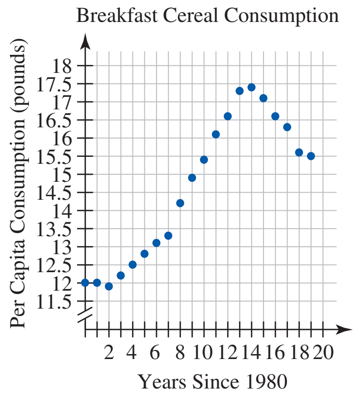


Figure 5.6

b. We use the graphing calculator to find the cubic regression model.

$$c(t) = -0.004718t^3 + 0.1165t^2 - 0.3585t + 12.17$$

where c is the per capita consumption of cereal (in pounds) and t is the number of years since 1980. (The cubic regression process is identical to that used for linear regression except that CubicReg is selected instead of LinReg.) The graph of the model is shown in Figure 5.7.

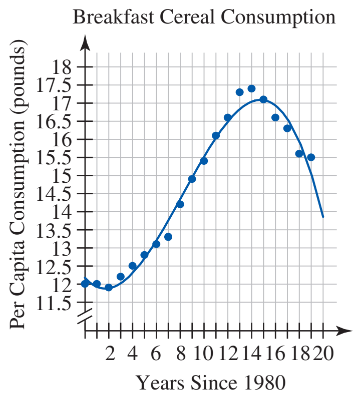


Figure 5.7

c. Since $t = 20$ represents the year 2000, we substitute this value into the function to predict the per capita consumption of cereal in 2000.

$$\begin{aligned} c(t) &= -0.004718t^3 + 0.1165t^2 - 0.3585t + 12.17 \\ c(20) &= -0.004718(20)^3 + 0.1165(20)^2 - 0.3585(20) + 12.17 \\ &= 13.86 \end{aligned}$$

In 2000, each person in the United States consumed nearly 14 pounds of cereal, on average, according to the model.

Polynomial Functions

Linear, quadratic, and cubic functions are all **polynomial functions**. Polynomial functions are formally defined as follows.

POLYNOMIAL FUNCTION

For whole number n, a function of the form

$$y = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1 x + a_0$$

with $a_n \neq 0$ is called a **polynomial function of degree** $\boldsymbol{n}$. Each $a_i x^i$ is called a **term**. The a_i are real-number values called the **coefficients** of the terms.

Note that the terms of polynomial functions have coefficients labeled a_n, a_{n-1}, and so on. Using this indexing system, we refer to the coefficient on the x^3-term as a_3, the coefficient of the x^2-term as a_2, and so on. The exponent of each term of a polynomial must be a nonnegative whole number. Table 5.4 shows examples of polynomial functions and Table 5.5 shows examples of nonpolynomial functions.

Table 5.4

Polynomial
$y = 2x - 5$
$y = -x^2 + 3x + 2$
$y = -\frac{2}{3}x^3 - 7x + \frac{2}{7}$
$y = 0.5x^5 - 2.5x^2 + 1.75x + 6.3$
$y = -5$

Table 5.5

Nonpolynomial	Reason
$y = \dfrac{2x - 5}{x^2 + 1}$	Not a sum of terms of the form $a_i x^i$.
$y = x^{0.5} + x^{-3.4}$	The exponents are not whole numbers.
$y = \sqrt{x} + x^2$	Since $\sqrt{x} = x^{1/2}$, one exponent is not a whole number.
$y = 2^x$	The exponent is a variable.
$y = 2x^{1/3} + 6x^{-1}$	The exponents are not whole numbers.

The graphs of polynomial functions are fairly predictable. We summarize the characteristics and appearance of the graphs of polynomial functions of the first through fifth degree in Table 5.6.

Table 5.6

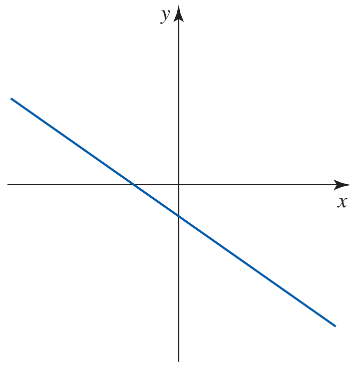
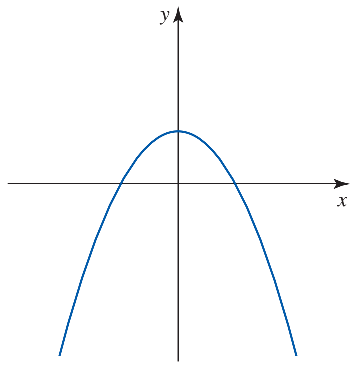
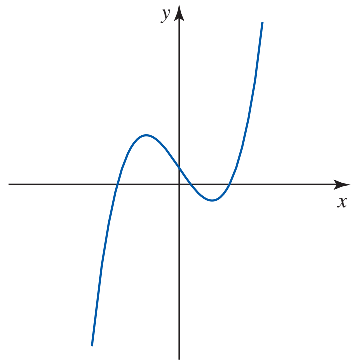
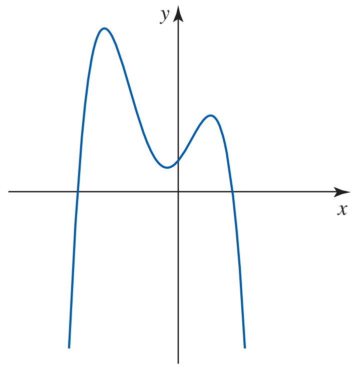
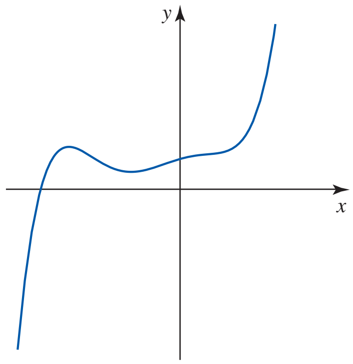
Function Name and Degree	Concavity, Inflection Points, and End Behavior	Constant Difference	Sample Graph
linear first degree	no concavity no inflection points one end $\to \infty$ one end $\to -\infty$	first	
quadratic second degree	concave up only or concave down only no inflection points both ends $\to \infty$ or both ends $\to -\infty$	second	
cubic third degree	concave up and concave down one inflection point one end $\to \infty$ one end $\to -\infty$	third	
quartic fourth degree	concavity changes zero or two times zero or two inflection points both ends $\to \infty$ or both ends $\to -\infty$	fourth	

Table 5.6 (*continued*)

Function Name and Degree	Concavity, Inflection Points, and End Behavior	Constant Difference	Sample Graph
quintic fifth degree	concavity changes one or three times one or three inflection points one end $\rightarrow \infty$ one end $\rightarrow -\infty$	fifth	

EXAMPLE 3 · Selecting a Higher-Order Polynomial Model

The per capita consumption of chicken in the United States has continued to increase since 1985; however, the rate of increase has varied. Create a scatter plot of the data in Table 5.7 and determine which polynomial function best models the situation. Then describe the relationship between the per capita consumption of chicken and time (in years) using the language of rate of change.

Table 5.7

Per Capita Consumption of Chicken (Boneless, Trimmed Weight)			Per Capita Consumption of Chicken (Boneless, Trimmed Weight)	
Years Since 1985 t	Consumption (pounds) C		Years Since 1985 t	Consumption (pounds) C
0	36.4		8	48.5
1	37.2		9	49.3
2	39.4		10	48.8
3	39.6		11	49.5
4	40.9		12	50.3
5	42.4		13	50.8
6	44.2		14	54.2
7	46.7			

Source: Statistical Abstract of the United States, 2001, Table 202

Solution We create the scatter plot of the data shown in Figure 5.8 and look for changes in concavity.

Although it is not always totally clear where the changes in concavity occur when looking at real-world data, we can look for trends and approximate. If we sketch a rough line graph as in Figure 5.9, we can better see that this data is initially concave up (roughly) but changes to concave down around 1990. Then, it changes to concave up again around 1996.

From Table 5.6 we see that the concave up, concave down, concave up pattern is best modeled using a quartic (fourth-degree polynomial) function. Using quartic regression, we determine that the function

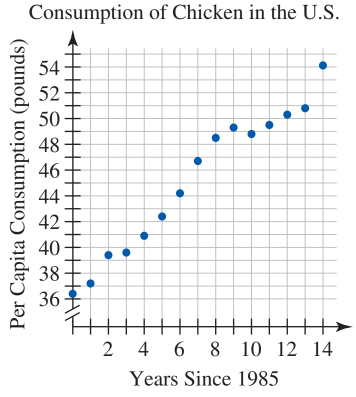


Figure 5.8

$$C(t) = 0.002623t^4 - 0.07545t^3 + 0.6606t^2 - 0.4688t + 36.92$$

best fits the data. We draw a graph of the model together with the scatter plot, as shown in Figure 5.10.

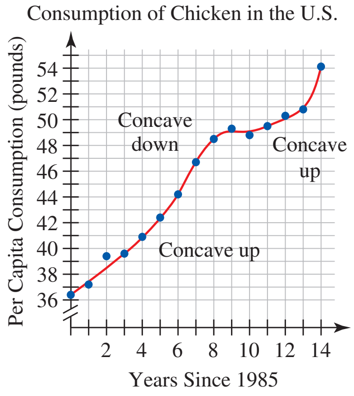


Figure 5.9

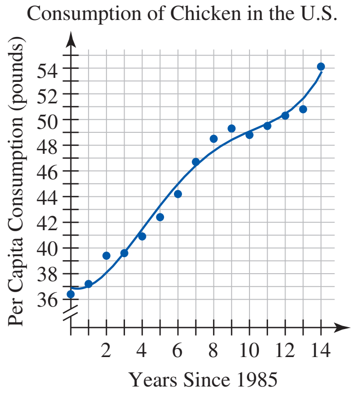


Figure 5.10

According to the model, per capita chicken consumption increased at an increasing rate from 1985 until approximately 1990. Then, from 1990 to approximately 1996, per capita chicken consumption continued to increase but at a decreasing rate. From 1996 to 1999, per capita chicken consumption again increased at an increasing rate.

■ End Behavior of Polynomial Functions

Polynomial functions have predictable long-run behavior, known as a function's **end behavior**.

End Behavior of a Polynomial Function

For any polynomial function, as x approaches $\pm\infty$, $f(x)$ approaches $\pm\infty$. That is, as the magnitude (absolute value) of x gets larger and larger, the magnitude of the function values will also get larger and larger. Symbolically we write

$$\text{as } x \to \pm\infty, \; f(x) \to \pm\infty$$

EXAMPLE 4 ■ Determining the End Behavior of a Polynomial Function

Use a table to determine the end behavior of $y = x^3 - x^2$.

Solution We create Table 5.8 and observe that as x approaches $-\infty$, the values of y approach $-\infty$. As x approaches ∞, the values of y approach ∞.

Table 5.8

x	y
−10,000	−1,000,100,000,000
−1000	−1,001,000,000
−100	−1,010,000
−10	−1100
−1	−2
1	0
10	900
100	990,000
1000	999,000,000
10,000	99,990,000,000,000

Symbolically, we write

$$\text{as } x \to -\infty, y \to -\infty \text{ and as } x \to \infty, y \to \infty$$

It is important to keep end behavior in mind when extrapolating with polynomial function models. We illustrate this idea in the following example, where we see that extrapolation can give inaccurate predictions.

EXAMPLE 5 ■ Extrapolating Using a Cubic Regression Model

The data in Table 5.9 show the projected Internet usage (in hours per person per year) in the United States.

Table 5.9

Projected Internet Usage: Hours per Person per Year (Based on 1995–1999 Data)	
Years Since 1995 t	Usage per Person (hours per year) H
0	5
1	10
2	34
3	61
4	99
5	135
6	162
7	187
8	208
9	228

Source: Statistical Abstract of the United States, 2001, Table 1125

a. Referring to a scatter plot of the data, determine if a cubic regression model fits the data. Then find and graph the cubic regression model that shows the relationship between Internet usage and time.

b. Predict the Internet usage (hours per person per year) in the year 2012. Discuss the accuracy of this prediction.

c. In light of the results from part (b), determine a practical domain for the cubic regression model.

Solution

a. We analyze the scatter plot of the data shown in Figure 5.11. The data is concave up initially but changes to concave down, so a cubic polynomial function is appropriate.

Using a graphing calculator, we find the cubic regression model

$$L(t) = -0.4514t^3 + 6.125t^2 + 6.171t + 1.994$$

where t is measured in years since 1995 and L is Internet usage per person per year (in hours). The model has a coefficient of determination of $r^2 = 0.9982$ so it is a great fit for the data. The graph of the model is shown in Figure 5.12.

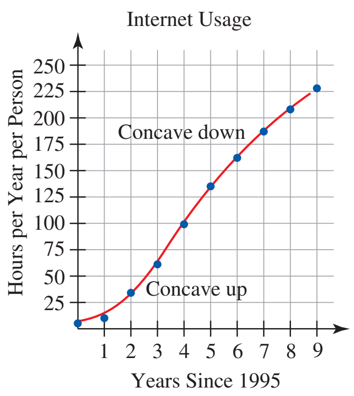


Figure 5.11

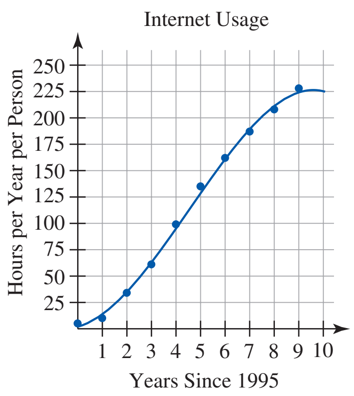


Figure 5.12

b. We use the cubic regression model to predict the amount of Internet usage in 2012. Since 2012 is 17 years since 1995, we evaluate L when $t = 17$.

$$\begin{aligned} L(17) &= -0.4514(17)^3 + 6.125(17)^2 + 6.171(17) + 1.994 \\ &= -341.2122 \end{aligned}$$

According to the model, people will spend a negative amount of time using the Internet, which is, of course, impossible. Observe from the graph in Figure 5.13 that the cubic model decreases quickly after 2005. Based on our life experience, we have no reason to anticipate that Internet usage will decrease.

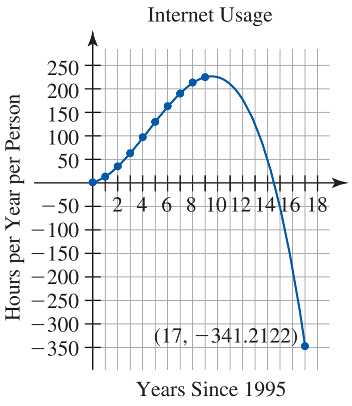


Figure 5.13

c. It is clear that when we extrapolate using the cubic model, we may obtain results that do not make sense. For this regression model, we estimate the practical domain to be [0, 10].

EXAMPLE 6 ■ **Graphing Polynomial Functions to Match a Verbal Description**

Graph a polynomial function f that meets the following criteria.

- Fourth degree.
- As $x \to \infty$, $f(x) \to \infty$.
- The graph of f has two inflection points.
- $f(x) = 0$ exactly twice.

Solution A function that is concave up, then concave down, then concave up will have exactly two inflection points. A function with exactly two x-intercepts will meet the condition $f(x) = 0$ exactly twice. A fourth-degree polynomial has the property that both ends approach the same value. Since we know that as $x \to \infty$, $f(x) \to \infty$, it must be true that as $x \to -\infty$, $f(x) \to \infty$. The graph in Figure 5.14 meets the criteria. (Note: There are other graphs that also meet this criteria.)

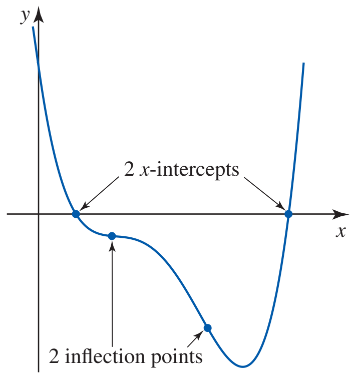


Figure 5.14

Relative Extrema of Polynomial Functions

A **relative maximum** occurs at the point where a graph changes from increasing to decreasing. A **relative minimum** occurs at the point where a graph changes from decreasing to increasing. The term **relative extrema** is used to refer to maxima and minima simultaneously. The graph of a polynomial function of degree n will have at most $n - 1$ relative extrema but it may have fewer.

EXAMPLE 7 ■ **Identifying Relative Extrema**

Graph the function $y = x^5 - 4x^4 + 4x^3$ and identify the points where the relative extrema occur.

Solution The function will have at most 4 relative extrema (since the degree is 5). Graphing the function as shown in Figure 5.15, we see that there is one point where a relative maximum occurs and one point where a relative minimum occurs.

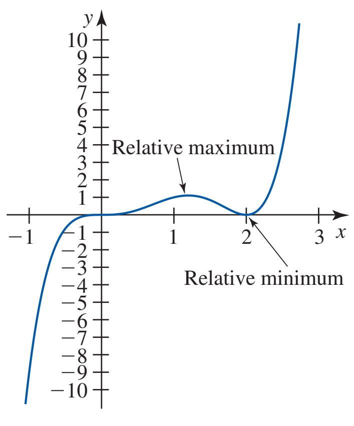


Figure 5.15

Inverses of Polynomial Functions

Not all polynomial functions have inverse functions; however, some do. Any polynomial function whose graph is strictly increasing or strictly decreasing will have an inverse function. Any polynomial function whose graph changes from increasing to decreasing or vice versa at any point in its domain will not have an inverse function.

