MATPMD2 : NETWORKS and GRAPH THEORY

6 Graph colouring

Definition A *graph colouring* is an assignment of colours to vertices such that no two adjacent vertices have the same colour. A k-colouring uses k colours. The *chromatic number* of a graph G, $\chi(G)$, is the least number k for which a k-colouring exists.

For a graph G, if we can obtain an k-colouring by inspection then the most we can say is that $\chi(G) \leq k$, since there may exist a colouring using fewer colours that we simply cannot find. Finding some lower bound for $\chi(G)$ would be helpful.

Examples

1. If $\chi(G) = 1$ then G is such that we can colour all vertices with one colour. It follows that we have no adjacent pairs, consequently no edges and so $G = nK_1$, a graph of n isolated vertices.

2. We have already seen that a bipartite graph has a two-colouring and we can say that $\chi(G) = 2$ if and only if G is bipartite.

3. Let G be the complete graph K_n. Such a graph contains all possible edges and so each of the n vertices is adjacent to $n-1$ other vertices. It follows that $\chi(K_n) = n$.

   Now suppose that G with n vertices is a graph which contains the complete graph K_r as a subgraph; that K_r is a subgraph induced on some r of the n vertices. Note that an induced subgraph which is complete is called a *clique*. We will need at least r colours to colour the vertices of K_r and so $\chi(G) \geq r$, we have a lower bound for $\chi(G)$.

4. Let $K_n - \lambda$ be the graph obtained from the complete graph K_n by deleting a uv edge λ. Now $K_n - \lambda$ contains K_{n-1} as a subgraph and so $\chi(G) \geq n-1$. Furthermore, suppose vertex u is coloured with one of $n-1$ colours, then vertex v can be coloured with the same colour since $v \not\sim u$ and so $\chi(G) \leq n-1$ and we conclude that $\chi(G) = n-1$.

5. Let G be an even cycle; a cycle with an even number of vertices. Then $G = C_{2k}$ and $\chi(C_{2k}) = 2$.

6. Let G be an odd cycle. Then $G = C_{2k+1}$ with chromatic number 3.

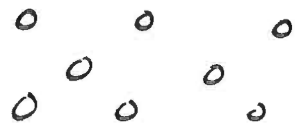
7. The chromatic number of any non-trivial tree [a tree other that K_1] is 2.

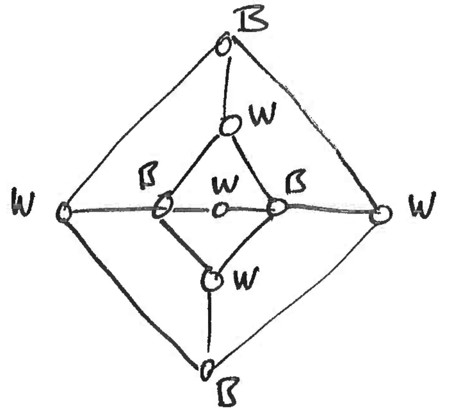
8. Let G be a graph. Fix a vertex $v \in V(G)$. so that $G - v$ is the graph obtained from G by deleting the vertex v. If $\chi(G - v) = k$ then $\chi(G) \leq k+1$ since replacing the vertex v will require at most one extra colour in any colouring.

9. Example

Graph colouring.

Examples.

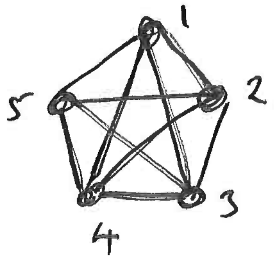
1) $\chi(G) = 1$; $G =$  , vertices all one colour.

2) 

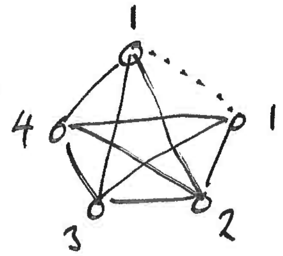
has a 2-colouring as shown and so is bipartite.

$\chi(G) = 2$

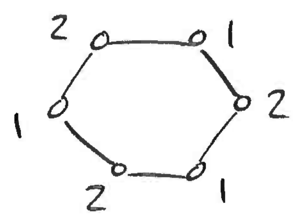
[Subgraph of $K_{4,5}$].

3) eg K_5 = 

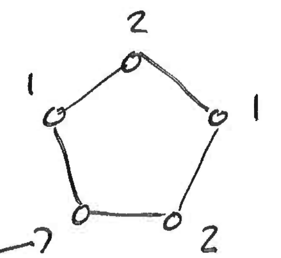
needs 5-colours.

