Instructors' Solutions

for

Mathematical Methods for Physics and Engineering (*third edition*)

K.F. Riley and M.P. Hobson

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Introduction

The second edition of *Mathematical Methods for Physics and Engineering* carried more than twice as many exercises, based on its various chapters, as did the first. In the Preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved we have, in the third edition, completely changed the way this matter is handled. All of the exercises from the second edition, plus a number of additional ones testing the newly-added material, have been included in penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, but *only to the odd-numbered exercises*. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus **odd-numbered** exercises, complete solutions are available, to both students and their teachers, in the form of a separate manual, K. F. Riley and M. P. Hobson, *Student Solutions Manual for Mathematical Methods for Physics and Engineering, 3rd edn.* (Cambridge: CUP, 2006). These full solutions are additional to the hints and outline answers given in the main text. For each exercise, the original question is reproduced and then followed by a fully-worked solution. For those exercises that make internal reference to the main text or to other (even-numbered) exercises not included in the manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone.

The remaining four hundred or so **even-numbered** exercises have no hints or answers, outlined or detailed, available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in the manual (though they may contain cross-references to the main text or to other exercises), form the body of the material on this website.

In many cases, in the manual as well as here, the solution given is even fuller than one that might be expected of a good student who has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is fomulated and have given the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result they are normally included in full; this should enable the instructor to determine whether a student's incorrect answer is due to a misunderstanding of principles or to a technical error.

In all new publications, on paper or on a website, errors and typographical mistakes are virtually unavoidable and we would be grateful to any instructor who brings instances to our attention.

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1

Preliminary algebra

Polynomial equations

1.2 *Determine how the number of real roots of the equation*

$$g(x) = 4x^3 - 17x^2 + 10x + k = 0$$

depends upon k. *Are there any cases for which the equation has exactly two distinct real roots*?

We first determine the positions of the turning points (if any) of $g(x)$ by equating its derivative $g'(x) = 12x^2 - 34x + 10$ to zero. The roots of $g'(x) = 0$ are given, either by factorising $g'(x)$, or by the standard formula,

$$\alpha_{1,2} = \frac{34 \pm \sqrt{1156 - 480}}{24},$$

as $\frac{5}{2}$ and $\frac{1}{3}$.

We now determine the values of $g(x)$ at these turning points; they are $g(\frac{5}{2}) = -\frac{75}{4} + k$ and $g(\frac{1}{3}) = \frac{43}{27} + k$. These will remain of opposite signs, as is required for three real roots, provided k remains in the range $-\frac{43}{27} < k < \frac{75}{4}$. If k is equal to one of these two extreme values, a graph of $g(x)$ just touches the x-axis and two of the roots become coincident, resulting in only two *distinct* real roots.

1.4 *Given that* $x = 2$ *is one root of*

$$g(x) = 2x^4 + 4x^3 - 9x^2 - 11x - 6 = 0,$$

use factorisation to determine how many real roots it has.

Given that $x = 2$ is one root of $g(x) = 0$, we write $g(x) = (x - 2)h(x)$ or, more explicitly,

$$2x^4 + 4x^3 - 9x^2 - 11x - 6 = (x - 2)(b_3x^3 + b_2x^2 + b_1x + b_0).$$

Equating the coefficients of successive (decreasing) powers of x, we obtain

$$b_3 = 2, \quad b_2 - 2b_3 = 4, \quad b_1 - 2b_2 = -9, \quad b_0 - 2b_1 = -11, \quad -2b_0 = -6.$$

These five equations have the consistent solution for the four unknowns b_i of $b_3 = 2$, $b_2 = 8$, $b_1 = 7$ and $b_0 = 3$. Thus $h(x) = 2x^3 + 8x^2 + 7x + 3$.

Clearly, since all of its coefficients are positive, $h(x)$ can have no zeros for positive values of x. A few tests with negative integer values of x (with the initial intention of making a rough sketch) reveal that $h(-3) = 0$, implying that $(x + 3)$ is a factor of $h(x)$. We therefore write

$$2x^3 + 8x^2 + 7x + 3 = (x + 3)(c_2x^2 + c_1x + c_0),$$

and, proceeding as previously, obtain $c_2 = 2$, $c_1 + 3c_2 = 8$, $c_0 + 3c_1 = 7$ and $3c_0 = 3$, with corresponding solution $c_2 = 2$, $c_1 = 2$ and $c_0 = 1$.

We now have that $g(x) = (x - 2)(x + 3)(2x^2 + 2x + 1)$. If we now try to determine the zeros of the quadratic term using the standard form (1.4) we find that, since $2^2 - (4 \times 2 \times 1)$, i.e. -4, is negative, its zeros are complex. In summary, the only real roots of $g(x) = 0$ are $x = 2$ and $x = -3$.

1.6 *Use the results of* (i) *equation (1.13),* (ii) *equation (1.12) and* (iii) *equation (1.14) to prove that if the roots of* $3x^3 - x^2 - 10x + 8 = 0$ *are* α_1, α_2 *and* α_3 *then*

(a) $\alpha_1^{-1} + \alpha_2^{-1} + \alpha_3^{-1} = 5/4$,
(b) $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 61/9$,
(c) $\alpha_1^3 + \alpha_2^3 + \alpha_3^3 = -125/27$.
(d) *Convince yourself that eliminating (say)* α_2 *and* α_3 *from* (i), (ii) *and* (iii) *does* not *give a simple explicit way of finding* α_1.

If the roots of $3x^3 - x^2 - 10x + 8 = 0$ are α_1, α_2 and α_3, then:

(i) from equation (1.13), $\alpha_1 + \alpha_2 + \alpha_3 = -\dfrac{-1}{3} = \dfrac{1}{3}$;

(ii) from equation (1.12), $\alpha_1\alpha_2\alpha_3 = (-1)^3\dfrac{8}{3} = -\dfrac{8}{3}$;

(iii) from equation (1.14), $\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1 = \dfrac{-10}{3} = -\dfrac{10}{3}$.

We now use these results in various combinations to obtain expressions for the given quantities:

(a) $\dfrac{1}{\alpha_1}+\dfrac{1}{\alpha_2}+\dfrac{1}{\alpha_3}=\dfrac{\alpha_2\alpha_3+\alpha_1\alpha_3+\alpha_2\alpha_1}{\alpha_1\alpha_2\alpha_3}=\dfrac{-(10/3)}{-(8/3)}=\dfrac{5}{4}$;

(b) $\alpha_1^2+\alpha_2^2+\alpha_3^2=(\alpha_1+\alpha_2+\alpha_3)^2-2(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)$

$=\left(\frac{1}{3}\right)^2-2\left(-\frac{10}{3}\right)=\frac{61}{9}$;

(c) $\alpha_1^3+\alpha_2^3+\alpha_3^3=$

$(\alpha_1+\alpha_2+\alpha_3)^3-3(\alpha_1+\alpha_2+\alpha_3)(\alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1)+3\alpha_1\alpha_2\alpha_3$

$=(\frac{1}{3})^3-3(\frac{1}{3})(-\frac{10}{3})+3(-\frac{8}{3})=-\frac{125}{27}$.

(d) No answer is given as it cannot be done. All manipulation is complicated and, at best, leads back to the original equation. Unfortunately, the 'convincing' will have to come from frustration, rather than from a proof by contradiction!

Trigonometric identities

1.8 *The following exercises are based on the half-angle formulae.*

(a) *Use the fact that* $\sin(\pi/6)=1/2$ *to prove that* $\tan(\pi/12)=2-\sqrt{3}$.
(b) *Use the result of* (a) *to show further that* $\tan(\pi/24)=q(2-q)$, *where* $q^2=2+\sqrt{3}$.

(a) Writing $\tan(\pi/12)$ as t and using (1.32), we have

$$\frac{1}{2}=\sin\frac{\pi}{6}=\frac{2t}{1+t^2},$$

from which it follows that $t^2-4t+1=0$.

The quadratic solution (1.6) then shows that $t=2\pm\sqrt{2^2-1}=2\pm\sqrt{3}$; there are two solutions because $\sin(5\pi/6)$ is also equal to $1/2$. To resolve the ambiguity, we note that, since $\pi/12<\pi/4$ and $\tan(\pi/4)=1$, we must have $t<1$; hence, the negative sign is the appropriate choice.

(b) Writing $\tan(\pi/24)$ as u and using (1.34) and the result of part (a), we have

$$2-\sqrt{3}=\frac{2u}{1-u^2}.$$

Multiplying both sides by $q^2=2+\sqrt{3}$, and then using $(2+\sqrt{3})(2-\sqrt{3})=1$, gives

$$1-u^2=2q^2u.$$

This quadratic equation has the (positive) solution

$$\begin{aligned} u &= -q^2 + \sqrt{q^4 + 1} \\ &= -q^2 + \sqrt{4 + 4\sqrt{3} + 3 + 1} \\ &= -q^2 + 2\sqrt{2 + \sqrt{3}} \\ &= -q^2 + 2q = q(2 - q), \end{aligned}$$

as stated in the question.

1.10 *If* $s = \sin(\pi/8)$, *prove that*

$$8s^4 - 8s^2 + 1 = 0,$$

and hence show that $s = [(2 - \sqrt{2})/4]^{1/2}$.

With $s = \sin(\pi/8)$, using (1.29) gives

$$\sin\frac{\pi}{4} = 2s(1 - s^2)^{1/2}.$$

Squaring both sides, and then using $\sin(\pi/4) = 1/\sqrt{2}$, leads to

$$\frac{1}{2} = 4s^2(1 - s^2),$$

i.e. $8s^4 - 8s^2 + 1 = 0$. This is a quadratic equation in $u = s^2$, with solutions

$$s^2 = u = \frac{8 \pm \sqrt{64 - 32}}{16} = \frac{2 \pm \sqrt{2}}{4}.$$

Since $\pi/8 < \pi/4$ and $\sin(\pi/4) = 1/\sqrt{2} = \sqrt{2/4}$, it is clear that the minus sign is the appropriate one. Taking the square root of both sides then yields the stated answer.

Coordinate geometry

1.12 *Obtain in the form (1.38), the equations that describe the following:*

(a) *a circle of radius* 5 *with its centre at* $(1, -1)$;
(b) *the line* $2x + 3y + 4 = 0$ *and the line orthogonal to it which passes through* $(1, 1)$;
(c) *an ellipse of eccentricity* 0.6 *with centre* $(1, 1)$ *and its major axis of length* 10 *parallel to the* y*-axis.*

(a) Using (1.42) gives $(x-1)^2+(y+1)^2=5^2$, i.e. $x^2+y^2-2x+2y-23=0$.

(b) From (1.24), a line orthogonal to $2x+3y+4=0$ must have the form $3x-2y+c=0$, and, if it is to pass through $(1,1)$, then $c=-1$. Expressed in the form (1.38), the pair of lines takes the form

$$0=(2x+3y+4)(3x-2y-1)=6x^2-6y^2+5xy+10x-11y-4.$$

(c) As the major semi-axis has length 5 and the eccentricity is 0.6, the minor semi-axis has length $5[1-(0.6)^2]^{1/2}=4$. The equation of the ellipse is therefore

$$\frac{(x-1)^2}{4^2}+\frac{(y-1)^2}{5^2}=1,$$

which can be written as $25x^2+16y^2-50x-32y-359=0$.

1.14 *For the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

with eccentricity e, the two points $(-ae,0)$ and $(ae,0)$ are known as its foci. Show that the sum of the distances from any *point on the ellipse to the foci is $2a$.*

[*The constancy of the sum of the distances from two fixed points can be used as an alternative defining property of an ellipse.*]

Let the sum of the distances be s. Then, for a point (x,y) on the ellipse,

$$s=[(x+ae)^2+y^2]^{1/2}+[(x-ae)^2+y^2]^{1/2},$$

where the positive square roots are to be taken.

Now, $y^2=b^2[1-(x/a)^2]$, with $b^2=a^2(1-e^2)$. Thus, $y^2=(1-e^2)(a^2-x^2)$ and

$$\begin{aligned}s&=(x^2+2aex+a^2e^2+a^2-a^2e^2-x^2+e^2x^2)^{1/2}\\&\qquad+(x^2-2aex+a^2e^2+a^2-a^2e^2-x^2+e^2x^2)^{1/2}\\&=(a+ex)+(a-ex)=2a.\end{aligned}$$

This result is independent of x and hence holds for *any* point on the ellipse.

Partial fractions

1.16 *Express the following in partial fraction form:*

$$\text{(a)}\ \frac{2x^3-5x+1}{x^2-2x-8},\qquad\qquad \text{(b)}\ \frac{x^2+x-1}{x^2+x-2}.$$

(a) For

$$f(x) = \frac{2x^3 - 5x + 1}{x^2 - 2x - 8},$$

we note that the degree of the numerator is higher than that of the denominator, and so we must first divide through by the latter. Write

$$2x^3 - 5x + 1 = (2x + s_0)(x^2 - 2x - 8) + (r_1 x + r_0).$$

Equating the coefficients of the powers of x: $0 = s_0 - 4$, $-5 = -16 - 2s_0 + r_1$, and $1 = -8s_0 + r_0$, giving $s_0 = 4$, $r_1 = 19$, and $r_0 = 33$. Thus,

$$f(x) = 2x + 4 + \frac{19x + 33}{x^2 - 2x - 8}.$$

The denominator in the final term factorises as $(x - 4)(x + 2)$, and so we write the term as

$$\frac{A}{x - 4} + \frac{B}{x + 2}.$$

Using the third method given in section 1.4:

$$A = \frac{19(4) + 33}{4 + 2} \quad \text{and} \quad B = \frac{19(-2) + 33}{-2 - 4}.$$

Thus,

$$f(x) = 2x + 4 + \frac{109}{6(x - 4)} + \frac{5}{6(x + 2)}.$$

(b) Since the highest powers of x in the denominator and numerator are equal, the partial–fraction expansion takes the form

$$f(x) = \frac{x^2 + x - 1}{x^2 + x - 2} = 1 + \frac{1}{x^2 + x - 2} = 1 + \frac{A}{x + 2} + \frac{B}{x - 1}.$$

Using the same method as above, we have

$$A = \frac{1}{-2 - 1}; \quad B = \frac{1}{1 + 2}.$$

Thus,

$$f(x) = 1 - \frac{1}{3(x + 2)} + \frac{1}{3(x - 1)}.$$

1.18 *Resolve the following into partial fractions in such a way that x does not appear in any numerator:*

(a) $\dfrac{2x^2 + x + 1}{(x - 1)^2(x + 3)}$, (b) $\dfrac{x^2 - 2}{x^3 + 8x^2 + 16x}$, (c) $\dfrac{x^3 - x - 1}{(x + 3)^3(x + 1)}$.

Since no factor x may appear in a numerator, all repeated factors appearing in the denominator give rise to as many terms in the partial fraction expansion as the power to which that factor is raised in the denominator.

(a) The denominator is already factorised but contains the repeated factor $(x-1)^2$. Thus the expansion will contain a term of the form $(x-1)^{-1}$, as well as one of the form $(x-1)^{-2}$. So,

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{A}{x+3} + \frac{B}{(x-1)^2} + \frac{C}{x-1}.$$

We can evaluate A and B using the third method given in section 1.4:

$$A = \frac{2(-3)^2-3+1}{(-3-1)^2} = 1 \quad \text{and} \quad B = \frac{2(1)^2+1+1}{1+3} = 1.$$

We now evaluate C by setting $x = 0$ (say):

$$\frac{1}{(-1)^2 3} = \frac{1}{3} + \frac{1}{(-1)^2} + \frac{C}{-1},$$

giving $C = 1$ and the full expansion as

$$\frac{2x^2+x+1}{(x-1)^2(x+3)} = \frac{1}{x+3} + \frac{1}{(x-1)^2} + \frac{1}{x-1}.$$

(b) Here the denominator needs factorising, but this is elementary,

$$\frac{x^2-2}{x^3+8x^2+16x} = \frac{x^2-2}{x(x+4)^2} = \frac{A}{x} + \frac{B}{(x+4)^2} + \frac{C}{x+4}.$$

Now, using the same method as in part (a):

$$A = \frac{0-2}{(0+4)^2} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-4)^2-2}{-4} = -\frac{7}{2}.$$

Setting $x = 1$ (say) determines C through

$$\frac{-1}{25} = -\frac{1}{8(1)} - \frac{7}{2(5)^2} + \frac{C}{5}.$$

Thus $C = 9/8$, and the full expression is

$$\frac{x^2-2}{x^3+8x^2+16x} = -\frac{1}{8x} - \frac{7}{2(x+4)^2} + \frac{9}{8(x+4)}.$$

(c)

$$\frac{x^3-x-1}{(x+3)^3(x+1)} = \frac{A}{x+1} + \frac{B}{(x+3)^3} + \frac{C}{(x+3)^2} + \frac{D}{x+3}.$$

As in parts (a) and (b), the third method in section 1.4 gives A and B as

$$A = \frac{(-1)^3-(-1)-1}{(-1+3)^3} = -\frac{1}{8} \quad \text{and} \quad B = \frac{(-3)^3-(-3)-1}{-3+1} = \frac{25}{2}.$$

Setting $x = 0$ requires that

$$\frac{-1}{27} = -\frac{1}{8} + \frac{25}{54} + \frac{C}{9} + \frac{D}{3} \qquad \text{i.e. } C + 3D = -\frac{27}{8}.$$

Setting $x = 1$ gives the additional requirement that

$$\frac{-1}{128} = -\frac{1}{16} + \frac{25}{128} + \frac{C}{16} + \frac{D}{4} \qquad \text{i.e. } C + 4D = -\frac{18}{8}.$$

Solving these two equations for C and D now yields $D = 9/8$ and $C = -54/8$. Thus,

$$\frac{x^3 - x - 1}{(x+3)^3(x+1)} = -\frac{1}{8(x+1)} + \frac{1}{8}\left[\frac{100}{(x+3)^3} - \frac{54}{(x+3)^2} + \frac{9}{x+3}\right].$$

If necessary, that the expansion is valid for all x (and not just for 0 and 1) can be checked by writing all of its terms so as to have the common denominator $(x+3)^3(x+1)$.

Binomial expansion

1.20 *Use a binomial expansion to evaluate $1/\sqrt{4.2}$ to five places of decimals, and compare it with the accurate answer obtained using a calculator.*

To use the binomial expansion, we need to express the inverse square root in the form $(1+a)^{-1/2}$ with $|a| < 1$. We do this as follows.

$$\begin{aligned}\frac{1}{\sqrt{4.2}} &= \frac{1}{(4+0.2)^{1/2}} = \frac{1}{2(1+0.05)^{1/2}} \\ &= \frac{1}{2}\left[1 - \frac{1}{2}(0.05) + \frac{3}{8}(0.05)^2 - \frac{15}{48}(0.05)^3 + \cdots\right] \\ &= 0.487949218.\end{aligned}$$

This four-term sum and the accurate value differ by about 8×10^{-7}.

Proof by induction and contradiction

1.22 *Prove by induction that*

$$1 + r + r^2 + \cdots + r^k + \cdots + r^n = \frac{1 - r^{n+1}}{1 - r}.$$

To prove that

$$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r},$$

assume that the result is valid for $n = N$, and consider the corresponding sum for $n = N + 1$, which is the original sum plus one additional term:

$$\begin{aligned}\sum_{k=0}^{N+1} r^k &= \sum_{k=0}^{N} r^k + r^{N+1} \\ &= \frac{1-r^{N+1}}{1-r} + r^{N+1}, \quad \text{using the assumption,} \\ &= \frac{1-r^{N+1}+r^{N+1}-r^{N+2}}{1-r} \\ &= \frac{1-r^{N+2}}{1-r}.\end{aligned}$$

This is the same form as in the assumption, except that N has been replaced by $N + 1$, and shows that the result is valid for $n = N + 1$ if it is valid for $n = N$.

But, since $(1 - r)/(1 - r) = 1$, the result is trivially valid for $n = 0$. It therefore follows that it is valid for all n.

1.24 *If a sequence of terms u_n satisfies the recurrence relation $u_{n+1} = (1 - x)u_n + nx$, with $u_1 = 0$, then show by induction that, for $n \geq 1$,*

$$u_n = \frac{1}{x}[nx - 1 + (1 - x)^n].$$

Assume that the stated result is valid for $n = N$, and consider the expression for the next term in the sequence:

$$\begin{aligned}u_{N+1} &= (1-x)u_N + Nx \\ &= \frac{1-x}{x}\left[Nx - 1 + (1-x)^N\right] + Nx, \quad \text{using the assumption,} \\ &= \frac{1}{x}\left[Nx - Nx^2 - 1 + x + (1-x)^{N+1} + Nx^2\right] \\ &= \frac{1}{x}\left[(N+1)x - 1 + (1-x)^{N+1}\right].\end{aligned}$$

This has the same form as in the assumption, except that N has been replaced by $N + 1$, and shows that the result is valid for $n = N + 1$ if it is valid for $n = N$.

The assumed result gives u_1 as $x^{-1}(x-1+1-x) = 0$ (i.e. as stated in the question), and so is valid for $n = 1$. It now follows, from the result proved earlier, that the given expression is valid for all $n \geq 1$.

1.26 *The quantities a_i in this exercise are all positive real numbers.*

(a) *Show that*

$$a_1 a_2 \leq \left(\frac{a_1 + a_2}{2}\right)^2.$$

(b) *Hence, prove by induction on m that*

$$a_1 a_2 \cdots a_p \leq \left(\frac{a_1 + a_2 + \cdots + a_p}{p}\right)^p,$$

where $p = 2^m$ with m a positive integer. Note that each increase of m by unity doubles the number of factors in the product.

(a) Consider $(a_1 - a_2)^2$ which is always non-negative:

$$\begin{aligned}
(a_1 - a_2)^2 &\geq 0, \\
a_1^2 - 2a_1a_2 + a_2^2 &\geq 0, \\
a_1^2 + 2a_1a_2 + a_2^2 &\geq 4a_1a_2, \\
(a_1 + a_2)^2 &\geq 4a_1a_2, \\
\left(\frac{a_1 + a_2}{2}\right)^2 &\geq a_1a_2.
\end{aligned}$$

(b) With $p = 2^m$, assume that

$$a_1 a_2 \cdots a_p \leq \left(\frac{a_1 + a_2 + \cdots + a_p}{p}\right)^p$$

is valid for some $m = M$. Write $P = 2^M$, $P' = 2P$, $b_1 = a_1 + a_2 + \cdots + a_P$ and $b_2 = a_{P+1} + a_{P+2} + \cdots + a_{P'}$. Note that both b_1 and b_2 consist of P terms.

Now consider the multiple product $u = a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'}$.

$$\begin{aligned}
u &\leq \left(\frac{a_1 + a_2 + \cdots + a_P}{P}\right)^P \left(\frac{a_{P+1} + a_{P+2} + \cdots + a_{P'}}{P}\right)^P \\
&= \left(\frac{b_1 b_2}{P^2}\right)^P,
\end{aligned}$$

where the assumed result has been applied twice, once to a set consisting of the first P numbers, and then for a second time to the remaining set of P numbers, $a_{P+1}, a_{P+2}, \ldots, a_{P'}$. We have also used the fact that, for positive real numbers, if $q \leq r$ and $s \leq t$ then $qs \leq rt$.

But, from part (a),

$$b_1 b_2 \leq \left(\frac{b_1 + b_2}{2}\right)^2.$$

Thus,

$$\begin{aligned} a_1 a_2 \cdots a_P a_{P+1} a_{P+2} \cdots a_{P'} &\leq \left(\frac{1}{P^2}\right)^P \left(\frac{b_1+b_2}{2}\right)^{2P} \\ &= \frac{(b_1+b_2)^{P'}}{(2P)^{2P}} \\ &= \left(\frac{b_1+b_2}{P'}\right)^{P'}. \end{aligned}$$

This shows that the result is valid for $P' = 2^{M+1}$ if it is valid for $P = 2^M$. But for $m = M = 1$ the postulated inequality is simply result (a), which was shown directly. Thus the inequality holds for all positive integer values of m.

1.28 *An arithmetic progression of integers a_n is one in which $a_n = a_0 + nd$, where a_0 and d are integers and n takes successive values $0, 1, 2, \ldots$.*

(a) *Show that if any one term of the progression is the cube of an integer, then so are infinitely many others.*
(b) *Show that no cube of an integer can be expressed as $7n+5$ for some positive integer n.*

(a) We proceed by the method of contradiction. Suppose $d > 0$. Assume that there is a finite, but non-zero, number of natural cubes in the arithmetic progression. Then there must be a largest cube. Let it be $a_N = a_0 + Nd$, and write it as $a_N = a_0 + Nd = m^3$. Now consider $(m+d)^3$:

$$\begin{aligned} (m+d)^3 &= m^3 + 3dm^2 + 3d^2m + d^3 \\ &= a_0 + Nd + d(3m^2 + 3dm + d^2) \\ &= a_0 + dN_1, \end{aligned}$$

where $N_1 = N + 3m^2 + 3dm + d^2$ is necessarily an integer, since N, m and d all are. Further, $N_1 > N$. Thus $a_{N_1} = a_0 + N_1 d$ is also the cube of a natural number and is greater than a_N; this contradicts the assumption that it is possible to select a largest cube in the series and establishes the result that, if there is one such cube, then there are infinitely many of them. A similar argument (considering the smallest term in the series) can be carried through if $d < 0$.

We note that the result is also formally true in the case in which $d = 0$; if a_0 is a natural cube, then so is every term, since they are all equal to a_0.

(b) Again, we proceed by the method of contradiction. Suppose that $7N + 5 = m^3$

for some pair of positive integers N and m. Consider the quantity

$$\begin{aligned}(m-7)^3 &= m^3 - 21m^2 + 147m - 343\\ &= 7N + 5 - 7(3m^2 - 21m + 49)\\ &= 7N_1 + 5,\end{aligned}$$

where $N_1 = N - 3m^2 + 21m - 49$ is an integer smaller than N. From this, it follows that if m^3 can be expressed in the form $7N+5$ then so can $(m-7)^3$, $(m-14)^3$, etc. Further, for some finite integer p, $(m-7p)$ must lie in the range $0 \le m - 7p \le 6$ and will have the property $(m-7p)^3 = 7N_p + 5$.

However, explicit calculation shows that, when expressed in the form $7n+q$, the cubes of the integers 0, 1, 2, $\cdots$, 6 have respective values of q of 0, 1, 1, 6, 1, 6, 6; none of these is equal to 5. This contradicts the conclusion that followed from our initial supposition and subsequent argument. It was therefore wrong to assume that there is a natural cube that can be expressed in the form $7N+5$.

[Note that it is not sufficient to carry out the above explicit calculations and then rely on the construct from part (a), as this does not guarantee to generate every cube.]

Necessary and sufficient conditions

1.30 *Prove that the equation $ax^2 + bx + c = 0$, in which a, b and c are real and $a > 0$, has two real distinct solutions IFF $b^2 > 4ac$.*

As is usual for IFF proofs, this answer will consist of two parts.

Firstly, assume that $b^2 > 4ac$. We can then write the equation as

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) = 0,$$

$$a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c = 0,$$

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} = \lambda^2.$$

Since $b^2 > 4ac$ and $a > 0$, λ is real, positive and non-zero. So, taking the square roots of both sides of the final equation gives

$$x = -\frac{b}{2a} \pm \frac{\lambda}{\sqrt{a}},$$

i.e. both roots are real and they are distinct; thus, the 'if' part of the proposition is established.

Now assume that both roots are real, α and β say, with $\alpha \neq \beta$. Then,

$$a\alpha^2 + b\alpha + c = 0,$$
$$a\beta^2 + b\beta + c = 0.$$

Subtraction of the two equations gives

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0 \quad \Rightarrow \quad b = -(\alpha + \beta)a, \text{ since } \alpha - \beta \neq 0.$$

Multiplying the first displayed equation by β and the second by α and then subtracting, gives

$$a(\alpha^2\beta - \beta^2\alpha) + c(\beta - \alpha) = 0 \quad \Rightarrow \quad c = \alpha\beta a, \text{ since } \alpha - \beta \neq 0.$$

Now, recalling that $\alpha \neq \beta$ and that $a > 0$, consider the inequality

$$\begin{aligned} 0 < (\alpha - \beta)^2 &= \alpha^2 - 2\alpha\beta + \beta^2 \\ &= (\alpha + \beta)^2 - 4\alpha\beta \\ &= \frac{b^2}{a^2} - 4\frac{c}{a} = \frac{b^2 - 4ac}{a^2}. \end{aligned}$$

This inequality shows that b^2 is necessarily greater than $4ac$, and so establishes the 'only if' part of the proof.

1.32 *Given that at least one of a and b, and at least one of c and d, are non-zero, show that $ad = bc$ is both a necessary and sufficient condition for the equations*

$$ax + by = 0,$$
$$cx + dy = 0,$$

to have a solution in which at least one of x and y is non-zero.

First, suppose that $ad = bc$ with at least one of a and b, and at least one of c and d, non-zero. Assume, for definiteness, that a and c are non-zero; if this is not the case, then the following proof is modified in an obvious way by interchanging the roles of a and b and/or of c and d, as necessary:

$$ax + by = 0 \Rightarrow x = -\frac{b}{a}y,$$
$$cx + dy = 0 \Rightarrow x = -\frac{d}{c}y.$$

Now

$$ad = bc \quad \Rightarrow \quad d = \frac{bc}{a} \quad \Rightarrow \quad \frac{d}{c} = \frac{b}{a},$$

where we have used, in turn, that $a \neq 0$ and $c \neq 0$. Thus the two solutions for x in terms of y are the same. Any non-zero value for y may be chosen, but that for x is then determined (and may be zero). This establishes that the condition is sufficient.

To show that it is a necessary condition, suppose that there is a non-trivial solution to the original equations and that, say, $x \neq 0$. Multiply the first equation by d and the second by b to obtain

$$dax + dby = 0,$$
$$bcx + bdy = 0.$$

Subtracting these equations gives $(ad - bc)x = 0$ and, since $x \neq 0$, it follows that $ad = bc$.

If $x = 0$ then $y \neq 0$, and multiplying the first of the original equations by c and the second by a leads to the same conclusion.

This completes the proof that the condition is both necessary and sufficient.

2

Preliminary calculus

2.2 *Find from first principles the first derivative of* $(x+3)^2$ *and compare your answer with that obtained using the chain rule.*

Using the definition of a derivative, we consider the difference between $(x+\Delta x+3)^2$ and $(x+3)^2$, and determine the following limit (if it exists):

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{(x+\Delta x+3)^2-(x+3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(x+3)^2+2(x+3)\Delta x+(\Delta x)^2]-(x+3)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(2(x+3)\Delta x+(\Delta x)^2}{\Delta x} \\ &= 2x+6. \end{aligned}$$

The limit does exist, and so the derivative is $2x+6$.

Rewriting the function as $f(x)=u^2$, where $u(x)=x+3$, and using the chain rule:

$$f'(x) = 2u \times \frac{du}{dx} = 2u \times 1 = 2u = 2x+6,$$

i.e. the same, as expected.

2.4 *Find the first derivatives of*

(a) $x/(a+x)^2$, (b) $x/(1-x)^{1/2}$, (c) $\tan x$, *as* $\sin x/\cos x$,

(d) $(3x^2+2x+1)/(8x^2-4x+2)$.

In each case, using (2.13) for a quotient:

(a) $$f'(x) = \frac{[(a+x)^2 \times 1] - [x \times 2(a+x)]}{(a+x)^4} = \frac{a^2 - x^2}{(a+x)^4} = \frac{a-x}{(a+x)^3};$$

(b) $$f'(x) = \frac{[(1-x)^{1/2} \times 1] - [x \times -\frac{1}{2}(1-x)^{-1/2}]}{1-x} = \frac{1-\frac{1}{2}x}{(1-x)^{3/2}};$$

(c) $$f'(x) = \frac{[\cos x \times \cos x] - [\sin x \times (-\sin x)]}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x;$$

(d) $$\begin{aligned} f'(x) &= \frac{[(8x^2-4x+2)\times(6x+2)] - [(3x^2+2x+1)\times(16x-4)]}{(8x^2-4x+2)^2} \\ &= \frac{x^3(48-48) + x^2(16-24+12-32) + \cdots}{(8x^2-4x+2)^2} \\ &\qquad \frac{\cdots + x(-8+12+8-16) + (4+4)}{(8x^2-4x+2)^2} \\ &= \frac{-28x^2-4x+8}{(8x^2-4x+2)^2} = \frac{-7x^2-x+2}{(4x^2-2x+1)^2}. \end{aligned}$$

2.6 *Show that the function* $y(x) = \exp(-|x|)$ *defined as*

$$\begin{aligned} \exp x \ &\textit{for } x < 0, \\ 1 \ &\textit{for } x = 0, \\ \exp(-x) \ &\textit{for } x > 0, \end{aligned}$$

is not *differentiable at* $x = 0$. *Consider the limiting process for both* $\Delta x > 0$ *and* $\Delta x < 0$.

For $x > 0$, let $\Delta x = \eta$. Then,

$$\begin{aligned} y'(x>0) &= \lim_{\eta\to 0} \frac{e^{-0-\eta} - 1}{\eta} \\ &= \lim_{\eta\to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{\eta} = -1. \end{aligned}$$

For $x < 0$, let $\Delta x = -\eta$. Then,

$$\begin{aligned} y'(x>0) &= \lim_{\eta\to 0} \frac{e^{0-\eta} - 1}{-\eta} \\ &= \lim_{\eta\to 0} \frac{1 - \eta + \frac{1}{2!}\eta^2 \cdots - 1}{-\eta} = 1. \end{aligned}$$

The two limits are not equal, and so $y(x)$ is not differentiable at $x = 0$.

2.8 *If* $2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$, *show that* $dy/dx = 16$ *when* $x = 1$.

For this equation neither x nor y can be made the subject of the equation, i.e neither can be written explicitly as a function of the other, and so we are forced to use implicit differentiation. Starting from

$$2y + \sin y + 5 = x^4 + 4x^3 + 2\pi$$

implicit differentiation, and the use of the chain rule when differentiating $\sin y$ with respect to x, gives

$$2\frac{dy}{dx} + \cos y\frac{dy}{dx} = 4x^3 + 12x^2.$$

When $x = 1$ the original equation reduces to $2y + \sin y = 2\pi$ with the obvious (and unique, as can be verified from a simple sketch) solution $y = \pi$. Thus, with $x = 1$ and $y = \pi$,

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{4+12}{2+\cos\pi} = 16.$$

2.10 *The function* $y(x)$ *is defined by* $y(x) = (1 + x^m)^n$.

(a) *Use the chain rule to show that the first derivative of* y *is* $nmx^{m-1}(1+x^m)^{n-1}$.
(b) *The binomial expansion (see section 1.5) of* $(1 + z)^n$ *is*

$$(1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{r!}z^r + \cdots.$$

Keeping only the terms of zeroth and first order in dx, *apply this result twice to derive result* (a) *from first principles.*
(c) *Expand* y *in a series of powers of* x *before differentiating term by term. Show that the result is the series obtained by expanding the answer given for* dy/dx *in part* (a).

(a) Writing $1 + x^m$ as u, $y(x) = u^n$, and so $dy/du = nu^{n-1}$, whilst $du/dx = mx^{m-1}$. Thus, from the chain rule,

$$\frac{dy}{dx} = nu^{n-1} \times mx^{m-1} = nmx^{m-1}(1+x^m)^{n-1}.$$

(b) From the defining process for a derivative,

$$
\begin{aligned}
y'(x) &= \lim_{\Delta x\to 0} \frac{[1+(x+\Delta x)^m]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[1+x^m(1+\frac{\Delta x}{x})^m]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{[1+x^m(1+\frac{m\Delta x}{x}+\cdots)]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{(1+x^m+mx^{m-1}\Delta x+\cdots)^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{\left[(1+x^m)\left(1+\frac{mx^{m-1}\Delta x}{1+x^m}+\cdots\right)\right]^n - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{(1+x^m)^n\left(1+\frac{mnx^{m-1}\Delta x}{1+x^m}+\cdots\right) - (1+x^m)^n}{\Delta x} \\
&= \lim_{\Delta x\to 0} \frac{mn(1+x^m)^{n-1}x^{m-1}\Delta x+\cdots}{\Delta x} \\
&= nmx^{m-1}(1+x^m)^{n-1},
\end{aligned}
$$

i.e. the same as the result in part (a).

(c) Expanding in a power series before differentiating:

$$
\begin{aligned}
y(x) &= 1 + nx^m + \frac{n(n-1)}{2!}\,x^{2m} + \cdots \\
&\qquad + \frac{n(n-1)\cdots(n-r+1)}{r!}\,x^{rm} + \cdots, \\
y'(x) &= m\,nx^{m-1} + \frac{2m\,n(n-1)}{2!}\,x^{2m-1} + \cdots \\
&\qquad + \frac{rm\,n(n-1)\cdots(n-r+1)}{r!}\,x^{rm-1} + \cdots.
\end{aligned}
$$

Now, expanding the result given in part (a) gives

$$
\begin{aligned}
y'(x) &= nmx^{m-1}(1+x^m)^{n-1} \\
&= nmx^{m-1}\left(1+\cdots+\frac{(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms}+\cdots\right) \\
&= nmx^{m-1}+\cdots+\frac{mn(n-1)(n-2)\cdots(n-s)}{s!}\,x^{ms+m-1}+\cdots.
\end{aligned}
$$

This is the same as the previous expansion of $y'(x)$ if, in the general term, the index is moved by one, i.e. $s = r - 1$.

2.12 *Find the positions and natures of the stationary points of the following functions:*

(a) $x^3 - 3x + 3$; (b) $x^3 - 3x^2 + 3x$; (c) $x^3 + 3x + 3$;
(d) $\sin ax$ *with* $a \neq 0$; (e) $x^5 + x^3$; (f) $x^5 - x^3$.

In each case, we need to determine the first and second derivatives of the function. The zeros of the 1st derivative give the positions of the stationary points, and the values of the 2nd derivatives at those points determine their natures.

(a) $$y = x^3 - 3x + 3; \quad y' = 3x^2 - 3; \quad y'' = 6x.$$

$y' = 0$ has roots at $x = \pm 1$, where the values of y'' are ± 6. Therefore, there is a minimum at $x = 1$ and a maximum at $x = -1$.

(b) $$y = x^3 - 3x^2 + 3x; \quad y' = 3x^2 - 6x + 3; \quad y'' = 6x - 6.$$

$y' = 0$ has a double root at $x = 1$, where the value of y'' is 0. Therefore, there is a point of inflection at $x = 1$, but no other stationary points. At the point of inflection, the tangent to the curve $y = y(x)$ is horizontal.

(c) $$y = x^3 + 3x + 3; \quad y' = 3x^2 + 3; \quad y'' = 6x.$$

$y' = 0$ has no real roots, and so there are no stationary points.

(d) $$y = \sin ax; \quad y' = a \cos ax; \quad y'' = -a^2 \sin ax.$$

$y' = 0$ has roots at $x = (n + \frac{1}{2})\pi/a$ for integer n. The corresponding values of y'' are $\mp a^2$, depending on whether n is even or odd. Therefore, there is a maximum for even n and a minimum where n is odd.

(e) $$y = x^5 + x^3; \quad y' = 5x^4 + 3x^2; \quad y'' = 20x^3 + 6x.$$

$y' = 0$ has, as its only real root, a double root at $x = 0$, where the value of y'' is 0. Thus, there is a (horizontal) point of inflection at $x = 0$, but no other stationary point.

(f) $$y = x^5 - x^3; \quad y' = 5x^4 - 3x^2; \quad y'' = 20x^3 - 6x.$$

$y' = 0$ has a double root at $x = 0$ and simple roots at $x = \pm(\frac{3}{5})^{1/2}$, where the respective values of y'' are 0 and $\pm 6(\frac{3}{5})^{1/2}$. Therefore, there is a point of inflection at $x = 0$, a maximum at $x = -(\frac{3}{5})^{1/2}$ and a minimum at $x = (\frac{3}{5})^{1/2}$.

Figure 2.1 The solutions to exercise 2.14.

2.14 *By finding their stationary points and examining their general forms, determine the range of values that each of the following functions $y(x)$ can take. In each case make a sketch-graph incorporating the features you have identified.*

(a) $y(x) = (x - 1)/(x^2 + 2x + 6)$.
(b) $y(x) = 1/(4 + 3x - x^2)$.
(c) $y(x) = (8 \sin x)/(15 + 8 \tan^2 x)$.

See figure 2.1 (a)–(c).

(a) Some simple points to calculate for

$$y = \frac{x - 1}{x^2 + 2x + 6}$$

are $y(0) = -\frac{1}{6}$, $y(1) = 0$ and $y(\pm\infty) = 0$, and, since the denominator has no real roots ($2^2 < 4 \times 1 \times 6$), there are no infinities. Its 1st derivative is

$$y' = \frac{-x^2 + 2x + 8}{(x^2 + 2x + 6)^2} = \frac{-(x + 2)(x - 4)}{(x^2 + 2x + 6)^2}.$$

Thus there are turning points only at $x = -2$, with $y(-2) = -\frac{1}{2}$, and at $x = 4$, with $y(4) = \frac{1}{10}$. The former must be a minimum and the latter a maximum. The range in which $y(x)$ lies is $-\frac{1}{2} \le y \le \frac{1}{10}$.

(b) Some simple points to calculate for

$$y = \frac{1}{4 + 3x - x^2}.$$

are $y(0) = \frac{1}{4}$ and $y(\pm\infty) = 0$, approached from negative values. Since the denominator can be written as $(4-x)(1+x)$, the function has infinities at $x = -1$ and $x = 4$ and is positive in the range of x between them.

The 1st derivative is

$$y' = \frac{2x - 3}{(4 + 3x - x^2)^2}.$$

Thus there is only one turning point; this is at $x = \frac{3}{2}$, with corresponding $y(\frac{3}{2}) = \frac{4}{25}$. Since $\frac{3}{2}$ lies in the range $-1 < x < 4$, at the ends of which the function $\to +\infty$, the stationary point must be a minimum. This sets a lower limit on the positive values of $y(x)$ and so the ranges in which it lies are $y < 0$ and $y \geq \frac{4}{25}$.

(c) The function

$$y = \frac{8 \sin x}{15 + 8 \tan^2 x}$$

is clearly periodic with period 2π.

Since $\sin x$ and $\tan^2 x$ are both symmetric about $x = \frac{1}{2}\pi$, so is the function. Also, since $\sin x$ is antisymmetric about $x = \pi$ whilst $\tan^2 x$ is symmetric, the function is antisymmetric about $x = \pi$.

Some simple points to calculate are $y(n\pi) = 0$ for all integers n. Further, since $\tan(n + \frac{1}{2})\pi = \infty$, $y((n + \frac{1}{2})\pi) = 0$. As the denominator has no real roots there are no infinities.

Setting the derivative of $y(x) \equiv 8u(x)/v(x)$ equal to zero, i.e. writing $vu' = uv'$, and expressing all terms as powers of $\cos x$ gives (using $\tan^2 z = \sec^2 z - 1$ and $\sin^2 z = 1 - \cos^2 z$)

$$\begin{aligned}
(15 + 8 \tan^2 x) \cos x &= 16 \sin x \tan x \sec^2 x, \\
15 + \frac{8}{\cos^2 x} - 8 &= \frac{16(1 - \cos^2 x)}{\cos^4 x}, \\
7 \cos^4 x + 24 \cos^2 x - 16 &= 0.
\end{aligned}$$

This quadratic equation for $\cos^2 x$ has roots of $\frac{4}{7}$ and -4. Only the first of these gives real values for $\cos x$ of $\pm\frac{2}{\sqrt{7}}$. The corresponding turning values of $y(x)$ are $\pm\frac{8}{7\sqrt{21}}$. The value of y always lies between these two limits.

2.16 *The curve* $4y^3 = a^2(x + 3y)$ *can be parameterised as* $x = a\cos 3\theta$, $y = a\cos\theta$.

(a) *Obtain expressions for* dy/dx (i) *by implicit differentiation and* (ii) *in parameterised form. Verify that they are equivalent.*
(b) *Show that the only point of inflection occurs at the origin. Is it a stationary point of inflection?*
(c) *Use the information gained in* (a) *and* (b) *to sketch the curve, paying particular attention to its shape near the points* $(-a, a/2)$ *and* $(a, -a/2)$ *and to its slope at the 'end points'* (a, a) *and* $(-a, -a)$.

(a) (i) Differentiating the equation of the curve implicitly:

$$12y^2\frac{dy}{dx} = a^2 + 3a^2\frac{dy}{dx}, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{a^2}{12y^2 - 3a^2}.$$

(ii) In parameterised form:

$$\frac{dy}{d\theta} = -a\sin\theta, \quad \frac{dx}{d\theta} = -3a\sin 3\theta, \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-a\sin\theta}{-3a\sin 3\theta}.$$

But, using the results from section 1.2, we have that

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \cos 2\theta\sin\theta\\ &= 2\sin\theta\cos^2\theta + (2\cos^2\theta - 1)\sin\theta\\ &= \sin\theta(4\cos^2\theta - 1),\end{aligned}$$

thus giving dy/dx as

$$\frac{dy}{dx} = \frac{1}{12\cos^2\theta - 3} = \frac{a^2}{12a^2\cos^2\theta - 3a^2},$$

with $a\cos\theta = y$, i.e. as in (i).

(b) At a point of inflection $y'' = 0$. For the given function,

$$\frac{d^2y}{dx^2} = \frac{d}{dy}\left(\frac{dy}{dx}\right) \times \frac{dy}{dx} = -\frac{a^2}{(12y^2 - 3a^2)^2} \times 24y \times \frac{a^2}{12y^2 - 3a^2}.$$

This can only equal zero at $y = 0$, when $x = 0$ also. But, when $y = 0$ it follows from (a)(i) that $dy/dx = 1/(-3) = -\frac{1}{3}$. As this is non-zero the point of inflection is not a stationary point.

(c) See figure 2.2. Note in particular that the curve has vertical tangents when $y = \pm a/2$ and that $dy/dx = \frac{1}{9}$ at $y = \pm a$, i.e. the tangents at the end points of the 'S'-shaped curve are not horizontal.

Figure 2.2 The parametric curve described in exercise 2.16.

2.18 *Show that the maximum curvature on the catenary* $y(x) = a\cosh(x/a)$ *is* $1/a$. *You will need some of the results about hyperbolic functions stated in subsection 3.7.6.*

The general expression for the curvature, ρ^{-1}, of the curve $y = y(x)$ is

$$\frac{1}{\rho} = \frac{y''}{(1 + y'^2)^{3/2}},$$

and so we begin by calculating the first two derivatives of y. Starting from $y = a\cosh(x/a)$, we obtain

$$y' = a\frac{1}{a}\sinh\frac{x}{a},$$
$$y'' = \frac{1}{a}\cosh\frac{x}{a}.$$

Therefore the curvature of the catenary at the point (x, y) is given by

$$\frac{1}{\rho} = \frac{\frac{1}{a}\cosh\frac{x}{a}}{\left[1 + \sinh^2\frac{x}{a}\right]^{3/2}} = \frac{1}{a}\frac{\cosh\frac{x}{a}}{\cosh^3\frac{x}{a}} = \frac{a}{y^2}.$$

To obtain this result we have used the identity $\cosh^2 z = 1 + \sinh^2 z$. We see that the curvature is maximal when y is minimal; this occurs when $x = 0$ and $y = a$. The maximum curvature is therefore $1/a$.

Figure 2.3 The coordinate system described in exercise 2.20.

2.20 *A two-dimensional coordinate system useful for orbit problems is the tangential-polar coordinate system (figure 2.3). In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. By proceeding as indicated below, show that the radius of curvature at P can be written in the form $\rho = r\,dr/dp$.*

Consider two neighbouring points P and Q on the curve. The normals to the curve through those points meet at C, with (in the limit $Q \to P$) $CP = CQ = \rho$. Apply the cosine rule to triangles OPC and OQC to obtain two expressions for c^2, one in terms of r and p and the other in terms of $r+\Delta r$ and $p+\Delta p$. By equating them and letting $Q \to P$ deduce the stated result.

We first note that $\cos OPC$ is equal to the sine of the angle between OP and the tangent at P, and that this in turn has the value p/r. Now, applying the cosine rule to the triangles OCP and OCQ, we have

$$\begin{aligned} c^2 &= r^2 + \rho^2 - 2r\rho\cos OPC = r^2 + \rho^2 - 2\rho p \\ c^2 &= (r+\Delta r)^2 + \rho^2 - 2(r+\Delta r)\rho\cos OQC \\ &= (r+\Delta r)^2 + \rho^2 - 2\rho(p+\Delta p). \end{aligned}$$

Subtracting and rearranging then yields

$$\rho = \frac{r\Delta r + \frac{1}{2}(\Delta r)^2}{\Delta p},$$

or, in the limit $Q \to P$, that $\rho = r(dr/dp)$.

2.22 *If $y = \exp(-x^2)$, show that $dy/dx = -2xy$ and hence, by applying Leibnitz' theorem, prove that for $n \geq 1$*

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0.$$

With $y(x) = \exp(-x^2)$,

$$\frac{dy}{dx} = -2x\exp(-x^2) = -2xy.$$

We now take the nth derivatives of both sides and use Leibnitz' theorem to find that of the product xy, noting that all derivatives of x beyond the first are zero:

$$y^{(n+1)} = -2[\,(y^{(n)})(x) + n(y^{(n-1)})(1) + 0\,].$$

i.e.

$$y^{(n+1)} + 2xy^{(n)} + 2ny^{(n-1)} = 0,$$

as stated in the question.

2.24 *Determine what can be learned from applying Rolle's theorem to the following functions $f(x)$:* (a)e^x; (b)$x^2 + 6x$; (c)$2x^2 + 3x + 1$; (d)$2x^2 + 3x + 2$; (e)$2x^3 - 21x^2 + 60x + k$. (f)*If $k = -45$ in* (e), *show that $x = 3$ is one root of $f(x) = 0$, find the other roots, and verify that the conclusions from* (e) *are satisfied.*

(a) Since the derivative of $f(x) = e^x$ is $f'(x) = e^x$, Rolle's theorem states that between any two consecutive roots of $f(x) = e^x = 0$ there must be a root of $f'(x) = e^x = 0$, i.e. another root of the same equation. This is clearly a contradiction and it is wrong to suppose that there is more than one root of $e^x = 0$. In fact, there are no finite roots of the equation and the only zero of e^x lies formally at $x = -\infty$.

(b) Since $f(x) = x(x + 6)$, it has zeros at $x = -6$ and $x = 0$. Therefore the (only) root of $f'(x) = 2x + 6 = 0$ must lie between these values; it clearly does, as $-6 < -3 < 0$.

(c) With $f(x) = 2x^2 + 3x + 1$ and hence $f'(x) = 4x + 3$, any roots of $f(x) = 0$ (actually -1 and $-\frac{1}{2}$) must lie on either side of the root of $f'(x) = 0$, i.e. $x = -\frac{3}{4}$. They clearly do.

(d) This is as in (c), but there are no real roots. However, it can be more generally stated that if there are two values of x that give $2x^2 + 3x + k$ equal values then they lie one on each side of $x = -\frac{3}{4}$.

(e) With $f(x) = 2x^3 - 21x^2 + 60x + k$,

$$f'(x) = 6x^2 - 42x + 60 = 6(x-5)(x-2)$$

and $f'(x) = 0$ has roots 2 and 5. Therefore, if $f(x) = 0$ has three real roots α_i with $\alpha_1 < \alpha_2 < \alpha_3$, then $\alpha_1 < 2 < \alpha_2 < 5 < \alpha_3$.

(f) When $k = -45$, $f(3) = 54 - 189 + 180 - 45 = 0$ and so $x = 3$ is a root of $f(x) = 0$ and $(x-3)$ is a factor of $f(x)$. Writing $f(x) = 2x^3 - 21x^2 + 60x - 45$ as $(x-3)(a_2x^2 + a_1x + a_0)$ and equating coefficients gives $a_2 = 2$, $a_1 = -15$ and $a_0 = 15$. The other two roots are therefore

$$\frac{15 \pm \sqrt{225 - 120}}{4} = \frac{1}{4}(15 \pm \sqrt{105}) = 1.19 \text{ or } 6.31.$$

Result (e) is verified in this case since $1.19 < 2 < 3 < 5 < 6.31$.

2.26 *Use the mean value theorem to establish bounds*

(a) *for* $-\ln(1-y)$, *by considering* $\ln x$ *in the range* $0 < 1 - y < x < 1$,
(b) *for* $e^y - 1$, *by considering* $e^x - 1$ *in the range* $0 < x < y$.

(a) The mean value theorem applied to $\ln x$ within limits $1 - y$ and 1 gives

$$\frac{\ln(1) - \ln(1-y)}{1 - (1-y)} = \frac{d}{dx}(\ln x) = \frac{1}{x} \qquad (*)$$

for some x in the range $1 - y < x < 1$. Now, since $1 - y < x < 1$ it follows that

$$\begin{aligned} & \frac{1}{1-y} > \frac{1}{x} > 1, \\ \Rightarrow\ & \frac{1}{1-y} > \frac{-\ln(1-y)}{y} > 1, \\ \Rightarrow\ & \frac{y}{1-y} > -\ln(1-y) > y. \end{aligned}$$

The second line was obtained by substitution from $(*)$.

(b) The mean value theorem applied to $e^x - 1$ within limits 0 and y gives

$$\frac{e^y - 1 - 0}{y - 0} = e^x \quad \text{for some } x \text{ in the range } 0 < x < y.$$

Now, since $0 < x < y$ it follows that

$$\begin{aligned} & 1 < e^x < e^y, \\ \Rightarrow\ & 1 < \frac{e^y - 1}{y} < e^y, \\ \Rightarrow\ & y < e^y - 1 < ye^y. \end{aligned}$$

Again, the second line was obtained by substitution for x from the mean value theorem result.

2.28 *Use Rolle's theorem to deduce that if the equation $f(x) = 0$ has a repeated root x_1 then x_1 is also a root of the equation $f'(x) = 0$.*

(a) *Apply this result to the 'standard' quadratic equation $ax^2 + bx + c = 0$, to show that a necessary condition for equal roots is $b^2 = 4ac$.*
(b) *Find all the roots of $f(x) = x^3 + 4x^2 - 3x - 18 = 0$, given that one of them is a repeated root.*
(c) *The equation $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2 = 0$ has a repeated integer root. How many real roots does it have altogether?*

If two roots of $f(x) = 0$ are x_1 and x_2, i.e. $f(x_1) = f(x_2) = 0$, then it follows from Rolle's theorem that there is some x_3 in the range $x_1 \le x_3 \le x_2$ for which $f'(x_3) = 0$. Now let $x_2 \to x_1$ to form the repeated root; x_3 must also tend to the limit x_1, i.e. x_1 is a root of $f'(x) = 0$ as well as of $f(x) = 0$.

(a) A quadratic equation $f(x) = ax^2 + bx + c = 0$ only has two roots and so if they are equal the common root α must also be a root of $f'(x) = 2ax + b = 0$, i.e. $\alpha = -b/2a$. Thus

$$a\frac{b^2}{4a^2} + b\frac{-b}{2a} + c = 0.$$

It then follows that $c - (b^2/4a) = 0$ and that $b^2 = 4ac$.

(b) With $f(x) = x^3 + 4x^2 - 3x - 18$, the repeated root must satisfy

$$f'(x) = 3x^2 + 8x - 3 = (3x - 1)(x + 3) = 0 \quad \text{i.e. } x = \frac{1}{3} \text{ or } x = -3.$$

Trying the two possibilities: $f(\frac{1}{3}) \neq 0$ but $f(-3) = -27 + 36 + 9 - 18 = 0$. Thus $f(x)$ must factorise as $(x + 3)^2(x - b)$, and comparing the constant terms in the two expressions for $f(x)$ immediately gives $b = 2$. Hence, $x = 2$ is the third root.

(c) Here $f(x) = x^4 + 4x^3 + 7x^2 + 6x + 2$. As previously, we examine $f'(x) = 0$, i.e. $f'(x) = 4x^3 + 12x^2 + 14x + 6 = 0$. This has to have an integer solution and, by inspection, this is $x = -1$. We can therefore factorise $f(x)$ as the product $(x + 1)^2(a_2x^2 + a_1x + a_0)$. Comparison of the coefficients gives immediately that $a_2 = 1$ and $a_0 = 2$. From the coefficients of x^3 we have $2a_2 + a_1 = 4$; hence $a_1 = 2$. Thus $f(x)$ can be written

$$f(x) = (x + 1)^2(x^2 + 2x + 2) = (x + 1)^2[(x + 1)^2 + 1].$$

The second factor, containing only positive terms, can have no real zeros and hence $f(x) = 0$ has only two real roots (coincident at $x = -1$).

2.30 *Find the following indefinite integrals:*

(a) $\int (4 + x^2)^{-1}\, dx$; (b) $\int (8 + 2x - x^2)^{-1/2}\, dx$ for $2 \le x \le 4$;
(c) $\int (1 + \sin\theta)^{-1}\, d\theta$; (d) $\int (x\sqrt{1-x})^{-1}\, dx$ for $0 < x \le 1$.

We make reference to the 12 standard forms given in subsection 2.2.3 and, where relevant, select the appropriate model.

(a) Using model 9,

$$\int \frac{1}{4 + x^2}\, dx = \frac{1}{2}\tan^{-1}\frac{x}{2} + c.$$

(b) We rearrange the integrand in the form of model 12:

$$\int \frac{1}{\sqrt{8 + 2x - x^2}}\, dx = \int \frac{1}{\sqrt{8 + 1 - (x-1)^2}}\, dx = \sin^{-1}\frac{x-1}{3} + c.$$

(c) See equation (2.35) and the subsequent text.

$$\begin{aligned}\int \frac{1}{1 + \sin\theta}\, d\theta &= \int \frac{1}{1 + \dfrac{2t}{1+t^2}}\, \frac{2}{1+t^2}\, dt \\ &= \int \frac{2}{(1+t)^2}\, dt \\ &= -\frac{2}{1+t} + c \\ &= -\frac{2}{1 + \tan\dfrac{\theta}{2}} + c.\end{aligned}$$

(d) To remove the square root, set $u^2 = 1 - x$; then $2u\, du = -dx$ and

$$\begin{aligned}\int \frac{1}{x\sqrt{1-x}}\, dx &= \int \frac{1}{(1-u^2)u} \times -2u\, du \\ &= \int \frac{-2}{1-u^2}\, du \\ &= \int \left(\frac{-1}{1-u} + \frac{-1}{1+u}\right) du \\ &= \ln(1-u) - \ln(1+u) + c \\ &= \ln\frac{1 - \sqrt{1-x}}{1 + \sqrt{1-x}} + c.\end{aligned}$$

2.32 *Express $x^2(ax+b)^{-1}$ as the sum of powers of x and another integrable term, and hence evaluate*

$$\int_0^{b/a} \frac{x^2}{ax+b}\,dx.$$

We need to write the numerator in such a way that every term in it that involves x contains a factor $ax+b$. Therefore, write x^2 as

$$x^2 = \frac{x}{a}(ax+b) - \frac{b}{a^2}(ax+b) + \frac{b^2}{a^2}.$$

Then,

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \int_0^{b/a} \left(\frac{x}{a} - \frac{b}{a^2} + \frac{b^2}{a^2(ax+b)}\right) dx \\ &= \left[\frac{x^2}{2a} - \frac{bx}{a^2} + \frac{b^2}{a^3}\ln(ax+b)\right]_0^{b/a} \\ &= \frac{b^2}{a^3}\left(\ln 2 - \frac{1}{2}\right).\end{aligned}$$

An alternative approach, consistent with the wording of the question, is to use the binomial theorem to write the integrand as

$$\frac{x^2}{ax+b} = \frac{x^2}{b}\left(1+\frac{ax}{b}\right)^{-1} = \frac{x^2}{b}\sum_{n=0}^{\infty}\left(-\frac{ax}{b}\right)^n.$$

Then the integral is

$$\begin{aligned}\int_0^{b/a} \frac{x^2}{ax+b}\,dx &= \frac{1}{b}\int_0^{b/a}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{b}\right)^n x^{n+2}\,dx \\ &= \frac{1}{b}\sum_{n=0}^{\infty}(-1)^n\left(\frac{a}{b}\right)^n \frac{1}{n+3}\left(\frac{b}{a}\right)^{n+3} \\ &= \frac{b^2}{a^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

That these two solutions are the same can be seen by writing $\ln 2 - \frac{1}{2}$ as

$$\begin{aligned}\ln 2 - \frac{1}{2} &= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots\right) - \frac{1}{2} \\ &= \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=0}^{\infty}\frac{(-1)^n}{n+3}.\end{aligned}$$

2.34 *Use logarithmic integration to find the indefinite integrals J of the following:*

(a) $\sin 2x/(1+4\sin^2 x)$;
(b) $e^x/(e^x - e^{-x})$;
(c) $(1+x\ln x)/(x\ln x)$;
(d) $[x(x^n+a^n)]^{-1}$.

To use logarithmic integration each integrand needs to be arranged as a fraction that has the derivative of the denominator appearing in the numerator.

(a) Either by noting that $\sin 2x = 2\sin x\cos x$ and so is proportional to the derivative of $\sin^2 x$ or by recognising that $\sin^2 x$ can be written in terms of $\cos 2x$ and constants and that $\sin 2x$ is then its derivative, we have

$$J = \int \frac{\sin 2x}{1+4\sin^2 x}\,dx$$
$$= \int \frac{2\sin x\cos x}{1+4\sin^2 x}\,dx = \frac{1}{4}\ln(1+4\sin^2 x) + c,$$

or

$$J = \int \frac{\sin 2x}{1+2(1-\cos 2x)}\,dx = \frac{1}{4}\ln(3-2\cos 2x) + c.$$

These two answers are equivalent since $3 - 2\cos 2x = 3 - 2(1-2\sin^2 x) = 1 + 4\sin^2 x$.

(b) This is straightforward if it is noticed that multiplying both numerator and denominator by e^x produces the required form:

$$J = \int \frac{e^x}{e^x - e^{-x}}\,dx = \int \frac{e^{2x}}{e^{2x}-1}\,dx = \frac{1}{2}\ln(e^{2x}-1) + c.$$

An alternative, but longer, method is to write the numerator as $\cosh x + \sinh x$ and the denominator as $2\sinh x$. This leads to $J = \frac{1}{2}(x + \ln\sinh x)$, which can be re-written as

$$J = \tfrac{1}{2}(\ln e^x + \ln\sinh x) = \tfrac{1}{2}\ln(e^x\sinh x) = \tfrac{1}{2}\ln(e^{2x}-1) + \tfrac{1}{2}\ln\tfrac{1}{2}.$$

The $\frac{1}{2}\ln\frac{1}{2}$ forms part of c.

(c) Here we must first divide the numerator by the denominator to produce two separate terms, and then twice apply the result that $1/z$ is the derivative of $\ln z$:

$$J = \int \frac{1+x\ln x}{x\ln x}\,dx = \int \left(\frac{1}{x\ln x} + 1\right)dx = \ln(\ln x) + x + c.$$

(d) To put the integrand in a form suitable for logaritmic integration, we must

first multiply both numerator and denominator by nx^{n-1} and then use partial fractions so that each denominator contains x only in the form x^m, of which mx^{m-1} is the derivative.

$$\begin{aligned} J &= \int \frac{dx}{x(x^n + a^n)} = \int \frac{nx^{n-1}}{nx^n(x^n + a^n)}\, dx \\ &= \frac{1}{na^n} \int \left(\frac{nx^{n-1}}{x^n} - \frac{nx^{n-1}}{x^n + a^n} \right) dx \\ &= \frac{1}{na^n} \left[n \ln x - \ln(x^n + a^n) \right] + c \\ &= \frac{1}{na^n} \ln \left(\frac{x^n}{x^n + a^n} \right) + c. \end{aligned}$$

2.36 *Find the indefinite integrals J of the following functions involving sinusoids:*

(a) $\cos^5 x - \cos^3 x$;
(b) $(1 - \cos x)/(1 + \cos x)$;
(c) $\cos x \sin x/(1 + \cos x)$;
(d) $\sec^2 x/(1 - \tan^2 x)$.

(a) As the integrand contains only odd powers of $\cos x$, take $\cos x$ out as a common factor and express the remainder in terms of $\sin x$, of which $\cos x$ is the derivative:

$$\begin{aligned} \cos^5 x - \cos^3 x &= \left[(1 - \sin^2 x)^2 - (1 - \sin^2 x) \right] \cos x \\ &= (\sin^4 x - \sin^2 x) \cos x. \end{aligned}$$

Hence,

$$J = \int (\sin^4 x - \sin^2 x) \cos x\, dx = \frac{1}{5} \sin^5 x - \frac{1}{3} \sin^3 x + c.$$

A more formal way of expressing this approach is to say 'set $\sin x = u$ with $\cos x\, dx = du$.'

(b) This integral can be found either by writing the numerator and denominator in terms of sinusoidal functions of $x/2$ or by making the substitution $t = \tan(x/2)$. Using first the half-angle identities, we have

$$\begin{aligned} J &= \int \frac{1 - \cos x}{1 + \cos x}\, dx = \int \frac{2 \sin^2 \frac{x}{2}}{2 \cos^2 \frac{x}{2}} \\ &= \int \tan^2 \frac{x}{2}\, dx = \int \left(\sec^2 \frac{x}{2} - 1 \right) dx = 2 \tan \frac{x}{2} - x + c. \end{aligned}$$

The second approach (see subsection 2.2.7) is

$$\begin{aligned} J &= \int \frac{1 - \dfrac{1-t^2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}} \frac{2\,dt}{1+t^2} \\ &= \int \frac{2t^2}{1+t^2}\,dt \\ &= \int 2\,dt - \int \frac{2}{1+t^2}\,dt \\ &= 2t - 2\tan^{-1} t + c = 2\tan\frac{x}{2} - x + c. \end{aligned}$$

(c) This integrand, containing only sinusoidal functions, can be converted to an algebraic one by writing $t = \tan(x/2)$ and expressing the functions appearing in the integrand in terms of it,

$$\begin{aligned} \frac{\cos x \sin x}{1+\cos x}\,dx &= \frac{\dfrac{1-t^2}{1+t^2}\dfrac{2t}{1+t^2}\dfrac{2}{1+t^2}}{1 + \dfrac{1-t^2}{1+t^2}}\,dt \\ &= \frac{2t(1-t^2)}{(1+t^2)^2}\,dt \\ &= 2t\left[\frac{A}{(1+t^2)^2} + \frac{B}{1+t^2}\right]\,dt, \end{aligned}$$

with $A + B(1+t^2) = 1 - t^2$, implying that $B = -1$ and $A = 2$.
And so, recalling that $1 + t^2 = \sec^2(x/2) = 1/[\cos^2(x/2)]$,

$$\begin{aligned} J &= \int \left(\frac{4t}{(1+t^2)^2} - \frac{2t}{1+t^2}\right)\,dt \\ &= -\frac{2}{1+t^2} - \ln(1+t^2) + c \\ &= -2\cos^2\frac{x}{2} + \ln(\cos^2\frac{x}{2}) + c. \end{aligned}$$

(d) We can either set $\tan x = u$ or show that the integrand is $\sec 2x$ and then use the result of exercise 2.35; here we use the latter method.

$$\int \frac{\sec^2 x}{1-\tan^2 x}\,dx = \int \frac{1}{\cos^2 x - \sin^2 x}\,dx = \int \sec 2x\,dx.$$

It then follows from the earlier result that $J = \frac{1}{2}\ln(\sec 2x + \tan 2x) + c$. This can also be written as $\frac{1}{2}\ln[(1+\tan x)/(1-\tan x)] + c$.

2.38 *Determine whether the following integrals exist and, where they do, evaluate them:*

(a) $\displaystyle\int_0^\infty \exp(-\lambda x)\,dx$; (b) $\displaystyle\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx$;

(c) $\displaystyle\int_1^\infty \frac{1}{x+1}\,dx$; (d) $\displaystyle\int_0^1 \frac{1}{x^2}\,dx$;

(e) $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$; (f) $\displaystyle\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx$.

(a) This is an infinite integral and so we must examine the result of letting the range of a finite integral go to infinity:

$$\int_0^\infty e^{-\lambda x}\,dx = \lim_{R\to\infty}\left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^R = \lim_{R\to\infty}\left[\frac{1}{\lambda} - \frac{e^{-\lambda R}}{\lambda}\right].$$

The limit as $R \to \infty$ does exist if $\lambda > 0$ and is then equal to λ^{-1}.

(b) This is also an infinite integral. However, because of the antisymmetry of the integrand, the integral is zero for all finite values of R. It therefore has a limit as $R \to \infty$ of zero, which is consequently the value of the integral.

$$\int_{-\infty}^\infty \frac{x}{(x^2+a^2)^2}\,dx = \lim_{R\to\infty}\left[\frac{-1}{2(x^2+a^2)}\right]_{-R}^R = \lim_{R\to\infty}[0] = 0.$$

(c) The integral is elementary over any finite range $(1, R)$ and so we must examine its behaviour as $R \to \infty$:

$$\int_1^\infty \frac{1}{x+1}\,dx = \lim_{R\to\infty}[\ln(1+x)]_1^R = \lim_{R\to\infty}\ln\frac{1+R}{2} = \infty.$$

The limit is not finite and so the integral does not exist.

(d) The integrand, $1/x^2$ is undefined at $x = 0$ and so we must examine the behaviour of the integral with lower limit ϵ as $\epsilon \to 0$.

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{\epsilon\to 0}\left[-\frac{1}{x}\right]_\epsilon^1 = \lim_{\epsilon\to 0}\left(-1+\frac{1}{\epsilon}\right) = \infty.$$

As the limit is not finite the integral does not exist.

(e) Again, a infinite quantity ($\cot 0$) appears in the integrand and the limit test has to be applied.

$$\begin{aligned}\int_0^{\pi/2} \cot\theta\,d\theta &= \int_0^{\pi/2} \frac{\cos\theta}{\sin\theta}\,d\theta \\ &= \lim_{\epsilon\to 0}[\ln(\sin\theta)]_\epsilon^{\pi/2} = \lim_{\epsilon\to 0}[0 - \ln(\sin\epsilon)] = -(-\infty).\end{aligned}$$

The limit is not finite and so the integral does not exist.

(f) Yet again, the integrand has an infinity (at $x = 1$) and the limit test has to be applied

$$\int_0^1 \frac{x}{(1-x^2)^{1/2}}\,dx = \lim_{z\to 1}\left[-(1-x^2)^{1/2}\right]_0^z = 0 + 1 = 1.$$

This time the limit does exist; the integral is defined and has value 1.

2.40 *Show, using the following methods, that the indefinite integral of* $x^3/(x+1)^{1/2}$ *is*

$$J = \tfrac{2}{35}(5x^3 - 6x^2 + 8x - 16)(x+1)^{1/2} + c.$$

(a) *Repeated integration by parts.*
(b) *Setting* $x + 1 = u^2$ *and determining* dJ/du *as* $(dJ/dx)(dx/du)$.

(a) Evaluating the successive integrals produced by the repeated integration by parts:

$$\begin{aligned}
\int \frac{x^3}{(x+1)^{1/2}}\,dx &= 2x^3\sqrt{x+1} - \int 3x^2\,2\sqrt{x+1}\,dx,\\
\int x^2\sqrt{x+1}\,dx &= \frac{2}{3}x^2(x+1)^{3/2} - \int 2x\,\frac{2}{3}(x+1)^{3/2}\,dx,\\
\int x(x+1)^{3/2}\,dx &= \frac{2}{5}x(x+1)^{5/2} - \int \frac{2}{5}(x+1)^{5/2}\,dx,\\
\int (x+1)^{5/2}\,dx &= \frac{2}{7}(x+1)^{7/2}.
\end{aligned}$$

And so, remembering to carry forward the multiplicative factors generated at each stage, we have

$$\begin{aligned}
J &= \sqrt{x+1}\left[2x^3 - 4x^2(x+1) + \frac{16}{5}x(x+1)^2 - \frac{32}{35}(x+1)^3\right] + c\\
&= \frac{2\sqrt{x+1}}{35}\left[5x^3 - 6x^2 + 8x - 16\right] + c.
\end{aligned}$$

(b) Set $x + 1 = u^2$, giving $dx = 2u\,du$, to obtain

$$\begin{aligned}
J &= \int \frac{(u^2-1)^3}{u}\,2u\,du\\
&= 2\int (u^6 - 3u^4 + 3u^2 - 1)\,du.
\end{aligned}$$

This integral is now easily evaluated to give

$$\begin{aligned} J &= 2\left(\frac{1}{7}u^7 - \frac{3}{5}u^5 + u^3 - u\right) + c \\ &= \frac{2u}{35}(5u^6 - 21u^4 + 35u^2 - 35) + c \\ &= \frac{2u}{35}\,[\,5(x^3 + 3x^2 + 3x + 1) - 21(x^2 + 2x + 1) + 35(x + 1) - 35\,] + c \\ &= \frac{2\sqrt{x+1}}{35}\,[\,5x^3 - 6x^2 + 8x - 16\,] + c. \end{aligned}$$

i.e. the same final result as for method (a).

2.42 *Define $J(m,n)$, for non-negative integers m and n, by the integral*

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta \sin^n\theta \, d\theta.$$

(a) *Evaluate $J(0,0)$, $J(0,1)$, $J(1,0)$, $J(1,1)$, $J(m,1)$, $J(1,n)$.*
(b) *Using integration by parts prove that, for m and n both > 1,*

$$J(m,n) = \frac{m-1}{m+n}J(m-2,n) \quad \text{and} \quad J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) *Evaluate (i) $J(5,3)$, (ii) $J(6,5)$, (iii) $J(4,8)$.*

(a) For these special values of m and/or n the integrals are all elementary, as follows.

$$\begin{aligned} J(0,0) &= \int_0^{\pi/2} 1\,d\theta = \frac{\pi}{2}, \\ J(0,1) &= \int_0^{\pi/2} \sin\theta\,d\theta = 1, \\ J(1,0) &= \int_0^{\pi/2} \cos\theta\,d\theta = 1, \\ J(1,1) &= \int_0^{\pi/2} \cos\theta\sin\theta\,d\theta = \left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = \frac{1}{2}, \\ J(m,1) &= \int_0^{\pi/2} \cos^m\theta\sin\theta\,d\theta = \frac{1}{m+1}, \\ J(1,n) &= \int_0^{\pi/2} \cos\theta\sin^n\theta\,d\theta = \frac{1}{n+1}. \end{aligned}$$

(b) In order to obtain a reduction formula, we 'sacrifice' one of the cosine factors

so that it can act as the derivative of a sine function, so allowing $\sin^n\theta$ to be integrated. The two extra powers of $\sin\theta$ generated by the integration by parts are then removed by writing them as $1-\cos^2\theta$.

$$\begin{aligned}
J(m,n) &= \int_0^{\pi/2} \cos^{m-1}\theta \sin^n\theta \cos\theta \, d\theta \\
&= \left[\frac{\cos^{m-1}\theta \sin^{n+1}\theta}{n+1}\right]_0^{\pi/2} \\
&\qquad - \int_0^{\pi/2} \frac{(m-1)\cos^{m-2}\theta(-\sin\theta)\sin^{n+1}\theta}{n+1}\, d\theta \\
&= 0 + \frac{m-1}{n+1}\int_0^{\pi/2} \cos^{m-2}\theta(1-\cos^2\theta)\sin^n\theta\, d\theta \\
&= \frac{m-1}{n+1}J(m-2,n) - \frac{m-1}{n+1}J(m,n). \\
J(m,n) &= \frac{m-1}{m+n}J(m-2,n).
\end{aligned}$$

Similarly, by 'sacrificing' a sine term to act as the derivative of a cosine term,

$$J(m,n) = \frac{n-1}{m+n}J(m,n-2).$$

(c) For these specific cases we apply the reduction formulae in (b) to reduce them to one of the forms evaluated in (a).

$$\text{(i)}\quad J(5,3) = \frac{2}{8}J(5,1) = \frac{2}{8}\,\frac{1}{6} = \frac{1}{24},$$

$$\text{(ii)}\quad J(6,5) = \frac{4}{11}\,\frac{2}{9}J(6,1) = \frac{4}{11}\,\frac{2}{9}\,\frac{1}{7} = \frac{8}{693},$$

$$\text{(iii)}\quad J(4,8) = \frac{3}{12}\,\frac{1}{10}J(0,8) = \frac{3}{12}\,\frac{1}{10}\,\frac{7}{8}\,\frac{5}{6}\,\frac{3}{4}\,\frac{1}{2}\,\frac{\pi}{2} = \frac{7\pi}{2048}.$$

2.44 *Evaluate the following definite integrals:*

(a) $\int_0^\infty xe^{-x}\,dx$; (b) $\int_0^1 \left[(x^3+1)/(x^4+4x+1)\right]\,dx$;
(c) $\int_0^{\pi/2}[a+(a-1)\cos\theta]^{-1}\,d\theta$ *with* $a > \frac{1}{2}$; (d) $\int_{-\infty}^\infty (x^2+6x+18)^{-1}\,dx$.

(a) Integrating by parts:

$$\int_0^\infty xe^{-x}\,dx = \left[-xe^{-x}\right]_0^\infty - \int_0^\infty -e^{-x}\,dx = 0 + \left[-e^{-x}\right]_0^\infty = 1.$$

(b) This is a logarithmic integration:

$$\int_0^1 \frac{x^3+1}{x^4+4x+1}\,dx = \frac{1}{4}\int_0^1 \frac{4x^3+4}{x^4+4x+1} = \frac{1}{4}\left[\ln(x^4+4x+1)\right]_0^1 = \frac{1}{4}\ln 6.$$

(c) Writing $t = \tan(\theta/2)$:

$$\begin{aligned}\int_0^{\pi/2} \frac{1}{a+(a-1)\cos\theta}\,d\theta &= \int_0^1 \frac{1}{a+(a-1)\left(\dfrac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2}\\ &= \int_0^1 \frac{2}{2a-1+t^2}\,dt\\ &= \frac{2}{\sqrt{2a-1}}\left[\tan^{-1}\frac{t}{\sqrt{2a-1}}\right]_0^1\\ &= \frac{2}{\sqrt{2a-1}}\tan^{-1}\frac{1}{\sqrt{2a-1}}.\end{aligned}$$

(d) The denominator has no real zeros ($6^2 < 4\times 1\times 18$) and so, completing the square, we have:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{1}{x^2+6x+18}\,dx &= \int_{-\infty}^{\infty} \frac{1}{(x+3)^2+9}\,dx\\ &= \frac{1}{3}\left[\tan^{-1}\left(\frac{x+3}{3}\right)\right]_{-\infty}^{\infty}\\ &= \frac{1}{3}\left[\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{3}.\end{aligned}$$

2.46 *Find positive constants a, b such that* $ax \le \sin x \le bx$ *for* $0 \le x \le \pi/2$. *Use this inequality to find (to two significant figures) upper and lower bounds for the integral*

$$I = \int_0^{\pi/2} (1+\sin x)^{1/2}\,dx.$$

Use the substitution $t = \tan(x/2)$ *to evaluate* I *exactly.*

Consider $f(x) = (\sin x)/x$. Its derivative is

$$f'(x) = \frac{x\cos x - \sin x}{x^2} = \frac{x-\tan x}{x^2}\cos x,$$

which is everwhere negative (or zero) in the given range. This shows that $f(x)$ is a monotonically decreasing function in that range and reaches its lowest value at the end of the range. This value must therefore be $\sin(\pi/2)/(\pi/2)$, i.e. $2/\pi$.

From the standard Maclaurin series for $\sin x$ (subsection 4.6.3)

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots,$$

and the limit of $f(x)$ as $x \to 0$ is 1. In summary,

$$\frac{2}{\pi} \le \frac{\sin x}{x} \le 1 \quad \text{for} \quad 0 \le x \le \frac{\pi}{2}.$$

It then follows that

$$\int_0^{\pi/2} (1 + \frac{2}{\pi}x)^{1/2}\, dx \le \int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx \le \int_0^{\pi/2} (1 + x)^{1/2}\, dx,$$

$$\begin{aligned}
\left[\frac{\pi}{2}\,\frac{2}{3}(1 + \frac{2}{\pi}x)^{3/2}\right]_0^{\pi/2} \le \quad & I \quad \le \left[\frac{2}{3}(1 + x)^{3/2}\right]_0^{\pi/2}, \\
\frac{\pi}{3}\left[(2)^{3/2} - 1\right] \le \quad & I \quad \le \frac{2}{3}\left[(1 + \frac{\pi}{2})^{3/2} - 1\right], \\
1.91 \le \quad & I \quad \le \quad 2.08.
\end{aligned}$$

For an exact evaluation we use the standard half-angle formulae:

$$t = \tan\frac{x}{2}, \qquad \sin x = \frac{2t}{1 + t^2}, \qquad dx = \frac{2}{1 + t^2}\, dt.$$

Substitution of these gives

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^1 \left(1 + \frac{2t}{1 + t^2}\right)^{1/2} \frac{2}{1 + t^2}\, dt \\
&= \int_0^1 \frac{2 + 2t}{(1 + t^2)^{3/2}}\, dt \\
&= \int_0^1 \frac{2}{(1 + t^2)^{3/2}}\, dt + 2\left[-\frac{1}{(1 + t^2)^{1/2}}\right]_0^1.
\end{aligned}$$

To evaluate the first integral we turn it back into one involving sinusoidal functions and write $t = \tan\theta$ with $dt = \sec^2\theta\, d\theta$. Then the original integral becomes

$$\begin{aligned}
\int_0^{\pi/2} (1 + \sin x)^{1/2}\, dx &= \int_0^{\pi/4} \frac{2\sec^2\theta}{\sec^3\theta}\, d\theta + 2\left[1 - \frac{1}{\sqrt{2}}\right] \\
&= \int_0^{\pi/4} 2\cos\theta\, d\theta + 2 - \sqrt{2} \\
&= 2[\sin\theta]_0^{\pi/4} + 2 - \sqrt{2} \\
&= \sqrt{2} - 0 + 2 - \sqrt{2} = 2.
\end{aligned}$$

An alternative evaluation can be made by setting $x = (\pi/2) - y$ and then writing $1 + \cos y$ in the form $2\cos^2(y/2)$. This gives the final value of 2 more directly.

Whichever method is used in (b), we note that, as it must (or at least should!) the exact value of the integral lies between our calculated bounds.

2.48 *Show that the total length of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$, which can be parameterised as $x = a\cos^3\theta$, $y = a\sin^3\theta$, is $6a$.*

We first check that $x^{2/3} + y^{2/3} = a^{2/3}$ can be parameterised as $x = a\cos^3\theta$ and $y = a\sin^3\theta$. This is so, since $a^{2/3}\cos^2\theta + a^{2/3}\sin^2\theta = a^{2/3}$ is an identity.

Now the element of length of the curve ds is given by $ds^2 = dx^2 + dy^2$ or, using the parameterisation,

$$\begin{aligned} ds &= \left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]^{1/2} d\theta \\ &= \left[\left(-3a\cos^2\theta\sin\theta\right)^2 + \left(3a\sin^2\theta\cos\theta\right)^2\right]^{1/2} d\theta \\ &= 3a\cos\theta\sin\theta\, d\theta. \end{aligned}$$

The total length of the asteroid curve is four times its length in the first quadrant and therefore given by

$$s = 4 \times 3a \int_0^{\pi/2} \cos\theta\sin\theta\, d\theta = 12a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 6a.$$

2.50 *The equation of a cardioid in plane polar coordinates is*

$$\rho = a(1 - \sin\phi).$$

Sketch the curve and find (i) *its area,* (ii) *its total length,* (iii) *the surface area of the solid formed by rotating the cardioid about its axis of symmetry and* (iv) *the volume of the same solid.*

For a sketch of the 'heart-shaped' (actually more apple-shaped) curve see figure 2.4.

To avoid any possible double counting, integrals will be taken from $\phi = \pi/2$ to $\phi = 3\pi/2$ and symmetry used for scaling up.

Figure 2.4 The cardioid discussed in exercise 2.50.

(i) Area. In plane polar coordinates this is straightforward.

$$\begin{aligned}\int \frac{1}{2}\rho^2\,d\phi &= 2\int_{\pi/2}^{3\pi/2} \frac{1}{2}a^2(1-\sin\phi)^2\,d\phi \\ &= a^2\int_{\pi/2}^{3\pi/2}(1-2\sin\phi+\sin^2\phi)\,d\phi \\ &= a^2(\pi-0+\tfrac{1}{2}\pi) \\ &= \frac{3\pi a^2}{2}.\end{aligned}$$

The third term in the integral was evaluated using the standard result that the average value of the square of a sinusoid over a whole number of quarter cycles is $\frac{1}{2}$.

(ii) Length. Since $ds^2 = d\rho^2 + \rho^2 d\phi^2$, the total length is

$$\begin{aligned}L &= 2\int_{\pi/2}^{3\pi/2}\left[\left(\frac{d\rho}{d\phi}\right)^2+\rho^2\right]^{1/2} d\phi \\ &= 2\int_{\pi/2}^{3\pi/2}(a^2\cos^2\phi + a^2 - 2a^2\sin\phi + a^2\sin^2\phi)^{1/2}\,d\phi \\ &= 2a\sqrt{2}\int_{\pi/2}^{3\pi/2}(1-\sin\phi)^{1/2}\,d\phi \\ &= 2a\sqrt{2}\int_0^{-\pi}(1-\cos\phi')^{1/2}(-d\phi') \quad \text{where} \quad \phi = \frac{1}{2}\pi - \phi'.\end{aligned}$$

Using the trigonometric half-angle formula $1 - \cos\theta = 2\sin^2(\theta/2)$, this integral is easily evaluated to give

$$\begin{aligned} L &= 2a\sqrt{2}\int_{-\pi}^{0} \sqrt{2}\sin\frac{\phi'}{2}\,d\phi' \\ &= 4a\left[-2\cos\frac{\phi'}{2}\right]_{-\pi}^{0} = -8a. \end{aligned}$$

The negative sign is irrelevant and merely reflects the (inappropriate) choice of taking the positive square root of $\sin^2(\phi'/2)$. The total length of the curve is thus $8a$.

(iii) Surface area of the solid of rotation.
The elemental circular strip at any given value of ρ and ϕ has a total length of $2\pi\rho\cos\phi$ and a width ds (on the surface) given by $(ds)^2 = (d\rho)^2 + (\rho d\phi)^2$. This strip contributes an elemental surface area $2\pi\rho\cos\phi\,ds$ and so the total surface area S of the solid is given by

$$\begin{aligned} S &= \int_{\pi/2}^{3\pi/2} 2\pi\rho\cos\phi\left[\left(\frac{d\rho}{d\phi}\right)^2 + \rho^2\right]^{1/2} d\phi \\ &= 2\sqrt{2}\pi a^2\int_{\pi/2}^{3\pi/2}(1-\sin\phi)^{3/2}\cos\phi\,d\phi \quad [\text{using the result from (ii)}] \\ &= 2\sqrt{2}\pi a^2\left[-\frac{2}{5}(1-\sin\phi)^{5/2}\right]_{\pi/2}^{3\pi/2} \\ &= -\frac{32\pi a^2}{5}. \end{aligned}$$

Again, the minus sign is irrelevant and arises because, in the range of ϕ used, the elemental strip radius is actually $-\rho\cos\phi$.

(iv) Volume of the solid of rotation.
The height above the origin of any point is $\rho\sin\phi$ and so, for $\pi/2 \le \phi \le 3\pi/2$, the thickness of any elemental disc is $-d(\rho\sin\phi)$ whilst its area is $\pi\rho^2\cos^2\phi$.

It should be noted that this formulation allows correctly for the 'missing' part of the body of revolution – as it were, for the air that surrounds the 'stalk of the apple'. Whilst ϕ is in the range $\pi/2 \le \phi \le 5\pi/6$ (the upper limit being found by maximising $y = \rho\sin\phi = a(1-\sin\phi)\sin\phi$), negative volume is being added to the solid, representing 'the air'. For $5\pi/6 \le \phi \le \pi$ the solid acquires volume as if there were no air core. For the rest of the range, $\pi \le \phi \le 3\pi/2$, such considerations do not arise.

The required volume is therefore given by the single integral

$$
\begin{aligned}
V &= -\int_{\pi/2}^{3\pi/2} \pi\rho^2 \cos^2\phi \, d(\rho \sin\phi) \\
&= -\int_{\pi/2}^{3\pi/2} \pi a^2 (1-\sin\phi)^2 \cos^2\phi \; a(\cos\phi - 2\sin\phi\cos\phi)\, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 2\sin\phi + \sin^2\phi)(1-2\sin\phi)\cos^3\phi \, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 4\sin\phi + 5\sin^2\phi - 2\sin^3\phi)(1-\sin^2\phi)\cos\phi \, d\phi \\
&= -\pi a^3 \int_{\pi/2}^{3\pi/2} (1 - 4\sin\phi + 4\sin^2\phi + 2\sin^3\phi - \cdots \\
&\qquad\qquad \cdots - 5\sin^4\phi + 2\sin^5\phi)\cos\phi \, d\phi \\
&= \pi a^3 (2 - 0 + \frac{8}{3} + 0 - 2 + 0) = \frac{8}{3}\pi a^3.
\end{aligned}
$$

3

Complex numbers and hyperbolic functions

3.2 *By considering the real and imaginary parts of the product* $e^{i\theta}e^{i\phi}$ *prove the standard formulae for* $\cos(\theta+\phi)$ *and* $\sin(\theta+\phi)$.

We apply Euler's equation, $e^{i\theta} = \cos\theta + i\sin\theta$, separately to the two sides of the identity

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$$

and obtain

$$\begin{aligned}\cos(\theta+\phi) + i\sin(\theta+\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi)\\ &\qquad + i(\cos\theta\sin\phi + \sin\theta\cos\phi).\end{aligned}$$

Equating the real and imaginary parts gives the standard results:

$$\begin{aligned}\cos(\theta+\phi) &= \cos\theta\cos\phi - \sin\theta\sin\phi,\\ \sin(\theta+\phi) &= \cos\theta\sin\phi + \sin\theta\cos\phi.\end{aligned}$$

It is worth noting the relationship between this method of proof and the purely geometrical one given in subsection 1.2.2. In the Argand diagram, $e^{i\theta}$ is represented by a unit vector making an angle θ (A in figure 1.2) with the x-axis. Multiplying by $e^{i\phi}$ corresponds to turning the vector through a further angle ϕ (B in the figure), without any change in length, thus giving the point (P in the figure) that is represented by $e^{i(\theta+\phi)}$.

3.4 *Find the locus in the complex z-plane of points that satisfy the following equations.*

(a) $z - c = \rho\left(\dfrac{1+it}{1-it}\right)$, *where c is complex, ρ is real and t is a real parameter that varies in the range* $-\infty < t < \infty$.

(b) $z = a + bt + ct^2$, *in which t is a real parameter and a, b, and c are complex numbers with b/c real.*

(a) We start by rationalising the complex fraction so that it is easier to see what it represents:

$$\begin{aligned} z - c &= \rho\left(\frac{1+it}{1-it}\right) \qquad\qquad -\infty < t < \infty \\ &= \rho\frac{1+it}{1-it}\,\frac{1+it}{1+it} \\ &= \rho\frac{(1-t^2)+2it}{1+t^2}. \end{aligned}$$

The real and imaginary parts of the RHS now have the forms of cosine and sine functions, respectively, as expressed by the $\tan(x/2)$ formulae (see subsection 2.2.7). The equation can therefore be written as

$$z - c = \rho(\cos 2\theta + i\sin 2\theta) = \rho e^{2i\theta},$$

where $\theta = \tan^{-1} t$. Since $-\infty < t < \infty$, θ must lie in the range $-\pi/2 < \theta < \pi/2$ and $-\pi < 2\theta < \pi$. Thus the locus of z is a circle centred on $z = c$ and of radius ρ.

(b) If we write $a = a_r + ia_i$, etc. then, since t is real, $x + iy = z = a + bt + ct^2$ gives

$$\begin{aligned} x &= a_r + b_r t + c_r t^2, \\ y &= a_i + b_i t + c_i t^2. \end{aligned}$$

Multiplying these equations by c_i and c_r, respectively, and subtracting eliminates the t^2 term and gives

$$c_i x - c_r y = a_r c_i - a_i c_r + t(c_i b_r - c_r b_i).$$

Now, since the ratio b/c is real the phases of b and c are equal or differ by π. Either way, $b_i c_r = c_i b_r$ with the consequence that the term in parentheses in the above equation vanishes and

$$y = \frac{c_i}{c_r}x + a_i - \frac{c_i a_r}{c_r},$$

i.e. the locus is a straight line of slope $\tan(\arg c)$ [or, equivalently, $\tan(\arg b)$]. The same conclusion can be reached by eliminating the t term.

3.6 *Find the equations in terms of x and y of the sets of points in the Argand diagram that satisfy the following:*

(a) $\operatorname{Re} z^2 = \operatorname{Im} z^2$;
(b) $(\operatorname{Im} z^2)/z^2 = -i$;
(c) $\arg[z/(z-1)] = \pi/2$.

(a) Straightforward substitution of $z = x + iy$ gives

$$\begin{aligned} \operatorname{Re} z^2 &= \operatorname{Im} z^2, \\ \operatorname{Re}(x^2 - y^2 + 2ixy) &= \operatorname{Im}(x^2 - y^2 + 2ixy), \\ x^2 - y^2 &= 2xy, \\ y^2 + 2xy - x^2 &= 0, \\ y &= (-1 \pm \sqrt{2})x. \end{aligned}$$

This is a pair of straight lines through the origin. Since the product of their slopes, $(-1+\sqrt{2})(-1-\sqrt{2})$, is equal to -1, the lines are orthogonal.

Geometrically, it is clear that the condition is equivalent to z^2 lying on the line of positive slope $\pi/4$. Thus z must lie on one of the two lines with slopes $\pi/8$ and $-3\pi/8$; that the tangents of these angles are $\sqrt{2}-1$ and $-\sqrt{2}-1$ confirms this conclusion.

(b) Since $\operatorname{Im} z^2$ is necessarily real, z^2 will have to be purely imaginary. Proceeding as in part (a):

$$\begin{aligned} \operatorname{Im} z^2 &= -iz^2, \\ 2xy &= -i(x^2 - y^2 + 2ixy), \\ 0 &= x^2 - y^2, \\ y &= \pm x. \end{aligned}$$

This is the pair of orthogonal lines that bisects the angles between the x- and y-axes.

We note that the imposed condition can only be satified non-trivially because of the particular constants involved; had the equation been, say, $(\operatorname{Im} z^2)/z^2 = -3i$, the real parts of the equality would not have cancelled and there would have been no solution; for the form $(\operatorname{Im} z^2) = -3iz^2$, the only solution would have been $z = 0$.

(c) Rearranging the given condition,

$$\arg \frac{z}{z-1} = \frac{\pi}{2},$$

we obtain

$$z = (z-1)\lambda e^{i\pi/2} = \lambda(z-1)i \quad \text{with} \quad \lambda > 0,$$
$$x + iy = i\lambda(x-1) - \lambda y.$$

Thus $x = -\lambda y$ and $y = \lambda(x-1)$, and so

$$x(x-1) = -y^2$$
$$(x - \tfrac{1}{2})^2 + y^2 = \tfrac{1}{4}.$$

This is a circle of radius $\frac{1}{2}$ centred on $(\frac{1}{2}, 0)$. Since $\lambda > 0$ and the circle lies entirely in $0 \le x \le 1$, both expressions for y, namely $-\lambda^{-1}x$ and $\lambda(x-1)$, must be negative. Hence the locus is the part of the circle that lies in $y < 0$. Plotting the points 0, 1 and z in the complex plane shows the relationship of this result to the classical geometric result about the 'angle in a semi-circle being a right angle'.

3.8 *The two sets of points $z = a$, $z = b$, $z = c$, and $z = A$, $z = B$, $z = C$ are the corners of two similar triangles in the Argand diagram. Express in terms of $a, b, \ldots, C$*

(a) *the equalities of corresponding angles, and*
(b) *the constant ratio of corresponding sides,*

in the two triangles.

By noting that any complex quantity can be expressed as

$$z = |z| \exp(i \arg z),$$

deduce that

$$a(B-C) + b(C-A) + c(A-B) = 0.$$

(a) The angle α between the two sides of the triangle that meet at $z = a$ is the difference between the arguments of $b - a$ and $c - a$. This, in turn, is equal to the argument of their ratio, i.e.

$$\alpha = \arg \frac{b-a}{c-a}.$$

Thus the equality of corresponding angles in the similar triangles is expressed by

$$\arg \frac{b-a}{c-a} = \arg \frac{B-A}{C-A}$$

and similar relations.

(b) The constant ratio of corresponding sides is expressed by

$$\left|\frac{b-a}{c-a}\right| = \left|\frac{B-A}{C-A}\right|.$$

Now, using the fact that $z = |z|\exp(i\arg z)$, we can write

$$\begin{aligned}\frac{b-a}{c-a} &= \left|\frac{b-a}{c-a}\right|\exp\left(i\arg\frac{b-a}{c-a}\right)\\ &= \left|\frac{B-A}{C-A}\right|\exp\left(i\arg\frac{B-A}{C-A}\right) = \frac{B-A}{C-A},\end{aligned}$$

where the two results obtained previously have been used to justify the second equality. Hence,

$$(C-A)b-(C-A)a = (B-A)c-(B-A)a.$$

Cancelling the term aA that appears on both sides of the equality and then rearranging gives

$$\begin{aligned} & b(C-A)-aC+c(A-B)+aB = 0,\\ \Rightarrow\quad & b(C-A)+a(B-C)+c(A-B) = 0,\end{aligned}$$

as stated in the question.

3.10 *The most general type of transformation between one Argand diagram, in the z-plane, and another, in the Z-plane, that gives one and only one value of Z for each value of z (and conversely) is known as the* general bilinear transformation *and takes the form*

$$z = \frac{aZ+b}{cZ+d}.$$

(a) *Confirm that the transformation from the Z-plane to the z-plane is also a general bilinear transformation.*

(b) *Recalling that the equation of a circle can be written in the form*

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda, \qquad \lambda \neq 1,$$

show that the general bilinear transformation transforms circles into circles (or straight lines). What is the condition that z_1, z_2 and λ must satisfy if the transformed circle is to be a straight line?

(a) To test whether this is so, we must make Z the subject of the transformation equation. Starting from the original form and rearranging:

$$\begin{aligned} z &= \frac{aZ+b}{cZ+d}, \\ czZ + zd &= aZ + b, \\ Z(cz-a) &= -dz + b, \\ Z &= \frac{-dz+b}{cz-a} \qquad (*), \end{aligned}$$

i.e. another general bilinear transformation, though with different, but related, parameters.

(b) Given the circle

$$\left|\frac{z-z_1}{z-z_2}\right| = \lambda, \quad \lambda \neq 1,$$

in the z-plane, it transforms into the curve given in the Z-plane by

$$\left|\frac{\dfrac{aZ+b}{cZ+d} - z_1}{\dfrac{aZ+b}{cZ+d} - z_2}\right| = \lambda,$$

This equation can be manipulated to make the multipliers of Z unity, as follows:

$$\begin{aligned} \left|\frac{(a-z_1c)Z + b - z_1d}{(a-z_2c)Z + b - z_2d}\right| &= \lambda, \\ \frac{|cz_1-a|}{|cz_2-a|}\left|\frac{-Z + \dfrac{b-z_1d}{cz_1-a}}{-Z + \dfrac{b-z_2d}{cz_2-a}}\right| &= \lambda, \\ \left|\frac{Z-Z_1}{Z-Z_2}\right| &= \left|\frac{cz_2-a}{cz_1-a}\right|\lambda = \mu. \end{aligned}$$

Thus, the transformed curve is also a circle (or a straight line). It is a straight line if Z is always equidistant from Z_1 and Z_2, i.e. if $\mu = 1$. The condition for this is

$$|a - cz_1| = \lambda|a - cz_2|.$$

We note, from $(*)$, that Z_1 and Z_2 are the points into which z_1 and z_2 are carried by the transformation.

3.12 *Denote the nth roots of unity by* $1, \omega_n, \omega_n^2, \ldots, \omega_n^{n-1}$.

(a) *Prove that*

$$rm\ (i)\ \sum_{r=0}^{n-1} \omega_n^r = 0, \qquad \text{(ii)}\ \prod_{r=0}^{n-1} \omega_n^r = (-1)^{n+1}.$$

(b) *Express* $x^2 + y^2 + z^2 - yz - zx - xy$ *as the product of two factors, each linear in* x, y *and* z, *with coefficients dependent on the third roots of unity (and those of the* x *terms arbitrarily taken as real).*

(a) In order to establish properties of the sums and products of the nth roots of unity we need to express their common defining property as a polynomial equation; this is $\omega^n - 1 = 0$. Now writing the polynomial as the product of factors typified by $(\omega - \omega_n^r)$ we have

$$\begin{aligned}\omega^n - 1 &= (\omega - 1)(\omega - \omega_n)(\omega - \omega_n^2)\cdots(\omega - \omega_n^{n-1})\\ &= \omega^n - \omega^{n-1}\sum_{r=0}^{n-1}\omega_n^r + \cdots + (-1)^n\prod_{r=0}^{n-1}\omega_n^r.\end{aligned}$$

Equating coefficients of

$$\begin{aligned}&\text{(i) } \omega^{n-1}, & 0 &= \sum_{r=0}^{n-1}\omega_n^r,\\ &\text{(ii) constants,} & -1 &= (-1)^n\prod_{r=0}^{n-1}\omega_n^r.\end{aligned}$$

Hence the stated results.

(b) Writing the given expression f in the required form with the x-coefficients both taken as $+1$, we have

$$\begin{aligned}f &= x^2 + y^2 + z^2 - yz - zx - xy\\ &= (x + \alpha y + \beta z)(x + \gamma y + \delta z), \quad \text{say,}\\ &= x^2 + \alpha\gamma y^2 + \beta\delta z^2 + (\alpha + \gamma)xy + (\beta + \delta)xz + (\alpha\delta + \beta\gamma)yz.\end{aligned}$$

The cube roots of unity have the explicit properties:

$$\begin{aligned}1 + \omega_3 + \omega_3^2 &= 0,\\ 1 \times \omega_3 \times \omega_3^2 &= 1.\end{aligned}$$

So, if we choose $\alpha = \omega_3$ then, from the coefficient of y^2 we must have $\gamma = \omega_3^2$. This also makes $\alpha + \gamma$, the coefficient of the xy term, equal to -1 as required.

Turning now to the choices for β and δ, we cannot take $\beta = \omega_3$ since the

coefficient of z^2 would then imply that $\delta = \omega_3^2$; this makes the coefficient of yz wrong (2 rather than -1). Therefore, take $\beta = \omega_3^2$ leading to the result $\delta = \omega_3$. This choice makes the coefficients of both z^2 and yz correct and leads to a factorisation of the form

$$x^2 + y^2 + z^2 - yz - zx - xy = (x + \omega_3 y + \omega_3^2 z)(x + \omega_3^2 y + \omega_3 z).$$

We note that the factors can be made to have the coefficents of y (say) equal to unity by multiplying the first by ω_3^2 and the second by ω_3; the net effect is to multiply by ω_3^3, i.e. by unity, and so the LHS is unaffected.

3.14 *The complex position vectors of two parallel interacting equal fluid vortices moving with their axes of rotation always perpendicular to the z-plane are z_1 and z_2. The equations governing their motions are*

$$\frac{dz_1^*}{dt} = -\frac{i}{z_1 - z_2}, \qquad \frac{dz_2^*}{dt} = -\frac{i}{z_2 - z_1}.$$

Deduce that (a) $z_1 + z_2$, (b) $|z_1 - z_2|$ *and* (c) $|z_1|^2 + |z_2|^2$ *are all constant in time, and hence describe the motion geometrically.*

(a) To obtain the time derivative of $z_1 + z_2$ we add the two equations and take the complex conjugate of the result.

$$\begin{aligned}\frac{d(z_1 + z_2)}{dt} &= \left[\frac{d(z_1^* + z_2^*)}{dt}\right]^* \\ &= \left(-\frac{i}{z_1 - z_2} - \frac{i}{z_2 - z_1}\right)^* \\ &= 0 \quad \text{i.e. } z_1 + z_2 \text{ is constant.}\end{aligned}$$

(b) It is easier to consider the time derivative of the square of $|z_1 - z_2|$, expressed as $(z_1 - z_2)(z_1^* - z_2^*)$:

$$\begin{aligned}\frac{d(|z_1 - z_2|^2)}{dt} &= \frac{d}{dt}\left[(z_1 - z_2)(z_1^* - z_2^*)\right] \\ &= (z_1 - z_2)\left(\frac{dz_1^*}{dt} - \frac{dz_2^*}{dt}\right) + (z_1^* - z_2^*)\left(\frac{dz_1}{dt} - \frac{dz_2}{dt}\right) \\ &= (z_1 - z_2)\left(-\frac{i}{z_1 - z_2} + \frac{i}{z_2 - z_1}\right) \\ &\qquad + (z_1^* - z_2^*)\left(\frac{i}{z_1^* - z_2^*} - \frac{i}{z_2^* - z_1^*}\right) \\ &= -2i + 2i = 0,\end{aligned}$$

i.e. $|z_1 - z_2|^2$ is constant, and so, therefore, is $|z_1 - z_2|$.

(c) We write $2|z_1|^2+2|z_2|^2$ as $|z_1+z_2|^2+|z_1-z_2|^2$. Since both the latter terms have been shown to be constants of the motion, $|z_1|^2+|z_2|^2$ must also be constant in time.

Since the geometrical centre of the pair of vertices is fixed [result (a)], as is their distance apart [result (b)], they must move in circular motion about a fixed point with the vortices at the opposite ends of a diameter.

3.16 *The polynomial $f(z)$ is defined by*

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

(a) *Show that the equation $f(z) = 0$ has roots of the form $z = \lambda i$ where λ is real, and hence factorize $f(z)$.*
(b) *Show further that the cubic factor of $f(z)$ can be written in the form $(z + a)^3 + b$, where a and b are real, and hence solve the equation $f(z) = 0$ completely.*

(a) Substitute $z = \lambda i$ in

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48,$$

to obtain

$$f(\lambda i) = i(\lambda^5 - 15\lambda^3 + 36\lambda) + (-6\lambda^4 + 34\lambda^2 - 48).$$

For λ to be a root, both parts of $f(\lambda i)$ must be zero, i.e.

$$\lambda = 0 \quad \text{or} \quad \lambda^2 = \frac{15 \pm \sqrt{225 - 144}}{2} = 12 \text{ or } 3,$$

and

$$3\lambda^4 - 17\lambda^2 + 24 = 0 \quad \text{i.e.} \quad \lambda^2 = \frac{17 \pm \sqrt{289 - 288}}{6} = 3 \text{ or } \frac{16}{6}.$$

Only $\lambda^2 = 3$ satisfies both. Thus two of the (five) roots are $z = \pm\sqrt{3}i$ and $(z - i\sqrt{3})(z + i\sqrt{3})$ are factors of $f(z)$.
By eye (or by equating coefficients or by long division),

$$f(z) = (z^2 + 3)(z^3 - 6z^2 + 12z - 16).$$

(b) If

$$\begin{aligned} z^3 - 6z^2 + 12z - 16 &= (z + a)^3 + b \\ &= z^3 + 3az^2 + 3a^2z + a^3 + b, \end{aligned}$$

then $a = -2$ and $b = -8$ provides a consistent solution. Thus, the three remaining roots are given by $(z-2)^3 - 8 = 0$, yielding

$$z = 2 + 2 = 4; \quad z = 2 + 2e^{2\pi i/3} = 1 + i\sqrt{3}; \quad z = 2 + 2e^{4\pi i/3} = 1 - i\sqrt{3}.$$

These, together with $z = \pm\sqrt{3}i$, are the five roots of the original equation.

3.18 *By considering* $(1 + \exp i\theta)^n$, *prove that*

$$\sum_{r=0}^{n} {}^nC_r \cos r\theta = 2^n \cos^n(\theta/2)\cos(n\theta/2),$$

$$\sum_{r=0}^{n} {}^nC_r \sin r\theta = 2^n \cos^n(\theta/2)\sin(n\theta/2),$$

where ${}^nC_r = n!/[r!(n-r)!]$.

To express $1 + \exp i\theta$ in terms of its real and imaginary parts, we first use Euler's equation and then the half-angle formulae:

$$1 + e^{i\theta} = 1 + \cos\theta + i\sin\theta = 2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}.$$

Thus,

$$\left(1 + e^{i\theta}\right)^n = \left(2\cos^2\frac{\theta}{2} + i2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^n = 2^n\left(\cos\frac{\theta}{2}\right)^n\left(e^{i\theta/2}\right)^n.$$

But, we also have from the binomial expansion that

$$(1 + e^{i\theta})^n = 1 + ne^{i\theta} + \cdots + {}^nC_r e^{ir\theta} + \cdots + e^{in\theta}.$$

Equating the real parts of the two equal expressions yields the result

$$\sum_{r=0}^{n} {}^nC_r \cos r\theta = 1 + n\cos\theta + \cdots + {}^nC_r \cos r\theta + \cdots + \cos n\theta$$
$$= 2^n\left(\cos\frac{\theta}{2}\right)^n \cos\frac{n\theta}{2}.$$

Similarly, by equating the imaginary parts, we obtain

$$\sum_{r=0}^{n} {}^nC_r \sin r\theta = n\sin\theta + \cdots + {}^nC_r \sin r\theta + \cdots + \sin n\theta$$
$$= 2^n\left(\cos\frac{\theta}{2}\right)^n \sin\frac{n\theta}{2}.$$

3.20 *Express* $\sin^4\theta$ *entirely in terms of the trigonometric functions of multiple angles and deduce that its average value over a complete cycle is* $\frac{3}{8}$.

We first express $\sin\theta$ in terms of complex exponentials and then compute its fourth power using a binomial expansion:

$$\begin{aligned}\sin^4\theta &= \left(\frac{e^{i\theta}-e^{-i\theta}}{2i}\right)^4\\ &= \frac{1}{2^4}\left(e^{i4\theta}-4e^{i2\theta}+6-4e^{-i2\theta}+e^{-i4\theta}\right).\end{aligned}$$

We now collect together the terms containing $e^{im\theta}$ and $e^{-im\theta}$ and write them in terms of sinusoids:

$$\begin{aligned}\sin^4\theta &= \tfrac{1}{16}(2\cos 4\theta - 8\cos 2\theta + 6)\\ &= \tfrac{1}{8}\cos 4\theta - \tfrac{1}{2}\cos 2\theta + \tfrac{3}{8}.\end{aligned}$$

Clearly, the average values of the first two terms over a complete cycle are both zero; so that of $\sin^4\theta$ is $\frac{3}{8}$.

Strictly speaking, the two final lines of equations are not necessary once it is noted that $e^{im\theta}$ has zero average value for all m *except* $m = 0$. The reader may like to show that the average value of $\sin^{2p}\theta$, with p a positive integer, is $(2p)!/[2^{2p}(p!)^2]$.

3.22 *Prove the following results involving hyperbolic functions.*

(a) *That*

$$\cosh x - \cosh y = 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right).$$

(b) *That, if* $y = \sinh^{-1}x$,

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0.$$

(a) When trying to prove equalities, it is generally better to start with the more complicated explicit expression and try to simplify it, than to start with a simpler explicit expression and have to guess how best to write it in a more complicated

way. We therefore start with RHS of the putative equality:

$$\begin{aligned} f(x,y) &= 2\sinh\left(\frac{x+y}{2}\right)\sinh\left(\frac{x-y}{2}\right) \\ &= 2\,\tfrac{1}{2}\left(e^{(x+y)/2} - e^{-(x+y)/2}\right)\tfrac{1}{2}\left(e^{(x-y)/2} - e^{-(x-y)/2}\right) \\ &= \tfrac{1}{2}\left(e^{2x/2} - e^{2y/2} - e^{-2y/2} + e^{-2x/2}\right) \\ &= \tfrac{1}{2}\left(e^{x} + e^{-x} - e^{y} - e^{-y}\right) \\ &= \cosh x - \cosh y, \end{aligned}$$

thus establishing the stated result.

(b) To establish the unknown derivative of this inverse function, we first convert it into a function for which we do know the derivative. With $y = \sinh^{-1} x$, we have $x = \sinh y$ and consequently that $dx/dy = \cosh y$. Thus

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\cosh y} = \frac{1}{(1+x^2)^{1/2}}, \\ \frac{d^2y}{dx^2} &= -\frac{x}{(1+x^2)^{3/2}}. \end{aligned}$$

For the second equality in the first line we have used the identity $\cosh^2 z = 1 + \sinh^2 z$.

Hence

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} = -\frac{x}{(1+x^2)^{1/2}} + \frac{x}{(1+x^2)^{1/2}} = 0,$$

as stated.

3.24 *Use the definitions and properties of hyperbolic functions to do the following:*

(a) *Solve* $\cosh x = \sinh x + 2\,\mathrm{sech}\ x$.
(b) *Show that the real solution* x *of* $\tanh x = \mathrm{cosech}\ x$ *can be written in the form* $x = \ln(u + \sqrt{u})$. *Find an explicit value for* u.
(c) *Evaluate* $\tanh x$ *when* x *is the real solution of* $\cosh 2x = 2\cosh x$.

(a) Expressing each term of

$$\cosh x = \sinh x + 2\,\mathrm{sech}\, x$$

in exponential form:

$$\begin{aligned}\tfrac{1}{2}(e^x + e^{-x}) &= \tfrac{1}{2}(e^x - e^{-x}) + \frac{4}{e^x + e^{-x}},\\ e^{-x} &= \frac{4}{e^x + e^{-x}},\\ 1 + e^{-2x} &= 4.\end{aligned}$$

Taking logarithms of this, after simplification, yields

$$x = \tfrac{1}{2}(-\ln 3) = -\ln\sqrt{3} = -0.5493.$$

(b) Expressing the defining equality in terms of sinh and cosh functions,

$$\tanh x = \operatorname{cosech} x \quad \Rightarrow \quad \sinh^2 x = \cosh x.$$

Writing $\cosh x$ as u, this equation can be put in the form $u^2 - 1 = u$. It follows from substituting this into the standard logarithmic expression for an inverse hyperbolic cosine that

$$x = \cosh^{-1} u = \ln(\sqrt{u^2 - 1} + u) = \ln(\sqrt{u} + u).$$

Since $u^2 - u - 1 = 0$, we also have that

$$u = \frac{1 \pm \sqrt{5}}{2}, \qquad \text{(i.e. the golden mean and minus its reciprocal!)}$$

with the positive sign being taken to make $\cosh x > 0$, as is required if x is to be real.

(c) Using a double angle formula for hyperbolic functions to replace $\cosh 2x$ in

$$\cosh 2x = 2\cosh x,$$

we have

$$\begin{aligned}2\cosh^2 x - 1 &= 2\cosh x,\\ \cosh x &= \frac{1 \pm \sqrt{3}}{2},\end{aligned}$$

with the positive sign needed for x to be real. It then follows that

$$\begin{aligned}\sinh x &= \pm\left(\frac{1 + 2\sqrt{3} + 3}{4} - 1\right)^{1/2} = \pm\left(\frac{\sqrt{3}}{2}\right)^{1/2},\\ \Rightarrow \quad \tanh x &= \frac{\pm 2\left(\frac{\sqrt{3}}{2}\right)^{1/2}}{1 + \sqrt{3}} = \pm\frac{(12)^{1/4}}{1 + \sqrt{3}}.\end{aligned}$$

3.26 *In the theory of special relativity, the relationship between the position and time coordinates of an event as measured in two frames of reference that have parallel x-axes can be expressed in terms of hyperbolic functions. If the coordinates are x and t in one frame and x' and t' in the other then the relationship take the form*

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = -x\sinh\phi + ct\cosh\phi.$$

Express x and ct in terms of x', ct' and ϕ and show that

$$x^2 - (ct)^2 = (x')^2 - (ct')^2.$$

We need to solve

$$x' = x\cosh\phi - ct\sinh\phi,$$
$$ct' = -x\sinh\phi + ct\cosh\phi.$$

for x and ct in turn.

Multiplying the first equation by $\cosh\phi$ and the second by $\sinh\phi$ and adding yields

$$x'\cosh\phi + ct'\sinh\phi = x\cosh^2\phi - x\sinh^2\phi = x.$$

Multiplying the first equation by $\sinh\phi$ and the second by $\cosh\phi$ and adding yields

$$x'\sinh\phi + ct'\cosh\phi = -ct\sinh^2\phi + ct\cosh^2\phi = ct.$$

Thus the inverse expressions are the same as the original ones except that ϕ is replaced by $-\phi$.

To show the stated equality in the two frames of (the Lorentz invariant) $x^2-(ct)^2$, we simply resubstitute for x and ct:

$$\begin{aligned} x^2 - (ct)^2 &= (x'\cosh\phi + ct'\sinh\phi)^2 - (x'\sinh\phi + ct'\cosh\phi)^2 \\ &= x'^2(\cosh^2\phi - \sinh^2\phi) + (ct')^2(\sinh^2\phi - \cosh^2\phi) \\ &= x'^2 + (ct')^2(-1) = x'^2 - (ct')^2. \end{aligned}$$

Thus, this form, '$x^2 - (ct)^2$' has a value that is independent of ϕ and so has the same value in all frames of reference; it is called a scalar invariant.

3.28 *The principal value of the logarithmic function of a complex variable is defined to have its argument in the range* $-\pi < \arg z \leq \pi$. *By writing* $z = \tan w$ *in terms of exponentials show that*

$$\tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right).$$

Use this result to evaluate

$$\tan^{-1}\left(\frac{2\sqrt{3}-3i}{7}\right).$$

We express $\tan w$ in terms of exponential functions by first writing it as $\sin w / \cos w$:

$$\begin{aligned}
z = \tan w &= \frac{-i(e^{iw} - e^{-iw})}{e^{iw} + e^{-iw}}, \\
z(e^{iw} + e^{-iw}) &= -ie^{iw} + ie^{-iw}, \\
(z+i)e^{2iw} &= -z + i, \\
e^{2iw} &= \frac{1+iz}{1-iz}, \\
\tan^{-1} z = w &= \frac{1}{2i} \ln \frac{1+iz}{1-iz}.
\end{aligned}$$

Now setting $z = (2\sqrt{3} - 3i)/7$ and recalling that $\ln z = \ln |z| + i \arg z$ gives

$$\begin{aligned}
\tan^{-1} \frac{2\sqrt{3}-3i}{7} &= \frac{1}{2i} \ln \frac{7 + i2\sqrt{3} + 3}{7 - i2\sqrt{3} - 3} \\
&= \frac{1}{2i} \ln \left(\frac{10 + i2\sqrt{3}}{4 - i2\sqrt{3}} \, \frac{4 + i2\sqrt{3}}{4 + i2\sqrt{3}} \right) \\
&= \frac{1}{2i} \ln \frac{28 + i28\sqrt{3}}{16 + 12} \\
&= \frac{1}{2i} \ln(1 + i\sqrt{3}) \\
&= \frac{1}{2i} \left(\ln 2 + i\frac{\pi}{3} \right) = \frac{\pi}{6} - i \ln \sqrt{2}.
\end{aligned}$$

4

Series and limits

4.2 *If you invest £1000 on the first day of each year, and interest is paid at 5% on your balance at the end of each year, how much money do you have after 25 years?*

An investment A (= *£1000*) made at the start of the pth year is worth Ar^{26-p} at the end of the 25th year, where $r = 1.05$. The total value of the investment at the end of 25 years is therefore

$$\sum_{p=1}^{p=25} Ar^{26-p} = Ar \sum_{q=0}^{q=24} r^q = \frac{Ar(r^{25}-1)}{r-1},$$

where we have set $25 - p = q$ and used the formula for the sum of a finite geometric series. Inserting numerical values yields £50 113 as the total value.

4.4 *Show that for testing the convergence of the series*

$$x + y + x^2 + y^2 + x^3 + y^3 + \cdots,$$

where $0 < x < y < 1$, *the D'Alembert ratio test fails but the Cauchy root test is successful.*

The ratio of successive terms has one of the two forms

$$\frac{x^{m+1}}{y^m} = x\left(\frac{x}{y}\right)^m \to 0 \quad \text{as} \quad m \to \infty,$$

or

$$\frac{y^m}{x^m} = \left(\frac{y}{x}\right)^m \to \infty \quad \text{as} \quad m \to \infty.$$

Since the ratio does not tend to a unique limit, the D'Alembert test fails.

However, since $x < y$, the nth term u_n of the series (whether $x^{(n+1)/2}$ or $y^{n/2}$) is $\leq y^{n/2}$. Thus $(u_n)^{1/n} \leq (y^{n/2})^{1/n} = y^{1/2} < 1$, and the Cauchy root test proves the convergence of the series.

That the series does converge is clear, as it is the sum, $x/(1-x) + y/(1-y)$, of two (absolutely) convergent series; but that is not the point of the question!

4.6 *By grouping and rearranging terms of the absolutely convergent series*

$$S = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

show that

$$S_{\mathrm{o}} = \sum_{n \text{ odd}}^{\infty} \frac{1}{n^2} = \frac{3S}{4}.$$

From the given result we pick out the terms making up the wanted series and note that what remains (the sum of the inverse squares of the even integers) is a multiple of the originally given series.

$$\begin{aligned} S &= \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \cdots \\ &= \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots + \frac{1}{2^2}\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) = S_{\mathrm{o}} + \tfrac{1}{4}S, \end{aligned}$$

and hence the stated result.

4.8 *The $N+1$ complex numbers ω_m are given by $\omega_m = \exp(2\pi i m/N)$ for $m = 0, 1, 2, \ldots, N$.*

(a) *Evaluate the following:*

$$\text{(i)} \sum_{m=0}^{N} \omega_m, \quad \text{(ii)} \sum_{m=0}^{N} \omega_m^2, \quad \text{(iii)} \sum_{m=0}^{N} \omega_m x^m.$$

(b) *Use these results to evaluate*

$$\text{(i)} \sum_{m=0}^{N} \left[\cos\left(\frac{2\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right], \quad \text{(ii)} \sum_{m=0}^{3} 2^m \sin\left(\frac{2\pi m}{3}\right).$$

(a)(i) This sum is a geometric series with common ratio $e^{2\pi i/N}$. Thus,

$$\sum_{m=0}^{N} \omega_m = e^{2\pi i 0/N} \frac{1 - e^{2\pi i(N+1)/N}}{1 - e^{2\pi i/N}} = 1 \frac{1 - e^{2\pi i/N}}{1 - e^{2\pi i/N}} = 1.$$

However, if $N = 1$ then the common ratio is unity and direct computation is needed:

$$\sum_{m=0}^{1} \omega_m = e^0 + e^{2\pi i} = 2.$$

(a)(ii) By a similar calculation to that in (i),

$$\sum_{m=0}^{N} \omega_m^2 = e^{4\pi i 0/N} \frac{1 - e^{4\pi i(N+1)/N}}{1 - e^{4\pi i/N}},$$

i.e. is unity unless $N = 1$ or $N = 2$, when the common ratio is unity.

For $N = 1$ there are two terms each equal to 1 and the sum equals 2.
For $N = 2$ there are three terms each equal to 1 and the sum equals 3.

(a)(iii) If $x = 1$ then the calculation is as in (i); we assume that $x \neq 1$. This sum is then a geometric series with common ratio $xe^{2\pi i/N}$. Thus,

$$\sum_{m=0}^{N} \omega_m x^m = x^0 e^{2\pi i 0/N} \frac{1 - x^{N+1} e^{2\pi i(N+1)/N}}{1 - xe^{2\pi i/N}} = \frac{1 - x^{N+1} e^{2\pi i/N}}{1 - xe^{2\pi i/N}}.$$

If $N = 1$ then this reduces to $(1 - x^2)/(1 - x)$, i.e. to $1 + x$.

(b)(i) We recognise $\cos(2\pi m/N)$ as the real part of ω_m and, by squaring the definition of ω_m, recognise $\cos(4\pi m/N)$ as the real part of ω_m^2. And so

$$\sum_{m=0}^{N} \left[\cos\left(\frac{2\pi m}{N}\right) - \cos\left(\frac{4\pi m}{N}\right)\right] = \mathrm{Re} \sum_{m=0}^{N} (\omega_m - \omega_m^2).$$

From the previous results, this has the value $2 - 2 = 0$ for $N = 1$, $1 - 3 = -2$ for $N = 2$ and $1 - 1 = 0$ for all $N \geq 3$.

(b)(ii) Taking $N = 3$

$$\begin{aligned}\sum_{m=0}^{3} 2^m \sin\left(\frac{2\pi m}{3}\right) &= \operatorname{Im} \sum_{m=0}^{3} \omega_m 2^m \\ &= \operatorname{Im} \frac{1 - 2^4 e^{2\pi i/3}}{1 - 2e^{2\pi i/3}} \\ &= \operatorname{Im} \frac{1 + 8 - i8\sqrt{3}}{1 + 1 - i\sqrt{3}} \\ &= \operatorname{Im} \frac{(9 - i8\sqrt{3})(2 + i\sqrt{3})}{7} \\ &= \frac{-16\sqrt{3} + 9\sqrt{3}}{7} \\ &= -\sqrt{3}.\end{aligned}$$

In the second line we used the result from part (a)(iii).

4.10 *Determine whether the following series converge (θ and p are positive real numbers):*

$$\text{(a)} \sum_{n=1}^{\infty} \frac{2\sin n\theta}{n(n+1)}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{2}{n^2}, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{1}{2n^{1/2}},$$

$$\text{(d)} \sum_{n=2}^{\infty} \frac{(-1)^n (n^2+1)^{1/2}}{n \ln n}, \qquad \text{(e)} \sum_{n=1}^{\infty} \frac{n^p}{n!}.$$

(a) As shown in the text, the series $\sum_{n=1}^{N} \dfrac{1}{n(n+1)}$ has a partial sum of $\dfrac{N}{N+1}$. This tends to the limit 1 as $N \to \infty$. Thus the series is (absolutely) convergent. The nth term of the given series is $\leq \dfrac{1}{n(n+1)}$ in magnitude. Thus, the given series is absolutely convergent and therefore also convergent.

(b) If, in the quotient test, we take $u_n = \dfrac{1}{n(n+1)}$ and $v_n = \dfrac{2}{n^2}$ then $\rho = \frac{1}{2}$. Since this is finite but non-zero and $\sum u_n$ converges, so must $\sum v_n$.

(c) We have already shown that $\sum n^{-1}$ diverges. Every term of the series $u_n = n^{-1/2}$ is not smaller than the corresponding term in $\sum n^{-1}$. It follows that $\sum n^{-1/2}$ must also diverge.

(d) As $n \to \infty$ the terms decrease to zero (albeit slowly) as $1/\ln n$. This together with the alternating sign of consecutive terms in enough to establish convergence using the alternating series test.

(e) The ratio of successive terms is

$$\frac{(n+1)^p}{(n+1)!}\,\frac{n!}{n^p} = \frac{\left(1+\frac{1}{n}\right)^p}{n+1} \to 0 \quad \text{for all } p > 0.$$

Thus, by d'Alembert's ratio test, the series converges.

4.12 *Determine whether the following series are convergent:*

$$\text{(a)} \sum_{n=1}^{\infty} \frac{n^{1/2}}{(n+1)^{1/2}}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{n^2}{n!}, \quad \text{(c)} \sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^{n/2}}, \quad \text{(d)} \sum_{n=1}^{\infty} \frac{n^n}{n!}.$$

(a) The individual terms tend to unity, rather than zero, as $n \to \infty$ and so the series must diverge.

(b) The succesive term ratio is

$$\frac{(n+1)^2}{(n+1)!}\,\frac{n!}{n^2} = \frac{1}{n+1}\left(1+\frac{1}{n}\right)^2 \to 0 \quad \textit{as} \quad n \to \infty.$$

Thus the series is convergent.

(c) The nth root of u_n is $(\ln n)/n^{1/2}$ which $\to 0$ as $n \to \infty$. Thus the series is convergent by the Cauchy root test.

(d) The succesive term ratio is

$$\frac{(n+1)^{n+1}}{(n+1)!}\,\frac{n!}{n^n} = \frac{n^{n+1}\left(1+\frac{1}{n}\right)^{n+1}}{(n+1)\,n^n} = \left(1+\frac{1}{n}\right)^n \to e \quad \text{as} \quad n \to \infty.$$

This ratio is > 1 and so the series diverges.

4.14 *Obtain the positive values of x for which the following series converges:*

$$\sum_{n=1}^{\infty} \frac{x^{n/2}e^{-n}}{n}.$$

Using the Cauchy root test:

$$\rho = \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{x^{n/2}e^{-n}}{n}\right)^{1/n} = \frac{x^{1/2}}{e}.$$

For convergence $\rho < 1$ and this requires $x < e^2$. Direct substitution shows that that the series diverges at the limit $x = e^2$.

4.16 *An extension to the proof of the integral test (subsection 4.3.2) shows that, if $f(x)$ is positive, continuous and monotonically decreasing, for $x \geq 1$, and the series $f(1) + f(2) + \cdots$ is convergent, then its sum does not exceed $f(1) + L$, where L is the integral*

$$\int_1^\infty f(x)\,dx.$$

Use this result to show that the sum $\zeta(p)$ of the Riemann zeta series $\sum n^{-p}$, with $p > 1$, is not greater than $p/(p-1)$.

The function $f(x)$ appropriate to the Riemann zeta series is $f(x) = x^{-p}$.

$$\int_1^\infty \frac{1}{x^p}\,dx = \left[-\frac{1}{p-1}\,\frac{1}{x^{p-1}}\right]_1^\infty = \frac{1}{p-1}.$$

This implies that

$$\sum_{n=1}^\infty \frac{1}{n^p} \leq \frac{1}{1^p} + \frac{1}{p-1} = \frac{p}{p-1}.$$

4.18 *Illustrate result* (iv) *of section 4.4 concerning Cauchy products by considering the double summation*

$$S = \sum_{n=1}^\infty \sum_{r=1}^n \frac{1}{r^2(n+1-r)^3}.$$

By examining the points in the nr-plane over which the double summation is to be carried out, show that S can be written as

$$S = \sum_{n=r}^\infty \sum_{r=1}^\infty \frac{1}{r^2(n+1-r)^3}.$$

Deduce that $S \leq 3$.

As shown in figure 4.1, the original summation runs along lines (shown solid) parallel to the r-axis. The same mesh of points can be covered by lines running parallel to the n-axis; these are shown broken in the figure. Thus we have

$$S = \sum_{n=1}^\infty \sum_{r=1}^n \frac{1}{r^2(n+1-r)^3} = \sum_{n=r}^\infty \sum_{r=1}^\infty \frac{1}{r^2(n+1-r)^3}.$$

Figure 4.1 The summation points used in exercise 4.18.

If we now set $n+1-r$ equal to s, the double summation becomes

$$S = \sum_{s=1}^{\infty}\sum_{r=1}^{\infty}\frac{1}{r^2 s^3} = \zeta(2)\zeta(3).$$

In view of the result of exercise 4.16, $\zeta(2) \le 2$ and $\zeta(3) \le 3/2$. Consequently $S \le 3$.

4.20 *Identify the series*

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n-1)!},$$

and then by integration and differentiation deduce the values S of the following series:

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n^2}{(2n)!}; \qquad \text{(b)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n}{(2n+1)!};$$

$$\text{(c)}\ \sum_{n=1}^{\infty}\frac{(-1)^{n+1}n\pi^{2n}}{4^n(2n-1)!}; \qquad \text{(d)}\ \sum_{n=0}^{\infty}\frac{(-1)^n(n+1)}{(2n)!}.$$

Writing out the first few terms of the given series,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}x^{2n}}{(2n-1)!} &= x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} + \cdots \\ &= x\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) \\ &= x\sin x, \qquad (*)\end{aligned}$$

allows ready identification of the series.

(a) Differentiate both sides of (∗):

$$\sin x + x\cos x = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n x^{2n-1}}{(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 4n^2 x^{2n-1}}{(2n)!}.$$

Now set $x = 1$ to obtain

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n^2}{(2n)!} = \tfrac{1}{4}(\sin 1 + \cos 1) = 0.345.$$

(b) Integrate both sides of (∗):

$$\int^{x} x \sin x \, dx = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)(2n-1)!} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n x^{2n+1}}{(2n+1)!}.$$

Now, the LHS can be explicitly integrated by parts to yield $\sin x - x\cos x$. Setting $x = 1$ in both this and the derived series then gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{(2n+1)!} = \tfrac{1}{2}(\sin 1 - \cos 1) = 0.151.$$

(c) This is clearly similar to the RHS of the equation obtained in (a) if both its denominator and numerator are divided by $2n$ and x is set to $\pi/2$. If this is done then the equation in (a) reads

$$\sin \tfrac{1}{2}\pi + \tfrac{1}{2}\pi \cos \tfrac{1}{2}\pi = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n \, \pi^{2n-1}}{(2n-1)! \, 2^{2n-1}}.$$

After multiplying both sides by $\frac{1}{2}\pi$ and noting that $\cos \frac{1}{2}\pi = 0$, this can be rearranged as

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n \pi^{2n}}{4^n (2n-1)!} = \tfrac{1}{2} \times 1 \times \tfrac{1}{2}\pi = \tfrac{1}{4}\pi.$$

(d) It is not obvious how to obtain this sum, but the lowered starting value for the summation index suggests a redefinition of it with $n \to n-1$. To achieve this result the equation in (a) needs to be differentiated again:

$$\begin{aligned} 2\cos x - x \sin x &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n(2n-1) x^{2n-2}}{(2n-1)!} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n+1} 2n \, x^{2n-2}}{(2n-2)!} \\ &= \sum_{s=0}^{\infty} \frac{(-1)^{s} 2(s+1) x^{2s}}{(2s)!}, \end{aligned}$$

where in the last line we have defined s as $n-1$. Finally, setting $x=1$ and rearranging gives

$$\sum_{n=0}^{\infty} \frac{(-1)^n(n+1)}{(2n)!} = \tfrac{1}{2}(2\cos 1 - \sin 1) = 0.120.$$

4.22 *Find the Maclaurin series for*

$$\text{(a)}\ \ \ln\left(\frac{1+x}{1-x}\right), \qquad \text{(b)}\ \ (x^2+4)^{-1}, \qquad \text{(c)}\ \ \sin^2 x.$$

(a) Using the Maclaurin series for $\ln(1+x)$,

$$\begin{aligned}
\ln\left(\frac{1+x}{1-x}\right) &= \ln(1+x) - \ln(1-x) \\
&= \left(x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots\right) - \left(-x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots\right) \\
&= 2x + 2\frac{x^3}{3} + \cdots \\
&= 2\sum_{n\,\text{odd}} \frac{x^n}{n}.
\end{aligned}$$

This series is convergent only for $-1 < x < 1$.

(b) Here the binomial expansion can be employed directly. This saves the trouble of having to find the expansion coefficients by differentiation.

$$\begin{aligned}
(x^2+4)^{-1} &= \frac{1}{4}\left(1 + \frac{x^2}{4}\right)^{-1} \\
&= \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{4^n} \\
&= \frac{1}{4}\sum_{n=0}^{\infty} (-1)^n \left(\frac{x}{2}\right)^{2n}.
\end{aligned}$$

(c) We calculate the derivatives of $\sin^2 x$ at $x = 0$.

$$\begin{aligned} y &= \sin^2 x, \\ y' &= 2\sin x\cos x = \sin 2x, \\ y'' &= 2\cos 2x, \\ y''' &= -4\sin 2x, \\ \vdots &= \quad \vdots \\ y^{(2m)} &= (-1)^{m+1}2^{2m-1}\cos 2x, \\ y^{(2m+1)} &= (-1)^m 2^{2m}\sin 2x. \end{aligned}$$

At $x = 0$ all odd derivatives vanish and $\cos 2x = 1$ in the even ones. So the Maclaurin expansion is

$$\sin^2 x = \sum_{m=0}^{\infty} \frac{(-1)^{m+1}2^{2m-1}x^{2m}}{(2m)!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2x)^{2n}}{2(2n)!}.$$

This result could also have been obtained by writing $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ and using the known Maclaurin series for a cosine function.

4.24 *Find the first three non-zero terms in the Maclaurin series for the following functions:*

(a) $(x^2+9)^{-1/2}$, (b) $\ln[(2+x)^3]$, (c) $\exp(\sin x)$,
(d) $\ln(\cos x)$, (e) $\exp[-(x-a)^{-2}]$, (f) $\tan^{-1} x$.

(a) We use the binomial expansion directly to avoid the need to find derivatives, but must first get the term inside the parentheses into the form $1 + \alpha x^2$.

$$\begin{aligned} (x^2+9)^{-1/2} &= \frac{1}{3}\left(1 + \frac{x^2}{9}\right)^{-1/2} \\ &= \frac{1}{3}\left[1 - \frac{x^2}{18} + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2!}\,\frac{x^4}{81} + \cdots\right] \\ &= \frac{1}{3} - \frac{x^2}{54} + \frac{x^4}{648} + \cdots. \end{aligned}$$

(b) This function needs a small amount of manipulation before the expansion of $\ln(1+\delta)$ is invoked.

$$\begin{aligned} \ln[(2+x)^3] &= 3\ln\left[2\left(1 + \frac{x}{2}\right)\right] \\ &= 3\ln 2 + 3\left[\frac{x}{2} - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \cdots\right] \\ &= \ln 8 + \frac{3x}{2} - \frac{3x^2}{8} + \cdots. \end{aligned}$$

(c) Write $f(x) = \exp(\sin x)$. Then

$$\begin{aligned} f(0) &= 1 \\ f' &= \cos x\, e^{\sin x}, \qquad f'(0) = 1, \\ f'' &= -\sin x\, e^{\sin x} + \cos^2 x\, e^{\sin x}, \qquad f''(0) = 1 \end{aligned}$$

Hence, by the normal Taylor expansion about $x = 0$,

$$f(x) = 1 + x + \frac{x^2}{2!} + \cdots .$$

(d) Here the calculation of the derivatives needed for the Taylor expansion rapidly becomes complicated. Further, as three non-vanishing terms are needed, quite high derivatives are involved.

$$\begin{aligned} y &= \ln(\cos x), \quad y(0) = 0, \\ y' &= -\frac{\sin x}{\cos x} = -\tan x, \quad y'(0) = 0, \\ y'' &= -\sec^2 x, \quad y''(0) = -1, \\ y''' &= -2\sec^2 x \tan x, \quad y'''(0) = 0, \\ y^{(4)} &= -4\sec^2 x \tan^2 x - 2\sec^4 x, \quad y^{(4)}(0) = -2, \\ y^{(5)} &= -8\sec^2 x \tan^3 x - 16\sec^4 x \tan x, \quad y^{(5)}(0) = 0. \end{aligned}$$

Calculating $y^{(6)}$ is complicated but the only term that does not vanish at $x = 0$ comes from differentiating the $\tan x$ factor in the final product in $y^{(5)}$. This term will be $-16\sec^6 x$ which makes the value of $y^{(6)}(0)$ equal to -16. Hence, finally,

$$y(x) = -\frac{x^2}{2!} - 2\frac{x^4}{4!} - 16\frac{x^6}{6!} + \cdots = -\frac{x^2}{2} - \frac{x^4}{12} - \frac{x^6}{45} + \cdots .$$

(e) Write $f(x) = \exp[-(x-a)^{-2}]$.

$$\begin{aligned} f(x) &= \exp[-(x-a)^{-2}], \quad f(0) = \exp(-a^{-2}), \\ f' &= \frac{2f}{(x-a)^3}, \quad f'(0) = -\frac{2}{a^3}\exp(-a^{-2}), \\ f'' &= \frac{2f'}{(x-a)^3} - \frac{6f}{(x-a)^4}, \quad f''(0) = \left(\frac{4}{a^6} - \frac{6}{a^4}\right)\exp(-a^{-2}). \end{aligned}$$

Thus, from the Taylor expansion, the Maclaurin series is

$$f(x) = \exp(-a^{-2})\left[1 - \frac{2x}{a^3} - \frac{3a^2-2}{a^6}x^2 + \cdots\right].$$

(f) If $f(x) = \tan^{-1} x$ then

$$f' = \frac{1}{1+x^2}, \quad f'' = -\frac{2x}{(1+x^2)^2}, \quad f''' = \frac{6x^2-2}{(1+x^2)^3}$$

$$f^{(4)} = \frac{24x - 24x^3}{(1+x^2)^4}, \quad f^{(5)} = \frac{24 - \cdots}{(1+x^2)^5}.$$

At $x = 0$ only the odd derivatives are non-zero (as they must be since $\tan^{-1} x$ is an odd function of x) and

$$\begin{aligned}\tan^{-1} x &= x - 2\frac{x^3}{3!} + 24\frac{x^5}{5!} + \cdots \\ &= x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots .\end{aligned}$$

4.26 *Determine whether the following functions $f(x)$ are* (i) *continuous, and* (ii) *differentiable at $x = 0$:*

(a) $f(x) = \exp(-|x|)$;
(b) $f(x) = (1 - \cos x)/x^2$ *for* $x \neq 0$, $f(0) = \frac{1}{2}$;
(c) $f(x) = x \sin(1/x)$ *for* $x \neq 0$, $f(0) = 0$;
(d) $f(x) = [4 - x^2]$, *where* $[y]$ *denotes the integer part of* y.

(a) Taking $\Delta > 0$

$$\exp(-|\Delta|) = 1 - \Delta + \frac{\Delta^2}{2!} - \cdots ,$$

whilst for $\Delta < 0$

$$\exp(-|\Delta|) = 1 + \Delta + \frac{\Delta^2}{2!} + \cdots .$$

In the limit $\Delta \to 0$ both series tend to the common limit 1. Thus the function is continuous at $x = 0$.

However, the derivative at $x = 0$ when $\Delta > 0$ is given by

$$\lim_{\Delta \to 0} \frac{1 - \Delta + \frac{\Delta^2}{2!} - \cdots - 1}{\Delta},$$

whilst that for $\Delta < 0$ is

$$\lim_{\Delta \to 0} \frac{1 + \Delta + \frac{\Delta^2}{2!} + \cdots - 1}{\Delta}.$$

The former limit has value -1 whilst the latter is $+1$. These are not equal and so $\exp(-|x|)$ is not differentiable at $x = 0$.

(b) Whether x is positive or negative,

$$\frac{1-\cos x}{x^2} = \frac{1-\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots\right)}{x^2}$$
$$= \frac{1}{2} - \frac{x^2}{4!} + \cdots .$$

As $x \to 0$ from either positive or negative values this expression tends to the value $\frac{1}{2}$. As this is the defined value of the function at $x = 0$, the function is continuous.

Proceeding as in (a), the first derivative of $f(x)$ is given for any Δ by

$$\lim_{\Delta\to 0} \frac{\frac{1}{2} - \frac{\Delta^2}{4!} + \cdots - \frac{1}{2}}{\Delta}.$$

This has the value zero, whether Δ is positive or negative. Thus, the function is differentiable at $x = 0$.

(c) Consider the modulus of $x\sin(1/x)$. This is $\leq |x|$ for all x and, as the latter $\to 0$ as $x \to 0$, $|x\sin(1/x)|$ must also $\to 0$ as $x \to 0$. Thus the function is continuous at $x = 0$.

However,

$$\frac{f(x)-f(0)}{x-0} = \frac{x\sin(1/x)-0}{x} = \sin\left(\frac{1}{x}\right).$$

This oscillates (increasingly rapidly) as $x \to 0$ between ± 1 and does not tend to a limit. Therefore the function is not differentiable at $x = 0$.

(d) For $|x| \leq 1$ the function $[4-x^2]$ has the (constant) value 3. However, at $x = 0$ it has the value 4. Thus the fuction is not continuous at $x = 0$. It also follows that it cannot be differentiable there.

4.28 *Evaluate the following limits:*

$$\text{(a)} \lim_{x\to 0} \frac{\sin 3x}{\sinh x}, \qquad \text{(b)} \lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x},$$
$$\text{(c)} \lim_{x\to 0} \frac{\tan x - x}{\cos x - 1}, \qquad \text{(d)} \lim_{x\to 0} \left(\frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5}\right).$$

Using L'Hôpital's rule whenever both the numerator and denominator of a fraction become zero in the limit, we have for cases (a)-(c):

(a) $$\lim_{x\to 0} \frac{\sin 3x}{\sinh x} = \lim_{x\to 0} \frac{3\cos 3x}{\cosh x} = 3.$$

(b)
$$\begin{aligned}
L &= \lim_{x\to 0} \frac{\tan x - \tanh x}{\sinh x - x} \\
&= \lim_{x\to 0} \frac{\sec^2 x - \operatorname{sech}^2 x}{\cosh x - 1} \\
&= \lim_{x\to 0} \frac{2\sec^2 x \tan x + 2\operatorname{sech}^2 x \tanh x}{\sinh x} \\
&= \lim_{x\to 0} \frac{2(\tan x + \tan^3 x + \tanh x - \tanh^3 x)}{\sinh x} \\
&= \lim_{x\to 0} \frac{2(\sec^2 x + 3\tan^2 x \sec^2 x + \operatorname{sech}^2 x - 3\tanh^2 x \operatorname{sech}^2 x)}{\cosh x} \\
&= 4.
\end{aligned}$$

(c)
$$\lim_{x\to 0} \frac{\tan x - x}{\cos x - 1} = \lim_{x\to 0} \frac{\sec^2 x - 1}{-\sin x} = \lim_{x\to 0} \frac{2\sec^2 x \tan x}{-\cos x} = 0.$$

(d) In order to avoid undetermined combinations such as $x \operatorname{cosec} x$, we arrange the function so that all terms in it tend to 0 or a finite quantity as $x \to 0$. For this particular case this also allows us to use known McLaurin expansions of the factors involved.

[Inspection of the function shows that both fractions behave as x^{-4} as $x \to 0$, but that their difference will be less divergent. Finding the limit (if it is finite) using L'Hôpital's rule will take up to six differentiations; hence our preference for Maclaurin expansions.]

$$\begin{aligned}
L &= \frac{\operatorname{cosec} x}{x^3} - \frac{\sinh x}{x^5} \\
&= \frac{x^2 - \sin x \sinh x}{x^5 \sin x} \\
&= \frac{x^2 - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)}{x^5\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)} \\
&= \frac{1 - \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)}{x^4\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)} \\
&= \frac{(1-1) + x^2\left(\frac{1}{3!} - \frac{1}{3!}\right) + x^4\left(-\frac{1}{5!} - \frac{1}{5!} + \frac{1}{(3!)^2}\right) + \cdots}{x^4\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right)},
\end{aligned}$$

which has as its limit, when $x \to 0$, the value

$$\frac{1}{(3!)^2} - \frac{2}{5!} = \frac{20-12}{720} = \frac{1}{90}.$$

4.30 *Use Taylor expansions to three terms to find approximations to* (a) $\sqrt[4]{17}$, *and* (b) $\sqrt[3]{26}$.

(a) For $f(x) = x^{1/4}$,

$$f'(x) = \frac{1}{4}\frac{1}{x^{3/4}} \quad \text{and} \quad f''(x) = -\frac{3}{16}\frac{1}{x^{7/4}}.$$

Thus, writing 17 as $16 + 1$,

$$\begin{aligned}(17)^{1/4} &= (16)^{1/4} + 1\,\frac{1}{4}\,\frac{1}{(16)^{3/4}} - \frac{1}{2}\,1^2\,\frac{3}{16}\,\frac{1}{(16)^{7/4}} + \cdots \\ &= 2 + \frac{1}{32} - \frac{3}{2 \times 16 \times 16 \times 8} + \cdots \\ &= 2.030\,518.\end{aligned}$$

The more accurate answer is 2.030 543.

(b) For $f(x) = x^{1/3}$,

$$f'(x) = \frac{1}{3}\frac{1}{x^{2/3}} \quad \text{and} \quad f''(x) = -\frac{2}{9}\frac{1}{x^{5/3}}.$$

Thus, writing 26 as $27 + (-1)$,

$$\begin{aligned}(26)^{1/3} &= (27)^{1/3} + (-1)\,\frac{1}{3}\,\frac{1}{(27)^{2/3}} - \frac{1}{2}\,(-1)^2\,\frac{2}{9}\,\frac{1}{(27)^{5/3}} + \cdots \\ &= 3 - \frac{1}{27} - \frac{1}{9 \times 27 \times 9} + \cdots \\ &= 2.962\,506.\end{aligned}$$

The more accurate answer is 2.962 496.

4.32 *Evaluate*

$$\lim_{x \to 0}\left[\frac{1}{x^3}\left(\operatorname{cosec} x - \frac{1}{x} - \frac{x}{6}\right)\right].$$

In order to evaluate the limit L, we avoid ill-defined products by multiplying both

numerator and denominator by $x \sin x$. (See also part (d) of exercise 4.28)

$$\begin{aligned}
L &= \lim_{x\to 0} \frac{1}{x^3}\left(\operatorname{cosec} x - \frac{1}{x} - \frac{x}{6}\right) \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left(x - \sin x - \frac{x^2}{6}\sin x\right) \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left[x - \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right) - \frac{x^2}{6}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)\right] \\
&= \lim_{x\to 0} \frac{1}{x^4 \sin x}\left[x(1-1) + x^3\left(\frac{1}{3!} - \frac{1}{6}\right) + x^5\left(-\frac{1}{5!} + \frac{1}{6\times 3!}\right) + \cdots\right] \\
&= \lim_{x\to 0}\left(\frac{x}{\sin x}\,\frac{-6+20}{720} + \cdots\right) \\
&= \frac{7}{360}.
\end{aligned}$$

4.34 *In a very simple model of a crystal, point-like atomic ions are regularly spaced along an infinite one-dimensional row with spacing R. Alternate ions carry equal and opposite charges $\pm e$. The potential energy of the ith ion in the electric field due to the jth ion is*

$$\frac{q_i q_j}{4\pi\epsilon_0 r_{ij}},$$

where q_i, q_j are the charges on the ions and r_{ij} is the distance between them.

Write down a series giving the total contribution V_i of the ith ion to the overall potential energy. Show that the series converges, and, if V_i is written as

$$V_i = \frac{\alpha e^2}{4\pi\epsilon_0 R},$$

find a closed-form expression for α, the Madelung constant for this (unrealistic) lattice.

The ion that is nR distant from the ith ion has charge $(-1)^n q_i$ and contributes

$$\frac{(-1)^n q_i}{4\pi\epsilon_0\,(nR)}$$

to the potential at the ith ion. The infinite sum of terms of this form converges by the alternating sign test and the total potential energy of the ith ion is

$$q_i \sum_{n=-\infty}^{-1} \frac{(-1)^n q_i}{4\pi\epsilon_0\,|n|R} + q_i \sum_{n=1}^{\infty} \frac{(-1)^n q_i}{4\pi\epsilon_0\, nR} = \frac{2q_i^2}{4\pi\epsilon_0 R}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \frac{2e^2 \ln 2}{4\pi\epsilon_0 R}.$$

The value of α is thus $-2\ln 2$.

4.36 *In quantum theory a certain method (the Born approximation) gives the (so-called) amplitude $f(\theta)$ for the scattering of a particle of mass m through an angle θ by a uniform potential well of depth V_0 and radius b (i.e. the potential energy of the particle is $-V_0$ within a sphere of radius b and zero elsewhere) as*

$$f(\theta) = \frac{2mV_0}{\hbar^2 K^3}(\sin Kb - Kb\cos Kb).$$

Here $\hbar$ is the Planck constant divided by 2π, the energy of the particle is $\hbar^2k^2/(2m)$ and K is $2k\sin(\theta/2)$.

Use l'Hôpital's rule to evaluate the amplitude at low energies, i.e. when k and hence K tend to zero, and so determine the low-energy total cross-section.

[*Note: the differential cross-section is given by $|f(\theta)|^2$ and the total cross-section by the integral of this over all solid angles, i.e. $2\pi\int_0^\pi |f(\theta)|^2 \sin\theta\, d\theta$.*]

We need to determine the value L given by

$$\begin{aligned}
L &= \lim_{K\to 0} \frac{\sin Kb - Kb\cos Kb}{K^3} \\
&= \lim_{K\to 0} \frac{\left(Kb - \dfrac{(Kb)^3}{3!} + \cdots\right) - Kb\left(1 - \dfrac{(Kb)^2}{2!} + \dfrac{(Kb)^4}{4!} - \cdots\right)}{K^3} \\
&= \lim_{K\to 0} \frac{(Kb)^3\left(-\dfrac{1}{3!} + \dfrac{1}{2!} + O(K^2b^2)\right)}{K^3} \\
&= \tfrac{1}{3}b^3.
\end{aligned}$$

Thus $f(\theta) = 2mV_0b^3/(3\hbar)^2$. This result is real and independent of θ (indicating that the scattering is isotropic). Consequently, the total cross-section is simply 4π times the square of $f(\theta)$, i.e. $4\pi[2mV_0b^3/(3\hbar)^2]^2$.

5

Partial differentiation

5.2 *Determine which of the following are exact differentials:*

(a) $(3x+2)y\,dx + x(x+1)\,dy$;
(b) $y\tan x\,dx + x\tan y\,dy$;
(c) $y^2(\ln x+1)\,dx + 2xy\ln x\,dy$;
(d) $y^2(\ln x+1)\,dy + 2xy\ln x\,dx$;
(e) $[x/(x^2+y^2)]\,dy - [y/(x^2+y^2)]\,dx$.

If $df = A\,dx + B\,dy$ then a necessary and sufficient condition that df is exact is that $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$. We apply this test in each case.

(a) $\dfrac{\partial[(3x+2)y]}{\partial y} = 3x+2,$

$\dfrac{\partial[x(x+1)]}{\partial x} = 2x+1,$ unequal $\Rightarrow$ not exact.

(b) $\dfrac{\partial(y\tan x)}{\partial y} = \tan x, \quad \dfrac{\partial(x\tan y)}{\partial x} = \tan y,$ unequal $\Rightarrow$ not exact.

(c) $\dfrac{\partial[y^2(\ln x+1)]}{\partial y} = 2y(\ln x+1),$

$\dfrac{\partial(2xy\ln x)}{\partial x} = 2y\ln x + 2xy\dfrac{1}{x} = 2y(\ln x+1),$ equal $\Rightarrow$ exact.

(d) $\dfrac{\partial[y^2(\ln x+1)]}{\partial x} = \dfrac{y^2}{x}, \ \dfrac{\partial(2xy\ln x)}{\partial y} = 2x\ln x,$ unequal $\Rightarrow$ not exact.

(e) $\dfrac{\partial}{\partial x}\left(\dfrac{x}{x^2+y^2}\right) = \dfrac{y^2-x^2}{(x^2+y^2)^2}, \quad \dfrac{\partial}{\partial y}\left(\dfrac{-y}{x^2+y^2}\right) = \dfrac{y^2-x^2}{(x^2+y^2)^2},$

as was shown in exercise 5.1(c). The equality of these two expressions shows that the differential is exact.

5.4 *Show that*

$$df = y(1 + x - x^2)\,dx + x(x + 1)\,dy$$

is not an exact differential.

Find the differential equation that a function $g(x)$ must satisfy if $d\phi = g(x)df$ is to be an exact differential. Verify that $g(x) = e^{-x}$ is a solution of this equation and deduce the form of $\phi(x, y)$.

If $df = A\,dx + B\,dy$ then a necessary and sufficient condition that df is exact is that $\dfrac{\partial A}{\partial y} = \dfrac{\partial B}{\partial x}$. We apply this test.

$$\frac{\partial[y(1 + x - x^2)]}{\partial y} = 1 + x - x^2, \qquad \frac{\partial[x(x + 1)]}{\partial x} = 2x + 1.$$

These are not equal and so the differential is not exact.

If $g(x)df$ is to be exact we must have

$$\begin{aligned}
\frac{\partial[gy(1 + x - x^2)]}{\partial y} &= \frac{\partial[gx(x + 1)]}{\partial x}, \\
g(1 + x - x^2) &= g'x(x + 1) + g(2x + 1), \\
g(-x - x^2) &= g'x(x + 1), \\
-g &= g'.
\end{aligned}$$

Clearly $g(x) = e^{-x}$, with $g'(x) = -e^{-x}$, satisfies this equation. The more general solution $g(x) = Ae^{-x}$ would do just as well from the point of view of making $d\phi$ an exact differential.

Accepting the stated form (with $A = 1$), we must now have that

$$\begin{aligned}
x(x + 1)e^{-x} &= \frac{\partial\phi}{\partial y}, \quad \Rightarrow \quad \phi(x, y) = xy(x + 1)e^{-x} + h(x). \\
y(1 + x - x^2)e^{-x} &= \frac{\partial\phi}{\partial x} = -xy(x + 1)e^{-x} + (2x + 1)ye^{-x} + h'(x).
\end{aligned}$$

The terms involving e^{-x} all cancel and thus $h'(x) = 0$, implying $h(x) = k$ (a constant) and $\phi(x, y) = xy(x + 1)e^{-x} + k$.

5.6 *A possible equation of state for a gas takes the form*

$$pV = RT \exp\left(-\frac{\alpha}{VRT}\right),$$

in which α *and* R *are constants. Calculate expressions for*

$$\left(\frac{\partial p}{\partial V}\right)_T, \qquad \left(\frac{\partial V}{\partial T}\right)_p, \qquad \left(\frac{\partial T}{\partial p}\right)_V,$$

and show that their product is -1*, as stated in section 5.4.*

The required differentiations are most easily carried out after taking the logarithms of both sides of the original equation (for typographical convenience we use P instead of p):

$$\ln P + \ln V - \ln R - \ln T = -\frac{\alpha}{VRT}. \qquad (*)$$

First, differentiating $(*)$ with respect to V with T held fixed:

$$\frac{1}{P}\left(\frac{\partial P}{\partial V}\right)_T + \frac{1}{V} = \frac{\alpha}{V^2RT},$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \left(\frac{\alpha}{V^2RT} - \frac{1}{V}\right)P$$

$$= \frac{P(\alpha - VRT)}{V^2RT}.$$

Next, differentiating $(*)$ with respect to T with P held fixed:

$$\frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P - \frac{1}{T} = \frac{\alpha}{VRT^2} + \frac{\alpha}{V^2RT}\left(\frac{\partial V}{\partial T}\right)_P,$$

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\dfrac{\alpha}{VRT^2} + \dfrac{1}{T}}{\dfrac{1}{V} - \dfrac{\alpha}{V^2RT}}$$

$$= \frac{V(\alpha + VRT)}{T(VRT - \alpha)}.$$

Finally, differentiating $(*)$ with respect to P with V held fixed.

$$\frac{1}{P} - \frac{1}{T}\left(\frac{\partial T}{\partial P}\right)_V = \frac{\alpha}{VRT^2}\left(\frac{\partial T}{\partial P}\right)_V,$$

$$\left(\frac{\partial T}{\partial P}\right)_V = \frac{1}{P\left(\dfrac{\alpha}{VRT^2} + \dfrac{1}{T}\right)}$$

$$= \frac{VRT^2}{P(\alpha + VRT)}.$$

These are the three partial derivatives and their product,

$$\left(\frac{\partial P}{\partial V}\right)_T \left(\frac{\partial V}{\partial T}\right)_P \left(\frac{\partial T}{\partial P}\right)_V,$$

is

$$\frac{P(\alpha - VRT)}{V^2RT}\frac{V(\alpha + VRT)}{T(VRT - \alpha)}\frac{VRT^2}{P(\alpha + VRT)} = -1,$$

as expected.

5.8 *In the xy-plane, new coordinates s and t are defined by*

$$s = \tfrac{1}{2}(x + y), \qquad t = \tfrac{1}{2}(x - y).$$

Transform the equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{\partial^2\phi}{\partial y^2} = 0$$

into the new coordinates and deduce that its general solution can be written

$$\phi(x, y) = f(x + y) + g(x - y),$$

where $f(u)$ and $g(v)$ are arbitrary functions of u and v respectively.

To make the transformation we need the partial derivatives

$$\frac{\partial s}{\partial x} = \frac{1}{2}, \quad \frac{\partial s}{\partial y} = \frac{1}{2}, \quad \frac{\partial t}{\partial x} = \frac{1}{2}, \quad \frac{\partial t}{\partial y} = -\frac{1}{2}.$$

Now, using the chain rule, we obtain for the partial differential operators

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial}{\partial t}\frac{\partial t}{\partial x} = \frac{1}{2}\frac{\partial}{\partial s} + \frac{1}{2}\frac{\partial}{\partial t},$$
$$\frac{\partial}{\partial y} = \frac{\partial}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial}{\partial t}\frac{\partial t}{\partial y} = \frac{1}{2}\frac{\partial}{\partial s} - \frac{1}{2}\frac{\partial}{\partial t}.$$

Thus, we have

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \frac{1}{4}\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} + \frac{\partial}{\partial t}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial s^2} + 2\frac{\partial^2}{\partial t\partial s} + \frac{\partial^2}{\partial t^2}\right),\\ \frac{\partial^2}{\partial y^2} &= \frac{1}{4}\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial s} - \frac{\partial}{\partial t}\right)\\ &= \frac{1}{4}\left(\frac{\partial^2}{\partial s^2} - 2\frac{\partial^2}{\partial t\partial s} + \frac{\partial^2}{\partial t^2}\right),\end{aligned}$$

and so, by subtraction of these two operators,

$$\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial t \partial s}.$$

On writing $\phi(x, y)$ as $\psi(s, t)$, the original equation becomes

$$\frac{\partial^2 \psi}{\partial t \partial s} = 0,$$

with a first integral (with respect to t) of

$$\frac{\partial \psi}{\partial s} = f_1(s),$$

where $f_1(s)$ is any arbitrary function of s (but not of t). A second integration, with respect to s, gives

$$\psi = \int^s f_1(u)\, du + f_2(t),$$

where $f_2(t)$ is any arbitrary function of t (but not of s).

Thus $\phi(x, y) = \psi(s, t)$ can be written as the sum of two arbitrary functions, one of s and the other of t, or, equivalently, of two (slightly different, because of the factors of $\frac{1}{2}$) arbitrary functions of $x + y$ and $x - y$, respectively.

5.10 *If* $x = e^u \cos\theta$ *and* $y = e^u \sin\theta$, *show that*

$$\frac{\partial^2 \phi}{\partial u^2} + \frac{\partial^2 \phi}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

where $f(x, y) = \phi(u, \theta)$.

The four partial derivatives needed to make the change of variables are:

$$\frac{\partial x}{\partial u} = e^u \cos\theta = x, \qquad \frac{\partial x}{\partial \theta} = -e^u \sin\theta = -y,$$
$$\frac{\partial y}{\partial u} = e^u \sin\theta = y, \qquad \frac{\partial y}{\partial \theta} = e^u \cos\theta = x,$$

giving (using the chain rule) the connections between the differential operators in the two sets of coordinates as

$$\frac{\partial}{\partial u} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \frac{\partial}{\partial \theta} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.$$

Now,

$$\begin{aligned}\frac{\partial^2}{\partial u^2} &= \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)\\ &= x^2\frac{\partial^2}{\partial x^2} + x\frac{\partial}{\partial x} + yx\frac{\partial^2}{\partial y\partial x} + xy\frac{\partial^2}{\partial x\partial y} + y^2\frac{\partial^2}{\partial y^2} + y\frac{\partial}{\partial y}.\\ \frac{\partial^2}{\partial \theta^2} &= \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)\\ &= y^2\frac{\partial^2}{\partial x^2} - x\frac{\partial}{\partial x} - xy\frac{\partial^2}{\partial y\partial x} - yx\frac{\partial^2}{\partial x\partial y} + x^2\frac{\partial^2}{\partial y^2} - y\frac{\partial}{\partial y}.\end{aligned}$$

Adding these two operators:

$$\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + xy\left(2\frac{\partial^2}{\partial x\partial y} - 2\frac{\partial^2}{\partial x\partial y}\right).$$

Thus,

$$\frac{\partial^2\phi}{\partial u^2} + \frac{\partial^2\phi}{\partial \theta^2} = (x^2 + y^2)\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right),$$

where $f(x, y) = \phi(u, \theta)$.

5.12 *Show that*

$$f(x, y) = x^3 - 12xy + 48x + by^2, \qquad b \neq 0,$$

has two, one, or zero stationary points according to whether $|b|$ *is less than, equal to, or greater than* 3.

At a stationary point, the total differential df must be zero whatever values the infinitesimal changes dx and dy take. This condition requires that

$$\begin{aligned}0 &= \frac{\partial f}{\partial x} = 3x^2 - 12y + 48,\\ 0 &= \frac{\partial f}{\partial y} = -12x + 2by.\end{aligned}$$

From the second of these $y = 6x/b$ (with $b \neq 0$), and the first equation can be written as

$$x^2 - \frac{24}{b}x + 16 = 0.$$

This is a quadratic equation for x and has two, one or zero real roots according

to whether

$$\left(\frac{24}{b}\right)^2 \text{ is greater than, or equal to, or less than } 4 \times 1 \times 16,$$

$$\text{i.e. } \left|\frac{24}{b}\right| \text{ is greater than, or equal to, or less than } 8,$$

$$\text{i.e. } |b| \text{ is less than, or equal to, or greater than } 3.$$

5.14 *Find the stationary points of the function*

$$f(x, y) = x^3 + xy^2 - 12x - y^2$$

and identify their nature.

As explained in the solution to exercise 5.12, stationary points occur when df is zero whatever values the infinitesimal changes dx and dy take. For the present question this implies that

$$0 = \frac{\partial f}{\partial x} = 3x^2 + y^2 - 12,$$
$$0 = \frac{\partial f}{\partial y} = 2xy - 2y.$$

From the second equation, either $y = 0$ or $x = 1$ and, correspondingly, from the first $x = \pm 2$ or $y = \pm 3$.

To determine the nature of the stationary points, we must calculate the second derivatives. The required derivatives are

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = 2y, \quad \frac{\partial^2 f}{\partial y^2} = 2(x - 1).$$

At $(2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = 12$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ and $\dfrac{\partial^2 f}{\partial y^2} = 2$. Since both unmixed second derivatives are positive and $2 \times 12 > 0$ this is a minimum with value -16.

At $(-2, 0)$, $\dfrac{\partial^2 f}{\partial x^2} = -12$, $\dfrac{\partial^2 f}{\partial x \partial y} = 0$ and $\dfrac{\partial^2 f}{\partial y^2} = -6$. Since both unmixed second derivatives are negative and $-6 \times -12 > 0$ this is a maximum with value 16.

At $(1, \pm 3)$, $\dfrac{\partial^2 f}{\partial x^2} = 6$, $\dfrac{\partial^2 f}{\partial x \partial y} = \pm 6$ and $\dfrac{\partial^2 f}{\partial y^2} = 0$. Since $0 \times 6 \not> 36$ these are saddle points with the common value -11.

5.16 *The temperature of a point (x, y, z) on the unit sphere is given by*

$$T(x, y, z) = 1 + xy + yz.$$

By using the method of Lagrange multipliers find the temperature of the hottest point on the sphere.

It is clear that the larger the absolute values that x, y and z can take, the larger the maximum temperature can be; the hottest point(s) of the sphere will therefore occur on its surface, i.e. the coordinates of the hottest point must satisfy $x^2 + y^2 + z^2 - 1 = 0$.

We incorporate this constraint using a Lagrange multiplier and consider

$$f(x, y, z) = T(x, y, z) + \lambda(x^2 + y^2 + z^2 - 1) = 1 + xy + yz + \lambda(x^2 + y^2 + z^2 - 1).$$

Its stationary values are given by

$$0 = \frac{\partial f}{\partial x} = y + 2\lambda x,$$
$$0 = \frac{\partial f}{\partial y} = x + z + 2\lambda y,$$
$$0 = \frac{\partial f}{\partial z} = y + 2\lambda z.$$

From symmetry $x = z$, leading to

$$0 = y + 2\lambda x \quad \text{and} \quad 0 = 2x + 2\lambda y.$$

Elimination of λ between these two equations gives $y^2 = 2x^2$, and then substitution for y and z in $x^2 + y^2 + z^2 = 1$ yields $x^2 = \frac{1}{4}$ and $x(= z) = \pm\frac{1}{2}$.

The four possible hottest spots are therefore $(\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{2})$ and $(-\frac{1}{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{2})$. Direct substitution shows that a maximum of $T = 1 + \frac{1}{\sqrt{2}}$ occurs at $\pm(\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2})$. [The two other stationary points give temperature minima.]

5.18 *Two horizontal corridors, $0 \le x \le a$ with $y \ge 0$, and $0 \le y \le b$ with $x \ge 0$, meet at right angles. Find the length L of the longest ladder (considered as a stick) that may be carried horizontally around the corner.*

Let the ends of the ladder touch the outside walls of the corner at the points $(a+\xi, 0)$ and $(0, b+\eta)$. Then the square of the length of the ladder is

$$L^2 = (a+\xi)^2 + (b+\eta)^2.$$

The longest ladder will touch the inside corner at (a, b) and simple geometry then requires that

$$\frac{\eta}{a} = \frac{b}{\xi}.$$

Thus we need to maximise L (or L^2) subject to $\eta\xi = ab$. We therefore consider

$$f(\xi, \eta) = (a+\xi)^2 + (b+\eta)^2 + \lambda\eta\xi.$$

Its stationary values occur when

$$0 = \frac{\partial f}{\partial \xi} = 2(a+\xi) + \lambda\eta,$$
$$0 = \frac{\partial f}{\partial \eta} = 2(b+\eta) + \lambda\xi.$$

Thus $2\xi(a+\xi) - 2\eta(b+\eta) = 0$; together with $\eta\xi = ab$, this gives as an equation for ξ,

$$2\xi a + 2\xi^2 - \frac{2ab}{\xi}\left(b + \frac{ab}{\xi}\right) = 0,$$
$$\Rightarrow \quad (\xi^3 - ab^2)(\xi + a) = 0.$$

The only physical solution to this is $\xi = (ab^2)^{1/3}$; the corresponding value of η is $ab/\xi = a^{2/3}b^{1/3}$. Then

$$\begin{aligned} L^2 &= \left(a + a^{1/3}b^{2/3}\right)^2 + \left(b + a^{2/3}b^{1/3}\right)^2 \\ &= a^2 + 3a^{4/3}b^{2/3} + 3a^{2/3}b^{4/3} + b^2 \\ &= \left(a^{2/3} + b^{2/3}\right)^3. \end{aligned}$$

Thus the longest ladder that may be carried horizontally round the corner is of length $\left(a^{2/3} + b^{2/3}\right)^{3/2}$.

5.20 *Show that the envelope of all concentric ellipses that have their axes along the x- and y-coordinate axes and that have the sum of their semi-axes equal to a constant L is the same curve (an astroid) as that found in the worked example in section 5.10.*

The equation of a typical ellipse with semi-axis a in the x-direction is

$$f(x, y, a) = \frac{x^2}{a^2} + \frac{y^2}{(L-a)^2} - 1 = 0. \qquad (*)$$

To find the envelope of all the ellipses we set $\partial f/\partial a = 0$. This gives

$$\frac{\partial f}{\partial a} = -\frac{2x^2}{a^3} + \frac{2y^2}{(L-a)^3} = 0.$$

Re-arranging this equation so as to provide expressions that can be used to eliminate a from $(*)$ we obtain

$$\frac{x^2}{y^2} = \frac{a^3}{(L-a)^3} \quad \Rightarrow \quad \frac{a}{L-a} = \frac{x^{2/3}}{y^{2/3}},$$

yielding

$$a = \frac{x^{2/3}}{x^{2/3} + y^{2/3}} L \quad \text{and} \quad L - a = \frac{y^{2/3}}{x^{2/3} + y^{2/3}} L.$$

Substituting these values into $f(x, y, a) = 0$ gives the equation of the envelope as

$$\frac{x^2(x^{2/3} + y^{2/3})^2}{x^{4/3}L^2} + \frac{y^2(x^{2/3} + y^{2/3})^2}{y^{4/3}L^2} = 1,$$
$$(x^{2/3} + y^{2/3})^2(x^{2/3} + y^{2/3}) = L^2,$$
$$x^{2/3} + y^{2/3} = L^{2/3},$$

i.e. an astroid.

5.22 *Prove that the envelope of the circles whose diameters are those chords of a given circle that pass through a fixed point on its circumference, is the cardioid*

$$r = a(1 + \cos\theta).$$

Here a is the radius of the given circle and (r, θ) are the polar coordinates of the envelope. Take as the system parameter the angle ϕ between a chord and the polar axis from which θ is measured.

The fixed circle, shown in figure 5.1, has diameter OD. Its chord OQ is a diameter of a typical member of the family of circles generated as Q is varied.

Since OQ is a diameter of the typical circle with parameter ϕ the angle OPQ is a right angle. The radial polar coordinate of P is therefore $r = c\cos(\theta - \phi)$. Similarly, since angle OQD is also a right angle, $c = 2a\cos\phi$.

Thus the equation of a typical circle is

$$f(r, \theta, \phi) = r - 2a\cos\phi\cos(\theta - \phi) = 0.$$

Figure 5.1 The fixed circle in exercise 5.22 has diameter OD and a general chord OQ. A typical member of the family of circles passes through O and has OQ as a diameter. The family is generated as Q is varied.

The envelope to this family of circles, in which ϕ is a parameter that is fixed for each circle, is given by $\partial f/\partial\phi = 0$, i.e.

$$2a\sin\phi\cos(\theta-\phi) - 2a\cos\phi\sin(\theta-\phi) = 0.$$

Using the compound-angle formula for $\sin(A+B)$, this can be simplified to

$$\sin(\phi-(\theta-\phi)) = 0,$$

and thus $\theta = 2\phi$. This gives the point on the circle with parameter ϕ at which the envelope touches it. The formal second solution, $\theta = 2\phi - \pi$, leads to a negative value for r and can be discarded.

The equation of the envelope itself is therefore obtained by eliminating ϕ between this condition and the equation of the circle:

$$\begin{aligned} r &= 2a\cos\tfrac{1}{2}\theta\cos(\theta-\tfrac{1}{2}\theta) \\ &= 2a\cos^2\tfrac{1}{2}\theta \\ &= a(1+\cos\theta). \end{aligned}$$

This curve is a cardioid, and a sketch of one is shown in figure 2.4, as part of the answer to exercise 2.50.

5.24 *In order to make a focussing mirror that concentrates parallel axial rays to one spot (or conversely forms a parallel beam from a point source) a parabolic shape should be adopted. If a mirror that is part of a circular cylinder or sphere were used, the light would be spread out along a curve. This curve is known as a* caustic *and is the envelope of the rays reflected from the mirror. Denoting by θ the angle which a typical incident axial ray makes with the normal to the mirror at the place where it is reflected, the geometry of reflection (the angle of incidence equals the angle of reflection) is shown in figure 5.2.*

Show that a parametric specification of the caustic is

$$x = R\cos\theta\left(\tfrac{1}{2} + \sin^2\theta\right), \qquad y = R\sin^3\theta,$$

where R is the radius of curvature of the mirror. The curve is, in fact, part of an epicycloid.

Denoting the points where the ray strikes the mirror and later crosses the axis by P and Q respectively, we see, by applying the sine rule to the triangle OPQ, that

$$\frac{OQ}{\sin\theta} = \frac{R}{\sin 2\theta}.$$

Thus, taking O as the origin, the equation of the reflected ray is

$$y = \tan 2\theta\left(x - \frac{R\sin\theta}{\sin 2\theta}\right).$$

Putting this into the standard form $f(x, y, \theta) = 0$, setting $\partial f/\partial\theta$ equal to zero, and then eliminating y from the two resulting equations gives

$$\begin{aligned}
0 &= f(x, y, \theta) = y\cos 2\theta - x\sin 2\theta + R\sin\theta,\\
0 &= \frac{\partial f}{\partial\theta} = -2y\sin 2\theta - 2x\cos 2\theta + R\cos\theta,\\
0 &= -x(\sin^2 2\theta + \cos^2 2\theta) + R\sin\theta\sin 2\theta + \tfrac{1}{2}R\cos\theta\cos 2\theta.
\end{aligned}$$

From the last of these

$$x = R\cos\theta\left[2\sin^2\theta + \tfrac{1}{2}(1 - 2\sin^2\theta)\right] = R\cos\theta(\tfrac{1}{2} + \sin^2\theta),$$

and re-substitution in $f(x, y, \theta) = 0$ then yields

$$\begin{aligned}
y\cos 2\theta &= R\cos\theta(\tfrac{1}{2} + \sin^2\theta)\sin 2\theta - R\sin\theta\\
&= R\cos^2\theta\sin\theta + 2R\cos^2\theta\sin^3\theta - R\sin\theta\\
&= R\sin^3\theta(2\cos^2\theta - 1).\\
\Rightarrow \quad y &= R\sin^3\theta.
\end{aligned}$$

These expressions for x and y are the stated parametric specification of the caustic.

Figure 5.2 The reflecting mirror discussed in exercise 5.24.

5.26 *Functions $P(V, T)$, $U(V, T)$ and $S(V, T)$ are related by*

$$T\,dS = d\,U + P\,dV,$$

where the symbols have the same meaning as in the previous question. P is known from experiment to have the form

$$P = \frac{T^4}{3} + \frac{T}{V},$$

in appropriate units. If

$$U = \alpha V T^4 + \beta T,$$

where α, β, are constants (or at least do not depend on T, V), deduce that α must have a specific value but β may have any value. Find the corresponding form of S.

Writing all the other quantities as functions of T and V, we express the total differential dS as

$$dS = \left(\frac{\partial S}{\partial V}\right)_T dV + \left(\frac{\partial S}{\partial T}\right)_V dT$$

and similarly for $d\,U$.

The given relationship then becomes

$$T\left(\frac{\partial S}{\partial V}\right)_T dV + T\left(\frac{\partial S}{\partial T}\right)_V dT = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT + P\,dV,$$

from which we can deduce that

$$T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P \quad \text{and} \quad T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

These two equations give us explicit expressions for $\left(\frac{\partial S}{\partial V}\right)_T$ and $\left(\frac{\partial S}{\partial T}\right)_V$.

Differentiating them again, the first with respect to T and the second with respect to V, and then using the fact that $\partial^2 S/\partial T \partial V$ and $\partial^2 S/\partial V \partial T$ must be equal, allows us to write

$$\left(\frac{\partial}{\partial T}\left[\frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T}\right]\right)_V = \left(\frac{\partial}{\partial V}\left[\frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V\right]\right)_T,$$
$$\left(\frac{\partial}{\partial T}\left[\alpha T^3 + \frac{T^3}{3} + \frac{1}{V}\right]\right)_V = \left(\frac{\partial}{\partial V}\left[4\alpha V T^2 + \frac{\beta}{T}\right]\right)_T,$$

i.e.

$$3\alpha T^2 + T^2 = 4\alpha T^2.$$

This necessary equality implies that we must have $\alpha = 1$, but it imposes no constraint on β.

Having now found explicit expressions for the two partial derivatives of S, integrating them will give two forms for $S(V, T)$. The two forms must be made to be mutually compatible, if they are not already so, by appropriate choices for the arbitrary functions of the non-integrated variable ($f(V)$ and $g(T)$, below) that are introduced by the integrations:

$$\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{T}\left(\frac{\partial U}{\partial T}\right)_V = 4VT^2 + \frac{\beta}{T},$$
$$\Rightarrow \quad S(V, T) = \tfrac{4}{3}VT^3 + \beta \ln T + f(V).$$

and

$$\left(\frac{\partial S}{\partial V}\right)_T = \frac{1}{T}\left(\frac{\partial U}{\partial V}\right)_T + \frac{P}{T} = T^3 + \frac{T^3}{3} + \frac{1}{V},$$
$$\Rightarrow \quad S(V, T) = \tfrac{4}{3}VT^3 + \ln V + g(T).$$

Clearly, $f(V)$ must be identified with $\ln V$ and $g(T)$ with $\beta \ln T$ to give as the full expression for S

$$S(V, T) = \tfrac{4}{3}VT^3 + \beta \ln T + \ln V + c,$$

where c is a constant and, as shown earlier, β is arbitrary.

5.28 *The entropy $S(H, T)$, the magnetisation $M(H, T)$ and the internal energy $U(H, T)$ of a magnetic salt placed in a magnetic field of strength H at temperature T are connected by the equation*

$$TdS = dU - HdM.$$

By considering $d(U - TS - HM)$ prove that

$$\left(\frac{\partial M}{\partial T}\right)_H = \left(\frac{\partial S}{\partial H}\right)_T.$$

For a particular salt

$$M(H, T) = M_0[1 - \exp(-\alpha H/T)].$$

Show that, at a fixed temperature, if the applied field is increased from zero to a strength such that the magnetization of the salt is $\frac{3}{4}M_0$ then the salt's entropy decreases *by an amount*

$$\frac{M_0}{4\alpha}(3 - \ln 4).$$

Given that $TdS = d\,U - HdM$, consider dF where $F = U - TS - HM$.

$$\begin{aligned} dF &= d(U - TS - HM) \\ &= d\,U - TdS - SdT - HdM - MdH \\ &= -SdT - MdH. \end{aligned}$$

It follows that

$$\left(\frac{\partial F}{\partial T}\right)_H = -S \quad \text{and} \quad \left(\frac{\partial F}{\partial H}\right)_T = -M.$$

Differentiating the first equation with respect to H and the second with respect to T yields

$$-\left(\frac{\partial S}{\partial H}\right)_T = \frac{\partial^2 F}{\partial H \partial T} = \frac{\partial^2 F}{\partial T \partial H} = -\left(\frac{\partial M}{\partial T}\right)_H,$$

thus establishing the equality stated in the question.

With

$$M(H, T) = M_0\left(1 - e^{-\alpha H/T}\right),$$

$$\left(\frac{\partial S}{\partial H}\right)_T = \left(\frac{\partial M}{\partial T}\right)_H = -\frac{M_0 \alpha H}{T^2} e^{-\alpha H/T}.$$

If the final field strength is H_1 then $e^{-\alpha H_1/T} = \frac{1}{4}$, or $H_1 = \alpha^{-1} T \ln 4$. The change

in entropy $\Delta S = S(H,T) - S(0,T)$ is given by

$$\begin{aligned}
\Delta S &= \int_0^{H_1} -\frac{M_0 \alpha H}{T^2} e^{-\alpha H/T}\, dH \\
&= -\frac{M_0 \alpha}{T^2} \left\{ \left[-\frac{HT}{\alpha} e^{-\alpha H/T} \right]_0^{H_1} + \int_0^{H_1} \frac{T}{\alpha} e^{-\alpha H/T}\, dH \right\} \\
&= -\frac{M_0 \alpha}{T^2} \left[-\frac{T^2}{4\alpha^2} \ln 4 + \left(\frac{T}{\alpha}\right)^2 \frac{3}{4} \right] \\
&= -\frac{M_0}{4\alpha}(3 - \ln 4),
\end{aligned}$$

i.e. the salt's entropy *decreases* by the stated amount.

5.30 *The integral*

$$\int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx$$

has the value $(\pi/\alpha)^{1/2}$. *Use this result to evaluate*

$$J(n) = \int_{-\infty}^{\infty} x^{2n} e^{-x^2}\, dx,$$

where n is a positive integer. Express your answer in terms of factorials.

We first observe that differentiating the given result with respect to α will introduce a factor of $-x^2$ into the integrand; doing so repeatedly will enable a factor of $(-1)^n x^{2n}$ to be generated. We therefore define a function $I(n,\alpha)$ by

$$I(n,\alpha) = \int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\, dx,$$

with $I(0,\alpha) = \sqrt{\pi/\alpha}$. The required $J(n)$ will be equal to $I(n,1)$.

We now carry out the n differentiations on the explicitly stated form of $I(0,\alpha)$ to generate an explicit form for $I(n,\alpha)$.

$$\begin{aligned}
I(n,\alpha) &= (-1)^n \frac{d^n I(0,\alpha)}{d\alpha^n} \\
&= (-1)^n \frac{d^n(\sqrt{\pi}\alpha^{-1/2})}{d\alpha^n} \\
&= (-1)^n \left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2n-1}{2}\right) \frac{\sqrt{\pi}}{\alpha^{n+1/2}} \\
&= \frac{(2n)!}{2^n\, 2^n\, n!} \frac{\sqrt{\pi}}{\alpha^{n+1/2}}.
\end{aligned}$$

Now, setting $\alpha = 1$, we obtain

$$J(n) = I(n, 1) = \frac{(2n)!\,\sqrt{\pi}}{4^n\, n!}.$$

5.32 *The functions $f(x,t)$ and $F(x)$ are defined by*

$$f(x,t) = e^{-xt},$$
$$F(x) = \int_0^x f(x,t)\,dt.$$

Verify by explicit calculation that

$$\frac{dF}{dx} = f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\,dt.$$

For the LHS

$$F(x) = \int_0^x e^{-xt}\,dt = \left[-\frac{e^{-xt}}{x}\right]_0^x = \frac{1 - e^{-x^2}}{x},$$

and hence

$$\frac{dF}{dx} = 2e^{-x^2} - \frac{1 - e^{-x^2}}{x^2}.$$

For the RHS, we start from

$$\frac{\partial f(x,t)}{\partial x} = -te^{-xt},$$

and so obtain

$$\begin{aligned}\int_0^x \frac{\partial f(x,t)}{\partial x}\,dt &= -\int_0^x te^{-xt}\,dt \\ &= -\left[-\frac{te^{-xt}}{x}\right]_0^x - \int_0^x \frac{e^{-xt}}{x}\,dt \\ &= \frac{xe^{-x^2}}{x} - 0 - \left[-\frac{e^{-xt}}{x^2}\right]_0^x \\ &= e^{-x^2} - \frac{1 - e^{-x^2}}{x^2}.\end{aligned}$$

Further $f(x,x) = e^{-x^2}$, and so

$$f(x,x) + \int_0^x \frac{\partial f(x,t)}{\partial x}\,dt = 2e^{-x^2} - \frac{1 - e^{-x^2}}{x^2} = \frac{dF}{dx},$$

as stated.

5.34 *Find the derivative with respect to x of the integral*

$$I(x) = \int_x^{3x} \exp xt\, dt.$$

Using the extension to Leibnitz' rule, we have

$$\begin{aligned}
I(x) &= \int_x^{3x} e^{xt}\, dt, \\
\frac{dI}{dx} &= \int_x^{3x} te^{xt}\, dt + 3\, e^{3x^2} - e^{x^2} \\
&= \left[\frac{te^{xt}}{x}\right]_x^{3x} - \int_x^{3x} \frac{e^{xt}}{x}\, dt + 3\, e^{3x^2} - e^{x^2} \\
&= 3\, e^{3x^2} - e^{x^2} - \left[\frac{e^{xt}}{x^2}\right]_x^{3x} + 3\, e^{3x^2} - e^{x^2} \\
&= 6\, e^{3x^2} - 2e^{x^2} - \frac{1}{x^2}(e^{3x^2} - e^{x^2}).
\end{aligned}$$

6

Multiple integrals

6.2 *Evaluate the volume integral of $x^2 + y^2 + z^2$ over the rectangular parallelepiped bounded by the six surfaces $x = \pm a$, $y = \pm b$, $z = \pm c$.*

This is a straightforward triple integral; the order of performing the integrations is arbitrary and for this integrand no particular one offers any special advantage.

$$\begin{aligned} I &= \int_{-a}^{a} dx \int_{-b}^{b} dy \int_{-c}^{c} (x^2 + y^2 + z^2)\, dz \\ &= 2 \int_{-a}^{a} dx \int_{-b}^{b} (cx^2 + cy^2 + \tfrac{1}{3}c^3)\, dy \\ &= 4 \int_{-a}^{a} \left(bcx^2 + \tfrac{1}{3}b^3c + \tfrac{1}{3}bc^3\right)\, dx \\ &= 8\left(\tfrac{1}{3}a^3bc + \tfrac{1}{3}ab^3c + \tfrac{1}{3}abc^3\right) \\ &= \tfrac{8}{3}abc(a^2 + b^2 + c^2). \end{aligned}$$

As would be expected, the result is symmetric in a, b and c.

6.4 *Evaluate the surface integral of $f(x, y)$ over the rectangle $0 \le x \le a$, $0 \le y \le b$ for the functions*

$$\text{(a) } f(x, y) = \frac{x}{x^2 + y^2}, \qquad \text{(b) } f(x, y) = (b - y + x)^{-3/2}.$$

(a) It is not clear which order of integration is to be preferred; integrating first

with respect to x will produce a logarithmic function whilst doing so with respect to y will generate an inverse tangent. We arbitrarily choose the former.

$$\begin{aligned} I &= \int_0^b dy \int_0^a \frac{x}{x^2+y^2}\,dx \\ &= \int_0^b \left[\tfrac{1}{2}\ln(x^2+y^2)\right]_0^a dy \\ &= \frac{1}{2}\int_0^b \ln\left(\frac{a^2+y^2}{y^2}\right) dy. \end{aligned}$$

In order to carry out the y integration we use the device of introducing an additional factor '1' into the integrand and then integrate by parts. By choosing this '1' as the term to be integrated, we obtain

$$\begin{aligned} I &= \frac{1}{2}\left[y\ln\left(\frac{a^2+y^2}{y^2}\right)\right]_0^b - \frac{1}{2}\int_0^b y\left(\frac{2y}{a^2+y^2} - \frac{2y}{y^2}\right) dy \\ &= \frac{b}{2}\ln\left(\frac{a^2+b^2}{b^2}\right) + \int_0^b \frac{a^2}{a^2+y^2}\,dy \\ &= \frac{b}{2}\ln\left(\frac{a^2+b^2}{b^2}\right) + a\tan^{-1}\left(\frac{b}{a}\right). \end{aligned}$$

Not surprisingly, the inverse tangent we avoided initially by our choice of integration order has popped up again!

(b) This integrand could be made to look more symmetric by writing $z = b - y$, but it is no more difficult to integrate it as it stands. We arbitrarily choose to perform the x-integration first.

$$\begin{aligned} \int_0^b dy \int_0^a \frac{1}{(b-y+x)^{3/2}}\,dx &= \int_0^b \left[\frac{-2}{(b-y+x)^{1/2}}\right]_{x=0}^{x=a} dy \\ &= 2\int_0^b \left[-\frac{1}{(b+a-y)^{1/2}} + \frac{1}{(b-y)^{1/2}}\right] dy \\ &= 2\left[2(b+a-y)^{1/2} - 2(b-y)^{1/2}\right]_0^b \\ &= 4\left[a^{1/2} - (b+a)^{1/2} + b^{1/2}\right]. \end{aligned}$$

In view of the opening comment, the symmetry of the answer with respect to a and b was to be expected.

6.6 *The function*

$$\Psi(r) = A\left(2 - \frac{Zr}{a}\right)e^{-Zr/2a}$$

gives the form of the quantum mechanical wavefunction representing the electron in a hydrogen-like atom of atomic number Z when the electron is in its first allowed spherically symmetric excited state. Here r is the usual spherical polar coordinate, but, because of the spherical symmetry, the coordinates θ and ϕ do not appear explicitly in Ψ. Determine the value that A (assumed real) must have if the wavefunction is to be correctly normalised, i.e. the volume integral of $|\Psi|^2$ over all space is equal to unity.

To evaluate the integral of $|\Psi|^2$ over all space we use spherical polar coordinates and, in this spherically symmetrical case, a volume element of $4\pi r^2\,dr$.

$$\begin{aligned}\int |\Psi|^2\,dV &= A^2\int_0^\infty 4\pi r^2\left(2 - \frac{Zr}{a}\right)^2 e^{-Zr/a}\,dr\\ &= 4\pi A^2\int_0^\infty\left(4r^2 - \frac{4Zr^3}{a} + \frac{Z^2r^4}{a^2}\right)e^{-Zr/a}\,dr\\ &= 4\pi A^2\left(4\frac{2!\,a^3}{Z^3} - 4\frac{3!\,Za^4}{aZ^4} + \frac{4!\,Z^2a^5}{a^2Z^5}\right)\\ &= \frac{32\pi A^2a^3}{Z^3}.\end{aligned}$$

Thus if the wavefunction is to be correctly normalised A must be taken as

$$A = \pm\frac{Z^{3/2}}{\sqrt{32\pi}\,a^{3/2}}.$$

6.8 *A planar figure is formed from uniform wire and consists of two semicircular arcs, each with its own closing diameter, joined so as to form a letter 'B'. The figure is freely suspended from its top left-hand corner. Show that the straight edge of the figure makes an angle θ with the vertical given by* $\tan\theta = (2+\pi)^{-1}$.

For each semi-circle, denote its radius by a, the linear density of the wire by ρ and the distance of its centre of gravity from its straight edge by d. Further, let the distance of the centre of gravity of the whole figure from its straight edge be $\bar{x}$. Then, since rotating a semi-circle about its straight edge produces a sphere, by Pappus' second theorem,

$$4\pi a^2 = 2\pi d \times \pi a, \qquad \Rightarrow \qquad d = \frac{2a}{\pi}.$$

Then, for the centre of gravity of the letter 'B',

$$\bar{x} = \frac{\left(2\pi a\rho \times \dfrac{2a}{\pi}\right) + (4a\rho \times 0)}{2\pi a\rho + 4a\rho} = \frac{4a}{2\pi + 4}.$$

When the wire letter is supended its centre of gravity will lie below the suspension point and the straight edge will make an angle θ with the vertical given by $\tan\theta = \bar{x}/2a$. Thus $\theta = \tan^{-1}[\,1/(\pi + 2)\,]$.

6.10 *A thin uniform circular disc has mass M and radius a.*

(a) *Prove that its moment of inertia about an axis perpendicular to its plane and passing through its centre is $\frac{1}{2}Ma^2$.*
(b) *Prove that the moment of inertia of the same disc about a diameter is $\frac{1}{4}Ma^2$.*

This is an example of the general result for planar bodies that the moment of inertia of the body about an axis perpendicular to the plane is equal to the sum of the moments of inertia about two perpendicular axes lying in the plane: in an obvious notation

$$I_z = \int r^2\,dm = \int (x^2 + y^2)\,dm = \int x^2\,dm + \int y^2\,dm = I_y + I_x.$$

Denote the mass per unit area of the disc by σ. Then, using plane polar coordinates (ρ, ϕ) or Cartesian coordinates (x, y), as appropriate, we find the moments of inertia of the disc about axes (a) perpendicular to its plane, and (b) about the y-axis as follows.

(a) $$I_\perp = \int_0^a \sigma\rho^2\, 2\pi\rho\, d\rho = \frac{2\pi\sigma a^4}{4} = \tfrac{1}{2}Ma^2.$$

(b) $$\begin{aligned} I_\parallel &= \int_{-a}^{a} \sigma x^2\, 2(a^2 - x^2)^{1/2}\, dx \\ &= 4\sigma \int_0^a x^2(a^2 - x^2)^{1/2}\, dx \\ &= 4\sigma \int_{\pi/2}^{0} a^2\cos^2\phi\; a\sin\phi\; (-a\sin\phi\, d\phi) \\ &= \sigma a^4 \int_0^{\pi/2} \sin^2 2\phi\, d\phi = \sigma a^4\, \tfrac{1}{2}\,\tfrac{\pi}{2} = \tfrac{1}{4}Ma^2. \end{aligned}$$

In the third line we set x equal to $a\cos\phi$.

6.12 *The shape of an axially symmetric hard-boiled egg, of uniform density ρ_0, is given in spherical polar coordinates by $r = a(2 - \cos\theta)$, where θ is measured from the axis of symmetry.*

(a) *Prove that the mass M of the egg is $M = \frac{40}{3}\pi\rho_0 a^3$.*
(b) *Prove that the egg's moment of inertia about its axis of symmetry is $\frac{342}{175}Ma^2$.*

(a) We need to consider slices of the egg perpendicular to the polar axis, the thickness of a typical slice being dz where $z = r\cos\theta$ and consequently

$$\begin{aligned} dz &= d(r\cos\theta) \\ &= d(2a\cos\theta - a\cos^2\theta) \\ &= 2a\sin\theta(\cos\theta - 1)\,d\theta \end{aligned}$$

Writing $\cos\theta$ as c in places to save space, the element of mass lying between z and $z + dz$ is

$$\begin{aligned} dm &= \pi\rho_0(r\sin\theta)^2\,dz \\ &= \pi\rho_0\, a^2\sin^2\theta\,(2-c)^2\; 2a\sin\theta(c-1)\,d\theta \\ &= 2\pi\rho_0 a^3\,(1-c^2)(2-c)^2(1-c)\,dc. \end{aligned}$$

This now has to be integrated between $c = 1$ and $c = -1$. Only those terms in the integrand which are even powers of c will give a non-zero contribution. Omitting the multiplicative constants, the integrand is

$$(1 - c - c^2 + c^3)(4 - 4c + c^2) = 4 + c^2 - 5c^4 + \text{ odd powers of } c.$$

Consequently, the value of the volume integral is

$$M = 2\pi\rho_0 a^3\; 2\left(4 + \frac{1}{3} - \frac{5}{5}\right) = \frac{40}{3}\pi\rho_0 a^3.$$

(b) The moment of inertia of a single slice of mass dm about the axis of symmetry is $\frac{1}{2}(r\sin\theta)^2\,dm$ and again this has to be integrated between $\theta = 0$ and $\theta = \pi$.

$$\begin{aligned} dI &= \tfrac{1}{2}(r\sin\theta)^2\,dm \\ &= \tfrac{1}{2}a^2(2-c)^2(1-c^2)\,2\pi\rho_0 a^3(1-c^2)(2-c)^2(1-c)\,dc \\ &= \pi\rho_0 a^5\,(1-c^2)^2(2-c)^4(1-c)\,dc \end{aligned}$$

The c-dependent terms in this integrand are

$$\begin{aligned} f(c) &= (1-c^2)^2(2-c)^4(1-c) \\ &= (1-c)(1-2c^2+c^4)(16-32c+24c^2-8c^3+c^4) \\ &= (1-c-2c^2+2c^3+c^4-c^5)(16-32c+24c^2-8c^3+c^4) \\ &= 16+c^2(24+32-32)+c^4(1+8-48-64+16) \\ &\qquad +c^6(-2-16+24+32)+c^8(1+8)+ \text{ odd powers of } c. \end{aligned}$$

As previously, this has to be integrated with respect to c between -1 and $+1$ with the odd powers of c contributing nothing. Thus the total moment of inertia is

$$\begin{aligned} I &= \pi\rho_0 a^5\, 2\left(16+\frac{24}{3}-\frac{87}{5}+\frac{38}{7}+\frac{9}{9}\right) \\ &= \frac{912}{35}\pi\rho_0 a^5 \\ &= \frac{342}{175}Ma^2. \end{aligned}$$

6.14 *By expressing both the integrand and the surface element in spherical polar coordinates, show that the surface integral*

$$\int \frac{x^2}{x^2+y^2}\,dS$$

over the surface $x^2+y^2=z^2$, $0 \le z \le 1$, *has the value* $\pi/\sqrt{2}$.

The surface S is an inverted cone of unit height and half angle $\pi/4$. Since a cone is a coordinate surface ($\theta =$ constant) in spherical polar coordinates, we change to that system.

In these coordinates the surface is given by $0 \le r \le \sqrt{2}$, $\theta = \pi/4$ and $0 \le \phi \le 2\pi$. The integrand is $[r^2\sin^2(\pi/4)\cos^2\phi]/[r^2\sin^2(\pi/4)]$ whilst the surface element is $dS = r\sin(\pi/4)\,d\phi\,dr$. Thus the integral I is given by

$$\begin{aligned} I &= \int_0^{2\pi}\cos^2\phi\,d\phi\int_0^{\sqrt{2}}\frac{r}{\sqrt{2}}\,dr \\ &= \frac{2\pi}{2}\left[\frac{r^2}{2\sqrt{2}}\right]_0^{\sqrt{2}} \\ &= \frac{\pi}{\sqrt{2}}. \end{aligned}$$

Figure 6.1 The parabolic coordinate curves discussed in exercise 6.16.

6.16 *Sketch the two families of curves*

$$y^2 = 4u(u - x), \qquad y^2 = 4v(v + x),$$

where u and v are parameters.

By transforming to the uv-plane, evaluate the integral of $y/(x^2 + y^2)^{1/2}$ *over that part of the quadrant* $x > 0$, $y > 0$ *bounded by the lines* $x = 0$, $y = 0$ *and the curve* $y^2 = 4a(a - x)$.

Sketches of typical curves are shown in figure 6.1. Each family is a set of non-intersecting parabolas with the x-axis as the axis of symmetry. However, each u-curve meets each v-curve in two places and vice versa.

The area over which the integral is to be taken has the points $(0, 0)$, $(a, 0)$ and $(0, 2a)$ as its 'corners' and is shown shaded in the figure for the case $a = 2$. We transform to the uv-plane where:

(i) The boundary $x = 0$, $y > 0$ becomes $4u^2 = y^2 = 4v^2$, i.e. $u = v$.
(ii) The boundary $x > 0$, $y = 0$ becomes $v = 0$.
(iii) The boundary $y^2 = 4a(a - x)$ becomes $u = a$.

In this plane the integration region is thus a right-angled triangle with vertices at $(0,0)$, $(a,0)$ and (a,a).

The equations of the transformation can be rewritten as $x = u - v$ and $y = 2\sqrt{uv}$, making the Jacobian

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & \sqrt{\dfrac{v}{u}} \\ -1 & \sqrt{\dfrac{u}{v}} \end{vmatrix} = \sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}}.$$

The integral can therefore be transformed to one over the triangular region and evaluated as follows.

$$\begin{aligned} I &= \int_S \frac{y}{x^2+y^2}\,dx\,dy \\ &= \int_0^a du \int_0^u \frac{2\sqrt{uv}}{(u^2-2uv+v^2+4uv)^{1/2}} \left(\sqrt{\frac{u}{v}} + \sqrt{\frac{v}{u}}\right) dv \\ &= \int_0^a du \int_0^u \frac{2(u+v)}{u+v}\,dv \\ &= \int_0^a 2u\,du \\ &= 2\left[\frac{u^2}{2}\right]_0^a = a^2. \end{aligned}$$

6.18 *Sketch the domain of integration for the integral*

$$I = \int_0^1 \int_{x=y}^{1/y} \frac{y^3}{x} \exp[y^2(x^2+x^{-2})]\,dx\,dy$$

and characterise its boundaries in terms of new variables $u = xy$ and $v = y/x$. Show that the Jacobian for the change from (x,y) to (u,v) is equal to $(2v)^{-1}$, and hence evaluate I.

The integration area is shown shaded in figure 6.2. In terms of the new variables, $u = xy$ and $v = y/x$, the original variables are $x = (u/v)^{1/2}$ and $y = (uv)^{1/2}$.
(i) The boundary $y = 0$, $0 < x < \infty$ becomes both $u = 0$ and $v = 0$.
(ii) The boundary $y = x$ becomes $v = 1$.
(iii) The boundary $x = 1/y$ becomes $u = 1$.

The Jacobian of the transformation is

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} = \frac{1}{2(uv)^{1/2}}\frac{u^{1/2}}{2v^{1/2}} - \frac{(-1)u^{1/2}}{2v^{3/2}}\frac{v^{1/2}}{2u^{1/2}} = \frac{1}{2v}.$$

Figure 6.2 The integration area for exercise 6.18 is shown shaded.

Making the change of variables and then integrating gives

$$
\begin{aligned}
I &= \int_0^1 \int_{x=y}^{x=1/y} \frac{y^3}{x} \exp\left[y^2(x^2+x^{-2})\right]\, dx\, dy \\
&= \int_0^1 \int_0^1 (uv)^{3/2} \left(\frac{v}{u}\right)^{1/2} \exp\left[uv\left(\frac{u}{v}+\frac{v}{u}\right)\right] \frac{1}{2v}\, du\, dv \\
&= \int_0^1 \int_0^1 \frac{uv}{2} \exp(u^2+v^2)\, du\, dv \\
&= \frac{1}{2}\int_0^1 u\exp(u^2)\, du \int_0^1 v\exp(v^2)\, dv \\
&= \tfrac{1}{8}\left[\exp(u^2)\right]_0^1 \left[\exp(v^2)\right]_0^1 \\
&= \tfrac{1}{8}(e-1)^2.
\end{aligned}
$$

6.20 *Define a coordinate system u, v whose origin coincides with that of the usual x, y system and whose u-axis coincides with the x-axis, whilst the v-axis makes an angle α with it. By considering the integral $I = \int \exp(-r^2)\, dA$, where r is the radial distance from the origin, over the area defined by $0 \le u < \infty$, $0 \le v < \infty$, prove that*

$$\int_0^\infty \int_0^\infty \exp(-u^2 - v^2 - 2uv\cos\alpha)\, du\, dv = \frac{\alpha}{2\sin\alpha}.$$

As can be seen from figure 6.3, the coordinates of a general point P that lies in

Figure 6.3 The coordinate system for exercise 6.20.

the area defined by positive values for u and v are related in the two systems by

$$x = v\cos\alpha + u \quad \text{and} \quad y = v\sin\alpha.$$

The Jacobian of a coordinate transformation between the two systems is therefore

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1 & 0 \\ \cos\alpha & \sin\alpha \end{vmatrix} = \sin\alpha.$$

We note that this value for the Jacobian does not depend upon the actual position of P.

Now, because of azimuthal symmetry, the integral of $\exp(-r^2)$ over the region of positive u and v (shown shaded in the figure) is $\alpha/2\pi$ of the same integral taken over the whole of the xy-space. This latter is

$$\int e^{-r^2}\,dA = \int\int e^{-(x^2+y^2)}\,dx\,dy = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy = (\sqrt{\pi})^2.$$

Expressed in terms of u and v, $r^2 = x^2 + y^2 = u^2 + v^2 + 2uv\cos\alpha$. As shown above, $dx\,dy = \sin\alpha\,du\,dv$ and so the integral over the shaded region takes the form

$$\int_0^{\infty}\int_0^{\infty} \exp(-u^2 - v^2 - 2uv\cos\alpha)\sin\alpha\,du\,dv.$$

This integral therefore has the value $(\alpha/2\pi)\times\pi$ and the stated result about the integral in the question follows when both the itegral and its value are divided by the constant $\sin\alpha$.

6.22 *The distances of the variable point P, which has coordinates x, y, z, from the fixed points $(0, 0, 1)$ and $(0, 0, -1)$ are denoted by u and v respectively. New variables ξ, η, ϕ are defined by*

$$\xi = \tfrac{1}{2}(u + v), \qquad \eta = \tfrac{1}{2}(u - v),$$

and ϕ is the angle between the plane $y = 0$ and the plane containing the three points. Prove that the Jacobian $\partial(\xi, \eta, \phi)/\partial(x, y, z)$ has the value $(\xi^2 - \eta^2)^{-1}$ and that

$$\int\int\int_{\text{all space}} \frac{(u-v)^2}{uv} \exp\left(-\frac{u+v}{2}\right) dx\, dy\, dz = \frac{16\pi}{3e}.$$

From straightforward algebraic geometry, u and v are given by the positive square roots of

$$u^2 = x^2 + y^2 + (z-1)^2 \quad \text{and} \quad v^2 = x^2 + y^2 + (z+1)^2.$$

The new variables and their ranges are

$$\begin{aligned} \xi &= \tfrac{1}{2}(u+v) \quad \text{over} \quad 1 \le \xi < \infty, \\ \eta &= \tfrac{1}{2}(u-v) \quad \text{over} \quad -1 \le \eta < 1, \\ \phi &= \tan^{-1}\frac{y}{x} \quad \text{over} \quad 0 \le \phi < 2\pi. \end{aligned}$$

We start by calculating

$$\frac{\partial \xi}{\partial x} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\right) = \frac{1}{2}\left(\frac{x}{u} + \frac{x}{v}\right) = \frac{x\xi}{uv}.$$

Similarly for $\partial\xi/\partial y$, $\partial\eta/\partial x$ and $\partial\eta/\partial y$.

The other required derivatives are

$$\begin{aligned} \frac{\partial \xi}{\partial z} &= \frac{1}{2}\left(\frac{z-1}{u} + \frac{z+1}{v}\right) = \frac{z\xi + \eta}{uv}, \\ \frac{\partial \eta}{\partial z} &= \frac{1}{2}\left(\frac{z-1}{u} - \frac{z+1}{v}\right) = \frac{-z\eta - \xi}{uv}, \\ \frac{\partial \phi}{\partial x} &= \frac{-y}{x^2}\,\frac{1}{1 + \frac{y^2}{x^2}} = -\frac{y}{x^2 + y^2}, \\ \frac{\partial \phi}{\partial y} &= \frac{1}{x}\,\frac{1}{1 + \frac{y^2}{x^2}} = \frac{x}{x^2 + y^2}. \end{aligned}$$

Collecting these together gives the Jacobian as

$$\frac{\partial(\xi,\eta,\phi)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{x\xi}{uv} & -\dfrac{x\eta}{uv} & -\dfrac{y}{x^2+y^2} \\ \dfrac{y\xi}{uv} & -\dfrac{y\eta}{uv} & \dfrac{x}{x^2+y^2} \\ \dfrac{z\xi+\eta}{uv} & -\dfrac{z\eta+\xi}{uv} & 0 \end{vmatrix}$$

$$= \frac{-1}{(uv)^2}\,\frac{1}{x^2+y^2} \begin{vmatrix} x\xi & x\eta & -y \\ y\xi & y\eta & x \\ z\xi+\eta & z\eta+\xi & 0 \end{vmatrix}$$

$$= \frac{-1}{(uv)^2}\,\frac{1}{x^2+y^2} \begin{vmatrix} 0 & x\eta & -y \\ 0 & y\eta & x \\ \dfrac{\eta^2-\xi^2}{\eta} & z\eta+\xi & 0 \end{vmatrix}$$

$$= \frac{-(\eta^2-\xi^2)(x^2+y^2)}{(uv)^2(x^2+y^2)} = \frac{\xi^2-\eta^2}{(uv)^2}.$$

But $uv = \xi^2 - \eta^2$ and so the Jacobian has the value $(\xi^2-\eta^2)^{-1}$. To obtain the third line of the above evaluation of the Jacobian we subtracted ξ/η times the 2nd column from the 1st column.

We now express the given integral in terms of ξ and η:

$$I = \int\int\int_{\text{all space}} \frac{(u-v)^2}{uv} \exp\left(-\frac{u+v}{2}\right) dx\,dy\,dz$$

$$= \int_0^{2\pi} d\phi \int_1^{\infty} d\xi \int_{-1}^{1} \frac{4\eta^2}{\xi^2-\eta^2} e^{-\xi} \frac{1}{(\xi^2-\eta^2)^{-1}}\, d\eta$$

$$= 8\pi \int_1^{\infty} e^{-\xi}\, d\xi \int_{-1}^{1} \eta^2\, d\eta$$

$$= 8\pi\, e^{-1}\, \frac{2}{3} = \frac{16\pi}{3e},$$

as stated in the question.

7

Vector algebra

7.2 *A unit cell of diamond is a cube of side A with carbon atoms at each corner, at the centre of each face and, in addition, at positions displaced by* $\frac{1}{4}A(\mathbf{i}+\mathbf{j}+\mathbf{k})$ *from each of those already mentioned;* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ *are unit vectors along the cube axes. One corner of the cube is taken as the origin of coordinates. What are the vectors joining the atom at* $\frac{1}{4}A(\mathbf{i}+\mathbf{j}+\mathbf{k})$ *to its four nearest neighbours? Determine the angle between the carbon bonds in diamond.*

The four nearest neighbours are positioned at

$$A(0,0,0),\ A(\tfrac{1}{2},\tfrac{1}{2},0),\ A(0,\tfrac{1}{2},\tfrac{1}{2}) \text{ and } A(\tfrac{1}{2},0,\tfrac{1}{2}).$$

The corresponding vectors joining them to the atom at $A(\frac{1}{4},\frac{1}{4},\frac{1}{4})$ are

$$\frac{A}{4}(-1,-1,-1),\quad \frac{A}{4}(1,1,-1),\quad \frac{A}{4}(-1,1,1),\quad \frac{A}{4}(1,-1,1).$$

The length of each vector is $\sqrt{3}A/4$ and so the angle between any two bonds (say the first and second) is

$$\theta = \cos^{-1}\frac{\frac{A^2}{16}(-1-1+1)}{\left(\frac{\sqrt{3}A}{4}\right)^2} = \cos^{-1}\left(\frac{-1}{3}\right) = 109.5^\circ.$$

7.4 *Find the angle between the position vectors to the points* $(3,-4,0)$ *and* $(-2,1,0)$ *and find the direction cosines of a vector perpendicular to both.*

If θ is the angle between the vectors $\mathbf{a} = (3,-4,0)$ and $\mathbf{b} = (-2,1,0)$ then its cosine is given by

$$\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{a\,b} = \frac{-6-4+0}{5\sqrt{5}} = \frac{-2}{\sqrt{5}},$$

giving $\theta = 153.4°$.

A vector perpendicular to both **a** and **b** is their cross product

$$\mathbf{a} \times \mathbf{b} = (0-0,\ 0-0,\ 3-8) = (0, 0, -5).$$

The normalised cross product is $(0, 0, -1)$ whose components therefore are the required direction cosines. Clearly $(0, 0, 1)$ is an equally valid vector perpendicular to both **a** and **b**.

7.6 *Use vector methods to prove that the lines joining the mid-points of the opposite edges of a tetrahedron $OABC$ meet at a point and that this point bisects each of the lines.*

Let the vertices of the tetrahedron have vector positions **0**, **a**, **b** and **c**. The mid-points of the pair of opposite sides OA and BC are $\frac{1}{2}(\mathbf{0} + \mathbf{a})$ and $\frac{1}{2}(\mathbf{b} + \mathbf{c})$, respectively. The mid-point of the line joining these two points is, similarly, $\frac{1}{2}[\frac{1}{2}(\mathbf{0} + \mathbf{a}) + \frac{1}{2}(\mathbf{b} + \mathbf{c})] = \frac{1}{4}(\mathbf{a} + \mathbf{b} + \mathbf{c})$.

From the symmetry of this expression it is clear that the same result would be obtained by considering the pair of sides OB and AC, or the pair of sides OC and AB. Thus the lines joining the mid-points of all pairs of opposite edges meet at this one point, which bisects each of them.

7.8 *Prove, by writing it out in component form, that*

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a},$$

and deduce the result, stated in (7.25), that the operation of forming the vector product is non-associative.

We compute only the x-component of each side of the equation. The corresponding results for other components can be obtained by cyclic permutation of x, y and z.

$$\begin{aligned}
\mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x) \\
[(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]_x &= (a_z b_x - a_x b_z)c_z - (a_x b_y - a_y b_x)c_y \\
&= b_x(a_z c_z + a_y c_y) - a_x(b_z c_z + b_y c_y) \\
&= b_x(a_z c_z + a_y c_y + a_x c_x) - a_x(b_x c_x + b_z c_z + b_y c_y) \\
&= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}]_x\,.
\end{aligned}$$

To obtain the penultimate line we both added and subtracted $a_x b_x c_x$ on the

RHS. This establishes the result for the x-component and hence for all three components.

We have shown that

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}.$$

Now consider

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = -(\mathbf{b} \times \mathbf{c}) \times \mathbf{a} = -(\mathbf{b} \cdot \mathbf{a})\mathbf{c} + (\mathbf{c} \cdot \mathbf{a})\mathbf{b} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{a})\mathbf{c}.$$

The last terms on the RHSs of the two equations are not equal, showing that $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} \neq \mathbf{a} \times (\mathbf{b} \times \mathbf{c})$.

7.10 *For four arbitrary vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$, *evaluate*

$$(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d})$$

in two different ways and so prove that

$$\mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,] - \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] + \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,] = \mathbf{0}.$$

Show that this reduces to the normal Cartesian representation of the vector $\mathbf{d}$, *i.e.* $d_x\mathbf{i} + d_y\mathbf{j} + d_z\mathbf{k}$, *if* $\mathbf{a}, \mathbf{b}$ *and* $\mathbf{c}$ *are taken as* $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$, *the Cartesian base vectors.*

Firstly, treating the given expression as the triple vector product of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c} \times \mathbf{d}$,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{b}\,[\,\mathbf{a} \cdot (\mathbf{c} \times \mathbf{d})] - \mathbf{a}\,[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{d})] \\ &= \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] - \mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,].\end{aligned}$$

Secondly, treating the given expression as the triple vector product of $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$,

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{d}) &= \mathbf{c}\,[\,\mathbf{d} \cdot (\mathbf{a} \times \mathbf{b})] - \mathbf{d}\,[\,\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})] \\ &= \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,]\end{aligned}$$

Now, equating these two expressions gives the stated result; explicitly,

$$\mathbf{a}\,[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,] - \mathbf{b}\,[\,\mathbf{c}, \mathbf{d}, \mathbf{a}\,] + \mathbf{c}\,[\,\mathbf{d}, \mathbf{a}, \mathbf{b}\,] - \mathbf{d}\,[\,\mathbf{a}, \mathbf{b}, \mathbf{c}\,] = 0.$$

Setting $\mathbf{a} = \mathbf{i}$, $\mathbf{b} = \mathbf{j}$ and $\mathbf{c} = \mathbf{k}$ reduces the above equation to

$$\mathbf{i}\,[\,\mathbf{j}, \mathbf{k}, \mathbf{d}\,] - \mathbf{j}\,[\,\mathbf{k}, \mathbf{d}, \mathbf{i}\,] + \mathbf{k}\,[\,\mathbf{d}, \mathbf{i}, \mathbf{j}\,] - \mathbf{d}\,[\,\mathbf{i}, \mathbf{j}, \mathbf{k}\,] = 0,$$

which, since $[\,\mathbf{i}, \mathbf{j}, \mathbf{k}\,] = 1$, reduces to

$$\mathbf{d} = \mathbf{i}\,d_x - \mathbf{j}\,(-d_y) + \mathbf{k}\,d_z = d_x\,\mathbf{i} + d_y\,\mathbf{j} + d_z\,\mathbf{k}.$$

7.12 *The plane P_1 contains the points A, B and C, which have position vectors $\mathbf{a} = -3\mathbf{i} + 2\mathbf{j}$, $\mathbf{b} = 7\mathbf{i} + 2\mathbf{j}$ and $\mathbf{c} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$ respectively. Plane P_2 passes through A and is orthogonal to the line BC, whilst plane P_3 passes through B and is orthogonal to the line AC. Find the coordinates of $\mathbf{r}$, the point of intersection of the three planes.*

Since both $\mathbf{b} - \mathbf{a}$ and $\mathbf{c} - \mathbf{a}$ lie in P_1, a normal to that plane is in the direction of

$$(\mathbf{b} - \mathbf{a}) \times (\mathbf{c} - \mathbf{a}) = (10, 0, 0) \times (5, 1, 2) = (0, -20, 10).$$

The equation of P_1 is therefore of the form $-2y + z = c$ and, since A lies on it, $c = -4$.

The specification for P_2 takes the form

$$(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{b} - \mathbf{c}) = 0 \quad \text{or} \quad \mathbf{r} \cdot (\mathbf{b} - \mathbf{c}) = \mathbf{a} \cdot (\mathbf{b} - \mathbf{c}).$$

Thus

$$(x, y, z) \cdot (5, -1, -2) = (-3, 2, 0) \cdot (5, -1, -2) = -17 \quad \text{or} \quad 5x - y - 2z = -17.$$

For P_3

$$(\mathbf{r} - \mathbf{b}) \cdot (\mathbf{c} - \mathbf{a}) = 0 \quad \text{or} \quad \mathbf{r} \cdot (\mathbf{c} - \mathbf{a}) = \mathbf{b} \cdot (\mathbf{c} - \mathbf{a}),$$

leading to

$$(x, y, z) \cdot (5, 1, 2) = (7, 2, 0) \cdot (5, 1, 2) = 37 \quad \text{or} \quad 5x + y + 2z = 37.$$

Solving the three equations for P_1, P_2 and P_3 simultaneously gives the coordinates of the point of intersection. By adding the equations for P_2 and P_3 we obtain $x = 2$. Then using either of these equations and that for P_1 yields $y = 7$ and $z = 10$.

7.14 *Two fixed points, A and B, in three-dimensional space have position vectors $\mathbf{a}$ and $\mathbf{b}$. Identify the plane P given by*

$$(\mathbf{a} - \mathbf{b}) \cdot \mathbf{r} = \tfrac{1}{2}(a^2 - b^2),$$

where a and b are the magnitudes of $\mathbf{a}$ and $\mathbf{b}$.
Show also that the equation

$$(\mathbf{a} - \mathbf{r}) \cdot (\mathbf{b} - \mathbf{r}) = 0$$

describes a sphere S of radius $|\mathbf{a} - \mathbf{b}|/2$. Deduce that the intersection of P and S is also the intersection of two spheres, centred on A and B and each of radius $|\mathbf{a} - \mathbf{b}|/\sqrt{2}$.

The normal to the plane P is clearly in the direction $\mathbf{a}-\mathbf{b}$ and furthermore the point $\mathbf{r}=\frac{1}{2}(\mathbf{a}+\mathbf{b})$ satisfies

$$(\mathbf{a}-\mathbf{b})\cdot\tfrac{1}{2}(\mathbf{a}+\mathbf{b})=\tfrac{1}{2}(a^2-b^2),$$

and so lies in the plane. The plane must therefore be orthogonal to the line joining A to B and pass through its mid-point.

From the given equation

$$\begin{aligned}(\mathbf{a}-\mathbf{r})\cdot(\mathbf{b}-\mathbf{r})&=0,\\ \mathbf{r}\cdot\mathbf{r}-(\mathbf{a}+\mathbf{b})\cdot\mathbf{r}+\mathbf{a}\cdot\mathbf{b}&=0,\\ \left[\mathbf{r}-\tfrac{1}{2}(\mathbf{a}+\mathbf{b})\right]^2&=-\mathbf{a}\cdot\mathbf{b}+[\tfrac{1}{2}(\mathbf{a}+\mathbf{b})]^2\\ &=[\tfrac{1}{2}(\mathbf{a}-\mathbf{b})]^2.\end{aligned}$$

Thus the equation describes a sphere S of radius $\frac{1}{2}|\mathbf{a}-\mathbf{b}|$ centred on the point $\frac{1}{2}(\mathbf{a}+\mathbf{b})$. It has AB as a diameter.

Now consider the (circular) intersection of P and S, given by solving their equations simultaneously:

$$\begin{aligned}\mathbf{a}\cdot\mathbf{r}-\mathbf{b}\cdot\mathbf{r}&=\tfrac{1}{2}(a^2-b^2),\\ r^2+\mathbf{a}\cdot\mathbf{b}-\mathbf{a}\cdot\mathbf{r}-\mathbf{b}\cdot\mathbf{r}&=0.\end{aligned}$$

Subtracting them gives

$$\begin{aligned}r^2+\mathbf{a}\cdot\mathbf{b}-2\mathbf{a}\cdot\mathbf{r}&=\tfrac{1}{2}(b^2-a^2),\\ (\mathbf{r}-\mathbf{a})^2&=\tfrac{1}{2}(b^2-a^2)-\mathbf{a}\cdot\mathbf{b}+a^2\\ &=\tfrac{1}{2}(b^2+a^2)-\mathbf{a}\cdot\mathbf{b}\\ &=\tfrac{1}{2}(\mathbf{b}-\mathbf{a})^2.\end{aligned}$$

Adding them gives

$$\begin{aligned}r^2+\mathbf{a}\cdot\mathbf{b}-2\mathbf{b}\cdot\mathbf{r}&=\tfrac{1}{2}(a^2-b^2),\\ (\mathbf{r}-\mathbf{b})^2&=\tfrac{1}{2}(a^2-b^2)-\mathbf{a}\cdot\mathbf{b}+b^2\\ &=\tfrac{1}{2}(b^2+a^2)-\mathbf{a}\cdot\mathbf{b}\\ &=\tfrac{1}{2}(\mathbf{a}-\mathbf{b})^2.\end{aligned}$$

The two deduced equations satisfied by the points that lie on the intersection of P and S are those of spheres of equal radius $|\mathbf{a}-\mathbf{b}|/\sqrt{2}$, one centred on A and the other on B. Thus the intersection of the plane and sphere is also the intersection of two equal (larger) spheres whose centres are A and B.