EXAMPLE 8 ■ Finding the Inverse of a Polynomial Function

Find the inverse of $f(x) = x^5$.

Solution

$$\begin{aligned} y &= x^5 \\ (y)^{1/5} &= (x^5)^{1/5} \\ y^{1/5} &= x \\ f^{-1}(y) &= y^{1/5} \end{aligned}$$

The inverse of $f(x) = x^5$ is $f^{-1}(y) = y^{1/5}$.

SUMMARY

In this section you learned how to determine which higher-polynomial function can best model a real-world situation. You also learned how to describe the graph of any polynomial function by using the language of rate of change.

5.1 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, numerical representations of three functions are shown in a table. Each function is either linear, quadratic, cubic, or none of these. Use successive differences to identify each function appropriately.

1.

x	$f(x)$	$g(x)$	$h(x)$
0	−1.00	0.00	2.50
2	−0.92	−0.88	5.00
4	−0.68	−3.04	7.50
6	−0.28	−5.76	10.0
8	0.28	−8.32	12.5
10	1.00	−10.00	15.0
12	1.88	−10.08	17.5
14	2.92	−7.84	20.0
16	4.12	−2.56	22.5
18	5.48	6.48	25.0

2.

x	$f(x)$	$g(x)$	$h(x)$
0	5.00	0.76	1.00
2	5.51	1.32	53.28
4	6.08	2.00	111.24
6	6.70	2.80	176.56
8	7.39	3.72	250.92
10	8.14	4.76	336.00
12	8.98	5.92	433.48
14	9.90	7.20	545.04
16	10.91	8.60	672.36
18	12.03	10.12	817.12

3.

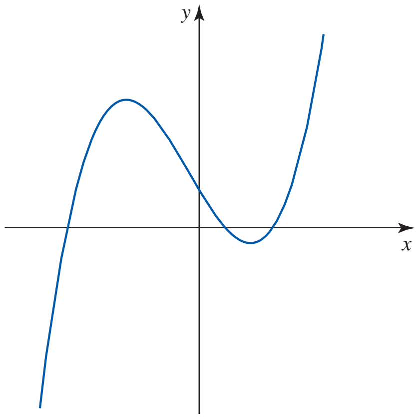
x	$f(x)$	$g(x)$	$h(x)$
−5	−105	−5	11.27
−4	−48	5	9.58
−3	−15	15	8.14
−2	0	25	6.92
−1	3	35	5.88
0	0	45	5.00
1	−3	55	4.25
2	0	65	3.61
3	15	75	3.07
4	48	85	2.61

4.

x	$f(x)$	$g(x)$	$h(x)$
0	0	0	0.09
1	4	9	0.10
2	0	16	0.11
3	−6	21	0.13
4	−8	24	0.14
5	0	25	0.17
6	24	24	0.20
7	70	21	0.25
8	144	16	0.33
9	252	9	0.50

In Exercises 5–8, examine the given graph and indicate the number of times the concavity changes. Use this result to determine which type of polynomial function is represented by the graph.

5.



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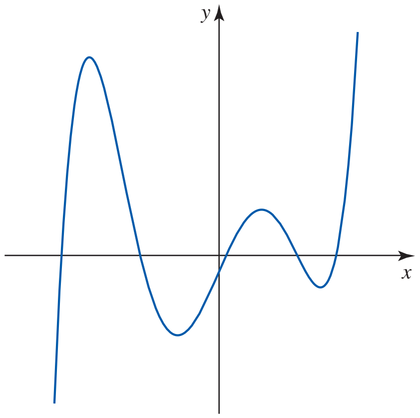

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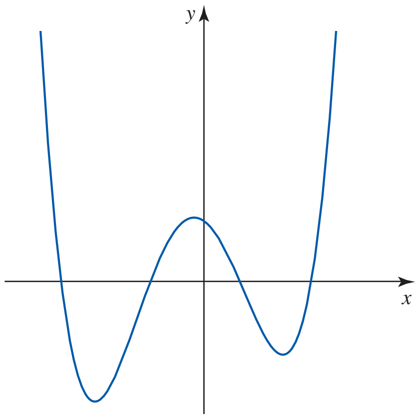

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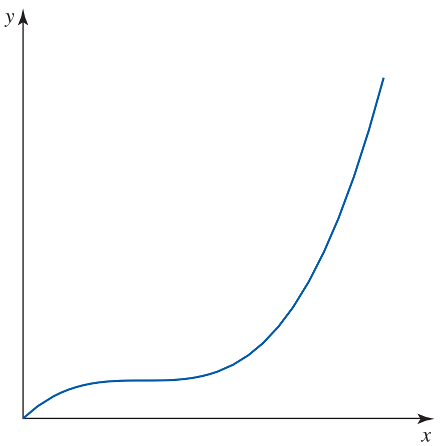


In Exercises 9–16,

a. Describe the end behavior of the given polynomial function.

b. Make a table of values that confirms the end behavior you described. Create your table in such a way that it shows what happens to function values as $x \to \infty$ and $x \to -\infty$.

9. $y = -2x^2 - 12x - 10$

10. $y = x^3 - 6x + 1$

11. $y = -3x^3 + 2x^2 - 2x - 3$

12. $y = x^4 - 2x^2 - 5$

13. $y = x^5 - 3x^3 + 2$

14. $y = 0.5x^2 - 10x - 5$

15. $y = -x^5 + x^4 + 2x^3 + 3x^2$

16. $y = x^6 - 10x^5 - x - 5$

In Exercises 17–22, sketch the graph of a function with the given description. Then classify the function by its degree.

17. The function is always increasing. It is initially concave up, then concave down, then concave up.

18. The function is initially increasing and concave up. It continues to increase but changes to be concave down. Finally, the function decreases.

19. The function is always increasing at a constant rate.

20. The function is initially decreasing and concave up. The function then changes to increasing but remains concave up. Finally, the function changes to concave down but remains increasing.

21. The function is always increasing and concave down.

22. The function is always increasing. It is initially concave down but changes to concave up before returning to concave down.

In Exercises 23–27, sketch the graph of a polynomial function that has the given characteristics.

23.
- Third degree
- As $x \to \infty$, $f(x) \to -\infty$
- 1 maximum and 1 minimum

24.
- Fourth degree
- As $x \to \infty$, $f(x) \to \infty$
- 2 minimums and 1 maximum
- $f(x) \neq 0$

25.
- Fourth degree
- As $x \to \infty$, $f(x) \to -\infty$
- 2 maximums and 1 minimum
- $f(x) = 0$ exactly twice

26.
- Fifth degree
- As $x \to \infty, f(x) \to \infty$
- As $x \to -\infty, f(x) \to -\infty$
- Always increasing

27.
- Second degree
- As $x \to \infty, f(x) \to -\infty$
- $f(x) = 0$ exactly once

SHOW YOU KNOW

28. Explain how to use successive differences to determine if a numerical representation of a function is linear, quadratic, cubic, quartic, etc.

29. Explain how to use rates of change and concavity to determine which polynomial type would best model a scatter plot of data.

30. How can we tell if a given function is a polynomial function?

31. Explain why the following function is or is not a polynomial function:

$$f(x) = 3x^2 - \sqrt{x} + 8$$

32. Explain why the following function is or is not a polynomial function:

$$f(x) = 2x^5 - x\sqrt{2} + \frac{1}{3}$$

33. How do we determine the end behavior of a polynomial function? Why is it important to understand this end behavior when modeling real-world data?

MAKE IT REAL

34. Drug Use Among Eighth Graders The data in the table show the percentage of eighth graders who admit to using the drug Ecstasy.

Years Since 1990 t	Drug Use (percent) D
6	3.4
7	3.2
8	2.7
9	2.7
10	5.2
11	4.3
12	3.2

Source: www.ojp.usdoj.gov

a. Make a scatter plot of these data.

b. Which type of polynomial function (quadratic, cubic, quartic) does the scatter plot best represent?

35. Adjustable Rate Mortgage In 2001, adjustable rate mortgages (ARMs) were offered at different rates at different times of the year. The table shows the average rate for each month in 2001.

Month t	Interest Rate (percent) P
1	6.69
2	6.49
3	6.35
4	6.24
5	6.17
6	6.08
7	6.07
8	5.89
9	5.74
10	5.48
11	5.38
12	5.55

Source: www.hsh.com

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a quartic function best models the data.

b. Use regression to find the quartic model equation.

36. U.S. Homicide Rate The U.S. homicide rate in the 1990s is shown in the table.

Years Since 1990 t	Homicides (per 100,000 people) H
0	9.4
1	9.8
2	9.3
3	9.5
4	9
5	8.2
6	7.4
7	6.8
8	6.3
9	5.7
10	5.5

Source: www.infoplease.lycos.com

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a cubic function best models the data.

b. Use regression to find the cubic model equation.

37. Average Hotel Room Rate The average hotel room rate is shown in the table.

Years Since 1990 t	Room Rate (dollars) R		Years Since 1990 t	Room Rate (dollars) R
0	57.96		5	66.65
1	58.08		6	70.93
2	58.91		7	75.31
3	60.53		8	78.62
4	62.86		9	81.33

Source: Statistical Abstract of the United States, 2001, Table 1266

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a cubic function best models the data.

b. Use regression to find the cubic model equation.

38. Ford Employee Earnings The average hourly earnings for a Ford Motor Company employee increased throughout the 1990s. However, the rate of increase was not constant, as shown in the table.

Years Since 1990 t	Hourly Wage (dollars) W		Years Since 1990 t	Hourly Wage (dollars) W
1	19.10		7	22.95
2	19.92		8	24.30
3	20.94		9	25.58
4	21.81		10	26.73
5	21.79		11	27.38
6	22.30			

Source: Ford Motor Company 2001 Annual Report, p. 71

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a quartic function best models the data.

b. Use regression to find the quartic model equation.

c. Describe the end behavior for the function model and discuss whether or not you think this end behavior will accurately predict future earnings.

39. Nonbusiness Bankruptcy Filings The number of nonbusiness Chapter 11 bankruptcies filed in the United States is given in the table.

Years Since 1998 t	Bankruptcies B
0	981
1	731
2	722
3	745
4	894
5	966
6	935

Source: Statistical Abstract of the United States, 2006, Table 749

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a cubic function best models the data.

b. Use regression to determine the cubic model equation.

40. Mail Order and Supermarket Prescriptions The data show the number of mail order prescriptions filled as a function of the number of supermarket prescriptions filled for selected years.

Years Since 1995 t	Prescriptions Filled at Supermarkets (millions) s	Prescriptions Filled by Mail Order (millions) m
0	221	86
2	269	109
3	306	123
4	357	134
5	394	146
6	418	161
7	444	174
8	462	189
9	470	214

Source: Statistical Abstract of the United States, 2006, Table 126

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a cubic function best models the data.

b. Use regression to find the cubic model equation.

c. Describe the relationship between the two quantities. Specifically, describe what the concave up and concave down portions of the graph tell us about the relationship between the quantities.

■ STRETCH YOUR MIND

Exercises 41–47 are intended to challenge your understanding of higher-order polynomial functions. In Exercises 41–45, describe the end behavior of the function when $a > 0$ and again for when $a < 0$.

41. $y = ax + b$

42. $y = ax^2 + bx + c$

43. $y = ax^3 + bx^2 + cx + d$

44. $y = ax^4 + bx^3 + cx^2 + dx + e$

45. $y = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$

46. Write a general statement that describes the maximum number of x-intercepts for the general polynomial

$$y = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

47. Write a general statement that describes the end behavior of the general polynomial

$$y = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x + a_0$$

SECTION 5.2

Power Functions

LEARNING OBJECTIVES

- Use power functions to model real-world situations
- Use the language of rate of change to describe the behavior of power functions
- Determine whether a power function represents direct or inverse variation

GETTING STARTED

Many species of animals migrate from one area to another for the purpose of feeding, giving birth, or escaping seasonal climate changes. For some species, migration occurs every season. For example, certain species of birds migrate south for winter and return north in the summer. For other species, migration happens just once in a lifetime. For example, Pacific salmon are born in fresh water, migrate to ocean waters, and then return to their freshwater birthplace to breed before dying. An article published in the *Journal of Zoology* revealed that the migration speed of animals that run, swim, and fly depends on the mass of the animal and can be modeled using *power functions*. (*Source:* Hedenstrom, A. (2003), "Scaling Migration Speed in Animals That Run, Swim and Fly," *Journal of Zoology 259*, 155–160)

In this section we use the real-world context of animal migration speed to learn about power functions. In addition to looking at the equations and graphs of power functions, we investigate the rates of changes of these functions. We also discuss direct and inverse variation.

Power Functions

Migrating land animals, such as caribou, are called *runners*. Runners migrate seasonally from one region to another. The migration speed of such animals, in kilometers per day, is dependent on the mass of the animal, given in kilograms. In Table 5.10, we see the migration speed for several different runners. The mass given is the median mass for each animal.

Table 5.10

Migration Speed of Runners		
Animal	**Mass (kg) *m***	**Migration Speed (km/day) *V***
Grey wolf	40	14.5
Gazelle	55	15.1
Dall sheep	140	16.8
Zebra	270	18.2
Wildebeest	275	18.3
Polar bear	500	19.6
Caribou	600	20.1
American buffalo	900	21.1
African elephant	5500	26.2

Source: Hedenstrom (2003)

To understand the relationship between the mass of the migrating land animal and its migration speed, let's consider the average rate of change between data points, shown in Table 5.11.

Table 5.11

Mass (kg) *m*	**Migration Speed (km/day)** *V*	**Average Rate of Change (km/day/kg)**
40	14.5	$\dfrac{0.6 \text{ km/day}}{15 \text{ kg}} = 0.04 \dfrac{\text{km per day}}{\text{kg}}$
55	15.1	$\dfrac{1.7 \text{ km/day}}{85 \text{ kg}} = 0.02 \dfrac{\text{km per day}}{\text{kg}}$
140	16.8	$\dfrac{1.4 \text{ km/day}}{130 \text{ kg}} = 0.01 \dfrac{\text{km per day}}{\text{kg}}$
270	18.2	$\dfrac{0.1 \text{ km/day}}{5 \text{ kg}} = 0.02 \dfrac{\text{km per day}}{\text{kg}}$
275	18.3	$\dfrac{1.3 \text{ km/day}}{225 \text{ kg}} = 0.006 \dfrac{\text{km per day}}{\text{kg}}$
500	19.6	$\dfrac{0.5 \text{ km/day}}{100 \text{ kg}} = 0.005 \dfrac{\text{km per day}}{\text{kg}}$
600	20.1	$\dfrac{1.0 \text{ km/day}}{300 \text{ kg}} = 0.003 \dfrac{\text{km per day}}{\text{km}}$
900	21.1	$\dfrac{5.1 \text{ km/day}}{4600 \text{ kg}} = 0.001 \dfrac{\text{km per day}}{\text{kg}}$
5500	26.2	

Observe that the rate of change is approaching zero. That is, as the mass of the animal increases, the number of additional kilometers per day that it travels when 1 additional kilogram is added to its mass approaches 0 kilometers. In other words, the migration speed initially increases quickly for every 1-kilogram increase in mass, but as the mass increases, the migration speed increases at an ever-decreasing rate.

The scatter plot in Figure 5.16 confirms our findings from examining the average rates of change. When the animal mass is small, even small changes in the mass of the animal result in relatively large increases in the migration speed. As the animal mass increases, the migration speed increases at a smaller rate.

Functions that demonstrate this behavior can often be modeled with a **power function**.

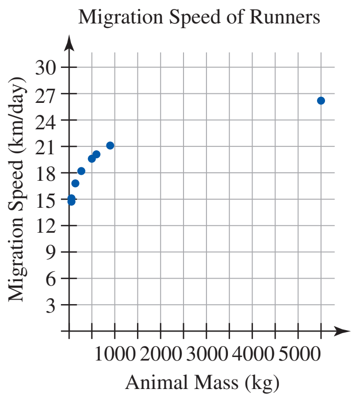


Figure 5.16

> ## Power Function
>
> A function with equation of the form
>
> $$y = ax^b$$
>
> where a and b are constants, is called a **power function**.

One main difference between a power function and a polynomial function is that in a power function the exponent, b, can take on any real-number value rather than just positive integer values. Also, a power function is a single-term function whereas a polynomial function may have multiple terms.

EXAMPLE 1 ■ Using a Power Function in a Real-World Context

Create a power function to describe the relationship between the mass of a runner and its migration speed as shown in Table 5.12. Use this model to determine the migration speed of the bull moose, which typically has a mass of around 650 kilograms.

Table 5.12

Migration Speed of Runners		
Animal	Mass (kg) m	Migration Speed (km/day) V
Grey wolf	40	14.5
Gazelle	55	15.1
Dall sheep	140	16.8
Zebra	270	18.2
Wildebeest	275	18.3
Polar bear	500	19.6
Caribou	600	20.1
American buffalo	900	21.1
African elephant	5500	26.2

Source: Hedenstrom (2003)

Solution Using the Technology Tip at the end of this section, we find a power regression model

$$V(m) = 9.31m^{0.12}$$

where m is the mass of the animal in kilograms and V is the migration speed in kilometers per day. We use this model to find the migration speed of a bull moose with a mass of 650 kilograms.

$$\begin{aligned} V(m) &= 9.31m^{0.12} \\ V(650) &= 9.31(650^{0.12}) \\ &= 20.25 \end{aligned}$$

According to the model, the migration speed of a bull moose is 20.25 kilometers per day.