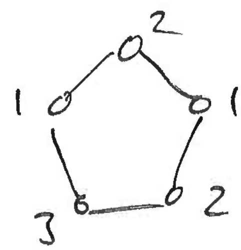
4) eg $K_5 - d$ = 

with a 4-colouring

5) eg C_6 = 

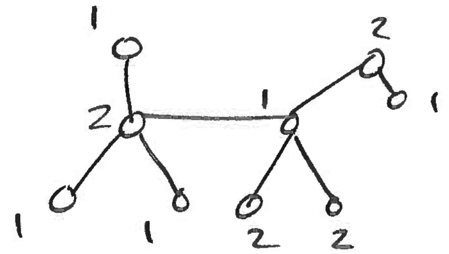
with a 2-colouring [so even cycles are bipartite].

6) eg C_5 = 



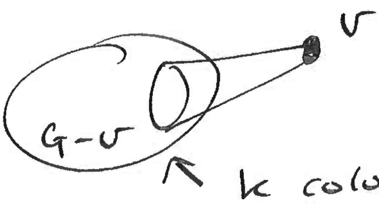
cannot be colours 1 or 2, need a third colour

7) Colouring a tree



a tree
with a 2-colouring

8) If $\chi(G-v)=k$

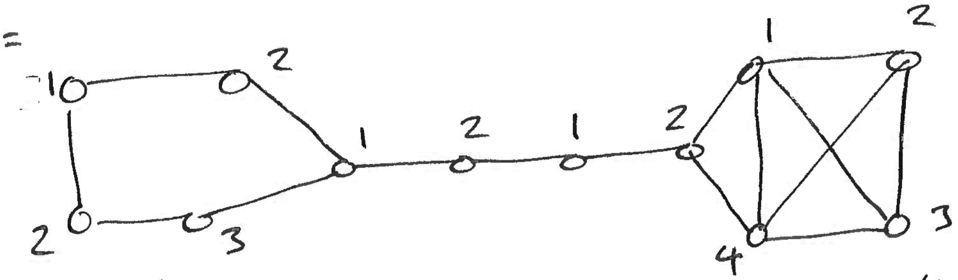


↖ k colours used in here

if all k-colours are needed for the neighbours of v then we need one more colour for v, otherwise we use one of the k colours for v

$\chi(G) \leq k+1$.

9) G=



5-cycle here
so $\chi(G) \geq 2$

K_4 clique here
so $\chi(G) \geq 4$
[better lower bound]

G contains clique K_4 so $\chi(G) \geq 4$.

G has a 4-colouring as shown so $\chi(G) \leq 4$.

Hence $\chi(G) = 4$.

Proposition Let G be a graph on n vertices which has maximum degree d. Then $\chi(G) \leq d+1$.

Proof
By induction on n, the number of vertices.
Clear for $n = 1$ since then $G = K_1$, $d = 0$ and $\chi(G) = 1 \leq 0 + 1 = d + 1$.
Suppose $n > 1$ and result holds for graphs with fewer than n vertices. Fix vertex $v \in V(G)$. Then $\chi(G - v) \leq d' + 1$ where d' is the maximum degree in $G - v$ and since $d' \leq d$ we have $\chi(G - v) \leq d + 1$.
If $\chi(G - v) \leq d$ then $\chi(G) \leq d + 1$ since replacing a vertex requires at most one more colour [see previous example], and we are done.
So suppose that $\chi(G - v) > d$. Then $d < \chi(G - v) \leq d + 1$ whence $\chi(G - v) = d + 1$ and so $G - v$ has a $(d + 1)$-colouring.
Note that $\deg(v) \leq d$ and so v has at most d neighbours. Not all $d + 1$ colours are used for the neighbours of v and so one of these $d + 1$ colours can be used for v. Hence $\chi(G) \leq d + 1$.

6.1 An application

We can apply a graph-colouring approach to a timetabling problem.

Example Suppose we have 9 students each doing a some combination of eight modules A, B, C, D, E, F, G, H. Each module has a final exam. What is the least number of exam time slots will we need?

student	1	2	3	4	5	6	7	8	9
modules	BHE	AGF	GDF	EB	AHE	FC	BCE	ADF	DGA

Solution
We construct a graph G on 8 vertices - one vertex for each module. Vertices are adjacent if those modules are taken by the same student. Thus for each student we construct a complete graph on the vertices representing their chosen modules. We want to partition the vertices into independent set - sets of isolated vertices. Each independent set can be assigned one time slot, so we want to find the least number of independent sets which will give us the least number of time slots required. Each independent set can be coloured with one colour so the least number of colours required to colour the graph will give us the least number of time slots required; this is the chromatic number of G.

Clearly G contains the clique K_3 since at least one student takes three modules, and so $\chi(G) \geq 3$. We look for a larger clique.
G contains K_4 as a subgraph on vertices $ADFG$ and so $\chi(G) \geq 4$. Furthermore the graph has a 4-colouring as shown:

colour	1	2	3	4
modules	A, C	B, D	E, F	H, G

Thus $\chi(G) = 4$ and the least number of time slots required is 4. An example allocation is shown in the table where colours 1,2,3,4 represent time slots 1,2,3,4.

Prop^{n} $\chi(G) \leq d+1$ where $d =$ max deg.

Pf/ Induction on n.