7.16 *The vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are coplanar and related by*

$$\lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c} = 0,$$

where λ, μ, ν *are not all zero. Show that the condition for the points with position vectors* $\alpha\mathbf{a}$, $\beta\mathbf{b}$ *and* $\gamma\mathbf{c}$ *to be collinear is*

$$\frac{\lambda}{\alpha} + \frac{\mu}{\beta} + \frac{\nu}{\gamma} = 0.$$

We assume that $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are not simply multiples of each other.

For collinearity of the three points we must have

$$\gamma\mathbf{c} = \theta\alpha\mathbf{a} + (1-\theta)\beta\mathbf{b}$$

for some θ. Thus

$$\lambda\mathbf{a} + \mu\mathbf{b} + \frac{\theta\alpha\nu}{\gamma}\mathbf{a} + \frac{(1-\theta)\beta\nu}{\gamma}\mathbf{b} = \mathbf{0},$$

implying that

$$\lambda + \frac{\theta\alpha\nu}{\gamma} = 0 \qquad \text{and} \qquad \mu + \frac{(1-\theta)\beta\nu}{\gamma} = 0.$$

Eliminating θ and then dividing through by $\beta\gamma$ gives

$$\gamma\mu + \beta\nu - \beta\nu\left(\frac{-\lambda\gamma}{\alpha\nu}\right) = 0 \quad \Rightarrow \quad \frac{\mu}{\beta} + \frac{\nu}{\gamma} + \frac{\lambda}{\alpha} = 0,$$

which is therefore a necessary condition for the collinearity of the three points.

7.18 *Four points* X_i, $i = 1, 2, 3, 4$, *taken for simplicity as all lying within the octant* $x, y, z \geq 0$, *have position vectors* $\mathbf{x}_i$. *Convince yourself that the direction of vector* $\mathbf{x}_n$ *lies within the sector of space defined by the directions of the other three vectors if*

$$\min_{\text{over } j}\left[\frac{\mathbf{x}_i \cdot \mathbf{x}_j}{|\mathbf{x}_i||\mathbf{x}_j|}\right],$$

considered for $i = 1, 2, 3, 4$ *in turn, takes its maximum value for* $i = n$, *i.e.* n *equals that value of* i *for which the largest of the set of angles which* $\mathbf{x}_i$ *makes with the other vectors is found to be the lowest. Determine whether any of the four points with coordinates*

$$X_1 = (3, 2, 2), \quad X_2 = (2, 3, 1), \quad X_3 = (2, 1, 3), \quad X_4 = (3, 0, 3)$$

lies within the tetrahedron defined by the origin and the other three points.

Suppose that, for some n, x_n lies within the sector defined by the other three vectors. Then each of the other three vectors must make a larger angle with at least one of the other remaining two than it does with x_n. Since a larger angle between unit vectors corresponds to a smaller value of their scalar product, s_{ij}, this requirement can be expressed as in the question. Clearly, at most one of the vectors can satisfy the geometrical condition; if none of the vectors does so then the same scalar product will appear as the minimum for two different values of i and be the largest such minimum.

For the given points the table of scalar products is as follows.

s_{ij}	X_1	X_2	X_3	X_4	Minimum
X_1	1	0.907	0.907	0.857	0.857
X_2	0.907	1	0.714	0.567	0.567
X_3	0.907	0.714	1	0.945	0.714
X_4	0.857	0.567	0.945	1	0.567

The largest minimum occurs uniquely in the line corresponding to X_1 whose direction is therefore contained in the sector defined by the directions of X_2, X_3 and X_4.

To establish whether X_1 lies inside the tetrahedron defined by the origin, X_2, X_3 and X_4, we need to determine whether or not it lies on the same side of the plane P, defined by X_2, X_3 and X_4, as the origin.
The normal to P is given by

$$(\mathbf{X}_3 - \mathbf{X}_2) \times (\mathbf{X}_4 - \mathbf{X}_2) = (0, -2, 2) \times (1, -3, 2) = (2, 2, 2) = \frac{2}{\sqrt{3}}(1, 1, 1)$$

and therefore, since it contains X_2, the equation of the plane is

$$f(x, y, z) = x + y + z - 6 = 0.$$

At the origin the value of $f(x, y, z)$ is $-6 < 0$, whilst at X_1 it is $f(3, 2, 2) = 7-6 > 0$; therefore the origin and X_1 are on opposite sides of P and it follows that X_1 does not lie inside the tetrahedron.

7.20 *Three non-coplanar vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *have as their respective reciprocal vectors the set* $\mathbf{a}'$, $\mathbf{b}'$ *and* $\mathbf{c}'$. *Show that the normal to the plane containing the points* $k^{-1}\mathbf{a}$, $l^{-1}\mathbf{b}$ *and* $m^{-1}\mathbf{c}$ *is in the direction of the vector* $k\mathbf{a}' + l\mathbf{b}' + m\mathbf{c}'$.

The plane containing $k^{-1}\mathbf{a}$, $l^{-1}\mathbf{b}$ and $m^{-1}\mathbf{c}$ is $\mathbf{r} = \alpha k^{-1}\mathbf{a} + \beta l^{-1}\mathbf{b} + \gamma m^{-1}\mathbf{c}$, where the scalar quantities α, β and γ satisfy the relationship $\alpha + \beta + \gamma = 1$.

The normal to this plane is in the direction

$$\begin{aligned}\mathbf{n} &= (l^{-1}\mathbf{b} - k^{-1}\mathbf{a}) \times (m^{-1}\mathbf{c} - k^{-1}\mathbf{a}) \\ &= (lm)^{-1}(\mathbf{b} \times \mathbf{c}) - (km)^{-1}(\mathbf{a} \times \mathbf{c}) - (lk)^{-1}(\mathbf{b} \times \mathbf{a}) \\ &= \frac{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}{klm}(k\mathbf{a}' + l\mathbf{b}' + m\mathbf{c}').\end{aligned}$$

To obtain the last line we used the definitions of the reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$ and $\mathbf{c}'$, as given in section 7.9.

7.22 *In subsection 7.6.2 we showed how the moment or torque of a force about an axis could be represented by a vector in the direction of the axis. The magnitude of the vector gives the size of the moment and the sign of the vector gives the sense. Similar representations can be used for angular velocities and angular momenta.*

(a) *The magnitude of the angular momentum about the origin of a particle of mass m moving with velocity $\mathbf{v}$ on a path that is a perpendicular distance d from the origin is given by $m|\mathbf{v}|d$. Show that if $\mathbf{r}$ is the position of the particle then the vector $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$ represents the angular momentum.*

(a) *Now consider a rigid collection of particles (or a solid body) rotating about an axis through the origin, the angular velocity of the collection being represented by $\boldsymbol{\omega}$.*

(i) *Show that the velocity of the ith particle is*

$$\mathbf{v}_i = \boldsymbol{\omega} \times \mathbf{r}_i$$

and that the total angular momentum $\mathbf{J}$ is

$$\mathbf{J} = \sum_i m_i[r_i^2\,\boldsymbol{\omega} - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i].$$

(ii) *Show further that the component of $\mathbf{J}$ along the axis of rotation can be written as $I\boldsymbol{\omega}$, where I, the moment of inertia of the collection about the axis or rotation, is given by*

$$I = \sum_i m_i\rho_i^2.$$

Interpret ρ_i geometrically.

(iii) *Prove that the total kinetic energy of the particles is $\frac{1}{2}I\omega^2$.*

(a) The magnitude of the angular momentum is $m|\mathbf{v}|d$ (see figure 7.1(a)) and, as drawn in the figure, its sense is downwards. Now consider $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$. This also has magnitude $J = m|\mathbf{v}|r\sin\theta = m|\mathbf{v}|d$, and, as shown in the figure, is directed downwards; it is therefore a vector expression for the angular momentum.

Figure 7.1 The vectors discussed in exercise 7.22. (a) The vector representation of angular momentum, $\mathbf{J} = \mathbf{r} \times m\mathbf{v}$. (b) The linear velocity of the ith particle in a rotating rigid body is given by $\boldsymbol{\omega}\times\mathbf{r}_i$.

(b)(i) As can be seen from figure (b), the velocity of the ith particle has magnitude ωd_i and is directed into the plane of the paper. Its velocity is therefore represented vectorially by $\mathbf{v}_i = \boldsymbol{\omega}\times\mathbf{r}_i$, since $d_i = r_i \sin\theta_i$.

Its angular momentum about the axis of $\boldsymbol{\omega}$ is, from part (a), given by

$$\mathbf{J}_i = \mathbf{r}_i \times m_i\mathbf{v}_i = \mathbf{r}_i \times m_i(\boldsymbol{\omega} \times \mathbf{r}_i).$$

The total angular momentum of the whole collection is consequently

$$\begin{aligned}\mathbf{J} &= \sum_i \mathbf{J}_i \\ &= \sum_i m_i[\mathbf{r}_i \times (\boldsymbol{\omega} \times \mathbf{r}_i)] \\ &= \sum_i m_i \left[\boldsymbol{\omega} r_i^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})\mathbf{r}_i\right]. \qquad \text{(see exercise 7.8)}\end{aligned}$$

(b)(ii) The component of $\mathbf{J}$ along the direction of $\boldsymbol{\omega}$ is

$$\begin{aligned}\frac{\mathbf{J}\cdot\boldsymbol{\omega}}{\omega} &= \frac{1}{\omega}\sum_i m_i \left[r_i^2\omega^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})^2\right] \\ &= \omega \sum_i m_i \left[r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2\right].\end{aligned}$$

This is of the form $I\boldsymbol{\omega}$, where

$$I = \sum_i m_i \left[r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2\right] = \sum_i m_i \rho_i^2.$$

Here ρ_i, which is independent of the magnitude of $\boldsymbol{\omega}$, is given by

$$\begin{aligned}\rho_i^2 &= r_i^2 - \left(\frac{\mathbf{r}_i \cdot \boldsymbol{\omega}}{\omega}\right)^2 \\ &= r_i^2 - r_i^2 \cos^2\theta_i \\ &= r_i^2 \sin^2\theta_i,\end{aligned}$$

i.e. ρ_i is the distance of the ith particle from the axis of rotation [denoted by d_i in figure (b)].

(b)(iii) The total kinetic energy of the particles is the sum of their individual kinetic energies, and so

$$\begin{aligned}T &= \sum_i \tfrac{1}{2} m_i v_i^2 \\ &= \frac{1}{2}\sum_i m_i(\boldsymbol{\omega} \times \mathbf{r}_i)\cdot(\boldsymbol{\omega} \times \mathbf{r}_i) \\ &= \frac{1}{2}\sum_i m_i\left[r_i^2\omega^2 - (\mathbf{r}_i \cdot \boldsymbol{\omega})^2\right] \\ &= \frac{1}{2} I\omega^2.\end{aligned}$$

To obtain the penultimate line we used the result of exercise 7.9.

7.24 *Without carrying out any further integration, use the results of the previous exercise (the parallel axis theorem), the worked example in subsection 6.3.4 and exercise 6.10 to prove that the moment of inertia of a uniform rectangular lamina, of mass M and sides a and b, about an axis perpendicular to its plane and passing through the point $(\alpha a/2, \beta b/2)$, with $-1 \le \alpha, \beta \le 1$, is*

$$\frac{M}{12}[a^2(1+3\alpha^2) + b^2(1+3\beta^2)].$$

In the worked example the moment of inertia (MI) about a side of length b was found to be $\frac{1}{3}Ma^2$.

By the parallel axis theorem the MI about a parallel axis through the centre of gravity, O, of the lamina is $\frac{1}{3}Ma^2 - M(\frac{1}{2}a)^2 = \frac{1}{12}Ma^2$.

By symmetry, the MI about an an axis passing through O and parallel to a side of length a will have the corresponding value $\frac{1}{12}Mb^2$.

By the perpendicular axes theorem established in exercise 6.10, the MI about an axis *normal* to the lamina and passing through O is equal to $\frac{1}{12}(a^2 + b^2)$.

A second use of the parallel axis theorem then gives the MI about an axis perpendicular to the lamina and passing through $(\alpha a/2, \beta b/2)$ as

$$\frac{1}{12}M(a^2+b^2)+M\left[\left(\frac{\alpha a}{2}\right)^2+\left(\frac{\beta b}{2}\right)^2\right]=\frac{M}{12}\left[a^2(1+3\alpha^2)+b^2(1+3\beta^2)\right].$$

7.26 *Systems that can be modelled as damped harmonic oscillators are widespread; pendulum clocks, car shock absorbers, tuning circuits in television sets and radios, and collective electron motions in plasmas and metals are just a few examples.*

In all these cases, one or more variables describing the system obey(s) an equation of the form

$$\ddot{x}+2\gamma\dot{x}+\omega_0^2x=P\cos\omega t,$$

where $\dot{x}=dx/dt$, etc. and the inclusion of the factor 2 is conventional. In the steady state (i.e. after the effects of any initial displacement or velocity have been damped out) the solution of the equation takes the form

$$x(t)=A\cos(\omega t+\phi).$$

By expressing each term in the form $B\cos(\omega t+\epsilon)$ and representing it by a vector of magnitude B making an angle ϵ with the x-axis, draw a closed vector diagram, at $t=0$, say, that is equivalent to the equation.

(a) *Convince yourself that whatever the value of ω (>0) ϕ must be negative $(-\pi<\phi\leq 0)$ and that*

$$\phi=\tan^{-1}\left(\frac{-2\gamma\omega}{\omega_0^2-\omega^2}\right).$$

(b) *Obtain an expression for A in terms of P, ω_0 and ω.*

Substituting $x(t)=A\cos(\omega t+\phi)$ into the differential equation:

$$\begin{aligned}
P\cos\omega t &= \ddot{x}+2\gamma\dot{x}+\omega_0^2x,\\
P\cos\omega t &= -\omega^2A\cos(\omega t+\phi)-2\gamma\omega A\sin(\omega t+\phi)+\omega_0^2A\cos(\omega t+\phi),\\
P\cos\omega t &= \omega^2A\cos(\omega t+\phi+\pi)+2\gamma\omega A\cos(\omega t+\phi+\tfrac{1}{2}\pi)\\
&\qquad +\omega_0^2A\cos(\omega t+\phi).
\end{aligned}$$

Now, set $t=0$ and represent each term as a vector with magnitude and phase as shown in figure 7.2.

Figure 7.2 The vector diagram for the equation in exercise 7.26.

(a) For the last of these equations to be valid the three vectors representing the terms on the RHS must have a resultant equal to that representing P on the LHS, i.e. the resultant must be real and positive. As can be seen, with $\phi > 0$ (illustrated by $\phi = \phi_1$ in the figure), no matter what value ω takes, the possible resultants (broken arrows) can never equal P.

(b) However, with $\phi < 0$ (illustrated by $\phi = \phi_2$), the three vectors from the RHS can have a resultant corresponding to P. When this happens, from the geometry of the quadrilateral, it can be seen that

$$|\tan\phi_2| = \frac{2\gamma\omega A}{\omega_0^2 A - \omega^2 A} \Rightarrow \phi = \tan^{-1}\left(\frac{-2\gamma\omega}{\omega_0^2 - \omega^2}\right),$$

and, from the geometry of a right-angled triangle, that

$$P^2 = (2\gamma\omega A)^2 + (\omega_0^2 A - \omega^2 A)^2,$$
$$\Rightarrow \quad A = \frac{P}{[(\omega_0^2 - \omega^2)^2 + 4\gamma^2\omega^2]^{1/2}}.$$

This is the amplitude of the response of the system when that of the sinusoidal input is P.

8

Matrices and vector spaces

8.2 *Evaluate the determinants*

$$\text{(a)} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \qquad \text{(b)} \begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix}$$

and

$$\text{(c)} \begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}.$$

(a) Using the elements and cofactors of the first row in a straightforward Laplace expansion, we have

$$\begin{aligned} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} &= a(bc-f)+h(fg-hc)+g(hf-gb) \\ &= abc+2fgh-af^2-bg^2-ch^2. \end{aligned}$$

(b) At each stage we subtract a suitable multiple of the first column from each other column so as to make the first entry in each of the other columns zero; then we use a Laplace expansion with a single term. Here this reduction is carried

out three times.

$$\begin{vmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & 4 & -2 \\ -2 & 1 & -2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 \\ 3 & -3 & -2 & -11 \\ -2 & 1 & 2 & 7 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 1 & -2 & 1 \\ -3 & -2 & -11 \\ 1 & 2 & 7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -3 & -8 & -8 \\ 1 & 4 & 6 \end{vmatrix}$$

$$= 1 \begin{vmatrix} -8 & -8 \\ 4 & 6 \end{vmatrix} = 1 \begin{vmatrix} -8 & 0 \\ 4 & 2 \end{vmatrix} = -8|2| = -16.$$

(c) In making this reduction we (i) subtract g times the third row from the first row, (ii) subtract the second row from the fourth, (iii) use the Laplace expansion, (iv) subtract the second column from the third, and (v) use a Laplace expansion followed by direct evaluation.

$$\begin{vmatrix} gc & ge & a+ge & gb+ge \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix} = \begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & b & b+f & b+d \end{vmatrix}$$

$$\begin{vmatrix} 0 & 0 & a & 0 \\ 0 & b & b & b \\ c & e & e & b+e \\ a & 0 & f & d \end{vmatrix} = a \begin{vmatrix} 0 & b & b \\ c & e & b+e \\ a & 0 & d \end{vmatrix}$$

$$= a \begin{vmatrix} 0 & b & 0 \\ c & e & b \\ a & 0 & d \end{vmatrix} = -ab \begin{vmatrix} c & b \\ a & d \end{vmatrix}$$

$$= ab(ab - cd).$$

8.4 *Consider the matrices*

$$\text{(a)}\ \mathsf{B} = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix}, \qquad \text{(b)}\ \mathsf{C} = \frac{1}{\sqrt{8}} \begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix}.$$

Are they (i) *real,* (ii) *diagonal,* (iii) *symmetric,* (iv) *antisymmetric,* (v) *singular,* (vi) *orthogonal,* (vii) *Hermitian,* (viii) *anti-Hermitian,* (ix) *unitary,* (x) *normal*?

(a) For matrix B:

Clearly, (i)-(iii) are not true whilst (iv) is.
(v) $|\mathsf{B}| = -i(i^2 - 0) + i(i^2) = 0$ and so B is singular.
(vi) From (v) it follows that B has no inverse. In particular, its transpose cannot be its inverse, i.e. B is not orthogonal.
(vii)

$$(\mathsf{B}^*)^T = \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & -i & i \\ i & 0 & -i \\ -i & i & 0 \end{pmatrix} = \mathsf{B},$$

i.e. B is Hermitian.
(viii) In view of (vii), B cannot be anti-Hermitian.
(ix) As in (vi), B cannot be unitary.
(x) Since B is Hermitian, it commutes with its Hermitian conjugate (itself) and is therefore normal.

(b) For matrix C:

C is clearly real, i.e. satisfies (i), and, equally clearly, satisfies none of (ii)-(iv).
(v)

$$\begin{aligned} |\mathsf{C}| = \left(\frac{1}{\sqrt{8}}\right)^3 \begin{vmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{vmatrix} &= \left(\frac{1}{\sqrt{8}}\right)^3 \begin{vmatrix} \sqrt{3} & -\sqrt{2} & 0 \\ 1 & \sqrt{6} & 0 \\ 2 & 0 & 4 \end{vmatrix} \\ &= \frac{1}{\sqrt{32}}(\sqrt{18} + \sqrt{2}) \\ &= \tfrac{1}{4}(3 + 1) = 1 \neq 0. \end{aligned}$$

Thus C is not singular.
(vi) Consider $\mathsf{C}^T\mathsf{C}$, which is given by

$$\frac{1}{8}\begin{pmatrix} \sqrt{3} & 1 & 2 \\ -\sqrt{2} & \sqrt{6} & 0 \\ -\sqrt{3} & -1 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{3} & -\sqrt{2} & -\sqrt{3} \\ 1 & \sqrt{6} & -1 \\ 2 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathsf{I}_3.$$

Thus C is orthogonal.
(vii) & (viii) In view of (i), (iii) and (iv), C cannot be either Hermitian or anti-Hermitian.
(ix) In view of (i) and (vi), C is unitary.
(x) In view of (i) and (vi), $\mathsf{C}^\dagger\mathsf{C} = \mathsf{C}^T\mathsf{C} = \mathsf{I} = \mathsf{C}\mathsf{C}^T = \mathsf{C}\mathsf{C}^\dagger$. Hence C is normal.

8.6 *This exercise considers a crystal whose unit cell has base vectors that are not necessarily mutually orthogonal.*

(a) *The basis vectors of the unit cell of a crystal, with the origin O at one corner, are denoted by $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$. The matrix G has elements G_{ij}, where $G_{ij} = \mathbf{e}_i \cdot \mathbf{e}_j$ and H_{ij} are the elements of the matrix $\mathsf{H} \equiv \mathsf{G}^{-1}$. Show that the vectors $\mathbf{f}_i = \sum_j H_{ij}\,\mathbf{e}_j$ are the reciprocal vectors and that $H_{ij} = \mathbf{f}_i \cdot \mathbf{f}_j$.*

(b) *If the vectors $\mathbf{u}$ and $\mathbf{v}$ are given by*

$$\mathbf{u} = \sum_i u_i \mathbf{e}_i, \qquad \mathbf{v} = \sum_i v_i \mathbf{f}_i,$$

obtain expressions for $|\mathbf{u}|$, $|\mathbf{v}|$, and $\mathbf{u} \cdot \mathbf{v}$.

(c) *If the basis vectors are each of length a and the angle between each pair is $\pi/3$, write down G and hence obtain H.*

(d) *Calculate* (i) *the length of the normal from O onto the plane containing the points $p^{-1}\mathbf{e}_1$, $q^{-1}\mathbf{e}_2$, $r^{-1}\mathbf{e}_3$, and* (ii) *the angle between this normal and $\mathbf{e}_1$.*

(a) With $\mathbf{f}_i$ defined by $\mathbf{f}_i = \sum_j H_{ij}\,\mathbf{e}_j$, consider

$$\mathbf{f}_i \cdot \mathbf{e}_k = \sum_j H_{ij}\,\mathbf{e}_j \cdot \mathbf{e}_k = \sum_j \left(G^{-1}\right)_{ij} G_{jk} = \left(G^{-1}G\right)_{ik} = \delta_{ik}.$$

Thus the $\mathbf{f}_i$ are the reciprocal vectors of the cell's base vectors.

Now consider

$$\begin{aligned}\mathbf{f}_i \cdot \mathbf{f}_j &= \sum_k H_{ik}\mathbf{e}_k \sum_m H_{jm}\mathbf{e}_m = \sum_{k,m} H_{ik}H_{jm}G_{km} \\ &= \sum_m H_{jm} \sum_k H_{ik}G_{km} = \sum_m H_{jm}\delta_{im} = H_{ji} = H_{ij}.\end{aligned}$$

(b) With $\mathbf{u} = \sum_i u_i\,\mathbf{e}_i$,

$$|\mathbf{u}|^2 = \sum_i u_i\,\mathbf{e}_i \sum_j u_j\,\mathbf{e}_j = \sum_{i,j} u_i\,G_{ij}\,u_j \Rightarrow |\mathbf{u}| = \left(\sum_{i,j} u_i\,G_{ij}\,u_j\right)^{1/2}.$$

Similarly,

$$|\mathbf{v}| = \left(\sum_{i,j} v_i\,H_{ij}\,v_j\right)^{1/2}.$$

For the scalar product of $\mathbf{u}$ and $\mathbf{v}$,

$$\mathbf{u} \cdot \mathbf{v} = \sum_i u_i\,\mathbf{e}_i \sum_j v_j\,\mathbf{f}_j = \sum_{i,j} u_i\,v_j\,\delta_{ij} = \sum_i u_i\,v_i.$$

(c) For $i = j$, $\mathbf{e}_i \cdot \mathbf{e}_j = a^2$ whilst, for $i \neq j$, $\mathbf{e}_i \cdot \mathbf{e}_j = a^2 \cos(\pi/3) = \frac{1}{2}a^2$. Thus

$$\mathsf{G} = \tfrac{1}{2}a^2 \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}.$$

The matrix $2\mathsf{G}/a^2$ has determinant 4 and all of its co-factors are either 3 or ± 1. The matrix H, computed using this data, is found to be

$$\mathsf{H} \equiv \mathsf{G}^{-1} = \frac{1}{2a^2} \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}.$$

(d)(i) The normal to the plane is in the direction

$$(q^{-1}\mathbf{e}_2 - p^{-1}\mathbf{e}_1) \times (r^{-1}\mathbf{e}_3 - p^{-1}\mathbf{e}_1) \propto \left[(qr)^{-1}\mathbf{f}_1 + (pr)^{-1}\mathbf{f}_2 + (qp)^{-1}\mathbf{f}_3\right]$$

i.e. in the direction $\mathbf{f} = p\,\mathbf{f}_1 + q\,\mathbf{f}_2 + r\,\mathbf{f}_3$.

A unit vector in this direction is

$$\hat{\mathbf{n}} = \frac{p\,\mathbf{f}_1 + q\,\mathbf{f}_2 + r\,\mathbf{f}_3}{(\mathbf{f} \cdot \mathbf{f})^{1/2}}$$

and the distance from the origin to the plane is the scalar product of this unit vector and the position vector of any one of the three points (necessarily, they all give the same answer). Using $p^{-1}\mathbf{e}_1$ and denoting (p, q, r) by v_i, we have the distance d as

$$\begin{aligned} d = \hat{\mathbf{n}} \cdot p^{-1}\mathbf{e}_1 &= \frac{p^{-1}p + 0 + 0}{(\mathbf{f} \cdot \mathbf{f})^{1/2}} \\ &= \frac{1}{(\sum_i v_i \mathbf{f}_i \sum_j v_j \mathbf{f}_j)^{1/2}} \\ &= \frac{1}{(\sum_{i,j} v_i H_{ij} v_j)^{1/2}} = \frac{1}{M}, \end{aligned}$$

where $M^2 = (2a^2)^{-1}[3(p^2 + q^2 + r^2) - 2(qr + rp + pq)]$.

(d)(ii) The angle θ between $\hat{\mathbf{n}}$ and $\mathbf{e}_1$ is given by

$$\theta = \cos^{-1} \frac{\hat{\mathbf{n}} \cdot \mathbf{e}_1}{|\mathbf{e}_1|} = \cos^{-1} \frac{p\,d}{a} = \cos^{-1} \frac{p}{aM}.$$

8.8 *A and B are real non-zero 3×3 matrices and satisfy the equation*

$$(\mathsf{AB})^{\mathrm{T}} + \mathsf{B}^{-1}\mathsf{A} = \mathsf{0}.$$

(a) *Prove that if B is orthogonal then A is antisymmetric.*
(b) *Without assuming that B is orthogonal, prove that A is singular.*

We have that $(\mathsf{AB})^{\mathrm{T}} = -\mathsf{B}^{-1}\mathsf{A}$.

(a) Given $\mathsf{B}^{\mathrm{T}}\mathsf{B} = \mathsf{I}$ (i.e. B is orthogonal),

$$\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} = -\mathsf{B}^{-1}\mathsf{A} \quad \Rightarrow \quad \mathsf{B}\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} = -\mathsf{B}\mathsf{B}^{-1}\mathsf{A} \quad \Rightarrow \quad \mathsf{A}^{\mathrm{T}} = -\mathsf{A},$$

i.e. A is antisymmetric.

(b) Since B^{-1} is defined, $|\mathsf{B}| \neq 0$.

$$\begin{aligned} \mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} &= -\mathsf{B}^{-1}\mathsf{A} \\ \mathsf{B}\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} &= -\mathsf{A} \\ |\mathsf{B}|\,|\mathsf{B}^{\mathrm{T}}|\,|\mathsf{A}^{\mathrm{T}}| &= |-\mathsf{A}| \\ |\mathsf{B}|^2\,|\mathsf{A}| &= (-1)^3|\mathsf{A}|, \quad \text{since } |\mathsf{B}^{\mathrm{T}}| = |\mathsf{B}|. \end{aligned}$$

In the last line the factor $(-1)^3$ arises because A is a 3×3 matrix. The two sides of the last equation have opposite (contradictory) signs unless $|\mathsf{A}| = 0$, i.e. unless A is singular.

8.10 *The four matrices S_x, S_y, S_z and I are defined by*

$$\mathsf{S}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{S}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$
$$\mathsf{S}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

where $i^2 = -1$. Show that $\mathsf{S}_x^2 = \mathsf{I}$ and $\mathsf{S}_x\mathsf{S}_y = i\mathsf{S}_z$, and obtain similar results by permutting x, y and z. Given that $\mathbf{v}$ is a vector with Cartesian components (v_x, v_y, v_z), the matrix $\mathsf{S}(\mathbf{v})$ is defined as

$$\mathsf{S}(\mathbf{v}) = v_x\mathsf{S}_x + v_y\mathsf{S}_y + v_z\mathsf{S}_z.$$

Prove that, for general non-zero vectors $\mathbf{a}$ and $\mathbf{b}$,

$$\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) = \mathbf{a}\cdot\mathbf{b}\,\mathsf{I} + i\,\mathsf{S}(\mathbf{a}\times\mathbf{b}).$$

Without further calculation, deduce that $\mathsf{S}(\mathbf{a})$ and $\mathsf{S}(\mathbf{b})$ commute if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

As we have only the definitions to work with, these basic products must be found by explicit matrix multiplication:

$$\mathsf{S}_x^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{I}.$$

$$\mathsf{S}_x\mathsf{S}_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i\mathsf{S}_z.$$

$$\mathsf{S}_y\mathsf{S}_z = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i\mathsf{S}_x.$$

Similarly $\mathsf{S}_y^2 = \mathsf{S}_z^2 = \mathsf{I}$ and $\mathsf{S}_z\mathsf{S}_x = i\mathsf{S}_y$. We also note that

$$\mathsf{S}_y\mathsf{S}_x = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -i\mathsf{S}_z,$$

i.e. that S_x and S_y anticommute. This applies to any pair of the matrices (excluding I of course).

We first note that if $\mathbf{0}$ is the zero vector then $\mathsf{S}(\mathbf{0}) = \mathsf{O}$, the zero matrix; conversely, if $\mathsf{S}(\mathbf{v}) = \mathsf{O}$ then $\mathbf{v} = \mathbf{0}$. Now consider the product of the two matrices $\mathsf{S}(\mathbf{a})$ and $\mathsf{S}(\mathbf{b})$.

$$\begin{aligned}
\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) &= (a_x\mathsf{S}_x + a_y\mathsf{S}_y + a_z\mathsf{S}_z)(b_x\mathsf{S}_x + b_y\mathsf{S}_y + b_z\mathsf{S}_z) \\
&= (a_xb_x + a_yb_y + a_zb_z)\mathsf{I} + a_xb_y(i\mathsf{S}_z) + a_yb_x(-i\mathsf{S}_z) + \cdots \\
&= (a_xb_x + a_yb_y + a_zb_z)\mathsf{I} + i(\mathbf{a}\times\mathbf{b})_z\mathsf{S}_z + \cdots \\
&= (\mathbf{a}\cdot\mathbf{b})\mathsf{I} + i\mathsf{S}(\mathbf{a}\times\mathbf{b}), \qquad \text{as stated in the question.}
\end{aligned}$$

Interchanging $\mathbf{a}$ and $\mathbf{b}$ gives $\mathsf{S}(\mathbf{b})\mathsf{S}(\mathbf{a}) = (\mathbf{b}\cdot\mathbf{a})\mathsf{I} + i\mathsf{S}(\mathbf{b}\times\mathbf{a})$. It then follows that

$$\mathsf{S}(\mathbf{a})\mathsf{S}(\mathbf{b}) - \mathsf{S}(\mathbf{b})\mathsf{S}(\mathbf{a}) = 2i\mathsf{S}(\mathbf{a}\times\mathbf{b}).$$

The matrix on the RHS is the zero matrix if and only if $\mathbf{a}\times\mathbf{b} = \mathbf{0}$, i.e. $\mathbf{a}$ and $\mathbf{b}$ are parallel vectors.

8.12 *Given a matrix*

$$\mathsf{A} = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α and β are non-zero complex numbers, find its eigenvalues and eigenvectors. Find the respective conditions for (a) *the eigenvalues to be real and* (b) *the eigenvectors to be orthogonal. Show that the conditions are jointly satisfied if and only if* A *is Hermitian.*

The eigenvalues λ of A are the roots of

$$\begin{vmatrix} 1-\lambda & \alpha & 0 \\ \beta & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0,$$

i.e. the values of λ that satisfy

$$(1-\lambda)\left[(1-\lambda)^2 - \alpha\beta\right] = 0.$$

This means that either $\lambda = 1$ or that

$$\lambda^2 - 2\lambda + 1 - \alpha\beta = 0,$$
$$\Rightarrow \quad \lambda = 1 \pm (\alpha\beta)^{1/2}.$$

For $\lambda = 1$ the corresponding eigenvector is obviously $\mathsf{e}^1 = (0,\ 0,\ 1)^{\mathrm{T}}$.

For $\lambda = 1 \pm (\alpha\beta)^{1/2}$, we have for eigenvector $(x,\ y,\ z)^{\mathrm{T}}$ that

$$\mp(\alpha\beta)^{1/2}x + \alpha y = 0,$$
$$\mp(\alpha\beta)^{1/2}z = 0.$$

Thus $\mathsf{e}^{2,3} = \left(\sqrt{\alpha},\ \pm\sqrt{\beta},\ 0\right)^{\mathrm{T}}$.

(a) For all the eigenvalues to be real, we need the product $\alpha\beta$ to be real and positive.

(b) For the eigenvectors to be mutually orthogonal we need (recall that α and β can be complex)

$$0 = (\mathsf{e}^2)^{\dagger} \cdot \mathsf{e}^3 = (\alpha^*)^{1/2}\alpha^{1/2} - (\beta^*)^{1/2}\beta^{1/2} \quad \Rightarrow \quad |\alpha| = |\beta|.$$

The orthogonality of e^3 to the other two is trivially obvious.

(i) If both conditions are satisfied and we write $\alpha = ce^{i\theta}$, then the first condition requires that the argument of β is $-\theta$ whilst the second requires its magnitude to be c. Thus $\beta = ce^{-i\theta}$ and $\beta = \alpha^*$, making A Hermitian.

(ii) If A is Hermitian, $\beta = \alpha^*$ and so $\alpha\beta$ is real and positive. The eigenvalues are then 1, $1 + |\alpha|$ and $1 - |\alpha|$, i.e. all real.

The corresponding eigenvectors have the forms $(0,\ 0,\ 1)^{\mathrm{T}}$, $\left(\sqrt{\alpha},\ \sqrt{\alpha^*},\ 0\right)^{\mathrm{T}}$ and $\left(\sqrt{\alpha},\ -\sqrt{\alpha^*},\ 0\right)^{\mathrm{T}}$, and are clearly mutually orthogonal.

Thus, the two conditions are jointly satisfied if and only if A is Hermitian.

8.14 *If a unitary matrix* U *is written as* $\mathsf{A} + i\mathsf{B}$, *where* A *and* B *are Hermitian with non-degenerate eigenvalues, show the following:*

(a) A *and* B *commute*;
(b) $\mathsf{A}^2 + \mathsf{B}^2 = \mathsf{I}$;
(c) *The eigenvectors of* A *are also eigenvectors of* B;
(d) *The eigenvalues of* U *have unit modulus (as is necessary for any unitary matrix).*

Given that $\mathsf{U} = \mathsf{A} + i\mathsf{B}$, with $\mathsf{A}^\dagger = \mathsf{A}$, $\mathsf{B}^\dagger = \mathsf{B}$ and $\mathsf{U}^\dagger\mathsf{U} = \mathsf{I}$, consider

$$\begin{aligned} \mathsf{I} = \mathsf{U}^\dagger\mathsf{U} &= (\mathsf{A}^\dagger - i\mathsf{B}^\dagger)(\mathsf{A} + i\mathsf{B}) \\ &= (\mathsf{A} - i\mathsf{B})(\mathsf{A} + i\mathsf{B}) \\ &= \mathsf{A}^2 + \mathsf{B}^2 + i(\mathsf{AB} - \mathsf{BA}). \\ \text{and} \qquad \mathsf{I} = \mathsf{U}\mathsf{U}^\dagger &= (\mathsf{A} + i\mathsf{B})(\mathsf{A}^\dagger - i\mathsf{B}^\dagger) \\ &= (\mathsf{A} + i\mathsf{B})(\mathsf{A} - i\mathsf{B}) \\ &= \mathsf{A}^2 + \mathsf{B}^2 + i(\mathsf{BA} - \mathsf{AB}). \end{aligned}$$

Comparison of the two results implies

(a) $\mathsf{BA} - \mathsf{AB} = \mathsf{O}$, i.e. A and B commute, and, consequently, (b) $\mathsf{A}^2 + \mathsf{B}^2 = \mathsf{I}$.

(c) Let x be an eigenvector of A with eigenvalue λ, i.e. $\mathsf{Ax} = \lambda\mathsf{x}$. Then

$$\mathsf{ABx} = \mathsf{BAx} = \mathsf{B}\lambda\mathsf{x} = \lambda\mathsf{Bx},$$

where we have used result (a) to justify the first equality.
Now, the above result shows that $\mathsf{y} = \mathsf{Bx}$ is an eigenvector of A with eigenvalue λ.
But the eigenvalues of A are non-degenerate and so y must be a multiple of x, i.e. $\mathsf{Bx} = \mu\mathsf{x}$ for some μ.
However, this is the statement that x is an eigenvector of B (as well as of A). Hence each eigenvector of A is also an eigenvector of B.

(d) Let x be an eigenvector of U with (complex) eigenvalue λ. We then have

$$\begin{aligned} (\mathsf{A} + i\mathsf{B})\mathsf{x} &= \lambda\mathsf{x}, \quad \text{take the hermitian conjugate,} \\ \mathsf{x}^\dagger(\mathsf{A}^\dagger - i\mathsf{B}^\dagger) &= \lambda^*\mathsf{x}^\dagger, \\ \mathsf{x}^\dagger(\mathsf{A}^2 + \mathsf{B}^2)\mathsf{x} &= \mathsf{x}^\dagger|\lambda|^2\mathsf{x}, \\ \mathsf{x}^\dagger\,\mathsf{I}\,\mathsf{x} &= \mathsf{x}^\dagger|\lambda|^2\mathsf{x}, \\ |\mathsf{x}|^2 &= |\lambda|^2|\mathsf{x}|^2. \end{aligned}$$

To obtain the third equation we multiplied the two LHS and the two RHS of the

first two equations together. Finally, since x is a non-zero vector it follows from the last equation that $|\lambda| = 1$.

8.16 *Find the eigenvalues and a set of eigenvectors of the matrix*

$$\begin{pmatrix} 1 & 3 & -1 \\ 3 & 4 & -2 \\ -1 & -2 & 2 \end{pmatrix}.$$

Verify that its eigenvectors are mutually orthogonal.

The eigenvalues must be the roots of

$$\begin{vmatrix} 1-\lambda & 3 & -1 \\ 3 & 4-\lambda & -2 \\ -1 & -2 & 2-\lambda \end{vmatrix} = 0.$$

Evaluating the determinant gives

$$\begin{aligned} (1-\lambda)(\lambda^2 - 6\lambda + 4) + 3(-4 + 3\lambda) - 1(-2 - \lambda) &= 0, \\ (1-\lambda)(\lambda^2 - 6\lambda + 4) - 10 + 10\lambda &= 0, \\ (1-\lambda)(\lambda^2 - 6\lambda - 6) &= 0. \end{aligned}$$

Thus $\lambda = 1$ or $\lambda = 3 \pm \sqrt{15}$.

Writing an eigenvector as $\mathsf{e} = (x,\ y,\ z)^{\mathrm{T}}$:

For $\lambda = 1$, $0x + 3y - z = 0$ and $3x + 3y - 2z = 0$ which imply

$$\mathsf{e}^1 = (1,\ 1,\ 3)^{\mathrm{T}}.$$

For $\lambda = 3 \pm \sqrt{15}$,

$$\begin{aligned} (-2 \mp \sqrt{15})x + 3y - z &= 0, \\ 3x + (1 \mp \sqrt{15})y - 2z &= 0. \end{aligned}$$

We now eliminate z and obtain

$$(-7 \mp 2\sqrt{15})x + (5 \pm \sqrt{15})y = 0.$$

Taking $\mathsf{e} = \left(5 \pm \sqrt{15},\ 7 \pm 2\sqrt{15},\ z\right)^{\mathrm{T}}$, the first equation gives

$$\begin{aligned} z &= (-2 \mp \sqrt{15})(5 \pm \sqrt{15}) + 3(7 \pm 2\sqrt{15}) \\ &= -4 \mp \sqrt{15}. \end{aligned}$$

Thus the three eigenvectors are

$$\begin{aligned}
\mathsf{e}^1 &= (1,\ 1,\ 3)^{\mathrm{T}}, \\
\mathsf{e}^2 &= \left(5+\sqrt{15},\ 7+2\sqrt{15},\ -4-\sqrt{15}\right)^{\mathrm{T}}, \\
\mathsf{e}^3 &= \left(5-\sqrt{15},\ 7-2\sqrt{15},\ -4+\sqrt{15}\right)^{\mathrm{T}}.
\end{aligned}$$

Their mutual orthogonality is established by considering the following scalar products.

$$\begin{aligned}
\mathsf{e}^1 \cdot \mathsf{e}^2 &= (5+7-12)+(1+2-3)\sqrt{15} = 0, \\
\mathsf{e}^1 \cdot \mathsf{e}^3 &= (5+7-12)+(-1-2+3)\sqrt{15} = 0, \\
\mathsf{e}^2 \cdot \mathsf{e}^3 &= (25-15)+(49-60)+(16-15) = 0,
\end{aligned}$$

i.e. they are mutually orthogonal. We note that, formally, the first factor in each scalar product should be the hermitian conjugate of the eigenvector; here this makes no difference as all components are real.

8.18 *Use the results of the first worked example in section 8.14 to evaluate, without repeated matrix multiplication, the expression* $\mathsf{A}^6\mathsf{x}$*, where* $\mathsf{x} = (2 \quad 4 \quad -1)^{\mathrm{T}}$ *and* A *is the matrix given in the example.*

A set of three (un-normalised, but that does not matter here) independent eigenvectors of A, and their corresponding eigenvalues, are, as given in section 8.14,

$$\begin{aligned}
\mathsf{x}^1 &= (1,\ 1,\ 0)^{\mathrm{T}} \quad \text{for} \quad \lambda = 2, \\
\mathsf{x}^2 &= (1,\ -1,\ 1)^{\mathrm{T}} \quad \text{for} \quad \lambda = 3, \\
\mathsf{x}^3 &= (1,\ -1,\ -2)^{\mathrm{T}} \quad \text{for} \quad \lambda = 2.
\end{aligned}$$

We first express the given vector x in terms of the eigenvectors as

$$\begin{pmatrix} 2 \\ 4 \\ -1 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} = 3\mathsf{x}^1 - \mathsf{x}^2.$$

We now use the fact that, for an eigenvector, $\mathsf{A}^n\mathsf{x} = \lambda^n\mathsf{x}$. This gives

$$\begin{aligned}
\mathsf{A}^6\mathsf{x} = 3\ 2^6\mathsf{x}^1 - 3^6\mathsf{x}^2 &= \left(3\ 2^6(1) - 3^6(1),\ 3\ 2^6(1) - 3^6(-1),\ 3\ 2^6(0) - 3^6(1)\right)^{\mathrm{T}} \\
&= (-537,\ 921,\ -729)^{\mathrm{T}}.
\end{aligned}$$

8.20 *Demonstrate that the matrix*

$$\mathsf{A} = \begin{pmatrix} 2 & 0 & 0 \\ -6 & 4 & 4 \\ 3 & -1 & 0 \end{pmatrix},$$

is defective, i.e. does not have three linearly independent eigenvectors, by showing the following:

(a) *its eigenvalues are degenerate and, in fact, all equal*;
(b) *any eigenvector has the form* $(\mu \quad (3\mu - 2v) \quad v)^{\mathrm{T}}$;
(c) *if two pairs of values,* μ_1, v_1 *and* μ_2, v_2*, define two independent eigenvectors* v_1 *and* v_2 *then* any *third similarly defined eigenvector* v_3 *can be written as a linear combination of* v_1 *and* v_2*, i.e.*

$$\mathsf{v}_3 = a\mathsf{v}_1 + b\mathsf{v}_2$$

where

$$a = \frac{\mu_3 v_2 - \mu_2 v_3}{\mu_1 v_2 - \mu_2 v_1} \quad \text{and} \quad b = \frac{\mu_1 v_3 - \mu_3 v_1}{\mu_1 v_2 - \mu_2 v_1}.$$

Illustrate (c) *using the example* $(\mu_1, v_1) = (1, 1)$, $(\mu_2, v_2) = (1, 2)$ *and* $(\mu_3, v_3) = (0, 1)$.

Show further that any matrix of the form

$$\begin{pmatrix} 2 & 0 & 0 \\ 6n-6 & 4-2n & 4-4n \\ 3-3n & n-1 & 2n \end{pmatrix}$$

is defective, with the same eigenvalues and eigenvectors as A.

(a) The eigenvalues of A are given by

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ -6 & 4-\lambda & 4 \\ 3 & -1 & -\lambda \end{vmatrix} = (2-\lambda)(-4\lambda + \lambda^2 + 4) = (2-\lambda)^3 = 0.$$

Thus A has three equal eigenvalues $\lambda = 2$.

(b) Using this value for λ, an eigenvector $(x,\ y,\ z)^{\mathrm{T}}$ must satisfy

$$\begin{aligned} 0x + 0y + 0z &= 0, \\ -6x + 2y + 4z &= 0, \\ 3x - y - 2z &= 0, \end{aligned}$$

leading to the conclusion that $\mathsf{v} = (\mu,\ 3\mu - 2v,\ v)^{\mathrm{T}}$, with μ and v arbitrary, will

be an eigenvector. Clearly any two components of a vector of this form can be chosen arbitrarily, but the third one is then determined.

(c) Given two eigenvectors

$$\mathsf{v}_1 = (\mu_1,\ 3\mu_1 - 2v_1,\ v_1)^{\mathrm{T}} \quad \text{and} \quad \mathsf{v}_2 = (\mu_2,\ 3\mu_2 - 2v_2,\ v_2)^{\mathrm{T}},$$

any third vector v_3 of the same form with parameters μ_3 and v_3 can be written as a linear combination of v_1 and v_2, as is shown by the following argument. Define the vector v and the numbers a and b by

$$\mathsf{v} = a\mathsf{v}_1 + b\mathsf{v}_2 \equiv \frac{\mu_3 v_2 - \mu_2 v_3}{\mu_1 v_2 - \mu_2 v_1}\mathsf{v}_1 + \frac{\mu_1 v_3 - \mu_3 v_1}{\mu_1 v_2 - \mu_2 v_1}\mathsf{v}_2.$$

Now consider the first component, say, of the vector on the RHS

$$\frac{\mu_3 v_2 - \mu_2 v_3}{\mu_1 v_2 - \mu_2 v_1}\mu_1 + \frac{\mu_1 v_3 - \mu_3 v_1}{\mu_1 v_2 - \mu_2 v_1}\mu_2 = \frac{\mu_1\mu_3 v_2 - \mu_2\mu_3 v_1}{\mu_1 v_2 - \mu_2 v_1} = \mu_3.$$

Similarly, the second component is $3\mu_3 - 2v_3$ and the third one v_3. In other words, $\mathsf{v} = \mathsf{v}_3$ and v_3 has been expressed explicitly as a linear combination of v_1 and v_2. This establishes that A does *not* have three linearly independent eigenvectors, i.e. it is defective.

(d) With

$$\begin{aligned}(\mu_1, v_1) &= (1,1) \text{ and } \mathsf{v}_1 = (1,\ 1,\ 1)^{\mathrm{T}},\\ (\mu_2, v_2) &= (1,2) \text{ and } \mathsf{v}_2 = (1,\ -1,\ 2)^{\mathrm{T}},\\ (\mu_3, v_3) &= (0,1) \text{ and } \mathsf{v}_3 = (0,\ -2,\ 1)^{\mathrm{T}},\end{aligned}$$

$$a = \frac{(0\times 2) - (1\times 1)}{(1\times 2) - (1\times 1)} = -1 \quad \text{and} \quad b = \frac{(1\times 1) - (0\times 1)}{(1\times 2) - (1\times 1)} = 1.$$

Thus

$$\mathsf{v} = -1\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} = \mathsf{v}_3,$$

as expected.

The eigenvalues of this more general matrix, $\mathsf{A}(n)$ say, are given by

$$\begin{aligned}0 &= \begin{vmatrix} 2-\lambda & 0 & 0 \\ 6n-6 & 4-2n-\lambda & 4-4n \\ 3-3n & n-1 & 2n-\lambda \end{vmatrix}\\ &= (2-\lambda)[(4-2n-\lambda)(2n-\lambda) - (n-1)(4-4n)]\\ &= (2-\lambda)(8n - 4\lambda - 4n^2 + 2n\lambda - 2n\lambda + \lambda^2 - 4n + 4 + 4n^2 - 4n)\\ &= (2-\lambda)(2-\lambda)^2.\end{aligned}$$

This shows that the eigenvalues of $\mathsf{A}(n)$ are the same as those of A.

The equations to be satisfied by the components of an eigenvector of $\mathsf{A}(n)$, $\mathsf{v} = (x,\ y,\ z)^{\mathrm{T}}$, are

$$\begin{aligned} 0x + 0y + 0z &= 0, \\ (6n-6)x + (2-2n)y + (4-4n)z &= 0, \\ (3-3n)x + (n-1)y + (2n-2)z &= 0. \end{aligned}$$

When the common factor $(n-1)$ has been cancelled from the second and third of these, the equations remaining are identical to those satisfied by the components of the eigenvectors of A. The eigenvectors will therefore be identical to those of A; it also follows that $\mathsf{A}(n)$ is defective.

8.22 *Use the stationary properties of quadratic forms to determine the maximum and minimum values taken by the expression*

$$Q = 5x^2 + 4y^2 + 4z^2 + 2xz + 2xy$$

on the unit sphere, $x^2 + y^2 + z^2 = 1$. *For what values of* x, y *and* z *do they occur?*

Since all vectors on the unit sphere have unit modulus the maximum and minimum values of Q will be equal to the largest and smallest of the eigenvalues of the associated symmetric matrix. These we find by considering

$$\begin{vmatrix} 5-\lambda & 1 & 1 \\ 1 & 4-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0,$$

$$\begin{aligned} (5-\lambda)(16 - 8\lambda + \lambda^2) - 4 + \lambda - 4 + \lambda &= 0, \\ \lambda^3 - 13\lambda^2 + 54\lambda - 72 &= 0. \end{aligned}$$

There is no concise automatic way to solve this cubic equation, but by inspection it is clear that if $\lambda = 3$ the top row of the determinant is equal to the sum of the other two, implying that $\lambda = 3$ is one root. The polynomial equation is now easily factorised as

$$(\lambda - 3)(\lambda - 4)(\lambda - 6) = 0,$$

showing that the maximum and minimum values of Q are 6 and 3.

The corresponding values of $(x,\ y,\ z)^{\mathrm{T}}$ are given by the associated (normalised) eigenvectors:

For $\lambda = 6$ (the maximum),

$$-x + y + z = 0,\ x - 2y = 0,\ x - 2z = 0 \Rightarrow \mathsf{x}_{\max} = \pm(6)^{-1/2}\,(2,\ 1,\ 1)^{\mathrm{T}}.$$

For $\lambda = 3$ (the minimum),

$$2x + y + z = 0,\ x + y = 0,\ x + z = 0 \Rightarrow \mathsf{x}_{\min} = \pm(3)^{-1/2}\,(1,\ -1,\ -1)^{\mathrm{T}}\,.$$

8.24 *Find the lengths of the semi-axes of the ellipse*

$$73x^2 + 72xy + 52y^2 = 100,$$

and determine its orientation.

This is a quadric surface with no z-dependence (an elliptical cylinder) and if its semi-axes are a and b then a^{-2} and b^{-2} are given by the eigenvalues of the associated matrix after the RHS has been made unity. The eigenvalues therefore satisfy

$$\begin{aligned} 0 &= \begin{vmatrix} 0.73 - \lambda & 0.36 \\ 0.36 & 0.52 - \lambda \end{vmatrix} \\ &= 0.3796 - 1.25\lambda + \lambda^2 - 0.1296 \\ &= \lambda^2 - 1.25\lambda + 0.25 \\ &= (\lambda - 1)(\lambda - 0.25). \end{aligned}$$

Thus $\lambda = a^{-2} = 1$, giving $a = 1$, and $\lambda = b^{-2} = 0.25$ yielding $b = 2$.

The eigenvector $(x,\ y)^{\mathrm{T}}$ corresponding to the major semi-axis ($b = 2$) has $(0.73 - 0.25)x + 0.36y = 0$, i.e. makes an angle $\tan^{-1}(-4/3)$ with the x-axis.

8.26 *Show that the quadratic surface*

$$5x^2 + 11y^2 + 5z^2 - 10yz + 2xz - 10xy = 4$$

is an ellipsoid with semi-axes of lengths 2, 1 *and* 0.5. *Find the direction of its longest axis.*

As previously, we need to solve the characteristic equation of the matrix associated with the quadric:

$$\begin{aligned} 0 &= \begin{vmatrix} 5 - \lambda & -5 & 1 \\ -5 & 11 - \lambda & -5 \\ 1 & -5 & 5 - \lambda \end{vmatrix} \\ &= (5 - \lambda)(55 - 16\lambda + \lambda^2 - 25) - 5(-5 + 25 - 5\lambda) + 1(25 - 11 + \lambda) \\ &= -\lambda^3 + 21\lambda^2 - 85\lambda + 64. \end{aligned}$$

Clearly, $\lambda = 1$ is one root of this equation, which can be written

$$-(\lambda - 1)(\lambda^2 - 20\lambda + 64) = -(\lambda - 1)(\lambda - 4)(\lambda - 16) = 0.$$

The eigenvalues are all positive and so the quadratic surface is an ellipsoid with semi-axes $(1/4)^{-1/2}$, $(4/4)^{-1/2}$ and $(16/4)^{-1/2}$, i.e. 2, 1 and 0.5. The longest axis corresponds to the smallest eigenvalue, $\lambda = 1$, and its direction $(x,\ y,\ z)^{\mathrm{T}}$ satisfies

$$(5 - 1)x - 5y + z = 0,$$
$$-5x + (11 - 1)y - 5z = 0.$$

The unit vector in this direction is $(3)^{-1/2}\,(1,\ 1,\ 1)^{\mathrm{T}}$.

8.28 *Find the eigenvalues, and sufficient of the eigenvectors, of the following matrices to be able to describe the quadratic surfaces associated with them.*

$$\text{(a)}\ \begin{pmatrix} 5 & 1 & -1 \\ 1 & 5 & 1 \\ -1 & 1 & 5 \end{pmatrix}, \quad \text{(b)}\ \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, \quad \text{(c)}\ \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 1 \end{pmatrix}.$$

In each case the eigenvalues and then the eigenvectors of the matrices can be found by the methods employed in the previous four exercises and the details will not be given here. The results and their interpretations are:

(a) The eigenvalues are 6, 6 and 3. Since they are all positive and two are equal, the surface is an ellipsoid with a circular cross-section perpendicular to the direction $(1, -1, 1)/\sqrt{3}$, which is the eigenvector corresponding to eigenvalue 3. If the maximimum radius of the circular cross-section is r, say, (when the section includes the origin) then the semi-axis of the ellipsoid in the direction of the axis of symmetery is of length $\sqrt{2}r$.

(b) The eigenvalues are 5, -1 and -1. Since two are negative and equal, the surface is a hyperboloid of revolution about an axis in the direction $(1, 1, 1)/\sqrt{3}$ (the direction of the eigenvector corresponding to the non-repeated eigenvalue). In tranformed coordinates the equation of the surface will take the form

$$\frac{y_1^2}{(1/\sqrt{5})^2} - \frac{y_2^2}{1^2} - \frac{y_3^2}{1^2} = a$$

from this it can be seen that the two halves of the hyperboloid are asymptotic to a cone of semi-angle $\tan^{-1}\sqrt{5}$ that passes through the origin and has the same symmetry axis as the hyperboloid.

(c) The eigenvalues are 6, 0 and 0. A zero eigenvalue formally implies an infinitely long semi-axis; in other words, the surface is a cylinder (not necessarily circular)

with the correponding eigenvector as the cylinder's axis. Here, there are two such eigenvalues and we have 'infinite cylinders in two directions', i.e the notional ellipsoid has degenerated into a pair of parallel planes. They are equidistant from the origin and have their normals in the direction of the eigenvector, $(1, 2, 1)$, corresponding to the only non-zero eigenvalue. The equation of the 'surface' in transformed coordinates becomes simply

$$\frac{y_1^2}{(1/\sqrt{6})^2} + 0y_2^2 + 0y_3^2 = a,$$

which describes the two planes $y_1 = \pm\sqrt{a/6}$.

8.30 *Find an orthogonal transformation that takes the quadratic form*

$$Q \equiv -x_1^2 - 2x_2^2 - x_3^2 + 8x_2x_3 + 6x_1x_3 + 8x_1x_2$$

into the form

$$\mu_1 y_1^2 + \mu_2 y_2^2 - 4y_3^2,$$

and determine μ_1 and μ_2 (see section 8.17).

Expressing Q as $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}$, the required transformation has the normalised eigenvectors e^i of A as its columns. So, we need to determine the eigenvalues and eigenvectors of A. Following the normal method:

$$\begin{aligned}
0 &= \begin{vmatrix} -1-\lambda & 4 & 3 \\ 4 & -2-\lambda & 4 \\ 3 & 4 & -1-\lambda \end{vmatrix} \\
&= -(1+\lambda)(\lambda^2 + 3\lambda - 14) + 4(16 + 4\lambda) + 3(22 + 3\lambda) \\
&= -\lambda^3 - 4\lambda^2 + 36\lambda + 144 \\
&= -(\lambda + 4)(\lambda^2 - 36) \\
&= -(\lambda + 4)(\lambda - 6)(\lambda + 6).
\end{aligned}$$

We were guided by the given answer when writing $\lambda + 4$ as a factor of the characteristic polynomial. The values of μ_1 and μ_2 are determined as 6 and -6.

We now need to find the three *normalised* eigenvectors $(x,\ y,\ z)^{\mathrm{T}}$.
For $\lambda = -4$:

$$3x + 4y + 3z = 0, \quad 4x + 2y + 4z = 0, \quad \Rightarrow \mathsf{e}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ -1)^{\mathrm{T}}.$$

For $\lambda = 6$:

$$-7x + 4y + 3z = 0, \quad 4x - 8y + 4z = 0, \quad \Rightarrow \mathsf{e}^1 = \frac{1}{\sqrt{3}}(1,\ 1,\ 1)^{\mathrm{T}}.$$

For $\lambda = -6$:

$$5x + 4y + 3z = 0, \quad 4x + 4y + 4z = 0, \quad \Rightarrow \mathsf{e}^2 = \frac{1}{\sqrt{6}}(1,\ -2,\ 1)^{\mathrm{T}}.$$

Thus, the required new coordinates are:

$$y_1 = \frac{1}{\sqrt{3}}(x_1 + x_2 + x_3),\ y_2 = \frac{1}{\sqrt{6}}(x_1 - 2x_2 + x_3),\ y_3 = \frac{1}{\sqrt{2}}(x_1 - x_3).$$

The labelling of the y_i is, of course, arbitrary.

8.32 *Do the following sets of equations have non-zero solutions? If so, find them.*

(a) $3x + 2y + z = 0, \qquad x - 3y + 2z = 0, \qquad 2x + y + 3z = 0.$
(b) $2x = b(y + z), \qquad x = 2a(y - z), \qquad x = (6a - b)y - (6a + b)z.$

(a) For the equations, written in the form

$$\mathsf{Ax} = \begin{pmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

to have a non-zero solution we must have $|\,\mathsf{A}\,| = 0$. But

$$|\,\mathsf{A}\,| = \begin{vmatrix} 3 & 2 & 1 \\ 1 & -3 & 2 \\ 2 & 1 & 3 \end{vmatrix} = 3(-11) + 2(1) + 1(7) = -24 \neq 0,$$

and so the equations have no non-trivial solutions.

(b) Rearranged in standard form, the equations read

$$\mathsf{Ax} = \begin{pmatrix} 2 & -b & -b \\ 1 & -2a & 2a \\ 1 & b - 6a & 6a + b \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Either by direct calculation or by observing that the sum of the first and third rows is equal to three time the second row, we conclude that $|\,\mathsf{A}\,| = 0$ and that a non-trivial solution is possible.

Arbitrarily taking $x = 1$, we require that

$$\begin{aligned} 2 - by - bz &= 0, \\ 1 - 2ay + 2az &= 0, \\ 4a + b - 2aby - 2aby &= 0, \\ \Rightarrow \quad y &= \frac{4a+b}{4ab}, \\ \Rightarrow \quad z &= \frac{2}{b} - y = \frac{4a-b}{4ab}. \end{aligned}$$

Thus the solution is any multiple of $(4ab,\ 4a+b,\ 4a-b)^{\mathrm{T}}$.

8.34 *Solve the following simultaneous equations for x_1, x_2 and x_3, using matrix methods:*

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1, \\ 3x_1 + 4x_2 + 5x_3 &= 2, \\ x_1 + 3x_2 + 4x_3 &= 3. \end{aligned}$$

We need to invert the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 1 & 3 & 4 \end{pmatrix},$$

whose determinant is $1(1) + 2(-7) + 3(5) = 2$. This is non-zero and so A has an inverse. The matrix of cofactors is

$$\mathsf{C} = \begin{pmatrix} 1 & -7 & 5 \\ 1 & 1 & -1 \\ -2 & 4 & -2 \end{pmatrix},$$

from which it follows that $\mathsf{A}^{-1} = (1/|\mathsf{A}|)\mathsf{C}^{\mathrm{T}}$ is given by

$$\mathsf{A}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix}.$$

Finally, rewriting the given equation $\mathsf{Ax} = \mathsf{y}$ as $\mathsf{x} = \mathsf{A}^{-1}\mathsf{y}$, we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 1 & -2 \\ -7 & 1 & 4 \\ 5 & -1 & -2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -3 \\ 7 \\ -3 \end{pmatrix}.$$

Thus $x_1 = -\frac{3}{2}$, $x_2 = \frac{7}{2}$ and $x_3 = -\frac{3}{2}$.

8.36 *Find the condition(s) on α such that the simultaneous equations*

$$\begin{aligned} x_1 + \alpha x_2 &= 1, \\ x_1 - x_2 + 3x_3 &= -1, \\ 2x_1 - 2x_2 + \alpha x_3 &= -2 \end{aligned}$$

have (a) *exactly one solution,* (b) *no solutions, or* (c) *an infinite number of solutions; give all solutions where they exist.*

As usual, and in the normal notation, we start by examining

$$|\,\mathsf{A}\,| = 1(6-\alpha) + \alpha(6-\alpha) = (1+\alpha)(6-\alpha).$$

(a) For exactly one solution we need $|\mathsf{A}| \neq 0$, i.e. $\alpha \neq -1$ and $\alpha \neq 6$. Then

$$\mathsf{A}^{-1} = \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha & -\alpha^2 & 3\alpha \\ 6-\alpha & \alpha & -3 \\ 0 & 2+2\alpha & -1-\alpha \end{pmatrix}$$

and

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha & -\alpha^2 & 3\alpha \\ 6-\alpha & \alpha & -3 \\ 0 & 2+2\alpha & -1-\alpha \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \\ &= \frac{1}{(1+\alpha)(6-\alpha)} \begin{pmatrix} 6-\alpha+\alpha^2-6\alpha \\ 6-\alpha-\alpha+6 \\ -2-2\alpha+2+2\alpha \end{pmatrix} \\ &= \left(\frac{1-\alpha}{1+\alpha}, \frac{2}{1+\alpha}, 0 \right)^{\mathrm{T}}. \end{aligned}$$

(b) and (c). For no or infinitely many solutions the matrix must have rank 2 or less, which requires that either $\alpha = -1$ or $\alpha = 6$.

With $\alpha = -1$ the equations become

$$\begin{aligned} x_1 - x_2 &= 1, \\ x_1 - x_2 + 3x_3 &= -1, \\ 2x_1 - 2x_2 - x_3 &= -2. \end{aligned}$$

Substituting from the first equation for $x_1 - x_2$ leaves two equations for x_3 which are contradictory; this is case (b) of no solution.

With $\alpha = 6$ the equations become

$$\begin{aligned} x_1 + 6x_2 &= 1, \\ x_1 - x_2 + 3x_3 &= -1, \\ 2x_1 - 2x_2 + 6x_3 &= -2. \end{aligned}$$

The last two equations are multiples of each other, but not of the first. Therefore there are infinitely many solutions containing one free parameter. Taking this as $x_2 = \beta$, the general solution is

$$(x_1,\ x_2,\ x_3)^{\mathrm{T}} = \left(1 - 6\beta,\ \beta,\ \tfrac{1}{3}(7\beta - 2)\right)^{\mathrm{T}}.$$

This solution is that for case (c) and corresponds to figure 8.1(a) in the main text; case (b) corresponds to figure 8.1(b).

8.38 *Make an LU decomposition of the matrix*

$$\mathsf{A} = \begin{pmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{pmatrix}.$$

Hence solve $\mathsf{Ax} = \mathsf{b}$ *for* (i) $\mathsf{b} = (-4 \quad 1 \quad 8 \quad -5)^{\mathrm{T}}$, *and* (ii) $\mathsf{b} = (-10 \quad 0 \quad -3 \quad -24)^{\mathrm{T}}$. *Deduce that* $\det \mathsf{A} = -160$ *and confirm this by direct calculation.*

To avoid a lot of subscripts we will use single lower-case letters as the elements of the upper- and lower-diagonal matrices. We also make the immediately-apparent entries in U.

We need

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ a & 1 & 0 & 0 \\ b & c & 1 & 0 \\ d & e & f & 1 \end{pmatrix} \begin{pmatrix} 2 & -3 & 1 & 3 \\ 0 & g & h & j \\ 0 & 0 & k & l \\ 0 & 0 & 0 & m \end{pmatrix} = \begin{pmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{pmatrix}.$$

From the 1st column of A:

$$a = \tfrac{1}{2}, \quad b = \tfrac{5}{2} \quad \text{and} \quad d = \tfrac{3}{2}.$$

Then, from the 2nd row:

$$\begin{aligned} -3a + g = 4 &\Rightarrow g = \tfrac{11}{2}, \\ a + h = -3 &\Rightarrow h = -\tfrac{7}{2}, \\ 3a + j = -3 &\Rightarrow j = -\tfrac{9}{2}. \end{aligned}$$

From the 2nd column:

$$-3b + gc = 3 \Rightarrow c = \tfrac{21}{11},$$
$$-3d + ge = -6 \Rightarrow e = -\tfrac{3}{11}.$$

From the 3rd row:

$$b + ch + k = -1 \Rightarrow k = -1 + \tfrac{92}{22} = \tfrac{35}{11},$$
$$3b + cj + l = -1 \Rightarrow l = -1 + \tfrac{24}{22} = \tfrac{1}{11}.$$

From the 3rd column:

$$d + he + fk = -3 \Rightarrow f = (\tfrac{-120}{20})/(\tfrac{35}{11}) = -\tfrac{12}{7}.$$

Finally, from the 4th row:

$$3d + je + fl + m = 1 \Rightarrow m = 1 - \tfrac{9}{2} - \tfrac{27}{22} + \tfrac{12}{77} = -\tfrac{32}{7}.$$

(i) We first solve $\mathsf{Ly} = \mathsf{b}$ as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{5}{2} & \frac{21}{11} & 1 & 0 \\ \frac{3}{2} & -\frac{3}{11} & -\frac{12}{7} & 1 \end{pmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ 8 \\ -5 \end{bmatrix}.$$

That $y_1 = -4$ and $y_2 = 3$ are immediately apparent. The third row gives $y_3 = 8 + 10 - \frac{63}{11} = \frac{135}{11}$, whilst the fourth yields $y_4 = -5 + 6 + \frac{9}{11} + \frac{12}{7}\,\frac{135}{11} = \frac{160}{7}$. The solution vector x is now deduced from $\mathsf{Ux} = \mathsf{y}$:

$$\begin{pmatrix} 2 & -3 & 1 & 3 \\ 0 & \frac{11}{2} & -\frac{7}{2} & -\frac{9}{2} \\ 0 & 0 & \frac{25}{11} & \frac{1}{11} \\ 0 & 0 & 0 & -\frac{32}{7} \end{pmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -4 \\ 3 \\ \frac{135}{11} \\ \frac{160}{7} \end{bmatrix}.$$

That $x_4 = -5$ is obvious. The third row gives $x_3 = \frac{11}{35}(\frac{135}{11} + \frac{5}{11}) = 4$, whilst the second yields $x_2 = \frac{2}{11}(3 + 14 - \frac{45}{2}) = -1$. Finally, the top row gives $x_1 = \frac{1}{2}(-4 - 3 - 4 + 15) = 2$.

(ii) This calculation proceeds just as in (i). The intermediate vector y is found to be $(-10 \quad 5 \quad \frac{137}{11} \quad \frac{96}{7})^{\mathrm{T}}$ and the solution vector $\mathsf{x} = (-1 \quad 1 \quad 4 \quad -3)^{\mathrm{T}}$.

The determinant of A is given by the product of the diagonal entries of the matrix U, i.e. $|\mathsf{A}| = 2 \times \frac{11}{2} \times \frac{35}{11} \times (-\frac{32}{7}) = -160$.

Confirming this by direct calculation:

$$\begin{vmatrix} 2 & -3 & 1 & 3 \\ 1 & 4 & -3 & -3 \\ 5 & 3 & -1 & -1 \\ 3 & -6 & -3 & 1 \end{vmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 7 & -5 & -3 & 6 \\ 7 & 0 & -1 & 2 \\ 9 & -15 & -3 & 10 \end{pmatrix}$$

$$= 1 \begin{vmatrix} 7 & -5 & 6 \\ 7 & 0 & 2 \\ 9 & -15 & 10 \end{vmatrix}$$

$$= 7(30) - 5(-52) + 6(-105) = -160.$$

At the first step, an appropriate multiple of the 3rd column was subtracted from each of the other columns.

8.40 *Find the equation satisfied by the squares of the singular values of the matrix associated with the following over-determined set of equations:*

$$\begin{aligned} 2x + 3y + z &= 0 \\ x - y - z &= 1 \\ 2x + y &= 0 \\ 2y + z &= -2. \end{aligned}$$

Show that one of the singular values is close to zero. Determine the two larger singular values by an appropriate iteration process and the smallest by indirect calculation.

The matrix and its (Hermitian) transpose associated with the set of equations are

$$\mathsf{A} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & -1 & -1 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix}, \quad \mathsf{A}^\dagger = \begin{pmatrix} 2 & 1 & 2 & 0 \\ 3 & -1 & 1 & 2 \\ 1 & -1 & 0 & 1 \end{pmatrix},$$

and their 3×3 product is

$$\mathsf{A}^\dagger \mathsf{A} = \begin{pmatrix} 9 & 7 & 1 \\ 7 & 15 & 6 \\ 1 & 6 & 3 \end{pmatrix}.$$

To determine the singular values of A we find the eigenvalues of this product:

$$
\begin{aligned}
0 &= \begin{vmatrix} 9-\lambda & 7 & 1 \\ 7 & 15-\lambda & 6 \\ 1 & 6 & 3-\lambda \end{vmatrix} \\
&= (9-\lambda)(9-18\lambda+\lambda^2) + 7(7\lambda-15) + 1(\lambda+27) \\
&= -(\lambda^3 - 27\lambda^2 + 121\lambda - 3).
\end{aligned}
$$

This is the equation satisfied by the squares of the singular values of A.

Using the properties of the three roots λ_i of the cubic equation, we conclude that, since their sum $\sum_i \lambda_i = 27$ whilst their product is only $\prod_i \lambda_i = 3$, at least one of the roots must be close to zero.

Using either the rearrangement iteration method,

$$x_{n+1} = \left(27x_n^2 - 121x_n + 3\right)^{1/3},$$

or the Newton-Raphson method,

$$x_{n+1} = x_n - \frac{x_n^3 - 27x_n^2 + 121x_n - 3}{3x_n^2 - 54x_n + 121},$$

the two larger roots are found to be 21.33521 and 5.639852. The third root can be found most accurately as $3/(21.33521 \times 5.639852) = 0.024938$.

The corresponding singular values are the square roots of these eigenvalues, namely, 4.6190, 2.3748 and 0.1579.

8.42 *Find the SVD form of the matrix*

$$\mathsf{A} = \begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}.$$

Use it to determine the best solution x *of the equation* Ax = b *when* (i) $\mathsf{b} = (6 \quad -39 \quad 15 \quad 18)^{\mathrm{T}}$, (ii) $\mathsf{b} = (9 \quad -42 \quad 15 \quad 15)^{\mathrm{T}}$, *showing that* (i) *has an exact solution, but that the best solution to* (ii) *has a residual of* $\sqrt{18}$.

We start by computing

$$\mathsf{A}^\dagger\mathsf{A} = \begin{pmatrix} 22 & 1 & 19 & -6 \\ 28 & -2 & -2 & 12 \\ -22 & -19 & -1 & 6 \end{pmatrix} \begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}$$
$$= \begin{pmatrix} 882 & 504 & -558 \\ 504 & 936 & -504 \\ -558 & -504 & 882 \end{pmatrix}.$$

And then find its eigenvalues:

$$|\mathsf{A}^\dagger\mathsf{A} - \lambda| = \begin{vmatrix} 882-\lambda & 504 & -558 \\ 504 & 936-\lambda & -504 \\ -558 & -504 & 882-\lambda \end{vmatrix}$$
$$= \begin{vmatrix} 324-\lambda & 0 & 324-\lambda \\ 504 & 936-\lambda & -504 \\ -558 & -504 & 882-\lambda \end{vmatrix}$$
$$= (324-\lambda)\left(\begin{vmatrix} 936-\lambda & -504 \\ -504 & 882-\lambda \end{vmatrix} + \begin{vmatrix} 504 & 936-\lambda \\ -558 & -504 \end{vmatrix} \right)$$
$$= (324-\lambda)(\lambda^2 - 1818\lambda + 571536 - 558\lambda + 268272)$$
$$= (324-\lambda)(\lambda^2 - 2376\lambda + 839808)$$
$$= (324-\lambda)(\lambda - 432)(\lambda - 1944).$$

This shows that the singular values ($\sqrt{\lambda}$) are $18\sqrt{6}$, $12\sqrt{3}$ and 18. We have, as usual, taken the singular values to be positive; this choice is reflected in the signs of the terms in the matrix U calculated later.

The corresponding normalised eigenvectors $(x_1,\ x_2,\ x_3)^{\mathrm{T}}$ satisfy:

$$-1062x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 - 1008x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^1 = \frac{1}{\sqrt{3}}(1,\ 1,\ -1)^{\mathrm{T}}.$$
$$450x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 + 504x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^2 = \frac{1}{\sqrt{6}}(1,\ -2,\ -1)^{\mathrm{T}}.$$
$$558x_1 + 504x_2 - 558x_3 = 0,$$
$$504x_1 + 504x_2 - 504x_3 = 0. \Rightarrow \mathsf{v}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ 1)^{\mathrm{T}}.$$

The next step is to calculate the (normalised) column vectors u^i from $(s_i)^{-1}\mathsf{A}\mathsf{v}^i =$

u^i:

$$\mathsf{u}^1 = \frac{1}{18\sqrt{6}}\frac{1}{\sqrt{3}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

$$\mathsf{u}^2 = \frac{1}{12\sqrt{3}}\frac{1}{\sqrt{6}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ 2 \\ 2 \\ -3 \end{bmatrix}.$$

$$\mathsf{u}^3 = \frac{1}{18}\frac{1}{\sqrt{2}}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Although we will not need its components for the present exercise, we now find the fourth base vector (to make U a unitary matrix). It has to be orthogonal to the three vectors just found; simple simultaneous equations show that, when normalised, it is $\mathsf{u}^4 = (1/\sqrt{18})(-1 \quad 2 \quad 2 \quad 3)^{\mathrm{T}}$.

Thus, finally, we are able to write $\mathsf{A} = \mathsf{USV}^\dagger$ explicitly as

$$\frac{1}{N}\begin{pmatrix} 4 & -1 & 0 & -1 \\ 1 & 2 & -3 & 2 \\ 1 & 2 & 3 & 2 \\ 0 & -3 & 0 & 3 \end{pmatrix}\begin{pmatrix} 18\sqrt{6} & 0 & 0 \\ 0 & 12\sqrt{3} & 0 \\ 0 & 0 & 18 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ 1 & -2 & -1 \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix},$$

where $N = \sqrt{18} \times \sqrt{6}$.

The best solution to $\mathsf{Ax} = \mathsf{b}$ is given by $\mathsf{x} = \mathsf{V\bar{S}U}^\dagger\mathsf{b}$. We therefore compute $\mathsf{R} = \mathsf{V\bar{S}U}^\dagger$ as (with N defined as previously)

$$\frac{1}{N}\begin{pmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ -\sqrt{2} & -1 & \sqrt{3} \end{pmatrix}\begin{pmatrix} \frac{1}{18\sqrt{6}} & 0 & 0 & 0 \\ 0 & \frac{1}{12\sqrt{3}} & 0 & 0 \\ 0 & 0 & \frac{1}{18} & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ 0 & -3 & 3 & 0 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{N}\begin{pmatrix} \frac{1}{18\sqrt{3}} & \frac{1}{12\sqrt{3}} & \frac{\sqrt{3}}{18} & 0 \\ \frac{1}{18\sqrt{3}} & -\frac{1}{6\sqrt{3}} & 0 & 0 \\ -\frac{1}{18\sqrt{3}} & -\frac{1}{12\sqrt{3}} & \frac{\sqrt{3}}{18} & 0 \end{pmatrix}\begin{pmatrix} 4 & 1 & 1 & 0 \\ -1 & 2 & 2 & -3 \\ 0 & -3 & 3 & 0 \\ -1 & 2 & 2 & 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}.$$

(i) With $\mathsf{b} = (6 \quad -39 \quad 15 \quad 18)^{\mathrm{T}}$ the best solution is

$$\mathsf{x} = \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}\begin{bmatrix} 6 \\ -39 \\ 15 \\ 18 \end{bmatrix}$$

$$\begin{aligned} x_1 &= \tfrac{1}{648}(30 + 390 + 390 - 162) = 1, \\ x_2 &= \tfrac{1}{648}(84 + 390 - 150 + 324) = 1, \\ x_3 &= \tfrac{1}{648}(-30 + 1014 + 150 + 162) = 2. \end{aligned}$$

Thus, the best solution is $(1,\ 1,\ 2)^{\mathrm{T}}$ and the residual vector given by

$$\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} - \begin{bmatrix} 6 \\ -39 \\ 15 \\ 18 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The residual vector is the zero vector and the best solution is an exact solution.

(ii) With $\mathsf{b} = (9 \quad -42 \quad 15 \quad 15)^{\mathrm{T}}$ the best solution is

$$\mathsf{x} = \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 5 & -10 & 26 & -9 \\ 14 & -10 & -10 & 18 \\ -5 & -26 & 10 & 9 \end{pmatrix}\begin{bmatrix} 9 \\ -42 \\ 15 \\ 15 \end{bmatrix} = \frac{1}{36}\begin{pmatrix} 40 \\ 37 \\ 74 \end{pmatrix}.$$

With $\frac{1}{36}(40,\ 37,\ 74)^{\mathrm{T}}$ as the best solution, the residual vector is

$$\frac{1}{36}\begin{pmatrix} 22 & 28 & -22 \\ 1 & -2 & -19 \\ 19 & -2 & -1 \\ -6 & 12 & 6 \end{pmatrix}\begin{pmatrix} 40 \\ 37 \\ 74 \end{pmatrix} - \begin{bmatrix} 9 \\ -42 \\ 15 \\ 15 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 2 \\ 3 \end{bmatrix}.$$

We conclude that the solution is not exact and that the residual (equal to the modulus of the residual vector) is $[(-1)^2 + 2^2 + 2^2 + 3^2]^{1/2} = \sqrt{18}$.

9

Normal modes

9.2 *A double pendulum, smoothly pivoted at A, consists of two light rigid rods, AB and BC, each of length l, which are smoothly jointed at B and carry masses m and αm at B and C respectively. The pendulum makes small oscillations in one plane under gravity; at time t, AB and BC make angles θ(t) and φ(t) respectively with the downward vertical. Find quadratic expressions for the kinetic and potential energies of the system and hence show that the normal modes have angular frequencies given by*

$$\omega^2 = \frac{g}{l}\left[1 + \alpha \pm \sqrt{\alpha(1+\alpha)}\right].$$

For $\alpha = 1/3$, show that in one of the normal modes the mid-point of BC does not move during the motion.

For small oscillations, the sideways displacements and consequent velocities of the masses are

$$x_1 = l\theta \quad \text{and} \quad \dot{x}_1 = l\dot{\theta}$$
$$x_2 = l\theta + l\phi \quad \text{and} \quad \dot{x}_2 = l\dot{\theta} + l\dot{\phi}$$

To first order in small quantities (i.e. ignoring any vertical components of velocity) the total kinetic energy of the system is therefore

$$\begin{aligned}\mathrm{KE} &= \tfrac{1}{2}m\dot{x}_1^2 + \tfrac{1}{2}\alpha m\dot{x}_2^2 \\ &= \tfrac{1}{2}ml^2[\dot{\theta}^2 + \alpha(\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2)],\end{aligned}$$

and the kinetic energy matrix

$$\mathsf{T} = \tfrac{1}{2}ml^2\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & \alpha \end{pmatrix}.$$

Remembering that the raising of the lower mass receives a contribution from that of the upper mass and working to second order in the displacements, the potential energy is

$$\begin{aligned}\mathrm{PE} &= mgl(1-\cos\theta) + \alpha mgl[(1-\cos\theta)+(1-\cos\phi)]\\ &\approx \tfrac{1}{2}mgl\theta^2 + \tfrac{1}{2}\alpha mgl(\theta^2+\phi^2).\end{aligned}$$

The potential energy matrix is therefore

$$\mathsf{V} = \tfrac{1}{2}mgl\begin{pmatrix} 1+\alpha & 0 \\ 0 & \alpha \end{pmatrix}.$$

The normal frequencies, determined by $|-\omega^2\mathsf{T}+\mathsf{V}| = 0$, are given by

$$\tfrac{1}{2}m\left|-l^2\omega^2\begin{pmatrix} 1+\alpha & \alpha \\ \alpha & \alpha \end{pmatrix} + gl\begin{pmatrix} 1+\alpha & 0 \\ 0 & \alpha \end{pmatrix}\right| = 0.$$

Writing $\omega^2 l/g$ as λ, this requirement is

$$\begin{vmatrix} (1+\alpha)-\lambda(1+\alpha) & -\lambda\alpha \\ -\lambda\alpha & \alpha-\lambda\alpha \end{vmatrix} = 0,$$

i.e. $\quad (1+\alpha)(1-\lambda)\alpha(1-\lambda)-\lambda^2\alpha^2 = \lambda^2-2(1+\alpha)\lambda+(1+\alpha) = 0$

The angular frequencies of the two normal modes are given by the roots of this quadratic equation as

$$\omega^2 = \frac{g\lambda}{l} = \frac{g}{l}[(1+\alpha)\pm\sqrt{\alpha(1+\alpha)}]. \qquad (*)$$

For $\alpha = \frac{1}{3}$ the two values of ω^2 become $\dfrac{2g}{l}$ and $\dfrac{2g}{3l}$, and the components of the solution vector must satisfy (using the second line of the matrix-vector equation)

$$\begin{aligned} -2\frac{1}{3}\theta + \left(\frac{1}{3}-\frac{2}{3}\right)\phi = 0 &\Rightarrow \phi = -2\theta \quad \text{when} \quad \lambda = 2,\\ -\frac{2}{3}\frac{1}{3}\theta + \left(\frac{1}{3}-\frac{2}{9}\right)\phi = 0 &\Rightarrow \phi = 2\theta \quad \text{when} \quad \lambda = \frac{2}{3}.\end{aligned}$$

For the mid-point of BC, $x = l\theta + \frac{1}{2}l\phi$. In the higher frequency mode, $\phi = -2\theta$ and $\dot{x} = l\dot{\theta} + \frac{1}{2}l(-2\dot{\theta}) = 0$, i.e. the mid-point does not move.

Note It is of some interest to check that $(*)$ gives the correct limits for small and large α. It obviously leads (correctly) to $\omega^2 = g/l$ (repeated) as $\alpha \to 0$. For $\alpha \to \infty$ one solution has an (unphysical) infinite frequency; the other has

$$\begin{aligned}\frac{l\omega^2}{g} &= \lim_{\alpha\to\infty} 1+\alpha-\sqrt{\alpha(1+\alpha)} = \lim_{\alpha\to\infty} 1+\alpha-\alpha\left(1+\frac{1}{\alpha}\right)^{1/2}\\ &= \lim_{\alpha\to\infty} 1+\alpha-\alpha\left(1+\frac{1}{2\alpha}+\cdots\right) = \frac{1}{2},\end{aligned}$$

Figure 9.1 The circuit and notation for example 9.4.

i.e. the correct value for a simple pendulum of length $2l$.

9.4 *Consider the circuit consisting of three equal capacitors and two different inductors shown in figure 9.1. For charges Q_i on the capacitors and currents I_i through the components, write down Kirchhoff's law for the total voltage change around each of two complete circuit loops. Note that, to within an unimportant constant, the conservation of current implies that $Q_3 = Q_1 - Q_2$ and hence express the loop equations in the form given in (9.7), namely*

$$\mathsf{A}\ddot{\mathsf{Q}} + \mathsf{B}\mathsf{Q} = 0.$$

Use this to show that the normal frequencies of the circuit are given by

$$\omega^2 = \frac{1}{CL_1L_2}\left[L_1 + L_2 \pm (L_1^2 + L_2^2 - L_1L_2)^{1/2}\right].$$

Obtain the same matrices and result by finding the total energy stored in the various capacitors (typically $Q^2/(2C)$) and in the inductors (typically $LI^2/2$).

For the special case $L_1 = L_2 = L$ determine the relevant eigenvectors and so describe the patterns of current flow in the circuit.

We apply Kirchhoff's law to a loop taken round the left-hand part of the circuit and to one taken round the whole circuit (one round the right-hand part does not give any further independent information as there are only two currents needed to specify the situtation).

$$L_1\dot{I}_1 + \frac{Q_1}{C} + \frac{Q_3}{C} = 0,$$
$$L_1\dot{I}_1 + \frac{Q_1}{C} + \frac{Q_2}{C} + L_2\dot{I}_2 = 0,$$

with $I_1 = \dot{Q}_1$, $I_2 = \dot{Q}_2$ and $I_1 - I_2 = \dot{Q}_3 = \dot{Q}_1 - \dot{Q}_2$.

Writing everything in terms of Q_1, Q_2 and their time derivatives,

$$L_1\ddot{Q}_1 + \frac{Q_1}{C} + \frac{Q_1 - Q_2}{C} = 0,$$
$$L_1\ddot{Q}_1 + \frac{Q_1}{C} + \frac{Q_2}{C} + L_2\ddot{Q}_2 = 0.$$

In matrix and vector form, $\mathsf{A}\ddot{\mathsf{Q}} + \mathsf{B}\mathsf{Q} = \mathsf{0}$, these equations read

$$\begin{pmatrix} L_1 & 0 \\ L_1 & L_2 \end{pmatrix}\begin{pmatrix} \ddot{Q}_1 \\ \ddot{Q}_2 \end{pmatrix} + \begin{pmatrix} 2C^{-1} & -C^{-1} \\ C^{-1} & C^{-1} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

To find the normal frequencies, we now have to solve $|\mathsf{B} - \omega^2\mathsf{A}| = \mathsf{0}$. After mutiplying through by C, this reads

$$\begin{vmatrix} 2 - \omega^2 L_1 C & -1 \\ 1 - \omega^2 L_1 C & 1 - \omega^2 L_2 C \end{vmatrix} = 0,$$
$$2 - (L_1 + 2L_2)C\omega^2 + L_1L_2C^2\omega^4 + 1 - L_1C\omega^2 = 0,$$
$$L_1L_2C^2\omega^4 - 2(L_1 + L_2)C\omega^2 + 3 = 0.$$

Hence the normal frequencies are

$$\omega^2 = \frac{(L_1 + L_2)C \pm \sqrt{(L_1 + L_2)^2C^2 - 3L_1L_2C^2}}{L_1L_2C^2}$$
$$= \frac{1}{CL_1L_2}\left[L_1 + L_2 \pm (L_1^2 + L_2^2 - L_1L_2)^{1/2}\right].$$

We now repeat this derivation, working in terms of stored energy, rather than the equations of motion.

The total 'kinetic energy' is the energy stored in the magnetic fields of the inductors (typically $\frac{1}{2}LI^2$). This is

$$T = \tfrac{1}{2}L_1I_1^2 + \tfrac{1}{2}L_2I_2^2.$$

The 'potential energy' term is the energy stored in the capacitors (typically $\frac{1}{2}C^{-1}Q^2$). This is

$$V = \tfrac{1}{2}C^{-1}[\,Q_1^2 + Q_2^2 + (Q_1 - Q_2)^2\,].$$

The charateristic equation determining (the squares of) the normal mode frequencies is therefore

$$\begin{vmatrix} 2C^{-1} - \omega^2 L_1 & -C^{-1} \\ -C^{-1} & 2C^{-1} - \omega^2 L_2 \end{vmatrix} = 0,$$
$$4C^{-2} - 2(L_1 + L_2)C^{-1}\omega^2 + L_1L_2\omega^4 - C^{-2} = 0.$$

After multiplication by C^2, this is the same equation as that obtained previously and has the same roots for ω^2.

If $L_1 = L_2 = L$ then one mode has $\omega^2 = (LC)^{-1}$ and the eigenvector is given by

$$(2-1)Q_1 - Q_2 = 0 \quad \Rightarrow \quad Q_1 = Q_2.$$

Under these circumstances $Q_3 = 0$ and no current flows through the central capacitor.

The other mode has $\omega^2 = 3(LC)^{-1}$; for this mode,

$$(2-3)Q_1 - Q_2 = 0 \quad \Rightarrow \quad Q_1 = -Q_2.$$

In this case, equal currents I (one clockwise, one anticlockwise) flow in the two loops and the current through the central capacitor is $2I$.

9.6 The simultaneous reduction to diagonal form of two real symmetric quadratic forms.

Consider the two real symmetric quadratic forms $\mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u}$ *and* $\mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u}$, *where* u^{T} *stands for the row matrix* $(x \quad y \quad z)$, *and denote by* u^n *those column matrices that satisfy*

$$\mathsf{B}\mathsf{u}^n = \lambda_n \mathsf{A}\mathsf{u}^n \qquad (*),$$

in which n is a label and the λ_n *are real, non-zero and all different.*

(a) *By multiplying* $(*)$ *on the left by* $(\mathsf{u}^m)^{\mathrm{T}}$ *and the transpose of the corresponding equation for* u^m *on the right by* u^n, *show that* $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n = 0$ *for* $n \neq m$.
(b) *By noting that* $\mathsf{A}\mathsf{u}^n = (\lambda_n)^{-1}\mathsf{B}\mathsf{u}^n$, *deduce that* $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}\mathsf{u}^n = 0$ *for* $m \neq n$.
(c) *It can be shown that the* u^n *are linearly independent; the next step is to construct a matrix* P *whose columns are the vectors* u^n.
(d) *Make a change of variables* $\mathsf{u} = \mathsf{P}\mathsf{v}$ *such that* $\mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u}$ *becomes* $\mathsf{v}^{\mathrm{T}}\mathsf{C}\mathsf{v}$, *and* $\mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u}$ *becomes* $\mathsf{v}^{\mathrm{T}}\mathsf{D}\mathsf{v}$. *Show that* C *and* D *are diagonal by showing that* $c_{ij} = 0$ *if* $i \neq j$ *and similarly for* d_{ij}.

Thus $\mathsf{u} = \mathsf{P}\mathsf{v}$ *or* $\mathsf{v} = \mathsf{P}^{-1}\mathsf{u}$ *reduces both quadratics to diagonal form.*

To summarise, the method is as follows:

(a) *find the* λ_n *that allow* $(*)$ *a non-zero solution, by solving* $|\mathsf{B} - \lambda\mathsf{A}| = 0$;
(b) *for each* λ_n *construct* u^n;
(c) *construct the non-singular matrix* P *whose columns are the vectors* u^n;
(d) *make the change of variable* $\mathsf{u} = \mathsf{P}\mathsf{v}$.

We are given that $\mathsf{A}^{\mathrm{T}} = \mathsf{A}$, $\mathsf{B}^{\mathrm{T}} = \mathsf{B}$ and $\lambda_m \neq \lambda_n$ if $m \neq n$.

(a) From (∗) and its transpose, with n replaced by m in the latter,

$$\begin{aligned}
\mathsf{B}\mathsf{u}^n &= \lambda_n \mathsf{A}\mathsf{u}^n \quad \text{and } (\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}^{\mathrm{T}} = \lambda_m (\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}^{\mathrm{T}}, \\
(\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}\mathsf{u}^n &= \lambda_n (\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n \quad \text{and } (\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}\mathsf{u}^n = \lambda_m (\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n, \\
\Rightarrow \quad (\lambda_n - \lambda_m)(\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n &= 0, \\
\Rightarrow \quad (\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n &= 0 \quad \text{if} \quad m \neq n.
\end{aligned}$$

(b) We next rearrange (∗) to read $\mathsf{A}\mathsf{u}^n = (\lambda_n)^{-1}\mathsf{B}\mathsf{u}^n$. This equation is of the same form as (∗) but with the roles of A and B interchanged. It therefore follows that, since $(\lambda_m)^{-1} \neq (\lambda_n)^{-1}$ for $m \neq n$, $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{B}\mathsf{u}^n = 0$ if $m \neq n$.

(c) We now change variables from u to v where the two variables are connected by $\mathsf{u} = \mathsf{P}\mathsf{v}$; here P is the matrix whose columns are the u^n. Thus

$$\mathsf{P} = (\mathsf{u}^1 \quad \mathsf{u}^2 \quad \mathsf{u}^3), \quad \text{i.e.} \quad P_{ij} = (\mathsf{u}^j)_i \quad \text{and} \quad P^{\mathrm{T}}_{ij} = (\mathsf{u}^i)_j.$$

Then,

$$Q_1 = \mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u} = (\mathsf{P}\mathsf{v})^{\mathrm{T}}\mathsf{A}(\mathsf{P}\mathsf{v}) = \mathsf{v}^{\mathrm{T}}\mathsf{P}^{\mathrm{T}}\mathsf{A}\mathsf{P}\mathsf{v} = \mathsf{v}^{\mathrm{T}}\mathsf{C}\mathsf{v},$$

where the elements of C are c_{ij} given by

$$\begin{aligned}
c_{ij} &= (\mathsf{P}^{\mathrm{T}}\mathsf{A}\mathsf{P})_{ij} \\
&= P^{\mathrm{T}}_{ik} A_{kl} P_{lj} \\
&= (\mathsf{u}^i)_k A_{kl} (\mathsf{u}^j)_l \\
&= (\mathsf{u}^i)^{\mathrm{T}}\mathsf{A}(\mathsf{u}^j) \\
&= 0 \quad \text{if} \quad i \neq j.
\end{aligned}$$

Similarly,

$$Q_2 = \mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u} = \mathsf{v}^{\mathrm{T}}\mathsf{D}\mathsf{v},$$

with $d_{ij} = 0$ if $i \neq j$.

Thus, the transformation $\mathsf{u} = \mathsf{P}\mathsf{v}$ (or $\mathsf{v} = \mathsf{P}^{-1}\mathsf{u}$) reduces both Q_1 and Q_2 to diagonal form. We note that, in general, P is not an orthogonal matrix, even if the vectors u^n are normalised.

9.8 *(*It is recommended that the reader does not attempt this question until exercise 9.6 has been studied.*)*

Find a real linear transformation that simultaneously reduces the quadratic forms

$$3x^2 + 5y^2 + 5z^2 + 2yz + 6zx - 2xy,$$

$$5x^2 + 12y^2 + 8yz + 4zx$$

to diagonal form.

With the two quadratic forms

$$Q_1 = \mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x} = 3x^2 + 5y^2 + 5z^2 + 2yz + 6zx - 2xy,$$
$$Q_2 = \mathsf{x}^{\mathrm{T}}\mathsf{B}\mathsf{x} = 5x^2 + 12y^2 + 8yz + 4zx,$$

we must find the vectors that satisfy $\mathsf{B}\mathsf{u} = \lambda\mathsf{A}\mathsf{u}$. To do this, we evaluate

$$\begin{aligned}
&|\mathsf{B} - \lambda\mathsf{A}| \\
&= \begin{vmatrix} 5-3\lambda & \lambda & 2-3\lambda \\ \lambda & 12-5\lambda & 4-\lambda \\ 2-3\lambda & 4-\lambda & -5\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5 & 36-14\lambda & 14-6\lambda \\ \lambda & 12-5\lambda & 4-\lambda \\ 2 & 40-16\lambda & 12-8\lambda \end{vmatrix} \\
&= \begin{vmatrix} 5 & -6+4\lambda & 14-6\lambda \\ \lambda & -2\lambda & 4-\lambda \\ 2 & 4+8\lambda & 12-8\lambda \end{vmatrix} \\
&= 5(24\lambda^2 - 52\lambda - 16) + \lambda(-16\lambda^2 - 8\lambda + 128) + 2(-16\lambda^2 + 50\lambda - 24) \\
&= -16\lambda^3 + 80\lambda^2 - 32\lambda - 128.
\end{aligned}$$

Setting this expression equal to zero gives the cubic equation satisfied by acceptable values of λ:

$$\begin{aligned}
\lambda^3 - 5\lambda^2 + 2\lambda + 8 &= 0, \\
(\lambda+1)(\lambda-2)(\lambda-4) &= 0, \\
&\Rightarrow \quad \lambda = -1 \text{ or } 2 \text{ or } 4.
\end{aligned}$$

The three required vectors u^i must have components that satisfy:

For $\lambda = -1$

$$\begin{aligned}
8x - y + 5z &= 0, \\
-x + 17y + 5z &= 0, \quad \Rightarrow \quad \mathsf{u}^1 = (2,\ 1,\ -3)^{\mathrm{T}}.
\end{aligned}$$

For $\lambda = 2$

$$\begin{aligned}
-x + 2y - 4z &= 0, \\
2x + 2y + 2z &= 0, \quad \Rightarrow \quad \mathsf{u}^2 = (-2,\ 1,\ 1)^{\mathrm{T}}.
\end{aligned}$$

For $\lambda = 4$

$$\begin{aligned}
-7x + 4y - 10z &= 0, \\
4x - 8y &= 0, \quad \Rightarrow \quad \mathsf{u}^3 = (2,\ 1,\ -1)^{\mathrm{T}}.
\end{aligned}$$

The final step is to form the transformation matrix P, using these three vectors as its columns:

$$\mathsf{P} = \begin{pmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ -3 & 1 & -1 \end{pmatrix}$$

and read off from its rows the required transformation

$$\begin{aligned} x &= 2\xi - 2\eta + 2\chi, \\ y &= \xi + \eta + \chi, \\ z &= -3\xi + \eta - \chi. \end{aligned}$$

[They are not needed for the question as set, but the transformed expressions are $16\xi^2 + 16\eta^2 + 4\chi^2$ and $-16\xi^2 + 32\eta^2 + 16\chi^2$; as expected, they contain no cross terms. Note that corresponding coefficients are in the ratio given by the associated eigenvalue. Explicitly: $-16/16 = -1$ for ξ; $32/16 = 2$ for η; $16/4 = 4$ for χ.]

9.10 *Use the Rayleigh–Ritz method to estimate the lowest oscillation frequency of a heavy chain of N links, each of length a $(= L/N)$, which hangs freely from one end. Consider simple calculable configurations such as all links but one vertical, or all links collinear, etc.*

Intuitively, having all links collinear should give a good estimate of the lowest oscillation frequency of the chain. However, the example discussed in the text, of a rod on the end of a string, suggests that in the true lowest-frequency mode the lower links will tend to be at a larger inclination to the vertical than are the upper ones.

With θ_i as the (small) angle the ith link makes with the vertical, the potential energy of the ith link is

$$mg\sum_{j=1}^{i-1} a(1-\cos\theta_j) + mg\frac{a}{2}(1-\cos\theta_i) \approx \frac{mga}{2}\left(\sum_{j=1}^{i-1}\theta_j^2 + \tfrac{1}{2}\theta_i^2\right).$$

The lateral velocity of the same link is

$$\sum_{j=1}^{i-1} a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i.$$

The link's kinetic energy is therefore

$$\frac{1}{2}m\left(\sum_{j=1}^{i-1} a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i\right)^2.$$

It also has some rotational kinetic energy about its own centre of mass but this is small compared to the two contributions considered above and can be ignored in an estimate such as this.

The two quadratic forms $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}$ and $\mathsf{x}^{\mathrm{T}}\mathsf{B}\mathsf{x}$ are, respectively, the total kinetic energy divided by ω^2 and the total potential energy. We now evaluate them with the trial configuration

$$\mathsf{x} = (\theta_1,\ \theta_2,\ \ldots\ ,\ \theta_N)^{\mathrm{T}} = \alpha(1,\ 1,\ \ldots\ ,\ 1)^{\mathrm{T}}.$$

The contribution of the ith link to the potential energy is

$$\frac{mga}{2}\left(\sum_{j=1}^{i-1}\theta_j^2 + \tfrac{1}{2}\theta_i^2\right) = \frac{mga}{2}(i-\tfrac{1}{2})\alpha^2,$$

and its contribution to $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}/\omega^2$ is

$$\frac{1}{2}m\left(\sum_{j=1}^{i-1}a\dot{\theta}_j + \tfrac{1}{2}a\dot{\theta}_i\right)^2 = \frac{ma^2}{2}(i-\tfrac{1}{2})^2\alpha^2.$$

For the whole chain:

$$\begin{aligned}
V &= \tfrac{1}{2}mga\alpha^2\textstyle\sum_{i=1}^{N}(i-\tfrac{1}{2})\\
&= \tfrac{1}{2}mga\alpha^2[\,\tfrac{1}{2}N(N+1)-\tfrac{1}{2}N\,] = \tfrac{1}{4}mga\alpha^2N^2.\\
\frac{T}{\omega^2} &= \tfrac{1}{2}ma^2\alpha^2\textstyle\sum_{i=1}^{N}(i-\tfrac{1}{2})^2\\
&= \tfrac{1}{2}ma^2\alpha^2[\,\tfrac{1}{6}N(N+1)(2N+1)-\tfrac{1}{2}N(N+1)+\tfrac{1}{4}N\,]\\
&\approx \tfrac{1}{6}ma^2\alpha^2N^3 \quad \text{for large } N.
\end{aligned}$$

Thus the estimate is

$$\lambda \approx \frac{\frac{1}{4}mga\alpha^2N^2}{\frac{1}{6}ma^2\alpha^2N^3} = \frac{3g}{2Na} = \frac{3g}{2L}.$$

10

Vector calculus

10.2 *At time $t = 0$, the vectors* $\mathbf{E}$ *and* $\mathbf{B}$ *are given by* $\mathbf{E} = \mathbf{E}_0$ *and* $\mathbf{B} = \mathbf{B}_0$, *where the fixed unit vectors* $\mathbf{E}_0$ *and* $\mathbf{B}_0$ *are orthogonal. The equations of motion are*

$$\frac{d\mathbf{E}}{dt} = \mathbf{E}_0 + \mathbf{B} \times \mathbf{E}_0,$$
$$\frac{d\mathbf{B}}{dt} = \mathbf{B}_0 + \mathbf{E} \times \mathbf{B}_0.$$

Find $\mathbf{E}$ *and* $\mathbf{B}$ *at a general time t, showing that after a long time the directions of* $\mathbf{E}$ *and* $\mathbf{B}$ *have almost interchanged.*

Use a coordinate system in which $\mathbf{E}_0$ has components $(1, 0, 0)$ and $\mathbf{B}_0$ has components $(0, 1, 0)$. Then

$$\mathbf{B} \times \mathbf{E}_0 = (0, B_z, -B_y) \quad \text{and} \quad \mathbf{E} \times \mathbf{B}_0 = (-E_z, 0, E_x)$$

and, on equating components in the equations of motion,

$$\frac{dE_x}{dt} = 1, \qquad \frac{dE_y}{dt} = B_z, \quad \frac{dE_z}{dt} = -B_y,$$
$$\frac{dB_x}{dt} = -E_z, \quad \frac{dB_y}{dt} = 1, \qquad \frac{dB_z}{dt} = E_x.$$

Recalling that $E_x(0) = 1 = B_y(0)$, we see that the first and fifth of these equations integrate to

$$E_x = t + 1 \qquad \text{and} \qquad B_y = t + 1.$$

We note that all other components have zero values at $t = 0$, leading to:

$$\begin{aligned}
\frac{dB_z}{dt} &= E_x = t + 1 \Rightarrow B_z = \tfrac{1}{2}t^2 + t, \\
\frac{dE_z}{dt} &= -B_y = -t - 1 \Rightarrow E_z = -\tfrac{1}{2}t^2 - t, \\
\frac{dE_y}{dt} &= B_z = \tfrac{1}{2}t^2 + t \Rightarrow E_y = \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2, \\
\frac{dB_x}{dt} &= -E_z = \tfrac{1}{2}t^2 + t \Rightarrow B_x = \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2.
\end{aligned}$$

Thus,

$$\mathbf{E}(t) = (t + 1, \tfrac{1}{6}t^3 + \tfrac{1}{2}t^2, -\tfrac{1}{2}t^2 - t) \quad \text{and} \quad \mathbf{B}(t) = (\tfrac{1}{6}t^3 + \tfrac{1}{2}t^2, t + 1, \tfrac{1}{2}t^2 + t).$$

So, after a long time, when the terms cubic in t dominate, $\mathbf{E}$ is almost along the y-direction and $\mathbf{B}$ is almost along the x-direction, i.e. the directions of $\mathbf{E}$ and $\mathbf{B}$ have almost interchanged.

10.4 *Use vector methods to find the maximum angle to the horizontal at which a stone may be thrown so as to ensure that it is always moving away from the thrower.*

The equation of motion of the stone is

$$\ddot{\mathbf{r}} = \mathbf{g} \quad \text{with} \quad \dot{\mathbf{r}}(0) = \mathbf{v}_0 \quad \text{and} \quad \mathbf{r}(0) = \mathbf{0}.$$

Integrating the equation with the given boundary conditions yields

$$\dot{\mathbf{r}} = \mathbf{g}t + \mathbf{v}_0 \quad \text{and} \quad \mathbf{r} = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{v}_0 t.$$

The requirement that the stone is always moving away from the thrower can be expressed as $\dot{\mathbf{r}} \cdot \mathbf{r} > 0$ for all t, i.e. that $\dot{\mathbf{r}} \cdot \mathbf{r} = 0$ has no real roots for $t > 0$:

$$\tfrac{1}{2}g^2t^3 + \tfrac{3}{2}\mathbf{v}_0 \cdot \mathbf{g}t^2 + v_0^2 t > 0$$

for all t, which requires that

$$\begin{aligned}
4 \times \tfrac{1}{2}g^2 \times v_0^2 &> (\tfrac{3}{2})^2(\mathbf{v}_0 \cdot \mathbf{g})^2, \\
\sqrt{\frac{8}{9}} &> \frac{\mathbf{v}_0 \cdot \mathbf{g}}{v_0 g}.
\end{aligned}$$

This means that the angle between the initial trajectory and the vertical must exceed $\cos^{-1} 0.9429 = 19.5°$. The maximum permitted angle to the horizontal is therefore $70.5°$.

10.6 *Prove that for a space curve* $\mathbf{r} = \mathbf{r}(s)$*, where* s *is the arc length measured along the curve from a fixed point, the triple scalar product*

$$\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3}$$

at any point on the curve has the value $\kappa^2\tau$*, where* κ *is the curvature and* τ *the torsion at that point.*

We start from the relationship

$$\hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}}$$

in which $\hat{\mathbf{b}}$, $\hat{\mathbf{t}}$ and $\hat{\mathbf{n}}$ are respectively the binormal, tanget and normal unit vectors at a point on the curve. This is differentiated with respect to s and use is made of the Frenet-Serret formulae. One of these is needed in the form $\kappa\hat{\mathbf{n}} = d\hat{\mathbf{t}}/ds = d^2\mathbf{r}/ds^2$ or, in terms of the radius of curvature ρ, $\hat{\mathbf{n}} = \rho\, d\hat{\mathbf{t}}/ds = \rho\, d^2\mathbf{r}/ds^2$.

$$\begin{aligned}
\hat{\mathbf{b}} &= \hat{\mathbf{t}} \times \hat{\mathbf{n}}, \\
\frac{d\hat{\mathbf{b}}}{ds} &= \left(\frac{d\hat{\mathbf{t}}}{ds} \times \hat{\mathbf{n}}\right) + \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right), \\
-\tau\hat{\mathbf{n}} &= (\kappa\hat{\mathbf{n}} \times \hat{\mathbf{n}}) + \left(\hat{\mathbf{t}} \times \frac{d\hat{\mathbf{n}}}{ds}\right), \\
-\tau\rho\frac{d^2\mathbf{r}}{ds^2} &= \mathbf{0} + \frac{d\mathbf{r}}{ds} \times \frac{d}{ds}\left(\rho\frac{d\hat{\mathbf{t}}}{ds}\right) \\
&= \frac{d\mathbf{r}}{ds} \times \frac{d}{ds}\left(\rho\frac{d^2\mathbf{r}}{ds^2}\right) \\
&= \rho\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) + \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right)\frac{d\rho}{ds}.
\end{aligned}$$

We now take the scalar product of this vector equation with $d^2\mathbf{r}/ds^2$ (sometimes written as $\kappa\hat{\mathbf{n}}$) and obtain

$$\begin{aligned}
-\tau\rho\kappa^2(\hat{\mathbf{n}} \cdot \hat{\mathbf{n}}) &= \rho\left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) \cdot \frac{d^2\mathbf{r}}{ds^2} + \frac{d\rho}{ds}\left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^2\mathbf{r}}{ds^2}, \\
-\tau\kappa^2 &= \left(\frac{d\mathbf{r}}{ds} \times \frac{d^3\mathbf{r}}{ds^3}\right) \cdot \frac{d^2\mathbf{r}}{ds^2} + 0, \\
\tau\kappa^2 &= \left(\frac{d\mathbf{r}}{ds} \times \frac{d^2\mathbf{r}}{ds^2}\right) \cdot \frac{d^3\mathbf{r}}{ds^3},
\end{aligned}$$

i.e. as stated in the question.

10.8 *The shape of the curving slip road joining two motorways that cross at right angles and are at vertical heights $z = 0$ and $z = h$ can be approximated by the space curve*

$$\mathbf{r} = \frac{\sqrt{2}h}{\pi} \ln\cos\left(\frac{z\pi}{2h}\right)\mathbf{i} + \frac{\sqrt{2}h}{\pi} \ln\sin\left(\frac{z\pi}{2h}\right)\mathbf{j} + z\mathbf{k}.$$

Show that the radius of curvature ρ of the slip road is $(2h/\pi)\operatorname{cosec}(z\pi/h)$ at height z and that the torsion $\tau = -1/\rho$. (To shorten the algebra, set $z = 2h\theta/\pi$ and use θ as the parameter.)

The slip road is given by

$$\mathbf{r} = A(\ln\cos\theta,\ \ln\sin\theta,\ \sqrt{2}\theta),$$

where $A = \sqrt{2}h/\pi$. It follows that

$$\frac{d\mathbf{r}}{d\theta} = A(-\tan\theta,\ \cot\theta,\ \sqrt{2})$$

and

$$\begin{aligned}\frac{ds}{d\theta} &= A(\tan^2\theta + \cot^2\theta + 2)^{1/2}\\ &= A(\sec^2\theta + \operatorname{cosec}^2\theta)^{1/2}\\ &= \frac{A}{(\sin^2\theta\cos^2\theta)^{1/2}} = \frac{A}{\sin\theta\cos\theta}.\end{aligned}$$

Next,

$$\begin{aligned}\hat{\mathbf{t}} &= \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{d\theta}\frac{d\theta}{ds}\\ &= A(-\tan\theta,\ \cot\theta,\ \sqrt{2})\frac{\sin\theta\cos\theta}{A}\\ &= (-\sin^2\theta,\ \cos^2\theta,\ \sqrt{2}\sin\theta\cos\theta).\end{aligned}$$

from which it follows that

$$\frac{1}{\rho}\hat{\mathbf{n}} = \frac{d\hat{\mathbf{t}}}{ds} = \frac{d\hat{\mathbf{t}}}{d\theta}\frac{d\theta}{ds} = \frac{\sin\theta\cos\theta}{A}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta).$$

Thus,

$$\frac{1}{\rho} = \frac{\sin\theta\cos\theta}{A}(\sin^2 2\theta + \sin^2 2\theta + 2\cos^2 2\theta)^{1/2} = \frac{\sqrt{2}\sin\theta\cos\theta}{A}$$

and, as was required to be shown,

$$\rho = \sqrt{2}A\operatorname{cosec} 2\theta = \frac{2h}{\pi}\operatorname{cosec}\left(\frac{\pi z}{h}\right).$$

With $\hat{\mathbf{n}}$ a unit vector in the direction of $d\hat{\mathbf{t}}/ds$, we have

$$\hat{\mathbf{n}} = \frac{1}{\sqrt{2}}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta) \quad \text{and}$$
$$\hat{\mathbf{t}} = (-\sin^2\theta,\ \cos^2\theta,\ \sqrt{2}\sin\theta\cos\theta),$$
$$\Rightarrow \quad \hat{\mathbf{b}} = \hat{\mathbf{t}} \times \hat{\mathbf{n}} = (\cos^2\theta,\ -\sin^2\theta,\ \sqrt{2}\sin\theta\cos\theta).$$
$$\text{and} \quad \frac{d\hat{\mathbf{b}}}{ds} = \frac{d\hat{\mathbf{b}}}{d\theta}\frac{d\theta}{ds} = \frac{\sin\theta\cos\theta}{A}(-\sin 2\theta,\ -\sin 2\theta,\ \sqrt{2}\cos 2\theta)$$
$$= \frac{\sqrt{2}\sin\theta\cos\theta}{A}\hat{\mathbf{n}}.$$

From this it follows that

$$\tau = -\hat{\mathbf{n}} \cdot \frac{d\hat{\mathbf{b}}}{ds} = \frac{-\sqrt{2}\sin\theta\cos\theta}{\sqrt{2}h/\pi} = -\frac{\pi}{2h}\sin\left(\frac{z\pi}{h}\right) = -\frac{1}{\rho}.$$

10.10 *Find the areas of the given surfaces using parametric coordinates.*

(a) *Using the parameterization $x = u\cos\phi$, $y = u\sin\phi$, $z = u\cot\Omega$, find the sloping surface area of a right circular cone of semi-angle Ω whose base has radius a. Verify that it is equal to $\frac{1}{2}\times$ perimeter of the base $\times$ slope height.*

(b) *Using the same parameterization as in (a) for x and y, and an appropriate choice for z, find the surface area between the planes $z = 0$ and $z = Z$ of the paraboloid of revolution $z = \alpha(x^2 + y^2)$.*

(a) With $x = u\cos\phi$, $y = u\sin\phi$, and $z = u\cot\Omega$,

$$\frac{\partial \mathbf{r}}{\partial u} = (\cos\phi,\ \sin\phi,\ \cot\Omega),$$
$$\frac{\partial \mathbf{r}}{\partial \theta} = (-u\sin\phi,\ u\cos\phi,\ 0),$$
$$\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \theta} = (-u\cos\phi\cot\Omega,\ -u\sin\phi\cot\Omega,\ u),$$
$$\left|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial \theta}\right| = u(1 + \cot^2\Omega)^{1/2} = u\,\mathrm{cosec}\,\Omega.$$

Thus $dS = u\,\mathrm{cosec}\,\Omega\,du\,d\phi$ and the total surface area is

$$S = \int_0^{2\pi} d\phi \int_0^a u\,\mathrm{cosec}\,\Omega\,du = \pi a^2 \mathrm{cosec}\,\Omega.$$

This can clearly be written as $\frac{1}{2} \times 2\pi a \times a\,\mathrm{cosec}\,\Omega$, i.e. $\frac{1}{2}\times$ perimeter of the base $\times$ slope height of the cone.

(b) With the given parameterization for x and y, we have

$$z = \alpha(x^2 + y^2) = \alpha u^2,$$

and so

$$\frac{d\mathbf{r}}{du} = (\cos\phi,\ \sin\phi,\ 2\alpha u),$$
$$\frac{d\mathbf{r}}{d\phi} = (-u\sin\phi,\ u\cos\phi,\ 0),$$
$$dS = \left|\frac{\partial\mathbf{r}}{\partial u} \times \frac{\partial\mathbf{r}}{\partial\phi}\right| du\, d\phi = |\,(-2\alpha u^2\cos\phi,\ -2\alpha u^2\sin\phi,\ u)\,|\, du\, d\phi$$
$$= (u^2 + 4\alpha^2u^4)^{1/2}\, du\, d\phi = u(1 + 4\alpha^2u^2)^{1/2}\, du\, d\phi.$$

The total area is thus

$$S = \int_0^{2\pi} d\phi \int_0^{(Z/\alpha)^{1/2}} u(1 + 4\alpha^2u^2)^{1/2}\, du$$
$$= 2\pi\left[\frac{2}{3}\,\frac{(1 + 4\alpha^2u^2)^{3/2}}{8\alpha^2}\right]_0^{(Z/\alpha)^{1/2}}$$
$$= \frac{\pi}{6\alpha^2}\left[(1 + 4\alpha Z)^{3/2} - 1\right].$$

This is only the curved surface area; if the plane end of the paraboloid is also counted, an additional $\pi Z/\alpha$ must be included.

10.12 *For the function*

$$z(x, y) = (x^2 - y^2)e^{-x^2-y^2},$$

find the location(s) at which the steepest gradient occurs. What are the magnitude and direction of that gradient? The algebra involved is easier if plane polar coordinates are used.

The function is antisymmetric under the interchange of x and y and so we need to consider explicitly only $x > 0$.

With $x = \rho\cos\phi$ and $y = \rho\sin\phi$ we can write $z(x, y) = f(\rho, \phi)$ as

$$f(\rho, \phi) = \rho^2(\cos^2\phi - \sin^2\phi)e^{-\rho^2} = \rho^2\cos 2\phi e^{-\rho^2}.$$

From this it follows that

$$\nabla f = \left(2(\rho - \rho^3)e^{-\rho^2}\cos 2\phi, -2\rho^2e^{-\rho^2}\sin 2\phi\right),$$
$$|\nabla f|^2 = 4\rho^2e^{-2\rho^2}[\,(1 - \rho^2)^2\cos^2 2\phi + \rho^2\sin^2 2\phi\,]$$
$$= 4se^{-2s}[\,(1 - s)^2\cos^2 2\phi + s\sin^2 2\phi\,],$$

where we have set $\rho^2 = s$ for brevity.

For the line of steepest gradient,

(i) $\partial|\nabla f|^2/\partial\phi = 0$ giving

$$-4(1-s)^2\cos 2\phi \sin 2\phi + 4s\sin 2\phi\cos 2\phi = 0,$$

i.e. $\sin 4\phi = 0$ and $\phi = n\pi/4$ *or* $s = (1-s)^2 \Rightarrow s = (3 \pm \sqrt{5})/2$.

(ii) $\partial|\nabla f|^2/\partial\rho = 0$ or, alternatively, $\partial|\nabla f|^2/\partial s = 0$. We therefore require that

$$(1-2s)[(1-s)^2\cos^2 2\phi + s\sin^2 2\phi] + s[-2(1-s)\cos^2 2\phi + \sin^2 2\phi] = 0. \qquad (*)$$

We now need to examine the various possible combinations of conditions.

For $\sin 2\phi = 0$, $\cos^2 2\phi = 1$ and $(*)$ reduces to

$$(1-2s)(1-s)^2 - 2s(1-s) = 0 \quad \Rightarrow \quad s = 1 \text{ or } s = \frac{5 \pm \sqrt{17}}{4}.$$

The corresponding values of $|\nabla f|^2$, obtained by direct substitution, are 0, 0.156 and 0.345.

For $\cos 2\phi = 0$, $\sin^2 2\phi = 1$ and $(*)$ reduces to

$$(1-2s)s + s = 0 \quad \Rightarrow \quad s = 0 \text{ or } s = 1.$$

The corresponding values of $|\nabla f|^2$, obtained by direct substitution, are 0 and $4e^{-2} = 0.541$.

In the third case, when $s = (1-s)^2$, $|\nabla f|^2$ has no ϕ dependence and takes the form $4s^2e^{-2s}$, which has values 0.146 and 0.272 at $s = (3 \pm \sqrt{5})/2$

The largest of these seven values is 0.541, obtained when $\cos 2\phi = 0$, i.e. when $\phi = \pm\pi/4$ and $s = 1$. Thus the steepest gradient occurs on the circle $\rho = 1$ at the points $x = \pm 1/\sqrt{2}$, $y = \pm 1/\sqrt{2}$. The gradient vector there is

$$\nabla f(\rho, \phi) = (0, -2e^{-2}\sin 2\phi)$$

and is therefore azimuthal along the lines $x \pm y = \pm\sqrt{2}$ and $x \pm y = \mp\sqrt{2}$.

10.14 *In the following exercises* **a** *is a vector field.*

(a) *Simplify*

$$\nabla \times \mathbf{a}(\nabla \cdot \mathbf{a}) \;+\; \mathbf{a} \times [\nabla \times (\nabla \times \mathbf{a})] \;+\; \mathbf{a} \times \nabla^2\mathbf{a}.$$

(b) *By explicitly writing out the terms in Cartesian coordinates prove that*

$$[\mathbf{c} \cdot (\mathbf{b} \cdot \nabla) - \mathbf{b} \cdot (\mathbf{c} \cdot \nabla)]\,\mathbf{a} = (\nabla \times \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{c}).$$

(c) *Prove that* $\mathbf{a} \times (\nabla \times \mathbf{a}) = \nabla(\frac{1}{2}a^2) - (\mathbf{a} \cdot \nabla)\mathbf{a}$.

(a) Using results given in the text for $\nabla \times (\phi\mathbf{a})$ and for $\nabla \times (\nabla \times \mathbf{a})$, the first two

terms can be expanded, as follows.

$$\nabla \times \mathbf{a}(\nabla \cdot \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) \times \mathbf{a} + (\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a}),$$
$$\mathbf{a} \times [\nabla \times (\nabla \times \mathbf{a})] = [\mathbf{a} \times \nabla(\nabla \cdot \mathbf{a})] - [\mathbf{a} \times \nabla^2\mathbf{a}].$$

Substituting these into the original expression gives

$$\nabla(\nabla \cdot \mathbf{a}) \times \mathbf{a} + (\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a}) + \mathbf{a} \times \nabla(\nabla \cdot \mathbf{a}) - \mathbf{a} \times \nabla^2\mathbf{a} + \mathbf{a} \times \nabla^2\mathbf{a}.$$

Thus the original expression is equal to $(\nabla \cdot \mathbf{a})(\nabla \times \mathbf{a})$.

(b) The first term on the LHS is

$$\begin{aligned}\mathbf{c} \cdot (\mathbf{b} \cdot \nabla)\mathbf{a} = {} & c_x b_x \frac{\partial a_x}{\partial x} + c_x b_y \frac{\partial a_x}{\partial y} + c_x b_z \frac{\partial a_x}{\partial z} \\ & + c_y b_x \frac{\partial a_y}{\partial x} + c_y b_y \frac{\partial a_y}{\partial y} + c_y b_z \frac{\partial a_y}{\partial z} \\ & + c_z b_x \frac{\partial a_z}{\partial x} + c_z b_y \frac{\partial a_z}{\partial y} + c_z b_z \frac{\partial a_z}{\partial z}.\end{aligned}$$

The second term has the same form, but with **b** and **c** interchanged. The difference between the two is therefore

$$\begin{aligned}& (c_x b_y - b_x c_y)\frac{\partial a_x}{\partial y} + (c_x b_z - b_x c_z)\frac{\partial a_x}{\partial z} + (c_y b_x - b_y c_x)\frac{\partial a_y}{\partial x} \\ & \quad + (c_y b_z - b_y c_z)\frac{\partial a_y}{\partial z} + (c_z b_x - b_z c_x)\frac{\partial a_z}{\partial x} + (c_z b_y - b_z c_y)\frac{\partial a_z}{\partial y}. \\ & = (b_x c_y - b_y c_x)\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right) + (b_y c_z - b_z c_y)\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) \\ & \quad + (b_z c_x - b_x c_z)\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) \\ & = (\mathbf{b} \times \mathbf{c})_z(\nabla \times \mathbf{a})_z + (\mathbf{b} \times \mathbf{c})_x(\nabla \times \mathbf{a})_x + (\mathbf{b} \times \mathbf{c})_y(\nabla \times \mathbf{a})_y \\ & = (\mathbf{b} \times \mathbf{c}) \cdot (\nabla \times \mathbf{a}),\end{aligned}$$

as stated.

(c) Consider the z-component of the LHS.

$$\begin{aligned}[\mathbf{a} \times (\nabla \times \mathbf{a})]_z &= a_x(\nabla \times \mathbf{a})_y - a_y(\nabla \times \mathbf{a})_x \\ &= a_x\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) - a_y\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) \\ &= a_x\frac{\partial a_x}{\partial z} + a_y\frac{\partial a_y}{\partial z} + a_z\frac{\partial a_z}{\partial z} - a_z\frac{\partial a_z}{\partial z} - a_x\frac{\partial a_z}{\partial x} - a_y\frac{\partial a_z}{\partial y} \\ &= \frac{1}{2}\frac{\partial}{\partial z}\left(a_x^2 + a_y^2 + a_z^2\right) - (\mathbf{a} \cdot \nabla)a_z \\ &= \left[\nabla(\tfrac{1}{2}a^2) - (\mathbf{a} \cdot \nabla)\mathbf{a}\right]_z \\ &= z\text{-component of RHS.}\end{aligned}$$

In the third line $a_z(\partial a_z/\partial z)$ was both added and subtracted. The corresponding results for the x- and y-components can be proved in the same way, thus establishing the vector result.

10.16 *Verify that (10.42) is valid for each component separately when* $\mathbf{a}$ *is the Cartesian vector* $x^2y\,\mathbf{i} + xyz\,\mathbf{j} + z^2y\,\mathbf{k}$, *by showing that each side of the equation is equal to* $z\,\mathbf{i} + (2x + 2z)\,\mathbf{j} + x\,\mathbf{k}$.

With $\mathbf{a} = (x^2y,\ xyz,\ z^2y)$ we have:

For the RHS
$$\begin{aligned}\nabla\cdot\mathbf{a} &= 2xy + xz + 2zy,\\ \nabla(\nabla\cdot\mathbf{a}) &= (2y + z,\ 2x + 2z,\ x + 2y),\\ \nabla^2\mathbf{a} &= (2y,\ 0,\ 2y),\\ \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a} &= (z,\ 2x + 2z,\ x).\end{aligned}$$

For the LHS
$$\begin{aligned}\nabla\times\mathbf{a} &= (z^2 - xy,\ 0,\ yz - x^2),\\ \nabla\times(\nabla\times\mathbf{a}) &= (z,\ 2z + 2x,\ x).\end{aligned}$$

This verifies that the vector equality is valid term-by-term for this vector field expressed in Cartesian coordinates.

10.18 *Evaluate the Laplacian of a vector field using two different coordinate systems as follows.*

(a) *For cylindrical polar coordinates* ρ, ϕ, z *evaluate the derivatives of the three unit vectors with respect to each of the coordinates, showing that only* $\partial\hat{\mathbf{e}}_\rho/\partial\phi$ *and* $\partial\hat{\mathbf{e}}_\phi/\partial\phi$ *are non-zero.*

  (i) *Hence evaluate* $\nabla^2\mathbf{a}$ *when* $\mathbf{a}$ *is the vector* $\hat{\mathbf{e}}_\rho$, *i.e. a vector of unit magnitude everywhere directed radially outwards from the* z*-axis.*

  (ii) *Note that it is trivially obvious that* $\nabla\times\mathbf{a} = \mathbf{0}$ *and hence that equation (10.41) requires that* $\nabla(\nabla\cdot\mathbf{a}) = \nabla^2\mathbf{a}$.

  (iii) *Evaluate* $\nabla(\nabla\cdot\mathbf{a})$ *and show that the latter equation holds, but that*
$$[\nabla(\nabla\cdot\mathbf{a})]_\rho \neq \nabla^2 a_\rho.$$

(b) *Rework the same problem in Cartesian coordinates (where, as it happens, the algebra is more complicated).*

Figure 10.1 The changes in the unit base vectors calculated in exercise 10.18.

(a) It is clear that $\hat{\mathbf{e}}_z$ does not depend upon the position at which it is evaluated (recall that the vectors are determined by their magnitudes and directions, and not by their absolute positions in space); consequently all of its derivatives are zero.
Equally, $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ are unaltered if only the value of z is changed; thus $\partial\hat{\mathbf{e}}_\rho/\partial z = \partial\hat{\mathbf{e}}_\phi/\partial z = \mathbf{0}$.
In the same way, $\partial\hat{\mathbf{e}}_\rho/\partial\rho = \partial\hat{\mathbf{e}}_\phi/\partial\rho = \mathbf{0}$ are both zero.

That leaves only possible variations of $\hat{\mathbf{e}}_\rho$ and $\hat{\mathbf{e}}_\phi$ with ϕ to consider. Figure 10.1, a section in any plane of constant z, shows these two unit vectors and the changes in them. When ϕ is changed to $\phi + d\phi$, $\hat{\mathbf{e}}_\rho$ changes direction by $d\phi$ and its vector position has been changed by an amount $1 \times d\phi$ in the azimuthal direction parallel to $\hat{\mathbf{e}}_\phi$, i.e.

$$d\hat{\mathbf{e}}_\rho = d\phi\,\hat{\mathbf{e}}_\phi \quad \Rightarrow \quad \frac{\partial\hat{\mathbf{e}}_\rho}{\partial\phi} = \hat{\mathbf{e}}_\phi.$$

The same change, $d\phi$, also causes $\hat{\mathbf{e}}_\phi$ to change direction and alter its vector position by an amount of magnitude $d\phi$. But *this* change is along the radial direction $\hat{\mathbf{e}}_\rho$ and directed towards the polar axis, i.e.

$$d\hat{\mathbf{e}}_\phi = -d\phi\,\hat{\mathbf{e}}_\rho \quad \Rightarrow \quad \frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} = -\hat{\mathbf{e}}_\rho.$$

(i) With $\mathbf{a} = (1,\ 0,\ 0)$,

$$\begin{aligned}
\nabla^2\mathbf{a} &= \nabla^2(1\,\hat{\mathbf{e}}_\rho) \\
&= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\hat{\mathbf{e}}_\rho}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial}{\partial\phi}\frac{\partial\hat{\mathbf{e}}_\rho}{\partial\phi} + \frac{\partial}{\partial z}\frac{\partial\hat{\mathbf{e}}_\rho}{\partial z} \\
&= \mathbf{0} + \frac{1}{\rho^2}\frac{\partial\hat{\mathbf{e}}_\phi}{\partial\phi} + \mathbf{0} \\
&= \frac{1}{\rho^2}(-\hat{\mathbf{e}}_\rho).
\end{aligned}$$

(ii) and (iii). Using the expressions for the divergence and gradient in cylindrical

polar coordinates, as given in the text, we have

$$\nabla \cdot \mathbf{a} = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho\, 1) + 0 + 0 = \frac{1}{\rho}.$$
$$\nabla(\nabla \cdot \mathbf{a}) = -\frac{1}{\rho^2}\,\hat{\mathbf{e}}_\rho + 0\,\hat{\mathbf{e}}_\phi + 0\,\hat{\mathbf{e}}_z = -\frac{1}{\rho^2}\,\hat{\mathbf{e}}_\rho.$$

In this case, in which $\nabla \times \mathbf{a}$, and hence $\nabla \times (\nabla \times \mathbf{a})$, are trivially zero everywhere, this verifies the more general result

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}.$$

However, even in this especially simple case, it is clear that $\nabla^2 a_\rho = \nabla^2 1 = 0$ whilst the ρ-component of $\nabla(\nabla \cdot \mathbf{a})$ is equal to $-1/\rho^2$; this shows that the equality does not hold component-by-component.

(b) In Cartesian coordinates, the same vector field takes the form

$$\mathbf{a} = \left(\frac{x}{(x^2+y^2)^{1/2}},\ \frac{y}{(x^2+y^2)^{1/2}},\ 0\right).$$

Straightforward but somewhat tedious differentiation gives the required partial derivatives of the x-component as

$$\frac{\partial a_x}{\partial x} = \frac{y^2}{(x^2+y^2)^{3/2}}, \quad \frac{\partial a_x}{\partial y} = \frac{-xy}{(x^2+y^2)^{3/2}},$$
$$\frac{\partial^2 a_x}{\partial x^2} = \frac{-3y^2x}{(x^2+y^2)^{5/2}}, \quad \frac{\partial^2 a_x}{\partial x^2} = \frac{2y^2x - x^3}{(x^2+y^2)^{5/2}}.$$

Together with the obvious $\partial^2 a_x/\partial z^2 = 0$, these results give

$$\nabla^2 a_x = \frac{-3y^2x}{(x^2+y^2)^{5/2}} + \frac{2y^2x - x^3}{(x^2+y^2)^{5/2}} = -\frac{x}{(x^2+y^2)^{3/2}}.$$

But,

$$\begin{aligned}
[\nabla(\nabla \cdot \mathbf{a})]_x &= \frac{\partial}{\partial x}\left(\frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{y^2}{(x^2+y^2)^{3/2}} + \frac{x^2}{(x^2+y^2)^{3/2}}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{1}{(x^2+y^2)^{1/2}}\right) = -\frac{x}{(x^2+y^2)^{3/2}},
\end{aligned}$$

thus re-establishing the result. Similarly $\nabla^2 a_y = [\nabla(\nabla \cdot \mathbf{a})]_y$.

10.20 *For a description in spherical polar coordinates with axial symmetry of the flow of a very viscous fluid, the components of the velocity field* **u** *are given in terms of the* stream function ψ *by*

$$u_r = \frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}, \qquad u_\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}.$$

Find an explicit expression for the differential operator E defined by

$$E\psi = -(r\sin\theta)(\nabla\times\mathbf{u})_\phi.$$

The stream function satisfies the equation of motion $E^2\psi = 0$ and, for the flow of a fluid past a sphere, takes the form $\psi(r,\theta) = f(r)\sin^2\theta$. Show that $f(r)$ satisfies the (ordinary) differential equation

$$r^4 f^{(4)} - 4r^2 f'' + 8rf' - 8f = 0.$$

Using the formulae given in the text, we have

$$\begin{aligned}
(\nabla\times\mathbf{u})_\phi &= \frac{r\sin\theta}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right] \\
&= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{-1}{\sin\theta}\frac{\partial\psi}{\partial r}\right) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)\right] \\
&= -\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right).
\end{aligned}$$

Hence, E is the operator

$$E = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right).$$

With $\psi(r,\theta) = f(r)\sin^2\theta$,

$$\begin{aligned}
E\psi &= f''\sin^2\theta + \frac{f\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{2\sin\theta\cos\theta}{\sin\theta}\right) \\
&= \left(f'' - \frac{2f}{r^2}\right)\sin^2\theta, \\
E^2\psi &= \frac{\partial^2}{\partial r^2}\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta \\
&= \sin^2\theta\left[f^{(4)} - \frac{\partial}{\partial r}\left(-\frac{4f}{r^3} + \frac{2f'}{r^2}\right)\right] - \frac{2\sin^2\theta}{r^2}\left(f'' - \frac{2f}{r^2}\right) \\
&= \sin^2\theta\left[f^{(4)} - \frac{12f}{r^4} + \frac{4f'}{r^3} - \frac{2f''}{r^2} + \frac{4f'}{r^3} - \frac{2f''}{r^2} + \frac{4f}{r^4}\right] = 0.
\end{aligned}$$

Simplifying then gives

$$f^{(4)} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0,$$

in agreement with the equation stated in the question.

10.22 *Non-orthogonal curvilinear coordinates are difficult to work with and should be avoided if at all possible, but the following example is provided to illustrate the content of section 10.10.*

In a new coordinate system for the region of space in which the Cartesian coordinate z satisfies $z \geq 0$, the position of a point $\mathbf{r}$ is given by (α_1, α_2, R), where α_1 and α_2 are respectively the cosines of the angles made by $\mathbf{r}$ with the x- and y-coordinate axes of a Cartesian system and $R = |\mathbf{r}|$. The ranges are $-1 \leq \alpha_i \leq 1$, $0 \leq R < \infty$.

(a) *Express $\mathbf{r}$ in terms of α_1, α_2, R and the unit Cartesian vectors $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$.*

(b) *Obtain expressions for the vectors $\mathbf{e}_i$ $(= \partial\mathbf{r}/\partial\alpha_1, \ldots)$ and hence show that the scale factors h_i are given by*

$$h_1 = \frac{R(1-\alpha_2^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_2 = \frac{R(1-\alpha_1^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2)^{1/2}}, \qquad h_3 = 1.$$

(c) *Verify formally that the system is not an orthogonal one.*

(d) *Show that the volume element of the coordinate system is*

$$dV = \frac{R^2\, d\alpha_1\, d\alpha_2\, dR}{(1-\alpha_1^2-\alpha_2^2)^{1/2}},$$

and demonstrate that this is always less than or equal to the corresponding expression for an orthogonal curvilinear system.

(e) *Calculate the expression for $(ds)^2$ for the system, and show that it differs from that for the corresponding orthogonal system by*

$$\frac{2\alpha_1\alpha_2 R^2}{1-\alpha_1^2-\alpha_2^2}\, d\alpha_1 d\alpha_2.$$

(a) Clearly, $x = R\alpha_1$, $y = R\alpha_2$ and, since $z^2 = R^2 - x^2 - y^2$, $z = (1-\alpha_1^2-\alpha_2^2)^{1/2}R$. To save space we will write $(1-\alpha_1^2-\alpha_2^2)^{1/2}$ as β and $(1-\alpha_1^2-\alpha_2^2)^{-1/2}$ as γ. We note that $\beta\gamma = 1$ and that $\partial\beta/\partial\alpha_i = -\gamma\alpha_i$. Thus,

$$\mathbf{r} = \alpha_1 R\,\mathbf{i} + \alpha_2 R\,\mathbf{j} + \beta R\,\mathbf{k}.$$

(b) The tangent vectors are

$$\mathbf{e}_1 = \frac{\partial \mathbf{r}}{\partial \alpha_1} = R\,\mathbf{i} + 0\,\mathbf{j} - R\gamma\alpha_1\,\mathbf{k},$$
$$\mathbf{e}_2 = \frac{\partial \mathbf{r}}{\partial \alpha_2} = 0\,\mathbf{i} + R\,\mathbf{j} - R\gamma\alpha_2\,\mathbf{k},$$
$$\mathbf{e}_3 = \frac{\partial \mathbf{r}}{\partial R} = \alpha_1\,\mathbf{i} + \alpha_2\,\mathbf{j} + \beta\,\mathbf{k},$$

and so the scale factors are

$$h_1^2 = R^2 + R^2\gamma^2\alpha_1^2 = R^2\gamma^2(1-\alpha_2^2) \Rightarrow h_1 = R\gamma(1-\alpha_2^2)^{1/2},$$
$$h_2^2 = R^2 + R^2\gamma^2\alpha_2^2 = R^2\gamma^2(1-\alpha_1^2) \Rightarrow h_2 = R\gamma(1-\alpha_1^2)^{1/2},$$
$$h_3^2 = \alpha_1^2 + \alpha_2^2 + \beta^2 = 1 \Rightarrow h_3 = 1.$$

(c) Consider the scalar products:

$$\mathbf{e}_1 \cdot \mathbf{e}_2 = 0 + 0 + R^2\gamma^2\alpha_1\alpha_2 \neq 0,$$
$$\mathbf{e}_1 \cdot \mathbf{e}_3 = \alpha_1 R + 0 - \beta\gamma\alpha_1 R = 0,$$
$$\mathbf{e}_2 \cdot \mathbf{e}_3 = 0 + \alpha_2 R - \beta\gamma\alpha_2 R = 0.$$

These show that, whilst both $\mathbf{e}_1$ and $\mathbf{e}_2$ are orthogonal to $\mathbf{e}_3$, they are not orthogonal to each other, i.e the system is not an orthogonal one.

(d) The volume element is

$$\begin{aligned} dV &= d\alpha_1\,d\alpha_2\,dR\,|(\mathbf{e}_1 \times \mathbf{e}_2)\cdot\mathbf{e}_3| \\ &= d\alpha_1\,d\alpha_2\,dR\,(R^2\gamma\alpha_1,\ R^2\gamma\alpha_2,\ R^2)\cdot(\alpha_1,\ \alpha_2,\ \beta) \\ &= d\alpha_1\,d\alpha_2\,dR\,R^2(\gamma\alpha_1^2 + \gamma\alpha_2^2 + \beta) \\ &= d\alpha_1\,d\alpha_2\,dR\,\gamma R^2. \end{aligned}$$

If the system were an orthogonal one, the elemental volume would be

$$dV_\perp = h_1 d\alpha_1\,h_2 d\alpha_2\,h_3 dR = R^2\gamma^2(1-\alpha_2^2)^{1/2}(1-\alpha_1^2)^{1/2}\,d\alpha_1\,d\alpha_2\,dR.$$

The ratio of the two is

$$\frac{dV}{dV_\perp} = \frac{1}{\gamma(1-\alpha_2^2)^{1/2}(1-\alpha_1^2)^{1/2}} = \frac{(1-\alpha_1^2-\alpha_2^2)^{1/2}}{(1-\alpha_1^2-\alpha_2^2+\alpha_1^2\alpha_2^2)^{1/2}}.$$

This is always less than or equal to unity, with equality only when α_1 and/or α_2 is equal to zero.

(e) We first note that $d\beta = -\gamma\alpha_1\, d\alpha_1 - \gamma\alpha_2\, d\alpha_2$. Then,

$$\begin{aligned}
(ds)^2 &= (R\, d\alpha_1 + \alpha_1\, dR)^2 + (R\, d\alpha_2 + \alpha_2\, dR)^2 + (R\, d\beta + \beta\, dR)^2 \\
&= (d\alpha_1)^2(R^2 + R^2\gamma^2\alpha_1^2) + (d\alpha_2)^2(R^2 + R^2\gamma^2\alpha_2^2) \\
&\qquad + (dR)^2(\alpha_1^2 + \alpha_2^2 + \beta^2) + (d\alpha_1\, d\alpha_2)(2R^2\gamma^2\alpha_1\alpha_2) \\
&\qquad + (d\alpha_1\, dR)(2R\alpha_1 - 2R\beta\gamma\alpha_1) + (d\alpha_2\, dR)(2R\alpha_2 - 2R\beta\gamma\alpha_2) \\
&= R^2\gamma^2(1-\alpha_2^2)(d\alpha_1)^2 + R^2\gamma^2(1-\alpha_1^2)(d\alpha_2)^2 \\
&\qquad + (dR)^2 + 2R^2\gamma^2\alpha_1\alpha_2\, d\alpha_1\, d\alpha_2 \\
&= h_1^2(d\alpha_1)^2 + h_2^2(d\alpha_2)^2 + h_3^2(dR)^2 + \frac{2R^2\alpha_1\alpha_2}{1-\alpha_1^2-\alpha_2^2}\, d\alpha_1\, d\alpha_2.
\end{aligned}$$

This establishes the stated result.

10.24 *In a Cartesian system, A and B are the points $(0,0,-1)$ and $(0,0,1)$ respectively. In a new coordinate system a general point P is given by (u_1, u_2, u_3) with $u_1 = \frac{1}{2}(r_1 + r_2)$, $u_2 = \frac{1}{2}(r_1 - r_2)$, $u_3 = \phi$; here r_1 and r_2 are the distances AP and BP and ϕ is the angle between the plane ABP and $y = 0$.*

(a) *Express z and the perpendicular distance ρ from P to the z-axis in terms of u_1, u_2, u_3.*
(b) *Evaluate $\partial x/\partial u_i$, $\partial y/\partial u_i$, $\partial z/\partial u_i$, for $i = 1, 2, 3$.*
(c) *Find the Cartesian components of $\hat{\mathbf{u}}_j$ and hence show that the new coordinates are mutually orthogonal. Evaluate the scale factors and the infinitesimal volume element in the new coordinate system.*
(d) *Determine and sketch the forms of the surfaces $u_i =$ constant.*
(e) *Find the most general function f of u_1 only that satisfies $\nabla^2 f = 0$.*

We have the following five defining equations:

$$\begin{aligned}
&\text{(i)} \quad r_1^2 = x^2 + y^2 + (z+1)^2, \\
&\text{(ii)} \quad r_2^2 = x^2 + y^2 + (z-1)^2, \\
&\text{(iii)} \quad r_1 + r_2 = 2u_1, \qquad 1 \le u_1 < \infty, \\
&\text{(iv)} \quad r_1 - r_2 = 2u_2, \qquad -1 \le u_2 \le 1, \\
&\text{(v)} \quad \phi = u_3.
\end{aligned}$$

(a) Multiplying (iii) by (iv) and subtracting (ii) from (i) gives the equality

$$4u_1u_2 = r_1^2 - r_2^2 = (z+1)^2 - (z-1)^2 = 4z \quad \Rightarrow \quad z = u_1u_2.$$

Writing $\rho^2 = x^2 + y^2$, the addition of (i) and (ii) gives

$$\begin{aligned} 2\rho^2 + 2z^2 + 2 = r_1^2 + r_2^2 &= (u_1 + u_2)^2 + (u_1 - u_2)^2 = 2u_1^2 + 2u_2^2, \\ \rho^2 &= u_1^2 + u_2^2 - u_1^2 u_2^2 - 1 \\ &= (u_1^2 - 1)(1 - u_2^2). \\ \rho\, d\rho &= (1 - u_2^2) u_1\, du_1 - (u_1^2 - 1) u_2\, du_2. \end{aligned}$$

(b) and (c)

$$\begin{aligned} \mathbf{r} &= \rho \cos u_3\, \mathbf{i} + \rho \sin u_3\, \mathbf{j} + u_1 u_2\, \mathbf{k}, \\ \mathbf{u}_1 &= \frac{\partial \mathbf{r}}{\partial u_1} = \frac{u_1(1 - u_2^2)}{\rho} \cos u_3\, \mathbf{i} + \frac{u_1(1 - u_2^2)}{\rho} \sin u_3\, \mathbf{j} + u_2\, \mathbf{k}, \\ \mathbf{u}_2 &= \frac{\partial \mathbf{r}}{\partial u_2} = -\frac{u_2(u_1^2 - 1)}{\rho} \cos u_3\, \mathbf{i} - \frac{u_2(u_1^2 - 1)}{\rho} \sin u_3\, \mathbf{j} + u_1\, \mathbf{k}, \\ \mathbf{u}_3 &= \frac{\partial \mathbf{r}}{\partial u_3} = -\rho \sin u_3\, \mathbf{i} + \rho \cos u_3\, \mathbf{j}. \end{aligned}$$

Next, consider the scalar products:

$$\begin{aligned} \mathbf{u}_1 \cdot \mathbf{u}_2 &= \frac{u_1 u_2 (1 - u_2^2)(u_1^2 - 1)}{\rho^2} (-\cos^2 u_3 - \sin^2 u_3) + u_1 u_2 \\ &= u_1 u_2 (1 - \cos^2 u_3 - \sin^2 u_3) = 0, \\ \mathbf{u}_1 \cdot \mathbf{u}_3 &= -(1 - u_2^2) u_1 \cos u_3 \sin u_3 + (1 - u_2^2) u_1 \sin u_3 \cos u_3 = 0, \\ \mathbf{u}_2 \cdot \mathbf{u}_3 &= (u_1^2 - 1) u_2 \cos u_3 \sin u_3 - (u_1^2 - 1) u_2 \sin u_3 \cos u_3 = 0. \end{aligned}$$

Thus, the new coordinates form an orthogonal system. Further,

$$\begin{aligned} h_1^2 &= \frac{(1 - u_2^2)^2 u_1^2}{\rho^2} (\cos^2 u_3 + \sin^2 u_3) + u_2^2 = \frac{u_1^2 - u_2^2}{u_1^2 - 1}, \\ h_2^2 &= \frac{(u_1^2 - 1)^2 u_2^2}{\rho^2} (\cos^2 u_3 + \sin^2 u_3) + u_1^2 = \frac{u_1^2 - u_2^2}{1 - u_2^2}, \\ h_3^2 &= \rho^2 (\sin^2 u_3 + \cos^2 u_3) = \rho^2, \end{aligned}$$

giving the scale factors as

$$h_1 = \sqrt{\frac{u_1^2 - u_2^2}{u_1^2 - 1}}, \qquad h_2 = \sqrt{\frac{u_1^2 - u_2^2}{1 - u_2^2}}, \qquad h_3 = \sqrt{(u_1^2 - 1)(1 - u_2^2)}.$$

The volume element is

$$dV = h_1 h_2 h_3\, du_1\, du_2\, du_3 = |u_1^2 - u_2^2|\, du_1\, du_2\, du_3.$$

(d) Since one definition of an ellipsoid is the locus of a point the sum of whose distances from two fixed points is a constant, the surfaces $u_1 = \frac{1}{2}(r_1 + r_2) = c$ must be ellipsoids, *all* with foci at $(0, 0, \pm 1)$. The range of c is $1 \le c < \infty$, with $c = 1$ corresponding to the line AB.

Similarly, $u_2 = \frac{1}{2}(r_1 - r_2) = c$, with $-1 \leq c \leq 1$, is a set confocal hyperboloids. The extreme values for c of $+1$ and -1 correspond to the parts of the z-axis $1 \leq z < \infty$ and $-1 \geq z > -\infty$, respectively.

The surfaces $u_3 =$ constant are clearly half-planes containing the z-axis.

(e) If $f = f(u_1)$ is a solution of $\nabla^2 f = 0$, then Laplace's equation reduces to

$$\begin{aligned} 0 &= \frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) \\ &= \frac{\partial}{\partial u_1}\left(\frac{\sqrt{u_1^2-1}\sqrt{u_1^2-1}\sqrt{1-u_2^2}}{\sqrt{1-u_2^2}}\frac{\partial f}{\partial u_1}\right). \end{aligned}$$

Integrating this and simplifying the factor containing square roots, now gives

$$\frac{\partial f}{\partial u_1} = \frac{k}{u_1^2 - 1} = \frac{k}{2(u_1 - 1)} - \frac{k}{2(u_1 + 1)}$$

which on further integration gives the most general function of u_1 that satisfies Laplace's equation as

$$f(u_1) = A \ln \frac{u_1 - 1}{u_1 + 1} + B,$$

where A and B are arbitrary constants.

11

Line, surface and volume integrals

11.2 *The vector field* $\mathbf{Q}$ *is defined as*

$$\mathbf{Q} = \left[3x^2(y+z) + y^3 + z^3\right]\mathbf{i} + \left[3y^2(z+x) + z^3 + x^3\right]\mathbf{j} + \left[3z^2(x+y) + x^3 + y^3\right]\mathbf{k}.$$

Show that $\mathbf{Q}$ *is a conservative field, construct its potential function and hence evaluate the integral* $J = \int \mathbf{Q}\cdot d\mathbf{r}$ *along any line connecting the point* A *at* $(1,-1,1)$ *to* B *at* $(2,1,2)$.

To test whether $\mathbf{Q}$ is conservative we consider the components of $\nabla \times \mathbf{Q}$, which are

$$(\nabla \times \mathbf{Q})_x = \frac{\partial Q_z}{\partial y} - \frac{\partial Q_y}{\partial z} = (3z^2 + 3y^2) - (3y^2 + 3z^2) = 0,$$
$$(\nabla \times \mathbf{Q})_y = \frac{\partial Q_x}{\partial z} - \frac{\partial Q_z}{\partial x} = (3x^2 + 3z^2) - (3z^2 + 3x^2) = 0,$$
$$(\nabla \times \mathbf{Q})_z = \frac{\partial Q_y}{\partial x} - \frac{\partial Q_x}{\partial y} = (3y^2 + 3x^2) - (3x^2 + 3y^2) = 0.$$

Hence, $\nabla \times \mathbf{Q} = \mathbf{0}$ which implies that $\mathbf{Q}$ is indeed a conservative field.

Let its potential function be $\phi(x,y,z) = f(x,y,z) + g(y,z) + h(z)$. Then, from the x-component of $\mathbf{Q}$,

$$\frac{\partial f}{\partial x} = 3x^2(y+z) + y^3 + z^3 \quad \Rightarrow \quad f(x,y,z) = x^3(y+z) + x(y^3 + z^3).$$

From its y-component,

$$x^3 + 3y^2x + \frac{\partial g}{\partial y} = 3y^2(z+x) + z^3 + x^3 \quad \Rightarrow \quad g(y,z) = y^3z + z^3y.$$

And finally, from its z-component,

$$x^3 + 3z^2x + y^3 + 3z^2y + \frac{\partial h}{\partial z} = 3z^2(x+y) + x^3 + y^3 \quad \Rightarrow \quad h(z) = c.$$

Thus,

$$\begin{aligned}\phi(x,y,z) &= x^3(y+z) + (y^3+z^3)x + y^3z + z^3y + c \\ &= yz(y^2+z^2) + zx(z^2+x^2) + xy(x^2+y^2) + c.\end{aligned}$$

Because the field is conservative, J is independent of the path taken and equal to $\phi(2,1,2) - \phi(1,-1,1) = (52+c) - (-2+c) = 54$.

11.4 *By making an appropriate choice for the functions $P(x,y)$ and $Q(x,y)$ that appear in Green's theorem in a plane, show that the integral of $x - y$ over the upper half of the unit circle centred on the origin has the value $-\frac{2}{3}$. Show the same result by direct integration in Cartesian coordinates.*

To obtain the integral of $x - y$ over the bounded region we must choose $Q(x,y)$ such that $\partial Q/\partial x$ is x, and $P(x,y)$ such that $\partial P/\partial y$ is y. Clearly $Q(x,y) = \frac{1}{2}x^2$ and $P(x,y) = \frac{1}{2}y^2$ will do. Green's theorem then reads

$$\frac{1}{2}\oint_C \left(y^2\,dx + x^2\,dy\right) = \int\int_R (x-y)\,dx\,dy.$$

We now evaluate the line integral on the LHS using $x = \cos\theta$ and $y = \sin\theta$ on the semi-circular part of the contour and ordinary integration with $y = 0$ on the straight-line portion joining $(-1,0)$ to $(1,0)$. Clearly, the latter contributes nothing, as both $y = 0$ and $dy = 0$.

With this parameterisation, the integral is

$$\begin{aligned}I &= \int_0^\pi \sin^2\theta(-\sin\theta\,d\theta) + \cos^2\theta(\cos\theta\,d\theta) \\ &= -\int_0^\pi (1-\cos^2\theta)\sin\theta\,d\theta + \int_0^\pi (1-\sin^2\theta)\cos\theta\,d\theta \\ &= \left[\cos\theta - \frac{1}{3}\cos^3\theta\right]_0^\pi + \left[\sin\theta - \frac{1}{3}\sin^3\theta\right]_0^\pi \\ &= -2 + \frac{2}{3} + 0 - 0 = -\frac{4}{3}.\end{aligned}$$

The integral of $x - y$ is therefore one-half of this, i.e. $-\frac{2}{3}$.

As a double integral in Cartesian coordinates, we have

$$\begin{aligned}\int_{-1}^{1} dx \int_{0}^{\sqrt{1-x^2}} (x-y)\,dy &= \int_{-1}^{1} \left[x\sqrt{1-x^2} - \frac{1}{2}(1-x^2) \right] dx \\ &= \left[-\frac{1}{3}(1-x^2)^{3/2} \right]_{-1}^{1} - \frac{1}{2}\left[x - \frac{x^3}{3} \right]_{-1}^{1} \\ &= 0 - 0 - 1 + \tfrac{1}{3} = -\tfrac{2}{3}.\end{aligned}$$

11.6 *By using parameterisations of the form $x = a\cos^n\theta$ and $y = a\sin^n\theta$ for suitable values of n, find the area bounded by the curves*

$$x^{2/5} + y^{2/5} = a^{2/5} \quad \textit{and} \quad x^{2/3} + y^{2/3} = a^{2/3}.$$

Consider first $x^{2/5} + y^{2/5} = a^{2/5}$, which is clearly parameterised by $x = a\cos^5\theta$ and $y = a\sin^5\theta$. As shown in the worked example in section 11.3, the area of a region R enclosed by a simple closed curve C is given by $A = \frac{1}{2}\oint_C (x\,dy - y\,dx) = \oint_C x\,dy = -\oint_C y\,dx$. Applying this to the present case,

$$\begin{aligned}A_1 &= \frac{1}{2}\oint (x\,dy - y\,dx) \\ &= \frac{1}{2}\int_0^{2\pi} [\,5a^2\cos^5\theta\sin^4\theta\cos\theta - 5a^2\sin^5\theta\cos^4\theta(-\sin\theta)\,]\,d\theta \\ &= \frac{1}{2}\int_0^{2\pi} 5a^2\cos^4\theta\sin^4\theta\,d\theta.\end{aligned}$$

In the same way, the area bounded by $x^{2/3} + y^{2/3} = a^{2/3}$ will be given by

$$A_2 = \frac{1}{2}\int_0^{2\pi} 3a^2\cos^2\theta\sin^2\theta\,d\theta.$$

Integrals of this sort were considered in exercise 2.42 where it was shown that

$$J(m,n) = \int_0^{\pi/2} \cos^m\theta\sin^n\theta\,d\theta = \frac{m-1}{m+n}J(m-2,n) = \frac{n-1}{m+n}J(m,n-2),$$

with $J(0,0) = \pi/2$. Hence,

$$\begin{aligned}A_1 &= \frac{5a^2}{2}\,4\,J(4,4) = 10a^2\frac{3}{8}J(2,4) \\ &= \frac{15a^2}{4}\frac{1}{6}J(0,4) = \frac{5a^2}{8}\frac{3}{4}J(0,2) \\ &= \frac{15a^2}{32}\frac{1}{2}J(0,0) = \frac{15\pi a^2}{128}.\end{aligned}$$

In the same way,

$$A_2 = \frac{3a^2}{2}\,4\,J(2,2) = 6a^2\frac{1}{4}J(0,2)$$
$$= \frac{3a^2}{2}\frac{1}{2}J(0,0) = \frac{3\pi a^2}{8}.$$

The area in the first quadrant enclosed between the two curves is therefore

$$\pi a^2\left(\frac{3}{8} - \frac{15}{128}\right) = \frac{33\pi a^2}{128}.$$

11.8 *Criticise the following 'proof' that* $\pi = 0$.

(a) *Apply Green's theorem in a plane to the two functions* $P(x, y) = \tan^{-1}(y/x)$ *and* $Q(x, y) = \tan^{-1}(x/y)$, *taking the region* R *to be the unit circle centred on the origin.*

(b) *The RHS of the equality so produced is*

$$\int\int_R \frac{y-x}{x^2+y^2}\,dx\,dy$$

which, either by symmetry considerations or by changing to plane polar co-ordinates, can be shown to have zero value.

(c) *In the LHS of the equality set* $x = \cos\theta$ *and* $y = \sin\theta$, *yielding* $P(\theta) = \theta$ *and* $Q(\theta) = \pi/2 - \theta$. *The line integral becomes*

$$\int_0^{2\pi}\left[\left(\frac{\pi}{2} - \theta\right)\cos\theta - \theta\sin\theta\right]\,d\theta,$$

which has value 2π.

(d) *Thus* $2\pi = 0$ *and the stated result follows.*

All of the mathematical steps are as indicated with, in part (b),

$$\frac{\partial P}{\partial y} = \frac{x}{x^2+y^2} \quad \text{and} \quad \frac{\partial Q}{\partial x} = \frac{y}{x^2+y^2},$$

and, in part (c),

$$P = \tan^{-1}\frac{\sin\theta}{\cos\theta} = \theta \quad \text{and} \quad Q = \tan^{-1}\frac{\cos\theta}{\sin\theta} = \tan^{-1}\cot\theta = \frac{\pi}{2} - \theta.$$

The non-zero contribution to the integral on the LHS comes from the integral of $\theta\,(-\sin\theta\,d\theta)$. Thus the false result does not arise from an algebraic or integration error.

However, the functions $P(x, y) = \tan^{-1}(y/x)$ and $Q(x, y) = \tan^{-1}(x/y)$, are not continuous (let alone differentiable!) at the origin. As this point is enclosed by the contour, the conditions for Green's theorem to apply are not met and the 'proof' is false.

11.10 *Find the vector area* **S** *of the part of the curved surface of the hyperboloid of revolution*

$$\frac{x^2}{a^2} - \frac{y^2 + z^2}{b^2} = 1$$

that lies in the region $z \geq 0$ *and* $a \leq x \leq \lambda a$.

The curved surface in question, together with the semicircular intersection of the hyperboloid with the plane $x = \lambda a$ and its hyperbolic intersection with the plane $z = 0$, make up a closed surface. Since the vector area of a closed surface vanishes, the vector area of the curved surface can be found by subtracting the vector areas of the other two plane surfaces from $\mathbf{0}$. Thus, $\mathbf{S} = -S_1\,\mathbf{i} + S_2\,\mathbf{k}$ where S_1 is the area of the semicircle and S_2 that of the hyperbolic intersection.

For S_1, $x = \lambda a$ and

$$\frac{\lambda^2 a^2}{a^2} - 1 = \frac{y^2 + z^2}{b^2},$$

i.e the radius of the semicircular intersection is $b\sqrt{\lambda^2 - 1}$ and the corresponding area is $S_1 = \frac{1}{2}\pi b^2(\lambda^2 - 1)$.

For S_2, $z = 0$ and $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and so, making the substitution $x = a\cosh\theta$ and writing $\cosh^{-1}\lambda$ as μ, we obtain

$$\begin{aligned} S_2 &= \int_a^{\lambda a} 2b\sqrt{\frac{x^2}{a^2} - 1}\,dx \\ &= \int_0^{\mu} 2b\sinh\theta\,(a\sinh\theta\,d\theta) \\ &= \int_0^{\mu} ab(\cosh 2\theta - 1)\,d\theta \\ &= ab\left[\frac{\sinh 2\theta}{2} - \theta\right]_0^{\mu} \\ &= ab(\lambda\sqrt{\lambda^2 - 1} - \cosh^{-1}\lambda). \end{aligned}$$

In summary, $\mathbf{S} = -\frac{1}{2}\pi b^2(\lambda^2 - 1)\,\mathbf{i} + ab(\lambda\sqrt{\lambda^2 - 1} - \cosh^{-1}\lambda)\,\mathbf{k}$.

11.12 *Show that the expression below is equal to the solid angle subtended by a rectangular aperture of sides* $2a$ *and* $2b$ *at a point a distance* c *from the aperture along the normal to its centre:*

$$\Omega = 4\int_0^b \frac{ac}{(y^2+c^2)(y^2+c^2+a^2)^{1/2}}\,dy.$$

By setting $y = (a^2+c^2)^{1/2}\tan\phi$, *change this integral into the form*

$$\int_0^{\phi_1} \frac{4ac\cos\phi}{c^2+a^2\sin^2\phi}\,d\phi,$$

where $\tan\phi_1 = b/(a^2+c^2)^{1/2}$, *and hence show that*

$$\Omega = 4\tan^{-1}\left[\frac{ab}{c(a^2+b^2+c^2)^{1/2}}\right].$$

The general expression for the solid angle subtended at the origin is

$$\Omega = \int_S \frac{\mathbf{r}\cdot d\mathbf{S}}{r^3}.$$

In the present case, taking the plane's normal along the z-axis, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + c\,\mathbf{k}$ and $d\mathbf{S} = dx\,dy\,\mathbf{k}$. Therefore

$$\Omega = 4\int_0^a\int_0^b \frac{c\,dx\,dy}{(c^2+x^2+y^2)^{3/2}}.$$

If we write $c^2+y^2 = p^2$ and $x = p\tan\theta$ with $\tan^{-1}(a/p) = \mu$, then this becomes

$$\begin{aligned}\Omega &= 4\int_0^b dy\int_0^\mu \frac{cp\sec^2\theta\,d\theta}{(p^2\sec^2\theta)^{3/2}}\\ &= 4\int_0^b dy\int_0^\mu \frac{c\cos\theta}{p^2}\,d\theta\\ &= 4\int_0^b \frac{c\sin\mu}{p^2}\,dy.\end{aligned}$$

Now,

$$\sin\mu = \frac{a}{(a^2+p^2)^{1/2}} = \frac{a}{(a^2+c^2+y^2)^{1/2}},$$

and so

$$\Omega = 4\int_0^b \frac{ac}{(y^2+c^2)(y^2+c^2+a^2)^{1/2}}\,dy,$$

as given in the question.

Next, as suggested, set $y = (a^2+c^2)^{1/2}\tan\phi$ and define ϕ_1 by $b = (a^2+c^2)^{1/2}\tan\phi_1$. Then,

$$\begin{aligned}
\Omega &= 4ac\int_0^{\phi_1}\frac{(a^2+c^2)^{1/2}\sec^2\phi\,d\phi}{[c^2+(a^2+c^2)\tan^2\phi\,](a^2+c^2)^{1/2}(\sec^2\phi)^{1/2}}\\
&= 4ac\int_0^{\phi_1}\frac{\sec\phi\,d\phi}{c^2+(a^2+c^2)\tan^2\phi}\\
&= 4ac\int_0^{\phi_1}\frac{\cos\phi\,d\phi}{a^2\sin^2\phi+c^2}\\
&= \frac{4ac}{a^2}\int_0^{\phi_1}\frac{\cos\phi\,d\phi}{\sin^2\phi+\dfrac{c^2}{a^2}}\\
&= \frac{4c}{a}\left[\frac{a}{c}\tan^{-1}\frac{a\sin\phi}{c}\right]_0^{\phi_1}\\
&= 4\tan^{-1}\frac{a\sin\phi_1}{c} = 4\tan^{-1}\frac{ab}{c(a^2+c^2+b^2)^{1/2}}.
\end{aligned}$$

This establishes the explicit expression for the solid angle subtended by the rectangle.

11.14 *A vector field* $\mathbf{a}$ *is given by* $(z^2+2xy)\,\mathbf{i}+(x^2+2yz)\,\mathbf{j}+(y^2+2zx)\,\mathbf{k}$. *Show that* $\mathbf{a}$ *is conservative and that the line integral* $\int\mathbf{a}\cdot d\mathbf{r}$ *along any line joining* $(1,1,1)$ *and* $(1,2,2)$ *has the value* 11.

We show that the field is conservative by showing that it is possible to construct a suitable potential function, as follows.

$$\begin{aligned}
\frac{\partial\phi}{\partial x} &= z^2+2xy \quad\Rightarrow\quad \phi(x,y,z) = xz^2+x^2y+f(y,z),\\
\frac{\partial\phi}{\partial y} &= x^2+2yz = x^2+\frac{\partial f}{\partial y} \quad\Rightarrow\quad f(y,z) = y^2z+g(z),\\
\frac{\partial\phi}{\partial z} &= y^2+2zx = 2xz+y^2+\frac{\partial g}{\partial z} \quad\Rightarrow\quad g(z) = c.
\end{aligned}$$

Thus

$$\phi(x,y,z) = xz^2+x^2y+y^2z+c$$

is a suitable potential function.

It follows that the line integral of $\mathbf{a}$ along *any* line joining $(1,1,1)$ to $(1,2,2)$ has the value $\phi(1,2,2)-\phi(1,1,1) = (14+c)-(3+c) = 11$.

11.16 *One of Maxwell's electromagnetic equations states that all magnetic fields* $\mathbf{B}$ *are solenoidal (i.e.* $\nabla\cdot\mathbf{B}=0$*). Determine whether each of the following vectors could represent a real magnetic field; where it could, try to find a suitable vector potential* $\mathbf{A}$*, i.e. such that* $\mathbf{B}=\nabla\times\mathbf{A}$*. (Hint: seek a vector potential that is parallel to* $\nabla\times\mathbf{B}$*.):*

(a) $\dfrac{B_0 b}{r^3}[(x-y)z\,\mathbf{i}+(x-y)z\,\mathbf{j}+(x^2-y^2)\,\mathbf{k}]$ *in Cartesians with* $r^2=x^2+y^2+z^2$*;*

(b) $\dfrac{B_0 b^3}{r^3}[\cos\theta\ \cos\phi\,\hat{\mathbf{e}}_r-\sin\theta\ \cos\phi\,\hat{\mathbf{e}}_\theta+\sin 2\theta\ \sin\phi\,\hat{\mathbf{e}}_\phi]$ *in spherical polars;*

(c) $B_0 b^2\left[\dfrac{z\rho}{(b^2+z^2)^2}\,\hat{\mathbf{e}}_\rho+\dfrac{1}{b^2+z^2}\,\hat{\mathbf{e}}_z\right]$ *in cylindrical polars.*

(a) We calculate $\nabla\cdot\mathbf{B}$ in Cartesian coordinates.

$$\begin{aligned}\frac{\nabla\cdot\mathbf{B}}{B_0 b}&=-\frac{3(x-y)zx}{r^5}+\frac{z}{r^3}-\frac{3(x-y)zy}{r^5}-\frac{z}{r^3}-\frac{3(x^2-y^2)z}{r^5}\\&=-\frac{6(x^2-y^2)z}{r^5}\neq 0\quad\Rightarrow\quad\mathbf{B}\text{ cannot be a real field.}\end{aligned}$$

(b) Working in spherical polar coordinates:

$$\begin{aligned}\nabla\cdot\mathbf{B}&=\frac{1}{r^2}\frac{\partial}{\partial r}(r^2B_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta B_\theta)+\frac{1}{r\sin\theta}\frac{\partial B_\phi}{\partial\phi}\\&=B_0b^3\left[-\frac{\cos\theta\cos\phi}{r^4}-\frac{\cos\phi\,2\sin\theta\cos\theta}{r^4\sin\theta}+\frac{\sin 2\theta\cos\phi}{r^4\sin\theta}\right]\\&=-\frac{B_0b^3\cos\theta\cos\phi}{r^4}\neq 0\quad\Rightarrow\quad\mathbf{B}\text{ cannot be a real field.}\end{aligned}$$

(c) $\mathbf{B}$ has no ϕ-dependence and so

$$\begin{aligned}\nabla\cdot\mathbf{B}&=\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho)+\frac{\partial B_z}{\partial z}\\\frac{\nabla\cdot\mathbf{B}}{B_0b^2}&=\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\frac{\rho\,z\,\rho}{(b^2+z^2)^2}\right)+\frac{\partial}{\partial z}\left(\frac{1}{b^2+z^2}\right)\\&=\frac{2z}{(b^2+z^2)^2}-\frac{2z}{(b^2+z^2)^2}=0\\&\Rightarrow\mathbf{B}\text{ could be a real magnetic field.}\end{aligned}$$

Following the hint (and with no ϕ component or ϕ-dependence in $\mathbf{B}$),

$$\begin{aligned}\frac{\nabla\times\mathbf{B}}{B_0b^2}&=\left(\frac{1}{\rho}(0-0),\ \frac{\partial B_\rho}{\partial z}-\frac{\partial B_z}{\partial\rho},\ \frac{1}{\rho}(0-0)\right)\\&=\left(0,\ \frac{\rho}{(b^2+z^2)^2}-\frac{4\rho z^2}{(b^2+z^2)^3}-0,\ 0\right).\end{aligned}$$

$\nabla \times \mathbf{B}$ has only a ϕ-component and so take $\mathbf{A} = (0, A_\phi, 0)$. We then require

$$\begin{aligned}
\frac{B_0 b^2 z\rho}{(b^2+z^2)^2} &= (\nabla \times \mathbf{A})_\rho = -\frac{\partial A_\phi}{\partial z} \\
\Rightarrow \quad A_\phi &= \frac{B_0 b^2 \rho}{2(b^2+z^2)} + g(\rho), \\
\frac{B_0 b^2}{b^2+z^2} &= (\nabla \times \mathbf{A})_z = \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A_\phi) \\
\Rightarrow \quad \rho A_\phi &= \frac{B_0 b^2 \rho^2}{2(b^2+z^2)} + f(z).
\end{aligned}$$

These two equations, taken together, imply that $\mathbf{A} = (0, A_\phi, 0)$, with

$$A_\phi = \frac{B_0 b^2 \rho}{2(b^2+z^2)}$$

a suitable component. To this $\mathbf{A}$ could be added *any* vector field that is the gradient of a scalar.

11.18 *A vector field* $\mathbf{a} = f(r)\mathbf{r}$ *is spherically symmetric and everywhere directed away from the origin. Show that* $\mathbf{a}$ *is irrotational but that it is also solenoidal only if* $f(r)$ *is of the form* Ar^{-3}.

In spherical polar coordinates, $\mathbf{a} = f(r)\mathbf{r} = rf(r)\hat{\mathbf{e}}_r$, and

$$\begin{aligned}
\nabla \times \mathbf{a} &= \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{\mathbf{e}}_r & r\hat{\mathbf{e}}_\theta & r\sin\theta\hat{\mathbf{e}}_\phi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ rf(r) & 0 & 0 \end{vmatrix} \\
&= 0\hat{\mathbf{e}}_r + \frac{1}{r\sin\theta}\frac{\partial(rf)}{\partial\phi}\hat{\mathbf{e}}_\theta - \frac{1}{r}\frac{\partial(rf)}{\partial\theta}\hat{\mathbf{e}}_\phi \\
&= \mathbf{0}.
\end{aligned}$$

Hence $\mathbf{a}$ is irrotational. For it also to be solenoidal requires that

$$0 = \nabla \cdot \mathbf{a} = \frac{1}{r^2}\frac{\partial}{\partial r}[r^2 rf(r)] + 0 + 0 \;\Rightarrow\; r^3 f(r) = A \;\Rightarrow\; f(r) = \frac{A}{r^3}.$$

11.20 *Obtain an expression for the value ϕ_P at a point P of a scalar function ϕ that satisfies $\nabla^2\phi = 0$ in terms of its value and normal derivative on a surface S that encloses it, by proceeding as follows.*

(a) *In Green's second theorem take ψ at any particular point Q as $1/r$, where r is the distance of Q from P. Show that $\nabla^2\psi = 0$ except at $r = 0$.*
(b) *Apply the result to the doubly connected region bounded by S and a small sphere Σ of radius δ centred on P.*
(c) *Apply the divergence theorem to show that the surface integral over Σ involving $1/\delta$ vanishes, and prove that the term involving $1/\delta^2$ has the value $4\pi\phi_P$.*
(d) *Conclude that*

$$\phi_P = -\frac{1}{4\pi}\int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\,dS + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS.$$

This important result shows that the value at a point P of a function ϕ that satisfies $\nabla^2\phi = 0$ everywhere within a closed surface S that encloses P may be expressed entirely *in terms of its value and normal derivative on S. This matter is taken up more generally in connection with Green's functions in chapter 21 and in connection with functions of a complex variable in section 24.10.*

Green's theorems apply to any suitably differentiable pair of functions, but here we apply them to a function ϕ that satisfies $\nabla^2\phi = 0$ and ψ, which has a value at any point Q equal to the reciprocal of its distance r from a fixed point P.

(a) Using spherical polar coordinates centred on P, we have

$$\nabla^2\psi = \nabla^2\left(\frac{1}{r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right) = \frac{1}{r^2}\frac{\partial(-1)}{\partial r} = 0.$$

Thus $\nabla^2\psi = 0$ except at $r = 0$ where the function is not differentiable.

(b) When these results are put into Green's second theorem applied to the doubly connected region bounded by S and a small sphere Σ of radius δ centred on P, the integrand in the volume integral vanishes, leading to

$$\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS + \int_\Sigma \frac{1}{r}\frac{\partial\phi}{\partial n}\,dS = \int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\,dS + \int_\Sigma \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right)\,dS.$$

(c) For the term on the LHS taken over the sphere Σ, the factor r^{-1} is a constant and equal to δ^{-1} and, by the divergence theorem, the surface integral of $\partial\phi/\partial n$ is equal to the volume integral of $\nabla^2\phi$. But this is zero and so the term vanishes.

For the term on the RHS taken over the sphere Σ,

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = -\left(-\frac{1}{r^2}\right)_{r=\delta} = \frac{1}{\delta^2}.$$

The additional minus sign arises because $\hat{\mathbf{n}}$ is the outward normal to the space and this is in the direction of decreasing r. The surface area is $4\pi\delta^2$ and so the value of the integral is $4\pi\phi_P$ in the limit of $\delta \to 0$.

(d) Taking all terms involving integrals over S to one side of the equation, it can be rearranged as

$$\phi_P = -\frac{1}{4\pi}\int_S \phi\frac{\partial}{\partial n}\left(\frac{1}{r}\right) dS + \frac{1}{4\pi}\int_S \frac{1}{r}\frac{\partial\phi}{\partial n}\, dS,$$

thus establishing the stated result.

11.22 *A rigid body of volume V and surface S rotates with angular velocity $\boldsymbol{\omega}$. Show that*

$$\boldsymbol{\omega} = -\frac{1}{2V}\oint_S \mathbf{u}\times d\mathbf{S},$$

where $\mathbf{u}(\mathbf{x})$ is the velocity of the point $\mathbf{x}$ on the surface S.

From result (11.22), which is proved in exercise 11.24, we have in general that

$$\int_V (\nabla\times\mathbf{b})\, dV = \oint_S d\mathbf{S}\times\mathbf{b}.$$

For the current application we set $\mathbf{b}$ equal to $\mathbf{u}(\mathbf{x}) = \boldsymbol{\omega}\times\mathbf{x}$, giving

$$\begin{aligned}\oint_S d\mathbf{S}\times\mathbf{u} &= \int_V \nabla\times(\boldsymbol{\omega}\times\mathbf{x})\, dV\\ &= \int_V [\,\boldsymbol{\omega}(\nabla\cdot\mathbf{x}) - \mathbf{x}(\nabla\cdot\boldsymbol{\omega}) + (\mathbf{x}\cdot\nabla)\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\nabla)\mathbf{x}\,]\, dV\\ &= \int_V [\,3\boldsymbol{\omega} - \mathbf{0} - \mathbf{0} - (\omega_x\,\mathbf{i} + \omega_y\,\mathbf{j} + \omega_z\,\mathbf{k})\,]\, dV\\ &= 3\boldsymbol{\omega}V - \boldsymbol{\omega}V.\end{aligned}$$

To obtain the second line we used the standard identity for $\nabla\times(\mathbf{a}\times\mathbf{b})$ (see table 10.1). Thus,

$$\boldsymbol{\omega} = -\frac{1}{2V}\oint_S \mathbf{u}\times d\mathbf{S}.$$

11.24 *Prove equation (11.22) and, by taking* $\mathbf{b} = zx^2\mathbf{i} + zy^2\mathbf{j} + (x^2 - y^2)\mathbf{k}$, *show that the two integrals*

$$I = \int x^2\,dV \quad \textit{and} \quad J = \int \cos^2\theta \sin^3\theta \cos^2\phi\,d\theta\,d\phi,$$

both taken over the unit sphere, must have the same value. Evaluate both directly to show that the common value is $4\pi/15$.

We have to prove that

$$\int_V (\nabla \times \mathbf{b})\,dV = \oint_S d\mathbf{S} \times \mathbf{b}.$$

Let $\mathbf{a} = \mathbf{b} \times \mathbf{c}$, where $\mathbf{c}$ is an arbitrary but fixed vector. Then, from the divergence theorem

$$\int_V \nabla \cdot \mathbf{a} = \oint_S \mathbf{a} \cdot d\mathbf{S},$$

$$\int_V \nabla \cdot (\mathbf{b} \times \mathbf{c}) = \oint_S (\mathbf{b} \times \mathbf{c}) \cdot d\mathbf{S},$$

$$\int_V [\,\mathbf{c} \cdot (\nabla \times \mathbf{b}) - \mathbf{b} \cdot (\nabla \times \mathbf{c})\,]\,dV = \oint_S (d\mathbf{S} \times \mathbf{b}) \cdot \mathbf{c}.$$

To obtain this last line we have used a result from table 10.1 and the cyclic property of a triple scalar product. But $\nabla \times \mathbf{c} = \mathbf{0}$ and so

$$\mathbf{c} \cdot \int_V (\nabla \times \mathbf{b})\,dV = \mathbf{c} \cdot \oint_S d\mathbf{S} \times \mathbf{b},$$

and, since $\mathbf{c}$ is also arbitrary, it follows that

$$\int_V (\nabla \times \mathbf{b})\,dV = \oint_S d\mathbf{S} \times \mathbf{b}.$$

With $\mathbf{b} = zx^2\,\mathbf{i} + zy^2\,\mathbf{j} + (x^2 - y^2)\,\mathbf{k}$,

$$\nabla \times \mathbf{b} = (-2y - y^2)\,\mathbf{i} + (x^2 - 2x)\,\mathbf{j}.$$

Clearly, on (anti-) symmetry grounds, $\int x\,dV = \int y\,dV = 0$ for integrals over the unit sphere and so $\int(\nabla \times \mathbf{b})\,dV$ has the form $(-I,\ I,\ 0)$ where $I = \int x^2\,dV = \int y^2\,dV$.

On the surface of the unit sphere, where $x = \sin\theta\cos\phi$, $y = \sin\theta\sin\phi$ and $z = \cos\theta$,

$$d\mathbf{S} = \sin\theta(\sin\theta\cos\phi\,\mathbf{i} + \sin\theta\sin\phi\,\mathbf{j} + \cos\theta\,\mathbf{k})\,d\theta\,d\phi,$$

$$\mathbf{b} = \cos\theta\sin^2\theta\cos^2\phi\,\mathbf{i} + \cos\theta\sin^2\theta\sin^2\phi\,\mathbf{j} + \sin^2\theta\cos 2\phi\,\mathbf{k}.$$

Thus, we have

$$\begin{aligned}\frac{d\mathbf{S}\times\mathbf{b}}{d\theta\,d\phi} &= \sin\theta(\sin^3\theta\sin\phi\cos 2\phi - \cos^2\theta\sin^2\theta\sin^2\phi)\,\mathbf{i}\\ &\quad + \sin\theta(\cos^2\theta\sin^2\theta\cos^2\phi - \sin^3\theta\cos\phi\cos 2\phi)\,\mathbf{j}\\ &\quad + \sin\theta(\cos\theta\sin^3\theta\cos\phi\sin^2\phi - \sin^3\theta\cos\theta\sin\phi\cos^2\phi)\,\mathbf{k}.\end{aligned}$$

The two terms in the $\mathbf{k}$-coordinate cancel each other when integrated over $0 \le \phi < 2\pi$. The first term in the $\mathbf{i}$-coordinate can be written as $g(\theta)(\sin 3\phi - \sin\phi)$ and therefore integrates to zero; similarly, the second term in the $\mathbf{j}$-coordinate does not contribute.

In summary, the integral $\oint d\mathbf{S}\times\mathbf{b}$ has the form $(-J,\ J,\ 0)$ where

$$J = \iint \cos^2\theta\sin^3\theta\sin^2\phi\,d\theta\,d\phi = \iint \cos^2\theta\sin^3\theta\cos^2\phi\,d\theta\,d\phi.$$

It follows that the integrals I and J defined in the question are equal.

It only remains to evaluate I and J. For I we have

$$\begin{aligned}I &= \int_V x^2\,dV\\ &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 r^2\sin^2\theta\cos^2\phi\,r^2\sin\theta\,dr\,d\phi\,d\theta\\ &= \int_0^{\pi}\int_0^{2\pi}\frac{1}{5}\sin^3\theta\cos^2\phi\,d\phi\,d\theta\\ &= \frac{\pi}{5}\int_0^{\pi}\sin\theta(1-\cos^2\theta)\,d\theta\\ &= \frac{\pi}{5}\left[2-\frac{2}{3}\right] = \frac{4\pi}{15}.\end{aligned}$$

For J the integral is

$$\begin{aligned}J &= \int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^3\theta\sin^2\phi\,d\theta\,d\phi\\ &= \pi\int_0^{\pi}(\cos^2\theta-\cos^4\theta)\sin\theta\,d\theta\\ &= \pi\left[\frac{2}{3}-\frac{2}{5}\right] = \frac{4\pi}{15},\end{aligned}$$

i.e. the same value as I.

11.26 *A vector field* $\mathbf{F}$ *is defined in cylindrical polar coordinates* ρ, θ, z *by*

$$\mathbf{F} = F_0\left(\frac{x\cos\lambda z}{a}\mathbf{i} + \frac{y\cos\lambda z}{a}\mathbf{j} + (\sin\lambda z)\mathbf{k}\right)$$
$$\equiv \frac{F_0\rho}{a}(\cos\lambda z)\mathbf{e}_\rho + F_0(\sin\lambda z)\mathbf{k},$$

where $\mathbf{i}$, $\mathbf{j}$ *and* $\mathbf{k}$ *are the unit vectors along the Cartesian axes and* $\mathbf{e}_\rho$ *is the unit vector* $(x/\rho)\mathbf{i} + (y/\rho)\mathbf{j}$.

(a) *Calculate, as a surface integral, the flux of* $\mathbf{F}$ *through the closed surface bounded by the cylinders* $\rho = a$ *and* $\rho = 2a$ *and the planes* $z = \pm a\pi/2$.
(b) *Evaluate the same integral using the divergence theorem.*

(a) The flux through the cylindrical surfaces

$$= \int_{-a\pi/2}^{a\pi/2} \frac{F_0}{a}\cos\lambda z\, dz \left[\int_0^{2\pi} \rho\,\rho\, d\phi\right]_{\rho=a}^{\rho=2a}$$
$$= \int_{-a\pi/2}^{a\pi/2} \frac{2\pi F_0}{a}[(2a)^2 - (a)^2]\cos\lambda z\, dz$$
$$= 6\pi F_0 a \int_{-a\pi/2}^{a\pi/2} \cos\lambda z\, dz$$
$$= \frac{12\pi F_0 a}{\lambda}\sin\left(\frac{\lambda\pi a}{2}\right).$$

The flux through the planes

$$= \pi(4a^2 - a^2)\left[F_0\sin\left(\frac{\lambda\pi a}{2}\right) - F_0\sin\left(-\frac{\lambda\pi a}{2}\right)\right]$$
$$= 6\pi a^2 F_0\sin\left(\frac{\lambda\pi a}{2}\right).$$

Adding these together gives

$$\text{Total flux } = 6\pi F_0\left(a^2 + \frac{2a}{\lambda}\right)\sin\left(\frac{\lambda\pi a}{2}\right).$$

(b) Using the Cartesian form

$$\nabla\cdot\mathbf{F} = \frac{F_0}{a}(\cos\lambda z + \cos\lambda z + a\lambda\cos\lambda z),$$

which is independent of ρ. Thus the ρ and ϕ integrations are trivial and the

volume integral of the divergence reduces to

$$\begin{aligned}\int_V \nabla\cdot\mathbf{F}\,dV &= \int_{-a\pi/2}^{a\pi/2} \frac{\pi F_0(4a^2-a^2)(2+a\lambda)}{a}\cos\lambda z\,dz\\ &= 3\pi F_0 a(2+a\lambda)\left[\frac{\sin\lambda z}{\lambda}\right]_{-a\pi/2}^{a\pi/2}\\ &= 6\pi F_0\left(\frac{2a}{\lambda}+a^2\right)\sin\left(\frac{\lambda\pi a}{2}\right), \quad \text{as in part (a).}\end{aligned}$$

11.28 *A vector force field* $\mathbf{F}$ *is defined in Cartesian coordinates by*

$$\mathbf{F} = F_0\left[\left(\frac{y^3}{3a^3}+\frac{y}{a}e^{xy/a^2}+1\right)\mathbf{i}+\left(\frac{xy^2}{a^3}+\frac{x+y}{a}e^{xy/a^2}\right)\mathbf{j}+\frac{z}{a}e^{xy/a^2}\mathbf{k}\right].$$

Use Stokes' theorem to calculate

$$\oint_L \mathbf{F}\cdot d\mathbf{r},$$

where L *is the perimeter of the rectangle* $ABCD$ *given by* $A=(0,a,0)$, $B=(a,a,0)$, $C=(a,3a,0)$ *and* $D=(0,3a,0)$.

The rectangle $ABCD$ lies in the plane $z=0$ and so to apply Stokes' theorem we need only the z-component of $\nabla\times\mathbf{F}$. this is given by

$$\begin{aligned}(\nabla\times\mathbf{F})_z &= F_0\left(\frac{y^2}{a^3}+\frac{1}{a}e^{xy/a^2}+\frac{(x+y)y}{a^3}e^{xy/a^2}\right)\\ &\quad - F_0\left(\frac{y^2}{a^3}+\frac{1}{a}e^{xy/a^2}+\frac{xy}{a^3}e^{xy/a^2}\right)\\ &= \frac{F_0y^2}{a^3}e^{xy/a^2}.\end{aligned}$$

So, by Stokes' theorem, the line integral has the same value as this component of curl $\mathbf{F}$ integrated over the area of the rectangle, i.e.

$$\oint_L \mathbf{F}\cdot d\mathbf{r} = \int_0^a dx\int_a^{3a} dy\,\frac{F_0y^2}{a^3}e^{xy/a^2}.$$

Since x appears in the integrand only in the form $e^{\lambda x}$, the x-integration is straightforward and is therefore carried out first to give

$$\oint_L \mathbf{F}\cdot d\mathbf{r} = \frac{F_0}{a^3}\int_a^{3a}\frac{a^2\left(e^{y/a}-1\right)y^2}{y}\,dy.$$

After simplification, this can be integrated by parts to yield the final value for the contour integral:

$$\begin{aligned}
\oint_L \mathbf{F} \cdot d\mathbf{r} &= \frac{F_0}{a} \int_a^{3a} (ye^{y/a} - y)\, dy \\
&= \frac{F_0}{a} \left\{ \left[aye^{y/a} \right]_a^{3a} - \int_a^{3a} ae^{y/a}\, dy - \left[\frac{y^2}{a} \right]_a^{3a} \right\} \\
&= \frac{F_0}{a} \left(3a^2e^3 - a^2e - a^2e^3 + a^2e - \frac{9}{2}a^2 + \frac{1}{2}a^2 \right) \\
&= F_0 a(2e^3 - 4).
\end{aligned}$$

12

Fourier series

12.2 *Derive the Fourier coefficients b_r in a similar manner to the derivation of the a_r in section 12.2.*

As explained in the text, the method of proof is almost identical to that given in section 12.2, the only differences being multiplying the Fourier expansion (12.4) through by $\sin(2\pi px/L)$ rather than by $\cos(2\pi px/L)$ and using the first (12.1) and last (12.3) of the orthogonality relations (rather than the first and second).

The given text may be taken as a model solution once these small changes have been allowed for. There is never a b_0 term, formally because $\sin(2\pi rx/L)$ is zero for all x if $r = 0$.

12.4 *By moving the origin of t to the centre of an interval in which $f(t) = +1$, i.e. by changing to a new independent variable $t' = t - \frac{1}{4}T$, express the square-wave function in the example in section 12.2 as a cosine series. Calculate the Fourier coefficients involved* (a) *directly and* (b) *by changing the variable in result (12.8).*

With the change in origin, the function becomes an even one and only cosine terms will be needed. However, the change of origin does not affect the average value of the function, which therefore remains equal to zero. This means that the value of A_0 in the cosine series will also be zero.

(a) By direct calculation:

$$\begin{aligned}
A_n &= \frac{2}{T}\int_{-T/2}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= \frac{4}{T}\int_{0}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= \frac{4}{T}\int_{0}^{T/4} f(t')\cos\frac{2\pi nt'}{T}\,dt' - \frac{4}{T}\int_{T/4}^{T/2} f(t')\cos\frac{2\pi nt'}{T}\,dt' \\
&= 4\left[\frac{\sin(2\pi nt'/T)}{2\pi n}\right]_0^{T/4} - 4\left[\frac{\sin(2\pi nt'/T)}{2\pi n}\right]_{T/4}^{T/2} \\
&= \frac{2}{\pi n}\left[\sin\frac{n\pi}{2} - \sin 0 - \sin n\pi + \sin\frac{n\pi}{2}\right] \\
&= \frac{4}{n\pi}(-1)^{(n-1)/2} \text{ for odd } n, \text{ and } = 0 \text{ for even } n.
\end{aligned}$$

(b) By changing the variable in the result for $f(t)$ derived in the text, and writing $2\pi/T$ as ω:

$$\begin{aligned}
f(t) &= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\omega t}{n}, \\
g(t') &= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\omega(t' + \frac{1}{4}T)}{n}, \\
&= \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{1}{n}[\sin(n\omega t')\cos(\tfrac{1}{4}n\omega T) + \cos(n\omega t')\sin(\tfrac{1}{4}n\omega T)].
\end{aligned}$$

Now, for n odd, $\cos(\frac{1}{4}n\omega T) = \cos(n\pi/2) = 0$ but $\sin(\frac{1}{4}n\omega T) = \sin(n\pi/2) = (-1)^{(n-1)/2}$. Thus, only the $\cos(n\omega t')$ terms survive and

$$g(t') = \frac{4}{\pi}\sum_{n\,\text{odd}} \frac{(-1)^{(n-1)/2}}{n}\cos(n\omega t').$$

This is, as it must be, the same result as that obtained by direct calculation.

12.6 *For the function*

$$f(x) = 1 - x, \qquad 0 \le x \le 1,$$

find (a) *the Fourier sine series and* (b) *the Fourier cosine series. Which would be better for numerical evaluation? Relate your answer to the relevant periodic continuations.*

(a) Sine series. In order to make the function both periodic and odd in x, it must

be continued in the range $-1 < x \leq 0$ as $f(x) = -1 - x$. The function thus has a discontinuity of 2 at $x = 0$. The Fourier coefficients are

$$\begin{aligned} b_n &= 2\,\frac{2}{2}\int_0^1 (1-x)\sin n\pi x\,dx \\ &= 2\left\{\left[-\frac{\cos n\pi x}{n\pi}\right]_0^1 + \left[\frac{x\cos n\pi x}{n\pi}\right]_0^1 - \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx\right\} \\ &= 2\left(-\frac{(-1)^n - 1}{n\pi} + \frac{(-1)^n}{n\pi} - 0\right) = \frac{2}{n\pi}. \end{aligned}$$

Thus the Fourier sine series for this function is

$$1 - x = f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n}.$$

(b) Cosine series. In order to make the function both periodic and even in x, it must be continued in the range $-1 < x \leq 0$ as $f(x) = 1 + x$. The function then has no discontinuity at $x = 0$. The Fourier coefficients are

$$\begin{aligned} a_n &= 2\,\frac{2}{2}\int_0^1 (1-x)\cos n\pi x\,dx \\ &= 2\left\{\left[\frac{\sin n\pi x}{n\pi}\right]_0^1 - \left[\frac{x\sin n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\sin n\pi x}{n\pi}\,dx\right\} \\ &= 2\left(0 - 0 + \left[-\frac{\cos n\pi x}{n^2\pi^2}\right]_0^1\right) \\ &= 2\left(-\frac{(-1)^n - 1}{n^2\pi^2}\right) \\ &= \frac{4}{n^2\pi^2} \text{ for } n \text{ odd, and } = 0 \text{ for positive even } n. \end{aligned}$$

For $n = 0$, the non-zero integral is $a_0 = 2\int_0^1 (1-x)\,dx = 1$, making the complete Fourier cosine series representation

$$1 - x = f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n\,\text{odd}}\frac{\cos n\pi x}{n^2}.$$

Because alternate terms (the positive even values of n) are missing and the series converges as n^{-2} (rather than as n^{-1}), it is clear that the cosine series is much superior for calculational purposes. This superiority is reinforced by the lack of a discontinuity in the continued function in case (b); the discontinuity in case (a) will bring additional computational difficulty as a result of the Gibbs' phenomenon.

12.8 *The function $y(x) = x\sin x$ for $0 \le x \le \pi$ is to be represented by a Fourier series of period 2π that is either even or odd. By sketching the function and considering its derivative, determine which series will have the more rapid convergence. Find the full expression for the better of these two series, showing that the convergence $\sim n^{-3}$ and that alternate terms are missing.*

As the period is to be 2π, the question is how to define $y(x)$ in the range $-\pi \le x \le 0$. The even and odd continuations would be

$$y_{\mathrm{e}}(x) = x\sin x \quad \text{and} \quad y_{\mathrm{o}}(x) = -x\sin x.$$

Both continuations make the function continuous at $x = 0$ and at $x = \pm\pi$. However, there is a difference in their derivatives. Both have zero derivative at $x = 0$, but, at $x = -\pi$, $y'_{\mathrm{e}} = \pi$ whilst $y'_{\mathrm{o}} = -\pi$. To avoid a discontinuity in the derivative, the derivative of the continuation must match that of $y(x)$ evaluated at $x = +\pi$. The value of the latter is $-\pi$, and so the odd continuation is the one to be preferred as it will give more rapid convergence and avoid problems arising from the Gibbs' phenomenon.

Thus the series is to be a sine series with

$$\begin{aligned} b_n &= \frac{2}{2\pi}\,2\int_0^{\pi} x\sin x\sin nx\,dx \\ &= \frac{1}{\pi}\int_0^{\pi} x[\cos(n-1)x - \cos(n+1)x]\,dx. \end{aligned}$$

Now, for integer p, except when $p = 0$,

$$\int_0^{\pi} x\cos px\,dx = \left[\frac{x\sin px}{p}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin px}{p}\,dx = 0 + \left[\frac{\cos px}{p^2}\right]_0^{\pi} = \frac{(-1)^p - 1}{p^2}.$$

When $p = 0$ the integral has value $\pi^2/2$.
Thus

$$\begin{aligned} b_n &= \frac{1}{\pi}\left[\frac{(-1)^{n-1}-1}{(n-1)^2} - \frac{(-1)^{n+1}-1}{(n+1)^2}\right] \\ &= \frac{1}{\pi}\left[\frac{-2}{(n-1)^2} + \frac{2}{(n+1)^2}\right], \text{ for } n \text{ even}, \\ &= -\frac{8n}{\pi(n^2-1)^2}, \text{ for } n \text{ even}, \\ b_n &= 0 - 0 \text{ for } n \text{ odd, except for } n = 1 \end{aligned}$$

When $n = 1$,

$$b_n = \frac{1}{\pi}\int_0^{\pi} x(1-\cos 2x)\,dx = \frac{1}{\pi}\left[\frac{\pi^2}{2} - 0\right] = \frac{\pi}{2}.$$

Finally, collecting together the results obtained,

$$y(x) = x\sin x = \frac{\pi}{2}\sin x - \frac{8}{\pi}\sum_{n\,\text{even}} \frac{n}{(n^2-1)^2}\sin nx.$$

The series converges as $n/(n^2)^2 \sim n^{-3}$.

12.10 *By integrating term by term the Fourier series found in the previous question (exercise 12.9) and using the Fourier series for $f(x) = x$, show that $\int \exp\ x\,dx = \exp\ x + c$. Why is it not possible to show that $d(\exp\ x)/dx = \exp\ x$ by differentiating the Fourier series of $f(x) = \exp\ x$ in a similar manner?*

The series for $\exp\ x$ (found in exercise 12.9) is

$$\exp\ x = (\sinh 1)\left\{1 + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}[\cos(n\pi x) - n\pi\sin(n\pi x)]\right\}.$$

Integrating this term by term gives

$$I = \int \exp\ x\,dx = (\sinh 1)\left\{x + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2}\left[\frac{\sin(n\pi x)}{n\pi} + \cos(n\pi x)\right]\right\}.$$

Now the function x can be expanded in $-1 \le x \le 1$ as a Fourier sine series with

$$b_n = \frac{2}{2}\int_{-1}^{1} x\sin(n\pi x)\,dx = \left[-\frac{x\cos(n\pi x)}{n\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{\cos(n\pi x)}{n\pi}\,dx = \frac{2(-1)^{n+1}}{n\pi} + 0.$$

Thus, we may show that I has the form stated in the question as follows:

$$\begin{aligned} I &= 2(\sinh 1)\sum_{n=1}^{\infty}\left[\left(\frac{(-1)^n}{n\pi(1+n^2\pi^2)} + \frac{(-1)^{n+1}}{n\pi}\right)\sin(n\pi x)\right. \\ &\qquad\qquad \left. + \frac{(-1)^n}{1+n^2\pi^2}\cos(n\pi x)\right] \\ &= 2(\sinh 1)\sum_{n=1}^{\infty}\left[\frac{(-1)^n(-n^2\pi^2)}{n\pi(1+n^2\pi^2)}\sin(n\pi x) + \frac{(-1)^n}{1+n^2\pi^2}\cos(n\pi x)\right] \\ &= \exp x - \sinh 1, \text{ by comparison with the original series.} \end{aligned}$$

If the original series is differentiated (with the aim of finding a series to represent the derivative of $\exp\ x$) it will contain the sum

$$2(\sinh 1)\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(n\pi)^2}{1+n^2\pi^2}\cos(n\pi x).$$

This sum does not converge since the terms do not $\to 0$ as $n \to \infty$. Consequently, no useful result is obtained.

12.12 *Find, without calculation, which terms will be present in the Fourier series for the periodic functions $f(t)$, of period T, that are given in the range $-T/2$ to $T/2$ by:*

(a) $f(t) = 2$ *for* $0 \le |t| < T/4$, $f = 1$ *for* $T/4 \le |t| < T/2$;
(b) $f(t) = \exp[-(t - T/4)^2]$;
(c) $f(t) = -1$ *for* $-T/2 \le t < -3T/8$ *and* $3T/8 \le t < T/2, f(t) = 1$ *for* $-T/8 \le t < T/8$; *the graph of f is completed by two straight lines in the remaining ranges so as to form a continuous function.*

If the Fourier series for $f(t)$ is written in the form

$$f(t) = \frac{a_0}{2} + \sum_{r=1}^{\infty} \left[a_r \cos\left(\frac{2\pi rt}{L}\right) + b_r \sin\left(\frac{2\pi rt}{L}\right) \right],$$

then the consequences of any symmetry properties that $f(t)$ may possess can be summarised by

- if $f(t)$ is even about $t = 0$ then all $b_r = 0$,
- if $f(t)$ is odd about $t = 0$ then all $a_r = 0$,
- if $f(t)$ is even about $t = T/4$ then $a_{2r+1} = 0$ and $b_{2r} = 0$,
- if $f(t)$ is odd about $t = T/4$ then $a_{2r} = 0$ and $b_{2r+1} = 0$,
- the average value of $f(t)$ over a complete cycle is $\frac{1}{2}a_0$.

Sketching the given functions shows the following.

(a) This is a function that:
(i) is even about $t = 0 \Rightarrow$ no sine terms are present;
(ii) has a non-zero average $\Rightarrow a_0$ $(= 3)$ present;
(iii) is odd about $t = T/4$ once the average value has been subtracted $\Rightarrow$ $a_{2n} = 0$.
Thus the series contains a constant and odd-n cosine terms.

(b) The periodic version of this function does not exhibit symmetry about any value of t; there is a discontinuity of $-(e^{-T^2/16} - e^{-9T^2/16})$ at $t = T/2$. Consequently, all terms are present.

(c) This is a function that:
(i) is even about $t = 0 \Rightarrow$ no sine terms are present;
(ii) has a zero average $\Rightarrow a_0 = 0$;
(iii) is odd about $t = T/4 \Rightarrow a_{2n} = 0$.
Thus the series consists of odd-n cosine terms only.

12.14 *Show that the Fourier series for the function* $y(x) = |x|$ *in the range* $-\pi \le x < \pi$ *is*

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}.$$

By integrating this equation term by term from 0 *to* x, *find the function* $g(x)$ *whose Fourier series is*

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Deduce the value of the sum S *of the series*

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots.$$

The function $y(x) = |x|$ is an even function and its Fourier series will therefore contain only cosine terms. They are given, for $n \ge 1$, by

$$\begin{aligned} a_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi}|x|\cos nx\,dx \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx \\ &= \left\{\left[\frac{x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}\frac{\sin nx}{n}\,dx\right\} \\ &= \frac{2}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} \\ &= -\frac{4}{\pi n^2} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.} \end{aligned}$$

The constant term is $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x\,dx = \pi$. Thus

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}.$$

We now consider the integral of $y(x)$ from 0 to x.

(i) For $x < 0$, $\displaystyle\int_0^x |x'|\,dx' = \int_0^x (-x')\,dx' = -\frac{1}{2}x^2$.

(ii) For $x > 0$, $\displaystyle\int_0^x |x'|\,dx' = \int_0^x x'\,dx' = \frac{1}{2}x^2$.

Integrating the series gives

$$\frac{\pi x}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Equating these two results and isolating the series gives

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3} = \begin{cases} \frac{1}{2}x(\pi+x) & \text{for } x \le 0, \\ \frac{1}{2}x(\pi-x) & \text{for } x \ge 0. \end{cases}$$

If we set $x = \pi/2$ in this result, the sine terms have values $(-1)^m$ and we obtain

$$\frac{1}{2}\frac{\pi}{2}\left(\pi - \frac{\pi}{2}\right) = \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m}{(2m+1)^3} = \frac{4}{\pi}S.$$

It follows that $S = \pi^3/32$.

12.16 *By finding a cosine Fourier series of period 2 for the function $f(t)$ that takes the form $f(t) = \cosh(t-1)$ in the range $0 \le t \le 1$, prove that*

$$\sum_{n=1}^{\infty}\frac{1}{n^2\pi^2+1} = \frac{1}{e^2-1}.$$

Deduce values for the sums $\sum(n^2\pi^2+1)^{-1}$ over odd n and even n separately.

In order to obtain a cosine series we must make an even continuation $f(t) = \cosh(t+1)$ for $-1 \le t \le 0$.

The constant term in the series is $a_0/2$ with

$$a_0 = 2\int_0^1 \cosh(t-1)\,dt = [\,2\sinh(t-1)\,]_0^1 = 2\sinh(1).$$

The general coeeficient is

$$\begin{aligned} a_n &= 2\int_0^1 \cosh(t-1)\cos(n\pi t)\,dt \\ &= 2\left[\frac{\cosh(t-1)\sin(n\pi t)}{n\pi}\right]_0^1 - 2\int_0^1 \frac{\sinh(t-1)\sin(n\pi t)}{n\pi}\,dt \\ &= 0 - 2\left[-\frac{\sinh(t-1)\cos(n\pi t)}{n^2\pi^2}\right]_0^1 - 2\int_0^1 \frac{\cosh(t-1)\cos(n\pi t)}{n^2\pi^2}\,dt \end{aligned}$$

Hence,

$$a_n\left(1+\frac{1}{n^2\pi^2}\right) = -\frac{2\sinh(-1)}{n^2\pi^2} \quad\Rightarrow\quad a_n = \frac{2\sinh(1)}{n^2\pi^2+1}.$$

The Fourier expansion for $f(t)$ is thus

$$\cosh(t-1) = \sinh(1)\left(1 + 2\sum_{n=1}^{\infty}\frac{\cos n\pi t}{1+n^2\pi^2}\right).$$

Setting $t = 0$ gives

$$\cosh(1) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2} \qquad (*)$$

$$\sum_{n=1}^{\infty}\frac{1}{1+n^2\pi^2} = \frac{\cosh(1)-\sinh(1)}{2\sinh(1)}$$

$$= \frac{e^{-1}}{e-e^{-1}} = \frac{1}{e^2-1}.$$

Now, to separate the contributions to the series from the odd and the even integers, we need an extra factor of $(-1)^n$ in each term. We get this, in the form $\cos n\pi$, by setting $t = 1$ and so obtain

$$\cosh(0) = \sinh(1) + 2\sinh(1)\sum_{n=1}^{\infty}\frac{(-1)^n}{1+n^2\pi^2} \qquad (**).$$

Adding $(*)$ and $(**)$ gives, with an obvious notation,

$$\frac{\cosh(1)+\cosh(0)}{\sinh(1)} = 2 + 4\,\Sigma_{\text{even}}$$

Re-arrangement and substitution of explicit expressions for the hyperbolic sinusoids give

$$\Sigma_{\text{even}} = \frac{1}{4}\frac{\frac{1}{2}(e+e^{-1})+1-(e-e^{-1})}{\frac{1}{2}(e-e^{-1})}$$

$$= \frac{3-e^2+2e}{4(e^2-1)} = \frac{(3-e)(1+e)}{4(e-1)(e+1)} = \frac{3-e}{4(e-1)}.$$

It then follows that

$$\Sigma_{\text{odd}} = \Sigma_{\text{all}} - \Sigma_{\text{even}}$$

$$= \frac{1}{e^2-1} - \frac{3-e}{4(e-1)} = \frac{4-2e+e^2-3}{4(e^2-1)}$$

$$= \frac{(1-e)^2}{4(e^2-1)} = \frac{e-1}{4(e+1)}.$$

12.18 *Express the function $f(x) = x^2$ as a Fourier sine series in the range $0 < x \le 2$ and show that it converges to zero at $x = \pm 2$.*

To ensure a sine series we take $f(x) = -x^2$ for $-2 < x \le 0$. This means that $f(-2) = -4$ and so, since $f(2) = +4$, we expect the series to converge to the average value of $\frac{1}{2}(-4+4) = 0$ at $x = \pm 2$.

The coefficients in the sine series $\sum b_n \sin(n\pi x/2)$ are

$$b_n = 2\,\frac{2}{4}\int_0^2 x^2 \sin\frac{n\pi x}{2}\,dx.$$

Setting $n\pi x/2 = y$ gives $b_n = 8I_n/(n\pi)^3$ with

$$\begin{aligned} I_n &= \int_0^{n\pi} y^2 \sin y\,dy \\ &= \left[-y^2\cos y\right]_0^{n\pi} + \int_0^{n\pi} 2y\cos y\,dy \\ &= (-1)^{n+1}n^2\pi^2 + \left[2y\sin y\right]_0^{n\pi} - \int_0^{n\pi} 2\sin y\,dy \\ &= (-1)^{n+1}n^2\pi^2 + 0 + \left[2\cos y\right]_0^{n\pi} \\ &= (-1)^{n+1}n^2\pi^2 + 2(-1)^n - 2. \end{aligned}$$

Thus,

$$\begin{aligned} b_n &= \frac{(-1)^{n+1}8}{n\pi} - \frac{32}{n^3\pi^3} \quad \text{for } n \text{ odd}, \\ &= \frac{(-1)^{n+1}8}{n\pi} \quad \text{for } n \text{ even}. \end{aligned}$$

For $x = \pm 2$ all terms in the series are zero and so this is the value of the expansion at these points. This is not simply as expected, but inevitable, because, for a pure Fourier sine series, the arguments of all the sine functions are bound to be of the form $n\pi$ at the end points of the period.

12.20 *Show that the Fourier series for* $|\sin\theta|$ *in the range* $-\pi \le \theta \le \pi$ *is given by*

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2m\theta}{4m^2-1}.$$

By setting $\theta = 0$ *and* $\theta = \pi/2$, *deduce values for*

$$\sum_{m=1}^{\infty}\frac{1}{4m^2-1} \quad \text{and} \quad \sum_{m=1}^{\infty}\frac{1}{16m^2-1}.$$

This is an even function about $\theta = 0$ with cosine coefficient a_0 given by

$$a_0 = \frac{2}{2\pi}\int_{-\pi}^{\pi}|\sin\theta|\,d\theta = \frac{1}{\pi}\,2\,\left[-\cos\theta\right]_0^{\pi} = \frac{4}{\pi}.$$

For $n \geq 1$, we have

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\, 2 \int_0^{\pi} \sin\theta \cos n\theta \, d\theta \\
&= \frac{2}{\pi} \int_0^{\pi} \tfrac{1}{2}[\sin(n+1)\theta - \sin(n-1)\theta]\, d\theta \\
&= \frac{1}{\pi}\left[-\frac{\cos(n+1)\theta}{n+1} + \frac{\cos(n-1)\theta}{n-1} \right]_0^{\pi} \\
&= \frac{1}{\pi}\left[-\frac{(-1)^{n+1}-1}{n+1} + \frac{(-1)^{n-1}-1}{n-1} \right]_0^{\pi} \\
&= -\frac{4}{\pi}\frac{1}{n^2-1} \text{ if } n \text{ is even, and } = 0 \text{ if } n \text{ is odd.}
\end{aligned}$$

Hence, writing $n = 2m$,

$$|\sin\theta| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2m\theta}{4m^2-1}.$$

Now, setting $\theta = 0$ yields

$$0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1} \quad \Rightarrow \quad \sum_{n=1}^{\infty}\frac{1}{4n^2-1} = \frac{1}{2}.$$

Setting $\theta = \pi/2$, instead, gives

$$1 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{4n^2-1}.$$

Adding this to the previous result then yields

$$\begin{aligned}
1 &= \frac{4}{\pi} - \frac{4}{\pi}\, 2 \sum_{n \text{ even}} \frac{1}{4n^2-1} \\
&= \frac{4}{\pi} - \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{1}{16m^2-1}, \\
\Rightarrow \quad \sum_{m=1}^{\infty}\frac{1}{16m^2-1} &= \frac{\pi}{8}\left(\frac{4}{\pi} - 1\right) = \frac{1}{2} - \frac{\pi}{8}.
\end{aligned}$$

12.22 *The repeating output from an electronic oscillator takes the form of a sine wave $f(t) = \sin t$ for $0 \le t \le \pi/2$; it then drops instantaneously to zero and starts again. The output is to be represented by a complex Fourier series of the form*

$$\sum_{n=-\infty}^{\infty} c_n e^{4nti}.$$

Sketch the function and find an expression for c_n. Verify that $c_{-n} = c_n^*$. Demonstrate that setting $t = 0$ and $t = \pi/2$ produces differing values for the sum

$$\sum_{n=1}^{\infty} \frac{1}{16n^2 - 1}.$$

Determine the correct value and check it using the quoted result of exercise 12.20.

As the period of the expansion is to be $\pi/2$ the complex expansion coefficients are given by

$$\begin{aligned}
\frac{\pi}{2} c_n &= \int_0^{\pi/2} \sin t \, e^{-i4nt} \\
&= \left[\frac{\sin t \, e^{-i4nt}}{-4ni} \right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\cos t \, e^{-i4nt}}{-4ni} \, dt \\
&= \frac{1}{-4ni} + \left[\frac{\cos t \, e^{-i4nt}}{4ni(-4ni)} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\sin t \, e^{-i4nt}}{4ni(-4ni)} \, dt.
\end{aligned}$$

It follows that

$$\frac{\pi}{2} c_n \left(1 - \frac{1}{16n^2} \right) = \frac{i}{4n} - \frac{1}{16n^2},$$

and hence that

$$c_n = \frac{2}{\pi} \frac{4ni - 1}{16n^2 - 1}.$$

It is obvious that

$$c_{-n}^* = \frac{2}{\pi} \frac{4(-n)(-i) - 1}{16n^2 - 1} = c_n.$$

The series representation is therefore

$$\sin(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{4ni - 1}{16n^2 - 1} \, e^{i4nt}. \qquad (*)$$

Setting $t = 0$, equating real parts, noting that the $n = 0$ term has value -1 and

that the $\pm n$-terms are equal for $n \neq 0$, together imply that

$$\sum_{n=-\infty}^{\infty} \frac{1}{16n^2-1} = 0 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{16n^2-1} = \frac{1}{2}.$$

But, setting $t = \pi/2$ and then equating real parts implies that

$$\frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{-1}{16n^2-1} = 1 \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{1}{16n^2-1} = -\frac{1}{2}\frac{\pi}{2}\left(1 - \frac{2}{\pi}\right) = \frac{1}{2} - \frac{\pi}{4}.$$

These results are clearly contradictory.

As there is a discontinuity in the function, the correct value is the mean of these two, namely $\frac{1}{2} - \frac{\pi}{8}$. This is the value obtained in solution 12.20.

12.24 *A string, anchored at $x = \pm L/2$, has a fundamental vibration frequency of $2L/c$, where c is the speed of transverse waves on the string. It is pulled aside at its centre point by a distance y_0 and released at time $t = 0$. Its subsequent motion can be described by the series*

$$y(x,t) = \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} \cos\frac{n\pi ct}{L}.$$

Find a general expression for a_n and show that only odd harmonics of the fundamental frequency are present in the sound generated by the released string. By applying Parseval's theorem, find the sum S of the series $\sum_0^\infty (2m+1)^{-4}$.

Since only cosine terms are present and the spatial terms have the form $\cos(n\pi x/L)$, the period of the continued function is $2L$. We take its forms beyond $x = \pm L/2$ as continuing that set by the physical string; they have the common value $-y_0$ at $x = \pm L$.

The values of the a_n are set by the initial displacement $y(x,0)$ and given by (writing π/L as k)

$$\begin{aligned}
a_n &= \frac{2}{2L}\int_{-L}^{L} y(x,0)\cos nkx\,dx \\
&= \frac{2y_0}{L}\int_0^L \left(1 - \frac{2x}{L}\right)\cos nkx\,dx \\
&= \frac{2y_0}{L}\left[\frac{\sin nkx}{nk}\right]_0^L - \frac{4y_0}{L^2}\left\{\left[\frac{x\sin nkx}{nk}\right]_0^L - \int_0^L \frac{\sin nkx}{nk}\,dx\right\} \\
&= \frac{4y_0}{L^2}\left[\frac{-\cos nkx}{n^2k^2}\right]_0^L \\
&= \frac{8y_0}{n^2\pi^2} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.}
\end{aligned}$$

This shows that only the odd harmonics are present. Because of the presence of n in the denominators of several expressions in the above calculation, the case $n = 0$ needs to be considered separately; however, it is clear that the average value of the continued function is zero and so $a_0 = 0$.

In order to apply Parseval's theorem we need to evaluate both the integral over one period of the square of the magnitude of the function, and the sum of the squares of the magnitudes of its Fourier coefficients:

(i)
$$\begin{aligned}\frac{1}{2L}\int_{-L}^{L}|y(x,0)|^2\,dx &= \frac{4y_0^2}{2L}\int_0^{L/2}\left(1-\frac{2x}{L}\right)^2 dx\\ &= \frac{2y_0^2}{L}\left[\frac{1}{3}\left(\frac{-L}{2}\right)\left(1-\frac{2x}{L}\right)^3\right]_0^{L/2}\\ &= \frac{1}{3}y_0^2.\end{aligned}$$

(ii)
$$\frac{1}{2}\sum_{n=1}^{\infty}a_n^2 = \frac{1}{2}\sum_{n\,\text{odd}}\frac{64y_0^2}{n^4\pi^4} = \frac{32y_0^2}{\pi^4}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4}.$$

Equating these two expressions shows that

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^4} = \frac{\pi^4}{96}.$$

12.26 *An odd function $f(x)$ of period 2π is to be approximated by a Fourier sine series having only m terms. The error in this approximation is measured by the square deviation*

$$E_m = \int_{-\pi}^{\pi}\left[f(x) - \sum_{n=1}^{m} b_n \sin nx\right]^2 dx.$$

By differentiating E_m with respect to the coefficients b_n, find the values of b_n that minimise E_m.

Sketch the graph of the function $f(x)$, where

$$f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0,\\ x(x-\pi) & \text{for } 0 \le x < \pi.\end{cases}$$

$f(x)$ is to be approximated by the first three terms of a Fourier sine series. What coefficients minimise E_3? What is the resulting value of E_3?

We minimise E_m by differentiating it with respect to b_j and setting the partial

derivative equal to zero.

$$
\begin{aligned}
E_m &= \int_{-\pi}^{\pi} \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right]^2 dx, \\
\frac{\partial E_m}{\partial b_j} &= \int_{-\pi}^{\pi} 2 \left[f(x) - \sum_{n=1}^{m} b_n \sin nx \right] \sin jx \, dx, \\
0 &= 2 \int_{-\pi}^{\pi} f(x) \sin jx \, dx - 2 \sum_{n=1}^{m} b_n \frac{1}{2} 2\pi \, \delta_{jn}, \\
\Rightarrow \quad b_j &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(jx) \, dx,
\end{aligned}
$$

i.e. b_j is the usual Fourier coefficient.

The function defined by

$$
f(x) = \begin{cases} -x(\pi + x) & \text{for } -\pi \le x < 0, \\ x(x - \pi) & \text{for } 0 \le x < \pi, \end{cases}
$$

is an odd function in x and therefore has a pure sine series Fourier expansion. The expansion coefficients are given by

$$
b_n = \frac{2}{\pi} \int_0^{\pi} x(x - \pi) \sin nx \, dx.
$$

Integrating by parts, we obtain

$$
\begin{aligned}
\frac{\pi b_n}{2} &= \int_0^{\pi} x^2 \sin nx \, dx - \int_0^{\pi} \pi x \sin nx \, dx \\
&= \left[\frac{-x^2 \cos nx}{n} \right]_0^{\pi} + \int_0^{\pi} \frac{2x \cos nx}{n} \, dx \\
&\quad - \left[\frac{-\pi x \cos nx}{n} \right]_0^{\pi} - \int_0^{\pi} \frac{\pi \cos nx}{n} \, dx \\
&= \frac{(-1)^{n+1} \pi^2}{n} + \left[\frac{2x \sin nx}{n^2} \right]_0^{\pi} - \int_0^{\pi} \frac{2 \sin nx}{n^2} \, dx + \frac{(-1)^{n+2} \pi^2}{n} + 0 \\
&= 0 + \frac{2}{n^2} \left[\frac{\cos nx}{n} \right]_0^{\pi},
\end{aligned}
$$

i.e.

$$
b_n = -\frac{8}{\pi n^3} \text{ for } n \text{ odd, and } = 0 \text{ for } n \text{ even.}
$$

Thus, the minimising coefficients are

$$
b_1 = -\frac{8}{\pi}, \quad b_2 = 0, \quad b_3 = -\frac{8}{27\pi}.
$$

As the full Fourier series reproduces $f(x)$ accurately, the error E_3 using these

three calculated coefficients must be

$$
\begin{aligned}
E_3 &= \int_{-\pi}^{\pi} \sum_{\text{odd } n=5}^{\infty} \left(-\frac{8}{\pi n^3} \sin nx \right)^2 dx \\
&= \sum_{\text{odd } n=5}^{\infty} \frac{64}{\pi^2 n^6} \frac{1}{2} 2\pi \\
&= \frac{64}{\pi} \sum_{m=2}^{\infty} \frac{1}{(2m+1)^6}.
\end{aligned}
$$

13

Integral transforms

13.2 *Use the general definition and properties of Fourier transforms to show the following.*

(a) *If $f(x)$ is periodic with period a then $\tilde{f}(k) = 0$ unless $ka = 2\pi n$ for integer n.*
(b) *The Fourier transform of $tf(t)$ is $i d\tilde{f}(\omega)/d\omega$.*
(c) *The Fourier transform of $f(mt + c)$ is*

$$\frac{e^{i\omega c/m}}{m}\tilde{f}\left(\frac{\omega}{m}\right).$$

(a) As f is periodic with period a,

$$f(x) = f(x - ma),$$

for any integer m. However, from the general translation property of Fourier transforms,

$$\tilde{f}(k) = \mathscr{F}[f(x)] = \mathscr{F}[f(x - ma)] = e^{-imka}\tilde{f}(k).$$

Thus

$$0 = \tilde{f}(k)(1 - e^{-imka}),$$

implying, in the particular case $m = 1$, that either $\tilde{f}(k) = 0$ or $ka = 2\pi n$ where n is an integer.