Let's check to see if the function $V(m) = 9.31m^{0.12}$ makes sense for the data showing the migration speed of runners. Recall that the scatter plot of the data was concave down, indicating that as the mass of the animal increases, the speed of the animal increases by a lesser amount. Table 5.13 shows that the function $V(m) = 9.31m^{0.12}$ is also concave down: As m increases, the value of $m^{0.12}$ increases by a lesser amount.

Table 5.13

m	$m^{0.12}$	Increase		m	$m^{0.12}$	Increase
0	0	1.213		30	1.504	0.028
5	1.213	0.105		35	1.532	0.025
10	1.318	0.066		40	1.557	0.022
15	1.384	0.049		45	1.579	0.020
20	1.433	0.039		50	1.599	
25	1.472	0.032				

Power Functions with $x \geq 0$ and $0 < b < 1$

In Example 1, we looked at the power function $V(m) = 9.31m^{0.12}$. Notice that the exponent b had the property that $0 < b < 1$ and that $m \geq 0$. Figure 5.17 illustrates the behavior of several power functions $y = ax^b$ with the characteristic that $x \geq 0$ and $0 < b < 1$.

We see that each power function contains the point $(1, a)$ since $1^b = 1$ for all b. Further, as b gets closer and closer to 1, the power function approaches the linear power function $y = ax^1$.

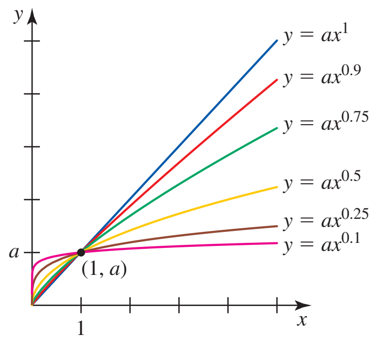


Figure 5.17

JUST IN TIME ■ SOLVING POWER FUNCTION EQUATIONS

To solve a power function equation $c = ax^b$ for x, apply the following steps.

General Procedure for $c = ax^b$

1. Divide each side by a.

$$\frac{c}{a} = x^b$$

2. Raise each side to $\frac{1}{b}$.

$$\left(\frac{c}{a}\right)^{1/b} = (x^b)^{1/b}$$

3. Simplify.

$$x = \left(\frac{c}{a}\right)^{1/b}$$

Specific Procedure for $58 = 27x^{0.75}$

1. $\frac{58}{27} = x^{0.75}$

2. $\left(\frac{58}{27}\right)^{1/0.75} = (x^{0.75})^{1/0.75}$

3. $x = \left(\frac{58}{27}\right)^{1/0.75} \approx 2.77$

When solving power function equations using the streamlined method shown here, it is important to remember the meaning of the rational exponent. For example, since $0.75 = \frac{3}{4}$, we have

$$\begin{aligned} x &= \left(\frac{58}{27}\right)^{1/0.75} \\ &= \left(\frac{58}{27}\right)^{\frac{1}{3/4}} \\ &= \left(\frac{58}{27}\right)^{4/3} \\ &= \sqrt[3]{\left(\frac{58}{27}\right)^4} \end{aligned}$$

EXAMPLE 2 ■ Using a Power Function in a Real-World Context

Using the Technology Tip at the end of this section, we determine a model for the migration speed of a swimmer (such as a whale, salmon, or shark): $V(m) = 5.41m^{0.16}$, where m is the mass of the swimmer in kilograms. Given that the bottlenose dolphin has a migration speed of 13 kilometers per day, use the model to determine the mass of the dolphin.

Solution In this case, we know the value of V, the migration speed, and need to find the mass, m.

$$V(m) = 5.41m^{0.16}$$
$$13 = 5.41m^{0.16}$$
$$\frac{13}{5.41} = m^{0.16}$$
$$m = \left(\frac{13}{5.41}\right)^{1/0.16}$$
$$m \approx 239.7$$

According to the model, the bottlenose dolphin that has a migration speed of 13 kilometers per day has a mass of 239.7 kilograms (about 530 pounds).

JUST IN TIME ■ RATIONAL EXPONENTS AND RADICALS

Rational exponents can be expressed using radical notation. For example,

$$x^{1/2} = \sqrt{x}$$
$$x^{1/3} = \sqrt[3]{x}$$
$$x^{1/4} = \sqrt[4]{x}$$

If the exponent is not a unit fraction like these, we use properties of exponents to first rewrite the expression.

$$x^{2/3} = (x^2)^{1/3} = \sqrt[3]{x^2}$$
$$x^{0.35} = x^{35/100} = x^{7/20} = (x^7)^{1/20} = \sqrt[20]{x^7}$$

These radical expressions can also be written as

$$\sqrt[3]{x^2} = (\sqrt[3]{x})^2$$
$$\sqrt[20]{x^7} = (\sqrt[20]{x})^7$$

In general,

$$x^{m/n} = \sqrt[n]{x^m} = (\sqrt[n]{x})^m$$

EXAMPLE 3 ■ Using a Power Function in a Real-World Context

Birds that migrate are classified by how they fly: flapping flight or thermal-soaring flight. In this example, we focus on birds that use flapping flight.

a. Use the data in Table 5.14 to create a scatter plot of migration speed versus mass.

b. Describe the relationship between the variables using the language of rate of change.

c. Find a power function model for the data.

Table 5.14

Migration Speed of Flyers		
Animal	Mass (kg) m	Migration Speed (km/day) V
Passerine	0.025	107.63
Arctic tern	0.110	81.22
Red knot shorebird	0.150	76.57
Duck	0.5	60.92
Canada goose	5	39.33
Tundra swan	7.26	36.64
Trumpeter swan	12.7	32.95

Source: Hedenstrom (2003)

Solution

a. Figure 5.18 shows the scatter plot of the data.

b. The migration speed decreases as the mass of the bird increases. The graph shows that when the mass is small, a small increase in mass results in a large decrease in migration speed. When the mass is larger, a similar increase in mass results in a significantly smaller decrease in migration speed.

c. We use the Technology Tip at the end of this section to find the model $s(m) = 53.4m^{-0.190}$ for the migration speed of flapping-flight migrators.

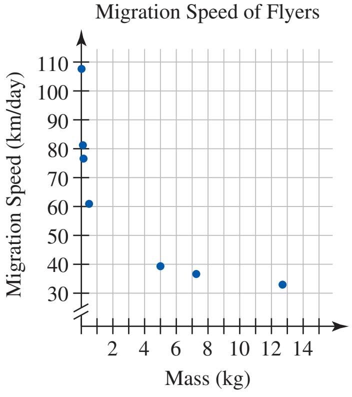


Figure 5.18

■ Power Functions with $x > 0$ and $b < 0$

Note that the power function model in Example 3 has a negative exponent. Figure 5.19 illustrates the behavior of such power functions with the characteristic that $x > 0$ and $b < 0$.

Each power function contains the point $(1, a)$. Also, the more negative the value of b, the more quickly the function values approach 0. Furthermore, as b approaches 0, the power function gets closer and closer to the linear power function $y = a$.

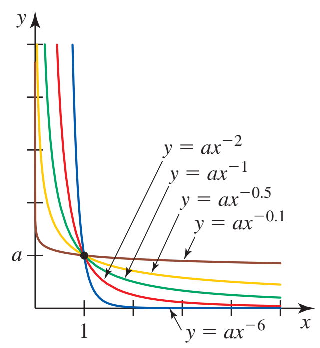


Figure 5.19

■ Direct and Inverse Variation

The power function $y = ax^b$ with $a > 0$ is an *increasing* function if $b > 0$ and a *decreasing* function if $b < 0$. For increasing power functions, the quantities x^b and y are said to vary *directly*. For decreasing power functions, the quantities x^b and y are said to vary *inversely*.

Direct Variation

A power function $y = ax^b$ with $b > 0$ represents direct variation. We say that "y varies directly with x^b" or "y is directly proportional to x^b." a is called the **constant of proportionality**.

A power function $y = ax^b$ with $b < 0$ may be rewritten as $y = ax^{-c}$ where c is a positive number equal to $|b|$. This alternate form is useful for interpreting inverse variation.

Inverse Variation

A power function $y = ax^{-c} = \dfrac{a}{x^c}$ with $c > 0$ represents inverse variation. We say that "y varies inversely with x^c" or "y is inversely proportional to x^c." a is called the **constant of proportionality**.

Many real-world quantities are directly proportional to each other, as shown in Table 5.15. Some quantities that are inversely proportional to each other are shown in Table 5.16.

Table 5.15

Related Quantities	Formula	In Words
Cost of a fuel purchase and the amount of fuel bought	$C = kg$	The cost of a fuel purchase is directly proportional to the amount of fuel purchased (in gallons). k is the constant of proportionality and represents the fuel price per gallon.
Area of a circle and its radius	$A = \pi r^2$	The area of a circle is directly proportional to the square of its radius. π is the constant of proportionality.
Blood flow in an artery and the radius of the artery	$F = kr^4$	The rate at which blood flows in an artery (in mL per minute) is directly proportional to the fourth power of the radius of the artery. k is the constant of proportionality.

Table 5.16

Related Quantities	Formula	In Words
Average earnings per hour when paid a fixed amount of money to complete a task and hours worked	$A = \dfrac{k}{x}$	The average earnings per hour, A, is inversely proportional to the amount of hours worked, x. k is the constant of proportionality and represents the fixed amount of money paid for the job.
Length of a 4-cubic-foot box with equal height and width and box width	$L = \dfrac{4}{w^2}$	The length of a box, L, with equal height and width, w, is inversely proportional to the square of the width with constant of proportionality 4.

JUST IN TIME ■ NEGATIVE EXPONENTS

Recall the following property of negative exponents:

$$x^{-p} = \frac{1}{x^p}$$

We can make sense of this rule by investigating patterns.

$$x^4 = x \cdot x \cdot x \cdot x$$

$$x^3 = \frac{x \cdot x \cdot x \cdot x}{x} = x \cdot x \cdot x$$

$$x^2 = \frac{x \cdot x \cdot x}{x} = x \cdot x$$

$$x^1 = \frac{x \cdot x}{x} = x$$

For each decrease in 1 of the exponent, we remove one factor of x.

$$x^0 = \frac{x}{x} = 1$$

Continuing to decrease the exponent by 1 and continuing to divide by x produces the following pattern.

$$x^{-1} = \frac{1}{x}$$

$$x^{-2} = \frac{\frac{1}{x}}{x} = \frac{1}{x^2}$$

$$x^{-3} = \frac{\frac{1}{x^2}}{x} = \frac{1}{x^3}$$

EXAMPLE 4 ■ Modeling an Inverse Variation Relationship

Table 5.17 shows the top five countries to which the United States exports cotton.

a. By analyzing a scatter plot of the data and investigating rates of change, explain why a power function may best model the data.

b. Determine a power regression model to represent the data and use it to describe the relationship between the amount of cotton exports and the ranking of the country.

c. According to the model, how much cotton would the United States export to a country whose rank is 6?

Table 5.17

Ranking of Country r	Cotton Exports (in thousand metric tons) C
1. China	1234
2. Turkey	478
3. Mexico	329
4. Indonesia	231
5. Thailand	160

Source: Statistical Abstract of the United States, 2007, Table 825

Solution

a. Using Figure 5.20, we analyze the scatter plot of the data to determine if a power function might model the situation. We observe the rate of change is initially very dramatic but lessens as the country rank increases. This is characteristic of a power function describing an inverse variation relationship between the quantities.

b. We use the Technology Tip at the end of this section to calculate the power regression model for this situation, and graph the model in Figure 5.21. We find the power function $C(r) = 1208r^{-1.229}$ models the situation well. We can also express this model as $C(r) = \dfrac{1208}{r^{1.229}}$ and say that the amount of cotton exports, C, varies inversely (with a constant of proportionality of 1208) with the rank of the country raised to the power of 1.229.

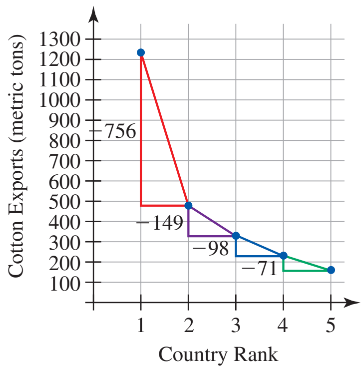


Figure 5.20

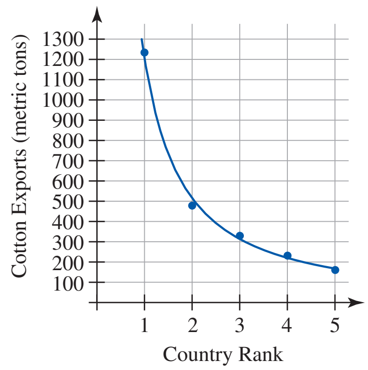


Figure 5.21

c. To determine the amount of cotton exported to the sixth-ranked country, we substitute $r = 6$ into the model.

$$C(r) = \frac{1208}{r^{1.229}}$$

$$C(6) = \frac{1208}{6^{1.229}}$$

$$C(6) \approx 133.6$$

According to the model, the sixth-ranked country receives about 133.6 metric tons of cotton from the United States.

■ Inverses of Power Functions

Power functions that are strictly increasing or strictly decreasing will have an inverse function. The process for finding the inverse is identical for all such functions. We calculate the inverse of $f(x) = ax^n$:

$$y = ax^n$$

$$\frac{y}{a} = x^n$$

$$\left(\frac{y}{a}\right)^{1/n} = (x^n)^{1/n}$$

$$x = \left(\frac{y}{a}\right)^{1/n}$$

$$f^{-1}(y) = \left(\frac{y}{a}\right)^{1/n}$$

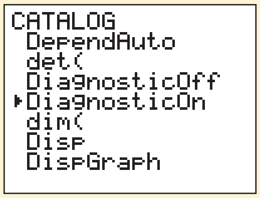
Although the term *inverse* is used when discussing *inverse variation* and *inverse function*, the two concepts are not synonymous.

SUMMARY

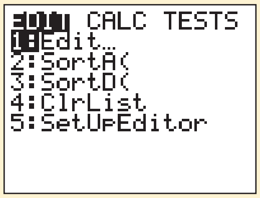
In this section you discovered how power functions behave and saw how to use a power function to model real-world situations. You also learned how to use the language of rate of change to describe a power function model. Finally, you learned that power functions can represent directly proportional or inversely proportional relationships.

TECHNOLOGY TIP ■ POWER REGRESSION

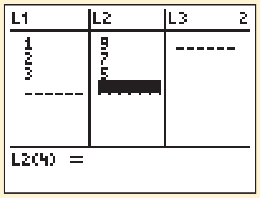
1. Press [2nd] then [0], scroll to `DiagnosticOn` and press [ENTER] twice. This will ensure that the correlation coefficient r and the coefficient of determination r^2 will appear.



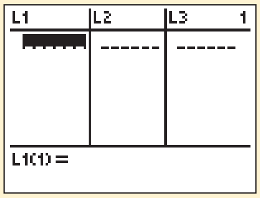
2. Bring up the Statistics Menu by pressing the [STAT] button.



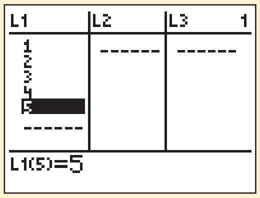
3. Bring up the List Editor by selecting `EDIT` and pressing [ENTER].



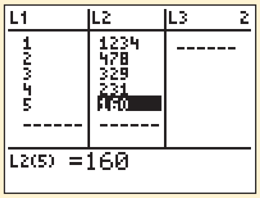
4. If there are data in the lists, clear the lists. Use the arrows to move the cursor to the list heading, `L1`, then press the [CLEAR] button and press [ENTER]. This clears all of the list data. Repeat for each list with data. (Warning: Be sure to use [CLEAR] instead of [DELETE]. [DELETE] removes the entire column.)



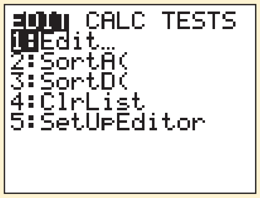
5. Enter the numeric values of the *inputs* in list `L1`, pressing [ENTER] after each entry.



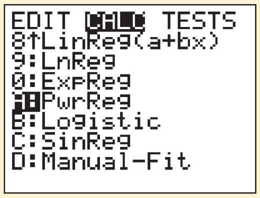
6. Enter the numeric values of the *outputs* in list `L2`, pressing [ENTER] after each entry.



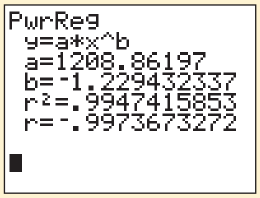
7. Return to the Statistics Menu by pressing the [STAT] button.



8. Bring up the Calculate Menu by using the arrows to select `CALC`. Use arrows to move down to `A:PwrReg`.



9. Calculate the power equation of the model by selecting `A:PwrReg` and pressing [ENTER] twice. The power regression model is $y = 1208.86x^{-1.22943}$ and has correlation coefficient $r = -0.9974$.