$n=1$: $G = K_1$, $d=0$, $\chi(K_1) = 1$ so true.

Spse true for $n>1$ and result holds for graphs with fewer than n vertices.

[Aim: show consequently true for n]

Fix $v \in V(G)$ with $\deg(v) \leq d =$ max degree of G.

Write d' for max deg $(G-v)$. Then $d' \leq d$ since deleting a vertex will not increase degree of graph.

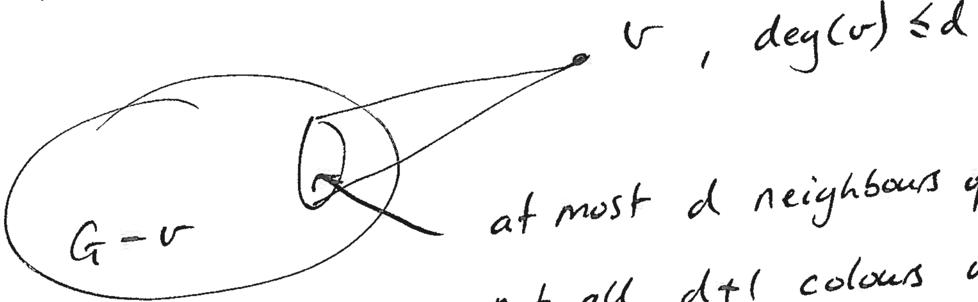
$G-v$ has fewer than n vertices so result holds by inductive hypothesis – $\chi(G-v) \leq d'+1 \leq d+1$.

$\chi(G-v) \leq d \Rightarrow \chi(G) \leq d+1$, done. (ex. 8)

So spse $\chi(G-v) \not\leq d$ i.e $\chi(G-v) > d$.

Then $d < \chi(G-v) \leq d+1 \Rightarrow \chi(G-v) = d+1$.

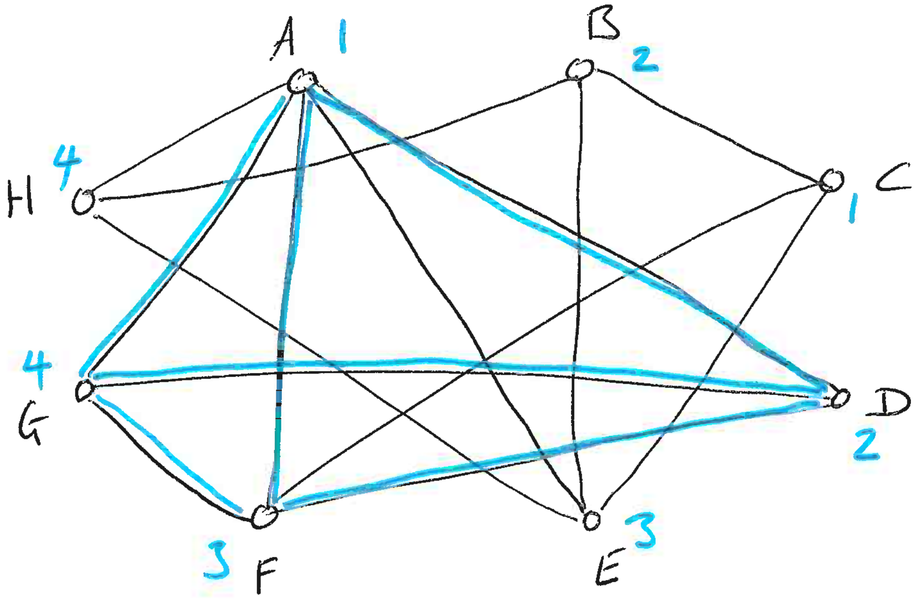
Then $G-v$ has ~~a~~ $(d+1)$-colouring



at most d neighbours of v, not all $d+1$ colours used in here so one can be used for v

Thus $\chi(G) \leq d+1$.

Timetabling example.



ADFG induces K_4 [need 4 colours for these vertices as they are pairwise adjacent] so $\chi(G) \geq 4$.

Graph has a 4-colouring as shown so $\chi(G) \leq 4$.

Hence $\chi(G) = 4$.

Possible allocation:

Time slots	1	2	3	4
modules	A, C	B, D	E, F	H, G

6.2 Chromatic polynomials

This will give us an alternative approach to finding the chromatic number of a graph.

Let $f(G,t)$ be the number of t-colourings of a labelled graph G. Then $\chi(G)$ is the least t such that $f(G,t) > 0$.

To obtain the chromatic polynomial of a graph we label the vertices of the graph with the number of choices of colours for that vertex, given choices already made. Then, informally, the chromatic polynomial is the product of the vertex labels.