(b) This result is immediate, since differentiating under the integral sign gives

$$i\frac{d\tilde{f}(\omega)}{d\omega} = \frac{i}{\sqrt{2\pi}}\frac{\partial}{\partial\omega}\left(\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} tf(t)\, e^{-i\omega t}\, dt.$$

(c) From the definition of a Fourier transform,

$$\mathscr{F}\,[\,f(mt+c)\,] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(mt+c)\,e^{-i\omega t}\,dt.$$

We make a change of integration variable by setting $mt + c = u$, with $dt = du/m$ and $-\infty < u < \infty$. This yields

$$\begin{aligned}\mathscr{F}\,[\,f(mt+c)\,] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i\omega(u-c)/m}\,\frac{du}{m}\\ &= \frac{e^{i\omega c/m}}{m}\,\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(u)\,e^{-i(\omega/m)u}\,du\\ &= \frac{e^{i\omega c/m}}{m}\,\tilde{f}\left(\frac{\omega}{m}\right),\end{aligned}$$

as stated in the question.

13.4 *Prove that the Fourier transform of the function $f(t)$ defined in the tf-plane by straight-line segments joining $(-T, 0)$ to $(0, 1)$ to $(T, 0)$, with $f(t) = 0$ outside $|t| < T$, is*

$$\tilde{f}(\omega) = \frac{T}{\sqrt{2\pi}}\mathrm{sinc}^2\left(\frac{\omega T}{2}\right),$$

where sinc x is defined as $(\sin x)/x$.

Use the general properties of Fourier transforms to determine the transforms of the following functions, graphically defined by straight-line segments and equal to zero outside the ranges specified:

(a) $(0, 0)$ *to* $(0.5, 1)$ *to* $(1, 0)$ *to* $(2, 2)$ *to* $(3, 0)$ *to* $(4.5, 3)$ *to* $(6, 0)$;
(b) $(-2, 0)$ *to* $(-1, 2)$ *to* $(1, 2)$ *to* $(2, 0)$;
(c) $(0, 0)$ *to* $(0, 1)$ *to* $(1, 2)$ *to* $(1, 0)$ *to* $(2, -1)$ *to* $(2, 0)$.

The function $f(t)$ is given algebraically by

$$f(t) = \begin{cases} 1 + \dfrac{t}{T} & \text{for } -T \le t \le 0,\\ 1 - \dfrac{t}{T} & \text{for } 0 \le t \le T.\end{cases}$$

Its Fourier transform is therefore

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-T}^{0}\left(1+\frac{t}{T}\right)e^{-i\omega t}\,dt + \frac{1}{\sqrt{2\pi}}\int_{0}^{T}\left(1-\frac{t}{T}\right)e^{-i\omega t}\,dt.$$

Setting $t = -u$ in the first integral and $t = u$ in the second yields

$$\begin{aligned}
\sqrt{2\pi}\tilde{f}(\omega) &= \int_T^0 \left(1 - \frac{u}{T}\right) e^{i\omega u}(-du) + \int_0^T \left(1 - \frac{u}{T}\right) e^{-i\omega u}\, du \\
&= \int_0^T \left(1 - \frac{u}{T}\right) 2\cos\omega u\, du \\
&= 2\left[\frac{\sin\omega u}{\omega}\right]_0^T - 2\left[\frac{u\sin\omega u}{\omega T}\right]_0^T + \frac{2}{T}\int_0^T \frac{\sin\omega u}{\omega}\, du \\
&= \frac{2\sin\omega T}{\omega} - 0 - \frac{2\sin\omega T}{\omega} + 0 + \frac{2}{T}\left[\frac{-\cos\omega u}{\omega^2}\right]_0^T \\
&= \frac{2}{T\omega^2}(1 - \cos\omega T) = \frac{4\sin^2(\frac{1}{2}\omega T)}{T\omega^2}, \\
\tilde{f}(\omega) &= \frac{T}{\sqrt{2\pi}}\,\mathrm{sinc}^2\left(\frac{\omega T}{2}\right),
\end{aligned}$$

where $\mathrm{sinc}(x) = \sin x/x$.

In addition to straightforward scaling, two of the other properties of Fourier transforms that are available are (i) $\mathscr{F}[f(t+a)] = e^{ia\omega}\tilde{f}(\omega)$ and (ii) $\mathscr{F}\left[f'(t)\right] = i\omega\tilde{f}(\omega)$.

(a) This function consists of three segments of the same shape as $f(x)$, but with each one scaled and shifted. The first segment is centred on $t = \frac{1}{2}$ and has $T = \frac{1}{2}$; its contribution to the transform is therefore

$$e^{-i\omega/2}\frac{1}{2\sqrt{2\pi}}\mathrm{sinc}^2\left(\frac{\omega}{4}\right).$$

The second segment is scaled by a factor of 2, is centred on $t = 2$ and has $T = 1$. The third is scaled by a factor of 3, is centred on $t = \frac{9}{2}$ and has $2T = 3$. The full function therefore has as its Fourier transform

$$\frac{1}{\sqrt{2\pi}}\left[e^{-i\omega/2}\frac{1}{2}\mathrm{sinc}^2\left(\frac{\omega}{4}\right) + e^{-i2\omega}2\mathrm{sinc}^2\left(\frac{\omega}{2}\right) + e^{-i9\omega/2}\frac{9}{2}\mathrm{sinc}^2\left(\frac{3\omega}{4}\right)\right]$$

$$= \frac{8}{\sqrt{2\pi}\omega^2}\left[e^{-i\omega/2}\sin^2\left(\frac{\omega}{4}\right) + e^{-i2\omega}\sin^2\left(\frac{\omega}{2}\right) + e^{-i9\omega/2}\sin^2\left(\frac{3\omega}{4}\right)\right]$$

(b) This function could be considered as the superposition of a 'triangle' of height 2 with $T = 2$ and two other triangles, each of unit height with $T = 1$, displaced from the first by ± 1. Alternatively, it could be considered as the difference between two 'triangles' centred on $t = 0$, one of height 4 with $T = 2$ and the other of height 2 with $T = 1$. Necessarily, both approaches give the same answer. Using

the second,

$$\begin{aligned}\tilde{f}_2(\omega) &= \frac{8}{\sqrt{2\pi}}\text{sinc}^2\omega - \frac{2}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega}{2}\right)\\ &= \frac{8}{\sqrt{2\pi}\omega^2}\sin^2\left(\frac{\omega}{2}\right)\left[4\cos^2\left(\frac{\omega}{2}\right)-1\right]\\ &= \frac{8}{\sqrt{2\pi}\omega^2}\sin^2\left(\frac{\omega}{2}\right)(2\cos\omega+1).\end{aligned}$$

(c) This function can be viewed as the superposition of a 'triangle' with $T = 1$ centred on $t = 1$ and one cycle of a unit square-wave function, also centred on $t = 1$. But, the unit square-wave function is exactly the derivative of the triangle function, i.e. $+1$ for $0 \le t \le 1$ and -1 for $1 \le t \le 2$. If the complete function were centred on $t = 0$, its Fourier transform would be

$$\tilde{f}_3(\omega) = \mathscr{F}\left[f(t)\right] + \mathscr{F}\left[f'(t)\right] = \tilde{f}(\omega) + i\omega\tilde{f}(\omega).$$

However, it is centred on $t = 1$ and so an extra factor of $e^{-i\omega}$ has to be included to give

$$\tilde{f}_3(\omega) = \frac{(1+i\omega)\,e^{-i\omega}}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega}{2}\right).$$

13.6 *By differentiating the definition of the Fourier sine transform $\tilde{f}_s(\omega)$ of the function $f(t) = t^{-1/2}$ with respect to ω, and then integrating the resulting expression by parts, find an elementary differential equation satisfied by $\tilde{f}_s(\omega)$. Hence show that this function is its own Fourier sine transform, i.e. $\tilde{f}_s(\omega) = Af(\omega)$, where A is a constant. Show that it is also its own Fourier cosine transform. Assume that the limit as $x \to \infty$ of $x^{1/2}\sin\alpha x$ can be taken as zero.*

Starting from the definition

$$\tilde{f}_s(\omega) = \sqrt{\frac{2}{\pi}}\int_0^\infty t^{-1/2}\sin\omega t\,dt,$$

and then differentiating under the integral sign, we have

$$\begin{aligned}\frac{d\tilde{f}_s(\omega)}{d\omega} &= \sqrt{\frac{2}{\pi}}\int_0^\infty t^{-1/2}\,t\,\cos\omega t\,dt,\\ &= \sqrt{\frac{2}{\pi}}\left[\frac{t^{1/2}\sin\omega t}{\omega}\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{2}t^{-1/2}\frac{\sin\omega t}{\omega}\,dt\\ &= 0 - 0 - \frac{1}{2\omega}\tilde{f}_s(\omega).\end{aligned}$$

The Fourier sine transform therefore satisfies the differential equation

$$\frac{d\tilde{f}_s(\omega)}{d\omega} + \frac{1}{2\omega}\tilde{f}_s(\omega) = 0 \quad \Rightarrow \quad \tilde{f}_s(\omega) = \frac{A}{\omega^{1/2}} = Af(\omega).$$

The Fourier cosine transform *must* behave in exactly the same way, although the constant A could be different, and the details will not be worked out here.

13.8 *Calculate the Fraunhofer spectrum produced by a diffraction grating, uniformly illuminated by light of wavelength $2\pi/k$, as follows. Consider a grating with $4N$ equal strips each of width a and alternately opaque and transparent. The aperture function is then*

$$f(y) = \begin{cases} A & \text{for } (2n+1)a \le y \le (2n+2)a, \quad -N \le n < N, \\ 0 & \text{otherwise.} \end{cases}$$

(a) *Show, for diffraction at angle θ to the normal to the grating, that the required Fourier transform can be written*

$$\widetilde{f}(q) = (2\pi)^{-1/2} \sum_{r=-N}^{N-1} \exp(-2iarq) \int_a^{2a} A \exp(-iqu)\, du,$$

where $q = k \sin\theta$.

(b) *Evaluate the integral and sum to show that*

$$\widetilde{f}(q) = (2\pi)^{-1/2} \exp(-iqa/2) \frac{A \sin(2qaN)}{q \cos(qa/2)},$$

and hence that the intensity distribution $I(\theta)$ in the spectrum is proportional to

$$\frac{\sin^2(2qaN)}{q^2 \cos^2(qa/2)}.$$

(c) *For large values of N, the numerator in the above expression has very closely spaced maxima and minima as a function of θ and effectively takes its mean value, $1/2$, giving a low-intensity background. Much more significant peaks in $I(\theta)$ occur when $\theta = 0$ or the cosine term in the denominator vanishes. Show that the corresponding values of $|\widetilde{f}(q)|$ are*

$$\frac{2aNA}{(2\pi)^{1/2}} \quad \text{and} \quad \frac{4aNA}{(2\pi)^{1/2}(2m+1)\pi} \quad \text{with } m \text{ integral.}$$

Note that the constructive interference makes the maxima in $I(\theta) \propto N^2$, not N. Of course, observable maxima only occur for $0 \le \theta \le \pi/2$.

(a) and (b) The required Fourier transform is given by

$$\begin{aligned}
\sqrt{2\pi}\tilde{f}(q) &= \int_{-\infty}^{\infty} f(y)\, e^{-iqy}\, dy \\
&= \sum_{n=-N}^{N-1} \int_{(2n+1)a}^{(2n+2)a} A\, e^{-iqy}\, dy, \text{ set } y = 2na + u, \\
&= \sum_{n=-N}^{N-1} e^{-iq2na} \int_{a}^{2a} A\, e^{-iqu}\, du \\
&= e^{iq2Na} \frac{1 - e^{-iq4Na}}{1 - e^{-iq2a}} \frac{A(e^{-iq2a} - e^{-iqa})}{-iq} \\
&= \frac{2i\sin(2qaN)}{e^{-iqa}2i\sin(qa)} \frac{e^{-i3qa/2}2iA[-\sin(qa/2)]}{-iq} \\
&= e^{-iqa/2} \frac{\sin(2qaN)}{2\sin(qa/2)\cos(qa/2)} \frac{2A\sin(qa/2)}{q}, \\
\tilde{f}(q) &= \frac{1}{\sqrt{2\pi}}\, e^{-iqa/2} \frac{A\sin(2qaN)}{q\cos(qa/2)}.
\end{aligned}$$

The intensity distribution is proportional to the squared modulus of this, i.e to

$$\frac{\sin^2(2qaN)}{q^2\cos^2(qa/2)}.$$

(c) For the significant peaks:

(i) At $\theta = 0$, when $q = k\sin\theta = 0$. Using the fact that for small ϕ $\sin\phi \approx \phi$,

$$|\tilde{f}(q)| = \frac{1}{\sqrt{2\pi}} \frac{A2qaN}{q\,1} = \frac{2aNA}{\sqrt{2\pi}}.$$

(ii) When $qa/2 = (m + \frac{1}{2})\pi$ with m an integer, i.e. $ka\sin\theta = (2m+1)\pi$. The modulus of the transform becomes

$$|\tilde{f}(q)| = \frac{1}{\sqrt{2\pi}} \frac{A\sin[N(4m+2)\pi]}{a^{-1}(2m+1)\pi\cos[(m+\frac{1}{2})\pi]}.$$

This has the form 0/0 and is indeterminate. To evaluate the ratio we set $qa = \psi$ and determine the limit of the ratio as $\psi \to (2m+1)\pi$ using l'Hôpital's rule.

$$\begin{aligned}
\sqrt{2\pi}|\tilde{f}| &= \left| \frac{Aa\sin(2\psi N)}{\psi\cos(\psi/2)} \right| \\
&= \left| \frac{2NAa\cos(2\psi N)}{\cos(\psi/2) + \frac{1}{2}\psi\sin(\psi/2)} \right| \\
&= \left| \frac{2NAa\,1}{\frac{1}{2}(2m+1)\pi(-1)^m} \right|, \\
|\tilde{f}| &= \frac{4NaA}{\sqrt{2\pi}(2m+1)\pi}.
\end{aligned}$$

13.10 *In many applications in which the frequency spectrum of an analogue signal is required, the best that can be done is to sample the signal $f(t)$ a finite number of times at fixed intervals and then use a* discrete Fourier transform F_k *to estimate discrete points on the (true) frequency spectrum $\tilde{f}(\omega)$.*

(a) *By an argument that is essentially the converse of that given in section 13.1, show that, if N samples f_n, beginning at $t = 0$ and spaced τ apart, are taken, then $\tilde{f}(2\pi k/(N\tau)) \approx F_k\tau$ where*

$$F_k = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n e^{-2\pi nki/N}.$$

(b) *For the function $f(t)$ defined by*

$$f(t) = \begin{cases} 1 & \text{for } 0 \le t < 1, \\ 0 & \text{otherwise,} \end{cases}$$

from which eight samples are drawn at intervals of $\tau = 0.25$, find a formula for $|F_k|$ and evaluate it for $k = 0, 1, \ldots, 7$.

(c) *Find the exact frequency spectrum of $f(t)$ and compare the actual and estimated values of $\sqrt{2\pi}|\tilde{f}(\omega)|$ at $\omega = k\pi$ for $k = 0, 1, \ldots, 7$. Note the relatively good agreement for $k < 4$ and the lack of agreement for larger values of k.*

(a) With the exact definition of the Fourier transform of $f(t)$ (taken as zero for $t < 0$) being given by the integral

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)\, e^{-i\omega t}\, dt,$$

we approximate it with the sum of the areas of a series of rectangles. Each has width τ but the height of the nth is determined by the sample value f_n.

$$\tilde{f}(\omega) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n \tau\, e^{-i\omega n\tau}.$$

For the sample frequencies $\omega = 2\pi k/(N\tau)$ this gives the estimated spectrum values as

$$\tilde{f}\left(\frac{2\pi k}{N\tau}\right) \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{N-1} f_n \tau\, e^{-i2\pi nk/N} \equiv F_k\tau.$$

(b) The eight sample values are $f_n = 1$ for $n = 0$, 1, 2 and 3, and $f_n = 0$ for $n =$ 4, 5, 6 and 7.

With $k = 0$, $\sqrt{2\pi}F_0 = 1 + 1 + 1 + 1 + 0 + 0 + 0 + 0 = 4$.

For a more general value of k

$$\begin{aligned} F_k &= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{3} e^{-i2\pi nk/8} \\ &= \frac{1}{\sqrt{2\pi}} \frac{1 - e^{-i4(k\pi/4)}}{1 - e^{-ik\pi/4}} \\ &= \frac{1}{\sqrt{2\pi}} \frac{e^{ik\pi/8}[1 - (-1)^k]}{e^{ik\pi/8} - e^{-ik\pi/8}}. \end{aligned}$$

To obtain the last line we have used $e^{ik\pi} = (-1)^k$ and arranged for the final expression to be real apart from a single overall phase factor. It follows that the modulus of F_k is given by

$$|F_k| = \frac{1}{\sqrt{2\pi}} \frac{1}{\sin k\pi/8} \qquad \text{for odd } k$$

and is equal to zero for even k.

(c) The exact frequency spectrum of $f(t)$ is

$$\begin{aligned} \tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t}\, dt \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega t}}{-i\omega} \right]_0^1 \\ &= \frac{e^{-i\omega/2}}{\sqrt{2\pi}} \frac{2\sin(\omega/2)}{\omega} \\ &= \frac{e^{-i\omega/2}}{\sqrt{2\pi}} \operatorname{sinc}\frac{\omega}{2}. \end{aligned}$$

Noting that $\tau = \frac{1}{4}$, giving $\omega = 2\pi k/(8 \times 0.25)$, the comparison of the magnitudes of the exact values $\sqrt{2\pi}\tilde{f}(\omega)$ and the estimated values $\sqrt{2\pi}F_k\tau$ is

$\omega = k\pi$	=	0	π	2π	3π	4π	5π	6π	7π
$\sqrt{2\pi}\lvert\tilde{f}(\omega)\rvert$	=	1	$\frac{2}{\pi}$	0	$\frac{2}{3\pi}$	0	$\frac{2}{5\pi}$	0	$\frac{2}{7\pi}$
	=	1	0.637	0	0.212	0	0.127	0	0.091
$\sqrt{2\pi}\lvert F_k\tau\rvert$	=	1	0.653	0	0.271	0	0.271	0	0.653

The lack of agreement for the higher frequencies ($k > 4$) is obvious.

13.12 *A signal obtained by sampling a function $x(t)$ at regular intervals T is passed through an electronic filter, whose response $g(t)$ to a unit δ-function input is represented in a tg-plot by straight lines joining $(0,0)$ to $(T,1/T)$ to $(2T,0)$ and is zero for all other values of t. The output of the filter is the convolution of the input, $\sum_{-\infty}^{\infty} x(t)\delta(t-nT)$, with $g(t)$.*
Using the convolution theorem, and the result given in exercise 13.4, show that the output of the filter can be written

$$y(t) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty} sinc^2\left(\frac{\omega T}{2}\right) e^{-i\omega[(n+1)T-t]}d\omega.$$

In order to use the convolution theorem we need the Fourier transforms of both the input signal $x(t)$ and the filter response $g(t)$. The former is

$$\tilde{x}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\omega t}\sum_{n=-\infty}^{\infty} x(t)\delta(t-nT)\,dt = \frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} x(nT)\,e^{-in\omega T}.$$

The latter is the same as that obtained in exercise 13.4, except that it is scaled by a factor $1/T$ and centred on $t = T$, rather that $t = 0$. The required transform is therefore

$$\tilde{g}(\omega) = \frac{e^{-i\omega T}}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega T}{2}\right).$$

The transform of the output is therefore the product of these two transforms multiplied by $\sqrt{2\pi}$.

Using the Fourier inversion theorem, we can therefore write the output of the filter as

$$\begin{aligned} y(t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sqrt{2\pi}\,\tilde{g}(\omega)\,\tilde{x}(\omega)\right] e^{i\omega t}\,d\omega \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\sqrt{2\pi}\frac{e^{-i\omega T}}{\sqrt{2\pi}}\text{sinc}^2\left(\frac{\omega T}{2}\right)\right. \\ &\qquad\qquad \left.\times\frac{1}{\sqrt{2\pi}}\sum_{n=-\infty}^{\infty} x(nT)\,e^{-in\omega T}\right] e^{i\omega t}\,d\omega \\ &= \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x(nT)\int_{-\infty}^{\infty}\text{sinc}^2\left(\frac{\omega T}{2}\right) e^{-i\omega[(n+1)T-t]}d\omega, \end{aligned}$$

as stated in the question.

13.14 *Prove the equality*

$$\int_0^\infty e^{-2at}\sin^2 at\,dt = \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4+\omega^4}\,d\omega.$$

We utilise the first result of the previous exercise (13.13) in the special case where $\gamma = p = a$, $f(t) = e^{-at}\sin at$ and consequently

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\frac{a}{(a+i\omega)^2+a^2}.$$

Applying Parseval's theorem,

$$\int_{-\infty}^\infty |f(t)|^2\,dt = \int_{-\infty}^\infty |\tilde{f}(\omega)|^2\,d\omega,$$

to this function and its transform:

$$\begin{aligned}\int_0^\infty e^{-2at}\sin^2 at\,dt &= \frac{1}{2\pi}\int_{-\infty}^\infty \left[\frac{a}{(a+i\omega)^2+a^2}\right]\left[\frac{a}{(a-i\omega)^2+a^2}\right]d\omega\\ &= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{a^2}{(a^2+\omega^2)^2+2a^2(a^2-\omega^2)+a^4}\,d\omega\\ &= \frac{1}{\pi}\int_0^\infty \frac{a^2}{4a^4+\omega^4}\,d\omega.\end{aligned}$$

13.16 *In quantum mechanics, two equal-mass particles having momenta* $\mathbf{p}_j = \hbar\mathbf{k}_j$ *and energies* $E_j = \hbar\omega_j$ *and represented by plane wavefunctions* $\phi_j = \exp[i(\mathbf{k}_j\cdot\mathbf{r}_j - \omega_j t)]$, $j = 1, 2$, *interact through a potential* $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$. *In first-order perturbation theory the probability of scattering to a state with momenta and energies* $\mathbf{p}'_j, E'_j$ *is determined by the modulus squared of the quantity*

$$M = \iiint \psi_\mathrm{f}^* V \psi_\mathrm{i}\,d\mathbf{r}_1\,d\mathbf{r}_2\,dt.$$

The initial state ψ_i *is* $\phi_1\phi_2$ *and the final state* ψ_f *is* $\phi'_1\phi'_2$.

(a) *By writing* $\mathbf{r}_1 + \mathbf{r}_2 = 2\mathbf{R}$ *and* $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$ *and assuming that* $d\mathbf{r}_1\,d\mathbf{r}_2 = d\mathbf{R}\,d\mathbf{r}$, *show that* M *can be written as the product of three one-dimensional integrals.*

(b) *From two of the integrals deduce energy and momentum conservation in the form of* δ*-functions.*

(c) *Show that* M *is proportional to the Fourier transform of* V, *i.e.* $\widetilde{V}(\mathbf{k})$ *where* $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}'_2 - \mathbf{p}'_1)$.

Putting in explicit expressions for the wavefunctions gives

$$\begin{aligned} M &= \iiint \psi_{\mathrm{f}}^{*} V \psi_{\mathrm{i}}\, d\mathbf{r}_1\, d\mathbf{r}_2\, dt \\ &= \iiint \exp[-i(\mathbf{k}_2' \cdot \mathbf{r}_2 - \omega_2' t)] \exp[-i(\mathbf{k}_1' \cdot \mathbf{r}_1 - \omega_1' t)]\, V(|\mathbf{r}_1 - \mathbf{r}_2|) \\ &\qquad \times \exp[i(\mathbf{k}_2 \cdot \mathbf{r}_2 - \omega_2 t)] \exp[i(\mathbf{k}_1 \cdot \mathbf{r}_1 - \omega_1 t)]\, d\mathbf{r}_1\, d\mathbf{r}_2\, dt. \end{aligned}$$

(a) Writing the integrand in terms of the centre-of-mass coordinate, $\mathbf{r}$, and the coordinate of the centre of mass, $\mathbf{R}$, given by

$$\mathbf{r}_1 + \mathbf{r}_2 = 2\mathbf{R} \quad \text{and} \quad \mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r},$$

with $d\mathbf{r}_1\, d\mathbf{r}_2 = d\mathbf{R}\, d\mathbf{r}$, we can express $\mathbf{r}_1$ as $\mathbf{r}_1 = \mathbf{R} + \frac{1}{2}\mathbf{r}$ and $\mathbf{r}_2$ as $\mathbf{r}_2 = \mathbf{R} - \frac{1}{2}\mathbf{r}$.

When these substitutions are made the integral becomes

$$\begin{aligned} M = &\int \exp[i(-\mathbf{k}_2' - \mathbf{k}_1' + \mathbf{k}_2 + \mathbf{k}_1) \cdot \mathbf{R}]\, d\mathbf{R} \\ &\times \int \exp[\tfrac{1}{2} i(\mathbf{k}_2' - \mathbf{k}_1' - \mathbf{k}_2 + \mathbf{k}_1) \cdot \mathbf{r}]\, V(r)\, d\mathbf{r} \\ &\times \int \exp[i(\omega_2' + \omega_1' - \omega_2 - \omega_1)t]\, dt. \end{aligned}$$

This is now the product of three 1-dimensional integrals.

(b) The first integral is, as shown in the text, a representation of the 3-dimensional δ-function and is equal to $(2\pi)^3 \delta(\mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_1' - \mathbf{k}_2')$. Since $\mathbf{p}_j = \hbar \mathbf{k}_j$, this is equivalent to $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$, i.e. to momentum conservation.

Similarly, the last of the three integrals produces a 1-dimensional δ-function, which, since $E_j = \hbar\omega_j$, is equivalent to energy conservation, namely $E_1' + E_2' = E_1 + E_2$.

(c) The second integral, containing $V(r)$, can be written as

$$\int V(r)\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d\mathbf{r},$$

where $\mathbf{k} = \frac{1}{2}(\mathbf{k}_1' - \mathbf{k}_2' - \mathbf{k}_1 + \mathbf{k}_2)$, i.e. where $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}_2' - \mathbf{p}_1')$; the integral is thus proportional to $\tilde{V}(\mathbf{k})$.

Note Since, from part (b), $(\mathbf{p}_2' - \mathbf{p}_2) = -(\mathbf{p}_1' - \mathbf{p}_1)$ and $2\hbar\mathbf{k}$ can be written as $2\hbar\mathbf{k} = (\mathbf{p}_1' - \mathbf{p}_1) - (\mathbf{p}_2' - \mathbf{p}_2)$, it follows that $\hbar\mathbf{k} = \mathbf{p}_1' - \mathbf{p}_1$. Thus the $\mathbf{k}$ appearing in $\tilde{V}(\mathbf{k})$ is the wave vector corresponding to the momentum transferred from one particle to the other.

13.18 *The equivalent duration and bandwidth, T_e and B_e, of a signal $x(t)$ are defined in terms of the latter and its Fourier transform $\tilde{x}(\omega)$:*

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt,$$
$$B_e = \frac{1}{\tilde{x}(0)} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, d\omega,$$

where neither $x(0)$ nor $\tilde{x}(0)$ is zero. Show that the product $T_e B_e = 2\pi$ (this is a form of uncertainty principle), and find the equivalent bandwidth of the signal

$$x(t) = \exp(-|t|/T).$$

For this signal, determine the fraction of the total energy that lies in the frequency range $|\omega| < B_e/4$. You will need the indefinite integral with respect to x of $(a^2 + x^2)^{-2}$, which is

$$\frac{x}{2a^2(a^2 + x^2)} + \frac{1}{2a^3} \tan^{-1} \frac{x}{a}.$$

With $\tilde{x}(\omega)$ being the Fourier transform of $x(t)$,

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt \quad \text{and} \quad B_e = \frac{1}{\tilde{x}(0)} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, d\omega,$$

we have that

$$\tilde{x}(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, e^{-i0t}\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x(t)\, dt = \frac{1}{\sqrt{2\pi}}\, x(0) T_e.$$

Consequently,

$$x(0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{x}(\omega)\, e^{i\omega 0}\, d\omega = \frac{1}{\sqrt{2\pi}}\, \tilde{x}(0) B_e = \frac{1}{\sqrt{2\pi}}\, B_e \frac{1}{\sqrt{2\pi}}\, x(0) T_e.$$

It then follows that $B_e T_e = 2\pi$.

For $x(t) = \exp(-|t|/T)$, the equivalent duration is

$$T_e = \frac{1}{x(0)} \int_{-\infty}^{\infty} x(t)\, dt = 2 \int_0^{\infty} e^{-t/T}\, dt = 2 \left[-T e^{-t/T} \right]_0^{\infty} = 2T.$$

The equivalent bandwidth is therefore $B_e = 2\pi/(2T) = \pi/T$.

The energy density spectrum is proportional to $|\tilde{x}(\omega)|^2$ and the fraction of the total energy lying within $|\omega| < B_e/4$ is

$$f = \frac{\displaystyle\int_{-B_e/4}^{B_e/4} |\tilde{x}(\omega)|^2\, d\omega}{\displaystyle\int_{-\infty}^{\infty} |\tilde{x}(\omega)|^2\, d\omega}.$$

Now,

$$\begin{aligned}
\tilde{x}(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{t/T}\, e^{-i\omega t}\, dt + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T}\, e^{-i\omega t}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t'/T} e^{i\omega t'}\, dt' + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T}\, e^{-i\omega t}\, dt \\
&= \frac{2}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-t/T} \cos(\omega t)\, dt \\
&= \frac{2}{\sqrt{2\pi}} \operatorname{Re} \left[\frac{e^{-t/T+i\omega t}}{-\frac{1}{T} + i\omega} \right]_{0}^{\infty} \\
&= \frac{2}{\sqrt{2\pi}} \frac{T^{-1}}{T^{-2} + \omega^2} = \frac{2T}{\sqrt{2\pi}} \frac{1}{1 + \omega^2 T^2}.
\end{aligned}$$

The fraction f is therefore given by

$$\begin{aligned}
f &= \frac{\displaystyle\int_{-\pi/(4T)}^{\pi/(4T)} (1 + \omega^2 T^2)^{-2}\, d\omega}{\displaystyle\int_{-\infty}^{\infty} (1 + \omega^2 T^2)^{-2}\, d\omega} = \frac{2\displaystyle\int_{0}^{\pi/4} (1 + x^2)^{-2}\, dx}{2\displaystyle\int_{0}^{\infty} (1 + x^2)^{-2}\, dx} \\
&= \left\{ \frac{\pi}{8[1 + (\pi/4)^2]} + \frac{1}{2} \tan^{-1} \frac{\pi}{4} \right\} \left\{ 0 + \frac{1}{2} \tan^{-1} \infty \right\}^{-1} \\
&= \frac{1}{2[1 + (\pi/4)^2]} + \frac{2}{\pi} \tan^{-1} \frac{\pi}{4} = 0.733.
\end{aligned}$$

13.20 *Prove that the cross-correlation $C(z)$ of the Gaussian and Lorentzian distributions*

$$f(t) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(-\frac{t^2}{2\tau^2}\right), \qquad g(t) = \left(\frac{a}{\pi}\right) \frac{1}{t^2 + a^2},$$

has as its Fourier transform the function

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau^2\omega^2}{2}\right) \exp(-a|\omega|).$$

Hence show that

$$C(z) = \frac{1}{\tau\sqrt{2\pi}} \exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \cos\left(\frac{az}{\tau^2}\right).$$

We need the Fourier transforms of both $f(t)$ and $g(t)$. That for f is derived in the

text as

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2\tau^2/2}.$$

That for g can be found from the result of either of the exercises 13.18 and 13.19 (or from the contour integral of a complex variable, as in chapter 24). If, in the final result of the previous exercise (13.19), we make the substitutions $\omega \to -t$, $\lambda \to a$ and $z \to \omega$, we obtain

$$\int_{-\infty}^{\infty} \frac{e^{-it\omega}}{a^2 + t^2}\, dt = \frac{\pi}{a}\, e^{-a|\omega|}.$$

From this it follows that

$$\tilde{g}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{a}{\pi}\, \frac{e^{-i\omega t}}{t^2 + a^2}\, dt = \frac{1}{\sqrt{2\pi}}\, e^{-a|\omega|}.$$

From the Wiener-Kinchin theorem, we can now state that the Fourier transform of the cross-correlation function is

$$\tilde{C}(\omega) = \sqrt{2\pi}\, \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2\tau^2/2}\, \frac{1}{\sqrt{2\pi}}\, e^{-a|\omega|} = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{\tau^2\omega^2}{2}\right) \exp(-a|\omega|).$$

The correlation function itself is obtained by forming the inverse tranform and evaluating it by 'completing the square'.

$$\begin{aligned}
C(z) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\, e^{-\omega^2\tau^2/2}\, e^{-a|\omega|}\, e^{i\omega z}\, d\omega \\
&= \frac{2}{2\pi}\, \mathrm{Re} \int_0^{\infty} \exp(-\tfrac{1}{2}\omega^2\tau^2 - a\omega + i\omega z)\, d\omega \\
&= \frac{1}{\pi}\, \mathrm{Re} \int_0^{\infty} \exp\left\{-\frac{1}{2}\tau^2 \left[\omega^2 + \frac{2(a - iz)\omega}{\tau^2} + \frac{(a - iz)^2}{\tau^4}\right]\right\} \\
&\qquad \times \exp\left[\frac{(a - iz)^2}{2\tau^2}\right] d\omega \\
&= \frac{1}{\pi}\left(\frac{1}{2}\frac{\sqrt{2\pi}}{\tau}\right) \mathrm{Re}\, \left[\exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \exp\left(\frac{-2aiz}{2\tau^2}\right)\right] \\
&= \frac{1}{\sqrt{2\pi}}\frac{1}{\tau} \exp\left(\frac{a^2 - z^2}{2\tau^2}\right) \cos\left(\frac{az}{\tau^2}\right),
\end{aligned}$$

as given in the question.

13.22 *Find the functions $y(t)$ whose Laplace transforms are the following:*

(a) $1/(s^2 - s - 2)$;
(b) $2s/[(s+1)(s^2+4)]$;
(c) $e^{-(\gamma+s)t_0}/[(s+\gamma)^2 + b^2]$.

To find the original functions we must express the transforms in terms of those given in table 13.1. Partial fraction expansions (chapter 1) are needed for (a) and (b).

(a) Factorising the denominator and expressing the transform as partial fractions:

$$\bar{f}(s) = \frac{1}{s^2 - s - 2} = \frac{1}{3(s-2)} - \frac{1}{3(s+1)},$$

and from the table of Laplace transforms and the liearity of the process of taking Laplace transforms, it follows that

$$f(t) = \frac{1}{3}(e^{2t} - e^{-t}).$$

(b) The quadratic term in the denominator cannot be factorised further without involving complex roots (and, in any case, transforms containing $(s^2 + a^2)^{-2}$ appear in the table) and so we express the transform in partial fractions as

$$\frac{2s}{(s+1)(s^2+4)} = \frac{A}{s+1} + \frac{Bs + C}{s^2 + 4},$$

$$2s = s^2(A+B) + s(B+C) + (4A + C),$$

$$\Rightarrow A = -\tfrac{2}{5}, \quad B = \tfrac{2}{5}, \quad C = \tfrac{8}{5}.$$

Thus, we may write

$$\bar{f}(s) = -\frac{2}{5(s+1)} + \frac{2s+8}{5(s^2+4)},$$

and from the table of Laplace transforms can read off that

$$f(t) = \tfrac{2}{5}(-e^{-t} + \cos 2t + 4\sin 2t).$$

(c) Apart from the factor $e^{-\gamma t_0}$ (which indicates a change of origin to $t = t_0$), $\bar{f}(s)$ is the product of the Laplace transforms of $\delta(t - t_0)$ and $b^{-1}e^{-\gamma t}\sin bt$. Thus, by

the convolution theorem,

$$\begin{aligned} f(t) &= \frac{e^{-\gamma t_0}}{b} \int_0^t e^{-\gamma u} \sin(bu)\, H(u)\, \delta(t-u-t_0)\, du \\ &= \frac{e^{-\gamma t_0}}{b} e^{-\gamma(t-t_0)} \sin[\, b(t-t_0)\,]\, H(t-t_0) \\ &= \frac{1}{b} e^{-\gamma t} \sin[\, b(t-t_0)\,]\, H(t-t_0). \end{aligned}$$

Note that $f(t) = 0$ for $t < t_0$.

13.24 *Find the solution (the so-called* impulse response *or* Green's function*) of the equation*

$$T\frac{dx}{dt} + x = \delta(t)$$

by proceeding as follows.

(a) *Show by substitution that*

$$x(t) = A(1 - e^{-t/T})H(t)$$

is a solution, for which $x(0) = 0$, *of*

$$T\frac{dx}{dt} + x = AH(t), \qquad (*)$$

where $H(t)$ *is the Heaviside step function.*

(b) *Construct the solution when the RHS of (*) is replaced by* $AH(t-\tau)$ *with* $dx/dt = x = 0$ *for* $t < \tau$, *and hence find the solution when the RHS is a rectangular pulse of duration* τ.

(c) *By setting* $A = 1/\tau$ *and taking the limit when* $\tau \to 0$, *show that the impulse response is* $x(t) = T^{-1}e^{-t/T}$.

(d) *Obtain the same result much more directly by taking the Laplace transform of each term in the original equation, solving the resulting algebraic equation and then using the entries in table 13.1.*

(a) For $t > 0$, consider $x(t) = A(1 - e^{-t/T})H(t)$, for which $x(0) = A(1-1) = 0$. Substitute it into $(*)$:

$$T\frac{A}{T}e^{-t/T}H(t) + A(1 - e^{-t/T})H(t) = AH(t),$$

which is clearly satisfied.

(b) With the RHS of $(*) = AH(t-\tau)$ the solution will be $x(t) = A(1 -$

$e^{-(t-\tau)/T})H(t-\tau)$ and, because of the linearity of the equation, the solution when the RHS is a rectangular pulse of duration τ is

$$x(t) = A(1 - e^{-t/T})H(t) - A(1 - e^{-(t-\tau)/T})H(t-\tau).$$

This follows because the rectangular pulse can be thought of as the linear superposition of a positive Heaviside function and an equal and opposite negative Heaviside function, the latter being delayed by an interval τ.

(c) We now make A equal to $1/\tau$, so that the area under the pulse is unity, whatever the value of τ, and consider the limiting form of $f(t)$ as $\tau \to 0$.

$$\begin{aligned}
&\lim_{\tau\to 0}\frac{1}{\tau}\left[(1-e^{-t/T})H(t) - (1-e^{-(t-\tau)/T})H(t-\tau)\right]\\
&= \lim_{\tau\to 0}\frac{1}{\tau}\left\{(1-e^{-t/T})H(t) - \left[1 - e^{-t/T}\left(1+\frac{\tau}{T}+O(\tau^2)\right)\right]H(t-\tau)\right\}\\
&= \lim_{\tau\to 0}\left\{(1-e^{-t/T})\frac{[H(t)-H(t-\tau)]}{\tau} + \frac{e^{-t/T}}{\tau}\,\frac{\tau}{T}H(t-\tau) + O(\tau^2)\right\}\\
&= \left(1-e^{-t/T}\right)_{t=0} + \frac{e^{-t/T}}{T}H(t)\\
&= \frac{e^{-t/T}}{T}H(t).
\end{aligned}$$

This is the impulse response.

(d) Laplace transforming the original equation and incorporating the initial value $x(0) = 0$ gives

$$Ts\bar{x} - 0 + \bar{x} = 1.$$

From this it follows that

$$\begin{aligned}
\bar{x} &= \frac{1}{1+sT} = \frac{1}{T(T^{-1}+s)},\\
\Rightarrow \quad x(t) &= \frac{e^{-t/T}}{T}H(t),
\end{aligned}$$

in agreement with the result in part (c).

13.26 *By writing $f(x)$ as an integral involving the δ-function $\delta(\xi - x)$ and taking the Laplace transforms of both sides, show that the transform of the solution of the equation*

$$\frac{d^4y}{dx^4} - y = f(x)$$

for which y and its first three derivatives vanish at $x = 0$ can be written as

$$\bar{y}(s) = \int_0^\infty f(\xi)\frac{e^{-s\xi}}{s^4 - 1}\,d\xi.$$

Use the properties of Laplace transforms and the entries in table 13.1 to show that

$$y(x) = \frac{1}{2}\int_0^x f(\xi)\,[\sinh(x - \xi) - \sin(x - \xi)]\,d\xi.$$

We first need to write $f(x)$ as an integral involving the δ-function $\delta(\xi - x)$ so that the only x-dependence of the RHS is on functions for which we know the explicit Laplace transform; we do not know that of $f(x)$. Thus, we take the transform of the equation in the form

$$\frac{d^4y}{dx^4} - y = \int_{-\infty}^\infty H(\xi)\,f(\xi)\,\delta(\xi - x)\,d\xi.$$

Since y and its first three derivatives vanish at $x = 0$ the transform of d^4y/dx^4 does not contain any terms involving s^3, s^2, s or a constant. The transform of $\delta(x - x_0)$ is e^{-sx_0}, and so the Laplace transform of the original equation reads

$$s^4\bar{y} - \bar{y} = \int_0^\infty f(\xi)e^{-s\xi}\,d\xi.$$

This can be rearranged to express $\bar{y}$ explicitly as

$$\bar{y}(s) = \int_0^\infty f(\xi)\frac{e^{-s\xi}}{s^4 - 1}\,d\xi,$$

which is the form stated in the question.

To find the form of $y(x)$, we begin by rewriting the integrand using partial fractions. The denominator could be written as the product of four linear factors, but, with one eye on the form of the quoted solution (and the other on table 13.1), we write it as the product of two quadratic functions leading to the partial fractions representation:

$$\bar{y}(s) = \frac{1}{2}\int_0^\infty f(\xi)\left(\frac{e^{-s\xi}}{s^2 - 1} - \frac{e^{-s\xi}}{s^2 + 1}\right)\,d\xi.$$

Now, using the table and recognising the implication of the factor $e^{-s\xi}$ so far as the arguments of the inverted functions are concerned:

$$\begin{aligned} y(x) &= \frac{1}{2}\int_0^\infty f(\xi)[\sinh(x-\xi)-\sin(x-\xi)]\,H(x-\xi)\,d\xi \\ &= \frac{1}{2}\int_0^x f(\xi)[\sinh(x-\xi)-\sin(x-\xi)]\,d\xi, \end{aligned}$$

i.e. as stated in the question.

13.28 *Show that the Laplace transform of $f(t-a)H(t-a)$, where $a \geq 0$, is $e^{-as}\bar{f}(s)$ and that, if $g(t)$ is a periodic function of period T, $\bar{g}(s)$ can be written as*

$$\frac{1}{1-e^{-sT}}\int_0^T e^{-st}g(t)\,dt.$$

(a) *Sketch the periodic function defined in $0 \leq t \leq T$ by*

$$g(t) = \begin{cases} 2t/T & 0 \leq t < T/2, \\ 2(1-t/T) & T/2 \leq t \leq T, \end{cases}$$

and, using the previous result, find its Laplace transform.

(b) *Show, by sketching it, that*

$$\frac{2}{T}[tH(t) + 2\sum_{n=1}^{\infty}(-1)^n(t-\tfrac{1}{2}nT)H(t-\tfrac{1}{2}nT)]$$

is another representation of $g(t)$ and hence derive the relationship

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}.$$

From the definition of a Laplace transform,

$$\mathscr{L}\,[f(t-a)H(t-a)] = \int_0^\infty f(t-a)H(t-a)\,e^{-st}\,dt.$$

We start by changing the integration variable to $u = t-a$, with a corresponding change in the integration limits:

$$\begin{aligned} \mathscr{L}\,[f(t-a)H(t-a)] &= \int_{-a}^\infty f(u)H(u)e^{-su}e^{-sa}\,du \\ &= \int_0^\infty f(u)e^{-su}e^{-sa}\,du \\ &= e^{-sa}\bar{f}(s). \end{aligned}$$

With $g(t)$ periodic, $g(t) = g(t-T)$ and the Laplace transform can be written as

$$\int_0^\infty H(t)g(t)\,e^{-st}\,dt = \int_0^T H(t)g(t)\,e^{-st}\,dt + \int_T^\infty H(t)g(t)\,e^{-st}\,dt.$$

However, it follows from the properties of the Heavisde function that

$$\int_T^\infty H(t)\cdots dt = \int_0^\infty H(t-T)\cdots dt$$

and so, using the previous result, we can rewrite the above equation as

$$\begin{aligned}\bar{g}(s) = \int_0^\infty g(t)\,e^{-st}\,dt &= \int_0^T g(t)\,e^{-st}\,dt + \int_0^\infty H(t-T)g(t-T)\,e^{-st}\,dt \\ &= \int_0^T g(t)\,e^{-st}\,dt + e^{-sT}\bar{g}(s) \\ \Rightarrow \quad \bar{g}(s) &= \frac{1}{1-e^{-sT}}\int_0^T g(t)\,e^{-st}\,dt.\end{aligned}$$

(a) The graph of $g(t)$ consists of a continuously repeating pattern of isosceles triangles, each of unit height and base width T. Any one of these triangles has the same shape as the function $g_a(x)$ found in Exercise 13.27, except that a has been replaced by $T/2$ and the height of the triangle is unity rather than a. Its Laplace transform is therefore

$$\frac{1}{(T/2)}\frac{1}{s^2}\left(1-e^{-sT/2}\right)^2.$$

From our earlier result it now follows that the Laplace transform of $g(t)$ is

$$\begin{aligned}\bar{g}(s) &= \frac{2}{Ts^2}\frac{\left(1-e^{-sT/2}\right)^2}{1-e^{-sT}} \\ &= \frac{2}{Ts^2}\frac{1-e^{-sT/2}}{1+e^{-sT/2}} \\ &= \frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right).\end{aligned}$$

(b) The contributions to

$$f(t) = \sum_{n=0}^\infty f_n(t) = \frac{2}{T}[tH(t) + 2\sum_{n=1}^\infty(-1)^n(t-\tfrac{1}{2}nT)H(t-\tfrac{1}{2}nT)]$$

are shown in the sketch (figure 13.1).

For $0 \le t \le T/2$, only f_0 contributes; it is identical to $g(t)$ with $f(T/2) = 1$.

For $T/2 \le t \le T$, f_0 and f_1 contribute with *net* slope $-2/T$ and $f_1 = -f_0$ at $t = T$, making $f(T) = 0$.

For $T \le t \le 3T/2$, $f_2(t) + f_1(t) = f_1(T)$, since the two terms contribute equal and opposite changes as t varies; the change in $f(t)$ is entirely due to that in f_0.

Figure 13.1 The contributions to the function considered in exercise 13.28. Their analytic forms in terms of the Heaviside function are $f_0 = 2tH(t)/T$, with slope $2/T$; $f_n = 4(-1)^n\,(t - \frac{1}{2}nT)H(t - \frac{1}{2}nT)/T$, with slope $4/T$.

For $3T/2 \le t \le 2T$, the change in $f(t)$ is due to those in f_0 and f_3, i.e. has a net slope $-2/T$, making $f(2T) = 0$.

This sequence is then repeated in successive blocks of length $2T$.

Hence $f(t)$ is an alternative representation of $g(t)$ with

$$\bar{f}(s) = \frac{2}{T}\left[\mathscr{L}\,[tH(t)] + 2\sum_{n=1}^{\infty}(-1)^n e^{-nTs/2}\mathscr{L}\,[tH(t)]\right],$$

where we have used the result from part (a). But,

$$\mathscr{L}\,[tH(t)] = \int_0^{\infty} t\,e^{-st}\,dt = \left[\frac{t\,e^{-st}}{-s}\right]_0^{\infty} - \left[\frac{e^{-st}}{s^2}\right]_0^{\infty} = \frac{1}{s^2}.$$

Thus,

$$\frac{2}{Ts^2}\tanh\left(\frac{sT}{4}\right) = \frac{2}{Ts^2}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-nTs/2}\right].$$

Finally, setting $sT = 4x$ gives

$$\tanh x = 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-2nx}.$$

14

First-order ordinary differential equations

In this chapter unspecified symbols appearing in solutions are arbitrary constants. Some of the constants may have specific relationships to earlier ones in the same solution, but this will not be indicated unless it has particular significance in the final answer.

14.2 *Solve the following equations by separation of the variables:*

(a) $y' - xy^3 = 0$;
(b) $y' \tan^{-1} x - y(1 + x^2)^{-1} = 0$;
(c) $x^2 y' + xy^2 = 4y^2$.

In each case we re-arrange the equation so that all terms involving y appear on one side of an equality sign and all those involving x appear on the other. To save space we write two equations on each line.

(a)
$$y' = xy^3, \quad \Rightarrow \quad \frac{dy}{y^3} = x\,dx,$$
$$\Rightarrow \quad -\frac{1}{2y^2} = \frac{1}{2}x^2 + A, \quad \Rightarrow \quad y = \frac{\pm 1}{\sqrt{c - x^2}}.$$

(b)
$$y' \tan^{-1} x = \frac{y}{1 + x^2}, \quad \Rightarrow \quad \frac{dy}{y} = \frac{dx}{(1 + x^2)\tan^{-1} x},$$
$$\Rightarrow \quad \ln y = \ln(\tan^{-1} x) + A, \quad \Rightarrow \quad y = c \tan^{-1} x.$$

(c)
$$x^2 y' = y^2(4 - x), \quad \Rightarrow \quad \frac{dy}{y^2} = \left(\frac{4}{x^2} - \frac{1}{x}\right) dx,$$
$$\Rightarrow \quad -\frac{1}{y} = -\frac{4}{x} - \ln x + c, \quad \Rightarrow \quad y = \frac{x}{4 + x \ln x - cx}.$$

14.4 *Find the values of* α *and* β *that make*

$$dF(x, y) = \left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) dx + (xy^\beta + 1)\, dy$$

an exact differential. For these values solve $dF(x, y) = 0$.

For the differential to be exact we need

$$\frac{\partial}{\partial y}\left(\frac{1}{x^2+2} + \frac{\alpha}{y}\right) = \frac{\partial}{\partial x}(xy^\beta + 1),$$
$$-\frac{\alpha}{y^2} = y^\beta.$$

Thus if $\alpha = -1$ and $\beta = -2$ then dF will be an exact differential. Integrating the equation then leads to

$$c' = F(x, y) = \int \left(\frac{x}{y^2} + 1\right) dy + g(x)$$
$$= -\frac{x}{y} + y + g(x),$$

where

$$\frac{1}{x^2+2} - \frac{1}{y} = \frac{\partial F}{\partial x} = -\frac{1}{y} + g'(x),$$

which implies that

$$g(x) = \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + c''.$$

Collecting these results together, we can give the solution as

$$c = F(x, y) = -\frac{x}{y} + y + \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x}{\sqrt{2}}\right).$$

14.6 *By finding an appropriate integrating factor, solve*

$$\frac{dy}{dx} = -\frac{2x^2 + y^2 + x}{xy}.$$

Arrange the equation in the form

$$xy\, dy + (2x^2 + y^2 + x)\, dx = 0.$$

Now apply the standard prescription for determining whether a suitable IF exists:

$$\frac{1}{xy}\left[\frac{\partial}{\partial y}(2x^2+y^2+x)-\frac{\partial}{\partial x}(xy)\right]=\frac{1}{xy}(2y-y)=\frac{1}{x}.$$

This is a function of x only, thus showing that one does and that the IF needed is

$$\mu(x)=\exp\left\{\int\frac{1}{x}\,dx\right\}=\exp(\ln x)=x.$$

The exact equation is thus

$$x^2y\,dy+(2x^3+xy^2+x^2)\,dx=0.$$

If this is to integrate to $f(x,y)=c$ then

$$\frac{\partial f}{\partial x}=2x^3+xy^2+x^2 \quad\Rightarrow\quad f(x,y)=\frac{1}{2}x^4+\frac{1}{2}x^2y^2+\frac{1}{3}x^3+g(y).$$

The further requirement that $\partial f/\partial y = x^2y$ shows that $g(y)=0$ and so, on multiplying through by 6, we obtain the solution

$$3x^4+2x^3+3x^2y^2=c.$$

14.8 *An electric circuit contains a resistance R and a capacitor C in series, and a battery supplying a time-varying electromotive force $V(t)$. The charge q on the capacitor therefore obeys the equation*

$$R\frac{dq}{dt}+\frac{q}{C}=V(t).$$

Assuming that initially there is no charge on the capacitor, and given that $V(t)=V_0\sin\omega t$, find the charge on the capacitor as a function of time.

In standard form the equation is

$$\frac{dq}{dt}+\frac{q}{RC}=\frac{V_0}{R}\sin\omega t.$$

The required IF is

$$\mu(t)=\exp\left\{\int^t\frac{1}{RC}\,du\right\}=e^{t/RC}\equiv e^{\omega_0 t},$$

thus defining ω_0.

Multiplying through by this IF and expressing the LHS as a total derivative gives

$$\frac{d}{dt}[e^{\omega_0 t}q(t)]=\frac{V_0\,e^{\omega_0 t}}{R}\sin\omega t.$$

Since $q(0) = 0$, this leads to

$$\begin{aligned}
q(t) &= \frac{V_0 e^{-\omega_0 t}}{R} \int_0^t e^{\omega_0 u} \sin \omega u \, du \\
&= \frac{V_0 e^{-\omega_0 t}}{R} \operatorname{Im} \left\{ \int_0^t e^{(\omega_0 + i\omega)u} \, du \right\} \\
&= \frac{V_0 e^{-\omega_0 t}}{R} \operatorname{Im} \left[\frac{e^{(\omega_0 + i\omega)t} - 1}{\omega_0 + i\omega} \right] \\
&= \frac{V_0 e^{-\omega_0 t}}{R(\omega_0^2 + \omega^2)} \operatorname{Im} \left[\omega_0 e^{(\omega_0 + i\omega)t} - i\omega e^{(\omega_0 + i\omega)t} - \omega_0 + i\omega \right] \\
&= \frac{V_0 e^{-\omega_0 t}}{R(\omega_0^2 + \omega^2)} \left[\omega_0 e^{\omega_0 t} \sin \omega t - \omega e^{\omega_0 t} \cos \omega t + \omega \right] \\
&= \frac{R^2 C^2 V_0}{R(1 + R^2 C^2 \omega^2)} \left[\frac{1}{RC} \sin \omega t - \omega \cos \omega t + \omega e^{-t/RC} \right] \\
&= \frac{C V_0}{1 + (RC\omega)^2} \left[\sin \omega t - \omega RC \cos \omega t + \omega RC e^{-t/RC} \right].
\end{aligned}$$

This gives the full time dependence of the charge on the capacitor. The first two terms give the long-term behaviour, whilst the final one is a transient arising from the intial conditions.

14.10 *Use the result of exercise 14.9 to find the law of force, acting towards the origin, under which a particle must move so as to describe the following trajectories:*

(a) *A circle of radius a that passes through the origin;*
(b) *An equiangular spiral, which is defined by the property that the angle α between the tangent and the radius vector is constant along the curve.*

(a) As shown in part (a) of figure 14.1, $p = r \sin \phi$ and, from simple geometry, $\sin \phi = \frac{1}{2} r/a$. It follows immediately that $r^2 = 2ap$ and

$$f = \frac{h^2}{mp^3} \frac{dp}{dr} = \frac{h^2}{m} \frac{8a^3}{r^6} \frac{2r}{2a} \propto \frac{1}{r^5}.$$

(b) By definition, and as shown in figure (b), $p = r \sin \alpha$ and therefore

$$f = \frac{h^2}{mp^3} \frac{dp}{dr} = \frac{h^2 \sin \alpha}{mr^3 \sin^3 \alpha} \propto \frac{1}{r^3}.$$

Note that for each case the constant h, which depends upon the initial conditions, will contain the parameter a or α; consequently only the r-dependence of f can be stated.

Figure 14.1 The trajectories discussed in exercise 14.10.

14.12 *A mass m is accelerated by a time-varying force $\alpha\exp(-\beta t)v^3$, where v is its velocity. It also experiences a resistive force ηv, where η is a constant, owing to its motion through the air. The equation of motion of the mass is therefore*

$$m\frac{dv}{dt} = \alpha\exp(-\beta t)v^3 - \eta v.$$

Find an expression for the velocity v of the mass as a function of time, given that it has an initial velocity v_0.

The equation can be written as

$$\frac{dv}{dt} + \frac{\eta}{m}v = \frac{\alpha e^{-\beta t}}{m}v^3,$$

which is Bernouilli's equation with $n = 3$. Therefore put $u = v^{1-3}$, i.e. $v = u^{-1/2}$; this leads to

$$-\frac{1}{2}\frac{1}{u^{3/2}}\frac{du}{dt} + \frac{\eta}{m}u^{-1/2} = \frac{\alpha e^{-\beta t}}{m}u^{-3/2},$$

$$\frac{du}{dt} - \frac{2\eta}{m}u = -\frac{2\alpha e^{-\beta t}}{m}.$$

The IF for this equation is clearly $e^{-2\eta t/m}$ and when applied gives

$$\frac{d}{dt}(ue^{-2\eta t/m}) = -\frac{2\alpha}{m}\exp\left[-\left(\beta + \frac{2\eta}{m}\right)t\right]$$

$$ue^{-2\eta t/m} = \frac{2\alpha}{\beta m + 2\eta}\exp\left[-\left(\beta + \frac{2\eta}{m}\right)t\right] + A,$$

or, in terms of v, $$\frac{1}{v^2} = \frac{2\alpha e^{-\beta t}}{\beta m + 2\eta} + Ae^{2\eta t/m}.$$

Using the initial velocity to determine the value of A then gives the solution at a general time t as $v(t)$ where

$$\frac{1}{v^2} = \frac{2\alpha}{\beta m + 2\eta}\left(e^{-\beta t} - e^{2\eta t/m}\right) + \frac{1}{v_0^2}e^{2\eta t/m}.$$

14.14 *Solve*

$$\frac{dy}{dx} = \frac{1}{x + 2y + 1}.$$

Since the only linear combination of x and y to appear is $x + 2y + 1$, we set it equal to v with $dv/dx = 1 + 2dy/dx$. The equation then becomes

$$\frac{1}{2}\frac{dv}{dx} - \frac{1}{2} = \frac{1}{v} \quad \Rightarrow \quad \frac{dv}{dx} = \frac{v+2}{v}.$$

We can now separate the variables and integrate:

$$dx = \left(1 - \frac{2}{v+2}\right) dv \quad \Rightarrow \quad x + c = v - 2\ln(v+2).$$

Re-substitution for v gives the final answer as

$$\begin{aligned} & x + c = x + 2y + 1 - 2\ln(x + 2y + 3), \\ \Rightarrow \quad & k + y = \ln(x + 2y + 3). \end{aligned}$$

14.16 *If $u = 1 + \tan y$, calculate $d(\ln u)/dy$; hence find the general solution of*

$$\frac{dy}{dx} = \tan x \cos y\ (\cos y + \sin y).$$

With $u = 1 + \tan y$, the derivative of $\ln u$ with respect to y is

$$\frac{d(\ln u)}{dy} = \frac{\sec^2 y}{1 + \tan y} = \frac{1}{\cos y(\cos y + \sin y)}.$$

Now, rearranging the equation given in the question:

$$\begin{aligned} \frac{dy}{\cos y(\cos y + \sin y)} &= \tan x\, dx, && [\text{separating variables}] \\ \ln(1 + \tan y) &= -\ln\cos x + A, && [\text{integrating, using the above result}] \\ \cos x(1 + \tan y) &= k, \end{aligned}$$

to give as the final solution

$$y = \tan^{-1}(k \sec x - 1).$$

14.18 *A reflecting mirror is made in the shape of the surface of revolution generated by revolving the curve $y(x)$ about the x-axis. In order that light rays emitted from a point source at the origin are reflected back parallel to the x-axis, the curve $y(x)$ must obey*

$$\frac{y}{x} = \frac{2p}{1-p^2},$$

where $p = dy/dx$. By solving this equation for x find the curve $y(x)$.

We first eliminate y, by differentiating it to obtain a first-order equation for p, as follows.

$$\begin{aligned}
y &= \frac{2p\,x}{1-p^2}, \\
p = \frac{dy}{dx} &= \frac{(1-p^2)(2p+2xp') - (2p\,x)(-2p\,p')}{(1-p^2)^2}, \\
(p-p^3)(1-p^2) &= 2p - 2p^3 + 2xp' - 2xp^2p' + 4xp^2p', \\
(1-p^2)(p-p^3-2p) &= (1+p^2)2xp', \\
p\,(p^2-1) &= 2xp'.
\end{aligned}$$

We now separate the variables, use partial fractions and integrate:

$$\begin{aligned}
\frac{dx}{x} &= \frac{2\,dp}{p\,(p-1)(p+1)} \\
&= \frac{-2\,dp}{p} + \frac{dp}{p-1} + \frac{dp}{p+1}, \\
\Rightarrow \quad A + \ln x &= -2\ln p + \ln(p-1) + \ln(p+1).
\end{aligned}$$

This can be arranged as

$$Bx = \frac{p^2-1}{p^2} \qquad \text{or} \qquad p = \frac{\pm 1}{\sqrt{1-Bx}}.$$

We now substitute for p in the original equation and obtain

$$\frac{y}{x} = \frac{2p^{-1}}{p^{-2}-1} = \frac{\pm 2\sqrt{1-Bx}}{1-Bx-1},$$

which, in turn, can be rearranged as

$$y = \mp\frac{2\sqrt{1-Bx}}{B} \qquad \text{or} \qquad y^2 = \frac{4}{B^2} - \frac{4x}{B}.$$

This is a parabola, symmetric about the x-axis and with its apex at $x = 1/B$. The way it faces depends upon the sign of B.

14.20 *Find a parametric solution of*

$$x\left(\frac{dy}{dx}\right)^2 + \frac{dy}{dx} - y = 0$$

as follows.

(a) *Write an equation for y in terms of $p = dy/dx$ and show that*

$$p = p^2 + (2px + 1)\frac{dp}{dx}.$$

(b) *Using p as the independent variable, arrange this as a linear first-order equation for x.*

(c) *Find an appropriate integrating factor to obtain*

$$x = \frac{\ln p - p + c}{(1-p)^2},$$

which, together with the expression for y obtained in (a)*, gives a parameterisation of the solution.*

(d) *Reverse the roles of x and y in steps* (a) *to* (c)*, putting $dx/dy = p^{-1}$, and show that essentially the same parameterisation is obtained.*

(a) Writing $p = dy/dx$, the equation becomes

$$\begin{aligned} y &= xp^2 + p, \\ p = \frac{dy}{dx} &= p^2 + 2xp\,p' + p' \\ &= p^2 + (2xp + 1)\frac{dp}{dx}. \end{aligned}$$

(b) In differential form, this equation reads

$$df = p\,(1-p)\,dx - (2xp + 1)\,dp = 0.$$

(c) We now apply the standard test for the existence of an IF for $f(x,p)\,dx + g(x,p)\,dp$:

$$\frac{1}{f}\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial p}\right) = \frac{1}{p\,(1-p)}[-2p - (1-2p)] = -\frac{1}{p\,(1-p)}.$$

As this is a function of p alone, an IF exists and is given by

$$\exp\left[\int \frac{-1}{p\,(1-p)}\,dp\right] = \exp\left[\int \left(-\frac{1}{p} - \frac{1}{1-p}\right)\,dp\right]$$
$$= \exp[-\ln p + \ln(1-p)] = \frac{1-p}{p}.$$

With this IF, the equation becomes

$$(1-p)^2\,dx - \frac{(2xp+1)(1-p)}{p}\,dp = 0,$$
$$d[(1-p)^2 x] - \frac{dp}{p} + dp = 0,$$
$$(1-p)^2 x - \ln p + p = c.$$

This gives $x = (c + \ln p - p)(1-p)^{-2}$ and, together with $y = p + p^2 x$, gives a full parameterisation, $x = x(p)$, $y = y(p)$, of the solution.

(d) Now set $dx/dy = p^{-1} = q$. A parallel calculation to that in part (b) gives

$$0 = xp^2 + p - y,$$
$$x = -\frac{1}{p} + \frac{y}{p^2} = -q + q^2 y,$$
$$q = \frac{dx}{dy} = -\frac{dq}{dy} + q^2 + 2qy\frac{dq}{dy},$$
$$0 = (2qy - 1)\,dq + (q^2 - q)\,dy.$$

As in part (c), consider

$$\frac{1}{q(q-1)}[2q - (2q-1)] = -\frac{1}{q(1-q)}.$$

It follows that relevant IF is $(1-q)/q$ and that when it is applied the equation becomes

$$-(1-q)^2\,dy + \frac{(1-q)(2qy-1)}{q}\,dq = 0.$$

This leads to

$$d[(1-q)^2 y] + \frac{dq}{q} - dq = 0,$$

and then to

$$y = \frac{c' - \ln q + q}{(1-q)^2}.$$

Together with $x = -q + q^2 y$, this expression for y gives essentially the same parameterisation as obtained previously. This can be verified, if necessary, by writing $q = p^{-1}$ and substituting in the second parameterisation; it will be found that $c = 2 + c'$ makes the two forms identical.

14.22 *The action of the control mechanism on a particular system for an input $f(t)$ is described, for $t \geq 0$, by the coupled first-order equations:*

$$\dot{y} + 4z = f(t),$$
$$\dot{z} - 2z = \dot{y} + \tfrac{1}{2}y.$$

Use Laplace transforms to find the response $y(t)$ of the system to a unit step input $f(t) = H(t)$, given that $y(0) = 1$ and $z(0) = 0$.

We start by taking the Laplace transforms of the two equations, at the same time incorporating the initial conditions.

$$s\bar{y} - 1 + 4\bar{z} = \bar{f} \quad \Rightarrow \quad 4\bar{z} = \bar{f} + 1 - s\bar{y}$$

and

$$s\bar{z} - 0 - 2\bar{z} = s\bar{y} - 1 + \tfrac{1}{2}\bar{y} \quad \Rightarrow \quad (s-2)\bar{z} = (s + \tfrac{1}{2})\bar{y} - 1.$$

Eliminating $\bar{z}$ from these algebraic equations gives

$$\begin{aligned} 4\frac{(s+\frac{1}{2})\bar{y} - 1}{s-2} &= \bar{f} + 1 - s\bar{y}, \\ \bar{y}[4s + 2 + s(s-2)] &= (s-2)(\bar{f}+1) + 4, \\ \bar{y} &= \frac{(\bar{f}+1)s + 2 - 2\bar{f}}{s^2 + 2s + 2}. \end{aligned}$$

This is the transform of the response to a general input $f(t)$.

For the particular input $f(t) = H(t)$, $\bar{f} = 1/s$ and

$$\begin{aligned} \bar{y} &= \frac{1 + s + 2 - 2s^{-1}}{s^2 + 2s + 2} \\ &= \frac{s^2 + 3s - 2}{s[(s+1)^2 + 1]} \\ &= \frac{A}{s} + \frac{Bs + C}{(s+1)^2 + 1}. \end{aligned}$$

Cross-multiplying and equating coefficients requires that $1 = A + B$, $3 = 2A + C$ and $-2 = 2A$. These have solution $A = -1$, $B = 2$ and $C = 5$ to give

$$\begin{aligned} \bar{y} &= -\frac{1}{s} + \frac{2(s+1)}{(s+1)^2 + 1} + \frac{3}{(s+1)^2 + 1}, \\ \Rightarrow \quad y(t) &= -1 + 2e^{-t}\cos t + 3e^{-t}\sin t. \end{aligned}$$

14.24 *Solve the following first-order equations for the boundary conditions given:*

(a) $y' - (y/x) = 1, \qquad y(1) = -1;$
(b) $y' - y\tan x = 1, \qquad y(\pi/4) = 3;$
(c) $y' - y^2/x^2 = 1/4, \quad y(1) = 1;$
(d) $y' - y^2/x^2 = 1/4, \quad y(1) = 1/2.$

(a) As in part (a) of the previous exercise (14.23), this equation needs an integrating factor, given in this case by

$$\mu(x) = \exp\left\{\int -\frac{1}{x}\,dx\right\} = \exp(-\ln x) = \frac{1}{x}.$$

We then have

$$\frac{d}{dx}\left(\frac{y}{x}\right) = \frac{1}{x} \quad \Rightarrow \quad \frac{y}{x} = \ln x + A.$$

Since $y(1) = -1$, A must have the value -1 and so $y = x\ln x - x$.

(b) Again, an IF is needed; this time given by

$$\mu(x) = \exp\left\{\int -\tan x\,dx\right\} = \exp(\ln\cos x) = \cos x.$$

The equation now reads

$$\frac{d}{dx}(y\cos x) = \cos x \quad \Rightarrow \quad y\cos x = \sin x + A.$$

The given boundary condition is that $3/\sqrt{2} = 1/\sqrt{2} + A$, establishing A as $\sqrt{2}$. The final answer is therefore $y = \tan x + \sqrt{2}\sec x$.

(c) This is not a linear equation, though it is homogeneous, and we therefore set $y = ux$. The equation then becomes

$$u + x\frac{du}{dx} = u^2 + \frac{1}{4},$$
$$x\frac{du}{dx} = \left(\frac{1}{2} - u\right)^2.$$

The equation is now separable and gives

$$\frac{du}{(\frac{1}{2} - u)^2} = \frac{dx}{x} \quad \Rightarrow \quad \frac{1}{\frac{1}{2} - u} = \ln x + A.$$

The boundary condition is that $u = 1/1 = 1$ when $x = 1$, implying that $A = -2$.

Now, substituting $u = y/x$ gives

$$\frac{1}{2} - \frac{y}{x} = \frac{1}{\ln x - 2} \quad \Rightarrow \quad y = \frac{x}{2 - \ln x} + \frac{x}{2}.$$

(d) For the boundary condition $y(1) = 1/2$ the method of solution used in part (c) fails, as it requires A to satisfy the impossible equation $1/0 = 0 + A$. We must therefore try to find a singular solution.

The first equation of the solution in part (c) would be automatically satisfied with u independent of x if

$$\frac{du}{dx} = 0 \quad \Rightarrow \quad u = u^2 + \frac{1}{4} \quad \Rightarrow \quad u = \frac{1}{2}.$$

The conclusion is that u is a constant and, since this satisfies the inital assumption that its derivative is zero, the 'circle is complete'. Hence there is a singular solution $u = \frac{1}{2}$, i.e. $y = \frac{1}{2}x$, that satisfies both the differential equation and the boundary condition.

14.26 *Solve the differential equation*

$$\sin x \frac{dy}{dx} + 2y \cos x = 1$$

subject to the boundary condition $y(\pi/2) = 1$.

Either

By inspection, the IF for this equation is $\sin x$.

or

After dividing through by $\sin x$ this becomes a standard first-order linear equation in need of the integrating factor

$$\exp\left\{\int \frac{2\cos x}{\sin x}\, dx\right\} = \exp(2 \ln \sin x) = \sin^2 x.$$

By either method, multiplying the original equation through by $\sin x$ or the standardised one by $\sin^2 x$, the exact equation is

$$\begin{aligned} \sin^2 x \frac{dy}{dx} + 2y \cos x \sin x &= \sin x, \\ \frac{d}{dx}(y \sin^2 x) &= \sin x, \\ y \sin^2 x &= -\cos x + A. \end{aligned}$$

The condition $y(\pi/2) = 1$ implies that $A = 1$ and hence

$$y = \frac{1 - \cos x}{\sin^2 x} = \frac{1 - \cos x}{1 - \cos^2 x} = \frac{1}{1 + \cos x}.$$

14.28 *Find the solution of*

$$(5x + y - 7)\frac{dy}{dx} = 3(x + y + 1).$$

The equation is not homogeneous and the two variables x and y appear in different linear combinations on the two sides of the equation. We therefore seek shifts in their origins that will make the expression for the derivative homogeneous, i.e. remove the constant terms from both its numerator and denominator. To do this we set

$$x = X + \alpha \qquad \text{and} \qquad y = Y + \beta.$$

We then require

$$3\alpha + 3\beta + 3 = 0 \quad \text{and} \quad 5\alpha + \beta - 7 = 0$$

These have the straightforward solution $\alpha = 2$ and $\beta = -3$; with these values the original equation reduces to

$$\frac{dX}{dY} = \frac{3X + 3Y}{5X + Y}.$$

This is now homogeneous and to solve it we set $Y = vX$ and obtain

$$\begin{aligned}
\frac{dY}{dX} &= v + X\frac{dv}{dX}, \\
X\frac{dv}{dX} &= \frac{dY}{dX} - v = \frac{3X + 3Y}{5X + Y} - v, \\
&= \frac{3 + 3v - 5v - v^2}{5 + v}.
\end{aligned}$$

We now separate the variables and use method (iii) for a partial fraction expansion, obtaining

$$\begin{aligned}
\frac{dX}{X} &= \frac{5 + v}{3 - 2v - v^2} = \frac{A}{3 + v} + \frac{B}{1 - v}, \\
&= \frac{1}{2(3 + v)} + \frac{3}{2(1 - v)}, \\
\Rightarrow \quad \ln X &= \tfrac{1}{2}\ln(3 + v) - \tfrac{3}{2}\ln(1 - v) + k.
\end{aligned}$$

Re-substituting for v, X and Y, gives

$$x - 2 = A\left(3 + \frac{y + 3}{x - 2}\right)^{1/2}\left(1 - \frac{y + 3}{x - 2}\right)^{-3/2} = A\frac{(3x + y - 3)^{1/2}(x - 2)}{(x - y - 5)^{3/2}}.$$

Finally, this result can be re-written as

$$(x - y - 5)^3 = B(3x + y - 3).$$

14.30 *Find the solution of*

$$(2\sin y - x)\frac{dy}{dx} = \tan y,$$

if (a) $y(0) = 0$, *and* (b) $y(0) = \pi/2$.

Since x appears only in the combination $x\,dy/dx$ it will probably make the solution simpler to take y as the independent variable and x as the dependent one. With this in mind, we re-arrange the equation as

$$\tan y\frac{dx}{dy} + x = 2\sin y,$$

or, in standard form, as

$$\frac{dx}{dy} + x\cot y = 2\cos y.$$

The IF is clearly $\exp(\ln\sin y) = \sin y$, and the equation can be written

$$\frac{d}{dy}(x\sin y) = \sin 2y,$$

$$x\sin y = -\tfrac{1}{2}\cos 2y + k.$$

(a) For $y(0) = 0$ we must have $k = \frac{1}{2}$, and the solution becomes

$$x = \frac{1 - \cos 2y}{2\sin y} = \frac{2\sin^2 y}{2\sin y} = \sin y.$$

(b) If $y(0) = \pi/2$ then $k = -\frac{1}{2}$ and the solution is

$$x = \frac{-1 - \cos 2y}{2\sin y} = \frac{-2\cos^2 y}{2\sin y} = -\cos y\cot y.$$

15

Higher-order ordinary differential equations

15.2 *Find the roots of the auxiliary equation for the following. Hence solve them for the boundary conditions stated.*

(a) $\dfrac{d^2f}{dt^2} + 2\dfrac{df}{dt} + 5f = 0 \quad \text{with } f(0) = 1, f'(0) = 0.$

(b) $\dfrac{d^2f}{dt^2} + 2\dfrac{df}{dt} + 5f = e^{-t}\cos 3t \quad \text{with } f(0) = 0, f'(0) = 0.$

The two equations have the same LHS and the trial function $f(x) = e^{mx}$ leads to the common auxiliary equation

$$m^2 + 2m + 5 = 0 \quad \Rightarrow \quad m = -1 \pm \sqrt{1-5} = -1 \pm 2i.$$

Thus the CF for both equations is $f(t) = e^{-t}(A\cos 2t + B\sin 2t)$.

(a) Since the RHS of the equation is zero, no particular integral is needed (formally it is $f(x) = 0$). For the CF the boundary conditions require

$$\begin{aligned} f(0) = 1 \quad &\Rightarrow \quad 1 = 1(A + 0) \quad \Rightarrow \quad A = 1, \\ f'(0) = 0 \quad &\Rightarrow \quad 0 = -2e^{-t}\sin 2t - e^{-t}\cos 2t \\ &\qquad\qquad - Be^{-t}\sin 2t + 2Be^{-t}\cos 2t, \text{ at } t = 0, \\ &\qquad = -1 + 2B \quad \Rightarrow \quad B = \tfrac{1}{2}. \end{aligned}$$

Resubstitution gives $\qquad f(t) = e^{-t}(\cos 2t + \tfrac{1}{2}\sin 2t).$

(b) Since the CF does not contain a term involving $e^{-t}\cos 3t$ we may try a linear

combination of $e^{-t}\cos 3t$ and $e^{-t}\sin 3t$ as the PI, as follows.

$$\begin{aligned} f &= e^{-t}(C\cos 3t + D\sin 3t), \\ f' &= -e^{-t}(C\cos 3t + D\sin 3t) + e^{-t}(-3C\sin 3t + 3D\cos 3t), \\ f'' &= e^{-t}(C\cos 3t + D\sin 3t) - 2e^{-t}(-3C\sin 3t + 3D\cos 3t) \\ &\quad - 9e^{-t}(C\cos 3t + D\sin 3t). \end{aligned}$$

When these are substituted into the equation, the coefficients of $e^{-t}\sin 3t$ require that

$$D + 6C - 9D - 2D - 6C + 5D = 0 \quad \Rightarrow \quad D = 0.$$

Those for $e^{-t}\cos 3t$ imply that

$$C - 0 - 9C - 2C + 0 + 5C = 1 \quad \Rightarrow \quad C = -\tfrac{1}{5}.$$

With this PI the general solution becomes

$$f(t) = e^{-t}(A\cos 2t + B\sin 2t) - \tfrac{1}{5}e^{-t}\cos 3t.$$

The boundary condition $f(0) = 0$ requires that $A = \frac{1}{5}$, and the condition on the derivative, $f'(0) = 0$, implies (after multiplying all through by 5 for convenience) that

$$\begin{aligned} 0 &= -e^{-t}(\cos 2t + 5B\sin 2t - \cos 3t) \\ &\qquad + e^{-t}(-2\sin 2t + 10B\cos 2t + 3\sin 3t) \text{ at } t = 0, \\ \Rightarrow \quad 0 &= -(1 + 0 - 1) + (-0 + 10B + 0) \quad \Rightarrow \quad B = 0. \end{aligned}$$

Thus, the final solution is

$$f(t) = \tfrac{1}{5}e^{-t}(\cos 2t - \cos 3t),$$

which can, if necessary, be checked by re-substitution.

15.4 *Solve the differential equation*

$$\frac{d^2f}{dt^2} + 6\frac{df}{dt} + 9f = e^{-t},$$

subject to the conditions $f = 0$ *and* $df/dt = \lambda$ *at* $t = 0$.

Find the equation satisfied by the positions of the turning points of $f(t)$ *and hence, by drawing suitable sketch graphs, determine the number of turning points the solution has in the range* $t > 0$ *if* (a) $\lambda = 1/4$, *and* (b) $\lambda = -1/4$.

The auxiliary equation and resulting CF are

$$\begin{aligned} m^2 + 6m + 9 = 0 \quad &\Rightarrow \quad m = -3 \text{ (repeated root)}, \\ &\Rightarrow \quad f(t) = (A + Bt)e^{-3t}. \end{aligned}$$

For a particular integral, since e^{-t} does not appear in the CF, we try $f = Ce^{-t}$ and obtain

$$Ce^{-t} - 6Ce^{-t} + 9Ce^{-t} = e^{-t} \quad \Rightarrow \quad C = \tfrac{1}{4}.$$

The general solution is therefore

$$f(t) = (A + Bt)e^{-3t} + \tfrac{1}{4}e^{-t}.$$

We now incorporate the boundary conditions:

$$\begin{aligned} f(0) = 0 \quad &\Rightarrow \quad 0 = (A + 0)1 + \tfrac{1}{4} \quad \Rightarrow \quad A = -\tfrac{1}{4}, \\ f'(0) = \lambda \quad &\Rightarrow \quad \lambda = -3A + B - 0 - \tfrac{1}{4} \quad \Rightarrow \quad B = \lambda - \tfrac{1}{2}. \end{aligned}$$

Therefore the solution matching these boundary conditions is

$$f(t) = [-\tfrac{1}{4} + (\lambda - \tfrac{1}{2})t]e^{-3t} + \tfrac{1}{4}e^{-t}.$$

The turning points of the solution are given by $f'(t) = 0$, i.e.

$$\begin{aligned} \tfrac{3}{4}e^{-3t} + (\lambda - \tfrac{1}{2})e^{-3t} - 3(\lambda - \tfrac{1}{2})te^{-3t} - \tfrac{1}{4}e^{-t} &= 0, \\ e^{-2t}\left[(\lambda + \tfrac{1}{4}) + (\tfrac{3}{2} - 3\lambda)t\right] &= \tfrac{1}{4}, \\ (4\lambda + 1) + (6 - 12\lambda)t &= e^{2t}. \end{aligned}$$

(a) For $\lambda = \frac{1}{4}$ the equation becomes $e^{2t} = 2 + 3t$. Consider the behaviours of the functions on either side of this equation:

At $t = 0$, $\quad 2 + 3t = 2 > 1 = e^{2t}$.

For large t, $\quad e^{2t} > 2 + 3t$.

Both functions are monotonic and thus there is one, and only one, solution to $e^{2t} = 2 + 3t$ in $t > 0$. It follows that the solution to the original differential equation has only one turning point in this range.

(b) For $\lambda = -\frac{1}{4}$ the equation becomes $e^{2t} = 9t$. Again consider the behaviours of the two sides of the equation.

At $t = 0$, $\quad e^{2t} = 1 > 0 = 9t$.

At $t = 1$, $\quad e^{2t} = e^2 < 9 = 9t$.

For large t, $\quad e^{2t} > 9t$.

Both functions are monotonic and thus there are two solutions to $e^{2t} = 9t$ in $t > 0$. It follows that the solution to the original differential equation has two turning points in this range.

15.6 *Determine the values of α and β for which the following functions are linearly dependent:*

$$\begin{aligned} y_1(x) &= x\cosh x + \sinh x, \\ y_2(x) &= x\sinh x + \cosh x, \\ y_3(x) &= (x+\alpha)e^x, \\ y_4(x) &= (x+\beta)e^{-x}. \end{aligned}$$

You will find it convenient to work with those linear combinations of the $y_i(x)$ that can be written the most compactly.

To make the working more compact, write

$$\begin{aligned} y_5(x) &= y_1 + y_2 = (x+1)(\cosh x + \sinh x) = (x+1)e^x, \\ y_6(x) &= y_1 - y_2 = (x-1)(\cosh x - \sinh x) = (x-1)e^{-x}. \end{aligned}$$

We notice that $y_3 = y_5$ if $\alpha = 1$ and that $y_4 = y_6$ if $\beta = -1$. With these values the functions are linearly dependent and so give the answer to the original question.

However, we will continue as if this had not been noticed and compute the Wronskian $W(y_3, y_4, y_5, y_6)$. For this we need the derivatives (using Leibnitz' theorem)

$$\begin{aligned} \frac{d^n}{dx^n}[(x+\gamma)e^x] &= (x+\gamma)e^x + ne^x = (x+\gamma+n)e^x, \\ \frac{d^n}{dx^n}[(x+\gamma)e^{-x}] &= (-1)^n(x+\gamma)e^{-x} + n(-1)^{n-1}e^{-x} \\ &= (-1)^n(x+\gamma-n)e^{-x}. \end{aligned}$$

Each column of the Wronskian will have a common factor of $e^{\pm x}$ and we take these outside the determinant, writing $W(y_3, y_4, y_5, y_6)$ as

$$\begin{aligned} W &= e^x\, e^{-x}\, e^x\, e^{-x} \begin{vmatrix} x+\alpha & x+\beta & x+1 & x-1 \\ x+\alpha+1 & -x-\beta+1 & x+2 & -x+2 \\ x+\alpha+2 & x+\beta-2 & x+3 & x-3 \\ x+\alpha+3 & -x-\beta+3 & x+4 & -x+4 \end{vmatrix} \\ &= \begin{vmatrix} \alpha-1 & \beta+1 & x+1 & x-1 \\ \alpha-1 & -\beta-1 & x+2 & -x+2 \\ \alpha-1 & \beta+1 & x+3 & x-3 \\ \alpha-1 & -\beta-1 & x+4 & -x+4 \end{vmatrix}. \end{aligned}$$

To obtain this last form, we have subtracted the third column from the first and the fourth from the second. The common factors $\alpha-1$ and $\beta+1$ can be taken out of the determinant which then becomes a function of x only. For the Wronskian

to vanish for all x (and hence make the functions dependent) requires either $\alpha = 1$ or $\beta = -1$ or both. In fact, the remaining determinant has the value -16, independent of the value of x, but all that matters for drawing our conclusion is that it is non-zero.

15.8 *The two functions $x(t)$ and $y(t)$ satisfy the simultaneous equations*

$$\frac{dx}{dt} - 2y = -\sin t,$$
$$\frac{dy}{dt} + 2x = 5\cos t.$$

Find explicit expressions for $x(t)$ and $y(t)$, given that $x(0) = 3$ and $y(0) = 2$. Sketch the solution trajectory in the xy-plane for $0 \le t < 2\pi$, showing that the trajectory crosses itself at $(0, 1/2)$ and passes through the points $(0, -3)$ and $(0, -1)$ in the negative x-direction.

By differentiating the first equation and then substituting for dy/dt from the second we obtain

$$\frac{d^2x}{dt^2} - 2\frac{dy}{dt} = -\cos t,$$
$$\frac{d^2x}{dt^2} - 2(5\cos t - 2x) = -\cos t,$$
$$\frac{d^2x}{dt^2} + 4x = 9\cos t.$$

The RHS is not contained in the CF and so the general solution is of the form

$$x(t) = A\cos 2t + B\sin 2t + C\cos t.$$

Substituting the PI part of this into the equation to find the value of C, gives $C(-1 + 4) = 9$ and hence $C = 3$. Further, since $x(0) = 3$, we must have $A = 0$ and it follows that

$$x(t) = B\sin 2t + 3\cos t.$$

Now,

$$\begin{aligned} y(t) &= \frac{1}{2}\left(\frac{dx}{dt} + \sin t\right) \\ &= B\cos 2t - \tfrac{3}{2}\sin t + \tfrac{1}{2}\sin t \\ &= B\cos 2t - \sin t. \end{aligned}$$

Since $y(0) = 2$, $B = 2$ and so, in summary,

$$x(t) = 2\sin 2t + 3\cos t,$$
$$y(t) = 2\cos 2t - \sin t.$$

Figure 15.1 The closed curve generated by the equations in exercise 15.8.

The (closed) curve is shown in figure 15.1.

It crosses the y-axis when

$$2\sin 2t + 3\cos t = 0, \text{ i.e when } 4\sin t\cos t + 3\cos t = 0.$$

This has solutions when $\cos t = 0$, i.e. $t = \frac{1}{2}\pi$ and $t = \frac{3}{2}\pi$, as well as when $\sin t = -\frac{3}{4}$. The latter corresponds to two values of t, but with only one corresponding y-value given by

$$y(t) = 2\cos 2t - \sin t = 2 - 4\sin^2 t - \sin t = 2 - 4\left(-\frac{3}{4}\right)^2 + \frac{3}{4} = \frac{1}{2}.$$

Thus the curve crosses itself at $(0, \frac{1}{2})$.

Finally, consider the two other points on the trajectory at which $x = 0$.

When $t = \frac{1}{2}\pi$, $y(t) = 2(-1) - 1 = -3$ and $dx/dt = 4\cos 2t - 3\sin t = -4 - 3 = -7$.

When $t = \frac{3}{2}\pi$, $y(t) = 2(-1) - (-1) = -1$ and $dx/dt = -4 + 3 = -1$.

In both cases dx/dt is negative, showing that the trajectory passes through the points $(0, -3)$ and $(0, -1)$ in the negative x-direction.

15.10 *Use the method of Laplace transforms to solve*

(a) $\dfrac{d^2f}{dt^2} + 5\dfrac{df}{dt} + 6f = 0, \qquad f(0) = 1,\ f'(0) = -4,$

(b) $\dfrac{d^2f}{dt^2} + 2\dfrac{df}{dt} + 5f = 0, \qquad f(0) = 1,\ f'(0) = 0.$

(a) Recalling that $\mathscr{L}\left[f'\right] = s\bar{f} - f(0)$ and $\mathscr{L}\left[f''\right] = s^2\bar{f} - sf(0) - f'(0)$, we have

$$\begin{aligned}
0 &= s^2\bar{f} - sf(0) - f'(0) + 5[s\bar{f} - f(0)] + 6\bar{f} \\
&= s^2\bar{f} - s + 4 + 5(s\bar{f} - 1) + 6\bar{f}, \\
\Rightarrow \quad \bar{f} &= \frac{s+1}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}, \\
&= \frac{-1}{s+2} + \frac{2}{s+3}, \text{ using one of the standard methods,} \\
\Rightarrow \quad f(t) &= 2e^{-3t} - e^{-2t}, \text{ from the look-up table.}
\end{aligned}$$

(b) Using the same method as in (a), we have

$$\begin{aligned}
0 &= s^2\bar{f} - sf(0) - f'(0) + 2[s\bar{f} - f(0)] + 5\bar{f} \\
&= s^2\bar{f} - s + 2(s\bar{f} - 1) + 5\bar{f}, \\
\Rightarrow \quad \bar{f} &= \frac{s+2}{s^2+2s+5} = \frac{s+2}{(s+1)^2+4} \\
&= \frac{(s+1)+1}{(s+1)^2+4}, \\
\Rightarrow \quad f(t) &= e^{-t}\cos 2t + \tfrac{1}{2}e^{-t}\sin 2t, \text{ from the look-up table.}
\end{aligned}$$

We note that this is the same result as that obtained in Exercise 15.2(a).

15.12 *Use Laplace transforms to solve, for $t \geq 0$, the differential equations*

$$\ddot{x} + 2x + y = \cos t,$$
$$\ddot{y} + 2x + 3y = 2\cos t,$$

which describe a coupled system that starts from rest at the equilibrium position. Show that the subsequent motion takes place along a straight line in the xy-plane. Verify that the frequency at which the system is driven is equal to one of the resonance frequencies of the system; explain why there is no *resonant behaviour in the solution you have obtained.*

We start by taking the Laplace transforms of the equations with all initial values and first derivatives equal to zero.

$$s^2\bar{x} + 2\bar{x} + \bar{y} = \frac{s}{1+s^2} \qquad (*),$$
$$s^2\bar{y} + 2\bar{x} + 3\bar{y} = \frac{2s}{1+s^2} \qquad (**).$$

Now consider the equation obtained by taking $2 \times (*) - (**)$.

$$\begin{aligned} \bar{x}(2s^2 + 4 - 2) + \bar{y}(2 - s^2 - 3) &= 0, \\ 2(s^2 + 1)\bar{x} - (s^2 + 1)\bar{y} &= 0, \\ 2\bar{x} - \bar{y} &= 0. \end{aligned}$$

This final equation, which is independent of s and hence of the t-dependence of x and y, means that $y(t)$ is a direct multiple of $x(t)$ and the motion takes place along a straight line in the x-y plane.

Setting $\bar{y} = 2\bar{x}$ in $(*)$ gives

$$(s^2 + 4)\bar{x} = \frac{s}{1 + s^2},$$

which, after rearrangement, gives the partial fraction expression for $\bar{x}$ as

$$\bar{x} = \frac{s}{3(s^2 + 1)} - \frac{s}{3(s^2 + 4)}.$$

This, in turn, implies (from the table of Laplace transforms) that

$$x(t) = \tfrac{1}{3}(\cos t - \cos 2t).$$

As in chapter 9 on Normal Modes, the natural frequencies of the system are given by

$$\begin{aligned} \begin{vmatrix} -\omega^2 + 2 & 1 \\ 2 & -\omega^2 + 3 \end{vmatrix} &= 0, \\ \omega^4 - 5\omega^2 + 4 &= 0, \\ (\omega^2 - 4)(\omega^2 - 1) &= 0. \end{aligned}$$

Thus the resonance frequencies are $\omega = 2$ and $\omega = 1$; the given driving frequency is the second of these.

However, for $\omega = 1$ the (x, y) eigenvector satisfies $(-1 + 2)x + (1)y = 0$, i.e. $y = -x$, whilst for $\omega = 2$ the (x, y) eigenvector satisfies $(-4 + 2)x + (1)y = 0$, i.e. $y = 2x$.

The driving terms in the given situation have frequency $\omega = 1$. But the solution obtained is purely that corresponding to $\omega = 2$ and contains no component of the $\omega = 1$ response. Consequently there is no resonant behaviour.

15.14 *For a lightly damped ($\gamma < \omega_0$) harmonic oscillator driven at its undamped resonance frequency ω_0, the displacement $x(t)$ at time t satisfies the equation*

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = F\sin\omega_0 t.$$

Use Laplace transforms to find the displacement at a general time if the oscillator starts from rest at its equilibrium position.

(a) *Show that ultimately the oscillation has amplitude $F/(2\omega_0\gamma)$ with a phase lag of $\pi/2$ relative to the driving force per unit mass F.*
(b) *By differentiating the original equation, conclude that if $x(t)$ is expanded as a power series in t for small t then the first non-vanishing term is $F\omega_0 t^3/6$. Confirm this conclusion by expanding your explicit solution.*

With no initial dispacement or motion, the Laplace transformed equation reads

$$s^2\bar{x} + 2\gamma s\bar{x} + \omega_0^2\bar{x} = \frac{F\omega_0}{s^2+\omega_0^2},$$

$$\frac{\bar{x}}{F\omega_0} = \frac{1}{(s^2+\omega_0^2)(s^2+2\gamma s+\omega_0^2)} = \frac{A+Bs}{s^2+\omega_0^2} + \frac{C+Ds}{(s+\gamma)^2+k^2},$$

where $k^2 = \omega_0^2 - \gamma^2$.

Cross-multiplying and equating the coefficients of the various powers of s yields

$$\begin{aligned} s^3 &: D + B = 0, \\ s^2 &: A + 2B\gamma + C = 0, \\ s^1 &: 2A\gamma + B\gamma^2 + Bk^2 + D\omega_0^2 = 0, \\ s^0 &: A\gamma^2 + Ak^2 + C\omega_0^2 = 1, \end{aligned}$$

with solutions

$$A = 0, \qquad D = -B = \frac{1}{2\gamma\omega_0^2}, \qquad C = \frac{1}{\omega_0^2}.$$

We can now rewrite the partial fraction expansion as

$$\frac{2\gamma\omega_0^2\bar{x}}{F\omega_0} = -\frac{s}{s^2+\omega_0^2} + \frac{(s+\gamma)+2\gamma-\gamma}{(s+\gamma)^2+k^2},$$

which integrates to

$$\frac{2\gamma\omega_0\, x(t)}{F} = -\cos\omega_0 t + e^{-\gamma t}\cos kt + \frac{\gamma}{k}e^{-\gamma t}\sin kt,$$

$$\text{i.e. } x(t) = \frac{F}{2\omega_0}\left[\frac{1}{\gamma}(e^{-\gamma t}\cos kt - \cos\omega_0 t) + \frac{1}{k}e^{-\gamma t}\sin kt\right].$$

This is the complete solution, valid for all times $t > 0$.

(a) As $t \to \infty$,

$$x(t) \approx -\frac{F}{2\omega_0\gamma}\cos\omega_0 t = \frac{F}{2\omega_0\gamma}\sin(\omega_0 t - \tfrac{1}{2}\pi).$$

Thus ultimately the oscillation has amplitude $F/(2\omega_0\gamma)$ with a phase lag of $\pi/2$ relative to the driving force F.

(b) At $t = 0$, both x and x' are zero, and so is $F\sin\omega_0 t$. It therefore follows from the original equation that $x''(0)$ is also zero. Thus, if $x(t)$ were expanded in a Taylor series about $t = 0$ the constant, linear and quadratic terms of the series would be missing.
Now consider the equation

$$x''' + 2\gamma x'' + \omega_0^2 x' = \omega_0 F\cos\omega_0 t,$$

obtained by differentiating the original one. At $t = 0$ this reduces to $x''' = \omega_0 F$, which is non-zero. Thus the leading term in the Taylor expansion of $x(t)$ is $F\omega_0 t^3/3!$.

From the explicit solution, the contributions to $2\omega_0 x(t)/F$ of the three terms, up to order t^3, are:

$$\begin{aligned} f_1 &= \frac{1}{\gamma}\left(1 - \frac{k^2t^2}{2!} + \cdots\right)\left(1 - \gamma t + \frac{\gamma^2t^2}{2!} - \frac{\gamma^3t^3}{3!} + \cdots\right) \\ &= \frac{1}{\gamma}\left(1 - \gamma t + \frac{(\gamma^2 - k^2)t^2}{2!} - \frac{(\gamma^3 - 3\gamma k^2)t^3}{3!} + \cdots\right), \end{aligned}$$

$$f_2 = -\frac{1}{\gamma}\left(1 - \frac{\omega_0^2t^2}{2!} + \cdots\right),$$

$$\begin{aligned} f_3 &= \frac{1}{k}\left(kt - \frac{k^3t^3}{3!} + \cdots\right)\left(1 - \gamma t + \frac{\gamma^2t^2}{2!} - \frac{\gamma^3t^3}{3!} + \cdots\right) \\ &= \frac{1}{k}\left(kt - k\gamma t^2 + \frac{(3k\gamma^2 - k^3)t^3}{3!} + \cdots\right). \end{aligned}$$

Recalling that $k^2 = \omega_0^2 - \gamma^2$, we see that, when these contributions are added together, the constant term and the linear and quadratic terms in t all vanish. The cubic term in $2\omega_0 x(t)/F$ is

$$-\frac{(\gamma^2 - 3k^2)t^3}{3!} + \frac{(3\gamma^2 - k^2)t^3}{3!} = \frac{2(\gamma^2 + k^2)t^3}{3!} = \frac{\omega_0^2t^3}{3},$$

and so the leading term in $x(t)$ is $F\omega_0 t^3/6$. This confirms our earlier conclusion based on the differential equation rather than its solution.

15.16 *In a particular scheme for modelling numerically one-dimensional fluid flow, the successive values, u_n, of the solution are connected for $n \geq 1$ by the difference equation*

$$c(u_{n+1} - u_{n-1}) = d(u_{n+1} - 2u_n + u_{n-1}),$$

where c and d are positive constants. The boundary conditions are $u_0 = 0$ and $u_M = 1$. Find the solution to the equation and show that successive values of u_n will have alternating signs if $c > d$.

We substitute the trial solution $u_n = A\lambda^n$ into the recurrence relation and obtain

$$\begin{aligned} -c(\lambda^{n+1} - \lambda^{n-1}) + d(\lambda^{n+1} - 2\lambda^n + \lambda^{n-1}) &= 0, \\ (d-c)\lambda^2 - 2d\lambda + (d+c) &= 0. \end{aligned}$$

This is a quadratic equation for λ, with solution

$$\lambda = \frac{d \pm \sqrt{d^2 - (d^2 - c^2)}}{d-c} = \frac{d \pm c}{d-c} = 1 \text{ or } \frac{d+c}{d-c}.$$

The general solution, formed by taking a linear superposition of the trial solutions corresponding to the allowed values of λ, is thus

$$u_n = A1^n + B\left(\frac{d+c}{d-c}\right)^n \equiv A + B\mu^n, \text{ defining } \mu.$$

Now, imposing the boundary conditions:

$$\begin{aligned} u_0 = 0 \quad &\Rightarrow \quad B = -A, \\ u_M = 1 \quad &\Rightarrow \quad A(1-\mu^M) = 1, \\ \Rightarrow \quad u_n \quad &= \quad \frac{1-\mu^n}{1-\mu^M}. \end{aligned}$$

This is the specific solution as a function of n.

If $c > d$ then μ is negative and has a magnitude > 1. The ratio of successive terms is

$$\frac{1-\mu^{n+1}}{1-\mu^n} = \frac{1}{(d-c)} \frac{[(d-c)^{n+1} - (d+c)^{n+1}]}{[(d-c)^n - (d+c)^n]}.$$

Since d and c are both positive, the terms in square brackets are necessarily both negative and the ratio has the same sign as $d - c$, i.e negative. Thus successive terms alternate in sign.

15.18 *Find an explicit expression for the u_n satisfying*

$$u_{n+1} + 5u_n + 6u_{n-1} = 2^n,$$

given that $u_0 = u_1 = 1$. Deduce that $2^n - 26(-3)^n$ is divisible by 5 for all integer n.

The characteristic equation of the recurrence relation, obtained by substituting $u_n = C\lambda^n$ into it with the RHS set equal to zero, is

$$\lambda^2 + 5\lambda + 6 = 0 \quad \Rightarrow \quad \lambda = -2 \text{ or } -3.$$

As neither value of λ is equal to 2, we may try $u_n = D\,2^n$ as a particular solution, leading to

$$D\,2^{n+1} + 5D\,2^n + 6D\,2^{n-1} = 2^n,$$
$$D(4 + 10 + 6) = 2, \quad \Rightarrow \quad D = \tfrac{1}{10}.$$

The general solution is thus

$$u_n = A(-2)^n + B(-3)^n + \frac{2^n}{10}.$$

Incorporating the two initial values:

$$u_0 = 1 \quad \Rightarrow \quad 1 = A + B + \tfrac{1}{10},$$
$$u_1 = 1 \quad \Rightarrow \quad 1 = -2A - 3B + \tfrac{2}{10},$$
$$\Rightarrow \quad A = \tfrac{35}{10} \text{ and } B = -\tfrac{26}{10}.$$

Thus, for general n,

$$u_n = \frac{1}{10}\left[35(-2)^n - 26(-3)^n + 2^n\right].$$

With these initial values and a recurrence relation that has integer coefficients (with that for the highest-index term equal to unity) all terms in the series *must* be integers. Thus, the expression in square brackets must divide by 10 for all $n \geq 2$, as well as for $n = 0$ and $n = 1$.

For $n > 0$, the first term in the bracket contains explicit factors of 2 and 5 and so divides by 10. We thus conclude that the sum of the remaining terms must also divide by 10, i.e. $2^n - 26(-3)^n$ divides by 10 and, therefore, also by 5. For $n = 0$, explicit evaluation of the expression gives -25, which is divisible by 5; this completes the proof.

15.20 *Consider the seventh-order recurrence relation*

$$u_{n+7} - u_{n+6} - u_{n+5} + u_{n+4} - u_{n+3} + u_{n+2} + u_{n+1} - u_n = 0.$$

Find the most general form of its solution, and show that:

(a) *if only the four initial values* $u_0 = 0$, $u_1 = 2$, $u_2 = 6$ *and* $u_3 = 12$, *are specified then the relation has one solution that cycles repeatedly through this set of four numbers*;
(b) *but if, in addition, it is required that* $u_4 = 20$, $u_5 = 30$ *and* $u_6 = 42$ *then the solution is unique, with* $u_n = n(n+1)$.

The characteristic equation is a seventh-order polynomial equation (but fortunately one with some obvious roots).

$$\begin{aligned} \lambda^7 - \lambda^6 - \lambda^5 + \lambda^4 - \lambda^3 + \lambda^2 + \lambda - 1 &= 0, \\ (\lambda - 1)(\lambda^6 - \lambda^4 - \lambda^2 + 1) &= 0, \\ (\lambda - 1)(\lambda^2 - 1)(\lambda^4 - 1) &= 0. \end{aligned}$$

The roots are therefore $\lambda = 1$ (triple), $\lambda = -1$ (double) and $\lambda = \pm i$. Consequently, the general solution is

$$(A + Bn + Cn^2)1^n + (D + En)(-1)^n + F(i)^n + G(-i)^n,$$

where the constants $A, B, \ldots, G$ must be consistent with any given values of particular u_n.

(a) If only the four initial values $u_0 = 0$, $u_1 = 2$, $u_2 = 6$ and $u_3 = 12$ are specified then we can choose all constants associated with linear or quadratic terms in n to be zero, i.e. $B = C = E = 0$ and solve for the remaining constants.

$$\begin{aligned} n = 0, &\qquad 0 = A + D + F + G, \\ n = 1, &\qquad 2 = A - D + iF - iG, \\ n = 2, &\qquad 6 = A + D - F - G, \\ n = 3, &\qquad 12 = A - D - iF + iG. \end{aligned}$$

Adding all the equations shows that $A = 5$, and adding the first and third shows that $A + D = 3$, i.e. $D = -2$. Putting these values into the first two equations then gives

$$F = -\tfrac{3}{2} + \tfrac{5}{2}i \quad \text{and} \quad G = -\tfrac{3}{2} - \tfrac{5}{2}i.$$

Thus the solution

$$u_n = 5 - 2(-1)^n - \tfrac{3}{2}(i)^n(1 + (-1)^n) + \tfrac{5}{2}(i)^{n+1}(1 - (-1)^n)$$

fits the first four given values and then cycles endlessly around them since $(-1)^n$, $(i)^n$ and $(-i)^n$ are all unchanged if n is increased by 4.

(b) With the first 7 values given and 7 unknown constants $A, B, \ldots, G$ to be determined, the solution will be unique (unless the determining equations turn out to be dependent). The simultaneous equations to be solved are:

$$\begin{aligned}
0 &= A + D + F + G,\\
2 &= A + B + C - D - E + iF - iG,\\
6 &= A + 2B + 4C + D + 2E - F - G,\\
12 &= A + 3B + 9C - D - 3E - iF + iG,\\
20 &= A + 4B + 16C + D + 4E + F + G,\\
30 &= A + 5B + 25C - D - 5E + iF - iG,\\
42 &= A + 6B + 36C + D + 6E - F - G.
\end{aligned}$$

It is clear from inspection and easily verified by substitution that they are satisfied by $B = C = 1$, with all other constants equal to zero. The direct solution of these equations, though tedious, gives the same result; it also provides assurance that the solution is unique. The general expression is therefore $u_n = n(n+1)$.

15.22 *Find the general solution of*

$$(x+1)^2\frac{d^2y}{dx^2} + 3(x+1)\frac{dy}{dx} + y = x^2.$$

This is Legendre's linear equation and, as a first step, we set $x + 1 = e^t$ with

$$\frac{dx}{dt} = e^t, \qquad \frac{d}{dx} = e^{-t}\frac{d}{dt}, \qquad \frac{d^2}{dx^2} = e^{-t}\frac{d}{dt}\left(e^{-t}\frac{d}{dt}\right).$$

These substitutions give

$$\begin{aligned}
e^{2t}e^{-t}\frac{d}{dt}\left(e^{-t}\frac{dy}{dt}\right) + 3e^te^{-t}\frac{dy}{dt} + y &= (e^t-1)^2,\\
e^t\left(e^{-t}\frac{d^2y}{dt^2} - e^{-t}\frac{dy}{dt}\right) + 3\frac{dy}{dt} + y &= (e^t-1)^2,\\
\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y &= (e^t-1)^2.
\end{aligned}$$

This reduced equation with constant coefficients has the characteristic equation $m^2+2m+1 = 0$, which has a repeated root and gives the CF as $y(t) = (A+Bt)e^{-t}$. This is not the same function as that in the equation's RHS (which contains a

constant and t-dependent terms e^{2t} and e^t); we may therefore try substituting the simplest PI of $Ce^{2t} + De^t + E$ to obtain

$$4Ce^{2t} + De^t + 4Ce^{2t} + 2De^t + Ce^{2t} + De^t + E = e^{2t} - 2e^t + 1.$$

Clearly $C = \frac{1}{9}$, $D = -\frac{1}{2}$ and $E = 1$ and, after re-substituting for t, we have the general solution of the original equation as

$$\begin{aligned} y(x) &= \frac{A + B\ln(x+1)}{x+1} + \frac{(x+1)^2}{9} - \frac{x+1}{2} + 1 \\ &= \frac{A + B\ln(x+1)}{x+1} + \frac{x^2}{9} - \frac{5x}{18} + \frac{11}{18}. \end{aligned}$$

As expected, since the differential equation is second-order, its solution contains two arbitrary constants.

15.24 *Use the method of variation of parameters to find the general solutions of*

(a) $\dfrac{d^2y}{dx^2} - y = x^n$, (b) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = 2xe^x$.

(a) The CF is clearly

$$y(x) = Ae^x + Be^{-x},$$

and so we take as the PI

$$y(x) = k_1(x)e^x + k_2(x)e^{-x}.$$

The two simultaneous equations generated using the method of variation of parameters are

$$\begin{aligned} k_1'e^x + k_2'e^{-x} &= 0, \\ k_1'e^x - k_2'e^{-x} &= x^n. \end{aligned}$$

Solving for k_1' and integrating gives

$$\begin{aligned} k_1' &= \frac{x^n e^{-x}}{2}, \\ k_1 &= \left[-\frac{x^n e^{-x}}{2}\right]^x + \int^x \frac{nx^{n-1}}{2} e^{-x}\,dx \\ &= -\frac{e^{-x}}{2}(x^n + nx^{n-1} + n(n-1)x^{n-2} + \cdots + n!) \\ &= -\frac{e^{-x}}{2} n! \sum_{m=0}^{n} \frac{x^m}{m!}. \end{aligned}$$

Similarly, k_2 is given by

$$\begin{aligned}
k_2' &= -\frac{x^n e^x}{2}, \\
k_2 &= -\left[\frac{x^n e^x}{2}\right]^x - \int^x \frac{nx^{n-1}}{2} e^x \, dx \\
&= -\frac{e^x}{2}(x^n - nx^{n-1} + n(n-1)x^{n-2} - \cdots + (-1)^n n!) \\
&= -\frac{e^x}{2} n!(-1)^n \sum_{m=0}^{n} \frac{(-x)^m}{m!}.
\end{aligned}$$

The full PI, $k_1(x)e^x + k_2(x)e^{-x}$, has no explicit exponential factors, since each term in it contains the product $e^x e^{-x}$. It takes the form

$$\begin{aligned}
y(x) &= -\frac{n!}{2} \sum_{m=0}^{n} \frac{x^m}{m!} - \frac{n!}{2}(-1)^n \sum_{m=0}^{n} \frac{(-x)^m}{m!} \\
&= -\frac{n!}{2} \sum_{m=0}^{n} \frac{x^m}{m!} [\, 1 + (-1)^{n+m} \,].
\end{aligned}$$

This n-th order polynomial is added to the CF, $y(x) = Ae^x + Be^{-x}$, to give the general solution.

(b) The auxiliary equation for the CF is $m^2 - 2m + 1 = 0$, which has repeated roots $m = 1$. Thus the Cf is $y(x) = (A + Bx)e^x$ and, since the RHS of the original equation is contained in this, the PI is to be taken as $y(x) = k_1(x)e^x + k_2(x)xe^x$. The simultaneous equations generated by the variation of parameters method are

$$\begin{aligned}
k_1' e^x + k_2' x e^x &= 0, \\
k_1' e^x + k_2'(e^x + xe^x) &= 2xe^x, \\
k_2' e^x &= 2xe^x \quad \Rightarrow \quad k_2(x) = x^2, \\
k_1' &= -k_2' x = -2x^2 \quad \Rightarrow \quad k_1(x) = -\tfrac{2}{3}x^3.
\end{aligned}$$

A PI is therefore

$$y(x) = -\tfrac{2}{3}x^3 e^x + x^2 \, xe^x = \tfrac{1}{3}x^3 e^x,$$

giving the general solution as

$$y(x) = (A + Bx + \tfrac{1}{3}x^3)e^x.$$

15.26 *Consider the equation*

$$F(x, y) = x(x+1)\frac{d^2y}{dx^2} + (2 - x^2)\frac{dy}{dx} - (2 + x)y = 0.$$

(a) *Given that* $y_1(x) = 1/x$ *is one of its solutions, find a second linearly independent one,*
  (i) *by setting* $y_2(x) = y_1(x)u(x)$, *and*
  (ii) *by noting the sum of the coefficients in the equation.*

(b) *Hence, using the variation of parameters method, find the general solution of*

$$F(x, y) = (x + 1)^2.$$

(a)(i) Set $y_2(x) = u(x)/x$ and substitute:

$$x(x+1)\left(\frac{2u}{x^3} - \frac{2u'}{x^2} + \frac{u''}{x}\right) + (2 - x^2)\left(-\frac{u}{x^2} + \frac{u'}{x}\right) - \frac{2+x}{x}u = 0,$$

$$(1 + x)u'' + \left[-\frac{2(1+x)}{x} + \frac{2 - x^2}{x}\right]u' + 0u = 0,$$

$$(1 + x)u'' - (2 + x)u' = 0.$$

Hence, on separating variables and integrating once, we have

$$\frac{u''}{u'} = \frac{2+x}{1+x} = 1 + \frac{1}{1+x},$$

$$\ln u' = x + \ln(1 + x),$$

$$u' = (1 + x)e^x.$$

A second integration then gives

$$u = e^x + [xe^x]^x - \int^x e^x\,dx,$$

$$= e^x + xe^x - e^x = xe^x,$$

i.e. $\quad y_2(x) = \dfrac{1}{x}xe^x = e^x$ is the second solution.

(a)(ii) The sum of the coefficients of the various terms in the linear equation is

$$x(x + 1) + (2 - x^2) - (2 + x) = 0.$$

It follows immediately (see subsection 15.3.6) that $y(x) = e^x$ is a solution of the equation, as we have already found.

(b) We already have the two independent solutions needed to form the CF, x^{-1} and e^x. So we take for the PI

$$y(x) = k_1(x)x^{-1} + k_2(x)e^x$$

and require that

$$\begin{aligned}
k_1'x^{-1} + k_2'e^x &= 0,\\
-k_1'x^{-2} + k_2'e^x &= \frac{(x+1)^2}{x(x+1)} = \frac{x+1}{x},\\
-k_1'\frac{1+x}{x^2} &= \frac{x+1}{x}, \qquad\qquad \text{by subtraction,}\\
\Rightarrow \quad k_1' &= -x \quad \Rightarrow \quad k_1(x) = -\tfrac{1}{2}x^2,\\
\Rightarrow \quad k_2' &= -\frac{e^{-x}(-x)}{x} = e^{-x} \quad \Rightarrow \quad k_2(x) = -e^{-x}.
\end{aligned}$$

The complete PI is thus

$$y(x) = -\frac{1}{2}\frac{x^2}{x} - e^{-x}e^x = -\frac{x}{2} - 1,$$

and the general solution of the inhomogeneous equation $F(x, y) = (x+1)^2$ is

$$y(x) = \frac{A}{x} + Be^x - \frac{x}{2} - 1,$$

for arbitrary constants A and B.

15.28 *Use the result of the previous exercise (15.27) to find the Green's function $G(x, \xi)$ that satisfies*

$$\frac{d^2G}{dx^2} + 3\frac{dG}{dx} + 2G = \delta(x - x),$$

in the interval $0 \le x, \xi \le 1$ with $G(0, \xi) = G(1, \xi) = 0$. Hence obtain integral expressions for the solution of

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \begin{cases} 0 & 0 < x < x_0, \\ 1 & x_0 < x < 1, \end{cases}$$

distinguishing between the cases (a) $x < x_0$, *and* (b) $x > x_0$.

The auxiliary equation is $m^2 + 3m + 2 = 0$ and two independent solutions are $y(x) = e^{-x}$ and $y(x) = e^{-2x}$. We need linear combinations of these that satisfy $y_1(0) = 0$ and $y_2(1) = 0$. The former is clearly satisfied by taking $y_1(x) = e^{-x} - e^{-2x}$. For the latter, take

$$y_2(x) = e^{-x} + \alpha e^{-2x} \text{ and require } y_2(1) = 0 \quad \Rightarrow \quad \alpha = -e.$$

Thus $y_2(x) = e^{-x} - e^{-2x+1}$ is the appropriate linear combination for the region containing $x = 1$.

The Wronskian of these two functions is

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= (e^{-x} - e^{-2x})(-e^{-x} + 2e^{-2x+1}) \\ &\qquad - (e^{-x} - e^{-2x+1})(-e^{-x} + 2e^{-2x}) \\ &= e^{-3x+1} - e^{-3x} \\ &= (e-1)e^{-3x}. \end{aligned}$$

Hence, using the result from the previous question in the main text, the Green's function is

$$G(x,\xi) = \begin{cases} G_1(x,\xi) = \dfrac{(e^{-x} - e^{-2x})(e^{-\xi} - e^{-2\xi+1})}{(e-1)e^{-3\xi}} & 0 < x < \xi \\ G_2(x,\xi) = \dfrac{(e^{-x} - e^{-2x+1})(e^{-\xi} - e^{-2\xi})}{(e-1)e^{-3\xi}} & \xi < x < 1. \end{cases}$$

Now, in general, the solution of the given equation with its RHS equal to $f(x)$ is

$$y(x) = \int_0^1 G(x,\xi) f(\xi)\, d\xi = \int_{x_0}^1 G(x,\xi)\, d\xi.$$

The second equality follows because $f(\xi) = 0$ for $\xi < x_0$.

(a) For $x < x_0$, the variable of integration ξ is greater than x throughout the integration range $x_0 \le \xi \le 1$ and so $G = G_1$ throughout, i.e.

$$y(x) = \int_{x_0}^1 G_1(x,\xi)\, d\xi.$$

(b) However, for $x > x_0$ the integral is divided into two parts. For $x_0 < \xi < x$, G_2 is the appropriate Green's function, whilst for the remainder of the integral range, $x < \xi < 1$, G_1 must be used. Thus

$$y(x) = \int_{x_0}^x G_2(x,\xi)\, d\xi + \int_x^1 G_1(x,\xi)\, d\xi.$$

15.30 *Show that the Green's function for the equation*

$$\frac{d^2y}{dx^2} + \frac{y}{4} = f(x),$$

subject to the boundary conditions $y(0) = y(\pi) = 0$, *is given by*

$$G(x,z) = \begin{cases} -2\cos\frac{1}{2}x \sin\frac{1}{2}z & 0 \le z \le x, \\ -2\sin\frac{1}{2}x \cos\frac{1}{2}z & x \le z \le \pi. \end{cases}$$

This result could be written down almost immediately using the general result of exercise 15.27. It could also be derived more specifically using the continuity/discontinuity conditions at $x = z$. By way of illustration, we will use yet another approach, based on the variation of parameters.

It is clear that the CF of the given equation is $y(x) = A\sin(x/2) + B\cos(x/2)$. So we take $y(x) = A(x)\sin(x/2) + B(x)\cos(x/2)$, with $A(\pi) = 0$ and $B(0) = 0$, and impose the usual constraints of the method, namely

$$A'\sin\frac{x}{2} + B'\cos\frac{x}{2} = 0,$$
$$\frac{1}{2}A'\cos\frac{x}{2} - \frac{1}{2}B'\sin\frac{x}{2} = f(x).$$

These simultaneous equations have the solution

$$A' = 2f(x)\cos\frac{x}{2} \quad \text{and} \quad B' = -2f(x)\sin\frac{x}{2}.$$

Integrating these two differential equations, incorporating the boundary values of A and B given above, and resubstituting in the assumed form of solution gives

$$y(x) = -\cos\frac{x}{2}\int_0^x 2f(z)\sin\frac{z}{2}\,dz - \sin\frac{x}{2}\int_x^\pi 2f(z)\cos\frac{z}{2}\,dz.$$

Note that the expression for $A(x)$ has to be obtained by integrating from π, where its value is known, to x.

From this integral form of solution the Green's function can be read off as

$$G(x,z) = \begin{cases} -2\cos\frac{1}{2}x\sin\frac{1}{2}z & 0 \le z \le x, \\ -2\sin\frac{1}{2}x\cos\frac{1}{2}z & x \le z \le \pi. \end{cases}$$

15.32 *Consider the equation*

$$\frac{d^2y}{dx^2} + f(y) = 0,$$

where $f(y)$ can be any function.

(a) *By multiplying through by dy/dx, obtain the general solution relating x and y.*

(b) *A mass m, initially at rest at the point $x = 0$, is accelerated by a force*

$$f(x) = A(x_0 - x)\left[1 + 2\ln\left(1 - \frac{x}{x_0}\right)\right].$$

Its equation of motion is $m\,d^2x/dt^2 = f(x)$. Find x as a function of time and show that ultimately the particle has travelled a distance x_0.

(a) We multiply the equation through by y' and integrate twice obtaining

$$\begin{aligned} y'y'' &= -fy', \\ \frac{1}{2}(y')^2 &= -\int f(y)\,\frac{dy}{dx}\,dx + A, \\ \frac{dy}{dx} &= \left(B - 2\int^y f(u)\,du\right)^{1/2}, \\ \int^x dx &= \int^y \frac{dz}{\left(B - 2\int^z f(u)\,du\right)^{1/2}}, \\ C + x &= \int^y \frac{dz}{\left(B - 2\int^z f(u)\,du\right)^{1/2}}. \end{aligned}$$

This is the general equation relating y to x, whatever the form of the function f.

(b) Here x is the dependent variable, the independent one being t.

To use the result of part (a) we first need to evaluate

$$\int A(x_0 - u)\left[1 + 2\ln\left(1 - \frac{u}{x_0}\right)\right]\,du.$$

Either by observation or by integration by parts the integrand is the derivative of $-A(x_0 - x)^2 \ln[(x_0 - x)/x_0]$. The force is related to the 'f' of part (a) by a factor of $-1/m$. Consequently the solution given in part (a) becomes

$$C + t = \int^x \frac{dz}{\left(B - \frac{2A}{m}(x_0 - z)^2 \ln[(x_0 - z)/x_0]\right)^{1/2}}$$

Referring back to the derivation in part (a), we see that if $y' = 0$ when $y = 0$ then $A = B = 0$; the corresponding situation, $\dot{x} = 0$ when $x = 0$, holds good here and B in the denominator of the integrand is zero. More obviously, since $x = 0$ when $t = 0$ we must have $C = 0$. Thus

$$t = \int^x \frac{dz}{\left(\frac{2A}{m}(x_0 - z)^2 \ln[x_0/(x_0 - z)]\right)^{1/2}}.$$

Note that we have inverted the argument of the logarithm.

All that remains is to evaluate the integral.

$$\begin{aligned}
\sqrt{\frac{2A}{m}}t &= \int^{x} \frac{dz}{(x_0 - z)\sqrt{\ln x_0 - \ln(x_0 - z)}} \\
&= \left[2\sqrt{\ln x_0 - \ln(x_0 - z)} \right]_0^x, \\
&= 2\sqrt{\ln x_0 - \ln(x_0 - x)}, \\
\frac{2At^2}{m} &= 4[\ln x_0 - \ln(x_0 - x)], \\
\frac{x_0 - x}{x_0} &= e^{-At^2/2m}, \\
x &= x_0 \left(1 - e^{-At^2/2m}\right).
\end{aligned}$$

Clearly as $t \to \infty$ the value of x approaches x_0.

15.34 *Find the general solution of the equation*

$$x\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} = Ax.$$

This third-order equation is one in which y does not appear and so we set $dy/dx = p$ and rewrite the equation as one of second order.

$$\begin{aligned}
x\frac{d^2p}{dx^2} + 2\frac{dp}{dx} &= Ax, \\
x^2\frac{d^2p}{dx^2} + 2x\frac{dp}{dx} &= Ax^2, \qquad \text{using the obvious IF,} \\
\frac{d}{dx}\left(x^2\frac{dp}{dx}\right) &= Ax^2.
\end{aligned}$$

Successive integrations then give

$$\begin{aligned}
x^2\frac{dp}{dx} &= \frac{Ax^3}{3} + B, \\
\frac{dp}{dx} &= \frac{Ax}{3} + \frac{B}{x^2}, \\
p &= \frac{Ax^2}{6} - \frac{B}{x} + C, \\
y &= \frac{Ax^3}{18} - B\ln x + Cx + D.
\end{aligned}$$

Recall that A is given in the question; there are only three arbitray constants, B, C and D, as is to be expected for the solution of a third-order equation.

15.36 *Find the form of the solutions of the equation*

$$\frac{dy}{dx}\frac{d^3y}{dx^3} - 2\left(\frac{d^2y}{dx^2}\right)^2 + \left(\frac{dy}{dx}\right)^2 = 0$$

that have $y(0) = \infty$.

[*You will need the result* $\int^z \operatorname{cosech} u\,du = -\ln(\operatorname{cosech} z + \coth z)$.]

Since y does not appear in the equation (the same is true of x) we set $dy/dx = p$ and reformulate it. The required derivatives are

$$\frac{d^2y}{dx^2} = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

$$\frac{d^3y}{dx^3} = \frac{dy}{dx}\frac{d}{dy}\left(p\frac{dp}{dy}\right) = p\left(\frac{dp}{dy}\right)^2 + p^2\frac{d^2p}{dy^2},$$

and the re-formulated equation is

$$0 = p^2\left(\frac{dp}{dy}\right)^2 + p^3\frac{d^2p}{dy^2} - 2p^2\left(\frac{dp}{dy}\right)^2 + p^2,$$

$$0 = p\frac{d^2p}{dy^2} - \left(\frac{dp}{dy}\right)^2 + 1.$$

This non-linear equation can be simplified by setting $dp/dy = q$ with, in the same way as above, $d^2p/dy^2 = q\,dq/dp$.

$$p\,q\frac{dq}{dp} - q^2 + 1 = 0,$$

$$\frac{q\,dq}{q^2-1} = \frac{dp}{p}, \qquad \text{separating variables,}$$

$$\Rightarrow \quad \tfrac{1}{2}\ln(q^2-1) = \ln p + A,$$

$$q^2 - 1 = B^2p^2.$$

Now set $Bp = \sinh\theta$, with

$$q = \frac{dp}{dy} = \frac{\cosh\theta}{B}\frac{d\theta}{dy},$$

to obtain

$$\frac{\cosh^2\theta}{B^2}\left(\frac{d\theta}{dy}\right)^2 - 1 = \sinh^2\theta,$$

$$\frac{\cosh^2\theta}{B^2}\left(\frac{d\theta}{dy}\right)^2 = \cosh^2\theta,$$

$$\Rightarrow \quad \theta = By + C.$$

Now,

$$B\frac{dy}{dx} = Bp = \sinh(By + C),$$

$$dx = \frac{B\,dy}{\sinh(By + C)}, \qquad \text{separating variables,}$$

$$\Rightarrow \quad x + c = -\ln[\operatorname{cosech}(By + C) + \coth(By + C)].$$

Since $y(0) = \infty$,

$$0 + c = -\ln[0 + 1] \quad \Rightarrow \quad c = 0$$

and the final answer is

$$\operatorname{cosech}(By + C) + \coth(By + C) = e^{-x}.$$

16

Series solutions of ordinary differential equations

16.2 *Find solutions, as power series in z, of the equation*

$$4zy'' + 2(1-z)y' - y = 0.$$

Identify one of the solutions and verify it by direct substitution.

Putting the equation in its standard form shows that $z = 0$ is a singular point of the equation but, as $2z(1-z)/4z$ and $-z^2/4z$ are finite as $z \to 0$, it is a regular singular point. We therefore substitute a Frobenius type solution,

$$y(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n \text{ with } a_0 \neq 0,$$

and obtain

$$4\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1} + 2(1-z)\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1} - \sum_{n=0}^{\infty} a_n z^{n+\sigma} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero gives the indicial equation as

$$4\sigma(\sigma-1)a_0 + 2\sigma a_0 = 0 \quad \Rightarrow \quad \sigma = 0,\ \tfrac{1}{2}.$$

These two indicial values do not differ by an integer and so we expect both to yield (independent) series solutions.

(a) $\sigma = 0$.

Equating the general coefficient of z^m to zero (with $\sigma = 0$),

$$4(m+1)ma_{m+1} + 2(m+1)a_{m+1} - 2ma_m - a_m = 0,$$

gives the recurrence relation as

$$a_{m+1} = \frac{2m+1}{2(m+1)(2m+1)} = \frac{a_m}{2(m+1)},$$
$$\Rightarrow \quad a_m = \frac{a_0}{2^m\, m!},$$
$$\Rightarrow \quad y(x) = a_0 z^0 \sum_{n=0}^{\infty} \frac{z^n}{2^n\, n!} = a_0 e^{z/2}.$$

Substituting this into the original equation yields

$$4a_0 z \frac{e^{z/2}}{4} + 2a_0(1-z)\frac{e^{z/2}}{2} - a_0 e^{z/2} = 0,$$

which is a valid equation and thus verifies the solution.

(b) $\sigma = \frac{1}{2}$

Equating the general coefficient of $z^{m+\frac{1}{2}}$ to zero (with $\sigma = \frac{1}{2}$),

$$4(m+\tfrac{3}{2})(m+\tfrac{1}{2})a_{m+1} + 2(m+\tfrac{3}{2})a_{m+1} - 2(m+\tfrac{1}{2})a_m - a_m = 0,$$

gives the recurrence relation as

$$a_{m+1} = \frac{2m+2}{(2m+3)(2m+2)}\, a_m = \frac{a_m}{2m+3},$$
$$\Rightarrow \quad a_m = \frac{a_0}{1\,3\,5\cdots(2m+1)} = \frac{2^m\, m!\, a_0}{(2m+1)!},$$

and the second series solution as

$$y(x) = a_0 \sum_{n=0}^{\infty} \frac{2^n\, n!\, z^{n+\frac{1}{2}}}{(2n+1)!}.$$

16.4 *Change the independent variable in the equation*

$$\frac{d^2 f}{dz^2} + 2(z-\alpha)\frac{df}{dz} + 4f = 0 \qquad (*)$$

from z to $x = z - \alpha$, and find two independent series solutions, expanded about $x = 0$, of the resulting equation. Deduce that the general solution of $(*)$ is

$$f(z,\alpha) = A(z-\alpha)e^{-(z-\alpha)^2} + B\sum_{m=0}^{\infty} \frac{(-4)^m m!}{(2m)!}(z-\alpha)^{2m},$$

with A and B arbitrary constants.

We start with $f'' + 2(z-\alpha)f' + 4f = 0$ and set $f(z) = g(x)$, where $x = z - \alpha$.

The differential operators are unchanged, i.e. $d/dx = d/dz$ and $d^2/dx^2 = d^2/dz^2$. The resulting equation for $g(x)$ is $g'' + 2xg' + 4g = 0$, where now a prime denotes differentiation with respect to x.

The origin $x = 0$ is an ordinary point of this equation and we can write its two solutions as ordinary power series (formally, its indicial roots are 0 and 1),

$$g(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Making this substitution yields

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^n + 4\sum_{n=0}^{\infty} a_n x^n = 0.$$

Equating the coefficient of x^m to zero then gives

$$a_{m+2} = -\frac{2m+4}{(m+2)(m+1)}\, a_m = -\frac{2}{m+1}\, a_m,$$

or, on re-indexing,

$$a_m = -\frac{2}{m-1} a_{m-2}.$$

Since this recurrence relation relates indices differing by 2, we can (as expected) obtain two linearly independent solutions by taking (i) $a_0 = 1$, $a_1 = 0$ and (ii) $a_0 = 0$, $a_1 = 0$.

(i) $a_0 = 1$, $a_1 = 0$

Here, only even powers of x appear and we can write $m = 2p$. The expansion coefficients are then given by

$$a_{2p} = -\frac{2}{2p-1}\, a_{2p-2} = \frac{(-1)^p\, 2^p\, 2^p\, p\,!}{(2p)!}\, a_0,$$

and one solution of the original equation is

$$f_0(z) = g_0(x) = a_0 \sum_{n=0}^{\infty} \frac{(-4)^n\, n!}{(2n)!} x^{2n} = a_0 \sum_{n=0}^{\infty} \frac{(-4)^n\, n!}{(2n)!} (z-\alpha)^{2n}.$$

(ii) $a_0 = 0$, $a_1 = 1$

Here, only odd powers of x appear and we can write $m = 2p + 1$. The expansion coefficients are then given by

$$a_{2p+1} = -\frac{2}{2p}\, a_{2p-1} = \frac{(-1)^p}{p\,!}\, a_1,$$

and a second solution of the original equation is

$$\begin{aligned} f_1(z) = g_1(x) &= a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{2n+1} \\ &= a_1 \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (z-\alpha)^{2n+1} \\ &= a_1 (z-\alpha) e^{-(z-\alpha)^2}. \end{aligned}$$

The general solution of (∗) is any linear combination of $f_0(z)$ and $f_1(z)$, as stated.

16.6 *Verify that $z = 0$ is a regular singular point of the equation*

$$z^2 y'' - \tfrac{3}{2} z y' + (1+z) y = 0,$$

and that the indicial equation has roots 2 and 1/2. Show that the general solution is

$$\begin{aligned} y(z) = 6a_0 z^2 &\sum_{n=0}^{\infty} \frac{(-1)^n (n+1) 2^{2n} z^n}{(2n+3)!} \\ &+ b_0 \left(z^{1/2} + 2z^{3/2} - \frac{z^{1/2}}{4} \sum_{n=2}^{\infty} \frac{(-1)^n 2^{2n} z^n}{n(n-1)(2n-3)!} \right). \end{aligned}$$

In standard form the equation is

$$y'' - \frac{3}{2z} y' + \frac{1+z}{z^2} = 0.$$

Now,

$$-\frac{3z}{2z} \to -\frac{3}{2} \text{ and } \frac{z^2(1+z)}{z^2} \to 1 \text{ as } z \to 0.$$

Both limits are finite and so $z = 0$ is a regular singular point of the equation. Substituting a Frobenius solution gives

$$\sum_{n=0}^{\infty} (\sigma+n)(\sigma+n-1) a_n z^{\sigma+n} - \frac{3}{2} \sum_{n=0}^{\infty} (\sigma+n) a_n z^{\sigma+n} + (1+z) \sum_{n=0}^{\infty} a_n z^{\sigma+n} = 0.$$

Equating the coefficient of z^σ to zero shows that

$$\begin{aligned} [\,\sigma(\sigma-1) - \tfrac{3}{2}\sigma + 1\,] a_0 &= 0, \\ \sigma^2 - \tfrac{5}{2}\sigma + 1 = 0 \quad &\Rightarrow \quad \sigma = 2,\ \tfrac{1}{2}. \end{aligned}$$

These roots do not differ by an integer and so we expect two linearly independent power series solutions.

(a) Setting $\sigma = 2$ and equating the coefficient of z^{m+2} to zero gives

$$\begin{aligned}(2+m)(1+m)a_m - \tfrac{3}{2}(2+m)a_m + a_m + a_{m-1} &= 0,\\ \Rightarrow \quad a_m(2+3m+m^2-3-\tfrac{3}{2}m+1) &= -a_{m-1},\\ \Rightarrow \quad a_m[\,m(\tfrac{3}{2}+m)\,] &= -a_{m-1}.\end{aligned}$$

Repeated application of this relation gives

$$\begin{aligned}a_m = -\frac{2}{m(2m+3)}\,a_{m-1} &= \frac{(-1)^m\,2^m}{m!\,[\,(2m+3)(2m+1)\cdots 5\,]}\,a_0\\ &= \frac{(-1)^m\,2^m\,3\,2^{m+1}\,(m+1)!}{m!\,(2m+3)!}\,a_0\\ &= \frac{6(-1)^m\,2^{2m}\,(m+1)}{(2m+3)!}\,a_0,\end{aligned}$$

$$\text{leading to } y_1(z) = Az^2\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}\,(n+1)\,z^n}{(2n+3)!}.$$

(b) Setting $\sigma = \frac{1}{2}$ and following the same procedure (for $z^{m+\frac{1}{2}}$) yields

$$\begin{aligned}(\tfrac{1}{2}+m)(-\tfrac{1}{2}+m)a_m - \tfrac{3}{2}(\tfrac{1}{2}+m)a_m + a_m + a_{m-1} &= 0,\\ a_m(4m^2-1-3-6m+4) &= -4a_{m-1},\end{aligned}$$

which reduces to

$$a_m = -\frac{2}{m(2m-3)}\,a_{m-1}.$$

In particular, for $m = 1$,

$$a_1 = -\frac{2}{1(-1)}\,a_0 = 2a_0.$$

For $m \geq 2$,

$$a_m = \frac{(-1)^{m-1}\,2^{m-1}\,2^{m-2}\,(m-2)!}{m!\,(2m-3)!}\,a_1 = \frac{(-1)^{m-1}\,2^{2m}\,2a_0}{8m(m-1)\,(2m-3)!}$$

and the full expression for the second independent solution is

$$y_2(z) = Bz^{1/2}\left(1 + 2z - \frac{1}{4}\sum_{n=0}^{\infty}\frac{(-1)^n\,2^{2n}\,z^n}{n(n-1)\,(2n-3)!}\right).$$

As stated in the question, the general solution is $y_1(z) + y_2(z)$ for arbitrary values of A and B.

16.8 *Consider a series solution of the equation*

$$zy'' - 2y' + yz = 0 \qquad (*)$$

about its regular singular point.

(a) *Show that its indicial equation has roots that differ by an integer but that the two roots nevertheless generate linearly independent solutions*

$$y_1(z) = 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\ 2nz^{2n+1}}{(2n+1)!},$$

$$y_2(z) = a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}(2n-1)z^{2n}}{(2n)!}.$$

(b) *Show that $y_1(z)$ is equal to $3a_0(\sin z - z\cos z)$ by expanding the sinusoidal functions. Then, using the Wronskian method, find an expression for $y_2(z)$ in terms of sinusoids. You will need to write z^2 as $(z/\sin z)(z\sin z)$ and integrate by parts to evaluate the integral involved.*

(c) *Confirm that the two solutions are linearly independent by showing that their Wronskian is equal to $-z^2$, as would be expected from the form of $(*)$.*

(a) Substituting a Frobenius solution gives

$$\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n z^{\sigma+n-1} - 2\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1} + \sum_{n=0}^{\infty} a_n z^{\sigma+n+1} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero shows that

$$[\,\sigma(\sigma-1) - 2\sigma\,]a_0 = 0, \quad \Rightarrow \quad \sigma = 0 \text{ or } 3.$$

Setting the coefficient of $z^{m+\sigma-1}$ to zero now gives the required recurrence relation:

$$(\sigma+m)(\sigma+m-1)a_m - 2(\sigma+m)a_m + a_{m-2} = 0,$$
$$(\sigma+m)(\sigma+m-3)a_m + a_{m-2} = 0,$$

i.e.

$$a_m = \frac{-1}{(\sigma+m)(\sigma+m-3)}\, a_{m-2}.$$

For $\sigma = 3$, $a_m = -a_{m-2}/[\,m(m+3)\,]$ and

$$y_1(z) = a_0 z^3 \left(1 - \frac{z^2}{(2)(5)} + \frac{z^4}{(2)(5)(4)(7)} - \cdots\right)$$
$$= 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, 2n\, z^{2n+1}}{(2n+1)!}.$$

For $\sigma = 0$, $a_m = -a_{m-2}/[\,m(m-3)\,]$ and

$$\begin{aligned} y_2(z) &= a_0 z^0 \left(1 + \frac{z^2}{2} - \frac{z^4}{(2)(4)(1)} + \cdots\right) \\ &= a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,(2n-1)\,z^{2n}}{(2n)!}. \end{aligned}$$

Thus, although the indicial roots, 0 and 3, differ by an integer, two linearly independent series solutions are produced.

(b) We write the given function in series form:

$$\begin{aligned} 3a_0(\sin z - z\cos z) &= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!} - 3a_0 z \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \\ &= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^n\,(1-2n-1)\,z^{2n+1}}{(2n+1)!} \\ &= 3a_0 \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2n\,z^{2n+1}}{(2n+1)!} \\ &= 3a_0 \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\,2n\,z^{2n+1}}{(2n+1)!} = y_1(z). \end{aligned}$$

This confirms that the the first series is a multiple of $\sin z - z\cos z$ and the Wronskian method now gives the second solution as

$$\begin{aligned} y_2(z) &= y_1(z) \int^z \frac{1}{y_1^2(u)} \exp\left\{-\int^u \left(\frac{-2}{v}\right) dv\right\} du \\ &= y_1(z) \int^z \frac{1}{y_1^2(u)} \exp(2\ln u)\, du \\ &= y_1(z) \int^z \frac{u^2}{(\sin u - u\cos u)^2}\, du \\ &= y_1(z) \int^z \frac{u}{\sin u}\, \frac{u\sin u}{(\sin u - u\cos u)^2}\, du \\ &= y_1(z) \left\{ \left[\frac{u}{\sin u}\, \frac{-1}{(\sin u - u\cos u)} \right]^z \right. \\ &\qquad \left. + \int^z \left(\frac{1}{\sin u} - \frac{u\cos u}{\sin^2 u}\right) \left(\frac{1}{\sin u - u\cos u}\right) du \right\} \\ &= -\frac{z}{\sin z} + y_1(z) \int^z \frac{1}{\sin^2 u}\, du \\ &= -\frac{z}{\sin z} + (\sin z - z\cos z)(-\cot z) \\ &= \frac{-z - \sin z\cos z + z\cos^2 z}{\sin z} \\ &= -z\sin z - \cos z. \end{aligned}$$

Comparing the constant terms (i.e. the z^0 term) in this solution and the series obtained earlier, shows that this sinusoidal form is the negative of the series solution. Thus the series $y_2(z) = \cos z + z \sin z$.

(c) From the way the sinusoidal form was calculated, the solutions must satisfy the Wronskian condition, but an explicit verification is as follows.

$$\begin{aligned} W(y_1, y_2) &= (\sin z - z\cos z)(-\sin z + \sin z + z\cos z) \\ &\quad - (\cos z + z\sin z)(\cos z - \cos z + z\sin z) \\ &= z\cos z\sin z - z^2\cos^2 z - z\sin z\cos z - z^2\sin^2 z \\ &= -z^2. \end{aligned}$$

This is in accord with what is expected from (*), where the factor multiplying y' is $-2/z$ when the equation is put in standard form. Explicitly,

$$W(y_1, y_2) = \exp\left\{-\int^z \left(\frac{-2}{v}\right) dv\right\} = z^2,$$

an arbitrary constant multiplying factor being irrelevant.

16.10 *Solve the equation*

$$z(1-z)\frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0,$$

as follows.

(a) *Identify and classify its singular points and determine their indices.*
(b) *Find one series solution in powers of z. Give a formal expression for a second linearly independent solution.*
(c) *Deduce the values of λ for which there is a polynomial solution $P_N(z)$ of degree N. Evaluate the first four polynomials, normalised in such a way that $P_N(0) = 1$.*

(a) In standard form the equation reads

$$y'' + \frac{1}{z}y' + \frac{\lambda y}{z(1-z)} = 0.$$

By inspection, there are singular points at $z = 0$ and $z = 1$. However, the denominators of the functions that make the points singular have only first-order zeros; consequently, the points are both regular singular points.

Substituting a Frobenius solution about $z = 0$ gives

$$z(1-z)\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n z^{\sigma+n-2} + (1-z)\sum_{n=0}^{\infty}(\sigma+n)a_n z^{\sigma+n-1} + \lambda\sum_{n=0}^{\infty}a_n z^{\sigma+n} = 0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero shows that

$$[\,\sigma(\sigma-1)+1\sigma\,]a_0 = 0, \quad \Rightarrow \quad \sigma = 0 \text{ (repeated root).}$$

For the point $z = 1$, we set $z - 1 = u$ with $f(u) = y(z)$ and obtain

$$(u+1)(-u)f'' + (-u)f' + \lambda f = 0.$$

If a Frobenius solution about $u = 0$ is substituted, the lowest power of u present will be $u^{\sigma-1}$; it will arise only from the first term and have coefficient $-\sigma(\sigma-1)a_0$. For this to be zero requires that $\sigma = 0$ or $\sigma = 1$; these are the indices of the point $u = 0$, i.e. of the point $z = 1$.

(b) For the solution about $z = 0$ we have a repeated indicial root $\sigma = 0$. We therefore need to use the derivative method to obtain two solutions. The recurrence relation generated by setting the coefficient of $z^{m+\sigma}$ to zero in the equation in part (a) is

$$(m+1+\sigma)(m+\sigma)a_{m+1} - (m+\sigma)(m+\sigma-1)a_m + (m+1+\sigma)a_{m+1} - (m+\sigma)a_m + \lambda a_m = 0,$$

$$\Rightarrow \quad a_{m+1} = \frac{(m+\sigma)^2 - \lambda}{(m+1+\sigma)^2}\,a_m.$$

The first solution is obtained by setting σ equal to 0 in this relation to yield

$$a_m = \frac{\prod_{r=0}^{m-1}(r^2-\lambda)}{(m!)^2}\,a_0$$

and

$$y_1(z) = a_0 + a_0\sum_{n=1}^{\infty}\frac{\prod_{r=0}^{n-1}(r^2-\lambda)}{(n!)^2}\,z^n.$$

The second solution $y_2(z)$ is obtained by differentiating the general solution with

respect to σ before setting $\sigma = 0$.

$$\frac{\partial y}{\partial \sigma} = \frac{d(z^\sigma)}{d\sigma}\sum_{n=0}^{\infty} a_n(\sigma)z^n + z^\sigma \sum_{n=1}^{\infty} \frac{da_n(\sigma)}{d\sigma} z^n,$$

$$y_2(z) = z^0 \ln z \sum_{n=0}^{\infty} a_n(0)z^n + z^0 \sum_{n=1}^{\infty} \frac{da_n(0)}{d\sigma} z^n$$

$$= y_1(z)\ln z + \sum_{n=1}^{\infty} \frac{\partial}{\partial \sigma} \prod_{r=0}^{n-1} \frac{(r+\sigma)^2 - \lambda}{(r+1+\sigma)^2}\bigg|_{\sigma=0} z^n.$$

(c) From the recurrence relation it is clear that the condition for the series solution to terminate is that $\lambda = (m+\sigma)^2$ for some integer m. Since $\sigma = 0$, this means that $\lambda = N^2$ for a polynomial of degree N. The first four polynomials, constucted using the recurrence relation, are

$$\begin{aligned}
\lambda = 0, \qquad & P_0(z) = 1, \\
\lambda = 1, \qquad & P_1(z) = 1 + \frac{-1}{1^2}z = 1 - z, \\
\lambda = 4, \qquad & P_2(z) = 1 + \frac{-4}{1^2}z + \frac{(-3)(-4)}{2^2}z^2 = 1 - 4z + 3z^2, \\
\lambda = 9, \qquad & P_3(z) = 1 + \frac{-9}{1^2}z + \frac{(-8)(-9)}{2^2}z^2 + \frac{(-5)(-8)(-9)}{6^2}z^3 \\
& \qquad = 1 - 9z + 18z^2 - 10z^3.
\end{aligned}$$

By choosing $a_0 = 1$ in each case, we have ensured that $P_N(0) = 1$ for all N.

16.12 *Find the radius of convergence of a series solution about the origin for the equation* $(z^2 + az + b)y'' + 2y = 0$ *in the following cases:*

$$\text{(a) } a = 5,\ b = 6; \qquad \text{(b) } a = 5,\ b = 7.$$

Show that if a and b are real and $4b > a^2$ *then the radius of convergence is always given by* $b^{1/2}$.

The two roots of $z^2 + az + b = 0$ give the singular points, z_1 and z_2, of the equation. The radius of convergence R of the series solution about the origin is equal to the smaller of their two moduli.

If $4b > a^2$ then the roots are necessarily complex conjugates and

$$R^2 = |z_1|^2 = |z_2|^2 = \left(\frac{-a}{2}\right)^2 + \left(\frac{\sqrt{4b - a^2}}{2}\right)^2 = b \quad \Rightarrow \quad R = \sqrt{b}.$$

This is case (b), for which therefore $R = b = 7$.

If $a^2 > 4b$ the roots are real and the smaller of their two magnitudes gives the value of R. In case (a) the roots are $\frac{1}{2}(-5 \pm \sqrt{25-24}) = -2$ or -3, implying that $R = 2$.

16.14 *Prove that the Laguerre equation*

$$z\frac{d^2y}{dz^2} + (1-z)\frac{dy}{dz} + \lambda y = 0$$

has polynomial solutions $L_N(z)$ if λ is a non-negative integer N, and determine the recurrence relationship for the polynomial coefficients. Hence show that an expression for $L_N(z)$, normalised in such a way that $L_N(0) = N!$, is

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2}{(N-n)!(n!)^2} z^n.$$

Evaluate $L_3(z)$ explicitly.

We assume that there is a polynomial solution $L_N(z) = \sum_{n=0}^{N} a_n z^n$ with $a_N \neq 0$ and substitute this form into the differential equation:

$$z\sum_{n=0}^{N} n(n-1)a_n z^{n-2} + (1-z)\sum_{n=0}^{N} n a_n z^{n-1} + \lambda \sum_{n=0}^{N} a_n z^n = 0.$$

Consideration of the coefficient of z^N shows that we require $\lambda = N$.

The recurrence relation comes from equating the coefficient of z^{m-1} to zero:

$$m(m-1)a_m + m a_m - (m-1)a_{m-1} + N a_{m-1} = 0,$$

$$a_m = \frac{m-1-N}{m^2} a_{m-1} = \frac{(-1)^n N!\, a_0}{(N-n)!\,(n!)^2} = \frac{(-1)^n (N!)^2}{(N-n)!\,(n!)^2},$$

where, in the last step, we have used the requirement that $a_0 = L_N(0) = N!$.

Hence

$$L_N(z) = \sum_{n=0}^{N} \frac{(-1)^n (N!)^2 z^n}{(N-n)!\,(n!)^2}.$$

Explicitly, for $N = 3$,

$$L_3(z) = 3! - \frac{6^2}{2!\,1^2} z + \frac{6^2}{1!\,2^2} z^2 - \frac{6^2}{0!\,6^2} z^3 = 6 - 18z + 9z^2 - z^3.$$

Essentially the same proof, but with a different normalisation of the polynomials, is given in the main text in section 18.7.

16.16 *Obtain the recurrence relations for the solution of Legendre's equation*

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0.$$

in inverse *powers of z, i.e. set $y(z) = \sum a_n z^{\sigma-n}$, with $a_0 \neq 0$. Deduce that if ℓ is an integer then the series with $\sigma = \ell$ will terminate and hence converge for all $z \neq 0$ whilst that with $\sigma = -(\ell+1)$ does not terminate and hence converges only for $|z| > 1$.*

We substitute $y = \sum_{n=0}^{\infty} a_n z^{\sigma-n}$ with $a_0 \neq 0$ into Legendre's equation

$$(1-z^2)y'' - 2zy' + \ell(\ell+1)y = 0.$$

and obtain

$$(1-z^2)\sum_{n=0}^{\infty}(\sigma-n)(\sigma-n-1)a_n z^{\sigma-n-2} - 2z\sum_{n=0}^{\infty}(\sigma-n)a_n z^{\sigma-n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n z^{\sigma-n} = 0.$$

For the terms containing z^{σ},

$$\begin{aligned} -\sigma(\sigma-1)a_0 - 2\sigma a_0 + \ell(\ell+1)a_0 &= 0 \\ \Rightarrow \quad -\sigma(\sigma-1+2) + \ell(\ell+1) &= 0 \\ \Rightarrow \quad \sigma &= \ell,\ -(\ell+1). \end{aligned}$$

These are the two indicial roots.

The recurrence relation is obtained by equating the coefficient of $z^{\sigma-m}$, i.e.

$$(\sigma-m+2)(\sigma-m+1)a_{m-2} - (\sigma-m)(\sigma-m-1)a_m - 2(\sigma-m)a_m + \ell(\ell+1)a_m,$$

to zero. The relation is thus

$$a_m = \frac{(\sigma-m+2)(\sigma-m+1)}{(\sigma-m)(\sigma-m-1+2) - \ell(\ell+1)}\, a_{m-2} \text{ with } m \geq 2.$$

For $\sigma = \ell$

$$\begin{aligned} a_n &= \frac{(\ell-n+2)(\ell-n+1)}{(\ell-n)(\ell-n-1+2) - \ell(\ell+1)}\, a_{n-2} \\ &= \frac{(\ell-n+2)(\ell-n+1)}{-n(\ell+1+\ell) + n^2}\, a_{n-2} \\ &= \frac{(\ell-n+2)(\ell-n+1)}{n(n-2\ell-1)}\, a_{n-2}. \end{aligned}$$

If ℓ is a positive integer then, irrespective of whether ℓ is even or odd, n will

pass through either $\ell + 1$ or $\ell + 2$ and at that point one of the factors in the numerator will become zero. The series of coefficients will then terminate producing a function with a finite number of terms, each of which is a positive power of z^{-1}; such a function must be finite for all (non-zero) z. Although the denominator of the recurrence relation would become zero when $n = 2\ell + 1$, the series will have terminated before that value of n is reached.

For $\sigma = -(\ell + 1)$

$$\begin{aligned} a_n &= \frac{(-\ell - 1 - n + 2)(-\ell - 1 - n + 1)}{(-\ell - 1 - n)(-\ell - 1 - n - 1 + 2) - \ell(\ell + 1)}\, a_{n-2} \\ &= \frac{(\ell + n)(\ell + n - 1)}{(\ell + n + 1)(\ell + n) - \ell(\ell + 1)}\, a_{n-2} \\ &= \frac{(\ell + n)(\ell + n - 1)}{n(n + 2\ell + 1)}\, a_{n-2}. \end{aligned}$$

This series will not terminate because $(\ell + n)(\ell + n - 1)$ cannot be equal to zero for $\ell > 0$ and $n \geq 2$. The denominator of the recurrence relation can never become zero.

Since the series is an infinite one in *inverse* powers of z, it will only converge for

$$\left|\frac{1}{z^2}\right| < \lim_{n\to\infty}\left|\frac{a_{n-2}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(\ell - n + 2)(\ell - n + 1)}{n(n - 2\ell - 1)}\right| = 1,$$

i.e. for $|z| > 1$.

17

Eigenfunction methods for differential equations

17.2 *Write the homogeneous Sturm-Liouville eigenvalue equation for which* $y(a) = y(b) = 0$ *as*

$$\mathcal{L}(y;\lambda) \equiv (py')' + qy + \lambda\rho y = 0,$$

where $p(x), q(x)$ *and* $\rho(x)$ *are continuously differentiable functions. Show that if* $z(x)$ *and* $F(x)$ *satisfy* $\mathcal{L}(z;\lambda) = F(x)$ *with* $z(a) = z(b) = 0$ *then*

$$\int_a^b y(x)F(x)\,dx = 0.$$

Demonstrate the validity of this general result by direct calculation for the specific case in which $p(x) = \rho(x) = 1$, $q(x) = 0$, $a = -1$, $b = 1$ *and* $z(x) = 1 - x^2$.

Write the integral I (omitting all arguments of functions) as follows:

$$\begin{aligned}
I &= \int_a^b yF\,dx \\
&= \int_a^b y\mathcal{L}(z;\lambda)\,dx \\
&= \int_a^b \left[\, y(pz')' + yqz + y\lambda\rho z \,\right] dx \\
&= \left[\, ypz' \,\right]_a^b - \int_a^b \left[\, y'pz' - z(qy + \lambda\rho y) \,\right] dx, \text{ with } y(a) = y(b) = 0, \\
&= 0 - \left[\, y'pz \,\right]_a^b + \int_a^b \left[\, (y'p)'z + z(qy + \lambda\rho y) \,\right] dx, \text{ with } z(a) = z(b) = 0, \\
&= 0 + \int_a^b z\mathcal{L}(y;\lambda)\,dx = \int_a^b z\,0\,dx = 0.
\end{aligned}$$

For the special case in which $p(x) = \rho(x) = 1$, $a = -1$, $b = 1$ and $q(x) = 0$, the equation reduces to

$$\begin{aligned} & y'' + \lambda y = 0 \text{ with } y(\pm 1) = 0. \\ \Rightarrow \quad & y(x) = A\cos(\sqrt{\lambda}x), \end{aligned}$$

with $\lambda = \dfrac{(2n+1)^2\pi^2}{4}$ and n a non-negative integer.

With the given form of $z(x)$,

$$\mathcal{L}(z;\lambda) = (1-x^2)'' + \lambda(1-x^2) = -2 + \lambda(1-x^2).$$

To verify the result we need to prove that $I = 0$, where

$$\begin{aligned} I &= \int_{-1}^{1} \cos(\mu x)[-2 + \mu^2(1-x^2)]\,dx, \text{ with } \mu = \sqrt{\lambda} = \frac{(2n+1)\pi}{2}, \\ &= \int_{-1}^{1} (\mu^2 - 2)\cos\mu x\,dx - \mu^2 \int_{-1}^{1} x^2 \cos\mu x\,dx \\ &= (\mu^2 - 2)\left[\frac{\sin\mu x}{\mu}\right]_{-1}^{1} - \mu^2 J \\ &= \frac{2(-1)^n(\mu^2-2)}{\mu} - \mu^2 J. \end{aligned}$$

Here J is the integral

$$\begin{aligned} J &= \int_{-1}^{1} x^2 \cos\mu x\,dx \\ &= \left[\frac{x^2\sin\mu x}{\mu}\right]_{-1}^{1} - 2\int_{-1}^{1} x\frac{\sin\mu x}{\mu}\,dx \\ &= \frac{2(-1)^n}{\mu} + \left[\frac{2x\cos\mu x}{\mu^2}\right]_{-1}^{1} - \int_{-1}^{1} \frac{2\cos\mu x}{\mu^2}\,dx \\ &= \frac{2(-1)^n}{\mu} + 0 - 0 - \left[\frac{2\sin\mu x}{\mu^3}\right]_{-1}^{1} \\ &= (-1)^n\left(\frac{2}{\mu} - \frac{4}{\mu^3}\right). \end{aligned}$$

Thus

$$I = (-1)^n\frac{2(\mu^2-2)}{\mu} - \mu^2(-1)^n\left(\frac{2}{\mu} - \frac{4}{\mu^3}\right) = 0,$$

as expected.

17.4 *Show that the equation*

$$y'' + a\delta(x)y + \lambda y = 0,$$

with $y(\pm\pi) = 0$ *and* a *real, has a set of eigenvalues* λ *satisfying*

$$\tan(\pi\sqrt{\lambda}) = \frac{2\sqrt{\lambda}}{a}.$$

Investigate the conditions under which negative eigenvalues, $\lambda = -\mu^2$ *with* μ *real, are possible.*

The problem is that of finding the Green's function $G(x, x_0)$ for the point $x_0 = 0$ over the range $-\pi \le x \le \pi$ with boundary values $y(\pm\pi) = 0$. We assume first that $\lambda > 0$. Continuity of the solution is needed at $x = 0$ but its derivative will have a step increase of magnitude $-ay(0)$.

Let

$$y(x) = \begin{cases} A\sin\nu x + B\cos\nu x, & -\pi \le x < 0, \\ C\sin\nu x + D\cos\nu x, & 0 \le x \le \pi. \end{cases},$$

where $\nu = \sqrt{\lambda}$. Then, continuity at $x = 0$ implies that $D = B$, whilst the step condition can be written

$$(\nu C + 0) - (\nu A + 0) = -a(0 + B).$$

The boundary values require

$$\frac{B}{A} = \tan\nu\pi = -\frac{D}{C} \quad \Rightarrow \quad A = -C.$$

Thus, substituting in the step condition gives

$$-\nu A - \nu A = -aA\tan\nu\pi \quad \Rightarrow \quad \tan\nu\pi = \frac{2\nu}{a}, \text{ i.e. } \tan\sqrt{\lambda}\pi = \frac{2\sqrt{\lambda}}{a}.$$

We note that, since the operator $\dfrac{d^2}{dx^2} + a\delta(x)$ is Hermitian, its eigenvalues can only be real. But this does not rule out the possibility of negative eigenvalues $\lambda = -\mu^2$ with μ real.

Putting the calculated values of B, C and D back into the assumed forms in part (a) shows that the explicit solution for that part is

$$y(x) = \begin{cases} E\sin[\sqrt{\lambda}(\pi + x)] & -\pi \le x < 0, \\ E\sin[\sqrt{\lambda}(\pi - x)] & 0 \le x \le \pi. \end{cases}$$

The corresponding result for $\lambda = -\mu^2$ is

$$y(x) = \begin{cases} E\sinh[\mu(\pi + x)] & -\pi \le x < 0, \\ E\sinh[\mu(\pi - x)] & 0 \le x \le \pi. \end{cases},$$

leading to the condition

$$\tanh \mu\pi = \frac{2\mu}{a}.$$

A simple sketch shows that this equation can only have a real solution for μ if the slope of $f(\mu) = \tanh(\mu\pi)$ at $\mu = 0$ is greater than the slope of $g(\mu) = 2\mu/a$ at the same place. The former slope is π and the latter $2/a$. Thus the condition for negative eigenvalues of the original equation is $a > 2/\pi$.

17.6 *Starting from the linearly independent functions* $1, x, x^2, x^3, \ldots$, *in the range* $0 \le x < \infty$, *find the first three orthonormal functions* ϕ_0, ϕ_1 *and* ϕ_2, *with respect to the weight function* $\rho(x) = e^{-x}$. *By comparing your answers with the Laguerre polynomials generated by the recurrence relation*

$$(n+1)L_{n+1} - (2n+1-x)L_n + nL_{n-1} = 0,$$

deduce the form of $\phi_3(x)$.

We aim to construct the orthonormal functions using the Gram–Schmidt procedure. To evaluate the integrals involved we will make repeated use of the general result

$$\int_0^\infty x^n e^{-x}\, dx = \Gamma(n+1) = n!.$$

Starting with $\phi_0 = 1$, all we need to check is its normalisation. Since

$$\int_0^\infty 1^2 e^{-x}\, dx = 1,$$

ϕ_0 is already correctly normalised.

We next calculate ϕ_1 as

$$\begin{aligned}\phi_1 &= x - \phi_0\langle\phi_0|x\rangle \\ &= x - 1\int_0^\infty 1\, z\, e^{-z}\, dz \\ &= x - 1,\end{aligned}$$

and check its normalisation:

$$\langle\phi_1|\phi_1\rangle = \int_0^\infty (x-1)^2 e^{-x}\, dx = \int_0^\infty (x^2 - 2x + 1)e^{-x}\, dx = 2! - 2(1!) + 1 = 1.$$

It too is already correctly normalised.

To find ϕ_2 we continue with the Gram–Schmidt construction using the expressions already derived for ϕ_0 and ϕ_1, as follows.

$$\begin{aligned}\phi_2(x) &= x^2 - 1\int_0^\infty 1\, z^2 e^{-z}\, dz - (x-1)\int_0^\infty (z-1)z^2 e^{-z}\, dz \\ &= x^2 - 2! - (x-1)\int_0^\infty (z^3 - z^2)e^{-z}\, dz \\ &= x^2 - 2 - (x-1)(3! - 2!) = x^2 - 4x + 2.\end{aligned}$$

Determining its normalisation constant is a little more complicated than for the first two functions, but to do so we evaluate

$$\begin{aligned}\langle\phi_2|\phi_2\rangle &= \int_0^\infty (x^4 + 16x^2 + 4 - 8x^3 + 4x^2 - 16x)e^{-x}\, dx \\ &= 4! + 20(2!) - 8(3!) - 16(1!) + 4 \\ &= 24 + 40 - 48 - 16 + 4 = 4.\end{aligned}$$

It is then clear that the correctly normalised ϕ_2 is $\phi_2(x) = \frac{1}{2}(x^2 - 4x + 2)$.

Next we explicitly generate the Laguerre polynomials using the recurrence relation

$$(n+1)L_{n+1} - (2n + 1 - x)L_n + nL_{n-1} = 0,$$

starting with $L_0(x) = 1$ (and L_{-1} conventionally equal to zero; it is multiplied by zero in any case). The equations for $n = 0, 1, 2$ read

$$\begin{aligned}L_1 - (0 + 1 - x)L_0 + 0 = 0 \quad &\Rightarrow \quad L_1(x) = 1 - x, \\ 2L_2 - (2 + 1 - x)L_1 + L_0 = 0 \quad &\Rightarrow \quad L_2(x) = \tfrac{1}{2}[(3 - x)(1 - x) - 1] \\ &\Rightarrow \quad L_2(x) = \tfrac{1}{2}[x^2 - 4x + 2], \\ 3L_3 - (4 + 1 - x)L_2 + 2L_1 = 0 \quad &\Rightarrow \quad 3L_3(x) = (5 - x)\tfrac{1}{2}(x^2 - 4x + 2) \\ &\qquad\qquad -2(1 - x) \\ &\Rightarrow \quad L_3(x) = \tfrac{1}{6}(-x^3 + 9x^2 - 18x + 6).\end{aligned}$$

Comparing these results with the corresponding $\phi_n(x)$ for $n = 0, 1, 2$ shows that the ϕ_n are the same as the L_n, but their relative signs alternate. Although it is not conclusive on the basis of only three comparisons, a connection $\phi_n(x) = (-1)^n L_n(x)$ seems plausible. This is, in fact, correct and indicates that $\phi_3(x) = (-1)^3\frac{1}{6}(-x^3 + 9x^2 - 18x + 6) = \frac{1}{6}(x^3 - 9x^2 + 18x - 6)$, a conclusion that can be checked by direct, but tedious, calculation.

17.8 *A particle moves in a parabolic potential in which its natural angular frequency of oscillation is* $1/2$. *At time* $t = 0$ *it passes through the origin with velocity* v *and is suddenly subjected to an additional acceleration of* $+1$ *for* $0 \leq t \leq \pi/2$, *and then* -1 *for* $\pi/2 < t \leq \pi$. *At the end of this period it is at the origin again. By making an eigenfunction expansion of the solution to the equation of motion, show that*

$$v = -\frac{8}{\pi}\sum_{m=0}^{\infty}\frac{1}{(4m+2)^2 - \frac{1}{4}} \approx -0.81.$$

The equation of motion is

$$\ddot{y} + \frac{1}{4}y = f(t) = \begin{cases} 1, & 0 \leq t < \pi/2, \\ -1, & \pi/2 \leq t < \pi, \end{cases}$$

with $y(0) = y(\pi) = 0$.

The eigenfunctions of the operator $\mathcal{L} = \dfrac{d^2}{dt^2} + \dfrac{1}{4}$ are obviously

$$y_n(t) = A_n \sin nt + B_n \cos nt$$

with corresponding eigenvalues $\lambda_n = n^2 - \frac{1}{4}$.

The boundary conditions, $y(0) = y(\pi) = 0$, require that n is a positive integer and that $B_n = 0$, i.e.

$$y_n(t) = A_n \sin nt = \sqrt{\frac{2}{\pi}}\sin nt \text{ (when normalised) with } n \geq 1.$$

If the required solution is $y(t) = \sum_n a_n y_n(t)$, then direct substitution yields

$$\sum_{n=1}^{\infty}(\tfrac{1}{4} - n^2)a_n y_n(t) = f(t).$$

Remembering that the y_n are sine functions, we apply the normal procedure for Fourier analysis, and obtain

$$a_m = \frac{1}{\frac{1}{4} - m^2}\int_0^{\pi} f(z)y_m(z)\,dz$$

and, consequently, that

$$y(t) = \sum_{n=1}^{\infty}\sqrt{\frac{2}{\pi}}\frac{\sin nt}{\frac{1}{4} - n^2}\sqrt{\frac{2}{\pi}}\int_0^{\pi} f(z)\sin(nz)\,dz.$$

Now, from the given data, $f(z) = +1$ for $t < \pi/2$ and $f(z) = -1$ for $t > \pi/2$. So,

$$\begin{aligned}
\int_0^{\pi} f(z)\sin(nz)\,dz &= \int_0^{\pi/2} \sin(nz)\,dz - \int_{\pi/2}^{\pi} \sin(nz)\,dz \\
&= \left[\frac{-\cos nz}{n}\right]_0^{\pi/2} - \left[\frac{-\cos nz}{n}\right]_{\pi/2}^{\pi} \\
&= \begin{cases} \dfrac{1}{n}(1-1) = 0, & \text{for } n \text{ odd}, \\ \dfrac{1}{2m}[-(-1)^m + 1 + 1 - (-1)^m], & \text{for } n = 2m, \end{cases} \\
&= \begin{cases} \dfrac{4}{2m} & \text{for } m \text{ odd, i.e. } n = 2m = 4r+2, \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

Thus,

$$y(t) = \frac{2}{\pi}\sum_{r=0}^{\infty} \frac{4\sin[(4r+2)t]}{(4r+2)[\frac{1}{4} - (4r+2)^2]},$$

and, by differentiation with respect to t and then setting $t = 0$,

$$v = \dot{y}(0) = -\frac{8}{\pi}\sum_{r=0}^{\infty} \frac{\cos(0)}{(4r+2)^2 - \frac{1}{4}} \approx -0.81,$$

as stated in the question.

17.10 *Consider the following two approaches to constructing a Green's function.*

(a) *Find those eigenfunctions $y_n(x)$ of the self-adjoint linear differential operator d^2/dx^2 that satisfy the boundary conditions $y_n(0) = y_n(\pi) = 0$, and hence construct its Green's function $G(x, z)$.*

(b) *Construct the same Green's function using a method based on the complementary function of the appropriate differential equation and the boundary coditions to be satisfied at the position of the δ-function, showing that it is*

$$G(x,z) = \begin{cases} x(z-\pi)/\pi, & 0 \le x \le z, \\ z(x-\pi)/\pi, & z \le x \le \pi. \end{cases}$$

(c) *By expanding the function given in* (b) *in terms of the eigenfunctions $y_n(x)$, verify that it is the same function as that derived in* (a).

Recalling that we have chosen to define the eigenvalue of a linear operator by

$$\mathcal{L}y_n = \lambda_n \rho y_n,$$

the eigenfunctions satisfying the given boundary conditions are

$$y_n(x) = \sqrt{\frac{2}{\pi}} \sin nx,$$

with corresponding eigenvalues $\lambda_n = -n^2$ for integer n. The Green's function is thus

$$G(x,z) = \sum_{n=0}^{\infty} \frac{1}{\lambda_n} y_n(x) y_n^*(z) = -\frac{2}{\pi} \sum_{n=0}^{\infty} \frac{1}{n^2} \sin nx \sin nz.$$

(b) The differential equation defining the Green's function is

$$y'' = \delta(x-z) \qquad \text{with} \qquad y(0) = y(\pi) = 0.$$

It's solution takes the form

$$H(x,z) = \begin{cases} A + Bx & 0 \le x \le z, \\ C + Dx & z < x \le \pi. \end{cases}$$

From the boundary conditions it is clear that $A = 0$ and that $C = -D\pi$. Continuity at $x = z$ implies $Bz = C + Dz$ whilst the required unit step in the derivative implies $D - B = 1$. Together, these give $C = -z$, $D = z/\pi$ and $B = (z/\pi) - 1$. Resubstitution then gives as the Green's function

$$H(x,z) = \begin{cases} \left(\dfrac{z}{\pi} - 1\right) x = \dfrac{x(z-\pi)}{\pi} & 0 \le x \le z, \\ -z + \dfrac{z}{\pi} x = \dfrac{z(x-\pi)}{\pi} & z < x \le \pi. \end{cases}$$

(c) This verification is tantamount to finding a Fourier sine-series for the answer found in part (b):

$$H(x,z) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2}{\pi}} \sin nx \qquad \text{with}$$

$$\begin{aligned}
\pi \frac{\pi}{2} \sqrt{\frac{2}{\pi}} a_n &= \int_0^z (z-\pi) x \sin nx \, dx + \int_z^{\pi} z(x-\pi) \sin nx \, dx \\
&= (z-\pi) \left\{ \left[\frac{-x \cos nx}{n} \right]_0^z + \int_0^z \frac{\cos nx}{n} \, dx \right\} \\
&\qquad - z\pi \left[\frac{-\cos nx}{n} \right]_z^{\pi} + z \left\{ \left[\frac{-x \cos nx}{n} \right]_z^{\pi} + \int_z^{\pi} \frac{\cos nx}{n} \, dx \right\} \\
&= (z-\pi) \left(-\frac{z \cos nz}{n} - 0 + \frac{\sin nz}{n^2} - 0 \right) + \frac{(-1)^n z\pi}{n} \\
&\qquad - \frac{z\pi \cos nz}{n} + z \left(-\frac{(-1)^n \pi}{n} + \frac{z \cos nz}{n} + 0 - \frac{\sin nz}{n^2} \right) \\
&= -\frac{\pi \sin nz}{n^2}.
\end{aligned}$$

Thus,

$$a_n = -\sqrt{\frac{2}{\pi}}\frac{\sin nz}{n^2},$$

and resubstituting this expression for a_n shows that

$$H(x,z) = -\frac{2}{\pi}\sum_{n=0}^{\infty}\frac{\sin nz\ \sin nx}{n^2},$$

so confirming that this is the same function as that derived in part (a).

17.12 *Show that the linear operator*

$$\mathcal{L} \equiv \tfrac{1}{4}(1+x^2)^2\frac{d^2}{dx^2} + \tfrac{1}{2}x(1+x^2)\frac{d}{dx} + a,$$

acting upon functions defined in $-1 \le x \le 1$ *and vanishing at the endpoints of the interval, is Hermitian with respect to the weight function* $(1+x^2)^{-1}$.

By making the change of variable $x = \tan(\theta/2)$, *find two even eigenfunctions,* $f_1(x)$ *and* $f_2(x)$, *of the differential equation*

$$\mathcal{L}u = \lambda u.$$

We take as our general functions $u(x)$ and $v(x)$ with $u(\pm 1) = v(\pm 1) = 0$. The operator $\mathcal{L}$ will be Hermitian with respect to the given weight function $w(x)$ if its adjoint, $\mathcal{L}^\dagger$ defined by

$$\int_{-1}^{1} v^*(\mathcal{L}u)w\,dx = \int_{-1}^{1}(\mathcal{L}^\dagger v)^* uw\,dx,$$

is equal to $\mathcal{L}$ and certain boundary contributions vanish.

Now consider

$$\begin{aligned} I &= \int_{-1}^{1}\frac{v^*\mathcal{L}u}{1+x^2}\,dx \\ &= \int_{-1}^{1} v^*\left[\frac{1}{4}(1+x^2)\frac{d^2u}{dx^2} + \frac{1}{2}x\frac{du}{dx} + \frac{au}{1+x^2}\right]dx \\ &= \int_{-1}^{1} v^*\left\{\left[\tfrac{1}{4}(1+x^2)u'\right]' + \frac{au}{1+x^2}\right\}dx. \end{aligned}$$

Integrating by parts, we obtain

$$\begin{aligned}
I &= \left[v^* \tfrac{1}{4}(1+x^2)u' \right]_{-1}^{1} - \int_{-1}^{1} \left\{ v^{*\prime} \tfrac{1}{4}(1+x^2)u' - \frac{auv^*}{1+x^2} \right\} dx \\
&= 0 - \left[v^{*\prime} \tfrac{1}{4}(1+x^2)u \right]_{-1}^{1} + \int_{-1}^{1} \left\{ \left[\tfrac{1}{4}(1+x^2)v^{*\prime} \right]' u + \frac{auv^*}{1+x^2} \right\} dx \\
&= 0 + \int_{-1}^{1} u \left\{ \left[\tfrac{1}{4}(1+x^2)v' \right]' + \frac{a^* v}{1+x^2} \right\}^* dx \\
&= \int_{-1}^{1} \frac{(\mathcal{L}v)^* u}{1+x^2}\, dx,
\end{aligned}$$

provided a is real. If so, $\mathcal{L} = \mathcal{L}^\dagger$ and $\mathcal{L}$ is Hermitian.

We now make a change of variable to $\theta = 2\tan^{-1} x$ with $-\pi/2 \le \theta \le \pi/2$, $f(x) = u(\theta)$ and

$$\frac{d\theta}{dx} = \frac{2}{1+x^2} = 2\cos^2(\theta/2).$$

The expression for $\mathcal{L}u$ becomes

$$\begin{aligned}
\mathcal{L}u &= \frac{1}{4}\sec^4(\theta/2)\, 2\cos^2(\theta/2)\, \frac{d}{d\theta}\left(2\cos^2(\theta/2)\, \frac{du}{d\theta} \right) + \frac{1}{2}\tan(\theta/2)\, 2\frac{du}{d\theta} + au \\
&= \sec^2(\theta/2) \left[-\cos(\theta/2)\sin(\theta/2)\frac{du}{d\theta} + \cos^2(\theta/2)\frac{d^2u}{d\theta^2} \right] + \tan(\theta/2)\, \frac{du}{d\theta} + au \\
&= \frac{d^2u}{d\theta^2} + au
\end{aligned}$$

Thus, we have to solve

$$\frac{d^2u}{d\theta^2} + au = \lambda u \text{ with } u(-\tfrac{1}{2}\pi) = u(\tfrac{1}{2}\pi) = 0 \text{ and } u(-\theta) = u(\theta).$$

In view of the boundary conditions we need solutions of the form

$$u(\theta) = A\cos(\sqrt{a-\lambda}\,\theta) \quad \text{with} \quad \sqrt{a-\lambda} = 2n+1.$$

(i) $n = 0$ and $\lambda = a - 1$.

$$f_1(x) = u(\theta) = A\cos\theta = A\frac{1-x^2}{1+x^2}.$$

(ii) $n = 1$ and $\lambda = a - 9$.

$$\begin{aligned}
f_2(x) = u(\theta) = B\cos 3\theta &= B(4\cos^3\theta - 3\cos\theta) \\
&= 4B\left(\frac{1-x^2}{1+x^2}\right)^3 - 3B\frac{1-x^2}{1+x^2}.
\end{aligned}$$

Both of these functions are functions of x^2 and therefore clearly even functions of x.

17.14 *Express the solution of Poisson's equation in electrostatics,*

$$\nabla^2\phi(\mathbf{r}) = -\rho(\mathbf{r})/\epsilon_0,$$

where ρ is the non-zero charge density over a finite part of space, in the form of an integral and hence identify the Green's function for the ∇^2 operator.

Consider the (infinitesimal) potential $d\phi(\mathbf{r})$ due to a small element of charge $dq = \rho(\mathbf{r}')\,dv'$ situated at the position $\mathbf{r}'$. This is clearly

$$d\phi(\mathbf{r}) = \frac{\rho(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.$$

Since Poisson's equation is linear, we may apply superposition and so obtain the total potential at position $\mathbf{r}$. This same potential must also be expressible in terms of the Green's function associated with Poisson's equation. Thus,

$$-\int G(\mathbf{r},\mathbf{r}')\frac{\rho(\mathbf{r}')}{\epsilon_0}\,dv' \equiv \phi(\mathbf{r}) = \int d\phi(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')\,dv'}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|}.$$

Hence, by inspection,

$$G(\mathbf{r},\mathbf{r}') = -\frac{1}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

18

Special functions

18.2 *Express the function*

$$f(\theta,\phi) = \sin\theta[\sin^2(\theta/2)\cos\phi + i\cos^2(\theta/2)\sin\phi] + \sin^2(\theta/2)$$

as a sum of spherical harmonics.

Since every spherical harmonic can only contain ϕ as a multiplicative factor of the form $e^{\pm im\phi}$, we must decompose the given expression into a sum of terms containing such factors. Further, as the spherical harmonics are expressed in term of θ, (rather than of $\theta/2$) we also express the given function in these terms.

$$\begin{aligned}
f(\theta,\phi) &= \sin\theta\left[\sin^2\frac{\theta}{2}\cos\phi + i\cos^2\frac{\theta}{2}\sin\phi\right] + \sin^2\frac{\theta}{2} \\
&= \sin\theta\left[\frac{1}{2}(1-\cos\theta)\cos\phi + \frac{i}{2}(1+\cos\theta)\sin\phi\right] + \frac{1}{2}(1-\cos\theta) \\
&= \frac{1}{2}(1-\cos\theta) + \frac{1}{4}(1-\cos\theta)\sin\theta\ (e^{i\phi} + e^{-i\phi}) \\
&\qquad + \frac{1}{4}(1+\cos\theta)\sin\theta(e^{i\phi} - e^{-i\phi}) \\
&= \frac{1}{2} - \frac{1}{2}\cos\theta + \frac{1}{2}\sin\theta\, e^{i\phi} - \frac{1}{2}\cos\theta\sin\theta\, e^{-i\phi} \\
&= \frac{1}{2}\sqrt{4\pi}Y_0^0 - \frac{1}{2}\sqrt{\frac{4\pi}{3}}Y_1^0 - \frac{1}{2}\sqrt{\frac{8\pi}{3}}\,Y_1^1 - \frac{1}{2}\sqrt{\frac{8\pi}{15}}\,Y_2^{-1} \\
&= \sqrt{\pi}\left(Y_0^0 - \sqrt{\frac{1}{3}}Y_1^0 - \sqrt{\frac{2}{3}}\,Y_1^1 - \sqrt{\frac{2}{15}}\,Y_2^{-1}\right).
\end{aligned}$$

18.4 *Carry through the following procedure as a proof of the result*

$$I_n = \int_{-1}^{1} P_n(z)P_n(z)\,dz = \frac{2}{2n+1}.$$

(a) *Square both sides of the generating-function definition of the Legendre polynomials,*

$$(1-2zh+h^2)^{-1/2} = \sum_{n=0}^{\infty} P_n(z)h^n.$$

(b) *Express the RHS as a sum of powers of h, obtaining expressions for the coefficients.*

(c) *Integrate the RHS from* -1 *to 1 and use the orthogonality property of the Legendre polynomials.*

(d) *Similarly integrate the LHS and expand the result in powers of h.*

(e) *Compare coefficients.*

We are required to evaluate

$$I_n = \int_{-1}^{1} P_n(z)P_n(z)\,dz.$$

We start with the generating function and apply the steps indicated:

$$\begin{aligned}
(1-2zh+h^2)^{-1/2} &= \sum_{n=0}^{\infty} P_n(z)h^n, \\
\frac{1}{1-2zh+h^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} P_n(z)P_m(z)\,h^{m+n}, \\
\int_{-1}^{1} \frac{dz}{1-2zh+h^2} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \int_{-1}^{1} P_n(z)P_m(z)\,dz\,h^{m+n}, \\
-\frac{1}{2h}\left[\ln(1-2zh+h^2)\right]_{-1}^{1} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} I_m\,\delta_{mn}\,h^{m+n},
\end{aligned}$$

using the orthogonality property. Thus,

$$\begin{aligned}
\sum_{m=0}^{\infty} I_m\,h^{2m} &= -\frac{1}{2h}\ln\frac{(1-h)^2}{(1+h)^2} \\
&= \frac{1}{h}\ln\frac{1+h}{1-h} \\
&= \frac{1}{h}\left(\sum_{n=0}^{\infty}\frac{(-1)^n h^{n+1}}{n+1} - \sum_{n=0}^{\infty}\frac{(-1)h^{n+1}}{n+1}\right) = \sum_{n\ \text{even}}^{\infty}\frac{2h^n}{n+1}.
\end{aligned}$$

Figure 18.1 The arrangement of charges and notation for exercise 18.6.

Hence, from equating the coefficients of h^{2m} (i.e. setting $n = 2m$), we have

$$I_m = \frac{2}{2m+1},$$

as stated in the question.

18.6 *A charge $+2q$ is situated at the origin and charges of $-q$ are situated at distances $\pm a$ from it along the polar axis. By relating it to the generating function for the Legendre polynomials, show that the electrostatic potential Φ at a point (r, θ, ϕ) with $r > a$ is given by*

$$\Phi(r, \theta, \phi) = \frac{2q}{4\pi\epsilon_0 r} \sum_{s=1}^{\infty} \left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta).$$

The situation is shown in figure 18.1. We superimpose the potentials due to the individual charges. That due to the charge $2q$ is simply $2q/(4\pi\epsilon_0 r)$. To obtain the distances r_1 and r_2 of the point (r, θ) from the negative charges we use the cosine rule:

$$r_1^2 = r^2 + a^2 - 2ar\cos\theta,$$
$$\frac{1}{r_1} = \frac{1}{r}\left[1 - \frac{2a}{r}\cos\theta + \left(\frac{a}{r}\right)^2\right]^{-1/2}.$$

This gives Φ_1 as

$$\Phi_1 = -\frac{q}{4\pi\epsilon_0 r}(1 - 2h\cos\theta + h^2)^{-1/2} = -\frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(\cos\theta)h^n,$$

where we have written $a/r = h$ and, having done so, identified the resulting expression as the generating function for Legendre polynomials. Similarly,

$$\Phi_2 = -\frac{q}{4\pi\epsilon_0 r}\sum_{n=0}^{\infty} P_n(-\cos\theta)h^n.$$

Since $P_n(-\cos\theta) = (-1)^n P_n(\cos\theta)$, when all three terms are added together we obtain

$$\begin{aligned}\Phi &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{n\text{ even}}^{\infty} P_n(\cos\theta)h^n + \frac{2q}{4\pi\epsilon_0 r}\\ &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{n\text{ even}\neq 0}^{\infty} P_n(\cos\theta)h^n\\ &= -\frac{2q}{4\pi\epsilon_0 r}\sum_{s=1}^{\infty}\left(\frac{a}{r}\right)^{2s} P_{2s}(\cos\theta),\end{aligned}$$

as stated in the question.

18.8 *The quantum mechanical wavefunction for a one-dimensional simple harmonic oscillator in its nth energy level is of the form*

$$\psi(x) = \exp(-x^2/2)H_n(x),$$

where $H_n(x)$ is the nth Hermite polynomial. The generating function for the polynomials is

$$G(x,h) = e^{2hx-h^2} = \sum_{n=0}^{\infty}\frac{H_n(x)}{n!}h^n.$$

(a) *Find $H_i(x)$ for $i = 1, 2, 3, 4$.*
(b) *Evaluate by direct calculation*

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x)H_q(x)\,dx,$$

(i) *for $p = 2$, $q = 3$;* (ii) *for $p = 2$, $q = 4$;* (iii) *for $p = q = 3$. Check your answers against the expected values $2^p\, p!\sqrt{\pi}\,\delta_{pq}$.*

[*You will find it convenient to use*

$$\int_{-\infty}^{\infty} x^{2n}e^{-x^2}\,dx = \frac{(2n)!\sqrt{\pi}}{2^{2n}n!}$$

for integer $n \geq 0$.]

(a) The generating function is

$$G(x,h) = \exp(2hx - h^2) = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} h^n,$$

from which it follows that

$$H_n(x) = \frac{\partial^n}{\partial h^n} \left[\exp(2hx - h^2)\right]\Big|_{h=0}.$$

We therefore calculate these derivatives:

$$\begin{aligned}
\frac{\partial G}{\partial h} &= (2x-2h)G \quad \Rightarrow \quad H_1(x) = 2x, \\
\frac{\partial^2 G}{\partial h^2} &= (2x-2h)^2 G - 2G \quad \Rightarrow \quad H_2(x) = 4x^2 - 2, \\
\frac{\partial^3 G}{\partial h^3} &= (2x-2h)^3 G + 2(-2)(2x-2h)G - 2(2x-2h)G \\
&= (2x-2h)^3 G - 6(2x-2h)G \quad \Rightarrow \quad H_3(x) = 8x^3 - 12x, \\
\frac{\partial^4 G}{\partial h^4} &= (2x-2h)^4 G + 3(-2)(2x-2h)^2 G - 6(2x-2h)^2 G + 12G \\
&= (2x-2h)^4 G - 12(2x-2h)^2 G + 12G \\
&\Rightarrow \quad H_4(x) = 16x^4 - 48x^2 + 12.
\end{aligned}$$

(b) Denote by J_n the integral

$$J_n = \int_{-\infty}^{\infty} x^n e^{-x^2}\, dx \text{ with } J_{2r} = \frac{(2r)!\sqrt{\pi}}{2^{2r}\, r!} \text{ and } J_{2r+1} = 0.$$

Further, define I_{pq} as

$$I_{pq} = \int_{-\infty}^{\infty} e^{-x^2} H_p(x) H_q(x)\, dx.$$

Then, for case (i)

$$\begin{aligned}
I_{23} &= \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(8x^3-12)\, dx \\
&= 32J_5 - 16J_3 - 48J_3 + 24J_1 = 0, \quad \text{as all subscripts are odd.}
\end{aligned}$$

For case (ii)

$$\begin{aligned}
I_{24} &= \int_{-\infty}^{\infty} e^{-x^2}(4x^2-2)(16x^4-48x^2+12)\, dx \\
&= 64J_6 - 192J_4 + 48J_2 - 32J_4 + 96J_2 - 24J_0 \\
&= 64\,\frac{6!\sqrt{\pi}}{2^6\, 3!} - 224\,\frac{4!\sqrt{\pi}}{2^4\, 2!} + 144\,\frac{2!\sqrt{\pi}}{2^2\, 1!} - 24\,\frac{0!\sqrt{\pi}}{2^0\, 0!} \\
&= \sqrt{\pi}(120 - 168 + 72 - 24) = 0.
\end{aligned}$$

Finally, for case (iii)

$$\begin{aligned} I_{33} &= \int_{-\infty}^{\infty} e^{-x^2}(8x^3 - 12x)^2\,dx \\ &= 64J_6 - 192J_4 + 144J_2 \\ &= 64\,\frac{6!\,\sqrt{\pi}}{2^6\,3!} - 192\,\frac{4!\,\sqrt{\pi}}{2^4\,2!} + 144\,\frac{2!\,\sqrt{\pi}}{2^2\,1!} \\ &= \sqrt{\pi}(120 - 144 + 72) = 48\sqrt{\pi}. \end{aligned}$$

The expected values are

$$\int_{-\infty}^{\infty} e^{-x^2} H_p(x)H_q(x)\,dx = 2^p p\,!\sqrt{\pi}\delta_{pq}.$$

This is equal to zero for $p \neq q$ and equal to $2^3\,3!\,\sqrt{\pi} = 48\sqrt{\pi}$ for $p = q = 3$. All three results agree with this.

18.10 *By choosing a suitable form for h in their generating function,*

$$G(z,h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)h^n,$$

show that integral repesentations of the Bessel functions of the first kind are given, for integral m, by

$$J_{2m}(z) = \frac{(-1)^m}{\pi}\int_0^{2\pi} \cos(z\cos\theta)\,\cos 2m\theta\,d\theta, \qquad m \geq 1,$$

$$J_{2m+1}(z) = \frac{(-1)^{m+1}}{\pi}\int_0^{2\pi} \cos(z\cos\theta)\,\sin(2m+1)\theta\,d\theta, \quad m \geq 0.$$

In the generating function equation,

$$G(z,h) = \exp\left[\frac{z}{2}\left(h - \frac{1}{h}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)h^n,$$

we set $h = ie^{i\theta}$ and obtain

$$\exp\left[\frac{z}{2}\left(ie^{i\theta} + ie^{-i\theta}\right)\right] = \sum_{n=-\infty}^{\infty} J_n(z)i^n e^{in\theta},$$

$$\exp[\,iz\cos\theta\,] = \sum_{n=-\infty}^{\infty} J_n(z)i^n(\cos n\theta + i\sin n\theta).$$

Our choice for h was prompted by the presence in the quoted answer of a sinusoidal function with a sinusoidal arguement – or, equivalently, for complex

variables, an exponential function with an exponential argument. Equating the real parts of both sides of the equality gives

$$\cos(z\cos\theta) = \sum_{m=-\infty}^{\infty} (-1)^m J_{2m} \cos 2m\theta + \sum_{m=-\infty}^{\infty} (-1)^{m+1} J_{2m+1} \sin(2m+1)\theta.$$

Now multiplying both sides of this equation by $\cos 2r\theta$ and integrating over θ from 0 to 2π gives (because of the mutual orthogonality of the sinusoidal functions) that

$$\int_0^{2\pi} \cos(z\cos\theta)\,\cos(2r\theta)\,d\theta = (-1)^r \pi\, J_{2r}(z) \text{ for } r \geq 1.$$

Hence the first result stated.

Similarly, multiplying through by $\sin(2r+1)\theta$ and integrating produces the second result.

18.12 *By making the substitution $z = (1-x)/2$ and suitable choices for a, b and c, convert the hypergeometric equation,*

$$z(1-z)\frac{d^2u}{dz^2} + [\,c - (a+b+1)z\,]\frac{du}{dz} - abu = 0,$$

into the Legendre equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \ell(\ell+1)y = 0.$$

Hence, using the hypergeometric series, generate the Legendre polynomials $P_\ell(x)$ for the integer values $\ell = 0, 1, 2, 3$. Comment on their normalisations.

From the substitution $z = (1-x)/2$, it follows that $1 - z = (1+x)/2$ and $d\,/dz = -2d\,/dx$. If $u(z) = y(x)$ then the hypergeometric equation becomes

$$\frac{1-x}{2}\,\frac{1+x}{2}\,(-2)^2\,\frac{d^2y}{dx^2} + \left[\,c - \frac{(a+b+1)(1-x)}{2}\,\right](-2)\,\frac{dy}{dx} - ab\,y = 0. \quad (*)$$

We now compare this with the Legendre equation. From the undifferentiated term we must have that the product $-ab = \ell(\ell+1)$, whilst, from the coefficient of dy/dx, we see that the sum $a+b$ must not depend upon ℓ. The only possibilities are $a = \ell$ with $b = -(\ell+1)$ and $a = -\ell$ with $b = \ell+1$. Noting that $a+b+1$ has the value 0 in the former case and 2 in the latter, we choose the second possibility, since we require a term containing x in the coefficient of dy/dx.

The equation (*) now becomes

$$(1-x^2)\frac{d^2y}{dx^2} - 2[\,c-(1-x)\,]\frac{dy}{dx} + \ell(\ell+1)y = 0.$$

All that remains to reproduce the Legendre equation is to choose $c = 1$. Thus, apart from a possible normalisation factor,

$$P_\ell(x) = F\left(-\ell, \ell+1, 1, \frac{1-x}{2}\right).$$

The corresponding hypergeometric function is therefore

$$1 + \frac{(-\ell)(\ell+1)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-\ell)(-\ell+1)(\ell+1)(\ell+2)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2 + \cdots.$$

Because of the factor $-\ell + n$ in the numerator of the $(n+2)$th term, each series terminates after $\ell + 1$ terms. For the specific values of ℓ:

$$\begin{aligned}
\ell = 0, \quad & P_0(x) = 1, \\
\ell = 1, \quad & P_1(x) = 1 + \frac{(-1)(2)}{1!\,1!}\left(\frac{1-x}{2}\right) = x, \\
\ell = 2, \quad & P_2(x) = 1 + \frac{(-2)(3)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-2)(-1)(3)(4)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2 \\
& \qquad = 1 - 3(1-x) + \tfrac{3}{2}(1-2x+x^2) \\
& \qquad = -\tfrac{1}{2} + \tfrac{3}{2}x^2 = \tfrac{1}{2}(3x^2-1), \\
\ell = 3, \quad & P_3(x) = 1 + \frac{(-3)(4)}{1!\,1!}\left(\frac{1-x}{2}\right) + \frac{(-3)(-2)(4)(5)}{2!\,(1+1)!}\left(\frac{1-x}{2}\right)^2 \\
& \qquad\qquad + \frac{(-3)(-2)(-1)(4)(5)(6)}{3!\,(1+2)!}\left(\frac{1-x}{2}\right)^3 \\
& \qquad = 1 - 6(1-x) + \tfrac{15}{2}(1-2x+x^2) \\
& \qquad\qquad - \tfrac{5}{2}(1-3x+3x^2-x^3) \\
& \qquad = (1 - 6 + \tfrac{15}{2} - \tfrac{5}{2}) + (6 - 15 + \tfrac{15}{2})x \\
& \qquad\qquad + (\tfrac{15}{2} - \tfrac{15}{2})x^2 + \tfrac{5}{2}x^3 \\
& \qquad = -\tfrac{3}{2}x + \tfrac{5}{2}x^3 = \tfrac{1}{2}(5x^3 - 3x).
\end{aligned}$$

These are the first four Legendre polynomials — usually found by other means. That they are all correctly normalised is the result of the arbitrary, but standard, requirement that $P_\ell(1) = 1$ (rather than, say, $\int_{-1}^{1} P_\ell^2\,dx = 1$). This requirement is automatically satisfied by the hypergeometric series since when $x = 1$ we have $z = 0$ and $F(a, b, c; 0) = 1$ for all a and b, and for all c, except possibly when c is a negative integer; here $c = 1$.

18.14 *Prove that, if m and n are both greater than −1, then*

$$I = \int_0^\infty \frac{u^m}{(au^2+b)^{(m+n+2)/2}}\,du = \frac{\Gamma[\frac{1}{2}(m+1)]\,\Gamma[\frac{1}{2}(n+1)]}{2a^{(m+1)/2}\,b^{(n+1)/2}\,\Gamma[\frac{1}{2}(m+n+2)]}.$$

Deduce the value of

$$J = \int_0^\infty \frac{(u+2)^2}{(u^2+4)^{5/2}}\,du.$$

Since the quoted answer strongly resembles a beta function and this is most easily connected to integrals over the range 0 to 1, we first take a factor au^2 out of the parentheses in the denominator and then make the change of variable $1 + \dfrac{b}{au^2} = \dfrac{1}{x}$. With this change,

$$u = \left[\frac{b}{a}\,\frac{x}{1-x}\right]^{1/2}, \quad du = \frac{1}{2}\left(\frac{b}{a}\right)^{1/2} \frac{dx}{x^{1/2}\,(1-x)^{3/2}}$$

and the integration limits (originally 0 and ∞) are 0 and 1. Thus,

$$\begin{aligned} I &= \int_0^\infty \frac{u^m}{(au^2+b)^{(m+n+2)/2}}\,du \\ &= \int_0^\infty \frac{u^m}{a^{(m+n+2)/2}(1+\frac{b}{au^2})^{(m+n+2)/2}u^{m+n+2}}\,du \\ &= \int_0^1 \frac{x^{(m+n+2)/2}\,a^{(n+2)/2}\,(1-x)^{(n+2)/2}\,b^{1/2}}{a^{(m+n+2)/2}\,b^{(n+2)/2}\,x^{(n+2)/2}\,2a^{1/2}\,x^{1/2}(1-x)^{3/2}}\,dx \\ &= \int_0^1 \frac{x^{(m-1)/2}\,(1-x)^{(n-1)/2}}{2a^{(m+1)/2}\,b^{(n+1)/2}}\,dx. \end{aligned}$$

This integral is a multiple of the beta function $B\left(\dfrac{m+1}{2}, \dfrac{n+1}{2}\right)$ and can therefore be expressed in terms of gamma functions as

$$I = \frac{\Gamma[\frac{1}{2}(m+1)]\,\Gamma[\frac{1}{2}(n+1)]}{2a^{(m+1)/2}\,b^{(n+1)/2}\,\Gamma[\frac{1}{2}(m+n+2)]}.$$

In the notation used above, we have for this given specific case that $a = 1$ and $b = 4$. J can be expressed as the sum of three integrals of the form considered there by expanding its numerator:

$$J = \int_0^\infty \frac{u^2}{(u^2+4)^{5/2}}\,du + \int_0^\infty \frac{4u}{(u^2+4)^{5/2}}\,du + \int_0^\infty \frac{4}{(u^2+4)^{5/2}}\,du.$$

The corresponding pairs of values of m and n are $m = 2,\ n = 1$ for the first term,

$m = 1,\ n = 2$ for the second and $m = 0,\ n = 3$ for the third. Thus the value of J is

$$\begin{aligned} J &= \frac{1}{2\ 1\ 4}\,\frac{\Gamma(\frac{3}{2})\,\Gamma(1)}{\Gamma(\frac{5}{2})} + \frac{4}{2\ 1\ 4^{3/2}}\,\frac{\Gamma(1)\,\Gamma(\frac{3}{2})}{\Gamma(\frac{5}{2})} + \frac{4}{2\ 1\ 4^{2}}\,\frac{\Gamma(\frac{1}{2})\,\Gamma(2)}{\Gamma(\frac{5}{2})} \\ &= \frac{3\Gamma(\frac{3}{2})\,\Gamma(1) + \Gamma(\frac{1}{2})\,\Gamma(2)}{8\,\Gamma(\frac{5}{2})} \\ &= \frac{3\ \frac{1}{2}\sqrt{\pi}\ 1 + \sqrt{\pi}\ 1}{8\ \frac{3}{4}\ \sqrt{\pi}} = \frac{5}{12}. \end{aligned}$$

18.16 *For $-1 < \operatorname{Re} z < 1$, use the definition and value of the beta function to show that*

$$z!\,(-z)! = \int_0^\infty \frac{u^z}{(1+u)^2}\,du.$$

Contour integration gives the value of the integral on the RHS of the above equation as $\pi z \operatorname{cosec} \pi z$. Use this to deduce the value of $(-\frac{1}{2})!$.

From the expression for the beta function in terms of gamma functions and the relationship between the gamma and factorial functions for $\operatorname{Re} z_i > -1$, we have

$$\frac{z_1!\,z_2!}{(z_1+z_2+1)!} = B(z_1+1, z_2+1) = \int_0^1 t^{z_1}(1-t)^{z_2}\,dt.$$

Since $-1 < \operatorname{Re} z < 1$, $-z$ is not a negative integer and so $(-z)!$ is defined and finite. Setting $z_1 = z$ and $z_2 = -z$, we obtain

$$z!\,(-z)! = (z - z + 1)! \int_0^1 t^z(1-t)^{-z}\,dt.$$

Making the change of integration variable

$$t = \frac{u}{1+u}, \text{ with } 1 - t = \frac{1}{1+u} \text{ and } dt = \frac{1}{(1+u)^2}\,du,$$

gives

$$\begin{aligned} z!\,(-z)! &= 1! \int_0^\infty \frac{u^z}{(1+u)^z}\,\frac{(1+u)^z}{1^z}\,\frac{1}{(1+u)^2}\,du \\ &= \int_0^\infty \frac{u^z}{(1+u)^2}\,du = \frac{\pi z}{\sin \pi z}, \qquad \text{(given)}. \end{aligned}$$

Now, setting $z = -\frac{1}{2}$ and using the general result $(z+1)! = (z+1)z!$, we have

$$\tfrac{1}{2}!\,(-\tfrac{1}{2})! = (-\tfrac{1}{2}+1)\,(-\tfrac{1}{2})!\,(-\tfrac{1}{2})!.$$

Since $(-\frac{1}{2}\pi)/\sin(-\frac{1}{2}\pi) = \pi/2$, it follows that $[(-\frac{1}{2})!]^2 = \pi$.

Now, $\frac{1}{2}! = \int_0^\infty u^{-1/2}e^{-u}\,du$ and is clearly positive, since the integrand is positive everywhere. Further, since $\frac{1}{2}! = \frac{1}{2}(-\frac{1}{2})!$ it follows that $(-\frac{1}{2})!$ has the same sign as $\frac{1}{2}!$, i.e. $(-\frac{1}{2})!$ is positive. Therefore $(-\frac{1}{2})! = \sqrt{\pi}$.

18.18 *Consider two series expansions of the error function as follows:*

(a) *Obtain a series expansion of the error function* $\mathrm{erf}(x)$ *in ascending powers of* x. *How many terms are needed to give a value correct to four significant figures for* $\mathrm{erf}(1)$?

(b) *Obtain an asymptotic expansion that can be used to estimate* $\mathrm{erfc}(x)$ *for large* $x(>0)$ *in the form of a series*

$$\mathrm{erfc}(x) = R(x) = e^{-x^2}\sum_{n=0}^{\infty}\frac{a_n}{x^n}.$$

Consider what bounds can be put on the estimate and at what point the infinite series should be terminated in a practical estimate. In particular, estimate $\mathrm{erfc}(1)$ *and test the answer for compatibility with that in part* (a).

(a) This series can be determined straightforwardly by expanding the integrand in a series of its own and then integrating term-by-term.

$$\begin{aligned}\mathrm{erf}(x) &= \frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}\,du\\ &= \frac{2}{\sqrt{\pi}}\int_0^x\left(1-u^2+\frac{u^4}{2!}-\frac{u^6}{3!}+\cdots\right)du\\ &= \frac{2}{\sqrt{\pi}}\left(x-\frac{x^2}{3}+\frac{x^5}{2!\,5}-\frac{x^7}{3!\,7}+\cdots\right).\end{aligned}$$

From tables, either directly or by setting $x = \sqrt{2}$ in the relationship $\frac{1}{2}\mathrm{erf}(x/\sqrt{2}) = \Phi(x) - \frac{1}{2}$ where $\Phi(x)$ is the (cumulative) Gaussian distribution function, we find that $\mathrm{erf}(1) = 0.8427$.

From the calculated series, the successive partial sums corresponding to $1, 2, 3, \ldots$ terms are 1.1284, 0.7523, 0.8651, 0.8382, 0.8434, 0.8426, 0.8427, 0.8427, ... Thus seven terms are needed to obtain the desired accuracy.

(b) We start with

$$\mathrm{erfc}(x) = R(x) = e^{-x^2}\sum_{n=0}^{\infty}\frac{a_n}{x^n}.$$

Now,

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du \quad \Rightarrow \quad \frac{dR}{dx} = -\frac{2}{\sqrt{\pi}}\, e^{-x^2}.$$

Substituting for $R(x)$ and differentiating gives, as the equation to be satisfied,

$$-2xe^{-x^2} \sum_{n=0}^{\infty} \frac{a_n}{x^n} + e^{-x^2} \sum_{n=1}^{\infty} \frac{(-n)a_n}{x^{n+1}} = -\frac{2}{\sqrt{\pi}}\, e^{-x^2}.$$

Equating the coefficients of x and the constant terms gives $a_0 = 0$ and $a_1 = 1/\sqrt{\pi}$, whilst equating inverse powers of x^{n-1} yields the recurrence relation

$$-2a_n + [-(n-2)]a_{n-2} = 0.$$

Thus, only the terms with n odd are present and

$$\begin{aligned} a_{2k+1} &= -\frac{(2k-1)}{2}\, a_{2k-1} = \cdots = \frac{(-1)^k (2k-1)(2k-2)\cdots 1}{2^k}\, a_1 \\ &= \frac{(-1)^k (2k-1)!!}{2^k \sqrt{\pi}}, \end{aligned}$$

where $(2k-1)!!$ denotes the product $1 \times 3 \times 5 \times \cdots \times (2k-1)$.

The explicit form of $R(x)$ is therefore

$$R(x) = \frac{e^{-x^2}}{\sqrt{\pi}} \left(\frac{1}{x} + \sum_{k=1}^{\infty} \frac{(-1)^k\, (2k-1)!!}{2^k\, x^{2k+1}} \right).$$

Examination, as $k \to \infty$, of the modulus of the ratio of successive terms in the sum, which is $(2k+1)/2x^2$, shows that the series does not converge for any finite fixed x. However, if the series in truncated at $k = K$ with value $R(x, K)$ then,

$$\begin{aligned} R(x, K-1) < \operatorname{erfc}(x) < R(x, K) \qquad &\text{if } K \text{ is even,} \\ R(x, K-1) > \operatorname{erfc}(x) > R(x, K) \qquad &\text{if } K \text{ is odd.} \end{aligned}$$

Thus successive pairs of values of the partial sum bracket the true value of $\operatorname{erfc}(x)$, but with the bracketting range ultimately getting larger (rather than smaller). Which value of K gives the tightest bounds on $\operatorname{erfc}(x)$ depends upon the value of x; the best pair of values for K are probably the two integers that bracket $x^2 - \frac{1}{2}$.

For erfc(1) we have the series

$$\operatorname{erfc}(1) \approx \frac{e^{-1}}{\sqrt{\pi}} \left(\frac{1}{1} - \frac{1}{2} + \frac{3}{4} - \frac{3 \times 5}{8} + \cdots \right).$$

The partial sums for 1, 2, … terms are $0.2076, 0.1038, 0.2594, -0.1627, \ldots$. The correct answer [see part (a)] is $1.000 - 0.8427 = 0.1573$. This behaviour of the partial sums is as expected, with the correct value always lying between any successive pair and the tightest bounds given by $K = 0$ (i.e. just the first term)

and $K = 1$; in fact it lies roughly mid-way between the two. Later terms cause the partial sum to swing with increasing amplitude on either side of the correct value.

18.20 *The Bessel function $J_\nu(z)$ can be considered as a special case of the solution $M(a, c; z)$ of the confluent hypergeometric equation, the connection being*

$$\lim_{a\to\infty} \frac{M(a, \nu+1; -z/a)}{\Gamma(\nu+1)} = z^{-\nu/2} J_\nu(2\sqrt{z}).$$

Prove this equality by writing each side in terms of an infinite series and showing that the series are the same.

The hypergeometric series can be written more compactly by introducing the notation

$$(c)_n \equiv c(c+1)(c+2)\cdots(c+n-1) \quad \text{with} \quad (c)_0 = 1,$$

for the n-factor product. We note that $\lim_{c\to\infty}(c)_n/c^n = 1$ and that

$$\Gamma(\nu+1)(\nu+1)_n = \Gamma(\nu+n+1).$$

On the one hand, with this notation,

$$\begin{aligned}
\lim_{a\to\infty} \frac{M(a, \nu+1; -z/a)}{\Gamma(\nu+1)} &= \frac{1}{\Gamma(\nu+1)} \lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(a)_n}{n!\,(\nu+1)_n} \left(-\frac{z}{a}\right)^n \\
&= \frac{1}{\Gamma(\nu+1)} \lim_{a\to\infty} \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,(\nu+1)_n} \left[\frac{(a)_n}{a^n}\right] \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n z^n}{n!\,\Gamma(\nu+n+1)}.
\end{aligned}$$

But, on the other hand, from the standard series for the Bessel function of order ν,

$$\begin{aligned}
z^{-\nu/2} J_\nu(2\sqrt{z}) &= z^{-\nu/2} \sum_{n=0}^{\infty} \frac{(-1)^n (\frac{1}{2}\, 2\sqrt{z})^{\nu+2n}}{n!\,\Gamma(\nu+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (\sqrt{z})^{-\nu+\nu+2n}}{n!\,\Gamma(\nu+n+1)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n (z)^n}{n!\,\Gamma(\nu+n+1)}.
\end{aligned}$$

Thus the two series expressions are the same and the equality is established.

18.22 *Show from its definition that the Bessel function of the second kind of integral order ν can be written as*

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial \mu} - (-1)^\nu \frac{\partial J_{-\mu}(z)}{\partial \mu}\right]_{\mu=\nu}.$$

Using the explicit series expression for $J_\mu(z)$, show that $\partial J_\mu(z)/\partial\mu$ can be written as

$$J_\nu(z)\ln\left(\frac{z}{2}\right) + g(\nu, z),$$

and deduce that $Y_\nu(z)$ can be expressed as

$$Y_\nu(z) = \frac{2}{\pi}J_\nu(z)\ln\left(\frac{z}{2}\right) + h(\nu, z),$$

where $h(\nu, z)$, like $g(\nu, z)$, is a power series in z.

Using the fact that, for integer ν, $J_{-\nu}(z) = (-1)^\nu J_\nu(z)$, direct substitution in the definition of $Y_\nu(z)$ produces the indeterminate equation

$$Y_\nu(z) \equiv \lim_{\mu\to\nu}\left[\frac{J_\mu(z)\cos\mu\pi - J_{-\mu}(z)}{\sin\mu\pi}\right] = \left[\frac{J_\nu(z)(-1)^\nu - (-1)^\nu J_\nu(z)}{\sin\nu\pi}\right] = \frac{0}{0}.$$

We therefore employ l'Hôpital's rule:

$$\begin{aligned} Y_\nu(z) &\equiv \lim_{\mu\to\nu}\left[\frac{J_\mu(z)\cos\mu\pi - J_{-\mu}(z)}{\sin\mu\pi}\right] \\ &= \lim_{\mu\to\nu}\left[\frac{-\pi\sin(\mu\pi)J_\mu(z) + \cos\mu\pi\dot{J}_\mu(z) - \dot{J}_{-\mu}(z)}{\pi\cos\mu\pi}\right], \end{aligned}$$

where $\dot{J}_\mu(z) = \dfrac{\partial J_\mu(z)}{\partial\mu}$. Thus,

$$Y_\nu(z) = \frac{1}{\pi}\left[\frac{\partial J_\mu(z)}{\partial\mu} - (-1)^\nu\frac{\partial J_{-\mu}(z)}{\partial\mu}\right]_{\mu=\nu}. \quad (*)$$

Now, we have as an explicit series represntation of $J_\mu(z)$

$$J_\mu(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(\mu+n+1)}\left(\frac{z}{2}\right)^{\mu+2n}.$$

We need the partial derivative of this with respect to μ and since μ appears in each term as an exponent, as well as part of a multiplicative factor, each term in the series will generate two terms in the derivative, one of which will contain a logarithm. This is a particular example of the general result that the derivative of x^μ with respect to μ is $x^\mu\ln x$. Carrying this through, the derivative with respect

to μ is given by

$$\begin{aligned}\frac{\partial J_\mu}{\partial \mu} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n} \left[\frac{1}{\Gamma(\mu+n+1)} \frac{\partial}{\partial \mu} \left(\frac{z}{2}\right)^{\mu} \right. \\ &\qquad \left. + \left(\frac{z}{2}\right)^{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{\Gamma(\mu+n+1)} \right) \right] \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{z}{2}\right)^{2n} \left[\frac{1}{\Gamma(\mu+n+1)} \left(\frac{z}{2}\right)^{\mu} \ln\left(\frac{z}{2}\right) \right. \\ &\qquad \left. + \left(\frac{z}{2}\right)^{\mu} \frac{\partial}{\partial \mu} \left(\frac{1}{\Gamma(\mu+n+1)} \right) \right].\end{aligned}$$

Hence, to obtain the second solution we set $\mu = v$:

$$\left(\frac{\partial J_\mu}{\partial \mu}\right)_{\mu=v} = J_v(z) \ln\left(\frac{z}{2}\right) + g(v, z),$$

where $g(v, z)$ is a power series in z. The coefficients in the power series are complicated, but well defined, functions of v and n involving Γ-functions and their derivatives.

Setting $\mu = -v$ produces the corresponding result

$$\begin{aligned}\left(\frac{\partial J_\mu}{\partial \mu}\right)_{\mu=-v} &= -J_{-v}(z) \ln\left(\frac{z}{2}\right) + g(-v, z) \\ &= -(-1)^v J_v(z) \ln\left(\frac{z}{2}\right) + g(-v, z).\end{aligned}$$

We note that the early terms in $g(-v, z)$ will be absent because of the presence of (the infinite quantities) $\Gamma(n + 1 - v)$ and their derivatives in the denominator.

Finally, on substituting in (*), we note that the two logarithmic terms contribute in the same sense (as opposed to cancelling) and we have

$$Y_v(z) = \frac{2}{\pi} J_v(z) \ln\left(\frac{z}{2}\right) + h(v, z),$$

where $h(v, z)$ is a power series in z.

18.24 *The solutions $y(x,a)$ of the equation*

$$\frac{d^2y}{dx^2} - (\tfrac{1}{4}x^2 + a)y = 0 \qquad (*)$$

are known as parabolic cylinder functions.

(a) *If $y(x,a)$ is a solution of $(*)$, determine which of the following are also solutions:* (i) $y(a,-x)$, (ii) $y(-a,x)$, (iii) $y(a,ix)$ *and* (iv) $y(-a,ix)$.

(b) *Show that one solution of $(*)$, even in x, is*

$$y_1(x,a) = e^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2),$$

where $M(\alpha,c,z)$ is the confluent hypergeometric function satisfying

$$z\frac{d^2M}{dz^2} + (c-z)\frac{dM}{dz} - \alpha M = 0.$$

You may assume (or prove) that a second solution, odd in x, is given by $y_2(x,a) = xe^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{3}{4}, \tfrac{3}{2}, \tfrac{1}{2}x^2)$.

(c) *Find, as an infinite series, an explicit expression for $e^{x^2/4}y_1(x,a)$.*

(d) *Using the results from part* (a) *show that $y_1(x,a)$ can also be written as*

$$y_1(x,a) = e^{x^2/4}M(-\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}x^2)$$

(e) *By making a suitable choice for a deduce that*

$$1 + \sum_{n=1}^{\infty}\frac{b_n\,x^{2n}}{(2n)!} = e^{x^2/2}\left(1 + \sum_{n=1}^{\infty}\frac{(-1)^n\,b_n\,x^{2n}}{(2n)!}\right),$$

where $b_n = \prod_{r=1}^{n}(2r - \frac{3}{2})$.

(a) When changing x to μx the second derivative of y is multiplied by μ^{-2} and the factor x^2 by μ^2. Thus

(i) The equation becomes $(-1)^{-2}y'' - (\frac{1}{4}(-1)^2x^2 + a)y = 0$, i.e is unaltered. Thus $y(a,-x)$ is also a solution.

(ii) The equation becomes $y'' - (\frac{1}{4}x^2 - a)y = 0$, i.e is a different equation. Thus $y(-a,x)$ is not a solution of $(*)$.

(iii) The equation becomes $(i^{-2})y'' - (\frac{1}{4}(i^2)x^2 + a)y = 0$. This is the same equation as in part (ii). Thus $y(a,ix)$ is not a solution of $(*)$.

(iv) The equation becomes $(i^{-2})y'' - (\frac{1}{4}(i^2)x^2 - a)y = 0$, i.e is unaltered. Thus $y(-a,ix)$ is a second solution of $(*)$.

(b) We first write $y_1(x,a)$ as $y_1(x,a) = e^{-x^2/4}u(x)$ and determine the equation $u(x)$

must satisfy. The function and derivatives needed are

$$\begin{aligned} y_1 &= e^{-x^2/4}u, \\ y_1' &= -\frac{x}{2}e^{-x^2/4}u + e^{-x^2/4}u', \\ y_1'' &= -\frac{1}{2}e^{-x^2/4}u + \frac{x^2}{4}\,e^{-x^2/4}u - 2\frac{x}{2}\,e^{-x^2/4}u' + e^{-x^2/4}u''. \end{aligned}$$

Thus, (cancelling a factor $e^{-x^2/4}$) substitution in (*) yields

$$\begin{aligned} -\frac{1}{2}u + \frac{x^2}{4}\,u - xu' + u'' - \frac{x^2}{4}\,u - au &= 0, \\ \Rightarrow \quad u'' - xu' - (a + \tfrac{1}{2})u &= 0. \qquad (**) \end{aligned}$$

Now, the equation satisfied by $M(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, z)$ is

$$z\frac{d^2M}{dz^2} + (\tfrac{1}{2} - z)\frac{dM}{dz} - (\tfrac{1}{2}a + \tfrac{1}{4})M = 0.$$

In this we set $z = \frac{1}{2}x^2$, with $d/dz = x^{-1}d/dx$, and write $M(z) = N(x)$, obtaining

$$\begin{aligned} \frac{x^2}{2}\,\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{dN}{dx}\right) + \left(\frac{1}{2} - \frac{x^2}{2}\right)\frac{1}{x}\,\frac{dN}{dx} - \left(\frac{1}{2}a + \frac{1}{4}\right)N &= 0, \\ \frac{x}{2}\left(-\frac{1}{x^2}\,N' + \frac{1}{x}\,N''\right) + \frac{1}{2x}N' - \frac{x}{2}N' - \left(\frac{1}{2}a + \frac{1}{4}\right)N &= 0, \\ N'' - xN' - (a + \tfrac{1}{2})N &= 0. \end{aligned}$$

This is the same equation as (**) thus establishing that $y_1(a, x)$ can be written as

$$y_1(x, a) = e^{-x^2/4}M(\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}x^2).$$

Since the confluent hypergeometric function is a polynomial function of its third argument the solution is clearly even in x.

(c) With the result established in part (b) we need only evaluate a typical term of the hypergeometric series for $M(\frac{1}{2}a + \frac{1}{4}, \frac{1}{2}, \frac{1}{2}x^2)$. The zeroth term is 1 and the nth term ($n > 0$) is

$$\begin{aligned} t_n &= \frac{(\frac{1}{2}a + \frac{1}{4})(\frac{1}{2}a + \frac{5}{4})\cdots(\frac{1}{2}a + \frac{1}{4} + n - 1)}{(\frac{1}{2})(\frac{3}{2})\cdots(\frac{2n-1}{2})}\,\frac{1}{n!}\,\left(\frac{x^2}{2}\right)^n \\ &= \frac{(a + \frac{1}{2})(a + \frac{5}{2})\cdots(a + 2n - \frac{3}{2})}{(1)(3)\cdots(2n-1)\,(2^n\,n!)}\,x^{2n}. \end{aligned}$$

Writing the numerator as a product and noting that $2^n\,n! = (2)(4)\cdots(2n)$, we can now write the whole series as

$$e^{x^2/4}y_1(x, a) = 1 + \sum_{n=1}^{\infty}\frac{\prod_{r=1}^{n}(a + 2r - \frac{3}{2})}{(2n)!}\,x^{2n}.$$

(d) In part (a)(iv) we showed that since $y_1(x, a)$ is a solution of $(*)$ so is $y_1(-a, ix)$, i.e

$$\begin{aligned} y_3(x, a) \equiv y_1(-a, ix) &= e^{-i^2x^2/4} M(-\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, \tfrac{1}{2}i^2x^2) \\ &= e^{x^2/4} M(-\tfrac{1}{2}a + \tfrac{1}{4}, \tfrac{1}{2}, -\tfrac{1}{2}x^2) \end{aligned}$$

must also be a solution of $(*)$.

Since we already have two linearly independent solutions of $(*)$, namely y_1 and y_2, and $(*)$ is only a 2nd-order equation, y_3 must be linearly dependent on y_1 and y_2. However, it is clearly an even function of x and so it must be a multiple λ of y_1. Further, since $M(\alpha, c, 0) = 1$ for all α and c, and $\exp(\pm 0^2/4) = 1$, we conclude from setting $x = 0$ that $\lambda = 1$ and consequently that $y_3(x, a) = y_1(x, a)$.

(e) Expressing this last result in term of the series representions of the parabolic cylinder functions gives the equality

$$\begin{aligned} y_1(x, a) &= e^{-x^2/4} \left[1 + \sum_{n=1}^{\infty} \frac{\prod_{r=1}^{n}(a + 2r - \frac{3}{2})}{(2n)!} x^{2n} \right] \\ = y_3(x, a) &= e^{x^2/4} \left[1 + \sum_{n=1}^{\infty} \frac{\prod_{r=1}^{n}(-a + 2r - \frac{3}{2})}{(2n)!} (-x^2)^n \right] \\ &= e^{x^2/4} \left[1 + \sum_{n=1}^{\infty} \frac{\prod_{r=1}^{n}(a + \frac{3}{2} - 2r)}{(2n)!} x^{2n} \right]. \end{aligned}$$

Now, choosing $a = 0$ and writing $\prod_{r=1}^{n}(2r - \frac{3}{2})$ as b_n reduces the equality of the first and third lines to

$$e^{-x^2/4} \left[1 + \sum_{n=1}^{\infty} \frac{b_n}{(2n)!} x^{2n} \right] = e^{x^2/4} \left[1 + \sum_{n=1}^{\infty} \frac{(-1)^n b_n}{(2n)!} x^{2n} \right]$$

from which the stated result follows immediately.

19

Quantum operators

19.2 *By expressing the operator L_z, corresponding to the z-component of angular momentum, in spherical polar coordinates (r, θ, ϕ), show that the angular momentum of a particle about the polar axis cannot be known at the same time as its azimuthal position around that axis.*

The expression for L_z in Cartesian coordinates is

$$L_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right),$$

the connections with spherical polar coordinates being

$$x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi, \qquad z = r\cos\theta,$$
$$r^2 = x^2 + y^2 + z^2, \quad \theta = \tan^{-1}\frac{(x^2+y^2)^{1/2}}{z}, \quad \phi = \tan^{-1}\frac{y}{x}.$$

Using the chain rule, we have

$$\begin{aligned}\frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial}{\partial\theta} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial\phi}\\ &= \frac{y}{r}\frac{\partial}{\partial r} + \frac{zy}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial\theta} + \frac{x}{x^2+y^2}\frac{\partial}{\partial\phi}.\end{aligned}$$

Similarly,

$$\frac{\partial}{\partial x} = \frac{x}{r}\frac{\partial}{\partial r} + \frac{zx}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial\theta} - \frac{y}{x^2+y^2}\frac{\partial}{\partial\phi}.$$

Thus,

$$x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} = \frac{xy-yx}{r}\frac{\partial}{\partial r} + \frac{z(xy-yx)}{r^2(x^2+y^2)^{1/2}}\frac{\partial}{\partial\theta} + \frac{x^2+y^2}{x^2+y^2}\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\phi}.$$

Thus, expressed in spherical polar coordinates, $L_z = -i\hbar\partial/\partial\phi$. To establish a relationship between the uncertainties, ΔL_z and $\Delta\phi$, in the z-component of the angular momentum and angular position about the z-axis, we need to evaluate the commutator of L_z and ϕ. This is done by considering

$$[L_z, \phi]\,|\psi\rangle = -i\hbar\frac{\partial}{\partial\phi}(\phi|\psi\rangle) + i\hbar\phi\frac{\partial}{\partial\phi}|\psi\rangle = -i\hbar|\psi\rangle,$$

i.e. $[L_z, \phi] = -i\hbar$. Since the commutator is a non-zero constant, comparison with the case of $[p_x, x]$ shows that $\Delta L_z \times \Delta\phi \geq \frac{1}{2}\hbar$, whatever the state $|\psi\rangle$ of the system. That is, if the value of the z-component of the angular momentum is known ($\Delta L_z = 0$), the value of ϕ (considered as a multivalued function) is completely unknown.