5.2 EXERCISES

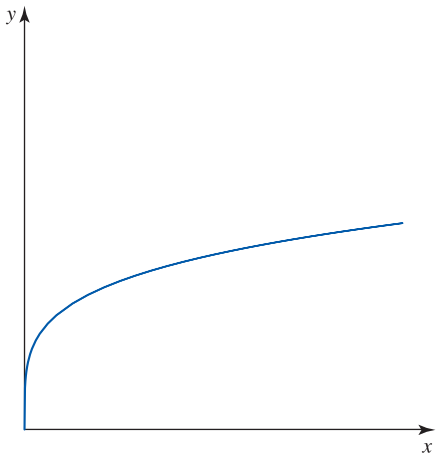
SKILLS AND CONCEPTS

In Exercises 1–10, a power function is given in tabular or graphical form. Determine whether the power function represents direct or inverse variation and explain how you know.

1.

x	y
0	0
2	3.138
4	4.925
6	6.410
8	7.728
10	8.934
12	10.06
14	11.12

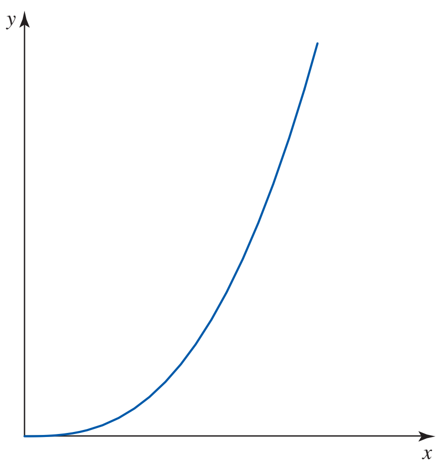
2.



3.

x	y
1	75.0
2	24.7
3	12.9
4	8.16
5	5.71
6	4.27

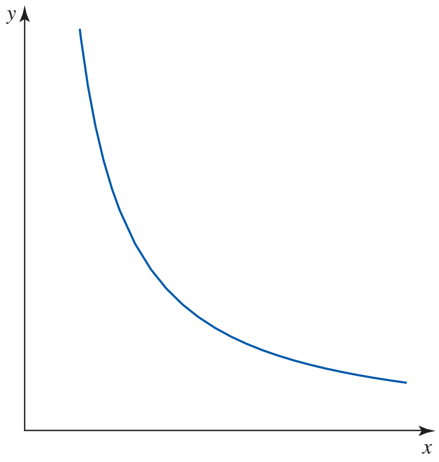
4.



5.

x	y
0	0
1	0.25
2	1.6245
3	4.85476
4	10.5561
5	19.2823
6	31.5463
7	47.8305

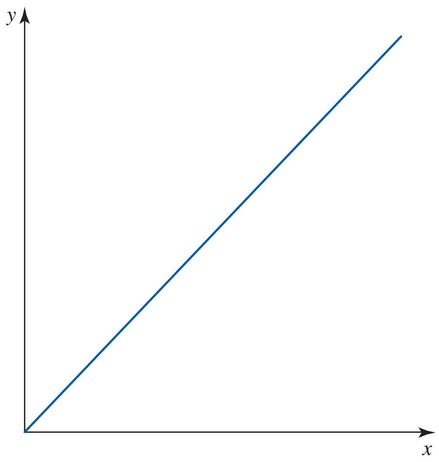
6.



7.

x	y
5	2.55402
10	1.19149
15	0.762765
20	0.555851
25	0.434868
30	0.355843
35	0.300342
40	0.259314

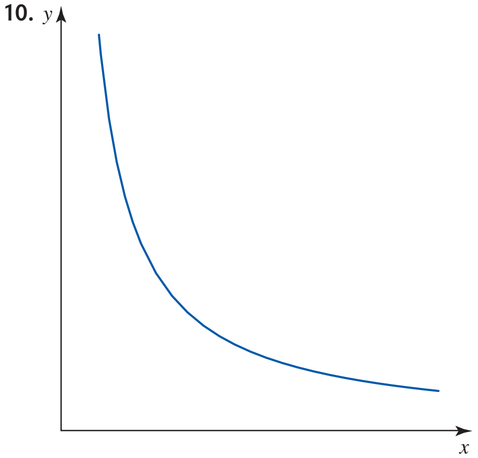
8.



9.

x	y
2	4
4	16
6	36
8	64
10	100
12	144

10.



In Exercises 11–15, write a power function representing the verbal statement.

11. The area, A, of a square is directly proportional to the square of the length of one of its sides, s.

12. A quantity W is inversely proportional to the square root of a quantity n.

13. Your weekly earnings, W, are directly proportional to the number of hours you work, h, with a constant of proportionality of 7.95.

14. The volume of a cylinder with a height of 4 centimeters is directly proportional to the square of the radius of the base of the cylinder with a constant of proportionality of 4π.

15. The time, t, in hours, needed to drive a distance of 400 miles is inversely proportional to the average speed, s, in miles per hour.

In Exercises 16–25, solve the given equation.

16. $2x^{3/2} = 10$

17. $5g^{-2} = 25$

18. $4\sqrt[3]{t^5} = 50$

19. $1.5m^{0.65} = 25$

20. $\dfrac{5}{w^{2.5}} = 0.5$

21. $2\sqrt[4]{t^7} = 25$

22. $10x^{-2.5} = 80$

23. $4.25v^{0.38} = 15$

24. $\dfrac{1}{f^{1/2}} = 0.75$

25. $x^{2/3} = 16$

In Exercises 26–30, complete each table so that it accurately represents the given verbal description of a power function. Then write the equation of the function.

26. The value of y is inversely proportional to the square of the value of x with a constant of proportionality of 3.

x	1	2	3	4
y				

27. The value of y is inversely proportional to the square root of the value of x with a constant of proportionality of 1.

x	1	4	16	25
y				

28. The value of y is directly proportional to the cube of the value of x with a constant of proportionality of -2.

x	0	1	2	3
y				

29. The value of y is directly proportional to the value of x raised to the 0.25 power with a constant of proportionality of 5.

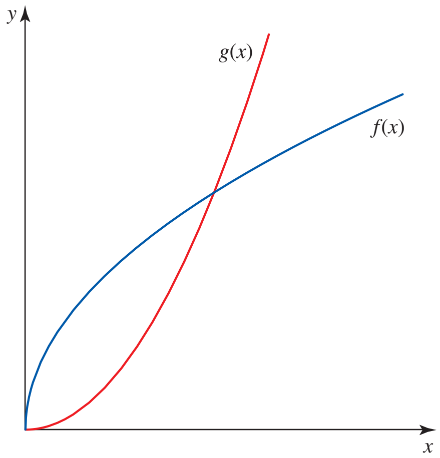
x	1	4	16	25
y				

30. The value of y is inversely proportional to the value of x raised to the 1.5 power with a constant of proportionality of 4.

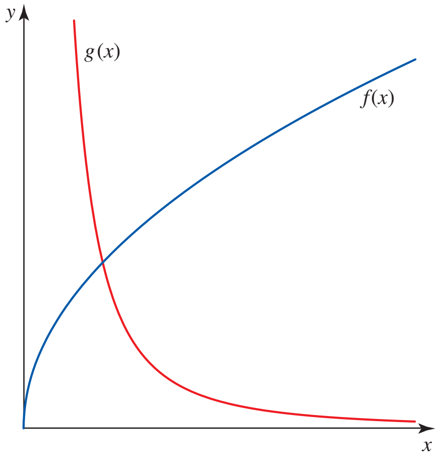
x	0	1	2	3
y				

SHOW YOU KNOW

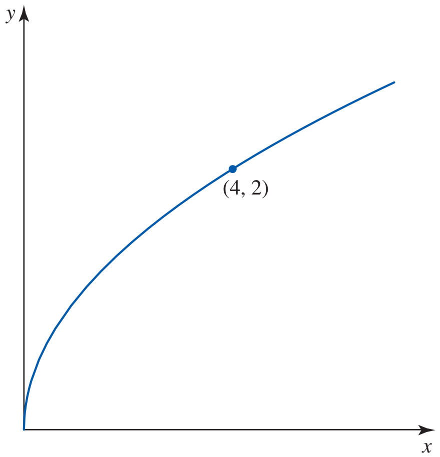
31. The graphs of two power functions are shown. One of the graphs represents the function $y = ax^n$ and the other represents the function $y = ax^{1/n}$ with n being an integer greater than 1. Explain which is which and how you can tell. Then find the coordinates of their point of intersection.



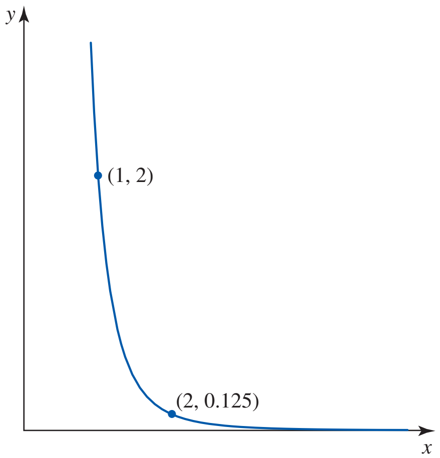
32. The graphs of two power functions are shown. One of the graphs represents the function $y = ax^{-n}$ and the other represents the function $y = ax^{1/n}$ with n being an integer greater than 1. Explain which is which and how you can tell. Then find the coordinates of their point of intersection.



33. Based on the information given in the following graph of a power function, can you determine the equation of the power function? Why or why not? If you can, find the equation of the power function. If not, describe the equation of the power function as much as you can.



34. Based on the information given in the following graph of a power function, can you determine the equation of the power function? Why or why not? If you can, find the equation of the power function. If not, describe the equation of the power function as much as you can.



35. Match each function with its graph. Explain how you were able to make each choice.

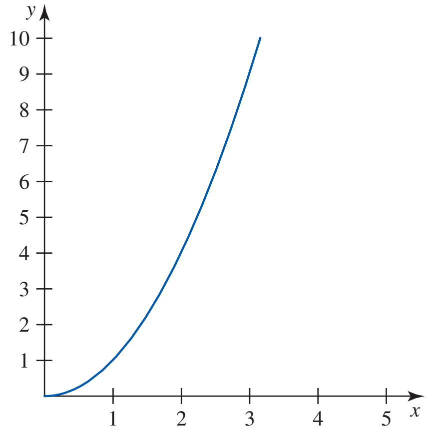
a. $y = x^{-2}$

b. $y = x^2$

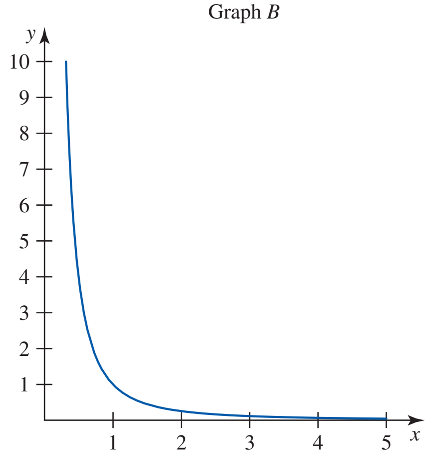
c. $y = x^{1/2}$

d. $y = x^{-1/2}$

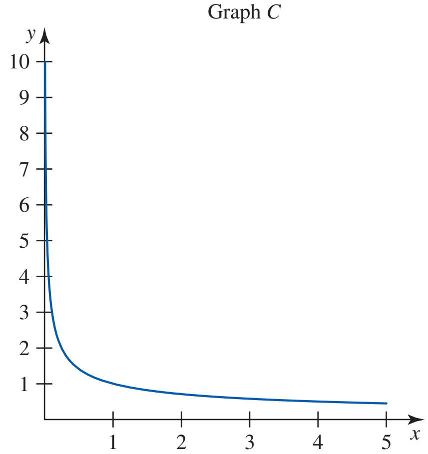
Graph *A*



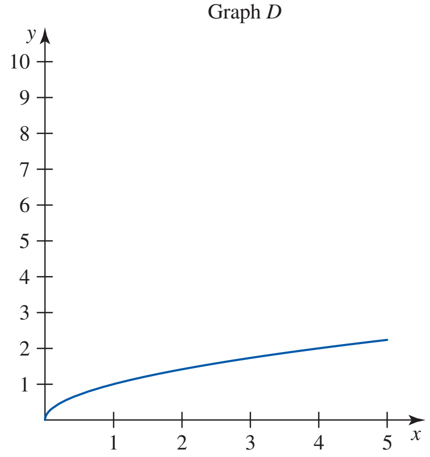
Graph *B*



Graph *C*



Graph D



■ MAKE IT REAL

36. Running Speed The table shows the maximum relative running speed, in meters per second, of several different animals compared to their body mass in kilograms.

a. Examine the data and explain why an inverse variation relationship might best model the situation.

b. Use a graphing calculator to find a power function model. Then use the model to describe the relationship between maximum relative running speed and body mass of these animals.

c. Use the power regression model to determine the mass of a zebra with a relative running speed of 8.16 meters per second.

Animal	Mass (kg) m	Relative Speed (m/s) V
Fallow deer	55	11.51
Desert warthog	85	12.38
Mountain goat	114	6.36
Antelope	227	6.65
Black wildebeest	300	10.97
Camel	550	3.03
European bison	865	5.4
Giraffe	1075	3.8
Black rhinoceros	1200	3.6
White rhinoceros	2000	1.79
Hippopotamus	3800	1.7
Asian elephant	4000	1.18
African elephant	6000	1.4

***Source:* *Journal of Experimental Biology 205*, 2897–2908**

37. Nail Sizes The common nail is used for many home improvement and construction projects. Nails are made in a variety of lengths to meet the specific needs of different projects. According to www.sizes.com, the rule of thumb is to use nails that are three times the thickness of the board being fastened with the nail.

a. This relationship between the length of the nail and the thickness of the board being fastened can be described using a power function. Will the power function be a direct or inverse variation? Explain.

b. What is the constant of proportionality?

c. Without using a calculator, write the power function that represents the rule of thumb described.

38. Nail Sizes The common nail comes in different lengths. In general, the longer the nail, the greater the diameter of the head of the nail. The table gives the length and diameter of a series of common nails.

Nail Length (inches) L	Nail Head Diameter (inches) D
1	0.0700
2	0.1130
3	0.1483
4	0.1920
5	0.2253
6	0.2625

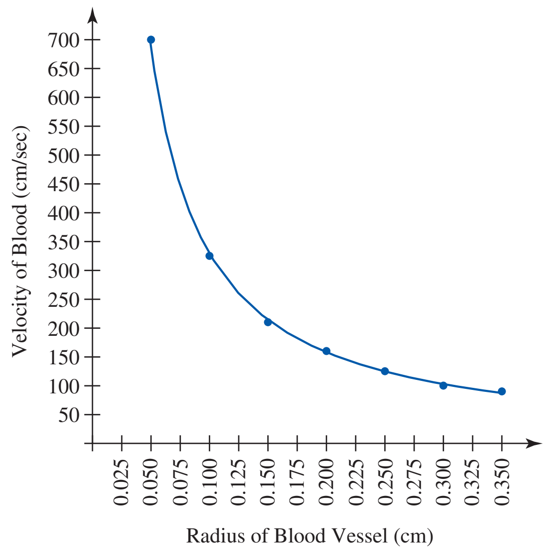
***Source:* www.sizes.com**

a. Assuming the data will be modeled by a power function $D = aL^b$, which of the following inequalities will the parameter b satisfy: $b < 0$, $0 < b < 1$, or $b > 1$? Explain.

b. Use a graphing calculator to find a power function to model the data. Then use the model to predict the nail head diameter of a nail with length 3.5 inches.

c. If you found a nail with head diameter 0.2492 inches, what is the length of the nail, according to the model?

39. Blood Velocity with Stenosis A stenosis is an abnormal narrowing in a blood vessel. As the blood vessel narrows, blood flow is often affected. The following graph models blood velocity relative to blood vessel radius.



***Source:* Adapted from data at www.personal.engin.umich.edu**

a. Explain why a power function model may be appropriate for this situation.

b. Will the power function model $y = ax^b$ have a value for b such that $b < 0$, $0 < b < 1$, or $b > 1$? Explain.

c. Use a graphing calculator to find a power function to model the relationship between the blood vessel radius and the velocity of blood.

d. What does the model suggest concerning blood flow as the blood vessel becomes increasingly large (less and less narrow)?

e. What does the model suggest concerning blood flow as the blood vessel becomes increasingly small (more and more narrow)?

40. **Wind Power** According to www.virtualsciencefair.org, the power generated by a windmill, in watts, is proportional (with a constant of proportionality of $\frac{1}{2}$) to the density of the air, the area swept out by the windmill rotor blade, and the cube of the velocity of the wind.

a. Write the given verbal description as a function where P is the power (W), D is the density of the air (kg/m^3), A is the area swept out by the rotor blade (m^2), and V is the velocity of the wind (m/s).

b. Suppose D and A are held constant. Describe the impact on P if the wind velocity, V, is doubled.

c. Suppose D and A are held constant. Describe the impact on P if the wind velocity, V, is tripled.

d. Suppose D and A are held constant. Describe the impact on P if the wind velocity, V, is cut in half.

41. **Terminal Velocity** Terminal velocity may be thought of as the maximum velocity obtained by a free-falling object. For example, when a skydiver first falls from the plane, she will free-fall faster and faster until terminal velocity is reached (unless the parachute is deployed first). Holding all other variables constant, terminal velocity is directly proportional to the square root of the mass (in kilograms) of the object falling.

a. Without using a calculator, write a power function for the terminal velocity, V, of a skydiver relative to the mass of the skydiver, m.

b. If the mass of the skydiver doubles, what will be the impact on the terminal velocity for that skydiver?

c. If the mass of the skydiver quadruples, what will be the impact on the terminal velocity for that skydiver?

d. If the mass of the skydiver is cut in half, what will be the impact on the terminal velocity for that skydiver?