Examples

1. Let $G = K_3$. We start with t colours. Call the vertices $v_!, v_2, v_3$.

   Then, starting with vertex v_1 we have t choices of colour for this vertex. Moving to v_2 we have $t-1$ choices since v_2 adjacent to v_1 means we cannot colour v_2 the same colour as v_1 and this reduces the number of choices by 1. Moving onto v_3. This vertex cannot be the same colour as either v_1 or v_2 since it is adjacent to both of these; v_1 and v_2 are different colours so the number of choices is reduced by 2. We have $t-2$ choices for v_3.

   Thus the chromatic polynomial is $f(K_3,t) = t(t-1)(t-2)$.

   Now $f(K_3,1) = 0,\ f(K_3,2) = 0$ but $f(K_3,3) \neq 0$ and so $\chi(K_3) = 3$

2. For the complete graph $f(K_n,t) = t(t-1)(t-2)\ldots(t-(n-1))$.

   For the complement of the complete graph $f(\overline{K_n},t) = t \times t \times t \ldots = t^n$.

3. For the broken wheel graph with k spokes, the chromatic polynomial is $t(t-1)(t-2)^{k-1}$.

4. Example: a tree.

5. Example

6. Example: inspection does not always work.

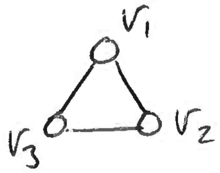
Deletion-contraction algorithm
When inspection does not work then we need a way to break down the graph into manageable bits and for this we need the deletion-contraction algorithm.

If uv is an edge of G then $f(G,t) = f(G-uv,t) - f(\epsilon_{uv}G,t)$ where $G-uv$ is G with the uv edge removed and $\epsilon_{uv}G$ is G with the edge uv contracted, i.e uv is contracted to one vertex w such that all neighbours of both u and v (except u and v) become the neighbours of w.

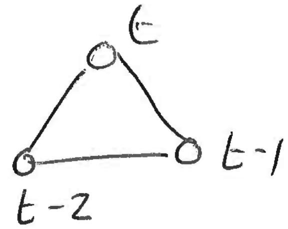
Examples

Chromatic polynomials.

Examples

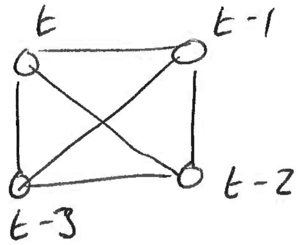
1. $K_3 =$ 

$$f(K_3,t) = \underbrace{t}_{\text{t choices for v_1}}\ \underbrace{(t-1)}_{\text{one less for v_2}}\ \underbrace{(t-2)}_{\text{two less for v_3}}$$



$\chi(K_3) = 3$ since $t = 3$ is least t for which $f(K_3,t)$ is positive.

2. K_n eg K_4



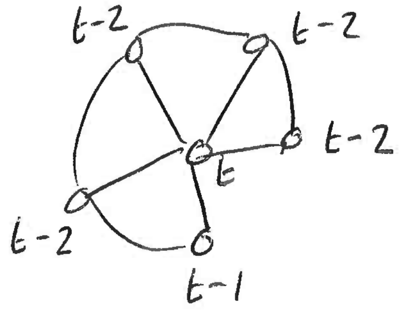
$$f(K_4) = t(t-1)(t-2)(t-3)$$
$$f(K_n) = t(t-1)(t-2)\dots(t-(n-1))$$

choices decreases by one as we move around vertices.

$\overline{K_n}$ = o o o .... o n isolated vertices

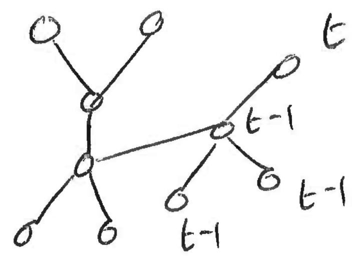
t choices for each one, $f(\overline{K_n}) = t^n$

3. Broken wheel graph.

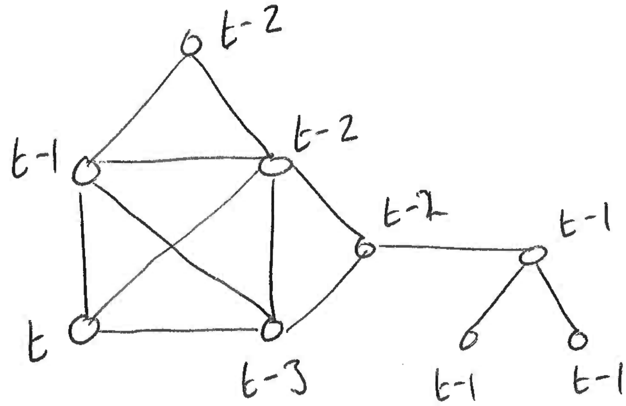


4) If T is a tree on n vertices then

$$F(T) = \underbrace{t}_{\text{initial choice}}(t-1)^{n-1}$$

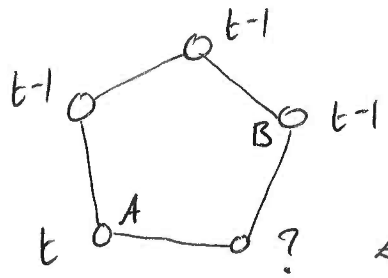


5)