[The transformation of the Cartesian expression for L_z is very much simpler if cylindrical, rather than spherical, polar coordinates are used, as the reader may wish to verify. The result is the same, as it must be, since ϕ has the same meaning in both systems of coordinates.]

19.4 *Show that the Pauli matrices*

$$\mathsf{S}_x = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{S}_y = \tfrac{1}{2}\hbar\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \mathsf{S}_z = \tfrac{1}{2}\hbar\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which are used as the operators corresponding to intrinsic spin of $\frac{1}{2}\hbar$ *in non-relativistic quantum mechanics, satisfy* $\mathsf{S}_x^2 = \mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$ *and have the same commutation properties as the components of orbital angular momentum. Deduce that any state* $|\psi\rangle$ *represented by the column vector* $(a,\ b)^{\mathrm{T}}$ *is an eigenstate of* S^2 *with eigenvalue* $3\hbar^2/4$.

We note that all the S_j are Hermitian and evaluate their various possible products.

$$\mathsf{S}_x^2 = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \frac{\hbar^2}{4}\mathsf{I},$$

$$\mathsf{S}_x\mathsf{S}_y = \frac{\hbar^2}{4}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \frac{i\hbar}{2}\mathsf{S}_z,$$

$$\mathsf{S}_y\mathsf{S}_x = \frac{\hbar^2}{4}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \frac{\hbar^2}{4}\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = -\frac{i\hbar}{2}\mathsf{S}_z.$$

Similarly, $\mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$ and

$$\mathsf{S}_y\mathsf{S}_z = \frac{i\hbar}{2}\mathsf{S}_x = -\mathsf{S}_z\mathsf{S}_y,$$

$$\mathsf{S}_z\mathsf{S}_x = \frac{i\hbar}{2}\mathsf{S}_y = -\mathsf{S}_x\mathsf{S}_z.$$

Thus,

$$\left[\mathsf{S}_x, \mathsf{S}_y\right] = \mathsf{S}_x\mathsf{S}_y - \mathsf{S}_y\mathsf{S}_x = \frac{i\hbar}{2}\,\mathsf{S}_z - \left(-\frac{i\hbar}{2}\,\mathsf{S}_z\right) = i\hbar\mathsf{S}_z,$$

and similarly

$$\left[\mathsf{S}_y, \mathsf{S}_z\right] = i\hbar\mathsf{S}_x \quad \text{and} \quad \left[\mathsf{S}_z, \mathsf{S}_x\right] = i\hbar\mathsf{S}_y.$$

Thus the commutators have the same structure as those for L_x, L_y and L_z in equation (19.27).

Since $\mathsf{S}_x^2 = \mathsf{S}_y^2 = \mathsf{S}_z^2 = \frac{1}{4}\hbar^2\mathsf{I}$, and $\mathsf{S}^2 = \mathsf{S}_x^2 + \mathsf{S}_y^2 + \mathsf{S}_z^2 = \frac{3}{4}\hbar^2\mathsf{I}$,

$$\mathsf{S}^2\begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{3}{4}\hbar^2\mathsf{I}\begin{pmatrix} a \\ b \end{pmatrix} = \tfrac{3}{4}\hbar^2\begin{pmatrix} a \\ b \end{pmatrix}$$

for any a and b,i.e. any such state is an eigenstate of S^2 with eigenvalue $3\hbar^2/4$.

19.6 *Operators A and B anti-commute. Evaluate $(A+B)^{2n}$ for a few values of n and hence propose an expression for c_{nr} in the expansion*

$$(A+B)^{2n} = \sum_{r=0}^{n} c_{nr}\, A^{2n-2r}\, B^{2r}.$$

Prove your proposed formula for general values of n, using the method of induction.

Show that

$$\cos(A+B) = \sum_{n=0}^{\infty}\sum_{r=0}^{n} d_{nr}\, A^{2n-2r} B^{2r},$$

where the d_{nr} are constants whose values you should determine.

By taking as A the matrix $\mathsf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, confirm that your answer is consistent with that obtained in exercise 19.5.

First a few trials, noting that $B^pA^q = (-1)^pAB^pA^{q-1} = \cdots = (-1)^{pq}A^qB^p$.

$n = 1 \quad (A+B)(A+B) = A^2+AB+BA+B^2 = A^2+B^2$, since $AB = -BA$.

$$\begin{aligned} n = 2 \quad (A+B)^4 &= (A^2+B^2)^2 \\ &= A^4 + A^2B^2 + B^2A^2 + B^4 \\ &= A^4 + A^2B^2 + (-1)^4A^2B^2 + B^4 \\ &= A^4 + 2A^2B^2 + B^4. \end{aligned}$$

$$
\begin{aligned}
n = 3 \qquad (A+B)^6 &= (A^2+B^2)(A^4+2A^2B^2+B^4) \\
&= A^6 + 2A^4B^2 + A^2B^4 + B^2A^4 + 2B^2A^2B^2 + B^6 \\
&= A^6 + 2A^4B^2 + A^2B^4 + (-1)^8A^4B^2 + 2(-1)^4A^2B^4 + B^6 \\
&= A^6 + 3A^4B^2 + 3A^2B^4 + B^6.
\end{aligned}
$$

The obvious indication is that

$$
(A+B)^{2n} = \sum_{r=0}^{n} {}^nC_r\, A^{2n-2r}\, B^{2r} \qquad (*).
$$

To prove this result for general n, we assume that it is true for a particular value of n and consider $(A+B)^{2n+2}$

$$
\begin{aligned}
&= (A^2+B^2)\sum_{r=0}^{n} {}^nC_r\, A^{2n-2r}\, B^{2r} \\
&= \sum_{r=0}^{n} {}^nC_r\, A^{2n+2-2r}\, B^{2r} + \sum_{r=0}^{n} {}^nC_r\,(-1)^{4n-4r}\, A^{2n-2r}\, B^{2r+2} \\
&= \sum_{r=0}^{n} {}^nC_r\, A^{2n+2-2r}\, B^{2r} + \sum_{s=1}^{n+1} {}^nC_{s-1}\, A^{2n-2s+2}\, B^{2s}, \text{ with } s = r+1, \\
&= {}^nC_0 A^{2n+2}B^0 + \sum_{r=1}^{n}\left[\left({}^nC_r + {}^nC_{r-1}\right) A^{2n+2-2r}\, B^{2r}\right] + {}^nC_n A^0 B^{2n+2}.
\end{aligned}
$$

Now

$$
\begin{aligned}
{}^nC_r + {}^nC_{r-1} &= \frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} \\
&= \frac{(n+1)!}{(n+1-r)!\,r!} = {}^{n+1}C_r
\end{aligned}
$$

whilst

$$
{}^nC_0 = 1 = {}^{n+1}C_0 \quad \text{and} \quad {}^nC_n = 1 = {}^{n+1}C_{n+1}.
$$

Thus,

$$
\begin{aligned}
(A+B)^{2n+2} &= {}^{n+1}C_0 A^{2n+2}B^0 + \sum_{r=1}^{n} {}^{n+1}C_r\, A^{2n+2-2r}\, B^{2r} \\
&\qquad + {}^{n+1}C_{n+1} A^0 B^{2n+2} \\
&= \sum_{r=0}^{n+1} {}^{n+1}C_r\, A^{2n+2-2r}\, B^{2r},
\end{aligned}
$$

i.e. the same form as $(*)$ but with $n \to n+1$, thus proving the form for general n, since it has already been shown to be valid for $n = 1$.

We calculate the cosine function from its defining series

$$\cos(A+B) = \sum_{n=0}^{\infty}(-1)^n \frac{(A+B)^{2n}}{(2n)!}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{(2n)!} \sum_{r=0}^{n} \frac{A^{2n-2r}\, B^{2r}}{(n-r)!\, r!}.$$

Thus

$$d_{nr} = \frac{(-1)^n\, n!}{(2n)!\,(n-r)!\, r!}.$$

Since in exercise 19.5 $\mathsf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, in order to use the given form of A we must take

$$\mathsf{B} = \mathsf{C} - \mathsf{A} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Now,

$$\mathsf{AB} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\mathsf{BA};$$

so they do anticommute, and we can apply our previous result. But

$$\mathsf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\text{and} \quad \mathsf{B}^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It follows that

$$\cos \mathsf{C} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} \frac{(-1)^n\, n!}{(2n)!\,(n-r)!\, r!}\, \mathsf{I}^{n-r}\, \mathsf{I}^r.$$

But

$$\sum_{r=0}^{n} \frac{n!}{(n-r)!\, r!} = (1+1)^n = 2^n,$$

and so

$$\cos \mathsf{C} = \sum_{n=0}^{\infty} \frac{(-1)^n\, 2^n}{(2n)!}\, \mathsf{I} = (\cos\sqrt{2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

as in exercise 19.5.

19.8 *For a system of N electrons in their ground state $|0\rangle$, the Hamiltonian is*

$$H = \sum_{n=1}^{N} \frac{p_{x_n}^2 + p_{y_n}^2 + p_{z_n}^2}{2m} + \sum_{n=1}^{N} V(x_n, y_n, z_n).$$

Show that $\left[p_{x_n}^2, x_n \right] = -2i\hbar p_{x_n}$, and hence that the expectation value of the double commutator $[\,[\,x, H\,], x\,]$, where $x = \sum_{n=1}^{N} x_n$ is given by

$$\langle 0 \,|\, [\,[\,x, H\,], x\,] \,|\, 0\rangle = \frac{N\hbar^2}{m}.$$

Now evaluate the expectation value using the eigenvalue properties of H, namely $H|r\rangle = E_r|r\rangle$, and deduce the sum rule for oscillation strengths,

$$\sum_{r=0}^{\infty} (E_r - E_0) |\langle r \,|\, x \,|\, 0\rangle|^2 = \frac{N\hbar^2}{2m}.$$

First we evaluate the commutator

$$\begin{aligned} \left[p_{x_n}^2, x_n \right] &= p_{x_n} \left[p_{x_n}, x \right] + \left[p_{x_n}, x \right] p_{x_n} \\ &= p_{x_n}(-i\hbar) + (-i\hbar) p_{x_n} = -2i\hbar p_{x_n}. \end{aligned}$$

Now, all variables with differing values of n, or referring to different coordinate directions even if n is the same, commute with each other whilst each x_m commutes with $V(x_n, y_n, z_n)$. Consequently the only non-zero terms in the commutator $[\,x, H\,]$ are terms like $\left[x_n, p_{x_n}^2/2m \right]$ which, as shown above, have the values $i\hbar p_{x_n}/m$. Thus,

$$D \equiv \langle 0 \,|\, [\,[\,x, H\,], x\,] \,|\, 0\rangle = \langle 0 \,|\, \sum_{n=1}^{N} \frac{i\hbar}{m} \left[p_{x_n}, x_n \right] \,|\, 0\rangle = \frac{i\hbar}{m}(-i\hbar)N = \frac{N\hbar^2}{m}.$$

We now evaluate D in a different way, making use of result (19.11):

$$\begin{aligned} D &= \langle 0 \,|\, (xH - Hx)x \,|\, 0\rangle - \langle 0 \,|\, x(xH - Hx) \,|\, 0\rangle \\ &= \sum_{r=0}^{\infty} \langle 0 \,|\, (xH - Hx) \,|\, r\rangle \langle r \,|\, x \,|\, 0\rangle - \sum_{r=0}^{\infty} \langle 0 \,|\, x \,|\, r\rangle \langle r \,|\, (xH - Hx) \,|\, 0\rangle \\ &= \sum_{r=0}^{\infty} \langle 0 \,|\, (xE_r - E_0 x) \,|\, r\rangle \langle r \,|\, x \,|\, 0\rangle - \sum_{r=0}^{\infty} \langle 0 \,|\, x \,|\, r\rangle \langle r \,|\, (xE_0 - E_r x) \,|\, 0\rangle \\ &= 2\sum_{r=0}^{\infty} (E_r - E_0) \langle 0 \,|\, x \,|\, r\rangle \langle r \,|\, x \,|\, 0\rangle = 2\sum_{r=0}^{\infty} (E_r - E_0) |\langle r \,|\, x \,|\, 0\rangle|^2 \end{aligned}$$

Equating the two expressions for D gives the stated result.

19.10 *For a system containing more than one particle, the total angular momentum J and its components are represented by operators that have completely analogous commutation relations to those for the operators for a single particle, i.e. J^2 has eigenvalue $j(j+1)\hbar^2$ and J_z has eigenvalue $m_j\hbar$ for the state $|j, m_j\rangle$. The usual orthonormality relationship $\langle j', m'_j \,|\, j, m_j\rangle = \delta_{j'j}\,\delta_{m'_j m_j}$ is also valid.*

A system consists of two (distinguishable) particles A and B. Particle A is in an $\ell = 3$ state and can have state functions of the form $|A, 3, m_A\rangle$ whilst B is in an $\ell = 2$ state with possible state functions $|B, 2, m_B\rangle$. The range of possible values for j is $|3-2| \le j \le |3+2|$, i.e. $1 \le j \le 5$, and the overall state function can be written as

$$|j, m_j\rangle = \sum_{m_A+m_B=m_j} C^{j\,m_j}_{m_A\,m_B}\,|\,A, 3, m_A\rangle\,|\,B, 2, m_B\rangle.$$

The numerical coefficients $C^{j\,m_j}_{m_A\,m_B}$ are known as Clebsch–Gordon *coefficients.*

Assume (as can be shown) that the ladder operators $U(AB)$ and $D(AB)$ for the system can be written as $U(A)+U(B)$ and $D(A)+D(B)$ respectively and that they lead to relationships equivalent to (19.34) and (19.35) with ℓ replaced by j and m by m_j.

(a) *Apply the operators to the (obvious) relationship*

$$|AB, 5, 5\rangle = |A, 3, 3\rangle\,|B, 2, 2\rangle$$

to show that

$$|AB, 5, 4\rangle = \sqrt{\tfrac{6}{10}}\,|A, 3, 2\rangle\,|B, 2, 2\rangle + \sqrt{\tfrac{4}{10}}\,|A, 3, 3\rangle\,|B, 2, 1\rangle.$$

(b) *Find, to within an overall sign, the real coefficients c and d in the expansion*

$$|AB, 4, 4\rangle = c|A, 3, 2\rangle\,|B, 2, 2\rangle + d|A, 3, 3\rangle\,|B, 2, 1\rangle$$

by requiring it to be orthogonal to $|AB, 5, 4\rangle$. Check your answer by considering $U(AB)|AB, 4, 4\rangle$.

(c) *Find, to within an overall sign and as efficiently as possible, an expression for $|AB, 4, -3\rangle$ as a sum of products of the form $|A, 3, m_A\rangle\,|B, 2, m_B\rangle$.*

(a) We start with $|AB, 5, 5\rangle = |A, 3, 3\rangle\,|B, 2, 2\rangle$ and apply $D(AB) = D(A) + D(B)$ to both sides, yielding

$$\begin{aligned}\sqrt{(5)(6)-(5)(4)}\,|AB, 5, 4\rangle &= \sqrt{(3)(4)-(3)(2)}\,|A, 3, 2\rangle\,|B, 2, 2\rangle \\ &\quad + \sqrt{(2)(3)-(2)(1)}\,|A, 3, 3\rangle\,|B, 2, 1\rangle \\ |AB, 5, 4\rangle &= \sqrt{\tfrac{6}{10}}\,|A, 3, 2\rangle\,|B, 2, 2\rangle + \sqrt{\tfrac{4}{10}}\,|A, 3, 3\rangle\,|B, 2, 1\rangle.\end{aligned}$$

(b) Since

$$|AB,4,4\rangle = c|A,3,2\rangle\,|B,2,2\rangle + d|A,3,3\rangle\,|B,2,1\rangle$$

must be orthogonal to $|AB,5,4\rangle$, we have (remembering the orthonormality relation $\langle j',m'_j\,|\,j,m_j\rangle = \delta_{j'j}\,\delta_{m'_j m_j}$)

$$\begin{aligned} 0 &= \langle AB,5,4\,|\,AB,4,4\rangle \\ &= \sqrt{\tfrac{6}{10}}\,c(1)(1) + \sqrt{\tfrac{6}{10}}\,d(0)(0) + \sqrt{\tfrac{4}{10}}\,c(0)(0) + \sqrt{\tfrac{4}{10}}\,d(1)(1). \end{aligned}$$

It must also be normalised, and so

$$1 = \langle AB,4,4\,|\,AB,4,4\rangle = c^2(1)(1) + cd(0)(0) + dc(0)(0) + d^2(1)(1).$$

Thus $c = \pm(4/10)^{1/2}$ and $d = \mp(6/10)^{1/2}$.

As a check, consider

$$\begin{aligned} &U(AB)|AB,4,4\rangle \\ &= [\,U(A) + U(B)\,]\left[\sqrt{\tfrac{4}{10}}\,|A,3,2\rangle\,|B,2,2\rangle - \sqrt{\tfrac{6}{10}}\,|A,3,3\rangle\,|B,2,1\rangle\right] \\ &= \sqrt{\tfrac{4}{10}}\left(\sqrt{(3)(4)-(2)(3)}|A,3,3\rangle\,|B,2,2\rangle + |\emptyset\rangle\right) \\ &\qquad - \sqrt{\tfrac{6}{10}}\left(|\emptyset\rangle + \sqrt{(2)(3)-(1)(2)}|A,3,3\rangle\,|B,2,2\rangle\right) \\ &= \left(\sqrt{\tfrac{4}{10}}\sqrt{6} - \sqrt{\tfrac{6}{10}}\sqrt{4}\right)|A,3,3\rangle\,|B,2,2\rangle \\ &= |\emptyset\rangle, \text{ as it should.} \end{aligned}$$

(c) We abbreviate our notation from $|A,3,m_A\rangle\,|B,2,m_B\rangle$ to $|m_A\rangle\,|m_B\rangle$, in that order.
We start with the known relationship that is 'closest' to $|AB,4,-3\rangle$, namely

$$|AB,5,-5\rangle = |-3\rangle\,|-2\rangle,$$

and apply $U(AB)$ to both sides, obtaining

$$\begin{aligned} \sqrt{(5)(6)-(-5)(-4)}|AB,5,-4\rangle &= \sqrt{(3)(4)-(-3)(-2)}\,|-2\rangle\,|-2\rangle \\ &\quad + \sqrt{(2)(3)-(-2)(-1)}\,|-3\rangle\,|-1\rangle, \\ |AB,5,-4\rangle &= \sqrt{\tfrac{6}{10}}\,|-2\rangle\,|-2\rangle + \sqrt{\tfrac{4}{10}}\,|-3\rangle\,|-1\rangle. \end{aligned}$$

The state $|AB,4,4\rangle$ must be orthogonal to this, but consist of a different linear combination of the same two states. As it must be normalised it can only be

$$|AB,4,-4\rangle = \sqrt{\tfrac{4}{10}}\,|-2\rangle\,|-2\rangle - \sqrt{\tfrac{6}{10}}\,|-3\rangle\,|-1\rangle.$$

Now use $U(AB)$ again on both sides:

$$\begin{aligned}
&\sqrt{(4)(5)-(-4)(-3)}\,|AB,4,-3\rangle \\
&= \sqrt{\tfrac{4}{10}}\,\Big(\sqrt{(3)(4)-(-2)(-1)}\,|-1\rangle\,|-2\rangle \\
&\qquad\qquad + \sqrt{(2)(3)-(-2)(-1)}\,|-2\rangle\,|-1\rangle\Big) \\
&\quad - \sqrt{\tfrac{6}{10}}\,\Big(\sqrt{(3)(4)-(-3)(-2)}\,|-2\rangle\,|-1\rangle \\
&\qquad\qquad + \sqrt{(2)(3)-(-1)(0)}\,|-3\rangle\,|0\rangle\Big).
\end{aligned}$$

Simplifying the RHS of this equation and dividing through by $\sqrt{8}$ then gives

$$\begin{aligned}
|AB,4,-3\rangle &= \sqrt{\tfrac{1}{2}}\,|-1\rangle\,|-2\rangle + \left(\sqrt{\tfrac{1}{5}} - \sqrt{\tfrac{9}{20}}\right)|-2\rangle\,|-1\rangle \\
&\qquad - \sqrt{\tfrac{9}{20}}\,|-3\rangle\,|0\rangle \\
&= \sqrt{\tfrac{1}{2}}\,|-1\rangle\,|-2\rangle - \sqrt{\tfrac{1}{20}}\,|-2\rangle\,|-1\rangle - \sqrt{\tfrac{9}{20}}\,|-3\rangle\,|0\rangle.
\end{aligned}$$

This is the required expansion and, as expected, it is automatically normalised:

$$\left(\sqrt{\tfrac{1}{2}}\right)^2 + \left(-\sqrt{\tfrac{1}{20}}\right)^2 + \left(-\sqrt{\tfrac{9}{20}}\right)^2 = 1.$$

20

Partial differential equations: general and particular solutions

20.2 *Find partial differential equations satisfied by the following functions $u(x, y)$ for all arbitrary functions f and all arbitrary constants a and b:*

(a) $u(x, y) = f(x^2 - y^2)$;
(b) $u(x, y) = (x - a)^2 + (y - b)^2$;
(c) $u(x, y) = y^n f(y/x)$;
(d) $u(x, y) = f(x + ay)$.

In each case we need to generate enough partial derivatives of u that the arbitrary functions and constants can be eliminated by re-substitution.

(a) $\dfrac{\partial u}{\partial x} = 2xf', \quad \dfrac{\partial u}{\partial y} = -2yf' \quad \Rightarrow \quad y\dfrac{\partial u}{\partial x} + x\dfrac{\partial u}{\partial y} = 0.$

(b) $\dfrac{\partial u}{\partial x} = 2(x-a), \quad \dfrac{\partial u}{\partial y} = 2(y-b) \quad \Rightarrow \quad \left(\dfrac{\partial u}{\partial x}\right)^2 + \left(\dfrac{\partial u}{\partial y}\right)^2 = 4u.$

(c)
$$u(x, y) = y^n f\left(\frac{y}{x}\right),$$
$$\frac{\partial u}{\partial x} = y^n f' \frac{\partial}{\partial x}\left(\frac{y}{x}\right) = -\frac{y^{n+1}}{x^2} f',$$
$$\frac{\partial u}{\partial y} = ny^{n-1} f + y^n f' \frac{\partial}{\partial y}\left(\frac{y}{x}\right) = ny^{n-1} f + \frac{y^n}{x} f'.$$

Substituting for f and f' from the first two equations into the third one gives

$$\frac{\partial u}{\partial y} = n\frac{u}{y} - \frac{x}{y}\frac{\partial u}{\partial x},$$

which can be rearranged as

$$\Rightarrow \quad y\frac{\partial u}{\partial y} + x\frac{\partial u}{\partial x} = nu.$$

(d) Since both the constant a and the form of the function f are to be eliminated, second partial derivatives will be required.

$$\frac{\partial u}{\partial x} = f' \quad \text{and} \quad \frac{\partial u}{\partial y} = af' \quad \Rightarrow \quad \frac{\partial u}{\partial y} = a\frac{\partial u}{\partial x}.$$

Differentiating again with respect to x, say, and then eliminating a between the two PDEs gives

$$\frac{\partial^2 u}{\partial x \partial y} = a\frac{\partial^2 u}{\partial x^2} \text{ and hence } \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial y}$$

If the second partial derivative is taken with respect to y (rather than x) the equivalent result is

$$\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2}.$$

20.4 *Find the most general solutions $u(x, y)$ of the following equations, consistent with the boundary conditions stated:*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0, \quad u(x, 0) = 1 + \sin x;$

(b) $i\dfrac{\partial u}{\partial x} = 3\dfrac{\partial u}{\partial y}, \quad u = (4 + 3i)x^2$ *on the line* $x = y$;

(c) $\sin x \sin y\dfrac{\partial u}{\partial x} + \cos x \cos y\dfrac{\partial u}{\partial y} = 0, \quad u = \cos 2y$ *on* $x + y = \pi/2$;

(d) $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0, \quad u = 2$ *on the parabola* $y = x^2$.

In each case, we need to determine a $p = p(x, y)$ such that the solution for general x and y is $u(x, y) = f(p)$. The form of p will be determined by the PDE and that of f by the (given) form that u takes on the relevant boundary.

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 0 \quad \Rightarrow \quad \dfrac{dx}{y} = -\dfrac{dy}{x} \quad \Rightarrow \quad x^2 + y^2 = p.$

The given boundary is $y = 0$ and on this line the expression for p is $p = x^2$. For this to match the given form, $1 + \sin x$, the form of $f(p)$ must be

$$f(p) = 1 + \sin p^{1/2}.$$

This then determines the form of $u(x, y) = f(p)$ for all x and y, not just for $y = 0$ and general x:

$$u(x, y) = 1 + \sin[(x^2 + y^2)^{1/2}].$$

The remaining parts of this exercise are tackled in an analogous way and are given with little commentary.

(b) $i\dfrac{\partial u}{\partial x} = 3\dfrac{\partial u}{\partial y} \quad \Rightarrow \quad \dfrac{dx}{i} = -\dfrac{dy}{3} \quad \Rightarrow \quad 3x+iy = p.$

On $x = y$, $p = (3 + i)x$ and

$$\begin{aligned} u(x, x) = (4 + 3i)x^2 \quad &\Rightarrow \quad f(p) = \alpha p^2, \\ \text{where } \alpha(3 + i)^2 \quad &= \quad 4 + 3i, \\ \alpha[(9 - 1) + 6i] \quad &= \quad 4 + 3i \quad \Rightarrow \quad \alpha = \tfrac{1}{2}, \\ \Rightarrow \quad u(x, y) \quad &= \quad \tfrac{1}{2}p^2 = \tfrac{1}{2}(9x^2 + 6ixy - y^2). \end{aligned}$$

(c) For $\sin x \sin y \dfrac{\partial u}{\partial x} + \cos x \cos y \dfrac{\partial u}{\partial y} = 0, \quad u = \cos 2y$ on $x + y = \pi/2$,

$$\begin{aligned} \frac{dx}{\sin x \sin y} &= \frac{dy}{\cos x \cos y}, \\ \frac{\cos x\, dx}{\sin x} &= \frac{\sin y\, dy}{\cos y}, \\ \ln(\sin x) &= -\ln(\cos y) + k, \\ \sin x \cos y &= p. \end{aligned}$$

On $x + y = \frac{1}{2}\pi$, $u(x, y) = \cos 2y$ and

$$\begin{aligned} p &= \sin(\tfrac{1}{2}\pi - y)\cos y = \cos^2 y, \\ f(p) &= \cos 2y = 2\cos^2 y - 1 = 2p - 1, \\ u(x, y) &= 2p - 1 = 2\sin x \cos y - 1. \end{aligned}$$

(d) For $\dfrac{\partial u}{\partial x} + 2x\dfrac{\partial u}{\partial y} = 0$ with $u(x, y) = 2$ on $y = x^2$,

$$\begin{aligned} \frac{dx}{1} &= \frac{dy}{2x}, \\ x^2 - y &= p. \end{aligned}$$

On $y = x^2$, $p = 0$ and $f(p) = g(p) + 2$, where $g(p)$ is *any* function for which $g(0) = 0$. The general solution is $u(x, y) = g(y - x^2) + 2$; this indeterminacy is related to the boundary curve being a characteristic of the PDE.

20.6 *Find the most general solutions $u(x, y)$ of the following equations consistent with the boundary conditions stated:*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u = x^2$ *on the line* $y = 0$;

(b) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u(1, 0) = 2$;

(c) $y^2\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = x^2y^2(x^3 + y^3)$, *no boundary conditions.*

(a) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u = x^2$ on the line $y = 0$.

The CF is given by

$$\frac{dx}{y} = -\frac{dy}{x} \quad \Rightarrow \quad x^2 + y^2 = p.$$

An obvious PI is $u(x, y) = -3y$ and, as *any* valid PI will do, the general solution is

$$u(x, y) = f(x^2 + y^2) - 3y.$$

On $y = 0$, $p = x^2$ and

$$x^2 = u(x, 0) = p \quad \Rightarrow \quad f(p) = p \quad \Rightarrow \quad u(x, y) = p - 3y = x^2 + y^2 - 3y.$$

(b) $y\dfrac{\partial u}{\partial x} - x\dfrac{\partial u}{\partial y} = 3x, \quad u(1, 0) = 2.$

As in part (a), the general solution is $u(x, y) = f(x^2 + y^2) - 3y$.

At $(1, 0)$, $p = 1$ and we require $2 = u(1, 0) = f(1) - 0$. Thus,

$$f(x^2 + y^2) = 2 + g(x^2 + y^2) \text{ where } g(1) = 0.$$

Thus the most general solution consistent with the (one-point) boundary condition is

$$\begin{aligned} u(x, y) &= 2 - 3y + g(x^2 + y^2) \\ &\text{or } 2 - 3y + (x^2 + y^2 - 1) + h(x^2 + y^2) \\ &\text{or } 2 - 3y + \sin[(x^2 + y^2)\pi] + j(x^2 + y^2) \\ &\text{or} \ldots, \end{aligned}$$

where any arbitrary function not written explicitly has value 0 when its argument has value 1.

(c) For $y^2\dfrac{\partial u}{\partial x} + x^2\dfrac{\partial u}{\partial y} = x^2y^2(x^3 + y^3)$ with no boundary conditions.

The CF is found from

$$\frac{dx}{y^2} = \frac{dy}{x^2} \quad \Rightarrow \quad x^3 - y^3 = p.$$

From the symmetry between x and y in the equation, we are led to try $u(x, y) = \alpha(x^n + y^n)$ for some n and α as a possible PI. Substituting this trial solution:

$$\alpha n y^2 x^{n-1} + \alpha n x^2 y^{n-1} = y^2 x^5 + x^2 y^5.$$

This is satisfied if $n = 6$ and $\alpha = 1/6$. Thus

$$u(x, y) = \tfrac{1}{6}(x^6 + y^6) + f(x^3 - y^3),$$

where $f(p)$ is any function of p. As there are no boundary conditions to be satisfied, there are no restrictions on the form of f, provided it is differentiable.

20.8 *A function $u(x, y)$ satisfies*

$$2\frac{\partial u}{\partial x} + 3\frac{\partial u}{\partial y} = 10,$$

and takes the value 3 on the line $y = 4x$. Evaluate $u(2, 4)$.

To find the CF we set

$$\frac{dx}{2} = \frac{dy}{3} \quad \Rightarrow \quad 3x - 2y = p.$$

An elementary PI, obvious from inspection, is $u = 5x$. Consequently the general solution is $u(x, y) = f(p) + 5x$.

On the line $y = 4x$, we have $p = 3x - 2(4x) = -5x$ and so

$$3 = u(x, 4x) = f(-5x) + 5x = f(-5x) - (-5x) \quad \Rightarrow \quad f(p) = p + 3.$$

This gives the form of $f(p) = u(x, y)$ everywhere, not just on the line $y = 4x$, and so re-expressing it in terms of x and y shows that

$$u(x, y) = f(3x - 2y) + 5x = 3x - 2y + 3 + 5x = 8x - 2y + 3.$$

We can now compute $u(2, 4)$ as

$$u(2, 4) = 16 - 8 + 3 = 11.$$

20.10 *Consider the partial differential equation*

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0. \qquad (*)$$

(a) *Find the function* $u(x, y)$ *that satisfies* $(*)$ *and the boundary condition* $u = \partial u/\partial y = 1$ *when* $y = 0$ *for all* x. *Evaluate* $u(0, 1)$.

(b) *In which region of the* xy*-plane would* u *be determined if the boundary condition were* $u = \partial u/\partial y = 1$ *when* $y = 0$ *for all* $x > 0$?

(a) For solutions of the form $u(x, y) = f(x + \lambda y)$, λ must satisfy

$$1 - 3\lambda + 2\lambda^2 = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2},\ 1.$$

Thus the general solution is $u(x, y) = g(x + \frac{1}{2}y) + f(x + y) \equiv g(p_1) + f(p_2)$.

With the boundary conditions $u = \dfrac{\partial u}{\partial y} = 1$ for $y = 0$ and all x, $p_1 = p_2 = x$ on the boundary and

$$1 = u(x, 0) = g(x) + f(x), \qquad (*)$$

$$1 = \frac{\partial u}{\partial y}(x, 0) = \tfrac{1}{2}g'(x) + f'(x).$$

From $(*)$, $\quad 0 = g'(x) + f'(x).$

Subtracting, $\quad 1 = -\frac{1}{2}g'(x).$

Integrating, $\quad g(x) = -2x + k \quad \Rightarrow \quad f(x) = 2x - k + 1$, from $(*)$.

Hence,

$$\begin{aligned} u(x, y) &= -2(x + \tfrac{1}{2}y) + k + 2(x + y) - k + 1 \\ &= y + 1 \\ \Rightarrow \quad u(0, 1) &= 2. \end{aligned}$$

(b) For $u = \dfrac{\partial u}{\partial y} = 1$ for $y = 0$ and $x > 0$, the validity of the solution obtained in part (a) is restricted to the region whose characteristic curves intersect the *positive* x-axis (as opposed to the *whole* x-axis). The characteristic curves in this case are the families of straight lines

$$x + \tfrac{1}{2}y = p_1 \qquad \text{and} \qquad x + y = p_2.$$

For both families, the lowest value of p_i for which the curve cuts the positive x-axis is 0. [For negative values of p_i the curves cut the negative x-axis.] The common slope of the first family is -2 and for the second family it is -1. The two lines with these slopes that pass through the origin determine the limit of the region of validity of the solution (both constraints must be satisfied). In terms of the conventional angle θ measured from the positive x-axis, $-\frac{1}{4}\pi < \theta < \frac{1}{2}\pi + \phi$,

where $\tan\phi = 2$. A rough sketch of typical characteristics will probably be found helpful.

20.12 *Solve*

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 14,$$

subject to $u = 2x + 1$ *and* $\partial u/\partial y = 4 - 6x$, *both on the line* $y = 0$.

For solutions of the form $u(x, y) = f(x + \lambda y)$ we require

$$6 - 5\lambda + \lambda^2 = 0 \quad \Rightarrow \quad \lambda = 2,\ 3.$$

One possible (trivial) PI is $u(x, y) = 7y^2$, making the general solution

$$u(x, y) = f(x + 2y) + g(x + 3y) + 7y^2.$$

Imposing the given boundary conditions

$$2x + 1 = u(x, 0) = f(x) + g(x), \qquad (*)$$
$$4 - 6x = \frac{\partial u}{\partial y}(x, 0) = 2f'(x) + 3g'(x),$$

Differentiating $(*)$ gives

$$2 = f' + g',$$

Eliminating f' from these two equations yields

$$-6x = g'(x),$$

from which it follows that

$$g(x) = -3x^2 + k$$
$$\Rightarrow \quad f(x) = 2x + 1 + 3x^2 - k.$$

Thus, the solution for general x and y is

$$\begin{aligned} u(x, y) &= 2(x + 2y) + 1 + 3(x + 2y)^2 - k - 3(x + 3y)^2 + k + 7y^2 \\ &= -8y^2 - 6xy + 2x + 4y + 1. \end{aligned}$$

It can be verified by re-substitution into the initial equation and checking the boundary conditions.

20.14 *Solve*

$$\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = x(2y + 3x).$$

For the homogeneous equation to have solutions of the form $u(x, y) = f(x + \lambda y)$ we require

$$\lambda + 3\lambda^2 = 0 \quad \Rightarrow \quad \lambda = 0,\ -3 \quad \Rightarrow \quad u(x, y) = f(x - 3y) + g(x).$$

This is the CF part of the solution.

For a PI we try $u(x, y) = Ax^m y^n$:

$$Amnx^{m-1}y^{n-1} + 3An(n-1)x^m y^{n-2} = 2xy + 3x^2.$$

Such an equation is not guaranteed to have a consistent solution for m and n, but in this case it has; it is satisfied by $m = 2$, $n = 2$ and $A = \frac{1}{2}$. The general solution is, therefore,

$$u(x, y) = f(x - 3y) + g(x) + \tfrac{1}{2}x^2y^2.$$

20.16 *An infinitely long string on which waves travel at speed c has an initial displacement*

$$y(x) = \begin{cases} \sin(\pi x/a), & -a \le x \le a, \\ 0, & |x| > a. \end{cases}$$

It is released from rest at time $t = 0$, and its subsequent displacement is described by $y(x, t)$.

By expressing the initial displacement as one explicit function incorporating Heaviside step functions, find an expression for $y(x, t)$ at a general time $t > 0$. In particular, determine the displacement as a function of time (a) *at $x = 0$,* (b) *at $x = a$, and* (c) *at $x = a/2$.*

The solution of the wave equation at a general time t can be expressed in terms of the initial displacement $\phi(x)$ by making the substitution

$$\phi(x) \to \tfrac{1}{2}[\,\phi(x - ct) + \phi(x + ct)\,]$$

and adding an integral of the initial velocity profile. In the present case there is no initial velocity and the integral contributes nothing.

The initial displacement profile, described piece-wise in the question, can be written as a single function of x by incorporating Heaviside functions as follows:

$$\phi(x) = \sin\left(\frac{\pi x}{a}\right)[\,H(x + a) - H(x - a)\,].$$

Crudely speaking, this formalism 'turns on' the sine function at $x = -a$ and turns it off again at $x = a$. It is only when x is between these limits that the expression in square brackets is non-zero.

Now making the substitution described above we obtain for a general time t that

$$\begin{aligned} y(x,t) &= \frac{1}{2}\sin\left(\frac{\pi(x-ct)}{a}\right)[H(x-ct+a)-H(x-ct-a)] \\ &\quad + \frac{1}{2}\sin\left(\frac{\pi(x+ct)}{a}\right)[H(x+ct+a)-H(x+ct-a)] \\ &= \frac{1}{2}\sin\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi ct}{a}\right)[H(x-ct+a)-H(x-ct-a) \\ &\qquad\qquad + H(x+ct+a)-H(x+ct-a)] \\ &\quad + \frac{1}{2}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi ct}{a}\right)[H(x+ct+a)-H(x+ct-a) \\ &\qquad\qquad - H(x-ct+a)+H(x-ct-a)]. \end{aligned}$$

Although this final expression is lengthy, its evaluation is fairly straightforward.

(a) At $x = 0$ the first term is zero for all t and the second contains the factor

$$[H(ct+a)-H(ct-a)-H(-ct+a)+H(-ct-a)] = [1-H(ct-a)-H(-ct+a)+0].$$

Whatever the sign of $ct - a$, one of the middle two terms in this bracket is -1 and the other is 0. Thus the bracket has total value 0 and the displacement is zero at all times.

(b) At $x = a$ the first term in $y(x,t)$ is zero for all t and the second contains the factor

$$[H(2a+ct)-H(ct)-H(2a-ct)+H(-ct)].$$

The first term in this is $+1$, the second -1 and the last is 0; the result therefore depends solely on whether or not $ct > 2a$. If it is, there is no displacement. If $0 \le ct \le 2a$ then the displacement is

$$\frac{1}{2}\cos\pi\sin\left(\frac{\pi ct}{a}\right)(-1) = \frac{1}{2}\sin\left(\frac{\pi ct}{a}\right).$$

(c) At $x = \frac{1}{2}a$ the second term in $y(x,t)$ is zero for all t; the first term contains the factor

$$[H(\frac{3}{2}a-ct)-H(-ct-\frac{1}{2}a)+H(\frac{3}{2}a+ct)-H(ct-\frac{1}{2}a)].$$

For $0 < 2ct < a$ this factor has the value $1-0+1-0 = 2$.
For $a < 2ct < 3a$ it has the value $1-0+1-1 = 1$.
For $3a < 2ct < \infty$ the bracket has the value $0-0+1-1 = 0$.

In summary the displacement at this value of x is $\cos(\pi ct/a)$ for $0 \le t \le a/2c$, $\frac{1}{2}\cos(\pi ct/a)$ for $a/2c \le t \le 3a/2c$, and 0 otherwise.

20.18 *Like the Schrödinger equation, the equation describing the transverse vibrations of a rod,*

$$a^4 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0,$$

has different orders of derivatives in its various terms. Show, however, that it has solutions of exponential form $u(x,t) = A\exp(\lambda x + i\omega t)$ *provided that the relation* $a^4\lambda^4 = \omega^2$ *is satisfied.*

Use a linear combination of such allowed solutions, expressed as the sum of sinusoids and hyperbolic sinusoids of λx*, to describe the transverse vibrations of a rod of length* L *clamped at both ends. At a clamped point both* u *and* $\partial u/\partial x$ *must vanish; show that this implies that* $\cos(\lambda L)\cosh(\lambda L) = 1$*, thus determining the frequencies* ω *at which the rod can vibrate.*

Direct substitution of $u(x,t) = A\exp(\lambda x + i\omega t)$ yields immediately that

$$a^4\lambda^4 u(x,t) + (i\omega)^2 u(x,t) = 0 \quad \Rightarrow \quad a^4\lambda^4 - \omega^2 = 0.$$

This gives $\pm\sqrt{\omega}/a$ and $\pm i\sqrt{\omega}/a$ as the four possible values of λ corresponding to any particular frequency ω. The four solutions were obtained as exponential functions, but we may work with any four independent linear combinations of them; for our purposes the four sinusoidal and hyperbolic sinusoids form a convenient set.

At each of the clamped ends, we apply both of the stated boundary conditions to a general expression for the (maximum) transverse displacement of the form

$$u(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x,$$

with

$$u'(x) = \lambda(A\cos\lambda x - B\sin\lambda x + C\cosh\lambda x + D\sinh\lambda x).$$

The four conditions will be enough to determine the four initially unknown constants, A, B, C and D.

At the $x = 0$ end of the rod:

$$u(0) = 0 \quad \Rightarrow \quad D = -B,$$
$$u'(0) = 0 \quad \Rightarrow \quad C = -A.$$

Hence,

$$u(x) = A(\sin\lambda x - \sinh\lambda x) + B(\cos\lambda x - \cosh\lambda x),$$
$$u'(x) = \lambda A(\cos\lambda x - \cosh\lambda x) + \lambda B(-\sin\lambda x - \sinh\lambda x).$$

Now, writing $\lambda L = \theta$, we have from the conditions at the other end $x = L$ that

$$A(\sin\theta - \sinh\theta) + B(\cos\theta - \cosh\theta) = 0,$$
$$\lambda A(\cos\theta - \cosh\theta) + \lambda B(-\sin\theta - \sinh\theta) = 0.$$

For consistency,
$$(\cos\theta - \cosh\theta)^2 + (\sin^2\theta - \sinh^2\theta) = 0,$$
$$\cos^2\theta - 2\cos\theta\cosh\theta + \cosh^2\theta + \sin^2\theta - \sinh^2\theta = 0,$$
$$2 - 2\cos\theta\cosh\theta = 0,$$

i.e. $\cos(\lambda L)\cosh(\lambda L) = 1$. For a given value of L this gives the spectrum of values of λ, and hence of ω, at which the rod can undergo free vibrations.

20.20 *A sheet of material of thickness w, specific heat capacity c and thermal conductivity k is isolated in a vacuum, but its two sides are exposed to fluxes of radiant heat of strengths J_1 and J_2. Ignoring short-term transients, show that the temperature difference between its two surfaces is steady at $(J_2 - J_1)w/2k$, whilst their average temperature increases at a rate $(J_2 + J_1)/cw$.*

As the short-term transients are being ignored, we need a solution of the diffusion equation, i.e. a solution of $k\partial^2 u/\partial x^2 = c\partial u/\partial t$, that does not involve time-dependent decaying exponentials. The required solution has the form

$$u(x,t) = \frac{\alpha c}{2k}x^2 + gx + \alpha t + \beta,$$

where $x = 0$ is one of the surfaces and g, α and β are constants to be determined. At each surface, the rate at which heat arrives must equal that at which it is carried into the material by the temperature gradient there. So

$$\text{at } x = 0,\ -k\frac{\partial u}{\partial x} = J_1 \quad \Rightarrow \quad -\alpha c\,0 - kg = J_1,$$
$$\text{at } x = w,\ -k\frac{\partial u}{\partial x} = -J_2 \quad \Rightarrow \quad -\alpha c\,w - kg = -J_2,$$

leading to $\alpha cw = J_1 + J_2$ and $g = -J_1/k$. Thus

$$u(x,t) = \frac{J_1 + J_2}{2kw}x^2 - \frac{J_1}{k}x + \frac{J_1 + J_2}{cw}t + \beta.$$

The temperature difference between the surfaces and the rate at which the average temperature rises are therefore given by

$$u(w,t) - u(0,t) = \frac{J_1 + J_2}{2kw}w^2 - \frac{J_1}{k}w = \frac{(J_2 - J_1)w}{2k},$$
$$\bar{u} = \frac{u(w,t) + u(0,t)}{2} = \frac{(J_1 + J_2)w}{4k} - \frac{J_1 w}{2k} + \frac{J_1 + J_2}{cw}t + \beta,$$
$$\frac{\partial \bar{u}}{\partial t} = 0 + \frac{J_1 + J_2}{cw} + 0.$$

20.22 *The daily and annual variations of temperature at the surface of the earth may be represented by sine-wave oscillations, with equal amplitudes and periods of 1 day and 365 days, respectively. Assume that for (angular) frequency ω the temperature at depth x in the earth is given by $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$, where λ and μ are constants.*

(a) *Use the diffusion equation to find the values of λ and μ.*
(b) *Find the ratio of the depths below the surface at which the two amplitudes have dropped to $1/20$ of their surface values.*
(c) *At what time of year is the soil coldest at the greater of these depths, assuming that the smoothed annual variation in temperature at the surface has a minimum on February 1st?*

(a) Substituting the form $u(x,t) = A\sin(\omega t + \mu x)\exp(-\lambda x)$ into the diffusion equation,

$$\kappa\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u(x,t)}{\partial t},$$

gives

$$\begin{aligned} A\kappa\,\big[-\mu^2\sin(\omega t+\mu x)e^{-\lambda x} + 2\mu(-\lambda)\cos(\omega t+\mu x)e^{-\lambda x} \\ +\ \lambda^2\sin(\omega t+\mu x)e^{-\lambda x}\big] = A\omega\cos(\omega t+\mu x)e^{-\lambda x}. \end{aligned}$$

From comparing coefficients it is clear that we need

$$\lambda^2 = \mu^2, \text{ and } 2\mu(-\lambda)\kappa = \omega \quad\Rightarrow\quad \lambda = -\mu = \left(\frac{\omega}{2\kappa}\right)^{1/2}.$$

(b) For the two sinusoids to be attenuated by the same factor they must have equal values of λx. Thus

$$\frac{x_\mathrm{d}}{x_\mathrm{y}} = \frac{\lambda_\mathrm{y}}{\lambda_\mathrm{d}} = \left(\frac{\omega_\mathrm{y}}{\omega_\mathrm{d}}\right)^{1/2} = \left(\frac{1}{365}\right)^{1/2}.$$

(c) At the greater depth x_y only the yearly variation is significant and its phase relative to that on the surface is $\mu_\mathrm{y}x_\mathrm{y}$. This is equal to $-\lambda_\mathrm{y}x_\mathrm{y}$ which, in turn, is equal to $-\ln 20$ (from the way x_y was defined). Thus, the temperature at this depth is $\ln 20/2\pi$ of a year behind that at the surface, i.e. it is at its coldest on 1 February + (0.477 × 365) days, about 23 July.

20.24 *This example gives a formal demonstration that the type of a second-order PDE (elliptic, parabolic or hyperbolic) cannot be changed by a new choice of independent variable. The algebra is somewhat lengthy, but straightforward.*

If a change of variable $\xi = \xi(x, y)$, $\eta = \eta(x, y)$ is made in

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x, y),$$

so that it reads

$$A'\frac{\partial^2 u}{\partial \xi^2} + B'\frac{\partial^2 u}{\partial \xi \partial \eta} + C'\frac{\partial^2 u}{\partial \eta^2} + D'\frac{\partial u}{\partial \xi} + E'\frac{\partial u}{\partial \eta} + F'u = R'(\xi, \eta),$$

show that

$$B'^2 - 4A'C' = (B^2 - 4AC)\left[\frac{\partial(\xi, \eta)}{\partial(x, y)}\right]^2.$$

Hence deduce the conclusion stated above.

To save space, we denote $\dfrac{\partial \xi}{\partial x}$ by ξ_x, etc.

By the chain rule, the differential operators with respect to x and y take the following forms when expressed in terms of the new variables:

$$\frac{\partial}{\partial x} = \xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial y} = \xi_y \frac{\partial}{\partial \xi} + \eta_y \frac{\partial}{\partial \eta}.$$

Then, with $u(x, y) = v(\xi, \eta)$, the second derivative of u with respect to x becomes

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \left(\xi_x \frac{\partial}{\partial \xi} + \eta_x \frac{\partial}{\partial \eta}\right)\left(\xi_x \frac{\partial v}{\partial \xi} + \eta_x \frac{\partial v}{\partial \eta}\right) \\ &= \xi_x^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_x \eta_x \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2 v}{\partial \eta^2} \\ &\quad + \xi_x \frac{\partial \xi_x}{\partial \xi}\frac{\partial v}{\partial \xi} + \eta_x \frac{\partial \xi_x}{\partial \eta}\frac{\partial v}{\partial \xi} + \xi_x \frac{\partial \eta_x}{\partial \xi}\frac{\partial v}{\partial \eta} + \eta_x \frac{\partial \eta_x}{\partial \eta}\frac{\partial v}{\partial \eta}.\end{aligned}$$

There are similar expressions for $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}$.

Since the nature of a second-order PDE is determined purely by the sign of $B^2 - 4AC$, for the purposes of this exercise we need only consider terms containing second derivatives (mixed or otherwise) of v, i.e. only three terms in each of $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}$ and no terms at all for $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial u}{\partial y}$ and u.

The relevant terms and their origins are thus

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} &: \qquad \xi_x^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_x \eta_x \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x^2 \frac{\partial^2 v}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial x \partial y} &: \qquad \xi_x \xi_y \frac{\partial^2 v}{\partial \xi^2} + (\xi_x \eta_y + \xi_y \eta_x) \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_x \eta_y \frac{\partial^2 v}{\partial \eta^2}, \\
\frac{\partial^2 u}{\partial y^2} &: \qquad \xi_y^2 \frac{\partial^2 v}{\partial \xi^2} + 2\xi_y \eta_y \frac{\partial^2 v}{\partial \xi \partial \eta} + \eta_y^2 \frac{\partial^2 v}{\partial \eta^2}.
\end{aligned}$$

The coefficients A', B' and C' of the transformed equations are therefore

$$\begin{aligned}
A' &= A\xi_x^2 + B\xi_x\xi_y + C\xi_y^2, \\
B' &= 2A\xi_x\eta_x + B(\eta_x\xi_y + \eta_y\xi_x) + 2C\xi_y\eta_y, \\
C' &= A\eta_x^2 + B\eta_x\eta_y + C\eta_y^2.
\end{aligned}$$

We now face the (messy) task of evaluating $D' = B'^2 - 4A'C'$.

$$\begin{aligned}
D' &= 4A^2\xi_x^2\eta_x^2 + 4C^2\xi_y^2\eta_y^2 + B^2(\eta_x\xi_y + \eta_y\xi_x)^2 \\
&\quad + 4AB\xi_x\eta_x(\eta_x\xi_y + \eta_y\xi_x) + 4BC\xi_y\eta_y(\eta_x\xi_y + \eta_y\xi_x) \\
&\quad + 8AC\xi_x\eta_x\xi_y\eta_y - 4A^2\xi_x^2\eta_x^2 - 4AB\xi_x\xi_y\eta_x^2 - 4AC\xi_y^2\eta_x^2 \\
&\quad - 4AB\xi_x^2\eta_x\eta_y - 4B^2\xi_x\xi_y\eta_x\eta_y - 4CB\xi_y^2\eta_x\eta_y \\
&\quad - 4AC\xi_x^2\eta_y^2 - 4BC\xi_x\xi_y\eta_y^2 - 4C^2\xi_y^2\eta_y^2 \\
&= B^2(\eta_x\xi_y + \eta_y\xi_x)^2 + 8AC\xi_x\eta_x\xi_y\eta_y \\
&\quad - 4AC\xi_y^2\eta_x^2 - 4B^2\xi_x\xi_y\eta_x\eta_y - 4AC\xi_x^2\eta_y^2 \\
&= B^2(\eta_x\xi_y - \eta_y\xi_x)^2 - 4AC(\eta_x^2\xi_y^2 - 2\xi_x\eta_x\xi_y\eta_y + \xi_x^2\eta_y^2) \\
&= (B^2 - 4AC)(\eta_x\xi_y - \eta_y\xi_x)^2 \\
&= (B^2 - 4AC)\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right]^2.
\end{aligned}$$

Since the square of the Jacobian is positive, D' has the same sign as $B^2 - 4AC$, showing that the equation type is not altered by the change of independent variables.

21

Partial differential equations: separation of variables and other methods

21.2 *A cube, made of material whose conductivity is k, has as its six faces the planes $x = \pm a$, $y = \pm a$ and $z = \pm a$, and contains no internal heat sources. Verify that the temperature distribution*

$$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

obeys the appropriate diffusion equation. Across which faces is there heat flow? What is the direction and rate of heat flow at the point $(3a/4, a/4, a)$ at time $t = a^2/(\kappa\pi^2)$?

The diffusion equation is

$$\kappa\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = \frac{\partial u}{\partial t}.$$

Substituting the given expression

$$u(x, y, z, t) = A \cos \frac{\pi x}{a} \sin \frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right)$$

into the equation gives

$$\kappa\left(-\frac{\pi^2}{a^2} + 0 - \frac{\pi^2}{a^2}\right) u = -\frac{2\kappa\pi^2}{a^2}\, u,$$

which is satisfied, i.e. the given temperature distribution obeys the relevant diffusion equation.

Heat will flow across a face if, at any point on it, the temperature gradient $\partial u/\partial n$, is not equal to zero; here $\mathbf{n}$ is the (local) outward normal to the face.

For the faces $x = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial x} = \pm\left(-\frac{\pi}{a}\right) A \sin\frac{\pi x}{a} \sin\frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right) = 0 \text{ at } x = \pm a.$$

Thus, although in general there is some heat flow in the x- direction, at the surfaces of the cube the rate of flow is zero.

Since u does not depend upon y, all derivatives with respect to y are zero. This means that there is no heat flow in the y-direction, not even in the body of the cube. In particular, for the faces $y = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial y} = 0 \text{ for all } x \text{ and } z\ ,$$

and no heat flows across any part of these two faces.

For the faces $z = \pm a$:

$$\frac{\partial u}{\partial n} = \pm\frac{\partial u}{\partial z} = \pm\left(\frac{\pi}{a}\right) A \cos\frac{\pi x}{a} \cos\frac{\pi z}{a} \exp\left(-\frac{2\kappa\pi^2 t}{a^2}\right) \neq 0 \text{ for general } x.$$

In summary, on the surface of the cube there is heat flow only across the faces $z = \pm a$.

For the point $(x, y, z) = \frac{1}{4}(3a, a, 4a)$, which lies in the face $z = a$, at time $t = a^2/(\kappa\pi^2)$,

$$\frac{\partial u}{\partial n} = +\frac{\partial u}{\partial z} = \left(\frac{\pi}{a}\right) A \left(\cos\frac{3\pi}{4}\right)(\cos\pi)\exp\left(-\frac{2\kappa\pi^2}{a^2}\,\frac{a^2}{\kappa\pi^2}\right)$$

and the heat flow is

$$-k\frac{\partial u}{\partial n} = -\frac{kA\pi}{a}\left(-\frac{1}{\sqrt{2}}\right)(-1)e^{-2}.$$

The heat flux *into* the cube is therefore $\dfrac{kA\pi e^{-2}}{\sqrt{2}a}$.

Note that k and κ are related by $\kappa = k/c$ where c is the specific heat (thermal capacity) of the material from which the cube is made.

21.4 *Schrödinger's equation for a non-relativistic particle in a constant potential region can be taken as*

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=i\hbar\frac{\partial u}{\partial t}.$$

(a) *Find a solution, separable in the four independent variables, that can be written in the form of a plane wave,*

$$\psi(x,y,z,t)=A\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)].$$

Using the relationships associated with de Broglie ($\mathbf{p}=\hbar\mathbf{k}$) and Einstein ($E=\hbar\omega$), show that the separation constants must be such that

$$p_x^2+p_y^2+p_z^2=2mE.$$

(b) *Obtain a different separable solution describing a particle confined to a box of side a (ψ must vanish at the walls of the box). Show that the energy of the particle can only take the quantised values*

$$E=\frac{\hbar^2\pi^2}{2ma^2}(n_x^2+n_y^2+n_z^2),$$

where n_x, n_y and n_z are integers.

(a) Take $u(x,y,z,t)=X(x)Y(y)Z(z)T(t)$. After substituting and dividing through by u, we obtain

$$-\frac{\hbar^2}{2m}\left(\frac{X''}{X}+\frac{Y''}{Y}+\frac{Z''}{Z}\right)=i\hbar\frac{T'}{T}\quad(*).$$

For a solution that can be written both as a plane wave and in the separable form $(*)$ we must have

$$\begin{aligned}\psi(x,y,z,t)&=A\exp[\,i(\mathbf{k}\cdot\mathbf{r}-\omega t)\,]\\&=Ae^{ik_xx}e^{ik_yy}e^{ik_zz}\,e^{-i\omega t}.\end{aligned}$$

This then implies that the separation constants k_i and ω satisfy

$$\begin{aligned}&-\frac{\hbar^2}{2m}(-k_x^2-k_y^2-k_z^2)=i\hbar(-i\omega),\\ \Rightarrow\quad&\frac{1}{2m}(p_x^2+p_y^2+p_z^2)=\hbar\omega=E,\end{aligned}$$

where we have used the de Broglie ($\mathbf{p}=\hbar\mathbf{k}$) and Einstein ($E=\hbar\omega$) relationships.

(b) For solutions that vanish on any of the walls of the box $x=0$, $x=a$, etc., we must have a product of sine waves of the form

$$u(x,y,z,t)=A\sin\left(\frac{n_x\pi x}{a}\right)\sin\left(\frac{n_y\pi y}{a}\right)\sin\left(\frac{n_z\pi z}{a}\right)\,e^{-i\omega t},$$

where the n_i are integers.

For this solution of (*) it follows that

$$-\frac{\hbar^2}{2m}\left[-\left(\frac{n_x\pi}{a}\right)^2-\left(\frac{n_y\pi}{a}\right)^2-\left(\frac{n_z\pi}{a}\right)^2\right]=i\hbar(-i\omega),$$

$$\frac{\hbar^2\pi^2}{2ma^2}(n_x^2+n_y^2+n_z^2)=\hbar\omega=E.$$

This shows that the possible values of E are quantised, since n_x, n_y and n_z can only take discrete integer values.

21.6 *Prove that the expression*

$$P_\ell^m(\mu)=(1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu), \qquad (*)$$

for the associated Legendre function $P_\ell^m(\mu)$ satisfies the appropriate equation,

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0, \qquad (**)$$

as follows.

(a) *Evaluate $dP_\ell^m(\mu)/d\mu$ and $d^2P_\ell^m(\mu)/d\mu^2$ using the form given in* (*) *and substitute them into* (**).
(b) *Differentiate Legendre's equation m times using Leibnitz' theorem.*
(c) *Show that the equations obtained in* (a) *and* (b) *are multiples of each other, and hence that the validity of* (b) *implies that of* (a).

To save space (and clutter) we will omit all references for (ordinary) Legendre functions to the fixed subscript ℓ and denote $\dfrac{d^m[P_\ell(\mu)]}{d\mu^m}$ by d^mP. Further, we will take $m>0$.

(a) From the given definition

$$\begin{aligned}
P_\ell^m(\mu)&=(1-\mu^2)^{m/2}d^mP,\\
(P_\ell^m)'&=-m\mu(1-\mu^2)^{(m/2)-1}d^mP+(1-\mu^2)^{m/2}d^{m+1}P\\
(P_\ell^m)''&=m(m-2)\mu^2(1-\mu^2)^{(m/2)-2}d^mP-m(1-\mu^2)^{(m/2)-1}d^mP\\
&\qquad-2m\mu(1-\mu^2)^{(m/2)-1}d^{m+1}P+(1-\mu^2)^{m/2}d^{m+2}P.
\end{aligned}$$

We now substitute these forms into the associated Legendre equation

$$\frac{d}{d\mu}\left[(1-\mu^2)\frac{dM}{d\mu}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-\mu^2}\right]M=0,$$

divide through by a factor $(1-\mu^2)^{m/2}$ and collect together the terms involving each particular derivative d^nP.

$$\begin{aligned}\big[m(m-2)\mu^2(1-\mu^2)^{-1} - m + 2m\mu^2(1-\mu^2)^{-1} \\ + \ell(\ell+1) - m^2(1-\mu^2)^{-1}\big]\, d^mP \\ + (-2m\mu - 2\mu)d^{m+1}P + (1-\mu^2)d^{m+2}P = 0,\end{aligned}$$

$$\begin{aligned}\left[\frac{m^2\mu^2 - 2m\mu^2 + 2m\mu^2 - m^2}{1-\mu^2} - m + \ell(\ell+1)\right] d^mP \\ - 2\mu(m+1)d^{m+1}P + (1-\mu^2)d^{m+2}P = 0,\end{aligned}$$

$$(1-\mu^2)d^{m+2}P - 2\mu(m+1)d^{m+1}P + [\ell(\ell+1) - m(m+1)]d^mP = 0.$$

This is an equation that must be valid if the given prescription generates associated Legendre functions, the latter being defined as being the solutions to the associated Legendre equation.

We now proceed to show that it is valid, taking Legendre's equation,

$$(1-\mu^2)P'' - 2\mu P' + \ell(\ell+1) = 0.$$

as our starting point.

(b) Using Leibnitz' theorem, we differentiate Legendre's equation m times and obtain

$$\begin{aligned}(1-\mu^2)d^{m+2}P + m(-2\mu)d^{m+1}P + \tfrac{1}{2}m(m-1)(-2)d^mP \\ -2\mu d^{m+1}P - 2md^mP + \ell(\ell+1)d^mP = 0,\end{aligned}$$

$$(1-\mu^2)d^{m+2}P - 2\mu(m+1)d^{m+1}P + [\ell(\ell+1) - m(m+)]d^mP = 0.$$

(c) We now note that the final equation obtained in part (b) is the same as the putative one obtained in part (a) and so, from the line of reasoning given in (a), we conclude that

$$P_\ell^m(\mu) = (1-\mu^2)^{|m|/2}\frac{d^{|m|}}{d\mu^{|m|}}P_\ell(\mu),$$

does indeed generate associated Legendre functions.

The solutions for negative m have the same forms as those for positive m but their signs and normalisations are defined by convention. As the equation is homogeneous they are still solutions of it.

21.8 *The motion of a very viscous fluid in the two-dimensional (wedge) region* $-\alpha < \phi < \alpha$ *can be described in* (ρ, ϕ) *coordinates by the (biharmonic) equation*

$$\nabla^2\nabla^2\psi \equiv \nabla^4\psi = 0,$$

together with the boundary conditions $\partial\psi/\partial\phi = 0$ *at* $\phi = \pm\alpha$*, which represent the fact that there is no radial fluid velocity close to either of the bounding walls because of the viscosity, and* $\partial\psi/\partial\rho = \pm\rho$ *at* $\phi = \pm\alpha$*, which impose the condition that azimuthal flow increases linearly with* r *along any radial line. Assuming a solution in separated-variable form, show that the full expression for* ψ *is*

$$\psi(\rho, \phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

The conditions to be met are

$$\begin{aligned}
&\nabla^4\psi = 0 && \text{with } \psi(r,\theta) = R(\rho)\Phi(\phi),\\
&\frac{\partial\psi}{\partial\phi} = 0, \quad \frac{\partial\psi}{\partial\rho} = -\rho, && \text{at } \phi = -\alpha,\\
&\frac{\partial\psi}{\partial\phi} = 0, \quad \frac{\partial\psi}{\partial\rho} = \rho, && \text{at } \phi = \alpha.
\end{aligned}$$

Since $\dfrac{\partial\psi}{\partial\rho} \propto \rho$, we need $R(\rho) \propto \rho^2$; let $R(\rho) = \rho^2$, with any multiplicative constant being absorbed into $\Phi(\phi)$.

With this choice of $R(\rho)$, $\nabla^2\psi$ takes the form

$$\begin{aligned}
\nabla^2\psi &= \Phi\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho\ 2\rho) + \frac{\rho^2}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2}\\
&= 4\Phi + \frac{d^2\Phi}{d\phi^2},
\end{aligned}$$

and $\nabla^4\psi = 0$ is

$$0 = \nabla^4\psi = 0 + \frac{1}{\rho^2}\frac{d^2}{d\phi^2}(4\Phi + \Phi'').$$

After this equation has been integrated twice we obtain

$$\Phi'' + 4\Phi = k\phi + c,$$

which has a CF of $C\cos 2\phi + D\sin 2\phi$ and a PI of $\frac{1}{4}(k\phi + c)$.

The general solution for Φ is therefore

$$\Phi = C\cos 2\phi + D\sin 2\phi + A\phi + B.$$

The boundary condition $\dfrac{\partial\psi}{\partial\phi} = 0$ requires Φ' to be zero at $\phi = \pm\alpha$, i.e.

$$-2C\sin 2\phi + 2D\cos 2\phi + A = 0 \text{ at both } \phi = \alpha \text{ and } \phi = -\alpha.$$

These two conditions jointly imply that

$$C = 0 \text{ and } A = -2D\cos 2\alpha.$$

Correspondingly, after substituting for A, the boundary condition on $\dfrac{\partial\psi}{\partial\rho}$ requires that both the equations

$$2\rho(D\sin 2\phi - 2D\cos 2\alpha\,\phi + B) = \pm\rho \text{ at } \phi = \pm\alpha,$$

are satisfied and so determines B and D as

$$B = 0 \text{ and } D = \frac{1}{2(\sin 2\alpha - 2\alpha\cos 2\alpha)}.$$

Thus, finally,

$$\psi(\rho,\phi) = \frac{\rho^2}{2}\,\frac{\sin 2\phi - 2\phi\cos 2\alpha}{\sin 2\alpha - 2\alpha\cos 2\alpha}.$$

21.10 *Consider possible solutions of Laplace's equation inside a circular domain, as follows:*

(a) *Find the solution in plane polar coordinates ρ, ϕ that takes the value $+1$ for $0 < \phi < \pi$ and the value -1 for $-\pi < \phi < 0$, when $\rho = a$.*

(b) *For a point (x, y) on or inside the circle $x^2 + y^2 = a^2$, identify the angles α and β defined by*

$$\alpha = \tan^{-1}\frac{y}{a+x} \qquad \textit{and} \qquad \beta = \tan^{-1}\frac{y}{a-x}.$$

Show that $u(x, y) = (2/\pi)(\alpha + \beta)$ is a solution of Laplace's equation that satisfies the boundary conditions given in (a).

(c) *Deduce a Fourier series expansion for the function*

$$\tan^{-1}\frac{\sin\phi}{1+\cos\phi} + \tan^{-1}\frac{\sin\phi}{1-\cos\phi}.$$

(a) The prescribed boundary values give an antisymmetric square-wave function for $-\pi < \phi \le \pi$. The sinusoidal terms in the general solution of the Laplace equation in plane polars are those used in a Fourier expansion. The required

Figure 21.1 The angles α and β defined in exercise 21.10.

solution is thus one that becomes a Fourier sine series on the circle $\rho = a$. In anticipation of part (b) we take a solution valid *inside* the circle, namely

$$u(\rho, \phi) = \sum_{n=1}^{\infty} A_n \rho^n \sin n\phi.$$

The Fourier sine series for a square-wave,

$$u(\rho, \phi) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{1}{n} \sin n\phi,$$

can be found in almost any textbook and the calculation will not be repeated here.

For the presumed form of $u(\rho, \phi)$ to coincide with the Fourier series on $\rho = a$, it is necessary that

$$A_n = \frac{4}{\pi n a^n} \text{ for } n \text{ odd and } A_n = 0 \text{ for } n \text{ even.}$$

The required solution is thus

$$u(\rho, \phi) = \frac{4}{\pi} \sum_{n \text{ odd}}^{\infty} \frac{\rho^n}{n a^n} \sin n\phi.$$

(b) As is clear from figure 21.1, the acute angles α and β are those made with the x-axis by the lines joining (x, y) to the extremes of the diameter of the circle that coincides with that axis. When (x, y) lies anywhere *on* the circle, $\alpha + \beta = \frac{1}{2}\pi$ (by

the usual 'angle in a semi-circle' property) and

$$u(x, y) = \frac{2(\alpha + \beta)}{\pi} = \frac{2}{\pi}\frac{\pi}{2} = 1.$$

Further,

$$\frac{\partial \alpha}{\partial x} = \frac{1}{1 + \left(\dfrac{y}{a+x}\right)^2}\frac{-y}{(a+x)^2} = -\frac{y}{(a+x)^2 + y^2},$$

$$\frac{\partial^2 \alpha}{\partial x^2} = \frac{2(a+x)y}{[(a+x)^2 + y^2]^2}, \text{ and similarly } \frac{\partial^2 \beta}{\partial x^2} = \frac{2(a-x)y}{[(a-x)^2 + y^2]^2},$$

$$\frac{\partial \alpha}{\partial y} = \frac{1}{1 + \left(\dfrac{y}{a+x}\right)^2}\frac{1}{a+x} = \frac{a+x}{(a+x)^2 + y^2},$$

$$\frac{\partial^2 \alpha}{\partial y^2} = \frac{-(a+x)2y}{[(a+x)^2 + y^2]^2}, \text{ and similarly } \frac{\partial^2 \beta}{\partial y^2} = -\frac{(a-x)2y}{[(a-x)^2 + y^2]^2}.$$

Hence $\nabla^2\alpha = 0$ and $\nabla^2\beta = 0$, clearly showing that $\nabla^2\left[\frac{2}{\pi}(\alpha + \beta)\right] = 0$. Thus $u(x, y)$ solves the Laplace equation and takes the values ± 1 on the upper and lower halves of the circle $\rho = a$, i.e. takes the boundary values given in part (a).

(c) Since the solution to Laplace's equation with a given set of Dirichlet boundary values is unique, the answers to (a) and (b) must coincide. Hence,

$$\frac{2}{\pi}\left(\tan^{-1}\frac{\rho\sin\phi}{a + \rho\cos\phi} + \tan^{-1}\frac{\rho\sin\phi}{a - \rho\cos\phi}\right) = \frac{4}{\pi}\sum_{n \text{ odd}}^{\infty}\frac{\rho^n}{na^n}\sin n\phi.$$

Finally, setting $\rho = a$, we have

$$\tan^{-1}\frac{\sin\phi}{1 + \cos\phi} + \tan^{-1}\frac{\sin\phi}{1 - \cos\phi} = \sum_{n \text{ odd}}^{\infty}\frac{2}{n}\sin n\phi.$$

21.12 *A membrane is stretched between two concentric rings of radii a and b ($b > a$). If the smaller ring is transversely distorted from the planar configuration by an amount $c|\phi|$, $-\pi \le \phi \le \pi$, show that the membrane then has a shape given by*

$$u(\rho, \phi) = \frac{c\pi}{2}\frac{\ln(b/\rho)}{\ln(b/a)} - \frac{4c}{\pi}\sum_{m \text{ odd}}\frac{a^m}{m^2(b^{2m} - a^{2m})}\left(\frac{b^{2m}}{\rho^m} - \rho^m\right)\cos m\phi.$$

A stationary membrane obeys the wave equation with the time derivative term set to zero, i.e. it obeys Laplace's equation $\nabla^2 u = 0$.

The most general single-valued solution of $\nabla^2 u = 0$ in plane-polar coordinates is

$$T(\rho, \phi) = C \ln \rho + D + \sum_{n=1}^{\infty} (A_n \cos n\phi + B_n \sin n\phi)(C_n \rho^n + D_n \rho^{-n}),$$

Since the given problem is symmetric in ϕ about $\phi = 0$, the solution will not contain any $\sin \phi$ terms. Further, on $\rho = b$ the average value of u is zero, implying that $0 = C \ln b + D$. In the same way, on $\rho = a$ the average value of u is $c\pi/2$, implying that $c\pi/2 = C \ln a + D$. Thus the first two terms of the solution can be written together as

$$u_1 = C \ln \rho - C \ln b = C \ln(\rho/b) = \frac{c\pi}{2(\ln a - \ln b)} \ln(\rho/b) = \frac{c\pi}{2} \frac{\ln(b/\rho)}{\ln(b/a)}.$$

The remainder of the solution is

$$u_2 = \sum_{n=1}^{\infty} \left(C_n \rho^n + D_n \rho^{-n}\right) \cos n\phi.$$

Using the mutual orthogonality of the $\cos m\phi$ functions for integer m, we may now obtain two equations linking C_m with D_m; one from $\rho = a$, the other from $\rho = b$.

$$\begin{aligned}
C_m a^m + D_m a^{-m} &= \frac{2}{2\pi} \int_{-\pi}^{\pi} c|\phi| \cos m\phi \, d\phi \\
&= \frac{2c}{\pi} \int_0^{\pi} \phi \cos m\phi \, d\phi \\
&= \frac{2c}{\pi} \left\{ \left[\frac{\phi \sin m\phi}{m} \right]_0^{\pi} - \int_0^{\pi} \frac{\sin m\phi}{m} \, d\phi \right\} \\
&= \frac{2c}{\pi} \left[\frac{\cos m\phi}{m^2} \right]_0^{\pi} \\
&= -\frac{4c}{\pi m^2} \text{ for } m \text{ odd}, = 0 \text{ for } m \text{ even}, \\
C_m b^m + D_m b^{-m} &= \frac{2}{2\pi} \int_{-\pi}^{\pi} 0 \cos m\phi \, d\phi = 0.
\end{aligned}$$

Thus $D_m = -C_m b^{2m}$ and $C_m a^m - C_m b^{2m} a^{-m} = -\dfrac{4c}{\pi m^2}$ for m odd.
Hence

$$\begin{aligned}
u_2 &= \sum_{n \text{ odd}} \left(\frac{-4c}{\pi n^2} \right) \left(\frac{1}{a^n - b^{2n} a^{-n}} \right) \left(\rho^n - b^{2n} \rho^{-n}\right) \cos n\phi \\
&= -\frac{4c}{\pi} \sum_{n \text{ odd}} \frac{a^n}{n^2 (b^{2n} - a^{2n})} \left(\frac{b^{2n}}{\rho^n} - \rho^n \right) \cos n\phi,
\end{aligned}$$

and $u = u_1 + u_2$ is as given in the question.

21.14 *A conducting spherical shell of radius a is cut round its equator and the two halves connected to voltages of $+V$ and $-V$. Show that an expression for the potential at the point (r, θ, ϕ) anywhere inside the two hemispheres is*

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!(4n+3)}{2^{2n+1} n!(n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

This problem is almost identical to the last worked example in the subsection *Laplace's equation in polar coordinates*, page 735. The only difference is that the two halves of the sphere are at potentials V and $-V$, rather than v_0 and 0. We can therefore take over that result by the change $v_0 \to 2V$ and then subtracting V from the complete solution; this latter change has the effect of removing the constant term and leaving a sum that contains *only* odd Legendre polynomials.

Their expansion coefficients are evaluated using the result from exercise 18.3; thus,

$$A_{2n+1} a^{2n+1} = \frac{2(2n+1)+1}{2} 2V \frac{(-1)^n (2n)!}{2^{2n+1} n!\,(n+1)!},$$

giving

$$u(r, \theta, \phi) = V \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!(4n+3)}{2^{2n+1} n!(n+1)!} \left(\frac{r}{a}\right)^{2n+1} P_{2n+1}(\cos\theta).$$

21.16 *A slice of biological material of thickness L is placed into a solution of a radioactive isotope of constant concentration C_0 at time $t = 0$. For a later time t find the concentration of radioactive ions at a depth x inside one of its surfaces if the diffusion constant is κ.*

The concentration is governed by the diffusion law

$$\kappa \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}.$$

Ultimately the concentration will be C_0 everywhere; this is formally, but trivially, a solution of the equation. To this must be added time-dependent solutions of the diffusion equation that represent the (decaying) transients and $\to 0$ as $t \to \infty$.

Writing $u(x,t) = C_0 + X(x)T(t)$ we obtain the usual separated variable equations

$$\frac{X''}{X} = \frac{T'}{\kappa T} = -\frac{\mu}{\kappa}, \qquad \mu > 0,$$

with the sign of acceptable separation constants chosen so as to ensure solutions decaying with time.

The solution for the time variation is trivial, $T(t) = T(0)e^{-\mu t}$. That for the spatial variation is of the form

$$X(x) = A \sin \sqrt{\frac{\mu}{\kappa}}\, x + B \cos \sqrt{\frac{\mu}{\kappa}}\, x.$$

At all times $u(0,t) = u(L,t) = C_0$ and so $X(0) = X(L) = 0$. This determines that $B = 0$ (whatever the value of μ) and imposes the condition that $\sqrt{\mu}L = n\pi\sqrt{\kappa}$ where n is an integer.

At this stage all positive integer values of n are possible and thus the general solution is a linear superposition of them:

$$u(x,t) = C_0 + \sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} \exp\left(-\frac{n^2\pi^2\kappa}{L^2}t\right).$$

At $t = 0$, before any diffusion has taken place, $u(x,0) = 0$ and so

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi x}{L} = -C_0.$$

This is, in fact, a Fourier expansion and we determine the coefficients A_n in the usual way, using the mutual orthogonality of sinusoidal functions. Multiplying both sides by $\sin(m\pi x/L)$ and integrating from 0 to L gives

$$\begin{aligned} A_m \frac{1}{2} L &= \int_0^L (-C_0) \sin \frac{m\pi x}{L}\, dx \\ &= C_0 \left[\frac{L}{m\pi} \cos \frac{m\pi x}{L} \right]_0^L \\ &= \begin{cases} -\dfrac{2C_0 L}{m\pi} & \text{for } m \text{ odd}, \\ 0 & \text{for } m \text{ even}. \end{cases} \end{aligned}$$

Inserting these values yields

$$u(x,t) = C_0 - \frac{4C_0}{\pi} \sum_{n \text{ odd}} \frac{1}{n} \sin \frac{n\pi x}{L} \exp\left(-\frac{n^2\pi^2\kappa t}{L^2}\right),$$

so giving the concentration at a general place and time.

21.18 *A sphere of radius a and thermal conductivity k_1 is surrounded by an infinite medium of conductivity k_2 in which far away the temperature tends to T_∞. A distribution of heat sources $q(\theta)$ embedded in the sphere's surface establish steady temperature fields $T_1(r,\theta)$ inside the sphere and $T_2(r,\theta)$ outside it. It can be shown, by considering the heat flow through a small volume that includes part of the sphere's surface, that*

$$k_1\frac{\partial T_1}{\partial r} - k_2\frac{\partial T_2}{\partial r} = q(\theta) \quad on \quad r = a.$$

Given that

$$q(\theta) = \frac{1}{a}\sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

find complete expressions for $T_1(r,\theta)$ and $T_2(r,\theta)$. What is the temperature at the centre of the sphere?

The general azimuthally symmetric solution in spherical polar coordinates of the time-independent diffusion equation, i.e. of Laplace's equation, is

$$T(r,\theta) = \sum_{\ell=0}^{\infty}\left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta).$$

Since T_1 covers a region including the origin, it must contain no inverse powers of r. Likewise, since T_2 covers a region including $r \to \infty$, it must not contain any positive powers of r. Thus

$$T_1(r,\theta) = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \text{ and } T_2(r,\theta) = \sum_{n=0}^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta) + T_\infty P_0.$$

The boundary conditions on $r = a$ are

$$T_1 = T_2 \quad \text{and} \quad k_1\frac{\partial T_1}{\partial r} - k_2\frac{\partial T_2}{\partial r} = \frac{1}{a}\sum_{n=0}^{\infty} q_n P_n(\cos\theta),$$

the RHS of the second one representing a 'Legendre expansion' of the distribution of heat sources, analogous to a Fourier series.

Since the P_n are mutually orthogonal we may equate their coefficients on the two sides of an equation. The first boudary condition therefore yields

$$A_0 = \frac{B_0}{a} + T_\infty \quad \text{and} \quad A_n a^n = \frac{B_n}{a^{n+1}}.$$

The second condition (for $n \geq 0$) is

$$k_1 n A_n a^{n-1} + k_2\frac{(n+1)B_n}{a^{n+2}} = \frac{q_n}{a},$$
$$\Rightarrow \quad [k_1 n + k_2(n+1)]A_n a^n = q_n.$$

These simultaneous equations give all the A_n and B_n and the temperatures in the two regions as

$$T_1(r,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{k_1 n + k_2(n+1)} \left(\frac{r}{a}\right)^n P_n(\cos\theta) + T_\infty,$$

$$T_2(r,\theta) = \sum_{n=0}^{\infty} \frac{q_n}{k_1 n + k_2(n+1)} \left(\frac{a}{r}\right)^{n+1} P_n(\cos\theta) + T_\infty.$$

The temperature at the centre of the sphere is $T_1(0,\theta) = \dfrac{q_0}{k_2} + T_\infty$. Perhaps surprisingly, this depends only on q_0 and k_2, and not on k_1. However, since q_0 is the only component-source that has a net non-zero output of heat (averaged over all directions), it is this and the rate at which the heat it produces is conducted to infinity that determine the level to which the temperature at the centre rises; hence k_2 is the controlling factor. How quickly the equilibrium state would be established starting from (say) a uniform temperature of T_∞ *would* be affected by the value of k_1, but that is not asked for here.

21.20 *Working in* spherical *polar coordinates* $\mathbf{r} = (r,\theta,\phi)$*, but for a system that has azimuthal symmetry around the polar axis, consider the following gravitational problem.*

(a) *Show that the gravitational potential due to a uniform disc of radius a and mass M, centred at the origin, is given for* $r < a$ *by*

$$\frac{2GM}{a}\left[1 - \frac{r}{a}P_1(\cos\theta) + \frac{1}{2}\left(\frac{r}{a}\right)^2 P_2(\cos\theta) - \frac{1}{8}\left(\frac{r}{a}\right)^4 P_4(\cos\theta) + \cdots\right],$$

and for $r > a$ *by*

$$\frac{GM}{r}\left[1 - \frac{1}{4}\left(\frac{a}{r}\right)^2 P_2(\cos\theta) + \frac{1}{8}\left(\frac{a}{r}\right)^4 P_4(\cos\theta) - \cdots\right],$$

where the polar axis is normal to the plane of the disc.

(b) *Reconcile the presence of a term* $P_1(\cos\theta)$*, which is odd under* $\theta \to \pi - \theta$*, with the symmetry with respect to the plane of the disc of the physical system.*

(c) *Deduce that the gravitational field near an infinite sheet of matter of constant density* ρ *per unit area is* $2\pi G\rho$*.*

We tackle this problem by first calculating directly the potential at a general point on the polar axis, i.e. on the central normal to the disc, and then choosing the constants in the general solution of Laplace's equation to make the two

expressions match. Finally (though it is often not explicitly stated) we appeal to the uniqueness theorem to claim that the solution so found is the correct one.

(a) An annulus of the disc of radius ρ and width $d\rho$ produces a potential $du(z)$ at a point on the polar axis distant z from the disc given by

$$du = \frac{2\pi\rho\,\sigma G}{(\rho^2+z^2)^{1/2}}\,d\rho,$$

where σ is the area density of the disc. The total potential at the point is thus

$$\begin{aligned} u(z) = \int_0^a \frac{2\pi\sigma G\,\rho}{(\rho^2+z^2)^{1/2}}\,d\rho &= 2\pi\sigma G\left[(\rho^2+z^2)^{1/2}\right]_0^a \\ &= 2\pi\frac{M}{\pi a^2}G[\,(a^2+z^2)^{1/2}-z\,]. \end{aligned}$$

For $r<a$, we now expand this function in positive powers of z and then replace z^n by $r^nP_n(\cos\theta)$, these two expressions being the same for $\theta=0$. The resulting expression $u(r,\theta)$ will be valid for all $r<a$ and all θ, not just on the axis:

$$\begin{aligned} u(z) &= \frac{2MGa}{a^2}\left(1+\frac{1}{2}\frac{z^2}{a^2}-\frac{1}{8}\frac{z^4}{a^4}+\cdots-\frac{z}{a}\right), \\ u(r,\theta) &= \frac{2MG}{a}\left(1+\frac{1}{2}\frac{r^2}{a^2}P_2(\cos\theta)-\frac{1}{8}\frac{r^4}{a^4}P_4(\cos\theta)+\cdots-\frac{r}{a}P_1(\cos\theta)\right). \end{aligned}$$

For $r>a$, the function has to be expanded in negative powers of z and then z^{-n-1} has to be replaced by $r^{-n-1}P_n(\cos\theta)$, these two expressions being the same for $\theta=0$. The resulting expression $u(r,\theta)$ will be valid for all $r>a$ and all θ, not just on the axis:

$$\begin{aligned} u(z) &= \frac{2MG}{a^2}\left(z+\frac{1}{2}\frac{a^2}{z}-\frac{1}{8}\frac{a^4}{z^3}+\frac{1}{16}\frac{a^6}{z^5}\cdots-z\right), \\ u(r,\theta) &= \frac{2MG}{a^2}\frac{a^2}{2r}\left(1-\frac{1}{4}\frac{a^2}{r^2}P_2(\cos\theta)+\frac{1}{8}\frac{a^4}{r^4}P_4(\cos\theta)-\cdots\right). \end{aligned}$$

(b) Under the change $\theta\to\pi-\theta$, z becomes negative and the appropriate form for $\left[(\rho^2+z^2)^{1/2}\right]_0^a$ is $(a^2+z^2)^{1/2}-(-z)$, i.e. $(a^2+z^2)^{1/2}+z$. When this is expressed in polar coordinates, $P_1(\cos\theta)$ changes sign under the interchange but r does not; neither do any of the $P_n(\cos\theta)$ when n is even. The derived expression is therefore still valid when $\theta>\pi/2$.

(c) The gravitational field is given by $-\partial u/\partial r$ in the direction $\theta=0$, where all $P_n=1$. An infinite sheet of matter is equivalent to a finite r and $a\to\infty$. Clearly the $r<a$ form is needed and M must be expressed as $M=\pi a^2\rho$:

$$\frac{\partial u}{\partial r} = \frac{2G\pi a^2\rho}{a}\left(-\frac{1}{a}P_1(1)+\frac{r}{a^2}P_2(1)+\cdots\right).$$

Figure 21.2 The charges as described in exercise 21.22 and the resulting spherical surface on which the potential is zero.

It then follows that

$$g = -\lim_{a\to\infty} \frac{\partial u}{\partial r} = 2\pi G\rho$$

is the gravitational field near an infinite sheet of constant-density matter.

21.22 *Point charges q and $-qa/b$ (with $a < b$) are placed, respectively, at a point P, a distance b from the origin O, and a point Q between O and P, a distance a^2/b from O. Show, by considering similar triangles QOS and SOP, where S is any point on the surface of the sphere centred at O and of radius a, that the net potential anywhere on the sphere due to the two charges is zero.*

Use this result (backed up by the uniqueness theorem) to find the force with which a point charge q placed a distance b from the centre of a spherical conductor of radius a ($< b$) is attracted to the sphere (i) *if the sphere is earthed, and* (ii) *if the sphere is uncharged and insulated.*

As can be seen from figure 21.2,

$$\phi_S = \frac{q}{4\pi\epsilon_0|PS|} - \frac{(a/b)q}{4\pi\epsilon_0|QS|}.$$

Now, the triangles QOS and SOP have sides in the ratio

$$\frac{OQ}{OS} = \frac{(a^2/b)}{a} = \frac{a}{b} = \frac{OS}{OP};$$

they also have the same included angle θ. This shows that they are similar

triangles and therefore that the ratio QS/PS is also equal to a/b. It now follows that $\phi_S = 0$.

(i) The potential at a general point R outside the earthed sphere must be given by

$$\phi(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{|\mathbf{r}_R - \mathbf{r}_P|} - \frac{a}{b|\mathbf{r}_R - \mathbf{r}_Q|}\right).$$

This is so because each term is a solution of Laplace's equation, $\phi(\mathbf{r}) = 0$ for $\mathbf{r}$ anywhere on the sphere and $\phi(\mathbf{r}) \to 0$ as $r \to \infty$; the uniqueness theorem shows that there is only one such solution and so this must be it.

A deduction from this is that a charge of $-aq/b$ placed at $\mathbf{r}_Q$ is the appropriate image charge for this situation. Since it produces the same potential distribution outside the sphere as the actual induced charge on the sphere does, it will also produce the same electrostatic field there. Consequently, the force with which the real charge at P will be attracted to the sphere is the same as that between the real charge and its image, i.e.

$$\begin{aligned} F_1 &= \frac{(a/b)q\,q}{4\pi\epsilon_0\,|\,PQ\,|^2} \\ &= \frac{aq^2}{4\pi\epsilon_0 b\left(b - \dfrac{a^2}{b}\right)^2} = \frac{abq^2}{4\pi\epsilon_0(b^2 - a^2)^2}. \end{aligned}$$

(ii) With the sphere uncharged and insulated it must still be an equipotential surface (though not necessarily at zero potential). However, it must now also have the property that zero net charge is contained in any surface that surrounds it.

These two requirements can be met by adding a further image charge at O, equal in magnitude but opposite in sign to that at Q. The additional charge affects all parts of the sphere equally, leaving it as an equipotential surface, but increases its potential by $(a/b)q/(4\pi\epsilon_0 a)$. The new force of attraction will be

$$F_2 = F_1 - \frac{(a/b)q^2}{4\pi\epsilon_0 b^2} = F_1 - \frac{aq^2}{4\pi\epsilon_0 b^3}.$$

Although we have not explicitly said so, the uniqueness theorem has again been invoked in arriving at this result.

21.24 *Electrostatic charge is distributed in a sphere of radius R centred on the origin. Determine the form of the resultant potential $\phi(\mathbf{r})$ at distances much greater than R, as follows.*

(a) *Express in the form of an integral over all space the solution of*

$$\nabla^2\phi = -\frac{\rho(\mathbf{r})}{\epsilon_0}.$$

(b) *Show that, for $r \gg r'$,*

$$|\mathbf{r}-\mathbf{r}'| = r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).$$

(c) *Use results* (a) *and* (b) *to show that $\phi(\mathbf{r})$ has the form*

$$\phi(\mathbf{r}) = \frac{M}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right).$$

Find expressions for M and $\mathbf{d}$, and identify them physically.

(a) The formal expression for the integral solution of Poisson's equation is

$$\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0|\,\mathbf{r}-\mathbf{r}'\,|}\,d\mathbf{r}',$$

where V is the sphere $r' \le R$. (b) When $r \gg r'$ we may expand the expression for $|\mathbf{r}-\mathbf{r}'|$ in powers of r'/r using the binomial theorem:

$$\begin{aligned}
|\,\mathbf{r}-\mathbf{r}'\,| &= [\,(\mathbf{r}-\mathbf{r}')\cdot(\mathbf{r}-\mathbf{r}')\,]^{1/2}\\
&= (r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2)^{1/2}\\
&= r\left(1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^{1/2}\\
&= r - \frac{\mathbf{r}\cdot\mathbf{r}'}{r} + \mathrm{O}\left(\frac{1}{r}\right).
\end{aligned}$$

(c) Putting this into the integral in part (a) and again using the binomial theorem:

$$\begin{aligned}
\phi(\mathbf{r}) &= \int_V \frac{\rho(\mathbf{r}')}{4\pi\epsilon_0\,r}\left[1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2} + \mathrm{O}\left(\frac{1}{r^2}\right)\right]^{-1} d\mathbf{r}'\\
&= \frac{M}{r} + \frac{\mathbf{d}\cdot\mathbf{r}}{r^3} + \mathrm{O}\left(\frac{1}{r^3}\right),
\end{aligned}$$

where

$$M = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\,d\mathbf{r}' \text{ and } \mathbf{d} = \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\mathbf{r}'\,d\mathbf{r}'.$$

The first term M is $(4\pi\epsilon_0)^{-1}$ times the total charge contained in the region of space bounded by the sphere of radius R, showing that from far enough away the charge appears like a point charge. The second term is similarly related to the dipole moment of the charge, and measures the grosser aspects of any deviation from spherical symmetry of the charge distribution.

21.26 *Find the Green's function for the three-dimensional Neumann problem*

$$\nabla^2\phi = 0 \quad \textit{for } z > 0 \qquad \textit{and} \qquad \frac{\partial\phi}{\partial z} = f(x, y) \quad \textit{on } z = 0.$$

Determine $\phi(x, y, z)$ *if*

$$f(x, y) = \begin{cases} \delta(y) & \textit{for } |x| < a, \\ 0 & \textit{for } |x| \geq a. \end{cases}$$

The general solution to the Poisson equation with Neumann boundary conditions is

$$u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) + \langle u(\mathbf{r})\rangle_S - \int_S G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}),$$

where $\langle u(\mathbf{r})\rangle_S$ is the average of u over the surface S.

In the present case the charge density ρ is zero (Laplace, rather than the more general Poisson), except of course at $\mathbf{r}_0$. Further, as one of the bounding surfaces is the hemisphere at infinity in the region $z > 0$ and $u = 0$ there, as well as on the plane $z = 0$, $\langle u(\mathbf{r})\rangle_S = 0$. Thus we are left with only the third term on the RHS and

$$u(\mathbf{r}_0) = -\int_S G(\mathbf{r}, \mathbf{r}_0) f(\mathbf{r})\, dS(\mathbf{r}).$$

Now, guided by the solution to the corresponding Dirichlet problem, we are able place an image charge outside the region (i.e. an image charge with $z < 0$) and so write a suitable form for the Green's function. One fundamental difference, however, is that the image charge must have the *same* sign as the charge at $\mathbf{r}_0$; this is because it is the normal derivative of G (rather than G itself) that must vanish on S_1, the plane $z = 0$. The explicit form of the Green's functions reads

$$\begin{aligned} G(\mathbf{r}, \mathbf{r}_0) &= -\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_0|} - \frac{1}{4\pi|\mathbf{r} + \mathbf{r}_0|} \qquad (*) \\ &= -\frac{1}{4\pi}\left\{\frac{1}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}}\right. \\ &\qquad \left. + \frac{1}{[(x - x_0)^2 + (y - y_0)^2 + (z + z_0)^2]^{1/2}}\right\}. \end{aligned}$$

Differentiating with respect to z then gives

$$\frac{\partial G}{\partial z} = -\frac{1}{4\pi}\left\{-\frac{2(z-z_0)}{2[\ldots+(z-z_0)^2]^{3/2}} - \frac{2(z+z_0)}{2[\ldots+(z+z_0)^2]^{3/2}}\right\},$$

$$\left.\frac{\partial G}{\partial z}\right|_{z=0} = \frac{1}{4\pi}\left\{\frac{-z_0}{[\ldots+z_0^2]^{3/2}} + \frac{z_0}{[\ldots+z_0^2]^{3/2}}\right\} = 0.$$

To find the dominant term in the normal derivative of G over the hemisphere S_2 at infinity, we may neglect $\mathbf{r}_0$ when differentiating $(*)$ and obtain

$$\begin{aligned}\frac{\partial G}{\partial r} &= -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{1}{|\mathbf{r}-\mathbf{r}_0|} + \frac{1}{|\mathbf{r}+\mathbf{r}_0|}\right)_{r\to\infty} \\ &\simeq -\frac{1}{4\pi}\frac{\partial}{\partial r}\left(\frac{2}{|\mathbf{r}|}\right)_{r\to\infty} \\ &= \frac{1}{2\pi r^2}.\end{aligned}$$

It follows that the surface integral over the hemisphere of this normal derivative is simply

$$\int_{S_2} \frac{\partial G}{\partial r}\, dS = \frac{\partial G}{\partial r}\, 2\pi r^2 = 1.$$

When this is added to the zero contribution arising from the integration of the zero derivative over S_1, a sum of unity is obtained, showing that the consistency condition for a Neumann Green's function is satisfied.

We now calculate $\phi(x, y, z)$ for the given distribution of $\partial\phi/\partial z$ on $z = 0$.

$$\begin{aligned}\phi(x_0, y_0, z_0) &= \int_{-a}^{a}\int_{-\infty}^{\infty} \frac{1}{4\pi}\left\{\frac{\delta(y)}{[\,(x-x_0)^2+(y-y_0)^2+(0-z_0)^2\,]^{1/2}}\right. \\ &\qquad \left. + \frac{\delta(y)}{[\,(x-x_0)^2+(y-y_0)^2+(0+z_0)^2\,]^{1/2}}\right\} dy\, dx \\ &= \frac{2}{4\pi}\int_{-a}^{a} \frac{dx}{[\,(x-x_0)^2+y_0^2+z_0^2\,]^{1/2}}.\end{aligned}$$

Now transform the integral by setting $x - x_0 = \sqrt{y_0^2 + z_0^2}\sinh\theta \equiv \mu\sinh\theta$.

$$\phi(x_0, y_0, z_0) = \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} \frac{\mu\cosh\theta\, d\theta}{\mu(\sinh^2\theta + 1)^{1/2}},$$

where $\mu \sinh\theta_1 = -a - x_0$ and $\mu \sinh\theta_2 = a - x_0$.

$$\begin{aligned}\phi(x_0, y_0, z_0) &= \frac{1}{2\pi}\int_{\theta_1}^{\theta_2} d\theta \\ &= \frac{1}{2\pi}(\theta_2 - \theta_1) \\ &= \frac{1}{2\pi}\left(\sinh^{-1}\frac{a - x_0}{\sqrt{y_0^2 + z_0^2}} + \sinh^{-1}\frac{a + x_0}{\sqrt{y_0^2 + z_0^2}}\right).\end{aligned}$$

This could, of course, be equally well written in terms of unsubscripted variables x, y and z. The subscript 0 is an artefact of the notation used to indicate which quantities were to be held constant during the integration over the Green's function; unsubscripted variables were used for the integration.

21.28 *Consider the PDE $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$, for which the differential operator $\mathcal{L}$ is given by*

$$\mathcal{L} = \nabla \cdot [\,p(\mathbf{r})\nabla\,] + q(\mathbf{r}),$$

where $p(\mathbf{r})$ and $q(\mathbf{r})$ are functions of position. By proving the generalised form of Green's theorem,

$$\int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\,dV = \oint_S p(\phi\nabla\psi - \psi\nabla\phi)\cdot\hat{\mathbf{n}}\,dS,$$

show that the solution of the PDE is given by

$$\begin{aligned}u(\mathbf{r}_0) = &\int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\,dV(\mathbf{r}) \\ &+ \oint_S p(\mathbf{r})\left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n}\right] dS(\mathbf{r}),\end{aligned}$$

where $G(\mathbf{r}, \mathbf{r}_0)$ is the Green's function satisfying $\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$.

First, consider the divergence of the quantity in parentheses appearing in the integrand on the RHS of the supposed generalised form of Green's theorem:

$$\begin{aligned}\text{divergence} &= \nabla\cdot(p\,\phi\nabla\psi - p\,\psi\nabla\phi) \\ &= p\,\phi\nabla^2\psi + \nabla\psi\cdot\nabla(p\,\phi) - p\,\psi\nabla^2\phi - \nabla\phi\cdot\nabla(p\,\psi) \\ &= p\,\phi\nabla^2\psi + p\,\nabla\psi\cdot\nabla\phi + \phi(\nabla\psi\cdot\nabla p\,) \\ &\qquad - p\,\psi\nabla^2\phi - p\,\nabla\phi\cdot\nabla\psi - \psi(\nabla\phi\cdot\nabla p\,) \\ &= p\,\phi\nabla^2\psi + \phi(\nabla\psi\cdot\nabla p\,) - p\,\psi\nabla^2\phi - \psi(\nabla\phi\cdot\nabla p\,).\end{aligned}$$

From the divergence theorem it follows that the surface integral on the RHS

must be equal to the volume integral of the expression in the last line of this equation.

Next, consider the LHS of the supposed equation:

$$\begin{aligned}
\text{LHS} &= \int_V (\phi\mathcal{L}\psi - \psi\mathcal{L}\phi)\, dV \\
&= \int_V \{\phi(\nabla\cdot[\,p\,\nabla\,] + q\,)\psi - \psi(\nabla\cdot[\,p\,\nabla\,] + q\,)\phi\}\, dV \\
&= \int_V \{\phi(\nabla\cdot[\,p\,\nabla\psi\,] - \psi\nabla\cdot[\,p\,\nabla\phi\,]\}\, dV \\
&= \int_V \{\phi(\nabla p\,\cdot\nabla\psi) + \phi p\,\nabla^2\psi - \psi(\nabla p\,\cdot\nabla\phi) - \psi p\,\nabla^2\phi\}\, dV.
\end{aligned}$$

Comparison of this result with that of the previous paragraph establishes the generalised form of Green's theorem.

Now, taking $\phi(\mathbf{r}) = u(\mathbf{r})$ with $\mathcal{L}u(\mathbf{r}) = \rho(\mathbf{r})$ and $\psi(\mathbf{r}) = G(\mathbf{r}, \mathbf{r}_0)$ with $\mathcal{L}G(\mathbf{r}, \mathbf{r}_0) = \delta(\mathbf{r} - \mathbf{r}_0)$, we have

$$\begin{aligned}
\int_V u(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}_0)\, dV - \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) \\
= \oint_S p(\mathbf{r}) \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}),
\end{aligned}$$

from which the stated result,

$$\begin{aligned}
u(\mathbf{r}_0) = \int_V G(\mathbf{r}, \mathbf{r}_0)\rho(\mathbf{r})\, dV(\mathbf{r}) \\
+ \oint_S p(\mathbf{r}) \left[u(\mathbf{r})\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0)\frac{\partial u(\mathbf{r})}{\partial n} \right] dS(\mathbf{r}),
\end{aligned}$$

follows immediately.

22

Calculus of variations

22.2 *Show that the lowest value of the integral*

$$\int_A^B \frac{(1+y'^2)^{1/2}}{y}\,dx,$$

where A is $(-1,1)$ and B is $(1,1)$, is $2\ln(1+\sqrt{2})$. Assume that the Euler–Lagrange equation gives a minimising curve.

If the integrand $F(y', y, x)$ does not contain x explicitly then a first integral of the E–L equation is that $F - y'\dfrac{\partial F}{\partial y'}$ is equal to a constant. Here,

$$\frac{(1+y'^2)^{1/2}}{y} - y'\left[\frac{y'}{y(1+y'^2)^{1/2}}\right] = C,$$

$$\frac{1}{y(1+y'^2)^{1/2}} = C.$$

On rearrangement, this gives

$$\frac{dy}{dx} = \pm\frac{(1-C^2y^2)^{1/2}}{Cy},$$

which can now be integrated:

$$\int \frac{y\,dy}{\sqrt{1-C^2y^2}} = \pm\int\frac{dx}{C},$$

$$\frac{\sqrt{1-C^2y^2}}{C^2} = \mp\frac{x}{C} + D.$$

Since the curve must pass through $(-1,1)$ and $(1,1)$, $D = 0$ and

$$\sqrt{1-C^2\,1} = \mp C \quad\Rightarrow\quad C = \pm\frac{1}{\sqrt{2}}.$$

Re-substituting these values and squaring both sides of the final equation shows that $x^2 + y^2 = 2$ is the minimising curve, with $2x + 2yy' = 0$ and hence that $y' = -x/y$.

The minimal integral has the value V given by

$$\begin{aligned} V &= \int_{-1}^{1} \frac{[\,1 + (x/y)^2\,]^{1/2}}{y}\,dx \\ &= \int_{-1}^{1} \frac{(x^2 + y^2)^{1/2}}{y^2}\,dx \\ &= \int_{-1}^{1} \frac{\sqrt{2}}{2 - x^2}\,dx \\ &= \frac{1}{2}\int_{-1}^{1} \left(\frac{1}{\sqrt{2} - x} + \frac{1}{\sqrt{2} + x} \right) dx \\ &= \frac{1}{2}\left[\ln \frac{\sqrt{2} + x}{\sqrt{2} - x} \right]_{-1}^{1} \\ &= 2\ln(\sqrt{2} + 1), \end{aligned}$$

as stated in the question.

22.4 *The Lagrangian for a π-meson is given by*

$$L(\mathbf{x}, t) = \tfrac{1}{2}(\dot{\phi}^2 - |\nabla\phi|^2 - \mu^2\phi^2),$$

where μ is the meson mass and $\phi(\mathbf{x}, t)$ is its wavefunction. Assuming Hamilton's principle find the wave equation satisfied by ϕ.

This is a situation in which there are four independent variables, x, y, z and t and so we apply the E–L equation

$$\frac{\partial L}{\partial \phi} = \frac{\partial}{\partial x}\left(\frac{\partial L}{\partial \phi_x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial L}{\partial \phi_y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial L}{\partial \phi_z}\right) + \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \phi_t}\right),$$

where $\phi_x = \dfrac{\partial \phi}{\partial x}$, etc. and $\phi_t = \dot{\phi}$.

With

$$L = \frac{1}{2}\left[\dot{\phi}^2 - \left(\frac{\partial \phi}{\partial x}\right)^2 - \left(\frac{\partial \phi}{\partial y}\right)^2 - \left(\frac{\partial \phi}{\partial z}\right)^2 - \mu^2\phi^2 \right],$$

the function that makes $\int L\,dx\,dy\,dz\,dt$ stationary satisfies

$$-\mu^2\phi = -\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right) + \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t}\right)$$

$$\mu^2\phi = \nabla^2\phi - \frac{\partial^2\phi}{\partial t^2}.$$

This is the equation satisfied by the meson's wavefunction. It is known as the Klein–Gordon equation; here it is expressed in units in which $\hbar = c = 1$, where $\hbar$ is the Planck constant and c is the speed of light *in vacuo*.

22.6 *For a system specified by the coordinates q and t, show that the equation of motion is unchanged if the Lagrangian $L(q, \dot{q}, t)$ is replaced by*

$$L_1 = L + \frac{d\phi(q,t)}{dt},$$

where ϕ is an arbitrary function. Deduce that the equation of motion of a particle that moves in one dimension subject to a force $-dV(x)/dx$ (x being measured from a point O) is unchanged if O is forced to move with a constant velocity v (x still being measured from O).

We start with the Lagrangian $L(q, \dot{q}, t)$ giving an equation of motion

$$\frac{\partial L}{\partial q} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right). \qquad (*)$$

Now consider

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}}\right) &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{d\phi}{dt}\right)\right] \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}}\left(\frac{\partial\phi}{\partial q}\dot{q} + \frac{\partial\phi}{\partial t}\right)\right] \\ &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{d}{dt}\left[\frac{\partial\phi}{\partial q} + 0 + 0\right], \qquad (**)\end{aligned}$$

since ϕ and $\dfrac{\partial\phi}{\partial t}$ do not contain $\dot{q}$.

Now, since $\phi = \phi(q, t)$,

$$\frac{d}{dt}\left(\frac{\partial\phi}{\partial q}\right) = \frac{\partial^2\phi}{\partial q^2}\dot{q} + \frac{\partial^2\phi}{\partial t\partial q} = \frac{\partial^2\phi}{\partial q^2}\dot{q} + \frac{\partial^2\phi}{\partial q\partial t} = \frac{\partial}{\partial q}\left(\frac{d\phi}{dt}\right).$$

Consequently, using $(*)$ to replace the first term on the RHS of $(**)$,

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} + \frac{\partial}{\partial q}\left(\frac{d\phi}{dt}\right) = \frac{\partial L_1}{\partial q}.$$

Thus, the equation of motion is unchanged by the addition to the Lagrangian.

When the point O is moved with constant velocity v, the potential function $V(x)$ is unchanged (as x is still measured from O) but the kinetic energy term, T, in the Lagrangian becomes

$$\begin{aligned} T_1 &= \tfrac{1}{2}m(\dot{x}+v)^2 \\ &= \tfrac{1}{2}m\dot{x}^2 + mv\dot{x} + \tfrac{1}{2}mv^2 \\ &= T + \frac{d}{dt}(mvx + \tfrac{1}{2}mv^2 t). \end{aligned}$$

And so the new Lagrangian is

$$L_1 = L + \frac{d}{dt}(mvx + \tfrac{1}{2}mv^2 t).$$

The additional term is of the form $d\phi(x,t)/dt$ previously considered and therefore the equations of motion are not changed.

22.8 *Derive the differential equations for the plane-polar coordinates r, ϕ of a particle of unit mass moving in a field of potential $V(r)$. Find the form of V if the path of the particle is given by $r = a\sin\phi$.*

In plane polar coordinates the kinetic energy of a particle of unit mass is $T = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2)$ and so the Lagrangian is

$$L = \tfrac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - V(r).$$

Hamilton's principle, that the integral of L with respect to time t (the independent variable) is stationary, gives the E–L equations for the two dependent variables r and ϕ in their usual form:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i},$$

which in this case implies that

$$\frac{d}{dt}(\dot{r}) = r\dot{\phi}^2 - \frac{\partial V}{\partial r} \quad \text{and} \quad \frac{d}{dt}(r^2\dot{\phi}) = 0.$$

The second of these expresses angular momentum conservation as $r^2\dot{\phi} = k$, whilst the first can be interpreted physically as saying that the actual radial acceleration is the difference between the outward centripetal acceleration and the inward one due to the potential field.

If the actual path is $r = a\sin\phi$, then this must be a solution of these coupled equations. The path in this form does not give the time dependence of either

variable and so we must aim to eliminate time and differentiations with respect to it. In particular, we need an expression for $\ddot{r}$ that does not contain time.

$$\dot{r} = a\cos\phi\,\dot{\phi} = \frac{ak}{r^2}\cos\phi,$$

$$\begin{aligned}\ddot{r} &= -\frac{2ak\cos\phi}{r^3}\,\dot{r} + \frac{ak(-\sin\phi)}{r^2}\,\dot{\phi}\\ &= -\frac{2a^2k^2\cos^2\phi}{r^5} - \frac{kr}{r^2}\,\frac{k}{r^2}\\ &= -\frac{2(a^2-r^2)k^2}{r^5} - \frac{k^2}{r^3}.\end{aligned}$$

Substitution in the radial equation now gives

$$-\frac{2(a^2-r^2)k^2}{r^5} - \frac{k^2}{r^3} = r\frac{k^2}{r^4} - \frac{\partial V}{\partial r} \quad\Rightarrow\quad \frac{\partial V}{\partial r} = \frac{2k^2a^2}{r^5}.$$

From this we conclude that

$$V(r) = -\frac{k^2a^2}{2r^4} + c$$

and that the potential is an inverse fourth-power law. This (admittedly unphysical) situation is of interest because the particle's orbit passes through the centre of force, and with infinite speed, in theory. This raises the question of relativistic effects . . . .

22.10 *Extend to the case of several dependent variables $y_i(x)$, the standard result about the first integral of the E–L equation when x does not appear explicitly in the general integrand $F(y_i', y_i, x)$. In particular, show that the first integral is*

$$F - \sum_{i=1}^{n} y_i'\frac{\partial F}{\partial y_i'} = \text{constant}.$$

For each of the dependent variables y_i, $i = 1, 2, \ldots, n$, we have

$$\frac{\partial F}{\partial y_i} = \frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right).$$

These n equations can be manipulated as follows.

$$\begin{aligned}y_i'\frac{\partial F}{\partial y_i} = y_i'\frac{d}{dx}\left(\frac{\partial F}{\partial y_i'}\right) &= \frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right) - y_i''\frac{\partial F}{\partial y_i'},\\ y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{\partial F}{\partial y_i} &= \frac{d}{dx}\left(y_i'\frac{\partial F}{\partial y_i'}\right),\\ \sum_{i=1}^{n}\left(y_i''\frac{\partial F}{\partial y_i'} + y_i'\frac{\partial F}{\partial y_i}\right) &= \frac{d}{dx}\left[\sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right)\right].\end{aligned}$$

But, since $F \neq F(x)$, the LHS is the total derivative of F, i.e. $\dfrac{dF}{dx}$. Thus

$$\frac{dF}{dx} = \frac{d}{dx}\left[\sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right)\right],$$
$$\Rightarrow \quad F - \sum_{i=1}^{n}\left(y_i'\frac{\partial F}{\partial y_i'}\right) = k,$$

is the first integral of the E–L equations.

22.12 *Light travels in the vertical xz-plane through a slab of material which lies between the planes $z = z_0$ and $z = 2z_0$ and in which the speed of light $v(z) = c_0 z/z_0$. Using Fermat's principle in the form that the travel time is minimised, show that the ray paths are arcs of circles.*

Deduce that, if a ray enters the material at $(0, z_0)$ at an angle to the vertical, $\pi/2 - \theta$, of more than $30°$ then it does not reach the far side of the slab.

We start with the three defining equations

$$v(z) = c_0\frac{z}{z_0}, \qquad (ds)^2 = (dx)^2 + (dz)^2$$

and

$$t = \int \frac{ds}{v} = \int \frac{(1+z'^2)^{1/2}}{v(z)}\,dx = \frac{z_0}{c_0}\int \frac{(1+z'^2)^{1/2}}{z}\,dx.$$

The independent variable x is not present in the integrand F and so a first integral of the E–L equation is $F - z'\dfrac{\partial F}{\partial z'} = k$:

$$\frac{(1+z'^2)^{1/2}}{z} - \frac{z'}{z}\frac{z'}{(1+z'^2)^{1/2}} = k,$$
$$\frac{1}{z(1+z'^2)^{1/2}} = k,$$
$$\frac{z\,dz}{(A-z^2)^{1/2}} = dx, \quad \text{where } A = k^{-1},$$
$$\Rightarrow \quad -(A-z^2)^{1/2} = x + B,$$
$$(x+B)^2 + z^2 = (\sqrt{A})^2.$$

This is a circle of radius $\sqrt{A}$ centred on $(-B, 0)$.

If the ray enters the slab at $(0, z_0)$ with $\dfrac{dz}{dx} = \tan\theta$, then

$$B^2 + z_0^2 = A \quad \text{and} \quad \frac{A - z_0^2}{z_0^2} = \left(\frac{dz}{dx}\right)^2_{x=0} = \tan^2\theta.$$

From these it follows that

$$\sqrt{A} = z_0 \sec\theta \quad \text{and} \quad B = z_0 \tan\theta.$$

The ray is horizontal when $z' = 0$, i.e. when $z^2 = A$, i.e. when $z = z_0 \sec\theta$. This will be below the top of the slab if $z_0 \sec\theta < 2z_0$, i.e. if $\cos\theta > \frac{1}{2}$. This requires $\theta < 60°$ and $\pi/2 - \theta$ to be more than $30°$. When this happens the ray will not reach the far side of the slab.

22.14 *In the brachistochrone problem of subsection 22.3.4 show that if the upper end-point can lie anywhere on the curve $h(x, y) = 0$ then the curve of quickest descent $y(x)$ meets $h(x, y) = 0$ at right angles.*

The slope m_h of the curve $h(x, y) = 0$ is given by

$$\frac{\partial h}{\partial x}\,dx + \frac{\partial h}{\partial y}\,dy = 0 \quad \Rightarrow \quad m_h = \frac{dy}{dx} = -\frac{\partial h}{\partial x}\bigg/\frac{\partial h}{\partial y}.$$

For the brachistochrone,

$$F(y, y', x) = F(y, y') = \sqrt{\frac{1 + y'^2}{y}},$$

giving

$$\frac{\partial F}{\partial y'} = \frac{1}{\sqrt{y}}\,\frac{y'}{\sqrt{1 + y'^2}}$$

and

$$F - y'\frac{\partial F}{\partial y'} = \frac{\sqrt{1 + y'^2}}{\sqrt{y}} - \frac{y'^2}{\sqrt{y}\sqrt{1 + y'^2}} = \frac{1}{\sqrt{y}\sqrt{1 + y'^2}}.$$

The end-point condition, equation (22.20), is

$$\begin{aligned}
\left(F - y'\frac{\partial F}{\partial y'}\right)\frac{\partial h}{\partial y} - \frac{\partial F}{\partial y'}\,\frac{\partial h}{\partial x} &= 0,\\
\frac{1}{\sqrt{y}\sqrt{1 + y'^2}}\,\frac{\partial h}{\partial y} - \frac{1}{\sqrt{y}}\,\frac{y'}{\sqrt{1 + y'^2}}\,\frac{\partial h}{\partial x} &= 0,\\
\frac{\partial h}{\partial y} - y'\frac{\partial h}{\partial x} &= 0,\\
y' = \frac{\partial h}{\partial y}\bigg/\frac{\partial h}{\partial x} &= -\frac{1}{m_h}.
\end{aligned}$$

The condition for curves of slopes m_1 and m_2 to meet at right angles is $m_1 m_2 = -1$. This condition is satisfied here and we conclude that the curves $y = y(x)$ and $h(x, y) = 0$ meet at right angles at the upper end point.

22.16 *Use the result*

$$\int_a^b (y_j' p y_i' - y_j q y_i)\, dx = \lambda_i \delta_{ij}$$

to evaluate

$$J = \int_{-1}^{1} (1 - x^2) P_m'(x) P_n'(x)\, dx,$$

where $P_m(x)$ is a Legendre polynomial of order m.

The result

$$\int_a^b (y_j' p y_i' - y_j q y_i)\, dx = \lambda_i \delta_{ij}, \qquad (*)$$

applies to normalised eigenfunctions of a Sturm–Liouville equation. Legendre's equation is such an equation, with $p(x) = (1 - x^2)$, $q(x) = 0$ and $\rho(x) = 1$. The limits are $a = -1$ and $b = 1$.

The normalised Legendre function corresponding to eigenvalue $m(m + 1)$ is

$$y_m(x) = \sqrt{\frac{2m+1}{2}} P_m(x),$$

and so $(*)$ reads

$$\int_{-1}^{1} \left[\sqrt{\frac{2m+1}{2}} P_m'(x)(1 - x^2) \sqrt{\frac{2n+1}{2}} P_n'(x) - 0 \right] dx = m(m+1)\delta_{mn}.$$

From this it follows immediately that

$$J = \int_{-1}^{1} (1 - x^2) P_m'(x) P_n'(x)\, dx = \frac{2m(m+1)}{2m+1}\, \delta_{mn}.$$

22.18 *Show that $y'' - xy + \lambda x^2 y = 0$ has a solution for which $y(0) = y(1) = 0$ and $\lambda \leq 147/4$.*

The equation is already in S–L form with $p = 1$, $q = -x$ and $\rho = x^2$. The boundary conditions require $y(0) = y(1) = 0$. The simplest polynomial that satisfies these conditions is $y(x) = x(1 - x)$ and so we use this as a trial function. For *any* trial function the lowest eigenvalue λ_0 must satisfy

$$\lambda_0 \leq \frac{\int (p y'^2 - q y^2)\, dx}{\int \rho y^2\, dx}.$$

With the trial function we have chosen, this means that

$$\begin{aligned}
\lambda_0 &\le \frac{\int_0^1 [\,(1)(1-2x)^2 - (-x)x^2(1-x)^2\,]\,dx}{\int_0^1 x^2x^2(1-x)^2\,dx} \\
&= \frac{\int_0^1 (x^5 - 2x^4 + x^3 + 4x^2 - 4x + 1)\,dx}{\int_0^1 (x^6 - 2x^5 + x^4)\,dx} \\
&= \frac{\frac{1}{6} - \frac{2}{5} + \frac{1}{4} + \frac{4}{3} - 2 + 1}{\frac{1}{7} - \frac{2}{6} + \frac{1}{5}} \\
&= \frac{10 - 24 + 15 + 80 - 120 + 60}{60}\,\frac{210}{30 - 70 + 42} \\
&= \frac{21}{60}\,\frac{210}{2} = \frac{147}{4}.
\end{aligned}$$

Thus, there must be a solution of the differential equation for which $y(0) = y(1) = 0$ and $\lambda \le 147/4$. In fact, the inequality sign must hold since the trial function used is *not* a solution to the given equation, as can be easily verified by substitution.

22.20 *Estimate the lowest eigenvalue λ_0 of the equation*

$$\frac{d^2y}{dx^2} - x^2y + \lambda y = 0, \qquad y(-1) = y(1) = 0,$$

using a quadratic trial function.

Following the normal procedure for an S–L equation with, in this case, $p = 1$, $q = -x^2$, $\rho = 1$ and a quadratic trial function $y(x) = 1 - x^2$ chosen to fit the boundary conditions, we obtain

$$\begin{aligned}
\lambda_0 &\le \frac{\int_{-1}^1 [\,4x^2 + x^2(1-x^2)^2\,]\,dx}{\int_{-1}^1 (1-x^2)^2\,dx} \\
&= \frac{\int_{-1}^1 (5x^2 - 2x^4 + x^6)\,dx}{\int_{-1}^1 (1 - 2x^2 + x^4)\,dx} \\
&= \frac{\frac{10}{3} - \frac{4}{5} + \frac{2}{7}}{2 - \frac{4}{3} + \frac{2}{5}} \\
&= \frac{350 - 84 + 30}{105}\,\frac{15}{30 - 20 + 6} \\
&= \frac{296}{105}\,\frac{15}{16} = \frac{37}{14}.
\end{aligned}$$

We also note that this problem can be recast to use the Rayleigh–Ritz principle

by writing the integrand in the numerator as $y(x)\left(-\dfrac{d^2}{dx^2}+x^2\right)y(x)$. With the same trial function, the same upper bound is obtained.

22.22 *Consider the problem of finding the lowest eigenvalue λ_0 of the equation*

$$(1+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+\lambda y=0, \qquad y(\pm 1)=0.$$

(a) *Recast the problem in variational form, and derive an approximation λ_1 to λ_0 by using the trial function $y_1(x)=1-x^2$.*
(b) *Show that an improved estimate λ_2 is obtained by using $y_2(x)=\cos(\pi x/2)$.*
(c) *Prove that the estimate $\lambda(\gamma)$ obtained by taking $y_1(x)+\gamma y_2(x)$ as the trial function is*

$$\lambda(\gamma)=\frac{64/15+64\gamma/\pi-384\gamma/\pi^3+(\pi^2/3+1/2)\gamma^2}{16/15+64\gamma/\pi^3+\gamma^2}.$$

Investigate $\lambda(\gamma)$ numerically as γ is varied, or, more simply, show that $\lambda(-1.80)=3.668$, an improvement on both λ_1 and λ_2.

The given equation is already in S–L form with $p(x)=1+x^2$, $q=0$ and $\rho=1$. We therefore take

$$I=\int_{-1}^{1}(1+x^2)y'^2\,dx \quad \text{and} \quad J=\int_{-1}^{1}y^2\,dx,$$

where $y(x)$ must satisfy $y(\pm 1)=0$, to estimate λ_0 as I/J.

(a) With trial function $y_1(x)=1-x^2$, we have as a first approximation

$$\begin{aligned}\lambda_1&=\frac{\int_{-1}^{1}(1+x^2)4x^2\,dx}{\int_{-1}^{1}(1-2x^2+x^4)\,dx}\\&=\frac{4\left(\frac{2}{3}+\frac{2}{5}\right)}{2-\frac{4}{3}+\frac{2}{5}}=\frac{4(10+6)}{30-20+6}=4.\end{aligned}$$

(b) The corresponding calculation for trial function $y_2(x)=\cos(\pi x/2)$ is

$$\begin{aligned}\lambda_2&=\frac{\int_{-1}^{1}(1+x^2)\frac{\pi^2}{4}\sin^2\left(\frac{\pi x}{2}\right)\,dx}{\int_{-1}^{1}\cos^2\left(\frac{\pi x}{2}\right)\,dx}\\&=\frac{\frac{\pi^2}{4}\left(1+\frac{1}{3}+\frac{2}{\pi^2}\right)}{1}=\frac{\pi^2}{3}+\frac{1}{2}=3.79,\ \text{(an improvement)}.\end{aligned}$$

To evaluate the integral we used

$$\begin{aligned}\int_{-1}^{1} x^2 \sin^2\left(\frac{\pi x}{2}\right)\, dx &= \frac{1}{2}\int_{-1}^{1} x^2(1-\cos\pi x)\, dx \\ &= \frac{1}{3} - \frac{1}{2}\left[\frac{x^2\sin\pi x}{\pi}\right]_{-1}^{1} + \int_{-1}^{1}\frac{2x\sin\pi x}{2\pi}\, dx \\ &= \frac{1}{3} + 0 + \frac{1}{\pi}\left[\frac{-x\cos\pi x}{\pi}\right]_{-1}^{1} + \frac{1}{\pi^2}\int_{-1}^{1}\cos\pi x\, dx \\ &= \frac{1}{3} + \frac{2}{\pi^2} + 0.\end{aligned}$$

(c) Taking as a third trial function the linear combination of $y_3 = y_1 + \gamma y_2$, where γ is an adjustable parameter, we have

$$y_3(x) = 1 - x^2 + \gamma\cos\left(\frac{\pi x}{2}\right) \quad \text{and} \quad y_3' = -2x - \frac{\pi\gamma}{2}\sin\left(\frac{\pi x}{2}\right).$$

To evaluate the integrals I and J we will need the following additional results.

$$\begin{aligned}\int_{-1}^{1}\cos\left(\frac{\pi x}{2}\right)\, dx &= \left[\frac{2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} = \frac{4}{\pi}, \\ \int_{-1}^{1} x\sin\left(\frac{\pi x}{2}\right)\, dx &= \left[-\frac{2x}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} + \int_{-1}^{1}\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\, dx \\ &= 0 + \frac{2}{\pi}\frac{4}{\pi} = \frac{8}{\pi^2}, \\ \int_{-1}^{1} x^2\cos\left(\frac{\pi x}{2}\right) &= \left[\frac{2x^2}{\pi}\sin\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} - \int_{-1}^{1}\frac{4x}{\pi}\sin\left(\frac{\pi x}{2}\right)\, dx \\ &= \frac{4}{\pi} - \frac{4}{\pi}\frac{8}{\pi^2}, \\ \int_{-1}^{1} x^3\sin\left(\frac{\pi x}{2}\right) &= \left[-\frac{2x^3}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_{-1}^{1} + \int_{-1}^{1}\frac{6x^2}{\pi}\cos\left(\frac{\pi x}{2}\right)\, dx \\ &= 0 + \frac{6}{\pi}\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right).\end{aligned}$$

We can now calculate I as

$$\begin{aligned}I &= \int_{-1}^{1}(1+x^2)\left[4x^2 + 2\pi\gamma x\sin\left(\frac{\pi x}{2}\right) + \frac{\pi^2\gamma^2}{4}\sin^2\left(\frac{\pi x}{2}\right)\right]\, dx \\ &= 4\left(\frac{2}{3}+\frac{2}{5}\right) + 2\pi\gamma\frac{8}{\pi^2} + 2\pi\gamma\,\frac{6}{\pi}\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) + \frac{\pi^2\gamma^2}{4}\left(1 + \frac{1}{3} + \frac{2}{\pi^2}\right) \\ &= \frac{64}{15} + \frac{64\gamma}{\pi}\left(1 - \frac{6}{\pi^2}\right) + \gamma^2\left(\frac{\pi^2}{3} + \frac{1}{2}\right).\end{aligned}$$

The corresponding calculation for J is

$$\begin{aligned} J &= \int_{-1}^{1} \left[(1-x^2)^2 + 2\gamma(1-x^2)\cos\left(\frac{\pi x}{2}\right) + \gamma^2 \cos^2\left(\frac{\pi x}{2}\right) \right] dx \\ &= 2 - \frac{4}{3} + \frac{2}{5} + 2\gamma\frac{4}{\pi} - 2\gamma\left(\frac{4}{\pi} - \frac{32}{\pi^3}\right) + \gamma^2 \\ &= \frac{16}{15} + \frac{64\gamma}{\pi^3} + \gamma^2. \end{aligned}$$

Inserting numerical values, we find that the estimate of λ_0 is

$$\frac{I}{J} = \frac{4.2667 + 7.9872\gamma + 3.7898\gamma^2}{1.0667 + 2.0641\gamma + \gamma^2}.$$

This reproduces results (a) and (b) for $\gamma = 0$ and $\gamma \gg 1$ respectively, as expected. However some numerical experimentation shows that the ratio drops to 3.6653 when $\gamma = -1.694$, thus providing a better upper limit than either (a) or (b).

22.24 *This is an alternative approach to the example in section 22.8. Using the notation of that section, the expectation value of the energy of the state ψ is given by $\int \psi^* H\psi\, dv$. Denote the eigenfunctions of H by ψ_i, so that $H\psi_i = E_i\psi_i$, and, since H is Hermitian, $\int \psi_j^* \psi_i\, dv = \delta_{ij}$.*

(a) *By writing any function ψ as $\sum c_j\psi_j$ and following an argument similar to that in section 22.7, show that*

$$E = \frac{\int \psi^* H\psi\, dv}{\int \psi^*\psi\, dv} \geq E_0,$$

the energy of the lowest state. This is the Rayleigh–Ritz principle.

(b) *Using the same trial function as in section 22.8, $\psi = \exp(-\alpha x^2)$, show that the same result is obtained.*

In order to find the energy E_0 of the lowest state, we seek to minimise

$$\langle H \rangle = \int \psi^* H\psi\, dv \text{ subject to } \int \psi^*\psi\, dv = 1.$$

(a) We begin by writing the trial function ψ as a linear combination of the eigenfunctions ψ_i of the Hamiltonian H; they satisfy $H\psi_i = E_i\psi_i$ and $\int \psi_j^*\psi_i\, dv = \delta_{ij}$. Thus

$$\psi = \sum_j c_j\psi_j,$$

where the c_j, as well as the ψ_j, can be complex. This results in an expression for $\langle H \rangle$ that is a double summation:

$$\begin{aligned}
\langle H \rangle &= \int \left(\sum_j c_j^* \psi_j^* \right) H \left(\sum_i c_i \psi_i \right) dv \\
&= \sum_{ij} \int c_j^* c_i \, \psi_j^* \, E_i \psi_i \, dv, \quad \text{using } H\psi_i = E_i \psi_i, \\
&= \sum_{ij} c_j^* c_i \, E_i \delta_{ij} \\
&= \sum_i |c_i|^2 E_i.
\end{aligned}$$

We also have for the normalisation integral of ψ that

$$\begin{aligned}
\int \psi^* \psi \, dv &= \int \left(\sum_j c_j^* \psi_j^* \right) \left(\sum_i c_i \psi_i \right) dv \\
&= \sum_{ij} \int c_j^* c_i \, \psi_j^* \psi_i \, dv, \\
&= \sum_{ij} c_j^* c_i \, \delta_{ij} \\
&= \sum_i |c_i|^2.
\end{aligned}$$

Now, since E_0 is the energy of the lowest state, $E_i \geq E_0$ for all i. Consequently

$$E = \frac{\int \psi^* H \psi \, dv}{\int \psi^* \psi \, dv} = \frac{\sum_i |c_i|^2 E_i}{\sum_i |c_i|^2} \geq \frac{\sum_i |c_i|^2 E_0}{\sum_i |c_i|^2} = E_0.$$

(b) In section 22.8 the Hamiltonian operator has the form

$$H = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + \frac{kx^2}{2}.$$

Denote the integral $\int_{-\infty}^{\infty} x^2 \exp(-2\alpha x^2) \, dx$ by J. Then, for the trial function $\psi = \exp(-\alpha x^2)$,

$$\begin{aligned}
\langle H \rangle &= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\alpha x^2} \frac{d^2}{dx^2} (e^{-\alpha x^2}) \, dx + \frac{k}{2} \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2} \, dx \\
&= -\frac{\hbar^2}{2m} \int_{-\infty}^{\infty} e^{-\alpha x^2} (4\alpha^2 x^2 - 2\alpha) e^{-\alpha x^2} \, dx + \frac{k}{2} J \\
&= -\frac{4\hbar^2 \alpha^2 J}{2m} + \frac{\hbar^2 \alpha}{m} \int_{-\infty}^{\infty} e^{-2\alpha x^2} \, dx + \frac{k}{2} J.
\end{aligned}$$

Now, from integrating the definition of J by parts,

$$J = \int_{-\infty}^{\infty} x^2 e^{-2\alpha x^2}\, dx$$
$$= \left[-\frac{x}{4\alpha} e^{-2\alpha x^2} \right]_{-\infty}^{\infty} + \frac{1}{4\alpha} \int_{-\infty}^{\infty} e^{-2\alpha x^2}\, dx,$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-2\alpha x^2}\, dx = 4\alpha J.$$

We can now conclude that

$$E = \frac{\langle H \rangle}{\int \psi^* \psi\, dv} = \frac{1}{4\alpha J} \left(-\frac{2\hbar^2 \alpha^2 J}{m} + \frac{4\hbar^2 \alpha^2 J}{m} + \frac{kJ}{2} \right) = \frac{\hbar^2 \alpha}{2m} + \frac{k}{8\alpha}.$$

This is exactly the same expression for E as that obtained in equation (22.34) and so when it is minimised with respect to α it gives the same result, $\frac{1}{2}\hbar(k/m)^{1/2}$, for the upper limit on the ground state energy.

22.26 *The Hamiltonian H for the hydrogen atom is*

$$-\frac{\hbar^2}{2m}\nabla^2 - \frac{q^2}{4\pi\epsilon_0 r}.$$

For a spherically symmetric state, as may be assumed for the ground state, the only relevant part of ∇^2 is that involving differentiation with respect to r.

(a) *Define the integrals J_n by*

$$J_n = \int_0^{\infty} r^n e^{-2\beta r}\, dr$$

and show that, for a trial wavefunction of the form $\exp(-\beta r)$ *with $\beta > 0$, $\int \psi^* H \psi\, dv$ and $\int \psi^* \psi\, dv$ can be expressed as $aJ_1 - bJ_2$ and cJ_2 respectively, where a, b, c are factors which you should determine.*

(b) *Show that the Rayleigh–Ritz estimate of E is minimised when β takes the value $mq^2/(4\pi\epsilon_0\hbar^2)$.*

(c) *Hence find an upper limit for the ground-state energy of the hydrogen atom. In fact,* $\exp(-\beta r)$ *is the correct form for the wavefunction and the limit gives the actual value.*

Working in spherical polar coordinates, the expression for $H\psi$, where ψ is a spherically symmetric state, takes the form

$$-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\psi}{dr} \right) - \frac{q^2 \psi}{4\pi\epsilon_0 r}.$$

If $\psi(r) = e^{-\beta r}$,

$$\begin{aligned}\langle H\rangle &= -\frac{\hbar^2}{2m}\int_0^\infty 4\pi r^2 \frac{e^{-\beta r}}{r^2}\frac{d}{dr}[\,r^2(-\beta)e^{-\beta r}\,]\,dr \\ &\qquad\qquad - \frac{q^2}{4\pi\epsilon_0}\int_0^\infty \frac{4\pi r^2 e^{-2\beta r}}{r}\,dr \\ &= \frac{2\pi\hbar^2\beta}{m}\int_0^\infty e^{-\beta r}(2r - r^2\beta)e^{-\beta r}\,dr - \frac{q^2}{\epsilon_0}J_1 \\ &= \frac{2\pi\hbar^2\beta}{m}(2J_1 - \beta J_2) - \frac{q^2}{\epsilon_0}J_1.\end{aligned}$$

Thus $a = \dfrac{4\pi\hbar^2\beta}{m} - \dfrac{q^2}{\epsilon_0}$ and $b = \dfrac{2\pi\hbar^2\beta^2}{m}$. Also

$$\int \psi^*\psi\,dv = \int_0^\infty 4\pi r^2 e^{-2\beta r}\,dr = 4\pi J_2 \quad\Rightarrow\quad c = 4\pi.$$

(b) The estimate of E is

$$\frac{\langle H\rangle}{\int \psi^*\psi\,dv} = \left(\frac{\hbar^2\beta}{m} - \frac{q^2}{4\pi\epsilon_0}\right)\frac{J_1}{J_2} - \frac{\hbar^2\beta^2}{2m}.$$

Now, integration by parts gives the relationship

$$J_2 = \int_0^\infty r^2 e^{-2\beta r}\,dr = \left[\frac{r^2 e^{-2\beta r}}{-2\beta}\right]_0^\infty - 2\int_0^\infty \frac{re^{-2\beta r}}{-2\beta}\,dr = \frac{1}{\beta}J_1.$$

Hence,

$$E_{\text{estim}} = \frac{\hbar^2\beta^2}{m} - \frac{q^2\beta}{4\pi\epsilon_0} - \frac{\hbar^2\beta^2}{2m} = \frac{\hbar^2\beta^2}{2m} - \frac{q^2\beta}{4\pi\epsilon_0}.$$

This is minimised when the parameter β is chosen to satisfy

$$0 = \frac{\partial E_{\text{estim}}}{\partial\beta} = \frac{\hbar^2\beta}{m} - \frac{q^2}{4\pi\epsilon_0} \quad\Rightarrow\quad \beta = \frac{q^2 m}{4\pi\epsilon_0\hbar^2}.$$

(c) The upper limit on the ground-state energy of the hydrogen atom provided by this form of trial function is therefore

$$\frac{\hbar^2}{2m}\frac{q^4m^2}{(4\pi\epsilon_0)^2\hbar^4} - \frac{q^4 m}{(4\pi\epsilon_0)^2\hbar^2} = -\frac{q^4 m}{2(4\pi\epsilon_0\hbar)^2}.$$

As noted in the question, the trial wavefunction happens to be of the correct form and the estimate obtained for the ground state energy is the actual one (within the limits of the model Hamiltonian used).

22.28 *A particle of mass m moves in a one-dimensional potential well of the form*

$$V(x) = -\mu \frac{\hbar^2\alpha^2}{m} \text{sech}^2 \alpha x,$$

where μ and α are positive constants. The expectation value $\langle E \rangle$ of the energy of the system is $\int \psi^* H \psi \, dx$, where the self-adjoint operator $H = -(\hbar^2/2m)d^2/dx^2 + V(x)$. Using trial wavefunctions of the form $y = A \,\text{sech}\, \beta x$, show the following:

(a) *for $\mu = 1$ there is an exact eigenfunction of H, with a corresponding $\langle E \rangle$ of half of the maximum depth of the well;*
(b) *for $\mu = 6$ the 'binding energy' of the ground state is at least $10\hbar^2\alpha^2/(3m)$.*

[*You will find it useful to note that for $u, v \geq 0$,* $\text{sech}\, u \,\text{sech}\, v \geq \text{sech}\,(u + v)$.]

To test for an exact eigenfunction we need to consider the relevant differential equation (here the Schrödinger equation). This is

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - \mu\frac{\hbar^2\alpha^2}{m}\text{sech}^2(\alpha x)\,\psi = E\psi.$$

With $y = A \,\text{sech}\, \beta x$ as a trial function,

$$\begin{aligned}
\psi(x) &= A \,\text{sech}\, \beta x, \\
\psi'(x) &= -A\beta \,\text{sech}\, \beta x \tanh \beta x, \\
\psi''(x) &= A\beta^2 \text{sech}\, \beta x \tanh^2 \beta x - A\beta^2 \text{sech}^3 \beta x.
\end{aligned}$$

So ψ will be a solution provided (cancelling $\text{sech}\, \beta x$ throughout)

$$\begin{aligned}
\beta^2 \tanh^2 \beta x - \beta^2 \text{sech}^2 \beta x + 2\mu\alpha^2 \text{sech}^2 \alpha x &= -\frac{2mE}{\hbar^2}, \\
\beta^2 - 2\beta^2 \text{sech}^2 \beta x + 2\mu\alpha^2 \text{sech}^2 \alpha x &= -\frac{2mE}{\hbar^2}.
\end{aligned}$$

(a) For $\mu = 1$ this equation is satisfied if $\beta = \alpha$ and $E = -\dfrac{\hbar^2\alpha^2}{2m}$. The binding energy, which is the negative of the total energy, is therefore $\dfrac{\hbar^2\alpha^2}{2m}$, i.e. half the maximum depth of the well, $\dfrac{\hbar^2\alpha^2}{m}\text{sech}^2(0)$.

(b) For $\mu = 6$ an exact solution of the given form is not possible, but an upper limit can be placed on the ground state energy.

First,

$$\int \psi^* \psi \, dx = \int_{-\infty}^{\infty} A^2 \text{sech}^2 \beta x \, dx = A^2 \left[\frac{\tanh \beta x}{\beta}\right]_{-\infty}^{\infty} = \frac{2A^2}{\beta}.$$

Next, writing $H = T + V$, we have from the previous expression for ψ'' that

$$\begin{aligned}
\langle T \rangle &= -\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} A\,\mathrm{sech}\,\beta x(A\beta^2\mathrm{sech}\,\beta x\tanh^2\beta x - A\beta^2\mathrm{sech}^3\beta x)\,dx \\
&= -\frac{\hbar^2\beta^2}{2m}A^2\int_{-\infty}^{\infty}\mathrm{sech}^2\beta x(\tanh^2\beta x - \mathrm{sech}^2\beta x)\,dx \\
&= -\frac{\hbar^2\beta^2}{2m}A^2\int_{-\infty}^{\infty}\mathrm{sech}^2\beta x(2\tanh^2\beta x - 1)\,dx \\
&= -\frac{\hbar^2\beta^2}{2m}A^2\left[\frac{2}{3\beta}\tanh^3\beta x - \frac{1}{\beta}\tanh\beta x\right]_{-\infty}^{\infty} \\
&= -\frac{\hbar^2\beta}{2m}A^2\left(\frac{4}{3} - 2\right) = \frac{\hbar^2\beta}{3m}A^2.
\end{aligned}$$

For the expectation of the contribution to H from the potential term we have

$$\begin{aligned}
\langle V \rangle &= -\frac{6\hbar^2\alpha^2}{m}\int_{-\infty}^{\infty} A^2\mathrm{sech}^2\alpha x\,\mathrm{sech}^2\beta x\,dx \\
&= -\frac{12\hbar^2\alpha^2}{m}\int_{0}^{\infty} A^2(\mathrm{sech}\,\alpha x\,\mathrm{sech}\,\beta x)^2\,dx \\
&\le -\frac{12\hbar^2\alpha^2}{m}\int_{0}^{\infty} A^2\mathrm{sech}^2[(\alpha+\beta)x]\,dx, \text{ using hint and } \langle V\rangle < 0, \\
&= -\frac{12\hbar^2\alpha^2}{m}A^2\left[\frac{\tanh[(\alpha+\beta)x]}{\alpha+\beta}\right]_0^{\infty} \\
&= -\frac{12\hbar^2\alpha^2}{m(\alpha+\beta)}A^2.
\end{aligned}$$

Thus, recalling that $\int \psi^*\psi\,dx = \dfrac{2A^2}{\beta}$,

$$E = \frac{\langle T\rangle + \langle V\rangle}{\int \psi^*\psi\,dx} \le \frac{\hbar^2\beta^2}{6m} - \frac{6\hbar^2\alpha^2\beta}{m(\alpha+\beta)}.$$

The upper limit is minimised with respect to β when β satisfies

$$\begin{aligned}
\frac{\beta}{3} - \frac{6\alpha^2}{\alpha+\beta} + \frac{6\alpha^2\beta}{(\alpha+\beta)^2} &= 0, \\
\beta(\alpha+\beta)^2 - 18\alpha^2(\alpha+\beta) + 18\alpha^2\beta &= 0, \\
\beta^3 + 2\alpha\beta^2 + \alpha^2\beta - 18\alpha^3 &= 0, \\
(\beta - 2\alpha)(\beta^2 + 4\alpha\beta + 9\alpha^2) &= 0.
\end{aligned}$$

Thus $\beta = 2\alpha$ or $\beta = -2\alpha \pm i\sqrt{5}\alpha$; only the first is a real turning point. With this choice

$$E \le \frac{4\hbar^2\alpha^2}{6m} - \frac{12\hbar^2\alpha^3}{3m\alpha} = -\frac{10}{3}\frac{\hbar^2\alpha^2}{m}.$$

Since this gives an upper limit on the ground state energy, and $V(x) \to 0$ as $x \to \pm\infty$, the binding energy of the ground state must be at least $\frac{10}{3}\frac{\hbar^2\alpha^2}{m}$.

23

Integral equations

23.2 *Solve*

$$\int_0^\infty f(t)\exp(-st)\,dt = \frac{a}{a^2+s^2}.$$

Although this is an integral equation, we note that its LHS is also the definition of the Laplace transform of $f(t)$.

The solution to the equation is therefore the inverse Laplace transform of $\dfrac{a}{a^2+s^2}$. This is given in standard tables, which show that $f(t) = \sin at$.

23.4 *Use the fact that its kernel is separable to solve for $y(x)$ the integral equation*

$$y(x) = A\cos(x+a) + \lambda\int_0^\pi \sin(x+z)y(z)\,dz.$$

[*This equation is an inhomogeneous extension of the homogeneous Fredholm equation (23.13), and is similar to equation (23.57).*]

The kernel is separable because the equation can be written

$$\begin{aligned} y(x) &= A\cos(x+a) + \lambda\int_0^\pi \sin(x+z)y(z)\,dz \\ &= A\cos(x+a) + \lambda\int_0^\pi [\sin x\cos z\; y(z) + \cos x\sin z\; y(z)]\,dz, \end{aligned}$$

i.e. the kernel consists of a sum of terms each of which is the direct product of a function of x and a function of z.

So, we take $y(x)$ as a linear sum of the functions of x that appear in the integrand, explicitly $y(x) = c_1 \sin x + c_2 \cos x$. When this form is substituted into both sides of the integral equation (with z as its argument on the RHS), we obtain

$$\begin{aligned} c_1 \sin x + c_2 \cos x = A \cos x \cos a - A \sin x \sin a \\ + \lambda \sin x \int_0^\pi (c_1 \cos z \sin z + c_2 \cos^2 z)\, dz \\ + \lambda \cos x \int_0^\pi (c_1 \sin^2 z + c_2 \sin z \cos z)\, dz. \end{aligned}$$

Equating the coefficients of $\sin x$ and $\cos x$, and recalling that the average value of the square of a sinusoid over a whole number of half periods is $\frac{1}{2}$, gives

$$\begin{aligned} c_1 &= -A \sin a + \tfrac{1}{2}\lambda\pi c_2, \\ c_2 &= A \cos a + \tfrac{1}{2}\lambda\pi c_1. \end{aligned}$$

Solving this pair of simultaneous equations then yields

$$\begin{aligned} c_1 \left(1 - \frac{\lambda^2\pi^2}{4}\right) &= A \left(\frac{\lambda\pi}{2} \cos a - \sin a\right), \\ \text{and} \quad c_2 \left(1 - \frac{\lambda^2\pi^2}{4}\right) &= A \left(\cos a - \frac{\lambda\pi}{2} \sin a\right). \end{aligned}$$

Thus, the final solution is

$$\begin{aligned} y(x) &= \frac{A\left(\frac{\lambda\pi}{2} \cos a - \sin a\right) \sin x + A\left(\cos a - \frac{\lambda\pi}{2} \sin a\right) \cos x}{1 - (\lambda\pi/2)^2} \\ &= A \frac{(\lambda\pi/2) \sin(x - a) + \cos(x + a)}{1 - (\lambda\pi/2)^2}. \end{aligned}$$

We note that setting $\alpha = a + \frac{1}{2}\pi$ in equation (23.57) converts that integral equation into the current one with $A = 1$. Doing the same in its solution gives

$$\begin{aligned} y(x) &= \frac{(\lambda\pi/2) \cos(x - a - \frac{1}{2}\pi) + \cos(x + a)}{1 - (\lambda\pi/2)^2} \\ &= \frac{(\lambda\pi/2) \sin(x - a) + \cos(x + a)}{1 - (\lambda\pi/2)^2}, \end{aligned}$$

in agreement with the current solution.

23.6 *Consider the inhomogeneous integral equation*

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y)\, dy.$$

for which the kernel $K(x, y)$ is real, symmetric and continuous in $a \le x \le b$, $a \le y \le b$.

(a) *If λ is one of the eigenvalues λ_i of the homogeneous equation*

$$f_i(x) = \lambda_i \int_a^b K(x, y) f_i(y)\, dy,$$

prove that the inhomogeneous equation can only a have non-trivial solution if $g(x)$ is orthogonal to the corresponding eigenfunction $f_i(x)$.

(b) *Show that the only values of λ for which*

$$f(x) = \lambda \int_0^1 xy(x + y) f(y)\, dy$$

has a non-trivial solution are the roots of the equation

$$\lambda^2 + 120\lambda - 240 = 0.$$

(c) *Solve*

$$f(x) = \mu x^2 + \int_0^1 2xy(x + y) f(y)\, dy.$$

(a) Suppose $f(x)$ is a solution of

$$f(x) = g(x) + \lambda \int_a^b K(x, y) f(y)\, dy$$

with $\lambda = \lambda_i$, then

$$\begin{aligned}\int_a^b f_i(x) f(x)\, dx &= \int_a^b f_i(x) g(x)\, dx + \lambda_i \int_a^b f_i(x)\, dx \int_a^b K(x, y) f(y)\, dy \\ &= \int_a^b f_i(x) g(x)\, dx + \int_a^b \left[\lambda_i \int_a^b K(y, x) f_i(x)\, dx \right] f(y)\, dy,\end{aligned}$$

since $K(x, y) = K(y, x)$. Thus

$$\begin{aligned}&\int_a^b f_i(x) f(x)\, dx = \int_a^b f_i(x) g(x)\, dx + \int_a^b f_i(y) f(y)\, dy, \\ \Rightarrow \quad &\int_a^b f_i(x) g(x)\, dx = 0.\end{aligned}$$

i.e. $g(x)$ being orthogonal to the eigenfunction $f_i(x)$ is a necessary condition for

the inhomogeneous equation to have a solution whenever λ is equal to one of the eigenvalues λ_i.

(b) The kernel, $K(x, y) = yx^2 + y^2x$, is both symmetric and degenerate. To solve the equation we set $f(x) = a_1x^2 + a_2x$ giving

$$\begin{aligned} a_1x^2 + a_2x &= \lambda \int_0^1 (x^2y + y^2x)(a_1y^2 + a_2y)\,dy \\ &= \lambda \left(a_1x^2\frac{1}{4} + a_1x\frac{1}{5} + a_2x^2\frac{1}{3} + a_2x\frac{1}{4} \right). \end{aligned}$$

Equating coefficients gives

$$a_1 = \frac{\lambda a_1}{4} + \frac{\lambda a_2}{3} \quad \text{and} \quad a_2 = \frac{\lambda a_1}{5} + \frac{\lambda a_2}{4}.$$

For a non-trivial solution for a_1 and a_2 we need

$$\begin{vmatrix} 1 - \frac{\lambda}{4} & -\frac{\lambda}{3} \\ -\frac{\lambda}{5} & 1 - \frac{\lambda}{4} \end{vmatrix} = 0,$$

$$1 - \frac{\lambda}{2} + \frac{\lambda^2}{16} - \frac{\lambda^2}{15} = 0,$$

$$240 - 120\lambda - \lambda^2 = 0, \qquad \text{as stated.}$$

(c) As in part (b), the kernel is both symmetric and degenerate. Further, in the notation of part (a), $\lambda = 2$; but this is not a root of the equation derived in part (b). We therefore set $f(x) = \mu x^2 + a_1x^2 + a_2x$ and obtain [in the same way as in (b)]

$$a_1x^2 + a_2x = 2\left[x^2(a_1 + \mu)\frac{1}{4} + x(a_1 + \mu)\frac{1}{5} + x^2a_2\frac{1}{3} + xa_2\frac{1}{4} \right].$$

Equating the coefficients of x^2 and x and rationalising, we obtain

$$\begin{aligned} 6a_1 - 3a_1 - 4a_2 &= 3\mu, \\ -4a_1 + 10a_2 - 5a_2 &= 4\mu, \end{aligned}$$

yielding $a_1 = -31\mu$ and $a_2 = -24\mu$ and the solution as

$$f(x) = -30\mu x^2 - 24\mu x = -6\mu x(5x + 4).$$

This can be checked by substitution.

23.8 *By taking its Laplace transform, and that of $x^n e^{-ax}$, obtain the explicit solution of*

$$f(x) = e^{-x}\left[x + \int_0^x (x-u)e^u f(u)\,du\right].$$

Verify your answer by substitution.

Integrating by parts, we find the Laplace transform

$$\mathscr{L}\left[x^n e^{-ax}\right] = \int_0^\infty x^n e^{-ax} e^{-sx}\,dx = \frac{n(n-1)\cdots 2\,1}{(a+s)^{n+1}} = \frac{n!}{(a+s)^{(n+1)}}.$$

Setting $e^x f(x) = p(x)$ and $x = q(x)$, we can write the equation (after multiplying through by e^x) as

$$p(x) = q(x) + \int_0^x q(x-u)p(u)\,du,$$

in which the integral is a convolution. Thus, when the equation is Laplace transformed, the convolution theorem can be invoked and the transformed equation written in the form

$$\begin{aligned} &\bar{p}(s) = \bar{q}(s) + \bar{p}(s)\bar{q}(s), \\ \Rightarrow\quad &\bar{p}(s) = \frac{\bar{q}(s)}{1-\bar{q}(s)}. \end{aligned}$$

Now $\bar{q}(s) = \mathscr{L}[x] = s^{-2}$ and so

$$\begin{aligned} &\bar{p}(s) = \frac{1}{s^2-1} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right), \\ \Rightarrow\quad &p(x) = \frac{1}{2}\left(\frac{x^0 e^x}{0!} - \frac{x^0 e^{-x}}{0!}\right) = \sinh x, \\ \Rightarrow\quad &f(x) = \tfrac{1}{2}(1-e^{-2x}). \end{aligned}$$

This is the solution to the integral equation.

Verification:

$$\begin{aligned} \tfrac{1}{2}(1-e^{-2x}) &= e^{-x}\left[x + \int_0^x (x-u)e^u \tfrac{1}{2}(1-e^{-2u})\,du\right], \\ \sinh x &= x + \int_0^x (x-u)\sinh u\,du \\ &= x + [x\cosh u]_0^x - [u\cosh u]_0^x + \int_0^x \cosh u\,du \\ &= x + x\cosh x - x - x\cosh x + 0 + \sinh x - 0 \\ &= \sinh x, \text{ as expected.} \end{aligned}$$

23.10 *Show that the equation*

$$f(x) = x^{-1/3} + \lambda \int_0^\infty f(y)\exp(-xy)\,dy$$

has a solution of the form $Ax^\alpha + Bx^\beta$. *Determine the values of* α *and* β *and show that those of* A *and* B *are*

$$\frac{1}{1-\lambda^2\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \qquad \text{and} \qquad \frac{\lambda\Gamma(\frac{2}{3})}{1-\lambda^2\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})},$$

where $\Gamma(z)$ *is the gamma function.*

We first find the Laplace transform of x^{-n} when $n < 1$ but is not necessarily integral. With $s > 0$,

$$\begin{aligned}
\mathscr{L}\left[x^{-n}\right] &= \int_0^\infty x^{-n}e^{-sx}\,dx, \text{ set } y = sx,\\
&= \int_0^\infty \frac{y^{-n}}{s^{-n}}\,e^{-y}\,\frac{dy}{s}\\
&= s^{n-1}\int_0^\infty y^{-n}e^{-y}\,dy\\
&= s^{n-1}\Gamma(1-n), \text{ provided } n < 1.
\end{aligned}$$

Next, we substitute the trial solution $f(x) = Ax^\alpha + Bx^\beta$ into the given equation:

$$\begin{aligned}
Ax^\alpha + Bx^\beta &= x^{-1/3} + \lambda\int_0^\infty (Ay^\alpha + By^\beta)e^{-xy}\,dy \qquad \text{(change variable to } u = xy)\\
&= x^{-1/3} + \lambda\left[A\Gamma(1+\alpha)x^{-\alpha-1} + B\Gamma(1+\beta)x^{-\beta-1}\right],
\end{aligned}$$

assuming that $\alpha, \beta > -1$. For this equation to be valid, one of α and β must be $-\frac{1}{3}$ and

$$\begin{aligned}
\textit{either} \quad & \alpha = -\alpha - 1, \quad \beta = -\beta - 1\\
\textit{or} \quad & \alpha = -\beta - 1, \quad \beta = -\alpha - 1.
\end{aligned}$$

The first of these, which requires both α and β to have the value $-\frac{1}{2}$, is inconsistent with the other condition, but both it and the second are satisfied if $\alpha = -\frac{1}{3}$ and $\beta = -\frac{2}{3}$ (or vice versa). The assumption that $\alpha, \beta > -1$ is then also justified.

Thus, with the choice $\alpha = -\frac{1}{3}$,

$$A = 1 + \lambda B\Gamma(\tfrac{1}{3}), \quad \text{and} \quad B = \lambda A\Gamma(\tfrac{2}{3}),$$

yielding

$$A = \frac{1}{1-\lambda^2\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})} \qquad \text{and} \qquad B = \frac{\lambda\Gamma(\frac{2}{3})}{1-\lambda^2\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})}.$$

23.12 *By considering functions of the form* $h(x) = \int_0^x (x-y)f(y)\,dy$, *show that the solution* $f(x)$ *of the integral equation*

$$f(x) = x + \tfrac{1}{2}\int_0^1 |x-y|f(y)\,dy$$

satisfies the equation $f''(x) = f(x)$.

By examining the special cases $x = 0$ *and* $x = 1$, *show that*

$$f(x) = \frac{2}{(e+3)(e+1)}[(e+2)e^x - ee^{-x}].$$

To deal with the modulus sign we divide the integral into two parts:

$$\begin{aligned} f(x) &= x + \tfrac{1}{2}\textstyle\int_0^1 |x-y|f(y)\,dy \\ &= x + \tfrac{1}{2}\textstyle\int_0^x (x-y)f(y)\,dy + \tfrac{1}{2}\int_x^1 (y-x)f(y)\,dy. \end{aligned}$$

Thus the first and second derivatives of $f(x)$ are given by

$$\begin{aligned} f'(x) &= 1 + \tfrac{1}{2}\left[(x-x)f(x) + \textstyle\int_0^x f(y)\,dy\right] \\ &\qquad + \tfrac{1}{2}\left[-(x-x)f(x) - \textstyle\int_x^1 f(y)\,dy\right] \\ &= 1 + \tfrac{1}{2}\textstyle\int_0^x f(y)\,dy - \tfrac{1}{2}\int_x^1 f(y)\,dy, \\ f''(x) &= \tfrac{1}{2}f(x) - [-\tfrac{1}{2}f(x)] = f(x). \end{aligned}$$

It follows from this differential equation that $f(x)$ must have the form $f(x) = Ae^x + Be^{-x}$.

Considering the integral equation in the special case $x = 0$:

$$\begin{aligned} A + B &= 0 + \frac{1}{2}\int_0^1 y(Ae^y + Be^{-y})\,dy \\ &= \frac{A}{2}\left\{[ye^y]_0^1 - \int_0^1 e^y\,dy\right\} + \frac{B}{2}\left\{[-ye^{-y}]_0^1 + \int_0^1 e^{-y}\,dy\right\} \\ &= \frac{A}{2}(e - 0 - e + 1) + \frac{B}{2}(-e^{-1} + 0 - e^{-1} + 1) \\ &= \frac{A}{2} + \frac{B}{2}(1 - 2e^{-1}), \end{aligned}$$

which gives the first relationship between A and B as

$$A = -B(1 + 2e^{-1}).$$

Now considering the case $x = 1$:

$$Ae + Be^{-1} = 1 + \frac{1}{2}\int_0^1 (1-y)(Ae^y + Be^{-y})\,dy,$$

$$\begin{aligned}
2Ae + 2Be^{-1} - 2 &= A\left\{e - 1 - [\,ye^y\,]_0^1 + \int_0^1 e^y\,dy\right\} \\
&\qquad + B\left\{1 - e^{-1} - \left[-ye^{-y}\right]_0^1 - \int_0^1 e^{-y}\,dy\right\} \\
&= A(e - 1 - e + 0 + e - 1) + B(1 - e^{-1} + e^{-1} - 0 + e^{-1} - 1) \\
&= A(e-2) + Be^{-1},
\end{aligned}$$

which gives the second relationship between A and B as

$$-2 = -(e+2)A - Be^{-1}.$$

Solving the two derived relationships as a pair of simultaneous equations, we obtain

$$A = \frac{2(1+2e^{-1})e}{(e+1)(e+3)} = \frac{2(e+2)}{(e+1)(e+3)},$$

$$\text{and} \quad B = \frac{-2}{e+4+3e^{-1}} = \frac{-2e}{(e+1)(e+3)}.$$

Thus, finally,

$$f(x) = \frac{2}{(e+3)(e+1)}[(e+2)e^x - ee^{-x}].$$

23.14 *For the integral equation*

$$y(x) = x^{-3} + \lambda\int_a^b x^2z^2y(z)\,dz,$$

show that the resolvent kernel is $5x^2z^2/[5-\lambda(b^5-a^5)]$ *and hence solve the equation. For what range of* λ *is the solution valid?*

We use the recurrence relation

$$K_n(x,z) = \int_a^b K(x,z_1)K_{n-1}(z_1,z)\,dz_1$$

to build up the terms of the infinite series representing the resolvent kernel

$$R(x,z:\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,z).$$

For this problem $y_0(x) = x^{-3}$ and $K(x,z) = x^2z^2$.

$$K_1(x,z) = x^2z^2,$$
$$K_2(x,z) = \int_a^b x^2u^2\,u^2z^2\,du = \frac{b^5-a^5}{5}\,x^2z^2,$$
$$K_3(x,z) = \int_a^b x^2u^2\left(\frac{b^5-a^5}{5}\right)u^2z^2\,du = \left(\frac{b^5-a^5}{5}\right)^2 x^2z^2.$$

Clearly,

$$K_n(x,z) = \left(\frac{b^5-a^5}{5}\right)^{n-1} x^2z^2,$$

and $$R(x,z;\lambda) = \sum_{n=0}^{\infty}\lambda^n K_{n+1}(x,z) = \sum_{n=0}^{\infty}\lambda^n\left(\frac{b^5-a^5}{5}\right)^n x^2z^2 = \frac{5x^2z^2}{5-\lambda(b^5-a^5)},$$

provided $|\lambda|\,|b^5-a^5| < 5$ (so that the series is convergent).

The solution to the integral equation, has the general form

$$y(x) = y_0(x) + \lambda\int_a^b R(x,z;\lambda)y_0(z)\,dz,$$

and, in this particular case,

$$y(x) = x^{-3} + \lambda\int_a^b \frac{5x^2z^2z^{-3}}{5-\lambda(b^5-a^5)}\,dz = x^{-3} + \frac{5\lambda\ln(b/a)x^2}{5-\lambda(b^5-a^5)}.$$

23.16 *This exercise shows that following formal theory is not necessarily the best way to get practical results!*

(a) *Determine the eigenvalues $\lambda_\pm$ of the kernel $K(x,z) = (xz)^{1/2}(x^{1/2}+z^{1/2})$ and show that the corresponding eigenfunctions have the forms*

$$y_\pm(x) = A_\pm(\sqrt{2}x^{1/2} \pm \sqrt{3}x),$$

where $A_\pm^2 = 5/(10 \pm 4\sqrt{6})$.

(b) *Use Schmidt–Hilbert theory to solve*

$$y(x) = 1 + \tfrac{5}{2}\int_0^1 K(x,z)y(z)\,dz.$$

(c) *As will have been apparent, the algebra involved in the formal method used in* (b) *is long and error-prone, and it is in fact much more straightforward to use a trial function $1 + \alpha x^{1/2} + \beta x$. Check your answer by doing so.*

(a) To find the eigenvalues, given the (supposed) forms of the eigenfunctions, we may substitute $y(x) = a_1 x + a_2 x^{1/2}$ and require consistency.

$$\begin{aligned} a_1 x + a_2 x^{1/2} &= \lambda \int_0^1 (xz^{1/2} + zx^{1/2})(a_1 z + a_2 z^{1/2})\, dz \\ \Rightarrow \quad a_1 &= \tfrac{2}{5}\lambda a_1 + \tfrac{1}{2}\lambda a_2, \\ a_2 &= \tfrac{1}{3}\lambda a_1 + \tfrac{2}{5}\lambda a_2. \end{aligned}$$

These two equations have a non-trivial solution if

$$\left(1 - \frac{2\lambda}{5}\right)^2 = \frac{\lambda^2}{6} \quad \Rightarrow \quad \lambda = \frac{5\sqrt{6}}{2\sqrt{6} \pm 5} = \pm 25\sqrt{6} - 60.$$

To find the corresponding eigenfunctions, set $a_1 = 1$, say. Then

$$\begin{aligned} a_2 &= \frac{2}{\lambda}\left(1 - \frac{2\lambda}{5}\right) \\ &= \frac{2}{\pm 25\sqrt{6} - 60} - \frac{4}{5} \\ &= \frac{2 \mp 20\sqrt{6} + 48}{\pm 25\sqrt{6} - 60} \\ &= \frac{(50 \mp 20\sqrt{6})(\pm 25\sqrt{6} + 60)}{3750 - 3600} \\ &= \frac{\pm 50\sqrt{6}}{150} = \pm\sqrt{\frac{2}{3}}. \end{aligned}$$

Thus the normalised eigenfunctions are

$$y_\pm(x) = A_\pm(\sqrt{3}x \pm \sqrt{2}x^{1/2}),$$

with

$$\begin{aligned} 1 &= \int_0^1 y_\pm^2\, dx \\ &= \int_0^1 A_\pm^2(\sqrt{3}x \pm \sqrt{2}x^{1/2})^2 dx \\ &= A_\pm^2\left(3\,\frac{1}{3} \pm 2\sqrt{6}\,\frac{2}{5} + 2\,\frac{1}{2}\right) \\ &= A_\pm^2\left(\frac{10 \pm 4\sqrt{6}}{5}\right) = A_\pm^2\left(\frac{2 \pm \sqrt{6}}{\sqrt{5}}\right)^2, \end{aligned}$$

giving the stated values for $A_\pm^2$.

(b) We first need to calculate

$$\begin{aligned}\langle y_\pm|f\rangle &= \int_0^1 A_\pm(\sqrt{3}x \pm \sqrt{2}x^{1/2})\,1\,dx \\ &= \frac{\sqrt{5}}{2\pm\sqrt{6}}\,\frac{3\sqrt{3}\pm 4\sqrt{2}}{6} \\ &= \frac{\sqrt{5}(3\sqrt{3}\pm 4\sqrt{2})(2\mp\sqrt{6})}{6(4-6)} = \frac{\sqrt{5}}{12}(2\sqrt{3}\pm\sqrt{2}).\end{aligned}$$

The solution is given by

$$\begin{aligned}y(x) &= f + \lambda\sum_i \frac{\langle y_\pm|f\rangle}{\lambda_i-\lambda}\,y_i \\ &= 1 + \frac{5}{2}\,\frac{\sqrt{5}}{12}\,\frac{2\sqrt{3}+\sqrt{2}}{25\sqrt{6}-60-\frac{5}{2}}\,\frac{\sqrt{5}}{2+\sqrt{6}}\,(\sqrt{3}x+\sqrt{2}x^{1/2}) \\ &\quad + \frac{5}{2}\,\frac{\sqrt{5}}{12}\,\frac{2\sqrt{3}-\sqrt{2}}{-25\sqrt{6}-60-\frac{5}{2}}\,\frac{\sqrt{5}}{2-\sqrt{6}}\,(\sqrt{3}x-\sqrt{2}x^{1/2}).\end{aligned}$$

The coefficient of x in this expression is

$$\begin{aligned}&\frac{25}{24}\sqrt{3}\left[\frac{2(2\sqrt{3}+\sqrt{2})}{(50\sqrt{6}-125)(2+\sqrt{6})} + \frac{2(2\sqrt{3}-\sqrt{2})}{(-50\sqrt{6}-125)(2-\sqrt{6})}\right] \\ &= \frac{\sqrt{3}}{12}\left[\frac{2\sqrt{3}+\sqrt{2}}{2-\sqrt{6}} + \frac{2\sqrt{3}-\sqrt{2}}{2+\sqrt{6}}\right] \\ &= \frac{\sqrt{3}}{12}\,\frac{4\sqrt{3}+2\sqrt{2}+6\sqrt{2}+2\sqrt{3}+4\sqrt{3}-2\sqrt{2}-6\sqrt{2}+2\sqrt{3}}{4-6} = -\frac{3}{2}.\end{aligned}$$

A similar (tedious) calculation shows that the coefficient of $x^{1/2}$ is $-4/3$, making the final solution

$$y(x) = 1 - \tfrac{3}{2}x - \tfrac{4}{3}x^{1/2}.$$

(c) Substituting the trial solution $1+\alpha x^{1/2}+\beta x$ directly into the equation gives

$$1+\alpha x^{1/2}+\beta x = 1 + \tfrac{5}{2}\int_0^1 (xz^{1/2}+zx^{1/2})\,(1+\alpha z^{1/2}+\beta z)\,dz.$$

Carrying out the integrations and equating the coefficients of x and $x^{1/2}$ then leads to

$$\begin{aligned}\alpha &= \tfrac{5}{2}\left(\tfrac{1}{2}+\tfrac{2}{5}\alpha+\tfrac{1}{3}\beta\right) \quad\Rightarrow\quad \beta = -\tfrac{3}{2}, \\ \beta &= \tfrac{5}{2}\left(\tfrac{2}{3}+\tfrac{1}{2}\alpha+\tfrac{2}{5}\beta\right) \quad\Rightarrow\quad \alpha = -\tfrac{4}{3}, \\ \Rightarrow\quad y(x) &= 1 - \tfrac{4}{3}x^{1/2} - \tfrac{3}{2}x.\end{aligned}$$

This is as in part (b) — but with much less effort!

24

Complex variables

24.2 *Find a function $f(z)$, analytic in a suitable part of the Argand diagram, for which*

$$Re\ f = \frac{\sin 2x}{\cosh 2y - \cos 2x}.$$

Where are the singularities of $f(z)$?

Let the required function be $f(z) = u + iv$, with $u = \sin 2x/(\cosh 2y - \cos 2x)$. Since y appears less often than x in the given expression, it will probably be easier to consider $\partial u/\partial y$ rather than $\partial u/\partial x$. This indicates that the relevant Cauchy–Riemann equation is

$$\frac{\partial u}{\partial y} = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} = -\frac{\partial v}{\partial x}.$$

Having differentiated w.r.t y, we now integrate w.r.t x:

$$v = \int \frac{2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}\,dx = -\frac{\sinh 2y}{\cosh 2y - \cos 2x} + f(y).$$

By inspection, or by substitution in the other C–R equation, $f(y)$ can be seen to be an ignorable constant. The required function is therefore

$$f(z) = \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}.$$

To determine what function of z this is, consider its form on the real axis where $y = 0$,

$$f(x) = \frac{\sin 2x}{1 - \cos 2x} = \frac{2\sin x \cos x}{2\sin^2 x} = \cot x \quad \Rightarrow \quad f(z) = \cot z.$$

This can be checked as follows.

$$\begin{aligned}
f(z) &= \frac{\cos(x+iy)}{\sin(x+iy)} \\
&= \frac{\cos x\cosh y - i\sin x\sinh y}{\sin x\cosh y + i\cos x\sinh y} \\
&= \frac{(\cos x\cosh y - i\sin x\sinh y)(\sin x\cosh y - i\cos x\sinh y)}{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y} \\
&= \frac{\sin x\cos x(\cosh^2 y - \sinh^2 y) - i\cosh y\sinh y(\cos^2 x + \sin^2 x)}{\cos^2 x(\sinh^2 y - \cosh^2 y) + \cosh^2 y} \\
&= \frac{\sin x\cos x - i\cosh y\sinh y}{\cosh^2 y - \frac{1}{2} - \cos^2 x + \frac{1}{2}} \\
&= \frac{\sin 2x - i\sinh 2y}{\cosh 2y - \cos 2x}.
\end{aligned}$$

Since $f(z) = \cot z$ the poles can only occur at the zeros of $\sin z$, i.e at $z = n\pi$ where n is an integer; $\cos n\pi \neq 0$ and so there will be a (simple) pole at each such point. The same conclusion is reached by studying $\cosh 2y - \cos 2x$. Since $\cosh 2y \geq 1$ and $\cos 2x \leq 1$, this denomiator can only vanish if both terms equal 1; this requires $y = 0$ and $x = n\pi$.

24.4 *Find the Taylor series expansion about the origin of the function $f(z)$ defined by*

$$f(z) = \sum_{r=1}^{\infty}(-1)^{r+1}\sin\left(\frac{pz}{r}\right)$$

where p is a constant. Hence verify that $f(z)$ is a convergent series for all z.

Because every term in the series is a sine function, all of its even derivatives will also be sine functions and therefore vanish at $z = 0$. The odd derivatives will consist entirely of cosine functions and the $(2n+1)$th derivative of a typical term will be

$$f_r^{(2n+1)}(z) = (-1)^{r+1}(-1)^n\left(\frac{p}{r}\right)^{2n+1}\cos\left(\frac{pz}{r}\right),$$

with

$$f_r^{(2n+1)}(0) = (-1)^{n+r+1}\left(\frac{p}{r}\right)^{2n+1}.$$

The Taylor series expansion is therefore

$$f(z) = \sum_{n=0}^{\infty}\frac{z^{2n+1}}{(2n+1)!}\sum_{r=1}^{\infty}(-1)^{n+r+1}\left(\frac{p}{r}\right)^{2n+1}.$$

giving the expansion coefficients as

$$a_{2n+1} = \frac{(-1)^{n+1}p^{2n+1}}{(2n+1)!}\sum_{r=1}^{\infty}\frac{(-1)^r}{r^{2n+1}},$$

with $a_{2n} = 0$.

As $n \to \infty$ the sum appearing in a_{2n+1} tends (rapidly) to -1 (only the $r = 1$ term contributing) and so the radius of convergence R is given by

$$\frac{1}{R^2} = \lim_{n\to\infty}\left|\frac{-p^{2n+3}(2n+1)!}{p^{2n+1}(2n+3)!}\right| = 0.$$

Thus $R = \infty$ and the series is convergent for all z.

24.6 *Identify the zeroes, poles and essential singularities of the following functions:*

(a) $\tan z$, (b) $[(z-2)/z^2]\sin[1/(1-z)]$, (c) $\exp(1/z)$,
(d) $\tan(1/z)$, (e) $z^{2/3}$.

(a) This function $\tan z = \dfrac{\sin z}{\cos z}$ has zeroes when

$$\sin z = \frac{1}{2i}(e^{ix-y} - e^{-ix+y}) = 0 \quad\Rightarrow\quad e^{ix}e^{-y} = e^{-ix}e^{y}.$$

The two terms can only be equal if they have equal magnitudes, i.e. $|e^{-y}| = |e^{y}| \quad\Rightarrow\quad y = 0$. We also need $e^{ix} = e^{-ix} \quad\Rightarrow\quad x = n\pi$, where n is an integer. Thus the zeroes of $\tan x$ occur at $z = n\pi$.

The poles of $\tan z$ will occur at the zeroes of $\cos z$. By a similar argument to that above, this needs $y = 0$ and $e^{ix} = -e^{-ix} = e^{i(-x+\pi+2n\pi)} \quad\Rightarrow\quad 2x = (2n+1)\pi$. Thus, the (simple) poles of $\tan z$ occur at $z = (n+\frac{1}{2})\pi$.

We note that both $\sin u$ and $\cos u$ have Maclaurin series that contain arbitrarily large powers of u and that they are not multiples of each other; we can conclude that their ratio will also have a Maclaurin series containing arbitrarily large powers of u. The same conclusion is reached by differentiating $\tan u$ and so constructing its Maclaurin series directly. Thus, when $z \to \infty$ is replaced by $z = 1/\xi$ with $\xi \to 0$, there will be arbitrarily large *inverse* powers of ξ in the series expansion; this establishes that $\xi = 0$ (i.e. $z = \infty$) is an essential singularity of $\tan z$.

For the remaining exercises we will not give such a detailed justification of our conclusions; most features are obvious and only the less obvious ones will be treated in any detail.

(b) This function, $\dfrac{z-2}{z^2}\sin\dfrac{1}{1-z}$, has obvious zeroes at $z=2$ and $z=\infty$. Equally clearly, at $z=0$ it has a 2nd-order pole. Further zeroes will occur when the sine term factor is zero; from the analysis in part (a), this will be when $(1-z)^{-1}=n\pi$, i.e. at $z=1-(n\pi)^{-1}$. The remaining singularity to classify is that at $z=1$. By a similar argument to that given in part (a), the Laurent expansion of the function about the point will have no largest negative power of $1-z$; the point is therefore an essential singularity.

(c) Since $\exp(0)=1$, the function is well behaved and analytic at ∞. The only non-analytic point is the origin, $z=0$, where the defining series for the exponential function generates a Laurent expansion with no largest negative power of z; the point is therefore an essential singularity.

(d) The singularities of $\tan(z^{-1})$ follow from those of $\tan z$ in part (a). They are therefore zeroes at $z=\infty$ and $(n\pi)^{-1}$, simple poles at $z=(n\pi+\frac{1}{2}\pi)^{-1}$ and an essential singularity at $z=0$.

(e) The origin, $z=0$ is both a zero and a branch point of the function $z^{2/3}$. To determine its behavior at ∞ we have to consider $1/\xi^{2/3}$ near $\xi=0$. There is clearly a singularity there, and, since the function cannot even be expressed as a Laurent series, the singularity is an essential singularity.

24.8 *Show that the transformation*

$$w=\int_0^z \frac{1}{(\zeta^3-\zeta)^{1/2}}\,d\zeta$$

transforms the upper half-plane into the interior of a square that has one corner at the origin of the w-plane and sides of length L, where

$$L=\int_0^{\pi/2}\operatorname{cosec}^{1/2}\theta\,d\theta.$$

This transformation is a Schwarz-Christoffel transformation of the upper half of the z-plane into a closed polygon. It can be written

$$w=\int_0^z \zeta^{-1/2}(\zeta-1)^{-1/2}(\zeta+1)^{-1/2}\,d\zeta.$$

Since each factor is raised to the same power, the interior angles at the corners of the polygon are all the same and given by $(\phi/\pi)-1=-\frac{1}{2}$, i.e. each $\phi=\pi/2$. To close the polygon a fourth vertex is needed (also with $\phi=\pi/2$); this must arise from transforming the point $x=\pm\infty$, $y=0$. Thus the four points on the x-axis that transform into the vertices of what is (for now) a rectangle are $x_1=-1$, $x_2=0$, $x_3=1$ and $x_4=\pm\infty$.

From the definition of the transform, the image of x_2 ($z = 0$) is clearly $w_2 = 0$. Thus one corner of the rectangle is at the origin in the w-plane. Further,

$$w_3 - w_2 = \int_0^1 \frac{1}{\zeta^{1/2}(\zeta^2 - 1)^{1/2}}\, d\zeta.$$

Setting $\zeta = \dfrac{1}{u}$ with $d\zeta = -\dfrac{1}{u^2}\, du$,

$$\begin{aligned} w_3 - w_2 &= \int_\infty^1 \frac{u^{1/2}}{(u^{-2} - 1)^{1/2}} \left(\frac{-1}{u^2}\right) du \\ &= \int_1^\infty \frac{u^{3/2}}{(1 - u^2)^{1/2} u^2}\, du \\ &= \int_1^\infty \frac{1}{(1 - u^2)^{1/2} u^{1/2}}\, du \\ &= \pm i(w_4 - w_3). \end{aligned}$$

Thus we have a rectangle with adjacent sides of equal length, i.e. a square.

The length of a side is given in magnitude by

$$L = \int_1^\infty \frac{1}{u^{1/2}(u^2 - 1)^{1/2}}\, du.$$

Setting $u = \operatorname{cosec}\theta$ with $du = -\operatorname{cosec}\theta \cot\theta\, d\theta$ and $u^2 - 1 = \cot^2\theta$, gives

$$\begin{aligned} L &= \int_{\pi/2}^0 \frac{-\operatorname{cosec}\theta \cot\theta}{\operatorname{cosec}^{1/2}\theta \cot\theta}\, d\theta \\ &= \int_0^{\pi/2} \operatorname{cosec}^{1/2}\theta\, d\theta, \end{aligned}$$

as stated in the question.

Many of the remaining exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar figures that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we show in figure 24.1 a number of typical contour types to which reference can be made.

24.10 *Show that, if a is a positive real constant, the function* $\exp(iaz^2)$ *is analytic and* $\to 0$ *as* $|z| \to \infty$ *for* $0 < \arg z \le \pi/4$. *By applying Cauchy's theorem to a suitable contour prove that*

$$\int_0^\infty \cos(ax^2)\, dx = \sqrt{\frac{\pi}{8a}}.$$

Figure 24.1 Typical contours for use in contour integration.

The function is explicitly a function of z and has no poles in the finite plane. By Cauchy's theorem, its integral around a closed loop will be zero. Writing z as $r\,e^{i\theta}$, we have

$$e^{iaz^2} = e^{iar^2(\cos 2\theta + i\sin 2\theta)} = e^{iar^2\cos 2\theta}e^{-ar^2\sin 2\theta}.$$

The real part of this, when $\theta = 0$, is the required integrand. Further, the function $\to 0$ as $r \to \infty$ provided $a\sin 2\theta$ is positive. Since a is positive, this requires $\sin 2\theta$ to be positive, i.e. $0 < \theta < \pi/2$.

To apply Cauchy's theorem we therefore need a closed contour which includes the positive real axis, $0 \le x < \infty$, and some part of the semi-circle at infinity in the first quadrant; from the above result, this part of the contour will contribute nothing. The contour needs to be completed by a path along which the line integral is known or can be evaluated by means other than contour integration.

In view of the i and the square of z appearing in the exponent in the integrand we consider a contour such as that shown in 24.1(a) with $\alpha = \pi/4$. On the closing part of the contour, $z = ue^{i\pi/4}$. Cauchy's theorem now reads:

$$\int_0^\infty e^{iax^2}\,dx + 0 + \int_\infty^0 \exp[\,ia(ue^{i\pi/4})^2\,]\,e^{i\pi/4}\,du = 0,$$

$$\int_0^\infty e^{iax^2}\,dx - \int_0^\infty e^{ia(u^2 i)}\,\frac{1+i}{\sqrt{2}}\,du = 0.$$

Taking the real parts of both sides gives the equality

$$\int_0^\infty \cos(ax^2)\,dx = \frac{1}{\sqrt{2}}\int_0^\infty e^{-au^2}\,du$$

$$= \frac{1}{\sqrt{2}}\,\frac{1}{2}\sqrt{\frac{\pi}{a}} = \sqrt{\frac{\pi}{8a}}\,.$$

In the last line we have used the standard result for the infinite integral of $\exp(-ax^2)$, which can be found in any textbook if not already known. Apart from a change of scale, the overall result is a special case of a Fresnel integral $C(x)$ in which the argument $x = \infty$

24.12 *By considering the real part of*

$$\int \frac{-iz^{n-1}\,dz}{1 - a(z + z^{-1}) + a^2},$$

where $z = \exp i\theta$ *and* n *is a non-negative integer, evaluate*

$$\int_0^\pi \frac{\cos n\theta}{1 - 2a\cos\theta + a^2}\,d\theta,$$

for a *real and* > 1.

The integrand can be rewritten in a form that establishes the positions of any poles it possesses:

$$\frac{-iz^{n-1}}{1 - a(z + z^{-1}) + a^2} = \frac{i}{a}\,\frac{z^n}{z^2 - (a + a^{-1})z + 1} = \frac{i}{a}\,\frac{z^n}{(z-a)(z-a^{-1})}.$$

We use a contour C of type (c) in figure 24.1 with $R = 1$ (i.e. the unit circle) and integrate $f(z)$ around it. As the above form shows, the integrand has poles at $z = a$ (outside the contour) and $z = a^{-1}$ (inside it) and so we need the residue only at the latter (simple) pole. It is

$$\frac{i}{a}\lim_{z\to a^{-1}}\left(\frac{z^n}{z-a}\right) = \frac{i}{a}\,\frac{1}{a^n(a^{-1}-a)} = \frac{i}{a}\,\frac{1}{a^{n-1}(1-a^2)}.$$

From the residue theorem it now follows that

$$\int_C \frac{i}{a}\,\frac{z^n\,dz}{(z-a)(z-a^{-1})}. = \frac{i}{a}\,\frac{2\pi i}{a^{n-1}(1-a^2)}.$$

On the unit circle, $z = e^{i\theta}$ and $dz = i\,e^{i\theta}\,d\theta$. Making this change of variable gives

$$\begin{aligned}\frac{-2\pi}{a^n(1-a^2)} &= \frac{i}{a}\int_0^{2\pi} \frac{e^{in\theta}\,i\,e^{i\theta}\,d\theta}{(e^{i\theta}-a)(e^{i\theta}-a^{-1})}\\ &= \int_0^{2\pi} \frac{e^{in\theta}\,d\theta}{(e^{i\theta}-a)(e^{-i\theta}-a)}\\ &= \int_0^{2\pi} \frac{e^{in\theta}\,d\theta}{a^2-2a\cos\theta+1}.\end{aligned}$$

On equating real parts,

$$\frac{2\pi}{a^n(a^2-1)} = \int_0^{2\pi} \frac{\cos n\theta\,d\theta}{a^2-2a\cos\theta+1}.$$

Finally, we note that the integrand is an even function of θ and so for the given limits of 0 and π the value of the integral is one-half of that calculated above, i.e. $\pi/(a^{n+2}-a^n)$.

24.14 *Prove that, for $\alpha > 0$, the integral*

$$\int_0^\infty \frac{t\sin\alpha t}{1+t^2}\,dt$$

has the value $(\pi/2)\exp(-\alpha)$.

We wish to evaluate

$$I = \int_0^\infty \frac{t\sin\alpha t}{1+t^2}\,dt = \frac{1}{2}\int_{-\infty}^\infty \frac{t\sin\alpha t}{1+t^2}\,dt = \frac{1}{2}\mathrm{Im}\int_{-\infty}^\infty \frac{t\,e^{i\alpha t}}{1+t^2}\,dt.$$

The complex function $f(z) = \dfrac{z}{1+z^2}$ has the properties:

(i) it is analytic in the upper half-plane except for a pole at $z = i$, and

(ii) $|f(z)| \sim |z^{-1}| \to 0$ as $|z| \to \infty$ in the upper half-plane.

Since $\alpha > 0$, all the conditions for Jordan's lemma are satisfied and we can usefully consider the integral

$$J = \int_C \frac{z\,e^{i\alpha z}}{1+z^2}\,dz,$$

where C is contour (d) in figure 24.1 with $R \to \infty$.

Jordan's lemma ensures that the integral along the semi-circle Γ goes to 0 as $R \to \infty$. The residue theorem then reads

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{x\, e^{i\alpha x}}{1+x^2}\, dx + 0 &= 2\pi i(\text{residue at } z = i)\\ &= 2\pi i \lim_{z\to i} \frac{(z-i)z\, e^{i\alpha z}}{(z-i)(z+i)}\\ &= 2\pi i \frac{ie^{-\alpha}}{2i}.\end{aligned}$$

Equating the imaginary parts of both side of the equation shows that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \pi e^{-\alpha}$$

and $I = \frac{1}{2}\pi e^{-\alpha}$, as stated in the question.

24.16 *Show that the principal value of the integral*

$$\int_{-\infty}^{\infty} \frac{\cos(x/a)}{x^2 - a^2}\, dx$$

is $-(\pi/a)\sin 1$.

The complex form of the integrand is

$$f(z) = \frac{e^{iz/a}}{z^2 - a^2};$$

this has two poles on the real axis, at $z = \pm a$. Consequently we need to work with a contour that has semicircular indentations into the upper half-plane at these points (see contour (e) in figure 24.1).

With this choice of contour, no poles are enclosed and the integral around the complete contour will be zero. However, the contributions from the separate parts of the contour are not individually zero. Since the conditions for Jordan's lemma are satisfied, the contribution from Γ *is* zero, but that still leaves the contributions from the indented semicircles as well as the principal value that we wish to evaluate.

Each semicircle contributes $-\pi i \times$ the residue at the corresponding (simple) pole, the minus sign arising from the fact that the semicircle is traversed in the negative sense. The residues are

$$\lim_{z\to -a} \frac{(z+a)e^{iz/a}}{(z+a)(z-a)} = \frac{e^{-i}}{-2a} \quad \text{and} \quad \lim_{z\to a} \frac{(z-a)e^{iz/a}}{(z+a)(z-a)} = \frac{e^{i}}{2a}.$$

The residue theorem (Cauchy's theorem in this case) then reads

$$-\frac{\pi i}{2a}(e^{i}-e^{-i})+P\int_{-\infty}^{\infty}\frac{\cos(x/a)}{x^2-a^2}\,dx+iP\int_{-\infty}^{\infty}\frac{\sin(x/a)}{x^2-a^2}\,dx=0.$$

Equating the real parts of the two sides of this equation yields the stated result,

$$P\int_{-\infty}^{\infty}\frac{\cos(x/a)}{x^2-a^2}\,dx=-\frac{\pi}{a}\sin 1.$$

24.18 *By applying the residue theorem around a wedge-shaped contour of angle $2\pi/n$, with one side along the real axis, prove that the integral*

$$\int_0^{\infty}\frac{dx}{1+x^n},$$

where n is real and ≥ 2, has the value $(\pi/n)\operatorname{cosec}(\pi/n)$.

The contour needed is that shown in figure 24.1 (a) with $\alpha = 2\pi/n$. The denominator of the complex integrand has zeroes when

$$z=\exp\left[\frac{(2m+1)i\pi}{n}\right],\ \text{for } m=0,1,\ldots,n-1.$$

Only one of these zeroes, the one at $z = e^{i\pi/n}$ with $m = 0$, lies within the sector contour, and none lie on it.

On OA, $z = x$ and $dz = dx$.

On AB, $z = R\,e^{i\theta}$ for $0 \leq \theta \leq 2\pi/n$ and $dz = iR\,e^{i\theta}\,d\theta$.

On BO, $z = ye^{2\pi i/n}$ and $dz = e^{2\pi i/n}\,dy$.

Applying the residue theorem to the contour integral gives

$$\int_0^R\frac{dx}{1+x^n}+\int_0^{2\pi/n}\frac{iR\,e^{i\theta}\,d\theta}{1+R^ne^{in\theta}}\,d\theta$$
$$+\int_R^0\frac{e^{2\pi i/n}\,dy}{1+y^ne^{2\pi i}}=2\pi i(\text{residue at } z=e^{i\pi/n}).$$

Letting $R \to \infty$ shows that the required integral I satisfies

$$I(1-e^{2\pi i/n})=2\pi i(\text{residue at } z=e^{i\pi/n}).$$

Now, since it is a simple pole at $z = e^{i\pi/n}$, the residue there is given by the inverse of the derivative of $1 + z^n$, i.e.

$$\text{residue}=\frac{1}{nz^{n-1}}=\frac{1}{ne^{[i\pi(n-1)]/n}}=-\frac{e^{i\pi/n}}{n}.$$

Thus,

$$\begin{aligned} I(1-e^{2\pi i/n}) &= \frac{-2\pi i}{n}\, e^{i\pi/n}, \\ I\,\frac{1}{2i}(e^{-i\pi/n}-e^{i\pi/n}) &= -\frac{\pi}{n}, \quad \text{after dividing through by } 2ie^{i\pi/n}, \\ \Rightarrow \quad I\sin\frac{\pi}{n} &= \frac{\pi}{n}, \\ \Rightarrow \quad I &= \frac{\pi}{n}\operatorname{cosec}\frac{\pi}{n}, \end{aligned}$$

as stated in the question.

24.20 *Show that*

$$\int_0^\infty \frac{\ln x}{x^{3/4}(1+x)}\,dx = -\sqrt{2}\pi^2.$$

Denote the required integral by I. The complex form of the integrand $f(z)$ is not single-valued and by choosing contour (f) of figure 24.1 we can capitalise on that fact. We first consider the behaviour of the integrand on the various parts of the contour.

(i) Around γ, $|zf(z)| \sim \dfrac{z^{1/4}\ln z}{1} \to 0$ as $|z| \to 0$.

(ii) Around Γ, $|zf(z)| \sim \dfrac{\ln z}{z^{3/4}} \to 0$ as $|z| \to \infty$.

(iii) On γ_1, $z = x$ and $f(z) = \dfrac{\ln x}{x^{3/4}(1+x)}$.

(iv) On γ_2, $z = xe^{2\pi i}$ and $f(z) = \dfrac{\ln x + i2\pi}{x^{3/4}e^{3\pi i/2}(1+xe^{2\pi i})}$.

The only pole *inside* the contour is a simple one at $z = e^{i\pi}$; that at $z = 0$ is excluded by the contour and its (non-) contribution is that calculated for γ. The residue at $z = e^{i\pi}$ is $(0+i\pi)e^{-3i\pi/4}$.

The residue theorem therefore reads [Note that $e^{-3\pi i/2} = i$]

$$\begin{aligned} 0 + I + 0 - Ie^{-3\pi i/2} - \int_0^\infty \frac{2\pi i\, e^{-3\pi i/2}}{x^{3/4}(1+x)}\,dx &= 2\pi i(i\pi e^{-3\pi i/4}), \\ I(1-i) + \int_0^\infty \frac{2\pi}{x^{3/4}(1+x)}\,dx &= -2\pi^2\left(\frac{-1-i}{\sqrt{2}}\right). \end{aligned}$$

Equating imaginary parts, $\qquad -I = \sqrt{2}\pi^2.$

As a bonus, we also deduce that

$$\int_0^\infty \frac{1}{x^{3/4}(1+x)}\,dx = \frac{1}{2\pi}(\sqrt{2}\pi^2 - I) = \sqrt{2}\pi.$$

24.22 *The equation of an ellipse in plane polar coordinates r, θ, with one of its foci at the origin, is*

$$\frac{\ell}{r} = 1 - \epsilon \cos\theta,$$

where ℓ is a length (that of the latus rectum) and ϵ $(0 < \epsilon < 1)$ is the eccentricity of the ellipse. Express the area of the ellipse as an integral around the unit circle in the complex plane, and show that the only singularity of the integrand inside the circle is a double pole at $z_0 = \epsilon^{-1} - (\epsilon^{-2} - 1)^{1/2}$.

By setting $z = z_0 + \xi$ and expanding the integrand in powers of ξ, find the residue at z_0 and hence show that the area is equal to $\pi\ell^2(1-\epsilon^2)^{-3/2}$.

[Note*: In terms of the semi-axes a and b of the ellipse, $\ell = b^2/a$ and $\epsilon^2 = (a^2 - b^2)/a^2$.*]

The area A is given by

$$A = \int_0^{2\pi} \frac{1}{2} r^2 \, d\theta = \frac{\ell^2}{2} \int_0^{2\pi} \frac{d\theta}{(1 - \epsilon\cos\theta)^2}.$$

Now, if we set $z = e^{i\theta}$, with $dz = i\,e^{i\theta}\,d\theta = iz\,d\theta$, the integral becomes a contour integral around the unit circle C [contour (c) in figure 24.1 with $R = 1$]. The area integral is then given by

$$\begin{aligned}
\frac{2A}{\ell^2} &= \int_C \frac{-iz^{-1}\,dz}{[1 - \frac{1}{2}\epsilon(z + z^{-1})]^2} \\
&= \frac{4}{\epsilon^2} \int_C \frac{-iz\,dz}{(z^2 - 2\epsilon^{-1}z + 1)^2} \\
\frac{A\epsilon^2}{2\ell^2} &= \int_C \frac{-iz\,dz}{(z - z_0)^2(z - z_1)^2}, \text{ where } z_{1,0} = \frac{1}{\epsilon} \pm \sqrt{\frac{1}{\epsilon^2} - 1}.
\end{aligned}$$

Since $0 < \epsilon < 1$, $\epsilon^{-1} > 1$ and only the pole at $z_0 = \epsilon^{-1} - (\epsilon^{-2} - 1)^{1/2}$ lies inside the circle $|z| = 1$. Clearly it is a double pole of the integrand.

To determine the residue at z_0 we set $z = z_0 + \xi$:

$$\begin{aligned}
\frac{-iz}{(z - z_0)^2(z - z_1)^2} &= \frac{-i(z_0 + \xi)}{\xi^2(z_0 - z_1 + \xi)^2} \\
&= \frac{-i(z_0 + \xi)}{\xi^2(z_0 - z_1)^2} \left[1 - \frac{2\xi}{z_0 - z_1} + \frac{6\xi^2}{2!\,(z_0 - z_1)^2} - \cdots \right].
\end{aligned}$$

The residue, equal to the coefficient of ξ^{-1}, is

$$\frac{-i}{(z_0 - z_1)^2} \left(1 - \frac{2z_0}{z_0 - z_1} \right) = \frac{i(z_0 + z_1)}{(z_0 - z_1)^3} = \frac{2i\epsilon^{-1}}{-8(\epsilon^{-2} - 1)^{3/2}}.$$

Thus, by the residue theorem,

$$\frac{A\epsilon^2}{2\ell^2} = 2\pi i \frac{i\epsilon^{-1}}{-4(\epsilon^{-2}-1)^{3/2}},$$

giving the area A as

$$A = \frac{\pi\ell^2\epsilon^{-1}}{\epsilon^2(\epsilon^{-2}-1)^{3/2}} = \frac{\pi\ell^2}{(1-\epsilon^2)^{3/2}}.$$

25

Applications of complex variables

Many of the exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar figures that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we showed in figure 24.1 of the previous chapter a number of typical contour types to which reference can be made.

25.2 *A long straight fence made of conducting wire mesh separates two fields and stands 1 metre high. Sometimes, on fine days, there is a vertical electric field over flat open countryside. Well away from the fence the strength of the field is* E_0. *By considering the effect of the transformation* $w = (1-z^2)^{1/2}$ *on the real and imaginary* z*-axes, find the strengths of the field* (a) *at a point one metre directly above the fence,* (b) *at ground level one metre to the side of the fence, and* (c) *at a point that is level with the top of the fence but one metre to the side of it. What is the direction of the field in case* (c)?

We first consider the situation of a uniform vertical field (conventionally in the positive y-direction) of strength E_0. The corresponding potential is $\phi = -E_0 y$ and, as this is the real part of $iE_0 z$, the appropriate complex potential is $f(z) = iE_0 z$.

Now consider what happens to the real z-axis ($y = 0$) under the transformation $w = (1 - z^2)^{1/2}$. For $-1 < x < 1$, $w = u + iv$ is real and covers $0 < u < 1$ twice, once in each direction. For $-\infty < x < -1$ and $1 < x < \infty$, w is imaginary and covers the v-axis in its entirety. Under the same transformation, the imaginary z-axis maps (twice) onto the part of the positive real w-axis given by $1 < u < \infty$.

Thus, apart from a rotation of the whole coordinate system through $\pi/2$, the transformation maps the equipotential (ground) surface $y = 0$ into another equipotential (ground) surface $u = 0$, but this time with a unit height 'fence' situated at $v = 0$, the fence being at the same potential as the surface.

Under the transformation, $w = (1 - z^2)^{1/2}$, or equivalently $z = (1 - w^2)^{1/2}$, the potential

$$f(z) = iE_0 z \to F(w) = iE_0(1 - w^2)^{1/2} = -E_0(w^2 - 1)^{1/2}.$$

The magnitude of the derivative $F' = dF/dw$ gives the strength of the field in the tranformed situation; the field's direction makes an angle $\pi - \arg F'$ with the u-axis, which corresponds to the upward vertical directly above the fence. The derivative is

$$\frac{dF}{dw} = -\frac{w\,E_0}{(w^2 - 1)^{1/2}}.$$

Its magnitude, and hence the strength of the field, is

$$\text{(a) for } w = 2 + 0i, \qquad \left| -\frac{2E_0}{(4-1)^{1/2}} \right| = \frac{2\,E_0}{\sqrt{3}},$$

$$\text{(b) for } w = 0 + 1i, \qquad \left| -\frac{iE_0}{(-1-1)^{1/2}} \right| = \frac{E_0}{\sqrt{2}},$$

$$\text{(c) for } w = 1 + 1i, \qquad \left| -\frac{(1+i)E_0}{(2i-1)^{1/2}} \right| = \frac{\sqrt{2}\,E_0}{5^{1/4}}.$$

In case (c),

$$\arg\left[-\frac{(1+i)E_0}{(2i-1)^{1/2}} \right] = \pi + \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\frac{2}{-1} = \frac{5\pi}{4} - 1.017.$$

Thus the direction of the field at $1 + i$ makes an angle $\pi - (\frac{5\pi}{4} - 1.017) = 0.232$ radians with the upward vertical.

Finally, we note that the equipotential surfaces are closely packed just above the top of the fence but separate as they spread out to become asymptotically parallel to the ground (without, of course, ever crossing each other).

25.4 *Find a complex potential in the z-plane appropriate to a physical situation in which the half-plane $x > 0$, $y = 0$ has zero potential and the half-plane $x < 0$, $y = 0$ has potential V.*

By making the transformation $w = a(z + z^{-1})/2$, with a real and positive, find the electrostatic potential associated with the half-plane $r > a$, $s = 0$ and the half-plane $r < -a$, $s = 0$ at potentials 0 and V respectively.

We require a function whose real or imaginary part takes the value 0 whenever $y = 0$ and $x > 0$, and takes a constant non-zero value whenever $y = 0$ but $x < 0$. The argument of z does this but, unfortunately, $\arg z$ is not an analytic function.

However $\arg z$ is (within a constant multiplier) the imaginary part of $\ln z$, which *is* an analytic function over (nearly all of) the complex plane. To get the scale correct and avoid problems with discontinuities across the negative x-axis, we need to take as the solution of Laplace's equation

$$\text{either } \phi(x, y) = \text{Im}\left(\frac{V}{\pi}\ln z\right) \text{ or } \phi(x, y) = \text{Re}\left(-i\frac{V}{\pi}\ln z\right)$$

with $z = |z|e^{i\theta}$ restricted by $0 \le \theta \le \pi$. The solution for the half-space $y < 0$ is to be given by symmetry, rather than by continuing the function into $-\pi < \theta < 0$.

Now consider the conformal transformation

$$r + is = w = \frac{a}{2}\left(z + \frac{1}{z}\right).$$

The half-plane $y = 0$, $0 < x < \infty$ becomes

$$r + is = \frac{a}{2}\left(x + \frac{1}{x}\right) \quad \Rightarrow \quad s = 0 \text{ and } a < r < \infty.$$

Similarly the half-plane $y = 0$, $-\infty < x < 0$ becomes

$$r + is = \frac{a}{2}\left(-|x| - \frac{1}{|x|}\right) \quad \Rightarrow \quad s = 0 \text{ and } -\infty < r < -a.$$

Thus the transformation maps the original half-planes (virtually touching at the origin) into two half-planes symmetrically separated about the origin by $2a$. This is as neeeded. To find the corresponding complex potential we must express $(V/\pi)\ln z$ in terms of w. We start by inverting the transformation,

$$z^2 - \frac{2w}{a}z + 1 = 0,$$

$$z = \frac{w}{a} \pm \sqrt{\frac{w^2}{a^2} - 1},$$

and then substitute for z,

$$F(w) = \frac{V}{\pi}\ln\left(\frac{w \pm \sqrt{w^2 - a^2}}{a}\right).$$

This, or $-i$ times it, (depending upon whether the imaginary or real part of the complex potential is taken as the physical potential) is the appropriate complex potential. Consideration of the particular case $w = is$ with $s > 0$, which must yield $+\frac{1}{2}V$ for the physical potential, shows that the $+$ sign for the square root is the correct choice. [Choosing the minus sign would make the expression in parentheses both imaginary and negative, and lead to a physical potential of $-\frac{1}{2}V$.]

25.6 *For the equation* $8z^3 + z + 1 = 0$*:*

(a) *show that all three roots lie between the circles* $|z| = 3/8$ *and* $|z| = 5/8$;
(b) *find the approximate location of the real root, and hence deduce that the complex ones lie in the first and fourth quadrants and have moduli greater than* 0.5.

(a) We start by considering $h(z) = 8z^3 + z + 1$ as $f(z) + g(z)$ where $f(z) = 8z^3$ and $g(z) = z + 1$. Now, on the circle $|z| = \frac{5}{8}$,

$$|8z^3| = \frac{1000}{512} = \frac{125}{64} \quad \text{and} \quad |z + 1| \leq \frac{5}{8} + 1 = \frac{104}{64}.$$

Thus $|f(z)| > |g(z)|$ at all points on the circle. It then follows from Rouché's theorem that $h(z)$ and $f(z)$ have the same number of zeroes inside the circle; $f(z)$ clearly has three (all at the origin), implying that all three zeroes of $h(z)$ lie within the same circle.

We next consider $h(z) = 8z^3+z+1$ as $f(z)+g(z)$ where $f(z) = 1$ and $g(z) = 8z^3+z$. On the circle $|z| = \frac{3}{8}$,

$$|1| = 1 \quad \text{and} \quad |8z^3 + z| \leq \frac{216}{512} + \frac{3}{8} = \frac{51}{64}.$$

Thus $|f(z)| > |g(z)|$ at all points on the circle. As before, it follows that $h(z)$ and $f(z)$ have the same number of zeroes inside the circle; $f(z)$ clearly has no zeroes inside the circle, and so, therefore, neither has $h(z)$.

Combining these results shows that all roots of the equation lie *between* the circles $|z| = 3/8$ and $|z| = 5/8$.

(b) Since the order of the cubic $h(z)$ is odd, the equation must have at least one real root; further, since the signs of all the coefficients in the cubic are the same, the real root must be negative. Let it be at x and write the equation in the form

$$8z^3 + z + 1 = 8(z - x)(z - \alpha - i\beta)(z - \alpha + i\beta) = 0.$$

Considering the coefficients of z^2 gives $-x - 2\alpha = 0$, showing that, since x is negative, α must be positive, i.e. the complex roots occur in the first and fourth quadrants.

At $x = -\frac{3}{8}$, $h(z) = \frac{13}{64}$, whilst at $z = -\frac{1}{2}$, $h(z) = -\frac{1}{2}$. Thus the real root lies between these two negative values of x. From the constant terms in the above expression of $h(z) = 0$ we deduce that $-8x(\alpha^2 + \beta^2) = 1$. Since $|x| < \frac{1}{2}$ it follows that $\alpha^2 + \beta^2 > \frac{1}{4}$ and that the modulus of either complex root $\sqrt{\alpha^2 + \beta^2} > 0.5$.

25.8 *The following is a method of determining the number of zeroes of an nth-degree polynomial $f(z)$ inside the contour C given by $|z| = R$:*

(a) *put $z = R(1+it)/(1-it)$, with $t = \tan(\theta/2)$ in the range $-\infty \le t \le \infty$;*
(b) *obtain $f(z)$ as*
$$\frac{A(t)+iB(t)}{(1-it)^n}\frac{(1+it)^n}{(1+it)^n};$$
(c) *it follows that* $\arg f(z) = \tan^{-1}(B/A) + n\tan^{-1} t$;
(d) *and that* $\Delta_C[\arg f(z)] = \Delta_C[\tan^{-1}(B/A)] + n\pi$;
(e) *determine* $\Delta_C[\tan^{-1}(B/A)]$ *by evaluating* $\tan^{-1}(B/A)$ *at* $t = \pm\infty$ *and finding the discontinuities in B/A by inspection or using a sketch graph.*

Then, by the principle of the argument, the number of zeroes inside C is given by the integer $(2\pi)^{-1}\Delta_C[\arg f(z)]$.

It can be shown that the zeroes of z^4+z+1 lie one in each quadrant. Use the above method to show that the zeroes in the second and third quadrants have $|z| < 1$.

(a) and (b). In this exercise we are concerned with the contour $|z| = 1$ and so we set

$$z = \frac{1+it}{1-it} = \frac{(1+it)^2}{(1-it)(1+it)} = \frac{1-t^2}{1+t^2} + i\frac{2t}{1+t^2}, \qquad t = \tan\frac{\theta}{2}, \qquad -\infty < t < \infty.$$

As θ increases from 0 to 2π, $z = \cos\theta + i\sin\theta$ and prescribes the unit circle, as can be verified from the half-angle identities. In terms of t increasing from $-\infty$ to $+\infty$, it is easier to think of θ increasing from $-\pi$ to $+\pi$, but the result is the same, as the circle is traversed once in the positive direction in either case.

The expression for $f(z)$ in terms of t is

$$\begin{aligned}
g(t) &= \frac{1}{(1-it)^4}[(1+it)^4 + (1+it)(1-it)^3 + (1-it)^4] \\
&= \frac{1}{(1-it)^4}[t^4(1-1+1) + t^3(-4i+i-3i+4i) \\
&\qquad + t^2(-6-3+3-6) + t(4i-3i+i-4i) + (1+1+1)] \\
&= \frac{1}{(1-it)^4}(t^4 - 2it^3 - 12t^2 - 2it + 3) \\
&= \frac{A(t)+iB(t)}{(1-it)^4}\frac{(1+it)^4}{(1+it)^4},
\end{aligned}$$

where $A(t) = t^4 - 12t^2 + 3$ and $B(t) = -2t^3 - 2t$.

(c) Since its denominator is real and its numerator contains two factors, the argument of $g(t)$ is the sum of the arguments of these two factors.

$$\begin{aligned}\arg g(t) &= \arg[A(t) + iB(t)] + \arg[(1+it)^4] \\ &= \tan^{-1}\frac{B}{A} + 4\arg(1+it) \\ &= \tan^{-1}\frac{B}{A} + 4\tan^{-1} t \\ &= \tan^{-1}\frac{B}{A} + 4\frac{\theta}{2}.\end{aligned}$$

(d) The change in $\arg g$ around C is therefore

$$\begin{aligned}\Delta_C(\arg g) &= \Delta_C\left(\tan^{-1}\frac{B}{A}\right) + 4\Delta_C\left(\frac{\theta}{2}\right) \\ &= \Delta_C\left(\tan^{-1}\frac{B}{A}\right) + 4\pi \\ &= \Delta_C\left(\tan^{-1}\frac{-2t^3-2t}{t^4-12t^2+3}\right) + 4\pi \qquad -\infty < t < \infty, \\ &\equiv \Delta_C(\alpha) + 4\pi, \qquad \text{thus defining } \alpha.\end{aligned}$$

(e) Taking account of the magnitude of α and the signs of the numerator and denominator of $\tan\alpha$ separately, we deduce that as $t \to -\infty$, $\alpha \to 0_+$ and as $t \to \infty$, $\alpha \to 0_-$. We also note that, for real t, the numerator of $\tan\alpha$ is zero only when $t = 0$ and that the denominator is zero when $t^2 = 6 \pm \sqrt{33}$. Thus, as t increases from $-\infty$ through $-(6+\sqrt{33})^{1/2}$, a graph of α increases from 0 and passes through $\pi/2$. However, it decreases through $\pi/2$ at $-(6-\sqrt{33})^{1/2}$ without reaching π in between. It then passes through 0 at $t = 0$. The rest of the graph is the antisymmetric reflection in the line $t = 0$ of the first half.

Thus there are no discontinuities in α and, as it starts and ends at 0, $\Delta_C(\alpha) = 0$. It follows that $\Delta_C(\arg g) = 4\pi$ and that there are 2 zeroes inside $|z| = 1$.

To determine in which quadrants the two zeroes of $f(z) = z^4 + z + 1$ occur requires some numerical work. On the circle $|z| = 1$,

$$\arg f(z) = \tan^{-1}\frac{\sin 4\theta + \sin\theta}{\cos 4\theta + \cos\theta + 1}.$$

For a contour such as (b) in figure 24.1 with $R = 1$, $\Delta_{OA}(\arg f) = 0$ since z, and hence f, is purely real. On BO, where $z = iy$, with $1 \geq y \geq 0$, the argument of f is equal to $\tan^{-1}[y/(y^4+1)]$ and varies smoothly from $\tan^{-1}\frac{1}{2}$ to 0. On AB it is given by the above expression. Numerical investigation (using a spreadsheet, say) shows that the variation from 0 at $\theta = 0$ to $\tan^{-1}\frac{1}{2}$ at $\theta = \pi/2$, whilst having several turning points, has no discontinuities. [By contrast, there is a discontinuity at $\theta \approx 2.095$ in the second quadrant.]

Putting these observations together shows that $\Delta_{OABO}[\arg f(z)] = 0$ and that the contour of unit radius encloses no zeroes. Since the zeroes of polynomials with real coefficients occur in complex conjugate pairs, it follows that a similar contour in the fourth quadrant also encloses no zeroes. Hence the two zeroes inside $|z| = 1$ must lie in the second and third quadrants.

25.10 *This exercise illustrates a method of summing some infinite series.*

(a) *Determine the residues at all the poles of the function*

$$f(z) = \frac{\pi \cot \pi z}{a^2 + z^2},$$

where a is a positive real constant.

(b) *By evaluating, in two different ways, the integral I of $f(z)$ along the straight line joining $-\infty - ia/2$ and $+\infty - ia/2$, show that*

$$\sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} = \frac{\pi \coth \pi a}{2a} - \frac{1}{2a^2}.$$

(c) *Deduce the value of $\sum_1^\infty n^{-2}$.*

(a) This function has simple poles at $z = n$ whenever n is an integer. At each one the residue is

$$\frac{\pi \cos \pi z}{d(\sin \pi z)/dz} \frac{1}{a^2 + z^2} = \frac{\pi \cos \pi z}{\pi \cos \pi z} \frac{1}{a^2 + n^2} = \frac{1}{a^2 + n^2}.$$

There are two other poles, at $z = \pm ia$, and they have (equal) residues of

$$\frac{\pi \cot(\pm i\pi a)}{\pm 2ia} = -\frac{\pi \coth(\pi a)}{2a}.$$

(b) As $|z| \to \infty$, $\cot \pi z$ is bounded in both the upper and lower half-planes (i.e. away from the real axis) and so the integrals along the semicircular paths Γ_1 and Γ_2 shown in figure 25.1 tend to zero as the radius of the circle $\to \infty$. Now take as a closed contour the line L, joining $-\infty - ia/2$ and $+\infty - ia/2$, together with Γ_1 and apply the residue theorem

$$I + 0 = 2\pi i \left[2 \sum_{n=1}^{\infty} \frac{1}{a^2 + n^2} + \frac{1}{a^2} - \frac{\pi \coth(\pi a)}{2a} \right].$$

Next take a closed contour consisting of L and Γ_2 and again apply the theorem (taking account of the sense of integration)

$$-I + 0 = 2\pi i \left[-\frac{\pi \coth(\pi a)}{2a} \right].$$

Figure 25.1 The contours used in exercise 25.10.

Adding these results gives

$$2\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}+\frac{1}{a^2}-\frac{2\pi\coth(\pi a)}{2a}=0,$$

i.e.

$$\sum_{n=1}^{\infty}\frac{1}{a^2+n^2}=\frac{\pi\coth(\pi a)}{2a}-\frac{1}{2a^2}.$$

(c) To deduce the value of $S=\sum_{n=1}^{\infty}n^{-2}$ we need to evaluate

$$\lim_{a\to 0}\left(\frac{\pi\coth(\pi a)}{2a}-\frac{1}{2a^2}\right).$$

For this we employ l'Hôpital's rule.

$$\begin{aligned}
S &= \lim_{a\to 0}\frac{a\pi\cosh(a\pi)-\sinh(a\pi)}{2a^2\sinh(a\pi)}\\
&= \lim_{a\to 0}\frac{\pi\cosh(a\pi)+a\pi^2\sinh(a\pi)-\pi\cosh(a\pi)}{4a\sinh(a\pi)+2a^2\pi\cosh(a\pi)}\\
&= \lim_{a\to 0}\frac{\pi^2\sinh(a\pi)+a\pi^3\cosh(a\pi)}{4\sinh(a\pi)+8\pi a\cosh(a\pi)+2a^2\pi^2\sinh(a\pi)}\\
&= \lim_{a\to 0}\frac{2\pi^3\cosh(a\pi)+a\pi^4\sinh(a\pi)}{4\pi\cosh(a\pi)+8\pi\cosh(a\pi)+12\pi^2 a\sinh(a\pi)+2a^2\pi^3\cosh(a\pi)}\\
&= \frac{2\pi^3}{12\pi}=\frac{\pi^2}{6}.
\end{aligned}$$

Not the most straightforward way to derive this particular result!

25.12 *Use the Bromwich inversion, and contours similar to that shown in figure 24.1*(g) *to find the functions of which the following are the Laplace transforms:*

(a) $s(s^2+b^2)^{-1}$;
(b) $n!(s-a)^{-(n+1)}$, *with* n *a positive integer and* $s>a$;
(c) $a(s^2-a^2)^{-1}$, *with* $s>|a|$.

Compare your answers with those given in a table of standard Laplace transforms.

(a) Inside the suggested contour $s(s^2+b^2)^{-1}$ has simple poles at $s=\pm ib$. At $s=ib$, the residue of $se^{sx}(s^2+b^2)^{-1}$ is $ib\,e^{ibx}/(ib+ib)=\frac{1}{2}e^{ibx}$. At $s=-ib$, the residue of $se^{sx}(s^2+b^2)^{-1}$ is $-ib\,e^{-ibx}/(-ib-ib)=\frac{1}{2}e^{-ibx}$. Thus the function $f(x)$ whose Laplace transform is $s(s^2+b^2)^{-1}$ is $\frac{1}{2}(e^{ibx}+e^{-ibx})=\cos bx$.

(b) For the inverse Laplace transform of $\dfrac{n!}{(s-a)^{n+1}}$ the contour formed by closing the Bromwich line in the left half-plane contains a pole in the right half-plane, at $z=a$. We therefore make a change of variable to $t=s-a$ with $t>0$ and again use a closed contour such as (g) in figure 24.1. The integrand becomes

$$\frac{n!e^{x(t+a)}}{t^{n+1}}=\frac{n!e^{ax}}{t^{n+1}}\sum_{r=0}^{\infty}\frac{(xt)^r}{r!}.$$

The only pole enclosed by the new contour is at $t=0$ and the coefficient of t^{-1} in the Taylor expansion about that point is

$$a_{-1}=\frac{n!\,e^{ax}\,x^n}{n!}=e^{ax}x^n.$$

This is the residue at that point and also the required function $f(x)=e^{ax}x^n$; note that the given function of s is its Laplace transform only for $s>a$.

(c) As in part (b), the integrand, in this case $ae^{sx}(s^2-a^2)^{-1}$, has a pole in the right half-plane. The same formal device as in part (b) could be used to convert the integral to one of type (g) with $\lambda=\epsilon>0$ for any small ϵ. However, we can obtain the correct result by ignoring this 'nicety' and merely noting that the integrand has simple poles with residues

$$\frac{ae^{ax}}{a+a}\ \text{at } s=a \qquad \text{and} \qquad \frac{ae^{-ax}}{-a-a}\ \text{at } s=-a.$$

The function of which $a(s^2-a^2)^{-1}$ is the Laplace transform is therefore $f(x)=\frac{1}{2}(e^{ax}-e^{-ax})=\sinh ax$. This conclusion is, however, valid only for $s>|a|$.

25.14 *A function $f(t)$ has the Laplace transform*

$$F(s) = \frac{1}{2i} \ln\left(\frac{s+i}{s-i}\right),$$

the complex logarithm being defined by a finite branch cut running along the imaginary axis from $-i$ to i.

(a) *Convince yourself that, for $t > 0$, $f(t)$ can be expressed as a closed contour integral that encloses only the branch cut.*
(b) *Calculate $F(s)$ on either side of the branch cut, evaluate the integral and hence determine $f(t)$.*
(c) *Confirm that the derivative with respect to s of the Laplace transform integral of your answer is the same as that given by dF/ds.*

From the standard Bromwich integral representation, $f(t)$ can be written

$$f(t) = \frac{1}{2\pi i} \int_{\lambda - i\infty}^{\lambda + i\infty} \frac{1}{2i} \ln\left(\frac{s+i}{s-i}\right) e^{st}\, ds.$$

For $t < 0$ the contour has to be closed, as usual, in the right half-plane [contour (h) of figure 24.1]. There are no poles with Re $s > \lambda$ and so the integral, and hence $f(t)$, are both zero for $t < 0$.

(a) For $t > 0$ the contour has to be closed in the left half-plane [contour (g)]. The integrand is

$$\frac{1}{2i}\left\{\ln\left|\frac{s+i}{s-i}\right| e^{st} + i[\arg(s+i) - \arg(s-i)]e^{st}\right\}.$$

This has no singularities in the left half-plane (excluding the imaginary axis) and so, by Cauchy's theorem, the contour can be deformed to be a line $z = -\lambda' + iy$ for any real $\lambda' > 0$. There are no poles on the imaginary axis beyond the cut and so the contour can be further deformed (shrunk) to one that just encloses the cut.

(b) We introduce plane-polar angles θ_1 and θ_2 centred respectively on the points $s = -i$ and $s = i$. Each is restricted to a range of 2π but where the zero of each is taken does not matter. Let a point P close to the cut but to the right of it, and given by $s = \epsilon + iy$, where $\epsilon > 0$ and $-1 < y < 1$, correspond to $\theta_1 = \phi_1$ and $\theta_2 = \phi_2$. Then

$$F(\epsilon + iy) = \frac{1}{2i}\left[\ln\left(\frac{1+y}{1-y}\right) + i\phi_1 - i\phi_2\right].$$

As P moves (upward) along a path beside the cut, rounds the point $s = i$ and moves (downwards) beside, but to the left of, the cut to reach $s = -\epsilon + iy$, the

value of θ_1 hardly changes and is finally the same as it started. However, θ_2 increases by 2π (as P rounds $s = i$). The new value of $F(s)$ is

$$F(-\epsilon + iy) = \frac{1}{2i}\left[\ln\left(\frac{1+y}{1-y}\right) + i\phi_1 - i(\phi_2 + 2\pi)\right].$$

In the evaluation of the integral y runs from -1 to 1 on the right of the cut and from 1 to -1 on the left. The two contributions from the terms containing logarithms clearly cancel and we are left with

$$\int_{-1}^{1} \frac{1}{2}(\phi_1 - \phi_2)e^{iyt} i\, dy + \int_{1}^{-1} \frac{1}{2}(\phi_1 - \phi_2 - 2\pi)\, e^{iyt} i\, dy.$$

The integrals involving ϕ_1 and ϕ_2 cancel and leave as the only contribution to the Bromwich integral

$$J = \int_{-1}^{1} \pi\, e^{iyt} i\, dy = \pi i \frac{e^{it} - e^{-it}}{it} = 2\pi i \frac{\sin t}{t}.$$

Hence $f(t)$, which is equal to $J/(2\pi i)$, has the form $\sin t/t$.

(c) Firstly, the derivative with respect to s of the Laplace transform of the solution just obtained is

$$\begin{aligned}
\frac{d}{ds}\int_0^\infty e^{-st}\frac{\sin t}{t}\, dt &= -\int_0^\infty e^{-st} \sin t\, dt \\
&= -\mathrm{Im} \int_0^\infty e^{-st}\, e^{it}\, dt \\
&= -\mathrm{Im} \left[\frac{e^{-st+it}}{-s+i}\right]_0^\infty \\
&= -\mathrm{Im} \left(\frac{1}{s-i}\right) = -\frac{1}{1+s^2}.
\end{aligned}$$

Secondly, the corresponding derivative of the given form for $F(s)$ is

$$\begin{aligned}
\frac{dF(s)}{ds} &= \frac{1}{2i}\frac{d}{ds}[\ln(s+i) - \ln(s-i)] \\
&= \frac{1}{2i}\left(\frac{1}{s+i} - \frac{1}{s-i}\right) \\
&= \frac{1}{2i}\frac{-2i}{s^2+1} = -\frac{1}{s^2+1}.
\end{aligned}$$

Thus, $\dfrac{d}{ds}\displaystyle\int_0^\infty e^{-st}\frac{\sin t}{t}\, dt = \frac{dF(s)}{ds}$, confirming the result stated in the question.