42. **Hydroplaning Speeds** When airplanes land on a wet runway, *hydroplaning* may occur—the tires on the airplane slide across the top of the water on the runway rather than roll along the pavement. The pilot does not have complete control of the airplane while it is hydroplaning. The minimum speeds at which a landing airplane will hydroplane depend on the air pressure in the tires, as shown in the table.

Tire Pressure (psi) P	Minimum Speed (mph) S
25	45
50	63.64
75	77.942
100	90
125	100.62
150	110.23

Source: www.flightsafety.org

a. Calculate the rate of change between each pair of consecutive data points.

b. Do the rates of change indicate a power function may be appropriate? Explain.

c. Use a graphing calculator to find a power function that models the minimum speed as a function of the tire pressure.

d. If the air pressure of the tires is 110 psi, what is the minimum speed at which hydroplaning is predicted to occur?

e. If the pilot will be landing at 95 mph, what must be the air pressure in the tires so the plane will not hydroplane?

43. **Pendulum Swing** The time it takes for a pendulum to complete one period (swing forward then back to its initial position) depends on the length of the pendulum. Data collected from a pendulum experiment by students, showing the period (in seconds) and the length of the pendulum (in meters), are given in the table.

Length of Pendulum (meters) L	Period (seconds) P
0.145	0.7601
0.2305	0.9583
0.3056	1.1116
0.3815	1.2347
0.434	1.3339
0.483	1.3639
0.6	1.5532

Source: phoenix.phys.clemson.edu

a. Using rates of change, show that the function $P(L)$, the period of the pendulum relative to the length of the pendulum, can be modeled using a power function.

b. Use a graphing calculator to find a power function that models the data.

c. In the short story "The Pit and the Pendulum" by Edgar Allen Poe, a 30-foot (about 9.1-meter) pendulum swings, ultimately to the demise of the main character. What does the model predict as the period of this pendulum?

d. If a pendulum's period is 2 seconds, what is the length of the pendulum according to the model?

44. Galileo Galilei Galileo Galilei is attributed with discovering that the distance traveled by a falling object is directly proportional to the square of the time that the object has been in the air. Prior to this, people thought that the time depended also on the mass of the object. Galileo showed that objects of different masses dropped at the same moment hit the ground at the same time. More specifically, the distance traveled by a falling object is proportional to the square of the time elapsed with a constant of proportionality of 16.

a. Write the equation that models the distance traveled d (in feet) as a function of time t (in seconds).

b. Use a graphing calculator to graph $d(t)$.

c. Use the model to find the average rate of change of a falling object over the first 2 seconds, the next 2 seconds, and the next 2 seconds. What can you say about the average rate of change of a falling object?

45. Light Intensity The intensity of a beam of light can provide valuable information about the source of the light. For example, astronomers sometimes measure the distance to a star by measuring the brightness of the star. In a classroom activity, students collected data showing the luminous intensity of light (in milliwatts per square centimeter) as a function of the distance (in meters) away from a light sensor, using a flashlight as the source of light.

Distance (meters) d	Light Intensity (mw/cm^2) L
0.5	0.462
0.6	0.341
0.7	0.280
0.8	0.203
0.9	0.161
1.0	0.164
1.1	0.111
1.2	0.097
1.3	0.091
1.4	0.091
1.5	0.080

a. Calculate the rates of change between consecutive data values. Do the rates of change indicate that a power function model may be appropriate for the data set?

b. Use a graphing calculator to find a power function that models the data.

c. For a light source 2 meters from the light sensor, use the model to predict the luminous intensity of the light.

d. If the luminous intensity of a light is known to be 0.25 mw/cm^2, how far from the light sensor is the light source?

46. Cell Phone Plans In 2007, T-Mobile offered the following My Faves™ for Individuals plan.

Minutes	Cost (dollars)
300	39.99
600	49.99
1000	59.99
1500	69.99

Source: **www.t-mobile.com**

a. Use a graphing calculator to find a power function to model the cost of the plan as a function of minutes.

b. According to the model, how many minutes would one get for $79.99?

■ STRETCH YOUR MIND

Exercises 47–51 are intended to challenge your understanding of power functions.

47. Graph several power functions of the form $y = ax^b$ and $b < 0$. Consider all values of x including $x < 0$. Write a description of the behavior of the resulting functions.

48. Graph several power functions of the form $y = ax^b$ and $0 < b < 1$. Consider all values of x including $x < 0$. Write a description of the behavior of the resulting functions.

49. Graph several power functions of the form $y = ax^b$ and $b > 1$. Consider all values of x including $x < 0$. Write a description of the behavior of the resulting functions.

50. Consider the following power functions: $f(x) = x^{0.25}$, $g(x) = x^{0.5}$, and $h(x) = x^{0.75}$. Write a convincing argument that shows that when $0 < x < 1$, $f(x) > g(x) > h(x)$ but when $x > 1$, $h(x) > g(x) > f(x)$.

51. Consider the following power functions: $f(x) = x^{-1}$, $g(x) = x^{-2}$, and $h(x) = x^{-4}$. Write a convincing argument that shows that when $0 < x < 1$, $f(x) < g(x) < h(x)$ but when $x > 1$, $h(x) < g(x) < f(x)$.

SECTION 5.3

Rational Functions

LEARNING OBJECTIVES

- Find and interpret the meaning of asymptotes in real-world applications
- Analyze rational function graphs and identify vertical and horizontal asymptotes as well as removable discontinuities
- Find the domain of rational functions
- Find the inverse of a rational function

GETTING STARTED

In 1989, the Exxon Valdez oil tanker struck a reef and spilled at least 11 million gallons of oil, polluting approximately 1180 miles of Alaskan coastline in Prince William Sound. This disastrous event has had far-reaching and long-lasting effects. As recently as 2006, an appeal in the 17-year-old federal court case was being convened to reconsider the punitive damages that ExxonMobil Corporation would be required to pay to the thousands of Alaskans whose lives were affected. (*Source:* www.answers.com/topic/exxon-valdez-oil-spill)

In this section, we investigate *rational functions* and see how real-world situations such as the cleanup costs of the Exxon Valdez oil spill can be modeled by such functions. Using the notion of rate of change, we look at vertical and horizontal asymptotes, removable discontinuities, vertical and horizontal intercepts, and domain.

■ Rational Functions

Recall that polynomial functions are defined as functions that have the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where n is a nonnegative integer. When one polynomial function is divided by another, a **rational function** is created.

Rational Function

A **rational function** is a function of the form

$$f(x) = \frac{p(x)}{q(x)}$$

where $p(x)$ and $q(x)$ are polynomial functions with $q(x) \neq 0$.

Cleanup efforts initially included relatively simple and inexpensive techniques, but the techniques became more complex and costly as time progressed. Rational functions can be used to model a variety of real-world phenomena, such as the cleanup cost for the Exxon Valdez oil spill in 1989. Assume that by employing a variety of cleanup techniques, pollutants from an oil spill can be removed. The initial removal of oil at sea is far easier and less expensive than the final stages of cleanup. In fact, it may be quite costly, or even impossible, to remove all of the pollutants. Explore North estimates the total cleanup cost for the Valdez spill was over 2.1 billion dollars. (*Source:* www.explorenorth.com) Based on this estimate and the cleanup processes involved, we created a rational function model for the Valdez oil spill cleanup cost: $C(p) = \dfrac{0.08p}{100 - p}$, which models the cleanup cost, C (in billions of dollars), to remove p percent of the pollutants. (Note: Because this model is based on a single data point and an awareness of the underlying behavior of cleanup costs, its accuracy may be limited.)

Vertical Asymptotes

Let's investigate the pollution cleanup function C by graphing it over a reasonable domain. Since p is the percentage of pollutants removed, anywhere from 0% to near 100% of the pollution could be removed. Therefore, we choose $0 \le p < 100$ as the domain. The graph is shown in Figure 5.22.

From the graph we see that this function appears to approach the red dashed vertical line, which is called a **vertical asymptote**. This line is not part of the graph of the function, but defines a kind of *boundary* for the function.

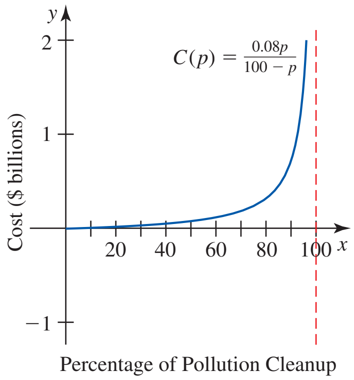


Figure 5.22

Vertical Asymptote

A **vertical asymptote** of a function $f(x)$ is a vertical line, $x = a$, that the graph of $f(x)$ approaches but does not cross. More formally, as x approaches a, $f(x)$ approaches $\pm\infty$. Symbolically, we write this as $x \to a, f(x) \to \pm\infty$.

Even though vertical asymptotes are not part of the graph of a rational function, they are often drawn because they are helpful in describing how the function behaves.

EXAMPLE 1 ■ Exploring Vertical Asymptotes in a Real-World Context

The cost of cleaning up p percent of the pollutants from an oil spill can be modeled by $C(p) = \dfrac{0.08p}{100 - p}$, where C is the cost in billions of dollars.

a. Create a table of values for C and ΔC in terms of p from $p = 0$ to $p = 100$ in increments of 25 percent.

b. Referring to the values of ΔC from part (a), describe what is happening to the cleanup cost as each additional 25 percent of pollutants is removed.

c. Explain why it is impossible to use the model to calculate the cost of removing 100 percent of the pollution.

d. Determine the equation of the vertical asymptote for $C(p)$ and justify why it is a vertical asymptote.

Solution

a. To create Table 5.18, we evaluate $C(p)$ for each of the values of p from 25 up to 100 percent. For instance, $C(25) = \dfrac{(0.08 \cdot 25)}{(100 - 25)} = \dfrac{2}{75} = 0.02667$. We cannot evaluate $C(100)$ because

$$\begin{aligned} C(100) &= \frac{(0.08 \cdot 100)}{(100 - 100)} \\ &= \frac{8}{0} \qquad \text{Division by 0 is undefined.} \\ &= \text{undefined} \end{aligned}$$

To fill in the ΔC column, we simply calculate the first differences.

Table 5.18

Percent of Pollution Removed p	Cost (\$ billions) C	Change in the Cleanup Cost (\$ billions) ΔC
0	0	0.02667
25	0.02667	0.05333
50	0.08000	0.16000
75	0.24000	unknown
100	undefined	

b. The rate of change shown in Table 5.18 tells us that the costs are continually increasing and at an increasing rate. We can see this as well on the graph shown in Figure 5.23. Notice the slope keeps getting steeper. The slopes of the tangent lines (red dashed lines) show us that the rate of change becomes extremely large near $p = 100$.

c. Recall that when we tried to calculate the cost of removing 100 percent of the pollutants, we got $C(100) =$ undefined. Consequently, we cannot determine the cost of removing 100 percent of the pollutants with the model.

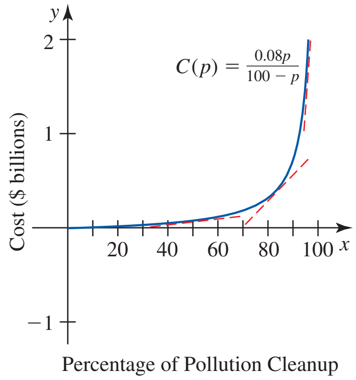


Figure 5.23

d. Since $C(p)$ is undefined at $p = 100$, $p = 100$ may be a vertical asymptote. We simply need to confirm that as $p \to 100$, $C(p) \to \infty$. To do this, we create a table of values for $C(p)$ and for the rate of change of $C(p)$ near $p = 100$. See Table 5.19.

Table 5.19

Percent of Pollution Removed P	Cost (\$ billions) C	Rate of Change (\$ billions per percent) $\frac{\Delta C}{\Delta p}$
93	1.06	0.19
94	1.25	0.27
95	1.52	0.40
96	1.92	0.67
97	2.59	1.33
98	3.92	4.00
99	7.92	cannot calculate
100	undefined	

Notice how rapidly the cost is increasing as p nears 100. Increasing the amount of pollution removed from 97 percent to 98 percent would cost 1.33 billion dollars.

Increasing the amount of pollution removed from 98 percent to 99 percent would cost an additional 4.00 billion dollars. We conclude that as $p \to 100$, $C(p) \to \infty$. Therefore, $p = 100$ is the vertical asymptote, shown as the red dashed vertical line in Figure 5.24. In the real-world context of oil cleanup, the cost to clean up a spill is relatively low at first but skyrockets as we get closer and closer to cleaning up 100 percent of the spill.

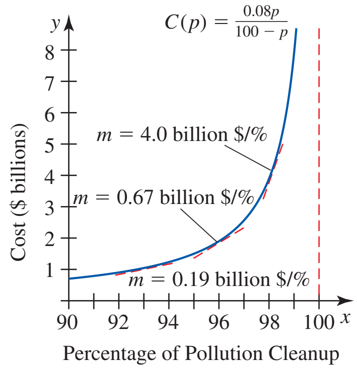


Figure 5.24

HOW TO: ■ FIND VERTICAL ASYMPTOTES

To find all vertical asymptotes of $f(x) = \dfrac{p(x)}{q(x)}$,

1. Factor the numerator $p(x)$ and denominator $q(x)$.
2. Cancel out any factors that the numerator and denominator have in common. (This puts the rational function in *simplified form*.)
3. Set the resultant denominator equal to zero and solve. The solutions to this equation are the vertical asymptotes of $f(x) = \dfrac{p(x)}{q(x)}$.

EXAMPLE 2 ■ Finding Vertical Asymptotes from a Function Equation

Determine the vertical asymptotes of the graph of $y = \dfrac{2x(x + 4)(x - 2)}{(x - 1)(x + 5)(x - 5)}$. Then explain the behavior of the function near its vertical asymptotes.

Solution Since the numerator and denominator do not have any factors in common, we simply need to determine the x-values that make the denominator equal 0: $x = 1$, $x = -5$, and $x = 5$. Consequently, vertical asymptotes occur at $x = 1$, $x = -5$, and $x = 5$. We confirm our conclusions with the graph in Figure 5.25. The asymptotes are indicated by the red dashed vertical lines.

Observe that as $x \to -5$ from the left, $y \to \infty$ and as $x \to -5$ from the right, $y \to -\infty$. As $x \to 1$ from the left, $y \to -\infty$ and as $x \to 1$ from the right, $y \to \infty$. As $x \to 5$ from the left, $y \to -\infty$ and as $x \to 5$ from the right, $y \to \infty$.

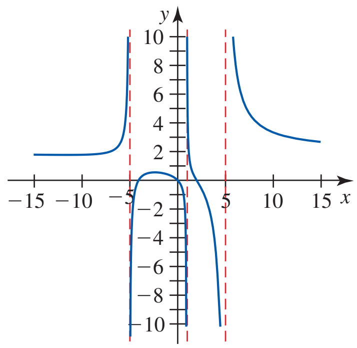


Figure 5.25

Removable Discontinuities (Holes)

To see what happens graphically if the numerator and denominator have a nonconstant factor in common, let's consider the rational function $f(x) = \dfrac{x^2 - x - 2}{x - 2}$. This function is undefined at $x = 2$ because the denominator equals zero when $x = 2$:

$$\begin{aligned} f(2) &= \frac{(2)^2 - 2 - 2}{2 - 2} \\ &= \frac{0}{0} \\ &= \text{undefined} \end{aligned}$$

However, the graph of f shown in Figure 5.26 appears linear, even at $x = 2$. To check this, we factor the numerator of f and simplify the function:

$$\begin{aligned} f(x) &= \frac{(x - 2)(x + 1)}{x - 2} \\ &= \left(\frac{x - 2}{x - 2}\right)(x + 1) \\ &= 1 \cdot (x + 1) \text{ for } x \neq 2 \\ &= x + 1 \end{aligned}$$

Thus the graph of f is the line $y = x + 1$ except at $x = 2$, where f is undefined. That is, although the line $y = x + 1$ contains the point (2, 3), the graph of f does not. We say the graph of f has a **removable discontinuity** (a hole) in it at the point (2, 3). We redraw the graph (Figure 5.27) to show this.

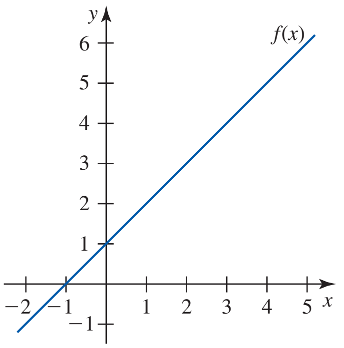


Figure 5.26

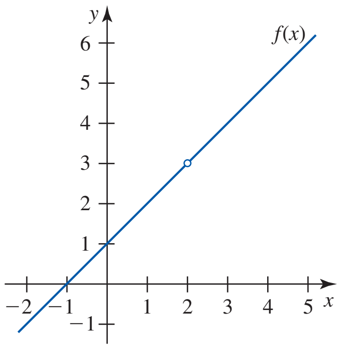


Figure 5.27

HOW TO: ■ FIND A REMOVABLE DISCONTINUITY (HOLE)

To find all removable discontinuities (holes) of $f(x) = \dfrac{p(x)}{q(x)}$,

1. Factor the numerator $p(x)$ and denominator $q(x)$.
2. Determine which factors the numerator and denominator have in common.
3. Set each of the common factors equal to zero and solve. The solutions to these equations are the removable discontinuities of $f(x) = \dfrac{p(x)}{q(x)}$.