$$f(G) = t(t-1)^4(t-2)^3(t-3), \quad \chi(G) = 4$$

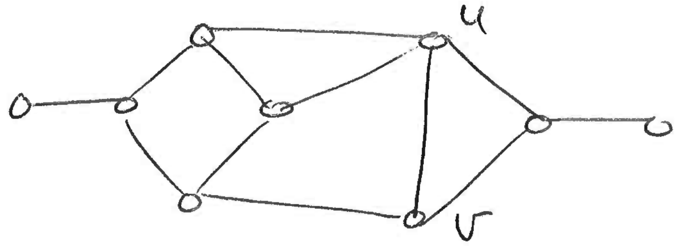
6)

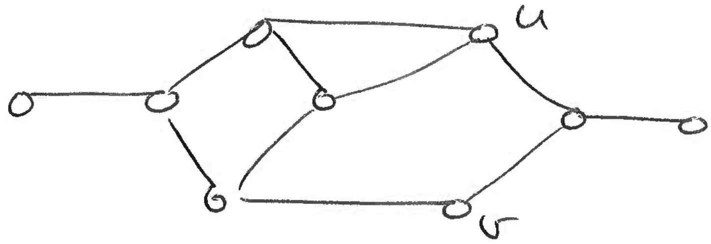


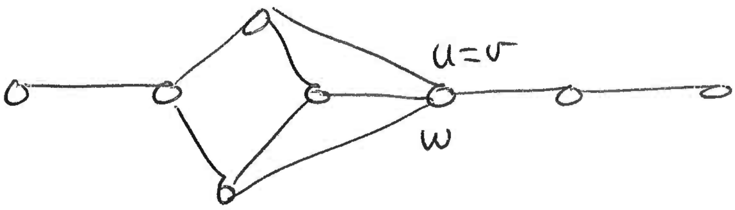
← $t-1$ choices if A, B same colour

$t-2$ choices if A, B different colours.

Notation

Say $G =$ 

$G - uv =$ 

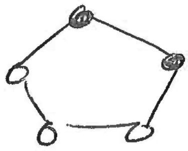
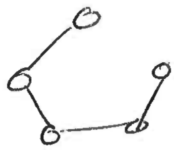
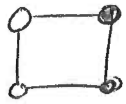
$\mathcal{E}_{uv} G =$ 

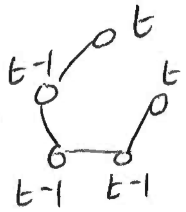
contract edge uv to vertex w

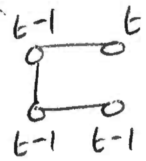
$n(w) = n(u) \cup n(v)$ [neighbours].

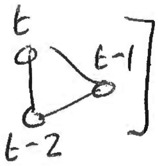
Deletion-contraction algorithm

$$f(G) = f(G - uv) - f(\mathcal{E}_{uv} G)$$

Example $G = C_5$ =  =  − 

= 

− [

− 

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$$f(C_5) = t(t-1)^4 - t(t-1)^3 + t(t-1)(t-2)$$
$$= \ldots = t(t-1)(t-2)(t^2-2t+2)$$

7 Connectivity

Recall that a graph is connected if for every u, v pair of vertices in the graph there exists a uv path. A connected graph has one component. If a graph is not connected then it is *unconnected* or *disconnected* and consists of more than one component.

Definition The *(vertex) connectivity* $\kappa(G)$ of graph G is the smallest number of vertices whose removal results in a disconnected or trivial graph. [The trivial graph is K_1.]
Recall that deletion of a vertex results in the deletion of every edge incident with that vertex. An edge is a vertex pair, deletion of one vertex from a pair will remove that pair.

Examples

(i) Consider cycle C_n. Removing one vertex will result in a path P_{n-1} which is connected. Removal of two adjacent vertices will result in a path P_{n-2}, also connected. Removal of two non-adjacent vertices will result in a disconnected graph: $P_i \cup P_j$, $i, j \geq 1$ where $i + j = n - 2$. Thus $\kappa(C_n) = 2$.

(ii) $\kappa(G) = 0$ is and only if G is already disconnected.

$\kappa(G) = 1$ if and only if G is connected and removal of a single vertex disconnects G. Such a vertex is called a *cut vertex*.

(iii) Consider the complete graph K_n. Removal of a vertex results in K_{n-1}, also connected. Removal of subsequent vertices will result in a successively smaller complete graphs. Removal of $n - 1$ vertices will result in K_1, the trivial graph and so $\kappa(K_n) = n - 1$. Note that any complete graph cannot be disconnected.

(iv) Let $G = K_n - \lambda$, the graph obtained from K_n by deleting an edge. Then $\kappa(K_n - \lambda) = n - 2 \, \forall \, n \geq 2$.