25.16 *Transverse vibrations of angular frequency ω on a string stretched with constant tension T are described by $u(x,t) = y(x)\,e^{-i\omega t}$ where*

$$\frac{d^2y}{dx^2} + \frac{\omega^2\,m(x)}{T}\,y(x) = 0.$$

Here, $m(x) = m_0 f(x)$ is the mass per unit length of the string and, in the general case, is a function of x. Find the first-order W.K.B. solution for $y(x)$.

Due to imperfections in its manufacturing process, a particular string has a small periodic variation in its linear density of the form $m(x) = m_0[\,1 + \epsilon\sin(2\pi x/L)\,]$, where $\epsilon \ll 1$. A progressive wave (i.e. one in which no energy is lost) travels in the positive x-direction along the string. Show that its amplitude fluctuates by $\pm\frac{1}{4}\epsilon$ of its value A_0 at $x = 0$ and that, to first order in ϵ, the phase of the wave is

$$\frac{\epsilon\,\omega\,L}{2\pi}\sqrt{\frac{m_0}{T}}\,\sin^2\frac{\pi x}{L}$$

ahead of what it would be if the string were uniform with $m(x) = m_0$.

We first write $\alpha^2 = m_0\omega^2/T$ and assume that $\alpha \gg 1$, so that the W.K.B. method is appropriate. We now try as a solution to

$$\frac{d^2y}{dx^2} + \alpha^2 f(x)y = 0 \qquad (*)$$

the form $y(x) = A(x)e^{i\alpha\phi(x)}$. The necessary derivatives are

$$\begin{aligned} y' &= A'e^{i\alpha\phi} + i\alpha A\phi' e^{i\alpha\phi}, \\ y'' &= A''e^{i\alpha\phi} + 2i\alpha A'\phi' e^{i\alpha\phi} + i\alpha A\phi'' e^{i\alpha\phi} - \alpha^2 A(\phi')^2 e^{i\alpha\phi}. \end{aligned}$$

Substituting these into $(*)$ and cancelling a factor of $e^{i\alpha\phi}$ throughout, yields

$$A'' + i\alpha(2A'\phi' + A\phi'') + \alpha^2[\,Af - A(\phi')^2\,] = 0.$$

The first W.K.B. approximations is obtained by setting the coefficients of α^2 and α (both $\gg 1$, but of different orders of magnitude) separately equal to zero (and assuming that A'' can be ignored).

$$\begin{aligned} (\phi')^2 = f \quad &\Rightarrow \quad \phi(x) = \int_0^x \sqrt{f(u)}\,du, \\ 2A'\phi' + A\phi'' = 0 \quad &\Rightarrow \quad \frac{2A'}{A} + \frac{\phi''}{\phi'} = 0 \quad \Rightarrow \quad \ln(A^2\phi') = k_1 \\ &\Rightarrow \quad A^2 = \frac{k_2}{f^{1/2}} \quad \Rightarrow \quad A(x) = \frac{c}{[\,m(x)\,]^{1/4}}. \end{aligned}$$

Collecting these results together gives the first-order W.K.B. solution as

$$y(x) = \frac{c}{[\,m(x)\,]^{1/4}}\exp\left[\,i\alpha\int_0^x \sqrt{f(u)}\,du\,\right],$$

with α as defined above.

We now substitute $m(x) = m_0[1 + \epsilon \sin(2\pi x/L)]$ in the above general result and expand the expressions containing $m(x)$ up to first order in ϵ:

$$\begin{aligned} y(x) &= \frac{c}{[m(x)]^{1/4}} \exp\left(i\alpha \int_0^x \sqrt{1 + \epsilon \sin \frac{2\pi u}{L}}\, du \right), \\ &= \frac{c}{m_0^{1/4}} (1 + \epsilon \sin \frac{2\pi x}{L})^{-1/4} \exp\left[i\alpha \int_0^x \left(1 + \frac{\epsilon}{2} \sin \frac{2\pi u}{L} \right) du \right] \\ &= \frac{c}{m_0^{1/4}} (1 - \frac{\epsilon}{4} \sin \frac{2\pi x}{L}) \exp\left\{ i\alpha \left[x + \frac{\epsilon L}{4\pi} \left(1 - \cos \frac{2\pi x}{L} \right) \right] \right\}. \end{aligned}$$

Thus, the amplitude at $x = 0$ is $A_0 = c/m_0^{1/4}$ and its variation at other values of x is up to $\pm\frac{1}{4}\epsilon A_0$.

With $m(x) = m_0$, the phase would be $\phi = \alpha x$ but, with the variation in linear density, it is ahead of this by

$$\alpha \frac{\epsilon L}{4\pi} \left(1 - \cos \frac{2\pi x}{L} \right) = \frac{\epsilon\, \omega\, L}{2\pi} \sqrt{\frac{m_0}{T}} \sin^2 \frac{\pi x}{L}.$$

25.18 *A W.K.B. solution of Bessel's equation of order zero,*

$$\frac{d^2 y}{dz^2} + \frac{1}{z}\frac{dy}{dz} + y = 0, \qquad (*)$$

valid for large $|z|$ *and* $-\pi/2 < \arg z < 3\pi/2$ *is* $y(z) = Az^{-1/2}e^{iz}$. *Obtain an improvement on this by finding a multiplier of* $y(z)$ *in the form of an asymptotic expansion in inverse powers of* z *as follows.*

(a) *Substitute for* $y(z)$ *in* $(*)$ *and show that the equation is satisfied to* $O(z^{-5/2})$.

(b) *Now replace the constant* A *by* $A(z)$ *and find the equation that must be satisfied by* $A(z)$. *Look for a solution of the form* $A(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^{-n}$ *where* $a_0 = 1$. *Show that* $\sigma = 0$ *is the only acceptable solution to the indicial equation and obtain a recurrence relation for the* a_n.

(c) *To within a (complex) constant, the expression* $y(z) = A(z)z^{-1/2}e^{iz}$ *is the asymptotic expansion of the Hankel function* $H_0^{(1)}(z)$. *Show that it is a divergent expansion for all values of* z *and estimate, in terms of* z, *the value of* N *such that* $\sum_{n=0}^{N} a_n z^{-n-1/2} e^{iz}$ *gives the best estimate of* $H_0^{(1)}(z)$.

(a) We first calculate the required derivatives appearing in Bessel's equation, which are given by

$$\frac{dy}{dz} = -\frac{A}{2z^{3/2}}\,e^{iz} + \frac{iA}{z^{1/2}}\,e^{iz},$$
$$\frac{d^2y}{dz^2} = \frac{(3/2)A}{2z^{5/2}}\,e^{iz} - 2\frac{iA}{2z^{3/2}}\,e^{iz} + \frac{i^2A}{z^{1/2}}\,e^{iz}.$$

Substituting these into (∗), and cancelling common factors A and e^{iz}, gives

$$\left(\frac{3}{4z^{5/2}} - \frac{i}{z^{3/2}} - \frac{1}{z^{1/2}}\right) + \left(-\frac{1}{2z^{5/2}} + \frac{i}{z^{3/2}}\right) + \frac{1}{z^{1/2}} = 0.$$

The equation is not satisfied exactly, the LHS having a value $z^{-5/2}/4$. Thus the error is $O(z^{-5/2})$.

(b) The first task is to find the equation satisfied by $A(z)$. We can make use of the result of part (a) by denoting the new $y(z)$ by $y(z) = A(z)y_1(z) \equiv A(z)z^{-1/2}e^{iz}$, where

$$\frac{d^2y_1}{dz^2} + \frac{1}{z}\frac{dy_1}{dz} + y_1 = \frac{e^{iz}}{4z^{5/2}}.$$

Using Leibnitz' theorem to calculate the LHS of (∗) gives

$$\begin{aligned}
& A''y_1 + 2A'y_1' + Ay_1'' + z^{-1}(A'y_1 + Ay_1') + Ay_1 = 0, \\
\Rightarrow\quad & A''y_1 + 2A'y_1' + \frac{1}{z}A'y_1 + \frac{Ae^{iz}}{4z^{5/2}} = 0, \\
\Rightarrow\quad & \frac{A''}{z^{1/2}} + A'\left(-\frac{2}{2z^{3/2}} + \frac{2i}{z^{1/2}} + \frac{1}{z^{3/2}}\right) + \frac{A}{4z^{5/2}} = 0, \\
\Rightarrow\quad & 4z^2A'' + 8iz^2A' + 1 = 0. \qquad (**)
\end{aligned}$$

This is the equation to be satisfied by $A(z)$ and so, setting $A(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^{-n}$, we have

$$4\sum_{n=0}^{\infty}(\sigma - n)(\sigma - n - 1)a_n z^{\sigma-n} + 8i\sum_{n=0}^{\infty}(\sigma - n)a_n z^{\sigma-n+1} + \sum_{n=0}^{\infty} a_n z^{\sigma-n} = 0.$$

The highest power of z present on the LHS is that of $z^{\sigma-n+1}$ when $n = 0$. Its coefficient is $8i(\sigma - 0)a_0$ and, since $a_0 \neq 0$, we must have $\sigma = 0$.

When this value is used the equation reduces to

$$4\sum_{n=0}^{\infty} n(n+1)a_n z^{-n} - 8i\sum_{n=0}^{\infty} na_n z^{-n+1} + \sum_{n=0}^{\infty} a_n z^{-n} = 0.$$

Equating the coefficients of z^{-n} gives

$$4n(n+1)a_n - 8i(n+1)a_{n+1} + a_n = 0 \quad\Rightarrow\quad a_{n+1} = -i\frac{(2n+1)^2}{8(n+1)}\,a_n.$$

(c) Successive terms t_n in the series (which has alternate real and imaginary terms when z is real) are related by

$$t_{n+1} = a_{n+1} z^{-(n+1)} = -i\frac{(2n+1)^2}{8(n+1)}\, a_n z^{-(n+1)} = -i\frac{(2n+1)^2}{8(n+1)}\, z^{-1} t_n.$$

The numerator of the fraction on the RHS varies as n^2 whilst the denominator is linear in n. Consequently, no matter how large $|z|$ is, $|t_{n+1}| > |t_n|$ for sufficiently large n, i.e. the expansion is divergent. The closest estimate to $H_0^{(1)}(z)$ given by a sum of the form $\sum_{n=0}^{N} a_n z^{-n-1/2} e^{iz}$ is obtained when N is chosen so that

$$\frac{(2N+1)^2}{8(N+1)}\,\frac{1}{|z|} \approx 1,$$

i.e. $4N^2/8N \approx |z|$, or $N \approx 2|z|$. For N larger than this, the additional terms, and hence the uncertainty in the value of the function, begin to get bigger, rather than smaller.

25.20 *Use the method of steepest descents to show that an approximate value for the integral*

$$F(z) = \int_{-\infty}^{\infty} \exp[\,iz(\tfrac{1}{5}t^5 + t)\,]\,dt,$$

where z is real and positive, is

$$\left(\frac{2\pi}{z}\right)^{1/2} \exp(-\beta z)\,\cos(\beta z - \tfrac{1}{8}\pi),$$

where $\beta = 4/(5\sqrt{2})$.

Although this is an integral with respect to the real variable t, we will consider it as one along the real axis in the complex t-plane and then distort its path so that it passes from $t = -\infty$ to $t = +\infty$ via one or more of the saddle points t_i of the complex function.

The saddle points are situated where the gradient of the integrand is zero. The values of t at which this happens are given by

$$0 = \frac{d}{dt}[\,iz(\tfrac{1}{5}t^5 + t)\,] = iz(t^4 + 1) \quad \Rightarrow \quad t_i = \pm e^{i\pi/4} \text{ or } \pm e^{3i\pi/4}.$$

We will use the two in the upper half plane, $t_1 = e^{3i\pi/4}$ and $t_2 = e^{i\pi/4}$.

The values of the exponents and their second derivatives at the saddles are

$$\begin{aligned}
f_1 = f(t_1) &= iz\left(\tfrac{1}{5}e^{15i\pi/4} + e^{3i\pi/4}\right) = \tfrac{4}{5}iz\, e^{3i\pi/4}, \\
f_2 = f(t_2) &= iz\left(\tfrac{1}{5}e^{5i\pi/4} + e^{i\pi/4}\right) = \tfrac{4}{5}iz\, e^{i\pi/4}, \\
f_1'' = f''(t_1) &= 4izt_1^3 = 4iz\, e^{9i\pi/4} \\
&\Rightarrow \quad A_1 = 4z \text{ and } \alpha_1 = \tfrac{9}{4}\pi + \tfrac{1}{2}\pi = \tfrac{3}{4}\pi, \\
f_2'' = f''(t_2) &= 4izt_2^3 = 4iz\, e^{3i\pi/4} \\
&\Rightarrow \quad A_2 = 4z \text{ and } \alpha_2 = \tfrac{3}{4}\pi + \tfrac{1}{2}\pi = \tfrac{5}{4}\pi.
\end{aligned}$$

We now need to determine the directions of the l.s.d. at the saddles and the senses in which they are traversed. At $t = t_1$ the directions θ of the l.s.d. are given in the usual way by $\sin(2\theta + \alpha_1) = 0$ and the appropriate pair of choices amongst these by the requirement that $\cos(2\theta + \alpha_1)$ is negative. With $\alpha_1 = 3\pi/4$, the two acceptable values of θ are $\pi/8$ and $9\pi/8$, with the contour, which starts at $-\infty$, clearly passing through the saddle and leaving it in the direction $\theta = \pi/8$. Since this lies in the range $-\frac{1}{2}\pi < \theta \leq \frac{1}{2}\pi$, the contribution to the approximate value of the integral from this saddle point will be positive.

A similar analysis at $t = t_2$ shows that the contour following the l.s.d. there approaches the saddle from the direction $\theta = 7\pi/8$ and leaves it in the direction $\theta = -\pi/8$, again making a positive contribution to the value of the integral.

Finally, we substitute these calculated data into the standard formula for the steepest descents estimation,

$$\pm\left(\frac{2\pi}{A_i}\right)^{1/2} \exp(f_i)\exp[\tfrac{1}{2}i(\pi - \alpha_i)],$$

and obtain, with $4/(5\sqrt{2})$ written as β,

$$\begin{aligned}
F_1(z) &= +\left(\frac{2\pi}{4z}\right)^{1/2} \exp\left[\frac{4iz}{5}\,\frac{(-1+i)}{\sqrt{2}}\right] \exp[\tfrac{1}{2}i(\pi - \tfrac{3}{4}\pi)] \\
&= \left(\frac{\pi}{2z}\right)^{1/2} \exp(-\beta z)\, \exp\left(-i\beta z + \frac{i\pi}{8}\right), \\
F_2(z) &= +\left(\frac{2\pi}{4z}\right)^{1/2} \exp\left[\frac{4iz}{5}\,\frac{(1+i)}{\sqrt{2}}\right] \exp[\tfrac{1}{2}i(\pi - \tfrac{5}{4}\pi)] \\
&= \left(\frac{\pi}{2z}\right)^{1/2} \exp(-\beta z)\, \exp\left(i\beta z - \frac{i\pi}{8}\right).
\end{aligned}$$

Adding these two contributions together gives the stated result

$$F(z) = \left(\frac{2\pi}{z}\right)^{1/2} \exp(-\beta z)\, \cos(\beta z - \tfrac{1}{8}\pi).$$

25.22 *The Bessel function $J_\nu(z)$ is given for $|\arg z| < \frac{1}{2}\pi$ by the integral around a contour C of the function*

$$g(z) = \frac{1}{2\pi i}\, t^{-(\nu+1)} \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right].$$

The contour starts and ends along the negative real t-axis and encircles the origin in the positive sense. It can be considered as made up of two contours. One of them, C_2, starts at $t = -\infty$, runs through the third quadrant to the point $t = -i$ and then approaches the origin in the fourth quadrant in a curve that is ultimately anti-parallel to the positive real axis. The other contour, C_1, is the mirror image of this in the real axis; it is confined to the upper half plane, passes through $t = i$ and is anti-parallel to the real t-axis at both of its extremities. The contribution to $J_\nu(z)$ from the curve C_k is $\frac{1}{2}H_\nu^{(k)}$, the function $H_\nu^{(k)}$ being known as a Hankel function.

Using the method of steepest descents, establish the leading term in an asymptotic expansion for $H_\nu^{(1)}$ for z real, large and positive. Deduce, without detailed calculation, the corresponding result for $H_\nu^{(2)}$. Hence establish the asymptotic form of $J_\nu(z)$ for the same range of z.

We first note that, in the neighbourhood of any saddle point, we will treat that part of the integrand that is not exponentiated, $(2\pi i)^{-1}t^{-(\nu+1)}$, by assigning it the value $(2\pi i)^{-1}t_0^{-(\nu+1)}$, where t_0 is the location of that saddle point.

The ends of contour C_1 are at $t = -\infty$ and $t = 0$ and we next check the values of the integrand at these two points, remembering that z is real and positive,

$$t = -\infty,\ \exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = \exp\left[\frac{z}{2}(-\infty + 0)\right] = 0,$$
$$t = 0,\ \exp\left[\frac{z}{2}(0 - \infty)\right] = 0.$$

These are both satisfactory in so far as not invalidating the method is concerned.

The saddle points of the integrand $f(t)$ are given by

$$\frac{d}{dt}\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right] = 0 \quad \Rightarrow \quad 1 + \frac{1}{t^2} = 0, \text{ i.e. } t = \pm i.$$

Clearly we need $t = +i$ for the C_1 contour; $t = -i$ will be appropriate to the C_2 contour giving $H_\nu^{(2)}$.

The other derivatives and values needed at $t_0 = i$ are

$$\frac{d^2}{dt^2}\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right] = \frac{z}{2}\frac{-2}{t^3} = -\frac{z}{t^3} = -iz, \text{ at } t = i,$$

$$f_0 \equiv f(t_0) = \frac{z}{2}\left(i - \frac{1}{i}\right) = iz,$$

$$t_0^{-(\nu+1)} = i^{-(\nu+1)} = e^{-i\pi(\nu+1)/2}.$$

In the standard notation, $f''(t_0) \equiv Ae^{i\alpha}$, we have $A = z$ and $\alpha = 3\pi/2$.

We now use the standard approach that if $t - t_0 = se^{i\theta}$ then

$$\left|\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]\right| = |\exp(f_0)|\,\exp[\tfrac{1}{2}As^2(\cos 2\theta + \alpha) + \mathrm{O}(s^3)].$$

and the line of steepest descents (l.s.d.) is given by the condition that the argument of $\exp\left[\frac{1}{2}z\left(t - t^{-1}\right)\right]$ is independent of s. This is $\sin(2\theta + \alpha) = 0$, leading to $\theta = \pm\frac{1}{4}\pi$ or $\theta = \pm\frac{3}{4}\pi$.

It is not immediately apparent which pair of directions from these four is the correct choice — for the contour to pass through the saddle point at $t = i$ from '(i) top right to bottom left' or '(ii) from bottom right to top left'. To determine which is correct, we evaluate $h(s) = \frac{1}{2}(t - t_0)^2 f''(t_0)$, where $t - t_0 = se^{i\theta}$, in each case

$$\begin{aligned}
\text{(i) } t - t_o &= se^{i\pi/4} \text{ (approaching) or } se^{-3i\pi/4} \text{ (leaving)}\\
&\Rightarrow h(s) = \tfrac{1}{2}s^2 e^{i\pi/2}(-iz) = \tfrac{1}{2}s^2 z, \text{ i.e. real and } > 0,\\
\text{(ii) } t - t_o &= se^{-i\pi/4} \text{ (approaching) or } se^{3i\pi/4} \text{ (leaving)}\\
&\Rightarrow h(s) = \tfrac{1}{2}s^2 e^{-i\pi/2}(-iz) = -\tfrac{1}{2}s^2 z, \text{ i.e. real and } < 0.
\end{aligned}$$

For approximating the integral by a Gaussian with its maximum at the saddle point, clearly we must have case (ii).

Finally, we use the standard form of the integral

$$I \approx \pm g(t_0)\left(\frac{2\pi}{A}\right)^{1/2}\exp(f_0)\exp[\tfrac{1}{2}i(\pi - \alpha)],$$

with the $\pm$ choice being resolved by the direction in which the l.s.d. passes through the saddle-point; it is positive if $|\theta| < \pi/2$ and negative otherwise. In this particular case, as we have just shown, the l.s.d. traverses the saddle-point in the direction $3\pi/4$ and the minus sign is appropriate. Putting in the specific values gives

$$\frac{1}{2}H_\nu^{(1)} = -\frac{1}{2\pi i}\left(\frac{2\pi}{z}\right)^{1/2}\exp(iz)\exp[\tfrac{1}{2}i(\pi - \tfrac{3}{2}\pi)]\,e^{-i\pi(\nu+1)/2},$$

$$H_\nu^{(1)} = \left(\frac{2}{\pi z}\right)^{1/2} e^{iz}e^{-i\pi/4}e^{-i\pi\nu/2}.$$

For $H_\nu^{(2)}$ we can deduce from symmetry/antisymmetry that $t_0 = -i$, $A = z$, $\alpha = \pi/2$, that the contour C_2 traverses the saddle point in the direction $+i\pi/4$ and, consequently, that the contribution is a positive one.

$$\begin{aligned} H_\nu^{(2)} &= +\frac{1}{\pi i}\left(\frac{2\pi}{z}\right)^{1/2} \exp(-iz)\exp[\tfrac{1}{2}i(\pi - \tfrac{1}{2}\pi)]\, e^{i\pi(\nu+1)/2} \\ &= \left(\frac{2}{\pi z}\right)^{1/2} e^{-iz} e^{i\pi/4} e^{i\pi\nu/2}. \end{aligned}$$

Now adding together the asymptotic forms of $\frac{1}{2}H_\nu^{(1)}$ and $\frac{1}{2}H_\nu^{(2)}$ to form that for $J_\nu(z)$ gives

$$\begin{aligned} J_\nu(z) &\sim \frac{1}{2}\left(\frac{2}{\pi z}\right)^{1/2} \left(e^{iz} e^{-i\pi/4} e^{-i\pi\nu/2} + e^{-iz} e^{i\pi/4} e^{i\pi\nu/2}\right) \\ &= \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{\pi}{4} - \frac{\nu\pi}{2}\right). \end{aligned}$$

26

Tensors

26.2 *The components of two vectors* **A** *and* **B** *and a second-order tensor* **T** *are given in one coordinate system by*

$$\mathsf{A} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In a second coordinate system, obtained from the first by rotation, the components of **A** *and* **B** *are*

$$\mathsf{A}' = \frac{1}{2}\begin{pmatrix} \sqrt{3} \\ 0 \\ 1 \end{pmatrix}, \quad \mathsf{B}' = \frac{1}{2}\begin{pmatrix} -1 \\ 0 \\ \sqrt{3} \end{pmatrix}.$$

Find the components of **T** *in this new coordinate system and hence evaluate, with a minimum of calculation,*

$$T_{ij}T_{ji}, \quad T_{ki}T_{jk}T_{ij}, \quad T_{ik}T_{mn}T_{ni}T_{km}.$$

Since we must have $x'_i = L_{ij}x_j$ and **A** and **B** have their components tranformed into the given values, L must have the form

$$\mathsf{L} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix}.$$

We determine a, b and c by requiring that L is orthogonal and has $|\mathsf{L}| = +1$.

$$\mathsf{L}\mathsf{L}^{\mathrm{T}} = \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & a \\ 0 & 0 & b \\ 1 & \sqrt{3} & c \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ a & b & c \end{pmatrix} = \mathsf{I},$$

giving $a = 0$, $b = \pm 2$ and $c = 0$. The determinant of L is $\frac{1}{8}(-3b - b + 0)$, thus requiring that $b = -2$. Hence the required orthoganal matrix L is

$$\mathsf{L} = \frac{1}{2}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}.$$

The third column of L could have been obtained by considering $\mathsf{A}' \times \mathsf{B}'$. The matrix product $\mathsf{T}' = \mathsf{L}\mathsf{T}\mathsf{L}^{\mathrm{T}}$ is given by

$$\begin{aligned}
\mathsf{T}' &= \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} 2 & \sqrt{3} & 0 \\ \sqrt{3} & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 1 \\ -1 & 0 & \sqrt{3} \\ 0 & -2 & 0 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} \sqrt{3} & -1 & 0 \\ 0 & 0 & -2 \\ 1 & \sqrt{3} & 0 \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 & 5 \\ -1 & 0 & 5\sqrt{3} \\ 0 & -4 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}.
\end{aligned}$$

As a check, we note that $\mathrm{Tr}\,\mathsf{T}' = 1 + 2 + 5 = 2 + 4 + 2 = \mathrm{Tr}\,\mathsf{T}$.

In this new coordinate system T is diagonal – and therefore very convenient for calculating the following invariants (tensors of order 0); their values are independent of the frame in which they are calculated.
$T_{ij}T_{ji} = T'_{ij}T'_{ji} = 1 + 4 + 25 = 30$.
$T_{ki}T_{jk}T_{ij} = T_{ij}T_{jk}T_{ki} = T'_{ij}T'_{jk}T'_{ki} = 1 + 8 + 125 = 134$.
$T_{ik}T_{mn}T_{ni}T_{km} = T_{ik}T_{km}T_{mn}T_{ni} = T'_{ik}T'_{km}T'_{mn}T'_{ni} = 1 + 16 + 625 = 642$.

26.4 *Show how to decompose the Cartesian tensor T_{ij} into three tensors,*

$$T_{ij} = U_{ij} + V_{ij} + S_{ij},$$

where U_{ij} is symmetric and has zero trace, V_{ij} is isotropic and S_{ij} has only three independent components.

We start by writing T_{ij} as the sum of its even and odd parts:

$$T_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}) + \tfrac{1}{2}(T_{ij} - T_{ji}) \equiv \tfrac{1}{2}(T_{ij} + T_{ji}) + S_{ij}.$$

Clearly S_{ij} has zeroes on its leading diagonal and $S_{ji} = -S_{ij}$; it therefore has only 3 independent components, S_{12}, S_{13} and S_{23}.

Now, if the trace T_{ii} is written as T_0, then $\frac{1}{3}T_0\delta_{ij}$ is an isotropic tensor V_{ij}. Subtracting this from the symmetric part of T_{ij} leaves

$$U_{ij} = \tfrac{1}{2}(T_{ij} + T_{ji}) - \tfrac{1}{3}T_0\delta_{ij}.$$

Since $U_{ij} = U_{ji}$, U_{ij} is symmetric. Further,

$$\text{Tr } U_{ij} = \tfrac{1}{2}(T_0 + T_0) - \tfrac{1}{3}T_0 3 = 0,$$

i.e. U_{ij} is traceless. This completes the decomposition.

If T_{ij} were a second-order tensor in n dimensions, S_{ij} would have $\frac{1}{2}n(n-1)$ independent components and the factor in V_{ij} would be $1/n$.

26.6 *Use tensor methods to establish the following vector identities:*

(a) $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}$;

(b) $\text{curl}\,(\phi\mathbf{u}) = \phi\,\text{curl}\,\mathbf{u} + (\text{grad}\,\phi) \times \mathbf{u}$;

(c) $\text{div}\,(\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \text{curl}\,\mathbf{u} - \mathbf{u} \cdot \text{curl}\,\mathbf{v}$;

(d) $\text{curl}\,(\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \text{grad})\mathbf{u} - (\mathbf{u} \cdot \text{grad})\mathbf{v} + \mathbf{u}\,\text{div}\,\mathbf{v} - \mathbf{v}\,\text{div}\,\mathbf{u}$;

(e) $\text{grad}\,\frac{1}{2}(\mathbf{u} \cdot \mathbf{u}) = \mathbf{u} \times \text{curl}\,\mathbf{u} + (\mathbf{u} \cdot \text{grad})\mathbf{u}$.

All of the expressions for vector operators in tensor notation are given in the text and should be known but, for convenience, the principal ones and the double epsilon identity are repeated here:

$$\begin{aligned}
(\mathbf{a} \times \mathbf{b})_i &= \epsilon_{ijk} a_j b_k, \\
\nabla \cdot \mathbf{a} &= \frac{\partial a_i}{\partial x_i}, \\
(\nabla \times \mathbf{a})_i &= \epsilon_{ijk} \frac{\partial a_k}{\partial x_j}, \\
\epsilon_{ijk}\epsilon_{klm} &= \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.
\end{aligned}$$

In addition, it should be remembered that ϵ_{ijk} is merely a number and can be moved from under a differentiation sign.

Where the identity is a vector equation, our proofs will consider only its ith component; but, as i is general, this will establish the full vector identity. Case (c) is a scalar identity.

(a) Consider the ith component:

$$\begin{aligned}
[(\mathbf{u}\times\mathbf{v})\times\mathbf{w}]_i &= \epsilon_{ijk}(\mathbf{u}\times\mathbf{v})_j w_k \\
&= \epsilon_{ijk}\epsilon_{jlm}u_l v_m w_k \\
&= (\delta_{kl}\delta_{im}-\delta_{km}\delta_{il})u_l v_m w_k \\
&= v_i u_l w_l - u_i v_m w_m \\
&= [(\mathbf{u}\cdot\mathbf{w})\mathbf{v}-(\mathbf{v}\cdot\mathbf{w})\mathbf{u}]_i
\end{aligned}$$

(b) Since ∇ is involved in this identity [and in those in parts (c)-(e)], we must take particular care with the order in which we write the differential operator and the functions (or products of functions) on which it might act. Again consider the ith component:

$$\begin{aligned}
[\nabla\times(\phi\mathbf{u})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\phi u_k) \\
&= \epsilon_{ijk}\phi\frac{\partial u_k}{\partial x_j}+\epsilon_{ijk}u_k\frac{\partial\phi}{\partial x_j} \\
&= \phi(\nabla\times\mathbf{u})_i+(\nabla\phi\times\mathbf{u})_i
\end{aligned}$$

(c) As noted above, this is a scalar quantity and any tensor expression for it must have all of its indices contracted.

$$\begin{aligned}
\nabla\cdot(\mathbf{u}\times\mathbf{v}) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk}u_j v_k) \\
&= \epsilon_{ijk}\frac{\partial u_j}{\partial x_i}v_k+\epsilon_{ijk}u_j\frac{\partial v_k}{\partial x_i} \\
&= v_k(\nabla\times\mathbf{u})_k-u_j(\nabla\times\mathbf{v})_j \\
&= \mathbf{v}\cdot(\nabla\times\mathbf{u})-\mathbf{u}\cdot(\nabla\times\mathbf{v}).
\end{aligned}$$

(d) For the ith component of this vector identity,

$$\begin{aligned}
[\nabla\times(\mathbf{u}\times\mathbf{v})]_i &= \epsilon_{ijk}\frac{\partial}{\partial x_j}(\mathbf{u}\times\mathbf{v})_k \\
&= \epsilon_{ijk}\frac{\partial}{\partial x_j}\epsilon_{klm}u_l v_m \\
&= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})\left(u_l\frac{\partial v_m}{\partial x_j}+v_m\frac{\partial u_l}{\partial x_j}\right) \\
&= u_i\frac{\partial v_j}{\partial x_j}+v_j\frac{\partial u_i}{\partial x_j}-u_j\frac{\partial v_i}{\partial x_j}-v_i\frac{\partial u_j}{\partial x_j} \\
&= u_i\nabla\cdot\mathbf{v}+\mathbf{v}\cdot\nabla u_i-\mathbf{u}\cdot\nabla v_i-v_i\nabla\cdot\mathbf{u} \\
&= [\mathbf{u}(\nabla\cdot\mathbf{v})+(\mathbf{v}\cdot\nabla)\mathbf{u}-(\mathbf{u}\cdot\nabla)\mathbf{v}-\mathbf{v}(\nabla\cdot\mathbf{u})]_i
\end{aligned}$$

(e) Since simplification and reduction are usually easier to recognise than the best way to make an expression more complicated but still valid, we start with the

most complicated of the terms in the identity:

$$\begin{aligned}[\mathbf{u}\times(\nabla\times\mathbf{u})]_i &= \epsilon_{ijk}u_j(\nabla\times\mathbf{u})_k \\ &= \epsilon_{ijk}u_j\,\epsilon_{klm}\frac{\partial u_m}{\partial x_l}.\end{aligned}$$

We can now employ the double ϵ formula to convert this expression into one containing Kronecker deltas.

$$\begin{aligned}[\mathbf{u}\times(\nabla\times\mathbf{u})]_i &= (\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl})u_j\frac{\partial u_m}{\partial x_l} \\ &= u_j\frac{\partial u_j}{\partial x_i}-u_j\frac{\partial u_i}{\partial x_j} \\ &= \frac{1}{2}\frac{\partial}{\partial x_i}(u_ju_j)-(\mathbf{u}\cdot\nabla)u_i \\ &= [\tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u})-(\mathbf{u}\cdot\nabla)\mathbf{u}]_i.\end{aligned}$$

This completes the proof.

26.8 *A column matrix* a *has components* a_x, a_y, a_z *and* A *is the matrix with elements* $A_{ij}=-\epsilon_{ijk}a_k$.

(a) *What is the relationship between column matrices* b *and* c *if* $\mathsf{Ab}=\mathsf{c}$?
(b) *Find the eigenvalues of* A *and show that* a *is one of its eigenvectors. Explain why this must be so.*

(a) The matrix equation $\mathsf{Ab}=\mathsf{c}$ will have the explicit form

$$\mathsf{c}=\mathsf{Ab}=\begin{pmatrix}0 & -a_z & a_y\\ a_z & 0 & -a_x\\ -a_y & a_x & 0\end{pmatrix}\begin{pmatrix}b_x\\ b_y\\ b_z\end{pmatrix}=\begin{pmatrix}-a_zb_y+a_yb_z\\ a_zb_x-a_xb_z\\ -a_yb_x+a_xb_y\end{pmatrix}=\mathsf{a}\times\mathsf{b},$$

i.e. $\mathsf{c}=\mathsf{a}\times\mathsf{b}$.

(b) The characteristic equation for A is

$$\begin{aligned}0=|\mathsf{A}-\lambda\mathsf{I}| &= \begin{vmatrix}-\lambda & -a_z & a_y\\ a_z & -\lambda & -a_x\\ -a_y & a_x & -\lambda\end{vmatrix} \\ &= -\lambda^3-a_x^2\lambda-a_z(a_xa_y+\lambda a_z)+a_y(a_za_x-\lambda a_y).\end{aligned}$$

Thus $\lambda=0$ or $-\lambda^2-a_x^2-a_z^2-a_y^2=0$. The second possibility gives $\lambda=\pm i(a_x^2+a_y^2+a_z^2)^{1/2}=\pm i|\mathbf{a}|$.

Now consider Aa. From part (a) this is

$$\begin{pmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix} = \begin{pmatrix} -a_z a_y + a_y a_z \\ a_z a_x - a_x a_z \\ -a_y a_x + a_x a_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0\mathsf{a}.$$

Thus a is an eigenvector of A corresponding to eigenvalue 0. That this must be so also follows from the general conclusion of part (a) that if $\mathsf{c} = \mathsf{Ab}$ then $\mathsf{c} = \mathsf{a} \times \mathsf{b}$; if $\mathsf{b} = \mathsf{a}$, then $\mathsf{c} = \mathsf{a} \times \mathsf{a} = \mathsf{0} = 0\mathsf{a}$.

26.10 *A symmetric second-order Cartesian tensor is defined by*

$$T_{ij} = \delta_{ij} - 3x_i x_j.$$

Evaluate the following surface integrals, each taken over the surface of the unit sphere:

$$\text{(a)} \int T_{ij}\, dS; \quad \text{(b)} \int T_{ik} T_{kj}\, dS; \quad \text{(c)} \int x_i T_{jk}\, dS.$$

We will need the following integrals over the unit sphere U:

$$\int_U 1\, dS = 4\pi,$$

$$\int_U x_i\, dS = 0, \text{ on antisymmetry grounds,}$$

$$\int_U x_i x_j\, dS = 0, \text{ on antisymmetry grounds for } i \neq j,$$

$$\int_U x_i^2\, dS = \frac{1}{3} \int_U (x_1^2 + x_2^2 + x_3^2)\, dS = \frac{1}{3} \int_U 1\, dS = \frac{4\pi}{3},$$

$$\int_U x_i^3\, dS = 0, \text{ on antisymmetry grounds.}$$

Thus, combining the third and fourth of these,

$$\int_U x_i x_j\, dS = \frac{4\pi}{3}\, \delta_{ij}.$$

We note that the integrands in (a) and (b) each have two uncontracted subscripts and that that in (c) has three. As the integrations are with respect to the scalar

quantity S, our answers must have corresponding properties.

(a) $$\int_U T_{ij}\,dS = \int_U \delta_{ij}\,dS - \int 3x_i x_j\,dS$$
$$= 4\pi\delta_{ij} - 3\,\frac{4\pi}{3}\delta_{ij} = 0 \quad \text{for all } i \text{ and } j.$$

(b) $$\int_U T_{ik}T_{kj}\,dS = \int_U (\delta_{ik} - 3x_i x_k)(\delta_{kj} - 3x_k x_j)\,dS$$
$$= \int_U (\delta_{ij} - 3x_i x_j - 3x_i x_j + 9x_i x_k x_k x_j)\,dS$$
$$= \int_U (\delta_{ij} + 3x_i x_j)\,dS, \text{ since } x_k x_k = 1 \text{ on } U,$$
$$= 4\pi\delta_{ij} + 3\,\frac{4\pi}{3}\delta_{ij} = 8\pi\delta_{ij}.$$

(c) $$\int_U x_i T_{jk}\,dS = \int_U (x_i\delta_{jk} - 3x_i x_j x_k)\,dS$$
$$= \begin{cases} 0 - 0 \text{ if } i \neq j \neq k \neq i, \text{ on antisymmetry grounds,} \\ 0 - \int_U x_k\,dS = 0 \text{ if } i = j \neq k \text{ or } j = k \neq i \text{ or } i = k \neq j, \\ 0 - \int_U x_k^3\,dS = 0 \text{ if } i = j = k. \end{cases}$$

26.12 *In four dimensions define second-order antisymmetric tensors F_{ij} and Q_{ij} and a first-order tensor S_i as follows:*

(a) $F_{23} = H_1$, $Q_{23} = B_1$ *and their cyclic permutations;*
(b) $F_{i4} = -D_i$, $Q_{i4} = E_i$ *for* $i = 1, 2, 3$*;*
(c) $S_4 = \rho$, $S_i = J_i$ *for* $i = 1, 2, 3$.

Then, taking x_4 as t and the other symbols to have their usual meanings in electromagnetic theory, show that the equations $\sum_j \partial F_{ij}/\partial x_j = S_i$ and $\partial Q_{jk}/\partial x_i + \partial Q_{ki}/\partial x_j + \partial Q_{ij}/\partial x_k = 0$ reproduce Maxwell's equations. In the latter i, j, k is any set of three subscripts selected from 1, 2, 3, 4, *but chosen in such a way that they are all different.*

We can write the defining equations for F and Q as

$$F_{ij} = \epsilon_{ijk}H_k \quad \text{and} \quad Q_{ij} = \epsilon_{ijk}B_k,$$

where none of i, j, k is equal to 4.

First, for $i = 1, 2, 3$,

$$J_i = S_i = \sum_j \frac{\partial F_{ij}}{\partial x_j} = \frac{\partial}{\partial x_j}(\epsilon_{ijk} H_k) + \frac{\partial F_{i4}}{\partial t} = (\nabla \times \mathbf{H})_i - \frac{\partial D_i}{\partial t}.$$

Thus the given equation is the ith component of the Maxwell equation

$$\nabla \times \mathbf{H} = \mathbf{J} + \dot{\mathbf{D}}.$$

Second, for $i = 4$, and noting that, since F is antisymmetric, $F_{44} = 0$,

$$\rho = S_4 = \sum_j \frac{\partial F_{4j}}{\partial x_j} = -\sum_{j=1}^{3} \frac{\partial F_{j4}}{\partial x_j} = \sum_{j=1}^{3} \frac{\partial D_j}{\partial x_j}.$$

This is the Maxwell equation $\nabla \cdot \mathbf{D} = \rho$.

For the equation involving Q, with i, j, k some non-repeating selection from $1, 2, 3$, we have, say,

$$\begin{aligned} \frac{\partial Q_{23}}{\partial x_1} + \frac{\partial Q_{31}}{\partial x_2} + \frac{\partial Q_{12}}{\partial x_3} &= 0, \\ \Rightarrow \quad \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} &= 0, \\ \nabla \cdot \mathbf{B} &= 0. \end{aligned}$$

This is one of Maxwell's equations. Reassigning $1, 2, 3$ amongst i, j, k produces the same equation.

Finally, for $k = 4$ and $i \neq j$ with neither equal to 4,

$$\frac{\partial Q_{j4}}{\partial x_i} + \frac{\partial Q_{4i}}{\partial x_j} + \frac{\partial Q_{ij}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial E_j}{\partial x_i} + \frac{\partial(-E_i)}{\partial x_j} + \frac{\partial}{\partial t}(\epsilon_{ijm} B_m) = 0.$$

We now contract this second equation with ϵ_{ijn} and use the double epsilon identity, remembering also that $\epsilon_{ijn} \partial E_j / \partial x_i$ is the nth component of $\nabla \times \mathbf{E}$:

$$\begin{aligned} (\nabla \times \mathbf{E})_n - (-\nabla \times \mathbf{E})_n + (\delta_{jj}\delta_{nm} - \delta_{jm}\delta_{nj}) \frac{\partial B_m}{\partial t} &= 0, \\ 2(\nabla \times \mathbf{E})_n + (3\delta_{nm} - \delta_{nm}) \frac{\partial B_m}{\partial t} &= 0, \\ \Rightarrow \quad (\nabla \times \mathbf{E})_n + (\dot{\mathbf{B}})_n &= 0, \quad \text{for } n = 1, 2, 3. \end{aligned}$$

This is the nth component of the Maxwell equation $(\nabla \times \mathbf{E}) + (\dot{\mathbf{B}}) = \mathbf{0}$.

26.14 *Assuming that the current density* $\mathbf{j}$ *and the electric field* $\mathbf{E}$ *appearing in equation (26.43),* $j_i = \sigma_{ij}E_j$*, are first-order Cartesian tensors, show explicitly that the electrical conductivity tensor* σ_{ij} *transforms according to the law appropriate to a second-order tensor.*

The rate W *at which energy is dissipated per unit volume, as a result of the current flow, is given by* $\mathbf{E}\cdot\mathbf{j}$*. Determine the limits between which* W *must lie for a given value of* $|\mathbf{E}|$ *as the direction of* $\mathbf{E}$ *is varied.*

This result follows immediately from the quotient law but to show it directly we proceed as follows.

$$\begin{aligned}
j_i &= \sigma_{ij}E_j, && \text{given,}\\
j'_i &= \sigma'_{ij}E'_j, && \text{in the transformed frame,}\\
L_{ip}j_p &= \sigma'_{ij}L_{jq}E_q, && j \text{ and } E \text{ are first-order tensors,}\\
L_{ip}\sigma_{pk}E_k &= \sigma'_{ij}L_{jq}E_q, && \text{substituting for } j_p,\\
L_{ir}L_{ip}\sigma_{pk}E_k &= L_{ir}L_{jq}\sigma'_{ij}E_q, && \text{multiply by } L_{ir} \text{ and sum,}\\
(\delta_{rp}\sigma_{pk} - L_{ir}L_{jk}\sigma'_{ij})E_k &= 0, && \mathsf{L} \text{ is orthogonal; relabel dummy suffix,}\\
\sigma_{rk} &= L_{ir}L_{jk}\sigma'_{ij}, && \text{since true for all } E_k.
\end{aligned}$$

This is a sufficient proof, but to put the final result in the usual form we continue with

$$L_{mr}L_{nk}\sigma_{rk} = L_{mr}L_{nk}L_{ir}L_{jk}\sigma'_{ij} = \delta_{mi}\delta_{nj}\sigma'_{ij} = \sigma'_{mn},$$

showing that σ is a second-order tensor.

The rate of dissipation is $W = E_i j_i = E_i\sigma_{ij}E_j$. The problem is to extremise this, subject to the constraint $E_kE_k = |\mathbf{E}|^2$, a constant. Introducing a Lagrange multiplier, we consider the extremes of

$$W' = E_i\sigma_{ij}E_j - \lambda E_kE_k.$$

They are given by

$$\frac{\partial W'}{\partial E_j} = 0 \quad\Rightarrow\quad 2(\sigma_{ij}E_i - \lambda E_j) = 0.$$

This shows that the extremising directions of $\mathbf{E}$ are the eigenvectors $\hat{\mathbf{E}}$ of σ. For these directions

$$\sigma_{ij}\hat{E}_i\hat{E}_j - \lambda\hat{E}_j\hat{E}_j = 0 \quad\Rightarrow\quad \hat{W} = \lambda|\mathbf{E}|^2.$$

Thus the maximum and minimum values of W are determined by the maximum and minimum eigenvalues of σ_{ij}.

26.16 *A rigid body consists of four particles of masses m, $2m$, $3m$, $4m$, respectively situated at the points (a, a, a), $(a, -a, -a)$, $(-a, a, -a)$, $(-a, -a, a)$ and connected together by a light framework.*

(a) *Find the inertia tensor at the origin and show that the principal moments of inertia are $20ma^2$ and $(20 \pm 2\sqrt{5})ma^2$.*
(b) *Find the principal axes and verify that they are orthogonal.*

The masses are

$$m \text{ at } a(1,1,1), \qquad 2m \text{ at } a(1,-1,-1),$$
$$3m \text{ at } a(-1,1,-1), \quad 4m \text{ at } a(-1,-1,1).$$

(a) The inertia tensor components (recall that the off-diagonal elements have an intrinsic minus sign) are calculated as

$$\begin{aligned}
I_{11} &= ma^2[\,1(2) + 2(2) + 3(2) + 4(2)\,] = 20ma^2,\\
I_{12} = I_{21} &= ma^2[\,1(-1) + 2(1) + 3(1) + 4(-1)\,] = 0,\\
I_{13} = I_{31} &= ma^2[\,1(-1) + 2(1) + 3(-1) + 4(1)\,] = 2ma^2,\\
I_{22} &= ma^2[\,1(2) + 2(2) + 3(2) + 4(2)\,] = 20ma^2,\\
I_{23} = I_{32} &= ma^2[\,1(-1) + 2(-1) + 3(1) + 4(1)\,] = 4ma^2,\\
I_{33} &= ma^2[\,1(2) + 2(2) + 3(2) + 4(2)\,] = 20ma^2.
\end{aligned}$$

Thus,

$$I = 2ma^2 \begin{pmatrix} 10 & 0 & 1 \\ 0 & 10 & 2 \\ 1 & 2 & 10 \end{pmatrix},$$

and the principal moments are given by $2ma^2\lambda$ where

$$\begin{aligned}
0 &= \begin{vmatrix} 10-\lambda & 0 & 1 \\ 0 & 10-\lambda & 2 \\ 1 & 2 & 10-\lambda \end{vmatrix}\\
&= (10-\lambda)(\lambda^2 - 20\lambda + 96) - 10 + \lambda\\
&= (10-\lambda)(\lambda^2 - 20\lambda + 95).
\end{aligned}$$

Thus $\lambda = 10$ or $\lambda = 10 \pm \sqrt{100-95}$ and the principal moments are $20ma^2$ and $(20 \pm 2\sqrt{5})ma^2$. It is clear that these add up to the trace of I (as a check).

(b) For $\lambda = 10$ the (unnormalised) axis vector, $\mathbf{v}_1$, satisfies

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_1 = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}.$$

For $\lambda = 10 \pm \sqrt{5}$ the (unnormalised) axis vectors, $\mathbf{v}_2$, and $\mathbf{v}_3$ satisfy

$$\begin{pmatrix} \mp\sqrt{5} & 0 & 1 \\ 0 & \mp\sqrt{5} & 2 \\ 1 & 2 & \mp\sqrt{5} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \mathbf{v}_{2,3} = \begin{pmatrix} \pm 1 \\ \pm 2 \\ \sqrt{5} \end{pmatrix}.$$

Further,

$$\mathbf{v}_1 \cdot \mathbf{v}_2 = (2 - 2 + 0) = 0, \ \mathbf{v}_1 \cdot \mathbf{v}_3 = (-2 + 2 + 0) = 0, \ \mathbf{v}_2 \cdot \mathbf{v}_3 = (-1 - 4 + 5) = 0,$$

showing that the axes vectors are mutually orthogonal.

26.18 *The paramagnetic tensor χ_{ij} of a body placed in a magnetic field, in which its energy density is $-\frac{1}{2}\mu_0 \mathbf{M} \cdot \mathbf{H}$ with $M_i = \sum_j \chi_{ij} H_j$, is*

$$\begin{pmatrix} 2k & 0 & 0 \\ 0 & 3k & k \\ 0 & k & 3k \end{pmatrix}.$$

Assuming depolarizing effects are negligible, find how the body will orientate itself if the field is horizontal, in the following circumstances:

(a) *the body can rotate freely;*
(b) *the body is suspended with the $(1, 0, 0)$ axis vertical;*
(c) *the body is suspended with the $(0, 1, 0)$ axis vertical.*

The equilibrium orientation of the body will be such as to minimise the total energy (per unit volume)

$$E = -\tfrac{1}{2}\mu_0 \mathbf{M} \cdot \mathbf{H} = -\tfrac{1}{2}\mu_0 \chi_{ij} H_i H_j,$$

subject to any constraints imposed by the method of suspension. We therefore need to maximise (assuming that $k > 0$) the quadratic form $\hat{\mathbf{n}}^{\mathrm{T}} \chi \hat{\mathbf{n}}$. This could be done by finding the eigenvalues and eigenvectors of χ or directly from the quadratic form. We will adopt the latter approach (and omit the factor k which is merely a scaling factor).

(a) If the body can rotate freely there are no constraints on the components n_i of the unit vector $\hat{\mathbf{n}}$ fixed in the body that aligns itself with the external field. We therefore maximise

$$Q = (n_1, n_2, n_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \end{pmatrix} = 2n_1^2 + 3n_2^2 + 3n_3^2 + 2n_2 n_3,$$

subject to $n_1^2 + n_2^2 + n_3^2 = 1$. Now, $Q = 2 + (n_2 + n_3)^2$ which is clearly maximal,

given the constraint, when $n_1 = 0$ and $n_2 = n_3 = 1/\sqrt{2}$. Thus the body aligns itself with the $(0, 1, 1)$ direction parallel to the field.

(b) With the $(1, 0, 0)$ axis vertical (and the field horizontal), $\hat{\mathbf{n}}$ must have the form $(0, n_2, n_3)$ with $n_2^2 + n_3^2 = 1$. We consider

$$Q = (0, n_2, n_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ n_2 \\ n_3 \end{pmatrix} = 3n_2^2 + 3n_3^2 + 2n_2 n_3.$$

This is $Q = 3 + 2n_2 n_3$ and, again, is clearly maximal, given the constraint, when $n_2 = n_3 = 1/\sqrt{2}$. Thus the equilibrium orientation is as in part (a).

(c) With the $(0, 1, 0)$ axis vertical (and the field horizontal), $\hat{\mathbf{n}}$ must have the form $(n_1, 0, n_3)$ with $n_1^2 + n_3^2 = 1$. We consider

$$Q = (n_1, 0, n_3) \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} n_1 \\ 0 \\ n_3 \end{pmatrix} = 2n_1^2 + 3n_3^2,$$

This is $Q = 2 + n_3^2$ and to obtain a maximum we must take $n_1 = 0$ and $n_3 = 1$. Thus the body aligns itself with the $(0, 0, 1)$ direction parallel to the field.

26.20 *For tin, the conductivity tensor is diagonal, with entries a, a, and b when referred to its crystal axes. A single crystal is grown in the shape of a long wire of length L and radius r, the axis of the wire making polar angle θ with respect to the crystal's 3-axis. Show that the resistance of the wire is*

$$\frac{L}{\pi r^2 ab} \left(a \cos^2 \theta + b \sin^2 \theta \right).$$

Since the conductivity tensor σ_{ij} is diagonal, the usual equation $J_i = \sigma_{ij} E_j$ can easily be inverted to read $E_i = \rho_{ij} J_j$ where ρ_{ij}, the 'resistance tensor', has the form

$$\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}.$$

The potential difference between the ends of the wire is given by

$$V = \mathbf{L} \cdot \mathbf{E} = L_i \rho_{ij} J_j,$$

where $\mathbf{L} = L(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. Now, although $\mathbf{E}$ is not necessarily parallel to the wire, the current density $\mathbf{J}$ must be and can be written

$$\mathbf{J} = \frac{I}{\pi r^2} (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta).$$

It follows that $V = \mathbf{L}^{\mathrm{T}}\rho\mathbf{J}$ can be expressed as

$$\frac{IL}{\pi r^2}(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)\begin{pmatrix} a^{-1} & 0 & 0 \\ 0 & a^{-1} & 0 \\ 0 & 0 & b^{-1} \end{pmatrix}\begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix},$$

and that the resistance of the wire is

$$R = \frac{V}{I} = \frac{L}{\pi r^2}\left(\frac{\sin^2\theta}{a} + \frac{\cos^2\theta}{b}\right) = \frac{L}{\pi r^2 ab}(b\sin^2\theta + a\cos^2\theta).$$

26.22 *For an isotropic elastic medium under dynamic stress, at time t the displacement u_i and the stress tensor p_{ij} satisfy*

$$p_{ij} = c_{ijkl}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \qquad \textit{and} \qquad \frac{\partial p_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2},$$

where c_{ijkl} is the isotropic tensor given in equation (26.47) and ρ is a constant. Show that both $\nabla\cdot\mathbf{u}$ and $\nabla\times\mathbf{u}$ satisfy wave equations and find the corresponding wave speeds.

Using the given equations and the form of the most general isotropic fourth-order tensor, we have

$$\begin{aligned} p_{ij} &= (\lambda\delta_{ij}\delta_{kl} + \eta\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk})\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right) \\ &= \lambda\delta_{ij}2\nabla\cdot\mathbf{u} + \eta\frac{\partial u_i}{\partial x_j} + \eta\frac{\partial u_j}{\partial x_i} + \nu\frac{\partial u_j}{\partial x_i} + \nu\frac{\partial u_i}{\partial x_j} \\ &= 2\lambda\delta_{ij}\nabla\cdot\mathbf{u} + (\eta+\nu)\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right). \end{aligned}$$

We now differentiate this equation with respect to x_j and sum over j. We also abbreviate $\eta + \nu$ to μ.

$$\begin{aligned} \frac{\partial p_{ij}}{\partial x_j} &= 2\lambda\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i}\right), \\ \Rightarrow \quad \rho\frac{\partial^2 u_i}{\partial t^2} &= 2\lambda\frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i} + \mu\left(\frac{\partial^2 u_i}{\partial x_j\partial x_j} + \frac{\partial(\nabla\cdot\mathbf{u})}{\partial x_i}\right). \end{aligned}$$

Next we differentiate this equation with respect to x_i and sum over i to obtain

$$\begin{aligned} \rho\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial t^2} &= 2\lambda\nabla^2(\nabla\cdot\mathbf{u}) + \mu[\nabla^2(\nabla\cdot\mathbf{u}) + \nabla^2(\nabla\cdot\mathbf{u})] \\ &= 2(\lambda+\mu)\nabla^2(\nabla\cdot\mathbf{u}). \end{aligned}$$

This is a wave equation for $(\nabla \cdot \mathbf{u})$ with wave speed $[2(\lambda+\eta+\nu)/\rho]^{1/2}$.

To find a similar equation for $\nabla \times \mathbf{u}$, we start from an expression for its ith component:

$$(\nabla \times \mathbf{u})_i = \epsilon_{ijk}\frac{\partial u_k}{\partial x_j},$$

$$\frac{\partial^2}{\partial t^2}(\nabla \times \mathbf{u})_i = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{\partial^2 u_k}{\partial t^2}\right) = \epsilon_{ijk}\frac{\partial}{\partial x_j}\left(\frac{1}{\rho}\frac{\partial p_{kl}}{\partial x_l}\right),$$

where the given time-dependent equation has been used to make the final step.

Now substituting the alternative expression for p_{ij} derived earlier, we have

$$\begin{aligned}
\rho\frac{\partial^2}{\partial t^2}(\nabla \times \mathbf{u})_i &= \epsilon_{ijk}\frac{\partial^2}{\partial x_j \partial x_l}\left[2\lambda\delta_{kl}\nabla\cdot\mathbf{u} + \mu\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right)\right] \\
&= \epsilon_{ijk}\frac{\partial^2}{\partial x_j \partial x_k}[2\lambda\nabla\cdot\mathbf{u}] + \mu\epsilon_{ijk}\frac{\partial^2}{\partial x_j \partial x_l}\left(\frac{\partial u_k}{\partial x_l}+\frac{\partial u_l}{\partial x_k}\right) \\
&= 0 + \mu\frac{\partial^2}{\partial x_l \partial x_l}\left(\epsilon_{ijk}\frac{\partial u_k}{\partial x_j}\right) + \mu\epsilon_{ijk}\frac{\partial^2(\nabla\cdot\mathbf{u})}{\partial x_j \partial x_k} \\
&= \mu\nabla^2(\nabla\times\mathbf{u})_i + 0.
\end{aligned}$$

This is a wave equation for the ith component of $(\nabla \times \mathbf{u})$ with a wave speed $[(\eta+\nu)/\rho]^{1/2}$ that is independent of i. To obtain the third and fourth lines of the final derivation we twice used the fact that, because ϵ_{ijk} is antisymmetric in j and k whilst $\partial^2/\partial x_j \partial x_k$ is symmetric in them, the contracted expression containing both is identically zero.

26.24 *Working in cylindrical polar coordinates ρ, ϕ, z, parameterise the straight line (geodesic) joining $(1,0,0)$ to $(1,\pi/2,1)$ in terms of s, the distance along the line. Show by substitution that the geodesic equations derived at the end of section 26.22 are satisfied.*

Clearly, the length of the line joining $(1,0,0)$ to $(1,\pi/2,1)$ in cylindrical polars, i.e. $(1,0,0)$ to $(0,1,1)$ in Caretesians, is $\sqrt{3}$. Points along the line are given in terms of s, the distance along the line, by

$$x = 1 - \frac{s}{\sqrt{3}}, \quad y = \frac{s}{\sqrt{3}}, \quad z = \frac{s}{\sqrt{3}}.$$

In cylindrical polars, z is as given and (defining the shorthand $(**)$ as in the first line below)

$$\rho = (x^2+y^2)^{1/2} = \frac{1}{\sqrt{3}}(3 - 2\sqrt{3}s + 2s^2)^{1/2} \equiv \frac{1}{\sqrt{3}}(**)^{1/2},$$

$$\phi = \tan^{-1}\frac{y}{x} = \tan^{-1}\frac{s}{\sqrt{3}-s}.$$

The required derivatives are

$$\begin{aligned}
\frac{d\rho}{ds} &= \frac{1}{2\sqrt{3}}\,\frac{4s-2\sqrt{3}}{(**)^{1/2}},\\
\frac{d^2\rho}{ds^2} &= \frac{1}{2\sqrt{3}}\,\frac{(**)^{1/2}(4)-2(2s-\sqrt{3})\frac{1}{2}(**)^{-1/2}(4s-2\sqrt{3})}{(**)}\\
&= \frac{1}{2\sqrt{3}}\,\frac{12-8\sqrt{3}s+8s^2-8s^2+8\sqrt{3}s-6}{(**)^{3/2}} = \frac{\sqrt{3}}{(**)^{3/2}},\\
\frac{d\phi}{ds} &= \frac{1}{1+\frac{s^2}{(\sqrt{3}-s)^2}}\,\frac{(\sqrt{3}-s)(1)-s(-1)}{(\sqrt{3}-s)^2} = \frac{\sqrt{3}}{(**)},\\
\frac{d^2\phi}{ds^2} &= \frac{-\sqrt{3}(4s-2\sqrt{3})}{(**)^2}.
\end{aligned}$$

Using these results, the first equation,

$$\frac{d^2\rho}{ds^2} - \rho\left(\frac{d\phi}{ds}\right)^2 = 0,$$

reads

$$\frac{\sqrt{3}}{(**)^{3/2}} - \frac{(**)^{1/2}}{\sqrt{3}}\,\frac{3}{(**)^2} = 0, \text{ which is satisfied.}$$

The second equation,

$$\frac{d^2\phi}{ds^2} + \frac{2}{\rho}\,\frac{d\rho}{ds}\,\frac{d\phi}{ds} = 0,$$

reads

$$\frac{-\sqrt{3}(4s-2\sqrt{3})}{(**)^2} + \frac{2\sqrt{3}}{(**)^{1/2}}\,\frac{(4s-2\sqrt{3})}{2\sqrt{3}(**)^{1/2}}\,\frac{\sqrt{3}}{(**)} = 0, \text{ which is also satisfied.}$$

The third equation $\dfrac{d^2z}{ds^2} = \dfrac{d^2}{ds^2}\left(\dfrac{s}{\sqrt{3}}\right) = 0$ is trivially satisfied, thus completing the verification.

26.26 *By writing down the expression for the square of the infinitesimal arc length* $(ds)^2$ *in spherical polar coordinates, find the components* g_{ij} *of the metric tensor in this coordinate system. Hence, using (26.97), find the expression for the divergence of a vector field* $\mathbf{v}$ *in spherical polars. Calculate the Christoffel symbols (of the second kind)* $\Gamma^i{}_{jk}$ *in this coordinate system.*

Since $(ds)^2 = (dr)^2 + r^2(d\theta)^2 + r^2 \sin^2\theta (d\phi)^2$, we have that

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}$$

and $g = |g_{ij}| = r^4 \sin^2\theta$. Further, for an orthogonal system, $g_{ij} = h_i \delta_{ij}$ with $g^{ij} = h_i^{-1}\delta_{ij}$.

For the divergence, as given in equation (26.97), we have

$$\begin{aligned}
\nabla \cdot \mathbf{v} = v^i_{;i} &= \frac{1}{\sqrt{g}} \frac{\partial}{\partial u^i}(\sqrt{g}\, v^i) \\
&= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial u^i}(r^2 \sin\theta\, v^i) \\
&= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial u^i}(r^2 \sin\theta\, g^{ik} v_k) \\
&= \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial u^i}[r^2 \sin\theta\, (v_i/h_i)], \text{ still summed over } i, \\
&= \frac{1}{r^2 \sin\theta} \left[\frac{\partial}{\partial r}(r^2 \sin\theta\, v_r) + \frac{\partial}{\partial \theta}(r \sin\theta\, v_\theta) + \frac{\partial}{\partial \phi}(r v_\phi) \right] \\
&= \frac{1}{r^2} \frac{\partial}{\partial r}(r^2 v_r) + \frac{1}{r \sin\theta} \frac{\partial}{\partial \theta}(\sin\theta\, v_\theta) + \frac{1}{r \sin\theta} \frac{\partial v_\phi}{\partial \phi},
\end{aligned}$$

which recovers the familiar form.

The Christoffel symbols of the second kind are calculated from

$$\Gamma^m_{ij} = \frac{1}{2} g^{mk} \left(\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} \right).$$

Because g^{mk} is diagonal, only terms with $m = k$ can contribute; $g_{11} = 1$, $g_{22} = r^2$ and $g_{33} = r^2 \sin^2\theta$. For each value of k we have $g^{kk} = (g_{kk})^{-1}$. Further, in the present case the only non-zero derivatives are

$$\frac{\partial g_{22}}{\partial u^1} = 2r, \quad \frac{\partial g_{33}}{\partial u^1} = 2r \sin^2\theta, \quad \frac{\partial g_{33}}{\partial u^2} = 2r^2 \sin\theta \cos\theta.$$

Using these expressions in the general formula gives, for those cases in which $i = j$,

$$\Gamma^m_{11} = \frac{1}{2} g^{mm} \left(\frac{\partial g_{1m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^m} \right) = 0, \text{ for all } m,$$

$$\begin{aligned}
\Gamma^m_{22} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^2} + \frac{\partial g_{m2}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^m} \right) \\
&= 0, \text{ for } m = 2, 3, \\
&= \tfrac{1}{2} g^{11} (0 + 0 - 2r) = -r, \text{ for } m = 1.
\end{aligned}$$

$$\begin{aligned}
\Gamma^m_{33} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{3m}}{\partial u^3} + \frac{\partial g_{m3}}{\partial u^3} - \frac{\partial g_{33}}{\partial u^m} \right) \\
&= 0, \text{ for } m = 3, \\
&= \tfrac{1}{2} g^{11} (0 + 0 - 2r \sin^2 \theta) = -r \sin^2 \theta, \text{ for } m = 1, \\
&= \tfrac{1}{2} g^{22} (0 + 0 - 2r^2 \sin\theta \cos\theta) = -\sin\theta \cos\theta, \text{ for } m = 2.
\end{aligned}$$

These account for 9 of the 27 possible Christoffel symbols. The other 18 are those in which i and j are different, but since the symbols are symmetric under i-j interchange, only 9 calculations are needed. They are as follows.

$$\begin{aligned}
\Gamma^m_{21} = \Gamma^m_{12} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^2} - \frac{\partial g_{12}}{\partial u^m} \right) \\
&= 0, \text{ for } m = 1, 3, \\
&= \tfrac{1}{2} g^{22} (2r + 0 - 0) = r^{-1}, \text{ for } m = 2,
\end{aligned}$$

$$\begin{aligned}
\Gamma^m_{31} = \Gamma^m_{13} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{3m}}{\partial u^1} + \frac{\partial g_{m1}}{\partial u^3} - \frac{\partial g_{13}}{\partial u^m} \right) \\
&= 0, \text{ for } m = 1, 2, \\
&= \tfrac{1}{2} g^{33} (2r \sin^2 \theta + 0 - 0) = r^{-1}, \text{ for } m = 3,
\end{aligned}$$

$$\begin{aligned}
\Gamma^m_{23} = \Gamma^m_{32} &= \frac{1}{2} g^{mm} \left(\frac{\partial g_{2m}}{\partial u^3} + \frac{\partial g_{m3}}{\partial u^2} - \frac{\partial g_{23}}{\partial u^m} \right) \\
&= 0, \text{ for } m = 1, 2, \\
&= \tfrac{1}{2} g^{33} (0 + 2r^2 \sin\theta \cos\theta - 0) = \cot\theta, \text{ for } m = 3.
\end{aligned}$$

This completes the calculation of all 27 Christoffel symbols for this coordinate system; only 9 of them are non-zero.

26.28 *A curve* $\mathbf{r}(t)$ *is parameterised by a scalar variable* t*. Show that the length of the curve between two points,* A *and* B*, is given by*

$$L = \int_A^B \sqrt{g_{ij}\frac{du^i}{dt}\frac{du^j}{dt}}\,dt.$$

Using the calculus of variations (see chapter 22), show that the curve $\mathbf{r}(t)$ *that minimises* L *satisfies the equation*

$$\frac{d^2u^i}{dt^2} + \Gamma^i{}_{jk}\frac{du^j}{dt}\frac{du^k}{dt} = \frac{\ddot{s}}{\dot{s}}\frac{du^i}{dt},$$

where s *is the arc length along the curve,* $\dot{s} = ds/dt$ *and* $\ddot{s} = d^2s/dt^2$*. Hence, show that if the parameter* t *is of the form* $t = as + b$*, where* a *and* b *are constants, then we recover the equation for a geodesic (26.101).*

[*A parameter which, like* t*, is the sum of a linear transformation of* s *and a translation is called an* affine *parameter.*]

Denoting derivatives with respect to t by a dot notation, the element of curve length is given by

$$(ds)^2 = g_{ij}\,du^i\,du^j = g_{ij}\,\dot{u}^i\,dt\,\dot{u}^j\,dt \quad\Rightarrow\quad \dot{s}^2 = g_{ij}\,\dot{u}^i\dot{u}^j.$$

Thus

$$L = \int_A^B ds = \int_A^B \dot{s}\,dt = \int_A^B (g_{ij}\,\dot{u}^i\dot{u}^j)^{1/2}\,dt \equiv \int_A^B F(u^i, \dot{u}^i, t)\,dt.$$

The Euler–Lagrange equation for minimising L is

$$\begin{aligned}
0 &= \frac{d}{dt}\left(\frac{\partial F}{\partial \dot{u}^k}\right) - \frac{\partial F}{\partial u^k}\\
&= \frac{d}{dt}\left(\frac{1}{2}\frac{2g_{ik}\,\dot{u}^i}{\dot{s}}\right) - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\frac{\partial g_{ij}}{\partial u^k}\\
&= -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i}{\dot{s}}\frac{dg_{ik}}{dt} - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\frac{\partial g_{ij}}{\partial u^k}\\
&= -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i}{\dot{s}}\frac{\partial g_{ik}}{\partial u^m}\dot{u}^m - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\frac{\partial g_{ij}}{\partial u^k}.
\end{aligned}$$

Now, in the double summation over dummy variables i and m in the third term on the RHS, we can set

$$\dot{u}^i\dot{u}^m\frac{\partial g_{ik}}{\partial u^m} = \dot{u}^i\dot{u}^m\frac{\partial g_{mk}}{\partial u^i}$$

and re-write the Euler–Lagrange equation as

$$0 = -\frac{\ddot{s}}{\dot{s}^2}\,g_{ik}\dot{u}^i + \frac{g_{ik}\,\ddot{u}^i}{\dot{s}} + \frac{\dot{u}^i\dot{u}^m}{\dot{s}}\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial u^m} + \frac{\partial g_{mk}}{\partial u^i}\right) - \frac{\dot{u}^i\,\dot{u}^j}{2\dot{s}}\frac{\partial g_{ij}}{\partial u^k}.$$

Next we multiply throughout by g^{lk} and contract over k to obtain

$$\begin{aligned} 0 &= -\frac{\ddot{s}}{\dot{s}}\,\delta_i^l \dot{u}^i + \delta_i^l\,\ddot{u}^i + \frac{1}{2}\dot{u}^i\dot{u}^j g^{lk}\left(\frac{\partial g_{ik}}{\partial u^j} + \frac{\partial g_{jk}}{\partial u^i} - \frac{\partial g_{ij}}{\partial u^k}\right) \\ &= -\frac{\ddot{s}}{\dot{s}}\,\dot{u}^l + \ddot{u}^l + \Gamma^l{}_{ij}\dot{u}^i\dot{u}^j, \end{aligned}$$

thus establishing the stated result.

If $t = as + b$, then $\dot{s} = a^{-1}$ and $\ddot{s} = 0$; further $\dfrac{d}{dt} = \dfrac{1}{a}\dfrac{d}{ds}$. With this substitution the minimising equation becomes

$$\frac{1}{a^2}\,\frac{d^2u^l}{ds^2} + \Gamma^l{}_{ij}\frac{1}{a}\,\frac{du^i}{ds}\,\frac{1}{a}\,\frac{du^j}{ds} = 0.$$

This is the equation for a geodesic as given in (26.101).

27

Numerical methods

27.2 *Using the Newton–Raphson procedure find, correct to three decimal places, the root nearest to 7 of the equation* $4x^3 + 2x^2 - 200x - 50 = 0$.

The Newton–Raphson scheme has 2nd-order convergence and so we expect rapid convergence if a reasonable first guess is made. The iteration scheme is

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} \\ &= x_n - \frac{4x_n^3 + 2x_n^2 - 200x_n - 50}{12x_n^2 + 4x_n - 200} \\ &= \frac{8x_n^3 + 2x_n^2 + 50}{12x_n^2 + 4x_n - 200}. \end{aligned}$$

Starting with $x_1 = 7$, $x_2 = 6.951923077$ and $x_3 = x_4 = \cdots = 6.951436674$ to 10 s.f. To 3 decimal places $x = 6.951$, a result achieved (but not verified) after only one iteration; as can be seen, a result accurate to 9 decimal places is achieved after only two iterations.

27.4 *The square root of a number* N *is to be determined by means of the iteration scheme*

$$x_{n+1} = x_n \left[1 - \left(N - x_n^2\right) f(N)\right].$$

Determine how to choose $f(N)$ *so that the process has second-order convergence. Given that* $\sqrt{7} \approx 2.65$*, calculate* $\sqrt{7}$ *as accurately as a single application of the formula will allow.*

Writing the iteration scheme in the standard form $x_{n+1} = F(x_n)$, we see that $F(x) = x - x(N - x^2)f(N)$. For second-order convergence we require that $F'(\sqrt{N}) = 0$. Now

$$\begin{aligned} F'(x) &= 1 - (N - x^2)f(N) + 2x^2 f(N), \\ F'(\sqrt{N}) = 0 &\Rightarrow f(N) = -\frac{1}{2N}, \end{aligned}$$

i.e an iteration scheme that has second-order convergence is

$$x_{n+1} = x_n \left(1 + \frac{N - x_n^2}{2N}\right).$$

With $x_1 = 2.65$ as a first approximation to $\sqrt{7}$, the second approximation is

$$x_2 = 2.65 \left(1 + \frac{7 - (2.65)^2}{14}\right) = 2.645741071.$$

To the same 10-figure accuracy, the correct answer is 2.645751311.

27.6 *The following table of values of a polynomial $p(x)$ of low degree contains an error. Identify and correct the erroneous value and extend the table up to $x = 1.2$.*

x	$p(x)$	x	$p(x)$
0.0	*0.000*	*0.5*	*0.165*
0.1	*0.011*	*0.6*	*0.216*
0.2	*0.040*	*0.7*	*0.245*
0.3	*0.081*	*0.8*	*0.256*
0.4	*0.128*	*0.9*	*0.243*

Since the function is a polynomial of low degree, we expect the n-th difference of the entries to be constant, where n is a relatively low number. To test for this we set out (in two overlapping parts) a table of calculable differences (working in units of 0.001) as follows:

x	0.0		0.1		0.2		0.3		0.4		0.5
$p(x)$	0		11		40		81		128		165
1st diff		11		29		41		47		37	
2nd diff			18		12		6		−10		14
3rd diff				−6		−6		−16		24	

x	0.5		0.6		0.7		0.8		0.9
$p(x)$	165		216		245		256		243
1st diff		51		29		11		−13	
2nd diff	14		−22		−18		−24		
3rd diff		−36		4		−6			

Several of the 3rd differences are equal at −6, suggesting that the others should have been but for the error in the given table. To test this we construct an inverted table assuming that all third differences are in fact −6.

x	0.0		0.1		0.2		0.3		0.4		0.5
3rd diff				−6		−6		−6		−6	
2nd diff			18		12		6		0		−6
1st diff		11		29		41		47		47	
$p(x)$	0		11		40		81		128		175

x	0.5		0.6		0.7		0.8		0.9
3rd diff		−6		−6		−6			
2nd diff	−6		−12		−18		−24		
1st diff		41		29		11		−13	
$p(x)$	175		216		245		256		243

This has reconstructed the original table except for the value of $p(0.5)$ which is now 0.175, rather than the erroneous value of 0.165.

We continue the table up to $x = 1.2$ by starting from $x = 0.8$, where all values and differences are known.

x	0.8		0.9		1.0		1.1		1.2
3rd diff		−6		−6		−6		−6	
2nd diff	−24		−30		−36		−42		
1st diff		−13		−43		−79		−121	
$p(x)$	256		243		200		121		0

The corrected and extended table now reads

x	$p(x)$	x	$p(x)$
0.0	0.000	0.7	0.245
0.1	0.011	0.8	0.256
0.2	0.040	0.9	0.243
0.3	0.081	1.0	0.200
0.4	0.128	1.1	0.121
0.5	0.175	1.2	0.000
0.6	0.216		

27.8 *A possible rule for obtaining an approximation to an integral is the* mid-point rule*, given by*

$$\int_{x_0}^{x_0+\Delta x} f(x)\,dx = \Delta x\, f(x_0 + \tfrac{1}{2}\Delta x) + \mathrm{O}(\Delta x^3).$$

Writing h for Δx, and evaluating all derivates at the mid-point of the interval $(x, x+\Delta x)$, use a Taylor series expansion to find, up to $O(h^5)$, the coefficients of the higher-order errors in both the trapezium and mid-point rules. Hence find a linear combination of these two rules that gives $O(h^5)$ accuracy for each step Δx.

With all derivatives evaluated at $x + \frac{1}{2}h$ the Taylor series up to terms in h^5 for $f_0 = f(x)$ and $f_1 = f(x+h)$ are

$$f_{1,0} = f \pm \frac{h}{2}f' + \frac{1}{2!}\left(\frac{h}{2}\right)^2 f'' \pm \frac{1}{3!}\left(\frac{h}{2}\right)^3 f^{(3)} + \frac{1}{4!}\left(\frac{h}{2}\right)^4 f^{(4)} \pm \frac{1}{5!}\left(\frac{h}{2}\right)^5 f^{(5)},$$

with the upper sign corresponding to f_1 and the lower to f_0.

By definition, the midpoint rule gives

$$I_{\mathrm{mid}} = hf.$$

as the integral for the interval $(x, x+h)$. For the same interval the trapezium rule is evaluated as

$$I_{\mathrm{trap}} = \frac{h}{2}(f_0 + f_1) = h\left[f + \frac{1}{2!}\frac{h^2}{4}f'' + \frac{1}{4!}\frac{h^4}{16}f^{(4)} + \mathrm{O}(h^6)\right].$$

The exact integral over the same interval is

$$\begin{aligned} I_{\mathrm{ex}} &= \int_{-h/2}^{h/2}\left(f + yf' + \frac{y^2}{2!}f'' + \frac{y^3}{3!}f^{(3)} + \frac{y^4}{4!}f^{(4)} + \frac{y^5}{5!}f^{(5)} + \cdots\right)dy \\ &= hf + \frac{2}{2!}\frac{1}{3}\frac{h^3}{8}f'' + \frac{2}{4!}\frac{1}{5}\frac{h^5}{32}f^{(4)} + \cdots. \end{aligned}$$

Thus, to $\mathrm{O}(h^5)$,

$$\begin{aligned} I_{\mathrm{ex}} &= hf + \frac{1}{24}h^3 f'', \\ I_{\mathrm{mid}} &= hf, \\ I_{\mathrm{trap}} &= hf + \frac{1}{8}h^3 f'', \end{aligned}$$

and the best linear combination of the trapezium and mid-point rules that approximates the exact result to this order is $I = \frac{1}{3}I_{\mathrm{trap}} + \frac{2}{3}I_{\mathrm{mid}}$.

27.10 *Using the points and weights given in table 27.9, answer the following questions.*

(a) *A table of unnormalised Hermite polynomials $H_n(x)$ has been spattered with ink blots and gives $H_5(x)$ as $32x^5 - ?x^3 + 120x$ and $H_4(x)$ as $?x^4 - ?x^2 + 12$, where the coefficients marked ? cannot be read. What should they read?*

(b) *What is the value of the integral*

$$I = \int_{-\infty}^{\infty} \frac{e^{-2x^2}}{1 + 4x^2 + 3x}\, dx,$$

as given by a 7-point integration routine?

(a) Since the integration points for an n-point Gauss–Hermite integration are those values of x that make $H_n(x) = 0$, the given sampling points for a 5-point routine are those that give the expression $32x^5 - ax^3 + 120x$ zero value. Thus

$$a = \frac{32x^5 + 120x}{x^3}, \text{ where } x \text{ is either of } 0.95857\ldots \text{ and } 2.0218\ldots.$$

As they must, both cases give the same value; that value is 160.

Let $H_4(x) = bx^4 - cx^2 + 12$. Then, since all four sampling points $\pm x_i$ for a 4-point Gauss–Hermite scheme must satisfy $H_4(x_i) = 0$, we can write

$$\begin{aligned} bx_1^4 - cx_1^2 + 12 &= 0, \\ bx_2^4 - cx_2^2 + 12 &= 0, \\ \Rightarrow \quad bx_1^4x_2^2 - cx_1^2x_2^2 + 12x_2^2 &= 0, \\ \text{and} \quad bx_2^4x_1^2 - cx_2^2x_1^2 + 12x_1^2 &= 0, \\ \Rightarrow \quad bx_1^2x_2^2(x_1^2 - x_2^2) + 12(x_2^2 - x_1^2) &= 0, \end{aligned}$$

$$\Rightarrow \quad b = \frac{12}{x_1^2x_2^2}, \text{ and } c = \frac{bx_i^4 + 12}{x_i^2}$$

for both $i = 1$ and $i = 2$. The first equation gives $b = 16$ and the second pair both yield $c = 48$.

(b) The denominator of the integrand is a quadratic form with '$4ac > b^2$'. It therefore has no real zeroes and we may use Gauss–Hermite integration. To cast the exponential in the appropriate form, we need to make the change of variable $y = \sqrt{2}\,x$; the exponential then has the form e^{-y^2}, as assumed in the quadrature formula. The integral becomes

$$I = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{2y^2 + (3/\sqrt{2})y + 1}\, \frac{dy}{\sqrt{2}} = \int_{-\infty}^{\infty} \frac{e^{-y^2}}{2\sqrt{2}y^2 + 3y + \sqrt{2}}\, dy.$$

This is now in the form $\int_{-\infty}^{\infty} e^{-y^2} g(y)\,dy$ to which the Gauss-Hermite procedure can be applied directly. Using the points and weights for a 7-point calculation gives a value of 1.1642.

27.12 *In normal use only a single application of n-point Gaussian quadrature is made, using a value of n that is estimated from experience to be 'safe'. However, it is instructive to examine what happens when n is changed in a controlled way.*

(a) *Evaluate the integral*

$$I_n = \int_2^5 \sqrt{7x - x^2 - 10}\,dx$$

using n-point Gauss–Legendre formulae for $n = 2, 3, \ldots, 6$. *Estimate (to* 4 *s.f.) the value* I_∞ *you would obtain for very large n and compare it with the result I obtained by exact integration. Explain why the variation of* I_n *with n is monotonically decreasing.*

(b) *Try to repeat the processes described in* (a) *for the integrals*

$$J_n = \int_2^5 \frac{1}{\sqrt{7x - x^2 - 10}}\,dx.$$

Why is it very difficult to estimate J_∞?

(a) Since the integral is not over the (finite) range $-1 \le x \le 1$, we must first make the transformation

$$z = \frac{2x - 5 - 2}{5 - 2} \qquad \Rightarrow \qquad x = \frac{7}{2} + \frac{3z}{2}.$$

This results in the integral being

$$I = \tfrac{5-2}{2} \int_{-1}^{1} g(z)\,dz$$

with $g(z) = f(x)$.

Using the tables of points and weights given in the text, we obtain the following results as n is varied.

n	2	3	4	5	6
I_n	3.674	3.581	3.556	3.546	3.541

Clearly, a constant value good to 4 s.f. has not been achieved but, either by rough plotting (preferably *versus* an inverse power of n) or by extrapolating the rate of change of the last two significant figures, we can estimate I_∞ as lying in the range 3.533 to 3.535.

To calculate the exact result, we note that $7x - x^2 - 10$ can be written (by 'completing the square') as $\frac{9}{4} - (x - \frac{7}{2})^2$ and so the substitution $x = \frac{7}{2} + \frac{3}{2}\sin\theta$ gives

$$\begin{aligned} I &= \int_{-\pi/2}^{\pi/2} \tfrac{3}{2}\sqrt{1-\sin^2\theta}\,\tfrac{3}{2}\cos\theta\,d\theta \\ &= \tfrac{9}{4}\int_{-\pi/2}^{\pi/2} \cos^2\theta\,d\theta = \tfrac{9}{4}\,\tfrac{1}{2}\,\pi = 3.5343 \end{aligned}$$

This does come within our estimated range, though, obviously, we could not be more certain of the value at a level better than ± 0.001 without using much higher values of n. [$n = 20$ gives $I_{20} = 3.5345$.]

The monotonic behaviour of I_n with n comes about because the integrand starts from zero, rises to a single maximum, and falls to zero again as x nears the end of its range. This, coupled with the fact that as n increases the sampling points are continually pushed further outwards from the middle of the range, means that smaller actual values of the integrand gain increasing weight in the sum, thus lowering it. Consequently, the series I_n approaches I monotonically from above.

(b) The procedure is exactly the same as in part (a); only the form of the integrand is different. The results are:

n	2	3	4	5	6	20
J_n	2.449	2.646	2.755	2.825	2.874	3.057

This time the results are monotonically increasing. This is no surprise, as the integrand now increases at the extremes of its range; indeed, it has an (integrable) singularity at each end point. These do not make the Gauss–Legendre sum infinite for any finite n since the sample points never include the end points of the range. However, they do prevent the estimates from converging rapidly for small values of n, thus making it virtually impossible to extrapolate to J_∞.

The accurate value of J is obtained by making the same substitution as in part (a) and produces

$$J = \int_{-\pi/2}^{\pi/2} \frac{\frac{3}{2}\cos\theta}{\frac{3}{2}\sqrt{1-\sin^2\theta}}\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

Thus we see that, even with n as high as 20, the second significant figure of the value of J is not yet established with any certainty.

Figure 27.1 The area to be estimated using the hit or miss Monte Carlo method of exercise 27.14.

27.14 *A, B and C are three circles of unit radius with centres in the xy-plane at* $(1, 2), (2.5, 1.5)$ *and* $(2, 3)$*, respectively. Devise a hit or miss Monte Carlo calculation to determine the size of the area that lies outside C but inside A and B, as well as inside the square centred on* $(2, 2.5)$ *that has sides of length 2 parallel to the coordinate axes. You should choose your sampling region so as to make the estimation as efficient as possible. Take the random number distribution to be uniform on* $(0, 1)$ *and determine the inequalities that have to be tested using the random numbers chosen.*

Figure 27.1 shows the three circles and the square, with the area whose size has to be determined shown shaded. The same shaded area is repeated in the top right-hand corner of the figure, but on a larger scale and labelled $PQRS$.

In order to make the estimation as efficient as possible, we sample in the minimal rectangular region that encloses $PQRS$. We must therefore find the coordinates of these 'corners'. Those of P are determined by the circle B and the square as $P = (2.5 - 1, 2.5 - 1) = (1.5, 1.5)$. The x-coordinate of Q is determined by the circle A as $1 + \sqrt{1 - (2 - 1.5)^2} = 1 + \sqrt{3}/2$; its y-coordinate is 1.5. It is therefore $(1 + \sqrt{3}/2, 1.5)$. The circles A and C determine R as $(2, 3 - 1) = (2, 2)$.

The coordinates of S are a little more difficult to find; they are determined by

the circles B and C as follows:

$$\begin{aligned}
(x-2.5)^2+(y-1.5)^2 &= 1,\\
(x-2)^2+(y-3)^2 &= 1,\\
-5x+4x+6.25-4-3y+6y+2.25-9 &= 0, \text{ by subtraction},\\
-x+3y-4.5 &= 0, \text{ substitute this into } C,\\
(3y-6.5)^2+(y-3)^2 &= 1,\\
10y^2-45y+50.25 &= 0,\\
y = \frac{45\pm\sqrt{2025-2010}}{20} &= 2.056 \quad (2.444, \text{ rejected}),\\
x = (3\times 2.056)-4.5 &= 1.669.
\end{aligned}$$

The minimal enclosing rectangle is thus $1.5 \le x \le 2$, $1.5 \le y \le 2.056$. We note that it is entirely contained within the square and so no subsequent test of individual sample points will be needed in this regard.

The sampling scheme is to select a pair of random numbers, (ξ_1, ξ_2), and then set $x = 1.5 + \alpha\xi_1$ and $y = 1.5 + \beta\xi_2$. The optimal values for α and β are 0.5 and 0.556 respectively. These values of x and y are then tested to see if they lie inside/outside the defining circles, using the inequalities:

$$\begin{aligned}
(\alpha\xi_1+1.5-1)^2+(\beta\xi_2+1.5-2)^2 &\le 1,\\
(\alpha\xi_1+1.5-2.5)^2+(\beta\xi_2+1.5-1.5)^2 &\le 1,\\
(\alpha\xi_1+1.5-2)^2+(\beta\xi_2+1.5-3)^2 &\ge 1.
\end{aligned}$$

If all three conditions are satisfied for n out of N pairs of randomly chosen numbers, (ξ_1, ξ_2), the area of $PQRS$ can be estimated as $n\alpha\beta/N$.

27.16 *Consider the application of the predictor–corrector method described near the end of subsection 27.6.3 to the equation*

$$\frac{dy}{dx} = x + y, \qquad y(0) = 0.$$

Show, by comparison with a Taylor series expansion, that the expression obtained for y_{i+1} in terms of x_i and y_i by applying the three steps indicated (without any repeat of the last two) is correct to $O(h^2)$. Using steps of $h = 0.1$, compute the value of $y(0.3)$ and compare it with the value obtained by solving the equation analytically.

Since $y' = x + y$, $y'' = 1 + y' = 1 + x + y$, the first few terms of the Taylor series

expansion of $y(x)$ are

$$\begin{aligned} y(x+h) &= y(x) + hy'(x) + \tfrac{1}{2}h^2 y''(x) + \cdots \\ &= y(x) + h(x+y) + \tfrac{1}{2}h^2(1+x+y) + \cdots . \end{aligned}$$

We now consider the predictor–corrector method and indicate intermediate (predicted) values by placing a bar over them. As in the main text, we denote the function of x and y (here $x+y$) that determines the derivative dy/dx by $f(x, y)$.

$$\begin{aligned} \overline{y_{i+1}} &= y_i + hf_i, \\ \overline{f_{i+1}} &= x_{i+1} + \overline{y_{i+1}} = x_{i+1} + y_i + h(x_i + y_i). \\ \text{Now, using } y_{i+1} &= y_i + \tfrac{1}{2}h(f_i + \overline{f_{i+1}}), \\ y_{i+1} &= y_i + \tfrac{1}{2}h[x_i + y_i + x_{i+1} + y_i + h(x_i + y_i)] \\ &= y_i + \tfrac{1}{2}h(x_i + x_{i+1}) + hy_i + \tfrac{1}{2}h^2(x_i + y_i) \\ \text{Now, since } x_{i+1} &= x_i + h, \\ y_{i+1} &= y_i + hx_i + hy_i + \tfrac{1}{2}h^2(1 + x_i + y_i). \end{aligned}$$

This coincides with the Taylor series expansion up to $O(h^2)$ and proves the stated result.

We calculate the required value of $y(0.3)$ in the following table (in which each column is completed before moving to the next and the last entry in each column becomes the third entry in the next):

i	0	1	2
x_i	0.0	0.1	0.2
y_i	0.00000	0.00500	0.02103
$h(x_i + y_i)$	0.00000	0.01050	0.02210
$\frac{1}{2}h^2(1 + x_i + y_i)$	0.00500	0.00553	0.00611
y_{i+1}	0.00500	0.02103	0.04924

The calculated value of $y(0.3)$ is thus 0.04924.

For the marginally reorganised initial equation

$$\frac{dy}{dx} - y = x,$$

we see by inspection that the CF is $y(x) = Ae^x$ and that a PI is $y(x) = -x - 1$. The given boundary condition, $y(0) = 0$, implies that $A = 1$ and that the exact solution is $y(x) = e^x - x - 1$. The correct value of $y(0.3)$ to 4 s.f. is therefore 0.04986.

27.18 *If $dy/dx = f(x, y)$ then show that*

$$\frac{d^2f}{dx^2} = \frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + f^2\frac{\partial^2 f}{\partial y^2} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f\left(\frac{\partial f}{\partial y}\right)^2.$$

Hence verify, by substitution and the subsequent expansion of arguments in Taylor series of their own, that the scheme given in (27.79) coincides with the Taylor expansion (27.68), i.e.

$$y_{i+1} = y_i + hy_i^{(1)} + \frac{h^2}{2!}y_i^{(2)} + \frac{h^3}{3!}y_i^{(3)} + \cdots.$$

up to terms in h^3.

The scheme (a third-order Runge–Kutta calculation) is

$$y_{i+1} = y_i + \tfrac{1}{6}(b_1 + 4b_2 + b_3),$$

where

$$\begin{aligned} b_1 &= hf(x_i, y_i), \\ b_2 &= hf(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}b_1), \\ b_3 &= hf(x_i + h,\ y_i + 2b_2 - b_1), \end{aligned}$$

To find the first and second total derivatives of $f(x, y)$ with respect to x we use the chain rule:

$$\begin{aligned} \frac{df}{dx} &= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} \\ &= \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}, \\ \frac{d^2f}{dx^2} &= \left(\frac{\partial}{\partial x} + f\frac{\partial}{\partial y}\right)\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &= \frac{\partial^2 f}{\partial x^2} + f\frac{\partial^2 f}{\partial y \partial x} + f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial y}\right)^2, \end{aligned}$$

i.e. as stated.

The (accurate) Taylor expansion is

$$\begin{aligned} y_{i+1} &= y_i + hy_i^{(1)} + \frac{h^2}{2!}y_i^{(2)} + \frac{h^3}{3!}y_i^{(3)} + \cdots \qquad (*) \\ &= y_i + hf_i + \frac{h^2}{2!}\left(\frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right) \\ &\quad + \frac{h^3}{3!}\left[\frac{\partial^2 f}{\partial x^2} + 2f\frac{\partial^2 f}{\partial x \partial y} + \frac{\partial f}{\partial x}\frac{\partial f}{\partial y} + f^2\frac{\partial^2 f}{\partial y^2} + f\left(\frac{\partial f}{\partial y}\right)^2\right] + \cdots \end{aligned}$$

We next need to find explicit expressions for the quantities b_1, b_2 and b_3 in terms of f and its various derivatives. The first is simple, $b_1 = hf_i$. For the other two, with f evaluated at points other than (x_i, y_i), we expand in local Taylor series, retaining only those terms that will be of order h^3 or less in the final expression. For b_2:

$$\begin{aligned} b_2 &= hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hf_i) \\ &= hf_i + h^2\left(\frac{1}{2}\frac{\partial f_i}{\partial x} + \frac{1}{2}f_i\frac{\partial f_i}{\partial y}\right) \\ &\quad + h^3\left(\frac{1}{2}\frac{1}{4}\frac{\partial^2 f_i}{\partial x^2} + \frac{1}{2}\frac{1}{4}\frac{\partial^2 f_i}{\partial y^2}f_i^2 + \frac{1}{2}\frac{1}{2}\frac{1}{2}2\frac{\partial^2 f_i}{\partial x\partial y}f_i\right). \end{aligned}$$

For b_3, still working to order h^3:

$$\begin{aligned} b_3 &= h\,f(\,x_i + h,\ y_i + 2hf(x_i + \tfrac{1}{2}h, y_i + \tfrac{1}{2}hf_i) - hf_i\,) \\ &= h\,f\left(x_i + h,\ y_i + 2h\left[f_i + \frac{1}{2}h\frac{\partial f_i}{\partial x} + \frac{1}{2}hf_i\frac{\partial f_i}{\partial y}\right] - hf_i\right) \\ &= h\,f\left(x_i + h,\ y_i + hf_i + h^2\frac{\partial f_i}{\partial x} + h^2 f_i\frac{\partial f_i}{\partial y} + \mathrm{O}(h^3)\right). \end{aligned}$$

We now need a two-variable Taylor expansion of this last function about (x_i, y_i). The leading term is clearly hf_i and the contributions from partial derivatives with respect to x only are

$$h^2\frac{\partial f_i}{\partial x} + \frac{1}{2}h^3\frac{\partial^2 f_i}{\partial x^2} + \mathrm{O}(h^4).$$

The contributions from partial derivatives with respect to y only are

$$h\left[hf_i + h^2\left(\frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right)\right]\frac{\partial f_i}{\partial y} + \mathrm{O}(h^4) + \frac{1}{2}h(hf_i)^2\frac{\partial^2 f_i}{\partial y^2} + \mathrm{O}(h^4),$$

whilst, to order h^3, the only contribution from the mixed derivatives is

$$h\,h\,hf_i\frac{\partial^2 f_i}{\partial x\partial y}.$$

Collecting these together gives b_3 as

$$\begin{aligned} b_3 &= hf_i + h^2\left(\frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right) \\ &\quad + h^3\left[\frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial f_i}{\partial x}\frac{\partial f_i}{\partial y} + f_i\left(\frac{\partial f_i}{\partial y}\right)^2 + \frac{1}{2}f_i^2\frac{\partial^2 f_i}{\partial y^2} + f_i\frac{\partial^2 f_i}{\partial x\partial y}\right]. \end{aligned}$$

Finally we must form the sum $\frac{1}{6}(b_1 + 4b_2 + b_3)$ and check it against the accurate Taylor expansion for $y_{i+1} - y_i$. The collected multipliers of the three powers of h

are:

$$h: \ \frac{1}{6}(f_i + 4f_i + f_i) = f_i,$$

$$h^2: \ \frac{1}{6}\left(\frac{4}{2}\frac{\partial f_i}{\partial x} + \frac{4}{2}f_i\frac{\partial f_i}{\partial y} + \frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right) = \frac{1}{2}\left(\frac{\partial f_i}{\partial x} + f_i\frac{\partial f_i}{\partial y}\right),$$

$$h^3: \ \frac{1}{6}\left[\frac{4}{8}\frac{\partial^2 f_i}{\partial x^2} + \frac{4}{8}\frac{\partial^2 f_i}{\partial y^2}f_i^2 + \frac{4}{4}\frac{\partial^2 f_i}{\partial x \partial y}f_i + \frac{1}{2}\frac{\partial^2 f_i}{\partial x^2} + \frac{\partial f_i}{\partial x}\frac{\partial f_i}{\partial y} + f_i\left(\frac{\partial f_i}{\partial y}\right)^2 + \frac{1}{2}f_i^2\frac{\partial^2 f_i}{\partial y^2} + f_i\frac{\partial^2 f_i}{\partial x \partial y}\right].$$

These can now be compared with the expression given in the last line of (*). The multipliers of h and h^2 have explicitly been shown to be equal and those of h^3 are also shown to be so when the final expression above is simplified; this establishes the validity of this third-order Runge–Kutta scheme.

27.20 *Set up a finite difference scheme to solve the ordinary differential equation*

$$x\frac{d^2\phi}{dx^2} + \frac{d\phi}{dx} = 0$$

in the range $1 \le x \le 4$ *and subject to the boundary conditions* $\phi(1) = 2$ *and* $d\phi/dx = 2$ *at* $x = 4$. *Using* N *equal increments,* Δx, *in* x, *obtain the general difference equation and state how the boundary conditions are incorporated into the scheme. Setting* Δx *equal to the (crude) value* 1, *obtain the relevant simultaneous equations and so obtain rough estimates for* $\phi(2), \phi(3)$ *and* $\phi(4)$.

Finally, solve the original equation analytically and compare your numerical estimates with the accurate values.

We will use central differences for both derivatives except at $x = 4$ where the backward difference is used to fit the boundary condition on $d\phi/dx$.

Defining Δx by $N\Delta x = 3$ and denoting $\phi(1 + j\Delta x)$ by ϕ_j, we have as boundary conditions, $\phi_0 = 2$ and $\phi_N = \phi_{N-1} + 2\Delta x$. We thus have to calculate ϕ_j for $j = 1, 2, \ldots, N-2$ using a difference representation of the differential equation taking the form

$$(1 + j\Delta x)\frac{\phi_{j+1} + \phi_{j-1} - 2\phi_j}{(\Delta x)^2} + \frac{\phi_{j+1} - \phi_{j-1}}{2\Delta x} = 0,$$

which can be re-arranged as

$$[2 + (2j+1)\Delta x]\phi_{j+1} - (4 + 4j\Delta x)\phi_j + [2 + (2j-1)\Delta x]\phi_{j-1} = 0.$$

Now setting $\Delta x = 1$ (and consequently $N = 3$) we have, for $j = 1, 2$, in turn,

$$5\phi_2 - 8\phi_1 + 3 \times 2 = 5\phi_2 - 8\phi_1 + 6 = 0,$$

$$7(\phi_2 + 2) - 12\phi_2 + 5\phi_1 = -5\phi_2 + 5\phi_1 + 14 = 0.$$

Solving this pair of simultaneous equations gives

$$\phi_1 = \frac{20}{3} = 6.67 \quad \text{and} \quad \phi_2 = \frac{142}{15} = 9.47.$$

Re-substitution of these values in the recurrence relation then gives the value of ϕ at the upper boundary, where only its slope has been specified, as

$$\phi_3 = 11.47.$$

For an analytical solution we write the equation as

$$\frac{\phi''}{\phi'} + \frac{1}{x} = 0 \quad \Rightarrow \quad \phi' x = k = 2 \times 4 = 8, \text{ using } \phi'(4) = 2,$$

$$\frac{d\phi}{dx} = \frac{8}{x} \quad \Rightarrow \quad \phi = c + 8 \ln x = 2 + 8 \ln x.$$

The boundary condition at $x = 4$ has already been incorporated into the first line above and that at $x = 1$ is used in the second. The accurately calculated values are therefore

$$\phi(2) = \phi_1 = 7.55, \quad \phi(3) = \phi_2 = 10.79, \quad \phi(4) = \phi_3 = 13.09.$$

The estimated values follow the same trend as the accurate ones but are consistently lower (except at $x = 1$ where they are forced to be equal). The major source of inaccuracy arises from forcing the difference between the estimated values of $\phi(4) = \phi_3$ and $\phi(3) = \phi_2$ to be 2; the accurate solution has an average slope in this range of $8 \ln(4/3) = 2.30$, i.e. significantly higher than 2.

27.22 *Use the isocline approach to sketch the family of curves that satisfies the non-linear first-order differential equation*

$$\frac{dy}{dx} = \frac{a}{\sqrt{x^2 + y^2}}.$$

At each point in the xy-plane, the equation determines the slope of the solution. A solution curve must therefore pass through a (continuous) series of points, at each of which its tangent has the relevant slope. A computer-generated plot, together with typical solution curves, is shown in figure 27.2.

Figure 27.2 Typical solutions $y = y(x)$, shown by solid lines, of $dy/dx = a(x^2 + y^2)^{-1/2}$. The short arrows give the direction that the tangent to any solution must have at that point.

27.24 *In the previous exercise (27.23) the difference scheme for solving*

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0,$$

in which A has been set equal to unity, was one-sided in both space (x) and time (t). A more accurate procedure (known as the Lax–Wendroff scheme) is

$$\frac{u(p, n+1) - u(p, n)}{\Delta t} + \frac{u(p+1, n) - u(p-1, n)}{2\Delta x} = \frac{\Delta t}{2}\left[\frac{u(p+1, n) - 2u(p, n) + u(p-1, n)}{(\Delta x)^2}\right].$$

(a) *Establish the orders of accuracy of the two finite difference approximations on the LHS of the equation.*

(b) *Establish the accuracy with which the expression in the brackets approximates* $\partial^2 u/\partial x^2$.

(c) *Show that the RHS of the equation is such as to make the whole difference scheme accurate to second order in both space and time.*

(a) and (b) We start with the (accurate) Taylor expansion in space for $u(p \pm 1, n)$;

$$u(p \pm 1, n) = u(p, n) \pm \Delta x \frac{\partial u(p, n)}{\partial x} + \frac{(\Delta x)^2}{2!} \frac{\partial^2 u(p, n)}{\partial x^2} \pm \frac{(\Delta x)^3}{3!} \frac{\partial^3 u(p, n)}{\partial x^3} +$$

The second term on the LHS of the Lax–Wendroff scheme is thus

$$\frac{u(p+1, n) - u(p-1, n)}{2\Delta x} = \frac{\partial u(p, n)}{\partial x} + \mathrm{O}\left((\Delta x)^2\right),$$

whilst its RHS is

$$\frac{\Delta t}{2} \left[\frac{\partial^2 u(p, n)}{\partial x^2} + \mathrm{O}\left((\Delta x)^2\right) \right].$$

Both are accurate to second order in Δx. We note at this point that the second spatial derivative does not actually appear on the RHS of the original equation; in the original equation the RHS is zero.

For the first term on the LHS we need a Taylor expansion in time:

$$u(p, n+1) = u(p, n) + \Delta t \frac{\partial u(p, n)}{\partial t} + \frac{(\Delta t)^2}{2!} \frac{\partial^2 u(p, n)}{\partial t^2} + \cdots.$$

Thus this term in the calculational scheme is

$$\frac{\partial u(p, n)}{\partial t} + \frac{\Delta t}{2!} \frac{\partial^2 u(p, n)}{\partial t^2} + \cdots.$$

So far as a representation of $\partial u / \partial t$ in the original equation is concerned, this is only accurate to first order in Δt and to make it accurate to second order we need to compensate for the term

$$\frac{\Delta t}{2!} \frac{\partial^2 u(p, n)}{\partial t^2}.$$

(c) However, from differentiating the original equation separately with respect to x and t, we have both

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial t \partial x} = 0 \quad \text{and} \quad \frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial x^2} = 0,$$

implying that

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}.$$

This equality (multiplied through by $\Delta t/2$) allows the unwanted terms on each side of the original equation to be cancelled, leaving the equation accurate to second order in both Δx and Δt.

27.26 *Consider the solution $\phi(x, y)$ of Laplace's equation in two dimensions using a relaxation method on a square grid with common spacing h. As in the main text, denote $\phi(x_0 + ih,\ y_0 + jh)$ by $\phi_{i,j}$. Further, define $\phi_{i,j}^{m,n}$ by*

$$\phi_{i,j}^{m,n} \equiv \frac{\partial^{m+n}\phi}{\partial x^m\, \partial y^n}$$

evaluated at $(x_0 + ih,\ y_0 + jh)$.

(a) *Show that*

$$\phi_{i,j}^{4,0} + 2\phi_{i,j}^{2,2} + \phi_{i,j}^{0,4} = 0.$$

(b) *Working up to terms of order h^5, find Taylor series expansions, expressed in terms of the $\phi_{i,j}^{m,n}$, for*

$$S_{\pm,0} = \phi_{i+1,j} + \phi_{i-1,j}$$
$$S_{0,\pm} = \phi_{i,j+1} + \phi_{i,j-1}.$$

(c) *Find a corresponding expansion, to the same order of accuracy, for $\phi_{i\pm1,j+1} + \phi_{i\pm1,j-1}$ and hence show that*

$$S_{\pm,\pm} = \phi_{i+1,j+1} + \phi_{i+1,j-1} + \phi_{i-1,j+1} + \phi_{i-1,j-1}$$

has the form

$$4\phi_{i,j}^{0,0} + 2h^2(\phi_{i,j}^{2,0} + \phi_{i,j}^{0,2}) + \frac{h^4}{6}(\phi_{i,j}^{4,0} + 6\phi_{i,j}^{2,2} + \phi_{i,j}^{0,4}).$$

(d) *Evaluate the expression $4(S_{\pm,0}+S_{0,\pm})+S_{\pm,\pm}$ and hence deduce that a possible relaxation scheme, good to the fifth order in h, is to recalculate each $\phi_{i,j}$ as the weighted mean of the current values of its four nearest neighbours (each with weight $\frac{1}{5}$) and its four next-nearest neighbours (each with weight $\frac{1}{20}$).*

(a) In the notation given, Laplace's equation takes the form

$$\phi_{i,j}^{2,0} + \phi_{i,j}^{0,2} = 0.$$

To save space and increase clarity we will omit subscripts that are i, j, but write them explicitly when they are not. Thus $\phi^{m,n} \equiv \phi_{i,j}^{m,n}$.

Differentiating Laplace's equation twice more with respect to x and y (separately) yields

$$\phi^{4,0} + \phi^{2,2} = 0 \text{ and } \phi^{2,2} + \phi^{0,4} = 0.$$

Adding these two equations yields the stated result, but we may, in addition, deduce several equalities to be used later. We start from $\phi^{2,0} = -\phi^{0,2}$ and obtain

in an obvious way that

$$\phi^{3,0} = -\phi^{1,2}, \qquad \phi^{0,3} = -\phi^{2,1},$$

and

$$\phi^{3,1} = -\phi^{1,3}, \qquad \phi^{4,0} = \phi^{0,4} = -\phi^{2,2},$$

and

$$\phi^{5,0} = \phi^{1,4} = -\phi^{3,2}, \qquad \phi^{0,5} = \phi^{4,1} = -\phi^{2,3}.$$

(b) The general Taylor series expansion for $\phi_{i\pm1,j}$ is

$$\phi_{i\pm1,j} = \sum_{m=0}^{\infty} \frac{(\pm h)^m}{m!}\,\phi^{m,0},$$

and, up to order h^5,

$$\begin{aligned} S_{\pm,0} &= \phi_{i+1,j} + \phi_{i-1,j} \\ &= 2\phi^{0,0} + h^2\phi^{2,0} + \tfrac{1}{12}h^4\phi^{4,0}. \end{aligned}$$

Similarly,

$$S_{0,\pm} = 2\phi^{0,0} + h^2\phi^{0,2} + \tfrac{1}{12}h^4\phi^{0,4}.$$

(c) The expansion of $\phi_{i\pm1,j+1}$ requires a 2-variable Taylor series and up to order h^5 takes the form

$$\begin{aligned} \phi_{i\pm1,j+1} = \phi^{0,0} &\pm h\phi^{1,0} + h\phi^{0,1} + \frac{h^2}{2!}(\phi^{2,0} \pm 2\phi^{1,1} + \phi^{0,2}) \\ &+ \frac{h^3}{3!}(\pm\phi^{3,0} + 3\phi^{2,1} \pm 3\phi^{1,2} + \phi^{0,3}) \\ &+ \frac{h^4}{4!}(\phi^{4,0} \pm 4\phi^{3,1} + 6\phi^{2,2} \pm 4\phi^{1,3} + \phi^{0,4}) \\ &+ \frac{h^5}{5!}(\pm\phi^{5,0} + 5\phi^{4,1} \pm 10\phi^{3,2} + 10\phi^{2,3} \pm 5\phi^{1,4} + \phi^{0,5}). \end{aligned}$$

Because of the equalities derived in part (a), this can be written more compactly as

$$\begin{aligned} \phi_{i\pm1,j+1} = \phi^{0,0} &\pm h\phi^{1,0} + h\phi^{0,1} + \frac{h^2}{2!}(\pm 2\phi^{1,1}) \\ &+ \frac{h^3}{3!}(\pm 2\phi^{1,2} + 2\phi^{2,1}) + \frac{h^4}{4!}(4\phi^{2,2}) + \frac{h^5}{5!}(\pm 4\phi^{3,2} + 4\phi^{2,3}). \end{aligned}$$

For the corresponding expansion of $\phi_{i\pm1,j-1}$ those terms for which the n in $\phi^{m,n}$ is odd, will change sign. When the two expansions are added together, such terms will cancel and leave

$$\phi_{i\pm1,j+1} + \phi_{i\pm1,j-1} = 2\phi^{0,0} \pm 2h\phi^{1,0} \pm \frac{2h^3}{3}\phi^{1,2} + \frac{h^4}{3}\phi^{2,2} \pm \frac{h^5}{15}\phi^{3,2}.$$

Hence,

$$\begin{aligned} S_{\pm,\pm} &= \phi_{i+1,\,j+1} + \phi_{i+1,\,j-1} + \phi_{i-1,\,j+1} + \phi_{i-1,\,j-1} \\ &= 4\phi^{0,0} + \tfrac{2}{3}h^4\phi^{2,2} + \mathrm{O}(h^6). \end{aligned}$$

This is consistent with the stated expression since $\phi^{2,0} + \phi^{0,2} = 0$ and $\phi^{4,0} + 6\phi^{2,2} + \phi^{0,4} = 4\phi^{2,2}$.

(d) Now consider the sum S given by

$$\begin{aligned} S &= 4(S_{\pm,o} + S_{0,\pm}) + S_{\pm,\pm} \\ &= 4\left[4\phi^{0,0} + h^2(\phi^{0,2} + \phi^{2,0}) + \frac{h^4}{12}(\phi^{4,0} + \phi^{0,4})\right] \\ &\qquad + 4\phi^{0,0} + \tfrac{2}{3}h^4\phi^{2,2} + \mathrm{O}(h^6) \\ &= 20\phi^{0,0} + 4h^2(\phi^{0,2} + \phi^{2,0}) + \frac{h^4}{3}(\phi^{4,0} + \phi^{0,4} + 2\phi^{2,2}) + \mathrm{O}(h^6) \\ &= 20\phi^{0,0} + 0 + 0 + \mathrm{O}(h^6). \end{aligned}$$

Thus

$$\begin{aligned} \phi^{0,0} &= \frac{4}{20}(S_{\pm,0} + S_{0,\pm}) + \frac{1}{20}S_{\pm,\pm} + \mathrm{O}(h^6) \\ &= \tfrac{1}{5}(\phi_{i+1,\,j} + \phi_{i-1,\,j} + \phi_{i,\,j+1} + \phi_{i,\,j-1}) \\ &\qquad + \tfrac{1}{20}(\phi_{i+1,\,j+1} + \phi_{i+1,\,j-1} + \phi_{i-1,\,j+1}\phi_{i-1,\,j-1}) + \mathrm{O}(h^6) \\ &= \tfrac{1}{5}\sum(\text{nearest neighbours}) \\ &\qquad + \tfrac{1}{20}\sum(\text{next nearest neighbours}) + \mathrm{O}(h^6). \end{aligned}$$

This could form the basis of a relaxation scheme, as described in the question.

28

Group theory

28.2 *Which of the following relationships between X and Y are equivalence relations? Give a proof of your conclusions in each case:*

(a) *X and Y are integers and $X - Y$ is odd;*
(b) *X and Y are integers and $X - Y$ is even;*
(c) *X and Y are people and have the same postcode;*
(d) *X and Y are people and have a parent in common;*
(e) *X and Y are people and have the same mother;*
(f) *X and Y are $n \times n$ matrices satisfying $Y = PXQ$, where P and Q are elements of a group $\mathcal{G}$ of $n \times n$ matrices.*

(a) Defining an odd integer as one that does not divide by 2 exactly to yield another integer, this relationship fails to pass the reflexivity test. The equation $X \sim X$ implies that $X - X = 0$ is odd. However, 0 *does* divide exactly by 2 and so $X \not\sim X$.

(b) With $X - Y$ required to be even, this relationship satisfies the reflexivity criterion. Further, since (i) $2n$ and $-2n$ are both even and (ii) $X - Z = (X - Y) + (Y - Z)$ and the sum of two even integers is even, the symmetry and transitivity requirements are also met. Thus this relationship is an equivalence relation. The partition is that of the integers into odd and even integers.

(c) Clearly an equivalence relation. The classes consist of all the people, amongst those considered, who have the same postcode.

(d) Although at first sight this may appear to be an equivalence relation, it is not necessarily a transitive relationship. For three people X, Y and Z for whom

$X \sim Y$ and $Y \sim Z$, if Y's parents both re-marry and X and Z are children of the two second marriages, then $X \not\sim Z$.

(e) Assuming that there is an agreed single definition of mother (so that, for example, a person cannot have two mothers, one natural and one the result of adoption or parental re-marriage) then this is an equivalence relation.

(f) Since the identity I belongs $\mathcal{G}$ and $X = IXI$, $X \sim X$, showing that the relationship is reflexive.

If $Y \sim X$ then $Y = PXQ \quad \Rightarrow \quad X = P^{-1}YQ^{-1}$. But as P and Q belong to $\mathcal{G}$ so do P^{-1} and Q^{-1}. Thus $X \sim Y$ and the relationship is symmetric.

If $Y \sim X$ and $Z \sim Y$ then $Y = PXQ$ and $Z = RYS$, where R and S are also elements of the group. Thus $Z = RPXQS$. However, as P, Q, R and S all belong to $\mathcal{G}$, so do RP and QS. It follows that $Z \sim X$ and that the relationship is transitive.

These three results together show that the relationship is an equivalence relation.

28.4 *Prove that the relationship $X \sim Y$, defined by $X \sim Y$ if Y can be expressed in the form*

$$Y = \frac{aX + b}{cX + d},$$

with a, b, c and d as integers, is an equivalence relation on the set of real numbers $\mathfrak{R}$. Identify the class that contains the real number 1.

(i) Reflexivity is shown by writing $X = \dfrac{1X + 0}{0X + 1}$.

(ii) Symmetry is shown by rewriting the defining equation as

$$cXY + dY = aX + b \quad \Rightarrow \quad X = \frac{-dY + b}{cY - a}.$$

(iii) For transitivity, we have

$$Y = \frac{aX + b}{cX + d} \quad \text{and} \quad Z = \frac{a'Y + b'}{c'Y + d'}$$

giving

$$Z = \frac{a'(aX + b) + b'(cX + d)}{c'(aX + b) + d'(cX + d)} = \frac{(a'a + b'c)X + (a'b + b'd)}{(c'a + d'c)X + (c'b + d'd)}.$$

All of these coefficients are integers and so transitivity is established.

Thus $X \sim Y$ satisfies the three requirements of an equivalence relation on $\mathfrak{R}$, the set of real numbers.

By setting $X = 1$ in the defining relationship, we can see that Y must have the

form of the ratio of (any) two integers. Thus the class to which 1 belongs is the set of rational numbers.

28.6 *Prove that the set $\mathcal{M}$ of matrices*

$$A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a, b, c are integers (mod 5) and $a \neq 0 \neq c$, forms a non-Abelian group under matrix multiplication.

Show that the subset containing elements of $\mathcal{M}$ that are of order 1 or 2 do not form a proper subgroup of $\mathcal{M}$,

(a) *using Lagrange's theorem,*
(b) *by direct demonstration that the set is not closed.*

Consider the product of two typical matrices in the set, A and B,

$$\mathsf{AB} = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} d & e \\ 0 & f \end{pmatrix} = \begin{pmatrix} ad & ae+bf \\ 0 & cf \end{pmatrix}.$$

Since 5 is prime, the product ad cannot equal 0 unless at least one of a and d is 0; but this is ruled out by the given form of the matrices. Similarly $cf \neq 0$. Thus the set is closed under matrix multiplication.

Matrix multiplication is associative.

The set has an obvious identity element $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The inverse of the typical matrix A is found in the usual way:

$$\mathsf{A}^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix},$$

as is easily verified. We use the fact that $a \neq 0 \neq c$ to deduce that $|\mathsf{A}| = ac \neq 0$ and hence justify the usual inversion calculation. We note that for $x = 1, 2, 3, 4$ the multiplicative inverses (mod 5) are $1, 3, 2, 4$ respectively and are well-defined and unique. In particular, this means that $1/ac$ is a well-defined integer (not the fraction it may appear to be at first sight).

These four results establish the set $\mathcal{M}$ as a group.

There are four choices each for the values of a and c and five choices for b; the group therefore has $4 \times 4 \times 5 = 80$ elements.

The only difference between the products AB and BA is in the $(1, 2)$ element. For

an Abelian group the two products need to be equal, i.e. $ae + bf = bd + ec$ (mod 5). But if, for example, $b \neq 0$, $a = c \neq 0$, $d \neq f$ and $e = 0$ then this needed equality is not satisfied. This is sufficent to show that the group is non-Abelian.

(a) The only element of order 1 is the identity. An element of order 2, i.e. $\mathsf{A}^2 = \mathsf{I}$, must have $a^2 = c^2 = 1$ and $ab + bc = 0$. The only possibilities for a and c are therefore 1 and 4.

If $a = c = 1$, then $2b = 0$ (mod 5) $\Rightarrow$ $b = 0$, $\mathsf{A} = \mathsf{I}$, already counted.
If $a = c = 4$, then $8b = 0$ (mod 5) $\Rightarrow$ $b = 0$, one such matrix.
If $a = 1$, $c = 4$, then $5b = 0$ (mod 5) $\Rightarrow$ b arbitrary, five such matrices.
If $a = 4$, $c = 1$, then $5b = 0$ (mod 5) $\Rightarrow$ b arbitrary, five such matrices.
Thus there is a total of 12 matrices of order 1 or 2 in the set.

But, by Lagrange's theorem, the order of any subgroup must divide the order of the group. As 12 does *not* divide 80, the subset cannot be a proper subgroup of $\mathcal{M}$.

(b) To find a counter-example to the closure of the set, consider the product of two matrices of the third type found in part (a):

$$\mathsf{C} = \begin{pmatrix} 1 & b \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & e \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & e + 4b \\ 0 & 1 \end{pmatrix}.$$

In view of its diagonal elements, this product can only be a member of the set if it is the identity [as shown in part(a)], thus requiring $e + 4b = 0$. So, to find a counter-example, we are led to consider a specific case, $b = 0$ and $e = 3$, which does not satisfy the requirement.

$$\mathsf{C} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}.$$

Now,

$$\mathsf{C}^2 = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus C has order > 2 and does not belong to the set; the set is not closed and cannot form a subgroup of $\mathcal{M}$.

28.8 *Show that, under matrix multiplication, matrices of the form*

$$\mathsf{M}(a_0, \mathsf{a}) = \begin{pmatrix} a_0 + a_1 i & -a_2 + a_3 i \\ a_2 + a_3 i & a_0 - a_1 i \end{pmatrix},$$

where a_0 and the components of column matrix $\mathsf{a} = (a_1 \quad a_2 \quad a_3)^{\mathrm{T}}$ are real numbers satisfying $a_0^2 + |\mathsf{a}|^2 = 1$, constitute a group. Deduce that, under the transformation $\mathsf{z} \to \mathsf{M}\mathsf{z}$, where z is any column matrix, $|\mathsf{z}|^2$ is invariant.

As usual, we take the associativity of matrix multiplication for granted.

An identity element is provided by $\mathsf{M}(1,\mathsf{0})$, i.e. $a_0 = 1,\ a_1 = a_2 = a_3 = 0$, which satisfy $a_0^2 + |\,\mathsf{a}\,|^2 = 1$.

We next note that if we write a typical matrix N in terms of complex numbers n_i as

$$\mathsf{N} = \begin{pmatrix} n_1 & -n_2^* \\ n_2 & n_1^* \end{pmatrix} \quad \text{with} \quad |\mathsf{N}| = (a_0^2 + a_1^2) + (a_2^2 + a_3^2) = |n_1|^2 + |n_2|^2 = 1,$$

then the product $\mathsf{P} = \mathsf{NM}$ of two such matrices is

$$\begin{aligned} \mathsf{P} &= \begin{pmatrix} n_1 & -n_2^* \\ n_2 & n_1^* \end{pmatrix} \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix} \\ &= \begin{pmatrix} n_1 m_1 - n_2^* m_2 & -n_1 m_2^* - n_2^* m_1^* \\ n_2 m_1 + n_1^* m_2 & -n_2 m_2^* + n_1^* m_1^* \end{pmatrix}. \end{aligned}$$

This is of the form

$$\begin{pmatrix} p_1 & -p_2^* \\ p_2 & p_1^* \end{pmatrix} \quad \text{with} \quad \begin{cases} p_1 = n_1 m_1 - n_2^* m_2 \\ p_2 = n_2 m_1 + n_1^* m_2 \end{cases}.$$

Further $|\mathsf{P}| = |\mathsf{N}|\,|\mathsf{M}| = 1 \times 1 = 1$, i.e. $p_1 p_1^* + p_2 p_2^* = 1$. Thus the set is closed.

It just remains to establish the existence of an inverse N^{-1} for each N within the set. The inverse is constructed in the normal way (recalling that $|\mathsf{N}| = 1$) as

$$\mathsf{N}^{-1} = \begin{pmatrix} n_1^* & n_2^* \\ -n_2 & n_1 \end{pmatrix}.$$

This is of the given form, with $n_1 \to n_1^*$ and $n_2 \to -n_2$. These changes correspond to $a_0 \to a_0$ and $a_i \to -a_i$ for $i = 1, 2, 3$, i.e. $\mathsf{a} \to -\mathsf{a}$.

Clearly, $|\mathsf{N}^{-1}| = |n_1|^2 + |n_2|^2$. This expression was shown to be equal to unity when we considered $|\mathsf{N}|$ earlier Thus, in summary,

$$[\mathsf{M}(a_0, \mathsf{a})]^{-1} = \mathsf{M}(a_0, -\mathsf{a}).$$

The set of matrices have now been shown to satisfy all the conditions for forming a group under matrix multiplication, and so they do so.

Under $\mathsf{z} \to \mathsf{Mz}$,

$$|\,\mathsf{z}\,|^2 = \mathsf{z}^\dagger \mathsf{z} \to \mathsf{z}^\dagger \mathsf{M}^\dagger \mathsf{Mz}.$$

But

$$\begin{aligned}
\mathsf{M}^\dagger\mathsf{M} &= \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix}^\dagger \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix} \\
&= \begin{pmatrix} m_1^* & m_2^* \\ -m_2 & m_1 \end{pmatrix} \begin{pmatrix} m_1 & -m_2^* \\ m_2 & m_1^* \end{pmatrix} \\
&= \begin{pmatrix} |m_1|^2 + |m_2|^2 & 0 \\ 0 & |m_2|^2 + |m_1|^2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{I}.
\end{aligned}$$

It follows that under the transformation $\mathsf{z} \to \mathsf{Mz}$ for a general matrix z,

$$|\,\mathsf{z}\,|^2 \to \mathsf{z}^\dagger\,\mathsf{I}\,\mathsf{z} = |\,\mathsf{z}\,|^2,$$

i.e. is invariant.

28.10 *The group of rotations (excluding reflections and inversions) in three dimensions that take a cube into itself is known as the group* 432 *(or O in the usual chemical notation). Show by each of the following methods that this group has* 24 *elements.*

(a) *Identify the distinct relevant axes and count the number of qualifying rotations about each.*
(b) *The orientation of the cube is determined if the directions of two of its body diagonals are given. Consider the number of distinct ways in which one body diagonal can be chosen to be 'vertical', say, and a second diagonal made to lie along a particular direction.*

(a) As always, the identity (do nothing) operation is *one* of the symmetries of the cube.

About the *three* normals through the centres of opposite faces of the cube, rotations of π take the cube into itself, as do the *six* rotations of $\pm\pi/2$ about the same normals.

Rotations of π about diagonals joining the centre points of opposite edges of the cube are further symmetry operations on the cube; there are *six* of these.

Finally, there are *eight* rotations of $\pm 2\pi/3$ about the cube's body diagonals.

These bring the total up to $1 + 3 + 6 + 6 + 8 = 24$ symmetry operations.

(b) The 'vertical' diagonal can be chosen in 4×2 ways (either end of each diagonal can be 'up'). There are then three equivalent rotational positions (notionally rotations through 0 and $\pm\frac{2}{3}\pi$) about the vertical, each bringing a different body diagonal into any specified position. Thus there are $4 \times 2 \times 3 = 24$ possibilities altogether.

28.12 *If $\mathcal{A}$ and $\mathcal{B}$ are two groups then their direct product, $\mathcal{A} \times \mathcal{B}$, is defined to be the set of ordered pairs (X, Y), with X an element of $\mathcal{A}$, Y an element of $\mathcal{B}$ and multiplication given by $(X, Y)(X', Y') = (XX', YY')$. Prove that $\mathcal{A} \times \mathcal{B}$ is a group.*

Denote the cyclic group of order n by $\mathcal{C}_n$ and the symmetry group of a regular n-sided figure (an n-gon) by $\mathcal{D}_n$ – thus $\mathcal{D}_3$ is the symmetry group of an equilateral triangle, as discussed in the text.

(a) *By considering the orders of each of their elements, show* (i) *that $\mathcal{C}_2 \times \mathcal{C}_3$ is isomorphic to $\mathcal{C}_6$, and* (ii) *that $\mathcal{C}_2 \times \mathcal{D}_3$ is isomorphic to $\mathcal{D}_6$.*
(b) *Are any of $\mathcal{D}_4$, $\mathcal{C}_8$, $\mathcal{C}_2 \times \mathcal{C}_4$, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ isomorphic?*

We consider the four requirements for $\mathcal{A} \times \mathcal{B}$ to be a group.

(1) Closure:

$$X \in \mathcal{A} \text{ and } X' \in \mathcal{A} \quad \Rightarrow \quad XX' \in \mathcal{A}, \text{ since } \mathcal{A} \text{ is a group.}$$

Similarly, $YY' \in \mathcal{B}$. Hence $(XX', YY') \in \mathcal{A} \times \mathcal{B}$, i.e the set is closed.

(2) Associativity holds since it does so in $\mathcal{A}$ and $\mathcal{B}$ separately.

(3) The identity:

$$I_{\mathcal{A}} \in \mathcal{A} \text{ and } I_{\mathcal{B}} \in \mathcal{B} \quad \Rightarrow \quad (I_{\mathcal{A}}, I_{\mathcal{B}}) \in \mathcal{A} \times \mathcal{B},$$

and

$$(I_{\mathcal{A}}, I_{\mathcal{B}})(X, Y) = (I_{\mathcal{A}}X, I_{\mathcal{B}}Y) = (X, Y).$$

Thus $(I_{\mathcal{A}}, I_{\mathcal{B}})$ is in $\mathcal{A} \times \mathcal{B}$ and is its identity.

(4) Inverse: If $X \in \mathcal{A}$ then $X^{-1} \in \mathcal{A}$ (since $\mathcal{A}$ is a group); similarly $Y^{-1} \in \mathcal{B}$. Thus

$$(X^{-1}, Y^{-1}) \in \mathcal{A} \times \mathcal{B}, \text{ and } (X, Y)(X^{-1}, Y^{-1}) = (XX^{-1}, YY^{-1}) = (I_{\mathcal{A}}, I_{\mathcal{B}}).$$

Thus each element of $\mathcal{A} \times \mathcal{B}$ has an inverse in the set.

These four results establish $\mathcal{A} \times \mathcal{B}$ as a group.

(a)(i) In $\mathcal{C}_6$ with generator P ($P^6 = I$), I has order 1, P and P^5 have order 6, P^2 and P^4 have order 3, whilst P^3 has order 2.

How consider $\mathcal{C}_2 \times \mathcal{C}_3$ which has $2 \times 3 = 6$ elements (the same as $\mathcal{C}_6$). With $X \in \mathcal{C}_2$ and $Y \in \mathcal{C}_3$, the elements (X, Y) and their orders are [X and Y are not the identity except where explicitly stated]:

$$\begin{aligned}
(I, I) &\text{ has order } 1 \quad 1 \text{ element;} \\
(I, Y) &\text{ has order } 3 \quad 2 \text{ elements;} \\
(X, I) &\text{ has order } 2 \quad 1 \text{ element;} \\
(X, Y) &\text{ has order } 6 \quad 1 \times 2 = 2 \text{ elements.}
\end{aligned}$$

Thus $\mathcal{C}_6$ and $\mathcal{C}_2 \times \mathcal{C}_3$ both have six elements and have the same numbers of elements of any particular order. Therefore they are isomorphic.

(ii) Consider first the set of symmetry operations on a regular hexagon (the group $\mathcal{D}_6$). In addition to the identity, it includes five rotations of $2\pi k/6$ ($k = 1, 2, \ldots, 5$) about an axis through the hexagon's centre and perpendicular to its plane. The two with $k = 1, 5$ have order 6, the two with $k = 2, 4$ have order 3, whilst that with $k = 3$ has order 2.

Further there are 3 reflection symmetries with respect to diagonals joining opposite corners of the hexagon, and 3 more with respect to diagonals joining the centres of opposite sides. Clearly, all of the reflection symmetries have order 2. In summary, $\mathcal{D}_6$ has twelve elements: one of order 1, seven of order 2, and two each of orders 3 and 6.

As shown in the text, the group $\mathcal{D}_3$ has six elements: one of order 1, two (rotations) of order 3, and three (reflections) of order 2. With the same notation as in part (a), the elements (X, Y) of $\mathcal{C}_2 \times \mathcal{D}_3$ and their orders are:

$$\begin{aligned}
(I, I) &\text{ has order } 1 \quad 1 \text{ element;} \\
(I, Y_{\text{rot}}) &\text{ has order } 3 \quad 2 \text{ elements;} \\
(I, Y_{\text{ref}}) &\text{ has order } 2 \quad 3 \text{ elements;} \\
(X, I) &\text{ has order } 2 \quad 1 \text{ element;} \\
(X, Y_{\text{rot}}) &\text{ has order } 6 \quad 1 \times 2 = 2 \text{ elements;} \\
(X, Y_{\text{ref}}) &\text{ has order } 2 \quad 1 \times 3 = 3 \text{ elements.}
\end{aligned}$$

Again, in summary, $\mathcal{C}_2 \times \mathcal{D}_3$ has twelve elements: one of order 1, seven of order 2, and two each of orders 3 and 6. Thus the groups $\mathcal{D}_6$ and $\mathcal{C}_2 \times \mathcal{D}_3$ are isomorphic.

(b) The groups $\mathcal{D}_4$, $\mathcal{C}_8$, $\mathcal{C}_2 \times \mathcal{C}_4$, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ each have eight elements [see the previous exercise (28.11) in the case of $\mathcal{D}_4$]. Thus each has the potential to be isomorphic to any other. The same exercise showed that $\mathcal{D}_4$ has five elements of order 2 and two of order 4, as well as the identity of order 1.

However, each cyclic group $\mathcal{C}_n$ must contain at least one element of order n and no element in any product cyclic group can have an order that is greater than the LCM of the orders of the individual groups. Thus $\mathcal{C}_8$ has an element of order 8, $\mathcal{C}_2 \times \mathcal{C}_2 \times \mathcal{C}_2$ has no element of order greater than 2, and $\mathcal{C}_2 \times \mathcal{C}_4$ has an element of order 4, but no greater. These results, together with the above observations about $\mathcal{D}_4$ mean that only $\mathcal{C}_2 \times \mathcal{C}_4$ and $\mathcal{D}_4$ could *possibly* be isomorphic. These have to be examined further.

$\mathcal{C}_4$ has one element of order 1, one of order 2 and two of order 4. (Elements of order 3 are not possible since 3 does not divide 4 exactly.) Forming the elements of $\mathcal{C}_2 \times \mathcal{C}_4$, we obtain:

$$\begin{aligned}
(I, I) &\text{ has order 1 } \ \text{1 element;}\\
(I, Y_2) &\text{ has order 2 } \ \text{1 element;}\\
(I, Y_4) &\text{ has order 4 } \ \text{2 elements;}\\
(X, I) &\text{ has order 2 } \ \text{1 element;}\\
(X, Y_2) &\text{ has order 2 } \ \text{1 element;}\\
(X, Y_4) &\text{ has order 4 } \ \text{2 elements.}
\end{aligned}$$

In summary, $\mathcal{C}_2 \times \mathcal{C}_4$ has one element of order 1, three of order 2 and four of order 4. This is not the same distribution as for $\mathcal{D}_4$ and so the two groups *cannot* be isomorphic.

28.14 *Show that if p is prime then the set of rational number pairs (a, b), excluding $(0, 0)$, with multiplication defined by*

$$(a, b) \bullet (c, d) = (e, f), \quad \text{where} \quad (a + b\sqrt{p})(c + d\sqrt{p}) = e + f\sqrt{p},$$

forms an Abelian group. Show further that the mapping $(a, b) \to (a, -b)$ is an automorphism.

From the given combination law

$$e = ac + bdp \qquad \text{and} \qquad f = ad + bc.$$

As a, b, c and d are all rational numbers, so are e and f. The set of rational number pairs is therefore closed. Associativity and commutativity are obvious and the number pair $I = (1, 0)$ clearly has the property that $IX = X$ for any rational number pair X. The existence of inverses is the only requirement remaining to be established in order to complete the proof that the set forms an Abelian group under the given combination law.

Now,

$$(c,d) \bullet (a,b) = (1,0) \quad \Rightarrow \quad c + d\sqrt{p} = \frac{1}{a + b\sqrt{p}} = \frac{a - b\sqrt{p}}{a^2 - b^2 p}.$$

Further, since a and b are rational (and $\sqrt{p}$ is not), $a^2 \neq b^2 p$. Although it is not zero, $a^2 - b^2 p$ is rational, and so, therefore are c and d. To summarise,

$$(a,b)^{-1} = (c,d) = \left(\frac{a}{a^2 - b^2 p}, \frac{-b}{a^2 - b^2 p} \right),$$

a rational number pair that is included in the set. The proof is now complete.

For the mapping $(a,b) \to (a,-b)$

$$[\,(a,b) \bullet (c,d)\,]' = (ac + bdp, ad + bc)' = (ac + bdp, -ad - bc),$$

whilst

$$(a,b)' \bullet (c,d)' = (a,-b) \bullet (c,-d) = (ac + bdp, -bc - ad).$$

Hence

$$[\,(a,b) \bullet (c,d)\,]' = (a,b)' \bullet (c,d)',$$

showing that the mapping is a homomorphism. The mapping is clearly one-to-one, making it an isomorphism and, finally, since the object and image sets are the same, it is an automorphism.

28.16 *For the group $\mathcal{G}$ with multiplication table 28.8 and proper subgroup $\mathcal{H} = \{I, A, B\}$, denote the coset $\{I, A, B\}$ by $\mathcal{C}_1$ and the coset $\{C, D, E\}$ by $\mathcal{C}_2$. Form the set of all possible products of a member of $\mathcal{C}_1$ with itself, and denote this by $\mathcal{C}_1\mathcal{C}_1$. Similarly compute $\mathcal{C}_2\mathcal{C}_2$, $\mathcal{C}_1\mathcal{C}_2$ and $\mathcal{C}_2\mathcal{C}_1$. Show that each product coset is equal to $\mathcal{C}_1$ or to $\mathcal{C}_2$ and that a 2×2 multiplication table can be formed, demonstrating that $\mathcal{C}_1$ and $\mathcal{C}_2$ are themselves the elements of a group of order 2. A subgroup like $\mathcal{H}$ whose cosets themselves form a group is a* normal subgroup.

The multiplication table is

	I	A	B	C	D	E
I	I	A	B	C	D	E
A	A	B	I	E	C	D
B	B	I	A	D	E	C
C	C	D	E	I	A	B
D	D	E	C	B	I	A
E	E	C	D	A	B	I

As can be seen, dividing the six elements into the two cosets based on the

subgroup $\{I, A, B\}$ corresponds to dividing the table into four quarters, as shown. The first three rows and the first three columns are labelled by the elements of $\mathcal{C}_1$ whilst the last three rows and columns are labelled by the elements of $\mathcal{C}_2$.

It is clear from inspection that the top LH quarter gives the set of products denoted in the question by $\mathcal{C}_1\mathcal{C}_1$; it is equally clear that it contains only elements that are in $\mathcal{C}_1$, thus $\mathcal{C}_1 \times \mathcal{C}_1 = \mathcal{C}_1$. Similarly, in the top RH quarter, $\mathcal{C}_1 \times \mathcal{C}_2 = \mathcal{C}_2$. In the same way, $\mathcal{C}_2 \times \mathcal{C}_1 = \mathcal{C}_2$ and $\mathcal{C}_2 \times \mathcal{C}_2 = \mathcal{C}_1$.

Thus we can also draw up a multiplication table for the cosets themselves

	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_2$
$\mathcal{C}_2$	$\mathcal{C}_2$	$\mathcal{C}_1$

This is the multiplication table of a group of order 2 in which $\mathcal{C}_1$ is the unit element.

28.18 *The group of reflection–rotation symmetries of a square is known as $\mathcal{D}_4$; let X be one of its elements. Consider a mapping $\Phi : \mathcal{D}_4 \to S_4$, the permutation group on four objects, defined by $\Phi(X)$ = the permutation induced by X on the set $\{x, y, d, d'\}$, where x and y are the two principal axes, and d and d' the two principal diagonals, of the square. For example, if R is a rotation by $\pi/2$, $\Phi(R) =$ (12)(34). Show that $\mathcal{D}_4$ is mapped onto a subgroup of S_4 and, by constructing the multiplication tables for $\mathcal{D}_4$ and the subgroup, prove that the mapping is a homomorphism.*

This group of symmetries and its multiplication table were considered and derived in exercise 28.11; we will use the same notation here.

Clearly, the set of principal axes and diagonals is left unchanged by both I and R^2 and so these two symmetries are mapped onto the permutation (1). As explained in the question, R interchanges the two principal axes and also interchanges the two principal diagonals, i.e. it causes the change $\{x, y, d, d'\} \to \{y, x, d', d\}$; it therefore maps onto (12)(34). The same is true of R^3. The reflections m_1 and m_2 about the principal axes leave those axes unchanged but interchange d and d'; they therefore both map onto the permutation (34). Similarly m_3 and m_4 are both mapped onto (12).

In summary, $\mathcal{D}_4$ is mapped by a two-to-one mapping onto the subgroup

$$\{(1), (12)(34), (34), (12)\}$$

of S_4. The entries in the multiplication table for this subgroup,

	(1)	(12)(34)	(34)	(12)
(1)	(1)	(12)(34)	(34)	(12)
(12)(34)	(12)(34)	(1)	(12)	(34)
(34)	(34)	(12)	(1)	(12)(34)
(12)	(12)	(34)	(12)(34)	(1)

are straightforward and can be verified by inspection; for example, the product [(12)(34)][(34)] interchanges the 3rd and 4th objects and then changes them back again whilst at the same time interchanging the 1st and 2nd. The net result is that the 1st and 2nd objects are interchanged and the 3rd and 4th are untouched, i.e. equivalent to (12).

This 4×4 table and the 4×4 table produced by combining the elements of $\mathcal{D}_4$ in appropriate pairs, according to their common effects on the axes and diagonals of the square, clearly have the common structure

	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	I	A
C	C	B	A	I

.

Here (1), I and R^2 are replaced by I; (12)(34), R and R^3 are replaced by A; (34), m_1 and m_2 are replaced by B; and (12), m_3 and m_4 are replaced by C. This shows that the mapping Φ is a homomorphism. It is not, however, an isomorphism as the mapping is not one-to-one; it is, in fact, an epimorphism.

28.20 *In the quaternion group $\mathcal{Q}$ the elements form the set*

$$\{1, -1, i, -i, j, -j, k, -k\},$$

with $i^2 = j^2 = k^2 = -1$, $ij = k$ and its cyclic permutations, and $ji = -k$ and its cyclic permutations. Find the proper subgroups of $\mathcal{Q}$ and the corresponding cosets. Show that all of the subgroups are normal subgroups. Show further that $\mathcal{Q}$ cannot be isomorphic to the group 4mm (C_{4v}) considered in exercise 28.11.

In order to establish the subgroups of the quaternion group we draw up its

multiplication table which reads as follows

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

From this table it can be seen that the proper subgroups are

$$\{1, -1\}, \quad \{1, -1, i, -i\}, \quad \{1, -1, j, -j\}, \quad \{1, -1, k, -k\}.$$

The cosets of the subgroup $\{1, -1\}$, which has order 2, are immediately found to be

$$\mathcal{C}_1 = \{1, -1\}, \quad \mathcal{C}_i = \{i, -i\}, \quad \mathcal{C}_j = \{j, -j\}, \quad \mathcal{C}_k = \{k, -k\}.$$

The multiplication table also shows that the product of two elements drawn one each from $\mathcal{C}_1$ and $\mathcal{C}_n$ always belongs to $\mathcal{C}_n$, i.e. $\mathcal{C}_1 \times \mathcal{C}_n = \mathcal{C}_n$ for $n = 1, i, j, k$. Similarly $\mathcal{C}_n \times \mathcal{C}_n = \mathcal{C}_1$. It is only slightly more complicated to see that $\mathcal{C}_i \times \mathcal{C}_j = \mathcal{C}_k$, and corresponding results obtained by interchanging any pair of subscripts, are valid. The cosets themselves obey a group multiplication table of the form

	$\mathcal{C}_1$	$\mathcal{C}_i$	$\mathcal{C}_j$	$\mathcal{C}_k$
$\mathcal{C}_1$	$\mathcal{C}_1$	$\mathcal{C}_i$	$\mathcal{C}_j$	$\mathcal{C}_k$
$\mathcal{C}_i$	$\mathcal{C}_i$	$\mathcal{C}_1$	$\mathcal{C}_k$	$\mathcal{C}_j$
$\mathcal{C}_j$	$\mathcal{C}_j$	$\mathcal{C}_k$	$\mathcal{C}_1$	$\mathcal{C}_i$
$\mathcal{C}_k$	$\mathcal{C}_k$	$\mathcal{C}_j$	$\mathcal{C}_i$	$\mathcal{C}_1$

With $\mathcal{C}_1$ as the identity, this is a group under coset multiplication and establishes $\{1, -1\}$ as a normal subgroup of $\mathcal{Q}$. We note that, unlike $\mathcal{Q}$ itself, the group of cosets is Abelian.

We now consider the three subgroups of order 4 and take $\{1, -1, i, -i\}$ as typical. Its cosets are $\mathcal{D}_i = \{1, -1, i, -i\}$ and $\mathcal{D}'_i = \{j, -j, k, -k\}$; these two cosets exhaust the group.

If we select two elements, one from each coset, and multiply them together we can only obtain one of the four quantities $\pm j$ and $\pm k$ (recall, that, for example, $(-i)(k) = j$). In other words, $\mathcal{D}_i \times \mathcal{D}'_i = \mathcal{D}'_i$. Similar, but even simpler, considerations show that $\mathcal{D}_i \times \mathcal{D}_i = \mathcal{D}_i$ and that $\mathcal{D}'_i \times \mathcal{D}'_i = \mathcal{D}_i$. Thus $\mathcal{D}_i$ and $\mathcal{D}'_i$ form a group of order 2 under coset multiplication, with $\mathcal{D}_i$ as its identity. This shows that the

subgroup $\{1, -1, i, -i\}$ is a normal subgroup; corresponding considerations and conclusions apply to the subgroups $\{1, -1, j, -j\}$ and $\{1, -1, k, -k\}$.

Finally, as shown in exercise 28.11, the group $4mm$ has five elements of order 2 (rotation by π and the four reflection symmetries). The quaternion group $\mathcal{Q}$ considered here has only one (-1); this rules out any possibility of isomorphism between the two groups.

28.22 *Show that the matrices*

$$\mathsf{M}(\theta, x, y) = \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix},$$

where $0 \le \theta < 2\pi$, $-\infty < x < \infty$, $-\infty < y < \infty$, *form a group under matrix multiplication.*

Show that those $\mathsf{M}(\theta, x, y)$ *for which* $\theta = 0$ *form a subgroup and identify its cosets. Show that the cosets themselves form a group.*

We start by noting that matrix multiplication is associative.

Next consider

$$\begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi & x' \\ \sin\phi & \cos\phi & y' \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) & X \\ \sin(\theta+\phi) & \cos(\theta+\phi) & Y \\ 0 & 0 & 1 \end{pmatrix},$$

where $X = x'\cos\theta - y'\sin\theta + x$ and $Y = x'\sin\theta + y'\cos\theta + y$. Written in terms of matrices of the given form, this reads

$$\mathsf{M}(\theta, x, y)\,\mathsf{M}(\phi, x', y') = \mathsf{M}(\theta + \phi, X, Y),$$

showing that the set is closed.

Clearly, the unit matrix $\mathsf{M}(0, 0, 0) = \mathsf{I}_3$ acts as an identity element.

To find an inverse for $\mathsf{M}(\theta, x, y)$, set $\phi = -\theta$ and $X = Y = 0$ in the above display

equation. This requires that

$$\begin{aligned}
& 0 = x' \cos\theta - y' \sin\theta + x, \\
& 0 = x' \sin\theta + y' \cos\theta + y, \\
\Rightarrow \quad & 0 = x'(\cos^2\theta + \sin^2\theta) + x\cos\theta + y\sin\theta, \\
\Rightarrow \quad & x' = -x\cos\theta - y\sin\theta, \\
\text{and} \quad & 0 = -y'(\cos^2\theta + \sin^2\theta) + x\sin\theta - y\cos\theta, \\
\Rightarrow \quad & y' = x\sin\theta - y\cos\theta.
\end{aligned}$$

Thus $\mathsf{M}(\theta, x, y)^{-1} = \mathsf{M}(-\theta, -x\cos\theta - y\sin\theta, x\sin\theta - y\cos\theta)$ and establishes the existence of an inverse within the set.
This completes the proof that the set forms a group under matrix multiplication.

For the subset $\mathsf{M}(0, x, y)$, associativity is as before, the identity, $\mathsf{M}(0, 0, 0)$, is included, $\mathsf{M}(0, x, y)^{-1} = \mathsf{M}(0, -x, -y)$ is of the correct form and closure is shown by $\mathsf{M}(0, x, y)\mathsf{M}(0, x', y') = \mathsf{M}(0, x+x', y+y')$. All four group conditions are satisfied and the subset forms a subgroup $\mathcal{N}$ of the group $\{\mathsf{M}(\theta, x, y)\}$.

Since $\mathsf{M}(0, x, y)\mathsf{M}(\theta, x', y') = \mathsf{M}(\theta, x + x', y + y')$, the cosets of $\mathcal{N}$ are

$$\mathcal{C}_\theta = \{\mathsf{M}(\theta, x, y), \text{ for all } -\infty < x, y < \infty\},$$

i.e. all members of any coset have the same value for θ.

If $\mathsf{M}(\theta_1, x_1, y_1)$ is any member of $\mathcal{C}_{\theta_1}$ and $\mathsf{M}(\theta_2, x_2, y_2)$ any member of $\mathcal{C}_{\theta_2}$, then

$$\begin{aligned}
& \mathsf{M}(\theta_1, x_1, y_1)\, \mathsf{M}(\theta_2, x_2, y_2) \\
= \; & \mathsf{M}(\theta_1 + \theta_2, x_1 + x_2\cos\theta_1 - y_2\sin\theta_1, y_1 + y_2\cos\theta_1 + x_2\sin\theta_1)
\end{aligned}$$

belongs to $\mathcal{C}_{\theta_1+\theta_2}$. In terms of coset multiplication, $\mathcal{C}_{\theta_1} \times \mathcal{C}_{\theta_2} = \mathcal{C}_{\theta_1+\theta_2}$; the product coset is contained in the set. The identity is $\mathcal{C}_0$ ($= \mathcal{N}$ itself) and $\mathcal{C}_\theta^{-1} = \mathcal{C}_{2\pi-\theta}$ is also in the set of cosets. The cosets therefore form a group.

29

Representation theory

29.2 *Using a square whose corners lie at coordinates* $(\pm 1, \pm 1)$*, form a natural representation of the dihedral group* $\mathcal{D}_4$*. Find the characters of the representation, and, using the information (and class order) in table 29.4 (p. 1102), express the representation in terms of irreps.*

Now form a representation in terms of eight 2×2 *orthogonal matrices, by considering the effect of each of the elements of* $\mathcal{D}_4$ *on a general vector* (x, y)*. Confirm that this representation is one of the irreps found using the natural representation.*

As in figure 29.1, we mark the corners of the square as 1, 2, 3 and 4 and describe the actions of the various symmetry operations by describing to which of the four fixed points $A = (1, 1)$, $B = (1, -1)$, $C = (-1, -1)$ and $D = (-1, 1)$ each of

Figure 29.1 The coordinate system and notation used in exercise 29.2.

the corners is carried. For example R, a clockwise rotation by $\pi/2$, carries corner 1 from A to B, corner 2 from B to C, etc. The corresponding matrix is

$$\mathsf{D}(R) = \begin{pmatrix} . & 1 & . & . \\ . & . & 1 & . \\ . & . & . & 1 \\ 1 & . & . & . \end{pmatrix}$$

This matrix is traceless, as will be those of the matrices corresponding to R^2 and R^3; thus the character for each of these is 0. The identity matrix is I_4 which has a trace, and hence a character, of 4.

The four matrices in the group corresponding to reflections are

$$\mathsf{D}(m_x) = \begin{pmatrix} . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \end{pmatrix} \quad \mathsf{D}(m_y) = \begin{pmatrix} . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \end{pmatrix}$$

$$\mathsf{D}(m_d) = \begin{pmatrix} 1 & . & . & . \\ . & . & . & 1 \\ . & . & 1 & . \\ . & 1 & . & . \end{pmatrix} \quad \mathsf{D}(m_{d'}) = \begin{pmatrix} . & . & 1 & . \\ . & 1 & . & . \\ 1 & . & . & . \\ . & . & . & 1 \end{pmatrix}$$

The first two (in the same class) have character 0, whilst the class consisting of m_d and $m_{d'}$ has character 2.

In summary the representation has characters $(4, 0, 0, 0, 2)$, the five classes being given in the same order as in table 29.4. Application of the standard formula (29.18), or direct inspection, shows that the only combination of irreps that gives the correct character totals for all classes is

$$\mathrm{A}_1 \oplus \mathrm{B}_2 \oplus \mathrm{E}.$$

The relevant character sum is

$$(4, 0, 0, 0, 2) = (1, 1, 1, 1, 1) + (1, 1, -1, -1, 1) + (2, -2, 0, 0, 0).$$

If the combination is calculated from the formula, the expansion coefficients for A_2 and B_1 are found to be zero; for each of the other three irreps they are unity.

As a second representation we consider what happens to a vector (x, y) under each of the symmetries contained in $\mathcal{D}_4$. For example, the rotation R, considered previously, takes (x, y) into $(y, -x)$ and so is represented by

$$\mathsf{D}(R) = \begin{pmatrix} . & 1 \\ -1 & . \end{pmatrix}.$$

The full set of symmetries and corresponding matrices is

$$\begin{array}{cccc} I & R & R^2 & R^3 \\ \begin{pmatrix} 1 & . \\ . & 1 \end{pmatrix}, & \begin{pmatrix} . & 1 \\ -1 & . \end{pmatrix}, & \begin{pmatrix} -1 & . \\ . & -1 \end{pmatrix}, & \begin{pmatrix} . & -1 \\ 1 & . \end{pmatrix}, \\ m_x & m_y & m_d & m_{d'} \\ \begin{pmatrix} -1 & . \\ . & 1 \end{pmatrix}, & \begin{pmatrix} 1 & . \\ . & -1 \end{pmatrix}, & \begin{pmatrix} . & 1 \\ 1 & . \end{pmatrix}, & \begin{pmatrix} . & -1 \\ -1 & . \end{pmatrix}. \end{array}$$

As expected (and required), symmetries in the same class have matrices with equal traces (i.e. have equal characters). The characters [in the same order $(I, R^2, R/R^3, m_x/m_y, m_d/m_{d'})$ as earlier] are $(2, -2, 0, 0, 0)$. This is exactly the character set for irrep E, found earlier in the natural representation. This second representation is two-dimensional but irreducible.

29.4 *Construct the character table for the irreps of the permutation group S_4 as follows.*

(a) *By considering the possible forms of its cycle notation, determine the number of elements in each conjugacy class of the permutation group S_4 and show that S_4 has five irreps. Give the logical reasoning that shows they must consist of two three-dimensional, one two-dimensional, and two one-dimensional irreps.*

(b) *By considering the odd and even permutations in the group S_4 establish the characters for one of the one-dimensional irreps.*

(c) *Form a natural matrix representation of 4×4 matrices based on a set of objects $\{a, b, c, d\}$, which may or may not be equal to each other, and, by selecting one example from each conjugacy class, show that this natural representation has characters* 4, 2, 1, 0, 0. *In the four-dimensional vector space in which each of the four coordinates takes on one of the four values a, b, c or d, the one-dimensional subspace consisting of the four points with coordinates of the form $\{a, a, a, a\}$ is invariant under the permutation group and hence transforms according to the invariant irrep* A_1. *The remaining three-dimensional subspace is irreducible; use this and the characters deduced above to establish the characters for one of the three-dimensional irreps,* T_1.

(d) *Complete the character table using orthogonality properties, and check the summation rule for each irrep. You should obtain table 29.1.*

(a) The group S_4 has $4! = 24$ elements; its possible cyclical forms and their

Irrep	Typical element and class size				
	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

Table 29.1 The character table for the permutation group S_4.

corresponding orders are

	structure	number	order
(i)	(1)(2)(3)(4)	${}^4C_0 = 1$	1
(ii)	(12)(3)(4)	${}^4C_2 = 6$	2
(iii)	(123)(4)	${}^4C_1\,{}^2C_1 = 8$	3
(iv)	(1234)	${}^3C_1\,{}^2C_1 = 6$	4
(v)	(12)(34)	${}^3C_1 = 3$	2

Now, as shown in the solution to exercise 28.9, elements in the same class must have the same order. This implies that here there are at least four classes. In fact, there are five.

To see this, we note that a permutation structure $(wx)(y)(z)$, involving only one pair interchange, is an odd permutation, whilst one with structure $(pq)(rs)$, which involves two interchanges, is an even one. Thus, since *any* permutation P has the same parity (odd or even) as its inverse P^{-1}, a relationship of the form

$$[\,(pq)(rs)\,] = P^{-1}[\,(wx)(y)(z)\,]P$$

would have even parity on the LHS and odd parity on the RHS; this is not possible and no such relationship can exist. Hence $(wx)(y)(z)$ and $(pq)(rs)$ belong to different classes, implying that there are five classes, and hence five irreps, in all.

Let the five irreps have dimensions n_i, $(i = 1, 2, \ldots, 5)$, with $n_1 = 1$ as the dimension of the identity irrep (which must be present). Then

$$\sum_{i=2}^{5} n_i^2 = 24 - 1 = 23.$$

Since $4 \times 2^2 = 16 < 23$, at least one n_i must be ≥ 3 (and clearly < 5). If one of the n_i were 4, we would require $\sum_{i=2}^{4} n_i^2 = 7$ which has no integral solutions with all $n_i \geq 1$. Thus, one n_i, say n_5, must be equal to 3, leaving $\sum_{i=2}^{4} n_i^2 = 14$. Now, since $3(2^2) < 14 <$ a second n_i, n_4 say, must equal 3. This now leaves $\sum_{i=2}^{3} n_i^2 = 5$, which has only one possible integral solution n_3(say)$= 2$ and $n_2 = 1$.

(b) We have taken the first irrep as A_1. A second one-dimensional irrep, $\mathsf{D}^{(2)}$, will be one in which the odd permutations in S_4 [cycle structures (ii) and (iv)] have -1 as their 1×1 matrix representation whilst the even permutations [structures (i), (iii) and (v)] are represented by $+1$. Then, in the order of classes used above, the character set for $\mathsf{D}^{(2)}$ is $(1, -1, 1, -1, 1)$. This irrep is normally denoted by A_2.

(c) We now consider the action of a typical element of each class (i)-(iv) on the set $\{a, b, c, d\}$ and determine in each case how many of the set are unchanged; this gives the trace of the corresponding (natural) permutation matrix and hence its character. It is clear that this is equal to the number of cycles of length one in the corresponding cycle notation. The character set is therefore $(4, 2, 1, 0, 0)$.

As noted in the question, the vector space on which the permutations act contains one invariant subspace and this must transform as the identity irrep A_1 with character set $(1, 1, 1, 1, 1)$. We are told that the subspace remaining when this invariant subspace is removed is irreducible and must therefore transform according to a three-dimensional irrep T_1. The character set for this space must be $(4-1,\ 2-1,\ 1-1,\ 0-1,\ 0-1) = (3, 1, 0, -1, -1)$.

At this point we have a partial character table

Irrep	Typical element and class size				
	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	-1	1	-1	1
E	2	a	b	c	d
T_1	3	1	0	-1	-1
T_2	3	w	x	y	z

Consider next the characters of E. From the summation rule,

$$4 + 6a^2 + 8b^2 + 6c^2 + 3d^2 = 24.$$

Since all the terms except $3d^2$ are necessarily even, $3d^2$ must be as well. Thus $d = 0$ or $d = \pm 2$.

If $d = 0$, then orthogonality of the character set with those of A_1 and A_2 requires

$$\begin{aligned} &2 + 6a + 8b + 6c = 0, \\ &2 - 6a + 8b - 6c = 0, \\ &\quad \Rightarrow \quad 4 + 16b = 0. \quad \text{Impossible, as } b \text{ must be integral.} \end{aligned}$$

So $d = \pm 2$, and, from the summation rule,

$$4 + 6a^2 + 8b^2 + 6c^2 + 12 = 24 \quad \Rightarrow \quad 6a^2 + 8b^2 + 6c^2 = 8,$$

for which the only integral solution is $a = c = 0$ with $b = +1$ or -1. From orthogonality with A_1

$$2 + 0 + 8b + 0 + 3d = 0,$$

which has no integral solution if $d = -2$, but gives $b = -1$ if $d = +2$. This is the only acceptable solution and, in summary, the characters for E are $(2, 0, -1, 0, 2)$.

Finally we turn to the character set for T_2 for which we have

$$9 + 6w^2 + 8x^2 + 6y^2 + 3z^2 = 24.$$

Since $24 - 9$ is odd and $6w^2 + 8x^2 + 6y^2$ is even, z must be odd with $|z| < \sqrt{15/3}$, i.e. $z = +1$ or -1. Hence $6w^2 + 8x^2 + 6y^2 = 12$, which can only have an integral solution if $x = 0$ and $|w| = |y| = 1$. Now, orthogonality with the characters of A_1 gives $3 + 6w + 6y + 3z = 0$, and with those of A_2 gives $3 - 6w - 6y + 3z = 0$. Hence $z = -1$ and $w = -y$. The remaining ambiguity is resolved using the orthogonality with $\chi^{(T_1)}$:

$$9 + 6w - 6y + 3 = 0 \quad \Rightarrow \quad 12w + 12 = 0 \quad \Rightarrow \quad w = -1 \text{ and } y = 1.$$

Hence $\chi^{(T_2)} = (3, -1, 0, 1, -1)$ and the table is complete.

The summation rule was used to establish $\chi^{(E)}$ and $\chi^{(T_2)}$ and is easily verified for A_2 and T_1.

29.6 *Consider a regular hexagon orientated so that two of its vertices lie on the x-axis. Find matrix representations of a rotation R through $2\pi/6$ and a reflection m_y in the y-axis by determining their effects on vectors lying in the xy-plane . Show that a reflection m_x in the x-axis can be written as $m_x = m_y R^3$ and that the 12 elements of the symmetry group of the hexagon are given by R^n or $R^n m_y$.*

Using the representations of R and m_y as generators, find a two-dimensional representation of the symmetry group, C_6, of the regular hexagon. Is it a faithful representation?

Under the rotation R,

$$(x, y) \to \left(x \cos \tfrac{\pi}{3} - y \sin \tfrac{\pi}{3}, y \cos \tfrac{\pi}{3} + x \sin \tfrac{\pi}{3}\right) = \tfrac{1}{2} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Under the reflection m_y

$$(x, y) \to (-x, y) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

whilst under the reflection m_x

$$(x, y) \to (x, -y) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

We now consider $m_y R^3$ whose matrix representation is

$$\begin{aligned} &\tfrac{1}{8} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \\ &= \tfrac{1}{8} \begin{pmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix} \\ &= \tfrac{1}{8} \begin{pmatrix} 8 & 0 \\ 0 & -8 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \end{aligned}$$

i.e. equal to that for m_x, which can also be written $R^3 m_y$ since R^3 and m_y commute.

Denoting the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

by $\mathsf{M}(\theta)$, we see that the representation of R is $\mathsf{M}(\pi/3)$. Now,

$$\begin{aligned} \mathsf{M}(\theta)\mathsf{M}(\phi) &= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix} \\ &= \begin{pmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{pmatrix} \\ &= \mathsf{M}(\theta+\phi). \end{aligned}$$

It follows that R^n is represented by $\mathsf{M}(n\pi/3)$ and that a rotation by $n\pi/3$ is generated by R^n. This accounts for all the rotational symmetries of the hexagon including, formally, the identity I for which $n = 0$.

The reflection symmetries are of two kinds; three are in axes joining opposite vertices of the hexagon and are exemplified by m_x; three are in axes joining the mid-points of opposite sides as in the case of m_y. In each case, the other two reflections can be obtained by applying either R^2 or R^4 after the reflections.

Figure 29.2 summarises the situation. The label against each of the 12 dots (e.g. $R^5 m_y$) shows the effect on the original ringed point, marked (x, y), of the 12 corresponding symmetry operations. In cases including a reflection, the effective reflection axis for the whole operation is marked with the same symbol. Thus, reflection in the axis (marked $R^5 m_y$) through the vertices in the first and third quadrants carries (x, y) to the point marked $R^5 m_y$. Each of the 12 operations can be expressed either as R^m or as $R^n m_y$.

Figure 29.2 The rotation and reflection symmetries of a regular hexagon as discussed in exercise 29.6. A point labelled by the name of any particular symmetry shows the effect of that symmetry on the ringed point (x, y). The axis labelled $R^n m_y$ is the reflection axis corresponding to the point marked $R^n m_y$.

To ease the calculation of the representation, we note that $R^3 = -I$ and obtain

$$R = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix} = -R^4,$$

$$R^2 = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ \sqrt{3} & -1 \end{pmatrix} = -R^5,$$

$$Rm_y = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = -R^4m_y,$$

$$R^2m_y = \tfrac{1}{4}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}\begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} = \tfrac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ -\sqrt{3} & -1 \end{pmatrix} = -R^5m_y.$$

To these eight we must add

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R^3 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$m_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad R^3m_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The twelve matrices are all distinct and so the representation is faithful.

29.8 *Sulphur hexafluoride is a molecule with the same structure as the crystalline compound in exercise 29.7, except that a sulphur atom is now the central atom. The following are the forms of some of the electronic orbitals of the sulphur atom, together with the irreps according to which they transform under the symmetry group* 432 *(or O).*

$$\begin{array}{ll} \Psi_s = f(r) & \mathrm{A}_1 \\ \Psi_{p_1} = zf(r) & \mathrm{T}_1 \\ \Psi_{d_1} = (3z^2 - r^2)f(r) & \mathrm{E} \\ \Psi_{d_2} = (x^2 - y^2)f(r) & \mathrm{E} \\ \Psi_{d_3} = xyf(r) & \mathrm{T}_2 \end{array}$$

The function x transforms according to the irrep T_1. *Use the above data to determine whether dipole matrix elements of the form* $J = \int \phi_1 x \phi_2 \, d\tau$ *can be non-zero for the following pairs of orbitals* ϕ_1, ϕ_2 *in a sulphur hexafluoride molecule:* (a) Ψ_{d_1}, Ψ_s; (b) Ψ_{d_1}, Ψ_{p_1}; (c) Ψ_{d_2}, Ψ_{d_1}; (d) Ψ_s, Ψ_{d_3}; (e) Ψ_{p_1}, Ψ_s.

For each dipole matrix element we need to determine whether the irrep A_1 is present in the reduction of the representation of the triple product of ϕ_1, ϕ_2 and x; only if it is, can the dipole matrix element be non-zero. To do this we require the character table for the group 432 produced in exercise 29.5, namely

Irrep	Typical element and class size				
	I	2_d	3	4_z	2_z
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

We also need to use the formula

$$m_\mu = \frac{1}{g} \sum_X \left[\chi^{(\mu)}(X)\right]^* \chi(X) = \frac{1}{g} \sum_i c_i \left[\chi^{(\mu)}(X_i)\right]^* \chi(X_i)$$

with μ set to A_1 [for which all $\chi^{(\mu)}(X) = 1$] in order to calculate whether or not A_1 is present.

In each case we obtain the character set for the dipole matrix element by multiplying together, for each class, the corresponding characters of the two orbitals and x. The resulting character set then has to be tested using the above formula to see whether A_1 is present.

(a) Ψ_{d_1}, Ψ_s:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_s	A_1	1	1	1	1	1
x	T_1	3	1	0	−1	−1
		6	0	0	0	−2

Thus, $m_{A_1} = \dfrac{1(6) + 3(-2)}{24} = 0 \quad \Rightarrow \quad$ No.

(b) Ψ_{d_1}, Ψ_{p_1}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_{p_1}	T_1	3	1	0	−1	−1
x	T_1	3	1	0	−1	−1
		18	0	0	0	2

Thus, $m_{A_1} = \dfrac{1(18) + 3(2)}{24} = 1 \quad \Rightarrow \quad$ Yes.

(c) Ψ_{d_2}, Ψ_{d_1}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{d_1}	E	2	0	−1	0	2
Ψ_{d_2}	E	2	0	−1	0	2
x	T_1	3	1	0	−1	−1
		12	0	0	0	−4

Thus, $m_{A_1} = \dfrac{1(12) + 3(-4)}{24} = 0 \quad \Rightarrow \quad$ No.

(d) Ψ_s, Ψ_{d_3}:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_s	A_1	1	1	1	1	1
Ψ_{d_3}	T_2	3	−1	0	1	−1
x	T_1	3	1	0	−1	−1
		9	−1	0	−1	1

Thus, $m_{A_1} = \dfrac{1(9) + 6(-1) + 6(-1) + 3(1)}{24} = 0 \quad \Rightarrow \quad$ No.

(e) Ψ_{p_1}, Ψ_s:

Orbital	Irrep	I	2_d	3	4_z	2_z
		1	6	8	6	3
Ψ_{p_1}	T_1	3	1	0	-1	-1
Ψ_s	A_1	1	1	1	1	1
x	T_1	3	1	0	-1	-1
		9	1	0	1	1

Thus, $m_{A_1} = \dfrac{1(9) + 6(1) + 6(1) + 3(1)}{24} = 1 \quad \Rightarrow \quad$ Yes.

In summary, only in cases (b) and (e) is there the possibility of a non-zero dipole matrix element. Further calculation, involving data not provided here, would be needed to determine whether these two matrix elements are in fact non-zero.

29.10 *Investigate the properties of an alternating group and construct its character table as follows.*

(a) *The set of even permutations of four objects (a proper subgroup of S_4) is known as the* alternating group *A_4. List its twelve members using cycle notation.*

(b) *Assume that all permutations with the same cycle structure belong to the same conjugacy class. Show that this leads to a contradiction and hence demonstrates that even if two permutations have the same cycle structure they do not necessarily belong to the same class.*

(c) *By evaluating the products*

$$p_1 = (123)(4) \bullet (12)(34) \bullet (132)(4) \quad \textit{and} \quad p_2 = (132)(4) \bullet (12)(34) \bullet (123)(4)$$

deduce that the three elements of A_4 with structure of the form (12)(34) *belong to the same class.*

(d) *By evaluating products of the form* $(1\alpha)(\beta\gamma) \bullet (123)(4) \bullet (1\alpha)(\beta\gamma)$, *where α, β and γ are various combinations of* 2, 3 *and* 4, *show that the class to which* (123)(4) *belongs contains at least four members. Show the same for* (124)(3).

(e) *By combining results* (b), (c) *and* (d), *deduce that A_4 has exactly four classes, and determine the dimensions of its irreps.*

(f) *Using the orthogonality properties of characters and noting that elements of the form* (124)(3) *have order* 3, *find the character table for A_4.*

(a) The twelve members of A_4 are those permutations that involve an even number (0 or 2) of pair interchanges. For future identification, and as a shorthand, we list them, each with a label, as:

$I = (1)(2)(3)(4)$,
$2\text{A} = (12)(34)$, $2\text{B} = (13)(24)$, $2\text{C} = (14)(23)$,
$3\text{A} = (123)(4)$, $3\text{B} = (124)(3)$, $3\text{C} = (134)(2)$, $3\text{D} = (234)(1)$,
$3\text{E} = (132)(4)$, $3\text{F} = (142)(3)$, $3\text{G} = (143)(2)$, $3\text{H} = (243)(1)$.

(b) If permutations with the same cycle structure all belonged to the same conjugacy class there would be 3 classes in A_4. This would imply that it has 3 irreps, one of which would have to be the identity irrep A_1. The dimensions of the other two would then have to satisfy

$$1 + n_2^2 + n_3^2 = 12.$$

This equation has no integral solutions for n_2 and n_3 and we conclude that the assumption that permutations with the same cycle structure all belong to the same conjugacy class leads to a contradiction and is therefore wrong.

(c) With the meaning of (pqr) as in the text, i.e. p is replaced by q, q is replaced by r and r is replaced by p, we evaluate the following products of the form $P^{-1}XP$:

$$\begin{aligned} p_1 &= (123)(4) \bullet (12)(34) \bullet (132)(4)\ abcd = (123)(4) \bullet (12)(34)\ cabd \\ &= (123)(4)\ acdb = cdab = (13)(24)\ abcd \\ p_2 &= (132)(4) \bullet (12)(34) \bullet (123)(4)\ abcd = (132)(4) \bullet (12)(34)\ bcad \\ &= (132)(4)\ cbda = dcba = (14)(23)\ abcd \end{aligned}$$

Thus (13)(24) and (14)(23) both belong to the same conjugacy class as (12)(34) and therefore to the same class as each other, i.e. all three permutations with structure $(pq)(rs)$ belong to the same class.

(d) These evaluations follow the same lines as in (c) and we summarise the results using the labels allocated in (a).

$(2\text{A})^{-1}3\text{A}\ 2\text{A} = 3\text{F}$, $(2\text{B})^{-1}3\text{A}\ 2\text{B} = 3\text{C}$, $(2\text{C})^{-1}3\text{A}\ 2\text{C} = 3\text{H}$.

Thus the class to which 3A=(123)(4) belongs also contains 3F, 3C and 3H, i.e. it has at least four members. Further,

$(2\text{A})^{-1}3\text{B}\ 2\text{A} = 3\text{E}$, $(2\text{B})^{-1}3\text{B}\ 2\text{B} = 3\text{D}$, $(2\text{C})^{-1}3\text{B}\ 2\text{C} = 3\text{G}$,

showing that the class containing 3B also contains at least three other members.

(e) As always, I is in a class by itself and, as we have shown, the class of permutations with structure $(pq)(rs)$ has 3 members. Permutations with structure $(pqr)(s)$ are contained in a maximum of two classes since we have already shown that there exist two classes with at least four members each — and this exhausts the group. These two sets of four cannot form one class of eight as this would reduce the total number of classes to three, and we have shown in (b) that this is not possible. We conclude that there are exactly four classes containing 1, 4, 4

and 3 members. It also follows that there are four irreps, whose dimensions must satisfy

$$1 + n_2^2 + n_3^2 + n_4^2 = 12 \quad \Rightarrow \quad n_4 = 3,\ n_2 = n_3 = 1,$$

as the only integer solution.

(f) The character set for A_1 is, of course, $(1, 1, 1, 1)$. Suppose that for the three-dimensional irrep T it is $(3, x, y, z)$. Then, firstly,

$$1|3|^2 + 4|x|^2 + 4|y|^2 + 3|z|^2 = 12 \quad \Rightarrow \quad x = y = 0,\ |z| = 1,$$

for an integer solution. Secondly, the orthogonality of the two irreps gives

$$1(1)(3) + 4(1)(0) + 4(1)(0) + 3(1)z = 0 \quad \Rightarrow \quad z = -1.$$

The character table thus takes the form

Irrep	Typical element and class size			
	(1)	(123)	(132)	(12)(34)
	1	4	4	3
A_1	1	1	1	1
A_2	1	a	b	c
A_3	1	d	e	f
T	3	0	0	-1

The orthogonality of T to each of A_2 and A_3 implies that $c = f = 1$.

This leaves only a, b, d and e to be determined. Since the elements in both the classes of which they are the characters have order 3, each character can only be the sum of a number of cube roots of unity (see the text). As they are further restricted by summation rules of the form $1 + 4|a|^2 + 4|b|^2 + 3 = 12$, $|a| \leq \sqrt{2}$ and similarly for the other three characters.

The cube roots of unity are 1, $\omega = \exp(2\pi i/3)$ and ω^2, with $1 + \omega + \omega^2 = 0$. There must be at least one character in each of the sets for A_2 and A_3 that is not unity – otherwise they become A_1. Let $a = \omega$; then, from the orthogonality with A_1, $1 + 4\omega + 4b + 3 = 0$ implying that $b = \omega^2$. Since all the character sets must be different, we must have for A_3 that $d = \omega^2$ and $e = \omega$, thus completing the character table for the group A_4 as

Irrep	Typical element and class size			
	(1)	(123)	(132)	(12)(34)
	1	4	4	3
A_1	1	1	1	1
A_2	1	ω	ω^2	1
A_3	1	ω^2	ω	1
T	3	0	0	-1

Figure 29.3 The three normal vibration modes of the equilateral array. Mode (a) is known as the 'breathing mode'. Modes (b) and (c) transform according to irrep E and have equal vibrational frequencies.

The orthogonality of the character sets for A_2 and A_3 may verified as follows:

$$\sum_X \left[\chi^{(A_2)}(X)\right]^* \chi^{(A_3)}(X) = 1(1)(1) + 4\omega^*\omega^2 + 4(\omega^2)^*\omega + 3(1)(1)$$
$$= 1 + 4e^{-i2\pi/3}e^{i4\pi/3} + 4e^{-i4\pi/3}e^{i2\pi/3} + 3$$
$$= 4 + 4[2\cos(2\pi/3)] = 4 + 4(-1) = 0.$$

29.12 *Demonstrate that equation (29.24) does indeed generate a set of vectors transforming according to an irrep λ, by sketching and superposing drawings of an equilateral triangle of springs and masses, based on that shown in figure 29.4.*

(a) *Make an initial sketch showing an arbitrary small mass displacement from, say, vertex C. Draw the results of operating on this initial sketch with each of the symmetry elements of the group 3m (C_{3v}).*

(b) *Superimpose the results, weighting them according to the characters of irrep* A_1 *(table 29.1 in section 29.6) and verify that the resultant is a symmetrical arrangement in which all three masses move symmetrically towards (or away from) the centroid of the triangle. The mode is illustrated in figure 29.4*(a).

(c) *Start again, now considering a displacement δ of C parallel to the x-axis. Form a similar superposition of sketches weighted according to the characters of irrep* E *(note that the reflections are not needed). The resultant contains some bodily displacement of the triangle, since this also transforms according to* E. *Show that the displacement of the centre of mass is $\bar{x} = \delta$, $\bar{y} = 0$. Subtract this out and verify that the remainder is of the form shown in figure 29.4*(c).

(d) *Using an initial displacement parallel to the y-axis, and an analogous procedure, generate the remaining normal mode, degenerate with that in* (c) *and shown in figure 29.4*(b).

Figure 29.4 The construction of the 'breathing mode' of an equilateral array of equal springs and masses, as discussed in exercise 29.12.

(a) Part (i) of figure 29.4 shows the triangle with an initial displacement at vertex C and the results of operating on this with rotations R and R^2 and reflections m_1, m_2 and m_3. Of course, the original is also the result of operating with the identity I.

(b) To use equation (29.24),

$$\Psi_i^{(\lambda)} = \sum_X \chi^{(\lambda)^*}(X) X \Psi_i$$

with $\lambda = \mathrm{A}_1$ we need the character set for A_1 which is $(1, 1, 1)$. Thus the six triangles shown in part (i) of the figure have to be superimposed, all with equal weights. This has been done in part (ii) of the figure and, after the two displacements at each vertex have been added vectorially, the result is shown to be that of the 'breathing mode'. In this mode all movements are directed away from (or towards) the centroid of the triangle.

(c) For the two-dimensional irrep E the character set is $(2, -1, 0)$, showing that

the projection operator will contain no contribution from the third class, namely the reflections.

The original displacement δ parallel to the x-axis and the results of operating on this with R and R^2 are shown in part (i) of figure 29.5. These have to be superimposed with the original (in the role of the result of I) having weight 2 and the results of the rotations having weight -1, i.e. the directions of the displacements are reversed. This is done in part (ii) of the figure. The superposition can be broken down into an overall bodily displacement of the triangle and displacements about its centroid as follows:

$$
\begin{aligned}
(\bar{x}, \bar{y}) &= \tfrac{1}{3}(2\delta + \delta\cos\frac{\pi}{3} + \delta\cos\frac{\pi}{3}, 0 + \delta\sin\frac{\pi}{3} - \delta\sin\frac{\pi}{3}) = (\delta, 0),\\
(x, y)_C &= (\bar{x}, \bar{y}) + (\delta, 0),\\
(x, y)_A &= (\bar{x}, \bar{y}) + (-\frac{1}{2}\delta, -\frac{\sqrt{3}}{2}\delta),\\
(x, y)_B &= (\bar{x}, \bar{y}) + (-\frac{1}{2}\delta, \frac{\sqrt{3}}{2}\delta).
\end{aligned}
$$

This breakdown is also shown in part (ii). Note that all the vibrational displacements are of magnitude δ.

(d) The final normal mode, degenerate with that in (c), is shown in part (iii) of figure 29.5. The construction parallels that in (c) and so only the calculational details are given. They are:

$$
\begin{aligned}
(\bar{x}, \bar{y}) &= \tfrac{1}{3}(\delta\cos\frac{\pi}{3} - \delta\cos\frac{\pi}{3}, 2\delta + \delta\sin\frac{\pi}{3} + \delta\sin\frac{\pi}{3}) = (0, \delta),\\
(x, y)_C &= (\bar{x}, \bar{y}) + (0, \delta),\\
(x, y)_A &= (\bar{x}, \bar{y}) + (\frac{\sqrt{3}}{2}\delta, -\frac{1}{2}\delta),\\
(x, y)_B &= (\bar{x}, \bar{y}) + (-\frac{\sqrt{3}}{2}\delta, -\frac{1}{2}\delta).
\end{aligned}
$$

This is mode (b) as given in the question.

Figure 29.5 The construction of each of the two degenerate normal modes of an equilateral array of equal springs and masses, as discussed in exercise 29.12. Construction of the first mode is shown in (i) and (ii); that of the second mode is shown in (iii).

30

Probability

30.2 *Given that events X, Y and Z satisfy*

$$(X \cap Y) \cup (Z \cap X) \cup \overline{(\bar{X} \cup \bar{Y})} = \overline{(Z \cup \bar{Y})} \cup \{[\overline{(\bar{Z} \cup \bar{X})} \cup (\bar{X} \cap Z)] \cap Y\},$$

prove that $X \supset Y$ and either $X \cap Z = \emptyset$ or $Y \supset Z$.

We start by simplifying both sides of the equation separately using the commutativity and distributional properties of $\cup$ and $\cap$ and de Morgan's laws. For the LHS:

$$\begin{aligned}(X \cap Y) \cup (Z \cap X) \cup \overline{(\bar{X} \cup \bar{Y})} &= [(Y \cup Z) \cap X] \cup (X \cap Y) \\ &= X \cap (Y \cup Z \cup Y) \\ &= X \cap (Y \cup Z).\end{aligned}$$

For the RHS we have

$$\begin{aligned}&\overline{(Z \cup \bar{Y})} \cup \{[\overline{(\bar{Z} \cup \bar{X})} \cup (\bar{X} \cap Z)] \cap Y\} \\ &= (\bar{Z} \cap Y) \cup \{[(Z \cap X) \cup (\bar{X} \cap Z)] \cap Y\} \\ &= (\bar{Z} \cap Y) \cup [Z \cap (X \cup \bar{X}) \cap Y] \\ &= (\bar{Z} \cap Y) \cup (Z \cap Y) \\ &= Y \cap (\bar{Z} \cup Z) = Y.\end{aligned}$$

Thus the equation reduces to $X \cap (Y \cup Z) = Y$. This implies that X contains everything that is in Y, i.e. $X \supset Y$, and that X contains no part of Z that is not also in Y. This latter requirement means that *either* Z is wholly contained in Y, i.e. $Y \supset Z$ *or* X and Z have no events in common, i.e. $X \cap Z = \emptyset$.

30.4 *Use the method of induction to prove equation (30.16), the probability addition law for the union of n general events.*

We are required to prove that

$$\begin{aligned}\Pr(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_i \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) \\ + \sum_{i,j,k} \Pr(A_i \cap A_j \cap A_k) - \cdots \\ \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n). \quad (*)\end{aligned}$$

We do so by first assuming that $(*)$ is true for some particular value of n and use this to prove that this implies that it is true for $n \to n+1$. The relationship is obvious for $n = 1$.

Let event B be the union of events $A_1, A_2, \ldots, A_n$ and apply

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B), \qquad (**)$$

with event A as A_{n+1}. The probability $\Pr(A)$ is simply $\Pr(A_{n+1})$ and $\Pr(B)$ is the assumed result for the probability of the union of n events. This leaves only the calculation of the final term $\Pr(A \cap B)$. This is given by

$$\begin{aligned}\Pr(B \cap A_{n+1}) &= \Pr[(A_1 \cup A_2 \cup \cdots \cup A_n) \cap A_{n+1}] \\ &= \Pr[(A_1 \cap A_{n+1}) \cup (A_2 \cap A_{n+1}) \cup \cdots \cup (A_n \cap A_{n+1})] \\ &\equiv \Pr(C_1 \cup C_2 \cup \cdots \cup C_n),\end{aligned}$$

where we have defined the set of n events C_i as $C_i = A_i \cap A_{n+1}$.

We now apply the result assumed valid for n events to the C_i and obtain

$$\begin{aligned}&\Pr(B \cap A_{n+1}) \\ &= \sum_i^n \Pr(C_i) - \sum_{i,j}^n \Pr(C_i \cap C_j) + \cdots + (-1)^{n+1} \Pr(C_1 \cap C_2 \cap \cdots \cap C_n) \\ &= \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[(A_i \cap A_{n+1}) \cap (A_j \cap A_{n+1})] + \\ &\quad \cdots + (-1)^{n+1} \Pr[(A_1 \cap A_{n+1}) \cap (A_2 \cap A_{n+1}) \cap \cdots \cap (A_n \cap A_{n+1})] \\ &= \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[(A_i \cap A_j) \cap A_{n+1})] + \\ &\quad \cdots + (-1)^{n+1} \Pr[(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1})].\end{aligned}$$

We now substitute for the various terms in (**) and obtain

$$\begin{aligned}
&\Pr(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) \\
&= \Pr(A_{n+1}) + \sum_i^n \Pr(A_i) - \sum_{i,j}^n \Pr(A_i \cap A_j) + \\
&\quad \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n) \\
&\quad -\Big\{ \sum_i^n \Pr(A_i \cap A_{n+1}) - \sum_{i,j}^n \Pr[(A_i \cap A_j) \cap A_{n+1})] + \\
&\quad \cdots + (-1)^{n+1} \Pr[(A_1 \cap A_2 \cap \cdots \cap A_n) \cap A_{n+1})] \Big\}.
\end{aligned}$$

Finally, collecting together similar terms and noting that, for example, $(P \cap Q) \cap R = P \cap Q \cap R$, we obtain

$$\begin{aligned}
&\Pr(A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}) \\
&= \sum_i^{n+1} \Pr(A_i) - \sum_{i,j}^{n+1} \Pr(A_i \cap A_j) + \sum_{i,j,k}^{n+1} \Pr(A_i \cap A_j \cap A_k) - \\
&\quad \cdots + (-1)^{n+2} \Pr(A_1 \cap A_2 \cap \cdots \cap A_{n+1}).
\end{aligned}$$

All summations now run over $i, j, \ldots = 1, 2, \ldots, n+1$ and this expression is the same as (*) but with n increased to $n+1$. This, together with our earlier observation that the formula is valid for $n = 1$ completes the proof by induction.

30.6 *$X_1, X_2, \ldots, X_n$ are independent identically distributed random variables drawn from a uniform distribution on* $[0, 1]$. *The random variables A and B are defined by*

$$A = \min(X_1, X_2, \ldots, X_n), \qquad B = \max(X_1, X_2, \ldots, X_n).$$

For any fixed k such that $0 \le k \le \frac{1}{2}$, find the probability p_n that both

$$A \le k \qquad \textit{and} \qquad B \ge 1 - k.$$

Check your general formula by considering directly the cases (a) $k = 0$, (b) $k = \frac{1}{2}$, (c) $n = 1$ *and* (d) $n = 2$.

There are four possible situations, according as to whether A is less than or greater than k and as to whether B is less than or greater than $1 - k$. We need to calculate the probability for just one of these combinations and we do so by finding the probabilities for the other three and subtracting them from unity.

If $A \ge k$ then all X_i must lie in $k \le X_i \le 1$. This has probability $(1-k)^n$. Similarly,

$$\Pr(B \le 1 - k) = (1 - k)^n \quad \text{and} \quad \Pr(A \ge k \text{ and } B \le 1 - k) = (1 - 2k)^n.$$

Now,

$$\Pr(A \geq k) = \Pr(A \geq k \text{ and } B \leq 1-k) + \Pr(A \geq k \text{ and } B \geq 1-k),$$
$$(1-k)^n = (1-2k)^n + \Pr(A \geq k \text{ and } B \geq 1-k),$$

and, substituting from this result into

$$\Pr(B \geq 1-k) = \Pr(B \geq 1-k \text{ and } A \geq k) + \Pr(B \geq 1-k \text{ and } A \leq k),$$

gives

$$1-(1-k)^n = [(1-k)^n - (1-2k)^n] + \Pr(B \geq 1-k \text{ and } A \leq k).$$

Thus

$$p_n = \Pr(B \geq 1-k \text{ and } A \leq k) = 1 - 2(1-k)^n + (1-2k)^n.$$

In retrospect, the structure of this answer is straightforward to understand. Its RHS is

$$\begin{aligned}&\Pr(A \text{ and } B \text{ have some values}) - \Pr(\text{all the } X_i \text{ lie in } k < X_i < 1)\\ &\quad - \Pr(\text{all the } X_i \text{ lie in } 0 < X_i < 1-k)\\ &\qquad + \Pr(\text{all the } X_i \text{ lie in } k < X_i < 1-k),\end{aligned}$$

the final term being added back in to account for the fact that the range $k < X_i < 1-k$ has been subtracted out twice (instead of once) by the previous two terms.

For the special cases given we have:

(a) $k = 0$, i.e $A \leq 0$ and $B \geq 1$. This clearly has zero probability, in agreement with $1 - 2(1-0)^n + (1-0)^n = 0$ for all n.

(b) $k = \frac{1}{2}$, i.e $A \leq \frac{1}{2}$ and $B \geq \frac{1}{2}$. This outcome requires the avoidance of a situation in which all the X_i are in one half of the range. For all of them to be in one half has probability $(\frac{1}{2})^n$; this has to be doubled as there are two possible half ranges. The probability of $A \leq \frac{1}{2}$ and $B \geq \frac{1}{2}$ is therefore $1 - (\frac{1}{2})^{n-1}$. The formula derived earlier gives $1 - 2(1-\frac{1}{2})^n + (1-1)^n = 1 - (\frac{1}{2})^{n-1}$, which is in agreement with this direct calculation.

(c) $n = 1$. Clearly a single random variable, which has to act as both the minimum and the maximum of its set, cannot satisfy both inequalities. The probability in this case must be zero, as given by the general formula $1 - 2(1-k) + (1-2k) = 0$ for any k.

(d) $n = 2$. In this case, the first of the two X_i has to be less than k or greater than $1-k$. This has probability $2k$. The second then has to fall in a particular one of these two ranges; this has probability k. The overall probability is therefore $2k^2$. The derived formula gives $1 - 2(1-2k+k^2) + (1-4k+4k^2) = 2k^2$, as expected.

30.8 *This exercise shows that the odds are hardly ever 'evens' when it comes to dice rolling.*

(a) *Gamblers A and B each roll a fair six-faced die, and B wins if his score is strictly greater than A's. Show that the odds are* 7 *to* 5 *in A's favour.*
(b) *Calculate the probabilities of scoring a total T from two rolls of a fair die for* $T = 2, 3, \ldots, 12$. *Gamblers C and D each roll a fair die twice and score respective totals* T_C *and* T_D, *D winning if* $T_D > T_C$. *Realising that the odds are not equal, D insists that C should increase her stake for each game. C agrees to stake £1.10 per game, as compared to D's £1.00 stake. Who will show a profit?*

(a) Out of the 36 equally likely outcomes for a single roll of each die, there are 6 in which the scores are equal and result in a win for A. The other 30 cases, in which the scores are unequal, provide 15 wins for A and 15 wins for B. Thus A wins in $6 + 15 = 21$ of the 36 cases and B in only 15 cases, giving A favourable odds of 7 to 5.

(b) The probability distribution $p(T)$ for the total T from two rolls of a die is

T	2	3	4	5	6	7	8	9	10	11	12
$36p(T)$	1	2	3	4	5	6	5	4	3	2	1

The probability that $T_D > T_C$ could be calculated for each T_D by adding all the probabilities for T_C up to $T_D - 1$ and the overall probability found by then weighting each sum by the probability of T_D. However, it is simpler to add up the probabilities for the two totals being equal, subtract this from unity and then take $\Pr(T_D > T_C)$ as one-half of what is left, i.e.

$$\begin{aligned}\Pr(T_D > T_C) &= \frac{1}{2}\left\{1 - \frac{1}{(36)^2}[\,2(1^2 + 2^2 + 3^2 + 4^2 + 5^2) + 6^2\,]\right\} \\ &= \frac{1}{2}\left[1 - \frac{146}{(36)^2}\right] = 0.4437.\end{aligned}$$

This is the chance that D wins and so C's expected return for a *£1.10* stake is *£2.10*$(1 - 0.4437) =$ *£1.168*, i.e. a profit of about 7 pence per game.

A straightforward calculation shows that C should stake just over *£1.25* to make the game 'fair'.

30.10 *As every student of probability theory will know, Bayesylvania is awash with natives, not all of whom can be trusted to tell the truth, and lost and apparently somewhat deaf travellers who ask the same question several times in an attempt to get directions to the nearest village.*

One such traveller finds himself at a T-junction in an area populated by the Asciis and Bisciis in the ratio 11 *to* 5. *As is well known, the Biscii always lie but the Ascii tell the truth three quarters of the time, giving independent answers to all questions, even to immediately repeated ones.*

(a) *The traveller asks one particular native twice whether he should go to the left or to the right to reach the local village. Each time he is told 'left'. Should he take this advice, and, if he does, what are his chances of reaching the village*?

(b) *The traveller then asks the same native the same question a third time and for a third time receives the answer 'left'. What should the traveller do now*? *Have his chances of finding the village been altered by asking the third question*?

If the native is an Ascii then the chance of getting the same answer twice is

$$\frac{3}{4}\frac{3}{4}+\frac{1}{4}\frac{1}{4}=\frac{5}{8}.$$

The same calculation shows that a repeated answer by an Ascii is 9 times more likely to be the truth than a lie. If the native is a Biscii, then a repeated answer is guaranteed; it is also guaranteed to be a lie.

With A denoting the event that the native is an Ascii and B that he/she is a Biscii, let E be the event that the same answer is given twice by the native. Then, by Bayes' theorem,

$$\begin{aligned}\Pr(A|E) &= \frac{\Pr(E|A)\ \Pr(A)}{\Pr(E|A)\ \Pr(A)+\Pr(E|B)\ \Pr(B)}\\ &= \frac{\frac{5}{8}\ \frac{11}{16}}{\frac{5}{8}\ \frac{11}{16}+1\ \frac{5}{16}}=\frac{55}{95}.\end{aligned}$$

The traveller can only be told the truth if the native is an Ascii and the two identical responses are the truth. The combined probability for this is $(55/95)\times(9/10)=99/190=52.11\%$. As this is more than 50%, the traveller should go *left*.

(b) The corresponding calculations for the event F of the same answer of 'left'

being given three times are:

$$\Pr(F|A) = \left(\frac{3}{4}\right)^3 + \left(\frac{1}{4}\right)^3 = \frac{28}{64},$$

with the truth 27 times more likely than a lie;

$$\Pr(A|F) = \frac{\frac{28}{64}\,\frac{11}{16}}{\frac{28}{64}\,\frac{11}{16} + 1\,\frac{5}{16}} = \frac{308}{628};$$

the overall probability of being told the truth is $(308/628) \times (27/28) = 297/628$. As this is less than half he should go *right* with a probability of $331/628 = 52.71\%$ of being correct — a very marginal improvement in his chances!

30.12 *Villages A, B, C and D are connected by overhead telephone lines joining AB, AC, BC, BD and CD. As a result of severe gales, there is a probability p (the same for each link) that any particular link is broken.*

(a) *Show that the probability that a call can be made from A to B is*

$$1 - p^2 - 2p^3 + 3p^4 - p^5.$$

(b) *Show that the probability that a call can be made from D to A is*

$$1 - 2p^2 - 2p^3 + 5p^4 - 2p^5.$$

We denote the probability that a link is intact by q, equal to $1 - p$.

(a) Situations with a total n of breaks have a probability of occuring equal to $p^n q^{5-n}$. For each value of n we need to identify the number m out of the ${}^5C_n$ possible combinations of breaks that will still allow a call to be made from A to B. Denote an intact link AB by its name and a break in AB by $\overline{AB}$.

n	probability	${}^5C_n$	acceptable break patterns	m
0	q^5	1	any (with 0 breaks)	1
1	pq^4	5	any with 1 break	5
2	p^2q^3	10	all except $\overline{AB} + \overline{AC}$	9
3	p^3q^2	10	AB with any of ${}^4C_3$ break patterns	4
			$AC + BC$	1
4	p^4q	5	only AB	1
5	p^5	1	none	0

The total probability of being able to make a call from A to B is

$$\begin{aligned} P_{AB} &= q^5 + 5pq^4 + 9p^2q^3 + 5p^3q^2 + p^4q \\ &= (p+q)^5 - p^2q^3 - 5p^3q^2 - 4p^4q - p^5 \\ &= 1 - p^2(q^3 + 5pq^2 + 4p^2q + p^3) \\ &= 1 - p^2[(q+p)^3 + 2pq^2 + p^2q] \\ &= 1 - p^2(1 + 2p - 4p^2 + 2p^3 + p^2 - p^3) \\ &= 1 - p^2 - 2p^3 + 3p^4 - p^5. \end{aligned}$$

(b) A similar count for a call from A to D gives

n	probability	${}^5C_n$	acceptable break patterns	m
0	q^5	1	any (with 0 breaks)	1
1	pq^4	5	any with 1 break	5
2	p^2q^3	10	all except $\overline{AB} + \overline{AC}$ and $\overline{BD} + \overline{CD}$	8
3	p^3q^2	10	only $AB + BD$ and $AC + CD$	2
4	p^4q	5	none	0
5	p^5	1	none	0

The total probability of being able to make a call from A to D is

$$\begin{aligned} P_{AD} &= q^5 + 5pq^4 + 8p^2q^3 + 2p^3q^2 \\ &= (p+q)^5 - 2p^2q^3 - 8p^3q^2 - 5p^4q - p^5 \\ &= 1 - p^2(2q^3 + 8pq^2 + 5p^2q + p^3) \\ &= 1 - p^2(2 - 6p + 6p^2 - 2p^3 + 8p - 16p^2 + 8p^3 + 5p^2 - 5p^3 + p^3) \\ &= 1 - 2p^2 - 2p^3 + 5p^4 - 2p^5. \end{aligned}$$

As they should, both calculations give unit probability if $p = 0$ and zero probability if $p = 1$.

30.14 *A certain marksman never misses his target, which consists of a disc of unit radius with centre O. The probability that any given shot will hit the target within a distance t of O is t^2 for $0 \le t \le 1$. The marksman fires n independendent shots at the target, and the random variable Y is the radius of the smallest circle with centre O that encloses all the shots. Determine the PDF for Y and hence find the expected area of the circle.*

The shot that is furthest from O is now rejected and the corresponding circle determined for the remaining $n-1$ shots. Show that its expected area is

$$\frac{n-1}{n+1}\pi.$$

Let the n shots be at distances r_i $(i = 1, 2, \dots, n)$ from O; Y is equal to the largest of these. The cumulative distribution function for each shot is $F(y) = y^2$ and the probability that all n shots lie within y of the centre is y^{2n}. But this is also the CDF for Y, whose PDF must therefore be its derivative, i.e. $2ny^{2n-1}$. The expected area for covering the n shots is thus

$$A_n = \int_0^1 \pi y^2 \, 2ny^{2n-1} \, dy = 2\pi n \int_0^1 y^{2n+1} \, dy = \frac{2\pi n}{2n+2} = \frac{n\pi}{n+1}.$$

As already found, the probability that the worst shot lies in the range y to $y + dy$ is $2ny^{2n-1} \, dy$. The other $n - 1$ shots are individually distributed as given in the question, but only over the region $0 < z < y$. Their common CDF is therefore z^2/y^2 (reaching unity at $z = y$). The CDF for all $n - 1$ shots is thus $(z^2/y^2)^{n-1}$ and the corresponding PDF is its derivative, $(2n-2)z^{2n-3}/y^{2n-2}$.

We now need to average the area of the circle covering the best $n - 1$ shots over all values of the radius of the worst shot. This gives

$$\begin{aligned} A_{n-1} &= \int_0^1 2ny^{2n-1} \, dy \int_0^y \pi z^2 \, \frac{(2n-2)z^{2n-3}}{y^{2n-2}} \, dz \\ &= 2n(2n-2)\pi \int_0^1 y \, dy \int_0^y z^{2n-1} \, dz \\ &= 2n(2n-2)\pi \int_0^1 \frac{y^{2n+1}}{2n} \, dy \\ &= (2n-2)\pi \, \frac{1}{2n+2} = \frac{n-1}{n+1} \, \pi. \end{aligned}$$

30.16 *Kittens from different litters do not get on with each other and fighting breaks out whenever two kittens from different litters are present together. A cage initially contains x kittens from one litter and y from another. To quell the fighting, kittens are removed at random, one at a time, until peace is restored. Show, by induction, that the expected number of kittens finally remaining is*

$$N(x, y) = \frac{x}{y+1} + \frac{y}{x+1}.$$

This result is trivially true if either of x or y is zero, as no kittens need to be removed. We therefore consider $x, y \geq 1$. It is also clear that $N(a, b)$ must be equal to $N(b, a)$, as this is only a matter of labelling the litters.

Let the cage contain x kittens from the first litter and $y + 1$ from the second and consider the removal of one randomly chosen kitten. With probability $(y +$

$1)/(x+y+1)$ it will be from the second litter and the expected final number of kittens will be $N(x,y)$. Correspondingly, with probability $x/(x+y+1)$ it will be from the first litter and the expected final number of kittens will be $N(x-1,y+1)$. Thus we have the recurrence relation

$$N(x,y+1) = \frac{y+1}{x+y+1}\,N(x,y) + \frac{x}{x+y+1}\,N(x-1,y+1).$$

Now *suppose* that

$$N(x,y) = \frac{x}{y+1} + \frac{y}{x+1} \qquad (*)$$

for all x, y such that $x+y=n$ and apply the assumption to the RHS of the above equation.

$$\begin{aligned}
N(x,y+1) &= \frac{y+1}{x+y+1}\left(\frac{x}{y+1}+\frac{y}{x+1}\right) \\
&\qquad + \frac{x}{x+y+1}\left(\frac{x-1}{y+2}+\frac{y+1}{x}\right) \\
&= \frac{1}{x+y+1}\left(x+\frac{y^2+y}{x+1}+\frac{x^2-x}{y+2}+y+1\right) \\
&= \frac{1}{x+y+1}\left[\left(x+\frac{x^2-x}{y+2}\right)+\left(y+1+\frac{y^2+y}{x+1}\right)\right] \\
&= \frac{1}{x+y+1}\left(\frac{xy+x+x^2}{y+2}+\frac{xy+x+2y+1+y^2}{x+1}\right) \\
&= \frac{x}{y+2}+\frac{yx+y^2+y+x+y+1}{(x+y+1)(x+1)} \\
&= \frac{x}{y+2}+\frac{y+1}{x+1} \qquad (**) \\
&= \frac{x}{(y+1)+1}+\frac{(y+1)}{x+1}.
\end{aligned}$$

Either by interchanging x and y in $(**)$ to obtain $N(y,x+1)$ and then using $N(a,b)=N(b,a)$, or by a similar calculation to the above, we can also show that

$$N(x+1,y) = \frac{(x+1)}{y+1} + \frac{y}{(x+1)+1}.$$

Combining the two results then shows that the assumption that $(*)$ is valid for $x+y=n$ implies that it is valid for $x+y=n+1$.

However, it is valid by direct inspection for $n=2$, since $N(2,0)=2=N(0,2)$ (no kitten need be removed) and $N(1,1)=1$ (one kitten must be removed). Thus $(*)$ is valid for all n and hence for all x and y. It is formally valid, by inspection, for $n=1$, but the proof given then involves negative arguments of N, albeit with N multiplied by zero.

30.18 *A particle is confined to the one-dimensional space $0 \le x \le a$ and classically it can be in any small interval dx with equal probability. However, quantum mechanics gives the result that the probability distribution is proportional to $\sin^2(n\pi x/a)$, where n is an integer. Find the variance in the particle's position in both the classical and quantum mechanical pictures and show that, although they differ, the latter tends to the former in the limit of large n, in agreement with the correspondence principle of physics.*

Classical Mechanics

Here, since the probability is uniformly distributed throughout the interval $0 \le x \le a$, we have

$$p(x)\,dx = \frac{1}{a}\,dx \quad \Rightarrow \quad \bar{x} = \frac{a}{2}.$$

The corresponding variance in the position of the particle is

$$V[X] = \int_0^a \left(x - \frac{a}{2}\right)^2 \frac{1}{a}\,dx = \frac{1}{a}\left[\frac{(x - \frac{1}{2}a)^3}{3}\right]_0^a = \frac{a^2}{12}.$$

Quantum Mechanics

The probability density is

$$p(x)\,dx = A\sin^2\frac{n\pi x}{a}\,dx, \text{ where } \int_0^a A\sin^2\frac{n\pi x}{a}\,dx = 1 \quad \Rightarrow \quad A = \frac{2}{a}.$$

The mean value of x is

$$\bar{x} = \frac{2}{a}\int_0^a x\sin^2\frac{n\pi x}{a}\,dx = \frac{a}{2}, \text{ by symmetry.}$$

We compute the variance as $\overline{x^2} - \bar{x}^2$.

$$\begin{aligned}
V(x) &= \frac{2}{a}\int_0^a x^2\sin^2\frac{n\pi x}{a}\,dx - \frac{a^2}{4} \\
&= \frac{1}{a}\int_0^a x^2\left(1 - \cos\frac{2n\pi x}{a}\right)dx - \frac{a^2}{4} \\
&= \frac{1}{a}\frac{a^3}{3} - \frac{1}{a}\left[\frac{ax^2}{2n\pi}\sin\frac{2n\pi x}{a}\right]_0^a + \frac{1}{a}\int_0^a \frac{a2x}{2n\pi}\sin\frac{2n\pi x}{a}\,dx - \frac{a^2}{4} \\
&= \frac{a^2}{12} - 0 + \frac{1}{n\pi}\left[-\frac{ax}{2n\pi}\cos\frac{2n\pi x}{a}\right]_0^a + \frac{1}{n\pi}\int_0^a \frac{a}{2n\pi}\cos\frac{2n\pi x}{a}\,dx \\
&= \frac{a^2}{12} - \frac{a^2}{2n^2\pi^2} + 0 + 0.
\end{aligned}$$

The classical and quantum results differ by an amount that depends upon n, but the latter tends to the former as $n \to \infty$.

30.20 *For a non-negative integer random variable X, in addition to the probability generating function $\Phi_X(t)$ defined in equation (30.71) it is possible to define the probability generating function*

$$\Psi_X(t) = \sum_{n=0}^{\infty} g_n t^n,$$

where g_n is the probability that $X > n$.

(a) *Prove that Φ_X and Ψ_X are related by*

$$\Psi_X(t) = \frac{1 - \Phi_X(t)}{1 - t}.$$

(b) *Show that $E[X]$ is given by $\Psi_X(1)$ and that the variance of X can be expressed as $2\Psi'_X(1) + \Psi_X(1) - [\Psi_X(1)]^2$.*

(c) *For a particular random variable X, the probability that $X > n$ is equal to α^{n+1} with $0 < \alpha < 1$. Use the results in* (b) *to show that $V[X] = \alpha(1-\alpha)^{-2}$.*

(a) We first note, from the definition of Ψ_X, that $g_0 = 1 - f_0$ and, for general n, that $g_n = f_{n+1} + f_{n+2} + \cdots$. Now consider

$$\begin{aligned}
(1-t)\Psi_X(t) &= (1-t)\sum_{n=0}^{\infty} g_n t^n \\
&= \sum_{n=1}^{\infty}(g_n - g_{n-1})t^n + g_0 \\
&= -\sum_{n=1}^{\infty} f_n t^n + 1 - f_0 \\
&= 1 - \sum_{n=0}^{\infty} f_n t^n \\
&= 1 - \Phi_X(t),
\end{aligned}$$

thus establishing the given result.

(b) We wish to express the standard result that $E[X] = \Phi'_X(1)$ in terms of $\Psi_X(t)$ and to do so differentiate the equation derived in (a) with respect to t and then set $t = 1$:

$$(1-t)\Psi'_X(t) - \Psi_X(t) = 0 - \Phi'_X(t) \quad \Rightarrow \quad \Psi_X(1) = \Phi'_X(1) = E[X].$$

For the variance, we need to obtain alternative expressions for the terms that appear in the general result

$$V[X] = \Phi''_X(1) + \Phi'_X(1) - [\Phi'_X(1)]^2.$$

The final two terms are already dealt with; for the first we differentiate the earlier result a second time and obtain

$$(1-t)\Psi''_X(t) - \Psi'_X(t) - \Psi'_X(t) = -\Phi''_X(t).$$

Setting $t = 1$ shows that $-\Phi''_X(1) = -2\Psi'_X(1)$. Substitution in the expression for the variance then shows that $V[X] = 2\Psi'_X(1) + \Psi_X(1) - [\Psi_X(1)]^2$.

(c) As the probability that $X > n$ is equal to α^{n+1} with $0 < \alpha < 1$, $g_n = \alpha^{n+1}$ and so

$$\Psi_X(t) = \sum_{n=0}^{\infty} \alpha^{n+1} t^n = \frac{\alpha}{1-\alpha t},$$

$$\Psi'_X(t) = \frac{\alpha^2}{(1-\alpha t)^2}.$$

The mean of the distribution is $\Psi_X(1) = \dfrac{\alpha}{1-\alpha}$ and the variance is given by

$$\begin{aligned} V[X] &= 2\Psi'_X(1) + \Psi_X(1) - [\Psi_X(1)]^2 \\ &= \frac{2\alpha^2}{(1-\alpha)^2} + \frac{\alpha}{1-\alpha} - \frac{\alpha^2}{(1-\alpha)^2} \\ &= \frac{2\alpha^2 + \alpha - \alpha^2 - \alpha^2}{(1-\alpha)^2} \\ &= \frac{\alpha}{(1-\alpha)^2}, \end{aligned}$$

as stated in the question.

30.22 *Use the formula obtained in subsection 30.8.2 for the moment generating function of the geometric distribution to determine the CGF $K_n(t)$ for the number of trials needed to record n successes. Evaluate the first four cumulants and use them to confirm the stated results for the mean and variance and to show that the distribution has skewness and kurtosis given, respectively, by*

$$\frac{2-p}{\sqrt{n(1-p)}} \qquad \text{and} \qquad 3 + \frac{6-6p+p^2}{n(1-p)}.$$

The MGF obtained in the text for the number of trials required to obtain the first success is

$$M(t) = \frac{pe^t}{1-qe^t},$$

and so it follows that the MGF for the number of trials needed to record n successes is

$$M_n(t) = \left(\frac{pe^t}{1 - qe^t}\right)^n.$$

The CGF of the distribution is therefore given by

$$\begin{aligned} K_n(t) &= \ln M_n(t) \\ &= n\ln(pe^t) - n\ln(1 - qe^t) \\ &= n\ln p + nt + n\sum_{r=1}^{\infty}\frac{(qe^t)^r}{r} \\ &= n\ln p + nt + n\sum_{r=1}^{\infty}\frac{q^r}{r}\sum_{s=0}^{\infty}\frac{(tr)^s}{s!}. \end{aligned}$$

This must be the same as

$$\kappa_1 t + \kappa_2\frac{t^2}{2!} + \kappa_3\frac{t^3}{3!} + \cdots,$$

where κ_i is the ith cumulant.

The coefficient of t^0 is

$$n\ln p + n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{1}{0!} = n\ln p - n\ln(1 - q) = 0,$$

as expected, since no CGF contains a constant term.

The coefficient of t^1 is $\kappa_1 = \mu_1 = \mu$ and given by

$$n + n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r}{1!} = n + \frac{nq}{1 - q} = \frac{n}{1 - q} = \frac{n}{p},$$

in agreement with the stated result for $n = 1$.

The coefficient of t^2 is $\kappa_2/(2!)$ with $\kappa_2 = \nu_2 = \sigma^2$ and given by

$$\begin{aligned} \frac{\kappa_2}{2!} &= n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r^2}{2!} = \frac{n}{2!}\sum_{r=1}^{\infty}rq^r = \frac{nq}{2!}\frac{d}{dq}\left(\sum_{r=1}^{\infty}q^r\right) \\ &= \frac{nq}{2!}\frac{d}{dq}\left(\frac{q}{1 - q}\right) = \frac{nq}{2!}\frac{1 - q + q}{(1 - q)^2}, \\ \kappa_2 &= \frac{nq}{p^2}, \end{aligned}$$

again in agreement with the stated result for $n = 1$.

The coefficient of t^3 is $\kappa_3/(3!)$ with $\kappa_3 = \nu_3$ and given by

$$\frac{\kappa_3}{3!} = n\sum_{r=1}^{\infty}\frac{q^r}{r}\frac{r^3}{3!} = \frac{n}{3!}\sum_{r=1}^{\infty}r^2q^r.$$

To evaluate this sum we make further use of the result that we have just derived,

$$\sum_{r=1}^{\infty} rq^r = \frac{q}{(1-q)^2},$$

by differentiating both sides with respect to q. This gives

$$\sum_{r=1}^{\infty} r^2 q^{r-1} = \frac{(1-q)^2 + 2q(1-q)}{(1-q)^4} = \frac{1+q}{(1-q)^3}.$$

The sum on the LHS is closely related to the one appearing in the expression for κ_3 and substituting for it gives

$$\kappa_3 = \frac{nq(1+q)}{p^3}.$$

The skewness, equal to $\nu_3/(\nu_2)^{3/2}$, therefore has the value

$$\gamma_3 = \frac{\kappa_3}{(\kappa_2)^{3/2}} = \frac{nq(1+q)}{p^3}\,\frac{p^3}{n^{3/2}q^{3/2}} = \frac{1+q}{\sqrt{nq}} = \frac{2-p}{\sqrt{n(1-p)}}.$$

The kurtosis γ_4 of the distribution is given by $\gamma_4 = \nu_4/(\nu_2)^2$ with $\nu_4 = \kappa_4 + 3(\nu_2)^2$. And so, to determine it, we need an explicit expression for κ_4. This is obtained from the coefficient of t^4, which is $\kappa_4/(4!)$ and given by

$$\frac{\kappa_4}{4!} = n\sum_{r=1}^{\infty} \frac{q^r}{r}\,\frac{r^4}{4!} = \frac{n}{4!}\sum_{r=1}^{\infty} r^3 q^r.$$

Differentiating the result obtained for $\sum_{r=1}^{\infty} r^2 q^r$ when finding κ_3, we deduce that

$$\begin{aligned}\sum_{r=1}^{\infty} r^3 q^{r-1} &= \frac{(1-q)^3(1+2q) + 3q(1+q)(1-q)^2}{(1-q)^6}\\ &= \frac{1+q-2q^2+3q+3q^2}{(1-q)^4}\\ &= \frac{1+4(1-p)+(1-p)^2}{p^4}.\end{aligned}$$

We conclude that

$$\begin{aligned}&\kappa_4 = \frac{n(1-p)(6-6p+p^2)}{p^4},\\ \Rightarrow\quad &\gamma_4 = 3 + \frac{n(1-p)(6-6p+p^2)}{n^2(1-p)^2} = 3 + \frac{6-6p+p^2}{n(1-p)}.\end{aligned}$$

30.24 *As assistant to a celebrated and imperious newspaper proprietor, you are given the job of running a lottery in which each of his five million readers will have an equal independent chance p of winning a million pounds; you have the job of choosing p. However, if nobody wins it will be bad for publicity, whilst, if more than two readers do so, the prize cost will more than offset the profit from extra circulation – in either case you will be sacked! Show that, however you choose p, there is more than a* 40% *chance you will soon be clearing your desk.*

The number of winners x will follow a Poisson distribution; let its mean be μ. I will keep my job provided the number of winners is 1 or 2. The probability $y(\mu)$ of this is

$$y(\mu) = \frac{\mu}{1!}\, e^{-\mu} + \frac{\mu^2}{2!}\, e^{-\mu}.$$

This is maximal when μ is chosen to satisfy

$$0 = \frac{dy}{d\mu} = -(\tfrac{1}{2}\mu^2 + \mu)e^{-\mu} + (1 + \mu)e^{-\mu} \quad \Rightarrow \quad \mu = \sqrt{2}.$$

The corresponding value of p is $\sqrt{2}/(5 \times 10^6)$ and the chance that I keep my job is

$$y(\sqrt{2}) = (\sqrt{2} + 1)e^{-\sqrt{2}} = 0.587,$$

i.e. a 41% chance that I will be clearing my desk.

30.26 *In the game of Blackball, at each turn Muggins draws a ball at random from a bag containing five white balls, three red balls and two black balls; after being recorded, the ball is replaced in the bag. A white ball earns him \$1 whilst a red ball gets him \$2; in either case he also has the option of leaving with his current winnings or of taking a further turn on the same basis. If he draws a black ball the game ends and he loses all he may have gained previously. Find an expression for Muggins' expected return if he adopts the strategy of drawing up to n balls if he has not been eliminated by then.*

Show that, as the entry fee to play is \$3, Muggins should be dissuaded from playing Blackball, but if that cannot be done what value of n would you advise him to adopt?

Suppose that Muggins draws all of the n balls dictated by his strategy and let the

respective numbers of the different colours be w, r and b, where $w+r+b=n$. If b is non-zero, his winnings s, given otherwise by $S=w+2r$, will be zero. Now,

$$\Pr(S=0)=1-\left(\frac{8}{10}\right)^n$$

$$\text{and}\quad \Pr(S=w+2r=n+r)=\left(\frac{8}{10}\right)^n\left[{}^nC_r\left(\frac{3}{8}\right)^r\left(\frac{5}{8}\right)^{n-r}\right].$$

Thus his expected return is (noting that for non-zero contributions $w+2r=n+r$)

$$\begin{aligned}
S(n)&=\left(\frac{8}{10}\right)^n\sum_{r=0}^{n}{}^nC_r\left(\frac{3}{8}\right)^r\left(\frac{5}{8}\right)^{n-r}(n+r)\\
&=\left(\frac{8}{10}\right)^n\left[n\left(\frac{3}{8}+\frac{5}{8}\right)^n+\sum_{r=0}^{n}{}^nC_r\,r\left(\frac{3}{8}\right)^r\left(\frac{5}{8}\right)^{n-r}\right]\\
&=\left(\frac{8}{10}\right)^n\left[n+\sum_{r=1}^{n}\frac{n!}{(r-1)!\,(n-r)!}\left(\frac{3}{8}\right)^r\left(\frac{5}{8}\right)^{n-r}\right]\\
&=\left(\frac{8}{10}\right)^n\left[n+\frac{3n}{8}\sum_{r=1}^{n}\frac{(n-1)!}{(r-1)!\,[n-1-(r-1)]!}\right.\\
&\qquad\qquad\left.\times\left(\frac{3}{8}\right)^{r-1}\left(\frac{5}{8}\right)^{[n-1-(r-1)]}\right]\\
&=\left(\frac{8}{10}\right)^n\left[n+\frac{3n}{8}\sum_{s=0}^{n-1}\frac{(n-1)!}{s!\,[n-1-s]!}\left(\frac{3}{8}\right)^s\left(\frac{5}{8}\right)^{[n-1-s]}\right]\\
&=\left(\frac{4}{5}\right)^n\frac{11n}{8}.
\end{aligned}$$

In hindsight, this should have been expected, since the average gain for a non-zero return is $n\times[(5\times1)+(3\times2)]/8=11n/8$. The bag could have been more easily treated as one containing 2 black balls and 8 non-black balls, each of the latter offering a return of \$11/8 if drawn.

To optimise this return n should be chosen so that $\ln y$, where $y=n(\frac{4}{5})^n$, is optimised, i.e.

$$\begin{aligned}
\ln y&=\ln n+n\ln\tfrac{4}{5},\\
0=\frac{dy}{dn}&=\frac{1}{n}+\ln\frac{4}{5},\\
\Rightarrow\quad n&=\frac{1}{\ln\frac{5}{4}}=4.48.
\end{aligned}$$

Since n must be integral, we calculate $S(4)=2.2528$ and $S(5)=2.2528$. These are equal as the calculated formula shows they must be. However, they are both less than 3 and Muggins would be well advised to keep his \$3 in his pocket; if he will

not take this advice then he should probably choose $n = 5$ and lose his money marginally more slowly.

30.28 *A husband and wife decide that their family will be complete when it includes two boys and two girls – but that this would then be enough! The probability that a new baby will be a girl is p. Ignoring the possibility of identical twins, show that the expected size of their family is*

$$2\left(\frac{1}{pq} - 1 - pq\right),$$

where $q = 1 - p$.

The 'experiment' will end after n 'trials' if the previous $n-1$ trials have produced either $n-2$ boys and 1 girl or $n-2$ girls and one boy, and the nth trial produces the girl or boy (respectively) needed to complete the desired family. These two situations have respective probabilities

$$({}^{n-1}C_1\, pq^{n-2})p \quad \text{and} \quad ({}^{n-1}C_1\, qp^{n-2})q.$$

Thus the probability that the size of the family is n is

$${}^{n-1}C_1\, p^2q^{n-2} + {}^{n-1}C_1\, q^2p^{n-2} = (n-1)(p^2q^{n-2} + q^2p^{n-2}).$$

Averaging this over all possible values of n (≥ 4) gives the expected size as

$$\langle n \rangle = \sum_{n=4}^{\infty} n(n-1)(p^2q^{n-2} + q^2p^{n-2}).$$

Now, from the formula for the sum of a geometric series, we have

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}.$$

Differentiating this (twice) with respect to r gives

$$\sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2},$$

$$\sum_{n=2}^{\infty} n(n-1)r^{n-2} = \frac{2}{(1-r)^3}.$$

So, as the minimum size of their family is 4,

$$\begin{aligned}
\langle n\rangle &= \sum_{n=4}^{\infty} n(n-1)(p^2q^{n-2}+q^2p^{n-2}) \\
&= p^2\left[\frac{2}{(1-q)^3}-(2)(1)q^0-(3)(2)q^1\right] \\
&\qquad\qquad + q^2\left[\frac{2}{(1-p)^3}-(2)(1)p^0-(3)(2)p^1\right] \\
&= p^2\left[\frac{2}{p^3}-2-6q\right]+q^2\left[\frac{2}{q^3}-2-6p\right] \\
&= 2\left[\frac{1}{p}+\frac{1}{q}-p^2-q^2-3pq(p+q)\right] \\
&= 2\left[\frac{p+q}{pq}-(1^2-2pq)-3pq(1)\right] \\
&= 2\left[\frac{1}{pq}-1-pq\right],
\end{aligned}$$

as given in the question.

30.30 *A shopper buys* 36 *items at random in a supermarket where, because of the sales tax imposed, the final digit (the number of pence) in the price is uniformly and randomly distributed from* 0 *to* 9. *Instead of adding up the bill exactly she rounds each item to the nearest* 10 *pence, rounding up or down with equal probability if the price ends in a '5'. Should she suspect a mistake if the cashier asks her for* 23 *pence more than she estimated*?

The probability distribution for the rounding (in pence) is

$$\Pr(i) = p_i = \begin{cases} \frac{1}{10} & -4 \le i \le 4 \\ \frac{1}{20} & i = 5,\ i = -5 \end{cases}$$

This clearly has mean $\mu = 0$ and the variance is given by

$$\sigma^2 = \sum_{i=-5}^{5} i^2 p_i - (\mu)^2 = 0 + \frac{2}{10}(1^2+2^2+3^2+4^2) + \frac{2}{20}\,5^2 - 0 = \frac{17}{2}.$$

The standard deviation for 36 items is therefore $\sqrt{36}\sigma = 6\sqrt{17/2} = 17.49$. The extra 23 pence asked for is therefore only $23/17.49 = 1.31$ s.d. and, for such a bill of items, this discrepancy can be expected to be exceeded (either way) about 20% of the time.