Horizontal Asymptotes

A rational function may have a *horizontal asymptote* as well as vertical asymptotes. The *horizontal asymptote* of a rational function is a horizontal line, $y = b$, that the function approaches as the independent variable approaches $-\infty$ or ∞. To find horizontal asymptotes, we need to know how the output values of the function behave as the input values approach $\pm\infty$.

EXAMPLE 3 ■ Exploring Horizontal Asymptotes in a Real-World Context

When a new drug is first marketed, it is usually under a patent that restricts drug sales to the pharmaceutical company that developed it. This gives the company a chance to recoup its development costs, which average around \$800,000,000 for research, testing, and equipment. (*Source:* www.wikipedia.org)

Suppose the total cost C, in dollars, of producing g grams of a new drug is given by the linear function $C(g) = 800{,}000{,}000 + 10g$, where $10g$ represents the additional costs associated with manufacturing each gram of the drug.

a. Evaluate $C(0)$ and explain what the numerical answer means in its real-world context.

b. Find the slope of $C(g)$ and explain what the numerical answer means in its real-world context.

c. Form the rational function $A(g) = \frac{C(g)}{g}$ and explain what $A(g)$ represents in its real-world context.

d. Discuss the rate of change of $A(g)$ as g increases and explain what this tells about the cost of the new drug.

e. Complete Table 5.20 for the given values of g. Then describe what happens to $A(g)$ as g approaches ∞. Explain what this means in terms of the real-world context.

Table 5.20

Grams of New Drug g	Average Cost (dollars per gram) $A(g) = \frac{C(g)}{g}$
0	
100	
1000	
10,000	
100,000	
1,000,000	
10,000,000	
100,000,000	
1,000,000,000	
10,000,000,000	
100,000,000,000	

f. Determine the horizontal asymptote for $A(g)$.

Solution

a. $C(0) = 800{,}000{,}000$. This means that the cost of producing 0 grams of the new drug is \$800,000,000. The \$800,000,000 is the initial or fixed cost of development that must be recouped (or lost) by the company.

b. The slope of $C(g)$ is 10 dollars per gram. That is, the drug costs 10 dollars per gram to produce.

c. We have

$$A(g) = \frac{C(g)}{g}\,\frac{\text{dollars}}{\text{grams}}$$

$$= \frac{800{,}000{,}000 + 10g}{g} \text{ dollars per gram}$$

The function $A(g)$ gives the average cost per gram of producing a total of g grams of the drug.

d. We create Table 5.21 and the graph in Figure 5.28 to show the rate of change of $A(g)$. As the number of grams produced increases, the average cost decreases but at a lesser and lesser rate.

Table 5.21

Grams of New Drug g	Average Cost (dollars per gram) $A(g) = \frac{C(g)}{g}$	Rate of Change $\frac{\Delta A}{\Delta g}$
0	undefined	cannot calculate
10	80,000,010	$\frac{-\$4{,}000{,}000}{1 \text{ gram}}$
20	40,000,010	$\frac{-\$1{,}333{,}333.33}{1 \text{ gram}}$
30	26,666,676.67	$\frac{-\$666{,}666.67}{1 \text{ gram}}$
40	20,000,010	$\frac{-\$400{,}000}{1 \text{ gram}}$
50	16,000,010	

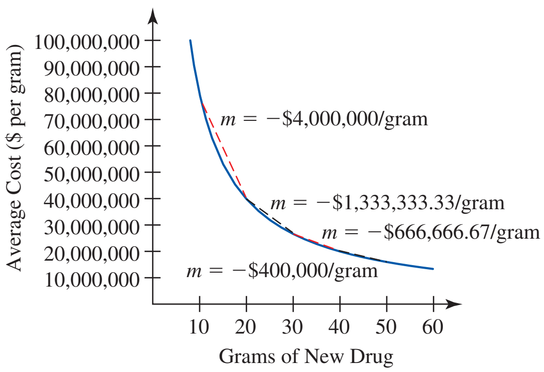


Figure 5.28

e. The completed table is shown in Table 5.22. As $g \to \infty$, $A(g) \to 10$. This means as the number of grams produced becomes very large, the average cost per gram approaches \$10.

Table 5.22

Grams of New Drug g	Average Cost (dollars per gram) $A(g) = \frac{C(g)}{g}$
0	undefined
100	\$8,000,010
1000	\$800,010
10,000	\$80,010
100,000	\$8010
1,000,000	\$810
10,000,000	\$90
100,000,000	\$18
1,000,000,000	\$10.80
10,000,000,000	\$10.08
100,000,000,000	$\approx$ \$10.00

f. Since $A(g) \to 10$ as $g \to \infty$, $y = 10$ is the horizontal asymptote for $A(g)$. We indicate this with a red dashed horizontal line on the graph shown in Figure 5.29.

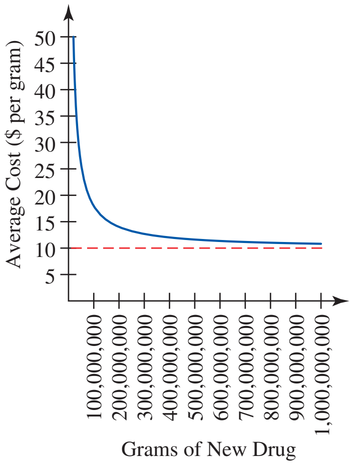


Figure 5.29

We now formally define the term **horizontal asymptote**.

Horizontal Asymptote

A **horizontal asymptote** of a function f is a horizontal line that the graph of f approaches as x approaches positive or negative infinity. More formally, a horizontal asymptote occurs at $y = b$ if and only if the graph of f approaches the line $y = b$ as x approaches either ∞ or $-\infty$.

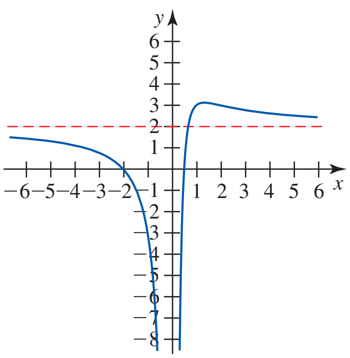


Figure 5.30

The graph of a rational function never crosses a vertical asymptote. However, the graphs of some rational functions do cross their horizontal asymptotes. The difference is that a vertical asymptote occurs where the function is undefined, whereas a horizontal asymptote represents a *limiting value* of the function as $x \to \pm\infty$. There is no reason the function cannot take on this limiting value for some finite x-value. For example, the graph of $h(x) = \frac{2x^2 + 3x - 2}{x^2}$ (Figure 5.30) crosses the line $y = 2$, its horizontal asymptote; however, the graph does not cross its vertical asymptote, the y-axis.

Another example is the graph shown in Figure 5.31. Note the graph moves back and forth (oscillates) over the horizontal line $y = 3$ so f crosses this line many times, but as $x \to \infty$, $f(x) \to 3$. We can see this from Table 5.23 as well. A similar type of situation can also occur for a function that approaches a limiting value as $x \to -\infty$, as shown in Figure 5.32.

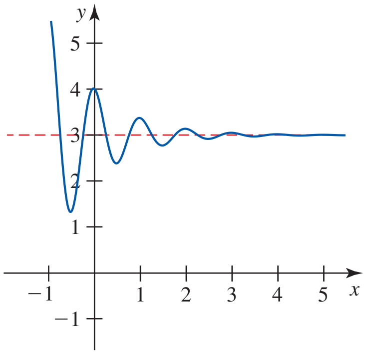


Figure 5.31

Table 5.23

x	y		x	y
0	4		3.5	2.969
0.5	2.393		4	3.018
1	3.367		4.5	2.988
1.5	2.776		5	3.006
2	3.135		5.5	2.995
2.5	2.917		6	3.002
3	3.049			

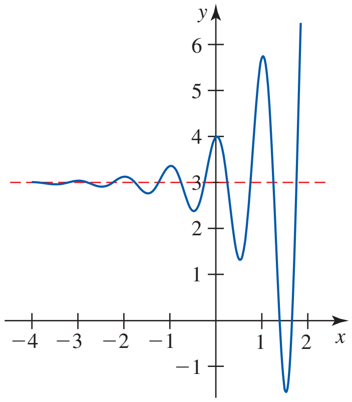


Figure 5.32

■ Finding Horizontal Asymptotes

Recall that the *leading coefficient* of a polynomial is the coefficient on the term with the largest exponent. Leading coefficients and the degrees of the polynomials in the numerator and denominator play a key role in determining the location of horizontal asymptotes on a rational function graph.

Due to the end behavior of polynominals, as the magnitude of x becomes increasingly large, the graph of $f(x) = \dfrac{ax^n + \cdots}{bx^m + \cdots}$ will be more and more influenced by the leading terms. In fact, for values of x near $\pm\infty$, $f(x) \approx \dfrac{ax^n}{bx^m}$. Consider the following three functions:

$$f(x) = \frac{3x^2 + 2x + 1}{4x^3 - 5}$$

$$g(x) = \frac{3x^2 + 2x + 1}{4x^2 - 5}$$

$$h(x) = \frac{3x^2 + 2x + 1}{4x - 5}$$

For large values of x, the value of each function may be approximated by the ratio of the leading terms.

$$f(x) \approx \frac{3x^2}{4x^3} \approx \frac{3}{4x} \qquad g(x) \approx \frac{3x^2}{4x^2} \approx \frac{3}{4} \qquad h(x) \approx \frac{3x^2}{4x} \approx \frac{3x}{4}$$

Observe that as $x \to \infty$, $f(x) \to 0$, $g(x) \to \dfrac{3}{4}$, and $h(x) \to \infty$. Consequently, f has a horizontal asymptote at $y = 0$, g has a horizontal asymptote at $y = \dfrac{3}{4}$, and h does not have a horizontal asymptote.

HOW TO: ■ FIND HORIZONTAL ASYMPTOTES

For a rational function $f(x) = \dfrac{p(x)}{q(x)} = \dfrac{ax^n + \cdots}{bx^m + \cdots}$, where a is the leading coefficient of the numerator and b is the leading coefficient of the denominator,

- If $n < m$ (i.e., if the degree of the numerator is less than that of the denominator), a horizontal asymptote occurs at $y = 0$.
- If $n = m$ (i.e., if the degree of the numerator is equal to that of the denominator), a horizontal asymptote occurs at $y = \dfrac{a}{b}$.
- If $n > m$ (i.e., if the degree of the numerator is greater than that of the denominator), the function does not have a horizontal asymptote.

EXAMPLE 4 ■ Finding a Horizontal Asymptote from an Equation

The ratio of the surface area of any cube to its volume is given by $r(s) = \dfrac{6s^2}{s^3}$, where s is the length of any side. Determine the horizontal asymptote of the function and interpret what it means in its real-world context.

Solution The degree of the numerator is 2 and the degree of the denominator is 3. Since $2 < 3$, this function has a horizontal asymptote at $r = 0$. Increasing the length of the side of any cube decreases its area to volume ratio. For large values of s, this ratio is near 0.

Recall that the x-intercepts of a function occur where $y = 0$ and the y-intercepts occur where $x = 0$. If we know the x-intercepts, y-intercepts, horizontal asymptote, and vertical asymptotes of a rational function, it is relatively easy to determine the basic shape of its graph by calculating just a few additional points.

EXAMPLE 5 ■ Drawing a Rational Function Graph by Hand

Draw the graph of $f(x) = \dfrac{3x^2 - 15x + 18}{2x^2 - 18}$.

Solution

- horizontal asymptote:

  Since the degrees of the numerator and denominator are equal ($m = n = 2$), the graph has a horizontal asymptote at $y = \dfrac{3}{2}$.
- y-intercept:

  $$f(0) = \frac{18}{-18} = -1 \text{ so there is a } y\text{-intercept at } (0, -1).$$

To find the vertical asymptotes and x-intercept, it is helpful to write the function in factored form.

$$f(x) = \frac{3x^2 - 15x + 18}{2x^2 - 18}$$
$$= \frac{3(x-2)(x-3)}{2(x+3)(x-3)}$$

Observe that the factor $x - 3$ occurs in both the numerator and denominator. As long as $x \neq 3$, $\frac{x-3}{x-3} = 1$. Therefore, we can rewrite the function in simplified form as

$$f(x) = \frac{3(x-2)}{2(x+3)}, \quad x \neq 3$$

We see that a removable discontinuity (hole) in the graph occurs at $x = 3$.

We now proceed to finding the x-intercept and vertical asymptotes.

- x-intercepts:

  x-intercepts occur at values of x that make the numerator of the simplified rational function equal to zero. So an x-intercept occurs at (2, 0).

- vertical asymptotes:

  Vertical asymptotes occur at values of x that make the denominator of the simplified rational function equal to zero. So a vertical asymptote occurs at $x = -3$.

In Table 5.24 we evaluate the function at a few other key points. We plot the points and then connect them with a smooth curve. At $x = 3$, we place an open dot on the curve to indicate that the function is discontinuous there, as shown in Figure 5.33.

Table 5.24

x	$f(x) = \frac{3(x-2)}{2(x+3)}, \quad x \neq 3$
−10	2.57
−4	9.00
−2	−6.00
10	0.92

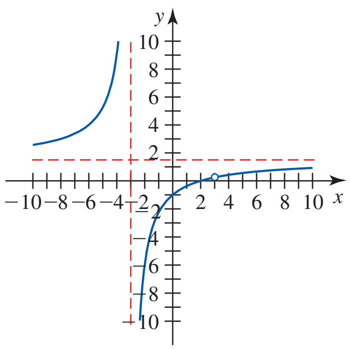


Figure 5.33

EXAMPLE 6 ■ Finding the Domain of a Rational Function

Find the domain of each of the following rational functions.

a. $f(x) = \frac{3x^2 - 5}{x + 4}$

b. $g(x) = \frac{2}{x^2 - 25}$

c. $h(x) = \frac{2x^4}{x^2 + 1}$

Solution

a. The domain of the rational function $f(x) = \frac{3x^2 - 5}{x + 4}$ consists of all real values of x except $x = -4$ since this value makes the denominator equal to zero.

b. The domain of the rational function $g(x) = \dfrac{2}{x^2 - 25}$ consists of all real numbers x except $x = 5$ and $x = -5$ since these values make the denominator equal to zero.

c. The domain of the rational function $h(x) = \dfrac{2x^4}{x^2 + 1}$ consists of all real numbers x. There is no value of x that makes the denominator equal to zero.

Inverses of Rational Functions

A rational function that is strictly increasing or strictly decreasing will have an inverse function. One that changes from increasing to decreasing or vice versa will not. Often the process of finding the inverse of a rational function is algebraically complex.

EXAMPLE 7 ■ Finding the Inverse of a Rational Function

Find the inverse of $f(x) = \dfrac{x + 2}{3x}$.

Solution

$$\begin{aligned}
y &= \frac{x+2}{3x} \\
3xy &= x + 2 \\
3xy - x &= 2 \\
x(3y - 1) &= 2 \\
x &= \frac{2}{3y - 1} \\
f^{-1}(y) &= \frac{2}{3y - 1}
\end{aligned}$$

SUMMARY

In this section you learned about rational functions and their graphs. You discovered how to determine the location of vertical and horizontal asymptotes as well as removable discontinuities (holes). You also modeled real-world contexts with rational functions. Finally, you learned how to determine the domain and the inverse of a rational function.

5.3 EXERCISES

SKILLS AND CONCEPTS

In Exercises 1–4, find the horizontal and vertical asymptotes, if any, for each function.

1. $f(x) = \dfrac{2x}{x + 4}$

2. $h(x) = \dfrac{3}{2x - 6}$

3. $g(x) = \dfrac{(1 - x)(2 + 3x)}{2x^2 + 1}$

4. $k(x) = \dfrac{2x - 2}{2x^2 + x - 3}$

In Exercises 5–8, what are the x-intercepts, y-intercepts, and horizontal and vertical asymptotes (if any)?

5. $f(x) = \dfrac{x - 3}{x - 5}$

6. $g(x) = \dfrac{x^2 - 16}{x^2 + 16}$

7. $k(x) = \dfrac{x^2 - 9}{x^3 + 9x^2}$

8. $m(x) = \dfrac{x(2 - x)}{(x^2 - 10x + 12)}$

9. Let $f(x) = \dfrac{1}{x - 2}$.

a. Complete the table for x-values close to 2. What happens to the values of $f(x)$ as $x \to 2$ from the right and from the left?

x	$f(x)$
1.00	
1.90	
1.99	
2.00	
2.01	
2.10	
3.00	

b. Complete the following tables. Use the tables of values to determine what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

x	$f(x)$
5	
50	
500	
5000	
50,000	

x	$f(x)$
−5	
−50	
−500	
−5000	
−50,000	

c. Without using a calculator, graph $f(x)$. Give the equations for the vertical and horizontal asymptotes.