To see why we consider an example $K_5 - \lambda$. Deletion of either vertex in the deleted edge will result in K_4 which cannot be disconnected; we would need to delete a further 3 vertices to obtain the trivial graph. However if we delete any other vertex we will get $K_4 - \lambda$ which is connected. Delete another vertex not in the deleted edge and we get $K_3 - \lambda$, still connected. Repeat and we get $K_2 - \lambda = 2K_1$ which is disconnected. Thus we have managed to disconnect $K_5 - \lambda$ by deleting 3 vertices.

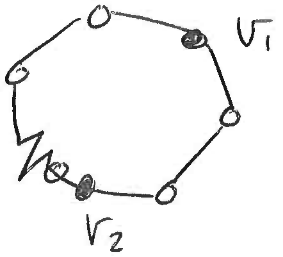
Formally: we can disconnect $K_n - \lambda$ by removing $n - 2$ vertices. If we remove fewer than $n - 2$ vertices this leaves $t \geq 3$ vertices and the remaining graph is K_t or $K_t - \lambda$, both connected.

Definition We say a graph G is *k-connected* if $\kappa(G) \geq k$. A k-connected graph cannot be disconnected unless at least k vertices are removed. However removal of any k vertices does not ensure the graph will become disconnected. If a graph is k-connected then there is a subset of the vertex set $V(G)$ of size k whose deletion will disconnect the graph.

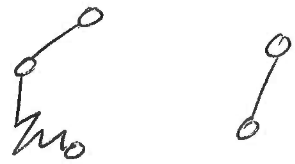
Connectivity

Vertex connectivity $K(G)$ [kappa G].

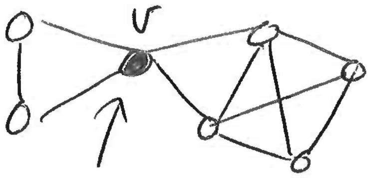
i) $G = C_n =$



Delete v_1, v_2 :

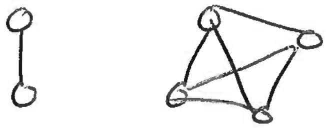


ii) $G =$



cut vertex

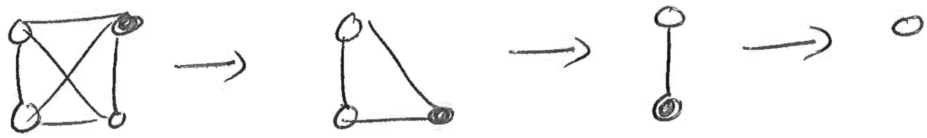
$G - v =$



$K(G) = 1$

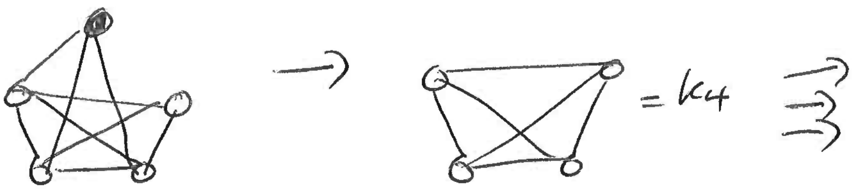
iii) $K(K_n) = n-1$

eg K_4



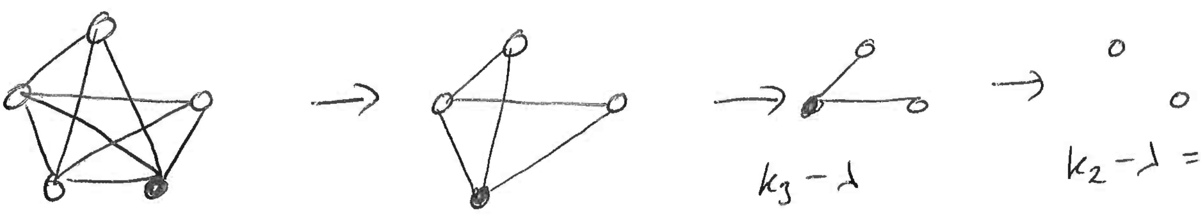
remove one vertex at a time $\rightarrow$ $\rightarrow$ trivial graph.

iv) $K_5 - \lambda$



$= K_4 \Rightarrow$ trivial graph

[4 vertices removed]



$K_4 - \lambda$

$K_3 - \lambda$

$K_2 - \lambda = 2K_1$

[3 vertices removed]

Definition The *edge connectivity* $\lambda(G)$ is the smallest number of edges whose removal disconnects G. Note that removal of an edge does not remove a vertex. The edge set of a graph is the subset of all possible unordered vertex pairs so deletion of an edge will remove a vertex pair from the edge set but will not affect the vertex set.

Examples

(i) Let G be the cycle C_n. Deleting a single edge results in P_n which is connected. Deleting two consecutive edges will result in the graph $K_1 \cup P_{n-1}$ which is disconnected. Deleting any two edges will result in the (disjoint) union of two paths. Thus $\lambda(C_n) = 2$.