30.32 *In a certain parliament the government consists of 75 New Socialites and the opposition consists of 25 Preservatives. Preservatives never change their mind, always voting against government policy without a second thought; New Socialites vote randomly, but with probability p that they will vote for their party leader's policies.*

Following a decision by the New Socialites' leader to drop certain manifesto commitments, N of his party decide to vote consistently with the opposition. The leader's advisors reluctantly admit that an election must be called if N is such that, at any vote on government policy, the chance of a simple majority in favour would be less than 80%. Given that $p = 0.8$, estimate the lowest value of N that would precipitate an election.

From interpolation in the tabulation of the CPF for the Gaussian distribution the value of z for which $\Phi(z) = 0.8$ is 0.841. It follows that the chance of a defeat becomes more than 20% when the difference between 50 and the predictable number of anti-government votes is reduced to 0.841 times the standard deviation in the number of pro-government votes.

The number of assured anti-government votes is $25 + N$ whilst the remaining number of unpredictable New Socialites is $75 - N$. For these members of parliament, voting is a series of Bernoulli trials with a probability that they will vote with the opposition of $1 - p$. The mean number of votes they will cast with the opposition is therefore $(1 - p)(75 - N)$. The standard deviations of the number of their votes cast either with or against the government are the same and equal to $\sqrt{(75 - N)p(1 - p)}$.

An election will be precipitated if

$$25 + N + (1 - p)(75 - N) + 0.841\sqrt{(75 - N)p(1 - p)} = 50$$

Setting $p = 0.8$ and rearranging gives

$$\begin{aligned} 0.841\sqrt{0.16(75 - N)} &= 10 - 0.8N, \\ (0.7073)(0.16)(75 - N) &= 100 - 16N + 0.64N^2, \\ 0.64N^2 - 15.89N + 91.51 &= 0, \\ \Rightarrow \quad N &= 9.08 \text{ or } 15.75. \end{aligned}$$

The second value corresponds to taking -0.841 standard deviations and is not relevant here. The conclusion is that 10 rebel New Socialites would be enough to precipitate an election.

30.34 *The random variables X and Y take integer values ≥ 1 such that $2x+y \leq 2a$, where a is an integer greater than 1. The joint probability within this region is given by*

$$\Pr(X = x, Y = y) = c(2x + y),$$

where c is a constant, and it is zero elsewhere.

Show that the marginal probability $\Pr(X = x)$ is

$$\Pr(X = x) = \frac{6(a-x)(2x+2a+1)}{a(a-1)(8a+5)},$$

and obtain expressions for $\Pr(Y = y)$, (a) *when y is even and* (b) *when y is odd. Show further that*

$$E[Y] = \frac{6a^2+4a+1}{8a+5}.$$

[*You will need the results about series involving the natural numbers given in subsection 4.2.5.*]

Since the boundary of the region is $2x + y \leq 2a$, the maximal value of y for any fixed x will be even. The marginal probability $\Pr(X = x)$ is obtained by summing over the probabilities for all the allowed values of y in the range $1 \leq y \leq 2a-2x$, i.e.

$$\begin{aligned}
\Pr(X = x) &= \sum_{y=1}^{2a-2x} c(2x+y) \\
&= 2cx(2a-2x) + c\,\tfrac{1}{2}(2a-2x)(2a-2x+1) \\
&= 2c(a-x)(2x+a-x+\tfrac{1}{2}) \\
&= c(a-x)(2x+2a+1).
\end{aligned}$$

However, the overall normalisation requires that $\sum_{x=1}^{a-1} \Pr(X = x)$ must be equal to unity:

$$\begin{aligned}
1 &= \sum_{x=1}^{a-1} \Pr(X = x) \\
&= \sum_{x=1}^{a-1} c(a-x)(2x+2a+1) = c\sum_{x=1}^{a-1} 2a^2 - 2x^2 + a - x \\
&= c\,[\,2a^2(a-1) - \tfrac{2}{6}(a-1)(a)(2a-1) + a(a-1) - \tfrac{1}{2}(a-1)a\,] \\
&= ca(a-1)[\,2a - \tfrac{1}{3}(2a-1) + 1 - \tfrac{1}{2}\,] \\
&= \tfrac{1}{6}ca(a-1)(8a+5).
\end{aligned}$$

The normalisation condition therefore requires that

$$c = \frac{6}{a(a-1)(8a+5)},$$

$$\Rightarrow \quad \Pr(X = x) = \frac{6(a-x)(2x+2a+1)}{a(a-1)(8a+5)}.$$

(a) If y is even then, for the calculation of the marginal probability $\Pr(Y = y)$, the largest value of x to be included in the sum over x lies on the boundary of the region at $x = \frac{1}{2}(2a - y)$. The sum is therefore

$$\begin{aligned}
\Pr(y) &= \sum_{x=1}^{(2a-y)/2} c(2x+y) \\
&= 2c\,\frac{1}{2}\,\frac{2a-y}{2}\,\frac{2a-y+2}{2} + cy\,\frac{2a-y}{2} \\
&= \frac{1}{4}c(2a-y)(2a-y+2+2y) \\
&= \frac{3(2a-y)(2a+y+2)}{2a(a-1)(8a+5)}.
\end{aligned}$$

(b) When y is odd the largest value of x does *not* lie on the boundary but is given by $\frac{1}{2}(2a - y - 1)$. Hence

$$\begin{aligned}
\Pr(y) &= \sum_{x=1}^{(2a-y-1)/2} c(2x+y) \\
&= 2c\,\frac{1}{2}\,\frac{2a-y-1}{2}\,\frac{2a-y+1}{2} + cy\,\frac{2a-y-1}{2} \\
&= \frac{1}{4}c(2a-y-1)(2a-y+1+2y) \\
&= \frac{3(2a-y-1)(2a+y+1)}{2a(a-1)(8a+5)}.
\end{aligned}$$

The mean value $E[\,Y\,]$ is equal to the sum $\sum_y y\,\Pr(Y = y)$, the minimum value of y being 1 and the maximum value $2a - 2$, i.e. there are an even number of terms. We group the values in pairs, $y = 2m - 1$ and $y = 2m$, for $m = 1, 2, \ldots, a - 1$. Denoting by k the constant $3/[\,2a(a-1)(8a+5)\,]$, we have that

$$\begin{aligned}
E[\,Y\,] &= \sum_{m=1}^{a-1}[\,k(2a-2m)(2a+2m)(2m-1) + \ k(2a-2m)(2a+2m+2)2m\,] \\
&= 4k\sum_{m=1}^{a-1}[\,(a^2-m^2)(2m-1) + (a^2-m^2+a-m)2m\,] \\
&= 4k\sum_{m=1}^{a-1}[\,-a^2 + (4a^2+2a)m - m^2 - 4m^3\,].
\end{aligned}$$

This sum may be evaluated by using the formulae for the sums of the powers of the natural numbers (see chapter 4), and reads

$$\begin{aligned}
E[\,Y\,] &= 4k\left[-a^2(a-1) + 2a(2a+1)\,\frac{1}{2}(a-1)a \right.\\
&\qquad\qquad \left. -\ \frac{1}{6}(a-1)a(2a-1) - 4\,\frac{1}{4}(a-1)^2a^2\right]\\
&= 4k(a-1)(-a^2 + 2a^3 + a^2 - \tfrac{1}{3}a^2 + \tfrac{1}{6}a - a^3 + a^2)\\
&= \frac{4k(a-1)a}{6}(6a^2 + 4a + 1)\\
&= \frac{3}{2a(a-1)(8a+5)}\,\frac{4(a-1)a}{6}(6a^2 + 4a + 1)\\
&= \frac{6a^2 + 4a + 1}{8a+5}.
\end{aligned}$$

30.36 *A discrete random variable X takes integer values $n = 0, 1, \ldots, N$ with probabilities p_n. A second random variable Y is defined as $Y = (X - \mu)^2$, where μ is the expectation value of X. Prove that the covariance of X and Y is given by*

$$Cov[X, Y] = \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 2\mu^3.$$

Now suppose that X takes all its possible values with equal probability and hence demonstrate that two random variables can be uncorrelated even though one is defined in terms of the other.

The covariance of X and Y is given by

$$\begin{aligned}
\mathrm{Cov}[X, Y] &= E[XY] - E[X]E[Y]\\
&= \sum_{n=0}^{N}([\,n(n-\mu)^2 p_n\,] - \mu \sum_{n=0}^{N}(n-\mu)^2 p_n\\
&= \sum_{n=0}^{N} n^3 p_n - 2\mu \sum_{n=0}^{N} n^2 p_n + \mu^2 \sum_{n=0}^{N} n p_n\\
&\qquad - \mu \sum_{n=0}^{N} n^2 p_n + 2\mu^2 \sum_{n=0}^{N} n p_n - \mu^3 \sum_{n=0}^{N} p_n.
\end{aligned}$$

But $\sum_{n=0}^{N} n p_n = \mu$ and $\sum_{n=0}^{N} p_n = 1$, and so

$$\mathrm{Cov}[X, Y] = \sum_{n=0}^{N} n^3 p_n - 3\mu \sum_{n=0}^{N} n^2 p_n + 2\mu^3.$$

Now suppose that $p_n = (N+1)^{-1}$ for all values of n. In this case, the mean $\mu = N/2$ and, using the sums of the first, second and third powers of the natural numbers derived in subsection 4.2.5, we have

$$\begin{aligned}\mathrm{Cov}[X,Y] &= \frac{N^2(N+1)^2}{4(N+1)} - \frac{3N}{2}\,\frac{N(N+1)(2N+1)}{6(N+1)} + \frac{2N^3}{8}\\ &= \frac{N^2}{4}[\,N+1-2N-1+N\,] = 0.\end{aligned}$$

Thus, as their covariance is zero, the random variables X and Y are uncorrelated – even though Y is defined in terms of X.

30.38 *A continuous random variable X is uniformly distributed over the interval $[-c,c]$. A sample of $2n+1$ values of X is selected at random and the random variable Z is defined as the* median *of that sample. Show that Z is distributed over $[-c,c]$ with probability density function*

$$f_n(z) = \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2-z^2)^n.$$

Find the variance of Z.

For the median of the sample of $2n+1$ values of X to lie in the interval $z \to z+dz$ we require that n values lie in the range $-c \le X < z$, n lie in the range $z+dz < X \le c$ and one is in the interval $z \le X \le z+dz$. We are thus considering a multinomial distribution and, as all the sample values in any one interval are equivalent, the probability density function is

$$\begin{aligned}f_n(z)\,dz &= \frac{(2n+1)!}{n!\,n!\,1!}\left(\frac{z+c}{2c}\right)^n\left(\frac{c-z}{2c}\right)^n\left(\frac{dz}{2c}\right)^1\\ &= \frac{(2n+1)!}{(n!)^2(2c)^{2n+1}}(c^2-z^2)^n\,dz\\ &\equiv A_n(c^2-z^2)^n\,dz,\end{aligned}$$

where A_n is defined (for any $n \ge 1$) by

$$A_n\int_{-c}^{c}(c^2-z^2)^n\,dz = 1.$$

Now, from symmetry, it is clear that $E[Z]=0$ and so the variance of Z is given

by

$$
\begin{aligned}
V[Z] &= E[Z^2] - (E[Z])^2 \\
&= A_n \int_{-c}^{c} z^2 (c^2 - z^2)^n \, dz - 0 \\
&= A_n \left[-\frac{z}{2} \frac{(c^2 - z^2)^{n+1}}{n+1} \right]_{-c}^{c} + \frac{A_n}{2} \int_{-c}^{c} \frac{(c^2 - z^2)^{n+1}}{n+1} \, dz \\
&= 0 + \frac{A_n}{2(n+1)} \frac{1}{A_{n+1}} \\
&= \frac{(2n+1)!}{2(n+1)\,(n!)^2\,(2c)^{2n+1}} \frac{[(n+1)!]^2\,(2c)^{2n+3}}{(2n+3)!} \\
&= \frac{(n+1)^2\,4c^2}{2(n+1)(2n+2)(2n+3)} \\
&= \frac{c^2}{2n+3}.
\end{aligned}
$$

We note that this result has been obtained without having to explicitly evaluate the integrals involved.

30.40 *The variables X_i, $i = 1, 2, \ldots, n$, are distributed as a multivariate Gaussian, with means μ_i and a covariance matrix V. If the X_i are required to satisfy the linear constraint $\sum_{i=1}^{n} c_i X_i = 0$, where the c_i are constants (and not all equal to zero), show that the variable*

$$\chi_n^2 = (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu)$$

follows a chi-squared distribution of order $n - 1$.

As shown in the text, the PDF of the multivariate Gaussian can be written

$$f(\mathsf{x}) = \frac{1}{(2\pi)^{n/2}\,|V|^{1/2}} \exp\left[-\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu) \right].$$

Now let S be the orthogonal matrix whose normalised columns are the eigenvectors of V with corresponding eigenvalues λ_i and define new variables y_i by $\mathsf{y} = \mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)$. Using the fact that $\mathsf{S}\mathsf{S}^{\mathrm{T}} = \mathsf{I}$, the argument of the exponential function becomes

$$
\begin{aligned}
-\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{V}^{-1} (\mathsf{x} - \mu) &= -\frac{1}{2} (\mathsf{x} - \mu)^{\mathrm{T}} \mathsf{S}\,\mathsf{S}^{\mathrm{T}} \mathsf{V}^{-1} \mathsf{S}\,\mathsf{S}^{\mathrm{T}} (\mathsf{x} - \mu) \\
&= -\frac{1}{2} [\,\mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)\,]^{\mathrm{T}}\, \mathsf{S}^{\mathrm{T}} \mathsf{V}^{-1} \mathsf{S}\, [\,\mathsf{S}^{\mathrm{T}}(\mathsf{x} - \mu)\,] \\
&= -\frac{1}{2}\, \mathsf{y}^{\mathrm{T}}\, \mathrm{diag}(\lambda_i^{-1})\, \mathsf{y}.
\end{aligned}
$$

A further scaling of the variables, $z_i = y_i/\sqrt{\lambda_i}$, reduces the argument to $-\frac{1}{2}\chi_n^2 = -\frac{1}{2}\sum_{i=1}^n z_i^2$ and

$$\begin{aligned} f(\mathsf{z})\, d^n\mathsf{z} &= f(z_1, z_2, \ldots, z_n)\, dz_1\, dz_2 \cdots dz_n \\ &= \frac{1}{(2\pi)^{n/2}\,|V|^{1/2}}\, \exp(-\frac{1}{2}\chi_n^2)\, dV_\chi, \end{aligned}$$

where $dz_1\, dz_2 \cdots dz_n$ is the infinitesimal volume enclosed by the intersection of the n-dimensional spherical shell of radius χ_n^2 and thickness $d\chi_n^2$ with the $(n-1)$-dimensional hyperplane $\sum_{i=1}^n c_i x_i = 0$.

From the way each y_i was constructed, both it and z_i have zero means; the scaling from y_i to z_i ensures that z_i has unit variance.

Under the successive transformations the linear constraint $\sum_{i=1}^n c_i x_i = 0$, with not all $c_i = 0$, will become another linear constraint $\sum_{i=1}^n c_i' Z_i = 0$, again with not all $c_i' = 0$ (since the λ_i are neither zero nor infinite).

The constraint can be incorporated by writing one z_k for which $c_k' \neq 0$ in terms of the others:

$$\chi_n^2 = z_1^2 + z_2^2 + \ldots + \left(-\sum_{j\neq k}^n \frac{c_j}{c_k} z_j\right)^2 + \ldots + z_n^2.$$

A further transformation can then be made to carry this into the form

$$\chi_n^2 = v_1^2 + v_2^2 + \cdots + v_{n-1}^2,$$

which can be considered as the square of the distance from the origin in the $(n-1)$-dimensional **V**-space. In this space the element of volume is

$$dV_\chi = A\chi_n^{n-2}\, d\chi_n = A\chi_n^{n-2}\frac{d(\chi_n^2)}{2\chi_n} = \frac{A}{2}(\chi_n^2)^{(n-3)/2}\, d\chi_n^2.$$

Collecting these results together gives

$$h(\chi_n^2)\, d\chi_n^2 \propto (\chi_n^2)^{(n-3)/2}\, \exp(-\frac{1}{2}\chi_n^2)\, d\chi_n^2,$$

i.e. χ_n^2 follows a chi-squared distribution of order $n-1$.

31

Statistics

31.2 *Measurements of a certain quantity gave the following values:* 296, 316, 307, 278, 312, 317, 314, 307, 313, 306, 320, 309. *Within what limits would you say there is a* 50% *chance that the correct value lies*?

Since all the other readings are within ± 12 of 308 and the reading of 278 is 30 away from this value, it should probably be rejected, as erroneous rather than a statistical fluctuation.

The other readings do not look as though they are Gaussian distributed and the best estimate is probably obtained by considering the distribution as approximating to a uniform distribution and using the inter-quartile range of the remaining 11 readings. Arranged in order, they are

$$296,\ 306,\ 307,\ 307,\ 309,\ 312,\ 313,\ 314,\ 316,\ 317,\ 320,$$

and their mean is 310.6.

This number of readings does not divide into four equal-sized groups and the perhaps over-cautious approach is to discard only two readings from each end of the range i.e. give the range in which the correct value lies with 50% probability as 307–316. An additional reading would probably have justified discarding three reading from each end.

31.4 *Two physical quantities x and y are connected by the equation*

$$y^{1/2} = \frac{x}{ax^{1/2} + b},$$

and measured pairs of values for x and y are as follows:

x:	10	12	16	20
y:	409	196	114	94.

Determine the best values for a and b by graphical means and (either by hand or by using a built-in calculator routine) by a least-squares fit to an appropriate straight line.

We aim to put this equation into a 'straight-line' form. One way to do this is to re-arrange it as

$$\frac{x}{y^{1/2}} = ax^{1/2} + b$$

and plot $(x/y^{1/2})$ against $x^{1/2}$. The slope of the graph will give a and its intercept on the $(x/y^{1/2})$-axis will give b. We therefore tabulate the required quantities:

x	10	12	16	20
y	409	196	114	94
$x^{1/2}$	3.16	3.46	4.00	4.47
$x/y^{1/2}$	0.494	0.857	1.499	2.063

Plotting the graph over the range $3.0 \leq x^{1/2} \leq 4.5$ gives a good straight line of slope $(2.09 - 0.31)/(4.50 - 3.00) = 1.19$. Thus $a = 1.19$. The fit to the line is sufficiently good that it is hard to estimate the uncertainty in a and a least-squares fit would result in a small but virtually meaningless value.

However, the measured values of $x^{1/2}$ are bunched in a range that is small compared to their distance from the $(x/y^{1/2})$-axis, where the intercept is b. Such a long graphical extrapolation could result in a serious error in the value of b. It is better to calculate b using the straight-line values at one point (say $x/y^{1/2} = 0.31$ at $x^{1/2} = 3.00$) and the slope just found:

$$b = 0.31 - (1.19 \times 3.00) = -3.26.$$

An alternative is to re-arrange the original equation as

$$\frac{x^{1/2}}{y^{1/2}} = a + \frac{b}{x^{1/2}}$$

and then plot values from the following table,

x	10	12	16	20
y	409	196	114	94
$x^{-1/2}$	0.316	0.288	0.250	0.223
$(x/y)^{1/2}$	0.156	0.247	1.375	2.461

over the range $0.200 \leq x^{-1/2} \leq 0.330$. An equally good straight-line fit is obtained with a slope, this time being equal to b (rather than a), of $(0.110-0.534)/(0.330-0.200) = -3.26$. A similar calculation to that used earlier now determines a as $0.534 + (3.26 \times 0.200) = 1.19$.

31.6 *Prove that the sample mean is the best* linear unbiased estimator *of the population mean μ as follows.*

(a) *If the real numbers $a_1, a_2, \ldots, a_n$ satisfy the constraint $\sum_{i=1}^n a_i = C$, where C is a given constant, show that $\sum_{i=1}^n a_i^2$ is minimised by $a_i = C/n$ for all i.*
(b) *Consider the linear estimator $\hat{\mu} = \sum_{i=1}^n a_i x_i$. Impose the conditions* (i) *that it is* unbiased, *and* (ii) *that it is as* efficient *as possible.*

(a) To minimise $S = \sum_{i=1}^n a_i^2$ subject to the constraint $\sum_{i=1}^n a_i = C$, we introduce a Lagrange multiplier and consider

$$\begin{aligned} T &= \sum_{i=1}^n a_i^2 - \lambda \sum_{i=1}^n a_i, \\ 0 = \frac{\partial T}{\partial a_i} &= 2a_i - \lambda \\ &\Rightarrow \quad a_i = \tfrac{1}{2}\lambda, \text{ for all } i. \end{aligned}$$

Re-substitution in the constraint gives $C = \frac{1}{2}n\lambda$, leading to $a_i = C/n$ for all i. The corresponding minimum value of S is C^2/n.

(b) If the sample values x_i are drawn from a population with mean μ and variance σ^2, consider the linear estimator $\hat{\mu} = \sum_{i=1}^n a_i x_i$. For the estimator to be *unbiased* we require that

$$\begin{aligned} 0 = \langle \hat{\mu} - \mu \rangle = \left\langle \sum_{i=1}^n a_i x_i - \mu \right\rangle &= \sum_{i=1}^n a_i \langle x_i \rangle - \mu \\ &= \sum_{i=1}^n a_i \mu - \mu = \mu \left(\sum_{i=1}^n a_i - 1 \right). \end{aligned}$$

Thus the first requirement is that $\sum_{i=1}^n a_i = 1$.

Now we add the further requirement of *efficiency* by minimising the variance of $\hat{\mu}$. The expression for the variance is

$$\begin{aligned}
\langle(\hat{\mu}-\mu)^2\rangle &= \left\langle\left(\sum_{i=1}^{n} a_i(\mu+z_i)-\mu\right)^2\right\rangle, \text{ with } \langle z_i\rangle = 0 \text{ and } \langle z_i^2\rangle = \sigma^2, \\
&= \left\langle\left(\sum_{i=1}^{n} a_i z_i + \sum_{i=1}^{n} a_i\mu - \mu\right)^2\right\rangle, \\
&= \left\langle\left(\sum_{i=1}^{n} a_i z_i + \mu - \mu\right)^2\right\rangle, \text{ since } \sum_{i=1}^{n} a_i = 1, \\
&= \left\langle\left(\sum_{i=1}^{n} a_i z_i\right)^2\right\rangle \\
&= \sum_{i=1}^{n} a_i^2\,\sigma^2, \text{ since } \langle z_i^2\rangle = \sigma^2 \text{ and the } z_i \text{ are independent.}
\end{aligned}$$

Now, from part (a), this expression is minimised subject to $\sum_{i=1}^{n} a_i = 1$ when $a_i = 1/n$ for all i, i.e. when $\hat{\mu}$ is taken as the mean of the sample. The minimum value for the variance is σ^2/n. This completes the proof that the sample mean is the best *linear unbiased estimator* of the population mean μ.

31.8 *Carry through the following proofs of statements made in subsections 31.5.2 and 31.5.3 about the ML estimators $\hat{\tau}$ and $\hat{\lambda}$.*

(a) *Find the expectation values of the ML estimators $\hat{\tau}$ and $\hat{\lambda}$ given respectively in (31.71) and (31.75). Hence verify equations (31.76), which show that, even though an ML estimator is unbiased, it does not follow that functions of it are also unbiased.*

(b) *Show that $E[\hat{\tau}^2] = (N+1)\tau^2/N$ and hence prove that $\hat{\tau}$ is a minimum-variance estimator of τ.*

(a) As shown in the text [equation (27.67)] the likelihood of the measured intervals x_k is

$$\prod_{k=1}^{N} \frac{1}{\tau}\exp\left(-\frac{x_k}{\tau}\right) = \frac{1}{\tau^N}\exp\left[-\frac{1}{\tau}(x_1+x_2+\cdots+x_N)\right].$$

The expectation value $E[\hat{\tau}]$ of the estimator $\hat{\tau} = N^{-1}\sum_{i=1}^{N} x_i$ is therefore

$$\frac{1}{N}\sum_{i=1}^{N}\int\cdots\int x_i\frac{1}{\tau^N}\exp\left[-\frac{1}{\tau}(x_1+x_2+\cdots+x_N)\right]dx_1\,dx_2\cdots dx_N.$$

In each term of the sum we can carry out the integrations over all the x_k variables except the one with $k = i$ (each gives τ) thereby reducing the sum to

$$\begin{aligned} E[\hat{\tau}] &= \frac{1}{N}\sum_{i=1}^{N} x_i \frac{1}{\tau}\, e^{-x_i/\tau}\, dx_i \\ &= \frac{1}{N}\sum_{i=1}^{N}\left(\left[-\frac{x_i}{\tau}\,\tau\, e^{-x_i/\tau}\right]_0^{\infty} + \int_0^{\infty} e^{-x_i/\tau}\, dx_i\right) \\ &= \frac{1}{N}\sum_{i=1}^{N}\tau = \frac{1}{N}\, N\tau = \tau, \text{ as expected.} \end{aligned}$$

We note that this estimator is unbiased and now turn to the expectation value of the estimator

$$\hat{\lambda} = \left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)^{-1} = \bar{x}^{-1}.$$

For typographical clarity we will omit explicit limits from the sum $\sum_{i=1}^{N} x_i$ where it appears in the equations that follow.

$$\begin{aligned} E[\hat{\lambda}] &= \int\cdots\int\left(\frac{N}{\sum x_i}\right)\lambda e^{-\lambda x_1}\,\lambda e^{-\lambda x_2}\cdots\lambda e^{-\lambda x_N}\, dx_1\, dx_2\cdots dx_N \\ &= \int \frac{N}{\sum x_i}\,\lambda^N\, e^{-\lambda\sum x_i}\, d^N x_i. \end{aligned}$$

To evaluate this integral we differentiate both sides of its definition with respect to λ. The RHS is a product of two functions of λ; differentiating it produces one term in which $\lambda^N \to N\lambda^{N-1}$ and the other produces a factor that cancels the $\sum x_i$ in the denominator. The result is

$$\begin{aligned} \frac{dE[\hat{\lambda}]}{d\lambda} &= \frac{N}{\lambda}\, E[\hat{\lambda}] - N\int \lambda^N\, e^{-\lambda\sum x_i}\, d^N x_i \\ &= \frac{N}{\lambda}\, E[\hat{\lambda}] - N, \end{aligned}$$

since the distribution function for each x_i is normalised (they are all the same). The integrating factor for this first-order equation is λ^{-N} giving

$$\frac{d}{d\lambda}\left(\frac{E}{\lambda^N}\right) = -\frac{N}{\lambda^N} \quad\Rightarrow\quad \frac{E}{\lambda^N} = \frac{N}{(N-1)\lambda^{N-1}} + c.$$

We must have $E[\hat{\lambda}] \to \lambda$ as $N \to \infty$ and so $c = 0$, yielding

$$E[\hat{\lambda}] = \frac{N}{N-1}\,\lambda.$$

Thus, although the bias tends to zero as $N \to \infty$, $\hat{\lambda}$ is a biased estimator of λ. Since it is directly given as the reciprocal of $\hat{\tau}$, the two results obtained, taken

together, show that even though an ML estimator is unbiased, it does not follow that functions of it are also unbiased.

(b) We start by using the Fisher inequality to determine the minimum variance that any estimator of τ *could* have; for this we need $\ln P(\mathbf{x}|\tau)$. This is given by

$$\ln P(\mathbf{x}|\tau) = \ln\left[\prod_{i=1}^{N}\left(\frac{e^{-x_i/\tau}}{\tau}\right)\right] = -\sum_{i=1}^{N}\left(\ln\tau + \frac{1}{\tau}x_i\right).$$

Hence,

$$\begin{aligned} E\left[-\frac{\partial^2}{\partial\tau^2}\ln P\right] &= E\left[\sum_{i=1}^{N}\left(-\frac{1}{\tau^2}+\frac{2}{\tau^3}x_i\right)\right] \\ &= -\frac{N}{\tau^2}+\frac{2N\tau}{\tau^3} = \frac{N}{\tau^2}. \end{aligned}$$

We have already shown that the estimator is unbiased; thus $\partial b/\partial\tau = 0$ and Fisher's inequality reads

$$V[\hat{\tau}] \geq \frac{1}{N/\tau^2} = \frac{\tau^2}{N}.$$

Next we compute

$$E[\hat{\tau}^2] = \int\cdots\int \frac{1}{N^2}\left(\sum x_i\right)^2 \frac{1}{\tau^N}\, e^{-(\sum x_i)/\tau}\, dx_1\, dx_2, \cdots dx_N.$$

We now separate off the N terms in the square of the sum that contain factors typified by x_i^2 from the $N(N-1)$ terms containing factors typified by $x_i x_j$ with $i \neq j$. All integrals over sample values not involving i, or i and j, (as the case may be) integrate to τ. Within each group all integrals have the same value and so we can write

$$\begin{aligned} E[\hat{\tau}^2] &= \frac{1}{N^2}\left[N\int_0^\infty \frac{x^2}{\tau}\, e^{-x/\tau}\, dx \right. \\ &\qquad \left. + \; N(N-1)\int_0^\infty\int_0^\infty \frac{x_1}{\tau}\frac{x_2}{\tau}\, e^{-x_1/\tau}\, e^{-x_2/\tau}\, dx_1\, dx_2\right] \\ &= \frac{1}{N^2}\left[2N\tau^2 + N(N-1)\tau^2\right] \\ &= \frac{N+1}{N}\,\tau^2. \end{aligned}$$

Finally, the variance of $\hat{\tau}$ is calculated as

$$V[\hat{\tau}] = E[\hat{\tau}^2] - (E[\hat{\tau}])^2 = \frac{N+1}{N}\,\tau^2 - \tau^2 = \frac{\tau^2}{N}.$$

This is equal to the minimum allowed by the Fisher inequality; thus $\hat{\tau}$ is a minimum-variance estimator of τ.

31.10 *This exercise is intended to illustrate the dangers of applying formalised estimator techniques to distributions that are not well behaved in a statistical sense.*

The following are five sets of 10 values, all drawn from the same Cauchy distribution with parameter a.

(i)	4.81	−1.24	1.30	−0.23	2.98
	−1.13	−8.32	2.62	−0.79	−2.85
(ii)	0.07	1.54	0.38	−2.76	−8.82
	1.86	−4.75	4.81	1.14	−0.66
(iii)	0.72	4.57	0.86	−3.86	0.30
	−2.00	2.65	−17.44	−2.26	−8.83
(iv)	−0.15	202.76	−0.21	−0.58	−0.14
	0.36	0.44	3.36	−2.96	5.51
(v)	0.24	−3.33	−1.30	3.05	3.99
	1.59	−7.76	0.91	2.80	−6.46

Ignoring the fact that the Cauchy distribution does not have a finite variance (or even a formal mean), show that $\hat{a}$, the ML estimator of a, has to satisfy

$$s(\hat{a}) = \sum_{i=1}^{10} \frac{1}{1 + x_i^2/\hat{a}^2} = 5. \quad (*)$$

Using a programmable calculator, spreadsheet or computer, find the value of $\hat{a}$ that satisfies (*) for each of the data sets and compare it with the value $a = 1.6$ used to generate the data. Form an opinion regarding the variance of the estimator.

Show further that if it is assumed that $(E[\hat{a}])^2 = E[\hat{a}^2]$ then $E[\hat{a}] = v_2^{1/2}$, where v_2 is the second (central) moment of the distribution, which for the Cauchy distribution is infinite!

The Cauchy distribution with parameter a has the form

$$f(x) = \frac{a}{\pi} \frac{1}{a^2 + x^2}.$$

It follows that the likelihood function for 10 sample values is

$$L(\mathbf{x}|a) = \left(\frac{a}{\pi}\right)^{10} \prod_{i=1}^{10} \frac{1}{a^2 + x_i^2},$$

and that the log-likelihood function

$$\ln L = -10 \ln \pi + 10 \ln a - \sum_{i=1}^{10} \ln(a^2 + x_i^2).$$

The equation satisfied by the ML estimator $\hat{a}$ is therefore

$$0 = \frac{\partial(\ln L)}{\partial a} = \frac{10}{a} - \sum_{i=1}^{10} \frac{2a}{a^2 + x_i^2} \quad \Rightarrow \quad s(\hat{a}) = \sum_{i=1}^{10} \frac{1}{1 + x_i^2/\hat{a}^2} = 5.$$

Using a simple spread sheet to calculate the sum on the LHS for various assumed values of a and then manual or automated interpolation to make the sum equal to 5, the following values for $\hat{a}$ are obtained for the five sets of data:

(i) 1.85, (ii) 1.66, (iii) 2.46, (iv) 0.68, (v) 2.44.

Although the estimates have the correct order of magnitude, there is clearly a very large (perhaps infinite) sampling variance. Even if all 50 samples are combined, the resulting estimated value for a of 1.84 is 0.24 away from that used to generate the data.

It is clear that for sets of N sample values (*) reads

$$\sum_{i=0}^{N} \frac{1}{\hat{a}^2 + x_i^2} = \frac{N}{2\hat{a}^2},$$

and we take this as the *definition* of $\hat{a}$. Multiplying both sides of this equation by $\prod_{k=1}^{N}(\hat{a}^2 + x_k^2)$, we obtain

$$\frac{N}{2\hat{a}^2} \prod_{k=1}^{N} (\hat{a}^2 + x_k^2) = \sum_{i=0}^{N} \prod_{k \neq i}^{N} (\hat{a}^2 + x_k^2).$$

Now we take expectation values over all the x_i, writing $E[\,x_i^r\,] = v_r$,

$$\frac{N}{2E[\,\hat{a}^2\,]} \left(E[\,\hat{a}^2\,] + v_2\right)^N = N \left(E[\,\hat{a}^2\,] + v_2\right)^{N-1}$$

$$E[\,\hat{a}^2\,] + v_2 = 2E[\,\hat{a}^2\,] \quad \Rightarrow \quad E[\,\hat{a}\,] = v_2^{1/2},$$

assuming that $E[\,\hat{a}\,]^2 = E[\hat{a}^2\,]$. As shown in exercise 31.8, this is not necessarily so, but any possible fractional bias is typically $\mathrm{O}(N^{-1})$. However, for the Cauchy distribution,

$$v_2 = \int_{-\infty}^{\infty} \frac{a}{\pi} \frac{x^2}{a^2 + x^2}\, dx = \infty.$$

This is rather more serious than an $\mathrm{O}(N^{-1})$ error and the statistically unsound procedure used leads to the false conclusion that the expected value of the estimator is infinite, when it ought to have a value equal to the finite parameter a of the sample distribution.

31.12 *On a certain (testing) steeplechase course there are* 12 *fences to be jumped and any horse that falls is not allowed to continue in the race. In a season of racing a total of* 500 *horses started the course and the following numbers fell at each fence:*

Fence:	1	2	3	4	5	6	7	8	9	10	11	12
Falls:	62	75	49	29	33	25	30	17	19	11	15	12

Use this data to determine the overall probability of a horse's falling at a fence, and test the hypothesis that it is the same for all horses and fences as follows.

(a) *Draw up a table of the expected number of falls at each fence on the basis of the hypothesis.*

(b) *Consider for each fence i the standardised variable*

$$z_i = \frac{\textit{estimated falls} - \textit{actual falls}}{\textit{standard deviation of estimated falls}}$$

and use it in an appropriate χ^2 *test.*

(c) *Show that the data indicates that the odds against all fences being equally testing are about* 40 *to* 1. *Identify the fences that are significantly easier or harder than the average.*

(a) The information as presented does not give statistically independent data for each fence, as a horse that falls at an early fence cannot attempt a later one. To extract the necessary data we extend the table by adding rows for the number of attempts at each fence and the number of successful jumps there.

Fence:	1	2	3	4	5	6	
Falls:	62	75	49	29	33	25	
Clearances:	438	363	314	285	252	227	
Attempts:	500	438	363	314	285	252	
Fence:	7	8	9	10	11	12	Total
Falls:	30	17	19	11	15	12	377
Clearances:	197	180	161	150	135	123	2825
Attempts:	227	197	180	161	150	135	3202

On the hypothesis that all fences are equally difficult the best estimator of the probability p of a fall at any particular fence i is $377/3202 = 0.1177$, independent of i. If the number of attempts at fence i is n_i then the expected number of falls at that fence is $x_i = pn_i$. Since each attempt is a Bernoulli trial the s.d. of x_i is given by $\sqrt{n_i p(1-p)} = 0.3223\sqrt{n_i}$.

(b) We may now draw up a further table of the expected number of falls and of

the standardised variable

$$z_i = \frac{\text{estimated falls} - \text{actual falls}}{\text{standard deviation of estimated falls}}$$

for each fence. The corresponding contribution to the overall χ^2 statistic is $\chi_i^2 = z_i^2$.

Fence:	1	2	3	4	5	6	
Falls:	62	75	49	29	33	25	
Estimated Falls:	58.9	51.6	42.7	37.0	33.6	29.7	
z_i:	-0.43	-3.47	-1.02	1.40	0.11	0.92	
χ_i^2:	0.2	12.0	1.0	2.0	0.0	0.8	
Fence:	7	8	9	10	11	12	Total
Falls:	30	17	19	11	15	12	377
Estimated Falls:	26.7	23.2	21.2	19.0	17.7	15.9	377
z_i:	-0.68	1.37	0.51	1.96	0.68	1.04	
χ_i:	0.5	1.9	0.3	3.8	0.5	1.1	24.1

Thus $\chi^2 = 24.1$ for $12 - 1 = 11$ degrees of freedom. This is close to the 99% limit and therefore it is *exceedingly unlikely* (odds of almost 100 to 1 against) that all fences are equally difficult and that the variations in the success rate are due to statistical fluctuations. Fence 2 is especially difficult, whilst fences 4, 8 and (particularly) 10 are easier than average.

A similar (slightly erroneous) calculation treating the number of falls as governed by a Poisson distribution (rather than each jump being a Bernoulli trial) gives a χ^2 value of 21.2 for 11 degrees of freedom and leads to the odds against uniform difficulty of the jumps of about 40 to 1.

31.14 *Three candidates X, Y and Z were standing for election to a vacant seat on their college's Student Committee. The members of the electorate (current first-year students, consisting of* 150 *men and* 105 *women) were each allowed to cross out the name of the candidate they least wished to be elected, the other two candidates then being credited with one vote each. The following data are known.*

(a) *X received* 100 *votes from men, whilst Y received* 65 *votes from women.*
(b) *Z received five more votes from men than X received from women.*
(c) *The total votes cast for X and Y were equal.*

Analyse this data in such a way that a χ^2 test can be used to determine whether voting was other than random (i) *amongst men, and* (ii) *amongst women.*

The numbers of votes cast for each candidate are not independent quantities

since for each vote a candidate receives another candidate also receives a vote. The independent quantities are the numbers of times each name has been crossed out. We must first determine the latter quantities. Suppose that the correlation table for crossings out is

	Not X	Not Y	Not Z	Total
Men	a	c	e	150
Women	b	d	f	105
Total	?	?	?	255

As the questions to be answered deal with men and women's voting patterns separately, we do not need to estimate overall percentages; the theoretical expectation of the result of random voting is $\frac{1}{3} \times 150 = 50$ crossings out by men and $\frac{1}{3} \times 105 = 35$ by women for each candidate. The corresponding variances, for what are essentially Bernoulii trials, are $\frac{1}{3} \times \frac{2}{3} \times 150$ and $\frac{1}{3} \times \frac{2}{3} \times 105$.

To determine the values in the table we know that $a+c+e = 150$ and $b+d+f = 105$. Further, from the information (a) - (c) provided:

(a) $c + e = 100$ and $b + f = 65$,
(b) $a + c = d + f + 5$,
(c) $c + d + e + f = a + b + e + f$.

From these it follows (in approximately deducible order) that $a = 50$, $d = 40$, $5 + c = f$, $c = b + 10$, $(c - 10) + c + 5 = 65 \Rightarrow c = 35$, $b = 25$, $f = 40$ and $e = 65$.

To test for random voting amongst the men we calculate

$$\chi^2 = \frac{(50-a)^2}{33.3} + \frac{(50-c)^2}{33.3} + \frac{(50-e)^2}{33.3} = 13.5$$

for $3 - 1 = 2$ d.o.f. Similarly for the women

$$\chi^2 = \frac{(35-b)^2}{23.3} + \frac{(35-d)^2}{23.3} + \frac{(35-f)^2}{23.3} = 6.4$$

for 2 d.o.f.
The χ^2 value for the men is significantly greater, at almost the 0.1% level, than would be expected for random voting, making the latter extremely unlikely. The corresponding value for women voters is only significant at about the 5% level and random voting cannot be ruled out.

Incidentally, X and Y, who each received 180 votes, tied for first place and a (more conventional) run-off was needed!

31.16 *The function $y(x)$ is known to be a quadratic function of x. The following table gives the measured values and uncorrelated standard errors of y measured at various values of x (in which there is negligible error):*

x	1	2	3	4	5
$y(x)$	3.5 ± 0.5	2.0 ± 0.5	3.0 ± 0.5	6.5 ± 1.0	10.5 ± 1.0

Construct the response matrix R *using as basis functions* 1, x, x^2. *Calculate the matrix* $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R}$ *and show that its inverse, the covariance matrix* V, *has the form*

$$\mathsf{V} = \frac{1}{9184}\begin{pmatrix} 12\,592 & -9708 & 1580 \\ -9708 & 8413 & -1461 \\ 1580 & -1461 & 269 \end{pmatrix}.$$

Use this matrix to find the best values, and their uncertainties, for the coefficients of the quadratic form for $y(x)$.

As the measured data has uncorrelated, but unequal, errors, the covariance matrix N, whilst being diagonal, will not be a multiple of the unit matrix; it will be

$$\mathsf{N} = \mathrm{diag}(0.25, 0.25, 0.25, 1.0, 1.0).$$

Using as base functions the three functions $h_1(x) = 1$, $h_2(x) = x$ and $h_3(x) = x^2$, we calculate the elements of the 5×3 response matrix $\mathsf{R}_{ij} = h_j(x_i)$. To save space we display its 3×5 transpose:

$$\mathsf{R}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{pmatrix}$$

Then

$$\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R} = \begin{pmatrix} 4 & 4 & 4 & 1 & 1 \\ 4 & 8 & 12 & 4 & 5 \\ 4 & 16 & 36 & 16 & 25 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \\ 1 & 5 & 25 \end{pmatrix}$$

$$= \begin{pmatrix} 14 & 33 & 97 \\ 33 & 97 & 333 \\ 97 & 333 & 1273 \end{pmatrix}.$$

The determinant of the square matrix $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R}$ is

$$\begin{aligned} &14[(97 \times 1273) - (333 \times 333)] + 33[(333 \times 97) - (33 \times 1273)] \\ &\qquad + 97[(33 \times 333) - (97 \times 97)] \\ &= 14 \times 12592 - 33 \times 9708 + 97 \times 1580 = 9184. \end{aligned}$$

This is non-zero and so the matrix has an inverse. It is tedious to calculate the inverse V by the standard methods and it is just as good for practical purposes to verify the given form for V, knowing that it is unique. The following matrix equation, $\mathsf{V}\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R} = \mathsf{I}_3$, can be verified numerically

$$\frac{1}{9184}\begin{pmatrix} 12\,592 & -9708 & 1580 \\ -9708 & 8413 & -1461 \\ 1580 & -1461 & 269 \end{pmatrix}\begin{pmatrix} 14 & 33 & 97 \\ 33 & 97 & 333 \\ 97 & 333 & 1273 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The best estimators $\hat{a}_1$, $\hat{a}_2$ and $\hat{a}_3$ for the coefficients in the quadratic form are now given by $\hat{\mathsf{a}} = \mathsf{V}\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{y}$, where y is the data column vector $(3.5,\ 2.0,\ 3.0,\ 6.5,\ 10.5)^{\mathrm{T}}$. The column vector $\hat{\mathsf{a}}$ is calculated as

$$\begin{pmatrix} 1.371 & -1.057 & 0.1720 \\ -1.057 & 0.9160 & -0.1591 \\ 0.1720 & -0.1591 & 0.0293 \end{pmatrix}\begin{pmatrix} 4 & 4 & 4 & 1 & 1 \\ 4 & 8 & 12 & 4 & 5 \\ 4 & 16 & 36 & 16 & 25 \end{pmatrix}\begin{pmatrix} 3.5 \\ 2.0 \\ 3.0 \\ 6.5 \\ 10.5 \end{pmatrix},$$

yielding the three components as 6.73, −4.34 and 1.03. The corresponding standard errors in these coefficients are given by the square roots of the diagonal elements of V, namely 1.17, 0.96 and 0.17.

Thus the best quadratic fit to the measured data, giving weight to the standard errors in them, is

$$y(x) = (6.73 \pm 1.17) - (4.34 \pm 0.96)x + (1.03 \pm 0.17)x^2.$$

The off-diagonal elements of V are not used here, but are closely related to the correlations between the fitted parameters.

31.18 *Prove that the expression given for the Student's t-distribution in equation (31.118) is correctly normalised.*

The given expression is

$$P(t|H_0) = \frac{1}{\sqrt{(N-1)\pi}}\frac{\Gamma\left(\frac{1}{2}N\right)}{\Gamma\left(\frac{1}{2}N - \frac{1}{2}\right)}\left(1 + \frac{t^2}{N-1}\right)^{-N/2},$$

Denoting the product of constants multiplying the t-dependent parentheses by $A(N)$, we require that

$$\int_{-\infty}^{\infty} P(t|H_0)\,dt = A(N)\int_{-\infty}^{\infty}\left(1 + \frac{t^2}{N-1}\right)^{-N/2} dt = 1.$$

Set $t = \sqrt{N-1}\tan\theta$ for $-\pi/2 \le \theta \le \pi/2$ giving

$$\begin{aligned}\int_{-\infty}^{\infty} P(t|H_0)\,dt &= A(N)\int_{-\pi/2}^{\pi/2}(1+\tan^2\theta)^{-N/2}\sqrt{N-1}\sec^2\theta\,d\theta \\ &= 2\sqrt{N-1}\,A(N)\int_0^{\pi/2}\sec^{-N+2}\theta\,d\theta \\ &= 2\sqrt{N-1}\,A(N)\int_0^{\pi/2}\cos^{N-2}\theta\,d\theta.\end{aligned}$$

Now, integrals of this form can be expressed in term of beta and gamma functions by

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} = B(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$$

It follows that

$$\begin{aligned}\int_0^{\pi/2}\cos^{N-2}\theta\,d\theta &= \frac{1}{2}B(\tfrac{1}{2},\tfrac{1}{2}N-\tfrac{1}{2}) \\ &= \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)} \\ &= \frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)}.\end{aligned}$$

Hence

$$\int_0^{\infty} P(t|H_0)\,dt = 2\sqrt{N-1}\,\frac{\Gamma(\frac{1}{2}N)}{\sqrt{(N-1)\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}\,\frac{\sqrt{\pi}\,\Gamma(\frac{1}{2}N-\frac{1}{2})}{2\Gamma(\frac{1}{2}N)} = 1,$$

as expected.

31.20 *It is claimed that the two following sets of values were obtained* (a) *by randomly drawing from a normal distribution that is* $N(0,1)$ *and then* (b) *randomly assigning each reading to one of two sets A and B:*

Set A:	−0.314	0.603	−0.551	−0.537	−0.160	−1.635	0.719
	0.610	0.482	−1.757	0.058			
Set B:	−0.691	1.515	−1.642	−1.736	1.224	1.423	1.165

Make tests, including t- and F-tests, to establish whether there is any evidence that either claims is, or both claims are, false.

(a) The mean and variance of the whole sample are −0.068 and 1.180, leading to an estimated standard deviation, including the Bessel correction for 18 readings,

of 1.12. These are obviously compatible with samples drawn from a $N(0,1)$ distribution, without the need for statistical tests.

(b) The means and sample variances of the two sets are: A, -0.226 and 0.741; B, 0.180 and 2.189, with estimated standard deviations of the populations from which they are drawn of 0.861 and 1.480 respectively.

The best estimator of $\hat{\sigma}$ for calculating t is

$$\hat{\sigma} = \left[\frac{(11 \times 0.741) + (7 \times 2.189)}{11 + 7 - 2}\right]^{1/2} = 1.21.$$

On the null hypothesis that the two samples are drawn from the same distribution, t is given by

$$t = \frac{0.180 - (-0.226)}{1.21}\left(\frac{11 \times 7}{11 + 7}\right)^{1/2} = 0.694.$$

This is for $11 + 7 - 2 = 16$ degrees of freedom. From the table $C_{16}(0.694) = 0.74$. Thus, this or a greater value of t (in magnitude) can be expected in marginally more than half of all cases (recall that here a two-tailed distribution is needed) and there is no evidence for a significant difference between the means of the two samples.

The value of the estimated variance ratio of the parent populations is

$$F = \frac{u^2}{v^2} = \frac{7 \times 2.189}{6}\,\frac{10}{11 \times 0.741} = 3.13.$$

For $n_1 = 6$ and $n_2 = 10$, this value is very close to the 95% confidence limit of 3.22. Thus it is rather unlikely that the allocation between the two groups was made at random – set B has significantly more readings that are more than one standard deviation from the mean for a $N(0,1)$ distribution than it should have.

THIRD EDITION

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Preface

The second edition of *Mathematical Methods for Physics and Engineering* carried more than twice as many exercises, based on its various chapters, as did the first. In the Preface we discussed the general question of how such exercises should be treated but, in the end, decided to provide hints and outline answers to all problems, as in the first edition. This decision was an uneasy one as, on the one hand, it did not allow the exercises to be set as totally unaided homework that could be used for assessment purposes, but, on the other, it did not give a full explanation of how to tackle a problem when a student needed explicit guidance or a model answer.

In order to allow both of these educationally desirable goals to be achieved, we have, in the third edition, completely changed the way this matter is handled. All of the exercises from the second edition, plus a number of additional ones testing the newly added material, have been included in penultimate subsections of the appropriate, sometimes reorganised, chapters. Hints and outline answers are given, as previously, in the final subsections, *but only to the odd-numbered exercises*. This leaves all even-numbered exercises free to be set as unaided homework, as described below.

For the four hundred plus *odd-numbered* exercises, complete solutions are available, to both students and their teachers, in the form of *this* manual; these are in addition to the hints and outline answers given in the main text. For each exercise, the original question is reproduced and then followed by a fully worked solution. For those original exercises that make internal reference to the text or to other (even-numbered) exercises not included in this solutions manual, the questions have been reworded, usually by including additional information, so that the questions can stand alone. Some further minor rewording has been included to improve the page layout.

In many cases the solution given is even fuller than one that might be expected

of a good student who has understood the material. This is because we have aimed to make the solutions instructional as well as utilitarian. To this end, we have included comments that are intended to show how the plan for the solution is formulated and have provided the justifications for particular intermediate steps (something not always done, even by the best of students). We have also tried to write each individual substituted formula in the form that best indicates how it was obtained, before simplifying it at the next or a subsequent stage. Where several lines of algebraic manipulation or calculus are needed to obtain a final result, they are normally included in full; this should enable the student to determine whether an incorrect answer is due to a misunderstanding of principles or to a technical error.

The remaining four hundred or so *even-numbered* exercises have no hints or answers (outlined or detailed) available for general access. They can therefore be used by instructors as a basis for setting unaided homework. Full solutions to these exercises, in the same general format as those appearing in this manual (though they may contain references to the main text or to other exercises), are available without charge to accredited teachers as downloadable pdf files on the password-protected website http://www.cambridge.org/9780521679718. Teachers wishing to have access to the website should contact solutions@cambridge.org for registration details.

As noted above, the original questions are reproduced in full, or in a suitably modified stand-alone form, at the start of each exercise. Reference to the main text is not needed provided that standard formulae are known (and a set of tables is available for a few of the statistical and numerical exercises). This means that, although it is not its prime purpose, this manual could be used as a test or quiz book by a student who has learned, or thinks that he or she has learned, the material covered in the main text.

In all new publications, errors and typographical mistakes are virtually unavoidable, and we would be grateful to any reader who brings instances to our attention. Finally, we are extremely grateful to Dave Green for his considerable and continuing advice concerning typesetting in LaTeX.

Ken Riley, Michael Hobson,
Cambridge, 2006

1

Preliminary algebra

Polynomial equations

1.1 *It can be shown that the polynomial*

$$g(x) = 4x^3 + 3x^2 - 6x - 1$$

has turning points at $x = -1$ *and* $x = \frac{1}{2}$ *and three real roots altogether. Continue an investigation of its properties as follows.*

(a) *Make a table of values of* $g(x)$ *for integer values of* x *between* -2 *and* 2. *Use it and the information given above to draw a graph and so determine the roots of* $g(x) = 0$ *as accurately as possible.*

(b) *Find one accurate root of* $g(x) = 0$ *by inspection and hence determine precise values for the other two roots.*

(c) *Show that* $f(x) = 4x^3 + 3x^2 - 6x - k = 0$ *has only one real root unless* $-5 \le k \le \frac{7}{4}$.

(a) Straightforward evaluation of $g(x)$ at integer values of x gives the following table:

x	-2	-1	0	1	2
$g(x)$	-9	4	-1	0	31

(b) It is apparent from the table alone that $x = 1$ is an exact root of $g(x) = 0$ and so $g(x)$ can be factorised as $g(x) = (x-1)h(x) = (x-1)(b_2x^2 + b_1x + b_0)$. Equating the coefficients of x^3, x^2, x and the constant term gives $4 = b_2$, $b_1 - b_2 = 3$, $b_0 - b_1 = -6$ and $-b_0 = -1$, respectively, which are consistent if $b_1 = 7$. To find the two remaining roots we set $h(x) = 0$:

$$4x^2 + 7x + 1 = 0.$$

The roots of this quadratic equation are given by the standard formula as

$$\alpha_{1,2} = \frac{-7 \pm \sqrt{49-16}}{8}.$$

(c) When $k = 1$ (i.e. the original equation) the values of $g(x)$ at its turning points, $x = -1$ and $x = \frac{1}{2}$, are 4 and $-\frac{11}{4}$, respectively. Thus $g(x)$ can have up to 4 subtracted from it or up to $\frac{11}{4}$ added to it and still satisfy the condition for three (or, at the limit, two) distinct roots of $g(x) = 0$. It follows that for k outside the range $-5 \le k \le \frac{7}{4}$, $f(x)$ $[= g(x) + 1 - k]$ has only one real root.

1.3 *Investigate the properties of the polynomial equation*

$$f(x) = x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0,$$

by proceeding as follows.

(a) *By writing the fifth-degree polynomial appearing in the expression for $f'(x)$ in the form $7x^5 + 30x^4 + a(x-b)^2 + c$, show that there is in fact only one positive root of $f(x) = 0$.*

(b) *By evaluating $f(1)$, $f(0)$ and $f(-1)$, and by inspecting the form of $f(x)$ for negative values of x, determine what you can about the positions of the real roots of $f(x) = 0$.*

(a) We start by finding the derivative of $f(x)$ and note that, because f contains no linear term, f' can be written as the product of x and a fifth-degree polynomial:

$$\begin{aligned} f(x) &= x^7 + 5x^6 + x^4 - x^3 + x^2 - 2 = 0, \\ f'(x) &= x(7x^5 + 30x^4 + 4x^2 - 3x + 2) \\ &= x[\,7x^5 + 30x^4 + 4(x - \tfrac{3}{8})^2 - 4(\tfrac{3}{8})^2 + 2\,] \\ &= x[\,7x^5 + 30x^4 + 4(x - \tfrac{3}{8})^2 + \tfrac{23}{16}\,]. \end{aligned}$$

Since, for positive x, every term in this last expression is necessarily positive, it follows that $f'(x)$ can have no zeros in the range $0 < x < \infty$. Consequently, $f(x)$ can have no turning points in that range and $f(x) = 0$ can have at most one root in the same range. However, $f(+\infty) = +\infty$ and $f(0) = -2 < 0$ and so $f(x) = 0$ has at least one root in $0 < x < \infty$. Consequently it has exactly one root in the range.

(b) $f(1) = 5$, $f(0) = -2$ and $f(-1) = 5$, and so there is at least one root in each of the ranges $0 < x < 1$ and $-1 < x < 0$.

There is no simple systematic way to examine the form of a general polynomial function for the purpose of determining where its zeros lie, but it is sometimes

helpful to group terms in the polynomial and determine how the sign of each group depends upon the range in which x lies. Here grouping successive pairs of terms yields some information as follows:

$$\begin{aligned} x^7 + 5x^6 &\text{ is positive for } x > -5, \\ x^4 - x^3 &\text{ is positive for } x > 1 \quad \text{and} \quad x < 0, \\ x^2 - 2 &\text{ is positive for } x > \sqrt{2} \quad \text{and} \quad x < -\sqrt{2}. \end{aligned}$$

Thus, all three terms are positive in the range(s) common to these, namely $-5 < x < -\sqrt{2}$ and $x > 1$. It follows that $f(x)$ is positive definite in these ranges and there can be no roots of $f(x) = 0$ within them. However, since $f(x)$ is negative for large negative x, there must be at least one root α with $\alpha < -5$.

1.5 *Construct the quadratic equations that have the following pairs of roots:* (a) $-6, -3$; (b) $0, 4$; (c) $2, 2$; (d) $3 + 2i, 3 - 2i$, *where* $i^2 = -1$.

Starting in each case from the 'product of factors' form of the quadratic equation, $(x - \alpha_1)(x - \alpha_2) = 0$, we obtain:

$$\begin{aligned} &\text{(a)} && (x+6)(x+3) = x^2 + 9x + 18 = 0; \\ &\text{(b)} && (x-0)(x-4) = x^2 - 4x = 0; \\ &\text{(c)} && (x-2)(x-2) = x^2 - 4x + 4 = 0; \\ &\text{(d)} && (x-3-2i)(x-3+2i) = x^2 + x(-3-2i-3+2i) \\ & && \qquad + (9 - 6i + 6i - 4i^2) \\ & && = x^2 - 6x + 13 = 0. \end{aligned}$$

Trigonometric identities

1.7 *Prove that*

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

by considering

(a) *the sum of the sines of* $\pi/3$ *and* $\pi/6$,
(b) *the sine of the sum of* $\pi/3$ *and* $\pi/4$.

(a) Using

$$\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right),$$

we have

$$\sin\frac{\pi}{3} + \sin\frac{\pi}{6} = 2\sin\frac{\pi}{4}\cos\frac{\pi}{12},$$
$$\frac{\sqrt{3}}{2} + \frac{1}{2} = 2\frac{1}{\sqrt{2}}\cos\frac{\pi}{12},$$
$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

(b) Using, successively, the identities

$$\sin(A+B) = \sin A\cos B + \cos A\sin B,$$
$$\sin(\pi-\theta) = \sin\theta$$
$$\text{and}\ \ \cos(\tfrac{1}{2}\pi-\theta) = \sin\theta,$$

we obtain

$$\sin\left(\frac{\pi}{3}+\frac{\pi}{4}\right) = \sin\frac{\pi}{3}\cos\frac{\pi}{4} + \cos\frac{\pi}{3}\sin\frac{\pi}{4},$$
$$\sin\frac{7\pi}{12} = \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} + \frac{1}{2}\frac{1}{\sqrt{2}},$$
$$\sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}},$$
$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}.$$

1.9 *Find the real solutions of*

(a) $3\sin\theta - 4\cos\theta = 2$,
(b) $4\sin\theta + 3\cos\theta = 6$,
(c) $12\sin\theta - 5\cos\theta = -6$.

We use the result that if

$$a\sin\theta + b\cos\theta = k$$

then

$$\theta = \sin^{-1}\left(\frac{k}{K}\right) - \phi,$$

where

$$K^2 = a^2 + b^2 \quad \text{and} \quad \phi = \tan^{-1}\frac{b}{a}.$$

Recalling that the inverse sine yields two values and that the individual signs of a and b have to be taken into account, we have

(a) $k = 2$, $K = \sqrt{3^2 + 4^2} = 5$, $\phi = \tan^{-1}(-4/3)$ and so

$$\theta = \sin^{-1}\tfrac{2}{5} - \tan^{-1}\tfrac{-4}{3} = 1.339 \text{ or } -2.626.$$

(b) $k = 6$, $K = \sqrt{4^2 + 3^2} = 5$. Since $k > K$ there is no solution for a real angle θ.

(c) $k = -6$, $K = \sqrt{12^2 + 5^2} = 13$, $\phi = \tan^{-1}(-5/12)$ and so

$$\theta = \sin^{-1}\tfrac{-6}{13} - \tan^{-1}\tfrac{-5}{12} = -0.0849 \text{ or } -2.267.$$

1.11 *Find all the solutions of*

$$\sin\theta + \sin 4\theta = \sin 2\theta + \sin 3\theta$$

that lie in the range $-\pi < \theta \leq \pi$. *What is the multiplicity of the solution* $\theta = 0$?

Using

$$\sin(A + B) = \sin A \cos B + \cos A \sin B,$$
$$\text{and} \quad \cos A - \cos B = -2\sin\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right),$$

and recalling that $\cos(-\phi) = \cos(\phi)$, the equation can be written successively as

$$2\sin\frac{5\theta}{2}\cos\left(-\frac{3\theta}{2}\right) = 2\sin\frac{5\theta}{2}\cos\left(-\frac{\theta}{2}\right),$$
$$\sin\frac{5\theta}{2}\left(\cos\frac{3\theta}{2} - \cos\frac{\theta}{2}\right) = 0,$$
$$-2\sin\frac{5\theta}{2}\sin\theta\sin\frac{\theta}{2} = 0.$$

The first factor gives solutions for θ of $-4\pi/5$, $-2\pi/5$, 0, $2\pi/5$ and $4\pi/5$. The second factor gives rise to solutions 0 and π, whilst the only value making the third factor zero is $\theta = 0$. The solution $\theta = 0$ appears in each of the above sets and so has multiplicity 3.

Coordinate geometry

1.13 *Determine the forms of the conic sections described by the following equations:*

(a) $x^2 + y^2 + 6x + 8y = 0$;
(b) $9x^2 - 4y^2 - 54x - 16y + 29 = 0$;
(c) $2x^2 + 2y^2 + 5xy - 4x + y - 6 = 0$;
(d) $x^2 + y^2 + 2xy - 8x + 8y = 0$.

(a) $x^2 + y^2 + 6x + 8y = 0$. The coefficients of x^2 and y^2 are equal and there is no xy term; it follows that this must represent a circle. Rewriting the equation in standard circle form by 'completing the squares' in the terms that involve x and y, each variable treated separately, we obtain

$$(x + 3)^2 + (y + 4)^2 - (3^2 + 4^2) = 0.$$

The equation is therefore that of a circle of radius $\sqrt{3^2 + 4^2} = 5$ centred on $(-3, -4)$.

(b) $9x^2 - 4y^2 - 54x - 16y + 29 = 0$. This equation contains no xy term and so the centre of the curve will be at $(\,54/(2 \times 9), 16/[2 \times (-4)]\,) = (3, -2)$, and in standardised form the equation is

$$9(x - 3)^2 - 4(y + 2)^2 + 29 - 81 + 16 = 0,$$

or

$$\frac{(x - 3)^2}{4} - \frac{(y + 2)^2}{9} = 1.$$

The minus sign between the terms on the LHS implies that this conic section is a hyperbola with asymptotes (the form for large x and y and obtained by ignoring the constant on the RHS) given by $3(x - 3) = \pm 2(y + 2)$, i.e. lines of slope $\pm\frac{3}{2}$ passing through its 'centre' at $(3, -2)$.

(c) $2x^2 + 2y^2 + 5xy - 4x + y - 6 = 0$. As an xy term is present the equation cannot represent an ellipse or hyperbola in standard form. Whether it represents two straight lines can be most easily investigated by taking the lines in the form $a_i x + b_i y + 1 = 0$, $(i = 1, 2)$ and comparing the product $(a_1 x + b_1 y + 1)(a_2 x + b_2 y + 1)$ with $-\frac{1}{6}(2x^2 + 2y^2 + 5xy - 4x + y - 6)$. The comparison produces five equations which the four constants a_i, b_i, $(i = 1, 2)$ must satisfy:

$$a_1 a_2 = \frac{2}{-6}, \quad b_1 b_2 = \frac{2}{-6}, \quad a_1 + a_2 = \frac{-4}{-6}, \quad b_1 + b_2 = \frac{1}{-6}$$

and

$$a_1 b_2 + b_1 a_2 = \frac{5}{-6}.$$

Combining the first and third equations gives $3a_1^2 - 2a_1 - 1 = 0$ leading to a_1 and a_2 having the values 1 and $-\frac{1}{3}$, in either order. Similarly, combining the second and fourth equations gives $6b_1^2 + b_1 - 2 = 0$ leading to b_1 and b_2 having the values $\frac{1}{2}$ and $-\frac{2}{3}$, again in either order.

Either of the two combinations ($a_1 = -\frac{1}{3}$, $b_1 = -\frac{2}{3}$, $a_2 = 1$, $b_2 = \frac{1}{2}$) and ($a_1 = 1$, $b_1 = \frac{1}{2}$, $a_2 = -\frac{1}{3}$, $b_2 = -\frac{2}{3}$) also satisfies the fifth equation [note that the two alternative pairings do not do so]. That a consistent set can be found shows that the equation does indeed represent a pair of straight lines, $x + 2y - 3 = 0$ and $2x + y + 2 = 0$.

(d) $x^2 + y^2 + 2xy - 8x + 8y = 0$. We note that the first three terms can be written as a perfect square and so the equation can be rewritten as

$$(x + y)^2 = 8(x - y).$$

The two lines given by $x + y = 0$ and $x - y = 0$ are orthogonal and so the equation is of the form $u^2 = 4av$, which, for Cartesian coordinates u, v, represents a parabola passing through the origin, symmetric about the v-axis ($u = 0$) and defined for $v \geq 0$. Thus the original equation is that of a parabola, symmetric about the line $x + y = 0$, passing through the origin and defined in the region $x \geq y$.

Partial fractions

1.15 *Resolve*

$$\text{(a)}\ \frac{2x+1}{x^2+3x-10}, \qquad \text{(b)}\ \frac{4}{x^2-3x}$$

into partial fractions using each of the following three methods:

(i) *Expressing the supposed expansion in a form in which all terms have the same denominator and then equating coefficients of the various powers of x.*
(ii) *Substituting specific numerical values for x and solving the resulting simultaneous equations.*
(iii) *Evaluation of the fraction at each of the roots of its denominator, imagining a factored denominator with the factor corresponding to the root omitted – often known as the 'cover-up' method.*

Verify that the decomposition obtained is independent of the method used.

(a) As the denominator factorises as $(x+5)(x-2)$, the partial fraction expansion must have the form

$$\frac{2x+1}{x^2+3x-10} = \frac{A}{x+5} + \frac{B}{x-2}.$$

(i)

$$\frac{A}{x+5}+\frac{B}{x-2}=\frac{x(A+B)+(5B-2A)}{(x+5)(x-2)}.$$

Solving $A+B=2$ and $-2A+5B=1$ gives $A=\frac{9}{7}$ and $B=\frac{5}{7}$.

(ii) Setting x equal to 0 and 1, say, gives the pair of equations

$$\frac{1}{-10}=\frac{A}{5}+\frac{B}{-2};\quad \frac{3}{-6}=\frac{A}{6}+\frac{B}{-1},$$

$$-1=2A-5B;\quad -3=A-6B,$$

with solution $A=\frac{9}{7}$ and $B=\frac{5}{7}$.

(iii)

$$A=\frac{2(-5)+1}{-5-2}=\frac{9}{7};\quad B=\frac{2(2)+1}{2+5}=\frac{5}{7}.$$

All three methods give the same decomposition.

(b) Here the factorisation of the denominator is simply $x(x-3)$ or, more formally, $(x-0)(x-3)$, and the expansion takes the form

$$\frac{4}{x^2-3x}=\frac{A}{x}+\frac{B}{x-3}.$$

(i)

$$\frac{A}{x}+\frac{B}{x-3}=\frac{x(A+B)-3A}{(x-0)(x-3)}.$$

Solving $A+B=0$ and $-3A=4$ gives $A=-\frac{4}{3}$ and $B=\frac{4}{3}$.

(ii) Setting x equal to 1 and 2, say, gives the pair of equations

$$\frac{4}{-2}=\frac{A}{1}+\frac{B}{-2};\quad \frac{4}{-2}=\frac{A}{2}+\frac{B}{-1},$$

$$-4=2A-B;\quad -4=A-2B,$$

with solution $A=-\frac{4}{3}$ and $B=\frac{4}{3}$.

(iii)

$$A=\frac{4}{0-3}=-\frac{4}{3};\quad B=\frac{4}{3-0}=\frac{4}{3}.$$

Again, all three methods give the same decomposition.

1.17 *Rearrange the following functions in partial fraction form:*

$$\text{(a)}\ \frac{x-6}{x^3-x^2+4x-4}, \qquad \text{(b)}\ \frac{x^3+3x^2+x+19}{x^4+10x^2+9}.$$

(a) For the function

$$f(x) = \frac{x-6}{x^3-x^2+4x-4} = \frac{g(x)}{h(x)}$$

the first task is to factorise the denominator. By inspection, $h(1) = 0$ and so $x-1$ is a factor of the denominator.

Write

$$x^3 - x^2 + 4x - 4 = (x-1)(x^2 + b_1 x + b_0).$$

Equating coefficients: $-1 = b_1 - 1$, $4 = -b_1 + b_0$ and $-4 = -b_0$, giving $b_1 = 0$ and $b_0 = 4$. Thus,

$$f(x) = \frac{x-6}{(x-1)(x^2+4)}.$$

The factor $x^2 + 4$ cannot be factorised further without using complex numbers and so we include a term with this factor as the denominator, but 'at the price of' having a linear term, and not just a number, in the numerator.

$$\begin{aligned} f(x) &= \frac{A}{x-1} + \frac{Bx+C}{x^2+4} \\ &= \frac{Ax^2 + 4A + Bx^2 + Cx - Bx - C}{(x-1)(x^2+4)}. \end{aligned}$$

Comparing the coefficients of the various powers of x in this numerator with those in the numerator of the original expression gives $A + B = 0$, $C - B = 1$ and $4A - C = -6$, which in turn yield $A = -1$, $B = 1$ and $C = 2$. Thus,

$$f(x) = -\frac{1}{x-1} + \frac{x+2}{x^2+4}.$$

(b) By inspection, the denominator of

$$\frac{x^3+3x^2+x+19}{x^4+10x^2+9}$$

factorises simply into $(x^2+9)(x^2+1)$, but neither factor can be broken down further. Thus, as in (a), we write

$$\begin{aligned} f(x) &= \frac{Ax+B}{x^2+9} + \frac{Cx+D}{x^2+1} \\ &= \frac{(A+C)x^3 + (B+D)x^2 + (A+9C)x + (B+9D)}{(x^2+9)(x^2+1)}. \end{aligned}$$

Equating coefficients gives

$$\begin{aligned} A + C &= 1, \\ B + D &= 3, \\ A + 9C &= 1, \\ B + 9D &= 19. \end{aligned}$$

From the first and third equations, $A = 1$ and $C = 0$. The second and fourth yield $B = 1$ and $D = 2$. Thus

$$f(x) = \frac{x+1}{x^2+9} + \frac{2}{x^2+1}.$$

Binomial expansion

1.19 *Evaluate those of the following that are defined:* (a) ${}^5C_3$, (b) ${}^3C_5$, (c) ${}^{-5}C_3$, (d) ${}^{-3}C_5$.

(a) ${}^5C_3 = \frac{5!}{3!\,2!} = 10$.

(b) ${}^3C_5$. This is not defined as $5 > 3 > 0$.

For (c) and (d) we will need to use the identity

$${}^{-m}C_k = (-1)^k \frac{m(m+1)\cdots(m+k-1)}{k!} = (-1)^k\ {}^{m+k-1}C_k.$$

(c) ${}^{-5}C_3 = (-1)^3\ {}^{5+3-1}C_3 = -\frac{7!}{3!\,4!} = -35$.

(d) ${}^{-3}C_5 = (-1)^5\ {}^{5+3-1}C_5 = -\frac{7!}{5!\,2!} = -21$.

Proof by induction and contradiction

1.21 *Prove by induction that*

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1) \qquad \text{and} \qquad \sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

To prove that

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1),$$

assume that the result is valid for $n = N$ and consider

$$\begin{aligned}\sum_{r=1}^{N+1} r &= \sum_{r=1}^{N} r + (N+1)\\ &= \tfrac{1}{2}N(N+1) + (N+1), \quad \text{using the assumption,}\\ &= (N+1)(\tfrac{1}{2}N+1)\\ &= \tfrac{1}{2}(N+1)(N+2).\end{aligned}$$

This is the same form as in the assumption except that N has been replaced by $N+1$; this shows that the result is valid for $n = N+1$ if it is valid for $n = N$. But the assumed result is trivially valid for $n = 1$ and is therefore valid for all n.

To prove that

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2,$$

assume that the result is valid for $n = N$ and consider

$$\begin{aligned}\sum_{r=1}^{N+1} r^3 &= \sum_{r=1}^{N} r^3 + (N+1)^3\\ &= \tfrac{1}{4}N^2(N+1)^2 + (N+1)^3, \quad \text{using the assumption,}\\ &= \tfrac{1}{4}(N+1)^2[N^2 + 4(N+1)]\\ &= \tfrac{1}{4}(N+1)^2(N+2)^2.\end{aligned}$$

This is the same form as in the assumption except that N has been replaced by $N+1$ and shows that the result is valid for $n = N+1$ if it is valid for $n = N$. But the assumed result is trivially valid for $n = 1$ and is therefore valid for all n.

1.23 *Prove that $3^{2n} + 7$, where n is a non-negative integer, is divisible by 8.*

As usual, we assume that the result is valid for $n = N$ and consider the expression with N replaced by $N+1$:

$$\begin{aligned}3^{2(N+1)} + 7 &= 3^{2N+2} + 7 + 3^{2N} - 3^{2N}\\ &= (3^{2N} + 7) + 3^{2N}(9-1).\end{aligned}$$

By the assumption, the first term on the RHS is divisible by 8; the second is clearly so. Thus $3^{2(N+1)} + 7$ is divisible by 8. This shows that the result is valid for $n = N+1$ if it is valid for $n = N$. But the assumed result is trivially valid for $n = 0$ and is therefore valid for all n.

1.25 *Prove by induction that*

$$\sum_{r=1}^{n} \frac{1}{2^r} \tan\left(\frac{\theta}{2^r}\right) = \frac{1}{2^n} \cot\left(\frac{\theta}{2^n}\right) - \cot\theta. \qquad (*)$$

Assume that the result is valid for $n = N$ and consider

$$\sum_{r=1}^{N+1} \frac{1}{2^r} \tan\left(\frac{\theta}{2^r}\right) = \frac{1}{2^N} \cot\left(\frac{\theta}{2^N}\right) - \cot\theta + \frac{1}{2^{N+1}} \tan\left(\frac{\theta}{2^{N+1}}\right).$$

Using the half-angle formula

$$\tan\phi = \frac{2r}{1-r^2}, \qquad \text{where } r = \tan\tfrac{1}{2}\phi,$$

to write $\cot(\theta/2^N)$ in terms of $t = \tan(\theta/2^{N+1})$, we have that the RHS is

$$\frac{1}{2^N}\left(\frac{1-t^2}{2t}\right) - \cot\theta + \frac{1}{2^{N+1}}\, t = \frac{1}{2^{N+1}}\left(\frac{1-t^2+t^2}{t}\right) - \cot\theta$$
$$= \frac{1}{2^{N+1}} \cot\left(\frac{\theta}{2^{N+1}}\right) - \cot\theta.$$

This is the same form as in the assumption except that N has been replaced by $N+1$ and shows that the result is valid for $n = N+1$ if it is valid for $n = N$.

But, for $n = 1$, the LHS of $(*)$ is $\frac{1}{2}\tan(\theta/2)$. The RHS can be written in terms of $s = \tan(\theta/2)$:

$$\frac{1}{2}\cot\left(\frac{\theta}{2}\right) - \cot\theta = \frac{1}{2s} - \frac{1-s^2}{2s} = \frac{s}{2},$$

i.e. the same as the LHS. Thus the result is valid for $n = 1$ and hence for all n.

1.27 *Establish the values of k for which the binomial coefficient ${}^pC_k$ is divisible by p when p is a prime number. Use your result and the method of induction to prove that $n^p - n$ is divisible by p for all integers n and all prime numbers p. Deduce that $n^5 - n$ is divisible by 30 for any integer n.*

Since

$${}^pC_k = \frac{p!}{k!(p-k)!},$$

its numerator will always contain a factor p. Therefore, the fraction will be divisible by p unless the denominator happens to contain a (cancelling) factor of p. Since p is prime, this latter factor cannot arise from the product of two or more terms in the denominator; nor can p have any factor that cancels with a

term in the denominator. Thus, for cancellation to occur, either $k!$ or $(p-k)!$ must contain a term p; this can only happen for $k = p$ or $k = 0$; for all other values of k, ${}^pC_k$ will be divisible by p.

Assume that $n^p - n$ is divisible by prime number p for $n = N$. Clearly this is true for $N = 1$ and any p. Now, using the binomial expansion of $(N+1)^p$, consider

$$\begin{aligned}(N+1)^p - (N+1) &= \sum_{k=0}^{p} {}^pC_k N^k - (N+1)\\ &= 1 + \sum_{k=1}^{p-1} {}^pC_k N^k + N^p - N - 1.\end{aligned}$$

But, as shown above, ${}^pC_k$ is divisible by p for all k in the range $1 \le k \le p-1$, and $N^p - N$ is divisible by p, by assumption. Thus $(N+1)^p - (N+1)$ is divisible by p if it is true that $N^p - N$ is divisible by p. Taking $N = 1$, for which, as noted above, the assumption is valid by inspection for any p, the result follows for all positive integers n and all primes p.

Now consider $f(n) = n^5 - n$. By the result just proved $f(n)$ is divisible by (prime number) 5. Further, $f(n) = n(n^4-1) = n(n^2-1)(n^2+1) = n(n-1)(n+1)(n^2+1)$. Thus the factorisation of $f(n)$ contains three consecutive integers; one of them must be divisible by 3 and at least one must be even and hence divisible by 2. Thus, $f(n)$ has the prime numbers 2, 3 and 5 as its divisors and must therefore be divisible by 30.

1.29 *Prove, by the method of contradiction, that the equation*

$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0,$$

in which all the coefficients a_i are integers, cannot have a rational root, unless that root is an integer. Deduce that any integral root must be a divisor of a_0 and hence find all rational roots of

(a) $x^4 + 6x^3 + 4x^2 + 5x + 4 = 0$,
(b) $x^4 + 5x^3 + 2x^2 - 10x + 6 = 0$.

Suppose that the equation has a rational root $x = p/q$, where integers p and q have no common factor and q is neither 0 nor 1. Then substituting the root and multiplying the resulting equation by q^{n-1} gives

$$\frac{p^n}{q} + a_{n-1}p^{n-1} + \cdots + a_1 pq^{n-2} + a_0 q^{n-1} = 0.$$

But the first term of this equation is not an integer (since p and q have no factor

in common) whilst each of the remaining terms is a product of integers and is therefore an integer. Thus we have an integer equal to (minus) a non-integer. This is a contradiction and shows that it was wrong to suppose that the original equation has a rational non-integer root.

From the general properties of polynomial equations we have that the product of the roots of the equation $\sum_{i=0}^{n} b_i x^i = 0$ is $(-1)^n b_0/b_n$. For our original equation, $b_n = 1$ and $b_0 = a_0$. Consequently, the product of its roots is equal to the integral value $(-1)^n a_0$. Since there are no non-integral rational roots it follows that any integral root must be a divisor of a_0.

(a) $x^4+6x^3+4x^2+5x+4 = 0$. This equation has integer coefficients and a leading coefficient equal to unity. We can thus apply the above result, which shows that its only possible rational roots are the six integers ± 1, ± 2 and ± 4. Of these, all positive values are impossible (since then every term would be positive) and trial and error will show that none of the negative values is a root either.

(b) $x^4 + 5x^3 + 2x^2 - 10x + 6 = 0$. In the same way as above, we deduce that for this equation the only possible rational roots are the eight values ± 1, ± 2, ± 3 and ± 6. Substituting each in turn shows that only $x = -3$ satisfies the equation.

Necessary and sufficient conditions

1.31 *For the real variable x, show that a sufficient, but not necessary, condition for $f(x) = x(x+1)(2x+1)$ to be divisible by 6 is that x is an integer.*

First suppose that x is an integer and consider $f(x)$ expressed as

$$f(x) = x(x+1)(2x+1) = x(x+1)(x+2) + x(x+1)(x-1).$$

Each term on the RHS consists of the product of three consecutive integers. In such a product one of the integers must divide by 3 and at least one of the other integers must be even. Thus each product separately divides by both 3 and 2, and hence by 6, and therefore so does their sum $f(x)$. Thus x being an integer *is a sufficient* condition for $f(x)$ to be divisible by 6.

That it is *not* a necessary condition can be shown by considering an equation of the form

$$f(x) = x(x+1)(2x+1) = 2x^3 + 3x^2 + x = 6m,$$

where m is an integer. As a specific counter-example consider the case $m = 4$. We note that $f(1) = 6$ whilst $f(2) = 30$. Thus there must be a root of the equation that lies strictly between the values 1 and 2, i.e a non-integer value of x that makes $f(x)$ equal to 24 and hence divisible by 6. This establishes the result that x being an integer is *not a necessary* condition for $f(x)$ to be divisible by 6.

1.33 *The coefficients a_i in the polynomial $Q(x) = a_4x^4 + a_3x^3 + a_2x^2 + a_1x$ are all integers. Show that $Q(n)$ is divisible by 24 for all integers $n \geq 0$ if and only if all of the following conditions are satisfied:*

(i) $2a_4 + a_3$ *is divisible by* 4;
(ii) $a_4 + a_2$ *is divisible by* 12;
(iii) $a_4 + a_3 + a_2 + a_1$ *is divisible by* 24.

This problem involves both proof by induction and proof of the 'if and only if' variety. Firstly, assume that the three conditions are satisfied:

$$\begin{aligned} 2a_4 + a_3 &= 4\alpha, \\ a_4 + a_2 &= 12\beta, \\ a_4 + a_3 + a_2 + a_1 &= 24\gamma, \end{aligned}$$

where α, β and γ are integers. We now have to prove that $Q(n) = a_4n^4 + a_3n^3 + a_2n^2 + a_1n$ is divisible by 24 for all integers $n \geq 0$. It is clearly true for $n = 0$, and we assume that it is true for $n = N$ and that $Q(N) = 24m$ for some integer m. Now consider $Q(N + 1)$:

$$\begin{aligned} Q(N+1) &= a_4(N+1)^4 + a_3(N+1)^3 + a_2(N+1)^2 + a_1(N+1) \\ &= a_4N^4 + a_3N^3 + a_2N^2 + a_1N + 4a_4N^3 + (6a_4 + 3a_3)N^2 \\ &\qquad +(4a_4 + 3a_3 + 2a_2)N + (a_4 + a_3 + a_2 + a_1) \\ &= 24m + 4a_4N^3 + 3(4\alpha)N^2 \\ &\qquad +[4a_4 + (12\alpha - 6a_4) + (24\beta - 2a_4)]N + 24\gamma \\ &= 24(m + \gamma + \beta N) + 12\alpha N(N+1) + 4a_4(N-1)N(N+1). \end{aligned}$$

Now $N(N + 1)$ is the product of two consecutive integers and so one must be even and contain a factor of 2; likewise $(N - 1)N(N + 1)$, being the product of three consecutive integers, must contain both 2 and 3 as factors. Thus every term in the expression for $Q(N + 1)$ divides by 24 and so, therefore, does $Q(N + 1)$. Thus the proposal is true for $n = N + 1$ if it is true for $n = N$, and this, together with our observation for $n = 0$, completes the 'if' part of the proof.

Now suppose that $Q(n) = a_4n^4 + a_3n^3 + a_2n^2 + a_1n$ is divisible by 24 for all integers $n \geq 0$. Setting n equal to 1, 2 and 3 in turn, we have

$$\begin{aligned} a_4 + a_3 + a_2 + a_1 &= 24p, \\ 16a_4 + 8a_3 + 4a_2 + 2a_1 &= 24q, \\ 81a_4 + 27a_3 + 9a_2 + 3a_1 &= 24r, \end{aligned}$$

for some integers p, q and r. The first of these equations is condition (iii). The

other conditions are established by combining the above equations as follows:

$$\begin{aligned} 14a_4 + 6a_3 + 2a_2 &= 24(q - 2p), \\ 78a_4 + 24a_3 + 6a_2 &= 24(r - 3p), \\ 36a_4 + 6a_3 &= 24(r - 3p - 3q + 6p), \\ 22a_4 - 2a_2 &= 24(r - 3p - 4q + 8p). \end{aligned}$$

The two final equations show that $6a_4 + a_3$ is divisible by 4 and that $11a_4 - a_2$ is divisible by 12. But, if $6a_4 + a_3$ is divisible by 4 then so is $(6 - 4)a_4 + a_3$, i.e. $2a_4 + a_3$. Similarly, $11a_4 - a_2$ being divisible by 12 implies that $12a_4 - (11a_4 - a_2)$, i.e. $a_4 + a_2$, is also divisible by 12. Thus, conditions (i) and (ii) are established and the 'only if' part of the proof is complete.

2

Preliminary calculus

2.1 *Obtain the following derivatives from first principles:*

(a) *the first derivative of* $3x + 4$;
(b) *the first, second and third derivatives of* $x^2 + x$;
(c) *the first derivative of* $\sin x$.

(a) From the definition of the derivative as a limit, we have

$$f'(x) = \lim_{\Delta x \to 0} \frac{[3(x + \Delta x) + 4] - (3x + 4)}{\Delta x} = \lim_{\Delta x \to 0} \frac{3\Delta x}{\Delta x} = 3.$$

(b) These are calculated similarly, but using each calculated derivative as the input function for finding the next higher derivative.

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 + (x + \Delta x)] - (x^2 + x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(x^2 + 2x\Delta x + (\Delta x)^2) + (x + \Delta x)] - (x^2 + x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[(2x\Delta x + (\Delta x)^2) + \Delta x]}{\Delta x} \\ &= 2x + 1; \end{aligned}$$

$$f''(x) = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x) + 1] - (2x + 1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = 2;$$

$$f'''(x) = \lim_{\Delta x \to 0} \frac{2 - 2}{\Delta x} = 0.$$

(c) We use the expansion formula for $\sin(A + B)$ and then the series definitions of the sine and cosine functions to write $\cos \Delta x$ and $\sin \Delta x$ as series involving

increasing powers of Δx.

$$\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(\sin x \cos \Delta x + \cos x \sin \Delta x) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\sin x\,(1 - \frac{(\Delta x)^2}{2!} + \cdots) + \cos x\,(\Delta x - \frac{(\Delta x)^3}{3!} + \cdots) - \sin x}{\Delta x} \\
&= \lim_{\Delta x \to 0} -\tfrac{1}{2}\Delta x \sin x + \cos x - \tfrac{1}{6}(\Delta x)^2 \cos x + \cdots \\
&= \cos x.
\end{aligned}$$

2.3 *Find the first derivatives of*

(a) $x^2 \exp x$, (b) $2 \sin x \cos x$, (c) $\sin 2x$, (d) $x \sin ax$,
(e) $(e^{ax})(\sin ax) \tan^{-1} ax$, (f) $\ln(x^a + x^{-a})$,
(g) $\ln(a^x + a^{-x})$, *(h)* x^x.

(a) $x^2 \exp x$ is the product of two functions, both of which can be differentiated simply. We therefore apply the product rule and obtain:

$$f'(x) = x^2 \frac{d(\exp x)}{dx} + \exp x \frac{d(x^2)}{dx} = x^2 \exp x + (2x) \exp x = (x^2 + 2x) \exp x.$$

(b) Again, the product rule is appropriate:

$$\begin{aligned}
f'(x) &= 2 \sin x \frac{d(\cos x)}{dx} + 2 \cos x \frac{d(\sin x)}{dx} \\
&= 2 \sin x(-\sin x) + 2 \cos x(\cos x) \\
&= 2(-\sin^2 x + \cos^2 x) = 2 \cos 2x.
\end{aligned}$$

(c) Rewriting the function as $f(x) = \sin u$, where $u(x) = 2x$, and using the chain rule:

$$f'(x) = \cos u \times \frac{du}{dx} = \cos u \times 2 = 2 \cos(2x).$$

We note that this is the same result as in part (b); this is not surprising as the two functions to be differentiated are identical, i.e. $2 \sin x \cos x \equiv \sin 2x$.

(d) Once again, the product rule can be applied:

$$f'(x) = x \frac{d(\sin ax)}{dx} + \sin ax \frac{d(x)}{dx} = xa \cos ax + \sin ax \times 1 = \sin ax + ax \cos ax.$$

(e) This requires the product rule for three factors:

$$\begin{aligned} f'(x) &= (e^{ax})(\sin ax)\frac{d(\tan^{-1} ax)}{dx} + (e^{ax})(\tan^{-1} ax)\frac{d(\sin ax)}{dx} \\ &\quad +(\sin ax)(\tan^{-1} ax)\frac{d(e^{ax})}{dx} \\ &= (e^{ax})(\sin ax)\left(\frac{a}{1+a^2x^2}\right) + (e^{ax})(\tan^{-1} ax)(a\cos ax) \\ &\quad +(\sin ax)(\tan^{-1} ax)(ae^{ax}) \\ &= ae^{ax}\left[\frac{\sin ax}{1+a^2x^2} + (\tan^{-1} ax)(\cos ax + \sin ax)\right]. \end{aligned}$$

(f) Rewriting the function as $f(x) = \ln u$, where $u(x) = x^a + x^{-a}$, and using the chain rule:

$$f'(x) = \frac{1}{u} \times \frac{du}{dx} = \frac{1}{x^a + x^{-a}} \times (ax^{a-1} - ax^{-a-1}) = \frac{a(x^a - x^{-a})}{x(x^a + x^{-a})}.$$

(g) Using logarithmic differentiation and the chain rule as in (f):

$$f'(x) = \frac{1}{a^x + a^{-x}} \times (\ln a\ a^x - \ln a\ a^{-x}) = \frac{\ln a(a^x - a^{-x})}{a^x + a^{-x}}.$$

(h) In order to remove the independent variable x from the exponent in $y = x^x$, we first take logarithms and then differentiate implicitly:

$$\begin{aligned} y &= x^x, \\ \ln y &= x \ln x, \\ \frac{1}{y}\frac{dy}{dx} &= \ln x + \frac{x}{x}, \quad \text{using the product rule,} \\ \frac{dy}{dx} &= (1 + \ln x)x^x. \end{aligned}$$

2.5 *Use the result that* $d[v(x)^{-1}]/dx = -v^{-2}dv/dx$ *to find the first derivatives of* (a) $(2x+3)^{-3}$, (b) $\sec^2 x$, (c) $\operatorname{cosech}^3 3x$, (d) $1/\ln x$, (e) $1/[\sin^{-1}(x/a)]$.

(a) Writing $(2x+3)^3$ as $v(x)$ and using the chain rule, we have

$$f'(x) = -\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{(2x+3)^6}\,[\,3(2x+3)^2\,(2)\,] = -\frac{6}{(2x+3)^4}.$$

(b) Writing $\cos^2 x$ as $v(x)$, we have

$$f'(x) = -\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{\cos^4 x}\,[\,2\cos x(-\sin x)\,] = 2\sec^2 x \tan x.$$

(c) Writing $\sinh^3 3x$ as $v(x)$, we have

$$f'(x) = -\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{\sinh^6 3x}\,[\,3\sinh^2 3x(\cosh 3x)(3)\,]$$
$$= -9\,\mathrm{cosech}^3 3x \coth 3x.$$

(d) Writing $\ln x$ as $v(x)$, we have

$$f'(x) = -\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{(\ln x)^2}\frac{1}{x} = -\frac{1}{x\ln^2 x}.$$

(e) Writing $\sin^{-1}(x/a)$ as $v(x)$, we have

$$f'(x) = -\frac{1}{v^2}\frac{dv}{dx} = -\frac{1}{[\,\sin^{-1}(x/a)\,]^2}\frac{1}{\sqrt{a^2-x^2}}.$$

2.7 *Find dy/dx if $x = (t-2)/(t+2)$ and $y = 2t/(t+1)$ for $-\infty < t < \infty$. Show that it is always non-negative, and make use of this result in sketching the curve of y as a function of x.*

We calculate dy/dx as $dy/dt \div dx/dt$:

$$\frac{dy}{dt} = \frac{(t+1)2 - 2t(1)}{(t+1)^2} = \frac{2}{(t+1)^2},$$
$$\frac{dx}{dt} = \frac{(t+2)(1) - (t-2)(1)}{(t+2)^2} = \frac{4}{(t+2)^2},$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{2}{(t+1)^2} \div \frac{4}{(t+2)^2} = \frac{(t+2)^2}{2(t+1)^2},$$

which is clearly positive for all t.

By evaluating x and y for a range of values of t and recalling that its slope is always positive, the curve can be plotted as in figure 2.1. Alternatively, we may eliminate t using

$$t = \frac{2x+2}{1-x} \quad \text{and} \quad t = \frac{y}{2-y},$$

to obtain the equation of the curve in x-y coordinates as

$$2(x+1)(2-y) = y(1-x),$$
$$xy - 4x + 3y - 4 = 0,$$
$$(x+3)(y-4) = 4 - 12 = -8.$$

Figure 2.1 The solution to exercise 2.7.

This shows that the curve is a rectangular hyperbola in the second and fourth quadrants with asymptotes, parallel to the x- and y-axes, passing through $(-3, 4)$.

2.9 *Find the second derivative of* $y(x) = \cos[(\pi/2) - ax]$. *Now set* $a = 1$ *and verify that the result is the same as that obtained by first setting* $a = 1$ *and simplifying* $y(x)$ *before differentiating.*

We use the chain rule at each stage and, either finally or initially, the equality of $\cos(\frac{1}{2}\pi - \theta)$ and $\sin\theta$:

$$y(x) = \cos\left(\frac{\pi}{2} - ax\right),$$
$$y'(x) = a\sin\left(\frac{\pi}{2} - ax\right),$$
$$y''(x) = -a^2\cos\left(\frac{\pi}{2} - ax\right).$$
$$\text{For } a = 1, \quad y''(x) = -\cos\left(\frac{\pi}{2} - x\right) = -\sin x.$$

Setting $a = 1$ initially, gives $y = \cos(\frac{1}{2}\pi - x) = \sin x$. Hence $y' = \cos x$ and $y'' = -\sin x$, yielding the same result as before.

2.11 *Show by differentiation and substitution that the differential equation*

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2+3)y = 0$$

has a solution of the form $y(x) = x^n \sin x$, *and find the value of* n.

The solution plan is to calculate the derivatives as functions of n and x and then, after substitution, require that the equation is identically satisfied for all x. This will impose conditions on n.

We have, by successive differentiation or by the use of Leibnitz' theorem, that

$$\begin{aligned} y(x) &= x^n \sin x, \\ y'(x) &= nx^{n-1}\sin x + x^n \cos x, \\ y''(x) &= n(n-1)x^{n-2}\sin x + 2nx^{n-1}\cos x - x^n \sin x. \end{aligned}$$

Substituting these into

$$4x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2+3)y = 0$$

gives

$$(4n^2 - 4n - 4n + 3)x^n \sin x + (-4+4)x^{n+2}\sin x + (8n-4)x^{n+1}\cos x = 0.$$

For this to be true for all x, both $4n^2 - 8n + 3 = (2n-3)(2n-1) = 0$ and $8n - 4 = 0$ have to be satisfied. If $n = \frac{1}{2}$, they *are* both satisfied, thus establishing $y(x) = x^{1/2}\sin x$ as a solution of the given equation.

2.13 *Show that the lowest value taken by the function* $3x^4 + 4x^3 - 12x^2 + 6$ *is* -26.

We need to calculate the first and second derivatives of the function in order to establish the positions and natures of its turning points:

$$\begin{aligned} y(x) &= 3x^4 + 4x^3 - 12x^2 + 6, \\ y'(x) &= 12x^3 + 12x^2 - 24x, \\ y''(x) &= 36x^2 + 24x - 24. \end{aligned}$$

Setting $y'(x) = 0$ gives $x(x+2)(x-1) = 0$ with roots 0, 1 and -2. The corresponding values of $y''(x)$ are -24, 36 and 72.

Since $y(\pm\infty) = \infty$, the lowest value of y is that corresponding to the lowest minimum, which can only be at $x = 1$ or $x = -2$, as y'' must be positive at a minimum. The values of $y(x)$ at these two points are $y(1) = 1$ and $y(-2) = -26$, and so the lowest value taken is -26.

2.15 *Show that* $y(x) = xa^{2x}\exp x^2$ *has no stationary points other than* $x = 0$, *if*

$$\exp(-\sqrt{2}) < a < \exp(\sqrt{2}).$$

Since the logarithm of a variable varies monotonically with the variable, the stationary points of the logarithm of a function of x occur at the same values of x as the stationary points of the function. As x appears as an exponent in the given function, we take logarithms before differentiating and obtain:

$$\ln y = \ln x + 2x\ln a + x^2,$$
$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2\ln a + 2x.$$

For a stationary point $dy/dx = 0$. Except at $x = 0$ (where y is also 0), this equation reduces to

$$2x^2 + 2x\ln a + 1 = 0.$$

This quadratic equation has no real roots for x if $4(\ln a)^2 < 4 \times 2 \times 1$, i.e. $|\ln a| < \sqrt{2}$; a result that can also be written as $\exp(-\sqrt{2}) < a < \exp(\sqrt{2})$.

2.17 *The parametric equations for the motion of a charged particle released from rest in electric and magnetic fields at right angles to each other take the forms*

$$x = a(\theta - \sin\theta), \qquad y = a(1 - \cos\theta).$$

Show that the tangent to the curve has slope $\cot(\theta/2)$. *Use this result at a few calculated values of* x *and* y *to sketch the form of the particle's trajectory.*

With the given parameterisation,

$$\frac{dx}{d\theta} = a - a\cos\theta,$$
$$\frac{dy}{d\theta} = a\sin\theta,$$
$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{d\theta}\frac{d\theta}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

Clearly, $y = 0$ whenever $\theta = 2n\pi$ with n an integer; dy/dx becomes infinite at the same points. The slope is zero whenever $\theta = (2n+1)\pi$ and the value of y is then $2a$. These results are plotted in figure 2.2.

Figure 2.2 The solution to exercise 2.17.

2.19 *The curve whose equation is* $x^{2/3}+y^{2/3}=a^{2/3}$ *for positive* x *and* y *and which is completed by its symmetric reflections in both axes is known as an astroid. Sketch it and show that its radius of curvature in the first quadrant is* $3(axy)^{1/3}$.

For the asteroid curve (see figure 2.3) and its first derivative in the first quadrant, where all fractional roots are positive, we have

$$\begin{aligned} x^{2/3}+y^{2/3} &= a^{2/3}, \\ \frac{2}{3x^{1/3}}+\frac{2}{3y^{1/3}}\frac{dy}{dx} &= 0, \\ \Rightarrow \quad \frac{dy}{dx} &= -\left(\frac{y}{x}\right)^{1/3}. \end{aligned}$$

Differentiating again,

$$\begin{aligned} \frac{d^2y}{dx^2} &= -\frac{1}{3}\left(\frac{y}{x}\right)^{-2/3}\left[\frac{-x(\frac{y}{x})^{1/3}-y}{x^2}\right] \\ &= \frac{1}{3}(x^{-2/3}y^{-1/3}+x^{-4/3}y^{1/3}) \\ &= \frac{1}{3}y^{-1/3}x^{-4/3}(x^{2/3}+y^{2/3}) \\ &= \frac{1}{3}y^{-1/3}x^{-4/3}a^{2/3}. \end{aligned}$$

Hence, the radius of curvature is

$$\rho = \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{\left[1+\left(\frac{y}{x}\right)^{2/3}\right]^{3/2}}{\frac{1}{3}y^{-1/3}x^{-4/3}a^{2/3}}$$

$$= 3(x^{2/3}+y^{2/3})^{3/2}x^{1/3}y^{1/3}a^{-2/3} = 3a^{1/3}x^{1/3}y^{1/3},$$

Figure 2.3 The astroid discussed in exercise 2.19.

as stated in the question.

2.21 *Use Leibnitz' theorem to find*

(a) *the second derivative of* $\cos x \sin 2x$,
(b) *the third derivative of* $\sin x \ln x$,
(c) *the fourth derivative of* $(2x^3 + 3x^2 + x + 2)e^{2x}$.

Leibnitz' theorem states that if $y(x) = u(x)v(x)$ and the rth derivative of a function $f(x)$ is denoted by $f^{(r)}$ then

$$y^{(n)} = \sum_{k=0}^{n} {}^{n}C_k \, u^{(k)} \, v^{(n-k)}.$$

So,

(a)
$$\begin{aligned}\frac{d^2(\cos x \sin 2x)}{dx^2} &= (-\cos x)(\sin 2x) + 2(-\sin x)(2\cos 2x) \\ &\quad + (\cos x)(-4\sin 2x) \\ &= -5\cos x \sin 2x - 4\sin x \cos 2x \\ &= 2\sin x[-5\cos^2 x - 2(2\cos^2 x - 1)] \\ &= 2\sin x(2 - 9\cos^2 x).\end{aligned}$$

(b)
$$\begin{aligned}\frac{d^3(\sin x \ln x)}{dx^3} &= (-\cos x)(\ln x) + 3(-\sin x)(x^{-1}) \\ &\quad + 3(\cos x)(-x^{-2}) + (\sin x)(2x^{-3}) \\ &= (2x^{-3} - 3x^{-1})\sin x - (3x^{-2} + \ln x)\cos x.\end{aligned}$$

(c) We note that the nth derivative of e^{2x} is $2^n e^{2x}$ and that the 4th derivative of a cubic polynomial is zero. And so,

$$\begin{aligned}
&\frac{d^4[(2x^3+3x^2+x+2)e^{2x}]}{dx^4}\\
&= (0)(e^{2x}) + 4(12)(2e^{2x}) + 6(12x+6)(4e^{2x})\\
&\qquad + 4(6x^2+6x+1)(8e^{2x}) + (2x^3+3x^2+x+2)(16e^{2x})\\
&= 16(2x^3+15x^2+31x+19)e^{2x}.
\end{aligned}$$

2.23 *Use the properties of functions at their turning points to do the following.*

(a) *By considering its properties near $x = 1$, show that $f(x) = 5x^4 - 11x^3 + 26x^2 - 44x + 24$ takes negative values for some range of x.*
(b) *Show that $f(x) = \tan x - x$ cannot be negative for $0 \le x < \pi/2$, and deduce that $g(x) = x^{-1}\sin x$ decreases monotonically in the same range.*

(a) We begin by evaluating $f(1)$ and find that $f(1) = 5 - 11 + 26 - 44 + 24 = 0$. This suggests that $f(x)$ will be positive on one side of $x = 1$ and negative on the other. However, to be sure of this we need to establish that $x = 1$ is *not* a turning point of $f(x)$. To do this we calculate its derivative there:

$$\begin{aligned}
f(x) &= 5x^4 - 11x^3 + 26x^2 - 44x + 24,\\
f'(x) &= 20x^3 - 33x^2 + 52x - 44,\\
f'(1) &= 20 - 33 + 52 - 44 = -5 \neq 0.
\end{aligned}$$

So, $f'(1)$ is negative and f is decreasing at this point, where its value is 0. Therefore $f(x)$ must be negative in the range $1 < x < \alpha$ for some $\alpha > 1$.

(b) The function $f(x) = \tan x - x$ is differentiable in the range $0 \le x < \pi/2$, and $f'(x) = \sec^2 x - 1 = \tan^2 x$ which is > 0 for all x in the range; taken together with $f(0) = 0$, this establishes the result.

For $g(x) = (\sin x)/x$, the rule for differentiating quotients gives

$$g'(x) = \frac{x\cos x - \sin x}{x^2} = -\frac{\cos x(\tan x - x)}{x^2}.$$

The term in parenthesis cannot be negative in the range $0 \le x < \pi/2$, and in the same range $\cos x > 0$. Thus $g'(x)$ is never positive in the range and $g(x)$ decreases monotonically [from its value of $g(0) = 1$].

2.25 *By applying Rolle's theorem to* $x^n \sin nx$, *where* n *is an arbitrary positive integer, show that* $\tan nx + x = 0$ *has a solution* α_1 *with* $0 < \alpha_1 < \pi/n$. *Apply the theorem a second time to obtain the nonsensical result that there is a real* α_2 *in* $0 < \alpha_2 < \pi/n$, *such that* $\cos^2(n\alpha_2) = -n$. *Explain why this incorrect result arises.*

Clearly, the function $f(x) = x^n \sin nx$ has zeroes at $x = 0$ and $x = \pi/n$. Therefore, by Rolle's theorem, its derivative,

$$f'(x) = nx^{n-1} \sin nx + nx^n \cos nx,$$

must have a zero in the range $0 < x < \pi/n$. But, since $x \neq 0$ and $n \neq 0$, this is equivalent to a root α_1 of $\tan nx + x = 0$ in the same range. To obtain this result we have divided $f'(x) = 0$ through by $\cos nx$; this is allowed, since $x = \pi/(2n)$, the value that makes $\cos nx = 0$, is not a solution of $f'(x) = 0$.

We now note that $g(x) = \tan nx + x$ has zeroes at $x = 0$ and $x = \alpha_1$. Applying Rolle's theorem again (blindly) then shows that $g'(x) = n \sec^2 nx + 1$ has a zero α_2 in the range $0 < \alpha_2 < \alpha_1 < \pi/n$, with $\cos^2(n\alpha_2) = -n$.

The false result arises because $\tan nx$ is not differentiable at $x = \pi/(2n)$, which lies in the range $0 < x < \pi/n$, and so the conditions for applying Rolle's theorem are not satisfied.

2.27 *For the function* $y(x) = x^2 \exp(-x)$ *obtain a simple relationship between* y *and* dy/dx *and then, by applying Leibnitz' theorem, prove that*

$$xy^{(n+1)} + (n + x - 2)y^{(n)} + ny^{(n-1)} = 0.$$

The required function and its first derivative are

$$\begin{aligned} y(x) &= x^2 e^{-x}, \\ y'(x) &= 2xe^{-x} - x^2 e^{-x} \\ &= 2xe^{-x} - y. \end{aligned}$$

Multiplying through by a factor x will enable us to express the first term on the RHS in terms of y and obtain

$$xy' = 2y - xy.$$

Now we apply Leibnitz' theorem to obtain the nth derivatives of both sides of this last equation, noting that the only non-zero derivative of x is the first derivative. We obtain

$$xy^{(n+1)} + n(1)y^{(n)} = 2y^{(n)} - [\,xy^{(n)} + n(1)y^{(n-1)}\,],$$

which can be rearranged as

$$xy^{(n+1)} + (n + x - 2)y^{(n)} + ny^{(n-1)} = 0,$$

thus completing the proof.

2.29 *Show that the curve $x^3 + y^3 - 12x - 8y - 16 = 0$ touches the x-axis.*

We first find an expression for the slope of the curve as a function of x and y. From

$$x^3 + y^3 - 12x - 8y - 16 = 0$$

we obtain, by implicit differentiation, that

$$3x^2 + 3y^2y' - 12 - 8y' = 0 \quad \Rightarrow \quad y' = \frac{3x^2 - 12}{8 - 3y^2}.$$

Clearly $y' = 0$ at $x = \pm 2$. At $x = 2$,

$$8 + y^3 - 24 - 8y - 16 = 0 \quad \Rightarrow \quad y \neq 0.$$

However, at $x = -2$,

$$-8 + y^3 + 24 - 8y - 16 = 0, \qquad \text{with one solution } y = 0.$$

Thus the point $(-2, 0)$ lies on the curve and $y' = 0$ there. It follows that the curve touches the x-axis at that point.

2.31 *Find the indefinite integrals J of the following ratios of polynomials:*

(a) $(x + 3)/(x^2 + x - 2)$;
(b) $(x^3 + 5x^2 + 8x + 12)/(2x^2 + 10x + 12)$;
(c) $(3x^2 + 20x + 28)/(x^2 + 6x + 9)$;
(d) $x^3/(a^8 + x^8)$.

(a) We first need to express the ratio in partial fractions:

$$\frac{x + 3}{x^2 + x - 2} = \frac{x + 3}{(x + 2)(x - 1)} = \frac{A}{x + 2} + \frac{B}{x - 1}.$$

Using any of the methods employed in exercise 1.15, we obtain the unknown

coefficients as $A = -\frac{1}{3}$ and $B = \frac{4}{3}$. Thus,

$$\begin{aligned}\int \frac{x+3}{x^2+x-2}\,dx &= \int \frac{-1}{3(x+2)}\,dx + \int \frac{4}{3(x-1)}\,dx \\ &= -\frac{1}{3}\ln(x+2) + \frac{4}{3}\ln(x-1) + c \\ &= \frac{1}{3}\ln\frac{(x-1)^4}{x+2} + c.\end{aligned}$$

(b) As the numerator is of higher degree than the denominator, we need to divide the numerator by the denominator and express the remainder in partial fractions before starting any integration:

$$\begin{aligned}x^3 + 5x^2 + 8x + 12 &= (\tfrac{1}{2}x + a_0)(2x^2 + 10x + 12) + (b_1x + b_0) \\ &= x^3 + (2a_0 + 5)x^2 + (10a_0 + 6 + b_1)x \\ &\quad + (12a_0 + b_0),\end{aligned}$$

yielding $a_0 = 0$, $b_1 = 2$ and $b_0 = 12$. Now, expressed as partial fractions,

$$\frac{2x+12}{2x^2+10x+12} = \frac{x+6}{(x+2)(x+3)} = \frac{4}{x+2} + \frac{-3}{x+3},$$

where, again, we have used one of the three methods available for determining coefficients in partial fraction expansions. Thus,

$$\begin{aligned}\int \frac{x^3+5x^2+8x+12}{2x^2+10x+12}\,dx &= \int \left(\frac{1}{2}x + \frac{4}{x+2} - \frac{3}{x+3}\right)dx \\ &= \tfrac{1}{4}x^2 + 4\ln(x+2) - 3\ln(x+3) + c.\end{aligned}$$

(c) By inspection,

$$3x^2 + 20x + 28 = 3(x^2 + 6x + 9) + 2x + 1.$$

Expressing the remainder after dividing through by x^2+6x+9 in partial fractions, and noting that the denominator has a double factor, we obtain

$$\frac{2x+1}{x^2+6x+9} = \frac{A}{(x+3)^2} + \frac{B}{x+3},$$

where $B(x+3) + A = 2x + 1$. This requires that $B = 2$ and $A = -5$. Thus,

$$\begin{aligned}\int \frac{3x^2+20x+28}{x^2+6x+9}\,dx &= \int \left[3 + \frac{2}{x+3} - \frac{5}{(x+3)^2}\right]dx \\ &= 3x + 2\ln(x+3) + \frac{5}{x+3} + c.\end{aligned}$$

(d) Noting the form of the numerator, we set $x^4 = u$ with $4x^3\,dx = du$. Then,

$$\int \frac{x^3}{a^8 + x^8}\,dx = \int \frac{1}{4(a^8 + u^2)}\,du$$
$$= \frac{1}{4a^4}\tan^{-1}\frac{u}{a^4} + c = \frac{1}{4a^4}\tan^{-1}\left(\frac{x^4}{a^4}\right) + c.$$

2.33 *Find the integral J of $(ax^2 + bx + c)^{-1}$, with $a \neq 0$, distinguishing between the cases* (i) $b^2 > 4ac$, (ii) $b^2 < 4ac$ *and* (iii) $b^2 = 4ac$.

In each case, we first 'complete the square' in the denominator, i.e. write it in such a form that x appears *only* in a term that is the square of a linear function of x. We then examine the overall sign of the terms that do not contain x; this determines the form of the integral. In case (iii) there is no such term. We write $b^2 - 4ac$ as $\Delta^2 > 0$, or $4ac - b^2$ as $\Delta'^2 > 0$, as needed.

(i) For $\Delta^2 = b^2 - 4ac > 0$,

$$J = \int \frac{dx}{a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right)\right]}$$
$$= \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta^2}{4a^2}}$$
$$= \frac{1}{a}\frac{a}{\Delta}\ln\frac{x + \frac{b}{2a} - \frac{\Delta}{2a}}{x + \frac{b}{2a} + \frac{\Delta}{2a}} + k$$
$$= \frac{1}{\Delta}\ln\frac{2ax + b - \Delta}{2ax + b + \Delta} + k.$$

(ii) For $-\Delta'^2 = b^2 - 4ac < 0$,

$$J = \int \frac{dx}{a\left[\left(x + \frac{b}{2a}\right)^2 - \left(\frac{b^2}{4a^2} - \frac{c}{a}\right)\right]}$$
$$= \frac{1}{a}\int \frac{dx}{\left(x + \frac{b}{2a}\right)^2 + \frac{\Delta'^2}{4a^2}}$$
$$= \frac{1}{a}\frac{2a}{\Delta'}\tan^{-1}\left(\frac{x + \frac{b}{2a}}{\frac{\Delta'}{2a}}\right) + k$$
$$= \frac{2}{\Delta'}\tan^{-1}\left(\frac{2ax + b}{\Delta'}\right) + k.$$

(iii) For $b^2 - 4ac = 0$,

$$\begin{aligned} J &= \int \frac{dx}{ax^2 + bx + \frac{b^2}{4a}} \\ &= \frac{1}{a} \int \frac{dx}{\left(x + \frac{b}{2a}\right)^2} \\ &= \frac{-1}{a\left(x + \frac{b}{2a}\right)} + k \\ &= -\frac{2}{2ax + b} + k. \end{aligned}$$

2.35 *Find the derivative of* $f(x) = (1 + \sin x)/\cos x$ *and hence determine the indefinite integral* J *of* $\sec x$.

We differentiate $f(x)$ as a quotient, i.e. using $d(u/v)/dx = (vu' - uv')/v^2$, and obtain

$$\begin{aligned} f(x) &= \frac{1 + \sin x}{\cos x}, \\ f'(x) &= \frac{\cos x(\cos x) - (1 + \sin x)(-\sin x)}{\cos^2 x} \\ &= \frac{1 + \sin x}{\cos^2 x} \\ &= \frac{f(x)}{\cos x}. \end{aligned}$$

Thus, since $\sec x = f'(x)/f(x)$, it follows that

$$\int \sec x \, dx = \ln[f(x)] + c = \ln\left(\frac{1 + \sin x}{\cos x}\right) + c = \ln(\sec x + \tan x) + c.$$

2.37 *By making the substitution* $x = a\cos^2\theta + b\sin^2\theta$, *evaluate the definite integrals* J *between limits* a *and* b $(> a)$ *of the following functions:*

(a) $[(x - a)(b - x)]^{-1/2}$;
(b) $[(x - a)(b - x)]^{1/2}$;
(c) $[(x - a)/(b - x)]^{1/2}$.

Wherever the substitution $x = a\cos^2\theta + b\sin^2\theta$ is made, the terms in parentheses

take the following forms:

$$x - a \to a\cos^2\theta + b\sin^2\theta - a = -a\sin^2\theta + b\sin^2\theta = (b-a)\sin^2\theta,$$
$$b - x \to b - a\cos^2\theta - b\sin^2\theta = -a\cos^2\theta + b\cos^2\theta = (b-a)\cos^2\theta,$$

and dx will be given by

$$dx = [2a\cos\theta(-\sin\theta) + 2b\sin\theta(\cos\theta)]\,d\theta = 2(b-a)\cos\theta\sin\theta\,d\theta.$$

The limits a and b will be replaced by 0 and $\pi/2$, respectively. We also note that the average value of the square of a sinusoid over any whole number of quarter cycles of its argument is one-half.

(a)
$$\begin{aligned} J_a &= \int_a^b \frac{dx}{[(x-a)(b-x)]^{1/2}} \\ &= \int_0^{\pi/2} \frac{2(b-a)\cos\theta\sin\theta}{[(b-a)\sin^2\theta\,(b-a)\cos^2\theta]^{1/2}}\,d\theta \\ &= \int_0^{\pi/2} 2\,d\theta = \pi. \end{aligned}$$

(b)
$$\begin{aligned} J_b &= \int_a^b [(x-a)(b-x)]^{1/2}\,dx \\ &= \int_0^{\pi/2} 2(b-a)^2\cos^2\theta\sin^2\theta\,d\theta \\ &= \frac{1}{2}(b-a)^2\int_0^{\pi/2}\sin^2 2\theta\,d\theta \\ &= \frac{1}{2}(b-a)^2\frac{1}{2}\frac{\pi}{2} = \frac{\pi(b-a)^2}{8}. \end{aligned}$$

(c)
$$\begin{aligned} J_c &= \int_a^b \sqrt{\frac{x-a}{b-x}}\,dx \\ &= \int_0^{\pi/2} \sqrt{\frac{(b-a)\sin^2\theta}{(b-a)\cos^2\theta}} \times 2(b-a)\cos\theta\sin\theta\,d\theta \\ &= \int_0^{\pi/2} 2(b-a)\sin^2\theta\,d\theta \\ &= \frac{\pi(b-a)}{2}. \end{aligned}$$

2.39 *Use integration by parts to evaluate the following:*

(a) $\int_0^y x^2 \sin x\, dx$; (b) $\int_1^y x \ln x\, dx$;

(c) $\int_0^y \sin^{-1} x\, dx$; (d) $\int_1^y \ln(a^2 + x^2)/x^2\, dx$.

If u and v are functions of x, the general formula for integration by parts is

$$\int_a^b uv'\, dx = [\,uv\,]_a^b - \int_a^b u'v\, dx.$$

Any given integrand $w(x)$ has to be written as $w(x) = u(x)v'(x)$ with $v'(x)$ chosen so that (i) it can be integrated explicitly, and (ii) it results in a u that has u' no more complicated than u itself. There are usually several possible choices but the one that makes both u and v as simple as possible is normally the best.

(a) Here the obvious choice at the first stage is $u(x) = x^2$ and $v'(x) = \sin x$. For the second stage, $u = x$ and $v' = \cos x$ are equally clear assignments.

$$\begin{aligned}\int_0^y x^2 \sin x dx &= \left[\,x^2(-\cos x)\,\right]_0^y - \int_0^y 2x(-\cos x)\, dx \\ &= -y^2 \cos y + [\,2x \sin x\,]_0^y - \int_0^y 2 \sin x\, dx \\ &= -y^2 \cos y + 2y \sin y + [\,2 \cos x\,]_0^y \\ &= (2 - y^2) \cos y + 2y \sin y - 2.\end{aligned}$$

(b) This integration is most straightforwardly carried out by taking $v'(x) = x$ and $u(x) = \ln x$ as follows:

$$\begin{aligned}\int_1^y x \ln x\, dx &= \left[\frac{x^2}{2} \ln x\right]_1^y - \int_1^y \frac{1}{x}\frac{x^2}{2}\, dx \\ &= \frac{y^2}{2} \ln y - \left[\frac{x^2}{4}\right]_1^y \\ &= \frac{1}{2}y^2 \ln y + \frac{1}{4}(1 - y^2).\end{aligned}$$

However, if you know that the integral of $\ln x$ is $x \ln x - x$, then the given integral can also be found by taking $v' = \ln x$ and $u = x$:

$$\begin{aligned}\int_1^y x \ln x\, dx &= [\,x(x \ln x - x)\,]_1^y - \int_1^y 1 \times (x \ln x - x)\, dx \\ &= y^2 \ln y - y^2 - 0 + 1 - \int_1^y x \ln x\, dx + \left[\frac{x^2}{2}\right]_1^y.\end{aligned}$$

After the limits have been substituted, the equation can be rearranged as

$$2\int_1^y x\ln x\,dx = y^2\ln y - y^2 + 1 + \frac{y^2}{2} - \frac{1}{2},$$
$$\int_1^y x\ln x\,dx = \frac{1}{2}y^2\ln y + \frac{1}{4}(1-y^2).$$

(c) Here we do not know the integral of $\sin^{-1}x$ (that is the problem!) but we do know its derivative. Therefore consider the integrand as $1\times\ \sin^{-1}x$, with $v'(x)=1$ and $u(x)=\sin^{-1}x$.

$$\begin{aligned}\int_0^y \sin^{-1}x\,dx &= \int_0^y 1\ \sin^{-1}x\,dx\\ &= \left[x\sin^{-1}x\right]_0^y - \int_0^y \frac{1}{\sqrt{1-x^2}}\,x\,dx\\ &= y\ \sin^{-1}y + \left[\sqrt{1-x^2}\right]_0^y\\ &= y\ \sin^{-1}y + \sqrt{1-y^2} - 1.\end{aligned}$$

(d) When the logarithm of a function of x appears as part of an integrand, it is normally helpful to remove its explicit appearance by making it the $u(x)$ part of an integration-by-parts formula. The reciprocal of the function, without any explicit logarithm, then appears in the resulting integral; this is usually easier to deal with. In this case we take $\ln(a^2+x^2)$ as $u(x)$.

$$\begin{aligned}\int_1^y \frac{\ln(a^2+x^2)}{x^2}dx &= \left[-\frac{\ln(a^2+x^2)}{x}\right]_1^y - \int_1^y \frac{2x}{a^2+x^2}\left(-\frac{1}{x}\right)dx\\ &= -\frac{\ln(a^2+y^2)}{y} + \ln(a^2+1) + \frac{2}{a}\left[\tan^{-1}\left(\frac{x}{a}\right)\right]_1^y\\ &= -\frac{\ln(a^2+y^2)}{y} + \ln(a^2+1)\\ &\quad + \frac{2}{a}\left[\tan^{-1}\left(\frac{y}{a}\right) - \tan^{-1}\left(\frac{1}{a}\right)\right].\end{aligned}$$

2.41 *The gamma function* $\Gamma(n)$ *is defined for all* $n > -1$ *by*

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx.$$

Find a recurrence relation connecting $\Gamma(n+1)$ *and* $\Gamma(n)$.

(a) *Deduce* (i) *the value of* $\Gamma(n+1)$ *when* n *is a non-negative integer, and* (ii) *the value of* $\Gamma\left(\frac{7}{2}\right)$, *given that* $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

(b) *Now, taking factorial* m *for* any m *to be defined by* $m! = \Gamma(m+1)$, *evaluate* $\left(-\frac{3}{2}\right)!$.

Integrating the defining equation by parts,

$$\Gamma(n+1) = \int_0^\infty x^n e^{-x}\,dx = \left[-x^n e^{-x}\right]_0^\infty + \int_0^\infty n x^{n-1} e^{-x}\,dx$$
$$= 0 + n\Gamma(n), \quad \text{for } n > 0,$$

i.e. $\Gamma(n+1) = n\Gamma(n)$.

(a)(i) Clearly $\Gamma(n+1) = n(n-1)(n-2)\cdots 2\,1\,\Gamma(1)$. But

$$\Gamma(1) = \int_0^\infty e^{-x}\,dx = 1.$$

Hence $\Gamma(n+1) = n!$.

(a)(ii) Applying the recurrence relation derived above,

$$\Gamma\left(\tfrac{7}{2}\right) = \tfrac{5}{2}\,\tfrac{3}{2}\,\tfrac{1}{2}\,\Gamma\left(\frac{1}{2}\right) = \tfrac{15}{8}\sqrt{\pi}.$$

(b) With this general definition of a factorial, we have

$$\left(-\tfrac{3}{2}\right)! = \Gamma\left(-\tfrac{1}{2}\right) = \frac{1}{-\frac{1}{2}}\Gamma\left(\tfrac{1}{2}\right) = -2\sqrt{\pi}.$$

2.43 *By integrating by parts twice, prove that* I_n *as defined in the first equality below for positive integers* n *has the value given in the second equality:*

$$I_n = \int_0^{\pi/2} \sin n\theta \cos\theta\,d\theta = \frac{n - \sin(n\pi/2)}{n^2 - 1}.$$

Taking $\sin n\theta$ as u and $\cos\theta$ as v and noting that with this choice $u'' = -n^2 u$

and $v'' = -v$, we expect that after two integrations by parts we will recover (a multiple of) I_n.

$$\begin{aligned} I_n &= \int_0^{\pi/2} \sin n\theta \cos\theta \, d\theta \\ &= [\sin n\theta \sin\theta]_0^{\pi/2} - \int_0^{\pi/2} n \cos n\theta \sin\theta \, d\theta \\ &= \sin\frac{n\pi}{2} - n\left\{[-\cos n\theta \cos\theta]_0^{\pi/2} - \int_0^{\pi/2} (-n\sin n\theta)(-\cos\theta)\, d\theta\right\} \\ &= \sin\frac{n\pi}{2} - n[-(-1) - nI_n]. \end{aligned}$$

Rearranging this gives

$$I_n(1-n^2) = \sin\frac{n\pi}{2} - n,$$

and hence the stated result.

2.45 *If J_r is the integral*

$$\int_0^\infty x^r \exp(-x^2)\, dx,$$

show that

(a) $J_{2r+1} = (r!)/2$,
(b) $J_{2r} = 2^{-r}(2r-1)(2r-3)\cdots(5)(3)(1)\, J_0$.

(a) We first derive a recurrence relationship for J_{2r+1}. Since we cannot integrate $\exp(-x^2)$ explicitly but can integrate $-2x\exp(-x^2)$, we extract the factor $-2x$ from the rest of the integrand and treat what is left ($-\frac{1}{2}x^{2r}$ in this case) as $u(x)$. This is the operation that has been carried out in the second line of what follows.

$$\begin{aligned} J_{2r+1} &= \int_0^\infty x^{2r+1} \exp(-x^2)\, dx \\ &= \int_0^\infty -\frac{x^{2r}}{2}(-2x)\exp(-x^2)\, dx \\ &= \left[-\frac{x^{2r}}{2}\exp(-x^2)\right]_0^\infty + \int_0^\infty \frac{2rx^{2r-1}}{2}\exp(-x^2)\, dx \\ &= 0 + rJ_{2r-1}. \end{aligned}$$

Applying the relationship r times gives

$$J_{2r+1} = r(r-1)\cdots 1\, J_1.$$

But

$$J_1 = \int_0^\infty x \exp(-x^2)\, dx = \left[-\frac{1}{2}\exp(-x^2)\right]_0^\infty = \frac{1}{2},$$

and so $J_{2r+1} = \frac{1}{2}r!$.

(b) Using the same method as in part (a) it can be shown that

$$J_{2r} = \frac{2r-1}{2} J_{2r-2}.$$

Hence,

$$J_{2r} = \frac{2r-1}{2}\,\frac{2r-3}{2} \cdots \frac{1}{2} J_0,$$

in agreement with the stated relationship.

2.47 *By noting that for* $0 \le \eta \le 1$, $\eta^{1/2} \ge \eta^{3/4} \ge \eta$, *prove that*

$$\frac{2}{3} \le \frac{1}{a^{5/2}} \int_0^a (a^2 - x^2)^{3/4}\, dx \le \frac{\pi}{4}.$$

We use the result that, if $g(x) \le f(x) \le h(x)$ for *all* x in the range $a \le x \le b$, then $\int g(x)\, dx \le \int f(x)\, dx \le \int h(x)\, dx$, where all integrals are between the limits a and b.

Set $\eta = 1 - (x/a)^2$ in the stated inequalities and integrate the result from 0 to a, giving

$$\int_0^a \left(1 - \frac{x^2}{a^2}\right)^{1/2} dx \ge \int_0^a \left(1 - \frac{x^2}{a^2}\right)^{3/4} dx \ge \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx.$$

Substituting $x = a\sin\theta$ and $dx = a\cos\theta\, d\theta$ in the first term and carrying out the elementary integration in the third term yields

$$\int_0^{\pi/2} a\cos^2\theta\, d\theta \ge \frac{1}{a^{3/2}} \int_0^a (a^2 - x^2)^{3/4}\, dx \ge \left[x - \frac{x^3}{3a^2}\right]_0^a,$$

$$\Rightarrow \quad a\frac{1}{2}\frac{\pi}{2} \ge \frac{1}{a^{3/2}} \int_0^a (a^2 - x^2)^{3/4}\, dx \ge \frac{2a}{3},$$

$$\Rightarrow \quad \frac{\pi}{4} \ge \frac{1}{a^{5/2}} \int_0^a (a^2 - x^2)^{3/4}\, dx \ge \frac{2}{3}.$$

2.49 *By noting that* $\sinh x < \frac{1}{2}e^x < \cosh x$, *and that* $1 + z^2 < (1 + z)^2$ *for* $z > 0$, *show that, for* $x > 0$, *the length* L *of the curve* $y = \frac{1}{2}e^x$ *measured from the origin satisfies the inequalities* $\sinh x < L < x + \sinh x$.

With $y = y' = \frac{1}{2}e^x$ and the element of curve length ds given by $ds = (1+y'^2)^{1/2}\,dx$, the total length of the curve measured from the origin is

$$L = \int_0^x ds = \int_0^x \left(1 + \tfrac{1}{4}e^{2x}\right)^{1/2} dx.$$

But, since all quantities are positive for $x \geq 0$,

$$\begin{aligned} \sinh x &< \tfrac{1}{2}e^x &&< \cosh x, \\ \Rightarrow \quad \sinh^2 x &< \tfrac{1}{4}e^{2x} &&< \cosh^2 x, \\ \cosh^2 x = 1 + \sinh^2 x &< 1 + \tfrac{1}{4}e^{2x} &&< 1 + \cosh^2 x < (1 + \cosh x)^2, \\ \Rightarrow \quad \cosh x &< \left(1 + \tfrac{1}{4}e^{2x}\right)^{1/2} &&< 1 + \cosh x. \end{aligned}$$

It then follows, from integrating each term in the double inequality, that

$$\begin{aligned} &\int_0^x \cosh x\,dx < L < \int_0^x (1 + \cosh x)\,dx, \\ \Rightarrow \quad &\sinh x < L < x + \sinh x, \end{aligned}$$

as stated in the question.

3

Complex numbers and hyperbolic functions

3.1 *Two complex numbers z and w are given by $z = 3 + 4i$ and $w = 2 - i$. On an Argand diagram, plot*

(a) $z + w$, (b) $w - z$, (c) wz, (d) z/w,
(e) $z^*w + w^*z$, (f) w^2, (g) $\ln z$, (h) $(1 + z + w)^{1/2}$.

With $z = 3 + 4i$, $w = 2 - i$ and, where needed, $i^2 = -1$:

(a) $z + w = 3 + 4i + 2 - i = 5 + 3i$;

(b) $w - z = 2 - i - 3 - 4i = -1 - 5i$;

(c) $wz = (2 - i)(3 + 4i) = 6 - 3i + 8i - 4i^2 = 10 + 5i$;

(d) $\dfrac{z}{w} = \dfrac{3 + 4i}{2 - i} = \dfrac{3 + 4i}{2 - i}\,\dfrac{2 + i}{2 + i} = \dfrac{6 + 8i + 3i + 4i^2}{4 - 2i + 2i - i^2} = \dfrac{2 + 11i}{5}$;

(e) $z * w + w * z = (3 - 4i)(2 - i) + (2 + i)(3 + 4i) = (2 - 11i) + (2 + 11i) = 4$;

(f) $w^2 = (2 - i)(2 - i) = 4 - 4i + i^2 = 3 - 4i$;

(g)
$$\begin{aligned}\ln z &= \ln|z| + i\arg z\\ &= \ln(3^2 + 4^2)^{1/2} + i\tan^{-1}\left(\tfrac{4}{3}\right)\\ &= \ln 5 + i\left[\tan^{-1}\left(\tfrac{4}{3}\right) + 2n\pi\right];\end{aligned}$$

(h)
$$\begin{aligned}(1 + z + w)^{1/2} &= (6 + 3i)^{1/2}\\ &= \left\{\sqrt{45}\exp\left[i\tan\left(\tfrac{3}{6}\right)\right]\right\}^{1/2}\\ &= \pm(45)^{1/4}\exp\left[i\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}\right)\right]\\ &= \pm 2.590\,(\cos 0.2318 + i\sin 0.2318)\\ &= \pm(2.521 + 0.595i).\end{aligned}$$

Figure 3.1 The solutions to exercise 3.1.

These results are plotted in figure 3.1. The answer to part (g) is multivalued and only five of the infinite number of possibilities are shown.

3.3 *By writing* $\pi/12 = (\pi/3) - (\pi/4)$ *and considering* $e^{i\pi/12}$, *evaluate* $\cot(\pi/12)$.

As we are expressing $\pi/12$ as the difference between two (familiar) angles, for which we know explicit formulae for their sines and cosines, namely

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{1}{\sqrt{2}},$$

we will need the formulae for $\cos(A - B)$ and $\sin(A - B)$. They are given by

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$
$$\text{and} \quad \sin(A - B) = \sin A \cos B - \cos A \sin B.$$

Applying these with $A = \pi/3$ and $B = \pi/4$,

$$\begin{aligned}
\exp\left(i\frac{\pi}{12}\right) &= \exp\left[i\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\right],\\
\cos\frac{\pi}{12} + i\sin\frac{\pi}{12} &= \cos\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{3}-\frac{\pi}{4}\right)\\
&= \cos\frac{\pi}{3}\cos\frac{\pi}{4} + \sin\frac{\pi}{3}\sin\frac{\pi}{4}\\
&\qquad + i\left(\sin\frac{\pi}{3}\cos\frac{\pi}{4} - \cos\frac{\pi}{3}\sin\frac{\pi}{4}\right)\\
&= \left(\frac{1}{2}\frac{1}{\sqrt{2}} + \frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}}\right) + i\left(\frac{\sqrt{3}}{2}\frac{1}{\sqrt{2}} - \frac{1}{2}\frac{1}{\sqrt{2}}\right).
\end{aligned}$$

Thus

$$\cot\frac{\pi}{12} = \frac{\cos(\pi/12)}{\sin(\pi/12)} = \frac{1+\sqrt{3}}{\sqrt{3}-1} = 2+\sqrt{3}.$$

3.5 *Evaluate*

(a) Re $(\exp 2iz)$, (b) Im $(\cosh^2 z)$, (c) $(-1+\sqrt{3}i)^{1/2}$,
(d) $|\exp(i^{1/2})|$, (e) $\exp(i^3)$, (f) Im (2^{i+3}), (g) i^i, (h) $\ln[(\sqrt{3}+i)^3]$.

All of these evaluations rely directly on the definitions of the various functions involved as applied to complex numbers; these should be known to the reader. There are too many to give every one individually at each step and, if the justification for any particular step is unclear, reference should be made to a textbook.

(a) Re $(\exp 2iz) = \text{Re}\,[\exp(2ix-2y)] = \exp(-2y)\cos 2x$.

(b)
$$\begin{aligned}
\text{Im}\,(\cosh^2 z) &= \text{Im}\,\left[\tfrac{1}{2}(\cosh 2z + 1)\right]\\
&= \tfrac{1}{2}\,\text{Im}\,[\cosh(2x+2iy)]\\
&= \tfrac{1}{2}\,\text{Im}\,(\cosh 2x\cosh 2iy + \sinh 2x\sinh 2iy)\\
&= \tfrac{1}{2}\,\text{Im}\,(\cosh 2x\cos 2y + i\sinh 2x\sin 2y)\\
&= \tfrac{1}{2}\,\sinh 2x\sin 2y.
\end{aligned}$$

(c)
$$\begin{aligned}
(-1+\sqrt{3}i)^{1/2} &= \left[(-1)^2 + (\sqrt{3})^2\right]^{1/4}\exp\left[i\tfrac{1}{2}(\tan^{-1}(\tfrac{\sqrt{3}}{-1}) + 2n\pi)\right]\\
&= \sqrt{2}\exp\left[i\tfrac{1}{2}(\tfrac{2}{3}\pi + 2n\pi)\right]\\
&= \sqrt{2}\exp\left(\tfrac{\pi i}{3}\right) \quad \text{or} \quad \sqrt{2}\exp\left(\tfrac{4\pi i}{3}\right).
\end{aligned}$$

(d) $$\begin{aligned}\left|\exp\left(i^{1/2}\right)\right| &= \left|\exp\left((e^{\frac{i\pi}{2}})^{1/2}\right)\right| \\ &= \left|\exp\left(e^{\frac{i\pi}{4}+in\pi}\right)\right| \\ &= \left|\exp\left[\cos(n+\tfrac{1}{4})\pi + i\sin(n+\tfrac{1}{4})\pi\right]\right| \\ &= \exp[\cos(n+\tfrac{1}{4})\pi] \\ &= \exp\left(\tfrac{1}{\sqrt{2}}\right) \quad \text{or} \quad \exp\left(-\tfrac{1}{\sqrt{2}}\right).\end{aligned}$$

(e) $$\begin{aligned}\exp\left(i^3\right) &= \exp\left(e^{3(\frac{i\pi}{2})}\right) = \exp(\cos\tfrac{3\pi}{2} + i\sin\tfrac{3\pi}{2}) \\ &= \exp(0-i) = \cos(-1) + i\sin(-1) = 0.540 - 0.841\,i.\end{aligned}$$

(f) $\mathrm{Im}\,(2^{i+3}) = \mathrm{Im}\,(8\times 2^i) = 8\,\mathrm{Im}\,(2^i) = 8\,\mathrm{Im}\,(e^{i\ln 2}) = 8\,\sin(\ln 2) = 5.11.$

(g) $i^i = \left[\exp i(\frac{1}{2}\pi + 2n\pi)\right]^i = \left[\exp i^2(\frac{1}{2}\pi + 2n\pi)\right] = \exp[-(2n+\frac{1}{2})\pi].$

(h) $$\begin{aligned}\ln\left[(\sqrt{3}+i)^3\right] &= 3\ln(\sqrt{3}+i) \\ &= 3\left(\ln 2 + i\tan^{-1}\tfrac{1}{\sqrt{3}}\right) \\ &= \ln 8 + 3i(\tfrac{\pi}{6} + 2n\pi) \\ &= \ln 8 + i(6n+\tfrac{1}{2})\pi.\end{aligned}$$

3.7 *Show that the locus of all points $z = x + iy$ in the complex plane that satisfy*

$$|z - ia| = \lambda|z + ia|, \quad \lambda > 0,$$

is a circle of radius $|2a\lambda/(1-\lambda^2)|$ centred on the point $z = ia[(1+\lambda^2)/(1-\lambda^2)]$. Sketch the circles for a few typical values of λ, including $\lambda < 1$, $\lambda > 1$ and $\lambda = 1$.

As we wish to find the locus in the x-y plane, we first express $|z \pm ia|$ explicitly in terms of x and y, remembering that a can be complex:

$$\begin{aligned}|x+iy-ia|^2 &= (x+iy-ia)(x-iy+ia^*) \\ &= x^2+y^2+|a|^2 - ia(x-iy) + ia^*(x+iy). \\ |x+iy+ia|^2 &= (x+iy+ia)(x-iy-ia^*) \\ &= x^2+y^2+|a|^2 + ia(x-iy) - ia^*(x+iy).\end{aligned}$$

Substituting in

$$|x+iy-ia|^2 = \lambda^2|x+iy+ia|^2$$

Figure 3.2 The solution to exercise 3.7.

gives, on dividing through by $1 - \lambda^2$,

$$x^2 - \frac{1+\lambda^2}{1-\lambda^2}(ia - ia^*)x + y^2 - \frac{1+\lambda^2}{1-\lambda^2}(a + a^*)y + |a|^2 = 0,$$

which can be rearranged as

$$\left(x + \frac{1+\lambda^2}{1-\lambda^2}\mathrm{Im}\, a\right)^2 + \left(y + \frac{1+\lambda^2}{1-\lambda^2}\mathrm{Re}\, a\right)^2 + |a|^2$$

$$- \left(\frac{1+\lambda^2}{1-\lambda^2}\right)^2 \left[(\mathrm{Im}\, a)^2 + (\mathrm{Re}\, a)^2\right] = 0.$$

This is of the form

$$(x-\alpha)^2 + (y-\beta)^2 = \left[\left(\frac{1+\lambda^2}{1-\lambda^2}\right)^2 - 1\right] |a|^2 = \frac{4\lambda^2}{(1-\lambda^2)^2}|a|^2,$$

where

$$\alpha + i\beta = \frac{1+\lambda^2}{1-\lambda^2}(-\mathrm{Im}\, a + i\mathrm{Re}\, a) = \frac{1+\lambda^2}{1-\lambda^2} ia.$$

Thus it is the equation of a circle of radius $|2\lambda/(1-\lambda^2)|a$ centred on the point $\alpha + i\beta$ as given above. See figure 3.2; note that a lies on the straight line (circle of infinite radius) corresponding to $\lambda = 1$. The circles centred on ia and $-ia$ have vanishingly small radii.

3.9 *For the real constant a find the loci of all points $z = x + iy$ in the complex plane that satisfy*

(a) $\operatorname{Re}\left\{\ln\left(\dfrac{z-ia}{z+ia}\right)\right\} = c, \qquad c > 0,$

(b) $\operatorname{Im}\left\{\ln\left(\dfrac{z-ia}{z+ia}\right)\right\} = k, \qquad 0 \le k \le \pi/2.$

Identify the two families of curves and verify that in case (b) all curves pass through the two points $\pm ia$.

(a) Recalling that

$$\ln z = \ln|z| + i\arg z$$

we have

$$\operatorname{Re}\left(\ln\frac{z-ia}{z+ia}\right) = \ln\left|\frac{z-ia}{z+ia}\right| = c, \quad c > 0,$$
$$|z-ia| = e^c|z+ia|, \quad e^c > 1.$$

As in exercise 3.7, this is a circle of radius $|2ae^c/(1-e^{2c})| = |a|\operatorname{cosech} c$ centred on the point $z = ia(1+e^{2c})/(1-e^{2c}) = ia\coth c$. As c varies this generates a family of circles whose centres lie on the y-axis above the point $z = ia$ (or below the point $z = ia$ if a is negative) and whose radii decrease as their centres approach that point. The curve corresponding to $c = 0$ is the x-axis.

(b) Using the principal value for the argument of a logarithm, we obtain

$$\operatorname{Im}\left(\ln\frac{z-ia}{z+ia}\right) = \arg\frac{z-ia}{z+ia} = k, \quad 0 \le k \le \frac{\pi}{2}.$$

Now, $$\frac{z-ia}{z+ia} = \frac{(z-ia)(z^*-ia)}{(z+ia)(z+ia)^*} = \frac{zz^* - ia(z+z^*) - a^2}{|z+ia|^2}.$$

Hence, $$k = \tan^{-1}\frac{-a(z+z^*)}{|z|^2 - a^2},$$
$$a(z+z^*) = (a^2 - |z|^2)\tan k,$$
$$2ax = a^2\tan k - (x^2+y^2)\tan k,$$
$$(x + a\cot k)^2 + y^2 = a^2(1+\cot^2 k).$$

This is a circle with centre $(-a\cot k, 0)$ and radius $a\operatorname{cosec} k$. As k varies the curves generate a family of circles whose centres lie on the negative x-axis (for $a > 0$) and whose radii decrease to a as their centres approach the origin. The curve corresponding to $k = 0$ is the y-axis.

The two points $z = \pm ia = (0, \pm a)$ lie on the curve if

$$(0 + a\cot k)^2 + a^2 = a^2(1 + \cot^2 k).$$

This is identically satisfied, verifying that all members of the family pass through the two points $z = \pm ia$.

3.11 *Sketch the parts of the Argand diagram in which*

(a) $Re\, z^2 < 0$, $|z^{1/2}| \leq 2$;
(b) $0 \leq \arg z^* \leq \pi/2$;
(c) $|\exp z^3| \to 0$ *as* $|z| \to \infty$.

What is the area of the region in which all three sets of conditions are satisfied?

Since we will need to study the signs of the real parts of certain powers of z, it will be convenient to consider z as $r\,e^{i\theta}$ with $0 \leq \theta \leq 2\pi$.

Condition (a) contains two specifications. Firstly, for the real part of z^2 to be negative, its argument must be greater than $\pi/2$ but less than $3\pi/2$. The argument of z itself, which is half that of z^2 (mod 2π), must therefore lie in one of the two ranges $\pi/4 < \arg z < 3\pi/4$ and $5\pi/4 < \arg z < 7\pi/4$. Secondly, since the modulus of any complex number is real and positive, $|z^{1/2}| \leq 2$ is equivalent to $|z| \leq 4$.

Since $\arg z^* = -\arg z$, condition (b) requires $\arg z$ to lie in the range $3\pi/2 \leq \theta \leq 2\pi$, i.e z to lie in the fourth quadrant.

Condition (c) will only be satisfied if the real part of z^3 is negative. This requires

$$(4n+1)\frac{\pi}{2} < 3\theta < (4n+3)\frac{\pi}{2}, \qquad n = 0, 1, 2.$$

The allowed regions for θ are thus alternate wedges of angular size $\pi/3$ with an allowed region starting at $\theta = \pi/6$. The allowed region overlapping those specified by conditions (a) and (b) is the wedge $3\pi/2 \leq \theta \leq 11\pi/6$.

All three conditions are satisfied in the region $3\pi/2 \leq \theta \leq 7\pi/4$, $|z| \leq 4$; see figure 3.3. This wedge has an area given by

$$\frac{1}{2}r^2\theta = \frac{1}{2}16\left(\frac{7\pi}{4} - \frac{3\pi}{2}\right) = 2\pi.$$

Figure 3.3 The defined region of the Argand diagram in exercise 3.11. Regions in which only one condition is satisfied are lightly shaded; those that satisfy two conditions are more heavily shaded; and the region satisfying all three conditions is most heavily shaded and outlined.

3.13 *Prove that* $x^{2m+1} - a^{2m+1}$, *where* m *is an integer* ≥ 1, *can be written as*

$$x^{2m+1} - a^{2m+1} = (x-a)\prod_{r=1}^{m}\left[x^2 - 2ax\cos\left(\frac{2\pi r}{2m+1}\right) + a^2\right].$$

For the sake of brevity, we shall denote $x^{2m+1} - a^{2m+1}$ by $f(x)$ and the $(2m+1)$th root of unity, $\exp[2\pi i/(2m+1)]$, by Ω.

Now consider the roots of the equation $f(x) = 0$. The $2m+1$ quantities of the form $x = a\Omega^r$ with $r = 0, 1, 2, \ldots, 2m$ are all solutions of this equation and, since it is a polynomial equation of order $2m+1$, they represent all of its roots. We can therefore reconstruct the polynomial $f(x)$ (which has unity as the coefficient of its highest power) as the product of factors of the form $(x - a\Omega^r)$:

$$f(x) = (x-a)(x-a\Omega)\cdots(x-a\Omega^m)(x-a\Omega^{m+1})\cdots(x-a\Omega^{2m}).$$

Now combine $(x - a\Omega^r)$ with $(x - a\Omega^{2m+1-r})$:

$$\begin{aligned} f(x) &= (x-a)\prod_{r=1}^{m}(x - a\Omega^r)(x - a\Omega^{2m+1-r}) \\ &= (x-a)\prod_{r=1}^{m}[x^2 - ax(\Omega^r + \Omega^{2m+1-r}) + a^2\Omega^{2m+1}] \\ &= (x-a)\prod_{r=1}^{m}[x^2 - ax(\Omega^r + \Omega^{-r}) + a^2], \quad \text{since } \Omega^{2m+1} = 1, \\ &= (x-a)\prod_{r=1}^{m}\left[x^2 - 2ax\cos\left(\frac{2\pi r}{2m+1}\right) + a^2\right]. \end{aligned}$$

This is the form given in the question.

3.15 *Solve the equation*

$$z^7 - 4z^6 + 6z^5 - 6z^4 + 6z^3 - 12z^2 + 8z + 4 = 0,$$

(a) *by examining the effect of setting* z^3 *equal to* 2, *and then*
(b) *by factorising and using the binomial expansion of* $(z + a)^4$.

Plot the seven roots of the equation on an Argand plot, exemplifying that complex roots of a polynomial equation always occur in conjugate pairs if the polynomial has real coefficients.

(a) Setting $z^3 = 2$ in $f(z)$ so as to leave no higher powers of z than its square, e.g. writing z^7 as $(z^3)^2 z = 4z$, gives

$$4z - 16 + 12z^2 - 12z + 12 - 12z^2 + 8z + 4 = 0,$$

which is satisfied identically. Thus $z^3 - 2$ is a factor of $f(z)$.

(b) Writing $f(z)$ as

$$f(z) = (z^3 - 2)(az^4 + bz^3 + cz^2 + dz + e) = 0$$

and equating the coefficients of the various powers of z gives $a = 1$, $b = -4$, $c = 6$, $d - 2a = -6$, $e - 2b = 6$, $-2c = -12$, $-2d = 8$ and $-2e = 4$. These imply (consistently) that $f(z)$ can be written as

$$f(z) = (z^3 - 2)(z^4 - 4z^3 + 6z^2 - 4z - 2).$$

We now note that the first four terms in the second set of parentheses are the

same as the corresponding terms in the expansion of $(z-1)^4$; only the constant term needs correction. Thus, we may write the original equation as

$$0 = f(z) = (z^3 - 2)[(z-1)^4 - 3],$$

$$\text{with solutions} \qquad z = 2^{1/3}e^{2n\pi i/3} \quad n = 0, 1, 2 \quad \text{or}$$
$$z - 1 = 3^{1/4}e^{2n\pi i/4} \quad n = 0, 1, 2, 3.$$

The seven roots are therefore

$$2^{1/3}, \quad 2^{1/3}\left(\frac{-1 \pm i\sqrt{3}}{2}\right), \quad 1 \pm 3^{1/4}, \quad 1 \pm 3^{1/4}i.$$

As is to be expected, each root that has a non-zero imaginary part occurs as one of a complex conjugate pair.

3.17 *The binomial expansion of* $(1+x)^n$ *can be written for a positive integer* n *as*

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^r,$$

where ${}^nC_r = n!/[r!(n-r)!]$.

(a) *Use de Moivre's theorem to show that the sum*

$$S_1(n) = {}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots + (-1)^m\, {}^nC_{2m}, \qquad n-1 \le 2m \le n,$$

has the value $2^{n/2}\cos(n\pi/4)$.

(b) *Derive a similar result for the sum*

$$S_2(n) = {}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots + (-1)^m\, {}^nC_{2m+1}, \qquad n-1 \le 2m+1 \le n,$$

and verify it for the cases $n = 6$, 7 *and* 8.

Since we seek the sum of binomial coefficients that contain either all even or all odd indices, we need to choose a value for x such that x^r has different characteristics depending upon whether r is even or odd. The quantity i has just such a property, being purely real when r is even and purely imaginary when r is odd. We therefore take $x = i$, write $1+i$ as $\sqrt{2}e^{i\pi/4}$ and apply de Moivre's theorem:

$$\begin{aligned}\left(\sqrt{2}e^{i\pi/4}\right)^n &= (1+i)^n \\ &= {}^nC_0 + i\,{}^nC_1 + i^2\,{}^nC_2 + \cdots \\ &= ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots) \\ &\qquad + i({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots).\end{aligned}$$

Thus $S_1(n) = ({}^nC_0 - {}^nC_2 + {}^nC_4 - \cdots + (-1)^m\, {}^nC_{2m})$, where $n - 1 \le 2m \le n$, has a value equal to that of the real part of $\left(\sqrt{2}e^{i\pi/4}\right)^n$. This is the real part of $2^{n/2}e^{in\pi/4}$, which, by de Moivre's theorem, is $2^{n/2}\cos(n\pi/4)$.

(b) The corresponding result for $S_2(n)$ is that it is equal to the imaginary part of $2^{n/2}e^{in\pi/4}$, which is $2^{n/2}\sin(n\pi/4)$.

We now verify this result for $n = 6$, 7 and 8 by direct calculation:

$$\begin{aligned}
S_2(6) &= {}^6C_1 - {}^6C_3 + {}^6C_5 = 6 - 20 + 6 = -8 = 2^3 \sin\frac{6\pi}{4},\\
S_2(7) &= {}^7C_1 - {}^7C_3 + {}^7C_5 - {}^7C_7\\
&= 7 - 35 + 21 - 1 = -8 = 2^{7/2}\sin\frac{7\pi}{4},\\
S_2(8) &= {}^8C_1 - {}^8C_3 + {}^8C_5 - {}^8C_7 = 8 - 56 + 56 - 8 = 0 = 2^4 \sin\frac{8\pi}{4}.
\end{aligned}$$

3.19 *Use de Moivre's theorem with $n = 4$ to prove that*

$$\cos 4\theta = 8\cos^4\theta - 8\cos^2\theta + 1,$$

and deduce that

$$\cos\frac{\pi}{8} = \left(\frac{2+\sqrt{2}}{4}\right)^{1/2}.$$

From de Moivre's theorem, $e^{i4\theta} = \cos 4\theta + i\sin 4\theta$. But, by the binomial theorem, we also have that

$$\begin{aligned}
e^{i4\theta} &= (\cos\theta + i\sin\theta)^4\\
&= \cos^4\theta + 4i\cos^3\theta\sin\theta - 6\cos^2\theta\sin^2\theta - 4i\cos\theta\sin^3\theta + \sin^4\theta.
\end{aligned}$$

Equating the real parts of the two equal expressions and writing $\sin^2\theta$ as $1-\cos^2\theta$,

$$\begin{aligned}
\cos 4\theta &= \cos^4\theta - 6\cos^2\theta(1-\cos^2\theta) + (1-\cos^2\theta)^2\\
&= 8\cos^4\theta - 8\cos^2\theta + 1.
\end{aligned}$$

Now set $\theta = \pi/8$ in this result and write $\cos(\pi/8)$ as c:

$$0 = \cos\frac{4\pi}{8} = 8c^4 - 8c^2 + 1.$$

Hence, as this is a quadratic equation in c^2,

$$c^2 = \frac{4 \pm \sqrt{16-8}}{8} \quad \text{and} \quad c = \cos\frac{\pi}{8} = \pm\left(\frac{2\pm\sqrt{2}}{4}\right)^{1/2}.$$

Since $0 < \pi/8 < \pi/2$, c must be positive. Further, as $\pi/8 < \pi/4$ and $\cos(\pi/4) = 1/\sqrt{2}$, c must be greater then $1/\sqrt{2}$. It is clear that the positive square roots are the appropriate ones in both cases.

3.21 *Use de Moivre's theorem to prove that*

$$\tan 5\theta = \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1},$$

where $t = \tan\theta$. *Deduce the values of* $\tan(n\pi/10)$ *for* $n = 1, 2, 3$ *and* 4.

Using the binomial theorem and de Moivre's theorem to expand $(e^{i\theta})^5$ in two different ways, we have, from equating the real and imaginary parts of the two results, that

$$\begin{aligned}
\cos 5\theta + i\sin 5\theta &= \cos^5\theta + i5\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta \\
&\quad -i10\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta, \\
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) \\
&\quad +5\cos\theta(1 - 2\cos^2\theta + \cos^4\theta) \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta, \\
\sin 5\theta &= 5(1 - 2\sin^2\theta + \sin^4\theta)\sin\theta \\
&\quad -10(1-\sin^2\theta)\sin^3\theta + \sin^5\theta \\
&= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.
\end{aligned}$$

Now, writing $\cos\theta$ as c, $\sin\theta$ as s and $\tan\theta$ as t, and further recalling that $c^{-2} = 1 + t^2$, we have

$$\begin{aligned}
\tan 5\theta &= \frac{16s^5 - 20s^3 + 5s}{16c^5 - 20c^3 + 5c} \\
&= \frac{16t^5 - 20t^3c^{-2} + 5tc^{-4}}{16 - 20c^{-2} + 5c^{-4}} \\
&= \frac{16t^5 - 20t^3(1+t^2) + 5t(1 + 2t^2 + t^4)}{16 - 20(1+t^2) + 5(1 + 2t^2 + t^4)} \\
&= \frac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}.
\end{aligned}$$

When θ is equal to $\dfrac{\pi}{10}$ or $\dfrac{3\pi}{10}$, $\tan 5\theta = \infty$, implying that

$$5t^4 - 10t^2 + 1 = 0 \quad \Rightarrow \quad t^2 = \frac{5 \pm \sqrt{25-5}}{5} \quad \Rightarrow \quad t = \pm\left(\frac{5 \pm \sqrt{20}}{5}\right)^{1/2}.$$

As both angles lie in the first quadrant the overall sign must be taken as positive in both cases, and it is clear that the positive square root in the numerator corresponds to $\theta = 3\pi/10$.

When θ is equal to $\dfrac{2\pi}{10}$ or $\dfrac{4\pi}{10}$, $\tan 5\theta = 0$, implying that

$$t^5 - 10t^3 + 5t = 0 \quad \Rightarrow \quad t^2 = 5 \pm \sqrt{25-5} \quad \Rightarrow \quad t = \pm\left(5 \pm \sqrt{20}\right)^{1/2}.$$

Again, as both angles lie in the first quadrant the overall sign must be taken as positive; it is also clear that the positive square root in the parentheses corresponds to $\theta = 4\pi/10$.

3.23 *Determine the conditions under which the equation*

$$a\cosh x + b\sinh x = c, \qquad c > 0,$$

has zero, one, or two real solutions for x. What is the solution if $a^2 = c^2 + b^2$?

We start by recalling that $\cosh x = \frac{1}{2}(e^x + e^{-x})$ and $\sinh x = \frac{1}{2}(e^x - e^{-x})$, and then rewrite the equation as a quadratic equation in e^x:

$$\begin{aligned} a\cosh x + b\sinh x - c &= 0, \\ (a+b)e^x - 2c + (a-b)e^{-x} &= 0, \\ (a+b)e^{2x} - 2ce^x + (a-b) &= 0. \end{aligned}$$

Hence,

$$e^x = \frac{c \pm \sqrt{c^2 - (a^2 - b^2)}}{a+b}.$$

For x to be real, e^x must be real and ≥ 0. Since $c > 0$, this implies that $a + b > 0$ and $c^2 + b^2 \geq a^2$. Provided these two conditions are satisfied, there are two roots if $c^2 + b^2 - a^2 < c^2$, i.e. if $b^2 < a^2$, but only one root if $c^2 + b^2 - a^2 > c^2$, i.e. if $b^2 > a^2$.

If $c^2 + b^2 = a^2$ then the double root is given by

$$\begin{aligned} e^x &= \frac{c}{a+b}, \\ e^{2x} = \frac{c^2}{(a+b)^2} &= \frac{a^2 - b^2}{(a+b)^2} = \frac{a-b}{a+b}, \\ x &= \frac{1}{2}\ln\frac{a-b}{a+b}. \end{aligned}$$

3.25 *Express* $\sinh^4 x$ *in terms of hyperbolic cosines of multiples of* x*, and hence find the real solutions of*

$$2\cosh 4x - 8\cosh 2x + 5 = 0.$$

In order to connect $\sinh^4 x$ to hyperbolic functions of other multiples of x, we need to express it in terms of powers of $e^{\pm x}$ and then to group the terms so as to make up those hyperbolic functions. Starting from

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}),$$

we have from the binomial theorem that

$$\sinh^4 x = \tfrac{1}{16}\left(e^{4x} - 4e^{2x} + 6 - 4e^{-2x} + e^{-4x}\right).$$

Terms containing related exponents nx and $-nx$ can now be grouped together and expressed as a linear sum of $\cosh nx$ and $\sinh nx$; here, because of the symmetry properties of the binomial coefficients, only the $\cosh nx$ combinations appear and yield

$$\sinh^4 x = \tfrac{1}{8}\cosh 4x - \tfrac{1}{2}\cosh 2x + \tfrac{3}{8}.$$

Now consider the relationship between this expression and the LHS of the given equation. They are clearly closely related; one is a multiple of the other, except in respect of the constant term. Making compensating corrections to the constant term allows us to rewrite the equation in terms of $\sinh^4 x$ as follows:

$$\begin{aligned} 2\cosh 4x - 8\cosh 2x + (6-1) &= 0, \\ 16\sinh^4 x - 1 &= 0, \\ \sinh^4 x &= \tfrac{1}{16}, \\ \sinh x &= \pm\tfrac{1}{2} \quad \text{(real solutions only).} \end{aligned}$$

We now use the explicit expression for the inverse hyperbolic sine, namely

$$\text{If } y = \sinh^{-1} z, \text{ then } y = \ln(\sqrt{1+z^2} + z),$$

to give in this case

$$x = \ln\left(\sqrt{1+\tfrac{1}{4}} \pm \tfrac{1}{2}\right) = 0.481 \text{ or } -0.481.$$

3.27 *A closed barrel has as its curved surface the surface obtained by rotating about the x-axis the part of the curve*

$$y = a[2 - \cosh(x/a)]$$

lying in the range $-b \le x \le b$, *where* $b < a\cosh^{-1} 2$. *Show that the total surface area, A, of the barrel is given by*

$$A = \pi a[9a - 8a\exp(-b/a) + a\exp(-2b/a) - 2b].$$

If s is the length of the curve defining the surface (measured from $x = 0$) then $ds^2 = dx^2 + dy^2$ and consequently $ds/dx = (1 + y'^2)^{1/2}$.

For this particular surface,

$$y = a\left(2 - \cosh\frac{x}{a}\right)$$

$$\text{and } \frac{dy}{dx} = -\sinh\frac{x}{a}.$$

It follows that

$$\begin{aligned}\frac{ds}{dx} &= \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{1/2}\\ &= \left(1 + \sinh^2\frac{x}{a}\right)^{1/2}\\ &= \cosh\frac{x}{a}.\end{aligned}$$

The curved surface area, A_1, is given by

$$\begin{aligned}A_1 &= 2\int_0^b 2\pi y\,ds\\ &= 2\int_0^b 2\pi y\frac{ds}{dx}\,dx\\ &= 4\pi a\int_0^b \left(2\cosh\frac{x}{a} - \cosh^2\frac{x}{a}\right)dx, \text{ use } \cosh^2 z = \tfrac{1}{2}(\cosh 2z + 1),\\ &= 4\pi a\int_0^b \left(2\cosh\frac{x}{a} - \frac{1}{2} - \frac{1}{2}\cosh\frac{2x}{a}\right)dx\\ &= 4\pi a\left[2a\sinh\frac{x}{a} - \frac{x}{2} - \frac{a}{4}\sinh\frac{2x}{a}\right]_0^b\\ &= \pi a\left(8a\sinh\frac{b}{a} - 2b - a\sinh\frac{2b}{a}\right).\end{aligned}$$

The area, A_2, of the two flat ends is given by

$$\begin{aligned} A_2 &= 2\pi a^2 \left(2 - \cosh\frac{b}{a}\right)^2 \\ &= 2\pi a^2 \left(4 - 4\cosh\frac{b}{a} + \cosh^2\frac{b}{a}\right). \end{aligned}$$

And so the total area is

$$\begin{aligned} A = \pi a \Big[& 4a\left(e^{b/a} - e^{-b/a}\right) - 2b - \frac{a}{2}\left(e^{2b/a} - e^{-2b/a}\right) \\ & + 8a - 4a\left(e^{b/a} + e^{-b/a}\right) + \frac{2a}{4}\left(e^{2b/a} + 2 + e^{-2b/a}\right)\Big] \\ = \pi a & \left(9a - 8ae^{-b/a} + ae^{-2b/a} - 2b\right). \end{aligned}$$

4

Series and limits

4.1 *Sum the even numbers between* 1000 *and* 2000 *inclusive.*

We must first express the given sum in terms of a summation for which we have an explicit form. The result that is needed is clearly

$$S_N = \sum_{n=1}^{N} n = \frac{1}{2}N(N+1),$$

and we must re-write the given summation in terms of sums of this form:

$$\begin{aligned}\sum_{n(\text{even})=1000}^{n=2000} n &= \sum_{m=500}^{m=1000} 2m \\ &= 2(S_{1000} - S_{499}) \\ &= 2\left(\tfrac{1}{2} \times 1000 \times 1001 - \tfrac{1}{2} \times 499 \times 500\right) \\ &= 751\,500.\end{aligned}$$

4.3 *How does the convergence of the series*

$$\sum_{n=r}^{\infty} \frac{(n-r)!}{n!}$$

depend on the integer r?

For $r \leq 1$, each term of the series is greater than or equal to the corresponding term of $\sum \frac{1}{n}$, which is known to be divergent (for a proof, see any standard textbook). Thus, by the comparison test, the given series is also divergent.

For $r \geq 2$, each term of the series is less than or equal to the corresponding term of $\sum_{1}^{\infty} \frac{1}{n(n+1)}$. By writing this latter sum as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n(n+1)} &= \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \left(\tfrac{1}{3} - \tfrac{1}{4}\right) + \cdots \\ &= 1 + \left(-\tfrac{1}{2} + \tfrac{1}{2}\right) + \left(-\tfrac{1}{3} + \tfrac{1}{3}\right) + \cdots \to 1,\end{aligned}$$

it is shown to be convergent. Thus, by the comparison test, the given series is also convergent when $r \geq 2$.

4.5 *Find the sum, S_N, of the first N terms of the following series, and hence determine whether the series are convergent, divergent or oscillatory:*

$$\text{(a)} \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right), \qquad \text{(b)} \sum_{n=0}^{\infty} (-2)^n, \qquad \text{(c)} \sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n}.$$

(a) We express this series as the difference between two series with similar terms and find that the terms cancel in pairs, leaving an explicit expression that contains only the last term of the first series and the first term of the second:

$$\sum_{n=1}^{N} \ln \frac{n+1}{n} = \sum_{n=1}^{N} \ln(n+1) - \sum_{n=1}^{N} \ln n = \ln(N+1) - \ln 1.$$

As $\ln(N+1) \to \infty$ as $N \to \infty$, the series diverges.

(b) Applying the normal formula for a geometric sum gives

$$\sum_{n=0}^{N-1} (-2)^n = \frac{1 - (-2)^N}{3}.$$

The series therefore oscillates infinitely.

(c) Denote the partial sum by S_N. Then,

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \frac{(-1)^{n+1} n}{3^n}, \\ \frac{1}{3} S_N &= \sum_{n=1}^{N} \frac{(-1)^{n+1} n}{3^{n+1}} = \sum_{s=2}^{N+1} \frac{(-1)^s (s-1)}{3^s} \\ &= \sum_{s=2}^{N+1} \frac{(-1)^s s}{3^s} - \sum_{s=2}^{N+1} \frac{(-1)^s}{3^s}. \end{aligned}$$

Separating off the last term of the first series on the RHS and adding S_N to both sides, with the S_N added to the RHS having its $n = 1$ term written explicitly, yields

$$\begin{aligned}\frac{4}{3}S_N &= \frac{(-1)^2 1}{3} + \sum_{n=2}^{N} \frac{(-1)^{n+1} n}{3^n} + \sum_{s=2}^{N} \frac{(-1)^s s}{3^s} \\ &\quad + \frac{(-1)^{N+1}(N+1)}{3^{N+1}} - \sum_{s=2}^{N+1} \frac{(-1)^s}{3^s} \\ &= \frac{1}{3} + \frac{(-1)^{N+1}(N+1)}{3^{N+1}} - \frac{1}{9}\,\frac{1-(-\frac{1}{3})^N}{1-(-\frac{1}{3})}.\end{aligned}$$

To obtain the last line we note that on the RHS the second and third terms (both summations) cancel and that the final term is a geometric series (with leading term $-\frac{1}{9}$). This result can be rearranged as

$$S_N = \frac{3}{16}\left[1 - \left(-\frac{1}{3}\right)^N\right] + \frac{3N}{4}\left(-\frac{1}{3}\right)^{N+1},$$

from which it is clear that the series converges to a sum of $\frac{3}{16}$.

4.7 *Use the difference method to sum the series*

$$\sum_{n=2}^{N} \frac{2n-1}{2n^2(n-1)^2}.$$

We try to write the nth term as the difference between two consecutive values of a partial-fraction function of n. Since the second power of n appears in the denominator the function will need two terms, An^{-2} and Bn^{-1}. Hence, we must have

$$\begin{aligned}\frac{2n-1}{2n^2(n-1)^2} &= \frac{A}{n^2} + \frac{B}{n} - \left[\frac{A}{(n-1)^2} + \frac{B}{n-1}\right] \\ &= \frac{A[-2n+1] + B[n(n-1)(n-1-n)]}{n^2(n-1)^2}.\end{aligned}$$

The powers of n in the numerators can be equated consistently if we take $A = -\frac{1}{2}$

and $B = 0$. Thus

$$\frac{2n-1}{2n^2(n-1)^2} = \frac{1}{2}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right].$$

We can now carry out the summation, in which the second component of each pair of terms cancels the first component of the next pair, leaving only the initial and very final components:

$$\begin{aligned}\sum_{n=2}^{N} \frac{2n-1}{2n^2(n-1)^2} &= \frac{1}{2}\sum_{n=2}^{N}\left[\frac{1}{(n-1)^2} - \frac{1}{n^2}\right] \\ &= \frac{1}{2}\left(\frac{1}{1} - \frac{1}{N^2}\right) \\ &= \tfrac{1}{2}(1 - N^{-2}).\end{aligned}$$

4.9 *Prove that*

$$\cos\theta + \cos(\theta+\alpha) + \cdots + \cos(\theta + n\alpha) = \frac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\cos(\theta + \tfrac{1}{2}n\alpha).$$

From de Moivre's theorem, the required sum, S, is the real part of the sum of the geometric series $\sum_{r=0}^{n} e^{i\theta}e^{ir\alpha}$. Using the formula for the partial sum of a geometric series, and multiplying by a factor that makes the denominator real, we have

$$\begin{aligned}S &= \mathrm{Re}\left(e^{i\theta}\frac{1-e^{i(n+1)\alpha}}{1-e^{i\alpha}}\,\frac{1-e^{-i\alpha}}{1-e^{-i\alpha}}\right) \\ &= \frac{\cos\theta - \cos[(n+1)\alpha+\theta] - \cos(\theta-\alpha) + \cos(\theta+n\alpha)}{2\times 2\sin^2\frac{1}{2}\alpha} \\ &= \frac{2\sin(\theta-\frac{1}{2}\alpha)\sin(-\frac{1}{2}\alpha) + 2\sin(n\alpha+\frac{1}{2}\alpha+\theta)\sin\frac{1}{2}\alpha}{4\sin^2\frac{1}{2}\alpha} \\ &= \frac{2\sin\frac{1}{2}\alpha\, 2\cos(\frac{1}{2}n\alpha+\theta)\ \sin[\frac{1}{2}(n+1)\alpha]}{4\sin^2\frac{1}{2}\alpha} \\ &= \frac{\sin\frac{1}{2}(n+1)\alpha}{\sin\frac{1}{2}\alpha}\cos(\theta+\tfrac{1}{2}n\alpha).\end{aligned}$$

In the course of this manipulation we have used the identity $1 - \cos\theta = 2\sin^2\frac{1}{2}\theta$ and the formulae for $\cos A - \cos B$ and $\sin A - \sin B$.

4.11 *Find the real values of x for which the following series are convergent:*

$$\text{(a)}\ \sum_{n=1}^{\infty}\frac{x^n}{n+1},\qquad \text{(b)}\ \sum_{n=1}^{\infty}(\sin x)^n,\qquad \text{(c)}\ \sum_{n=1}^{\infty}n^x,$$

$$\text{(d)}\ \sum_{n=1}^{\infty}e^{nx},\qquad \text{(e)}\ \sum_{n=2}^{\infty}(\ln n)^x.$$

(a) Using the ratio test:

$$\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=\lim_{n\to\infty}\frac{x^{n+1}}{n+2}\,\frac{n+1}{x^n}=x.$$

Thus the series is convergent for all $|x|<1$. At $x=1$ the series diverges, as shown in any standard text, whilst at $x=-1$ it converges by the alternating series test. Thus we have convergence for $-1\le x<1$.

(b) For all x other than $x=(2m\pm\frac{1}{2})\pi$, where m is an integer, $|\sin x|<1$ and so convergence is assured by the ratio test. At $x=(2m+\frac{1}{2})\pi$ the series diverges, whilst at $x=(2m-\frac{1}{2})\pi$ it oscillates finitely.

(c) This is the Riemann zeta series with p written as $-x$. Thus the series converges for all $x<-1$.

(d) The ratio of successive terms is e^x (independent of n) and for this to be less than unity in magnitude requires x to be negative. Thus the series is convergent when $x<0$.

(e) The sum $S=\sum_{n=2}^{\infty}(\ln n)^x$ is clearly divergent for all $x>-1$ (by comparison with $\sum n^{-1}$). So we define a positive X by $-X=x<-1$ and consider

$$S_1=\sum_{k=1}^{\infty}\sum_{r_k=M_{k-1}+1}^{M_k}\frac{1}{(\ln M_k)^X},$$

where M_k is the lowest integer such that $\ln M_k>k$. The notation is such that when $e^{k-1}<n<e^k$ then $n=M_{k-1}+r_k$.

For each fixed k, every term in the second (finite) summation is smaller than the corresponding term in S (because $n<M_k$). But, since all the terms in such a summation are equal, the value of the sum is simply $(M_k-M_{k-1})/(\ln M_k)^X$. Thus,

$$S_1=\sum_{k=1}^{\infty}\frac{M_k-M_{k-1}}{(\ln M_k)^X}=\sum_{k=1}^{\infty}\frac{(1-e^{-1})M_k}{(\ln M_k)^X}.$$

Now, the ratio of successive terms in this final summation is

$$\frac{M_{k+1}}{(\ln M_{k+1})^X}\,\frac{(\ln M_k)^X}{M_k}\to\frac{e}{\ln e}\quad\text{as}\quad k\to\infty.$$

This limit is > 1, and thus S_1 diverges for all X; hence, by the comparison test, so does S.

4.13 *Determine whether the following series are absolutely convergent, convergent or oscillatory:*

$$\text{(a)} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{5/2}}, \quad \text{(b)} \sum_{n=1}^{\infty} \frac{(-1)^n(2n+1)}{n}, \quad \text{(c)} \sum_{n=0}^{\infty} \frac{(-1)^n|x|^n}{n!},$$

$$\text{(d)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2+3n+2}, \quad \text{(e)} \sum_{n=1}^{\infty} \frac{(-1)^n 2^n}{n^{1/2}}.$$

(a) The sum $\sum n^{-5/2}$ is convergent (by comparison with $\sum n^{-2}$) and so $\sum(-1)^n n^{-5/2}$ is absolutely convergent.

(b) The magnitude of the individual terms $\to 2$ and not to zero; thus the series cannot converge. In fact it oscillates finitely about the value $-(1+\ln 2)$.

(c) The magnitude of the successive-term ratio is

$$\left|\frac{u_{n+1}}{u_n}\right| = \frac{|x|^{n+1}}{(n+1)!}\,\frac{n!}{|x|^n} = \frac{|x|}{n} \to 0 \quad \text{for all } x.$$

Thus, the series is absolutely convergent for all finite x.

(d) The polynomial in the denominator has all positive signs and a non-zero constant term; it is therefore always strictly positive. Thus, to test for absolute convergence, we need to replace the numerator by its absolute value and consider $\sum_{n=0}^{N}(n^2+3n+2)^{-1}$:

$$\sum_{n=0}^{N} \frac{1}{n^2+3n+2} = \sum_{n=0}^{N}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) = 1 - \frac{1}{N+2} \to 1 \quad \text{as} \quad N \to \infty.$$

Thus the given series is absolutely convergent.

(e) The magnitude of the individual terms does not tend to zero; in fact, it grows monotonically. The effect of the alternating signs is to make the series oscillate infinitely.

4.15 *Prove that*

$$\sum_{n=2}^{\infty} \ln\left[\frac{n^r+(-1)^n}{n^r}\right]$$

is absolutely convergent for $r = 2$, but only conditionally convergent for $r = 1$.

In each case divide the sum into two sums, one for n even and one for n odd.

(i) For $r = 2$, consider first the even series:

$$\sum_{n\,\text{even}} \ln\left(\frac{n^2+1}{n^2}\right) = \sum_{n\,\text{even}} \ln\left(1+\frac{1}{n^2}\right)$$
$$= \sum_{n\,\text{even}} \left(\frac{1}{n^2} - \frac{1}{2n^4} + \cdots\right).$$

The nth logarithmic term is positive for all n but, as shown above, less than n^{-2}. It follows from the comparison test that the series is (absolutely) convergent.

For the odd series we consider

$$\ln\left[\frac{(2m+1)^2-1}{(2m+1)^2}\right] = \ln\frac{4m^2+4m}{4m^2+4m+1}$$
$$= -\ln\left(1+\frac{1}{4m(m+1)}\right).$$

By a similar argument to that above, each term is negative but greater than $-[4m(m+1)]^{-1}$. Again, the comparison test shows that the series is (absolutely) convergent.

Thus the original series, being the sum of two absolutely convergent series, is also absolutely convergent.

(ii) For $r = 1$ we have to consider $\ln[(n \pm 1)/n]$, whose expansion contains a term $\pm n^{-1}$ and other inverse powers of n. The summations over the other powers converge and cannot cancel the divergence arising from $\sum \pm n^{-1}$. Thus both the even and odd series diverge; consequently the original series cannot be absolutely convergent.

However, if we group together consecutive pairs of terms, $n = 2m$ and $n = 2m+1$, then we see that

$$\sum_{n=2}^{\infty} \ln\left[\frac{n+(-1)^n}{n}\right] = \sum_{m=1}^{\infty}\left[\ln\frac{2m+1}{2m} + \ln\frac{2m+1-1}{2m+1}\right]$$
$$= \sum_{m=1}^{\infty} \ln 1 = \sum_{m=1}^{\infty} 0 = 0,$$

i.e. the terms cancel in pairs and the series is conditionally convergent to zero.

4.17 *Demonstrate that rearranging the order of its terms can make a conditionally convergent series converge to a different limit by considering the series* $\sum(-1)^{n+1}n^{-1} = \ln 2 = 0.693$. *Rearrange the series as*

$$S = \tfrac{1}{1} + \tfrac{1}{3} - \tfrac{1}{2} + \tfrac{1}{5} + \tfrac{1}{7} - \tfrac{1}{4} + \tfrac{1}{9} + \tfrac{1}{11} - \tfrac{1}{6} + \tfrac{1}{13} + \cdots$$

and group each set of three successive terms. Show that the series can then be written

$$\sum_{m=1}^{\infty} \frac{8m-3}{2m(4m-3)(4m-1)},$$

which is convergent (by comparison with $\sum n^{-2}$*) and contains only positive terms. Evaluate the first of these and hence deduce that* S *is not equal to* $\ln 2$.

Proceeding as indicated, we have

$$\begin{aligned} S &= \left(\frac{1}{1} + \frac{1}{3} - \frac{1}{2}\right) + \left(\frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right) + \left(\frac{1}{9} + \frac{1}{11} - \frac{1}{6}\right) + \cdots \\ &= \sum_{m=1}^{\infty} \left(\frac{1}{4m-3} + \frac{1}{4m-1} - \frac{1}{2m}\right) \\ &= \sum_{m=1}^{\infty} \frac{(8m^2-2m) + (8m^2-6m) - (16m^2-16m+3)}{2m(4m-3)(4m-1)} \\ &= \sum_{m=1}^{\infty} \frac{8m-3}{2m(4m-3)(4m-1)}. \end{aligned}$$

As noted, this series is convergent and contains only positive terms. The first of these terms ($m = 1$) is $5/6 = 0.833$. This, by itself, is greater than the known sum (0.693) of the original series. Thus S cannot be equal to $\ln 2$.

4.19 *A Fabry–Pérot interferometer consists of two parallel heavily silvered glass plates; light enters normally to the plates, and undergoes repeated reflections between them, with a small transmitted fraction emerging at each reflection. Find the intensity* $|B|^2$ *of the emerging wave, where*

$$B = A(1-r)\sum_{n=0}^{\infty} r^n e^{in\phi},$$

with r *and* ϕ *real.*

This is a simple geometric series but with a complex common ratio $re^{i\phi}$. Thus

we have

$$B = A(1-r)\sum_{n=0}^{\infty} r^n e^{in\phi}$$
$$= A\frac{1-r}{1-re^{i\phi}}.$$

To obtain the intensity $|B|^2$ we multiply this result by its complex conjugate, recalling that r and ϕ are real, but A may not be:

$$|\,B\,|^2 = \frac{|A|^2(1-r)^2}{(1-re^{i\phi})(1-re^{-i\phi})}$$
$$= \frac{|A|^2(1-r)^2}{1-2r\cos\phi + r^2}.$$

4.21 *Starting from the Maclaurin series for* $\cos x$*, show that*

$$(\cos x)^{-2} = 1 + x^2 + \frac{2x^4}{3} + \cdots.$$

Deduce the first three terms in the Maclaurin series for $\tan x$.

From the Maclaurin series for (or definition of) $\cos x$,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots.$$

Using the binomial expansion of $(1+z)^{-2}$, we have

$$(\cos x)^{-2} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^{-2}$$
$$= 1 - 2\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right) + \frac{2\,3}{2!}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)^2 + \cdots$$
$$= 1 + x^2 + x^4\left(-\frac{2}{4!} + \frac{2\,3}{2!\,2!\,2!}\right) + \mathrm{O}(x^6)$$
$$= 1 + x^2 + \tfrac{2}{3}x^4 + \cdots.$$

We now integrate both sides of the expansion from 0 to x, noting that $(\cos x)^{-2} \equiv \sec^2 x$ and that this integrates to $\tan x$. Thus

$$\tan x = \int_0^x \sec^2 u\,du = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots.$$

4.23 *If $f(x) = \sinh^{-1} x$, and its nth derivative $f^{(n)}(x)$ is written as*

$$f^{(n)} = \frac{P_n}{(1+x^2)^{n-1/2}},$$

where $P_n(x)$ is a polynomial (of order $n-1$), show that the $P_n(x)$ satisfy the recurrence relation

$$P_{n+1}(x) = (1+x^2)P_n'(x) - (2n-1)xP_n(x).$$

Hence generate the coefficients necessary to express $\sinh^{-1} x$ as a Maclaurin series up to terms in x^5.

With $f(x) = \sinh^{-1} x$,

$$x = \sinh f \quad \Rightarrow \quad \frac{dx}{df} = \cosh f \quad \Rightarrow \quad \frac{df}{dx} = \frac{1}{\cosh f} = \frac{1}{(1+x^2)^{1/2}}.$$

Thus $P_1(x) = 1$; we will need this as a starting value for the recurrence relation.

With the definition of $P_n(x)$ given,

$$\begin{aligned} f^{(n)} &= \frac{P_n}{(1+x^2)^{n-1/2}}, \\ f^{(n+1)} &= \frac{P_n'}{(1+x^2)^{n-1/2}} - \frac{(n-\frac{1}{2})\,2x\,P_n}{(1+x^2)^{n+1/2}} \\ &= \frac{(1+x^2)P_n' - (2n-1)xP_n}{(1+x^2)^{n+1-1/2}}. \end{aligned}$$

It then follows that

$$P_{n+1}(x) = (1+x^2)P_n'(x) - (2n-1)xP_n(x).$$

With $P_1 = 1$, as shown,

$$\begin{aligned} P_2 &= (1+x^2)0 - (2-1)x\,1 = -x, \\ P_3 &= (1+x^2)(-1) - (4-1)x(-x) = 2x^2 - 1, \\ P_4 &= (1+x^2)(4x) - (6-1)x(2x^2-1) = 9x - 6x^3, \\ P_5 &= (1+x^2)(9-18x^2) - (8-1)x(9x-6x^3) = 24x^4 - 72x^2 + 9. \end{aligned}$$

The corresponding values of $f^{(n)}(0) = P_n(0)/(1+0^2)^{n-1/2}$ can then be used to express the Maclaurin series for $\sinh^{-1} x$ as

$$\sinh^{-1} x = f(0) + \sum_{n=1}^{\infty} \frac{f^n(0)x^n}{n!} = x - \frac{x^3}{3!} + \frac{9x^5}{5!} - \cdots.$$

4.25 *By using the logarithmic series, prove that if a and b are positive and nearly equal then*

$$\ln\frac{a}{b} \simeq \frac{2(a-b)}{a+b}.$$

Show that the error in this approximation is about $2(a-b)^3/[3(a+b)^3]$.

Write $a+b=2c$ and $a-b=2\delta$. Then

$$\begin{aligned}
\ln\frac{a}{b} &= \ln a - \ln b \\
&= \ln(c+\delta) - \ln(c-\delta) \\
&= \ln c + \ln\left(1+\frac{\delta}{c}\right) - \ln c - \ln\left(1-\frac{\delta}{c}\right) \\
&= \left(\frac{\delta}{c} - \frac{\delta^2}{2c^2} + \frac{\delta^3}{3c^3} - \cdots\right) - \left(-\frac{\delta}{c} - \frac{\delta^2}{2c^2} - \frac{\delta^3}{3c^3} - \cdots\right) \\
&= \frac{2\delta}{c} + \frac{2}{3}\left(\frac{\delta}{c}\right)^3 + \cdots \\
&= \frac{2(a-b)}{a+b} + \frac{2}{3}\left(\frac{a-b}{a+b}\right)^3 + \cdots,
\end{aligned}$$

i.e. as stated in the question.

We note that other approximations are possible, and equally valid, e.g. setting $b=a+\epsilon$ leading to $-(\epsilon/a)[1-\epsilon/2a+\epsilon^2/3a^2-\cdots]$, but the given one, expanding symmetrically about $c=(a+b)/2$, contains no quadratic terms in $(a-b)$, only cubic and higher terms.

4.27 *Find the limit as* $x \to 0$ *of* $[\sqrt{1+x^m} - \sqrt{1-x^m}]/x^n$, *in which m and n are positive integers.*

Using the binomial expansions of the terms in the numerator,

$$\begin{aligned}
\frac{\sqrt{1+x^m} - \sqrt{1-x^m}}{x^n} &= \frac{1+\frac{1}{2}x^m + \cdots - (1-\frac{1}{2}x^m + \cdots)}{x^n} \\
&= \frac{x^m + \cdots}{x^n} \\
&= x^{m-n} + \cdots.
\end{aligned}$$

Thus the limit of the function as $x \to 0$ is 0 for $m > n$, 1 for $m = n$ and ∞ for $m < n$.

4.29 *Find the limits of the following functions:*

(a) $\dfrac{x^3+x^2-5x-2}{2x^3-7x^2+4x+4}$, *as* $x \to 0$, $x \to \infty$ *and* $x \to 2$;

(b) $\dfrac{\sin x - x\cosh x}{\sinh x - x}$, *as* $x \to 0$;

(c) $\displaystyle\int_x^{\pi/2} \left(\frac{y\cos y - \sin y}{y^2}\right) dy$, *as* $x \to 0$.

(a) Denote the ratio of polynomials by $f(x)$. Then

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{x^3+x^2-5x-2}{2x^3-7x^2+4x+4} = \frac{-2}{4} = -\frac{1}{2};$$

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{1+x^{-1}-5x^{-2}-2x^{-3}}{2-7x^{-1}+4x^{-2}+4x^{-3}} = \frac{1}{2};$$

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{x^3+x^2-5x-2}{2x^3-7x^2+4x+4} = \frac{0}{0}.$$

This final value is indeterminate and so, using l'Hôpital's rule, consider instead

$$\lim_{x\to 2} f(x) = \lim_{x\to 2} \frac{3x^2+2x-5}{6x^2-14x+4} = \frac{11}{0} = \infty.$$

(b) Using l'Hôpital's rule repeatedly,

$$\begin{aligned}
\lim_{x\to 0} \frac{\sin x - x\cosh x}{\sinh x - x} &= \lim_{x\to 0} \frac{\cos x - \cosh x - x\sinh x}{\cosh x - 1} \\
&= \lim_{x\to 0} \frac{-\sin x - \sinh x - \sinh x - x\cosh x}{\sinh x} \\
&= \lim_{x\to 0} \frac{-\cos x - 2\cosh x - \cosh x - x\sinh x}{\cosh x} = -4.
\end{aligned}$$

(c) Before taking the limit we need to find a closed form for the integral. So,

$$\begin{aligned}
\lim_{x\to 0} \int_x^{\pi/2} \left(\frac{y\cos y - \sin y}{y^2}\right) dy &= \lim_{x\to 0} \int_x^{\pi/2} \frac{d}{dy}\left(\frac{\sin y}{y}\right) dy \\
&= \lim_{x\to 0} \left[\frac{\sin y}{y}\right]_x^{\pi/2} \\
&= \lim_{x\to 0} \left(\frac{2}{\pi} - \frac{\sin x}{x}\right) \\
&= \lim_{x\to 0} \left[\frac{2}{\pi} - \frac{1}{x}\left(x - \frac{x^3}{3!} + \cdots\right)\right] \\
&= \frac{2}{\pi} - 1.
\end{aligned}$$

4.31 *Using a first-order Taylor expansion about $x = x_0$, show that a better approximation than x_0 to the solution of the equation*

$$f(x) = \sin x + \tan x = 2$$

is given by $x = x_0 + \delta$, where

$$\delta = \frac{2 - f(x_0)}{\cos x_0 + \sec^2 x_0}.$$

(a) *Use this procedure twice to find the solution of $f(x) = 2$ to six significant figures, given that it is close to $x = 0.9$.*
(b) *Use the result in (a) to deduce, to the same degree of accuracy, one solution of the quartic equation*

$$y^4 - 4y^3 + 4y^2 + 4y - 4 = 0.$$

(a) We write the solution to $f(x) = \sin x + \tan x = 2$ as $x = x_0 + \delta$. Substituting this form and retaining the first-order terms in δ in the Taylor expansions of $\sin x$ and $\tan x$ we obtain

$$\sin x_0 + \delta \cos x_0 + \cdots + \tan x_0 + \delta \sec^2 x_0 + \cdots = 2$$

$$\delta = \frac{2 - \sin x_0 - \tan x_0}{\cos x_0 + \sec^2 x_0}.$$

With $x_0 = 0.9$,

$$\delta_1 = \frac{2 - 0.783327 - 1.260158}{0.621610 + 2.587999} = \frac{-0.043485}{3.209609} = -0.013548,$$

making the first improved approximation $x_1 = x_0 + \delta_1 = 0.886452$.

Now, using x_1 instead of x_0 and repeating the process gives

$$\delta_2 = \frac{2 - 0.774833 - 1.225682}{0.632165 + 2.502295} = \frac{-5.15007 \times 10^{-4}}{3.13446} = -1.6430 \times 10^{-4},$$

making the second improved approximation $x_2 = x_1 + \delta_2 = 0.886287$. The method used up to here does not *prove* that this latest answer is accurate to six significant figures, but a further application of the procedure shows that $\delta_3 \approx 3 \times 10^{-7}$.

(b) In order to make use of the result in part (a) we need to make a change of variable that converts the geometric equation into an algebraic one. Since $\tan x$ can be expressed in terms of $\sin x$, if we set $y = \sin x$ in the equation

$\sin x + \tan x = 2$, it will become an algebraic equation:

$$\begin{aligned}
\sin x + \tan x = \sin x + \frac{\sin x}{\cos x} &= 2, \\
\Rightarrow \quad y + \frac{y}{\sqrt{1-y^2}} &= 2, \\
\frac{y^2}{1-y^2} &= (2-y)^2, \\
y^2 &= (1-y^2)(4-4y+y^2) \\
&= -y^4 + 4y^3 - 3y^2 - 4y + 4, \\
0 &= y^4 - 4y^3 + 4y^2 + 4y - 4.
\end{aligned}$$

This is the equation that is to be solved. Thus, since $x = 0.886287$ is an approximation to the solution of $\sin x + \tan x = 2$, $y = \sin x = 0.774730$ is an approximation to one of the solutions of $y^4 - 4y^3 + 4y^2 + 4y - 4 = 0$ to the same degree of accuracy.

We note that an equally plausible change of variable is to set $y = \tan x$, with $\sin x$ expressed as $\tan x / \sec x$, i.e. as $y/\sqrt{1+y^2}$. With this substitution the resulting algebraic equation is the quartic $y^4 - 4y^3 + 4y^2 - 4y + 4 = 0$ (very similar to, but not exactly the same as, the given quartic equation). The reader may wish to verify this. By a parallel argument to that above, $y = \tan 0.886287 = 1.225270$ is an approximate solution of this second quartic equation.

4.33 *In quantum theory, a system of oscillators, each of fundamental frequency ν and interacting at temperature T, has an average energy $\bar{E}$ given by*

$$\bar{E} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}},$$

where $x = h\nu/kT$, h and k being the Planck and Boltzmann constants, respectively. Prove that both series converge, evaluate their sums, and show that at high temperatures $\bar{E} \approx kT$, whilst at low temperatures $\bar{E} \approx h\nu \exp(-h\nu/kT)$.

In the expression

$$\bar{E} = \frac{\sum_{n=0}^{\infty} nh\nu e^{-nx}}{\sum_{n=0}^{\infty} e^{-nx}},$$

the ratio of successive terms in the series in the numerator is given by

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)h\nu e^{-(n+1)x}}{nh\nu e^{-nx}}\right| = \left|\frac{n+1}{n} e^{-x}\right| \to e^{-x} \quad \text{as} \quad n \to \infty,$$

where $x = h\nu/kT$. Since $x > 0$, $e^{-x} < 1$, and the series is convergent by the ratio test.

The series in the denominator is a geometric series with common ratio $r = e^{-x}$. This is < 1 and so the series converges with sum

$$S(x) = 1 + e^{-x} + e^{-2x} + \cdots + e^{-nx} + \cdots = \frac{1}{1 - e^{-x}}.$$

Now consider

$$-\frac{dS(x)}{dx} = e^{-x} + 2e^{-2x} + \cdots + ne^{-nx} + \cdots .$$

The series on the RHS, when multiplied by $h\nu$, gives the numerator in the expression for $\bar{E}$; the numerator therefore has the value

$$-\frac{dS(x)}{dx} = -\frac{d}{dx}\left(\frac{1}{1 - e^{-x}}\right) = \frac{e^{-x}}{(1 - e^{-x})^2}.$$

Hence,

$$\bar{E} = \frac{h\nu\, e^{-x}}{(1 - e^{-x})^2}\,\frac{1 - e^{-x}}{1} = \frac{h\nu}{e^{x} - 1}.$$

At high temperatures, $x \ll 1$ and

$$\bar{E} = \frac{h\nu}{\left(1 + \frac{h\nu}{kT} + \cdots\right) - 1} \approx kT.$$

At low temperatures, $x \gg 1$ and $e^{x} \gg 1$. Thus the -1 in the denominator can be neglected and $\bar{E} \approx h\nu \exp(-h\nu/kT)$.

4.35 *One of the factors contributing to the high relative permittivity of water to static electric fields is the permanent electric dipole moment, p, of the water molecule. In an external field E the dipoles tend to line up with the field, but they do not do so completely because of thermal agitation corresponding to the temperature, T, of the water. A classical (non-quantum) calculation using the Boltzmann distribution shows that the average polarisability per molecule, α, is given by*

$$\alpha = \frac{p}{E}(\coth x - x^{-1}),$$

where $x = pE/(kT)$ and k is the Boltzmann constant.

At ordinary temperatures, even with high field strengths ($10^4\,\mathrm{V\,m^{-1}}$ or more), $x \ll 1$. By making suitable series expansions of the hyperbolic functions involved, show that $\alpha = p^2/(3kT)$ to an accuracy of about one part in $15x^{-2}$.

As $x \ll 1$, we have to deal with a function that is the difference between two terms that individually tend to infinity as $x \to 0$. We will need to expand each in a series and consider the leading non-cancelling terms. The coth function will

have to be expressed in terms of the series for the sinh and cosh functions, as follows:

$$
\begin{aligned}
\alpha &= \frac{p}{E}\left(\coth x - \frac{1}{x}\right), \quad \text{with} \quad x = \frac{pE}{kT}, \\
&= \frac{p}{E}\left(\frac{\cosh x}{\sinh x} - \frac{1}{x}\right) \\
&= \frac{p}{E}\left[\frac{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots}{x\left(1 + \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)} - \frac{1}{x}\right] \\
&= \frac{p}{Ex}\left\{\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots\right)\left[1 - \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)\right.\right. \\
&\qquad \left.\left. + \left(\frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right)^2 + \cdots\right] - 1\right\} \\
&= \frac{p}{Ex}\left[0 + x^2\left(\frac{1}{2!} - \frac{1}{3!}\right) + x^4\left(-\frac{1}{5!} + \frac{1}{(3!)^2} - \frac{1}{2!\,3!} + \frac{1}{4!}\right) + \cdots\right] \\
&= \frac{px}{E}\left(\frac{1}{3} - \frac{x^2}{45} + \cdots\right).
\end{aligned}
$$

Thus the polarisability $\approx px/3E = p^2/3kT$, with the correction term being a factor of about $x^2/15$ smaller.

5

Partial differentiation

5.1 *Using the appropriate properties of ordinary derivatives, perform the following.*

(a) *Find all the first partial derivatives of the following functions* $f(x, y)$*:*
(i) x^2y, (ii) $x^2 + y^2 + 4$, (iii) $\sin(x/y)$, (iv) $\tan^{-1}(y/x)$,
(v) $r(x, y, z) = (x^2 + y^2 + z^2)^{1/2}$.

(b) *For* (i), (ii) *and* (v), *find* $\partial^2 f/\partial x^2$, $\partial^2 f/\partial y^2$ *and* $\partial^2 f/\partial x \partial y$.

(c) *For* (iv) *verify that* $\partial^2 f/\partial x \partial y = \partial^2 f/\partial y \partial x$.

These are all straightforward applications of the definitions of partial derivatives.

(a) (i) $$\frac{\partial f}{\partial x} = \frac{\partial(x^2y)}{\partial x} = 2xy; \quad \frac{\partial f}{\partial y} = \frac{\partial(x^2y)}{\partial y} = x^2.$$

(ii) $$\frac{\partial f}{\partial x} = \frac{\partial(x^2 + y^2 + 4)}{\partial x} = 2x; \quad \frac{\partial f}{\partial y} = \frac{\partial(x^2 + y^2 + 4)}{\partial y} = 2y.$$

(iii) $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} \sin\left(\frac{x}{y}\right) = \cos\left(\frac{x}{y}\right)\frac{1}{y};$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} \sin\left(\frac{x}{y}\right) = \cos\left(\frac{x}{y}\right)\frac{-x}{y^2}.$$

(iv) $$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1 + \frac{y^2}{x^2}}\frac{-y}{x^2} = -\frac{y}{x^2 + y^2};$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[\tan^{-1}\left(\frac{y}{x}\right)\right] = \frac{1}{1 + \frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2 + y^2}.$$

(v) $$\frac{\partial r}{\partial x} = \frac{\partial(x^2 + y^2 + z^2)^{1/2}}{\partial x} = \frac{\frac{1}{2} \times 2x}{(x^2 + y^2 + z^2)^{1/2}} = \frac{x}{r};$$

similarly for $\dfrac{\partial r}{\partial y}$ and $\dfrac{\partial r}{\partial z}$.

(b) (i) $\dfrac{\partial^2(x^2y)}{\partial x^2} = \dfrac{\partial(2xy)}{\partial x} = 2y; \quad \dfrac{\partial^2(x^2y)}{\partial y^2} = \dfrac{\partial(x^2)}{\partial y} = 0;$

$$\frac{\partial^2(x^2y)}{\partial x \partial y} = \frac{\partial(x^2)}{\partial x} = 2x.$$

(ii) $\dfrac{\partial^2(x^2+y^2+4)}{\partial x^2} = \dfrac{\partial(2x)}{\partial x} = 2; \quad \dfrac{\partial^2(x^2+y^2+4)}{\partial y^2} = \dfrac{\partial(2y)}{\partial y} = 2;$

$$\frac{\partial^2(x^2+y^2+4)}{\partial x \partial y} = \frac{\partial(2y)}{\partial x} = 0.$$

(v)
$$\frac{\partial^2(x^2+y^2+z^2)^{1/2}}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{x}{r}\right) = \frac{1}{r} - \frac{x}{r^2}\frac{\partial r}{\partial x}$$
$$= \frac{1}{r} - \frac{x}{r^2}\frac{x}{r} = \frac{y^2+z^2}{r^3};$$

similarly for $\dfrac{\partial^2 r}{\partial y^2}$;

$$\frac{\partial^2(x^2+y^2+z^2)^{1/2}}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{y}{r}\right) = -\frac{y}{r^2}\frac{x}{r} = -\frac{xy}{r^3}.$$

(c)
$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) = -\frac{(x^2+y^2) - y\,2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

and

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) = \frac{(x^2+y^2) - x\,2x}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2},$$

thus verifying the general result for this particular case.

5.3 *Show that the differential*

$$df = x^2\,dy - (y^2+xy)\,dx$$

is not exact, but that $dg = (xy^2)^{-1}df$ *is exact.*

If $df = A\,dx + B\,dy$ then a necessary and sufficient condition for df to be exact is

$$\frac{\partial A(x,y)}{\partial y} = \frac{\partial B(x,y)}{\partial x}.$$

Here $A = -(y^2+xy)$ and $B = x^2$, and so we calculate

$$\frac{\partial(x^2)}{\partial x} = 2x \quad \text{and} \quad \frac{\partial(-y^2-xy)}{\partial y} = -2y - x.$$

These are not equal and so df is *not* an exact differential.

However, for dg, $A = -(y^2 + xy)/(xy^2)$ and $B = x^2/(xy^2)$. Taking the appropriate partial derivatives gives

$$\frac{\partial}{\partial x}\left(\frac{x^2}{xy^2}\right) = \frac{1}{y^2} \quad \text{and} \quad \frac{\partial}{\partial y}\left(\frac{-y^2 - xy}{xy^2}\right) = 0 + \frac{1}{y^2}.$$

These are equal, implying that dg *is* an exact differential and that the original inexact differential has $1/xy^2$ as its integrating factor.

5.5 *The equation $3y = z^3 + 3xz$ defines z implicitly as a function of x and y. Evaluate all three second partial derivatives of z with respect to x and/or y. Verify that z is a solution of*

$$x\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = 0.$$

By successive partial differentiations of

$$3y = z^3 + 3xz \qquad (*)$$

and its derivatives with respect to (wrt) x and y, we obtain the following.

Of $(*)$ wrt x $\qquad 0 = 3z^2\dfrac{\partial z}{\partial x} + 3z + 3x\dfrac{\partial z}{\partial x},$

(i) $\qquad \Rightarrow \quad \dfrac{\partial z}{\partial x} = -\dfrac{z}{x + z^2}.$

Of $(*)$ wrt y $\qquad 3 = 3z^2\dfrac{\partial z}{\partial y} + 3x\dfrac{\partial z}{\partial y},$

(ii) $\qquad \Rightarrow \quad \dfrac{\partial z}{\partial y} = \dfrac{1}{x + z^2}.$

For the second derivatives:

differentiating (i) wrt x

$$\begin{aligned}
\frac{\partial^2 z}{\partial x^2} &= -\frac{(x + z^2)\frac{\partial z}{\partial x} - z\left(1 + 2z\frac{\partial z}{\partial x}\right)}{(x + z^2)^2} \\
&= \frac{(z^2 - x)\frac{\partial z}{\partial x} + z}{(x + z^2)^2} \\
&= \frac{(z^2 - x)(-z) + z(x + z^2)}{(x + z^2)^3}, \quad \text{using (i)}, \\
&= \frac{2xz}{(x + z^2)^3};
\end{aligned}$$

differentiating (i) wrt y

$$\begin{aligned}\frac{\partial^2 z}{\partial y \partial x} &= -\frac{(x+z^2)\frac{\partial z}{\partial y} - z\,2z\frac{\partial z}{\partial y}}{(x+z^2)^2} \\ &= \frac{(z^2-x)\frac{\partial z}{\partial y}}{(x+z^2)^2} \\ &= \frac{z^2-x}{(x+z^2)^3}, \qquad \text{using (ii);}\end{aligned}$$

differentiating (ii) wrt y

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{-1}{(x+z^2)^2}\,2z\,\frac{\partial z}{\partial y} \\ &= \frac{-2z}{(x+z^2)^3}, \qquad \text{using (ii).}\end{aligned}$$

We now have that

$$x\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x^2} = \frac{-2zx}{(x+z^2)^3} + \frac{2zx}{(x+z^2)^3} = 0,$$

i.e. z is a solution of the given partial differential equation.

5.7 *The function $G(t)$ is defined by*

$$G(t) = F(x,y) = x^2 + y^2 + 3xy,$$

where $x(t) = at^2$ and $y(t) = 2at$. Use the chain rule to find the values of (x,y) at which $G(t)$ has stationary values as a function of t. Do any of them correspond to the stationary points of $F(x,y)$ as a function of x and y?

Using the chain rule,

$$\begin{aligned}\frac{dG}{dt} &= \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} \\ &= (2x+3y)2at + (2y+3x)2a \\ &= 2at(2at^2+6at) + 2a(4at+3at^2) \\ &= 2a^2t(2t^2+9t+4) \\ &= 2a^2t(2t+1)(t+4).\end{aligned}$$

Thus dG/dt has zeroes at $t=0$, $t=-\frac{1}{2}$ and $t=-4$; the corresponding values of (x,y) are $(0,0)$, $(\frac{1}{4}a,-a)$ and $(16a,-8a)$.

Considered as a function of x and y, $F(x,y)$ has stationary points when

$$\begin{aligned}\frac{\partial F}{\partial x} &= 2x+3y = 0, \\ \frac{\partial F}{\partial y} &= 3x+2y = 0.\end{aligned}$$

The *only* solution to this pair of equations is $(x,y) = (0,0)$, which corresponds to

(only) one of the points found previously. This stationary point is a saddle point at the origin and is the only stationary point of $F(x, y)$.

The stationary points of $G(t)$ as a function of t are a maximum of $5a^2/16$ at $(\frac{1}{4}a, -a)$, a minimum of $-64a^2$ at $(16a, -8a)$, and a point of inflection at the origin. The first two are not stationary points of $F(x, y)$ for general values of x and y. They only appear to be so because the parameterisation, which restricts the search to the (one-dimensional) line defined by the parabola $y^2 = 4ax$, does not take into account the values of $F(x, y)$ at points close to, but not on, the line.

5.9 *The function $f(x, y)$ satisfies the differential equation*

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = 0.$$

By changing to new variables $u = x^2 - y^2$ and $v = 2xy$, show that f is, in fact, a function of $x^2 - y^2$ only.

In order to use the equations

$$\frac{\partial f}{\partial x_j} = \sum_{i=1}^{n} \frac{\partial f}{\partial u_i}\frac{\partial u_i}{\partial x_j}$$

that govern a change of variables, we need the partial derivatives

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y, \quad \frac{\partial v}{\partial y} = 2x.$$

Then, with $f(x, y)$ written as $g(u, v)$,

$$\frac{\partial f}{\partial x} = 2x\frac{\partial g}{\partial u} + 2y\frac{\partial g}{\partial v},$$
$$\frac{\partial f}{\partial y} = -2y\frac{\partial g}{\partial u} + 2x\frac{\partial g}{\partial v}.$$

Thus,

$$y\frac{\partial f}{\partial x} + x\frac{\partial f}{\partial y} = (2xy - 2xy)\frac{\partial g}{\partial u} + 2(y^2 + x^2)\frac{\partial g}{\partial v}$$

and the equation reduces to

$$\frac{\partial g}{\partial v} = 0 \qquad \Rightarrow \qquad g = g(u), \text{ i.e. } f(x, y) = g(x^2 - y^2) \text{ only.}$$

5.11 *Find and evaluate the maxima, minima and saddle points of the function*

$$f(x, y) = xy(x^2 + y^2 - 1).$$

The required derivatives are given by

$$\frac{\partial f}{\partial x} = 3x^2y + y^3 - y, \qquad \frac{\partial f}{\partial y} = x^3 + 3y^2x - x,$$

$$\frac{\partial^2 f}{\partial x^2} = 6xy, \qquad \frac{\partial^2 f}{\partial x \partial y} = 3x^2 + 3y^2 - 1, \qquad \frac{\partial^2 f}{\partial y^2} = 6xy.$$

Any stationary points must satisfy both of the equations

$$\frac{\partial f}{\partial x} = y(3x^2 + y^2 - 1) = 0,$$
$$\frac{\partial f}{\partial y} = x(x^2 + 3y^2 - 1) = 0.$$

If $x = 0$ then $y = 0$ or ± 1. If $y = 0$ then $x = 0$ or ± 1.

Otherwise, adding and subtracting the factors in parentheses gives

$$4(x^2 + y^2) = 2,$$
$$2(x^2 - y^2) = 0.$$

These have the solutions $x = \pm\frac{1}{2}$, $y = \pm\frac{1}{2}$.

Thus the nine stationary points are $(0, 0)$, $(0, \pm 1)$, $(\pm 1, 0)$, $\pm(\frac{1}{2}, \frac{1}{2})$ and $\pm(\frac{1}{2}, -\frac{1}{2})$. The corresponding values for $f(x, y)$ are 0 for the first five, $-\frac{1}{8}$ for the next two and $\frac{1}{8}$ for the final two.

For the first five cases, $\partial^2 f/\partial^2 x = \partial^2 f/\partial^2 y = 0$, whilst $\partial^2 f/\partial x \partial y = -1$ or 2. Since $(-1)^2 > 0 \times 0$ and $2^2 > 0 \times 0$, these points are all saddle points.

At $\pm(\frac{1}{2}, \frac{1}{2})$, $\partial^2 f/\partial^2 x = \partial^2 f/\partial^2 y = \frac{3}{2}$, whilst $\partial^2 f/\partial x \partial y = \frac{1}{2}$. Since $(\frac{1}{2})^2 < \frac{3}{2} \times \frac{3}{2}$, these two points are either maxima or minima (i.e. not saddle points) and the positive signs for $\partial^2 f/\partial^2 x$ and $\partial^2 f/\partial^2 y$ indicate that they are, in fact, minima.

At $\pm(\frac{1}{2}, -\frac{1}{2})$, $\partial^2 f/\partial^2 x = \partial^2 f/\partial^2 y = -\frac{3}{2}$, whilst $\partial^2 f/\partial x \partial y = \frac{1}{2}$. Since $(\frac{1}{2})^2 < -\frac{3}{2} \times -\frac{3}{2}$, these two points are also either maxima or minima; the common negative sign for $\partial^2 f/\partial^2 x$ and $\partial^2 f/\partial^2 y$ indicates that they are maxima.

5.13 *Locate the stationary points of the function*

$$f(x, y) = (x^2 - 2y^2)\exp[-(x^2 + y^2)/a^2],$$

where a is a non-zero constant.

Sketch the function along the x- and y-axes and hence identify the nature and values of the stationary points.

To find the stationary points, we set each of the two first partial derivatives,

$$\frac{\partial f}{\partial x} = \left[2x - \frac{2x}{a^2}(x^2 - 2y^2)\right]\exp\left(-\frac{x^2 + y^2}{a^2}\right),$$
$$\frac{\partial f}{\partial y} = \left[-4y - \frac{2y}{a^2}(x^2 - 2y^2)\right]\exp\left(-\frac{x^2 + y^2}{a^2}\right),$$

equal to zero:

$$\frac{\partial f}{\partial x} = 0 \quad \Rightarrow \quad x = 0 \text{ or } x^2 - 2y^2 = a^2;$$
$$\frac{\partial f}{\partial y} = 0 \quad \Rightarrow \quad y = 0 \text{ or } x^2 - 2y^2 = -2a^2.$$

Since $a \neq 0$, possible solutions for (x, y) are $(0, 0)$, $(0, \pm a)$ and $(\pm a, 0)$. The corresponding values are $f(0, 0) = 0$, $f(0, \pm a) = -2a^2e^{-1}$ and $f(\pm a, 0) = a^2e^{-1}$. These results, taken together with the observation that $|f(x, y)| \to 0$ as either or both of $|x|$ and $|y| \to \infty$, show that $f(x, y)$ has maxima at $(\pm a, 0)$, minima at $(0, \pm a)$ and a saddle point at the origin.

Sketches of $f(x, 0)$ and $f(0, y)$, whilst hardly necessary, illustrate rather than confirm these conclusions.

5.15 *Find the stationary values of*

$$f(x, y) = 4x^2 + 4y^2 + x^4 - 6x^2y^2 + y^4$$

and classify them as maxima, minima or saddle points. Make a rough sketch of the contours of f in the quarter plane $x, y \geq 0$.

The required derivatives are as follows:

$$\frac{\partial f}{\partial x} = 8x + 4x^3 - 12xy^2, \qquad \frac{\partial f}{\partial y} = 8y - 12x^2y + 4y^3,$$

$$\frac{\partial^2 f}{\partial x^2} = 8 + 12x^2 - 12y^2, \qquad \frac{\partial^2 f}{\partial x \partial y} = -24xy, \qquad \frac{\partial^2 f}{\partial y^2} = 8 - 12x^2 + 12y^2.$$

Figure 5.1 The contours found in exercise 5.15.

Any stationary points must satisfy both of the equations

$$\frac{\partial f}{\partial x} = 4x(2 + x^2 - 3y^2) = 0,$$
$$\frac{\partial f}{\partial y} = 4y(2 - 3x^2 + y^2) = 0.$$

If $x = 0$ then $4y(2 + y^2) = 0$, implying that $y = 0$ also, since $2 + y^2 = 0$ has no real solutions. Conversely, $y = 0$ implies $x = 0$. Further solutions exist if both expressions in parentheses equal zero; this requires $x^2 = y^2 = 1$.

Thus the stationary points are $(0, 0)$, $(1, 1)$, $(-1, 1)$, $(1, -1)$ and $(-1, -1)$, with corresponding values 0, 4, 4, 4 and 4.

At $(0, 0)$, $\partial^2 f/\partial^2 x = \partial^2 f/\partial^2 y = 8$, whilst $\partial^2 f/\partial x \partial y = 0$. Since $0^2 < 8 \times 8$, this point is a minimum.

In the other four cases, $\partial^2 f/\partial^2 x = \partial^2 f/\partial^2 y = 8$, whilst $\partial^2 f/\partial x \partial y = \pm 24$. Since $(24)^2 > 8 \times 8$, these four points are all saddle points.

It will probably be helpful when sketching the contours (figure 5.1) to determine the behaviour of $f(x, y)$ along the line $x = y$ and to note the symmetry about it. In particular, note that $f(x, y) = 0$ at both the origin and the point $(\sqrt{2}, \sqrt{2})$.

5.17 *A rectangular parallelepiped has all eight vertices on the ellipsoid*

$$x^2 + 3y^2 + 3z^2 = 1.$$

Using the symmetry of the parallelepiped about each of the planes $x = 0$, $y = 0$, $z = 0$, write down the surface area of the parallelepiped in terms of the coordinates of the vertex that lies in the octant $x, y, z \geq 0$. Hence find the maximum value of the surface area of such a parallelepiped.

Let S be the surface area and (x, y, z) the coordinates of one of the corners of the parallelepiped with x, y and z all positive. Then we need to maximise $S = 8(xy + yz + zx)$ subject to x, y and z satisfying $x^2 + 3y^2 + 3z^2 = 1$.

Consider

$$f(x, y, z) = 8(xy + yz + zx) + \lambda(x^2 + 3y^2 + 3z^2),$$

where λ is a Lagrange undetermined multipier. Then, setting each of the first partial derivatives separately to zero, we have the simultaneous equations

$$\begin{aligned} 0 &= \frac{\partial f}{\partial x} = 8y + 8z + 2\lambda x, \\ 0 &= \frac{\partial f}{\partial y} = 8x + 8z + 6\lambda y, \\ 0 &= \frac{\partial f}{\partial z} = 8x + 8y + 6\lambda z. \end{aligned}$$

From symmetry, $y = z$, leading to

$$\begin{aligned} 0 &= 16y + 2\lambda x, \\ 0 &= 8x + 8y + 6\lambda y. \end{aligned}$$

Thus, rejecting the trivial solution $x = 0$, $y = 0$, we conclude that $\lambda = -8y/x$, leading to $x^2 + xy - 6y^2 = (x - 2y)(x + 3y) = 0$. The only solution to this quadratic equation with x, y and z all positive is $x = 2y = 2z$. Substituting this into the equation of the ellipse gives

$$(2y)^2 + 3y^2 + 3y^2 = 1 \qquad \Rightarrow \qquad y = \frac{1}{\sqrt{10}}.$$

The value of S is then given by

$$S = 8\left(\frac{2}{10} + \frac{1}{10} + \frac{2}{10}\right) = 4.$$

5.19 *A barn is to be constructed with a uniform cross-sectional area A throughout its length. The cross-section is to be a rectangle of wall height h (fixed) and width w, surmounted by an isosceles triangular roof that makes an angle θ with the horizontal. The cost of construction is α per unit height of wall and β per unit (slope) length of roof. Show that, irrespective of the values of α and β, to minimise costs w should be chosen to satisfy the equation*

$$w^4 = 16A(A - wh),$$

and θ made such that $2\tan 2\theta = w/h$.

The cost *always* includes $2\alpha h$ for the vertical walls, which can therefore be ignored in the minimisation procedure. The rest of the calculation will be solely concerned with minimising the roof area, and the optimum choices for w and θ will be independent of β, the actual cost per unit length of the roof.

The cost of the roof is $2\beta \times \frac{1}{2}w\sec\theta$, but w and θ are constrained by the requirement that

$$A = wh + \frac{1}{2}w\frac{w}{2}\tan\theta.$$

So we consider $G(w, \theta)$, where

$$G(w, \theta) = \beta w\sec\theta - \lambda(wh + \tfrac{1}{4}w^2\tan\theta),$$

and the implications of equating its partial derivatives to zero. The first derivative to be set to zero is

$$\begin{aligned} \frac{\partial G}{\partial \theta} &= \beta w\sec\theta\tan\theta - \frac{\lambda}{4}w^2\sec^2\theta, \\ \Rightarrow \quad 0 &= \beta\sin\theta - \tfrac{1}{4}\lambda w, \\ \Rightarrow \quad \lambda &= \frac{4\beta\sin\theta}{w}. \end{aligned}$$

A second equation is provided by differentiation with respect to w and yields

$$\frac{\partial G}{\partial w} = \beta\sec\theta - \lambda h - \tfrac{1}{2}\lambda w\tan\theta.$$

Setting $\partial G/\partial w = 0$, multiplying through by $\cos\theta$ and substituting for λ, we obtain

$$\begin{aligned} \beta - 2\beta\sin^2\theta &= \frac{4\beta\sin\theta\, h\cos\theta}{w}, \\ w\cos 2\theta &= 2h\sin 2\theta, \\ \tan 2\theta &= \frac{w}{2h}. \end{aligned}$$

This is the second result quoted.

The overall area constraint can be written

$$\tan\theta = \frac{4(A - wh)}{w^2}.$$

From these two results and the double angle formula $\tan 2\phi = 2\tan\phi/(1-\tan^2\phi)$, it follows that

$$\begin{aligned}
\frac{w}{2h} &= \tan 2\theta \\
&= \frac{\dfrac{8(A - wh)}{w^2}}{1 - \dfrac{16(A - wh)^2}{w^4}}, \\
16wh(A - wh) &= w^4 - 16(A - wh)^2, \\
w^4 &= 16A(A - wh).
\end{aligned}$$

This is the first quoted result, and we note that, as expected, both optimum values are independent of β.

5.21 *Find the area of the region covered by points on the lines*

$$\frac{x}{a} + \frac{y}{b} = 1,$$

where the sum of any line's intercepts on the coordinate axes is fixed and equal to c.

The equation of a typical line with intercept a on the x-axis is

$$f(x, y, a) = \frac{x}{a} + \frac{y}{c - a} - 1 = 0.$$

To find the envelope of the lines we set $\partial f/\partial a = 0$. This gives

$$\frac{\partial f}{\partial a} = -\frac{x}{a^2} + \frac{y}{(c - a)^2} = 0.$$

Hence,

$$\begin{aligned}
(c - a)\sqrt{x} &= a\sqrt{y}, \\
a &= \frac{c\sqrt{x}}{\sqrt{x} + \sqrt{y}}.
\end{aligned}$$

Substituting this value into $f(x, y, a) = 0$ gives the equation of the envelope as

$$\begin{aligned}
\frac{x(\sqrt{x} + \sqrt{y})}{c\sqrt{x}} + \frac{y}{c - \frac{c\sqrt{x}}{\sqrt{x}+\sqrt{y}}} &= 1, \\
\sqrt{x}(\sqrt{x} + \sqrt{y}) + \sqrt{y}(\sqrt{x} + \sqrt{y}) &= c, \\
\sqrt{x} + \sqrt{y} &= \sqrt{c}.
\end{aligned}$$

This is a curve (not a straight line) whose end-points are $(c, 0)$ on the x-axis and $(0, c)$ on the y-axis. All points on lines with the given property lie below this envelope curve (except for one point on each line, which lies on the curve). Consequently, the area covered by the points is that bounded by the envelope and the two axes. It has the value

$$\begin{aligned}\int_0^c y\,dx &= \int_0^c (\sqrt{c} - \sqrt{x})^2\,dx \\ &= \int_0^c (c - 2\sqrt{c}\,\sqrt{x} + x)\,dx \\ &= c^2 - \tfrac{4}{3}\sqrt{c}\,c^{3/2} + \tfrac{1}{2}c^2 = \tfrac{1}{6}c^2.\end{aligned}$$

5.23 *A water feature contains a spray head at water level at the centre of a round basin. The head is in the form of a small hemisphere perforated by many evenly distributed small holes, through which water spurts out at the same speed, v_0, in all directions.*

(a) *What is the shape of the 'water bell' so formed?*
(b) *What must be the minimum diameter of the bowl if no water is to be lost?*

The system has cylindrical symmetry and so we work with cylindrical polar coordinates ρ and z.

For a jet of water emerging from the spray head at an angle θ to the vertical, the equations of motion are

$$\begin{aligned} z &= v_0 \cos\theta\, t - \tfrac{1}{2}gt^2, \\ \rho &= v_0 \sin\theta\, t. \end{aligned}$$

Eliminating the time, t, and writing $\cot\theta = \alpha$, we have

$$\begin{aligned} z &= \frac{\rho\, v_0 \cos\theta}{v_0 \sin\theta} - \frac{1}{2}g\frac{\rho^2}{v_0^2 \sin^2\theta}, \\ \Rightarrow \quad 0 &= z - \rho\cot\theta + \frac{g\rho^2}{2v_0^2}\operatorname{cosec}^2\theta, \end{aligned}$$

i.e. the trajectory of this jet is given by

$$f(\rho, z, \alpha) = z - \rho\alpha + \frac{g\rho^2}{2v_0^2}(1 + \alpha^2) = 0.$$

To find the envelope of all these trajectories as θ (and hence α) is varied, we set

$\partial f/\partial\alpha$ equal to zero:

$$0 = \frac{\partial f}{\partial \alpha} = 0 - \rho + \frac{2\alpha g\rho^2}{2v_0^2},$$
$$\Rightarrow \quad \alpha = \frac{v_0^2}{g\rho}.$$

Hence, the equation of the envelope, and thus of the water bell, is

$$g(\rho, z) = z - \frac{v_0^2}{g} + \frac{g\rho^2}{2v_0^2}\left(1 + \frac{v_0^4}{g^2\rho^2}\right) = 0,$$
$$\Rightarrow \quad z = \frac{v_0^2}{2g} - \frac{g\rho^2}{2v_0^2}.$$

(a) This is the equation of a parabola whose apex is at $z = v_0^2/2g$, $\rho = 0$. It follows that the water bell has the shape of an inverted paraboloid of revolution.

(b) When $z = 0$, ρ has the value v_0^2/g, and hence the minimum value needed for the diameter of the bowl is given by $2\rho = 2v_0^2/g$.

5.25 *By considering the differential*

$$dG = d(U + PV - ST),$$

where G is the Gibbs free energy, P the pressure, V the volume, S the entropy and T the temperature of a system, and given further that U, the internal energy, satisfies

$$dU = TdS - PdV,$$

derive a Maxwell relation connecting $(\partial V/\partial T)_P$ and $(\partial S/\partial P)_T$.