10. Let $h(x) = \dfrac{1}{(x + 3)^2}$.

a. Complete the table for x-values close to −3. What happens to the values of $h(x)$ as x approaches 3 from the left and from the right?

x	$h(x)$
−4.00	
−3.10	
−3.01	
−3.00	
−2.99	
−2.90	
−2.00	

b. Complete the following tables. Use the tables of values to determine what happens to the values of $h(x)$ as $x \to \infty$ and as $x \to -\infty$.

x	$h(x)$
5	
50	
500	
5000	
50,000	

x	$h(x)$
−5	
−50	
−500	
−5000	
−50,000	

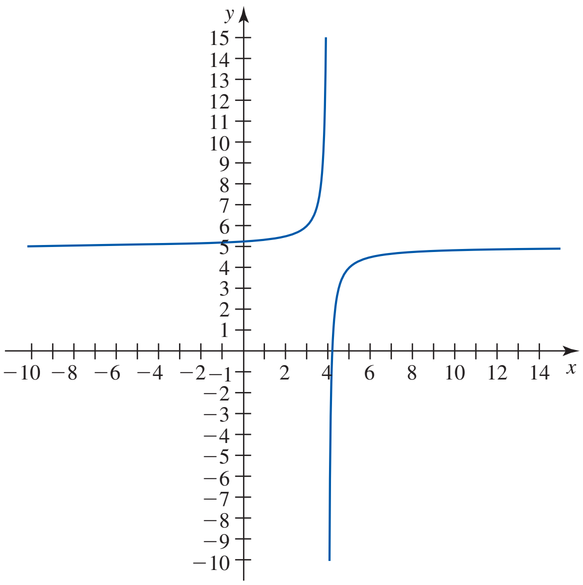
c. Without using a calculator, graph $h(x)$. Give the equations for the vertical and horizontal asymptotes.

11. Use the graph of $h(x)$ to describe the following transformations.

a. $y = -h(x) - 3$

b. $y = h(x - 2) + 4$

c. $y = -2h(x + 3) - 5$



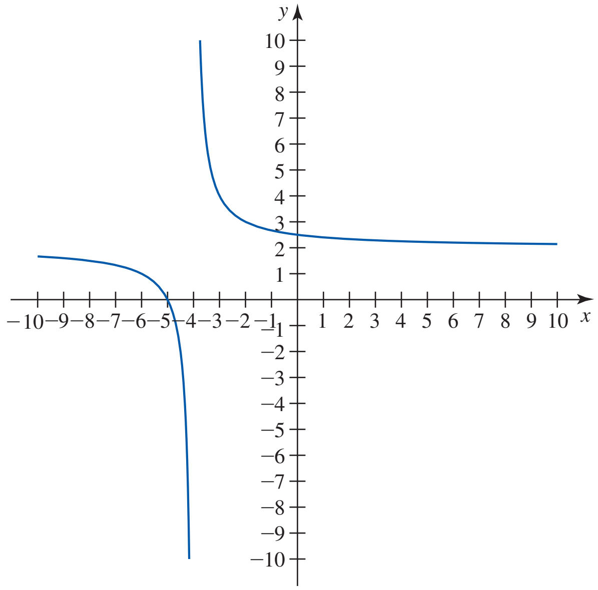
Exercises 12–13 show a transformation of $y = \dfrac{2}{x}$.

a. Find a possible formula for the graph.

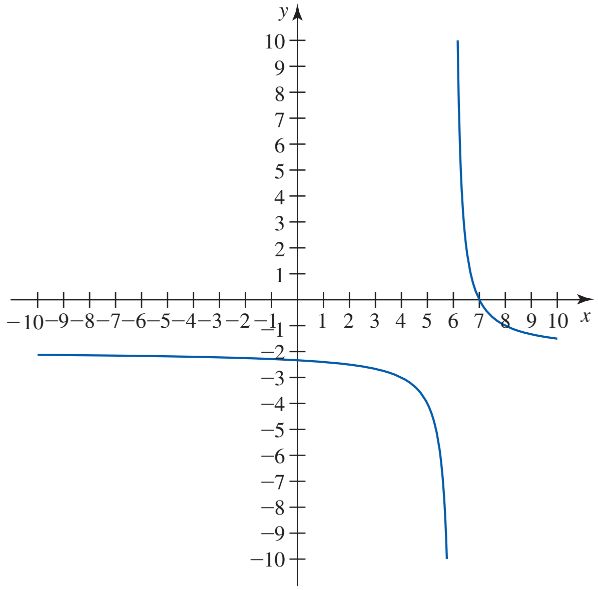
b. Give the coordinates of any x- and y-intercepts.

c. Find the equations for the vertical and horizontal asymptotes.

12.



13.



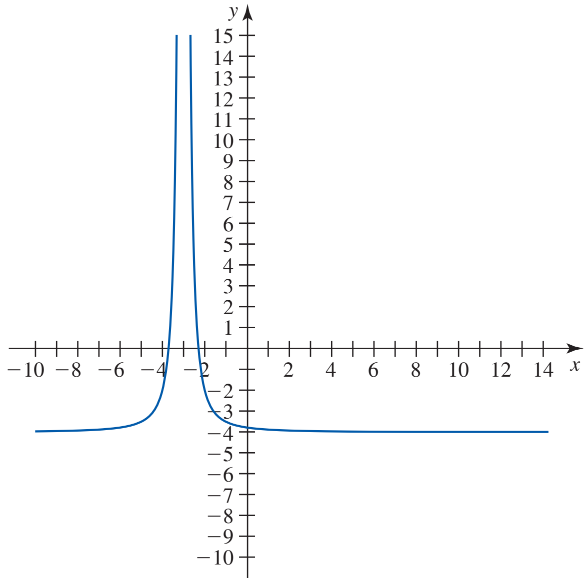
Exercises 14–15 show a transformation of $y = \frac{2}{x^2}$.

a. Find a possible formula for the graph.

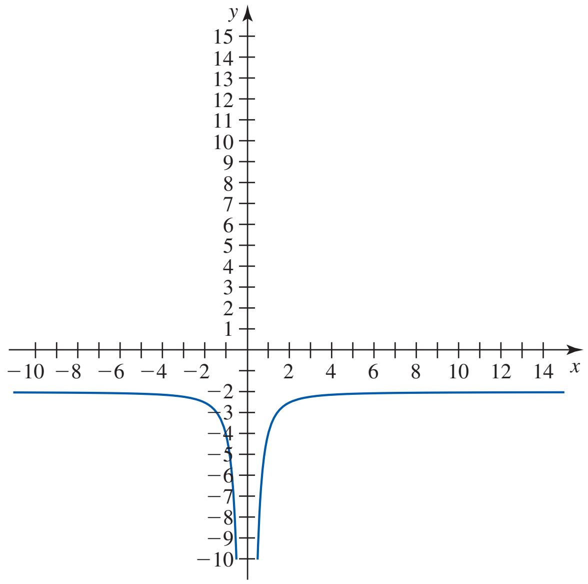
b. Give the coordinates of any x- and y-intercepts.

c. Find the equations for the vertical and horizontal asymptotes.

14.



15.



16. For the following table, give the coordinates of the x-intercept and the equation of the vertical asymptote.

x	y
3.0	3.0
3.1	2.8
3.2	2.5
3.3	2.1
3.4	1.7
3.5	1.0
3.6	0
3.7	−1.7
3.8	−5.0
3.9	−15.0
4.0	undefined
4.1	25
4.2	15

17. For the following table, give the coordinates of the y-intercept and the equation of the horizontal asymptote.

x	y
−6.0	−3.000
0.0	0.000
6.0	0.600
12.0	0.750
16.0	0.800
21.0	0.840
36.0	0.900
46.0	0.920
96.0	0.960
196.0	0.980
396.0	0.990
1096.0	0.996
4096.0	0.999

In Exercises 18–19, sketch a possible graph for each function described.

18. The function f has one vertical asymptote at $x = -2$ and a horizontal asymptote at $y = 2$. The graph has a horizontal intercept at (4, 0) and a vertical intercept at (0, −6). The graph is concave down for $-2 < x < 4$ and has an inflection point at (6, 1).

19. The function g has two vertical asymptotes, one at $x = -1$ and one at $x = 5$. There is a horizontal asymptote at $y = 4$ as $x \to -\infty$ and at $y = -6$ as $x \to \infty$. The graph is concave down for $-1 < x < 5$ and has an x-intercept at (1, 0). Finally, the graph is concave up for $x < -1$ and concave down for $x > 5$.

For Exercises 20–21, create a table of values that satisfies each condition for the functions described.

20. The function f has an x-intercept at 4, a y-intercept at −2, is concave down for $x > 0$, and has a horizontal asymptote at $y = 1$.

21. The function g has a vertical asymptote at $x = -10$, is concave up for $x > -10$, and has an x-intercept at $x = -8$.

In Exercises 22–23, write the formula for a rational function that satisfies each condition for the functions described.

22. The function g has two vertical asymptotes, the line $x = 6$ and the line $x = -2$.

23. A rational function has only one vertical asymptote and no positive function values.

In Exercises 24–27, graph the function and label all of the important features including any x- and y-intercepts, vertical and horizontal asymptotes, and removable discontinuities (holes).

24. $f(x) = \dfrac{2x + 1}{x - 3}$

25. $f(x) = \dfrac{4 - 3x}{2x + 1}$

26. $f(x) = \dfrac{3x^2 - 3x - 6}{x^2 + 8x + 16}$

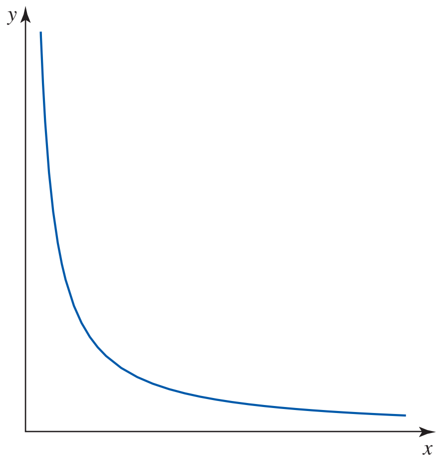
27. $f(x) = \dfrac{x^2 - 2x - 3}{2x^2 - x - 10}$

28. A function f is defined by the following graph. Which of the following describes the behavior of f?

A. As the value of x approaches 0, the value of f increases.

B. As the value of x increases, the value of f approaches 0.

C. As the value of x approaches 0, the value of f approaches 0.



29. Which of the following best describes the behavior of the function f defined by $f(x) = \dfrac{x^2}{x - 2}$?

A. As the value of x gets very large, the value of f approaches 2.

B. As the value of x gets very large, the value of f increases.

C. As the value of x approaches 2, the value of f approaches 0.

■ SHOW YOU KNOW

30. What is the definition of a rational function?

31. Use tables, symbols, graphs, and words to explain the behavior of a function near a vertical asymptote.

32. Use tables, symbols, graphs, and words to explain the behavior of a function near a horizontal asymptote.

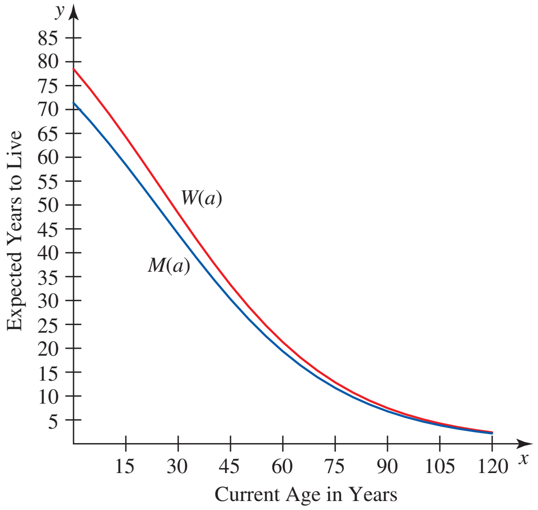
33. What role do asymptotes play in rational functions? How do we determine if and where they exist?

34. What is meant by the term *removable discontinuity*? Under what conditions will one exist?

35. Describe a real-world interpretation of a vertical and horizontal asymptote.

■ MAKE IT REAL

36. Life Expectancies in the United States The graphs of M and W illustrate the life expectancies for men, M, and women, W, in the United States based on their current age a.



Source: **Annuity Advantage Life Expectancy Tables, www.annuityadvantage.com/lifeexpectancy.htm**

a. Estimate $M(0)$ and $W(0)$ and explain what each answer means in this context.

b. Estimate $M(20)$ and $W(20)$ and explain what each answer means in this context.

c. Estimate the limiting value for each function.

d. Using the language of rate of change, explain what the horizontal asymptote of each function means in terms of men's and women's life expectancy.

37. Travel Time The speed limit on most interstate highways in Arizona through rural areas is 75 mph. (***Source:*** **www.dot.state.az.us/highways/Traffic/Speed.asp)**

a. If a driver maintains an average speed of 75 mph, how long will it take to make a 150-mile journey?

b. Write an equation that defines travel time, t, as a function of the average speed, s.

c. As the average speed increases, what happens to the travel time? Explain what this means in this real-world context.

d. When there are fatal accidents, highway patrol officers slow or stop the traffic on the interstate. Create a table for the time it takes to travel 150 miles when traveling at $s = 20$, 15, 10, 5, 4, 3, 2, and 1 miles per hour.

e. Taking into account your answers to parts (a)–(d), sketch a graph of $t(s)$ on the interval $0 < s < 100$.

f. Find the equations for the horizontal and vertical asymptotes and explain what these asymptotes mean in this context.

38. Depreciation A new car depreciates in value very quickly right after it is driven off the lot. Suppose a new car has the initial value of C dollars, and the value of the car as scrap metal is T dollars. If the life of the car is N years, then the average amount, D, by which the car depreciates in value each year is given by the multivariable function

$$D(C, T, N) = \frac{C - T}{N}.$$

a. If the purchase price of the car is \$34,000 and the value of the scrap metal is \$2000, write a formula for D as a function of N.

b. Using your answer to part (a), complete the following table of values for D.

N	D
1	
3	
5	
7	
9	
11	
13	
15	

c. Give the practical domain of $D(N)$.

d. Sketch a graph of $D(N)$.

e. Explain what the vertical and horizontal asymptotes of $D(N)$ mean in terms of the car's value.

39. Sound Intensity The loudness (or intensity) of any sound is a function of the listener's distance from the source of the sound. In general, the relationship between the intensity, I, in decibels (dB) and the distance, d, in feet can be modeled by the function $I(d) = \frac{k}{d^2}$. The constant k is determined by the source of the sound and the surroundings. A reasonable value for the human voice is $k = 1495$. (***Source:*** **www.glenbrook.k12.il.us)**

a. Complete the following table for $I(d)$.

d	I
0.1	
0.5	
1	
2	
5	
10	
20	
30	

b. Give a practical domain for $I(d)$.

c. Sketch a graph of $I(d)$.

d. As you move closer to the person speaking, what happens to the intensity of the sound? Be sure to include the idea of rate of change in your explanation.

40. Body Mass Index Doctors, physical therapists, and fitness specialists often use a person's body mass index to determine whether the person should lose weight. The formula for the body mass index, B, is the multivariable function $B(w, h) = \frac{705w}{h^2}$, where w is weight (in pounds) and h is height (in inches). (*Source:* Centers for Disease Control; www.cdc.gov/nccdphp/dnpa/bmi/index.htm)

a. Give the formula for a 190-pound person's body mass index.

b. Complete the table for $B(h)$ for a 190-pound person.

h	*B*
60	
62	
64	
66	
68	
70	
72	
74	
76	

c. Sketch a graph of the function $B(h)$ for a 190-pound person.

d. Explain what happens to B as h increases. Include the notion of rate of change in your explanation. Why does this make sense in the context of this situation?

41. Blood Alcohol Levels In 1992, the National Highway Traffic Safety Administration recommended that states adopt 0.08% blood alcohol concentration as the legal measure of drunk driving. (*Source:* www.nhtsa.dot.gov) A regular 12-ounce beer is 5% alcohol by volume and the normal bloodstream contains 5 liters (169 ounces) of fluid. A person's maximum blood alcohol concentration, C, can be approximately modeled by the multivariable function $C(w, n) = \frac{600n}{w(169 + 0.6n)}$, where n is the number of beers consumed in one hour and w is a person's body weight in pounds.

a. How many beers can a person have and legally drive if the person weighs 200 pounds? 110 pounds?

b. Does a person's weight have any impact on the upper limit of beers that can be consumed legally?

42. Walking Speed A heart patient recovering from surgery is on a rehabilitation program that includes the use of a treadmill to increase his cardiovascular fitness. His pace of walking/running is defined by the function $r(t) = \frac{5280}{t}$, where r is measured in feet per minute and t is the time (in minutes) it takes to complete the workout.

a. Graph the function $r(t)$ and label the independent and dependent axes.

b. Give the domain of the function $r(t)$.

c. Why would some of the domain values not make sense in a real-world context?

d. Explain what happens to $r(t)$ as t gets larger and larger. What does this mean in terms of the heart patient?

43. Google Share Price On November 22, 2006, the share price of Google reached a new high of \$513 per share. This share price reflected an increase of \$181.45 over its 52-week low price. Investors who purchased at the 52-week low price and sold at the new high earned an astounding 54.7% return on their investment. One question many investors ask before they buy shares in a company is "If I buy today, what is the lowest share price at which I can sell and still break even?" The answer takes into account the commissions that are paid when shares are bought or sold. USAA Investment Management Company charges its occasional investors \$19.95 per trade whether buying or selling shares. Suppose a new investor buys a number of Google shares at \$513 per share through this company.

a. Determine the total amount of money needed (including the transaction fees) to buy and sell 1 share, 10 shares, 100 shares, and 1000 shares.

b. Determine the selling price needed for the investor to make exactly enough money to cover the transaction cost of buying and selling the Google shares (the break-even share price) for 1 share, 10 shares, 100 shares, and 1000 shares.

c. As the number of shares initially purchased increases, what happens to the break-even share price?