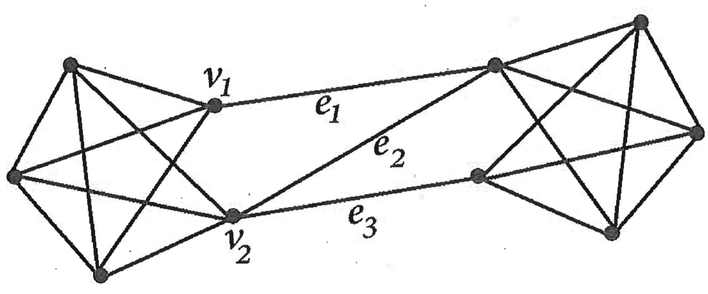
(ii) $\lambda(G) = 0$ if and only if G is disconnected or trivial.

$\lambda(G) = 1$ if and only if G is connected and contains an edge whose removal disconnects G, called a *bridge*.

(iii) $\lambda(K_n) = n - 1$. Consider K_n. Deletion of all edges incident with a single vertex will disconnect that vertex resulting in a disconnected graph. Clearly removing fewer edges would not disconnect the graph. Since K_n is complete it has all possible edges; the degree of each vertex is $n - 1$ so we must remove $n - 1$ edges to disconnect the graph.

Remark Always $\kappa(G) \leq \lambda(G) \leq \delta(G)$ where $\delta(G) = \min\,\{\deg\,(v) \colon v \in V(G)\}$.

Example
Verify that the inequality $\kappa(G) \leq \lambda(G) \leq \delta(G)$ is satisfied for the following graph G.



This graph consists of two $K_5 - \lambda$ joined by three edges $e_1,\ e_2,\ e_3$.
All vertices lie on a 10-cycle and so we need to delete at least 2 vertices to disconnect the graph since $\kappa(C_n) = 2$. Removal of vertices $v_1,\ v_2$ will disconnect the graph and so $\kappa(G) = 2$. [Deleting fewer vertices will not disconnect the graph.]
Removal of edges e_1, e_2, e_3 will disconnect G and so $\lambda(G) \leq 3$. However removal of fewer than 3 edges will not disconnect G. To see why consider the graph $G - e_i,\ \ (i = 1, 2, 3)$. In $G - e_i$ every edge lies in some cycle and we would need to remove an additional 2 edges in order to disconnect G. Thus $\lambda(G) = 3$.
Lastly $\delta(G) = 4$, the least degree in G and we can see that

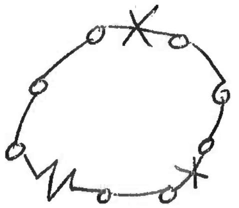
$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

is satisfied.

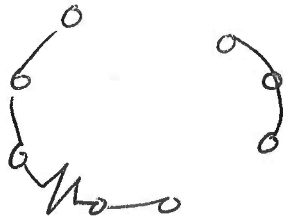
Edge connectivity $\lambda(G)$.

Examples

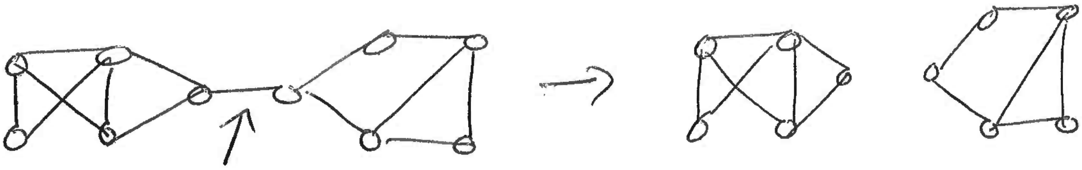
i) $C_n =$



removal of 2 edges disconnects graph



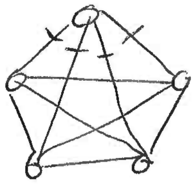
ii)



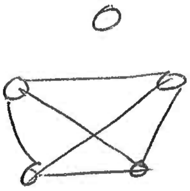
BRIDGE
or cut edge

iii) $\lambda(K_n) = n-1$

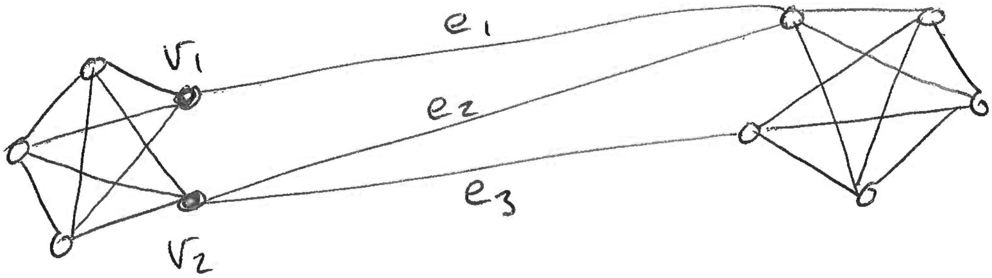
eg $K_5 =$



disconnect by removing 4 edges



Example



$$K(G) \leq \lambda(G) \leq \delta(G)$$

$$2 \leq 3 \leq 4$$

7.0.1 For interest only

Definition Let $s, t \in V(G)$. The set U of vertices of G is an $s-t$ *separating set* if s and t lie in different components of $G-U$, the graph obtained from G by deleting all vertices in U.