Given that $dU = TdS - PdV$, we have that

$$\begin{aligned} dG &= d(U + PV - ST) \\ &= dU + PdV + VdP - S\,dT - TdS \\ &= VdP - S\,dT. \end{aligned}$$

Hence,

$$\left(\frac{\partial G}{\partial P}\right)_T = V \quad \text{and} \quad \left(\frac{\partial G}{\partial T}\right)_P = -S.$$

It follows that

$$\left(\frac{\partial V}{\partial T}\right)_P = \frac{\partial^2 G}{\partial T \partial P} = \frac{\partial^2 G}{\partial P \partial T} = -\left(\frac{\partial S}{\partial P}\right)_T.$$

This is the required Maxwell thermodynamic relation.

5.27 *As implied in exercise 5.25 on the thermodynamics of a simple gas, the quantity $dS = T^{-1}(dU + PdV)$ is an exact differential. Use this to prove that*

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P.$$

In the van der Waals model of a gas, P obeys the equation

$$P = \frac{RT}{V-b} - \frac{a}{V^2},$$

where R, a and b are constants. Further, in the limit $V \to \infty$, the form of U becomes $U = cT$, where c is another constant. Find the complete expression for $U(V, T)$.

Writing the total differentials in $dS = T^{-1}(dU + PdV)$ in terms of partial derivatives with respect to V and T gives

$$T\left(\frac{\partial S}{\partial V}\right)_T dV + T\left(\frac{\partial S}{\partial T}\right)_V dT = \left(\frac{\partial U}{\partial V}\right)_T dV + \left(\frac{\partial U}{\partial T}\right)_V dT + PdV,$$

from which it follows that

$$T\left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial U}{\partial V}\right)_T + P \quad (*) \quad \text{and} \quad T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V.$$

Differentiating the first of these with respect to T and the second with respect to V, and then combining the two equations so obtained, gives

$$\left(\frac{\partial S}{\partial V}\right)_T + T\frac{\partial^2 S}{\partial T \partial V} = \frac{\partial^2 U}{\partial T \partial V} + \left(\frac{\partial P}{\partial T}\right)_V,$$

$$T\frac{\partial^2 S}{\partial V \partial T} = \frac{\partial^2 U}{\partial V \partial T},$$

$$\Rightarrow \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial P}{\partial T}\right)_V.$$

The equation $(*)$ can now be written in the required form:

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{\partial P}{\partial T}\right)_V - P.$$

For the van der Waals model gas,

$$P = \frac{RT}{V-b} - \frac{a}{V^2},$$

and we can substitute for P in the previous result to give

$$\left(\frac{\partial U}{\partial V}\right)_T = T\left(\frac{R}{V-b}\right) - \left(\frac{RT}{V-b} - \frac{a}{V^2}\right) = \frac{a}{V^2},$$

which integrates to

$$U(V,T) = -\frac{a}{V} + f(T).$$

Since $U \to cT$ as $V \to \infty$ for all T, the unknown function, $f(T)$, must be simply $f(T) = cT$. Thus, the full expression for $U(V,T)$ is

$$U(V,T) = cT - \frac{a}{V}.$$

We note that, in the limit $V \to \infty$, van der Waals' equation becomes $PV = RT$ and thus recognise c as the specific heat at constant volume of a perfect gas.

5.29 *By finding dI/dy, evaluate the integral*

$$I(y) = \int_0^\infty \frac{e^{-xy}\sin x}{x}\,dx.$$

Hence show that

$$J = \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

Since the integral is over positive values of x, its convergence requires that $y \geq 0$. We first express the $\sin x$ factor as a complex exponential:

$$\begin{aligned} I(y) &= \int_0^\infty \frac{e^{-xy}\sin x}{x}\,dx \\ &= \operatorname{Im}\int_0^\infty \frac{e^{-xy+ix}}{x}\,dx. \end{aligned}$$

And now differentiate under the integral sign:

$$\begin{aligned} \frac{dI}{dy} &= \operatorname{Im}\int_0^\infty \frac{(-x)e^{-xy+ix}}{x}\,dx \\ &= \operatorname{Im}\left[\frac{-e^{-xy+ix}}{-y+i}\right]_0^\infty \\ &= \operatorname{Im}\left(\frac{1}{-y+i}\right) \\ &= -\frac{1}{1+y^2}. \end{aligned}$$

This differential equation expresses how the integral varies as a function of y.

But, as we can see immediately that for $y = \infty$ the integral must be zero, we can find its value for non-infinite y by integrating the differential equation:

$$I(y) - I(\infty) = \int_{\infty}^{y} \frac{-1}{1+y^2}\,dy = -\tan^{-1} y + \tan^{-1}\infty = \frac{\pi}{2} - \tan^{-1} y.$$

In the limit $y \to 0$ this becomes

$$J = \int_0^{\infty} \frac{\sin x}{x}\,dx = I(0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}.$$

5.31 *The function $f(x)$ is differentiable and $f(0) = 0$. A second function $g(y)$ is defined by*

$$g(y) = \int_0^{y} \frac{f(x)\,dx}{\sqrt{y-x}}.$$

Prove that

$$\frac{dg}{dy} = \int_0^{y} \frac{df}{dx}\frac{dx}{\sqrt{y-x}}.$$

For the case $f(x) = x^n$, prove that

$$\frac{d^n g}{dy^n} = 2(n!)\sqrt{y}.$$

Integrating the definition of $g(y)$ by parts:

$$\begin{aligned} g(y) &= \int_0^{y} \frac{f(x)\,dx}{\sqrt{y-x}} \\ &= \left[-2f(x)\sqrt{y-x}\right]_0^y + \int_0^y 2\frac{df}{dx}\sqrt{y-x}\,dx \\ &= 2\int_0^y \frac{df}{dx}\sqrt{y-x}, \end{aligned}$$

where we have used $f(0) = 0$ in setting the definite integral to zero.

Now, differentiating $g(y)$ with respect to both its upper limit and its integrand, we obtain

$$\frac{dg}{dy} = 2\frac{df}{dx}\sqrt{y-y} + 2\int_0^y \frac{1}{2}\frac{df}{dx}\frac{1}{\sqrt{y-x}} = \int_0^y \frac{df}{dx}\frac{1}{\sqrt{y-x}}.$$

This result, showing that the construction of the derivative of g from the derivative of f is the same as that of g from f, applies to any function that satisfies $f(0) = 0$

and so applies to x^n and all of its derivatives. It follows that

$$\begin{aligned}\frac{d^n g}{dy^n} &= \int_0^y \frac{d^n f}{dx^n}\frac{1}{\sqrt{y-x}}\,dx \\ &= \int_0^y \frac{n!}{\sqrt{y-x}}\,dx \\ &= \left[\frac{n!(-1)\sqrt{y-x}}{\frac{1}{2}}\right]_0^y \\ &= 2(n!)\sqrt{y}.\end{aligned}$$

5.33 *If*

$$I(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\ln x}\,dx, \qquad \alpha > -1,$$

what is the value of $I(0)$*? Show that*

$$\frac{d}{d\alpha}x^\alpha = x^\alpha \ln x,$$

and deduce that

$$\frac{d}{d\alpha}I(\alpha) = \frac{1}{\alpha+1}.$$

Hence prove that $I(\alpha) = \ln(1+\alpha)$.

Since the integrand is singular at $x = 1$, we need to define $I(0)$ as a limit:

$$I(0) = \lim_{y\to 1}\int_0^y \frac{x^0 - 1}{\ln x}\,dx = \lim_{y\to 1}\int_0^y 0\,dx = \lim_{y\to 1} 0 = 0,$$

i.e. $I(0) = 0$.

With $z = x^\alpha$, we have

$$\ln z = \alpha \ln x \quad\Rightarrow\quad \frac{1}{z}\frac{dz}{d\alpha} = \ln x$$

$$\Rightarrow\quad \frac{dz}{d\alpha} = z\ln x \quad\Rightarrow\quad \frac{d}{d\alpha}x^\alpha = x^\alpha \ln x.$$

The derivative of $I(\alpha)$ is then

$$\begin{aligned}\frac{dI}{d\alpha} &= \int_0^1 \frac{1}{\ln x}\,x^\alpha \ln x\,dx \\ &= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 \\ &= \frac{1}{\alpha+1}.\end{aligned}$$

Finally, intergation gives

$$I(\alpha) - I(0) = \int_0^\alpha \frac{d\beta}{\beta + 1},$$
$$I(\alpha) - 0 = \ln(1 + \alpha).$$

To obtain this final line we have used our first result that $I(0) = 0$.

5.35 *The function $G(t, \xi)$ is defined for $0 \le t \le \pi$ by*

$$G(t, \xi) = -\cos t \sin \xi \quad \text{for } \xi \le t,$$
$$= -\sin t \cos \xi \quad \text{for } \xi > t.$$

Show that the function $x(t)$ defined by

$$x(t) = \int_0^\pi G(t, \xi) f(\xi)\, d\xi$$

satisfies the equation

$$\frac{d^2x}{dt^2} + x = f(t),$$

where $f(t)$ can be any *arbitrary (continuous) function. Show further that $x(0) = [dx/dt]_{t=\pi} = 0$, again for any $f(t)$, but that the* value *of $x(\pi)$ does depend upon the form of $f(t)$.*

[*The function $G(t, \xi)$ is an example of a Green's function, an important concept in the solution of differential equations.*]

The explicit integral expression for $x(t)$ is

$$x(t) = \int_0^\pi G(t, \xi) f(\xi)\, d\xi$$
$$= -\int_0^t \cos t \, \sin \xi \, f(\xi)\, d\xi - \int_t^\pi \sin t \, \cos \xi \, f(\xi)\, d\xi.$$

We now form its first two derivatives using Leibnitz' rule:

$$\frac{dx}{dt} = -\cos t[\sin t \, f(t)] + \sin t \int_0^t \sin \xi \, f(\xi)\, d\xi$$
$$+ \sin t[\cos t \, f(t)] - \cos t \int_t^\pi \cos \xi \, f(\xi)\, d\xi$$
$$= \sin t \int_0^t \sin \xi \, f(\xi)\, d\xi - \cos t \int_t^\pi \cos \xi \, f(\xi)\, d\xi.$$

$$\begin{aligned}\frac{d^2x}{dt^2} &= \cos t \int_0^t \sin\xi\, f(\xi)\, d\xi + \sin t[\sin t\, f(t)] \\ &\qquad + \sin t \int_t^\pi \cos\xi\, f(\xi)\, d\xi + \cos t[\cos t\, f(t)] \\ &= -x(t) + f(t)(\sin^2 t + \cos^2 t).\end{aligned}$$

This shows that

$$\frac{d^2x}{dt^2} + x = f(t)$$

for *any* continuous function $f(x)$.

When $t = 0$ the first integral in the expression for $x(t)$ has zero range and the second is multiplied by $\sin 0$; consequently $x(0) = 0$.

When $t = \pi$ the second integral in the expression for dx/dt has zero range and the first is multiplied by $\sin\pi$; consequently $[dx/dt]_{t=\pi} = 0$.

However, when $t = \pi$, although the second integral in the expression for $x(t)$ is multiplied by $\sin\pi$ and contributes nothing, the first integral is not zero in general and its value *will* depend upon the form of $f(t)$.

6

Multiple integrals

6.1 *Identify the curved wedge bounded by the surfaces $y^2 = 4ax$, $x + z = a$ and $z = 0$, and hence calculate its volume V.*

As will readily be seen from a rough sketch, the wedge consists of that part of a parabolic cylinder, parallel to the z-axis, that is cut off by two planes, one parallel to the y-axis and the other the coordinate plane $z = 0$.

For the first stage of the multiple integration, the volume can be divided equally easily into 'vertical columns' or into horizontal strips parallel to the y-axis. Thus there are two equivalent and equally obvious ways of proceeding.

Either

$$\begin{aligned} V &= \int_0^a dx \int_{-\sqrt{4ax}}^{\sqrt{4ax}} dy \int_0^{a-x} dz \\ &= \int_0^a 2\sqrt{4ax}(a-x)\,dx \\ &= 4\sqrt{a}\left[\tfrac{2}{3}ax^{3/2} - \tfrac{2}{5}x^{5/2}\right]_0^a = \tfrac{16}{15}a^3; \end{aligned}$$

or

$$\begin{aligned} V &= \int_0^a dz \int_0^{a-z} dx \int_{-\sqrt{4ax}}^{\sqrt{4ax}} dy \\ &= \int_0^a dz \int_0^{a-z} 2\sqrt{4ax}\,dx \\ &= 4\sqrt{a}\int \tfrac{2}{3}(a-z)^{3/2}\,dz \\ &= \frac{8\sqrt{a}}{3}\left[-\tfrac{2}{5}(a-z)^{5/2}\right]_0^a = \tfrac{16}{15}a^3. \end{aligned}$$

6.3 *Find the volume integral of x^2y over the tetrahedral volume bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.*

The bounding surfaces of the integration volume are symmetric in x, y and z and, on these grounds, there is nothing to choose between the various possible orders of integration.

However, the integrand does not contain z and so there is some advantage in carrying out the z-integration first. Its value can simply be set equal to the length of the z-interval and the dimension of the integral will have been reduced by one 'at a stroke'.

$$\begin{aligned}
I &= \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} x^2 y\, dz \\
&= \int_0^1 dx \int_0^{1-x} x^2 y(1-x-y)\, dy \\
&= \int_0^1 \left[x^2(1-x)\frac{(1-x)^2}{2} - x^2\frac{(1-x)^3}{3} \right] dx \\
&= \frac{1}{6}\int_0^1 x^2(1-3x+3x^2-x^3)\, dx \\
&= \frac{1}{6}\left(\frac{1}{3} - \frac{3}{4} + \frac{3}{5} - \frac{1}{6}\right) \\
&= \frac{1}{6}\,\frac{20-45+36-10}{60} = \frac{1}{360}.
\end{aligned}$$

6.5 *Calculate the volume of an ellipsoid as follows:*

(a) *Prove that the area of the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is πab.

(b) *Use this result to obtain an expression for the volume of a slice of thickness dz of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Hence show that the volume of the ellipsoid is $4\pi abc/3$.

(a) Dividing the ellipse into thin strips parallel to the y-axis, we may write its

area as

$$\text{area} = 2\int_{-a}^{a} y\,dx = 2\int_{-a}^{a} b\sqrt{1-\left(\frac{x}{a}\right)^2}\,dx.$$

Set $x = a\cos\phi$ with $dx = -a\sin\phi\,d\phi$. Then

$$\text{area} = 2b\int_{\pi}^{0} \sin\phi(-a\sin\phi)\,d\phi = 2ab\int_{0}^{\pi} \sin^2\phi\,d\phi = 2ab\,\frac{\pi}{2} = \pi ab.$$

(b) Consider slices of the ellipsoid, of thickness dz, taken perpendicular to the z-axis. Each is an ellipse whose bounding curve is given by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 - \frac{z^2}{c^2}$$

and is thus a scaled-down version of the ellipse considered in part (a) with semi-axes $a(1-(z/c)^2)^{1/2}$ and $b(1-(z/c)^2)^{1/2}$. Its area is therefore $\pi a(1-(z/c)^2)^{1/2}b(1-(z/c)^2)^{1/2}$ and its volume dV is this multiplied by dz. Thus, the total volume V of the ellipsoid is given by

$$\int_{-c}^{c} \pi ab\left(1-\frac{z^2}{c^2}\right)dz = \pi ab\left[z - \frac{1}{3}\frac{z^3}{c^2}\right]_{-c}^{c} = \frac{4\pi abc}{3}.$$

6.7 *In quantum mechanics the electron in a hydrogen atom in some particular state is described by a wavefunction Ψ, which is such that $|\Psi|^2\,dV$ is the probability of finding the electron in the infinitesimal volume dV. In spherical polar coordinates $\Psi = \Psi(r,\theta,\phi)$ and $dV = r^2\sin\theta\,dr\,d\theta\,d\phi$. Two such states are described by*

$$\Psi_1 = \left(\frac{1}{4\pi}\right)^{1/2}\left(\frac{1}{a_0}\right)^{3/2} 2e^{-r/a_0},$$

$$\Psi_2 = -\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta\; e^{i\phi}\left(\frac{1}{2a_0}\right)^{3/2}\frac{re^{-r/2a_0}}{a_0\sqrt{3}}.$$

(a) *Show that each Ψ_i is normalised, i.e. the integral over all space $\int|\Psi|^2\,dV$ is equal to unity – physically, this means that the electron must be somewhere.*

(b) *The (so-called) dipole matrix element between the states 1 and 2 is given by the integral*

$$p_x = \int \Psi_1^* qr\sin\theta\cos\phi\;\Psi_2\,dV,$$

where q is the charge on the electron. Prove that p_x has the value $-2^7qa_0/3^5$.

We need to show that the volume integral of $|\Psi_i|^2$ is equal to unity, and begin

by noting that, since ϕ is not explicitly mentioned, or appears only in the form $e^{i\phi}$, the ϕ integration of $|\Psi|^2$ yields a factor of 2π in each case. For Ψ_1 we have

$$\begin{aligned}\int |\Psi_1|^2\, dV &= \int |\Psi_1|^2 r^2 \sin\theta\, d\theta\, d\phi\, dr \\ &= \frac{1}{4\pi}\frac{4}{a_0^3}\, 2\pi \int_0^\infty r^2 e^{-2r/a_0}\, dr \int_0^\pi \sin\theta\, d\theta \\ &= \frac{2}{a_0^3}\int_0^\infty 2r^2 e^{-2r/a_0}\, dr \\ &= \frac{4}{a_0^3}\,\frac{a_0}{2}\, 2\,\frac{a_0}{2}\, 1\,\frac{a_0}{2} = 1.\end{aligned}$$

The last line has been obtained using repeated integration by parts.

For Ψ_2, the corresponding calculation is

$$\begin{aligned}\int |\Psi_2|^2\, dV &= \int |\Psi_2|^2 r^2 \sin\theta\, d\theta\, d\phi\, dr \\ &= \frac{2\pi}{64\pi\, a_0^5}\int_0^\infty r^4 e^{-r/a_0}\, dr \int_0^\pi \sin^3\theta\, d\theta \\ &= \frac{1}{32\, a_0^5}\int_0^\infty r^4 e^{-r/a_0}\, dr \int_0^\pi (1-\cos^2\theta)\sin\theta\, d\theta \\ &= \frac{1}{32\, a_0^5}\, 4!\, a_0^5 \left(2-\frac{2}{3}\right) = 1.\end{aligned}$$

Again, the r-integral was calculated using integration by parts. In summary, both functions are correctly normalised.

(b) The dipole matrix element has important physical properties, but for the purposes of this exercise it is simply an integral to be evaluated according to a formula, as follows:

$$\begin{aligned}p_x &= \int \Psi_1^* q r \sin\theta\cos\phi\ \Psi_2\, r^2 \sin\theta\, d\theta\, d\phi\, dr \\ &= \frac{-q}{8\pi a_0^4}\int_0^\pi \sin^3\theta\, d\theta \int_0^{2\pi} \cos\phi(\cos\phi + i\sin\phi)\, d\phi \int_0^\infty r^4 e^{-3r/2a_0}\, dr \\ &= -\frac{q}{8\pi a_0^4}\left(2-\frac{2}{3}\right)(\pi + i0)\, 4!\left(\frac{2a_0}{3}\right)^5 \\ &= -\frac{2^7}{3^5} q a_0.\end{aligned}$$

6.9 *A certain torus has a circular vertical cross-section of radius a centred on a horizontal circle of radius c* ($> a$).

(a) *Find the volume V and surface area A of the torus, and show that they can be written as*

$$V = \frac{\pi^2}{4}(r_o^2 - r_i^2)(r_o - r_i), \qquad A = \pi^2(r_o^2 - r_i^2),$$

where r_o *and* r_i *are, respectively, the outer and inner radii of the torus.*

(b) *Show that a vertical circular cylinder of radius c, coaxial with the torus, divides A in the ratio*

$$\pi c + 2a \ : \ \pi c - 2a.$$

(a) The inner and outer radii of the torus are $r_i = c - a$ and $r_o = c + a$, from which it follows that $r_o^2 - r_i^2 = 4ac$ and that $r_o - r_i = 2a$.

The torus is generated by sweeping the centre of a circle of radius a, area πa^2 and circumference $2\pi a$ around a circle of radius c. Therefore, by Pappus' first theorem, the volume of the torus is given by

$$V = \pi a^2 \times 2\pi c = 2\pi^2 a^2 c = \frac{\pi^2}{4}(r_o^2 - r_i^2)(r_o - r_i),$$

whilst, by his second theorem, its surface area is

$$A = 2\pi a \times 2\pi c = 4\pi^2 ac = \pi^2(r_o^2 - r_i^2).$$

(b) The vertical cylinder divides the perimeter of a cross-section of the torus into two equal parts. The distance from the cylinder of the centroid of either half is given by

$$\bar{x} = \frac{\int x\, ds}{\int ds} = \frac{\int_{-\pi/2}^{\pi/2} a\cos\phi \; a\, d\phi}{\int_{-\pi/2}^{\pi/2} a\, d\phi} = \frac{2a}{\pi}.$$

It therefore follows from Pappus' second theorem that

$$A_o = \pi a \times 2\pi\left(c + \frac{2a}{\pi}\right) \text{ and } A_i = \pi a \times 2\pi\left(c - \frac{2a}{\pi}\right),$$

leading to the stated result.

6.11 *In some applications in mechanics the moment of inertia of a body about a single point (as opposed to about an axis) is needed. The moment of inertia, I, about the origin of a uniform solid body of density ρ is given by the volume integral*

$$I = \int_V (x^2 + y^2 + z^2)\rho \, dV.$$

Show that the moment of inertia of a right circular cylinder of radius a, length $2b$ and mass M about its centre is given by

$$M\left(\frac{a^2}{2} + \frac{b^2}{3}\right).$$

Since the cylinder is easily described in cylindrical polar coordinates (ρ, ϕ, z), we convert the calculation to one using those coordinates and denote the density by ρ_0 to avoid confusion:

$$\begin{aligned} I &= \int_V (x^2 + y^2 + z^2)\rho_0 \, dV \\ &= \rho_0 \int_V (\rho^2 + z^2)\rho \, d\phi \, d\phi \, dz \\ &= \rho_0 \int_0^{2\pi} d\phi \int_0^a \rho \, d\rho \int_{-b}^b (\rho^2 + z^2) \, dz \\ &= 2\pi\rho_0 \int_0^a \rho \left(2b\rho^2 + \frac{2b^3}{3}\right) d\rho \\ &= 2\pi\rho_0 \left(2b\frac{a^4}{4} + \frac{2b^3}{3}\frac{a^2}{2}\right). \end{aligned}$$

Now $M = \pi a^2 \times 2b \times \rho_0$, and so the moment of inertia about the origin can be expressed as

$$I = M\left(\frac{a^2}{2} + \frac{b^2}{3}\right).$$

6.13 *In spherical polar coordinates r, θ, ϕ the element of volume for a body that is symmetrical about the polar axis is $dV = 2\pi r^2 \sin\theta \, dr \, d\theta$, whilst its element of surface area is $2\pi r \sin\theta[(dr)^2 + r^2(d\theta)^2]^{1/2}$. A particular surface is defined by $r = 2a\cos\theta$, where a is a constant and $0 \le \theta \le \pi/2$. Find its total surface area and the volume it encloses, and hence identify the surface.*

With the surface of the body defined by $r = 2a\cos\theta$, for calculating its total volume the radial integration variable r' lies in the range $0 \le r' \le 2a\cos\theta$. Hence

$$\begin{aligned} V &= \int_0^{\pi/2} 2\pi \sin\theta \, d\theta \int_0^{2a\cos\theta} r'^2 \, dr' \\ &= 2\pi \int_0^{\pi/2} \sin\theta \frac{(2a\cos\theta)^3}{3} \, d\theta \\ &= \frac{16\pi a^3}{3} \int_0^{\pi/2} \cos^3\theta \sin\theta \, d\theta \\ &= \frac{16\pi a^3}{3} \left[-\frac{\cos^4\theta}{4} \right]_0^{\pi/2} \\ &= \tfrac{4}{3}\pi a^3. \end{aligned}$$

The additional strip of surface area resulting from a change from θ to $\theta + d\theta$ is $2\pi r \sin\theta \, d\ell$, where $d\ell$ is the length of the generating curve that lies in this infinitesimal range of θ. This is given by

$$\begin{aligned} (d\ell)^2 &= (dr)^2 + (r\,d\theta)^2 \\ &= (-2a\sin\theta\,d\theta)^2 + (2a\cos\theta\,d\theta)^2 \\ &= 4a^2\,(d\theta)^2 \end{aligned}$$

The integral becomes one-dimensional with

$$\begin{aligned} S &= 2\pi \int_0^{\pi/2} 2a\cos\theta \sin\theta \, 2a \, d\theta \\ &= 8\pi a^2 \left[\frac{\sin^2\theta}{2} \right]_0^{\pi/2} \\ &= 4\pi a^2. \end{aligned}$$

With a volume of $\frac{4}{3}\pi a^3$ and a surface area of $4\pi a^2$, the surface is probably that of a sphere of radius a, with the origin at the 'lowest' point of the sphere. This conclusion is confirmed by the fact that the triangle formed by the two ends of the vertical diameter of the sphere and any point on its surface is a right-angled triangle in which $r/2a = \cos\theta$.

6.15 *By transforming to cylindrical polar coordinates, evaluate the integral*

$$I = \int\int\int \ln(x^2 + y^2)\, dx\, dy\, dz$$

over the interior of the conical region $x^2 + y^2 \le z^2$, $0 \le z \le 1$.

The volume element $dx\,dy\,dz$ becomes $\rho\,d\rho\,d\phi\,dz$ in cylindrical polar coordinates

and the integrand contains a factor $\rho \ln \rho^2 = 2\rho \ln \rho$. This is dealt with using integration by parts and the integral becomes

$$\begin{aligned}
I &= \int\int\int 2\rho \ \ln \rho \, d\rho \, d\phi \, dz \quad \text{over} \quad \rho \le z, \ 0 \le z \le 1, \\
&= 2 \int_0^{2\pi} d\phi \int_0^1 dz \int_0^z \rho \ln \rho \, d\rho \\
&= 2 \ 2\pi \int_0^1 \left(\left[\frac{\rho^2 \ln \rho}{2} \right]_0^z - \int_0^z \frac{1}{\rho} \frac{\rho^2}{2} \, d\rho \right) dz \\
&= 4\pi \int_0^1 \left(\frac{1}{2} z^2 \ln z - \frac{1}{4} z^2 \right) dz \\
&= 2\pi \left(\left[\frac{z^3 \ln z}{3} \right]_0^1 - \int_0^1 \frac{1}{z} \frac{z^3}{3} \, dz \right) - \pi \left[\frac{z^3}{3} \right]_0^1 \\
&= 2\pi \left(0 - \left[\frac{z^3}{9} \right]_0^1 \right) - \frac{\pi}{3} \\
&= -\frac{2\pi}{9} - \frac{\pi}{3} = -\frac{5\pi}{9}.
\end{aligned}$$

Although the integrand contains no explicit minus signs, a negative value for the integral is to be expected, since $1 \ge z^2 \ge x^2 + y^2$ and $\ln(x^2 + y^2)$ is therefore negative.

6.17 *By making two successive simple changes of variables, evaluate*

$$I = \int\int\int x^2 \, dx \, dy \, dz$$

over the ellipsoidal region

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$$

We start by making a scaling change aimed at producing an integration volume that has more amenable properties than an ellipsoid, namely a sphere. To do this, set $\xi = x/a$, $\eta = y/b$ and $\zeta = z/c$; the integral then becomes

$$\begin{aligned}
I &= \int\int\int a^2 \xi^2 \, a \, d\xi \, b \, d\eta \, c \, d\zeta \quad \text{over} \quad \xi^2 + \eta^2 + \zeta^2 \le 1 \\
&= a^3 bc \int\int\int \xi^2 \, d\xi \, d\eta \, d\zeta.
\end{aligned}$$

With the integration volume now a sphere it is sensible to change to spherical polar variables: $\xi = r \cos\theta$, $\eta = r \sin\theta \cos\phi$ and $\zeta = r \sin\theta \sin\phi$, with volume

Figure 6.1 The integration area for exercise 6.19.

element $d\xi\, d\eta\, d\zeta = r^2 \sin\theta\, dr\, d\theta\, d\phi$. Note that we have chosen to orientate the polar axis along the old x-axis, rather than along the more conventional z-axis.

$$\begin{aligned} I &= a^3bc \int_0^{2\pi} d\phi \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta \int_0^1 r^4\, dr \\ &= a^3bc\, 2\pi\, \frac{2}{3}\, \frac{1}{5} \\ &= \tfrac{4}{15}\pi a^3 bc. \end{aligned}$$

6.19 *Sketch that part of the region* $0 \le x,\ 0 \le y \le \pi/2$ *which is bounded by the curves* $x = 0,\ y = 0,\ \sinh x \cos y = 1$ *and* $\cosh x \sin y = 1$. *By making a suitable change of variables, evaluate the integral*

$$I = \int\int (\sinh^2 x + \cos^2 y) \sinh 2x \sin 2y\, dx\, dy$$

over the bounded subregion.

The integration area is shaded in figure 6.1. We are guided in making a choice of new variables by the equations defining the 'awkward' parts of the subregion's boundary curve. Ideally, the new variables should each be constant along one or more of the curves making up the boundary. This consideration leads us to make a change to new variables, $u = \sinh x \cos y$ and $v = \cosh x \sin y$. We then find the following.

(i) The boundary $y = 0$ becomes $v = 0$.
(ii) The boundary $x = 0$ becomes $u = 0$.
(iii) The boundary $\sinh x \cos y = 1$ becomes $u = 1$.
(iv) The boundary $\cosh x \sin y = 1$ becomes $v = 1$.

With this choice for the change, all four parts of the boundary can be characterised as being lines along which one of the coordinates is constant.

The Jacobian relating $dx\,dy$ to $du\,dv$, i.e. $du\,dv = \dfrac{\partial(u,v)}{\partial(x,y)}\,dx\,dy$, is

$$\begin{aligned}\frac{\partial(u,v)}{\partial(x,y)} &= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\\ &= (\cosh x\cos y)(\cosh x\cos y) - (-\sinh x\sin y)(\sinh x\sin y)\\ &= (\sinh^2 x + 1)\cos^2 y + \sinh^2 x\sin^2 y\\ &= \sinh^2 x + \cos^2 y.\end{aligned}$$

The Jacobian required for the change of variables in the current case is the inverse of this.

Making the change of variables, and recalling that $\sin 2z = 2\sin z\cos z$, and similarly for $\sinh 2z$, gives

$$\begin{aligned}I &= \int\int(\sinh^2 x + \cos^2 y)\sinh 2x\sin 2y\,dx\,dy\\ &= \int_0^1\int_0^1(\sinh^2 x + \cos^2 y)\,(4uv)\,\frac{du\,dv}{\sinh^2 x + \cos^2 y}\\ &= 4\int_0^1 u\,du\int_0^1 v\,dv\\ &= 4\left[\frac{u^2}{2}\right]_0^1\left[\frac{v^2}{2}\right]_0^1 = 1.\end{aligned}$$

This is the simple answer to a superficially difficult integral!

6.21 *As stated in some of the exercises in chapter 5, the first law of thermodynamics can be expressed as*

$$dU = TdS - PdV.$$

By calculating and equating $\partial^2U/\partial Y\partial X$ and $\partial^2U/\partial X\partial Y$, where X and Y are an unspecified pair of variables (drawn from P, V, T and S), prove that

$$\frac{\partial(S,T)}{\partial(X,Y)} = \frac{\partial(V,P)}{\partial(X,Y)}.$$

Using the properties of Jacobians, deduce that

$$\frac{\partial(S,T)}{\partial(V,P)} = 1.$$

Starting from

$$dU = TdS - PdV,$$

the partial derivatives of U with respect to X and Y are

$$\frac{\partial U}{\partial X} = T\frac{\partial S}{\partial X} - P\frac{\partial V}{\partial X} \quad \text{and} \quad \frac{\partial U}{\partial Y} = T\frac{\partial S}{\partial Y} - P\frac{\partial V}{\partial Y}.$$

We next differentiate these two expressions to obtain two (equal) second derivatives. Note that, since X and Y can be any pair drawn from P, V, T and S, we must differentiate all four terms on the RHS as products, giving rise to two terms each. The resulting equations are

$$\frac{\partial^2 U}{\partial Y \partial X} = T\frac{\partial^2 S}{\partial Y \partial X} + \frac{\partial T}{\partial Y}\frac{\partial S}{\partial X} - P\frac{\partial^2 V}{\partial Y \partial X} - \frac{\partial P}{\partial Y}\frac{\partial V}{\partial X},$$
$$\frac{\partial^2 U}{\partial X \partial Y} = T\frac{\partial^2 S}{\partial X \partial Y} + \frac{\partial T}{\partial X}\frac{\partial S}{\partial Y} - P\frac{\partial^2 V}{\partial X \partial Y} - \frac{\partial P}{\partial X}\frac{\partial V}{\partial Y}.$$

Equating the two expressions, and then cancelling the terms that appear on both side of the equality, yields

$$\begin{aligned} & \frac{\partial T}{\partial Y}\frac{\partial S}{\partial X} - \frac{\partial P}{\partial Y}\frac{\partial V}{\partial X} = \frac{\partial T}{\partial X}\frac{\partial S}{\partial Y} - \frac{\partial P}{\partial X}\frac{\partial V}{\partial Y}, \\ \Rightarrow \quad & \frac{\partial T}{\partial Y}\frac{\partial S}{\partial X} - \frac{\partial T}{\partial X}\frac{\partial S}{\partial Y} = \frac{\partial P}{\partial Y}\frac{\partial V}{\partial X} - \frac{\partial P}{\partial X}\frac{\partial V}{\partial Y}, \\ \Rightarrow \quad & \frac{\partial(S,T)}{\partial(X,Y)} = \frac{\partial(V,P)}{\partial(X,Y)}. \end{aligned}$$

Now, using this result and the properties of Jacobians ($J_{pr} = J_{pq}J_{qr}$ and $J_{pq} = [J_{qp}]^{-1}$), we can write

$$\begin{aligned} \frac{\partial(S,T)}{\partial(V,P)} &= \frac{\partial(S,T)}{\partial(X,Y)}\frac{\partial(X,Y)}{\partial(V,P)} \\ &= \frac{\partial(S,T)}{\partial(X,Y)}\left[\frac{\partial(V,P)}{\partial(X,Y)}\right]^{-1} \\ &= \frac{\partial(S,T)}{\partial(X,Y)}\left[\frac{\partial(S,T)}{\partial(X,Y)}\right]^{-1} \\ &= 1. \end{aligned}$$

6.23 *This is a more difficult question about 'volumes' in an increasing number of dimensions.*

(a) *Let R be a real positive number and define K_m by*

$$K_m = \int_{-R}^{R} \left(R^2 - x^2\right)^m \, dx.$$

Show, using integration by parts, that K_m satisfies the recurrence relation

$$(2m+1)K_m = 2mR^2 K_{m-1}.$$

(b) *For integer n, define $I_n = K_n$ and $J_n = K_{n+1/2}$. Evaluate I_0 and J_0 directly and hence prove that*

$$I_n = \frac{2^{2n+1}(n!)^2 R^{2n+1}}{(2n+1)!} \qquad \text{and} \qquad J_n = \frac{\pi(2n+1)!R^{2n+2}}{2^{2n+1}n!(n+1)!}.$$

(c) *A sequence of functions $V_n(R)$ is defined by*

$$V_0(R) = 1,$$
$$V_n(R) = \int_{-R}^{R} V_{n-1}\left(\sqrt{R^2 - x^2}\right) dx, \qquad n \geq 1.$$

Prove by induction that

$$V_{2n}(R) = \frac{\pi^n R^{2n}}{n!}, \qquad V_{2n+1}(R) = \frac{\pi^n 2^{2n+1} n! R^{2n+1}}{(2n+1)!}.$$

(d) *For interest,*

(i) *show that $V_{2n+2}(1) < V_{2n}(1)$ and $V_{2n+1}(1) < V_{2n-1}(1)$ for all $n \geq 3$;*

(ii) *hence, by explicitly writing out $V_k(R)$ for $1 \leq k \leq 8$ (say), show that the 'volume' of the totally symmetric solid of unit radius is a maximum in five dimensions.*

(a) Taking the second factor in the integrand to be unity and integrating by parts, we have

$$\begin{aligned} K_m &= \int_{-R}^{R} \left(R^2 - x^2\right)^m dx \\ &= \left[x(R^2 - x^2)^m\right]_{-R}^{R} - \int_{-R}^{R} mx(R^2 - x^2)^{m-1}(-2x)\, dx \\ &= 0 + 2m \int_{-R}^{R} (R^2 - x^2)^{m-1}(x^2 - R^2 + R^2)\, dx \\ &= -2mK_m + 2mR^2 K_{m-1}, \end{aligned}$$

i.e. $\qquad (2m+1)K_m = 2mR^2 K_{m-1}. \qquad (*)$

(b) With $I_n = K_n$ and $J_N = K_{n+1/2}$,

$$\begin{aligned} I_0 &= \int_{-R}^{R} 1\,dx = 2R \quad \text{and} \\ J_0 &= \int_{-R}^{R} (R^2 - x^2)^{1/2}\,dx, \qquad (\text{now set } x = R\cos\theta) \\ &= \int_0^{\pi} R^2 \sin\theta \sin\theta\,d\theta \\ &= \tfrac{1}{2}\pi R^2. \end{aligned}$$

Using the recurrence relation (*) then gives

$$\begin{aligned} I_n &= \frac{2n}{2n+1}\,\frac{2n-2}{2n-1}\cdots\frac{2}{3}\,R^{2n}\,I_0 \\ &= \frac{2^{n+1}\,n!\,(2^n\,n!)}{(2n+1)!}R^{2n+1} \\ &= \frac{2^{2n+1}(n!)^2 R^{2n+1}}{(2n+1)!}. \end{aligned}$$

Here, and below, we have written $(2n+1)(2n-1)\cdots 3$ in the form $(2n+1)!/(2^n\,n!)$. For J_n the corresponding calculation is

$$\begin{aligned} J_n &= \frac{2n+1}{2n+2}\,\frac{2n-1}{2n}\cdots\frac{3}{4}\,R^{2n}\,J_0 \\ &= \frac{(2n+1)!}{(2^{n+1}/2)(n+1)!}\,\frac{R^{2n}}{(2^n\,n!)}\,\frac{\pi R^2}{2} \\ &= \frac{\pi\,(2n+1)!\,R^{2n+2}}{2^{2n+1}\,n!\,(n+1)!}. \end{aligned}$$

(c) This is the most difficult part of the question as, although we proceed by induction on n, the general form of the expression for $n = N + 1$ is not the same as that for $n = N$. In fact it is the same as that for $n = N - 1$. Thus we will find two interleaving series of forms and have to prove the induction procedure for even and odd values of N separately. We start by assuming that

$$V_{2n}(R) = \frac{\pi^n R^{2n}}{n!}, \qquad V_{2n+1}(R) = \frac{\pi^n 2^{2n+1} n! R^{2n+1}}{(2n+1)!}.$$

For $n = 0$, the second expression gives $V_1(R) = (\pi^0\,2\;0!\,R)/1! = 2R$, whilst, for $n = 1$, the first gives $V_2(R) = \pi^1 R^2/1! = \pi R^2$; both of these are clearly valid.

Now, taking $n = 2N$, we compute $V_{2N+1}(R)$ from $V_{2N}(R)$ as

$$\begin{aligned} V_{2N+1}(R) &= \int_{-R}^{R} V_{2N}(\sqrt{R^2 - x^2})\, dx \\ &= \int_{-R}^{R} \frac{\pi^N}{N!}(R^2 - x^2)^{2N/2}\, dx \\ &= \frac{\pi^N}{N!} I_N \\ &= \frac{\pi^N\, 2^{2N+1}(N!)^2 R^{2N+1}}{N!\,(2N+1)!}, \end{aligned}$$

i.e. in agreement with the assumption about $V_{2n+1}(R)$.

Next, taking $n = 2N + 1$ we compute $V_{2N+2}(R)$ from $V_{2N+1}(R)$ as

$$\begin{aligned} V_{2N+2}(R) &= \int_{-R}^{R} V_{2N+1}(\sqrt{R^2 - x^2})\, dx \\ &= \frac{\pi^N 2^{2N+1} N!}{(2N+1)!} \int_{-R}^{R} (\sqrt{R^2 - x^2})^{2N+1}\, dx \\ &= \frac{\pi^N 2^{2N+1} N!}{(2N+1)!} J_N \\ &= \frac{\pi^N 2^{2N+1} N!}{(2N+1)!}\, \frac{\pi\,(2N+1)!\, R^{2N+2}}{2^{2N+1}\, N!\,(N+1)!} \\ &= \frac{\pi^{N+1} R^{2N+2}}{(N+1)!}, \end{aligned}$$

i.e. in agreement with the assumption about $V_{2n}(R)$.

Thus the two definitions generate each other consistently and, as has been shown, are directly verifiable for $N = 1$ and $N = 2$. This completes the proof.

(d)(i) Using the formulae just proved

$$\frac{V_{2n+2}(1)}{V_{2n}(1)} = \frac{\pi^{n+1} n!}{(n+1)!\,\pi^n} = \frac{\pi}{n+1} < 1 \quad \text{for} \quad n \geq 3,$$

$$\begin{aligned} \frac{V_{2n+1}(1)}{V_{2n-1}(1)} &= \frac{\pi^n\, 2^{2n+1}\, n!}{(2n+1)!}\, \frac{(2n-1)!}{\pi^{n-1} 2^{2n-1}(n-1)!} \\ &= \frac{2\pi}{2n+1} < 1 \quad \text{for} \quad n \geq 3. \end{aligned}$$

(ii) These two results show that the 'volumes' of all totally symmetric solids of unit radius in n dimensions are smaller than those in five or six dimensions if $n > 6$. Explicit calculations give the following for the first eight:

$$2, \quad \pi, \quad 4\pi/3, \quad \pi^2/2, \quad 8\pi^2/15, \quad \pi^3/6, \quad 16\pi^3/105, \quad \pi^4/24.$$

The largest of these is $V_5(1) = 8\pi^2/15 = 5.26$.

7

Vector algebra

7.1 *Which of the following statements about general vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are true*?

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$;
(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$;
(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$;
(d) $\mathbf{d} = \lambda\mathbf{a} + \mu\mathbf{b}$ *implies* $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = 0$;
(e) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c}$ *implies* $\mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = c\,|\mathbf{a} - \mathbf{b}|$;
(f) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]$.

All of the tests below are made using combinations of the common properties of the various types of vector products and justifications for individual steps are therefore not given. If the properties used are not recognised, they can be found in and learned from almost any standard textbook.

(a) $\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b}) = -\mathbf{c} \cdot (\mathbf{b} \times \mathbf{a}) = -(\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c} \neq (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{c}$.

(b) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \neq \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$.

(c) $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{cb}) - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$, a standard result.

(d) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{d} = (\mathbf{a} \times \mathbf{b}) \cdot (\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{a} + \mu(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b}$
$= \lambda\,0 + \mu\,0 = 0$.

(e) $\mathbf{a} \times \mathbf{c} = \mathbf{b} \times \mathbf{c} \Rightarrow (\mathbf{a} - \mathbf{b}) \times \mathbf{c} = \mathbf{0} \Rightarrow \mathbf{a} - \mathbf{b} \parallel \mathbf{c}$
$\Rightarrow (\mathbf{a} - \mathbf{b}) \cdot \mathbf{c} = c\,|\mathbf{a} - \mathbf{b}| \Rightarrow \mathbf{c} \cdot \mathbf{a} - \mathbf{c} \cdot \mathbf{b} = c\,|\mathbf{a} - \mathbf{b}|$.

(f) $(\mathbf{a} \times \mathbf{b}) \times (\mathbf{c} \times \mathbf{b}) = \mathbf{b}\,[\,\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})] - \mathbf{a}\,[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{b})]$
$= \mathbf{b}\,[\,\mathbf{a} \cdot (\mathbf{c} \times \mathbf{b})] - 0 = \mathbf{b}\,[\,\mathbf{b} \cdot (\mathbf{a} \times \mathbf{c})] = -\mathbf{b}\,[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]$
$\neq \mathbf{b}\,[\,\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]$.

Thus only (c), (d) and (e) are true.

7.3 *Identify the following surfaces:*

(a) $|\mathbf{r}| = k$; (b) $\mathbf{r} \cdot \mathbf{u} = l$; (c) $\mathbf{r} \cdot \mathbf{u} = m|\mathbf{r}|$ *for* $-1 \leq m \leq +1$;
(d) $|\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}| = n$.

Here k, l, m and n are fixed scalars and $\mathbf{u}$ *is a fixed unit vector.*

(a) All points on the surface are a distance k from the origin. The surface is therefore a sphere of radius k centred on the origin.

(b) This is the standard vector equation of a plane whose normal is in the direction $\mathbf{u}$ and whose distance from the origin is l.

(c) This is the surface generated by all vectors that make an angle $\cos^{-1} m$ with the fixed unit vector $\mathbf{u}$. The surface is therefore the cone of semi-angle $\cos^{-1} m$ that has the direction of $\mathbf{u}$ as its axis and the origin as its vertex.

(d) Since $(\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ is the component of $\mathbf{r}$ that is parallel to $\mathbf{u}$, $\mathbf{r} - (\mathbf{r} \cdot \mathbf{u})\mathbf{u}$ is the component perpendicular to $\mathbf{u}$. As this latter component is constant for all points on the surface, the surface must be a circular cylinder of radius n that has its axis parallel to $\mathbf{u}$.

7.5 *A, B, C and D are the four corners, in order, of one face of a cube of side 2 units. The opposite face has corners E, F, G and H, with AE, BF, CG and DH as parallel edges of the cube. The centre O of the cube is taken as the origin and the x-, y- and z- axes are parallel to AD, AE and AB, respectively. Find the following:*

(a) *the angle between the face diagonal AF and the body diagonal AG;*
(b) *the equation of the plane through B that is parallel to the plane CGE;*
(c) *the perpendicular distance from the centre J of the face $BCGF$ to the plane OCG;*
(d) *the volume of the tetrahedron $JOCG$.*

(a) Unit vectors in the directions of the two diagonals have components

$$\mathbf{f} - \mathbf{a} = \frac{(0, 2, 2)}{\sqrt{8}} \quad \text{and} \quad \mathbf{g} - \mathbf{a} = \frac{(2, 2, 2)}{\sqrt{12}}.$$

Taking the scalar product of these two unit vectors gives the angle between them as

$$\theta = \cos^{-1} \frac{0 + 4 + 4}{\sqrt{96}} = \cos^{-1} \sqrt{\frac{2}{3}}.$$

(b) The direction of a normal $\mathbf{n}$ to the plane CGE is in the direction of the cross product of any two non-parallel vectors that lie in the plane. These can be taken as those from C to G and from C to E:

$$(\mathbf{g}-\mathbf{c})\times(\mathbf{e}-\mathbf{c}) = (0,2,0)\times(-2,2,-2) = (-4,0,4).$$

The equation of the plane is therefore of the form

$$c = \mathbf{n}\cdot\mathbf{r} = -4x + 0y + 4z = -4x + 4z.$$

Since it passes through $\mathbf{b} = (-1,-1,1)$, the value of c must be 8 and the equation of the plane is $z - x = 2$.

(c) The direction of a normal $\mathbf{n}$ to the plane OCG is given by

$$\mathbf{c}\times\mathbf{g} = (1,-1,1)\times(1,1,1) = (-2,0,2).$$

The equation of the plane is therefore of the form

$$c = \mathbf{n}\cdot\mathbf{r} = -2x + 0y + 2z = -2x + 2z.$$

Since it passes through the origin, the value of c must be 0 and the equation of the plane written in the form $\hat{\mathbf{n}}\cdot\mathbf{r} = p$ is

$$-\frac{x}{\sqrt{2}} + \frac{z}{\sqrt{2}} = 0.$$

The distance of J from this plane is $\hat{\mathbf{n}}\cdot\mathbf{j}$, where $\mathbf{j} = (0,0,1)$. The distance is thus $-0 + (1/\sqrt{2}) = 1/\sqrt{2}$.

(d) The volume of the tetrahedron $= \frac{1}{3}$(base area $\times$ height perpendicular to the base). The area of triangle OCG is $\frac{1}{2}|\mathbf{c}\times\mathbf{g}|$ and the perpendicular height of the tetrahedron is the component of $\mathbf{j}$ in the direction of $\mathbf{c}\times\mathbf{g}$. Thus the volume is

$$V = \frac{1}{3}\left|\frac{1}{2}(\mathbf{c}\times\mathbf{g})\cdot\mathbf{j}\right| = \frac{1}{6}|(-2,0,2)\cdot(0,0,1)| = \frac{1}{3}.$$

7.7 *The edges OP, OQ and OR of a tetrahedron $OPQR$ are vectors $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$, respectively, where $\mathbf{p} = 2\mathbf{i} + 4\mathbf{j}$, $\mathbf{q} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $\mathbf{r} = 4\mathbf{i} - 2\mathbf{j} + 5\mathbf{k}$. Show that OP is perpendicular to the plane containing OQR. Express the volume of the tetrahedron in terms of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ and hence calculate the volume.*

The plane containing OQR has a normal in the direction $\mathbf{q}\times\mathbf{r} = (2,-1,3)\times(4,-2,5) = (1,2,0)$. This is parallel to $\mathbf{p}$ since $\mathbf{q}\times\mathbf{r} = \frac{1}{2}\mathbf{p}$. The volume of the tetrahedron is therefore one-third times $\frac{1}{2}|\mathbf{q}\times\mathbf{r}|$ times $|\mathbf{p}|$, i.e. $\frac{1}{6}|(1,2,0)|\sqrt{20} = \frac{5}{3}$.

7.9 *Prove Lagrange's identity, i.e.*

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) = (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}).$$

We treat the expression on the LHS as the triple scalar product of the three vectors $\mathbf{a} \times \mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ and use the cyclic properties of triple scalar products:

$$\begin{aligned}(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= \mathbf{d} \cdot [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}] \\ &= \mathbf{d} \cdot [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}] \\ &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{b}) - (\mathbf{b} \cdot \mathbf{c})(\mathbf{d} \cdot \mathbf{a}).\end{aligned}$$

In going from the first to the second line we used the standard result

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}$$

to replace $(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}$. This result, if not known, can be proved by writing it out in component form as follows.

Consider only the x-component of each side of the equation. The corresponding results for other components can be obtained by cyclic permutation of x, y and z.

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (a_y b_z - a_z b_y,\ a_z b_x - a_x b_z,\ a_x b_y - a_y b_x) \\ [(\mathbf{a} \times \mathbf{b}) \times \mathbf{c}]_x &= (a_z b_x - a_x b_z)c_z - (a_x b_y - a_y b_x)c_y \\ &= b_x(a_z c_z + a_y c_y) - a_x(b_z c_z + b_y c_y) \\ &= b_x(a_z c_z + a_y c_y + a_x c_x) - a_x(b_x c_x + b_z c_z + b_y c_y) \\ &= [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{b} \cdot \mathbf{c})\mathbf{a}]_x .\end{aligned}$$

To obtain the penultimate line we both added and subtracted $a_x b_x c_x$ on the RHS. This establishes the result for the x-component and hence for all three components.

7.11 *Show that the points* $(1, 0, 1)$, $(1, 1, 0)$ *and* $(1, -3, 4)$ *lie on a straight line. Give the equation of the line in the form*

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{b}.$$

To show that the points lie on a line, we need to show that their position vectors are linearly dependent. That this is so follows from noting that

$$(1, -3, 4) = 4(1, 0, 1) - 3(1, 1, 0).$$

This can also be written

$$(1,-3,4) = (1,0,1) + 3[(1,0,1) - (1,1,0)] = (1,0,1) + 3(0,-1,1).$$

The equation of the line is therefore

$$\mathbf{r} = \mathbf{a} + \lambda(-\mathbf{j} + \mathbf{k}),$$

where $\mathbf{a}$ is the vector position of *any* point on the line, e.g. $\mathbf{i} + \mathbf{k}$ or $\mathbf{i} + \mathbf{j}$ or $\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ or many others. Of course, choosing different points for $\mathbf{a}$ will entail using different values of λ to describe the same point $\mathbf{r}$ on the line. For example,

$$\begin{aligned}(1,-5,6) &= (1,0,1) + 5(0,-1,1)\\ \text{or } &= (1,1,0) + 6(0,-1,1)\\ \text{or } &= (1,-3,4) + 2(0,-1,1).\end{aligned}$$

7.13 *Two planes have non-parallel unit normals* $\hat{\mathbf{n}}$ *and* $\hat{\mathbf{m}}$ *and their closest distances from the origin are* λ *and* μ*, respectively. Find the vector equation of their line of intersection in the form* $\mathbf{r} = v\mathbf{p} + \mathbf{a}$.

The equations of the two planes are

$$\hat{\mathbf{n}} \cdot \mathbf{r} = \lambda \qquad \text{and} \qquad \hat{\mathbf{m}} \cdot \mathbf{r} = \mu.$$

The line of intersection lies in both planes and is thus perpendicular to both normals; it therefore has direction $\mathbf{p} = \hat{\mathbf{n}} \times \hat{\mathbf{m}}$. Consequently the equation of the line takes the form $\mathbf{r} = v\mathbf{p} + \mathbf{a}$, where $\mathbf{a}$ is any one point lying on it. One such point is the one in which the line meets the plane containing $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$; we take this point as $\mathbf{a}$. Since $\mathbf{a}$ also lies in both of the original planes, we must have

$$\hat{\mathbf{n}} \cdot \mathbf{a} = \lambda \quad \text{and} \quad \hat{\mathbf{m}} \cdot \mathbf{a} = \mu.$$

If we now write $\mathbf{a} = x\,\hat{\mathbf{n}} + y\,\hat{\mathbf{m}}$, these two conditions become

$$\begin{aligned}\lambda &= \hat{\mathbf{n}} \cdot \mathbf{a} = x + y(\hat{\mathbf{n}} \cdot \hat{\mathbf{m}}),\\ \mu &= \hat{\mathbf{m}} \cdot \mathbf{a} = x(\hat{\mathbf{n}} \cdot \hat{\mathbf{m}}) + y.\end{aligned}$$

It then follows that

$$x = \frac{\lambda - \mu(\hat{\mathbf{n}} \cdot \hat{\mathbf{m}})}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{m}})^2} \quad \text{and} \quad y = \frac{\mu - \lambda(\hat{\mathbf{n}} \cdot \hat{\mathbf{m}})}{1 - (\hat{\mathbf{n}} \cdot \hat{\mathbf{m}})^2},$$

thus determining $\mathbf{a}$. Both $\mathbf{p}$ and $\mathbf{a}$ are therefore determined in terms of λ, μ, $\hat{\mathbf{n}}$ and $\hat{\mathbf{m}}$, and so consequently is the line of intersection of the planes.

7.15 *Let O, A, B and C be four points with position vectors* $\mathbf{0}$, $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$, *and denote by* $\mathbf{g} = \lambda\mathbf{a} + \mu\mathbf{b} + \nu\mathbf{c}$ *the position of the centre of the sphere on which they all lie.*

(a) *Prove that λ, μ and ν simultaneously satisfy*

$$(\mathbf{a}\cdot\mathbf{a})\lambda + (\mathbf{a}\cdot\mathbf{b})\mu + (\mathbf{a}\cdot\mathbf{c})\nu = \tfrac{1}{2}a^2$$

and two other similar equations.

(b) *By making a change of origin, find the centre and radius of the sphere on which the points* $\mathbf{p} = 3\mathbf{i}+\mathbf{j}-2\mathbf{k}$, $\mathbf{q} = 4\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, $\mathbf{r} = 7\mathbf{i}-3\mathbf{k}$ *and* $\mathbf{s} = 6\mathbf{i}+\mathbf{j}-\mathbf{k}$ *all lie.*

(a) Each of the points O, A, B and C is the same distance from the centre G of the sphere. In particular, $OG = OA$, i.e.

$$\begin{aligned}
|\mathbf{g}-\mathbf{0}|^2 &= |\mathbf{a}-\mathbf{g}|^2,\\
g^2 &= a^2 - 2\mathbf{a}\cdot\mathbf{g} + g^2,\\
\mathbf{a}\cdot\mathbf{g} &= \tfrac{1}{2}a^2,\\
\mathbf{a}\cdot(\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}) &= \tfrac{1}{2}a^2,\\
(\mathbf{a}\cdot\mathbf{a})\lambda + (\mathbf{a}\cdot\mathbf{b})\mu + (\mathbf{a}\cdot\mathbf{c})\nu &= \tfrac{1}{2}a^2.
\end{aligned}$$

Two similar equations can be obtained from $OG = OB$ and $OG = OC$.

(b) To use the previous result we make P, say, the origin of a new coordinate system in which

$$\begin{aligned}
\mathbf{p}' &= \mathbf{p}-\mathbf{p} = (0,0,0),\\
\mathbf{q}' &= \mathbf{q}-\mathbf{p} = (1,2,-1),\\
\mathbf{r}' &= \mathbf{r}-\mathbf{p} = (4,-1,-1),\\
\mathbf{s}' &= \mathbf{s}-\mathbf{p} = (3,0,1).
\end{aligned}$$

The centre, G, of the sphere on which P, Q, R and S lie is then given by

$$\mathbf{g}' = \lambda\mathbf{q}' + \mu\mathbf{r}' + \nu\mathbf{s}',$$

where
$$\begin{aligned}
(\mathbf{q}'\cdot\mathbf{q}')\lambda + (\mathbf{q}'\cdot\mathbf{r}')\mu + (\mathbf{q}'\cdot\mathbf{s}')\nu &= \tfrac{1}{2}\mathbf{q}'\cdot\mathbf{q}',\\
(\mathbf{r}'\cdot\mathbf{q}')\lambda + (\mathbf{r}'\cdot\mathbf{r}')\mu + (\mathbf{r}'\cdot\mathbf{s}')\nu &= \tfrac{1}{2}\mathbf{r}'\cdot\mathbf{r}',\\
(\mathbf{s}'\cdot\mathbf{q}')\lambda + (\mathbf{s}'\cdot\mathbf{r}')\mu + (\mathbf{s}'\cdot\mathbf{s}')\nu &= \tfrac{1}{2}\mathbf{s}'\cdot\mathbf{s}',
\end{aligned}$$

i.e.
$$\begin{aligned}
6\lambda + 3\mu + 2\nu &= 3,\\
3\lambda + 18\mu + 11\nu &= 9,\\
2\lambda + 11\mu + 10\nu &= 5.
\end{aligned}$$

These equations have the solution

$$\lambda = \frac{5}{18}, \qquad \mu = \frac{5}{9}, \qquad \nu = -\frac{1}{6}.$$

Thus, the centre of the sphere can be calculated as

$$\mathbf{g}' = \frac{5}{18}(1,2,-1) + \frac{5}{9}(4,-1,-1) - \frac{1}{6}(3,0,1) = (2,0,-1).$$

Its radius is therefore $|G'O'| = |\mathbf{g}'| = \sqrt{5}$ and its centre in the original coordinate system is at $\mathbf{g}' + \mathbf{p} = (5,1,-3)$.

7.17 *Using vector methods:*

(a) *Show that the line of intersection of the planes* $x + 2y + 3z = 0$ *and* $3x + 2y + z = 0$ *is equally inclined to the* x*- and* z*-axes and makes an angle* $\cos^{-1}(-2/\sqrt{6})$ *with the* y*-axis.*

(b) *Find the perpendicular distance between one corner of a unit cube and the major diagonal not passing through it.*

(a) The origin O is clearly in both planes. A second such point can be found by setting $z = 1$, say, and solving the pair of simultaneous equations to give $x = 1$ and $y = -2$, i.e. $(1,-2,1)$ is in both planes. The direction cosines of the line of intersection, OP, are therefore

$$\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right),$$

i.e. the line is equally inclined to the x- and z-axes and makes an angle $\cos^{-1}(-2/\sqrt{6})$ with the y-axis.

The same conclusion can be reached by reasoning as follows. The line of intersection of the two planes must be orthogonal to the normal of either plane. Therefore it is in the direction of the cross product of the two normals and is given by

$$(1,2,3) \times (3,2,1) = (-4,8,-4) = -4\sqrt{6}\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right).$$

(b) We first note that all three major diagonals not passing through a corner come equally close to it. Taking the corner to be at the origin and the diagonal to be the one that passes through $(0,1,1)$ [and $(1,0,0)$], the equation of the diagonal is

$$(x,y,z) = (0,1,1) + \frac{\lambda}{\sqrt{3}}(1,-1,-1).$$

Using the result that the distance d of the point $\mathbf{p}$ from the line $\mathbf{r} = \mathbf{a} + \lambda\hat{\mathbf{b}}$ is given by

$$d = |(\mathbf{p} - \mathbf{a}) \times \hat{\mathbf{b}}|,$$

the distance of $(0,0,0)$ from the line of the diagonal is

$$\left| [(0,0,0) - (0,1,1)] \times \frac{1}{\sqrt{3}}(1,-1,-1) \right| = \frac{1}{\sqrt{3}}|(0,-1,1)| = \sqrt{\frac{2}{3}}.$$

7.19 *The vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$ *are not coplanar. Verify that the expressions*

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \quad \mathbf{b}' = \frac{\mathbf{c} \times \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}, \quad \mathbf{c}' = \frac{\mathbf{a} \times \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}$$

define a set of reciprocal vectors $\mathbf{a}'$, $\mathbf{b}'$ *and* $\mathbf{c}'$ *with the following properties:*

(a) $\mathbf{a}' \cdot \mathbf{a} = \mathbf{b}' \cdot \mathbf{b} = \mathbf{c}' \cdot \mathbf{c} = 1$;
(b) $\mathbf{a}' \cdot \mathbf{b} = \mathbf{a}' \cdot \mathbf{c} = \mathbf{b}' \cdot \mathbf{a} \quad \text{etc} = 0$;
(c) $[\mathbf{a}', \mathbf{b}', \mathbf{c}'] = 1/[\mathbf{a}, \mathbf{b}, \mathbf{c}]$;
(d) $\mathbf{a} = (\mathbf{b}' \times \mathbf{c}')/[\mathbf{a}', \mathbf{b}', \mathbf{c}']$.

Direct substitutions and the expansion formula for a triple vector product (proved in 7.9) enable the verifications to be made as follows. We make repeated use of the general result $(\mathbf{p} \times \mathbf{q}) \cdot \mathbf{p} = 0 = (\mathbf{p} \times \mathbf{q}) \cdot \mathbf{q}$.

(a) $$\mathbf{a}' \cdot \mathbf{a} = \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{a}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 1. \quad \text{Similarly for } \mathbf{b}' \cdot \mathbf{b} \text{ and } \mathbf{c}' \cdot \mathbf{c}.$$

(b) $$\mathbf{a}' \cdot \mathbf{b} = \frac{(\mathbf{b} \times \mathbf{c}) \cdot \mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = 0. \quad \text{Similarly for } \mathbf{a}' \cdot \mathbf{c},\ \mathbf{b}' \cdot \mathbf{a} \text{ etc.}$$

(c) $$\begin{aligned} [\mathbf{a}', \mathbf{b}', \mathbf{c}'] &= \frac{\mathbf{a}' \cdot \{(\mathbf{c} \times \mathbf{a}) \times (\mathbf{a} \times \mathbf{b})\}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2} \\ &= \frac{\mathbf{a}' \cdot \{[\mathbf{b} \cdot (\mathbf{c} \times \mathbf{a})]\,\mathbf{a} - [\mathbf{a} \cdot (\mathbf{c} \times \mathbf{a})]\,\mathbf{b}\}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2} \\ &= \frac{1\,[\mathbf{b}, \mathbf{c}, \mathbf{a}] - 0\,(\mathbf{a}' \cdot \mathbf{b})}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2}, \quad \text{using results (a) and (b),} \\ &= \frac{1}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]}. \end{aligned}$$

(d) $$\begin{aligned} \frac{\mathbf{b}' \times \mathbf{c}'}{[\mathbf{a}', \mathbf{b}', \mathbf{c}']} &= \frac{[\mathbf{b}, \mathbf{c}, \mathbf{a}]\,\mathbf{a} - 0\,\mathbf{b}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]^2\,[\mathbf{a}', \mathbf{b}', \mathbf{c}']}, && \text{as in part (c),} \\ &= \mathbf{a}, && \text{from result (c).} \end{aligned}$$

7.21 *In a crystal with a face-centred cubic structure, the basic cell can be taken as a cube of edge a with its centre at the origin of coordinates and its edges parallel to the Cartesian coordinate axes; atoms are sited at the eight corners and at the centre of each face. However, other basic cells are possible. One is the rhomboid shown in figure 7.1, which has the three vectors* $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{d}$ *as edges.*

(a) *Show that the volume of the rhomboid is one-quarter that of the cube.*
(b) *Show that the angles between pairs of edges of the rhomboid are* $60°$ *and that the corresponding angles between pairs of edges of the rhomboid defined by the reciprocal vectors to* $\mathbf{b}$, $\mathbf{c}$, $\mathbf{d}$ *are each* $109.5°$. *(This rhomboid can be used as the basic cell of a body-centred cubic structure, more easily visualised as a cube with an atom at each corner and one at its centre.)*
(c) *In order to use the Bragg formula,* $2d\sin\theta = n\lambda$, *for the scattering of X-rays by a crystal, it is necessary to know the perpendicular distance d between successive planes of atoms; for a given crystal structure, d has a particular value for each set of planes considered. For the face-centred cubic structure find the distance between successive planes with normals in the* $\mathbf{k}$, $\mathbf{i}+\mathbf{j}$ *and* $\mathbf{i}+\mathbf{j}+\mathbf{k}$ *directions.*

(a) From the figure it is easy to see that the edges of the rhomboid are the vectors $\mathbf{b} = \frac{1}{2}a(0,1,1)$, $\mathbf{c} = \frac{1}{2}a(1,0,1)$, and $\mathbf{d} = \frac{1}{2}a(1,1,0)$. The volume V of the rhomboid is therefore given by

$$\begin{aligned} V &= |\,[\mathbf{b}, \mathbf{c}, \mathbf{d}]\,| \\ &= |\mathbf{b}\cdot(\mathbf{c}\times\mathbf{d})| \\ &= \tfrac{1}{8}a^3\,|(0,1,1)\cdot(-1,1,1)| \\ &= \tfrac{1}{4}a^3, \end{aligned}$$

i.e. one-quarter that of the cube.

(b) To find the angle between two edges of the rhomboid we calculate the scalar product of two unit vectors, one along each edge; its value is $1\times 1\times\cos\phi$, where ϕ is the angle between the edges. Unit vectors along the edges of the rhomboid are

$$\hat{\mathbf{b}} = \frac{1}{\sqrt{2}}(0,1,1), \quad \hat{\mathbf{c}} = \frac{1}{\sqrt{2}}(1,0,1), \quad \hat{\mathbf{d}} = \frac{1}{\sqrt{2}}(1,1,0).$$

The scalar product of *any* pair of these particular vectors has the value $\frac{1}{2}$, e.g.

$$\hat{\mathbf{b}}\cdot\hat{\mathbf{c}} = \tfrac{1}{2}(0+0+1) = \tfrac{1}{2}.$$

Thus the angle between any pair of edges is $\cos^{-1}(\frac{1}{2}) = 60°$.

Figure 7.1 A face-centred cubic crystal.

The reciprocal vectors are, for example,

$$\mathbf{b}' = \frac{\mathbf{c} \times \mathbf{d}}{[\,\mathbf{b}, \mathbf{c}, \mathbf{d}\,]} = \frac{a^2}{4}\,\frac{(-1,1,1)}{(a^3/4)} = \frac{1}{a}(-1,1,1) = \frac{1}{a}(-\mathbf{i} + \mathbf{j} + \mathbf{k}),$$

where in the second equality we have used the result of part (a). Similarly, or by cyclic permutation, $\mathbf{c}' = a^{-1}(\mathbf{i} - \mathbf{j} + \mathbf{k})$ and $\mathbf{d}' = a^{-1}(\mathbf{i} + \mathbf{j} - \mathbf{k})$.
The angle between any pair of reciprocal vectors has the value 109.5°, e.g.

$$\theta = \cos^{-1}\left(\frac{\mathbf{b}' \cdot \mathbf{c}'}{|\mathbf{b}'||\mathbf{c}'|}\right) = \cos^{-1}\left(\frac{a^{-2}(-1-1+1)}{(\sqrt{3}\,a^{-1})^2}\right) = \cos^{-1}(-\tfrac{1}{3}) = 109.5^\circ.$$

Other pairs yield the same value.

(c) Planes with normals in the $\mathbf{k}$ direction are clearly separated by $\frac{1}{2}a$.

A plane with its normal in the direction $\mathbf{i} + \mathbf{j}$ has an equation of the form

$$\frac{1}{\sqrt{2}}(1,1,0) \cdot (x,y,z) = p,$$

where p is the perpendicular distance of the origin from the plane. Since the plane with the smallest positive value of p passes through $(\frac{1}{2}a, 0, \frac{1}{2}a)$, p has the value $a/\sqrt{8}$, which is therefore the distance between successive planes with normals in the direction $\mathbf{i} + \mathbf{j}$.

Planes with their normals in the direction $\mathbf{i} + \mathbf{j} + \mathbf{k}$ have equations of the form

$$\frac{1}{\sqrt{3}}(1,1,1) \cdot (x,y,z) = p.$$

For the plane P_1 containing $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{d}$ we have (for $\mathbf{b}$, say)

$$\frac{1}{\sqrt{3}}(1,1,1) \cdot (0, \tfrac{1}{2}a, \tfrac{1}{2}a) = p_1,$$

giving $p_1 = a/\sqrt{3}$. Similarly for the plane P_2 containing $\mathbf{c}+\mathbf{d}$, $\mathbf{b}+\mathbf{d}$ and $\mathbf{b}+\mathbf{c}$ we have (for $\mathbf{c}+\mathbf{d}$, say)

$$\frac{1}{\sqrt{3}}(1,1,1)\cdot(a,\tfrac{1}{2}a,\tfrac{1}{2}a) = p_2,$$

giving $p_2 = 2a/\sqrt{3}$. Thus the distance, d, between successive planes with normals in the direction $\mathbf{i}+\mathbf{j}+\mathbf{k}$ is the difference between these two values, i.e. $d = p_2 - p_1 = a/\sqrt{3}$.

7.23 *By proceeding as indicated below, prove the* parallel axis theorem*, which states that, for a body of mass M, the moment of inertia I about any axis is related to the corresponding moment of inertia I_0 about a parallel axis that passes through the centre of mass of the body by*

$$I = I_0 + Ma_\perp^2,$$

where $a_\perp$ is the perpendicular distance between the two axes. Note that I_0 can be written as

$$\int (\hat{\mathbf{n}}\times\mathbf{r})\cdot(\hat{\mathbf{n}}\times\mathbf{r})\,dm,$$

where $\mathbf{r}$ is the vector position, relative to the centre of mass, of the infinitesimal mass dm and $\hat{\mathbf{n}}$ is a unit vector in the direction of the axis of rotation. Write a similar expression for I in which $\mathbf{r}$ is replaced by $\mathbf{r}' = \mathbf{r}-\mathbf{a}$, where $\mathbf{a}$ is the vector position of any point on the axis to which I refers. Use Lagrange's identity and the fact that $\int \mathbf{r}\,dm = \mathbf{0}$ (by the definition of the centre of mass) to establish the result.

Figure 7.2 shows the vectors involved in describing the physical arrangement. With

$$\begin{aligned} I_0 &= \int (\hat{\mathbf{n}}\times\mathbf{r})\cdot(\hat{\mathbf{n}}\times\mathbf{r})\,dm \\ &= \int \left[(\hat{\mathbf{n}}\cdot\hat{\mathbf{n}})(\mathbf{r}\cdot\mathbf{r}) - (\hat{\mathbf{n}}\cdot\mathbf{r})^2\right]\,dm, \end{aligned}$$

the moment of inertia of the same mass distribution about a parallel axis passing

Figure 7.2 The vectors used in the proof of the parallel axis theorem in exercise 7.23.

through $\mathbf{a}$ is given by

$$
\begin{aligned}
I &= \int (\hat{\mathbf{n}} \times \mathbf{r}') \cdot (\hat{\mathbf{n}} \times \mathbf{r}')\, dm \\
&= \int [\hat{\mathbf{n}} \times (\mathbf{r} - \mathbf{a})] \cdot [\hat{\mathbf{n}} \times (\mathbf{r} - \mathbf{a})]\, dm \\
&= \int \left\{ (\hat{\mathbf{n}} \cdot \hat{\mathbf{n}})[(\mathbf{r} - \mathbf{a}) \cdot (\mathbf{r} - \mathbf{a})] - [\hat{\mathbf{n}} \cdot (\mathbf{r} - \mathbf{a})]^2 \right\}\, dm, \\
&= \int \left[r^2 - 2\mathbf{a} \cdot \mathbf{r} + a^2 - (\hat{\mathbf{n}} \cdot \mathbf{r})^2 + 2(\hat{\mathbf{n}} \cdot \mathbf{r})(\hat{\mathbf{n}} \cdot \mathbf{a}) - (\hat{\mathbf{n}} \cdot \mathbf{a})^2 \right]\, dm \\
&= I_0 - 2\mathbf{a} \cdot \mathbf{0} + 2(\hat{\mathbf{n}} \cdot \mathbf{a})(\hat{\mathbf{n}} \cdot \mathbf{0}) + \int \left[a^2 - (\hat{\mathbf{n}} \cdot \mathbf{a})^2 \right]\, dm \\
&= I_0 + a_\perp^2 M.
\end{aligned}
$$

When obtaining the penultimate line we (twice) used the fact that O is the centre of mass of the body and so, by definition, $\int \mathbf{r}\, dm = \mathbf{0}$. To obtain the final line we noted that $\hat{\mathbf{n}} \cdot \mathbf{a}$ is the component of $\mathbf{a}$ parallel to $\hat{\mathbf{n}}$ and so $a^2 - (\hat{\mathbf{n}} \cdot \mathbf{a})^2$ is the square of the component of $\mathbf{a}$ perpendicular to $\hat{\mathbf{n}}$.

7.25 *Define a set of (non-orthogonal) base vectors* $\mathbf{a} = \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + \mathbf{k}$ *and* $\mathbf{c} = \mathbf{i} + \mathbf{j}$.

(a) *Establish their reciprocal vectors and hence express the vectors* $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, $\mathbf{q} = \mathbf{i} + 4\mathbf{j}$ *and* $\mathbf{r} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$ *in terms of the base vectors* $\mathbf{a}$, $\mathbf{b}$ *and* $\mathbf{c}$.

(b) *Verify that the scalar product* $\mathbf{p} \cdot \mathbf{q}$ *has the same value,* -5, *when evaluated using either set of components.*

The new base vectors are $\mathbf{a} = (0, 1, 1)$, $\mathbf{b} = (1, 0, 1)$ and $\mathbf{c} = (1, 1, 0)$.

(a) The corresponding reciprocal vectors are thus

$$\mathbf{a}' = \frac{\mathbf{b} \times \mathbf{c}}{[\mathbf{a}, \mathbf{b}, \mathbf{c}]} = \frac{(-1, 1, 1)}{2} = \tfrac{1}{2}(-1, 1, 1),$$

and similarly for $\mathbf{b}' = \frac{1}{2}(1, -1, 1)$ and $\mathbf{c}' = \frac{1}{2}(1, 1, -1)$.

The coefficient of (say) $\mathbf{a}$ in the expression for (say) $\mathbf{p}$ is $\mathbf{a}' \cdot \mathbf{p} = -2$. The coefficient of $\mathbf{b}$ is $\mathbf{b}' \cdot \mathbf{p} = 3$, etc. Building up each of $\mathbf{p}$, $\mathbf{q}$ and $\mathbf{r}$ in this way, we find that their coordinates in terms of the new basis $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ are $\mathbf{p} = (-2, 3, 0)$, $\mathbf{q} = (\frac{3}{2}, -\frac{3}{2}, \frac{5}{2})$ and $\mathbf{r} = (2, -1, -1)$.

(b) The new basis vectors, which are neither orthogonal nor normalised, have the properties $\mathbf{a} \cdot \mathbf{a} = \mathbf{b} \cdot \mathbf{b} = \mathbf{c} \cdot \mathbf{c} = 2$ and $\mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} = \mathbf{a} \cdot \mathbf{b} = 1$. Thus the scalar product $\mathbf{p} \cdot \mathbf{q}$, calculated in the new basis, has the value

$$2\left(-3 - \tfrac{9}{2} + 0\right) + 1\left(3 - 5 + \tfrac{9}{2} + \tfrac{15}{2} + 0 + 0\right) = -15 + 10 = -5.$$

Using the original basis, $\mathbf{p} \cdot \mathbf{q} = 3 - 8 + 0 = -5$, verifying that the scalar product has the same value in both sets of coordinates.

7.27 *According to alternating current theory, the currents and potential differences in the components of the circuit shown in figure 7.3 are determined by Kirchhoff's laws and the relationships*

$$I_1 = \frac{V_1}{R_1}, \qquad I_2 = \frac{V_2}{R_2}, \qquad I_3 = i\omega C V_3, \qquad V_4 = i\omega L I_2.$$

The factor $i = \sqrt{-1}$ in the expression for I_3 indicates that the phase of I_3 is $90°$ ahead of V_3. Similarly the phase of V_4 is $90°$ ahead of I_2.

Measurement shows that V_3 has an amplitude of $0.661V_0$ and a phase of $+13.4°$ relative to that of the power supply. Taking $V_0 = 1\,\mathrm{V}$ and using a series of vector plots for potential differences and currents (they could all be on the same plot if suitable scales were chosen), determine all unknown currents and potential differences and find values for the inductance of L and the resistance of R_2.

[*Scales of* $1\,\mathrm{cm} = 0.1\,\mathrm{V}$ *for potential differences and* $1\,\mathrm{cm} = 1\,\mathrm{mA}$ *for currents are convenient.*]

Using the suggested scales, we construct the vectors shown in figure 7.4 in the following order:

(1) V_0 joining $(0, 0)$ to $(10, 0)$;
(2) V_3 of length 6.61 and phase $+13.4°$;
(3) $V_1 = V_0 - V_3$;

Figure 7.3 The oscillatory electric circuit in exercise 7.27. The power supply has angular frequency $\omega = 2\pi f = 400\pi\ \mathrm{s}^{-1}$.

Figure 7.4 The vector solution to exercise 7.27.

(4) I_1 parallel to V_1 and $(0.1 \times 1000)/50 = 2$ times as long;
(5) I_3, 90° ahead of V_3 in phase and $(0.1 \times 1000) \times 400\pi \times 10^{-5} = 1.26$ times as long;
(6) $I_2 = I_1 - I_3$;
(7) draw a parallel to I_2 through the origin;
(8) drop a perpendicular from V_3 onto this parallel to I_2;
(9) since $V_3 = V_2 + V_4$ and $V_2 \parallel I_2$, whilst $V_4 \perp I_2$, the foot of the perpendicular

gives V_2;
(10) $V_4 = V_3 - V_2$.

The corresponding steps are labelled in the figure, which is somewhat reduced from its actual size.

Finally, $R_2 = V_2/I_2$ and $L = (V_4 \times 0.1 \times 1000)/(400\pi \times I_2)$.

The accurate solutions (obtained by calculation rather than drawing) are:
$I_1 = (7.76, -23.2°)$, $I_2 = (14.36, -50.8°)$, $I_3 = (8.30, 103.4°)$;
$V_1 = (0.388, -23.2°)$, $V_2 = (0.287, -50.8°)$, $V_4 = (0.596, 39.2°)$;
$L = 33$ mH, $R_2 = 20\ \Omega$.

8

Matrices and vector spaces

8.1 *Which of the following statements about linear vector spaces are true? Where a statement is false, give a counter-example to demonstrate this.*

(a) *Non-singular $N \times N$ matrices form a vector space of dimension N^2.*
(b) *Singular $N \times N$ matrices form a vector space of dimension N^2.*
(c) *Complex numbers form a vector space of dimension 2.*
(d) *Polynomial functions of x form an infinite-dimensional vector space.*
(e) *Series $\{a_0, a_1, a_2, \ldots, a_N\}$ for which $\sum_{n=0}^{N} |a_n|^2 = 1$ form an N-dimensional vector space.*
(f) *Absolutely convergent series form an infinite-dimensional vector space.*
(g) *Convergent series with terms of alternating sign form an infinite-dimensional vector space.*

We first remind ourselves that for a set of entities to form a vector space, they must pass five tests: (i) closure under commutative and associative addition; (ii) closure under multiplication by a scalar; (iii) the existence of a null vector in the set; (iv) multiplication by unity leaves any vector unchanged; (v) each vector has a corresponding negative vector.

(a) False. The matrix O_N, the $N \times N$ null matrix, required by (iii) is *not* non-singular and is therefore not in the set.

(b) Consider the sum of $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. The sum is the unit matrix which is not singular and so the set is not closed; this violates requirement (i). The statement is false.

(c) The space is closed under addition and multiplication by a scalar; multiplication by unity leaves a complex number unchanged; there is a null vector ($= 0 + i0$)

and a negative complex number for each vector. All the necessary conditions are satisfied and the statement is true.

(d) As in the previous case, all the conditions are satisfied and the statement is true.

(e) This statement is false. To see why, consider $b_n = a_n + a_n$ for which $\sum_{n=0}^{N} |b_n|^2 = 4 \neq 1$, i.e. the set is not closed (violating (i)), or note that there is no zero vector with unit norm (violating (iii)).

(f) True. Note that an absolutely convergent series remains absolutely convergent when the signs of all of its terms are reversed.

(g) False. Consider the two series defined by

$$a_0 = \tfrac{1}{2}, \quad a_n = 2(-\tfrac{1}{2})^n \text{ for } n \geq 1; \quad b_n = -(-\tfrac{1}{2})^n \text{ for } n \geq 0.$$

The series that is the sum of $\{a_n\}$ and $\{b_n\}$ does not have alternating signs and so closure (required by (i)) does not hold.

8.3 *Using the properties of determinants, solve with a minimum of calculation the following equations for x:*

$$\text{(a)} \quad \begin{vmatrix} x & a & a & 1 \\ a & x & b & 1 \\ a & b & x & 1 \\ a & b & c & 1 \end{vmatrix} = 0, \qquad \text{(b)} \quad \begin{vmatrix} x+2 & x+4 & x-3 \\ x+3 & x & x+5 \\ x-2 & x-1 & x+1 \end{vmatrix} = 0.$$

(a) In view of the similarities between some rows and some columns, the property most likely to be useful here is that if a determinant has two rows/columns equal (or multiples of each other) then its value is zero.
(i) We note that setting $x = a$ makes the first and fourth columns multiples of each other and hence makes the value of the determinant 0; thus $x = a$ is one solution to the equation.
(ii) Setting $x = b$ makes the second and third rows equal, and again the determinant vanishes; thus b is another root of the equation.
(iii) Setting $x = c$ makes the third and fourth rows equal, and yet again the determinant vanishes; thus c is also a root of the equation.

Since the determinant contains no x in its final column, it is a cubic polynomial in x and there will be exactly three roots to the equation. We have already found all three!

(b) Here, the presence of x multiplied by unity in every entry means that subtracting rows/columns will lead to a simplification. After (i) subtracting the first

column from each of the others, and then (ii) subtracting the first row from each of the others, the determinant becomes

$$\begin{vmatrix} x+2 & 2 & -5 \\ x+3 & -3 & 2 \\ x-2 & 1 & 3 \end{vmatrix} = \begin{vmatrix} x+2 & 2 & -5 \\ 1 & -5 & 7 \\ -4 & -1 & 8 \end{vmatrix}$$
$$= (x+2)(-40+7) + 2(-28-8) - 5(-1-20)$$
$$= -33(x+2) - 72 + 105$$
$$= -33x - 33.$$

Thus $x = -1$ is the only solution to the original (linear!) equation.

8.5 *By considering the matrices*

$$\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix},$$

show that $\mathsf{AB} = \mathsf{0}$ *does* not *imply that either* A *or* B *is the zero matrix but that it does imply that at least one of them is singular.*

We have

$$\mathsf{AB} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus AB is the zero matrix O without either $\mathsf{A} = \mathsf{O}$ or $\mathsf{B} = \mathsf{O}$.

However, $\mathsf{AB} = \mathsf{O} \Rightarrow |\mathsf{A}||\mathsf{B}| = |\mathsf{O}| = 0$ and therefore either $|\mathsf{A}| = 0$ or $|\mathsf{B}| = 0$ (or both).

8.7 *Prove the following results involving Hermitian matrices:*

(a) *If* A *is Hermitian and* U *is unitary then* $\mathsf{U}^{-1}\mathsf{AU}$ *is Hermitian.*
(b) *If* A *is anti-Hermitian then* $i\mathsf{A}$ *is Hermitian.*
(c) *The product of two Hermitian matrices* A *and* B *is Hermitian if and only if* A *and* B *commute.*
(d) *If* S *is a real antisymmetric matrix then* $\mathsf{A} = (\mathsf{I} - \mathsf{S})(\mathsf{I} + \mathsf{S})^{-1}$ *is orthogonal. If* A *is given by*

$$\mathsf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

then find the matrix S *that is needed to express* A *in the above form.*
(e) *If* K *is skew-hermitian, i.e.* $\mathsf{K}^\dagger = -\mathsf{K}$, *then* $\mathsf{V} = (\mathsf{I} + \mathsf{K})(\mathsf{I} - \mathsf{K})^{-1}$ *is unitary.*

The general properties of matrices that we will need are $(\mathsf{A}^\dagger)^{-1} = (\mathsf{A}^{-1})^\dagger$ and

$$(\mathsf{AB}\cdots\mathsf{C})^{\mathrm{T}} = \mathsf{C}^{\mathrm{T}}\cdots\mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}}, \qquad (\mathsf{AB}\cdots\mathsf{C})^\dagger = \mathsf{C}^\dagger\cdots\mathsf{B}^\dagger\mathsf{A}^\dagger.$$

(a) Given that $\mathsf{A} = \mathsf{A}^\dagger$ and $\mathsf{U}^\dagger\mathsf{U} = \mathsf{I}$, consider

$$(\mathsf{U}^{-1}\mathsf{AU})^\dagger = \mathsf{U}^\dagger\mathsf{A}^\dagger(\mathsf{U}^{-1})^\dagger = \mathsf{U}^{-1}\mathsf{A}(\mathsf{U}^\dagger)^{-1} = \mathsf{U}^{-1}\mathsf{A}(\mathsf{U}^{-1})^{-1} = \mathsf{U}^{-1}\mathsf{AU},$$

i.e. $\mathsf{U}^{-1}\mathsf{AU}$ is Hermitian.

(b) Given $\mathsf{A}^\dagger = -\mathsf{A}$, consider

$$(i\mathsf{A})^\dagger = -i\mathsf{A}^\dagger = -i(-\mathsf{A}) = i\mathsf{A},$$

i.e. $i\mathsf{A}$ is Hermitian.

(c) Given $\mathsf{A} = \mathsf{A}^\dagger$ and $\mathsf{B} = \mathsf{B}^\dagger$.

(i) Suppose $\mathsf{AB} = \mathsf{BA}$, then

$$(\mathsf{AB})^\dagger = \mathsf{B}^\dagger\mathsf{A}^\dagger = \mathsf{BA} = \mathsf{AB},$$

i.e. AB is Hermitian.

(ii) Now suppose that $(\mathsf{AB})^\dagger = \mathsf{AB}$. Then

$$\mathsf{BA} = \mathsf{B}^\dagger\mathsf{A}^\dagger = (\mathsf{AB})^\dagger = \mathsf{AB},$$

i.e. A and B commute.

Thus, AB is Hermitian $\Longleftrightarrow$ A and B commute.

(d) Given that S is real and $\mathsf{S}^{\mathrm{T}} = -\mathsf{S}$ with $\mathsf{A} = (\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1}$, consider

$$\begin{aligned}
\mathsf{A}^{\mathrm{T}}\mathsf{A} &= [(\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1}]^{\mathrm{T}}[(\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1}] \\
&= [(\mathsf{I}+\mathsf{S})^{-1}]^{\mathrm{T}}(\mathsf{I}+\mathsf{S})(\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1} \\
&= (\mathsf{I}-\mathsf{S})^{-1}(\mathsf{I}+\mathsf{S}-\mathsf{S}-\mathsf{S}^2)(\mathsf{I}+\mathsf{S})^{-1} \\
&= (\mathsf{I}-\mathsf{S})^{-1}(\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1} \\
&= \mathsf{I}\,\mathsf{I} = \mathsf{I},
\end{aligned}$$

i.e. A is orthogonal.

If $\mathsf{A} = (\mathsf{I}-\mathsf{S})(\mathsf{I}+\mathsf{S})^{-1}$, then $\mathsf{A}+\mathsf{AS} = \mathsf{I}-\mathsf{S}$ and $(\mathsf{A}+\mathsf{I})\mathsf{S} = \mathsf{I}-\mathsf{A}$, giving

$$\begin{aligned}
\mathsf{S} &= (\mathsf{A}+\mathsf{I})^{-1}(\mathsf{I}-\mathsf{A}) \\
&= \begin{pmatrix} 1+\cos\theta & \sin\theta \\ -\sin\theta & 1+\cos\theta \end{pmatrix}^{-1} \begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} \\
&= \frac{1}{2+2\cos\theta}\begin{pmatrix} 1+\cos\theta & -\sin\theta \\ \sin\theta & 1+\cos\theta \end{pmatrix}\begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} \\
&= \frac{1}{4\cos^2(\theta/2)}\begin{pmatrix} 0 & -2\sin\theta \\ 2\sin\theta & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -\tan(\theta/2) \\ \tan(\theta/2) & 0 \end{pmatrix}.
\end{aligned}$$

(e) This proof is almost identical to the first section of part (d) but with S replaced by $-\mathsf{K}$ and transposed matrices replaced by hermitian conjugate matrices.

8.9 *The* commutator $[\mathsf{X}, \mathsf{Y}]$ *of two matrices is defined by the equation*

$$[\mathsf{X}, \mathsf{Y}] = \mathsf{XY} - \mathsf{YX}.$$

Two anticommuting matrices A *and* B *satisfy*

$$\mathsf{A}^2 = \mathsf{I}, \qquad \mathsf{B}^2 = \mathsf{I}, \qquad [\mathsf{A}, \mathsf{B}] = 2i\mathsf{C}.$$

(a) *Prove that* $\mathsf{C}^2 = \mathsf{I}$ *and that* $[\mathsf{B}, \mathsf{C}] = 2i\mathsf{A}$.
(b) *Evaluate* $[[[\mathsf{A}, \mathsf{B}], [\mathsf{B}, \mathsf{C}]], [\mathsf{A}, \mathsf{B}]]$.

(a) From $\mathsf{AB} - \mathsf{BA} = 2i\mathsf{C}$ and $\mathsf{AB} = -\mathsf{BA}$ it follows that $\mathsf{AB} = i\mathsf{C}$. Thus,

$$-\mathsf{C}^2 = i\mathsf{C}i\mathsf{C} = \mathsf{ABAB} = \mathsf{A}(-\mathsf{AB})\mathsf{B} = -(\mathsf{AA})(\mathsf{BB}) = -\mathsf{I}\,\mathsf{I} = -\mathsf{I},$$

i.e. $\mathsf{C}^2 = \mathsf{I}$. In deriving the above result we have used the associativity of matrix multiplication.

For the commutator of B and C,

$$\begin{aligned}
[\mathsf{B}, \mathsf{C}] &= \mathsf{BC} - \mathsf{CB} \\
&= \mathsf{B}(-i\mathsf{AB}) - (-i)\mathsf{ABB} \\
&= -i(\mathsf{BA})\mathsf{B} + i\mathsf{AI} \\
&= -i(-\mathsf{AB})\mathsf{B} + i\mathsf{A} \\
&= i\mathsf{A} + i\mathsf{A} = 2i\mathsf{A}.
\end{aligned}$$

(b) To evaluate this multiple-commutator expression we must work outwards from the innermost 'explicit' commutators. There are three such commutators at the first stage. We also need the result that $[\mathsf{C}, \mathsf{A}] = 2i\mathsf{B}$; this can be proved in the same way as that for $[\mathsf{B}, \mathsf{C}]$ in part (a), or by making the cyclic replacements $A \to B \to C \to A$ in the assumptions and their consequences, as proved in part (a). Then we have

$$\begin{aligned}
[[[\mathsf{A}, \mathsf{B}], [\mathsf{B}, \mathsf{C}]], [\mathsf{A}, \mathsf{B}]] &= [[2i\mathsf{C}, 2i\mathsf{A}], 2i\mathsf{C}] \\
&= -4[[\mathsf{C}, \mathsf{A}], 2i\mathsf{C}] \\
&= -4[2i\mathsf{B}, 2i\mathsf{C}] \\
&= (-4)(-4)[\mathsf{B}, \mathsf{C}] = 32i\mathsf{A}.
\end{aligned}$$

8.11 *A general triangle has angles α, β and γ and corresponding opposite sides a, b and c. Express the length of each side in terms of the lengths of the other two sides and the relevant cosines, writing the relationships in matrix and vector form, using the vectors having components a, b, c and $\cos\alpha, \cos\beta, \cos\gamma$. Invert the matrix and hence deduce the cosine-law expressions involving α, β and γ.*

By considering each side of the triangle as the sum of the projections onto it of the other two sides, we have the three simultaneous equations:

$$\begin{aligned} a &= b\cos\gamma + c\cos\beta, \\ b &= c\cos\alpha + a\cos\gamma, \\ c &= b\cos\alpha + a\cos\beta. \end{aligned}$$

Written in matrix and vector form, $\mathsf{A}\mathbf{x} = \mathbf{y}$, they become

$$\begin{pmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{pmatrix} \begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The matrix A is non-singular, since $|\mathsf{A}| = 2abc \neq 0$, and therefore has an inverse given by

$$\mathsf{A}^{-1} = \frac{1}{2abc} \begin{pmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix}.$$

And so, writing $\mathbf{x} = \mathsf{A}^{-1}\mathbf{y}$, we have

$$\begin{pmatrix} \cos\alpha \\ \cos\beta \\ \cos\gamma \end{pmatrix} = \frac{1}{2abc} \begin{pmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

From this we can read off the cosine-law equation

$$\cos\alpha = \frac{1}{2abc}(-a^3 + ab^2 + ac^2) = \frac{b^2 + c^2 - a^2}{2bc},$$

and the corresponding expressions for $\cos\beta$ and $\cos\gamma$.

8.13 *Using the Gram–Schmidt procedure:*

(a) *construct an orthonormal set of vectors from the following:*

$$\mathsf{x}_1 = (0 \quad 0 \quad 1 \quad 1)^{\mathrm{T}}, \qquad \mathsf{x}_2 = (1 \quad 0 \quad -1 \quad 0)^{\mathrm{T}},$$
$$\mathsf{x}_3 = (1 \quad 2 \quad 0 \quad 2)^{\mathrm{T}}, \qquad \mathsf{x}_4 = (2 \quad 1 \quad 1 \quad 1)^{\mathrm{T}};$$

(b) *find an orthonormal basis, within a four-dimensional Euclidean space, for the subspace spanned by the three vectors*

$$(1 \quad 2 \quad 0 \quad 0)^{\mathrm{T}}, \quad (3 \quad -1 \quad 2 \quad 0)^{\mathrm{T}}, \quad (0 \quad 0 \quad 2 \quad 1)^{\mathrm{T}}.$$

The general procedure is to construct the orthonormal base set $\{\hat{\mathsf{z}}_i\}$ using the iteration procedure

$$\mathsf{z}_n = \mathsf{x}_n - \sum_{r=1}^{n-1} [\hat{\mathsf{z}}_r^{\dagger} \mathsf{x}_n] \hat{\mathsf{z}}_r \text{ with } \mathsf{z}_1 = \mathsf{x}_1.$$

The vector $\hat{\mathsf{z}}$ is the vector z after normalisation and the expression in square brackets is the (complex) inner product of $\hat{\mathsf{z}}_r$ and x_n.

(a) We start with $\hat{\mathsf{z}}_1 = 2^{-1/2}\mathsf{x}_1 = 2^{-1/2}[0 \quad 0 \quad 1 \quad 1]^{\mathrm{T}}$.

Next we calculate $(\hat{\mathsf{z}}_1)^{\dagger}\mathsf{x}_2$ as $-2^{-1/2}$ and then form z_2 as

$$\mathsf{z}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} - \frac{-1}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}.$$

The normalised vector $\hat{\mathsf{z}}_2$ is $6^{-1/2}(2 \quad 0 \quad -1)^{\mathrm{T}}$ 1.

Proceeding in this way, but without detailed description, we obtain

$$\mathsf{z}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{6}}\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ 2 \\ -\frac{1}{3} \\ \frac{1}{3} \end{bmatrix}.$$

The normalised vector $\hat{\mathsf{z}}_3$ is $(39)^{-1/2}(-1 \quad 6 \quad -1)^{\mathrm{T}}$ 1.

Finally,

$$\mathsf{z}_4 = \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{6}}\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} - \frac{4}{\sqrt{39}}\frac{1}{\sqrt{39}} \begin{bmatrix} -1 \\ 6 \\ -1 \\ 1 \end{bmatrix}.$$

The normalised vector $\hat{\mathsf{z}}_4$ is $(13)^{-1/2}(2 \quad 1 \quad 2)^{\mathrm{T}} - 1$.

[Note that if the only requirement had been to find an orthonormal set of base vectors then the obvious $(1 \quad 0 \quad 0 \quad 0)^T$, $(0 \quad 1 \quad 0 \quad 0)^T$, etc. could have been chosen.]

(b) The procedure is as in part (a) except that we require only three orthonormal vectors. However, we *must* begin with the given vectors so as to ensure that the correct subspace is spanned.

We start with $\hat{\mathsf{z}}_1 = 5^{-1/2}\mathsf{x}_1 = 5^{-1/2}[1 \quad 2 \quad 0 \quad 0]^{\mathrm{T}}$.

Next we calculate $(\hat{\mathsf{z}}_1)^\dagger \mathsf{x}_2$ as $-5^{-1/2}$ and then form z_2 as

$$\mathsf{z}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 0 \end{bmatrix} - \frac{1}{\sqrt{5}}\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{14}{5} \\ -\frac{7}{5} \\ 2 \\ 0 \end{bmatrix}.$$

The normalised vector $\hat{\mathsf{z}}_2$ is $(345)^{-1/2}(14 \quad -7 \quad 10)^{\mathrm{T}}\,0$.

As the final base vector for the subspace we obtain

$$\mathsf{z}_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} - 0\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \frac{20}{\sqrt{345}}\frac{1}{\sqrt{345}} \begin{bmatrix} 14 \\ -7 \\ 10 \\ 0 \end{bmatrix} = \frac{1}{345} \begin{bmatrix} -280 \\ 140 \\ 490 \\ 345 \end{bmatrix}.$$

Thus, the normalised vector $\hat{\mathsf{z}}_3$ is $(18285)^{-1/2}(-56 \quad 28 \quad 98)^{\mathrm{T}}\,69$. The fact that three orthonormal vectors can be found shows that the subspace is 3-dimensional and that the three original vectors are not linearly dependent.

8.15 *Determine which of the matrices below are mutually commuting, and, for those that are, demonstrate that they have a complete set of eigenvectors in common:*

$$\mathsf{A} = \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix},$$
$$\mathsf{C} = \begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix}, \quad \mathsf{D} = \begin{pmatrix} 14 & 2 \\ 2 & 11 \end{pmatrix}.$$

To establish the result we need to examine all pairs of products.

$$\begin{aligned}
\mathsf{AB} &= \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix} \\
&= \begin{pmatrix} -10 & 70 \\ 70 & -115 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 8 \\ 8 & -11 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} = \mathsf{BA}. \\
\mathsf{AC} &= \begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix}\begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix} \\
&= \begin{pmatrix} -34 & -70 \\ -72 & 65 \end{pmatrix} \neq \begin{pmatrix} -34 & -72 \\ -70 & 65 \end{pmatrix} \\
&= \begin{pmatrix} -9 & -10 \\ -10 & 5 \end{pmatrix}\begin{pmatrix} 6 & -2 \\ -2 & 9 \end{pmatrix} = \mathsf{CA}.
\end{aligned}$$

Continuing in this way, we find:

$$\begin{aligned}
\mathsf{AD} &= \begin{pmatrix} 80 & -10 \\ -10 & 95 \end{pmatrix} = \mathsf{DA}. \\
\mathsf{BC} &= \begin{pmatrix} -89 & 30 \\ 38 & -135 \end{pmatrix} \neq \begin{pmatrix} -89 & 38 \\ 30 & -135 \end{pmatrix} = \mathsf{CB}. \\
\mathsf{BD} &= \begin{pmatrix} 30 & 90 \\ 90 & -105 \end{pmatrix} = \mathsf{DB}. \\
\mathsf{CD} &= \begin{pmatrix} -146 & -128 \\ -130 & 35 \end{pmatrix} \neq \begin{pmatrix} -146 & -130 \\ -128 & 35 \end{pmatrix} = \mathsf{DC}.
\end{aligned}$$

These results show that whilst A, B and D are mutually commuting, none of them commutes with C.

We could use any of the three mutually commuting matrices to find the common set (actually a pair, as they are 2×2 matrices) of eigenvectors. We arbitrarily choose A. The eigenvalues of A satisfy

$$\begin{aligned}
\begin{vmatrix} 6-\lambda & -2 \\ -2 & 9-\lambda \end{vmatrix} &= 0, \\
\lambda^2 - 15\lambda + 50 &= 0, \\
(\lambda - 5)(\lambda - 10) &= 0.
\end{aligned}$$

For $\lambda = 5$, an eigenvector $(x, y)^{\mathrm{T}}$ must satisfy $x - 2y = 0$, whilst, for $\lambda = 10$, $4x + 2y = 0$. Thus a pair of independent eigenvectors of A are $(2, 1)^{\mathrm{T}}$ and $(1, -2)^{\mathrm{T}}$. Direct substitution verifies that they are also eigenvectors of B and D with pairs of eigenvalues 5, -15 and 15, 10, respectively.

8.17 *Find three real orthogonal column matrices, each of which is a simultaneous eigenvector of*

$$\mathsf{A} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \mathsf{B} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

We first note that

$$\mathsf{AB} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = \mathsf{BA}.$$

The two matrices commute and so they *will* have a common set of eigenvectors.

The eigenvalues of A are given by

$$\begin{vmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{vmatrix} = (1-\lambda)(\lambda^2 - 1) = 0,$$

i.e. $\lambda = 1$, $\lambda = 1$ and $\lambda = -1$, with corresponding eigenvectors $\mathsf{e}^1 = (1,\ y_1,\ 1)^{\mathrm{T}}$, $\mathsf{e}^2 = (1,\ y_2,\ 1)^{\mathrm{T}}$ and $\mathsf{e}^3 = (1,\ 0,\ -1)^{\mathrm{T}}$. For these to be mutually orthogonal requires that $y_1 y_2 = -2$.

The third vector, e^3, is clearly an eigenvector of B with eigenvalue $\mu_3 = -1$. For e^1 or e^2 to be an eigenvector of B with eigenvalue μ requires

$$\begin{pmatrix} 0-\mu & 1 & 1 \\ 1 & 0-\mu & 1 \\ 1 & 1 & 0-\mu \end{pmatrix} \begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix};$$

i.e. $\quad -\mu + y + 1 = 0,$

and $\quad 1 - \mu y + 1 = 0,$

giving $\quad -\dfrac{2}{y} + y + 1 = 0,$

$$\Rightarrow \quad y^2 + y - 2 = 0,$$

$$\Rightarrow \quad y = 1 \quad \text{or} \quad -2.$$

Thus, $y_1 = 1$ with $\mu_1 = 2$, whilst $y_2 = -2$ with $\mu_2 = -1$.

The common eigenvectors are thus

$$\mathsf{e}^1 = (1,\ 1,\ 1)^{\mathrm{T}}, \quad \mathsf{e}^2 = (1,\ -2,\ 1)^{\mathrm{T}}, \quad \mathsf{e}^3 = (1,\ 0,\ -1)^{\mathrm{T}}.$$

We note, as a check, that $\sum_i \mu_i = 2 + (-1) + (-1) = 0 = \mathrm{Tr}\ \mathsf{B}$.

8.19 *Given that* A *is a real symmetric matrix with normalised eigenvectors* e^i*, obtain the coefficients* α_i *involved when column matrix* x*, which is the solution of*

$$\mathsf{A}\mathsf{x} - \mu\mathsf{x} = \mathsf{v}, \qquad (*)$$

is expanded as $\mathsf{x} = \sum_i \alpha_i \mathsf{e}^i$*. Here* μ *is a given constant and* v *is a given column matrix.*

(a) *Solve* $(*)$ *when*

$$\mathsf{A} = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix},$$

$\mu = 2$ *and* $\mathsf{v} = (1 \quad 2 \quad 3)^{\mathrm{T}}$.

(b) *Would* $(*)$ *have a solution if* (i) $\mu = 1$ *and* $\mathsf{v} = (1 \quad 2 \quad 3)^{\mathrm{T}}$, (ii) $\mathsf{v} = (2 \quad 2 \quad 3)^{\mathrm{T}}$? *Where it does, find it.*

Let $\mathsf{x} = \sum_i \alpha_i \mathsf{e}^i$, where $\mathsf{A}\mathsf{e}^i = \lambda_i \mathsf{e}^i$. Then

$$\begin{aligned} \mathsf{A}\mathsf{x} - \mu\mathsf{x} &= \mathsf{v}, \\ \sum_i \mathsf{A}\alpha_i\mathsf{e}^i - \sum_i \mu\alpha_i\mathsf{e}^i &= \mathsf{v}, \\ \sum_i \left(\lambda_i\alpha_i\mathsf{e}^i - \mu\alpha_i\mathsf{e}^i\right) &= \mathsf{v}, \\ \alpha_j &= \frac{(\mathsf{e}^j)^\dagger \mathsf{v}}{\lambda_j - \mu}. \end{aligned}$$

To obtain the last line we have used the mutual orthogonality of the eigenvectors. We note, in passing, that if $\mu = \lambda_j$ for any j there is no solution unless $(\mathsf{e}^j)^\dagger \mathsf{v} = 0$.

(a) To obtain the eigenvalues of the given matrix A, consider

$$0 = |\mathsf{A} - \lambda\mathsf{I}| = (3-\lambda)(4 - 4\lambda + \lambda^2 - 1) = (3-\lambda)(3-\lambda)(1-\lambda).$$

The eigenvalues, and a possible set of corresponding normalised eigenvectors, are therefore,

$$\begin{aligned} &\text{for} \quad \lambda = 3, \ \mathsf{e}^1 = (0,\ 0,\ 1)^{\mathrm{T}}\,; \\ &\text{for} \quad \lambda = 3, \ \mathsf{e}^2 = 2^{-1/2}(1,\ 1,\ 0)^{\mathrm{T}}\,; \\ &\text{for} \quad \lambda = 1, \ \mathsf{e}^3 = 2^{-1/2}(1,\ -1,\ 0)^{\mathrm{T}}\,. \end{aligned}$$

Since $\lambda = 3$ is a degenerate eigenvalue, there are infinitely many acceptable pairs of orthogonal eigenvectors corresponding to it; any pair of vectors of the form (a_i, a_i, b_i) with $2a_1a_2 + b_1b_2 = 0$ will suffice. The pair given is just about the simplest choice possible.

With $\mu = 2$ and $\mathsf{v} = (1,\ 2,\ 3)^{\mathrm{T}}$,

$$\alpha_1 = \frac{3}{3-2}, \quad \alpha_2 = \frac{3/\sqrt{2}}{3-2}, \quad \alpha_3 = \frac{-1/\sqrt{2}}{1-2}.$$

Thus the solution vector is

$$\mathsf{x} = 3\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{3}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}.$$

(b) If $\mu = 1$ then it is equal to the third eigenvalue and a solution is only possible if $(\mathsf{e}^3)^\dagger \mathsf{v} = 0$.

For (i) $\mathsf{v} = (1,\ 2,\ 3)^{\mathrm{T}}$, $(\mathsf{e}^3)^\dagger \mathsf{v} = -1/\sqrt{2}$ and so no solution is possible.

For (ii) $\mathsf{v} = (2,\ 2,\ 3)^{\mathrm{T}}$, $(\mathsf{e}^3)^\dagger \mathsf{v} = 0$, and so a solution is possible. The other scalar products needed are $(\mathsf{e}^1)^\dagger \mathsf{v} = 3$ and $(\mathsf{e}^2)^\dagger \mathsf{v} = 2\sqrt{2}$. For this vector v the solution to the equation is

$$\mathsf{x} = \frac{3}{3-1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \frac{2\sqrt{2}}{3-1}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \frac{3}{2} \end{pmatrix}.$$

[The solutions to both parts can be checked by resubstitution.]

8.21 *By finding the eigenvectors of the Hermitian matrix*

$$\mathsf{H} = \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix},$$

construct a unitary matrix U *such that* $\mathsf{U}^\dagger \mathsf{H} \mathsf{U} = \Lambda$, *where* Λ *is a real diagonal matrix.*

We start by finding the eigenvalues of H using

$$\begin{vmatrix} 10-\lambda & 3i \\ -3i & 2-\lambda \end{vmatrix} = 0,$$

$$20 - 12\lambda + \lambda^2 - 3 = 0,$$

$$\lambda = 1 \quad \text{or} \quad 11.$$

As expected for an hermitian matrix, the eigenvalues are real.

For $\lambda = 1$ and normalised eigenvector $(x,\ y)^{\mathrm{T}}$,

$$9x + 3iy = 0 \qquad \Rightarrow \qquad \mathsf{x}^1 = (10)^{-1/2}\,(1,\ 3i)^{\mathrm{T}}.$$

For $\lambda = 11$ and normalised eigenvector $(x,\ y)^{\mathrm{T}}$,

$$-x + 3iy = 0 \qquad \Rightarrow \qquad \mathsf{x}^2 = (10)^{-1/2}\,(3i,\ 1)^{\mathrm{T}}\,.$$

Again as expected, $(\mathsf{x}^1)^\dagger \mathsf{x}^2 = 0$, thus verifying the mutual orthogonality of the eigenvectors. It should be noted that the normalisation factor is determined by $(\mathbf{x}^i)^\dagger \mathbf{x}^i = 1$ (and *not* by $(\mathbf{x}^i)^{\mathrm{T}} \mathbf{x}^i = 1$).

We now use these normalised eigenvectors of H as the columns of the matrix U and check that it is unitary:

$$\mathsf{U} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix}, \quad \mathsf{U}^\dagger = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix},$$

$$\mathsf{U}\mathsf{U}^\dagger = \frac{1}{10} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \mathsf{I}.$$

U has the further property that

$$\begin{aligned} \mathsf{U}^\dagger \mathsf{H} \mathsf{U} &= \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} \begin{pmatrix} 10 & 3i \\ -3i & 2 \end{pmatrix} \frac{1}{\sqrt{10}} \begin{pmatrix} 1 & 3i \\ 3i & 1 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 1 & -3i \\ -3i & 1 \end{pmatrix} \begin{pmatrix} 1 & 33i \\ 3i & 11 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 110 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 11 \end{pmatrix} = \Lambda. \end{aligned}$$

That the diagonal entries of Λ are the eigenvalues of H is in accord with the general theory of normal matrices.

8.23 *Given that the matrix*

$$\mathsf{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

has two eigenvectors of the form $(1 \quad y \quad 1)^{\mathrm{T}}$, *use the stationary property of the expression* $J(\mathsf{x}) = \mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}/(\mathsf{x}^{\mathrm{T}}\mathsf{x})$ *to obtain the corresponding eigenvalues. Deduce the third eigenvalue.*

Since A is real and symmetric, each eigenvalue λ is real. Further, from the first component of $\mathsf{A}\mathbf{x} = \lambda\mathbf{x}$, we have that $2 - y = \lambda$, showing that y is also real. Considered as a function of a general vector of the form $(1,\ y,\ 1)^{\mathrm{T}}$, the quadratic

form $\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x}$ can be written explicitly as

$$\begin{aligned}
\mathsf{x}^{\mathrm{T}}\mathsf{A}\mathsf{x} &= (1\ y\ 1)\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ y \\ 1 \end{pmatrix} \\
&= (1\ y\ 1)\begin{pmatrix} 2-y \\ 2y-2 \\ 2-y \end{pmatrix} \\
&= 2y^2 - 4y + 4.
\end{aligned}$$

The scalar product $\mathsf{x}^{\mathrm{T}}\mathsf{x}$ has the value $2+y^2$, and so we need to find the stationary values of

$$I = \frac{2y^2 - 4y + 4}{2 + y^2}.$$

These are given by

$$\begin{aligned}
0 = \frac{dI}{dy} &= \frac{(2+y^2)(4y-4) - (2y^2 - 4y + 4)2y}{(2+y^2)^2} \\
0 &= 4y^2 - 8, \\
y &= \pm\sqrt{2}.
\end{aligned}$$

The corresponding eigenvalues are the values of I at the stationary points, explicitly:

$$\begin{aligned}
\text{for} \quad y = \sqrt{2}, \qquad \lambda_1 &= \frac{2(2) - 4\sqrt{2} + 4}{2+2} = 2 - \sqrt{2}; \\
\text{for} \quad y = -\sqrt{2}, \qquad \lambda_2 &= \frac{2(2) + 4\sqrt{2} + 4}{2+2} = 2 + \sqrt{2}.
\end{aligned}$$

The final eigenvalue can be found using the fact that the sum of the eigenvalues is equal to the trace of the matrix; so

$$\lambda_3 = (2+2+2) - (2-\sqrt{2}) - (2+\sqrt{2}) = 2.$$

8.25 *The equation of a particular conic section is*

$$Q \equiv 8x_1^2 + 8x_2^2 - 6x_1x_2 = 110.$$

Determine the type of conic section this represents, the orientation of its principal axes, and relevant lengths in the directions of these axes.

The eigenvalues of the matrix $\begin{pmatrix} 8 & -3 \\ -3 & 8 \end{pmatrix}$ associated with the quadratic form on the LHS (without any prior scaling) are given by

$$\begin{aligned} 0 &= \begin{vmatrix} 8-\lambda & -3 \\ -3 & 8-\lambda \end{vmatrix} \\ &= \lambda^2 - 16\lambda + 55 \\ &= (\lambda - 5)(\lambda - 11). \end{aligned}$$

Referred to the corresponding eigenvectors as axes, the conic section (an ellipse since both eigenvalues are positive) will take the form

$$5y_1^2 + 11y_2^2 = 110 \quad \text{or, in standard form,} \quad \frac{y_1^2}{22} + \frac{y_2^2}{10} = 1.$$

Thus the semi-axes are of lengths $\sqrt{22}$ and $\sqrt{10}$; the former is in the direction of the vector $(x_1, x_2)^{\mathrm{T}}$ given by $(8-5)x_1 - 3x_2 = 0$, i.e. it is the line $x_1 = x_2$. The other principal axis will be the line at right angles to this, namely the line $x_1 = -x_2$.

8.27 *Find the direction of the axis of symmetry of the quadratic surface*

$$7x^2 + 7y^2 + 7z^2 - 20yz - 20xz + 20xy = 3.$$

The straightforward, but longer, solution to this exercise is as follows.

Consider the characteristic polynomial of the matrix associated with the quadratic surface, namely,

$$\begin{aligned} f(\lambda) &= \begin{vmatrix} 7-\lambda & 10 & -10 \\ 10 & 7-\lambda & -10 \\ -10 & -10 & 7-\lambda \end{vmatrix} \\ &= (7-\lambda)(-51 - 14\lambda + \lambda^2) + 10(30 + 10\lambda) - 10(-30 - 10\lambda) \\ &= -\lambda^3 + 21\lambda^2 + 153\lambda + 243. \end{aligned}$$

If the quadratic surface has an axis of symmetry, it must have two equal major axes (perpendicular to it), and hence the characteristic equation must have a repeated root. This same root will therefore also be a root of $df/d\lambda = 0$, i.e. of

$$\begin{aligned} -3\lambda^2 + 42\lambda + 153 &= 0, \\ \lambda^2 - 14\lambda - 51 &= 0, \\ \lambda &= 17 \quad \text{or} \quad -3. \end{aligned}$$

Substitution shows that -3 is a root (and therefore a double root) of $f(\lambda) = 0$, but that 17 is not. The non-repeated root can be calculated as the trace of the matrix minus the repeated roots, i.e. $21 - (-3) - (-3) = 27$. It is the eigenvector that corresponds to this eigenvalue that gives the direction $(x,\ y,\ z)^{\mathrm{T}}$ of the axis of symmetry. Its components must satisfy

$$(7 - 27)x + 10y - 10z = 0,$$
$$10x + (7 - 27)y - 10z = 0.$$

The axis of symmetry is therefore in the direction $(1,\ 1,\ -1)^{\mathrm{T}}$.

A more subtle solution is obtained by noting that setting $\lambda = -3$ makes *all three* of the rows (or columns) of the determinant multiples of each other, i.e. it reduces the determinant to rank one. Thus -3 is a repeated root of the characteristic equation and the third root is $21 - 2(-3) = 27$. The rest of the analysis is as above.

We note in passing that, as two eigenvalues are negative and equal, the surface is the hyperboloid of revolution obtained by rotating a (two-branched) hyperbola about its axis of symmetry. Referred to this axis and two others forming a mutually orthogonal set, the equation of the quadratic surface takes the form $-3\chi^2 - 3\eta^2 + 27\zeta^2 = 3$ and so the tips of the two 'nose cones' ($\chi = \eta = 0$) are separated by $\frac{2}{3}$ of a unit.

8.29 *This exercise demonstrates the reverse of the usual procedure of diagonalising a matrix.*

(a) *Rearrange the result* $\mathsf{A}' = \mathsf{S}^{-1}\mathsf{A}\mathsf{S}$ *(which shows how to make a change of basis that diagonalises* A*) so as to express the original matrix* A *in terms of the unitary matrix* S *and the diagonal matrix* A'*. Hence show how to construct a matrix* A *that has given eigenvalues and given (orthogonal) column matrices as its eigenvectors.*

(b) *Find the matrix that has as eigenvectors* $(1\quad 2\quad 1)^{\mathrm{T}}$, $(1\quad -1\quad 1)^{\mathrm{T}}$ *and* $(1\quad 0\quad -1)^{\mathrm{T}}$ *and corresponding eigenvalues* λ, μ *and* ν.

(c) *Try a particular case, say* $\lambda = 3$, $\mu = -2$ *and* $\nu = 1$, *and verify by explicit solution that the matrix so found does have these eigenvalues.*

(a) Since S is unitary, we can multiply the given result on the left by S and on the right by $\mathsf{S}^{\dagger}$ to obtain

$$\mathsf{S}\mathsf{A}'\mathsf{S}^{\dagger} = \mathsf{S}\mathsf{S}^{-1}\mathsf{A}\mathsf{S}\mathsf{S}^{\dagger} = (\mathsf{I})\,\mathsf{A}\,(\mathsf{I}) = \mathsf{A}.$$

More explicitly, in terms of the eigenvalues and normalised eigenvectors x^i of A,

$$\mathsf{A} = (\mathsf{x}^1 \quad \mathsf{x}^2 \quad \cdots \quad \mathsf{x}^n)\Lambda(\mathsf{x}^1 \quad \mathsf{x}^2 \quad \cdots \quad \mathsf{x}^n)^\dagger.$$

Here Λ is the diagonal matrix that has the eigenvalues of A as its diagonal elements.

Now, given normalised orthogonal column matrices and n specified values, we can use this result to construct a matrix that has the column matrices as eigenvectors and the values as eigenvalues.

(b) The normalised versions of the given column vectors are

$$\frac{1}{\sqrt{6}}(1,\ 2,\ 1)^{\mathrm{T}}, \quad \frac{1}{\sqrt{3}}(1,\ -1,\ 1)^{\mathrm{T}}, \quad \frac{1}{\sqrt{2}}(1,\ 0,\ -1)^{\mathrm{T}},$$

and the orthogonal matrix S can be constructed using these as its columns:

$$\mathsf{S} = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} \end{pmatrix}.$$

The required matrix A can now be formed as $\mathsf{S}\Lambda\mathsf{S}^\dagger$:

$$\begin{aligned}
\mathsf{A} &= \frac{1}{6}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} \end{pmatrix}\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ \sqrt{3} & 0 & -\sqrt{3} \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ 2 & -\sqrt{2} & 0 \\ 1 & \sqrt{2} & -\sqrt{3} \end{pmatrix}\begin{pmatrix} \lambda & 2\lambda & \lambda \\ \sqrt{2}\mu & -\sqrt{2}\mu & \sqrt{2}\mu \\ \sqrt{3}\nu & 0 & -\sqrt{3}\nu \end{pmatrix} \\
&= \frac{1}{6}\begin{pmatrix} \lambda+2\mu+3\nu & 2\lambda-2\mu & \lambda+2\mu-3\nu \\ 2\lambda-2\mu & 4\lambda+2\mu & 2\lambda-2\mu \\ \lambda+2\mu-3\nu & 2\lambda-2\mu & \lambda+2\mu+3\nu \end{pmatrix}.
\end{aligned}$$

(c) Setting $\lambda = 3$, $\mu = -2$ and $\nu = 1$, as a particular case, gives A as

$$\mathsf{A} = \frac{1}{6}\begin{pmatrix} 2 & 10 & -4 \\ 10 & 8 & 10 \\ -4 & 10 & 2 \end{pmatrix}.$$

We complete the exercise by solving for the eigenvalues of A in the usual way. To avoid working with fractions, and any confusion with the value $\lambda = 3$ used

when constructing A, we will find the eigenvalues of $6\mathsf{A}$ and denote them by η.

$$
\begin{aligned}
0 &= |\,6\mathsf{A} - \eta\mathsf{I}\,| \\
&= \begin{vmatrix} 2-\eta & 10 & -4 \\ 10 & 8-\eta & 10 \\ -4 & 10 & 2-\eta \end{vmatrix} \\
&= (2-\eta)(\eta^2 - 10\eta - 84) + 10(10\eta - 60) - 4(132 - 4\eta) \\
&= -\eta^3 + 12\eta^2 + 180\eta - 1296 \\
&= -(\eta - 6)(\eta^2 - 6\eta - 216) \\
&= -(\eta - 6)(\eta + 12)(\eta - 18).
\end{aligned}
$$

Thus $6\mathsf{A}$ has eigenvalues 6, -12 and 18; the values for A itself are 1, -2 and 3, as expected.

8.31 *One method of determining the nullity (and hence the rank) of an $M \times N$ matrix A is as follows.*

- *Write down an augmented transpose of A, by adding on the right an $N \times N$ unit matrix and thus producing an $N \times (M + N)$ array B.*
- *Subtract a suitable multiple of the first row of B from each of the other lower rows so as to make $B_{i1} = 0$ for $i > 1$.*
- *Subtract a suitable multiple of the second row (or the uppermost row that does not start with M zero values) from each of the other lower rows so as to make $B_{i2} = 0$ for $i > 2$.*
- *Continue in this way until all remaining rows have zeros in the first M places. The number of such rows is equal to the nullity of A, and the N rightmost entries of these rows are the components of vectors that span the null space. They can be made orthogonal if they are not so already.*

Use this method to show that the nullity of

$$
\mathsf{A} = \begin{pmatrix} -1 & 3 & 2 & 7 \\ 3 & 10 & -6 & 17 \\ -1 & -2 & 2 & -3 \\ 2 & 3 & -4 & 4 \\ 4 & 0 & -8 & -4 \end{pmatrix}
$$

is 2 and that an orthogonal base for the null space of A is provided by any two column matrices of the form $(2 + \alpha_i \quad -2\alpha_i \quad 1 \quad \alpha_i)^{\mathrm{T}}$, for which the α_i $(i = 1, 2)$ are real and satisfy $6\alpha_1\alpha_2 + 2(\alpha_1 + \alpha_2) + 5 = 0$.

We first construct B as

$$\mathsf{B} = \begin{pmatrix} -1 & 3 & -1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 3 & 10 & -2 & 3 & 0 & 0 & 1 & 0 & 0 \\ 2 & -6 & 2 & -4 & -8 & 0 & 0 & 1 & 0 \\ 7 & 17 & -3 & 4 & -4 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now, following the bulleted steps in the question, we obtain, successively,

$$\mathsf{B}_1 = \begin{pmatrix} -1 & 3 & -1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 19 & -5 & 9 & 12 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 38 & -10 & 18 & 24 & 7 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\mathsf{B}_2 = \begin{pmatrix} -1 & 3 & -1 & 2 & 4 & 1 & 0 & 0 & 0 \\ 0 & 19 & -5 & 9 & 12 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 0 & 1 \end{pmatrix}.$$

Since there are two rows that have all zeros in the first five places, the nullity of A is 2, and hence its rank is $4 - 2 = 2$.

The same two rows show that the null space is spanned by the vectors $(2 \quad 0 \quad 1 \quad 0)^{\mathrm{T}}$ and $(1 \quad -2 \quad 0 \quad 1)^{\mathrm{T}}$ and, therefore, by any two linear combinations of them of the general form $(2+\alpha_i \quad -2\alpha_i \quad 1 \quad \alpha_i)^{\mathrm{T}}$ for $i = 1, 2$, where α_i is any real number. If the basis is to be orthogonal then the scalar product of the two vectors must be zero, i.e.

$$\begin{aligned} (2+\alpha_1)(2+\alpha_2) + 4\alpha_1\alpha_2 + 1 + \alpha_1\alpha_2 &= 0, \\ 6\alpha_1\alpha_2 + 2(\alpha_1 + \alpha_2) + 5 &= 0. \end{aligned}$$

Thus α_1 may be chosen arbitrarily, but α_2 is then determined.

8.33 *Solve the simultaneous equations*

$$\begin{aligned} 2x + 3y + z &= 11, \\ x + y + z &= 6, \\ 5x - y + 10z &= 34. \end{aligned}$$

To eliminate z, (i) subtract the second equation from the first and (ii) subtract 10 times the second equation from the third.

$$\begin{aligned} x + 2y &= 5, \\ -5x - 11y &= -26. \end{aligned}$$

To eliminate x add 5 times the first equation to the second

$$-y = -1.$$

Thus $y = 1$ and, by resubstitution, $x = 3$ and $z = 2$.

8.35 *Show that the following equations have solutions only if $\eta = 1$ or 2, and find them in these cases:*

$$x + y + z = 1, \qquad \text{(i)}$$
$$x + 2y + 4z = \eta, \qquad \text{(ii)}$$
$$x + 4y + 10z = \eta^2. \qquad \text{(iii)}$$

Expressing the equations in the form $\mathsf{Ax} = \mathsf{b}$, we first need to evaluate $|\mathsf{A}|$ as a preliminary to determining A^{-1}. However, we find that $|\mathsf{A}| = 1(20 - 16) + 1(4 - 10) + 1(4 - 2) = 0$. This result implies both that A is singular and has no inverse, and that the equations must be linearly dependent.

Either by observation or by solving for the combination coefficients, we see that for the LHS this linear dependence is expressed by

$$2 \times \text{(i)} + 1 \times \text{(iii)} - 3 \times \text{(ii)} = 0.$$

For a consistent solution, this must also be true for the RHSs, i.e.

$$2 + \eta^2 - 3\eta = 0.$$

This quadratic equation has solutions $\eta = 1$ and $\eta = 2$, which are therefore the only values of η for which the original equations have a solution. As the equations are linearly dependent, we may use any two to find these allowed solutions; for simplicity we use the first two in each case.

For $\eta = 1$,

$$x + y + z = 1, \quad x + 2y + 4z = 1 \Rightarrow \mathsf{x}^1 = (1 + 2\alpha,\ -3\alpha,\ \alpha)^{\mathrm{T}}.$$

For $\eta = 2$,

$$x + y + z = 1, \quad x + 2y + 4z = 2 \Rightarrow \mathsf{x}^2 = (2\alpha,\ 1 - 3\alpha,\ \alpha)^{\mathrm{T}}.$$

In both cases there is an infinity of solutions as α may take any finite value.

8.37 *Make an LU decomposition of the matrix*

$$\mathsf{A} = \begin{pmatrix} 3 & 6 & 9 \\ 1 & 0 & 5 \\ 2 & -2 & 16 \end{pmatrix}$$

and hence solve $\mathsf{Ax} = \mathsf{b}$, *where* (i) $\mathsf{b} = (21 \quad 9 \quad 28)^{\mathrm{T}}$, (ii) $\mathsf{b} = (21 \quad 7 \quad 22)^{\mathrm{T}}$.

Using the notation

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ L_{21} & 1 & 0 \\ L_{31} & L_{32} & 1 \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{pmatrix},$$

and considering rows and columns alternately in the usual way for an LU decomposition, we require the following to be satisfied.

1st row: $U_{11} = 3, \quad U_{12} = 6, \quad U_{13} = 9.$
1st col: $L_{21}U_{11} = 1, \quad L_{31}U_{11} = 2 \Rightarrow L_{21} = \frac{1}{3}, \quad L_{31} = \frac{2}{3}.$
2nd row: $L_{21}U_{12} + U_{22} = 0, \quad L_{21}U_{13} + U_{23} = 5 \Rightarrow U_{22} = -2, \quad U_{23} = 2.$
2nd col: $L_{31}U_{12} + L_{32}U_{22} = -2 \Rightarrow L_{32} = 3.$
3rd row: $L_{31}U_{13} + L_{32}U_{23} + U_{33} = 16 \Rightarrow U_{33} = 4.$

Thus

$$\mathsf{L} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 3 & 1 \end{pmatrix} \quad \text{and} \quad \mathsf{U} = \begin{pmatrix} 3 & 6 & 9 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

To solve $\mathsf{Ax} = \mathsf{b}$ with $\mathsf{A} = \mathsf{LU}$, we first determine y from $\mathsf{Ly} = \mathsf{b}$ and then solve $\mathsf{Ux} = \mathsf{y}$ for x.

(i) For $\mathsf{Ax} = (21,\ 9,\ 28)^{\mathrm{T}}$, we first solve

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ \frac{2}{3} & 3 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 21 \\ 9 \\ 28 \end{pmatrix}.$$

This can be done, almost by inspection, to give $\mathsf{y} = (21,\ 2,\ 8)^{\mathrm{T}}$.

We can now write $\mathsf{U}\mathbf{x} = \mathbf{y}$ explicitly as

$$\begin{pmatrix} 3 & 6 & 9 \\ 0 & -2 & 2 \\ 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 21 \\ 2 \\ 8 \end{pmatrix}$$

to give, equally easily, that the solution to the original matrix equation is $\mathsf{x} = (-1,\ 1,\ 2)^{\mathrm{T}}$.

(ii) To solve $\mathsf{Ax} = (21,\ 7,\ 22)^{\mathrm{T}}$ we use exactly the same forms for L and U, but the new values for the components of b, to obtain $\mathsf{y} = (21,\ 0,\ 8)^{\mathrm{T}}$ leading to the solution $\mathsf{x} = (-3,\ 2,\ 2)^{\mathrm{T}}$.

8.39 *Use the Cholesky separation method to determine whether the following matrices are positive definite. For each that is, determine the corresponding lower diagonal matrix* L :

$$\mathsf{A} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ 3 & -1 & 1 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix}.$$

The matrix A is real and so we seek a real lower-diagonal matrix L such that $\mathsf{LL}^{\mathrm{T}} = \mathsf{A}$. In order to avoid a lot of subscripts, we use lower-case letters as the non-zero elements of L:

$$\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix} \begin{pmatrix} a & b & d \\ 0 & c & e \\ 0 & 0 & f \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & 3 & -1 \\ 3 & -1 & 1 \end{pmatrix}.$$

Firstly, from A_{11}, $a^2 = 2$. Since an overall negative sign multiplying the elements of L is irrelevant, we may choose $a = +\sqrt{2}$. Next, $ba = A_{12} = 1$, implying that $b = 1/\sqrt{2}$. Similarly, $d = 3/\sqrt{2}$.

From the second row of A we have

$$b^2 + c^2 = 3 \Rightarrow c = \sqrt{\tfrac{5}{2}},$$

$$bd + ce = -1 \Rightarrow e = \sqrt{\tfrac{2}{5}}(-1 - \tfrac{3}{2}) = -\sqrt{\tfrac{5}{2}}.$$

And, from the final row,

$$d^2 + e^2 + f^2 = 1 \Rightarrow f = (1 - \tfrac{9}{2} - \tfrac{5}{2})^{1/2} = \sqrt{-6}.$$

That f is imaginary shows that A is not a positive definite matrix.

The corresponding argument (keeping the same symbols but with different numerical values) for the matrix B is as follows.

Firstly, from A_{11}, $a^2 = 5$. Since an overall negative sign multiplying the elements of L is irrelevant, we may choose $a = +\sqrt{5}$. Next, $ba = B_{12} = 0$, implying that $b = 0$. Similarly, $d = \sqrt{3}/\sqrt{5}$.

From the second row of B we have

$$b^2 + c^2 = 3 \Rightarrow c = \sqrt{3},$$

$$bd + ce = 0 \Rightarrow e = \sqrt{\tfrac{1}{3}}(0 - 0) = 0.$$

And, from the final row,

$$d^2 + e^2 + f^2 = 3 \Rightarrow f = (3 - \tfrac{3}{5} - 0)^{1/2} = \sqrt{\tfrac{12}{5}}.$$

Thus all the elements of L have been calculated and found to be real and, in summary,

$$\mathsf{L} = \begin{pmatrix} \sqrt{5} & 0 & 0 \\ 0 & \sqrt{3} & 0 \\ \sqrt{\frac{3}{5}} & 0 & \sqrt{\frac{12}{5}} \end{pmatrix}.$$

That $\mathsf{L}\mathsf{L}^{\mathrm{T}} = \mathsf{B}$ can be confirmed by substitution.

8.41 *Find the SVD of*

$$\mathsf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix},$$

showing that the singular values are $\sqrt{3}$ *and* 1.

With

$$\mathsf{A} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{A}^{\dagger} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix},$$

$$\mathsf{A}^{\dagger}\mathsf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

which has eigenvalues given by $(2-\lambda)(2-\lambda) - 1 = 0$. The roots of this equation are $\lambda_1 = 3$ and $\lambda_2 = 1$, showing that the singular values s_i of A are $\sqrt{3}$ and $\sqrt{1}$.

The normalised eigenvectors $(x_1,\ x_2)^{\mathrm{T}}$ corresponding to these eigenvalues satisfy

$$(2-3)x_1 + x_2 = 0 \Rightarrow \mathsf{v}^1 = \frac{1}{\sqrt{2}}(1,\ 1)^{\mathrm{T}},$$
$$(2-1)x_1 + x_2 = 0 \Rightarrow \mathsf{v}^2 = \frac{1}{\sqrt{2}}(1,\ -1)^{\mathrm{T}}.$$

The next step is to calculate the (normalised) column vectors u^i from $(s_i)^{-1}\mathsf{A}\mathsf{v}^i = \mathsf{u}^i$:

$$\mathsf{u}^1 = \frac{1}{\sqrt{3}}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix},$$

and

$$\mathsf{u}^2 = \frac{1}{\sqrt{1}}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

For the third column vector we need one orthogonal to both u^1 and u^2; this can be obtained from their cross product and is $\mathsf{u}^3 = (1/\sqrt{3})(1,\ 1,\ 1)^{\mathrm{T}}$.

Finally, we can write A in SVD form:

$$\mathsf{A} = \mathsf{USV}^\dagger = \frac{1}{\sqrt{6}}\begin{pmatrix} -1 & \sqrt{3} & \sqrt{2} \\ 2 & 0 & \sqrt{2} \\ -1 & -\sqrt{3} & \sqrt{2} \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

where U and V are unitary. Both the unitarity and the decomposition can be checked by direct multiplication.

8.43 *Four experimental measurements of particular combinations of three physical variables, x, y and z, gave the following inconsistent results:*

$$\begin{aligned} 13x + 22y - 13z &= 4, \\ 10x - 8y - 10z &= 44, \\ 10x - 8y - 10z &= 47, \\ 9x - 18y - 9z &= 72. \end{aligned}$$

Find the SVD best values for x, y and z. Denoting the equations by $\mathsf{Ax} = \mathsf{b}$, identify the null space of A and hence obtain the general SVD solution.

The method of finding the SVD follows that of exercise 8.41.

We start by computing

$$\begin{aligned} \mathsf{A}^\dagger\mathsf{A} &= \begin{pmatrix} 13 & 10 & 10 & 9 \\ 22 & -8 & -8 & -18 \\ -13 & -10 & -10 & -9 \end{pmatrix}\begin{pmatrix} 13 & 22 & -13 \\ 10 & -8 & -10 \\ 10 & -8 & -10 \\ 9 & -18 & -9 \end{pmatrix} \\ &= \begin{pmatrix} 450 & -36 & -450 \\ -36 & 936 & 36 \\ -450 & 36 & 450 \end{pmatrix}. \end{aligned}$$

We next find its eigenvalues:

$$
\begin{aligned}
|\mathsf{A}^{\dagger}\mathsf{A} - \lambda| &= \begin{vmatrix} 450-\lambda & -36 & -450 \\ -36 & 936-\lambda & 36 \\ -450 & 36 & 450-\lambda \end{vmatrix} \\
&= \begin{vmatrix} -\lambda & 0 & -\lambda \\ -36 & 936-\lambda & 36 \\ -450 & 36 & 450-\lambda \end{vmatrix} \\
&= -\lambda(\lambda^2 - 1836\lambda + 839808) \\
&= -\lambda(\lambda - 864)(\lambda - 972).
\end{aligned}
$$

This shows that the singular values s_i are $\sqrt{972} = 18\sqrt{3}$, $\sqrt{864} = 12\sqrt{6}$ and 0.

The corresponding normalised eigenvectors $(x_1,\ x_2,\ x_3)^{\mathrm{T}}$, used to construct the orthogonal matrix V, satisfy

$$
\begin{aligned}
&-522x_1 - 36x_2 - 450x_3 = 0, \\
&\quad -36x_1 - 36x_2 + 36x_3 = 0 \Rightarrow \mathsf{v}^1 = \frac{1}{\sqrt{6}}(1,\ -2,\ -1)^{\mathrm{T}}\ ; \\
&-414x_1 - 36x_2 - 450x_3 = 0, \\
&\quad -36x_1 + 72x_2 + 36x_3 = 0 \Rightarrow \mathsf{v}^2 = \frac{1}{\sqrt{3}}(1,\ 1,\ -1)^{\mathrm{T}}\ ; \\
&\ 450x_1 - 36x_2 - 450x_3 = 0, \\
&\quad -36x_1 + 936x_2 + 36x_3 = 0 \Rightarrow \mathsf{v}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ 1)^{\mathrm{T}}.
\end{aligned}
$$

The singular value 0 implies that v^3 will be a vector in (and spanning) the null space of A, which therefore has rank 2 (rather than 3, as would be generally expected in this case).

For the non-zero singular values we now calculate the (normalised) column vectors u^i from $(s_i)^{-1}\mathsf{A}\mathsf{v}^i = \mathsf{u}^i$:

$$
\mathsf{u}^1 = \frac{1}{18\sqrt{3}}\frac{1}{\sqrt{6}}\begin{pmatrix} 13 & 22 & -13 \\ 10 & -8 & -10 \\ 10 & -8 & -10 \\ 9 & -18 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} -1 \\ 2 \\ 2 \\ 3 \end{bmatrix};
$$

$$
\mathsf{u}^2 = \frac{1}{12\sqrt{6}}\frac{1}{\sqrt{3}}\begin{pmatrix} 13 & 22 & -13 \\ 10 & -8 & -10 \\ 10 & -8 & -10 \\ 9 & -18 & -9 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{3\sqrt{2}}\begin{bmatrix} 4 \\ 1 \\ 1 \\ 0 \end{bmatrix}.
$$

Although we will not need their components for the present exercise, we now find

the third and fourth base vectors (to make U a unitary matrix). They must be solutions of $\mathsf{A}^\dagger \mathsf{u}^i = 0$; simple simultaneous equations show that, when normalised, two suitable vectors are

$$\mathsf{u}^3 = \frac{1}{\sqrt{2}}(0, -1, 1, 0)^{\mathrm{T}} \quad \text{and} \quad \mathsf{u}^4 = \frac{1}{\sqrt{18}}(1, -2, -2, 3)^{\mathrm{T}}.$$

Thus, we are able to write $\mathsf{A} = \mathsf{USV}^\dagger$ explicitly as

$$\frac{1}{N}\begin{pmatrix} -1 & 4 & 0 & 1 \\ 2 & 1 & -3 & -2 \\ 2 & 1 & 3 & -2 \\ 3 & 0 & 0 & 3 \end{pmatrix}\begin{pmatrix} 18\sqrt{3} & 0 & 0 \\ 0 & 12\sqrt{6} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & -2 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} \\ \sqrt{3} & 0 & \sqrt{3} \end{pmatrix},$$

where $N = \sqrt{18} \times \sqrt{6}$.

We now compute $\mathsf{R} = \mathsf{V}\bar{\mathsf{S}}\mathsf{U}^\dagger$ as (with N defined as before)

$$\frac{1}{N}\begin{pmatrix} 1 & \sqrt{2} & \sqrt{3} \\ -2 & \sqrt{2} & 0 \\ -1 & -\sqrt{2} & \sqrt{3} \end{pmatrix}\begin{pmatrix} \frac{1}{18\sqrt{3}} & 0 & 0 & 0 \\ 0 & \frac{1}{12\sqrt{6}} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 & 2 & 3 \\ 4 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & -2 & -2 & 3 \end{pmatrix}$$

$$= \frac{1}{N}\begin{pmatrix} \frac{1}{18\sqrt{3}} & \frac{1}{12\sqrt{3}} & 0 & 0 \\ -\frac{1}{9\sqrt{3}} & \frac{1}{12\sqrt{3}} & 0 & 0 \\ -\frac{1}{18\sqrt{3}} & -\frac{1}{12\sqrt{3}} & 0 & 0 \end{pmatrix}\begin{pmatrix} -1 & 2 & 2 & 3 \\ 4 & 1 & 1 & 0 \\ 0 & -3 & 3 & 0 \\ 1 & -2 & -2 & 3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{108}}\frac{1}{36\sqrt{3}}\begin{pmatrix} 10 & 7 & 7 & 6 \\ 16 & -5 & -5 & -12 \\ -10 & -7 & -7 & -6 \end{pmatrix}.$$

The best SVD solution is thus given by

$$\mathsf{Rb} = \frac{1}{648}\begin{pmatrix} 10 & 7 & 7 & 6 \\ 16 & -5 & -5 & -12 \\ -10 & -7 & -7 & -6 \end{pmatrix}\begin{bmatrix} 4 \\ 44 \\ 47 \\ 72 \end{bmatrix} = \begin{pmatrix} 1.711 \\ -1.937 \\ -1.711 \end{pmatrix}.$$

As noted previously, the null space of A is spanned by the vector $\mathsf{x}^3 = \frac{1}{\sqrt{2}}(1,\ 0,\ 1)^{\mathrm{T}}$. The general SVD solution is therefore

$$(1.71 + \lambda,\ -1.94,\ -1.71 + \lambda)^{\mathrm{T}}.$$

9

Normal modes

9.1 *Three coupled pendulums swing perpendicularly to the horizontal line containing their points of suspension, and the following equations of motion are satisfied:*

$$-m\ddot{x}_1 = cmx_1 + d(x_1 - x_2),$$
$$-M\ddot{x}_2 = cMx_2 + d(x_2 - x_1) + d(x_2 - x_3),$$
$$-m\ddot{x}_3 = cmx_3 + d(x_3 - x_2),$$

where x_1, x_2 and x_3 are measured from the equilibrium points; m, M and m are the masses of the pendulum bobs; and c and d are positive constants. Find the normal frequencies of the system and sketch the corresponding patterns of oscillation. What happens as $d \to 0$ or $d \to \infty$?

In a normal mode all three coordinates x_i oscillate with the same frequency and with fixed relative phases. When this is represented by solutions of the form $x_i = X_i \cos \omega t$, where the X_i are fixed constants, the equations become, in matrix and vector form,

$$\begin{pmatrix} cm + d - m\omega^2 & -d & 0 \\ -d & cM + 2d - M\omega^2 & -d \\ 0 & -d & cm + d - m\omega^2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = \mathbf{0}.$$

For there to be a non-trivial solution to these simultaneous homogeneous equa-

tions, we need

$$
\begin{aligned}
0 &= \begin{vmatrix} (c-\omega^2)m+d & -d & 0 \\ -d & (c-\omega^2)M+2d & -d \\ 0 & -d & (c-\omega^2)m+d \end{vmatrix} \\
&= \begin{vmatrix} (c-\omega^2)m+d & 0 & -(c-\omega^2)m-d \\ -d & (c-\omega^2)M+2d & -d \\ 0 & -d & (c-\omega^2)m+d \end{vmatrix} \\
&= [(c-\omega^2)m+d]\,\{[(c-\omega^2)M+2d]\,[(c-\omega^2)m+d]-d^2-d^2\} \\
&= (cm-m\omega^2+d)(c-\omega^2)[Mm(c-\omega^2)+2dm+dM].
\end{aligned}
$$

Thus, the normal (angular) frequencies are given by

$$\omega^2 = c, \quad \omega^2 = c + \frac{d}{m} \quad \text{and} \quad \omega^2 = c + \frac{2d}{M} + \frac{d}{m}.$$

If the solution column matrix is $\mathsf{X} = (X_1,\ X_2,\ X_3)^{\mathrm{T}}$, then

(i) for $\omega^2 = c$, the components of X must satisfy

$$
\begin{aligned}
dX_1 - dX_2 &= 0, \\
-dX_1 + 2dX_2 - dX_3 &= 0, \quad \Rightarrow \quad \mathsf{X}^1 = (1,\ 1,\ 1)^{\mathrm{T}}\,;
\end{aligned}
$$

(ii) for $\omega^2 = c + \dfrac{d}{m}$, we have

$$
\begin{aligned}
-dX_2 &= 0, \\
-dX_1 + \left(-\frac{dM}{m} + 2d\right) X_2 - dX_3 &= 0, \quad \Rightarrow \quad \mathsf{X}^2 = (1,\ 0,\ -1)^{\mathrm{T}}\,;
\end{aligned}
$$

(iii) for $\omega^2 = c + \dfrac{2d}{M} + \dfrac{d}{m}$, the components must satisfy

$$
\begin{aligned}
\left[\left(-\frac{2d}{M} - \frac{d}{m}\right) m + d\right] X_1 - dX_2 &= 0, \\
-dX_2 + \left[\left(-\frac{2d}{M} - \frac{d}{m}\right) m + d\right] X_3 &= 0, \quad \Rightarrow \quad \mathsf{X}^3 = \left(1,\ -\frac{2m}{M},\ 1\right)^{\mathrm{T}}.
\end{aligned}
$$

The corresponding patterns are shown in figure 9.1.

If $d \to 0$, the three oscillations decouple and each pendulum swings independently with angular frequency $\sqrt{c}$.

If $d \to \infty$, the three pendulums become rigidly coupled. The second and third modes have (theoretically) infinite frequency and therefore zero amplitude. The only sustainable mode is the one shown as case (b) in the figure; one in which all the pendulums swing as a single entity with angular frequency $\sqrt{c}$.

Figure 9.1 The normal modes, as viewed from above, of the coupled pendulums in exercise 9.1.

9.3 *Find the normal frequencies of a system consisting of three particles of masses $m_1 = m$, $m_2 = \mu m$, $m_3 = m$ connected in that order in a straight line by two equal light springs of force constant k. Describe the corresponding modes of oscillation.*

Now consider the particular case in which $\mu = 2$.

(a) *Show that the eigenvectors derived above have the expected orthogonality properties with respect to both the kinetic energy matrix* A *and the potential energy matrix* B.

(b) *For the situation in which the masses are released from rest with initial displacements (relative to their equilibrium positions) of $x_1 = 2\epsilon$, $x_2 = -\epsilon$ and $x_3 = 0$, determine their subsequent motions and maximum displacements.*

Let the coordinates of the particles, x_1, x_2, x_3, be measured from their equilibrium positions, at which the springs are neither extended nor compressed.

The kinetic energy of the system is simply

$$T = \tfrac{1}{2}m\left(\dot{x}_1^2 + \mu\,\dot{x}_2^2 + \dot{x}_3^2\right),$$

whilst the potential energy stored in the springs takes the form

$$V = \tfrac{1}{2}k\left[(x_2 - x_1)^2 + (x_3 - x_2)^2\right].$$

The kinetic- and potential-energy symmetric matrices are thus

$$\mathsf{A} = \frac{m}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad \mathsf{B} = \frac{k}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

To find the normal frequencies we have to solve $|\mathsf{B} - \omega^2\mathsf{A}| = 0$. Thus, writing $m\omega^2/k = \lambda$, we have

$$\begin{aligned} 0 &= \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\mu\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} \\ &= (1-\lambda)(2-\mu\lambda-2\lambda+\mu\lambda^2-1)+(-1+\lambda) \\ &= (1-\lambda)\lambda(-\mu-2+\mu\lambda), \end{aligned}$$

which leads to $\lambda = 0$, 1 or $1 + 2/\mu$.

The normalised eigenvectors corresponding to the first two eigenvalues can be found by inspection and are

$$\mathsf{x}^1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathsf{x}^2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

The components of the third eigenvector must satisfy

$$-\frac{2}{\mu}x_1 - x_2 = 0 \quad \text{and} \quad x_2 - \frac{2}{\mu}x_3 = 0.$$

The normalised third eigenvector is therefore

$$\mathsf{x}^3 = \frac{1}{\sqrt{2+(4/\mu^2)}}\left(1,\ -\frac{2}{\mu},\ 1\right)^{\mathrm{T}}.$$

The physical motions associated with these normal modes are as follows.

The first, with $\lambda = \omega = 0$ and all the x_i equal, merely describes bodily translation of the whole system, with no (i.e. zero-frequency) internal oscillations.

In the second solution, the central particle remains stationary, $x_2 = 0$, whilst the other two oscillate with equal amplitudes in antiphase with each other. This motion has frequency $\omega = (k/m)^{1/2}$, the same as that for the oscillations of a single mass m suspended from a single spring of force constant k.

The final and most complicated of the three normal modes has angular frequency $\omega = \{[(\mu+2)/\mu](k/m)\}^{1/2}$, and involves a motion of the central particle which is in antiphase with that of the two outer ones and which has an amplitude $2/\mu$ times as great. In this motion the two springs are compressed and extended in turn. We also note that in the second and third normal modes the centre of mass of the system remains stationary.

Now setting $\mu = 2$, we have as the three normal (angular) frequencies 0, Ω and $\sqrt{2}\Omega$, where $\Omega^2 = k/m$. The corresponding (unnormalised) eigenvectors are

$$\mathsf{x}^1 = (1,\ 1,\ 1)^{\mathrm{T}}, \quad \mathsf{x}^2 = (1,\ 0,\ -1)^{\mathrm{T}}, \quad \mathsf{x}^3 = (1,\ -1,\ 1)^{\mathrm{T}}.$$

(a) The matrices A and B have the forms

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

To verify the standard orthogonality relations we need to show that the quadratic forms $(\mathsf{x}^i)^\dagger \mathsf{A}\mathsf{x}^j$ and $(\mathsf{x}^i)^\dagger \mathsf{B}\mathsf{x}^j$ have zero value for $i \neq j$. Direct evaluation of all the separate cases is as follows:

$$\begin{aligned}
(\mathsf{x}^1)^\dagger \mathsf{A}\mathsf{x}^2 &= 1 + 0 - 1 = 0, \\
(\mathsf{x}^1)^\dagger \mathsf{A}\mathsf{x}^3 &= 1 - 2 + 1 = 0, \\
(\mathsf{x}^2)^\dagger \mathsf{A}\mathsf{x}^3 &= 1 + 0 - 1 = 0, \\
(\mathsf{x}^1)^\dagger \mathsf{B}\mathsf{x}^2 &= (\mathsf{x}^1)^\dagger (1,\ 0,\ -1)^{\mathrm{T}} = 1 + 0 - 1 = 0, \\
(\mathsf{x}^1)^\dagger \mathsf{B}\mathsf{x}^3 &= (\mathsf{x}^1)^\dagger (2,\ -4,\ 2)^{\mathrm{T}} = 2 - 4 + 2 = 0, \\
(\mathsf{x}^2)^\dagger \mathsf{B}\mathsf{x}^3 &= (\mathsf{x}^2)^\dagger (2,\ -4,\ 2)^{\mathrm{T}} = 2 + 0 - 2 = 0.
\end{aligned}$$

If $(\mathsf{x}^i)^\dagger \mathsf{A}\mathsf{x}^j$ has zero value then so does $(\mathsf{x}^j)^\dagger \mathsf{A}\mathsf{x}^i$ (and similarly for B). So there is no need to investigate the other six possibilities and the verification is complete.

(b) In order to determine the behaviour of the system we need to know which modes are present in the initial configuration. Each contributory mode will subsequently oscillate with its own frequency. In order to carry out this initial decomposition we write

$$(2\epsilon,\ -\epsilon,\ 0)^{\mathrm{T}} = a(1,\ 1,\ 1)^{\mathrm{T}} + b(1,\ 0,\ -1)^{\mathrm{T}} + c(1,\ -1,\ 1)^{\mathrm{T}},$$

from which it is clear that $a = 0$, $b = \epsilon$ and $c = \epsilon$. As each mode vibrates with its own frequency, the subsequent displacements are given by

$$\begin{aligned}
x_1 &= \epsilon(\cos \Omega t + \cos \sqrt{2}\Omega t), \\
x_2 &= -\epsilon \cos \sqrt{2}\Omega t, \\
x_3 &= \epsilon(-\cos \Omega t + \cos \sqrt{2}\Omega t).
\end{aligned}$$

Since Ω and $\sqrt{2}\Omega$ are not rationally related, at some times the two modes will, for all practical purposes (but not mathematically), be in phase and, at other times, be out of phase. Thus the maximum displacements will be $x_1(\max) = 2\epsilon$, $x_2(\max) = \epsilon$ and $x_3(\max) = 2\epsilon$.

Figure 9.2 The circuit and notation for exercise 9.5.

9.5 *It is shown in physics and engineering textbooks that circuits containing capacitors and inductors can be analysed by replacing a capacitor of capacitance C by a 'complex impedance' $1/(i\omega C)$ and an inductor of inductance L by an impedance $i\omega L$, where ω is the angular frequency of the currents flowing and $i^2 = -1$.*

Use this approach and Kirchhoff's circuit laws to analyse the circuit shown in figure 9.2 and obtain three linear equations governing the currents I_1, I_2 and I_3. Show that the only possible frequencies of self-sustaining currents satisfy either (a) $\omega^2 LC = 1$ *or* (b) $3\omega^2 LC = 1$. *Find the corresponding current patterns and, in each case, by identifying parts of the circuit in which no current flows, draw an equivalent circuit that contains only one capacitor and one inductor.*

We apply Kirchhoff's laws to the three closed loops $PQUP$, $SUTS$ and $TURT$ and obtain, respectively,

$$\frac{1}{i\omega C}I_1 + i\omega L(I_1 - I_3) + i\omega L(I_1 - I_2) = 0,$$
$$i\omega L(I_2 - I_1) + \frac{1}{i\omega C}I_2 = 0,$$
$$i\omega L(I_3 - I_1) + \frac{1}{i\omega C}I_3 = 0.$$

For these simultaneous homogeneous linear equations to be consistent, it is necessary that

$$0 = \begin{vmatrix} \frac{1}{i\omega C} + 2i\omega L & -i\omega L & -i\omega L \\ -i\omega L & \frac{1}{i\omega C} + i\omega L & 0 \\ -i\omega L & 0 & \frac{1}{i\omega C} + i\omega L \end{vmatrix} = \begin{vmatrix} \lambda - 2 & 1 & 1 \\ 1 & \lambda - 1 & 0 \\ 1 & 0 & \lambda - 1 \end{vmatrix},$$

where, after dividing all entries by $-i\omega L$, we have written the combination

$(LC\omega^2)^{-1}$ as λ to save space. Expanding the determinant gives

$$\begin{aligned} 0 &= (\lambda-2)(\lambda-1)^2-(\lambda-1)-(\lambda-1) \\ &= (\lambda-1)(\lambda^2-3\lambda+2-2) \\ &= \lambda(\lambda-1)(\lambda-3). \end{aligned}$$

Only the non-zero roots are of practical physical interest, and these are $\lambda = 1$ and $\lambda = 3$.

(a) The first of these eigenvalues has an eigenvector $\mathsf{I}^1 = (I_1,\ I_2,\ I_3)^{\mathrm{T}}$ that satisfies

$$\begin{aligned} -I_1+I_2+I_3 &= 0, \\ I_1 &= 0 \quad\Rightarrow\quad \mathsf{I}^1 = (0,\ 1,\ -1)^{\mathrm{T}}. \end{aligned}$$

Thus there is no current in PQ and the capacitor in that link can be ignored. Equal currents circulate, in opposite directions, in the other two loops and, although the link TU carries both, there is no transfer between the two loops. Each loop is therefore equivalent to a capacitor of capacitance C in parallel with an inductor of inductance L.

(b) The second eigenvalue has an eigenvector $\mathsf{I}^2 = (I_1,\ I_2,\ I_3)^{\mathrm{T}}$ that satisfies

$$\begin{aligned} I_1+I_2+I_3 &= 0, \\ I_1+2I_2 &= 0 \quad\Rightarrow\quad \mathsf{I}^2 = (-2,\ 1,\ 1)^{\mathrm{T}}. \end{aligned}$$

In this mode there is no current in TU and the circuit is equivalent to an inductor of inductance $L+L$ in parallel with a capacitor of capacitance $3C/2$; this latter capacitance is made up of C in parallel with the capacitance equivalent to two capacitors C in series, i.e. in parallel with $\frac{1}{2}C$. Thus, the equivalent single components are an inductance of $2L$ and a capacitance of $3C/2$.

9.7 *A double pendulum consists of two identical uniform rods, each of length ℓ and mass M, smoothly jointed together and suspended by attaching the free end of one rod to a fixed point. The system makes small oscillations in a vertical plane, with the angles made with the vertical by the upper and lower rods denoted by θ_1 and θ_2, respectively. The expressions for the kinetic energy T and the potential energy V of the system are (to second order in the θ_i)*

$$\begin{aligned} T &\approx Ml^2\left(\tfrac{8}{3}\dot\theta_1^2+2\dot\theta_1\dot\theta_2+\tfrac{2}{3}\dot\theta_2^2\right), \\ V &\approx Mgl\left(\tfrac{3}{2}\theta_1^2+\tfrac{1}{2}\theta_2^2\right). \end{aligned}$$

Determine the normal frequencies of the system and find new variables ξ and η that will reduce these two expressions to diagonal form, i.e. to

$$a_1\dot\xi^2+a_2\dot\eta^2 \qquad \textit{and} \qquad b_1\xi^2+b_2\eta^2.$$

To find the new variables we will use the following result. If the reader is not familiar with it, a standard textbook should be consulted.

If $Q_1 = \mathsf{u}^{\mathrm{T}}\mathsf{A}\mathsf{u}$ and $Q_2 = \mathsf{u}^{\mathrm{T}}\mathsf{B}\mathsf{u}$ are two real symmetric quadratic forms and u^n are those column matrices that satisfy

$$\mathsf{B}\mathsf{u}^n = \lambda_n \mathsf{A}\mathsf{u}^n,$$

then the matrix P whose columns are the vectors u^n is such that the change of variables $\mathsf{u} = \mathsf{P}\mathsf{v}$ reduces both quadratic forms simultaneously to sums of squares, i.e. $Q_1 = \mathsf{v}^{\mathrm{T}}\mathsf{C}\mathsf{v}$ and $Q_2 = \mathsf{v}^{\mathrm{T}}\mathsf{D}\mathsf{v}$, with both C and D diagonal.
Further points to note are:
(i) that for the u^i as determined above, $(\mathsf{u}^m)^{\mathrm{T}}\mathsf{A}\mathsf{u}^n = 0$ if $m \neq n$ and similarly if A is replaced by B;
(ii) that P is not in general an orthogonal matrix, even if the vectors u^n are normalised.
(iii) In the special case that A is the identity matrix I: the above procedure is the same as diagonalising B; P *is* an orthogonal matrix if normalised vectors are used; mutual orthogonality of the eigenvectors takes on its usual form.

This exercise is a physical example to which the above mathematical result can be applied, the two real symmetric (actually positive-definite) matrices being the kinetic and potential energy matrices.

$$\mathsf{A} = \begin{pmatrix} \frac{8}{3} & 1 \\ 1 & \frac{2}{3} \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} \frac{3}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \quad \text{with} \quad \lambda_i = \frac{\omega_i^2 l}{g}.$$

We find the normal frequencies by solving

$$\begin{aligned} 0 &= |\mathsf{B} - \lambda\mathsf{A}| \\ &= \begin{vmatrix} \frac{3}{2} - \frac{8}{3}\lambda & -\lambda \\ -\lambda & \frac{1}{2} - \frac{2}{3}\lambda \end{vmatrix} \\ &= \tfrac{3}{4} - \tfrac{7}{3}\lambda + \tfrac{16}{9}\lambda^2 - \lambda^2 \\ \Rightarrow \quad 0 &= 28\lambda^2 - 84\lambda + 27. \end{aligned}$$

Thus, $\lambda = 2.634$ or $\lambda = 0.3661$, and the normal frequencies are $(2.634g/l)^{1/2}$ and $(0.3661g/l)^{1/2}$.

The corresponding column vectors u^i have components that satisfy the following.
(i) For $\lambda = 0.3661$,

$$\left(\tfrac{3}{2} - \tfrac{8}{3}\,0.3661\right)\theta_1 - 0.3661\theta_2 = 0 \quad \Rightarrow \quad \mathsf{u}^1 = (1,\ 1.431)^{\mathrm{T}}.$$

(ii) For $\lambda = 2.634$,

$$\left(\tfrac{3}{2} - \tfrac{8}{3}\,2.634\right)\theta_1 - 2.634\theta_2 = 0 \quad \Rightarrow \quad \mathsf{u}^2 = (1,\ -2.097)^{\mathrm{T}}.$$

We can now construct P as

$$\mathsf{P} = \begin{pmatrix} 1 & 1 \\ 1.431 & -2.097 \end{pmatrix}$$

and define new variables (ξ, η) by $(\theta_1, \theta_2)^{\mathrm{T}} = \mathsf{P}(\xi, \eta)^{\mathrm{T}}$. When the substitutions $\theta_1 = \xi + \eta$ and $\theta_2 = 1.431\xi - 2.097\eta \equiv \alpha\xi - \beta\eta$ are made into the expressions for T and V, they both take on diagonal forms. This can be checked by computing the coefficients of $\xi\eta$ in the two expressions. They are as follows.

$$\text{For } V: \quad 3 - \alpha\beta = 0, \qquad \text{and} \qquad \text{for } T: \quad \frac{16}{3} + 2(\alpha - \beta) - \frac{4}{3}\alpha\beta = 0.$$

As an example, the full expression for the potential energy becomes $V = Mg\ell\,(2.524\,\xi^2 + 3.699\,\eta^2)$.

9.9 *Three particles each of mass m are attached to a light horizontal string having fixed ends, the string being thus divided into four equal portions, each of length a and under a tension T. Show that for small transverse vibrations the amplitudes x^i of the normal modes satisfy $\mathsf{B}\mathsf{x} = (ma\omega^2/T)\mathsf{x}$, where B is the matrix*

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

Estimate the lowest and highest eigenfrequencies using trial vectors $(3,\ 4,\ 3)^{\mathrm{T}}$ and $(3,\ -4,\ 3)^{\mathrm{T}}$. Use also the exact vectors $\left(1,\ \sqrt{2},\ 1\right)^{\mathrm{T}}$ and $\left(1,\ -\sqrt{2},\ 1\right)^{\mathrm{T}}$ and compare the results.

For the ith mass, with displacement y_i, the force it experiences as a result of the tension in the string connecting it to the $(i+1)$th mass is the resolved component of that tension perpendicular to the equilibrium line, i.e. $f = \dfrac{y_{i+1} - y_i}{a}T$. Similarly the force due to the tension in the string connecting it to the $(i-1)$th mass is $f = \dfrac{y_{i-1} - y_i}{a}T$. Because the ends of the string are fixed the notional zeroth and fourth masses have $y_0 = y_4 = 0$.

The equations of motion are, therefore,

$$\begin{aligned} m\ddot{x}_1 &= \frac{T}{a}[(0 - x_1) + (x_2 - x_1)], \\ m\ddot{x}_2 &= \frac{T}{a}[(x_1 - x_2) + (x_3 - x_2)], \\ m\ddot{x}_3 &= \frac{T}{a}[(x_2 - x_3) + (0 - x_3)]. \end{aligned}$$

If the displacements are written as $x_i = X_i \cos\omega t$ and $\mathsf{x} = (X_1,\ X_2,\ X_3)^{\mathrm{T}}$, then

these equations become

$$-\frac{ma\omega^2}{T}X_1 = -2X_1 + X_2,$$
$$-\frac{ma\omega^2}{T}X_2 = X_1 - 2X_2 + X_3,$$
$$-\frac{ma\omega^2}{T}X_3 = X_2 - 2X_3.$$

This set of equations can be written as $\mathsf{Bx} = \dfrac{ma\omega^2}{T}\mathsf{x}$, with

$$\mathsf{B} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The Rayleigh–Ritz method shows that any estimate λ of $\dfrac{\mathsf{x}^{\mathrm{T}}\mathsf{Bx}}{\mathsf{x}^{\mathrm{T}}\mathsf{x}}$ always lies between the lowest and highest possible values of $ma\omega^2/T$.

Using the suggested *trial* vectors gives the following estimates for λ.

(i) For $\mathsf{x} = (3,\ 4,\ 3)^{\mathrm{T}}$

$$\begin{aligned}\lambda &= [(3,\ 4,\ 3)\mathsf{B}\,(3,\ 4,\ 3)^{\mathrm{T}}]/34 \\ &= [(3,\ 4,\ 3)\,(2,\ 2,\ 2)^{\mathrm{T}}]/34 \\ &= 20/34 = 0.588.\end{aligned}$$

(ii) For $\mathsf{x} = (3,\ -4,\ 3)^{\mathrm{T}}$

$$\begin{aligned}\lambda &= [(3,\ -4,\ 3)\mathsf{B}\,(3,\ -4,\ 3)^{\mathrm{T}}]/34 \\ &= [(3,\ -4,\ 3)\,(10,\ -14,\ 10)^{\mathrm{T}}]/34 \\ &= 116/34 = 3.412.\end{aligned}$$

Using, instead, the *exact* vectors yields the exact values of λ as follows.

(i) For the eigenvector corresponding to the lowest eigenvalue, $\mathsf{x} = (1,\ \sqrt{2},\ 1)^{\mathrm{T}}$,

$$\begin{aligned}\lambda &= \left[(1,\ \sqrt{2},\ 1)\mathsf{B}(1,\ \sqrt{2},\ 1)^{\mathrm{T}}\right]/4 \\ &= \left[(1,\ \sqrt{2},\ 1)(2-\sqrt{2},\ 2\sqrt{2}-2,\ 2-\sqrt{2})^{\mathrm{T}}\right]/4 \\ &= 2-\sqrt{2} = 0.586.\end{aligned}$$

(ii) For the eigenvector corresponding to the highest eigenvalue, $\mathsf{x} = (1,\ -\sqrt{2},\ 1)^{\mathrm{T}}$,

$$\begin{aligned}\lambda &= \left[(1,\ -\sqrt{2},\ 1)\mathsf{B}(1,\ -\sqrt{2},\ 1)^{\mathrm{T}}\right]/4 \\ &= \left[(1,\ -\sqrt{2},\ 1)(2+\sqrt{2},\ -2\sqrt{2}-2,\ 2+\sqrt{2})^{\mathrm{T}}\right]/4 \\ &= 2+\sqrt{2} = 3.414.\end{aligned}$$

As can be seen, the (crude) trial vectors give excellent approximations to the lowest and highest eigenfrequencies.

10

Vector calculus

10.1 *Evaluate the integral*

$$\int \left[\, \mathbf{a}(\dot{\mathbf{b}} \cdot \mathbf{a} + \mathbf{b} \cdot \dot{\mathbf{a}}) + \dot{\mathbf{a}}(\mathbf{b} \cdot \mathbf{a}) - 2(\dot{\mathbf{a}} \cdot \mathbf{a})\mathbf{b} - \dot{\mathbf{b}}|\mathbf{a}|^2 \,\right] dt$$

in which $\dot{\mathbf{a}}$ *and* $\dot{\mathbf{b}}$ *are the derivatives of* $\mathbf{a}$ *and* $\mathbf{b}$ *with respect to* t.

In order to evaluate this integral, we need to group the terms in the integrand so that each is a part of the total derivative of a product of factors. Clearly, the first three terms are the derivative of $\mathbf{a}(\mathbf{b} \cdot \mathbf{a})$, i.e.

$$\frac{d}{dt}[\,\mathbf{a}(\mathbf{b} \cdot \mathbf{a})\,] = \dot{\mathbf{a}}(\mathbf{b} \cdot \mathbf{a}) + \mathbf{a}(\dot{\mathbf{b}} \cdot \mathbf{a}) + \mathbf{a}(\mathbf{b} \cdot \dot{\mathbf{a}}).$$

Similarly,

$$\frac{d}{dt}[\,\mathbf{b}(\mathbf{a} \cdot \mathbf{a})\,] = \dot{\mathbf{b}}(\mathbf{a} \cdot \mathbf{a}) + \mathbf{b}(\dot{\mathbf{a}} \cdot \mathbf{a}) + \mathbf{b}(\mathbf{a} \cdot \dot{\mathbf{a}}).$$

Hence,

$$\begin{aligned} I &= \int \left\{ \frac{d}{dt}[\,\mathbf{a}(\mathbf{b} \cdot \mathbf{a})\,] - \frac{d}{dt}[\,\mathbf{b}(\mathbf{a} \cdot \mathbf{a})\,] \right\} dt \\ &= \mathbf{a}(\mathbf{b} \cdot \mathbf{a}) - \mathbf{b}(\mathbf{a} \cdot \mathbf{a}) + \mathbf{h} \\ &= \mathbf{a} \times (\mathbf{a} \times \mathbf{b}) + \mathbf{h}, \end{aligned}$$

where $\mathbf{h}$ is the (vector) constant of integration. To obtain the final line above, we used a special case of the expansion of a vector triple product.

10.3 *The general equation of motion of a (non-relativistic) particle of mass m and charge q when it is placed in a region where there is a magnetic field* **B** *and an electric field* **E** *is*

$$m\ddot{\mathbf{r}} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B});$$

here **r** *is the position of the particle at time t and* $\dot{\mathbf{r}} = d\mathbf{r}/dt$*, etc. Write this as three separate equations in terms of the Cartesian components of the vectors involved.*

For the simple case of crossed uniform fields $\mathbf{E} = E\mathbf{i}$, $\mathbf{B} = B\mathbf{j}$*, in which the particle starts from the origin at* $t = 0$ *with* $\dot{\mathbf{r}} = v_0\mathbf{k}$*, find the equations of motion and show the following:*

(a) *if* $v_0 = E/B$ *then the particle continues its initial motion;*
(b) *if* $v_0 = 0$ *then the particle follows the space curve given in terms of the parameter* ξ *by*

$$x = \frac{mE}{B^2q}(1 - \cos\xi), \qquad y = 0, \qquad z = \frac{mE}{B^2q}(\xi - \sin\xi).$$

Interpret this curve geometrically and relate ξ *to t. Show that the total distance travelled by the particle after time t is given by*

$$\frac{2E}{B}\int_0^t \left|\sin\frac{Bqt'}{2m}\right| dt'.$$

Expressed in Cartesian coordinates, the components of the vector equation read

$$\begin{aligned} m\ddot{x} &= qE_x + q(\dot{y}B_z - \dot{z}B_y), \\ m\ddot{y} &= qE_y + q(\dot{z}B_x - \dot{x}B_z), \\ m\ddot{z} &= qE_z + q(\dot{x}B_y - \dot{y}B_x). \end{aligned}$$

For $E_x = E$, $B_y = B$ and all other field components zero, the equations reduce to

$$m\ddot{x} = qE - qB\dot{z}, \qquad m\ddot{y} = 0, \qquad m\ddot{z} = qB\dot{x}.$$

The second of these, together with the initial conditions $y(0) = \dot{y}(0) = 0$, implies that $y(t) = 0$ for all t. The final equation can be integrated directly to give

$$m\dot{z} = qBx + mv_0, \qquad (*)$$

which can now be substituted into the first to give a differential equation for x:

$$m\ddot{x} = qE - qB\left(\frac{qB}{m}x + v_0\right),$$

$$\Rightarrow \quad \ddot{x} + \left(\frac{qB}{m}\right)^2 x = \frac{q}{m}(E - v_0B).$$

(i) If $v_0 = E/B$ then the equation for x is that of simple harmonic motion and

$$x(t) = A\cos\omega t + B\sin\omega t,$$

where $\omega = qB/m$. However, in the present case, the initial conditions $x(0) = \dot{x}(0) = 0$ imply that $x(t) = 0$ for all t. Thus, there is no motion in either the x- or the y-direction and, as is then shown by (*), the particle continues with its initial speed v_0 in the z-direction.

(ii) If $v_0 = 0$, the equation of motion is

$$\ddot{x} + \omega^2 x = \frac{qE}{m},$$

which again has sinusoidal solutions but has a non-zero RHS. The full solution consists of the same complementary function as in part (i) together with the simplest possible particular integral, namely $x = qE/m\omega^2$. It is therefore

$$x(t) = A\cos\omega t + B\sin\omega t + \frac{qE}{m\omega^2}.$$

The initial condition $x(0) = 0$ implies that $A = -qE/(m\omega^2)$, whilst $\dot{x}(0) = 0$ requires that $B = 0$. Thus,

$$\begin{aligned} x &= \frac{qE}{m\omega^2}(1-\cos\omega t), \\ \Rightarrow \quad \dot{z} &= \frac{qB}{m}x = \omega\frac{qE}{m\omega^2}(1-\cos\omega t) = \frac{qE}{m\omega}(1-\cos\omega t). \end{aligned}$$

Since $z(0) = 0$, straightforward integration gives

$$z = \frac{qE}{m\omega}\left(t - \frac{\sin\omega t}{\omega}\right) = \frac{qE}{m\omega^2}(\omega t - \sin\omega t).$$

Thus, since $qE/m\omega^2 = mE/B^2q$, the path is of the given parametric form with $\xi = \omega t$. It is a cycloid in the plane $y = 0$; the x-coordinate varies in the restricted range $0 \le x \le 2qE/(m\omega^2)$, whilst the z-coordinate continually increases, though not at a uniform rate.

The element of path length is given by $ds^2 = dx^2 + dy^2 + dz^2$. In this case, writing $qE/(m\omega) = E/B$ as μ,

$$\begin{aligned} ds &= \left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt \\ &= \left[\mu^2\sin^2\omega t + \mu^2(1-\cos\omega t)^2\right]^{1/2} dt \\ &= \left[2\mu^2(1-\cos\omega t)\right]^{1/2} dt = 2\mu|\sin\tfrac{1}{2}\omega t|\, dt. \end{aligned}$$

Thus the total distance travelled after time t is given by

$$s = \int_0^t 2\mu|\sin\tfrac{1}{2}\omega t'|\, dt' = \frac{2E}{B}\int_0^t \left|\sin\frac{qBt'}{2m}\right| dt'.$$

10.5 *If two systems of coordinates with a common origin O are rotating with respect to each other, the measured accelerations differ in the two systems. Denoting by* $\mathbf{r}$ *and* $\mathbf{r}'$ *position vectors in frames $OXYZ$ and $OX'Y'Z'$, respectively, the connection between the two is*

$$\ddot{\mathbf{r}}' = \ddot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),$$

where $\boldsymbol{\omega}$ *is the angular velocity vector of the rotation of $OXYZ$ with respect to $OX'Y'Z'$ (taken as fixed). The third term on the RHS is known as the Coriolis acceleration, whilst the final term gives rise to a centrifugal force.*

Consider the application of this result to the firing of a shell of mass m from a stationary ship on the steadily rotating earth, working to the first order in ω ($= 7.3 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$). If the shell is fired with velocity $\mathbf{v}$ *at time $t = 0$ and only reaches a height that is small compared with the radius of the earth, show that its acceleration, as recorded on the ship, is given approximately by*

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times (\mathbf{v} + \mathbf{g}t),$$

where $m\mathbf{g}$ is the weight of the shell measured on the ship's deck.

The shell is fired at another stationary ship (a distance $\mathbf{s}$ *away) and* $\mathbf{v}$ *is such that the shell would have hit its target had there been no Coriolis effect.*

(a) *Show that without the Coriolis effect the time of flight of the shell would have been $\tau = -2\mathbf{g} \cdot \mathbf{v}/g^2$.*

(b) *Show further that when the shell actually hits the sea it is off-target by approximately*

$$\frac{2\tau}{g^2}[(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v}](\mathbf{g}\tau + \mathbf{v}) - (\boldsymbol{\omega} \times \mathbf{v})\tau^2 - \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})\tau^3.$$

(c) *Estimate the order of magnitude Δ of this miss for a shell for which the initial speed v is $300\,\mathrm{m\,s^{-1}}$, firing close to its maximum range ($\mathbf{v}$ makes an angle of $\pi/4$ with the vertical) in a northerly direction, whilst the ship is stationed at latitude $45°$ North.*

As the Earth is rotating steadily $\dot{\boldsymbol{\omega}} = \mathbf{0}$, and for the mass at rest on the deck,

$$m\ddot{\mathbf{r}}' = m\mathbf{g} + \mathbf{0} + 2\boldsymbol{\omega} \times \dot{\mathbf{0}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}).$$

This, including the centrifugal effect, defines $\mathbf{g}$ which is assumed constant throughout the trajectory.

For the moving mass ($\ddot{\mathbf{r}}'$ is unchanged),

$$m\mathbf{g} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) = m\ddot{\mathbf{r}} + 2m\boldsymbol{\omega} \times \dot{\mathbf{r}} + m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}),$$

i.e.
$$\ddot{\mathbf{r}} = \mathbf{g} - 2\omega \times \dot{\mathbf{r}}.$$

Now, $\omega\dot{r} \ll g$ and so to zeroth order in ω

$$\ddot{\mathbf{r}} = \mathbf{g} \quad \Rightarrow \quad \dot{\mathbf{r}} = \mathbf{g}t + \mathbf{v}.$$

Resubstituting this into the Coriolis term gives, to first order in ω,

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times (\mathbf{v} + \mathbf{g}t).$$

(a) With no Coriolis force,

$$\dot{\mathbf{r}} = \mathbf{g}t + \mathbf{v} \quad \text{and} \quad \mathbf{r} = \tfrac{1}{2}\mathbf{g}t^2 + \mathbf{v}t.$$

Let $\mathbf{s} = \frac{1}{2}\mathbf{g}\tau^2 + \mathbf{v}\tau$ and use the observation that $\mathbf{s} \cdot \mathbf{g} = 0$, giving

$$\tfrac{1}{2}g^2\tau^2 + \mathbf{v} \cdot \mathbf{g}\tau = 0 \quad \Rightarrow \quad \tau = -\frac{2\mathbf{v} \cdot \mathbf{g}}{g^2}.$$

(b) With Coriolis force,

$$\begin{aligned}
\ddot{\mathbf{r}} &= \mathbf{g} - 2(\boldsymbol{\omega} \times \mathbf{g})t - 2(\boldsymbol{\omega} \times \mathbf{v}),\\
\dot{\mathbf{r}} &= \mathbf{g}t - (\boldsymbol{\omega} \times \mathbf{g})t^2 - 2(\boldsymbol{\omega} \times \mathbf{v})t + \mathbf{v},\\
\mathbf{r} &= \tfrac{1}{2}\mathbf{g}t^2 - \tfrac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})t^3 - (\boldsymbol{\omega} \times \mathbf{v})t^2 + \mathbf{v}t. \qquad (*)
\end{aligned}$$

If the shell hits the sea at time T in the position $\mathbf{r} = \mathbf{s} + \Delta$, then $(\mathbf{s} + \Delta) \cdot \mathbf{g} = 0$, i.e.

$$\begin{aligned}
0 = (\mathbf{s} + \Delta) \cdot \mathbf{g} &= \tfrac{1}{2}g^2\, T^2 - 0 - (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}\, T^2 + \mathbf{v} \cdot \mathbf{g}\, T,\\
\Rightarrow \quad -\mathbf{v} \cdot \mathbf{g} &= T(\tfrac{1}{2}g^2 - (\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}),\\
\Rightarrow \quad T &= -\frac{\mathbf{v} \cdot \mathbf{g}}{\frac{1}{2}g^2}\left[1 - \frac{(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}}{\frac{1}{2}g^2}\right]^{-1}\\
&\approx \tau\left(1 + \frac{2(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}}{g^2} + \cdots\right).
\end{aligned}$$

Working to first order in ω, we may put $T = \tau$ in those terms in $(*)$ that involve another factor ω, namely $\boldsymbol{\omega} \times \mathbf{v}$ and $\boldsymbol{\omega} \times \mathbf{g}$. We then find, to this order, that

$$\begin{aligned}
\mathbf{s} + \Delta &= \frac{1}{2}\mathbf{g}\left(\tau^2 + \frac{4(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}}{g^2}\tau^2 + \cdots\right) - \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})\tau^3\\
&\quad -(\boldsymbol{\omega} \times \mathbf{v})\tau^2 + \mathbf{v}\tau + 2\frac{(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}}{g^2}\mathbf{v}\tau\\
&= \mathbf{s} + \frac{(\boldsymbol{\omega} \times \mathbf{v}) \cdot \mathbf{g}}{g^2}(2\mathbf{g}\tau^2 + 2\mathbf{v}\tau) - \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})\tau^3 - (\boldsymbol{\omega} \times \mathbf{v})\tau^2.
\end{aligned}$$

Hence, as stated in the question,

$$\Delta = \frac{2\tau}{g^2}[(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v}](\mathbf{g}\tau + \mathbf{v}) - (\boldsymbol{\omega} \times \mathbf{v})\tau^2 - \frac{1}{3}(\boldsymbol{\omega} \times \mathbf{g})\tau^3.$$

(c) With the ship at latitude 45° and firing the shell at close to 45° to the local

horizontal, $\mathbf{v}$ and $\boldsymbol{\omega}$ are almost parallel and the $\boldsymbol{\omega} \times \mathbf{v}$ term can be set to zero. Further, with $\mathbf{v}$ in a northerly direction, $(\mathbf{g} \times \boldsymbol{\omega}) \cdot \mathbf{v} = 0$.

Thus we are left with only the cubic term in τ. In this,

$$\tau = \frac{2 \times 300 \cos(\pi/4)}{9.8} = 43.3 \text{ s},$$

and $\boldsymbol{\omega} \times \mathbf{g}$ is in a westerly direction (recall that ω is directed northwards and $\mathbf{g}$ is directed downwards, towards the origin) and of magnitude $7\,10^{-5}\,9.8\,\sin(\pi/4) = 4.85\,10^{-4}$ m s^{-3}. Thus the miss is by approximately

$$-\tfrac{1}{3} \times 4.85\,10^{-4} \times (43.3)^3 = -13 \text{ m},$$

i.e. some 10 – 15 m to the East of its intended target.

10.7 *For the twisted space curve $y^3 + 27axz - 81a^2y = 0$, given parametrically by*

$$x = au(3 - u^2), \qquad y = 3au^2, \qquad z = au(3 + u^2),$$

show that the following hold:

(a) *$ds/du = 3\sqrt{2}a(1 + u^2)$, where s is the distance along the curve measured from the origin;*
(b) *the length of the curve from the origin to the Cartesian point $(2a, 3a, 4a)$ is $4\sqrt{2}a$;*
(c) *the radius of curvature at the point with parameter u is $3a(1 + u^2)^2$;*
(d) *the torsion τ and curvature κ at a general point are equal;*
(e) *any of the Frenet–Serret formulae that you have not already used directly are satisfied.*

(a) We must first calculate

$$\frac{d\mathbf{r}}{du} = (3a - 3au^2,\ 6au,\ 3a + 3au^2),$$

from which it follows that

$$\frac{ds}{du} = \left(\frac{d\mathbf{r}}{du} \cdot \frac{d\mathbf{r}}{du}\right)^{1/2} = 3a(1 - 2u^2 + u^4 + 4u^2 + 1 + 2u^2 + u^4)^{1/2}$$
$$= 3\sqrt{2}a(1 + u^2).$$

(b) The point $(2a, 3a, 4a)$ is given by $u = 1$; the origin is $u = 0$. The length of the curve from the origin to the point is therefore given by

$$s = \int_0^1 3\sqrt{2}a(1 + u^2)\,du = 3\sqrt{2}a\left[u + \frac{u^3}{3}\right]_0^1 = 4\sqrt{2}a.$$

(c) Using

$$\hat{\mathbf{t}} = \frac{d\mathbf{r}}{ds} = \frac{d\mathbf{r}}{du}\frac{du}{ds} = \frac{3a}{3\sqrt{2}a(1+u^2)}(1-u^2,\ 2u,\ 1+u^2),$$

we find that

$$\begin{aligned}\frac{d\hat{\mathbf{t}}}{ds} &= \frac{d\hat{\mathbf{t}}}{du}\frac{du}{ds}\\ &= \frac{1}{3\sqrt{2}a(1+u^2)}\frac{1}{\sqrt{2}}\left[\frac{d}{du}\left(\frac{1-u^2}{1+u^2}\right),\ \frac{d}{du}\left(\frac{2u}{1+u^2}\right),\ \frac{d}{du}\left(\frac{1+u^2}{1+u^2}\right)\right]\\ &= \frac{1}{6a(1+u^2)^3}\,(-4u,\ 2-2u^2,\ 0).\end{aligned}$$

We now recall that $d\hat{\mathbf{t}}/ds = \kappa\hat{\mathbf{n}}$, where κ is the curvature and the principal normal $\hat{\mathbf{n}}$ is a unit vector in the same direction as $d\hat{\mathbf{t}}/ds$. Thus

$$\frac{1}{\rho} = \kappa = \left|\frac{d\hat{\mathbf{t}}}{ds}\right| = \frac{2(4u^2+1-2u^2+u^4)^{1/2}}{6a(1+u^2)^3} = \frac{1}{3a(1+u^2)^2}.$$

(d) From part (c) we have the two results

$$\begin{aligned}\hat{\mathbf{t}} &= \frac{1}{\sqrt{2}(1+u^2)}(1-u^2,\ 2u,\ 1+u^2),\\ \hat{\mathbf{n}} &= \frac{1}{1+u^2}(-2u,\ 1-u^2,\ 0),\end{aligned}$$

and so the binormal $\hat{\mathbf{b}}$ is given by

$$\begin{aligned}\hat{\mathbf{b}} &= \hat{\mathbf{t}}\times\hat{\mathbf{n}}\\ &= \frac{1}{\sqrt{2}(1+u^2)^2}\left[u^4-1,\ -2u(1+u^2),\ (1+u^2)^2\right]\\ &= \frac{1}{\sqrt{2}}\left(\frac{u^2-1}{u^2+1},\ \frac{-2u}{u^2+1},\ 1\right).\end{aligned}$$

From this it follows that

$$\begin{aligned}\frac{d\hat{\mathbf{b}}}{ds} &= \frac{d\hat{\mathbf{b}}}{du}\frac{du}{ds}\\ &= \frac{1}{3\sqrt{2}a(1+u^2)}\frac{1}{\sqrt{2}}\left(\frac{4u}{(1+u^2)^2},\ \frac{2(u^2-1)}{(1+u^2)^2},\ 0\right).\end{aligned}$$

Comparing this with $-\tau\hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ as given above, shows that

$$\tau = \frac{2}{6a(1+u^2)^2}.$$

But

$$\kappa = \frac{1}{\rho} = \frac{1}{3a(1+u^2)^2},$$

thus establishing the result that τ equals κ for this curve.

(e) The remaining Frenet–Serret formula is

$$\frac{d\hat{\mathbf{n}}}{ds} = \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}}.$$

Consider the two sides of the equation separately:

$$\begin{aligned}
\text{LHS} &= \frac{d\hat{\mathbf{n}}}{ds} = \frac{d\hat{\mathbf{n}}}{du}\frac{du}{ds} \\
&= \frac{1}{3\sqrt{2}a(1+u^2)}\left[\frac{d}{du}\left(\frac{-2u}{1+u^2}\right), \frac{d}{du}\left(\frac{1-u^2}{1+u^2}\right), 0\right] \\
&= \frac{1}{3\sqrt{2}a(1+u^2)}\left[\frac{2u^2-2}{(1+u^2)^2}, \frac{-4u}{(1+u^2)^2}, 0\right] \\
&= \frac{1}{3\sqrt{2}a(1+u^2)^3}(2u^2-2,\ -4u,\ 0); \\
\text{RHS} &= \tau\hat{\mathbf{b}} - \kappa\hat{\mathbf{t}} = \kappa(\hat{\mathbf{b}} - \hat{\mathbf{t}}) \\
&= \frac{\kappa}{\sqrt{2}(1+u^2)}[u^2 - 1 - (1-u^2),\ -2u - 2u,\ 1 + u^2 - (1+u^2)] \\
&= \frac{1}{3\sqrt{2}a(1+u^2)^3}(2u^2-2,\ -4u,\ 0).
\end{aligned}$$

Thus, the two sides are equal and the unused formula is verified.

10.9 *In a magnetic field, field lines are curves to which the magnetic induction* $\mathbf{B}$ *is everywhere tangential. By evaluating* $d\mathbf{B}/ds$*, where* s *is the distance measured along a field line, prove that the radius of curvature at any point on a line is given by*

$$\rho = \frac{B^3}{|\mathbf{B} \times (\mathbf{B}\cdot\nabla)\mathbf{B}|}.$$

We start with the three simple vector relationships

$$\frac{d\mathbf{r}}{ds} = \hat{\mathbf{t}}, \quad \frac{d\hat{\mathbf{t}}}{ds} = \frac{1}{\rho}\hat{\mathbf{n}} \quad \text{and} \quad \mathbf{B} = B\hat{\mathbf{t}},$$

and note that

$$d\mathbf{B} = \frac{\partial\mathbf{B}}{\partial x}dx + \frac{\partial\mathbf{B}}{\partial y}dy + \frac{\partial\mathbf{B}}{\partial z}dz = (d\mathbf{r}\cdot\nabla)\mathbf{B}.$$

Differentiating the third relationship with respect to s gives

$$\frac{d\mathbf{B}}{ds} = \frac{dB}{ds}\hat{\mathbf{t}} + B\frac{d\hat{\mathbf{t}}}{ds}.$$

We can replace the LHS of this equation with

$$\frac{d\mathbf{B}}{ds} = \left(\frac{d\mathbf{r}\cdot\nabla}{ds}\right)\mathbf{B} = (\hat{\mathbf{t}}\cdot\nabla)\mathbf{B} = \frac{\mathbf{B}\cdot\nabla}{B}\mathbf{B}$$

and obtain

$$\frac{\mathbf{B}\cdot\nabla}{B}\mathbf{B} = \frac{dB}{ds}\hat{\mathbf{t}} + \frac{B}{\rho}\hat{\mathbf{n}}.$$

Finally, we take the cross product of this equation with $\hat{\mathbf{t}}$ and obtain

$$\begin{aligned}
\hat{\mathbf{t}}\times\frac{\mathbf{B}\cdot\nabla}{B}\mathbf{B} &= \mathbf{0} + \frac{B}{\rho}\hat{\mathbf{t}}\times\hat{\mathbf{n}},\\
\frac{\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}}{B^2} &= \frac{B}{\rho}\hat{\mathbf{b}},\\
\frac{|\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}|}{B^2} &= \frac{B}{\rho} \quad\Rightarrow\quad \rho = \frac{B^3}{|\mathbf{B}\times(\mathbf{B}\cdot\nabla)\mathbf{B}|}.
\end{aligned}$$

In the penultimate line we have given the unit vector $\hat{\mathbf{t}}\times\hat{\mathbf{n}}$ its usual symbol $\hat{\mathbf{b}}$ (for binormal), though the only property that is needed here is that it has unit length. To obtain the final line, we took the modulus of both sides of the equation on the previous one.

10.11 *Parameterising the hyperboloid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

by $x = a\cos\theta\sec\phi$, $y = b\sin\theta\sec\phi$, $z = c\tan\phi$, *show that an area element on its surface is*

$$dS = \sec^2\phi\left[c^2\sec^2\phi\left(b^2\cos^2\theta + a^2\sin^2\theta\right) + a^2b^2\tan^2\phi\right]^{1/2}\,d\theta\,d\phi.$$

Use this formula to show that the area of the curved surface $x^2 + y^2 - z^2 = a^2$ *between the planes* $z = 0$ *and* $z = 2a$ *is*

$$\pi a^2\left(6 + \frac{1}{\sqrt{2}}\sinh^{-1}2\sqrt{2}\right).$$

With $x = a\cos\theta\sec\phi$, $y = b\sin\theta\sec\phi$ and $z = c\tan\phi$, the tangent vectors to the surface are given in Cartesian coordinates by

$$\begin{aligned}
\frac{d\mathbf{r}}{d\theta} &= (-a\sin\theta\sec\phi,\ b\cos\theta\sec\phi,\ 0),\\
\frac{d\mathbf{r}}{d\phi} &= (a\cos\theta\sec\phi\tan\phi,\ b\sin\theta\sec\phi\tan\phi,\ c\sec^2\phi),
\end{aligned}$$

and the element of area by

$$\begin{aligned} dS &= \left| \frac{d\mathbf{r}}{d\theta} \times \frac{d\mathbf{r}}{d\phi} \right| d\theta\, d\phi \\ &= \left| (bc\cos\theta\sec^3\phi,\ ac\sin\theta\sec^3\phi,\ -ab\sec^2\phi\tan\phi) \right| d\theta\, d\phi \\ &= \sec^2\phi\left[c^2\sec^2\phi\left(b^2\cos^2\theta + a^2\sin^2\theta\right) + a^2b^2\tan^2\phi\right]^{1/2} d\theta\, d\phi. \end{aligned}$$

We set $b = c = a$ and note that the plane $z = 2a$ corresponds to $\phi = \tan^{-1} 2$. The ranges of integration are therefore $0 \le \theta < 2\pi$ and $0 \le \phi \le \tan^{-1} 2$, whilst

$$dS = \sec^2\phi(a^4\sec^2\phi + a^4\tan^2\phi)^{1/2}\, d\theta\, d\phi,$$

i.e. it is independent of θ.

To evaluate the integral of dS, we set $\tan\phi = \sinh\psi/\sqrt{2}$, with

$$\sec^2\phi\, d\phi = \frac{1}{\sqrt{2}}\cosh\psi\, d\psi \quad \text{and} \quad \sec^2\phi = 1 + \tfrac{1}{2}\sinh^2\psi.$$

The upper limit for ψ will be given by $\Psi = \sinh^{-1} 2\sqrt{2}$; we note that $\cosh\Psi = 3$. Integrating over θ and making the above substitutions yields

$$\begin{aligned} S &= 2\pi\int_0^\Psi \frac{1}{\sqrt{2}}\cosh\psi\, d\psi\, a^2\left(1 + \frac{1}{2}\sinh^2\psi + \frac{1}{2}\sinh^2\psi\right)^{1/2} \\ &= \sqrt{2}\pi a^2\int_0^\Psi \cosh^2\psi\, d\psi \\ &= \frac{\sqrt{2}\pi a^2}{2}\int_0^\Psi (\cosh 2\psi + 1)\, d\psi \\ &= \frac{\sqrt{2}\pi a^2}{2}\left[\frac{\sinh 2\psi}{2} + \psi\right]_0^\Psi \\ &= \frac{\pi a^2}{\sqrt{2}}\left[\sinh\psi\cosh\psi + \psi\right]_0^\Psi \\ &= \frac{\pi a^2}{\sqrt{2}}[(2\sqrt{2})(3) + \sinh^{-1} 2\sqrt{2}] = \pi a^2\left(6 + \frac{1}{\sqrt{2}}\sinh^{-1} 2\sqrt{2}\right). \end{aligned}$$

10.13 *Verify by direct calculation that*

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}).$$

The proof of this standard result for the divergence of a vector product is most

easily carried out in Cartesian coordinates though, of course, the result is valid in any three-dimensional coordinate system.

$$\begin{aligned}
\text{LHS} &= \nabla \cdot (\mathbf{a} \times \mathbf{b}) \\
&= \frac{\partial}{\partial x}(a_y b_z - a_z b_y) + \frac{\partial}{\partial y}(a_z b_x - a_x b_z) + \frac{\partial}{\partial z}(a_x b_y - a_y b_x) \\
&= a_x \left(-\frac{\partial b_z}{\partial y} + \frac{\partial b_y}{\partial z} \right) + a_y \left(\frac{\partial b_z}{\partial x} - \frac{\partial b_x}{\partial z} \right) + a_z \left(-\frac{\partial b_y}{\partial x} + \frac{\partial b_x}{\partial y} \right) \\
&\quad + b_x \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z} \right) + b_y \left(-\frac{\partial a_z}{\partial x} + \frac{\partial a_x}{\partial z} \right) + b_z \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y} \right) \\
&= -\mathbf{a} \cdot (\nabla \times \mathbf{b}) + \mathbf{b} \cdot (\nabla \times \mathbf{a}) = \text{RHS}.
\end{aligned}$$

10.15 *Evaluate the Laplacian of the function*

$$\psi(x, y, z) = \frac{zx^2}{x^2 + y^2 + z^2}$$

(a) *directly in Cartesian coordinates, and* (b) *after changing to a spherical polar coordinate system. Verify that, as they must, the two methods give the same result.*

(a) In Cartesian coordinates we need to evaluate

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}.$$

The required derivatives are

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= \frac{2xz(y^2 + z^2)}{(x^2 + y^2 + z^2)^2}, & \frac{\partial^2 \psi}{\partial x^2} &= \frac{(y^2 + z^2)(2zy^2 + 2z^3 - 6x^2 z)}{(x^2 + y^2 + z^2)^3}, \\
\frac{\partial \psi}{\partial y} &= \frac{-2x^2 yz}{(x^2 + y^2 + z^2)^2}, & \frac{\partial^2 \psi}{\partial y^2} &= -\frac{2zx^2(x^2 + z^2 - 3y^2)}{(x^2 + y^2 + z^2)^3}, \\
\frac{\partial \psi}{\partial z} &= \frac{x^2(x^2 + y^2 - z^2)}{(x^2 + y^2 + z^2)^2}, & \frac{\partial^2 \psi}{\partial z^2} &= -\frac{2zx^2(3x^2 + 3y^2 - z^2)}{(x^2 + y^2 + z^2)^3}.
\end{aligned}$$

Thus, writing $r^2 = x^2 + y^2 + z^2$,

$$\begin{aligned}
\nabla^2 \psi &= \frac{2z[(y^2 + z^2)(y^2 + z^2 - 3x^2) - 4x^4]}{(x^2 + y^2 + z^2)^3} \\
&= \frac{2z[(r^2 - x^2)(r^2 - 4x^2) - 4x^4]}{r^6} \\
&= \frac{2z(r^2 - 5x^2)}{r^4}.
\end{aligned}$$

(b) In spherical polar coordinates,

$$\psi(r,\theta,\phi) = \frac{r\cos\theta\, r^2 \sin^2\theta \cos^2\phi}{r^2} = r\cos\theta \sin^2\theta\cos^2\phi.$$

The three contributions to $\nabla^2\psi$ in spherical polars are

$$\begin{aligned}
(\nabla^2\psi)_r &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)\\
&= \frac{2}{r}\cos\theta\sin^2\theta\cos^2\phi,
\end{aligned}$$

$$\begin{aligned}
(\nabla^2\psi)_\theta &= \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)\\
&= \frac{1}{r}\frac{\cos^2\phi}{\sin\theta}\frac{\partial}{\partial\theta}\left[\sin\theta\frac{\partial}{\partial\theta}(\cos\theta\sin^2\theta)\right]\\
&= \frac{\cos^2\phi}{r}(4\cos^3\theta - 8\sin^2\theta\cos\theta),
\end{aligned}$$

$$\begin{aligned}
(\nabla^2\psi)_\phi &= \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}\\
&= \frac{\cos\theta}{r}(-2\cos^2\phi + 2\sin^2\phi).
\end{aligned}$$

Thus, the full Laplacian in spherical polar coordinates reads

$$\begin{aligned}
\nabla^2\psi &= \frac{\cos\theta}{r}(2\sin^2\theta\cos^2\phi + 4\cos^2\theta\cos^2\phi\\
&\qquad - 8\sin^2\theta\cos^2\phi - 2\cos^2\phi + 2\sin^2\phi)\\
&= \frac{\cos\theta}{r}(4\cos^2\phi - 10\sin^2\theta\cos^2\phi - 2\cos^2\phi + 2\sin^2\phi)\\
&= \frac{\cos\theta}{r}(2 - 10\sin^2\theta\cos^2\phi)\\
&= \frac{2r\cos\theta(r^2 - 5r^2\sin^2\theta\cos^2\phi)}{r^4}.
\end{aligned}$$

Rewriting this last expression in terms of Cartesian coordinates, one finally obtains

$$\nabla^2\psi = \frac{2z(r^2 - 5x^2)}{r^4},$$

which establishes the equivalence of the two approaches.

10.17 *The (Maxwell) relationship between a time-independent magnetic field* $\mathbf{B}$ *and the current density* $\mathbf{J}$ *(measured in SI units in* $\mathrm{A\,m^{-2}}$*) producing it,*

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$$

can be applied to a long cylinder of conducting ionised gas which, in cylindrical polar coordinates, occupies the region $\rho < a$.

(a) *Show that a uniform current density* $(0, C, 0)$ *and a magnetic field* $(0, 0, B)$, *with* B *constant* $(= B_0)$ *for* $\rho > a$ *and* $B = B(\rho)$ *for* $\rho < a$, *are consistent with this equation. Given that* $B(0) = 0$ *and that* $\mathbf{B}$ *is continuous at* $\rho = a$, *obtain expressions for* C *and* $B(\rho)$ *in terms of* B_0 *and* a.

(b) *The magnetic field can be expressed as* $\mathbf{B} = \nabla \times \mathbf{A}$, *where* $\mathbf{A}$ *is known as the vector potential. Show that a suitable* $\mathbf{A}$ *can be found which has only one non-vanishing component,* $A_\phi(\rho)$, *and obtain explicit expressions for* $A_\phi(\rho)$ *for both* $\rho < a$ *and* $\rho > a$. *Like* $\mathbf{B}$, *the vector potential is continuous at* $\rho = a$.

(c) *The gas pressure* $p(\rho)$ *satisfies the hydrostatic equation* $\nabla p = \mathbf{J} \times \mathbf{B}$ *and vanishes at the outer wall of the cylinder. Find a general expression for* p.

(a) In cylindrical polars with $\mathbf{B} = (0, 0, B(\rho))$, for $\rho \le a$ we have

$$\mu_0(0,\ C,\ 0) = \nabla \times \mathbf{B} = \left(\frac{1}{\rho}\frac{\partial B}{\partial \phi},\ -\frac{\partial B}{\partial \rho},\ 0 \right).$$

As expected, $\partial B/\partial \phi = 0$. The azimuthal component of the equation gives

$$-\frac{\partial B}{\partial \rho} = \mu_0 C \quad \text{for} \quad \rho \le a \quad \Rightarrow \quad B(\rho) = B(0) - \mu_0 C \rho.$$

Since $\mathbf{B}$ has to be differentiable at the origin of ρ and have no ϕ-dependence, $B(0)$ must be zero. This, together with $B = B_0$ for $\rho > a$ requires that $C = -B_0/(a\mu_0)$ and $B(\rho) = B_0\rho/a$ for $0 \le \rho \le a$.

(b) With $\mathbf{B} = \nabla \times \mathbf{A}$, consider $\mathbf{A}$ of the form $\mathbf{A} = (0,\ A(\rho),\ 0)$. Then

$$\begin{aligned}(0,\ 0,\ B(\rho)) &= \frac{1}{\rho}\left(\frac{\partial}{\partial z}(\rho A),\ 0,\ \frac{\partial}{\partial \rho}(\rho A) \right) \\ &= \left(0,\ 0,\ \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho A) \right).\end{aligned}$$

We now equate the only non-vanishing component on each side of the above equation, treating inside and outside the cylinder separately.

For $0 < \rho \leq a$,

$$\begin{aligned}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A) &= \frac{B_0\rho}{a},\\ \rho A &= \frac{B_0\rho^3}{3a} + D,\\ A(\rho) &= \frac{B_0\rho^2}{3a} + \frac{D}{\rho}.\end{aligned}$$

Since $A(0)$ must be finite (so that A is differentiable there), $D = 0$.

For $\rho > a$,

$$\begin{aligned}\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A) &= B_0,\\ \rho A &= \frac{B_0\rho^2}{2} + E,\\ A(\rho) &= \frac{1}{2}B_0\rho + \frac{E}{\rho}.\end{aligned}$$

At $\rho = a$, the continuity of $\mathbf{A}$ requires

$$\frac{B_0a^2}{3a} = \frac{1}{2}B_0a + \frac{E}{a} \Rightarrow E = -\frac{B_0a^2}{6}.$$

Thus, to summarise,

$$\begin{aligned}&A(\rho) = \frac{B_0\rho^2}{3a} \quad \text{for} \quad 0 \leq \rho \leq a,\\ \text{and} \qquad &A(\rho) = B_0\left(\frac{\rho}{2} - \frac{a^2}{6\rho}\right) \quad \text{for} \quad \rho \geq a.\end{aligned}$$

(c) For the gas pressure $p(\rho)$ in the region $0 < \rho \leq a$, we have $\nabla p = \mathbf{J} \times \mathbf{B}$. In component form,

$$\left(\frac{dp}{d\rho},\ 0,\ 0\right) = \left(0,\ -\frac{B_0}{a\mu_0},\ 0\right) \times \left(0,\ 0,\ \frac{B_0\rho}{a}\right),$$

with $p(a) = 0$.

$$\frac{dp}{d\rho} = -\frac{B_0^2\rho}{\mu_0a^2} \quad \Rightarrow \quad p(\rho) = \frac{B_0^2}{2\mu_0}\left[1 - \left(\frac{\rho}{a}\right)^2\right].$$

10.19 *Maxwell's equations for electromagnetism in free space (i.e. in the absence of charges, currents and dielectric or magnetic media) can be written*

$$\text{(i)}\ \nabla\cdot\mathbf{B}=0,\qquad \text{(ii)}\ \nabla\cdot\mathbf{E}=0,$$
$$\text{(iii)}\ \nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=\mathbf{0},\quad \text{(iv)}\ \nabla\times\mathbf{B}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\mathbf{0}.$$

A vector $\mathbf{A}$ *is defined by* $\mathbf{B}=\nabla\times\mathbf{A}$, *and a scalar* ϕ *by* $\mathbf{E}=-\nabla\phi-\partial\mathbf{A}/\partial t$. *Show that if the condition*

$$\text{(v)}\ \nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\phi}{\partial t}=0$$

is imposed (this is known as choosing the Lorentz gauge), then $\mathbf{A}$ *and* ϕ *satisfy wave equations as follows.*

$$\text{(vi)}\ \nabla^2\phi-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}=0,$$
$$\text{(vii)}\ \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\mathbf{0}.$$

The reader is invited to proceed as follows.

(a) *Verify that the expressions for* $\mathbf{B}$ *and* $\mathbf{E}$ *in terms of* $\mathbf{A}$ *and* ϕ *are consistent with* (i) *and* (iii).
(b) *Substitute for* $\mathbf{E}$ *in* (ii) *and use the derivative with respect to time of* (v) *to eliminate* $\mathbf{A}$ *from the resulting expression. Hence obtain* (vi).
(c) *Substitute for* $\mathbf{B}$ *and* $\mathbf{E}$ *in* (iv) *in terms of* $\mathbf{A}$ *and* ϕ. *Then use the gradient of* (v) *to simplify the resulting equation and so obtain* (vii).

(a) Substituting for $\mathbf{B}$ in (i),

$$\nabla\cdot\mathbf{B}=\nabla\cdot(\nabla\times\mathbf{A})=0,\quad \text{as it is for } \textit{any} \text{ vector } \mathbf{A}.$$

Substituting for $\mathbf{E}$ and $\mathbf{B}$ in (iii),

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t}=-(\nabla\times\nabla\phi)-\nabla\times\frac{\partial\mathbf{A}}{\partial t}+\frac{\partial}{\partial t}(\nabla\times\mathbf{A})=\mathbf{0}.$$

Here we have used the facts that $\nabla\times\nabla\phi=\mathbf{0}$ for any scalar, and that, since $\partial/\partial t$ and ∇ act on different variables, the order in which they are applied to $\mathbf{A}$ can be reversed. Thus (i) and (iii) are automatically satisfied if $\mathbf{E}$ and $\mathbf{B}$ are represented in terms of $\mathbf{A}$ and ϕ.

(b) Substituting for **E** in (ii) and taking the time derivative of (v),

$$0 = \nabla \cdot \mathbf{E} = -\nabla^2\phi - \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}),$$

$$0 = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

Adding these equations gives

$$0 = -\nabla^2\phi + \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}.$$

This is result (vi), the wave equation for ϕ.

(c) Substituting for **B** and **E** in (iv) and taking the gradient of (v),

$$\nabla \times (\nabla \times \mathbf{A}) - \frac{1}{c^2}\left(-\frac{\partial}{\partial t}\nabla\phi - \frac{\partial^2\mathbf{A}}{\partial t^2}\right) = \mathbf{0},$$

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\phi) + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mathbf{0}.$$

From (v), $$\nabla(\nabla \cdot \mathbf{A}) + \frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\phi) = \mathbf{0}.$$

Subtracting these gives $$-\nabla^2\mathbf{A} + \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = \mathbf{0}.$$

In the second line we have used the vector identity

$$\nabla^2\mathbf{F} = \nabla(\nabla \cdot \mathbf{F}) - \nabla \times (\nabla \times \mathbf{F})$$

to replace $\nabla \times (\nabla \times \mathbf{A})$. The final equation is result (vii).

10.21 *Paraboloidal coordinates u, v, ϕ are defined in terms of Cartesian coordinates by*

$$x = uv\cos\phi, \qquad y = uv\sin\phi, \qquad z = \tfrac{1}{2}(u^2 - v^2).$$

Identify the coordinate surfaces in the u, v, ϕ system. Verify that each coordinate surface (u = constant, say) intersects every coordinate surface on which one of the other two coordinates (v, say) is constant. Show further that the system of coordinates is an orthogonal one and determine its scale factors. Prove that the u-component of $\nabla \times \mathbf{a}$ is given by

$$\frac{1}{(u^2+v^2)^{1/2}}\left(\frac{a_\phi}{v} + \frac{\partial a_\phi}{\partial v}\right) - \frac{1}{uv}\frac{\partial a_v}{\partial \phi}.$$

To find a surface of constant u we eliminate v from the given relationships:

$$x^2 + y^2 = u^2v^2 \quad \Rightarrow \quad 2z = u^2 - \frac{x^2+y^2}{u^2}.$$

This is an inverted paraboloid of revolution about the z-axis. The range of z is $-\infty < z \leq \frac{1}{2}u^2$.

Similarly, the surface of constant v is given by

$$2z = \frac{x^2 + y^2}{v^2} - v^2.$$

This is also a paraboloid of revolution about the z-axis, but this time it is not inverted. The range of z is $-\frac{1}{2}v^2 \leq z < \infty$.

Since every constant-u paraboloid has some part of its surface in the region $z > 0$ and every constant-v paraboloid has some part of its surface in the region $z < 0$, it follows that every member of the first set intersects each member of the second, and vice-versa.

The surfaces of constant ϕ, $y = x \tan \phi$, are clearly (half-) planes containing the z-axis; each cuts the members of the other two sets in parabolic lines.

We now determine (the Cartesian components of) the tangential vectors and test their orthogonality:

$$\begin{aligned}
\mathbf{e}_1 &= \frac{\partial \mathbf{r}}{\partial u} = (v \cos \phi,\ v \sin \phi,\ u), \\
\mathbf{e}_2 &= \frac{\partial \mathbf{r}}{\partial v} = (u \cos \phi,\ u \sin \phi,\ -v), \\
\mathbf{e}_3 &= \frac{\partial \mathbf{r}}{\partial \phi} = (-uv \sin \phi,\ uv \cos \phi,\ 0), \\
\mathbf{e}_1 \cdot \mathbf{e}_2 &= uv(\cos \phi \cos \phi + \sin \phi \sin \phi) - uv = 0, \\
\mathbf{e}_2 \cdot \mathbf{e}_3 &= u^2 v(-\cos \phi \sin \phi + \sin \phi \cos \phi) = 0, \\
\mathbf{e}_1 \cdot \mathbf{e}_3 &= uv^2(-\cos \phi \sin \phi + \sin \phi \cos \phi) = 0.
\end{aligned}$$

This shows that all pairs of tangential vectors are orthogonal and therefore that the coordinate system is an orthogonal one. Its scale factors are given by the magnitudes of these tangential vectors:

$$\begin{aligned}
h_u^2 &= |\mathbf{e}_1|^2 = (v \cos \phi)^2 + (v \sin \phi)^2 + u^2 = u^2 + v^2, \\
h_v^2 &= |\mathbf{e}_2|^2 = (u \cos \phi)^2 + (u \sin \phi)^2 + v^2 = u^2 + v^2, \\
h_\phi^2 &= |\mathbf{e}_3|^2 = (uv \sin \phi)^2 + (uv \cos \phi)^2 = u^2 v^2.
\end{aligned}$$

Thus

$$h_u = h_v = \sqrt{u^2 + v^2}, \qquad h_\phi = uv.$$

The u-component of $\nabla \times \mathbf{a}$ is given by

$$\begin{aligned}[\nabla \times \mathbf{a}]_u &= \frac{h_u}{h_u h_v h_\phi}\left[\frac{\partial}{\partial v}(h_\phi a_\phi) - \frac{\partial}{\partial \phi}(h_v a_v)\right] \\ &= \frac{1}{uv\sqrt{u^2+v^2}}\left[\frac{\partial}{\partial v}(uva_\phi) - \frac{\partial}{\partial \phi}(\sqrt{u^2+v^2}a_v)\right] \\ &= \frac{1}{\sqrt{u^2+v^2}}\left(\frac{a_\phi}{v} + \frac{\partial a_\phi}{\partial v}\right) - \frac{1}{uv}\frac{\partial a_v}{\partial \phi},\end{aligned}$$

as stated in the question.

10.23 *Hyperbolic coordinates u, v, ϕ are defined in terms of Cartesian coordinates by*

$$x = \cosh u \cos v \cos \phi, \qquad y = \cosh u \cos v \sin \phi, \qquad z = \sinh u \sin v.$$

Sketch the coordinate curves in the $\phi = 0$ plane, showing that far from the origin they become concentric circles and radial lines. In particular, identify the curves $u = 0$, $v = 0$, $v = \pi/2$ and $v = \pi$. Calculate the tangent vectors at a general point, show that they are mutually orthogonal and deduce that the appropriate scale factors are

$$h_u = h_v = (\cosh^2 u - \cos^2 v)^{1/2}, \qquad h_\phi = \cosh u \cos v.$$

Find the most general function $\psi(u)$ of u only that satisfies Laplace's equation $\nabla^2\psi = 0$.

In the plane $\phi = 0$, i.e. $y = 0$, the curves $u =$ constant have x and z connected by

$$\frac{x^2}{\cosh^2 u} + \frac{z^2}{\sinh^2 u} = 1.$$

This general form is that of an ellipse, with foci at $(\pm 1, 0)$. With $u = 0$, it is the line joining the two foci (covered twice). As $u \to \infty$, and $\cosh u \approx \sinh u$ the form becomes that of a circle of very large radius.

The curves $v =$ constant are expressed by

$$\frac{x^2}{\cos^2 v} - \frac{z^2}{\sin^2 v} = 1.$$

These curves are hyperbolae that, for large x and z and fixed v, approximate $z = \pm x \tan v$, i.e. radial lines. The curve $v = 0$ is the part of the x-axis $1 \le x \le \infty$ (covered twice), whilst the curve $v = \pi$ is its reflection in the z-axis. The curve $v = \pi/2$ *is* the z-axis.

In Cartesian coordinates a general point and its derivatives with respect to u, v and ϕ are given by

$$\begin{aligned}
\mathbf{r} &= \cosh u \cos v \cos\phi\,\mathbf{i} + \cosh u \cos v \sin\phi\,\mathbf{j} + \sinh u \sin v\,\mathbf{k},\\
\mathbf{e_1} &= \frac{\partial \mathbf{r}}{\partial u} = \sinh u \cos v \cos\phi\,\mathbf{i} + \sinh u \cos v \sin\phi\,\mathbf{j} + \cosh u \sin v\,\mathbf{k},\\
\mathbf{e_2} &= \frac{\partial \mathbf{r}}{\partial v} = -\cosh u \sin v \cos\phi\,\mathbf{i} - \cosh u \sin v \sin\phi\,\mathbf{j} + \sinh u \cos v\,\mathbf{k},\\
\mathbf{e_3} &= \frac{\partial \mathbf{r}}{\partial \phi} = \cosh u \cos v(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}).
\end{aligned}$$

Now consider the scalar products:

$$\begin{aligned}
\mathbf{e}_1 \cdot \mathbf{e}_2 &= \sinh u \cos v \cosh u \sin v(-\cos^2\phi - \sin^2\phi + 1) = 0,\\
\mathbf{e}_1 \cdot \mathbf{e}_3 &= \sinh u \cos^2 v \cosh u(-\sin\phi\cos\phi + \sin\phi\cos\phi) = 0,\\
\mathbf{e}_2 \cdot \mathbf{e}_3 &= \cosh^2 u \sin v \cos v(\sin\phi\cos\phi - \sin\phi\cos\phi) = 0.
\end{aligned}$$

As each is zero, the system is an orthogonal one.

The scale factors are given by $|\mathbf{e}_i|$ and are thus found from:

$$\begin{aligned}
|\mathbf{e}_1|^2 &= \sinh^2 u \cos^2 v(\cos^2\phi + \sin^2\phi) + \cosh^2 u \sin^2 v\\
&= (\cosh^2 u - 1)\cos^2 v + \cosh^2 u(1 - \cos^2 v)\\
&= \cosh^2 u - \cos^2 v;\\
|\mathbf{e}_2|^2 &= \cosh^2 u \sin^2 v(\cos^2\phi + \sin^2\phi) + \sinh^2 u \cos^2 v\\
&= \cosh^2 u(1 - \cos^2 v) + (\cosh^2 u - 1)\cos^2 v\\
&= \cosh^2 u - \cos^2 v;\\
|\mathbf{e}_3|^2 &= \cosh^2 u \cos^2 v(\sin^2\phi + \cos^2\phi) = \cosh^2 u \cos^2 v.
\end{aligned}$$

The immediate deduction is that

$$h_u = h_v = (\cosh^2 u - \cos^2 v)^{1/2}, \qquad h_\phi = \cosh u \cos v.$$

An alternative form for h_u and h_v is $(\sinh^2 u + \sin^2 v)^{1/2}$.

If a solution of Laplace's equation is to be a function, $\psi(u)$, of u only, then all differentiation with respect to v and ϕ can be ignored. The expression for $\nabla^2\psi$ reduces to

$$\begin{aligned}
\nabla^2\psi &= \frac{1}{h_u h_v h_\phi}\left[\frac{\partial}{\partial u}\left(\frac{h_v h_\phi}{h_u}\frac{\partial\psi}{\partial u}\right)\right]\\
&= \frac{1}{\cosh u \cos v(\cosh^2 u - \cos^2 v)}\left[\frac{\partial}{\partial u}\left(\cosh u \cos v \frac{\partial\psi}{\partial u}\right)\right].
\end{aligned}$$

Laplace's equation itself is even simpler and reduces to

$$\frac{\partial}{\partial u}\left(\cosh u \frac{\partial\psi}{\partial u}\right) = 0.$$

This can be rewritten as

$$\frac{\partial \psi}{\partial u} = \frac{k}{\cosh u} = \frac{2k}{e^u + e^{-u}} = \frac{2ke^u}{e^{2u} + 1},$$

$$d\psi = \frac{Ae^u\, du}{1 + (e^u)^2} \quad \Rightarrow \quad \psi = B \tan^{-1} e^u + c.$$

This is the most general function of u only that satisfies Laplace's equation.

11

Line, surface and volume integrals

11.1 *The vector field* $\mathbf{F}$ *is defined by*

$$\mathbf{F} = 2xz\mathbf{i} + 2yz^2\mathbf{j} + (x^2 + 2y^2z - 1)\mathbf{k}.$$

Calculate $\nabla \times \mathbf{F}$ *and deduce that* $\mathbf{F}$ *can be written* $F = \nabla\phi$. *Determine the form of* ϕ.

With $\mathbf{F}$ as given, we calculate the curl of $\mathbf{F}$ to see whether or not it is the zero vector:

$$\nabla \times \mathbf{F} = (4yz - 4yz,\ 2x - 2x,\ 0 - 0) = \mathbf{0}.$$

The fact that it is implies that $\mathbf{F}$ can be written as $\nabla\phi$ for some scalar ϕ.

The form of $\phi(x, y, z)$ is found by integrating, in turn, the components of $\mathbf{F}$ until consistency is achieved, i.e. until a ϕ is found that has partial derivatives equal to the corresponding components of $\mathbf{F}$:

$$\begin{aligned}
2xz = F_x = \frac{\partial\phi}{\partial x} \quad &\Rightarrow \quad \phi(x, y, z) = x^2z + g(y, z),\\
2yz^2 = F_y = \frac{\partial}{\partial y}[x^2z + g(y, z)] \quad &\Rightarrow \quad g(y, z) = y^2z^2 + h(z),\\
x^2 + 2y^2z - 1 = F_z \quad &= \quad \frac{\partial}{\partial z}[x^2z + y^2z^2 + h(z)]\\
&\Rightarrow \quad h(z) = -z + k.
\end{aligned}$$

Hence, to within an unimportant constant, the form of ϕ is

$$\phi(x, y, z) = x^2z + y^2z^2 - z.$$

11.3 *A vector field* $\mathbf{F}$ *is given by* $\mathbf{F} = xy^2\mathbf{i} + 2\mathbf{j} + x\mathbf{k}$ *and* L *is a path parameterised by* $x = ct$, $y = c/t$, $z = d$ *for the range* $1 \le t \le 2$. *Evaluate the three integrals*

$$\text{(a)} \quad \int_L \mathbf{F}\,dt, \qquad \text{(b)} \quad \int_L \mathbf{F}\,dy, \qquad \text{(c)} \quad \int_L \mathbf{F}\cdot d\mathbf{r}.$$

Although all three integrals are along the same path L, they are not necessarily of the same type. The vector or scalar nature of the integral is determined by that of the integrand when it is expressed in a form containing the infinitesimal dt.