44. Video Rentals and Sales The table provides information about video rentals (in millions of dollars) and video sales (in millions of dollars) for select years.

Years Since 1990 *t*	Video Rentals (\$ millions) *r*	Video Sales (\$ millions) *s*
1	8400	3600
2	9100	4000
4	9500	5500
6	9300	7300
10	8250	10800

The dollar value of video rentals can be modeled by the function $r(t) = -54.3t^2 + 562t + 8030$ million dollars. The dollar value of video sales can be modeled by $s(t) = 19.5t^2 + 603t + 2860$ million dollars. The population of the United States over the same time period can be modeled by $P(t) = 0.0067t^2 + 2.56t + 250$, where t is years since 1990.

a. Write a formula for the total revenue from video rentals and video sales.

b. Write a formula for the average annual cost per person in the United States for videos. Graph this formula.

c. Use the formula from part (b) to estimate the average annual video cost per person in the year 2004.

d. When will the average annual cost per person be \$65?

e. Describe the concavity of this annual cost per person graph. Explain what this concavity means in practical terms.

45. Wait Time An area of mathematics known as *queuing theory* addresses the issue of how a business can provide adequate customer service by analyzing the wait time of customers in line. (*Source:* **http://en.wikipedia.org/wiki/Queueing_theory**) When the ticket window opens for a concert of a very popular music group, there are already 150 people in line. People arrive and join the line at a rate of 50 people per hour. Customer service representatives start serving at a rate of 90 people per hour but, due to fatigue, slow down by 2 people per hour as time passes. The number of people in line at time t hours is

$$\begin{aligned} P(t) &= 150 + (50 - (90 - 2t))t \\ &= 150 - 40t + 2t^2 \end{aligned}$$

The representatives are serving people at a rate of $90 - 2t$ people per hour. To determine the wait time, we divide the number of people in line by the rate at which they are being served. That is,

$$R(t) = \frac{150 - 40t + 2t^2}{90 - 2t} \text{ hours}$$

a. What is a practical domain for the model $P(t)$?

b. What will be the wait time for someone who arrives 2 hours after the ticket window opens?

c. How long after the ticket window opens should a person arrive if they want to wait only 0.5 hours?

d. How long will it take for the length of the line to reach 0 people?

■ STRETCH YOUR MIND

Exercises 46–49 are designed to challenge your understanding of rational functions.

46. The function $f(x) = \dfrac{-3x^2 + 2}{x - 1}$ has a special kind of asymptote known as "slant" or "oblique." Graph $f(x)$. Then determine the equation of the asymptote for $f(x)$ algebraically.

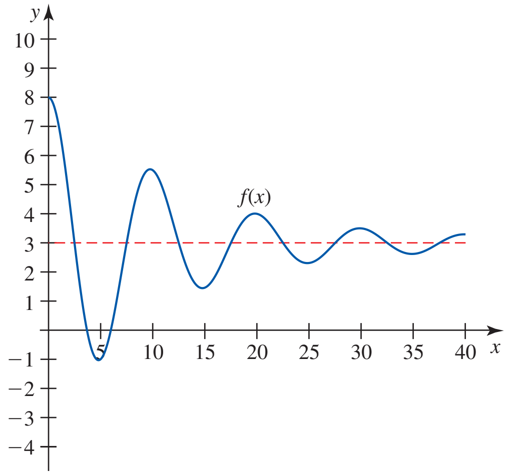
47. Are the following statements true or false? Explain your answer.

a. "An asymptote of a function $f(x)$ is a straight line."

b. "An asymptote of $f(x)$ can be approached but never reached or crossed by $f(x)$."

48. The function $f(x) = \dfrac{15x(x + 3)}{(x^2 + 1)(x - 4)(x + 3)}$ has a vertical asymptote, a horizontal asymptote, and a removable discontinuity. Find each. Also, by inspecting the graph of $f(x)$ explain over what intervals $f(x)$ is concave up and concave down. Finally, estimate its point(s) of inflection.

49. Create a table of values that could demonstrate what the values of the function $f(x)$ are doing in the following graph. Explain how the table and the graph demonstrate the same thing.



CHAPTER 5 Study Sheet

As a result of your work in this chapter, you should be able to answer the following questions, which are focused on the "big ideas" of this chapter.

SECTION 5.1

1. How can we use the idea of successive differences to determine if a numerical representation of a function is linear, quadratic, cubic, quartic, etc.?
2. How can we use the idea of rate of change and concavity to determine which polynomial type might best model a scatter plot of data?
3. What are the characteristics of a polynomial function? That is, what makes a polynomial a polynomial?
4. How can we determine the end behavior of a polynomial function? Why is it important, when modeling, to understand this end behavior?

SECTION 5.2

5. How can we use the idea of rate of change to describe the behavior of a power function?
6. What are the differences between direct and inverse variation?
7. Generally, what is the behavior of any power function with $b > 1$? $0 < b < 1$? $b < 0$?

SECTION 5.3

8. What is the definition of a rational function?
9. What role do asymptotes play in rational functions? What different types are there? How do we determine if and where they exist?
10. What is meant by a removable discontinuity? Under what conditions will one exist?
11. Describe a real-world interpretation of a vertical and horizontal asymptote.

REVIEW EXERCISES

■ SECTION 5.1 ■

In Exercises 1–2, numerical representations of three functions are shown in a table. The functions are either linear, quadratic, cubic, or none of these. Using the idea of rate of change, identify each function appropriately.

1.

x	$f(x)$	$g(x)$	$h(x)$
−9	170	14	−34.65
−7	104	12	−18.90
−5	54	10	−8.75
−3	20	8	−3.00
−1	2	6	−0.45
1	0	4	0.10
3	14	2	−0.15
5	44	0	0.00
7	90	−2	1.75
9	152	−4	6.30

2.

x	$f(x)$	$g(x)$	$h(x)$
−5	−9	200	50
−4	−7	108	36
−3	−5	48	24
−2	−3	14	14
−1	−1	0	6
0	1	0	0
1	3	8	−4
2	5	18	−6
3	7	24	−6
4	9	20	−4

In Exercises 3–7,

a. Describe the end behavior of the given polynomial function.

b. Make a table of values that confirms the end behavior you described. Create your table in such a way that it shows what happens to function values as $x \to \infty$ and $x \to -\infty$.

3. $y = x^2 + 2x - 8$

4. $y = x^3 - 2x^2 + x - 2$

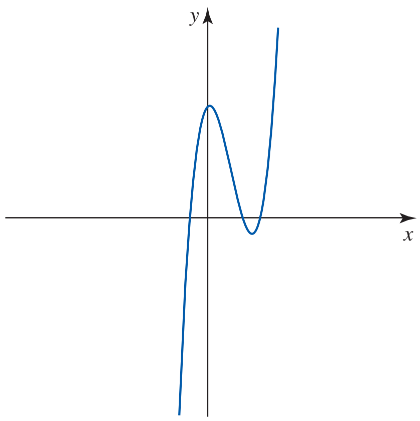
5. $y = -x^5 + 3x^2 + 2x - 7$

6. $y = -x^4 - 2x^3 + 3x^2 - x + 2$

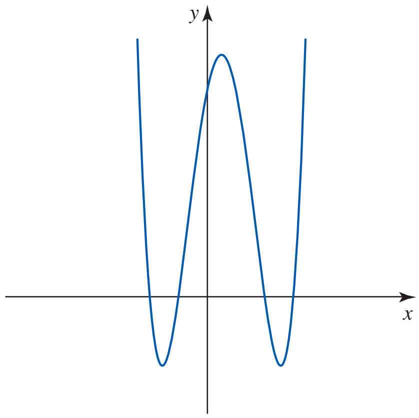
7. $y = x^6$

In Exercises 8–10, determine the minimum degree of the polynomial function by observing the number of changes in concavity in the graph.

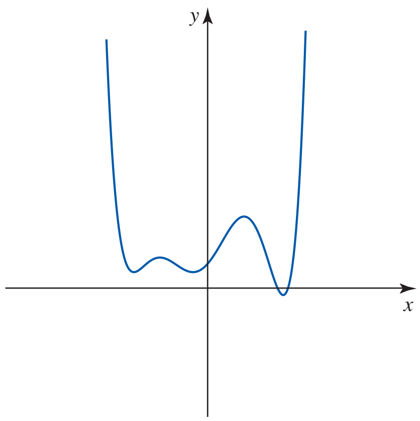
8.



9.



10.



In Exercises 11–12, apply your knowledge of higher-order polynomial functions to model real-world situations.

11. Cigarette Use among Eighth Graders The data in the table show the percentage of eighth graders who admit to using cigarettes.

Years Since 1990 t	Cigarette Use (percent) C
1	44.0
2	45.2
3	45.3
4	46.1
5	49.2
6	47.3
7	45.7
8	44.1
9	40.5
10	36.6
11	31.4
12	28.4

Source: www.ojp.usdoj.gov

a. Make a scatter plot of these data.

b. Analyze the scatter plot and explain which type of polynomial function (based on degree) might best model the situation. Base your decision on the rate of change observed in the scatter plot.

c. Use regression to find a function model equation of the polynomial type you identified in (b). Then graph the function.

d. Use your model to predict when the percentage of eighth graders who use cigarettes will be 0. Do you think this result is reasonable? Why or why not?

12. Egg Production in the United States The data in the table show the egg production in the United States for selected years.

Years Since 1990 t	Egg Production (billions) E
0	68.1
5	74.8
7	77.5
8	79.8
9	82.9
10	84.7
11	86.1
12	87.3
13	87.5
14	89.1

Source: Statistical Abstract of the United States, 2006, Table 842

a. Make a scatter plot of these data and, using the idea of rate of change, explain why a cubic function best models the data. Use regression to find the model.

b. Describe the end behavior for the function model and discuss whether or not you think this end behavior will accurately predict future egg production amounts.

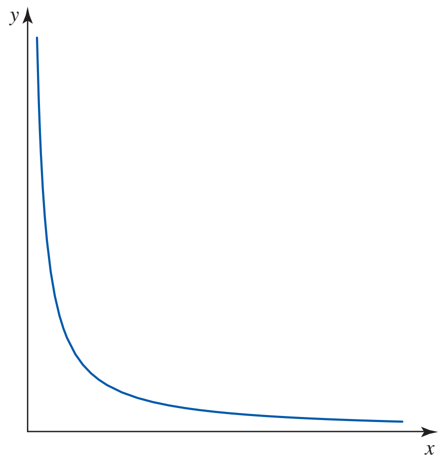
SECTION 5.2

In Exercises 13–16, a power function is given in tabular or graphical form. Determine whether the power function represents direct or inverse variation and explain how you know.

13.

x	y
0	0.00
2	4.76
4	11.31
6	18.78
8	26.91
10	35.57
12	44.67

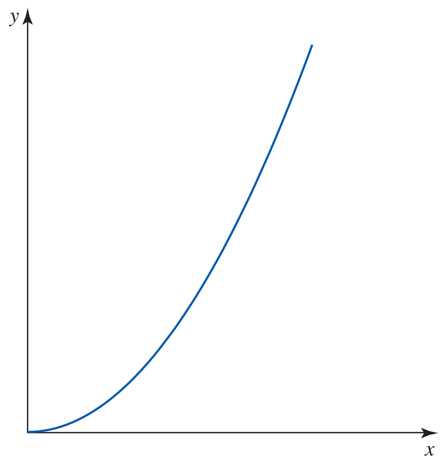
14.



15.

x	y
1	5.00
2	2.50
3	1.67
4	1.25
5	1.00
6	0.83

16.



In Exercises 17–21, write a power function representing the verbal statement.

17. The luminosity, L, of a star is directly proportional to the fourth power of its surface temperature, T. (*Source:* www.astronomynotes.com)

18. The cost, C, of putting 10 gallons of gas into a gas tank is directly proportional to the price per gallon, p.

19. The amount of force, F, acting on a certain object from the gravity of Earth at sea level is directly proportional to the object's mass, m. The gravitational constant, -9.8 m/s^2, is the constant of proportionality.

20. The area, A, of a rectangle is directly proportional to its width if its length is fixed to be 7 centimeters.

21. The average speed, s (in miles per hour), needed to drive a distance of 1200 miles is inversely proportional to the time, t (in hours), spent traveling.

In Exercises 22–23, use your knowledge of power functions to answer the questions about real-world situations.

22. Top Oil Exporters The table shows the 10 countries with the largest daily export rates of oil, measured in million barrels per day.

a. Examine the data and determine why an inverse variation relationship might best model the situation.

b. Find a power regression and use it to describe the relationship between the rank of the country and its daily export of oil.

c. Use the power regression model to determine the amount of oil exported daily by the country that ranks 11th in oil exports.

Top 10 Countries c	Oil Exports (in million bbl/day) E
1. Saudi Arabia	7.920
2. Russia	7.000
3. Norway	3.466
4. United Arab Emirates & Iran	2.500
5. Venezuela	2.100
6. Kuwait	1.970
7. Mexico	1.863
8. Canada	1.600
9. Iraq	1.500
10. United Kingdom	1.498

Source: CIA—The World Factbook, 2006

23. Public Debt of Nations The table shows the 10 countries with the largest public debt, measured as the percent of the gross domestic product.

a. Examine the data and determine why an inverse variation relationship might best model the situation.

b. Find a power regression and use it to describe the relationship between the rank of the country and the amount of its public debt.

c. Use the power regression model to determine the public debt of a country that ranks 11th.

Top 10 Countries c	Public Debt (% of GDP) P
1. Lebanon	209.0
2. Japan	176.2
3. Seychelles	166.1
4. Jamaica	133.3
5. Zimbabwe	108.4
6. Italy	107.8
7. Greece	104.6
8. Egypt	102.9
9. Singapore	100.6
10. Belgium	90.3

Source: CIA—The World Factbook, 2006

■ SECTION 5.3 ■

In Exercises 24–25, find the horizontal and vertical asymptotes, if any, of each function.

24. $f(x) = \dfrac{3x}{4x - 1}$

25. $h(x) = \dfrac{3}{(x - 6)(2x - 2)}$

In Exercises 26–27, what are the x-intercepts, y-intercepts, and horizontal and vertical intercepts (if any)?

26. $f(x) = \dfrac{x - 2}{x + 4}$

27. $m(x) = \dfrac{x(x + 4)}{(x^2 + 2x - 8)}$

28. Let $f(x) = \dfrac{5}{x - 6}$.

a. Complete the table for x-values close to 6. What happens to the values of $f(x)$ as $x \to 6$ from the right and from the left?

x	5	5.9	5.99	6	6.01	6.1	7
$f(x)$							

b. Complete the following tables. Use the tables of values to determine what happens to the values of $f(x)$ as $x \to \infty$ and as $x \to -\infty$?

x	1	10	100	1000	10,000
$f(x)$					

x	-1	-10	-100	-1000	$-10,000$
$f(x)$					

c. Without using a calculator, graph $f(x)$. Give the equations for the vertical and horizontal asymptotes.

29. For the following data, there is one vertical asymptote. Give its equation and explain.

x	y
6.6	−16
6.7	−22
6.8	−34
6.9	−68
7	undefined
7.1	71
7.2	36
7.3	24
7.4	19

30. For the following data, there is a horizontal asymptote. Give its equation and explain.

x	y
−20	−4.091
−15	−4.118
−10	−4.167
−5	−4.286
0	−5
5	−15

31. A function f has one vertical asymptote at $x = 4$ and a horizontal asymptote at $y = 0$. The graph has no horizontal intercepts and a vertical intercept at $(0, -10)$. The graph is concave down for $-6 < x < 4$ and concave down for $4 < x < 14$. Sketch a possible graph for the function described.

32. Electrical Appliances Every electrical appliance has two costs associated with its use: the purchase price and the operating cost. The following table shows a typical appliance's annual cost based on national averages.

Appliance	**Average Cost per Year in Electricity** C
Home computer	$10
Television	$16
Microwave	$17
Dishwasher	$51
Clothes dryer	$75
Washing machine	$79
Refrigerator	$92

Source: **www1.eere.energy.gov**

a. For a new refrigerator that costs $1150, determine the total annual cost if the refrigerator lasts for 15 years. Assume the only costs associated with the refrigerator are its purchase price and the cost for electricity.

b. Develop a function, $C(y)$, that gives the annual cost, C, of a refrigerator as a function of the number of years, y, the refrigerator lasts.

c. Use a calculator to graph $C(y)$.

d. Determine the equation for the vertical and horizontal asymptote of this function and explain the significance of each with respect to the refrigerator and its cost.

e. If a company offers a refrigerator that costs $1700, but says that it will last at least 25 years, is the refrigerator worth the difference in cost?

Make It Real Project

What to Do

1. Find a set of at least six data points from an area of personal interest. Choose data that appear to change concavity. Also try to choose data that does not involve time as the independent variable.
2. Draw a scatter plot of the data and explain which polynomial or power function might best model the situation.
3. Find a regression model for your data.
4. Using the idea of rate of change, describe the relationship between the two quantities.
5. Use the model to predict the value of the function at an unknown point and explain why you think the prediction is or is not accurate.
6. Determine if your regression model has an inverse function. If not, explain why and determine a meaningful way to restrict the domain of your function so that the inverse would be a function.
7. Interpret the meaning of the function in the context of your situation. If possible, compute the inverse function (if it is a quadratic, power, or rational function).
8. Explain how a consumer and/or a businessperson could benefit from the model.