Definition Two $s-t$ paths are *disjoint* if they have no vertices in common other than s and t.

Remark If U is an $s-t$ separating set, and if $\mathcal{D}$ is a collection of disjoint $s-t$ paths we have Menger's theorem:

$$\min |U| = \max |\mathcal{D}|.$$

Proposition G is k-connected if and only if for every pair s, t of vertices in G there exists k disjoint $s-t$ paths in G

7.1 Algebraic connectivity

Definition The *degree matrix* is a diagonal matrix where the diagonal entries are the degrees of each vertex.

Definition The *Laplacian matrix* for a simple graph G is defined as $L = D - A(G)$ where D is the degree matrix, and $A(G)$ is the adjacency matrix.

Definition The *algebraic connectivity* of a graph G, $a(G)$, is the second smallest eigenvalue of the Laplacian matrix.
[It can be shown that, for a connected graph, the smallest eigenvalue of L is zero with corresponding eigenvector **j**.]

Example Find the eigenvalues of the Laplacian matrix for the graph with adjacency matrix. State the algebraic connectivity of G.

$$A = \begin{matrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{matrix}$$

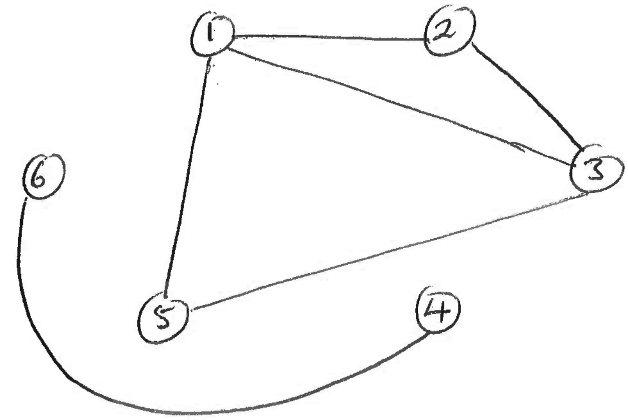
Solution
The row sums give the degree sequence: $3, 2, 3, 1, 2, 1$ in corresponding order and so

$$L = D - A = \begin{matrix} 3 & -1 & -1 & 0 & -1 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ -1 & 0 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{matrix}$$

with eigenvalues 0,0,2,2,4,4 - according to an online eigenvalue calculator: www.arndt-bruenner.de
Thus $a(G) = 0$; very low connectivity.

Algebraic connectivity

Example



Note graph is not connected so hardly surprising that $a(G)=0$.

Intuitively connectivity should increase with the addition of edges, with the complete graph with all possible edges having the maximum possible algebraic connectivity.

Example
Let $G = K_n$ with Laplacian matrix $L = (n-1)I - A$. The eigenvalues are n with multiplicity $n-1$ and 0 with multiplicity 1. Thus the algebraic connectivity $a(K_n)$ is n.
Since the eigenvalues of L are non-negative we conclude that if G is a graph on n vertices,

$$0 \leq a(G) \leq n.$$

Some measure of algebraic connectivity between 0 and 1 may be more useful which suggests the idea of normalizing this measure by dividing by n:

$$0 \leq \frac{a(G)}{n} \leq 1.$$

Remark: proof omitted
Let $G - U_k$ be a graph obtained from G by removing k vertices. Then

$$a(G) \leq a(G - U_k) + k$$

Remark: proof omitted
Let $\mu_1, \mu_2, \ldots, \mu_{n-1}, \mu_n$ be an ordered list of the eigenvalues of L. Then $a(G) = \mu_{n-1} \leq \kappa(G)$.

Example
It can be shown that $a(K_{r,s}) = \min\{r, s\}$. As an example, $a(K_{2,5}) = 2 \leq \kappa(K_{2,5}) = 2$.
Now $K_{1,4}$ can be obtained from $K_{2,5}$ by removing 2 vertices and we have

$$2 = a(K_{2,5}) \leq 1 + 2 = a(K_{1,4}) + k$$

Further examples

(i) $a(G - \lambda) \leq a(G)$.

(ii) $a(G) \leq a(G + \lambda) \leq a(G) + 2$.

(iii) $a(P_n) = 2\left(1 - \cos\frac{\pi}{n}\right)$, and $a(C_n) = 2\left(1 - \cos\frac{2\pi}{n}\right)$.

Theorem: proof omitted
Suppose that G is a k-regular graph with second smallest eigenvalue λ_{n-1}. Then $a(G) = k - \lambda_{n-1}$.