(a) This is a vector integral and contains three separate integrations. We express each of the integrands in terms of t, according to the parameterisation of the integration path L, before integrating:

$$\begin{aligned}\int_L \mathbf{F}\,dt &= \int_1^2 \left(\frac{c^3}{t}\,\mathbf{i} + 2\,\mathbf{j} + ct\,\mathbf{k}\right)\,dt \\ &= \left[c^3 \ln t\,\mathbf{i} + 2t\,\mathbf{j} + \frac{1}{2}ct^2\,\mathbf{k}\right]_1^2 \\ &= c^3 \ln 2\,\mathbf{i} + 2\,\mathbf{j} + \frac{3}{2}c\,\mathbf{k}.\end{aligned}$$

(b) This is a similar vector integral but here we must also replace the infinitesimal dy by the infinitesimal $-c\,dt/t^2$ before integrating:

$$\begin{aligned}\int_L \mathbf{F}\,dy &= \int_1^2 \left(\frac{c^3}{t}\,\mathbf{i} + 2\,\mathbf{j} + ct\,\mathbf{k}\right)\left(\frac{-c}{t^2}\right)\,dt \\ &= \left[\frac{c^4}{2t^2}\,\mathbf{i} + \frac{2c}{t}\,\mathbf{j} - c^2 \ln t\,\mathbf{k}\right]_1^2 \\ &= -\frac{3c^4}{8}\,\mathbf{i} - c\,\mathbf{j} - c^2 \ln 2\,\mathbf{k}.\end{aligned}$$

(c) This is a scalar integral and before integrating we must take the scalar product of $\mathbf{F}$ with $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$ to give a single integrand:

$$\begin{aligned}\int_L \mathbf{F}\cdot d\mathbf{r} &= \int_1^2 \left(\frac{c^3}{t}\,\mathbf{i} + 2\,\mathbf{j} + ct\,\mathbf{k}\right)\cdot\left(c\,\mathbf{i} - \frac{c}{t^2}\,\mathbf{j} + 0\,\mathbf{k}\right)dt \\ &= \int_1^2 \left(\frac{c^4}{t} - \frac{2c}{t^2}\right)\,dt \\ &= \left[c^4 \ln t + \frac{2c}{t}\right]_1^2 \\ &= c^4 \ln 2 - c.\end{aligned}$$

11.5 *Determine the point of intersection P, in the first quadrant, of the two ellipses*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \textit{and} \quad \frac{x^2}{b^2}+\frac{y^2}{a^2}=1.$$

Taking $b < a$, consider the contour L that bounds the area in the first quadrant that is common to the two ellipses. Show that the parts of L that lie along the coordinate axes contribute nothing to the line integral around L of $x\,dy - y\,dx$. Using a parameterisation of each ellipse of the general form $x = X\cos\phi$ and $y = Y\sin\phi$, evaluate the two remaining line integrals and hence find the total area common to the two ellipses.

Note: The line integral of $x\,dy - y\,dx$ around a general closed convex contour is equal to twice the area enclosed by that contour.

From the symmetry of the equations under the interchange of x and y, the point P must have $x = y$. Thus,

$$x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=1 \quad \Rightarrow \quad x=\frac{ab}{(a^2+b^2)^{1/2}}.$$

Denoting as curve C_1 the part of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

that lies on the boundary of the common region, we parameterise it by $x = a\cos\theta_1$ and $y = b\sin\theta_1$. Curve C_1 starts from P and finishes on the y-axis. At P,

$$a\cos\theta_1 = x = \frac{ab}{(a^2+b^2)^{1/2}} \quad \Rightarrow \quad \tan\theta_1 = \frac{a}{b}.$$

It follows that θ_1 lies in the range $\tan^{-1}(a/b) \le \theta_1 \le \pi/2$. Note that θ_1 is *not* the angle between the x-axis and the line joining the origin O to the corresponding point on the curve; for example, when the point is P itself then $\theta_1 = \tan^{-1} a/b$, whilst the line OP makes an angle of $\pi/4$ with the x-axis.

Similarly, referring to that part of

$$\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$$

that lies on the boundary of the common region as curve C_2, we parameterise it by $x = b\cos\theta_2$ and $y = a\sin\theta_2$ with $0 \le \theta_2 \le \tan^{-1}(b/a)$.

On the x-axis, both y and dy are zero and the integrand, $x\,dy - y\,dx$, vanishes.

Similarly, the integrand vanishes at all points on the y-axis. Hence,

$$\begin{aligned}
I &= \oint_L (x\,dy - y\,dx) \\
&= \int_{C_2} (x\,dy - y\,dx) + \int_{C_1} (x\,dy - y\,dx) \\
&= \int_0^{\tan^{-1}(b/a)} [\,ab(\cos\theta_2\cos\theta_2) - ab\sin\theta_2(-\sin\theta_2)\,]\,d\theta_2 \\
&\qquad + \int_{\tan^{-1}(a/b)}^{\pi/2} [\,ab(\cos\theta_1\cos\theta_1) - ab\sin\theta_1(-\sin\theta_1)\,]\,d\theta_1 \\
&= ab\tan^{-1}\frac{b}{a} + ab\left(\frac{\pi}{2} - \tan^{-1}\frac{a}{b}\right) \\
&= 2ab\tan^{-1}\frac{b}{a}.
\end{aligned}$$

As noted in the question, the area enclosed by L is equal to $\frac{1}{2}$ of this value, i.e. the total common area in all four quadrants is

$$4 \times \frac{1}{2} \times 2ab\tan^{-1}\frac{b}{a} = 4ab\tan^{-1}\frac{b}{a}.$$

Note that if we let $b \to a$ then the two ellipses become identical circles and we recover the expected value of πa^2 for their common area.

11.7 *Evaluate the line integral*

$$I = \oint_C \left[y(4x^2 + y^2)\,dx + x(2x^2 + 3y^2)\,dy\right]$$

around the ellipse $x^2/a^2 + y^2/b^2 = 1$.

As it stands this integral is complicated and, in fact, it is the sum of two integrals. The form of the integrand, containing powers of x and y that can be differentiated easily, makes this problem one to which Green's theorem in a plane might usefully be applied. The theorem states that

$$\oint_C (P\,dx + Q\,dy) = \int\int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy,$$

where C is a closed contour enclosing the convex region R.

In the notation used above,

$$P(x, y) = y(4x^2 + y^2) \quad \text{and} \quad Q(x, y) = x(2x^2 + 3y^2).$$

It follows that

$$\frac{\partial P}{\partial y} = 4x^2 + 3y^2 \quad \text{and} \quad \frac{\partial Q}{\partial x} = 6x^2 + 3y^2,$$

leading to

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2x^2.$$

This can now be substituted into Green's theorem and the y-integration carried out immediately as the integrand does not contain y. Hence,

$$\begin{aligned} I &= \int\int_R 2x^2\,dx\,dy \\ &= \int_{-a}^{a} 2x^2\,2b\left(1-\frac{x^2}{a^2}\right)^{1/2} dx \\ &= 4b\int_{\pi}^{0} a^2\cos^2\phi\ \sin\phi\ (-a\sin\phi\,d\phi), \text{ on setting } x = a\cos\phi, \\ &= -ba^3\int_{\pi}^{0}\sin^2(2\phi)\,d\phi = \tfrac{1}{2}\pi ba^3. \end{aligned}$$

In the final line we have used the standard result for the integral of the square of a sinusoidal function.

11.9 *A single-turn coil C of arbitrary shape is placed in a magnetic field* $\mathbf{B}$ *and carries a current I. Show that the couple acting upon the coil can be written as*

$$\mathbf{M} = I\int_C(\mathbf{B}\cdot\mathbf{r})\,d\mathbf{r} - I\int_C \mathbf{B}(\mathbf{r}\cdot d\mathbf{r}).$$

For a planar rectangular coil of sides $2a$ and $2b$ placed with its plane vertical and at an angle ϕ to a uniform horizontal field $\mathbf{B}$*, show that* $\mathbf{M}$ *is, as expected,* $4abBI\cos\phi\,\mathbf{k}$.

For an arbitrarily shaped coil the total couple acting can only be found by considering that on an infinitesimal element and then integrating this over the whole coil. The force on an element $d\mathbf{r}$ of the coil is $d\mathbf{F} = I\,d\mathbf{r}\times\mathbf{B}$, and the moment of this force about the origin is $d\mathbf{M} = \mathbf{r}\times\mathbf{F}$. Thus the total moment is given by

$$\begin{aligned} \mathbf{M} &= \oint_C \mathbf{r}\times(I\,d\mathbf{r}\times\mathbf{B}) \\ &= I\oint_C(\mathbf{r}\cdot\mathbf{B})\,d\mathbf{r} - I\oint_C \mathbf{B}(\mathbf{r}\cdot d\mathbf{r}). \end{aligned}$$

To obtain this second form we have used the vector identity

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}.$$

To determine the couple acting on the rectangular coil we work in Cartesian

coordinates with the z-axis vertical and choose the orientation of axes in the horizontal plane such that the edge of the rectangle of length $2a$ is in the x-direction. Then

$$\mathbf{B} = B\cos\phi\,\mathbf{i} + B\sin\phi\,\mathbf{j}.$$

In the first term in $\mathbf{M}$,
(i) for the horizontal sides

$$\mathbf{r} = x\,\mathbf{i} \pm b\,\mathbf{k}, \quad d\mathbf{r} = dx\,\mathbf{i}, \quad \mathbf{r}\cdot\mathbf{B} = xB\cos\phi,$$

$$\int (\mathbf{r}\cdot\mathbf{B})\,d\mathbf{r} = B\cos\phi\,\mathbf{i}\left(\int_{-a}^{a} x\,dx + \int_{a}^{-a} x\,dx\right) = \mathbf{0};$$

(ii) for the vertical sides

$$\mathbf{r} = \pm a\,\mathbf{i} + z\,\mathbf{k}, \quad d\mathbf{r} = dz\,\mathbf{k}, \quad \mathbf{r}\cdot\mathbf{B} = \pm aB\cos\phi,$$

$$\int (\mathbf{r}\cdot\mathbf{B})\,d\mathbf{r} = B\cos\phi\,\mathbf{k}\left(\int_{-b}^{b}(+a)\,dz + \int_{b}^{-b}(-a)\,dz\right) = 4abB\cos\phi\,\mathbf{k}.$$

For the second term in $\mathbf{M}$, since the field is uniform it can be taken outside the integral as a (vector) constant. On the horizontal sides the remaining integral is

$$\int \mathbf{r}\cdot d\mathbf{r} = \pm\int_{-a}^{a} x\,dx = 0.$$

Similarly, the contribution from the vertical sides vanishes and the whole of the second term contributes nothing in this particular configuration.

The total moment is thus $4abB\cos\phi\,\mathbf{k}$, as expected.

11.11 *An axially symmetric solid body with its axis AB vertical is immersed in an incompressible fluid of density ρ_0. Use the following method to show that, whatever the shape of the body, for $\rho = \rho(z)$ in cylindrical polars the Archimedean upthrust is, as expected, $\rho_0 gV$, where V is the volume of the body.*

Express the vertical component of the resultant force ($-\int p\,d\mathbf{S}$, where p is the pressure) on the body in terms of an integral; note that $p = -\rho_0 gz$ and that for an annular surface element of width dl, $\mathbf{n}\cdot\mathbf{n}_z\,dl = -d\rho$. Integrate by parts and use the fact that $\rho(z_A) = \rho(z_B) = 0$.

We measure z negatively from the water's surface $z = 0$ so that the hydrostatic pressure is $p = -\rho_0 gz$. By symmetry, there is no net horizontal force acting on the body.

The upward force, F, is due to the net vertical component of the hydrostatic pressure acting upon the body's surface:

$$\begin{aligned} F &= -\hat{\mathbf{n}}_z \cdot \int p\, d\mathbf{S} \\ &= -\hat{\mathbf{n}}_z \cdot \int (-\rho_0 g z)(2\pi\rho\, \hat{\mathbf{n}}\, dl), \end{aligned}$$

where $2\pi\rho\, dl$ is the area of the strip of surface lying between z and $z + dz$ and $\hat{\mathbf{n}}$ is the outward unit normal to that surface.

Now, from geometry, $\hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}$ is equal to minus the sine of the angle between dl and dz and so $\hat{\mathbf{n}}_z \cdot \hat{\mathbf{n}}\, dl$ is equal to $-d\rho$. Thus,

$$\begin{aligned} F &= 2\pi\rho_0 g \int_{z_A}^{z_B} \rho z(-d\rho) \\ &= -2\pi\rho_0 g \int_{z_A}^{z_B} \left(\rho \frac{\partial \rho}{\partial z}\right) z\, dz \\ &= -2\pi\rho_0 g \left\{ \left[z\frac{\rho^2}{2} \right]_{z_A}^{z_B} - \int_{z_A}^{z_B} \frac{\rho^2}{2}\, dz \right\}. \end{aligned}$$

But $\rho(z_A) = \rho(z_B) = 0$, and so the first contribution vanishes, leaving

$$F = \rho_0 g \int_{z_A}^{z_B} \pi\rho^2\, dz = \rho_0 g V,$$

where V is the volume of the solid. This is the mathematical form of Archimedes' principle. Of course, the result is also valid for a closed body of arbitrary shape, $\rho = \rho(z, \phi)$, but a different method would be needed to prove it.

11.13 *A vector field* $\mathbf{a}$ *is given by* $-zxr^{-3}\mathbf{i} - zyr^{-3}\mathbf{j} + (x^2 + y^2)r^{-3}\mathbf{k}$, *where* $r^2 = x^2 + y^2 + z^2$. *Establish that the field is conservative* (a) *by showing that* $\nabla \times \mathbf{a} = \mathbf{0}$, *and* (b) *by constructing its potential function* ϕ.

We are told that

$$\mathbf{a} = -\frac{zx}{r^3}\,\mathbf{i} - \frac{zy}{r^3}\,\mathbf{j} + \frac{x^2 + y^2}{r^3}\,\mathbf{k},$$

with $r^2 = x^2 + y^2 + z^2$. We will need to differentiate r^{-3} with respect to x, y and z, using the chain rule, and so note that $\partial r/\partial x = x/r$, etc.

(a) Consider $\nabla \times \mathbf{a}$, term-by-term:

$$
\begin{aligned}
[\nabla \times \mathbf{a}]_x &= \frac{\partial}{\partial y}\left(\frac{x^2+y^2}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{-zy}{r^3}\right) \\
&= \frac{-3(x^2+y^2)y}{r^4\,r} + \frac{2y}{r^3} + \frac{y}{r^3} - \frac{3(zy)z}{r^4\,r} \\
&= \frac{3y}{r^5}(-x^2-y^2+x^2+y^2+z^2-z^2) = 0; \\
[\nabla \times \mathbf{a}]_y &= \frac{\partial}{\partial z}\left(\frac{-zx}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{x^2+y^2}{r^3}\right) \\
&= \frac{3(zx)z}{r^4\,r} - \frac{x}{r^3} - \frac{2x}{r^3} + \frac{3(x^2+y^2)x}{r^4\,r} \\
&= \frac{3x}{r^5}(z^2-x^2-y^2-z^2+x^2+y^2) = 0; \\
[\nabla \times \mathbf{a}]_z &= \frac{\partial}{\partial x}\left(\frac{-zy}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{-zx}{r^3}\right) \\
&= \frac{3(zy)x}{r^4\,r} - \frac{3(zx)y}{r^4\,r} = 0.
\end{aligned}
$$

Thus all three components of $\nabla \times \mathbf{a}$ are zero, showing that $\mathbf{a}$ is a conservative field.

(b) To construct its potential function we proceed as follows:

$$
\begin{aligned}
\frac{\partial \phi}{\partial x} &= \frac{-zx}{(x^2+y^2+z^2)^{3/2}} \Rightarrow \phi = \frac{z}{(x^2+y^2+z^2)^{1/2}} + f(y,z), \\
\frac{\partial \phi}{\partial y} &= \frac{-zy}{(x^2+y^2+z^2)^{3/2}} = \frac{-zy}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial f}{\partial y} \Rightarrow f(y,z) = g(z), \\
\frac{\partial \phi}{\partial z} &= \frac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}} \\
&= \frac{1}{(x^2+y^2+z^2)^{1/2}} + \frac{-z\,z}{(x^2+y^2+z^2)^{3/2}} + \frac{\partial g}{\partial z} \\
&\Rightarrow g(z) = c.
\end{aligned}
$$

Thus,

$$
\phi(x,y,z) = c + \frac{z}{(x^2+y^2+z^2)^{1/2}} = c + \frac{z}{r}.
$$

The very fact that we can construct a potential function $\phi = \phi(x,y,z)$ whose derivatives are the components of the vector field shows that the field is conservative.

11.15 *A force* $\mathbf{F}(\mathbf{r})$ *acts on a particle at* $\mathbf{r}$. *In which of the following cases can* $\mathbf{F}$ *be represented in terms of a potential? Where it can, find the potential.*

(a) $\mathbf{F} = F_0\left[\mathbf{i} - \mathbf{j} - \dfrac{2(x-y)}{a^2}\mathbf{r}\right]\exp\left(-\dfrac{r^2}{a^2}\right)$;

(b) $\mathbf{F} = \dfrac{F_0}{a}\left[z\mathbf{k} + \dfrac{(x^2+y^2-a^2)}{a^2}\mathbf{r}\right]\exp\left(-\dfrac{r^2}{a^2}\right)$;

(c) $\mathbf{F} = F_0\left[\mathbf{k} + \dfrac{a(\mathbf{r}\times\mathbf{k})}{r^2}\right]$.

(a) We first write the field entirely in terms of the Cartesian unit vectors using $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ and then attempt to construct a suitable potential function ϕ:

$$\begin{aligned}\mathbf{F} &= F_0\left[\mathbf{i} - \mathbf{j} - \frac{2(x-y)}{a^2}\mathbf{r}\right]\exp\left(-\frac{r^2}{a^2}\right)\\ &= \frac{F_0}{a^2}\left[(a^2 - 2x^2 + 2xy)\,\mathbf{i} + (-a^2 - 2xy + 2y^2)\,\mathbf{j}\right.\\ &\qquad\qquad + \left.(-2xz + 2yz)\,\mathbf{k}\right]\exp\left(-\frac{r^2}{a^2}\right).\end{aligned}$$

Since the partial derivative of $\exp(-r^2/a^2)$ with respect to any Cartesian coordinate u is $\exp(-r^2/a^2)(-2r/a^2)(u/r)$, the z-component of $\mathbf{F}$ appears to be the most straightforward to tackle first:

$$\begin{aligned}\frac{\partial\phi}{\partial z} &= \frac{F_0}{a^2}(-2xz + 2yz)\exp\left(-\frac{r^2}{a^2}\right)\\ &\Rightarrow \phi(x,y,z) = F_0(x-y)\exp\left(-\frac{r^2}{a^2}\right) + f(x,y)\\ &\equiv \phi_1(x,y,z) + f(x,y).\end{aligned}$$

Next we examine the derivatives of $\phi = \phi_1 + f$ with respect to x and y to see how closely they generate F_x and F_y:

$$\begin{aligned}\frac{\partial\phi_1}{\partial x} &= F_0\left[\exp\left(-\frac{r^2}{a^2}\right) + (x-y)\exp\left(-\frac{r^2}{a^2}\right)\left(\frac{-2x}{a^2}\right)\right]\\ &= \frac{F_0}{a^2}(a^2 - 2x^2 + 2xy)\exp(-r^2/a^2) = F_x \quad \text{(as given)},\\ \text{and}\quad \frac{\partial\phi_1}{\partial y} &= F_0\left[-\exp\left(-\frac{r^2}{a^2}\right) + (x-y)\exp\left(-\frac{r^2}{a^2}\right)\left(\frac{-2y}{a^2}\right)\right]\\ &= \frac{F_0}{a^2}(-a^2 - 2xy + 2y^2)\exp(-r^2/a^2) = F_y \quad \text{(as given)}.\end{aligned}$$

Thus, to within an arbitrary constant, $\phi_1(x,y,z) = F_0(x-y)\exp\left(-\dfrac{r^2}{a^2}\right)$ is a

suitable potential function for the field, without the need for any additional function $f(x, y)$.

(b) We follow the same line of argument as in part (a). First, expressing $\mathbf{F}$ in terms of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$,

$$\begin{aligned}\mathbf{F} &= \frac{F_0}{a}\left[z\,\mathbf{k} + \frac{x^2+y^2-a^2}{a^2}\mathbf{r}\right]\exp\left(-\frac{r^2}{a^2}\right)\\ &= \frac{F_0}{a^3}\left[x(x^2+y^2-a^2)\,\mathbf{i} + y(x^2+y^2-a^2)\,\mathbf{j}\right.\\ &\qquad\left. +\ z(x^2+y^2)\,\mathbf{k}\right]\exp\left(-\frac{r^2}{a^2}\right),\end{aligned}$$

and then constructing a possible potential function ϕ. Again starting with the z-component:

$$\begin{aligned}\frac{\partial\phi}{\partial z} &= \frac{F_0 z}{a^3}(x^2+y^2)\exp\left(-\frac{r^2}{a^2}\right),\\ \Rightarrow\quad \phi(x,y,z) &= -\frac{F_0}{2a}(x^2+y^2)\exp\left(-\frac{r^2}{a^2}\right) + f(x,y)\\ &\equiv \phi_1(x,y,z) + f(x,y),\end{aligned}$$

Then, $$\frac{\partial\phi_1}{\partial x} = -\frac{F_0}{2a}\left[2x - \frac{2x(x^2+y^2)}{a^2}\right]\exp\left(-\frac{r^2}{a^2}\right) = F_x \quad \text{(as given)},$$

and $$\frac{\partial\phi_1}{\partial y} = -\frac{F_0}{2a}\left[2y - \frac{2y(x^2+y^2)}{a^2}\right]\exp\left(-\frac{r^2}{a^2}\right) = F_y \quad \text{(as given)}.$$

Thus, $\phi_1(x,y,z) = \dfrac{F_0}{2a}(x^2+y^2)\exp\left(-\dfrac{r^2}{a^2}\right)$, as it stands, is a suitable potential function for $\mathbf{F}(\mathbf{r})$ and establishes the conservative nature of the field.

(c) Again we express F in Cartesian components:

$$\mathbf{F} = F_0\left[\mathbf{k} + \frac{a(\mathbf{r}\times\mathbf{k})}{r^2}\right] = \frac{ay}{r^2}\,\mathbf{i} - \frac{ax}{r^2}\,\mathbf{j} + \mathbf{k}.$$

That the z-component of $\mathbf{F}$ has no dependence on y whilst its y-component does depend upon z suggests that the x-component of $\nabla\times\mathbf{F}$ may not be zero. To test this out we compute

$$(\nabla\times\mathbf{F})_x = \frac{\partial(1)}{\partial y} - \frac{\partial}{\partial z}\left(\frac{-ax}{r^2}\right) = 0 - \frac{2axz}{r^4} \neq 0,$$

and find that it is not. To have even one component of $\nabla\times\mathbf{F}$ non-zero is sufficient to show that $\mathbf{F}$ is not conservative and that no potential function can be found. There is no point in searching further!

The same conclusion can be reached by considering the implication of $\mathbf{F}_z = \mathbf{k}$, namely that any possible potential function has to have the form $\phi(x,y,z) =$

$z + f(x, y)$. However, $\partial\phi/\partial x$ is known to be $-ay/r^2 = -ay/(x^2 + y^2 + z^2)$. This yields a contradiction, as it requires $\partial f(x, y)/\partial x$ to depend on z, which is clearly impossible.

11.17 *The vector field* $\mathbf{f}$ *has components* $y\mathbf{i} - x\mathbf{j} + \mathbf{k}$ *and* γ *is a curve given parametrically by*

$$\mathbf{r} = (a - c + c\cos\theta)\mathbf{i} + (b + c\sin\theta)\mathbf{j} + c^2\theta\mathbf{k}, \quad 0 \le \theta \le 2\pi.$$

Describe the shape of the path γ *and show that the line integral* $\int_\gamma \mathbf{f} \cdot d\mathbf{r}$ *vanishes. Does this result imply that* $\mathbf{f}$ *is a conservative field?*

As θ increases from 0 to 2π, the x- and y-components of $\mathbf{r}$ vary sinusoidally and in quadrature about fixed values $a - c$ and b. Both variations have amplitude c and both return to their initial values when $\theta = 2\pi$. However, the z-component increases monotonically from 0 to a value of $2\pi c^2$. The curve γ is therefore one loop of a circular spiral of radius c and pitch $2\pi c^2$. Its axis is parallel to the z-axis and passes through the points $(a - c,\ b,\ z)$.

The line element $d\mathbf{r}$ has components $(-c\sin\theta\, d\theta, c\cos\theta\, d\theta, c^2\, d\theta)$ and so the line integral of f along γ is given by

$$\begin{aligned}
\int_\gamma \mathbf{f} \cdot d\mathbf{r} &= \int_0^{2\pi} \left[y(-c\sin\theta) - x(c\cos\theta) + c^2\right]\, d\theta \\
&= \int_0^{2\pi} \left[-c(b + c\sin\theta)\sin\theta - c(a - c + c\cos\theta)\cos\theta + c^2\right]\, d\theta \\
&= \int_0^{2\pi} \left(-bc\sin\theta - c^2\sin^2\theta - c(a - c)\cos\theta - c^2\cos^2\theta + c^2\right)\, d\theta \\
&= 0 - \pi c^2 - 0 - \pi c^2 + 2\pi c^2 = 0.
\end{aligned}$$

However, this does not imply that $\mathbf{f}$ is a conservative field since (i) γ is not a closed loop, and (ii) even if it were, the line integral has to vanish for *every* loop, not just for a particular one.

Further,

$$\nabla \times \mathbf{f} = (0 - 0,\ 0 - 0,\ -1 - 1) = (0,\ 0,\ -2) \neq \mathbf{0},$$

showing explicitly that $\mathbf{f}$ is not conservative.

11.19 *Evaluate the surface integral* $\int \mathbf{r} \cdot d\mathbf{S}$, *where* $\mathbf{r}$ *is the position vector, over that part of the surface* $z = a^2 - x^2 - y^2$ *for which* $z \geq 0$, *by each of the following methods.*

(a) *Parameterise the surface as* $x = a \sin\theta \cos\phi$, $y = a \sin\theta \sin\phi$, $z = a^2 \cos^2\theta$, *and show that*

$$\mathbf{r} \cdot d\mathbf{S} = a^4(2\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\,d\theta\,d\phi.$$

(b) *Apply the divergence theorem to the volume bounded by the surface and the plane* $z = 0$.

(a) With $x = a \sin\theta \cos\phi$, $y = a \sin\theta \sin\phi$, $z = a^2 \cos^2\theta$, we first check that this does parameterise the surface appropriately:

$$a^2 - x^2 - y^2 = a^2 - a^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = a^2(1 - \sin^2\theta) = a^2\cos^2\theta = z.$$

We see that it does so for the relevant part of the surface, i.e. that which lies above the plane $z = 0$ with $0 \leq \theta \leq \pi/2$. It would not do so for the part with $z < 0$ for which $x^2 + y^2$ has to be greater than a^2; this is not catered for by the given parameterisation.

Having carried out this check, we calculate expressions for $d\mathbf{S}$ and hence $\mathbf{r} \cdot d\mathbf{S}$ in terms of θ and ϕ as follows:

$$\mathbf{r} = a\sin\theta\cos\phi\,\mathbf{i} + a\sin\theta\sin\phi\,\mathbf{j} + a^2\cos^2\theta\,\mathbf{k},$$

and the tangent vectors at the point (θ, ϕ) on the surface are given by

$$\frac{\partial\mathbf{r}}{\partial\theta} = a\cos\theta\cos\phi\,\mathbf{i} + a\cos\theta\sin\phi\,\mathbf{j} - 2a^2\cos\theta\sin\theta\,\mathbf{k},$$
$$\frac{\partial\mathbf{r}}{\partial\phi} = -a\sin\theta\sin\phi\,\mathbf{i} + a\sin\theta\cos\phi\,\mathbf{j}.$$

The corresponding vector element of surface area is thus

$$\begin{aligned} d\mathbf{S} &= \frac{\partial\mathbf{r}}{\partial\theta} \times \frac{\partial\mathbf{r}}{\partial\phi} \\ &= 2a^3\cos\theta\sin^2\theta\cos\phi\,\mathbf{i} + 2a^3\cos\theta\sin^2\theta\sin\phi\,\mathbf{j} + a^2\cos\theta\sin\theta\,\mathbf{k}, \end{aligned}$$

giving $\mathbf{r} \cdot d\mathbf{S}$ as

$$\begin{aligned} \mathbf{r} \cdot d\mathbf{S} &= 2a^4\cos\theta\sin^3\theta\cos^2\phi + 2a^4\cos\theta\sin^3\theta\sin^2\phi + a^4\cos^3\theta\sin\theta \\ &= 2a^4\cos\theta\sin^3\theta + a^4\cos^3\theta\sin\theta. \end{aligned}$$

This is to be integrated over the ranges $0 \le \phi < 2\pi$ and $0 \le \theta \le \pi/2$ as follows:

$$\begin{aligned}\int \mathbf{r}\cdot d\mathbf{S} &= a^4 \int_0^{2\pi} d\phi \int_0^{\pi/2} (2\sin^3\theta\cos\theta + \cos^3\theta\sin\theta)\,d\theta \\ &= 2\pi a^4 \left(2\left[\frac{\sin^4\theta}{4}\right]_0^{\pi/2} + \left[\frac{-\cos^4\theta}{4}\right]_0^{\pi/2}\right) \\ &= 2\pi a^4\left(\frac{2}{4}+\frac{1}{4}\right) = \frac{3\pi a^4}{2}.\end{aligned}$$

(b) The divergence of the vector field $\mathbf{r}$ is 3, a constant, and so the surface integral $\int \mathbf{r}\cdot d\mathbf{S}$ taken over the complete surface Σ (including the part that lies in the plane $z = 0$) is, by the divergence theorem, equal to three times the volume V of the region bounded by Σ. Now,

$$V = \int_0^{a^2} \pi\rho^2\,dz = \int_0^{a^2} \pi(a^2 - z)\,dz = \pi(a^4 - \tfrac{1}{2}a^4) = \tfrac{1}{2}\pi a^4,$$

and so $\int_\Sigma \mathbf{r}\cdot d\mathbf{S} = 3\pi a^4/2$.

However, on the part of the surface lying in the plane $z = 0$, $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + 0\,\mathbf{k}$, whilst $d\mathbf{S} = -dS\,\mathbf{k}$. Consequently the scalar product $\mathbf{r}\cdot d\mathbf{S} = 0$; in words, for any point on this face its position vector is orthogonal to the normal to the face. The surface integral over this face therefore contributes nothing to the total integral and the value obtained is that due to the curved surface alone, in agreement with the result in (a).

11.21 *Use the result*

$$\int_V \nabla\phi\,dV = \oint_S \phi\,d\mathbf{S},$$

together with an appropriately chosen scalar function ϕ, to prove that the position vector $\bar{\mathbf{r}}$ of the centre of mass of an arbitrarily shaped body of volume V and uniform density can be written

$$\bar{\mathbf{r}} = \frac{1}{V}\oint_S \tfrac{1}{2}r^2\,d\mathbf{S}.$$

The position vector of the centre of mass is defined by

$$\bar{\mathbf{r}}\int_V \rho\,dV = \int_V \mathbf{r}\rho\,dV.$$

Now, we note that $\mathbf{r}$ can be written as $\nabla(\frac{1}{2}r^2)$. Thus, cancelling the constant ρ, we

have

$$\bar{\mathbf{r}}\,V = \int_V \nabla(\tfrac{1}{2}r^2)\,dV n = \oint_S \tfrac{1}{2}r^2\,d\mathbf{S}$$

$$\Rightarrow \quad \bar{\mathbf{r}} = \frac{1}{V}\oint_S \tfrac{1}{2}r^2\,d\mathbf{S}.$$

This result provides an alternative method of finding the centre of mass $\bar{z}\mathbf{k}$ of the uniform hemisphere $r = a$, $0 \le \theta \le \pi/2$, $0 \le \phi < 2\pi$. The curved surface contributes $3a/4$ to $\bar{z}$ and the plane surface contributes $-3a/8$, giving $\bar{z} = 3a/8$.

11.23 *Demonstrate the validity of the divergence theorem:*

(a) *by calculating the flux of the vector*

$$\mathbf{F} = \frac{\alpha\mathbf{r}}{(r^2 + a^2)^{3/2}}$$

through the spherical surface $|\mathbf{r}| = \sqrt{3}a$*;*

(b) *by showing that*

$$\nabla \cdot \mathbf{F} = \frac{3\alpha a^2}{(r^2 + a^2)^{5/2}}$$

and evaluating the volume integral of $\nabla \cdot \mathbf{F}$ *over the interior of the sphere* $|\mathbf{r}| = \sqrt{3}a$*. The substitution* $r = a\tan\theta$ *will prove useful in carrying out the integration.*

(a) The field is radial with

$$\mathbf{F} = \frac{\alpha\,\mathbf{r}}{(r^2 + a^2)^{3/2}} = \frac{\alpha\,r}{(r^2 + a^2)^{3/2}}\hat{\mathbf{e}}_r.$$

The total flux is therefore given by

$$\Phi = \left.\frac{4\pi r^2\,\alpha\,r}{(r^2 + a^2)^{3/2}}\right|_{r=a\sqrt{3}} = \frac{4\pi a^3\,\alpha\,3\sqrt{3}}{8a^3} = \frac{3\sqrt{3}\pi\alpha}{2}.$$

(b) From the divergence theorem, the total flux over the surface of the sphere is equal to the volume integral of its divergence within the sphere. The divergence is given by

$$\begin{aligned}\nabla \cdot \mathbf{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{r^2\,\alpha\,r}{(r^2 + a^2)^{3/2}}\right)\\ &= \frac{1}{r^2}\left[\frac{3\alpha r^2}{(r^2 + a^2)^{3/2}} - \frac{3\alpha r^4}{(r^2 + a^2)^{5/2}}\right]\\ &= \frac{3\alpha a^2}{(r^2 + a^2)^{5/2}},\end{aligned}$$

and on integrating over the sphere, we have

$$\begin{aligned}\int_V \nabla\cdot\mathbf{F}\,dV &= \int_0^{\sqrt{3}a} \frac{3\alpha a^2}{(r^2+a^2)^{5/2}} 4\pi r^2\,dr, \text{ set } r = a\tan\theta,\ 0\le\theta\le\tfrac{\pi}{3},\\ &= 12\pi\alpha a^2 \int_0^{\pi/3} \frac{a^2\tan^2\theta\, a\sec^2\theta}{a^5\sec^5\theta}\,d\theta\\ &= 12\pi\alpha\int_0^{\pi/3}\sin^2\theta\cos\theta\,d\theta\\ &= 12\pi\alpha\left[\frac{\sin^3\theta}{3}\right]_0^{\pi/3} = 12\pi\alpha\frac{\sqrt{3}}{8} = \frac{3\sqrt{3}\pi\alpha}{2}, \quad \text{as in (a).}\end{aligned}$$

The equality of the results in parts (a) and (b) is in accordance with the divergence theorem.

11.25 *In a uniform conducting medium with unit relative permittivity, charge density ρ, current density $\mathbf{J}$, electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$, Maxwell's electromagnetic equations take the form (with $\mu_0\epsilon_0 = c^{-2}$)*

$$\text{(i) } \nabla\cdot\mathbf{B} = 0, \qquad \text{(ii) } \nabla\cdot\mathbf{E} = \rho/\epsilon_0,$$

$$\text{(iii) } \nabla\times\mathbf{E} + \dot{\mathbf{B}} = \mathbf{0}, \quad \text{(iv) } \nabla\times\mathbf{B} - (\dot{\mathbf{E}}/c^2) = \mu_0\mathbf{J}.$$

The density of stored energy in the medium is given by $\frac{1}{2}(\epsilon_0 E^2 + \mu_0^{-1}B^2)$. Show that the rate of change of the total stored energy in a volume V is equal to

$$-\int_V \mathbf{J}\cdot\mathbf{E}\,dV - \frac{1}{\mu_0}\oint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S},$$

where S is the surface bounding V.

[*The first integral gives the ohmic heating loss, whilst the second gives the electromagnetic energy flux out of the bounding surface. The vector $\mu_0^{-1}(\mathbf{E}\times\mathbf{B})$ is known as the Poynting vector.*]

The total stored energy is equal to the volume integral of the energy density. Let R be its rate of change. Then, differentiating under the integral sign, we have

$$\begin{aligned}R &= \frac{d}{dt}\int_V \left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) dV\\ &= \int_V \left(\epsilon_0\mathbf{E}\cdot\dot{\mathbf{E}} + \frac{1}{\mu_0}\mathbf{B}\cdot\dot{\mathbf{B}}\right) dV.\end{aligned}$$

Now using (iv) and (iii), we have

$$\begin{aligned}
R &= \int_V \left[\epsilon_0 \mathbf{E}\cdot(-\mu_0 c^2 \mathbf{J} + c^2 \nabla\times\mathbf{B}) - \frac{1}{\mu_0}\mathbf{B}\cdot(\nabla\times\mathbf{E})\right] dV \\
&= -\int_V \mathbf{E}\cdot\mathbf{J}\,dV + \int_V \left[\epsilon_0 c^2\,\mathbf{E}\cdot(\nabla\times\mathbf{B}) - \frac{1}{\mu_0}\,\mathbf{B}\cdot(\nabla\times\mathbf{E})\right] dV \\
&= -\int_V \mathbf{E}\cdot\mathbf{J}\,dV - \frac{1}{\mu_0}\int_V \nabla\cdot(\mathbf{E}\times\mathbf{B})\,dV \\
&= -\int_V \mathbf{E}\cdot\mathbf{J}\,dV - \frac{1}{\mu_0}\oint_S (\mathbf{E}\times\mathbf{B})\cdot d\mathbf{S}, \quad \text{by the divergence theorem.}
\end{aligned}$$

To obtain the penultimate line we used the vector identity

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b}).$$

11.27 *The vector field* $\mathbf{F}$ *is given by*

$$\mathbf{F} = (3x^2yz + y^3z + xe^{-x})\mathbf{i} + (3xy^2z + x^3z + ye^x)\mathbf{j} + (x^3y + y^3x + xy^2z^2)\mathbf{k}.$$

Calculate (a) *directly, and* (b) *by using Stokes' theorem the value of the line integral* $\int_L \mathbf{F}\cdot d\mathbf{r}$, *where* L *is the (three-dimensional) closed contour* $OABCDEO$ *defined by the successive vertices* $(0,0,0)$, $(1,0,0)$, $(1,0,1)$, $(1,1,1)$, $(1,1,0)$, $(0,1,0)$, $(0,0,0)$.

(a) This calculation is a piece-wise evaluation of the line integral, made up of a series of scalar products of the length of a straight piece of the contour and the component of $\mathbf{F}$ parallel to it (integrated if that component varies along the particular straight section).

On OA, $y = z = 0$ and $F_x = xe^{-x}$;

$$I_1 = \int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = 1 - 2e^{-1}.$$

On AB, $x = 1$ and $y = 0$ and $F_z = 0$; the integral I_2 is zero.

On BC, $x = 1$ and $z = 1$ and $F_y = 3y^2 + 1 + ey$;

$$I_3 = \int_0^1 (3y^2 + 1 + ey)\,dy = 1 + 1 + \tfrac{1}{2}e.$$

On CD, $x = 1$ and $y = 1$ and $F_z = 1 + 1 + z^2$;

$$I_4 = \int_1^0 (1 + 1 + z^2)\,dz = -1 - 1 - \tfrac{1}{3}.$$

On DE, $y = 1$ and $z = 0$ and $F_x = xe^{-x}$;

$$I_5 = \int_1^0 xe^{-x}\,dx = -1 + 2e^{-1}.$$

On EO, $x = z = 0$ and $F_y = ye^0$;

$$I_6 = \int_1^0 ye^0\,dy = -\tfrac{1}{2}.$$

Adding up these six contributions shows that the complete line integral has the value $\dfrac{e}{2} - \dfrac{5}{6}$.

(b) As a simple sketch shows, the given contour is three-dimensional. However, it is equivalent to two plane square contours, one $OADEO$ (denoted by S_1) lying in the plane $z = 0$ and the other $ABCDA$ (S_2) lying in the plane $x = 1$; the latter is traversed in the negative sense. The common segment AD does not form part of the original contour but, as it is traversed in opposite senses in the two constituent contours, it (correctly) contributes nothing to the line integral.

To use Stokes' theorem we first need to calculate

$$\begin{aligned}
(\nabla \times \mathbf{F})_x &= x^3 + 3y^2x + 2yxz^2 - 3xy^2 - x^3 = 2yxz^2,\\
(\nabla \times \mathbf{F})_y &= 3x^2y + y^3 - 3x^2y - y^3 - y^2z^2 = -y^2z^2,\\
(\nabla \times \mathbf{F})_z &= 3y^2z + 3x^2z + ye^x - 3x^2z - 3y^2z = ye^x.
\end{aligned}$$

Now, S_1 has its normal in the positive z-direction and so only the z-component of $\nabla \times \mathbf{F}$ is needed in the first surface integral of Stokes' theorem. Likewise only the x-component of $\nabla \times \mathbf{F}$ is needed in the second integral, but its value must be subtracted because of the sense in which its contour is traversed:

$$\begin{aligned}
\int_{OABCDEO} (\nabla \times \mathbf{F}) \cdot d\mathbf{r} &= \int_{S_1} (\nabla \times \mathbf{F})_z\,dx\,dy - \int_{S_2} (\nabla \times \mathbf{F})_x\,dy\,dz\\
&= \int_0^1 \int_0^1 ye^x\,dx\,dy - \int_0^1 \int_0^1 2y \times 1 \times z^2\,dy\,dz\\
&= \frac{1}{2}(e-1) - 2\,\frac{1}{2}\,\frac{1}{3} = \frac{e}{2} - \frac{5}{6}.
\end{aligned}$$

As they must, the two methods give the same value.

12

Fourier series

12.1 *Prove the orthogonality relations that form the basis of the Fourier series representation of functions.*

All of the results are based on the values of the integrals

$$S(n) = \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi nx}{L}\right) dx \quad \text{and} \quad C(n) = \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi nx}{L}\right) dx$$

for integer values of n. Since in all cases with $n \geq 1$ the integrand goes through a whole number of complete cycles, the 'area under the curve' is zero. For the case $n = 0$, the integrand in $S(n)$ is zero and so therefore is $S(0)$; for $C(0)$ the integrand is unity and the value of $C(0)$ is L.

We now apply these observations to integrals whose integrands are the products of two sinusoidal functions with arguments that are multiples of a fundamental frequency. The integration interval is equal to the period of that fundamental frequency. To express the integrands in suitable forms, repeated use will be made of the expressions for the sums and differences of sinusoidal functions.

We consider first the product of a sine function and a cosine function:

$$\begin{aligned} I_1 &= \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right) \cos\left(\frac{2\pi px}{L}\right) \\ &= \int_{x_0}^{x_0+L} \frac{1}{2}\left[\sin\left(\frac{2\pi(r+p)x}{L}\right) + \sin\left(\frac{2\pi(r-p)x}{L}\right)\right] dx \\ &= \frac{1}{2}[S(r+p) + S(r-p)] = 0, \text{ for all } r \text{ and } p. \end{aligned}$$

Next, we consider the product of two cosines:

$$\begin{aligned} I_2 &= \int_{x_0}^{x_0+L} \cos\left(\frac{2\pi rx}{L}\right)\cos\left(\frac{2\pi px}{L}\right) \\ &= \int_{x_0}^{x_0+L} \frac{1}{2}\left[\cos\left(\frac{2\pi(r+p)x}{L}\right) + \cos\left(\frac{2\pi(r-p)x}{L}\right)\right] dx \\ &= \frac{1}{2}[C(r+p) + C(r-p)] = 0, \end{aligned}$$

unless $r = p > 0$ when $I_2 = \frac{1}{2}L$. If r and p are both zero, then the integrand is unity and $I_2 = L$.

Finally, for the product of two sine functions:

$$\begin{aligned} I_3 &= \int_{x_0}^{x_0+L} \sin\left(\frac{2\pi rx}{L}\right)\sin\left(\frac{2\pi px}{L}\right) \\ &= \int_{x_0}^{x_0+L} \frac{1}{2}\left[\cos\left(\frac{2\pi(r-p)x}{L}\right) - \cos\left(\frac{2\pi(r+p)x}{L}\right)\right] dx \\ &= \frac{1}{2}[C(r-p) - C(r+p)] = 0, \end{aligned}$$

unless $r = p > 0$ when $I_3 = \frac{1}{2}L$. If either of r and p is zero, then the integrand is zero and $I_3 = 0$.

In summary, all of the integrals have zero value except for those in which the integrand is the square of a single sinusoid. In these cases the integral has value $\frac{1}{2}L$ for all integers r $(= p)$ that are > 0. For r $(= p)$ equal to zero, the $\sin^2$ integral has value zero and the $\cos^2$ integral has value L.

12.3 *Which of the following functions of x could be represented by a Fourier series over the range indicated?*

(a) $\tanh^{-1}(x)$, $\quad -\infty < x < \infty$;
(b) $\tan x$, $\quad -\infty < x < \infty$;
(c) $|\sin x|^{-1/2}$, $\quad -\infty < x < \infty$;
(d) $\cos^{-1}(\sin 2x)$, $\quad -\infty < x < \infty$;
(e) $x\sin(1/x)$, $\quad -\pi^{-1} < x \leq \pi^{-1}$, *cyclically repeated.*

The Dirichlet conditions that a function must satisfy before it can be represented by a Fourier series are:

(i) the function must be periodic;
(ii) it must be single-valued and continuous, except possibly at a finite number of finite discontinuities;

(iii) it must have only a finite number of maxima and minima within one period;
(iv) the integral over one period of $|f(x)|$ must converge.

We now test the given functions against these:

(a) $\tanh^{-1}(x)$ is not a periodic function, since it is only defined for $-1 \leq x \leq 1$ and changes (monotonically) from $-\infty$ to $+\infty$ as x varies over this restricted range. This function therefore fails condition (i) and *cannot* be represented as a Fourier series.

(b) $\tan x$ is a periodic function but its discontinuities are not finite, nor is its absolute modulus integrable. It therefore fails tests (ii) and (iv) and *cannot* be represented as a Fourier series.

(c) $|\sin x|^{-1/2}$ is a periodic function of period π and, although it becomes infinite at $x = n\pi$, there are no infinite discontinuities. Near $x = 0$, say, it behaves as $|x|^{-1/2}$ and its absolute modulus is therefore integrable. There is only one minimum in any one period. The function therefore satisfies all four Dirichlet conditions and *can* be represented as a Fourier series.

(d) $\cos^{-1}(\sin 2x)$ is clearly a multi-valued function and fails condition (ii); it *cannot* be represented as a Fourier series.

(e) $x\sin(1/x)$, for $-\pi^{-1} < x \leq \pi^{-1}$ (cyclically repeated) is clearly cyclic (by definition), continuous, bounded, single-valued and integrable. However, since $\sin(1/x)$ oscillates with unlimited frequency near $x = 0$, there are an infinite number of maxima and minima in any region enclosing $x = 0$. Condition (iii) is therefore not satisfied and the function *cannot* be represented as a Fourier series.

12.5 *Find the Fourier series of the function $f(x) = x$ in the range $-\pi < x \leq \pi$. Hence show that*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}.$$

This is an odd function in x and so a sine series with period 2π is appropriate. The coefficient of $\sin nx$ will be given by

$$\begin{aligned} b_n &= \frac{2}{2\pi}\int_{-\pi}^{\pi} x\sin nx\,dx \\ &= \frac{1}{\pi}\left\{\left[-\frac{x\cos nx}{n}\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{\cos nx}{n}\,dx\right\} \\ &= \frac{1}{\pi}\left[-\frac{\pi(-1)^n - (-\pi)(-1)^n}{n} + 0\right] = \frac{2(-1)^{n+1}}{n}. \end{aligned}$$

Thus, $$x = f(x) = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx.$$

We note in passing that although this series is convergent, as it must be, it has poor (i.e. n^{-1}) convergence; this can be put down to the periodic version of the function having a discontinuity (of 2π) at the end of each basic period.

To obtain the sum of a series from such a Fourier representation, we must make a judicious choice for the value of x – making such a choice is rather more of an art than a science! Here, setting $x = \pi/2$ gives

$$\begin{aligned} \frac{\pi}{2} &= 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\pi/2)}{n} \\ &= 2\sum_{n\,\text{odd}} \frac{(-1)^{n+1}(-1)^{(n-1)/2})}{n}, \\ \Rightarrow \quad \frac{\pi}{4} &= \frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots. \end{aligned}$$

12.7 *For the function*

$$f(x) = 1 - x, \qquad 0 \le x \le 1,$$

a Fourier sine series can be found by continuing it in the range $-1 < x \le 0$ *as* $f(x) = -1 - x$. *The function thus has a discontinuity of 2 at* $x = 0$. *The series is*

$$1 - x = f(x) = \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{\sin n\pi x}{n}. \qquad (*)$$

In order to obtain a cosine series, the continuation has to be $f(x) = 1 + x$ *in the range* $-1 < x \le 0$. *The function then has no discontinuity at* $x = 0$ *and the corresponding series is*

$$1 - x = f(x) = \frac{1}{2} + \frac{4}{\pi^2}\sum_{n\,\text{odd}} \frac{\cos n\pi x}{n^2}. \qquad (**)$$

For these continued functions and series, consider (i) *their derivatives and* (ii) *their integrals. Do they give meaningful equations? You will probably find it helpful to sketch all the functions involved.*

(i) *Derivatives*

(a) The sine series. With the continuation given, the derivative df/dx has the value -1 everywhere, except at the origin where the function is not defined

(though $f(0) = 0$ seems the only possible choice), continuous or differentiable. Differentiating the given series (*) for $f(x)$ yields

$$\frac{df}{dx} = 2\sum_{n=1}^{\infty} \cos n\pi x.$$

This series does not converge and the equation is not meaningful.

(b) The cosine series. With the stated continuation for $f(x)$ the derivative is $+1$ for $-1 < x \leq 0$ and is -1 for $0 \leq x \leq 1$. It is thus the negative of an odd (about $x = 0$) unit square-wave, whose Fourier series is

$$-\frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\pi x}{n}.$$

This is confirmed by differentiating (**) term by term to obtain the same result:

$$\frac{df}{dx} = \frac{4}{\pi^2}\sum_{n\,\text{odd}} \frac{-n\pi \sin n\pi x}{n^2} = -\frac{4}{\pi}\sum_{n\,\text{odd}} \frac{\sin n\pi x}{n}.$$

(ii) *Integrals*

Since integrals contain an arbitrary constant of integration, we will define $F(-1) = 0$, where $F(x)$ is the indefinite integral of $f(x)$.

(a) The sine series. For $-1 \leq x \leq 0$,

$$F_a(x) = F(-1) + \int_{-1}^{x} (-1 - x)\,dx = -x - \tfrac{1}{2}x^2 - \tfrac{1}{2}.$$

For $0 \leq x \leq 1$,

$$F_a(x) = F(0) + \int_{0}^{x} (1 - x)\,dx = -\tfrac{1}{2} + \left[x - \tfrac{1}{2}x^2\right]_0^x = x - \tfrac{1}{2}x^2 - \tfrac{1}{2}.$$

This is a continuous function and, like all indefinite integrals, is 'smoother' than the function from which it is derived; this latter property will be reflected in the improved convergence of the derived series. Integrating term by term we find that its Fourier series is given by

$$\begin{aligned} F_a(x) &= \frac{2}{\pi}\int_{-1}^{x} \sum_{n=1}^{\infty} \frac{\sin n\pi x'}{n}\,dx' \\ &= \frac{2}{\pi}\sum_{n=1}^{\infty} \left[-\frac{\cos n\pi x'}{\pi n^2}\right]_{-1}^{x} \\ &= \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n - \cos n\pi x}{n^2} \\ &= -\frac{1}{6} - \frac{2}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos n\pi x}{n^2}, \end{aligned}$$

a series that has n^{-2} convergence. Here we have used the result that $\sum_{n=1}^{\infty}(-1)^n n^{-2} = -\pi^2/12$.

(b) The cosine series. The corresponding indefinite integral in this case is

$$F_b(x) = x + \tfrac{1}{2}x^2 + \tfrac{1}{2} \quad \text{for} \quad -1 \le x \le 0,$$
$$F_b(x) = x - \tfrac{1}{2}x^2 + \tfrac{1}{2} \quad \text{for} \quad 0 \le x \le 1,$$

and the corresponding integrated series, which has even better convergence (n^{-3}), is given by

$$\frac{1}{2}(x+1) + \frac{4}{\pi^3}\sum_{n\,\text{odd}} \frac{\sin n\pi x}{n^3}.$$

However, to have a true Fourier series expression, we must substitute a Fourier series for the $x/2$ term that arises from integrating the constant ($\frac{1}{2}$) in (**). This series must be that for $x/2$ across the complete range $-1 \le x \le 1$, and so neither (*) nor (**) can be rearranged for the purpose. A straightforward calculation (see exercise 12.25 part (b), if necessary) yields the poorly convergent sine series

$$x = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin n\pi x,$$

and makes the final expression for $F_b(x)$

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin n\pi x + \frac{4}{\pi^3}\sum_{n\,\text{odd}} \frac{\sin n\pi x}{n^3}.$$

As will be apparent from a simple sketch, the first series in the above expression dominates; all of its terms are present and it has only n^{-1} convergence. The second series has alternate terms missing and its convergence $\sim n^{-3}$.

12.9 *Find the Fourier coefficients in the expansion of* $f(x) = \exp x$ *over the range* $-1 < x < 1$. *What value will the expansion have when* $x = 2$?

Since the Fourier series will have period 2, we can say immediately that at $x = 2$ the series will converge to the value it has at $x = 0$, namely 1.

As the function $f(x) = \exp x$ is neither even nor odd, its Fourier series will contain

both sine and cosine terms. The cosine coefficients are given by

$$\begin{aligned} a_n &= \frac{2}{2}\int_{-1}^{1} e^x \cos(n\pi x)\,dx \\ &= [\cos(n\pi x)\,e^x]_{-1}^{1} + \int_{-1}^{1} n\pi \sin(n\pi x)\,e^x\,dx \\ &= (-1)^n(e^1 - e^{-1}) + [n\pi\,\sin(n\pi x)\,e^x]_{-1}^{1} \\ &\qquad\qquad - \int_{-1}^{1} n^2\pi^2\,\cos(n\pi x)\,e^x\,dx \\ &= 2(-1)^n \sinh 1 - n^2\pi^2 a_n, \end{aligned}$$

$$\Rightarrow \quad a_n = \frac{2(-1)^n \sinh 1}{1 + n^2\pi^2}.$$

Similarly, the sine coefficients are given by

$$\begin{aligned} b_n &= \frac{2}{2}\int_{-1}^{1} e^x \sin(n\pi x)\,dx \\ &= [\sin(n\pi x)\,e^x]_{-1}^{1} - \int_{-1}^{1} n\pi\,\cos(n\pi x)\,e^x\,dx \\ &= 0 + [-n\pi\,\cos(n\pi x)\,e^x]_{-1}^{1} - \int_{-1}^{1} n^2\pi^2\,\sin(n\pi x)\,e^x\,dx \\ &= 2(-1)^{n+1} n\pi \sinh 1 - n^2\pi^2 b_n, \end{aligned}$$

$$\Rightarrow \quad b_n = \frac{2(-1)^{n+1} n\pi \sinh 1}{1 + n^2\pi^2}.$$

12.11 *Consider the function* $f(x) = \exp(-x^2)$ *in the range* $0 \le x \le 1$. *Show how it should be continued to give as its Fourier series a series (the actual form is not wanted)* (a) *with only cosine terms,* (b) *with only sine terms,* (c) *with period* 1 *and* (d) *with period* 2.

Would there be any difference between the values of the last two series at (i) $x = 0$, (ii) $x = 1$?

The function and its four continuations are shown as (a)–(d) in figure 12.1. Note that in the range $0 \le x \le 1$, all four graphs are identical.

Where a continued function has a discontinuity at the ends of its basic period, the series will yield a value at those end-points that is the average of the function's values on the two sides of the discontinuity. Thus for continuation (c) both (i) $x = 0$ and (ii) $x = 1$ are end-points, and the value of the series there will be

Figure 12.1 The solution to exercise 12.11, showing the continuations of $\exp(-x^2)$ in $0 \le x \le 1$ to give: (a) cosine terms only; (b) sine terms only; (c) period 1; (d) period 2.

$(1 + e^{-1})/2$. For continuation (d), $x = 0$ is an end-point, and the series will have value $\frac{1}{2}(1 + e^{-4})$. However, $x = 1$ is not a point of discontinuity, and the series will have the expected value of e^{-1}.

12.13 *Consider the representation as a Fourier series of the displacement of a string lying in the interval* $0 \le x \le L$ *and fixed at its ends, when it is pulled aside by* y_0 *at the point* $x = L/4$. *Sketch the continuations for the region outside the interval that will*

(a) *produce a series of period* L,
(b) *produce a series that is antisymmetric about* $x = 0$, *and*
(c) *produce a series that will contain only cosine terms.*
(d) *What are* (i) *the periods of the series in* (b) *and* (c) *and* (ii) *the value of the '*a_0*-term' in* (c)?
(e) *Show that a typical term of the series obtained in* (b) *is*

$$\frac{32y_0}{3n^2\pi^2} \sin\frac{n\pi}{4} \sin\frac{n\pi x}{L}.$$

Parts (a), (b) and (c) of figure 12.2 show the three required continuations. Condition (b) will result in a series containing only sine terms, whilst condition (c) requires the continued function to be symmetric about $x = 0$.

(d) (i) The period in both cases, (b) and (c), is clearly $2L$.

(ii) The average value of the displacement is found from 'the area under the triangular curve' to be $(\frac{1}{2}Ly_0)/L = \frac{1}{2}y_0$, and this is the value of the 'a_0-term'.

(e) For the antisymmetric continuation there will be no cosine terms. The sine

Figure 12.2 Plucked string with fixed ends: (a)–(c) show possible mathematical continuations; (b) is antisymmetric about 0 and (c) is symmetric.

term coefficients (for a period of $2L$) are given by

$$
\begin{aligned}
b_n &= 2\,\frac{2}{2L}\int_0^L f(x)\sin(nkx)\,dx, \text{ where } k = 2\pi/2L = \pi/L,\\
&= \frac{2y_0}{L}\left[\int_0^{L/4}\frac{4x}{L}\sin(nkx)\,dx + \int_{L/4}^{L}\left(\frac{4}{3}-\frac{4x}{3L}\right)\sin(nkx)\,dx\right]\\
&= \frac{8y_0}{3L^2}\left[\int_0^{L/4}3x\sin(nkx)\,dx + \int_{L/4}^{L}(L-x)\sin(nkx)\,dx\right]\\
&= \frac{8y_0}{3L^2}\left\{\left[-\frac{3x\cos(nkx)}{nk}\right]_0^{L/4} + \int_0^{L/4}\frac{3\cos(nkx)}{nk}\,dx\right.\\
&\qquad \left. + \left[-\frac{L\cos(nkx)}{nk}\right]_{L/4}^{L} + \left[\frac{x\cos(nkx)}{nk}\right]_{L/4}^{L} - \int_{L/4}^{L}\frac{\cos(nkx)}{nk}\,dx\right\}.
\end{aligned}
$$

Integrating by parts then yields

$$
\begin{aligned}
b_n &= \frac{8y_0}{3L^2}\left\{-\frac{3L\cos(n\pi/4)}{4n(\pi/L)} - 0 + \left[\frac{3\sin(nkx)}{n^2k^2}\right]_0^{L/4} - \frac{L\cos(n\pi)}{n(\pi/L)}\right.\\
&\qquad \left. + \frac{L\cos(n\pi/4)}{n(\pi/L)} + \frac{L\cos(n\pi)}{n(\pi/L)} - \frac{L\cos(n\pi/4)}{4n(\pi/L)} - \left[\frac{\sin(nkx)}{n^2k^2}\right]_{L/4}^{L}\right\}\\
&= \frac{8y_0}{3L^2}\left[\frac{3L^2\sin(n\pi/4)}{n^2\pi^2} - \frac{L^2\sin(n\pi)}{n^2\pi^2} + \frac{L^2\sin(n\pi/4)}{n^2\pi^2}\right] = \frac{32y_0}{3n^2\pi^2}\sin\left(\frac{n\pi}{4}\right).
\end{aligned}
$$

A typical term is therefore

$$\frac{32y_0}{3n^2\pi^2}\sin\left(\frac{n\pi}{4}\right)\sin\left(\frac{n\pi x}{L}\right).$$

We note that every fourth term ($n = 4m$ with m an integer) will be missing.

12.15 *The Fourier series for the function $y(x) = |x|$ in the range $-\pi \le x < \pi$ is*

$$y(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\cos(2m+1)x}{(2m+1)^2}.$$

By integrating this equation term by term from 0 to x, find the function $g(x)$ whose Fourier series is

$$\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Using these results, determine, as far as possible by inspection, the form of the functions of which the following are the Fourier series:

(a)

$$\cos\theta + \frac{1}{9}\cos 3\theta + \frac{1}{25}\cos 5\theta + \cdots;$$

(b)

$$\sin\theta + \frac{1}{27}\sin 3\theta + \frac{1}{125}\sin 5\theta + \cdots;$$

(c)

$$\frac{L^2}{3} - \frac{4L^2}{\pi^2}\left[\cos\frac{\pi x}{L} - \frac{1}{4}\cos\frac{2\pi x}{L} + \frac{1}{9}\cos\frac{3\pi x}{L} - \cdots\right].$$

[*You may find it helpful to first set $x = 0$ in the quoted result and so obtain values for $S_o = \sum(2m+1)^{-2}$ and other sums derivable from it.*]

First, define

$$S = \sum_{\text{all } n\neq 0} n^{-2}, \quad S_o = \sum_{\text{odd } n} n^{-2}, \quad S_e = \sum_{\text{even } n\neq 0} n^{-2}.$$

Clearly, $S_e = \frac{1}{4}S$.

Now set $x = 0$ in the quoted result to obtain

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi}{2} - \frac{4}{\pi}S_o.$$

Thus, $S_o = \pi^2/8$. Further, $S = S_o + S_e = S_o + \frac{1}{4}S$; it follows that $S = \pi^2/6$ and, by subtraction, that $S_e = \pi^2/24$.

We now consider the integral of $y(x) = |x|$ from 0 to x.
(i) For $x < 0$, $\displaystyle\int_0^x |x|\,dx = \int_0^x (-x)\,dx = -\frac{1}{2}x^2$.
(ii) For $x > 0$, $\displaystyle\int_0^x |x|\,dx = \int_0^x x\,dx = \frac{1}{2}x^2$.
Integrating the series term by term gives

$$\frac{\pi x}{2} - \frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3}.$$

Equating these two results and isolating the series gives

$$\begin{aligned}\frac{4}{\pi}\sum_{m=0}^{\infty}\frac{\sin(2m+1)x}{(2m+1)^3} &= \tfrac{1}{2}x(\pi - x) \text{ for } x \geq 0,\\ &= \tfrac{1}{2}x(\pi + x) \text{ for } x \leq 0.\end{aligned}$$

Questions (a)–(c) are to be solved largely through inspection and so detailed working is not (cannot be) given.

(a) Straightforward substitution of θ for x and rearrangement of the original Fourier series give $g_1(\theta) = \frac{1}{4}\pi(\frac{1}{2}\pi - |\theta|)$.

(b) Straightforward substitution of θ for x and rearrangement of the integrated Fourier series give $g_2(\theta) = \frac{1}{8}\pi\theta(\pi - |\theta|)$.

(c) This contains only cosine terms and is therefore an even function of x. Its average value (given by the a_0 term) is $\frac{1}{3}L^2$. Setting $x = 0$ gives

$$\begin{aligned}f(0) &= \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(1 - \frac{1}{4} + \frac{1}{9} - \cdots\right)\\ &= \frac{L^2}{3} - \frac{4L^2}{\pi^2}(S_o - S_e)\\ &= \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(\frac{\pi^2}{8} - \frac{\pi^2}{24}\right) = 0.\end{aligned}$$

Setting $x = L$ gives

$$\begin{aligned}f(L) &= \frac{L^2}{3} - \frac{4L^2}{\pi^2}\left(-1 - \frac{1}{4} - \frac{1}{9} - \cdots\right)\\ &= \frac{L^2}{3} - \frac{4L^2}{\pi^2}(-S) = L^2.\end{aligned}$$

All of this evidence suggests that $f(x) = x^2$ (which it is).

12.17 *Find the (real) Fourier series of period 2 for $f(x) = \cosh x$ and $g(x) = x^2$ in the range $-1 \le x \le 1$. By integrating the series for $f(x)$ twice, prove that*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2(n^2\pi^2+1)} = \frac{1}{2}\left(\frac{1}{\sinh 1} - \frac{5}{6}\right).$$

Since both functions are even, we need consider only constants and cosine terms. The series for x^2 can be calculated directly or, more easily, by using the result of the final part of exercise 12.15 with L set equal to 1:

$$g(x) = x^2 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \pi n x \text{ for } -1 \le x \le 1.$$

For $f(x) = \cosh x$,

$$\begin{aligned} a_0 &= \frac{2}{2}\, 2 \int_0^1 \cosh x \, dx = 2\sinh(1), \\ a_n &= \frac{2}{2}\, 2 \int_0^1 \cosh x \cos(n\pi x)\, dx \\ &= 2\left[\frac{\cosh x \sin(n\pi x)}{n\pi}\right]_0^1 - 2\int_0^1 \frac{\sinh x \sin(n\pi x)}{n\pi}\, dx \\ &= 0 + 2\left[\frac{\sinh x \cos(n\pi x)}{n^2\pi^2}\right]_0^1 - \frac{a_n}{n^2\pi^2}. \end{aligned}$$

Rearranging this gives

$$a_n = \frac{(-1)^n 2\sinh(1)}{1+n^2\pi^2}.$$

Thus,

$$\cosh x = \sinh(1)\left(1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} \cos n\pi x\right).$$

We now integrate this expansion twice from 0 to x (anticipating that we will recover a hyperbolic cosine function plus some additional terms). Since $\sinh(0) = \sin(m\pi 0) = 0$, the first integration yields

$$\sinh x = \sinh(1)\left(x + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n\pi(1+n^2\pi^2)} \sin n\pi x\right).$$

For the second integration we use $\cosh(0) = \cos(m\pi 0) = 1$ to obtain

$$\cosh(x) - 1 = \sinh(1)\left(\frac{1}{2}x^2 + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2\pi^2(1+n^2\pi^2)}[\cos(n\pi x) - 1]\right).$$

However, this expansion must be the same as the original expansion for $\cosh(x)$ after a Fourier series has been substituted for the $\frac{1}{2}\sinh(1)x^2$ term. The coefficients of $\cos n\pi x$ in the two expressions must be equal; in particular, the equality of the constant terms (formally $\cos n\pi x$ with $n = 0$) requires that

$$\sinh(1) - 1 = \frac{1}{2}\sinh(1)\frac{1}{3} + 2\sinh(1)\sum_{n=1}^{\infty}\frac{(-1)^{n+2}}{n^2\pi^2(1+n^2\pi^2)},$$

i.e.

$$\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2\pi^2(n^2\pi^2+1)} = \frac{1}{2}\left(\frac{1}{\sinh\ 1} - \frac{5}{6}\right),$$

as stated in the question.

12.19 *Demonstrate explicitly for the odd (about $x = 0$) square-wave function that Parseval's theorem is valid. You will need to use the relationship*

$$\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}.$$

Show that a filter that transmits frequencies only up to $8\pi/T$ will still transmit more than 90% of the power in a square-wave voltage signal of period T.

As stated in the solution to exercise 12.7, and in virtually every textbook, the odd square-wave function has only the odd harmonics present in its Fourier sine series representation. The coefficient of the $\sin(2m+1)\pi x$ term is

$$b_{2m+1} = \frac{4}{(2m+1)\pi}.$$

For a periodic function of period L whose complex Fourier coefficients are c_r, or whose cosine and sine coefficients are a_r and b_r, respectively, Parseval's theorem for one function states that

$$\begin{aligned}\frac{1}{L}\int_{x_0}^{x_0+L}|f(x)|^2dx &= \sum_{r=-\infty}^{\infty}|c_r|^2 \\ &= \left(\tfrac{1}{2}a_0\right)^2 + \tfrac{1}{2}\sum_{r=1}^{\infty}(a_r^2 + b_r^2),\end{aligned}$$

and therefore requires in this particular case, in which all the a_r are zero and $L = 2$, that

$$\frac{1}{2}\sum_{m=0}^{\infty}\frac{16}{(2m+1)^2\pi^2} = \frac{1}{2}\sum_{n=1}^{\infty}b_n^2 = \frac{1}{2}\int_{-1}^{1}|\pm 1|^2\,dx = 1.$$

Since

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} = \frac{\pi^2}{8},$$

this reduces to the identity

$$\frac{1}{2}\,\frac{16}{\pi^2}\,\frac{\pi^2}{8} = 1.$$

The power at any particular frequency in an electrical signal is proportional to the square of the amplitude at that frequency, i.e. to $|b_n|^2$ in the present case. If the filter passes only frequencies up to $8\pi/T = 4\omega$, then only the $n = 1$ and the $n = 3$ components will be passed. They contribute a fraction

$$\left(\frac{1}{1} + \frac{1}{9}\right) \div \frac{\pi^2}{8} = 0.901$$

of the total, i.e. more than 90%.

12.21 *Find the complex Fourier series for the periodic function of period 2π defined in the range $-\pi \le x \le \pi$ by $y(x) = \cosh x$. By setting $x = 0$ prove that*

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+1} = \frac{1}{2}\left(\frac{\pi}{\sinh \pi} - 1\right).$$

We first note that, although $\cosh x$ is an even function of x, e^{-inx} is neither even nor odd. Consequently it will not be possible to convert the integral into one over the range $0 \le x \le \pi$. The complex Fourier coefficients c_n $(-\infty < n < \infty)$ are therefore calculated as

$$\begin{aligned}
c_n &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \cosh x\, e^{-inx}\, dx \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\left(e^{-inx+x} + e^{-inx-x}\right)\, dx \\
&= \frac{1}{4\pi}\left[\frac{e^{(1-in)x}}{1-in}\right]_{-\pi}^{\pi} + \frac{1}{4\pi}\left[\frac{e^{(-1-in)x}}{-1-in}\right]_{-\pi}^{\pi} \\
&= \frac{1}{4\pi}\,\frac{(1+in)(-1)^n(2\sinh\pi) - (1-in)(-1)^n(-2\sinh\pi)}{1+n^2} \\
&= \frac{(-1)^n 4\sinh(\pi)}{4\pi(1+n^2)}.
\end{aligned}$$

Thus,

$$\cosh x = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh\pi}{\pi(1+n^2)}\, e^{inx}.$$

We now set $x = 0$ on both sides of the equation:

$$1 = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh \pi}{\pi(1+n^2)},$$

$$\Rightarrow \quad \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh \pi}.$$

Separating out the $n = 0$ term, and noting that $(-1)^n = (-1)^{-n}$, now gives

$$1 + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{1+n^2} = \frac{\pi}{\sinh \pi}$$

and hence the stated result.

12.23 *The complex Fourier series for the periodic function generated by* $f(t) = \sin t$ *for* $0 \le t \le \pi/2$, *and repeated in every subsequent interval of* $\pi/2$, *is*

$$\sin(t) = \frac{2}{\pi} \sum_{n=-\infty}^{\infty} \frac{4ni - 1}{16n^2 - 1} e^{i4nt}.$$

Apply Parseval's theorem to this series and so derive a value for the sum of the series

$$\frac{17}{(15)^2} + \frac{65}{(63)^2} + \frac{145}{(143)^2} + \cdots + \frac{16n^2+1}{(16n^2-1)^2} + \cdots.$$

Applying Parseval's theorem (see solution 12.19) in a straightforward manner to the given equation:

$$\frac{2}{\pi} \int_0^{\pi/2} \sin^2(t)\, dt = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{4ni-1}{16n^2-1} \frac{-4ni-1}{16n^2-1},$$

$$\frac{2}{\pi} \frac{1}{2} \frac{\pi}{2} = \frac{4}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{16n^2+1}{(16n^2-1)^2},$$

$$\frac{\pi^2}{8} = 1 + 2\sum_{n=1}^{\infty} \frac{16n^2+1}{(16n^2-1)^2},$$

$$\Rightarrow \quad \sum_{n=1}^{\infty} \frac{16n^2+1}{(16n^2-1)^2} = \frac{\pi^2 - 8}{16}.$$

To obtain the second line we have used the standard result that the average value of the square of a sinusoid is $1/2$.

12.25 *Show that Parseval's theorem for two real functions whose Fourier expansions have cosine and sine coefficients a_n, b_n and α_n, β_n takes the form*

$$\frac{1}{L}\int_0^L f(x)g^*(x)\,dx = \frac{1}{4}a_0\alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n).$$

(a) *Demonstrate that for $g(x) = \sin mx$ or $\cos mx$ this reduces to the definition of the Fourier coefficients.*

(b) *Explicitly verify the above result for the case in which $f(x) = x$ and $g(x)$ is the square-wave function, both in the interval $-1 \le x \le 1$.*

If c_n and γ_n are the complex Fourier coefficients for the real functions $f(x)$ and $g(x)$ that have real Fourier coefficients a_n, b_n and α_n, β_n, respectively, then

$$c_n = \tfrac{1}{2}(a_n - ib_n) \quad \text{and} \quad \gamma_n = \alpha_n - i\beta_n,$$
$$c_{-n} = \tfrac{1}{2}(a_n + ib_n) \quad \text{and} \quad \gamma_{-n} = \alpha_n + i\beta_n.$$

The two functions can be written as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi inx}{L}\right),$$
$$g(x) = \sum_{n=-\infty}^{\infty} \gamma_n \exp\left(\frac{2\pi inx}{L}\right). \qquad (*)$$

Thus,

$$f(x)g^*(x) = \sum_{n=-\infty}^{\infty} c_n g^*(x) \exp\left(\frac{2\pi inx}{L}\right).$$

Integrating this equation with respect to x over the interval $(0, L)$ and dividing by L, we find

$$\begin{aligned}\frac{1}{L}\int_0^L f(x)g^*(x)\,dx &= \sum_{n=-\infty}^{\infty} c_n \frac{1}{L}\int_0^L g^*(x)\exp\left(\frac{2\pi inx}{L}\right)dx \\ &= \sum_{n=-\infty}^{\infty} c_n \left[\frac{1}{L}\int_0^L g(x)\exp\left(\frac{-2\pi inx}{L}\right)dx\right]^* \\ &= \sum_{n=-\infty}^{\infty} c_n \gamma_n^*.\end{aligned}$$

To obtain the last line we have used the inverse of relationship $(*)$.

Dividing up the sum over all n into a sum over positive n, a sum over negative n

and the $n = 0$ term, and then substituting for c_n and γ_n, gives

$$\begin{aligned}\frac{1}{L}\int_0^L f(x)g^*(x)\,dx &= \frac{1}{4}\sum_{n=1}^{\infty}(a_n - ib_n)(\alpha_n + i\beta_n)\\ &\quad + \frac{1}{4}\sum_{n=1}^{\infty}(a_n + ib_n)(\alpha_n - i\beta_n) + \frac{1}{4}a_0\alpha_0\\ &= \frac{1}{4}\sum_{n=1}^{\infty}(2a_n\alpha_n + 2b_n\beta_n) + \frac{1}{4}a_0\alpha_0\\ &= \frac{1}{2}\sum_{n=1}^{\infty}(a_n\alpha_n + b_n\beta_n) + \frac{1}{4}a_0\alpha_0,\end{aligned}$$

i.e. the stated result.

(a) For $g(x) = \sin mx$, $\beta_m = 1$ and all other α_n and β_n are zero. The above equation then reduces to

$$\frac{1}{L}\int_0^L f(x)\sin(mx)\,dx = \frac{1}{2}b_n,$$

which is the normal definition of b_n. Similarly, setting $g(x) = \cos mx$ leads to the normal definition of a_n.

(b) For the function $f(x) = x$ in the interval $-1 < x \leq 1$, the sine coefficients are

$$\begin{aligned}b_n &= \frac{2}{2}\int_{-1}^{1} x\sin n\pi x\,dx\\ &= 2\int_0^1 x\sin n\pi x\,dx\\ &= 2\left\{\left[\frac{-x\cos n\pi x}{n\pi}\right]_0^1 + \int_0^1 \frac{\cos n\pi x}{n\pi}\,dx\right\}\\ &= 2\left\{\frac{(-1)^{n+1}}{n\pi} + \left[\frac{\sin n\pi x}{n^2\pi^2}\right]_0^1\right\}\\ &= \frac{2(-1)^{n+1}}{n\pi}.\end{aligned}$$

As stated in exercise 12.19, for the (antisymmetric) square-wave function $\beta_n = 4/(n\pi)$ for odd n and $\beta_n = 0$ for even n.

Now the integral

$$\frac{1}{L}\int_0^L f(x)g^*(x)\,dx = \frac{1}{2}\left[\int_{-1}^{0}(-1)x\,dx + \int_0^1(+1)x\,dx\right] = \frac{1}{2},$$

whilst

$$\frac{1}{2}\sum_{n=1}^{\infty} b_n\beta_n = \frac{1}{2}\sum_{n\,\text{odd}}\frac{4}{n\pi}\,\frac{2(-1)^{n+1}}{n\pi} = \frac{4}{\pi^2}\sum_{n\,\text{odd}}\frac{1}{n^2} = \frac{4}{\pi^2}\,\frac{\pi^2}{8} = \frac{1}{2}.$$

The value of the sum $\sum n^{-2}$ for odd n is taken from S_o in the solution to exercise 12.15. Thus, the two sides of the equation agree, verifying the validity of Parseval's theorem in this case.

13

Integral transforms

13.1 *Find the Fourier transform of the function* $f(t) = \exp(-|t|)$.

(a) *By applying Fourier's inversion theorem prove that*

$$\frac{\pi}{2}\exp(-|t|) = \int_0^\infty \frac{\cos\omega t}{1+\omega^2}\,d\omega.$$

(b) *By making the substitution* $\omega = \tan\theta$, *demonstrate the validity of Parseval's theorem for this function.*

As the function $|t|$ is not representable by the same integrable function throughout the integration range, we must divide the range into two sections and use different explicit expressions for the integrand in each:

$$\begin{aligned}
\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|t|}\,e^{-i\omega t}\,dt \\
&= \frac{1}{\sqrt{2\pi}}\int_0^\infty e^{-(1+i\omega)t}\,dt + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} e^{(1-i\omega)t}\,dt \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{1+i\omega} + \frac{1}{1-i\omega}\right) \\
&= \frac{1}{\sqrt{2\pi}}\frac{2}{1+\omega^2}.
\end{aligned}$$

(a) Substituting this result into the inversion theorem gives

$$\exp^{-|t|} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2}{\sqrt{2\pi}(1+\omega^2)}\,e^{i\omega t}\,d\omega.$$

Equating the real parts on the two sides of this equation and noting that the

resulting integrand is symmetric in ω, shows that

$$\exp^{-|t|} = \frac{2}{\pi}\int_0^\infty \frac{\cos\omega t}{(1+\omega^2)}\,d\omega,$$

as given in the question.

(b) For Parseval's theorem, which states that

$$\int_{-\infty}^{\infty} |f(t)|^2\,dt = \int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2\,d\omega,$$

we first evaluate

$$\begin{aligned}\int_{-\infty}^{\infty} |f(t)|^2\,dt &= \int_{-\infty}^{0} e^{2t}\,dt + \int_0^\infty e^{-2t}\,dt\\ &= 2\int_0^\infty e^{-2t}\,dt\\ &= 2\left[\frac{e^{-2t}}{-2}\right]_0^\infty = 1.\end{aligned}$$

The second integral, over ω, is

$$\begin{aligned}\int_{-\infty}^{\infty} |\tilde{f}(\omega)|^2\,d\omega &= 2\int_0^\infty \frac{2}{\pi(1+\omega^2)^2}\,d\omega, \quad \text{set } \omega \text{ equal to } \tan\theta,\\ &= \frac{4}{\pi}\int_0^{\pi/2} \frac{1}{\sec^4\theta}\sec^2\theta\,d\theta\\ &= \frac{4}{\pi}\int_0^{\pi/2} \cos^2\theta\,d\theta = \frac{4}{\pi}\frac{1}{2}\frac{\pi}{2} = 1,\end{aligned}$$

i.e. the same as the first one, thus verifying the theorem for this function.

13.3 *Find the Fourier transform of $H(x-a)e^{-bx}$, where $H(x)$ is the Heaviside function.*

The Heaviside function $H(x)$ has value 0 for $x < 0$ and value 1 for $x \geq 0$. Write $H(x-a)e^{-bx} = h(x)$ with b assumed > 0. Then,

$$\begin{aligned}\tilde{h}(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(x-a)e^{-bx}\,e^{-ikx}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\int_a^\infty e^{-bx-ikx}\,dx\\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-bx-ikx}}{-b-ik}\right]_a^\infty\\ &= \frac{1}{\sqrt{2\pi}}\frac{e^{-ba}e^{-ika}}{b+ik} = e^{-ika}\frac{e^{-ba}}{\sqrt{2\pi}}\frac{b-ik}{b^2+k^2}.\end{aligned}$$

This same result could be obtained by setting $y = x - a$, finding the transform of $e^{-ba}e^{-by}$, and then using the translation property of Fourier transforms.

13.5 *By taking the Fourier transform of the equation*

$$\frac{d^2\phi}{dx^2} - K^2\phi = f(x),$$

show that its solution, $\phi(x)$, can be written as

$$\phi(x) = \frac{-1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{ikx}\widetilde{f}(k)}{k^2 + K^2}\, dk,$$

where $\widetilde{f}(k)$ is the Fourier transform of $f(x)$.

We take the Fourier transform of each term of

$$\frac{d^2\phi}{dx^2} - K^2\phi = f(x)$$

to give

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d^2\phi}{dx^2}\, e^{-ikx}\, dx - K^2\tilde{\phi}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx.$$

Since ϕ must vanish at $\pm\infty$, the first term can be integrated twice by parts with no contributions at the end-points. This gives the full equation as

$$-k^2\tilde{\phi}(k) - K^2\tilde{\phi}(k) = \tilde{f}(k).$$

Now, by the Fourier inversion theorem,

$$\begin{aligned}\phi(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\phi}(k)\, e^{ikx}\, dk \\ &= -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{\tilde{f}(k)\, e^{ikx}}{k^2 + K^2}\, dk.\end{aligned}$$

Note

The principal advantage of this Fourier approach to a set of one or more linear differential equations is that the differential operators act only on exponential functions whose exponents are linear in x. This means that the derivatives are no more than multiples of the original function and what were originally differential equations are turned into algebraic ones. As the differential equations are linear the algebraic equations can be solved explicitly for the transforms of their solutions, and the solutions themselves may then be found using the inversion theorem. The 'price' to be paid for this great simplification is that the inversion integral may not be tractable analytically, but, as a last resort, numerical integration can always be employed.

13.7 *Find the Fourier transform of the unit rectangular distribution*

$$f(t) = \begin{cases} 1 & |t| < 1 \\ 0 & \textit{otherwise.} \end{cases}$$

Determine the convolution of f with itself and, without further integration, deduce its transform. Deduce that

$$\int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2}\, d\omega = \pi,$$
$$\int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4}\, d\omega = \frac{2\pi}{3}.$$

The function to be transformed is unity in the range $-1 \le t \le 1$ and so

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{-i\omega} - e^{i\omega}}{-i\omega} \right] = \frac{2 \sin \omega}{\sqrt{2\pi}\omega}.$$

Denote by $p(t)$ the convolution of f with itself and, in the second line of the calculation below, change the integration variable from s to $u = t - s$:

$$\begin{aligned} p(t) &\equiv \int_{-\infty}^{\infty} f(t-s) f(s)\, ds = \int_{-1}^{1} f(t-s)\, 1\, ds \\ &= \int_{t+1}^{t-1} f(u)(-du) = \int_{t-1}^{t+1} f(u) du. \end{aligned}$$

It follows that

$$p(t) = \begin{cases} (t+1) - (-1) & 0 > t > -2 \\ 1 - (t-1) & 2 > t > 0 \end{cases} = \begin{cases} 2 - |t| & 0 < |t| < 2, \\ 0 & \text{otherwise.} \end{cases}$$

The transform of p is given directly by the convolution theorem [which states that if $h(t)$, given by $h = f * g$, is the convolution of f and g, then $\tilde{h} = \sqrt{2\pi}\, \tilde{f}\, \tilde{g}$] as

$$\tilde{p}(\omega) = \sqrt{2\pi}\, \frac{2 \sin \omega}{\sqrt{2\pi}\omega}\, \frac{2 \sin \omega}{\sqrt{2\pi}\omega} = \frac{4}{\sqrt{2\pi}}\, \frac{\sin^2 \omega}{\omega^2}.$$

Noting that the two integrals to be evaluated have as integrands the squares of functions that are essentially the known transforms of simple functions, we are led to apply Parseval's theorem to each. Applying the theorem to $f(t)$ and $p(t)$ yields

$$\int_{-\infty}^{\infty} \frac{4 \sin^2 \omega}{2\pi\omega^2}\, d\omega = \int_{-\infty}^{\infty} |f(t)|^2\, dt = 2 \quad \Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin^2 \omega}{\omega^2} = \pi,$$

and
$$\int_{-\infty}^{\infty} \frac{16}{2\pi} \frac{\sin^4 \omega}{\omega^4}\, d\omega = \int_{-2}^{0} (2+t)^2\, dt + \int_{0}^{2} (2-t)^2\, dt$$
$$= \left[\frac{(2+t)^3}{3}\right]_{-2}^{0} - \left[\frac{(2-t)^3}{3}\right]_{0}^{2}$$
$$= \frac{8}{3} + \frac{8}{3},$$
$$\Rightarrow \quad \int_{-\infty}^{\infty} \frac{\sin^4 \omega}{\omega^4}\, d\omega = \frac{2\pi}{3}.$$

13.9 *By finding the complex Fourier* series *for its LHS show that either side of the equation*
$$\sum_{n=-\infty}^{\infty} \delta(t+nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-2\pi nit/T}$$
can represent a periodic train of impulses. By expressing the function $f(t+nX)$, in which X is a constant, in terms of the Fourier transform $\tilde{f}(\omega)$ *of $f(t)$, show that*
$$\sum_{n=-\infty}^{\infty} f(t+nX) = \frac{\sqrt{2\pi}}{X} \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2n\pi}{X}\right) e^{2\pi nit/X}.$$
This result is known as the Poisson summation formula.

Denote by $g(t)$ the periodic function $\sum_{n=-\infty}^{\infty} \delta(t+nT)$ with $2\pi/T = \omega$. Its complex Fourier coefficients are given by
$$c_n = \frac{1}{T}\int_0^T g(t)\, e^{-in\omega t}\, dt = \frac{1}{T}\int_0^T \delta(t)\, e^{-in\omega t}\, dt = \frac{1}{T}.$$
Thus, by the inversion theorem, its Fourier *series* representation is
$$g(t) = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{in\omega t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{-in\omega t} = \sum_{n=-\infty}^{\infty} \frac{1}{T} e^{-i2n\pi t/T},$$
showing that both this sum and the original one are representations of a periodic train of impulses.

In this result,
$$\sum_{n=-\infty}^{\infty} \delta(t+nT) = \frac{1}{T} \sum_{n=-\infty}^{\infty} e^{-2\pi nit/T},$$
we now make the changes of variable $t \to \omega$, $n \to -n$ and $T \to 2\pi/X$ and obtain
$$\sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{X}\right) = \frac{X}{2\pi} \sum_{n=-\infty}^{\infty} e^{inX\omega}. \quad (*)$$

If we denote $f(t+nX)$ by $f_{nX}(t)$ then, by the translation theorem, we have $\tilde{f}_{nX}(\omega) = e^{inX\omega}\tilde{f}(\omega)$ and

$$\begin{aligned}
f(t+nX) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}_{nX}(\omega)\, e^{i\omega t}\, d\omega \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{inX\omega}\tilde{f}(\omega)\, e^{i\omega t}\, d\omega, \\
\sum_{n=-\infty}^{\infty} f(t+nX) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t} \sum_{n=-\infty}^{\infty} e^{inX\omega}\, d\omega, \text{ use } (*) \text{ above,} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{i\omega t}\, \frac{2\pi}{X} \sum_{n=-\infty}^{\infty} \delta\left(\omega - \frac{2\pi n}{X}\right) d\omega \\
&= \frac{\sqrt{2\pi}}{X} \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{X}\right) e^{i2\pi nt/X}.
\end{aligned}$$

In the final line we have made use of the properties of a δ-function when it appears as a factor in an integrand.

13.11 *For a function $f(t)$ that is non-zero only in the range $|t| < T/2$, the full frequency spectrum $\tilde{f}(\omega)$ can be constructed, in principle exactly, from values at discrete sample points $\omega = n(2\pi/T)$. Prove this as follows.*

(a) *Show that the coefficients of a complex Fourier* series *representation of $f(t)$ with period T can be written as*

$$c_n = \frac{\sqrt{2\pi}}{T}\tilde{f}\left(\frac{2\pi n}{T}\right).$$

(b) *Use this result to represent $f(t)$ as an infinite sum in the defining integral for $\tilde{f}(\omega)$, and hence show that*

$$\tilde{f}(\omega) = \sum_{n=-\infty}^{\infty} \tilde{f}\left(\frac{2\pi n}{T}\right) \operatorname{sinc}\left(n\pi - \frac{\omega T}{2}\right),$$

where $\operatorname{sinc} x$ *is defined as* $(\sin x)/x$.

(a) The complex coefficients for the Fourier series for $f(t)$ are given by

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, e^{-i2\pi nt/T}\, dt.$$

But, we also know that the Fourier transform of $f(t)$ is given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} f(t)\, e^{-i\omega t}\, dt.$$

Comparison of these two equations shows that $c_n = \frac{1}{T}\sqrt{2\pi}\tilde{f}\left(\frac{2\pi n}{T}\right)$.

(b) Using the Fourier series representation of $f(t)$, the frequency spectrum at a general frequency ω can now be constructed as

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2} f(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-T/2}^{T/2}\left[\sum_{n=-\infty}^{\infty}\frac{1}{T}\sqrt{2\pi}\tilde{f}\left(\frac{2\pi n}{T}\right)e^{i2\pi nt/T}\right]e^{-i\omega t}\,dt \\ &= \frac{1}{T}\sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{T}\right)\frac{2\sin\left(\frac{2\pi n}{2}-\frac{\omega T}{2}\right)}{\frac{2\pi n}{T}-\omega} = \sum_{n=-\infty}^{\infty}\tilde{f}\left(\frac{2\pi n}{T}\right)\operatorname{sinc}\left(n\pi-\frac{\omega T}{2}\right).\end{aligned}$$

This final formula gives a prescription for calculating the frequency spectrum $\tilde{f}(\omega)$ of $f(t)$ for *any* ω, given the spectrum at the (admittedly infinite number of) discrete values $\omega = 2\pi n/T$. The sinc functions give the weights to be assigned to the known discrete values; of course, the weights vary as ω is varied, with, as expected, the largest weights for the nth contribution occurring when ω is close to $2\pi n/T$.

13.13 *Find the Fourier transform specified in part* (a) *and then use it to answer part* (b).

(a) *Find the Fourier transform of*

$$f(\gamma, p, t) = \begin{cases} e^{-\gamma t}\sin pt & t > 0 \\ 0 & t < 0, \end{cases}$$

where γ (> 0) *and* p *are constant parameters.*

(b) *The current* $I(t)$ *flowing through a certain system is related to the applied voltage* $V(t)$ *by the equation*

$$I(t) = \int_{-\infty}^{\infty} K(t-u)V(u)\,du,$$

where

$$K(\tau) = a_1 f(\gamma_1, p_1, \tau) + a_2 f(\gamma_2, p_2, \tau).$$

The function $f(\gamma, p, t)$ *is as given in part* (a) *and all the* a_i, γ_i (> 0) *and* p_i *are fixed parameters. By considering the Fourier transform of* $I(t)$, *find the relationship that must hold between* a_1 *and* a_2 *if the total net charge* Q *passed through the system (over a very long time) is to be zero for an arbitrary applied voltage.*

(a) Write the given sine function in terms of exponential functions. Its Fourier transform is then easily calculated as

$$\begin{aligned}\tilde{f}(\omega,\gamma,p) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{(-\gamma-i\omega+ip)t}-e^{(-\gamma-i\omega-ip)t}}{2i}\,dt\\ &= \frac{1}{\sqrt{2\pi}}\frac{1}{2i}\left(\frac{-1}{-\gamma-i\omega+ip}+\frac{1}{-\gamma-i\omega-ip}\right)\\ &= \frac{1}{\sqrt{2\pi}}\frac{p}{(\gamma+i\omega)^2+p^2}.\end{aligned}$$

(b) Since the current is given by the convolution

$$I(t)=\int_{-\infty}^{\infty}K(t-u)V(u)\,du,$$

the convolution theorem implies that the Fourier transforms of I, K and V are related by $\tilde{I}(\omega)=\sqrt{2\pi}\,\tilde{K}(\omega)\,\tilde{V}(\omega)$ with, from part (a),

$$\tilde{K}(\omega)=\frac{1}{\sqrt{2\pi}}\left[\frac{a_1p_1}{(\gamma_1+i\omega)^2+p_1^2}+\frac{a_2p_2}{(\gamma_2+i\omega)^2+p_2^2}\right].$$

Now, by expressing $I(t')$ in its Fourier integral form, we can write

$$Q(\infty)=\int_{-\infty}^{\infty}I(t')\,dt'=\int_{-\infty}^{\infty}dt'\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\tilde{I}(\omega)\,e^{i\omega t'}\,d\omega.$$

But $\int_{-\infty}^{\infty}e^{i\omega t'}\,dt'=2\pi\delta(\omega)$ and so

$$\begin{aligned}Q(\infty) &= \int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\tilde{I}(\omega)\,2\pi\delta(\omega)\,d\omega\\ &= \frac{2\pi}{\sqrt{2\pi}}\tilde{I}(0)=\sqrt{2\pi}\,\sqrt{2\pi}\,\tilde{K}(0)\,\tilde{V}(0)\\ &= 2\pi\,\frac{1}{\sqrt{2\pi}}\left[\frac{a_1p_1}{\gamma_1^2+p_1^2}+\frac{a_2p_2}{\gamma_2^2+p_2^2}\right]\tilde{V}(0).\end{aligned}$$

For $Q(\infty)$ to be zero for an arbitrary $V(t)$, we must have

$$\frac{a_1p_1}{\gamma_1^2+p_1^2}+\frac{a_2p_2}{\gamma_2^2+p_2^2}=0,$$

and so this is the required relationship.

13.15 *Show that the Fourier transform of $tf(t)$ is $id\tilde{f}(\omega)/d\omega$.*

A linear amplifier produces an output that is the convolution of its input and its response function. The Fourier transform of the response function for a particular amplifier is

$$\tilde{K}(\omega) = \frac{i\omega}{\sqrt{2\pi}(\alpha + i\omega)^2}.$$

Determine the time variation of its output $g(t)$ when its input is the Heaviside step function.

This result is immediate, since differentiating the definition of a Fourier transform (under the integral sign) gives

$$i\frac{d\tilde{f}(\omega)}{d\omega} = \frac{i}{\sqrt{2\pi}}\frac{\partial}{\partial\omega}\left(\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt\right) = \frac{-i^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} tf(t)\,e^{-i\omega t}\,dt,$$

i.e. the transform of $tf(t)$.

Since the amplifier's output is the convolution of its input and response function, we will need the Fourier transforms of both to determine that of its output (using the convolution theorem). We already have that of its response function.

The input Heaviside step function $H(t)$ has a Fourier transform

$$\tilde{H}(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} H(t)\,e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\int_{0}^{\infty} e^{-i\omega t}\,dt = \frac{1}{\sqrt{2\pi}}\frac{1}{i\omega}.$$

Thus, using the convolution theorem,

$$\begin{aligned}
\tilde{g}(\omega) &= \sqrt{2\pi}\,\frac{i\omega}{\sqrt{2\pi}(\alpha + i\omega)^2}\,\frac{1}{\sqrt{2\pi}}\,\frac{1}{i\omega} \\
&= \frac{1}{\sqrt{2\pi}}\,\frac{1}{(\alpha + i\omega)^2} \\
&= \frac{i}{\sqrt{2\pi}}\,\frac{\partial}{\partial\omega}\left(\frac{1}{\alpha + i\omega}\right) \\
&= i\frac{\partial}{\partial\omega}\left\{\mathscr{F}\left[e^{-\alpha t}H(t)\right]\right\} \\
&= \mathscr{F}\left[te^{-\alpha t}H(t)\right],
\end{aligned}$$

where we have used the 'library' result to recognise the transform of a decaying exponential in the penultimate line and the result proved above in the final step. The output of the amplifier is therefore of the form $g(t) = te^{-\alpha t}$ for $t > 0$ when its input takes the form of the Heaviside step function.

13.17 *In quantum mechanics, two equal-mass particles having momenta* $\mathbf{p}_j = \hbar\mathbf{k}_j$ *and energies* $E_j = \hbar\omega_j$ *and represented by plane wavefunctions* $\phi_j = \exp[i(\mathbf{k}_j \cdot \mathbf{r}_j - \omega_j t)]$, $j = 1, 2$, *interact through a potential* $V = V(|\mathbf{r}_1 - \mathbf{r}_2|)$. *In first-order perturbation theory the probability of scattering to a state with momenta and energies* $\mathbf{p}'_j, E'_j$ *is determined by the modulus squared of the quantity*

$$M = \iiint \psi_{\mathrm{f}}^* V \psi_{\mathrm{i}}\, d\mathbf{r}_1\, d\mathbf{r}_2\, dt.$$

The initial state ψ_{i} *is* $\phi_1\phi_2$ *and the final state* ψ_{f} *is* $\phi'_1\phi'_2$. *It can be shown that* M *is proportional to the Fourier transform of* V, *i.e. to* $\widetilde{V}(\mathbf{k})$, *where* $2\hbar\mathbf{k} = (\mathbf{p}_2 - \mathbf{p}_1) - (\mathbf{p}'_2 - \mathbf{p}'_1)$.

For some ion–atom scattering processes, the spherically symmetric potential $V(\mathbf{r})$ *may be approximated by* $V = |\mathbf{r}_1 - \mathbf{r}_2|^{-1} \exp(-\mu|\mathbf{r}_1 - \mathbf{r}_2|)$. *Show that the probability that the ion will scatter from, say,* $\mathbf{p}_1$ *to* $\mathbf{p}'_1$ *is proportional to* $(\mu^2 + k^2)^{-2}$, *where* $k = |\mathbf{k}|$ *and* $\mathbf{k}$ *is as given above.*

We start by showing how to reduce the three-dimensional Fourier transform to a one-dimensional one whenever $V(\mathbf{r})$ is spherically symmetrical, i.e. $V(\mathbf{r}) = V(r)$. This result will be a general one and is not restricted to this particular example.

Choose spherical polar coordinates in which the vector $\mathbf{k}$ of the Fourier transform lies along the polar axis ($\theta = 0$); this can be done since $V(\mathbf{r})$ is spherically symmetric. We then have

$$d^3\mathbf{r} = r^2 \sin\theta\, dr\, d\theta\, d\phi \qquad \text{and} \qquad \mathbf{k} \cdot \mathbf{r} = kr\cos\theta,$$

where $k = |\mathbf{k}|$. The Fourier transform is given by

$$\begin{aligned}
\widetilde{V}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int V(\mathbf{r})\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, d^3\mathbf{r} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr \int_0^\pi d\theta \int_0^{2\pi} d\phi\, V(r) r^2 \sin\theta\, e^{-ikr\cos\theta} \\
&= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\, 2\pi V(r) r^2 \int_0^\pi d\theta\, \sin\theta\, e^{-ikr\cos\theta}.
\end{aligned}$$

The integral over θ may be evaluated straightforwardly by noting that

$$\frac{d}{d\theta}(e^{-ikr\cos\theta}) = ikr\sin\theta\, e^{-ikr\cos\theta}.$$

This enables us to carry through the angular integration over θ and so reduce

the multiple integral to a one-dimensional integral over the radial coordinate:

$$\begin{aligned}\widetilde{V}(\mathbf{k}) &= \frac{1}{(2\pi)^{3/2}} \int_0^\infty dr\, 2\pi V(r) r^2 \left[\frac{e^{-ikr\cos\theta}}{ikr}\right]_{\theta=0}^{\theta=\pi} \\ &= \frac{1}{(2\pi)^{3/2}} \int_0^\infty 4\pi r^2 V(r) \left(\frac{\sin kr}{kr}\right) dr \\ &= \frac{1}{(2\pi)^{3/2} k} \int_0^\infty 4\pi V(r) r \sin kr\, dr.\end{aligned}$$

The ion–atom interaction potential in this particular example is $V(r) = r^{-1}\exp(-\mu r)$. As this is spherically symmetric, we may apply the result just derived to it. Substituting for $V(r)$ gives

$$\begin{aligned}M \propto \widetilde{V}(\mathbf{k}) &\propto \frac{1}{k} \int_0^\infty \frac{e^{-\mu r}}{r} r \sin kr\, dr \\ &= \frac{1}{k} \operatorname{Im} \int_0^\infty e^{-\mu r + ikr}\, dr \\ &= \frac{1}{k} \operatorname{Im} \left[\frac{-1}{-\mu + ik}\right] \\ &= \frac{1}{k} \frac{k}{\mu^2 + k^2}.\end{aligned}$$

Since the probability of the ion scattering from $\mathbf{p}_1$ to $\mathbf{p}_1'$ is proportional to the modulus squared of M, the probability is $\propto |M|^2 \propto (\mu^2 + k^2)^{-2}$.

13.19 *Calculate directly the auto-correlation function $a(z)$ for the product $f(t)$ of the exponential decay distribution and the Heaviside step function,*

$$f(t) = \frac{1}{\lambda} e^{-\lambda t} H(t).$$

Use the Fourier transform and energy spectrum of $f(t)$ to deduce that

$$\int_{-\infty}^{\infty} \frac{e^{i\omega z}}{\lambda^2 + \omega^2}\, d\omega = \frac{\pi}{\lambda} e^{-\lambda|z|}.$$

By definition,

$$\begin{aligned} a(z) &= \int_{-\infty}^{\infty} \frac{1}{\lambda} e^{-\lambda t} H(t) \, \frac{1}{\lambda} e^{-\lambda(t+z)} H(t+z) \, dt \\ &= \frac{e^{-\lambda z}}{\lambda^2} \int_{z_0}^{\infty} e^{-2\lambda t} \, dt, \end{aligned}$$

where $z_0 = 0$ for $z > 0$ and $z_0 = |z|$ for $z < 0$; so

$$\begin{aligned} a(z) &= \frac{e^{-\lambda z}}{\lambda^2} \left[\frac{e^{-2\lambda t}}{-2\lambda} \right]_{z_0}^{\infty} \\ &= \frac{e^{-\lambda(z+2z_0)}}{2\lambda^3} = \frac{e^{-\lambda|z|}}{2\lambda^3}. \end{aligned}$$

The Fourier transform of $f(t)$ is given by

$$\tilde{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \frac{1}{\lambda} \, e^{-\lambda t} \, e^{-i\omega t} \, dt = \frac{1}{\sqrt{2\pi}\lambda(\lambda + i\omega)}.$$

The special case of the Wiener–Kinchin theorem in which both functions are the same shows that the inverse Fourier transform of the energy spectrum, $\sqrt{2\pi}|\tilde{f}(\omega)|^2$, is equal to the auto-correlation function, i.e.

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sqrt{2\pi} \, \frac{e^{i\omega z}}{2\pi\lambda^2(\lambda^2 + \omega^2)} \, d\omega = \frac{e^{-\lambda|z|}}{2\lambda^3},$$

from which the stated result follows immediately.

13.21 *Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$, by setting $x^2 = ts$ in the result*

$$\int_0^{\infty} \exp(-x^2) \, dx = \tfrac{1}{2}\sqrt{\pi}.$$

Setting $x^2 = st$, and hence $2x\,dx = s\,dt$ and $dx = s\,dt/(2\sqrt{st})$, we obtain

$$\int_0^{\infty} e^{-st} \, \frac{\sqrt{s}}{2} \, t^{-1/2} \, dt = \frac{\sqrt{\pi}}{2},$$

$$\Rightarrow \quad \mathscr{L}\left[t^{-1/2}\right] \equiv \int_0^{\infty} t^{-1/2} \, e^{-st} \, dt = \sqrt{\frac{\pi}{s}}.$$

Integrating the LHS of this result by parts yields

$$\left[\, e^{-st} \, 2t^{1/2} \, \right]_0^{\infty} - \int_0^{\infty} (-s) \, e^{-st} \, 2t^{1/2} \, dt = \sqrt{\frac{\pi}{s}}.$$

The first term vanishes at both limits, whilst the second is a multiple of the required Laplace transform of $t^{1/2}$. Hence,

$$\mathscr{L}\left[t^{1/2}\right] \equiv \int_0^\infty e^{-st}\, t^{1/2}\, dt = \frac{1}{2s}\sqrt{\frac{\pi}{s}}.$$

13.23 *Use the properties of Laplace transforms to prove the following without evaluating any Laplace integrals explicitly:*

(a) $\mathscr{L}\left[t^{5/2}\right] = \frac{15}{8}\sqrt{\pi}s^{-7/2}$;
(b) $\mathscr{L}\left[(\sinh at)/t\right] = \frac{1}{2}\ln\left[(s+a)/(s-a)\right], \quad s > |a|$;
(c) $\mathscr{L}\left[\sinh at \cos bt\right] = a(s^2 - a^2 + b^2)[(s-a)^2 + b^2]^{-1}[(s+a)^2 + b^2]^{-1}$.

(a) We use the general result for Laplace transforms that

$$\mathscr{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n \bar{f}(s)}{ds^n}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

If we take $n = 2$, then $f(t)$ becomes $t^{1/2}$, for which we found the Laplace transform in exercise 13.21:

$$\begin{aligned}\mathscr{L}\left[t^{5/2}\right] = \mathscr{L}\left[t^2\, t^{1/2}\right] &= (-1)^2 \frac{d^2}{ds^2}\left(\frac{\sqrt{\pi}s^{-3/2}}{2}\right) \\ &= \frac{\sqrt{\pi}}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)s^{-7/2} = \frac{15\sqrt{\pi}}{8}s^{-7/2}.\end{aligned}$$

(b) Here we apply a second general result for Laplace transforms which states that

$$\mathscr{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(u)\, du,$$

provided $\lim_{t\to 0}[f(t)/t]$ exists, which it does in this case.

$$\begin{aligned}\mathscr{L}\left[\frac{\sinh(at)}{t}\right] &= \int_s^\infty \frac{a}{u^2 - a^2}\, du, \quad u > |a|, \\ &= \frac{1}{2}\int_s^\infty \left(\frac{1}{u-a} - \frac{1}{u+a}\right) du \\ &= \frac{1}{2}\ln\left(\frac{s+a}{s-a}\right), \quad s > |a|.\end{aligned}$$

(c) The translation property of Laplace transforms can be used here to deal with

the $\sinh(at)$ factor, as it can be expressed in terms of exponential functions:

$$\begin{aligned}\mathscr{L}\left[\sinh(at)\cos(bt)\right] &= \mathscr{L}\left[\tfrac{1}{2}e^{at}\cos(bt)\right] - \mathscr{L}\left[\tfrac{1}{2}e^{-at}\cos(bt)\right]\\ &= \frac{1}{2}\frac{s-a}{(s-a)^2+b^2} - \frac{1}{2}\frac{s+a}{(s+a)^2+b^2}\\ &= \frac{1}{2}\frac{(s^2-a^2)2a+2ab^2}{[(s-a)^2+b^2][(s+a)^2+b^2]}\\ &= \frac{a(s^2-a^2+b^2)}{[(s-a)^2+b^2][(s+a)^2+b^2]}.\end{aligned}$$

The result is valid for $s > |a|$.

13.25 *This exercise is concerned with the limiting behaviour of Laplace transforms.*

(a) *If $f(t) = A + g(t)$, where A is a constant and the indefinite integral of $g(t)$ is bounded as its upper limit tends to ∞, show that*

$$\lim_{s\to 0} s\bar{f}(s) = A.$$

(a) *For $t > 0$, the function $y(t)$ obeys the differential equation*

$$\frac{d^2y}{dt^2} + a\frac{dy}{dt} + by = c\cos^2\omega t,$$

where a, b and c are positive constants. Find $\bar{y}(s)$ and show that $s\bar{y}(s) \to c/2b$ as $s \to 0$. Interpret the result in the t-domain.

(a) From the definition,

$$\begin{aligned}\bar{f}(s) &= \int_0^\infty [A + g(t)]\, e^{-st}\, dt\\ &= \left[\frac{A\,e^{-st}}{-s}\right]_0^\infty + \lim_{T\to\infty}\int_0^T g(t)\, e^{-st}\, dt,\\ s\bar{f}(s) &= A + s\lim_{T\to\infty}\int_0^T g(t)\, e^{-st}\, dt.\end{aligned}$$

Now, for $s \geq 0$,

$$\left|\lim_{T\to\infty}\int_0^T g(t)\, e^{-st}\, dt\right| \leq \left|\lim_{T\to\infty}\int_0^T g(t)\, dt\right| < B, \text{ say.}$$

Thus, taking the limit $s \to 0$,

$$\lim_{s\to 0} s\bar{f}(s) = A \pm \lim_{s\to 0} sB = A.$$

(b) We will need

$$\mathscr{L}\left[\cos^2 \omega t\right] = \mathscr{L}\left[\tfrac{1}{2}\cos 2\omega + \tfrac{1}{2}\right] = \frac{s}{2(s^2+4\omega^2)} + \frac{1}{2s}.$$

Taking the transform of the differential equation yields

$$-y'(0) - sy(0) + s^2\bar{y} + a[-y(0) + s\bar{y}] + b\bar{y} = c\left[\frac{s}{2(s^2+4\omega^2)} + \frac{1}{2s}\right].$$

This can be rearranged as

$$s\bar{y} = \frac{c\left(\dfrac{s^2}{2(s^2+4\omega^2)} + \dfrac{1}{2}\right) + sy'(0) + asy(0) + s^2y(0)}{s^2 + as + b}.$$

In the limit $s \to 0$, this tends to $(c/2)/b = c/(2b)$, a value independent of that of a and the initial values of y and y'.

The $s = 0$ component of the transform corresponds to long-term values, when a steady state has been reached and rates of change are negligible. With the first two terms of the differential equation ignored, it reduces to $by = c\cos^2 \omega t$, and, as the average value of $\cos^2 \omega t$ is $\frac{1}{2}$, the solution is the more or less steady value of $y = \frac{1}{2}c/b$.

13.27 *The function $f_a(x)$ is defined as unity for $0 < x < a$ and zero otherwise. Find its Laplace transform $\bar{f}_a(s)$ and deduce that the transform of $xf_a(x)$ is*

$$\frac{1}{s^2}\left[1 - (1+as)e^{-sa}\right].$$

Write $f_a(x)$ in terms of Heaviside functions and hence obtain an explicit expression for

$$g_a(x) = \int_0^x f_a(y)f_a(x-y)\,dy.$$

Use the expression to write $\bar{g}_a(s)$ in terms of the functions $\bar{f}_a(s)$ and $\bar{f}_{2a}(s)$, and their derivatives, and hence show that $\bar{g}_a(s)$ is equal to the square of $\bar{f}_a(s)$, in accordance with the convolution theorem.

From their definitions,

$$\bar{f}_a(s) = \int_0^a 1\,e^{-sx}\,dx = \frac{1}{s}(1 - e^{-sa}),$$

$$\int_0^a x\,f_a(x)\,e^{-sx}\,dx = -\frac{d\bar{f}_a}{ds} = \frac{1}{s^2}(1 - e^{-sa}) - \frac{a}{s}e^{-sa}$$

$$= \frac{1}{s^2}\left[1 - (1+as)e^{-sa}\right]. \qquad (*)$$

In terms of Heaviside functions,

$$f(x) = H(x) - H(x-a),$$

and so the expression for $g_a(x) = \int_0^x f_a(y) f_a(x-y)\,dy$ is

$$\int_{-\infty}^{\infty} [\,H(y) - H(y-a)\,]\,[\,H(x-y) - H(x-y-a)\,]\,dy.$$

This can be expanded as the sum of four integrals, each of which contains the common factors $H(y)$ and $H(x-y)$, implying that, in all cases, unless x is positive and greater than y, the integral has zero value. The other factors in the four integrands are generated analogously to the terms of the expansion $(a-b)(c-d) = ac - ad - bc + bd$:

$$\begin{aligned}
&\int_{-\infty}^{\infty} H(y)H(x-y)\,dy \\
&\quad - \int_{-\infty}^{\infty} H(y)H(x-y-a)\,dy \\
&\qquad - \int_{-\infty}^{\infty} H(y-a)H(x-y)\,dy \\
&\qquad\quad + \int_{-\infty}^{\infty} H(y-a)H(x-y-a)\,dy.
\end{aligned}$$

In all four integrals the integrand is either 0 or 1 and the value of each integral is equal to the length of the y-interval in which the integrand is non-zero.

- The first integral requires $0 < y < x$ and therefore has value x for $x > 0$.
- The second integral requires $0 < y < x - a$ and therefore has value $x - a$ for $x > a$ and 0 for $x < a$.
- The third integral requires $a < y < x$ and therefore has value $x - a$ for $x > a$ and 0 for $x < a$.
- The final integral requires $a < y < x - a$ and therefore has value $x - 2a$ for $x > 2a$ and 0 for $x < 2a$.

Collecting these together:

$$\begin{array}{ll}
x < 0 & g_a(x) = 0 - 0 - 0 + 0 = 0, \\
0 < x < a & g_a(x) = x - 0 - 0 + 0 = x, \\
a < x < 2a & g_a(x) = x - (x-a) - (x-a) + 0 = 2a - x, \\
2a < x & g_a(x) = x - (x-a) - (x-a) + (x-2a) = 0.
\end{array}$$

Consequently, the transform of $g_a(x)$ is given by

$$\begin{aligned}
\bar{g}_a(s) &= \int_0^a xe^{-sx}\,dx + \int_a^{2a} (2a - x)e^{-sx}\,dx \\
&= -\int_0^{2a} xe^{-sx}\,dx + 2\int_0^a xe^{-sx}\,dx + 2a\int_a^{2a} e^{-sx}\,dx \\
&= -\frac{1}{s^2}\left[1 - (1 + 2as)e^{-2sa}\right] + \frac{2}{s^2}\left[1 - (1 + as)e^{-sa}\right] \\
&\qquad + \frac{2a}{s}(e^{-sa} - e^{-2sa}) \\
&= \frac{1}{s^2}(1 - 2e^{-sa} + e^{-2sa}) \\
&= \frac{1}{s^2}(1 - e^{-as})^2 = [\bar{f}_a(s)]^2,
\end{aligned}$$

which is as expected. In order to adjust the integral limits in the second line, we both added and subtracted

$$\int_0^a (-x)e^{-sx}\,dx.$$

In the third line we used the result (*) twice, once as it stands and once with a replaced by $2a$.

14

First-order ordinary differential equations

14.1 *A radioactive isotope decays in such a way that the number of atoms present at a given time, $N(t)$, obeys the equation*

$$\frac{dN}{dt} = -\lambda N.$$

If there are initially N_0 atoms present, find $N(t)$ at later times.

This is a straightforward separable equation with a well known solution:

$$\frac{dN}{dt} = -\lambda N.$$

Separating the variables, $$\frac{dN}{N} = -\lambda\, dt.$$

Integrating, $$\ln N(t) - \ln N(0) = -\lambda(t - 0).$$

Thus, since $N(0) = N_0$, we have that, at a later time,

$$N(t) = N_0 e^{-\lambda t}.$$

14.3 *Show that the following equations either are exact or can be made exact, and solve them:*

(a) $y(2x^2y^2 + 1)y' + x(y^4 + 1) = 0$;
(b) $2xy' + 3x + y = 0$;
(c) $(\cos^2 x + y \sin 2x)y' + y^2 = 0$.

In general, given an equation expressed in the form $A\,dx + B\,dy = 0$, we consider

the function

$$h(x, y) = \frac{1}{B}\left[\frac{\partial A}{\partial y} - \frac{\partial B}{\partial x}\right].$$

If this expression is zero, then the equation is exact and can be integrated as it stands to give a solution of the form $f(x, y) = c$. Even if $g(x, y)$ is non-zero, if it is a function of x alone then

$$\mu(x) = \exp\left\{\int g(x)\,dx\right\}$$

provides an integrating factor (IF) that will make the equation exact. Similar considerations apply if $g(x, y)$ is a function of y alone. If g does actually depend on both x and y, then, in general, no further progress can be made using this method.

(a) Following the above procedure, we consider

$$h(x, y) = \frac{1}{2x^2y^3 + y}\left[\frac{\partial}{\partial y}(xy^4 + x) - \frac{\partial}{\partial x}(2x^2y^3 + y)\right] = \frac{4xy^3 - 4xy^3}{2x^2y^3 + y} = 0.$$

It follows that the equation is exact and can be integrated as it stands:

$$\begin{aligned} c = f(x, y) &= \int (2x^2y^3 + y)\,dy + g(x) \\ &= \frac{1}{2}x^2y^4 + \frac{1}{2}y^2 + g(x), \text{ where} \\ xy^4 + x = \frac{\partial f}{\partial x} &= xy^4 + 0 + g'(x), \quad \Rightarrow \quad g(x) = \frac{1}{2}x^2 + k, \\ \Rightarrow \quad c = f(x, y) &= \frac{1}{2}(x^2y^4 + y^2 + x^2). \end{aligned}$$

The common factor of $\frac{1}{2}$ on the RHS can, of course, be absorbed into the constant on the LHS and has no particular significance.

(b) Again following the procedure, we consider

$$h(x, y) = \frac{1}{2x}\left[\frac{\partial}{\partial y}(3x + y) - \frac{\partial}{\partial x}(2x)\right] = -\frac{1}{2x}.$$

This is non-zero and implies that the equation is not exact. However, it is a function of x alone and so there is an IF given by

$$\mu(x) = \exp\left\{\int -\frac{1}{2x}\,dx\right\} = \exp(-\frac{1}{2}\ln x) = \frac{1}{x^{1/2}}.$$

The exact equation is thus

$$2x^{1/2}\,dy + (3x^{1/2} + yx^{-1/2})\,dx = 0,$$

and this can now be integrated:

$$\begin{aligned} c = f(x,y) &= \int 2x^{1/2}\,dy + g(x) \\ &= 2x^{1/2}y + g(x), \text{ where} \\ 3x^{1/2} + yx^{-1/2} = \frac{\partial f}{\partial x} &= x^{-1/2}y + g'(x), \quad \Rightarrow \quad g(x) = 2x^{3/2} + k, \\ \Rightarrow \quad c = f(x,y) &= 2(x^{1/2}y + x^{3/2}). \end{aligned}$$

Again, the overall numerical multiplicative factor on the RHS has no particular significance.

(c) Following the same general procedure,

$$\begin{aligned} h(x,y) &= \frac{1}{\cos^2 x + y\sin 2x}\left[\frac{\partial}{\partial y}(y^2) - \frac{\partial}{\partial x}(\cos^2 x + y\sin 2x)\right] \\ &= \frac{1}{\cos^2 x + y\sin 2x}(2y + \sin 2x - 2y\cos 2x) \\ &= \frac{4y\sin^2 x + 2\sin x\cos x}{\cos^2 x + y\sin 2x} \\ &= \frac{2\sin x(2y\sin x + \cos x)}{\cos x(\cos x + 2y\sin x)} = 2\tan x. \end{aligned}$$

This is non-zero and implies that the equation is not exact. However, it is a function of x alone and so there is an IF given by

$$\mu(x) = \exp\left\{\int 2\tan x\,dx\right\} = \exp(-2\ln\cos x) = \frac{1}{\cos^2 x}.$$

The exact equation is thus

$$(1 + 2y\tan x)\,dy + y^2\sec^2 x\,dx = 0,$$

and this can now be integrated:

$$\begin{aligned} c = f(x,y) &= \int (1 + 2y\tan x)\,dy + g(x) \\ &= y + y^2\tan x + g(x), \text{ where} \\ y^2\sec^2 x = \frac{\partial f}{\partial x} &= 0 + y^2\sec^2 x + g'(x), \quad \Rightarrow \quad g(x) = k, \\ \Rightarrow \quad c = f(x,y) &= y + y^2\tan x. \end{aligned}$$

14.5 *By finding a suitable integrating factor, solve the following equations:*

(a) $(1-x^2)y' + 2xy = (1-x^2)^{3/2}$;
(b) $y' - y\cot x + \operatorname{cosec} x = 0$;
(c) $(x+y^3)y' = y$ *(treat y as the independent variable).*

(a) In standard form this is

$$y' + \frac{2xy}{1-x^2} = (1-x^2)^{1/2}.$$

The IF for this standard form is

$$\mu(x) = \exp\left\{\int \frac{2x}{1-x^2}\,dx\right\} = \exp[-\ln(1-x^2)] = \frac{1}{1-x^2},$$

i.e. $(1-x^2)^{-2}$ for the original form. Applying it gives

$$\begin{aligned}
\frac{y'}{1-x^2} + \frac{2xy}{(1-x^2)^2} &= \frac{1}{(1-x^2)^{1/2}},\\
\frac{d}{dx}\left(\frac{y}{1-x^2}\right) &= \frac{1}{(1-x^2)^{1/2}},\\
\frac{y}{1-x^2} &= \sin^{-1}x + k,\\
\Rightarrow \quad y &= (1-x^2)(\sin^{-1}x + k).
\end{aligned}$$

(b) In standard form this is

$$y' - \frac{y\cos x}{\sin x} = -\frac{1}{\sin x}.$$

The IF for this standard form is given by

$$\mu(x) = \exp\left\{-\int \frac{\cos x}{\sin x}\,dx\right\} = \exp[-\ln(\sin x)] = \frac{1}{\sin x}.$$

Applying it gives

$$\begin{aligned}
\frac{y'}{\sin x} - \frac{y\cos x}{\sin^2 x} &= -\frac{1}{\sin^2 x},\\
\frac{d}{dx}\left(\frac{y}{\sin x}\right) &= -\operatorname{cosec}^2 x,\\
\frac{y}{\sin x} &= \cot x + k,\\
\Rightarrow \quad y &= \cos x + k\sin x.
\end{aligned}$$

(c) Rearranging this to make y the independent variable,

$$\frac{dx}{dy} - \frac{x}{y} = y^2.$$

By inspection (or by the standard method) the IF is y^{-1}, yielding

$$\begin{aligned}
\frac{1}{y}\frac{dx}{dy} - \frac{x}{y^2} &= \frac{y^2}{y} \\
\frac{d}{dy}\left(\frac{x}{y}\right) &= y, \\
\frac{x}{y} &= \frac{1}{2}y^2 + k, \\
\Rightarrow \quad x &= \tfrac{1}{2}y^3 + ky.
\end{aligned}$$

14.7 *Find, in the form of an integral, the solution of the equation*

$$\alpha\frac{dy}{dt} + y = f(t)$$

for a general function $f(t)$. Find the specific solutions for

(a) $f(t) = H(t)$,
(b) $f(t) = \delta(t)$,
(c) $f(t) = \beta^{-1}e^{-t/\beta}H(t)$ *with* $\beta < \alpha$.

For case (c), *what happens if* $\beta \to 0$?

The IF needed for the standard form is $\exp[\int \alpha^{-1}\,dt]$, i.e. $e^{t/\alpha}$. The equation then reads

$$\begin{aligned}
e^{t/\alpha}\frac{dy}{dt} + \frac{y\,e^{t/\alpha}}{\alpha} &= \frac{f(t)\,e^{t/\alpha}}{\alpha}, \\
\frac{d}{dt}\left(y\,e^{t/\alpha}\right) &= \frac{f(t)\,e^{t/\alpha}}{\alpha}, \\
y(t) &= e^{-t/\alpha}\int^{t} \frac{f(t')\,e^{t'/\alpha}}{\alpha}\,dt'.
\end{aligned}$$

We now apply this general result to the three specific cases.

(a) $f(t) = H(t)$, the Heaviside function. This is zero for $t < 0$ and so we can take the integral as running from 0 to t. The value of $H(t)$ for $t > 0$ is unity. Hence,

$$y(t) = e^{-t/\alpha}\int_0^t \frac{e^{t'/\alpha}}{\alpha}\,dt' = e^{-t/\alpha}[\,e^{t/\alpha} - 1\,] = 1 - e^{-t/\alpha}.$$

(b) With $f(t) = \delta(t)$, the integration will be trivial:

$$y(t) = e^{-t/\alpha}\int^{t} \frac{\delta(t')\,e^{t'/\alpha}}{\alpha}\,dt' = e^{-t/\alpha} \times \frac{1}{\alpha} = \frac{e^{-t/\alpha}}{\alpha}.$$

(c) For $f(t) = \beta^{-1}e^{-t/\beta}H(t)$, with $\beta < \alpha$, we have

$$\begin{aligned} y(t) &= e^{-t/\alpha}\int_0^t \frac{e^{t'/\alpha}\,e^{-t'/\beta}}{\alpha\beta}\,dt' \\ &= e^{-t/\alpha}\left[\frac{e^{(\alpha^{-1}-\beta^{-1})t'}}{\alpha\beta(\alpha^{-1}-\beta^{-1})}\right]_0^t \\ &= \frac{e^{-t/\beta}}{\beta-\alpha} - \frac{e^{-t/\alpha}}{\beta-\alpha} \\ &= \frac{e^{-t/\alpha}-e^{-t/\beta}}{\alpha-\beta}. \end{aligned}$$

As $\beta \to 0$, $f(t)$ becomes very strongly peaked near $t = 0$, but with the area under the peak remaining constant at unity. In the limit, the input $f(t)$ becomes a δ-function, the same as that in case (b). It can also be seen that in the same limit the solution $y(t)$ for case (c) tends to that for case (b), as is to be expected.

14.9 *A two-dimensional coordinate system that is useful for orbit problems is the tangential–polar coordinate system. In this system a curve is defined by r, the distance from a fixed point O to a general point P of the curve, and p, the perpendicular distance from O to the tangent to the curve at P. It can be shown that the instantaneous radius of curvature of the curve is given by $\rho = r\,dr/dp$.*

Using tangential–polar coordinates, consider a particle of mass m moving under the influence of a force f directed towards the origin O. By resolving forces along the instantaneous tangent and normal, prove that

$$f = -mv\frac{dv}{dr} \qquad \text{and} \qquad mv^2 = fp\frac{dr}{dp}.$$

Show further that $h = mpv$ is a constant of the motion and that the law of force can be deduced from

$$f = \frac{h^2}{mp^3}\frac{dp}{dr}.$$

Denote by ϕ the angle between the radius vector and the tangent to the orbit at any instant. Then, firstly, we note that $\cos\phi = dr/ds$, where s is the distance moved along the orbit curve and, secondly, that $p = r\sin\phi$.

Now we equate the tangential component of the central force $-f\cos\phi$ to the rate of change of the tangential momentum:

$$-f\frac{dr}{ds} = -f\cos\phi = m\frac{dv}{dt} = m\frac{dv}{ds}\frac{ds}{dt} = mv\frac{dv}{ds}.$$

Hence,

$$f = -mv\frac{dv}{ds}\frac{ds}{dr} = -mv\frac{dv}{dr}.$$

This is the first of the results.

Equating the normal component of the central force to that needed to keep the particle moving in an orbit with instantaneous radius of curvature $\rho = r\,dr/dp$ gives

$$\frac{mv^2}{\rho} = f\sin\phi = f\frac{p}{r} \quad \Rightarrow \quad mv^2 = f\,\frac{p}{r}\,r\,\frac{dr}{dp} = f\,p\,\frac{dr}{dp}.$$

Eliminating f from the two equations yields

$$\begin{aligned} mv^2 = -mvp\frac{dv}{dp} \quad &\Rightarrow \quad mv + mp\frac{dv}{dp} = 0 \\ &\Rightarrow \quad h \equiv mpv \text{ is a constant of the motion.} \end{aligned}$$

It follows that

$$f = \frac{mv^2}{p}\frac{dp}{dr} = \frac{h^2}{mp^3}\frac{dp}{dr},$$

from which the law of force can be deduced once p is given as a function of r.

14.11 *Solve*

$$(y - x)\frac{dy}{dx} + 2x + 3y = 0.$$

We first test whether the equation is exact, or can be made so with the help of an integrating factor. To do this, we write the equation as

$$(y - x)\,dy + (2x + 3y)\,dx = 0$$

and consider

$$h_x(x, y) = \frac{1}{y - x}\left[\frac{\partial}{\partial y}(2x + 3y) - \frac{\partial}{\partial x}(y - x)\right] = \frac{4}{y - x}.$$

This is not a function of x alone. Equally

$$h_y(x, y) = \frac{1}{2x + 3y}\left[-\frac{\partial}{\partial y}(2x + 3y) + \frac{\partial}{\partial x}(y - x)\right] = \frac{-4}{2x + 3y}$$

is not a function of y alone. We conclude that there is no straightforward IF and that another method has to be tried.

We note that the equation is homogeneous in x and y and so we set $y = vx$, with $\dfrac{\partial y}{\partial x} = v + x\dfrac{\partial v}{\partial x}$, and obtain

$$\begin{aligned}
v + x\frac{\partial v}{\partial x} &= -\frac{2+3v}{v-1}, \\
x\frac{\partial v}{\partial x} &= \frac{-2-3v-v^2+v}{v-1} = -\frac{v^2+2v+2}{v-1}, \\
\frac{dx}{x} &= \frac{(1-v)\,dv}{v^2+2v+2} \\
&= \frac{2}{(v+1)^2+1} - \frac{v+1}{(v+1)^2+1}, \\
\Rightarrow \quad \ln Ax &= 2\tan^{-1}(v+1) - \tfrac{1}{2}\ln[1+(v+1)^2], \\
\ln\left\{Bx^2[1+(v+1)^2]\right\} &= 4\tan^{-1}(v+1).
\end{aligned}$$

On setting $v = y/x$ this becomes

$$B[x^2+(y+x)^2] = \exp\left[4\tan^{-1}\left(\frac{y+x}{x}\right)\right],$$

the final form of the solution.

14.13 *One of the properties of Laplace transforms is that the transform of the nth derivative of a function $f(t)$ is given by*

$$\mathscr{L}\left[\frac{d^n f}{dt^n}\right] = s^n\bar{f} - s^{n-1}f(0) - s^{n-2}\frac{df}{dt}(0) - \cdots - \frac{d^{n-1}f}{dt^{n-1}}(0), \qquad \text{for } s > 0.$$

Using this and the result about the Laplace transform of $tf(t)$ obtained in exercise 13.25, show, for a function $y(t)$ that satisfies

$$t\frac{dy}{dt} + (t-1)y = 0 \qquad (*)$$

with $y(0)$ finite, that $\bar{y}(s) = C(1+s)^{-2}$ for some constant C.

Given that

$$y(t) = t + \sum_{n=2}^{\infty} a_n t^n,$$

determine C and show that $a_n = (-1)^{n-1}/(n-1)!$. Compare this result with that obtained by integrating $(*)$ directly.

Using the stated property of derivatives with $n = 1$ and the result from the

exercise, we Laplace transform the equation and obtain

$$\begin{aligned}
-\frac{d}{ds}[s\bar{y}(s) - y(0)] - \frac{d\bar{y}(s)}{ds} - \bar{y}(s) &= 0,\\
-s\frac{d\bar{y}}{ds} - \bar{y} + 0 - \frac{d\bar{y}}{ds} - \bar{y} &= 0,\\
(1+s)\frac{d\bar{y}}{ds} + 2\bar{y} &= 0,\\
\frac{d\bar{y}}{\bar{y}} + \frac{2\,ds}{1+s} &= 0,\\
\Rightarrow \quad \ln\bar{y} + 2\ln(1+s) &= k,\\
\Rightarrow \quad \bar{y} &= \frac{C}{(1+s)^2}.
\end{aligned}$$

As a power series, $\bar{y}(s)$ takes the form

$$\begin{aligned}
\bar{y}(s) &= \frac{C}{s^2}\left(1 + \frac{1}{s}\right)^{-2}\\
&= \frac{C}{s^2}\left(1 - \frac{2}{s} + \frac{(-2)(-3)}{2!}\frac{1}{s^2} + \cdots\right)\\
&= \frac{C}{s^2} + \sum_{n=1}^{\infty}\frac{(-1)^n(n+1)}{s^{n+2}}.
\end{aligned}$$

But, transforming the given solution,

$$y(t) = t + \sum_{m=2}^{\infty} a_m t^m,$$

yields

$$\bar{y} = \frac{1}{s^2} + \sum_{m=2}^{\infty} a_m \frac{m!}{s^{m+1}}.$$

Comparing coefficients in the two expressions for $\bar{y}$ shows that $C = 1$ and that $a_{m+1} = (-1)^m/m!$, i.e. $a_m = (-1)^{m-1}/(m-1)!$.

Direct integration of $(*)$ by separating the variables gives

$$\begin{aligned}
0 &= \frac{dy}{y} + \left(1 - \frac{1}{t}\right)dt,\\
\Rightarrow \quad A &= \ln y + t - \ln t,\\
\Rightarrow \quad y &= Bte^{-t}\\
&= Bt + B\sum_{n=1}^{\infty}\frac{(-1)^n t^{n+1}}{n!}\\
&= Bt + B\sum_{m=2}^{\infty}\frac{(-1)^{m-1}t^m}{(m-1)!}.
\end{aligned}$$

With B determined by the linear term as unity, the two solutions agree.

14.15 *Solve*

$$\frac{dy}{dx} = -\frac{x+y}{3x+3y-4}.$$

Since x and y only appear in the combination $x+y$ we set $v = x+y$ with $dv/dx = 1 + dy/dx$. The equation and its solution then become

$$\begin{aligned}
\frac{dv}{dx} &= 1 - \frac{v}{3v-4}, \\
dx &= \frac{3v-4}{2v-4}\,dv = \left(\frac{3}{2} + \frac{2}{2v-4}\right) dv, \\
\Rightarrow \quad x + k &= \tfrac{3}{2}v + \ln(v-2) = \tfrac{3}{2}(x+y) + \ln(x+y-2), \\
\ln(x+y-2) &= k - \tfrac{1}{2}(x+3y).
\end{aligned}$$

Although the initial equation might look as if it could be made exact with an integrating factor, applying the method descibed in exercise 14.3 shows that this not so; $B^{-1}[\partial A/\partial y - \partial B/\partial x]$ is neither zero nor a function of only one of the variables.

14.17 *Solve*

$$x(1-2x^2y)\frac{dy}{dx} + y = 3x^2y^2,$$

given that $y(1) = 1/2$.

Though this is clearly not a homogeneous equation, we test whether it might be an isobaric one by giving x a weight 1 and y a weight m and then seeing whether a suitable value for m can be found. From the presence of the term $1 - 2x^2y$ it is clear that the only possible value of m is -2, since $2x^2y$ must have the same weight as unity, namely weight 0. For this value of m the three terms in the equation have weights

$$1 + 0 + (-2) - 1, \quad -2, \quad 2 + 2(-2).$$

These are all the same (at -2) and so the equation is isobaric.

To find its solution we set $y = vx^m = vx^{-2}$ with

$$\frac{dy}{dx} = -\frac{2v}{x^3} + \frac{1}{x^2}\frac{dv}{dx}.$$

Substituting in the original equation produces

$$
\begin{aligned}
x(1-2v)\left(-\frac{2v}{x^3}+\frac{1}{x^2}\frac{dv}{dx}\right)+\frac{v}{x^2} &= \frac{3x^2v^2}{x^4},\\
(1-2v)\left(-2v+x\frac{dv}{dx}\right)+v &= 3v^2,\\
(1-2v)x\frac{dv}{dx} &= v(1-v),\\
\frac{1-2v}{v(1-v)}\,dv &= \frac{dx}{x},\\
\left(\frac{1}{v}-\frac{1}{1-v}\right)dv &= \frac{dx}{x},\\
\Rightarrow \quad \ln v+\ln(1-v) &= \ln x + A \quad \Rightarrow \quad v(1-v)=Cx.
\end{aligned}
$$

Expressing this in terms of the original variables by substituting $v = yx^2$ gives $yx^2(1-yx^2) = Cx$, with $\frac{1}{2}(1-\frac{1}{2}) = C$. Thus, after cancelling x from both sides, the solution is

$$4yx(1-yx^2) = 1.$$

14.19 *Find the curve with the property that at each point on it the sum of the intercepts on the x- and y-axes of the tangent to the curve (taking account of sign) is equal to 1.*

At a point (X, Y) on the curve, the tangent to the curve is the straight line given by

$$y - Y = p(x - X),$$

where p is the slope of the tangent. This meets the axis $y = 0$ at $x = X - (Y/p)$ and the axis $x = 0$ at $y = Y - pX$. Thus, taking account of signs (i.e. some intercepts could be negative), the condition to be satisfied is

$$X - \frac{Y}{p} + Y - pX = 1.$$

Since (X, Y) lies on the required curve, the curve has an equation that satisfies

$$x - \frac{y}{p} + y - px = 1 \quad \Rightarrow \quad y = \frac{1-x+px}{1-p^{-1}} \quad (*).$$

Differentiating both sides of $(*)$ with respect to x, we now eliminate y by using

the fact that its derivative with respect to x is p:

$$p = \frac{(1-p^{-1})(-1+p+xp') - (1-x+px)p^{-2}p'}{(1-p^{-1})^2},$$
$$p(p-1)^2 = (p^2-p)(p-1) + p'[x(p^2-p) - 1 + x - px].$$

The LHS and the first term on the RHS are equal, and so we have that either $p' = 0$ or

$$x(p^2 - 2p + 1) - 1 = 0,$$
$$\Rightarrow \quad x = \frac{1}{(p-1)^2},$$
$$\Rightarrow \quad p = 1 \pm \frac{1}{\sqrt{x}}.$$

From this and (∗) it follows that

$$y = \frac{p[(1-x)+px]}{p-1} = \frac{\left(1 \pm \frac{1}{\sqrt{x}}\right)(1-x+x\pm\sqrt{x})}{\pm\frac{1}{\sqrt{x}}}$$
$$= (\pm\sqrt{x}+1)(1\pm\sqrt{x}).$$

As expected, the solution is symmetric between x and y; this is demonstrated by the following rearrangement of the form just obtained:

$$y = (1 \pm \sqrt{x})^2,$$
$$\pm\sqrt{y} = 1 \pm \sqrt{x} \qquad (\pm \text{ signs not correlated}),$$
$$\pm\sqrt{y} - 1 = \pm\sqrt{x},$$
$$(1 \mp \sqrt{y})^2 = x.$$

Because of the square roots involved, a real curve exists only for x and y both positive, i.e. in the first quadrant. That curve is $\sqrt{x} + \sqrt{y} = 1$.

The singular solution $p' = 0$ (ignored earlier) corresponds to a set of curves, on each of which the slope is a constant. Any one such curve is a *straight* line joining the axial points $(\theta, 0)$ and $(0, 1-\theta)$ for any arbitrary real θ; the tangent at any point on such a 'curve' is always the curve itself, whose intercepts, θ and $1-\theta$, sum to unity.

14.21 *Using the substitutions $u = x^2$ and $v = y^2$, reduce the equation*

$$xy\left(\frac{dy}{dx}\right)^2 - (x^2 + y^2 - 1)\frac{dy}{dx} + xy = 0$$

to Clairaut's form. Hence show that the equation represents a family of conics and the four sides of a square.

Writing $dy/dx = p$ and $dv/du = q$, we have

$$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = 2yp, \quad q = \frac{dv}{du} = \frac{yp}{x}, \quad p = \frac{x}{y}q.$$

Making the substitutions yields

$$xy\frac{x^2}{y^2}q^2 - (u + v - 1)\frac{x}{y}q + xy = 0.$$

We now multiply by $\dfrac{y}{x}$ and substitute again:

$$uq^2 - (u + v - 1)q + v = 0,$$

$$v(1 - q) - uq + q + uq^2 = 0,$$

$$v = uq + \frac{q}{q - 1}, \text{ Clairaut's form } (*).$$

As the equation now has Clairaut's form it has two solutions.

(i) The first is

$$v = cu + \frac{c}{c - 1},$$

$$y^2 - cx^2 = \frac{c}{c - 1}.$$

- For $c > 1$, this is a hyperbola of the form $y^2 - \alpha^2 x^2 = \beta^2$.
- For $1 > c > 0$, it is a hyperbola of the form $x^2 - \alpha^2 y^2 = \beta^2$.
- For $c < 0$, the conic is an ellipse of the form $y^2 + \alpha^2 x^2 = \beta^2$.

In each case $\alpha > \beta > 0$.

(ii) The second (singular) solution is given by

$$\frac{d}{dq}\left(\frac{q}{q - 1}\right) + u = 0,$$

$$\frac{-1}{(q - 1)^2} + u = 0,$$

$$q = 1 \pm \frac{1}{\sqrt{u}}.$$

Substituting this into (∗) expressed in terms of x and y then gives

$$\begin{aligned} y^2 &= x^2\left(1 \pm \frac{1}{x}\right) + \frac{1 \pm \dfrac{1}{x}}{\pm\dfrac{1}{x}} \\ &= x^2 \pm x \pm x + 1 \\ &= (x \pm 1)^2, \\ y &= \pm(x \pm 1). \end{aligned}$$

These lines are the four sides of the square that has corners at $(0, \pm 1)$ and $(\pm 1, 0)$.

14.23 *Find the general solutions of the following:*

$$\text{(a)}\ \frac{dy}{dx} + \frac{xy}{a^2 + x^2} = x, \qquad \text{(b)}\ \frac{dy}{dx} = \frac{4y^2}{x^2} - y^2.$$

(a) With dy/dx appearing in the first term and y in the second (and nowhere else), this is a linear first-order ODE and therefore has an IF given by

$$\mu(x) = \exp\left\{\int \frac{x}{a^2 + x^2}\right\} dx = \exp[\tfrac{1}{2}\ln(a^2 + x^2)] = (a^2 + x^2)^{1/2}.$$

When multiplied through by this, the equation becomes

$$\begin{aligned} &\frac{d}{dx}[(a^2 + x^2)^{1/2} y] = x(a^2 + x^2)^{1/2}, \\ \Rightarrow\quad &(a^2 + x^2)^{1/2} y = \tfrac{2}{3}\,\tfrac{1}{2}(a^2 + x^2)^{3/2} + A, \\ \Rightarrow\quad &y = \frac{a^2 + x^2}{3} + \frac{A}{(a^2 + x^2)^{1/2}}. \end{aligned}$$

(b) The RHS can be written as the product of one function of x and another one of y; the equation is therefore separable:

$$\begin{aligned} &\frac{dy}{y^2} = \left(\frac{4}{x^2} - 1\right) dx, \\ \Rightarrow\quad &-\frac{1}{y} = -\frac{4}{x} - x + A, \\ \Rightarrow\quad &y = \frac{x}{x^2 + Bx + 4}, \end{aligned}$$

where $B = -A$ and is the arbitrary integration constant.

14.25 *An electronic system has two inputs, to each of which a constant unit signal is applied, but starting at different times. The equations governing the system thus take the form*

$$\dot{x} + 2y = H(t),$$
$$\dot{y} - 2x = H(t-3).$$

Initially (at $t = 0$), $x = 1$ and $y = 0$; find $x(t)$ at later times.

Since we have coupled equations, working with their Laplace transforms suggests itself. This will convert the equations into simultaneous algebraic equations – though there may be some difficulty in converting the solution back into t-space.

The transform of the Heaviside function is s^{-1}, and so the two transformed equations (incorporating the initial conditions and using the translation property of Laplace transforms) are

$$s\bar{x} - 1 + 2\bar{y} = \frac{1}{s},$$
$$s\bar{y} - 0 - 2\bar{x} = \frac{1}{s}e^{-3s}.$$

Since it is $x(t)$ that we require, we eliminate $\bar{y}$ to obtain

$$s^2\bar{x} - s + \frac{2}{s}e^{-3s} + 4\bar{x} = 1,$$

from which

$$\bar{x} = \frac{s^2 + s - 2e^{-3s}}{s(s^2+4)},$$
$$= \frac{s+1}{s^2+4} + \left[-\frac{1}{2s} + \frac{s}{2(s^2+4)}\right]e^{-3s}.$$

For the first term in square brackets, the coefficient in the partial fractions expansion was determined by considering the limit $s \to 0$; that for the second term was found by inspection.

Now, using a look-up table if necessary, we find that, in t-space, the function corresponding to the $\bar{x}$ found above is

$$x(t) = \tfrac{1}{2}\sin 2t + \cos 2t - \tfrac{1}{2}H(t-3) + \tfrac{1}{2}H(t-3)\cos 2(t-3).$$

14.27 *Find the complete solution of*

$$\left(\frac{dy}{dx}\right)^2 - \frac{y}{x}\frac{dy}{dx} + \frac{A}{x} = 0,$$

where A is a positive constant.

At first sight this non-linear equation may appear to be homogeneous, but the term A/x rules this out. Since it is non-linear, we set $dy/dx = p$ and rearrange the equation to make y, which then appears only once, the subject:

$$p^2 - \frac{y}{x}p + \frac{A}{x} = 0,$$

$$xp - y + \frac{A}{p} = 0,$$

$$y = xp + \frac{A}{p}. \qquad (*)$$

This is now recognised as Clairaut's equation with $F(p) = A/p$. Its general solution is therefore given by

$$y = cx + \frac{A}{c} \qquad \text{for arbitrary } c.$$

It also has a singular solution (containing no arbitrary constants) given by

$$\frac{d}{dp}\left(\frac{A}{p}\right) + x = 0, \quad \Rightarrow \quad p = \sqrt{\frac{A}{x}} \quad \Rightarrow \quad y = x\sqrt{\frac{A}{x}} + \frac{A}{\sqrt{A/x}} = 2\sqrt{Ax}.$$

The final result was obtained by substituting for p in $(*)$.

14.29 *Find the solution $y = y(x)$ of*

$$x\frac{dy}{dx} + y - \frac{y^2}{x^{3/2}} = 0,$$

subject to $y(1) = 1$.

After being divided through by x, this equation is in the form of a Bernoulli equation with $n = 2$, i.e. it is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Here, $P(x) = x^{-1}$ and $Q(x) = x^{-5/2}$. So we set $v = y^{1-2} = y^{-1}$ and obtain

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

The equation then becomes

$$
\begin{aligned}
-\frac{1}{v^2}\frac{dv}{dx}+\frac{1}{vx} &= \frac{1}{v^2x^{5/2}},\\
\frac{dv}{dx}-\frac{v}{x} &= -\frac{1}{x^{5/2}}, \text{ for which the IF is } 1/x,\\
\frac{d}{dx}\left(\frac{v}{x}\right) &= -\frac{1}{x^{7/2}},\\
\frac{v}{x} &= \frac{2}{5}\frac{1}{x^{5/2}}+\frac{3}{5}, \text{ using } y(1)=1,\\
\frac{1}{y} &= \frac{2}{5}\frac{1}{x^{3/2}}+\frac{3x}{5},\\
y &= \frac{5x^{3/2}}{2+3x^{5/2}}.
\end{aligned}
$$

The equation can also be treated as an isobaric one with $m=\frac{3}{2}$; the substitution $y=vx^{3/2}$ is made and the equation is reduced to the separable form

$$\frac{dv}{v(2v-5)}=\frac{dx}{2x}.$$

After the LHS has been expressed in partial fractions, the integration can be carried out. The boundary condition, $v(1)=1$, determines the constant of integration and after resubstituting $yx^{-3/2}$ for v, the same answer as obtained earlier is recovered, as it must be.

14.31 *Find the family of solutions of*

$$\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2+\frac{dy}{dx}=0$$

that satisfy $y(0)=0$.

As the equation contains only derivatives, we write $dy/dx=p$ and $d^2y/dx^2=dp/dx$; this will reduce the equation to one of first order:

$$\frac{dp}{dx}+p^2+p=0.$$

Separating the variables:

$$\frac{dp}{p(p+1)}=-dx.$$

We now integrate and express the integrand in partial fractions:

$$\int \left(\frac{1}{p} - \frac{1}{p+1} \right) dp = -\int dx,$$
$$\ln(p) - \ln(p+1) = A - x,$$
$$\Rightarrow \quad \frac{p}{p+1} = Be^{-x},$$
$$\Rightarrow \quad p = \frac{e^{-x}}{C - e^{-x}}.$$

Now $p = dy/dx$ and so

$$\begin{aligned} \frac{dy}{dx} &= \frac{e^{-x}}{C - e^{-x}}, \\ y &= \ln(C - e^{-x}) + D \\ &= \ln(C - e^{-x}) - \ln(C - 1), \text{ since we require } y(0) = 0, \\ &= \ln \frac{C - e^{-x}}{C - 1}. \end{aligned}$$

This is as far as y can be determined since only one boundary condition is given for a second-order equation. As C is varied the solution generates a family of curves satisfying the original equation.

A variety of other forms of solution are possible and equally valid, the actual form obtained depending on where in the calculation the boundary condition is incorporated. They include

$$e^y = F(1 - e^{-x}) + 1, \quad y = \ln[G - (G-1)e^{-x}], \quad y = \ln(e^{-K} + 1 - e^{-x}) + K.$$

15

Higher-order ordinary differential equations

15.1 *A simple harmonic oscillator, of mass m and natural frequency ω_0, experiences an oscillating driving force $f(t) = ma\cos\omega t$. Therefore, its equation of motion is*

$$\frac{d^2x}{dt^2} + \omega_0^2 x = a\cos\omega t,$$

where x is its position. Given that at $t = 0$ we have $x = dx/dt = 0$, find the function $x(t)$. Describe the solution if ω is approximately, but not exactly, equal to ω_0.

To find the full solution given the initial conditions, we need the complete general solution made up of a complementary function (CF) and a particular integral (PI). The CF is clearly of the form $A\cos\omega_0 t + B\sin\omega_0 t$ and, in view of the form of the RHS, we try $x(t) = C\cos\omega t + D\sin\omega t$ as a PI. Substituting this gives

$$-\omega^2 C\cos\omega t - \omega^2 D\sin\omega t + \omega_0^2 C\cos\omega t + \omega_0^2 D\sin\omega t = a\cos\omega t.$$

Equating coefficients of the independent functions $\cos\omega t$ and $\sin\omega t$ requires that

$$-\omega^2 C + \omega_0^2 C = a \quad\Rightarrow\quad C = \frac{a}{\omega_0^2 - \omega^2},$$
$$-\omega^2 D + \omega_0^2 D = 0 \quad\Rightarrow\quad D = 0.$$

Thus, the general solution is

$$x(t) = A\cos\omega_0 t + B\sin\omega_0 t + \frac{a}{\omega_0^2 - \omega^2}\cos\omega t.$$

The initial conditions impose the requirements

$$x(0) = 0 \quad\Rightarrow\quad 0 = A + \frac{a}{\omega_0^2 - \omega^2},$$
$$\text{and } \dot{x}(0) = 0 \quad\Rightarrow\quad 0 = \omega_0 B.$$

Incorporating the implications of these into the general solution gives

$$x(t) = \frac{a}{\omega_0^2 - \omega^2}(\cos\omega t - \cos\omega_0 t)$$
$$= \frac{2a\,\sin[\frac{1}{2}(\omega+\omega_0)t]\,\sin[\frac{1}{2}(\omega_0-\omega)t]}{(\omega_0+\omega)(\omega_0-\omega)}.$$

For $\omega_0 - \omega = \epsilon$ with $|\epsilon|t \ll 1$,

$$x(t) \approx \frac{2a\sin\omega_0 t\,\frac{1}{2}\epsilon t}{2\omega_0\,\epsilon} = \frac{at}{2\omega_0}\sin\omega_0 t.$$

Thus, for moderate t, $x(t)$ is a sine wave of linearly increasing amplitude.

Over a long time, $x(t)$ will vary between $\pm 2a/(\omega_0^2 - \omega^2)$ with sizeable intervals between the two extremes, i.e. it will show beats of amplitude $2a/(\omega_0^2 - \omega^2)$.

15.3 *The theory of bent beams shows that at any point in the beam the 'bending moment' is given by K/ρ, where K is a constant (that depends upon the beam material and cross-sectional shape) and ρ is the radius of curvature at that point. Consider a light beam of length L whose ends, $x = 0$ and $x = L$, are supported at the same vertical height and which has a weight W suspended from its centre. Verify that at any point x ($0 \le x \le L/2$ for definiteness) the net magnitude of the bending moment (bending moment = force × perpendicular distance) due to the weight and support reactions, evaluated on either side of x, is $Wx/2$.*

If the beam is only slightly bent, so that $(dy/dx)^2 \ll 1$, where $y = y(x)$ is the downward displacement of the beam at x, show that the beam profile satisfies the approximate equation

$$\frac{d^2y}{dx^2} = -\frac{Wx}{2K}.$$

By integrating this equation twice and using physically imposed conditions on your solution at $x = 0$ and $x = L/2$, show that the downward displacement at the centre of the beam is $WL^3/(48K)$.

The upward reaction of the support at each end of the beam is $\frac{1}{2}W$.

At the position x the moment on the left is due to

(i) the support at $x = 0$ providing a clockwise moment of $\frac{1}{2}Wx$.

The moment on the right is due to

(ii) the support at $x = L$ providing an anticlockwise moment of $\frac{1}{2}W(L-x)$;

(iii) the weight at $x = \frac{1}{2}L$ providing a clockwise moment of $W(\frac{1}{2}L - x)$.

The net clockwise moment on the right is therefore $W(\frac{1}{2}L - x) - \frac{1}{2}W(L-x) = -\frac{1}{2}Wx$, i.e. equal in magnitude, but opposite in sign, to that on the left.

The radius of curvature of the beam is $\rho = [1+(-y')^2]^{3/2}/(-y'')$, but if $|y'| \ll 1$ this simplifies to $-1/y''$ and the equation of the beam profile satisfies

$$\frac{Wx}{2} = M = \frac{K}{\rho} = -K\frac{d^2y}{dx^2}.$$

We now need to integrate this, taking into account the boundary conditions $y(0) = 0$ and, on symmetry grounds, $y'(\frac{1}{2}L) = 0$:

$$\begin{aligned} y' &= -\frac{Wx^2}{4K} + A, \text{ with } y'(\tfrac{1}{2}L) = 0 \quad \Rightarrow \quad A = \frac{WL^2}{16K}, \\ y' &= \frac{W}{4K}\left(\frac{L^2}{4} - x^2\right), \\ y &= \frac{W}{4K}\left(\frac{L^2x}{4} - \frac{x^3}{3} + B\right), \text{ with } y(0) = 0 \quad \Rightarrow \quad B = 0. \end{aligned}$$

The centre is lowered by

$$y(\tfrac{1}{2}L) = \frac{W}{4K}\left(\frac{L^2}{4}\frac{L}{2} - \frac{1}{3}\frac{L^3}{8}\right) = \frac{WL^3}{48K}.$$

Note that the derived analytic form for $y(x)$ is not applicable in the range $\frac{1}{2}L \le x \le L$; the beam profile is symmetrical about $x = \frac{1}{2}L$, but the expression $\frac{1}{4}L^2x - \frac{1}{3}x^3$ is not invariant under the substitution $x \to L - x$.

15.5 *The function $f(t)$ satisfies the differential equation*

$$\frac{d^2f}{dt^2} + 8\frac{df}{dt} + 12f = 12e^{-4t}.$$

For the following sets of boundary conditions determine whether it has solutions, and, if so, find them:

(a) $f(0) = 0, \quad f'(0) = 0, \quad f(\ln\sqrt{2}) = 0;$
(b) $f(0) = 0, \quad f'(0) = -2, \quad f(\ln\sqrt{2}) = 0.$

Three boundary conditions have been given, and, as this is a second-order linear equation for which only two independent conditions are needed, they may be inconsistent. The plan is to solve it using two of the conditions and then test whether the third one is compatible.

The auxiliary equation for obtaining the CF is

$$\begin{aligned} m^2 + 8m + 12 = 0 \quad &\Rightarrow \quad m = -2 \text{ or } m = -6 \\ &\Rightarrow \quad f(t) = Ae^{-6t} + Be^{-2t}. \end{aligned}$$

Since the form of the RHS, Ce^{-4t}, is not included in the CF, we can try it as the particular integral:

$$16C - 32C + 12C = 12 \quad \Rightarrow \quad C = -3.$$

The general solution is therefore

$$f(t) = Ae^{-6t} + Be^{-2t} - 3e^{-4t}.$$

(a) For boundary conditions $f(0) = 0, \quad f'(0) = 0, \quad f(\ln\sqrt{2}) = 0$:

$$\begin{aligned} f(0) = 0 \quad &\Rightarrow \quad A + B - 3 = 0, \\ f'(0) = 0 \quad &\Rightarrow \quad -6A - 2B + 12 = 0, \\ &\Rightarrow \quad A = \tfrac{3}{2}, \quad B = \tfrac{3}{2}. \\ \text{Hence, } f(t) \quad &= \quad \tfrac{3}{2}e^{-6t} + \tfrac{3}{2}e^{-2t} - 3e^{-4t}. \end{aligned}$$

Recalling that $e^{-(\ln\sqrt{2})} = 1/\sqrt{2}$, we evaluate

$$f(\ln\sqrt{2}) = \frac{3}{2}\frac{1}{8} + \frac{3}{2}\frac{1}{2} - 3\frac{1}{4} = \frac{3}{16} \neq 0.$$

Thus the boundary conditions are inconsistent and there is no solution.

(b) For boundary conditions $f(0) = 0, \quad f'(0) = -2, \quad f(\ln\sqrt{2}) = 0$, we proceed as before:

$$\begin{aligned} f(0) = 0 \quad &\Rightarrow \quad A + B - 3 = 0, \\ f'(0) = 0 \quad &\Rightarrow \quad -6A - 2B + 12 = -2, \\ &\Rightarrow \quad A = 2, \quad B = 1. \\ \text{Hence, } f(t) \quad &= \quad 2e^{-6t} + e^{-2t} - 3e^{-4t}. \end{aligned}$$

We again evaluate

$$f(\ln\sqrt{2}) = 2\frac{1}{8} + \frac{1}{2} - 3\frac{1}{4} = 0.$$

This time the boundary conditions are consistent and there is a unique solution as given above.

15.7 *A solution of the differential equation*

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 4e^{-x}$$

takes the value 1 *when* $x = 0$ *and the value* e^{-1} *when* $x = 1$. *What is its value when* $x = 2$?

The auxiliary equation, $m^2 + 2m + 1 = 0$, has repeated roots $m = -1$, and so the general CF has the special form $y(x) = (A + Bx)e^{-x}$.

Turning to the PI, we note that the form of the RHS of the original equation is contained in the CF, and (to make matters worse) so is x times the RHS. We therefore need to take x^2 times the RHS as a trial PI:

$$y(x) = Cx^2e^{-x}, \quad y' = C(2x - x^2)e^{-x}, \quad y'' = C(2 - 4x + x^2)e^{-x}.$$

Substituting these into the original equation shows that

$$2Ce^{-x} = 4e^{-x} \quad \Rightarrow \quad C = 2$$

and that the full general solution is given by

$$y(x) = (A + Bx)e^{-x} + 2x^2e^{-x}.$$

We now determine the unknown constants using the information given about the solution. Since $y(0) = 1$, $A = 1$. Further, $y(1) = e^{-1}$ requires

$$e^{-1} = (1 + B)e^{-1} + 2e^{-1} \quad \Rightarrow \quad B = -2.$$

Finally, we conclude that $y(x) = (1-2x+2x^2)e^{-x}$ and, therefore, that $y(2) = 5e^{-2}$.

15.9 *Find the general solutions of*

(a) $\dfrac{d^3y}{dx^3} - 12\dfrac{dy}{dx} + 16y = 32x - 8,$

(b) $\dfrac{d}{dx}\left(\dfrac{1}{y}\dfrac{dy}{dx}\right) + (2a\coth 2ax)\left(\dfrac{1}{y}\dfrac{dy}{dx}\right) = 2a^2,$

where a is a constant.

(a) As this is a third-order equation, we expect three terms in the CF.

Since it is linear with constant coefficients, we can make use of the auxiliary equation, which is

$$m^3 - 12m + 16 = 0.$$

By inspection, $m = 2$ is one root; the other two can be found by factorisation:

$$m^3 - 12m + 16 = (m - 2)(m^2 + 2m - 8) = (m - 2)(m + 4)(m - 2) = 0.$$

Thus we have one repeated root ($m = 2$) and one other ($m = -4$) leading to a CF of the form

$$y(x) = (A + Bx)e^{2x} + Ce^{-4x}.$$

As the RHS contains no exponentials, we try $y(x) = Dx + E$ for the PI. We then need $16D = 32$ and $-12D + 16E = -8$, giving $D = 2$ and $E = 1$.

The general solution is therefore

$$y(x) = (A + Bx)e^{2x} + Ce^{-4x} + 2x + 1.$$

(b) The equation is already arranged in the form

$$\frac{dg(y)}{dx} + h(x)g(y) = j(x)$$

and so needs only an integrating factor to allow the first integration step to be made. For this equation the IF is

$$\exp\left\{\int 2a \coth 2ax \, dx\right\} = \exp(\ln \sinh 2ax) = \sinh 2ax.$$

After multiplication through by this factor, the equation can be written

$$\sinh 2ax \frac{d}{dx}\left(\frac{1}{y}\frac{dy}{dx}\right) + (2a \cosh 2ax)\left(\frac{1}{y}\frac{dy}{dx}\right) = 2a^2 \sinh 2ax,$$
$$\frac{d}{dx}\left(\sinh 2ax \frac{1}{y}\frac{dy}{dx}\right) = 2a^2 \sinh 2ax.$$

Integrating this gives

$$\sinh 2ax \frac{1}{y}\frac{dy}{dx} = \frac{2a^2}{2a}\cosh 2ax + A,$$
$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = a \coth 2ax + \frac{A}{\sinh 2ax}.$$

Integrating again, $\quad \ln y = \frac{1}{2}\ln(\sinh 2ax) + \int \frac{A}{\sinh 2ax}\, dx + B$

$$= \frac{1}{2}\ln(\sinh 2ax) + \frac{A}{2a}\ln(|\tanh ax|) + B,$$
$$\Rightarrow \quad y = C(\sinh 2ax)^{1/2}\,(|\tanh ax|)^D.$$

The indefinite integral of $(\sinh 2ax)^{-1}$ appearing in the fourth line can be verified by differentiating $y = \ln|\tanh ax|$ in the form $y = \frac{1}{2}\ln(\tanh^2 ax)$ and recalling that

$$\cosh ax \sinh ax = \frac{1}{2}\sinh 2ax.$$

15.11 *The quantities $x(t)$, $y(t)$ satisfy the simultaneous equations*

$$\ddot{x} + 2n\dot{x} + n^2 x = 0,$$
$$\ddot{y} + 2n\dot{y} + n^2 y = \mu\dot{x},$$

where $x(0) = y(0) = \dot{y}(0) = 0$ and $\dot{x}(0) = \lambda$. Show that

$$y(t) = \tfrac{1}{2}\mu\lambda t^2 \left(1 - \tfrac{1}{3}nt\right) \exp(-nt).$$

For these two coupled equations, in which an 'output' from the first acts as the 'driving input' for the second, we take Laplace transforms and incorporate the boundary conditions:

$$(s^2\bar{x} - 0 - \lambda) + 2n(s\bar{x} - 0) + n^2\bar{x} = 0,$$
$$(s^2\bar{y} - 0 - 0) + 2n(s\bar{y} - 0) + n^2\bar{y} = \mu(s\bar{x} - 0).$$

From the first transformed equation,

$$\bar{x} = \frac{\lambda}{s^2 + 2ns + n^2}.$$

Substituting this into the second transformed equation gives

$$\begin{aligned}
\bar{y} &= \frac{\mu s\bar{x}}{(s+n)^2} = \frac{\mu\lambda s}{(s+n)^2(s+n)^2} \\
&= \frac{\mu\lambda}{(s+n)^3} - \frac{\mu\lambda n}{(s+n)^4}, \\
\Rightarrow \quad y(t) &= \mu\lambda\left(\frac{t^2}{2!}e^{-nt} - \frac{nt^3}{3!}e^{-nt}\right), \text{ from the look-up table,} \\
&= \frac{1}{2}\mu\lambda t^2\left(1 - \frac{nt}{3}\right)e^{-nt},
\end{aligned}$$

i.e. as stated in the question.

15.13 *Two unstable isotopes A and B and a stable isotope C have the following decay rates per atom present: $A \to B$, $3\,\mathrm{s}^{-1}$; $A \to C$, $1\,\mathrm{s}^{-1}$; $B \to C$, $2\,\mathrm{s}^{-1}$. Initially a quantity x_0 of A is present but there are no atoms of the other two types. Using Laplace transforms, find the amount of C present at a later time t.*

Using the name symbol to represent the corresponding number of atoms and

taking Laplace transforms, we have

$$\begin{aligned}
\frac{dA}{dt} = -(3+1)A \quad &\Rightarrow \quad s\bar{A} - x_0 = -4\bar{A} \\
&\Rightarrow \quad \bar{A} = \frac{x_0}{s+4}, \\
\frac{dB}{dt} = 3A - 2B \quad &\Rightarrow \quad s\bar{B} = 3\bar{A} - 2\bar{B} \\
&\Rightarrow \quad \bar{B} = \frac{3x_0}{(s+2)(s+4)}, \\
\frac{dC}{dt} = A + 2B \quad &\Rightarrow \quad s\bar{C} = \bar{A} + 2\bar{B} \\
&\Rightarrow \quad \bar{C} = \frac{x_0(s+2) + 6x_0}{s(s+2)(s+4)}.
\end{aligned}$$

Using the 'cover-up' method for finding the coefficients of a partial fraction expansion without repeated factors, e.g. the coefficient of $(s+2)^{-1}$ is $[(-2+8)x_0]/[(-2)(-2+4)] = -6x_0/4$, we have

$$\bar{C} = \frac{x_0(s+8)}{s(s+2)(s+4)} = \frac{x_0}{s} - \frac{6x_0}{4(s+2)} + \frac{4x_0}{8(s+4)}$$

$$\Rightarrow \quad C(t) = x_0\left(1 - \tfrac{3}{2}e^{-2t} + \tfrac{1}{2}e^{-4t}\right).$$

This is the required expression.

15.15 *The 'golden mean', which is said to describe the most aesthetically pleasing proportions for the sides of a rectangle (e.g. the ideal picture frame), is given by the limiting value of the ratio of successive terms of the Fibonacci series u_n, which is generated by*

$$u_{n+2} = u_{n+1} + u_n,$$

with $u_0 = 0$ and $u_1 = 1$. Find an expression for the general term of the series and verify that the golden mean is equal to the larger root of the recurrence relation's characteristic equation.

The recurrence relation is second order and its characteristic equation, obtained by setting $u_n = A\lambda^n$, is

$$\lambda^2 - \lambda - 1 = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2}(1 \pm \sqrt{5}).$$

The general solution is therefore

$$u_n = A\left(\frac{1+\sqrt{5}}{2}\right)^n + B\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

The initial values (boundary conditions) determine A and B:

$$\begin{aligned} u_0 = 0 \quad &\Rightarrow \quad B = -A, \\ u_1 = 1 \quad &\Rightarrow \quad A\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right) = 1 \quad \Rightarrow \quad A = \frac{1}{\sqrt{5}}, \\ \text{Hence, } u_n \quad &= \quad \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \end{aligned}$$

If we write $(1-\sqrt{5})/(1+\sqrt{5}) = r < 1$, the ratio of successive terms in the series is

$$\begin{aligned} \frac{u_{n+1}}{u_n} &= \frac{\frac{1}{2}[(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}]}{(1+\sqrt{5})^n - (1-\sqrt{5})^n} \\ &= \frac{\frac{1}{2}[1+\sqrt{5} - (1-\sqrt{5})r^n]}{1-r^n} \\ &\to \frac{1+\sqrt{5}}{2} \text{ as } n \to \infty; \end{aligned}$$

i.e. the limiting ratio is the same as the larger value of λ.

This result is a particular example of the more general one that the ratio of successive terms in a series generated by a recurrence relation tends to the largest (in absolute magnitude) of the roots of the characteristic equation. Here there are only two roots, but for an Nth-order relation there will be N roots.

15.17 *The first few terms of a series u_n, starting with u_0, are $1, 2, 2, 1, 6, -3$. The series is generated by a recurrence relation of the form*

$$u_n = Pu_{n-2} + Qu_{n-4},$$

where P and Q are constants. Find an expression for the general term of the series and show that, in fact, the series consists of two interleaved series given by

$$\begin{aligned} u_{2m} &= \tfrac{2}{3} + \tfrac{1}{3}4^m, \\ u_{2m+1} &= \tfrac{7}{3} - \tfrac{1}{3}4^m, \end{aligned}$$

for $m = 0, 1, 2, \ldots$.

We first find P and Q using

$$\begin{aligned} n = 4 \qquad & 6 = 2P + Q, \\ n = 5 \quad & -3 = P + 2Q, \quad \Rightarrow \quad Q = -4 \text{ and } P = 5. \end{aligned}$$

The recurrence relation is thus

$$u_n = 5u_{n-2} - 4u_{n-4}.$$

To solve this we try $u_n = A + B\lambda^n$ for arbitrary constants A and B and obtain

$$\begin{aligned} A + B\lambda^n &= 5A + 5B\lambda^{n-2} - 4A - 4B\lambda^{n-4}, \\ \Rightarrow \quad 0 &= \lambda^4 - 5\lambda^2 + 4 \\ &= (\lambda^2 - 1)(\lambda^2 - 4) \quad \Rightarrow \quad \lambda = \pm 1, \pm 2. \end{aligned}$$

The general solution is $\quad u_n = A + B(-1)^n + C2^n + D(-2)^n.$

We now need to solve the simultaneous equations for A, B, C and D provided by the values of $u_0, \ldots, u_3$:

$$\begin{aligned} 1 &= A + B + C + D, \\ 2 &= A - B + 2C - 2D, \\ 2 &= A + B + 4C + 4D, \\ 1 &= A - B + 8C - 8D. \end{aligned}$$

These have the straightforward solution

$$A = \frac{3}{2}, \qquad B = -\frac{5}{6}, \qquad C = \frac{1}{12}, \qquad D = \frac{1}{4},$$

and so

$$u_n = \frac{3}{2} - \frac{5}{6}(-1)^n + \frac{1}{12}2^n + \frac{1}{4}(-2)^n.$$

When n is even and equal to $2m$,

$$u_{2m} = \frac{3}{2} - \frac{5}{6} + \frac{4^m}{12} + \frac{4^m}{4} = \frac{2}{3} + \frac{4^m}{3}.$$

When n is odd and equal to $2m + 1$,

$$u_{2m+1} = \frac{3}{2} + \frac{5}{6} + \frac{4^m}{6} - \frac{4^m}{2} = \frac{7}{3} - \frac{4^m}{3}.$$

In passing, we note that the fact that both P and Q, and all of the given values $u_0, \ldots, u_4$, are integers, and hence that all terms in the series are integers, provides an indirect proof that $4^m + 2$ is divisible by 3 (without remainder) for all non-negative integers m. This can be more easily proved by induction, as the reader may like to verify.

15.19 *Find the general expression for the u_n satisfying*

$$u_{n+1} = 2u_{n-2} - u_n$$

with $u_0 = u_1 = 0$ and $u_2 = 1$, and show that they can be written in the form

$$u_n = \frac{1}{5} - \frac{2^{n/2}}{\sqrt{5}} \cos\left(\frac{3\pi n}{4} - \phi\right),$$

where $\tan\phi = 2$.

The characteristic equation (which will be a cubic since the recurrence relation is third order) and its solution are given by

$$\begin{aligned} \lambda^{n+1} &= 2\lambda^{n-2} - \lambda^n, \\ \lambda^3 + \lambda^2 - 2 &= 0, \\ (\lambda - 1)(\lambda^2 + 2\lambda + 2) = 0 \quad &\Rightarrow \quad \lambda = 1 \text{ or } \lambda = -1 \pm i. \end{aligned}$$

Thus the general solution of the recurrence relation, which has the generic form $A\lambda_1^n + B\lambda_2^n + C\lambda_3^n$, is

$$\begin{aligned} u_n &= A + B(-1+i)^n + C(-1-i)^n \\ &= A + B\,2^{n/2}e^{i3\pi n/4} + C\,2^{n/2}e^{i5\pi n/4}. \end{aligned}$$

To determine A, B and C we use

$$\begin{aligned} u_0 = 0, \qquad & 0 = A + B + C, \\ u_1 = 0, \qquad & 0 = A + B\,2^{1/2}e^{i3\pi/4} + C\,2^{1/2}e^{i5\pi/4} \\ & \quad = A + B(-1+i) + C(-1-i), \\ u_2 = 1, \qquad & 1 = A + B\,2e^{i6\pi/4} + C\,2e^{i10\pi/4} = A + 2B(-i) + 2C(i). \end{aligned}$$

Adding twice each of the first two equations to the last one gives $5A = 1$. Substituting this into the first and last equations then leads to

$$B + C = -\frac{1}{5} \qquad \text{and} \qquad -B + C = \frac{2}{5i},$$

from which it follows that

$$\begin{aligned} B &= \frac{-1+2i}{10} = \frac{\sqrt{5}}{10}e^{i(\pi-\phi)} \\ \text{and} \quad C &= \frac{-1-2i}{10} = \frac{\sqrt{5}}{10}e^{i(\pi+\phi)}, \end{aligned}$$

where $\tan\phi = 2/1 = 2$.

Thus, collecting these results together, we have

$$\begin{aligned} u_n &= \frac{1}{5} + \frac{2^{n/2}\sqrt{5}}{10}(e^{i3\pi n/4}e^{i(\pi-\phi)} + e^{i5\pi n/4}e^{i(\pi+\phi)}) \\ &= \frac{1}{5} - \frac{2^{n/2}\sqrt{5}}{10}(e^{i3\pi n/4}e^{-i\phi} + e^{-i3\pi n/4}e^{i\phi}) \\ &= \frac{1}{5} - \frac{2^{n/2}\sqrt{5}}{10}\left[2\cos\left(\frac{3\pi n}{4} - \phi\right)\right] \\ &= \frac{1}{5} - \frac{2^{n/2}}{\sqrt{5}}\cos\left(\frac{3\pi n}{4} - \phi\right), \end{aligned}$$

i.e. the form of solution given in the question.

15.21 *Find the general solution of*

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = x,$$

given that $y(1) = 1$ *and* $y(e) = 2e$.

This is Euler's equation and can be solved either by a change of variables, $x = e^t$, or by trying $y = x^\lambda$; we will adopt the second approach. Doing so in the homogeneous equation (RHS set to zero) gives

$$x^2\,\lambda(\lambda-1)x^{\lambda-2} - x\,\lambda x^{\lambda-1} + x^\lambda = 0.$$

The CF is therefore obtained when λ satisfies

$$\lambda(\lambda-1) - \lambda + 1 = 0 \quad\Rightarrow\quad (\lambda-1)^2 = 0 \quad\Rightarrow\quad \lambda = 1 \text{ (repeated)}.$$

Thus, one solution is $y = x$; the other linearly independent solution implied by the repeated root is $x\ln x$ (see a textbook if this is not known).

There is now a further complication as the RHS of the original equation (x) is contained in the CF. We therefore need an extra factor of $\ln x$ in the trial PI, beyond those already in the CF. (This corresponds to the extra power of t needed in the PI if the transformation to a linear equation with constant coefficients is made via the $x = e^t$ change of variable.) As a consequence, the PI to be tried is $y = Cx(\ln x)^2$:

$$x^2\left[2C\frac{\ln x}{x} + \frac{2C}{x}\right] - x\left[Cx\frac{2\ln x}{x} + C(\ln x)^2\right] + Cx(\ln x)^2 = x.$$

This implies that $C = \frac{1}{2}$ and gives the general solution as

$$y(x) = Ax + Bx\ln x + \tfrac{1}{2}x(\ln x)^2.$$

It remains only to determine the unknown constants A and B; this is done using the two given values of $y(x)$. The boundary condition $y(1) = 1$ requires that $A = 1$, and $y(e) = 2e$ implies that $B = \frac{1}{2}$; the solution is now completely determined as

$$y(x) = x + \tfrac{1}{2}x \ln x(1 + \ln x).$$

15.23 *Prove that the general solution of*

$$(x-2)\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + \frac{4y}{x^2} = 0$$

is given by

$$y(x) = \frac{1}{(x-2)^2}\left[k\left(\frac{2}{3x} - \frac{1}{2}\right) + cx^2\right].$$

This equation is not of any plausible standard form, and the only solution method is to try to make it into an exact equation. If this is possible the order of the equation will be reduced by one.

We first multiply through by x^2 and then note that the resulting factor $3x^2$ in the second term can be written as $[x^2(x-2)]' + 4x$, i.e. as the derivative of the function multiplying y'' together with another simple function. This latter can be combined with the undifferentiated term and allow the whole equation to be written as an exact equation:

$$\frac{d}{dx}\left[x^2(x-2)\frac{dy}{dx}\right] + 4x\frac{dy}{dx} + 4y = 0,$$

$$\frac{d}{dx}\left[x^2(x-2)\frac{dy}{dx}\right] + \frac{d(4xy)}{dx} = 0,$$

$$\Rightarrow \quad x^2(x-2)\frac{dy}{dx} + 4xy = k.$$

Either by inspection or by use of the standard formula, the IF is $(x-2)/x^4$ and leads to

$$\frac{d}{dx}\left[\frac{(x-2)^2}{x^2}y\right] = \frac{k(x-2)}{x^4},$$

$$\Rightarrow \quad \frac{(x-2)^2}{x^2}y = k\left(-\frac{1}{2x^2} + \frac{2}{3x^3}\right) + c,$$

$$\Rightarrow \quad y = \frac{1}{(x-2)^2}\left(-\frac{k}{2} + \frac{2k}{3x} + cx^2\right).$$

15.25 *Find the Green's function that satisfies*

$$\frac{d^2G(x,\xi)}{dx^2} - G(x,\xi) = \delta(x-\xi) \qquad \textit{with} \qquad G(0,\xi) = G(1,\xi) = 0.$$

It is clear from inspection that the CF has solutions of the form $e^{\pm x}$. The other pair of solutions that may suggest themselves are $\sinh x$ and $\cosh x$, but these are merely independent linear combinations of the same two functions.

As both boundary conditions are given at finite values of x (rather than at $x \to \pm\infty$) and both are of the form $y(x) = 0$, it is more convenient to work with those particular linear combinations of e^x and e^{-x} that vanish at the boundary points. The only common linear combination of these two functions that vanishes at a finite value of x is a sinh function. To construct one that vanishes at $x = x_0$ the argument of the sinh function must be made to be $x - x_0$. For the present case the appropriate combinations are

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \text{and} \quad \sinh(1-x) = \left(\frac{e}{2}\right)e^{-x} - \left(\frac{1}{2e}\right)e^x.$$

Thus, with $0 \le \xi \le 1$, we take

$$G(x,\xi) = \begin{cases} A(\xi)\sinh x & x < \xi, \\ B(\xi)\sinh(1-x) & x > \xi. \end{cases}$$

The continuity requirement on $G(x,\xi)$ at $x = \xi$ and the unit discontinuity requirement on its derivative at the same point give

$$\begin{aligned} A\sinh\xi - B\sinh(1-\xi) &= 0 \\ \text{and} \qquad -B\cosh(1-\xi) - A\cosh\xi &= 1, \end{aligned}$$

leading to

$$\begin{aligned} A\sinh\xi\cosh(1-\xi) + A\cosh\xi\sinh(1-\xi) &= -\sinh(1-\xi), \\ A[\sinh(\xi+1-\xi)] &= -\sinh(1-\xi). \end{aligned}$$

Hence,

$$A = -\frac{\sinh(1-\xi)}{\sinh 1} \qquad \text{and} \qquad B = -\frac{\sinh\xi}{\sinh 1},$$

giving the full Green's function as

$$G(x,\xi) = \begin{cases} -\dfrac{\sinh(1-\xi)}{\sinh 1}\sinh x & x < \xi, \\ -\dfrac{\sinh\xi}{\sinh 1}\sinh(1-x) & x > \xi. \end{cases}$$

15.27 *Show generally that if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of*

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0,$$

with $y_1(0) = 0$ and $y_2(1) = 0$, then the Green's function $G(x, \xi)$ for the interval $0 \le x, \xi \le 1$ and with $G(0, \xi) = G(1, \xi) = 0$ can be written in the form

$$G(x, \xi) = \begin{cases} y_1(x)y_2(\xi)/W(\xi) & 0 < x < \xi, \\ y_2(x)y_1(\xi)/W(\xi) & \xi < x < 1, \end{cases}$$

where $W(x) = W[y_1(x), y_2(x)]$ is the Wronskian of $y_1(x)$ and $y_2(x)$.

As usual, we start by writing the general solution as a weighted sum of the linearly independent solutions, whilst leaving the possibility that the weights may be different for different x-ranges:

$$G(x, \xi) = \begin{cases} A(\xi)y_1(x) + B(\xi)y_2(x) & 0 < x < \xi, \\ C(\xi)y_1(x) + D(\xi)y_2(x) & \xi < x < 1. \end{cases}$$

Imposing the boundary conditions and using $y_1(0) = y_2(1) = 0$,

$$\begin{aligned} 0 = G(0, \xi) = A(\xi)y_1(0) + B(\xi)y_2(0) \quad &\Rightarrow \quad B(\xi) = 0, \\ 0 = G(1, \xi) = C(\xi)y_1(1) + D(\xi)y_2(1) \quad &\Rightarrow \quad C(\xi) = 0. \end{aligned}$$

The continuity requirement on $G(x, \xi)$ at $x = \xi$ and the unit discontinuity requirement on its derivative at the same point give

$$\begin{aligned} A(\xi)y_1(\xi) - D(\xi)y_2(\xi) &= 0, \\ A(\xi)y_1'(\xi) - D(\xi)y_2'(\xi) &= -1, \end{aligned}$$

leading to

$$\begin{aligned} A(\xi)[y_1y_2' - y_2y_1'] = y_2 \quad &\Rightarrow \quad A(\xi) = \frac{y_2(\xi)}{W(\xi)}, \\ D(\xi) = \frac{y_1(\xi)}{y_2(\xi)}A(\xi) = \frac{y_1(\xi)}{W(\xi)}&. \end{aligned}$$

Thus,

$$G(x, \xi) = \begin{cases} y_1(x)y_2(\xi)/W(\xi) & 0 < x < \xi, \\ y_2(x)y_1(\xi)/W(\xi) & \xi < x < 1. \end{cases}$$

This result is perfectly general for linear second-order equations of the type stated and can be a quick way to find the corresponding Green's function, provided the solutions that vanish at the end-points can be identified easily. Exercise 15.25 is a particular example of this general result.

15.29 *The equation of motion for a driven damped harmonic oscillator can be written*

$$\ddot{x} + 2\dot{x} + (1 + \kappa^2)x = f(t),$$

with $\kappa \neq 0$. If it starts from rest with $x(0) = 0$ and $\dot{x}(0) = 0$, find the corresponding Green's function $G(t, \tau)$ and verify that it can be written as a function of $t - \tau$ only. Find the explicit solution when the driving force is the unit step function, i.e. $f(t) = H(t)$. Confirm your solution by taking the Laplace transforms of both it and the original equation.

The auxiliary equation is

$$m^2 + 2m + (1 + \kappa^2) = 0 \quad \Rightarrow \quad m = -1 \pm i\kappa,$$

and the CF is $x(t) = Ae^{-t}\cos\kappa t + Be^{-t}\sin\kappa t$.
Let

$$G(t, \tau) = \begin{cases} A(\tau)e^{-t}\cos\kappa t + B(\tau)e^{-t}\sin\kappa t & 0 < t < \tau, \\ C(\tau)e^{-t}\cos\kappa t + D(\tau)e^{-t}\sin\kappa t & t > \tau. \end{cases}$$

The boundary condition $x(0) = 0$ implies that $A = 0$, and

$$\dot{x}(0) = 0 \quad \Rightarrow \quad B(-e^{-t}\sin\kappa t + \kappa e^{-t}\cos\kappa t) = 0 \quad \Rightarrow \quad B = 0.$$

Thus $G(t, \tau) = 0$ for $t < \tau$.

The continuity of G at $t = \tau$ gives

$$Ce^{-\tau}\cos\kappa\tau + De^{-\tau}\sin\kappa\tau = 0 \quad \Rightarrow \quad D = -\frac{C\cos\kappa\tau}{\sin\kappa\tau}.$$

The unit discontinuity in the derivative of G at $t = \tau$ requires (using $s = \sin\kappa\tau$ and $c = \cos\kappa\tau$ as shorthand)

$$\begin{aligned} Ce^{-\tau}(-c - \kappa s) + De^{-\tau}(-s + \kappa c) - 0 &= 1, \\ C\left[-c - \kappa s - \frac{c}{s}(-s + \kappa c)\right] &= e^{\tau}, \\ C(-sc - \kappa s^2 + cs - \kappa c^2) &= se^{\tau}, \end{aligned}$$

giving

$$C = -\frac{e^{\tau}\sin\kappa\tau}{\kappa} \quad \text{and} \quad D = \frac{e^{\tau}\cos\kappa\tau}{\kappa}.$$

Thus, for $t > \tau$,

$$\begin{aligned} G(t, \tau) &= \frac{e^{\tau}}{\kappa}(-\sin\kappa\tau\cos\kappa t + \cos\kappa\tau\sin\kappa t)e^{-t} \\ &= \frac{e^{-(t-\tau)}}{\kappa}\sin\kappa(t - \tau). \end{aligned}$$

This form verifies that the Green's function is a function only of the difference $t-\tau$ and not of t and τ separately.

The explicit solution to the given equation when $f(t) = H(t)$ is thus

$$\begin{aligned}
x(t) &= \int_0^\infty G(t,\tau)f(\tau)\,d\tau \\
&= \int_0^t G(t,\tau)H(\tau)\,d\tau, \text{ since } G(t,\tau) = 0 \text{ for } \tau > t, \\
&= \frac{1}{\kappa}\int_0^t e^{-(t-\tau)}\sin\kappa(t-\tau)\,d\tau \\
&= \frac{e^{-t}}{\kappa}\mathrm{Im}\int_0^t e^{\tau+i\kappa(t-\tau)}\,d\tau \\
&= \frac{e^{-t}}{\kappa}\mathrm{Im}\left[\frac{e^{i\kappa t}e^{\tau-i\kappa\tau}}{1-i\kappa}\right]_{\tau=0}^{\tau=t} \\
&= \frac{e^{-t}}{\kappa}\mathrm{Im}\left[\frac{e^{t}-e^{i\kappa t}}{1-i\kappa}\right].
\end{aligned}$$

Now multiplying both numerator and denominator by $1+i\kappa$ to make the latter real gives

$$\begin{aligned}
x(t) &= \frac{e^{-t}}{\kappa(1+\kappa^2)}\mathrm{Im}\,[(e^t - e^{i\kappa t})(1+i\kappa)] \\
&= \frac{e^{-t}}{\kappa(1+\kappa^2)}[\kappa(e^t-\cos\kappa t)-\sin\kappa t] \\
&= \frac{1}{1+\kappa^2}\left(1-e^{-t}\cos\kappa t-\frac{1}{\kappa}e^{-t}\sin\kappa t\right).
\end{aligned}$$

The Laplace transform of this solution is given by

$$\begin{aligned}
\bar{x} &= \frac{1}{1+\kappa^2}\left(\frac{1}{s}-\frac{s+1}{(s+1)^2+\kappa^2}-\frac{1}{\kappa}\frac{\kappa}{(s+1)^2+\kappa^2}\right) \\
&= \frac{(s+1)^2+\kappa^2-s(s+1)-s}{(1+\kappa^2)s[(s+1)^2+\kappa^2]} \\
&= \frac{1}{s[(s+1)^2+\kappa^2]}.
\end{aligned}$$

The Laplace transform of the original equation with the given initial conditions reads

$$[s^2\bar{x}-0s-0]+2[s\bar{x}-0]+(1+\kappa^2)\bar{x} = \frac{1}{s},$$

again showing that

$$\bar{x} = \frac{1}{s[s^2+2s+1+\kappa^2]} = \frac{1}{s[(s+1)^2+\kappa^2]},$$

and so confirming the solution reached using the Green's function approach.

15.31 *Find the Green's function $x = G(t, t_0)$ that solves*

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = \delta(t - t_0)$$

under the initial conditions $x = dx/dt = 0$ at $t = 0$. Hence solve

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = f(t),$$

where $f(t) = 0$ for $t < 0$. Evaluate your answer explicitly for $f(t) = Ae^{-\beta t}$ $(t > 0)$.

It is clear that one solution, $x(t)$, to the homogeneous equation has $\ddot{x} = -\alpha\dot{x}$ and is therefore $x(t) = Ae^{-\alpha t}$. The equation is of second order and therefore has a second solution; this is the trivial (but perfectly valid) x is a constant. The CF is thus $x(t) = Ae^{-\alpha t} + B$.

Let

$$G(t, t_0) = \begin{cases} Ae^{-\alpha t} + B, & 0 \le t \le t_0, \\ Ce^{-\alpha t} + D, & t > t_0. \end{cases}$$

Now, the initial conditions give

$$\begin{aligned} x(0) = 0 \quad &\Rightarrow \quad A + B = 0, \\ \dot{x}(0) = 0 \quad &\Rightarrow \quad -\alpha A = 0 \quad \Rightarrow \quad A = B = 0. \end{aligned}$$

Thus $G(t, t_0) = 0$ for $0 \le t \le t_0$.

The continuity/discontinuity conditions determine C and D through

$$\begin{aligned} &Ce^{-\alpha t_0} + D - 0 = 0, \\ &-\alpha Ce^{-\alpha t_0} - 0 = 1, \quad \Rightarrow \quad C = -\frac{e^{\alpha t_0}}{\alpha} \text{ and } D = \frac{1}{\alpha}. \end{aligned}$$

It follows that $\qquad G(t, t_0) = \dfrac{1}{\alpha}[1 - e^{-\alpha(t - t_0)}]$ for $t > t_0$.

The general formalism now gives the solution of

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} = f(t)$$

as

$$x(t) = \int_0^t \frac{1}{\alpha}[1 - e^{-\alpha(t-\tau)}]f(\tau)\,d\tau.$$

With $f(t) = Ae^{-\beta t}$ this becomes

$$\begin{aligned} x(t) &= \int_0^t \frac{1}{\alpha}[1 - e^{-\alpha(t-\tau)}]Ae^{-\beta\tau}\,d\tau \\ &= \frac{A}{\alpha}\int_0^t (e^{-\beta\tau} - e^{-\alpha t}e^{(\alpha-\beta)\tau})\,d\tau \\ &= A\left[\frac{1-e^{-\beta t}}{\alpha\beta} - \frac{e^{-\beta t} - e^{-\alpha t}}{\alpha(\alpha-\beta)}\right] \\ &= A\left[\frac{\alpha - \beta - \alpha e^{-\beta t} + \beta e^{-\alpha t}}{\beta\alpha(\alpha-\beta)}\right] \\ &= A\left[\frac{\alpha(1-e^{-\beta t}) - \beta(1-e^{-\alpha t})}{\beta\alpha(\alpha-\beta)}\right]. \end{aligned}$$

This is the required explicit solution.

15.33 *Solve*

$$2y\frac{d^3y}{dx^3} + 2\left(y + 3\frac{dy}{dx}\right)\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = \sin x.$$

The only realistic hope for this non-linear equation is to try to arrange it as an exact equation! We note that the second and fourth terms can be written as the derivative of a product, and that adding and subtracting $2y'y''$ will enable the first term to be written in a similar way. We therefore rewrite the equation as

$$\begin{aligned} \frac{d}{dx}\left(2y\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(2y\frac{dy}{dx}\right) + (6-2)\frac{dy}{dx}\frac{d^2y}{dx^2} &= \sin x, \\ \frac{d}{dx}\left(2y\frac{d^2y}{dx^2}\right) + \frac{d}{dx}\left(2y\frac{dy}{dx}\right) + \frac{d}{dx}\left[2\left(\frac{dy}{dx}\right)^2\right] &= \sin x. \end{aligned}$$

This second form is obtained by noting that the final term on the LHS of the first equation happens to be an exact differential. Thus the whole of the LHS is an exact differential and one stage of integration can be carried out:

$$2y\frac{d^2y}{dx^2} + 2y\frac{dy}{dx} + 2\left(\frac{dy}{dx}\right)^2 = -\cos x + A.$$

We now note that the first and third terms of this integrated equation can be combined as the derivative of a product, whilst the second term is the derivative

of y^2. This allows a further step of integration:

$$\begin{aligned}
\frac{d}{dx}\left(2y\frac{dy}{dx}\right) + 2y\frac{dy}{dx} &= -\cos x + A,\\
\frac{d}{dx}\left(2y\frac{dy}{dx}\right) + \frac{d(y^2)}{dx} &= -\cos x + A,\\
\Rightarrow \quad 2y\frac{dy}{dx} + y^2 &= -\sin x + Ax + B,\\
\frac{d(y^2)}{dx} + y^2 &= -\sin x + Ax + B.
\end{aligned}$$

At this stage an integrating factor is needed. However, as the LHS consists of the sum of the differentiated and undifferentiated forms of the same function, the required IF is simply e^x. After multiplying through by this, we obtain

$$\begin{aligned}
\frac{d}{dx}\left(e^x y^2\right) &= -e^x \sin x + Axe^x + Be^x,\\
\Rightarrow \quad y^2 &= e^{-x}\left[C + \int^x (B + Au - \sin u)e^u\,du\right]\\
&= Ce^{-x} + B + A(x-1) - \tfrac{1}{2}(\sin x - \cos x).
\end{aligned}$$

The last term in this final solution is obtained by considering

$$\begin{aligned}
\int^x e^u \sin u\,du &= \text{Im} \int^x e^{(1+i)u}\,du\\
&= \text{Im}\left[\frac{e^{(1+i)u}}{1+i}\right]^x\\
&= \text{Im}\left[\tfrac{1}{2}(1-i)e^{(1+i)x}\right]\\
&= \tfrac{1}{2}e^x(\sin x - \cos x).
\end{aligned}$$

15.35 *Express the equation*

$$\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + (4x^2+6)y = e^{-x^2}\sin 2x$$

in canonical form and hence find its general solution.

In the standard shortened notation, we have

$$a_1(x) = 4x, \qquad a_0(x) = 4x^2 + 6, \qquad f(x) = e^{-x^2}\sin 2x.$$

Then, with $y(x)$ expressed as $y(x) = u(x)v(x)$, in order to have an equation with no v' term in it, we choose $u(x)$ as

$$u(x) = \exp\left\{-\tfrac{1}{2}\textstyle\int^x a_1(z)\,dz\right\} = \exp\left\{-\tfrac{1}{2}\textstyle\int^x 4z\,dz\right\} = e^{-x^2}.$$

The equation is then reduced to

$$\frac{d^2v}{dx^2} + g(x)v = h(x),$$

where

$$g(x) = a_0(x) - \tfrac{1}{4}[a_1(x)]^2 - \tfrac{1}{2}a_1'(x) = 4x^2 + 6 - 4x^2 - 2 = 4$$

and

$$\begin{aligned} h(x) = f(x)\exp\left\{\tfrac{1}{2}\textstyle\int a_1(z)\,dz\right\} &= (e^{-x^2}\sin 2x)\exp\left\{\tfrac{1}{2}\textstyle\int 4z\,dz\right\} \\ &= (e^{-x^2}\sin 2x)\,e^{x^2} = \sin 2x. \end{aligned}$$

For this particular case the reduced equation is

$$v'' + 4v = \sin 2x.$$

This has CF $A\cos 2x + B\sin 2x$ but, because the RHS is contained in the CF, we need to try as a PI $y(x) = C(x)\cos 2x + D(x)\sin 2x$. Substituting this shows that C and D must satisfy

$$C''\cos 2x - 4C'\sin 2x + D''\sin 2x + 4D'\cos 2x = \sin 2x,$$

yielding the pair of simultaneous equations

$$\begin{aligned} C'' + 4D' &= 0, \\ -4C' + D'' &= 1. \end{aligned}$$

Any solution will suffice, and the simplest is $C(x) = -\frac{1}{4}x$ with $D(x) = 0$.

We can now write the general solution and express it in terms of the original variables:

$$\begin{aligned} v(x) &= (A - \tfrac{1}{4}x)\cos 2x + B\sin 2x, \\ y(x) = u(x)v(x) &= [(A - \tfrac{1}{4}x)\cos 2x + B\sin 2x]e^{-x^2}. \end{aligned}$$

15.37 *Consider the equation*

$$x^p y'' + \frac{n+3-2p}{n-1}x^{p-1}y' + \left(\frac{p-2}{n-1}\right)^2 x^{p-2}y = y^n,$$

in which $p \neq 2$ and $n > -1$ but $n \neq 1$. For the boundary conditions $y(1) = 0$ and $y'(1) = \lambda$, show that the solution is $y(x) = v(x)x^{(p-2)/(n-1)}$, where $v(x)$ is given by

$$\int_0^{v(x)} \frac{dz}{\left[\lambda^2 + 2z^{n+1}/(n+1)\right]^{1/2}} = \ln x.$$

To start, we test whether the equation is isobaric by giving y a weight m relative to x. The weights of the four terms are then

$$m-2+p, \quad m-1+p-1, \quad m+p-2, \quad mn.$$

These are all equal, provided m is chosen to satisfy $m+p-2=mn$, i.e. $m=(p-2)/(n-1)$. Thus the equation is isobaric.

Now set $y(x)=v(x)x^m$, noting that $y(1)=0 \Rightarrow v(1)=0$. As derivatives we have

$$y' = v'x^m + mvx^{m-1}, \quad y'' = v''x^m + 2mv'x^{m-1} + m(m-1)vx^{m-2}.$$

We further note that, since $y'(1)=\lambda$ implies $v'(1)+mv(1)=\lambda$, we must have $v'(1)=\lambda$.

Substituting the derivatives into the equation, rewriting the constants in terms of m and dividing through by x^{p+m-2} gives

$$\begin{aligned} x^2v'' + 2mxv' + m(m-1)v + (1-2m)(xv'+mv) + m^2v &= v^nx^0, \\ x^2v'' + xv' + [\,m(m-1)+m-2m^2+m^2\,]v &= v^n, \\ x^2v'' + xv' &= v^n. \end{aligned}$$

To solve this non-linear equation we set $x=e^t$ and $v(x)=u(t)$. The operator d/dx becomes $e^{-t}d/dt$. The initial conditions are that $u(0)=0$ and

$$\frac{du}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \lambda e^0 \text{ at } t=0.$$

The equation itself transforms to

$$\begin{aligned} e^{2t}e^{-t}\frac{d}{dt}\left(e^{-t}\frac{du}{dt}\right) + e^te^{-t}\frac{du}{dt} &= u^n, \\ u'' - u' + u' &= u^n, \\ u'' &= u^n, \\ u'u'' &= u'u^n, \\ \frac{1}{2}\left(\frac{du}{dt}\right)^2 &= \frac{u^{n+1}}{n+1} + k. \end{aligned}$$

Since $u'(0)=\lambda$ and $u(0)=0$, it follows that $k=\frac{1}{2}\lambda^2$ and that

$$\frac{du}{dt} = \left(\frac{2u^{n+1}}{n+1} + \lambda^2\right)^{1/2}.$$

Integrating this gives

$$\int_0^{u(t)} \left(\frac{2z^{n+1}}{n+1} + \lambda^2\right)^{-1/2} dz = t - 0,$$

and, by changing back to the original variables,

$$\int_0^{v(x)} \left(\frac{2z^{n+1}}{n+1} + \lambda^2 \right)^{-1/2} dz = \ln x.$$

For any given x, this equation determines $v(x)$.

The solution $y(x)$ to the original equation is then given by

$$y(x) = v(x)x^{(p-2)/(n-1)}.$$

16

Series solutions of ordinary differential equations

16.1 *Find two power series solutions about $z = 0$ of the differential equation*

$$(1 - z^2)y'' - 3zy' + \lambda y = 0.$$

Deduce that the value of λ for which the corresponding power series becomes an Nth-degree polynomial $U_N(z)$ is $N(N + 2)$. Construct $U_2(z)$ and $U_3(z)$.

If the equation is imagined divided through by $(1 - z^2)$ it is straightforward to see that, although $z = \pm 1$ are singular points of the equation, the point $z = 0$ is an *ordinary* point. We therefore expect two (uncomplicated!) series solutions with indicial values $\sigma = 0$ and $\sigma = 1$.

(a) $\sigma = 0$ and $y(z) = \sum_{n=0}^{\infty} a_n z^n$ with $a_0 \neq 0$.

Substituting and equating the coefficients of z^m,

$$(1 - z^2)\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} - 3\sum_{n=0}^{\infty} n a_n z^n + \lambda \sum_{n=0}^{\infty} a_n z^n = 0,$$

$$(m + 2)(m + 1)a_{m+2} - m(m - 1)a_m - 3ma_m + \lambda a_m = 0,$$

gives as the recurrence relation

$$a_{m+2} = \frac{m(m-1) + 3m - \lambda}{(m+2)(m+1)}\, a_m = \frac{m(m+2) - \lambda}{(m+1)(m+2)}\, a_m.$$

Since this recurrence relation connects alternate coefficients a_m, and $a_0 \neq 0$, only the coefficients with even indices are generated. All such coefficients with index higher than m will become zero, and the series will become an Nth-degree polynomial $U_N(z)$, if $\lambda = m(m + 2) = N(N + 2)$ for some (even) m appearing in the series; here, this means any positive even integer N.

To construct $U_2(z)$ we need to take $\lambda = 2(2+2) = 8$. The recurrence relation gives a_2 as

$$a_2 = \frac{0-8}{(0+1)(0+2)}a_0 = -4a_0 \quad \Rightarrow \quad U_2(z) = a_0(1-4z^2).$$

(b) $\sigma = 1$ and $y(z) = z\sum_{n=0}^{\infty} a_n z^n$ with $a_0 \neq 0$.

Substituting and equating the coefficients of z^{m+1},

$$(1-z^2)\sum_{n=0}^{\infty}(n+1)na_n z^{n-1} - 3\sum_{n=0}^{\infty}(n+1)a_n z^{n+1} + \lambda\sum_{n=0}^{\infty} a_n z^{n+1} = 0,$$
$$(m+3)(m+2)a_{m+2} - (m+1)ma_m - 3(m+1)a_m + \lambda a_m = 0,$$

gives as the recurrence relation

$$a_{m+2} = \frac{m(m+1)+3(m+1)-\lambda}{(m+2)(m+3)}a_m = \frac{(m+1)(m+3)-\lambda}{(m+2)(m+3)}a_m.$$

Again, all coefficients with index higher than m will become zero, and the series will become an Nth-degree polynomial $U_N(z)$, if $\lambda = (m+1)(m+3) = N(N+2)$ for some (even) m appearing in the series; here, this means any positive odd integer N.

To construct $U_3(z)$ we need to take $\lambda = 3(3+2) = 15$. The recurrence relation gives a_2 as

$$a_2 = \frac{3-15}{(0+2)(0+3)}a_0 = -2a_0.$$

Thus,

$$U_3(z) = a_0(z - 2z^3).$$

16.3 *Find power series solutions in z of the differential equation*

$$zy'' - 2y' + 9z^5 y = 0.$$

Identify closed forms for the two series, calculate their Wronskian, and verify that they are linearly independent. Compare the Wronskian with that calculated from the differential equation.

Putting the equation in its standard form shows that $z = 0$ is a singular point of the equation but, as $-2z/z$ and $9z^7/z$ are finite as $z \to 0$, it is a regular singular point. We therefore substitute a Frobenius type solution,

$$y(z) = z^{\sigma}\sum_{n=0}^{\infty} a_n z^n \text{ with } a_0 \neq 0,$$

and obtain

$$\sum_{n=0}^{\infty}(n+\sigma)(n+\sigma-1)a_n z^{n+\sigma-1}$$
$$-2\sum_{n=0}^{\infty}(n+\sigma)a_n z^{n+\sigma-1}+9\sum_{n=0}^{\infty}a_n z^{n+\sigma+5}=0.$$

Equating the coefficient of $z^{\sigma-1}$ to zero gives the indicial equation as

$$\sigma(\sigma-1)a_0-2\sigma a_0=0 \quad\Rightarrow\quad \sigma=0,\ 3.$$

These differ by an integer and may or may not yield two independent solutions. The larger root, $\sigma=3$, will give a solution; the smaller one, $\sigma=0$, may not.

(a) $\sigma=3$.

Equating the general coefficient of z^{m+2} to zero (with $\sigma=3$) gives

$$(m+3)(m+2)a_m-2(m+3)a_m+9a_{m-6}=0.$$

Hence the recurrence relation is

$$a_m=-\frac{9a_{m-6}}{m(m+3)},$$
$$\Rightarrow\quad a_{6p}=-\frac{9}{6p\,(6p+3)}\,a_{6p-6}=-\frac{a_{6p-6}}{2p\,(2p+1)}=\frac{(-1)^p a_0}{(2p+1)!}.$$

The first solution is therefore given by

$$y_1(x)=a_0 z^3\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{6n}=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{3(2n+1)}=a_0\sin z^3.$$

(b) $\sigma=0$.

Equating the general coefficient of z^{m-1} to zero (with $\sigma=0$) gives

$$m(m-1)a_m-2ma_m+9a_{m-6}=0.$$

Hence the recurrence relation is

$$a_m=-\frac{9a_{m-6}}{m(m-3)},$$
$$\Rightarrow\quad a_{6p}=-\frac{9}{6p\,(6p-3)}\,a_{6p-6}=-\frac{a_{6p-6}}{2p\,(2p-1)}=\frac{(-1)^p a_0}{(2p)!}.$$

A second solution is thus

$$y_2(x)=a_0\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{6n}=a_0\cos z^3.$$

We see that $\sigma=0$ does, in fact, produce a (different) series solution. This is because the recurrence relation relates a_n to a_{n+6} and does not involve a_{n+3};

the relevance here of considering the subscripted index '$m+3$' is that '3' is the difference between the two indicial values.

We now calculate the Wronskian of the two solutions, $y_1 = a_0 \sin z^3$ and $y_2 = b_0 \cos z^3$:

$$\begin{aligned} W(y_1, y_2) &= y_1 y_2' - y_2 y_1' \\ &= a_0 \sin z^3(-3b_0 z^2 \sin z^3) - b_0 \cos z^3(3a_0 z^2 \cos z^3) \\ &= -3a_0 b_0 z^2 \neq 0. \end{aligned}$$

The fact that the Wronskian is non-zero shows that the two solutions are linearly independent.

We can also calculate the Wronskian from the original equation in its standard form,

$$y'' - \frac{2}{z} y' + 9z^4 y = 0,$$

as

$$W = C \exp\left\{-\int^z \frac{-2}{u}\, du\right\} = C \exp(2 \ln z) = Cz^2.$$

This is in agreement with the Wronskian calculated from the solutions, as it must be.

16.5 *Investigate solutions of Legendre's equation at one of its singular points as follows.*

(a) *Verify that $z = 1$ is a regular singular point of Legendre's equation and that the indicial equation for a series solution in powers of $(z-1)$ has a double root $\sigma = 0$.*
(b) *Obtain the corresponding recurrence relation and show that a polynomial solution is obtained if ℓ is a positive integer.*
(c) *Determine the radius of convergence R of the $\sigma = 0$ series and relate it to the positions of the singularities of Legendre's equation.*

(a) In standard form, Legendre's equation reads

$$y'' - \frac{2z}{1-z^2} y' + \frac{\ell(\ell+1)}{1-z^2} y = 0.$$

This has a singularity at $z = 1$, but, since

$$\frac{-2z(z-1)}{1-z^2} \to 1 \text{ and } \frac{\ell(\ell+1)(z-1)^2}{1-z^2} \to 0 \text{ as } z \to 1,$$

i.e. both limits are finite, the point is a regular singular point.

We next change the origin to the point $z = 1$ by writing $u = z - 1$ and $y(z) = f(u)$. The transformed equation is

$$f'' - \frac{2(u+1)}{-u(u+2)} f' + \frac{\ell(\ell+1)}{-u(u+2)} y = 0$$

or $$-u(u+2)f'' - 2(u+1)f' + \ell(\ell+1)f = 0.$$

The point $u = 0$ is a regular singular point of this equation and so we set $f(u) = u^\sigma \sum_{n=0}^{\infty} a_n u^n$ and obtain

$$-u(u+2)\sum_{n=0}^{\infty}(\sigma+n)(\sigma+n-1)a_n u^{\sigma+n-2}$$
$$-2(u+1)\sum_{n=0}^{\infty}(\sigma+n)a_n u^{\sigma+n-1} + \ell(\ell+1)\sum_{n=0}^{\infty} a_n u^{\sigma+n} = 0.$$

Equating to zero the coefficient of $u^{\sigma-1}$ gives

$$-2\sigma(\sigma-1)a_0 - 2\sigma a_0 = 0 \quad \Rightarrow \quad \sigma^2 = 0;$$

i.e. the indicial equation has a double root $\sigma = 0$.

(b) To obtain the recurrence relation we set the coefficient of u^m equal to zero for general m:

$$-m(m-1)a_m - 2(m+1)ma_{m+1} - 2ma_m - 2(m+1)a_{m+1} + \ell(\ell+1)a_m = 0.$$

Tidying this up gives

$$2(m+1)(m+1)a_{m+1} = [\ell(\ell+1) - m^2 + m - 2m]a_m,$$
$$\Rightarrow \quad a_{m+1} = \frac{\ell(\ell+1) - m(m+1)}{2(m+1)^2} a_m.$$

From this it is clear that, if ℓ is a positive integer, then $a_{\ell+1}$ and all further a_n are zero and that the solution is a polynomial (of degree ℓ).

(c) The limit of the ratio of successive terms in the series is given by

$$\left|\frac{a_{n+1}u^{n+1}}{a_n u^n}\right| = \left|\frac{u[\ell(\ell+1) - m(m+1)]}{2(m+1)^2}\right| \to \frac{|u|}{2} \text{ as } m \to \infty.$$

For convergence this limit needs to be < 1, i.e. $|u| < 2$. Thus the series converges in a circle of radius 2 centred on $u = 0$, i.e. on $z = 1$. The value 2 is to be expected, as it is the distance from $z = 1$ of the next nearest (actually the only other) singularity of the equation (at $z = -1$), excluding $z = 1$ itself.

16.7 *The first solution of Bessel's equation for $\nu = 0$ is*

$$J_0(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+1)} \left(\frac{z}{2}\right)^{2n}.$$

Use the derivative method to show that

$$J_0(z)\ln z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\sum_{r=1}^{n} \frac{1}{r}\right) \left(\frac{z}{2}\right)^{2n}$$

is a second solution.

Bessel's equation with $\nu = 0$ reads

$$zy'' + y' + zy = 0.$$

The recurrence relations that gave rise to the first solution, $J_0(z)$, were $(\sigma+1)^2 a_1 = 0$ and $(\sigma+n)^2 a_n + a_{n-2} = 0$ for $n \geq 2$. Thus, in a general form as a function of σ, the solution is given by

$$y(\sigma, z) = a_0 z^{\sigma} \left\{ 1 - \frac{z^2}{(\sigma+2)^2} + \frac{z^4}{(\sigma+2)^2(\sigma+4)^2} - \cdots \right.$$
$$\left. + \frac{(-1)^n z^{2n}}{[(\sigma+2)(\sigma+4)\ldots(\sigma+2n)]^2} + \cdots \right\}.$$

Setting $\sigma = 0$ reproduces the first solution given above.

To obtain a second independent solution, we must differentiate the above expression with respect to σ, before setting σ equal to 0:

$$\frac{\partial y}{\partial \sigma} = \ln z\, J_0(z) + \sum_{n=1}^{\infty} \frac{da_{2n}(\sigma)}{d\sigma} z^{\sigma+2n} \text{ at } \sigma = 0.$$

Now

$$\left.\frac{da_{2n}(\sigma)}{d\sigma}\right|_{\sigma=0} = \frac{d}{d\sigma}\left\{\frac{(-1)^n}{[(\sigma+2)(\sigma+4)\ldots(\sigma+2n)]^2}\right\}_{\sigma=0}$$
$$= \frac{(-1)^n(-2)}{[\ldots]^3}\left(\frac{[\ldots]}{\sigma+2} + \frac{[\ldots]}{\sigma+4} + \cdots + \frac{[\ldots]}{\sigma+2n}\right)$$
$$= \frac{(-2)(-1)^n}{[\ldots]^2} \sum_{r=1}^{n} \frac{1}{\sigma+2r}$$
$$= \frac{-2(-1)^n}{2^{2n}(n!)^2} \sum_{r=1}^{n} \frac{1}{2r}, \qquad \text{at } \sigma = 0.$$

Substituting this result, we obtain the second series as

$$J_0(z)\ln z - \sum_{n=1}^{\infty} \frac{(-1)^n}{(n!)^2}\left(\sum_{r=1}^{n}\frac{1}{r}\right)\left(\frac{z}{2}\right)^{2n}.$$

This is the form given in the question.

16.9 *Find series solutions of the equation $y'' - 2zy' - 2y = 0$. Identify one of the series as $y_1(z) = \exp z^2$ and verify this by direct substitution. By setting $y_2(z) = u(z)y_1(z)$ and solving the resulting equation for $u(z)$, find an explicit form for $y_2(z)$ and deduce that*

$$\int_0^x e^{-v^2}\,dv = e^{-x^2}\sum_{n=0}^{\infty}\frac{n!}{2(2n+1)!}(2x)^{2n+1}.$$

(a) The origin is an ordinary point of the equation and so power series solutions will be possible. Substituting $y(z) = \sum_{n=0}^{\infty} a_n z^n$ gives

$$\sum_{n=0}^{\infty} n(n-1)a_n z^{n-2} - 2\sum_{n=0}^{\infty} n a_n z^n - 2\sum_{n=0}^{\infty} a_n z^n = 0.$$

Equating to zero the coefficient of z^{m-2} yields the recurrence relation

$$a_m = \frac{2m-2}{m(m-1)}\,a_{m-2} = \frac{2}{m}\,a_{m-2}.$$

The solution with $a_0 = 1$ and $a_1 = 0$ is therefore

$$\begin{aligned} y_1(z) &= 1 + \frac{2z^2}{2} + \frac{2^2 z^4}{(2)(4)} + \cdots + \frac{2^n z^{2n}}{2^n\, n!} + \cdots \\ &= \sum_{n=0}^{\infty}\frac{z^{2n}}{n!} = \exp z^2. \end{aligned}$$

Putting this result into the original equation,

$$(4z^2 + 2)\exp z^2 - 2z\,2z\exp z^2 - 2\exp z^2 = 0,$$

shows directly that it is a valid solution.

The solution with $a_0 = 0$ and $a_1 = 1$ takes the form

$$\begin{aligned} y_2(z) &= z + \frac{2z^3}{3} + \frac{2^2 z^5}{(3)(5)} + \cdots + \frac{2^n\, 2^n\, n!\, z^{2n+1}}{(2n+1)!} + \cdots \\ &= \sum_{n=0}^{\infty}\frac{n!\,(2z)^{2n+1}}{2(2n+1)!}. \end{aligned}$$

We now set $y_2(z) = u(z)y_1(z)$ and substitute it into the original equation. As they must, the terms in which u is undifferentiated cancel and leave

$$u'' \exp z^2 + 2u'(2z \exp z^2) - 2zu' \exp z^2 = 0.$$

It follows that

$$\frac{u''}{u'} = -2z \quad \Rightarrow \quad u' = Ae^{-z^2} \quad \Rightarrow \quad u(x) = A \int^x e^{-v^2}\, dv.$$

Hence, setting the two derived forms for a second solution equal to each other, we have

$$\sum_{n=0}^{\infty} \frac{n!\,(2x)^{2n+1}}{2(2n+1)!} = y_2(x) = y_1(x)u(x) = e^{x^2} A \int^x e^{-v^2}\, dv.$$

For arbitrary small x, only the $n = 0$ term in the series is significant and takes the value $2x/2 = x$, whilst the integral is $A \int^x 1\, dv = Ax$. Thus $A = 1$ and the equality

$$\int_0^x e^{-v^2}\, dv = e^{-x^2} \sum_{n=0}^{\infty} \frac{n!\,(2x)^{2n+1}}{2(2n+1)!}$$

holds for all x.

16.11 *Find the general power series solution about $z = 0$ of the equation*

$$z\frac{d^2y}{dz^2} + (2z-3)\frac{dy}{dz} + \frac{4}{z}y = 0.$$

The origin is clearly a singular point of this equation but, since $z(2z-3)/z$ and $4z^2/z^2$ are finite as $z \to 0$, it is a regular singular point. The equation will therefore have at least one Frobenius-type solution of the form $y(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n$.

The indicial equation for the solution can be read off directly from $z^2y'' + z(2z-3)y' + 4y = 0$ as

$$\sigma(\sigma-1) - 3\sigma + 4 = (\sigma-2)^2 = 0 \quad \Rightarrow \quad \sigma = 2 \text{ (repeated root)}.$$

The recurrence relation in terms of a general σ is needed and is provided by setting the coefficient of $z^{m+\sigma}$ equal to 0:

$$(m+\sigma)(m-1+\sigma)a_m + 2(m-1+\sigma)a_{m-1} - 3(m+\sigma)a_m + 4a_m = 0.$$

This relation can be simplified and then applied repeatedly to give a_m in terms of

a_0 and hence an explicit expression for $y(\sigma, z)$:

$$\begin{aligned} a_m &= \frac{-2(m-1+\sigma)}{(m+\sigma)^2-(m+\sigma)-3(m+\sigma)+4}\, a_{m-1} \\ &= \frac{-2(m-1+\sigma)}{(m+\sigma-2)^2}\, a_{m-1} \qquad\qquad \text{for } m \geq 1 \\ &= (-2)^m \frac{(m-1+\sigma)(m-2+\sigma)\ldots\sigma}{(m-2+\sigma)^2(m-3+\sigma)^2\ldots(\sigma-1)^2}\, a_0 \\ &= (-2)^m \frac{(m-1+\sigma)}{(m-2+\sigma)(m-3+\sigma)\ldots\sigma(\sigma-1)^2}\, a_0. \end{aligned}$$

Because of the form of the recurrence relation, we write the $n = 0$ and $n = 1$ terms explicitly:

$$\begin{aligned} y(\sigma, z) = a_0 z^\sigma &- \frac{2\sigma}{(\sigma-1)^2}\, a_0 z^{\sigma+1} \\ &+ z^\sigma \sum_{n=2}^{\infty} \frac{(n-1+\sigma)(-2z)^n}{(n-2+\sigma)(n-3+\sigma)\ldots\sigma(\sigma-1)^2}. \end{aligned}$$

We also need the derivative of this with respect to σ. As always, the derivative consists of two terms, the first of which is $y(\sigma, z)\ln z$. The second, in this case, is

$$\begin{aligned} &\frac{2(\sigma+1)}{(\sigma-1)^3}\, a_0 z^{\sigma+1} + a_0 z^\sigma \sum_{n=2}^{\infty} \frac{(n-1+\sigma)(-2z)^n}{(n-2+\sigma)(n-3+\sigma)\ldots\sigma(\sigma-1)^2} \\ &\quad \times \left[\frac{1}{n-1+\sigma} - \frac{1}{n-2+\sigma} - \frac{1}{n-3+\sigma} - \cdots - \frac{1}{\sigma} - \frac{2}{\sigma-1}\right]. \end{aligned}$$

The factor in square brackets is obtained by considering $a_n(\sigma)$ as the product of factors of the form $(\sigma+\alpha)^\beta$; differentiation of the product with respect to σ produces a sum of terms, each of which is the original product divided by $(\sigma+\alpha)$, for some α, and multiplied by the corresponding β. In the actual expression, β takes the values $+1$ (once), -1 (on $n-1$ occasions) and -2 (once).

To obtain two independent solutions, we finally set $\sigma = 2$ and $a_0 = 1$ obtaining

$$\begin{aligned} y_1(z) &= \sum_{n=0}^{\infty} \frac{(n+1)\,(-2)^n\, z^{n+2}}{n!}, \\ y_2(z) &= y_1(z)\ln z + 6a_0 z^3 \\ &\quad + \sum_{n=2}^{\infty} \frac{(n+1)\,(-2)^n\, z^{n+2}}{n!} \left[\frac{1}{n+1} - \frac{1}{n} - \frac{1}{n-1} - \cdots - \frac{1}{2} - 2\right]. \end{aligned}$$

The general solution is any linear combination of $y_1(z)$ and $y_2(z)$.

16.13 *For the equation $y'' + z^{-3}y = 0$, show that the origin becomes a regular singular point if the independent variable is changed from z to $x = 1/z$. Hence find a series solution of the form $y_1(z) = \sum_0^\infty a_n z^{-n}$. By setting $y_2(z) = u(z)y_1(z)$ and expanding the resulting expression for du/dz in powers of z^{-1}, show that $y_2(z)$ is a second solution with asymptotic form*

$$y_2(z) = c\left[z + \ln z - \tfrac{1}{2} + \mathrm{O}\left(\frac{\ln z}{z}\right)\right],$$

where c is an arbitrary constant.

With the equation in its original form, it is clear that, since $z^2/z^3 \to \infty$ as $z \to 0$, the origin is an irregular singular point. However, if we set $1/z = \xi$ and $y(z) = Y(\xi)$, with

$$\frac{d\xi}{dz} = -\frac{1}{z^2} = -\xi^2 \quad \Rightarrow \quad \frac{d}{dz} = -\xi^2\frac{d}{d\xi},$$

then

$$\begin{aligned} -\xi^2\frac{d}{d\xi}\left(-\xi^2\frac{dY}{d\xi}\right) + \xi^3 Y &= 0, \\ \xi^2\frac{d^2Y}{d\xi^2} + 2\xi\frac{dY}{d\xi} + \xi Y &= 0, \\ Y'' + \frac{2}{\xi}Y' + \frac{1}{\xi}Y &= 0. \end{aligned}$$

By inspection, $\xi = 0$ is a regular singular point of this equation, and its indicial equation is

$$\sigma(\sigma - 1) + 2\sigma = 0 \quad \Rightarrow \quad \sigma = 0,\ -1.$$

We start with the larger root, $\sigma = 0$, as this is 'guaranteed' to give a valid series solution and assume a solution of the form $Y(\xi) = \sum_{n=0}^\infty a_n\xi^n$, leading to

$$\sum_{n=0}^\infty n(n-1)a_n\xi^{n-1} + 2\sum_{n=0}^\infty na_n\xi^{n-1} + \sum_{n=0}^\infty a_n\xi^n = 0.$$

Equating to zero the coefficient of ξ^{m-1} gives the recurrence relation

$$a_m = \frac{-a_{m-1}}{m(m+1)} \quad \Rightarrow \quad a_m = \frac{(-1)^m}{(m+1)\,(m!)^2}\,a_0$$

and the series solution in inverse powers of z,

$$y_1(z) = a_0\sum_{n=0}^\infty \frac{(-1)^n}{(n+1)\,(n!)^2\,z^n}.$$

To find the second solution we set $y_2(z) = f(z)y_1(z)$. As usual (and as intended),

all terms with f undifferentiated vanish when this is substituted in the original equation. What is left is

$$0 = f''(z)y_1(z) + 2f'(z)y_1'(z),$$

which on rearrangement yields

$$\frac{f''}{f'} = -\frac{2y_1'}{y_1}.$$

This equation, although it contains a second derivative, is in fact only a first-order equation (for f'). It can be integrated directly to give

$$\ln f' = -2\ln y_1 + c.$$

After exponentiation, this equation can be written as

$$\begin{aligned}\frac{df}{dz} = \frac{A}{y_1^2(z)} &= \frac{A}{a_0^2}\left(1 - \frac{1}{2\times 1^2 z} + \frac{1}{3\times 2^2 z^2} - \cdots\right)^{-2} \\ &= \frac{A}{a_0^2}\left[1 + \frac{1}{z} + \mathrm{O}\left(\frac{1}{z^2}\right)\right],\end{aligned}$$

where $A = e^c$.

Hence, on integrating a second time, one obtains

$$f(z) = \frac{A}{a_0^2}\left(z + \ln z + \mathrm{O}\left(\frac{1}{z}\right)\right),$$

which in turn implies

$$\begin{aligned}y_2(z) &= \frac{A}{a_0^2}\left[z + \ln z + \mathrm{O}\left(\frac{1}{z}\right)\right] a_0\left(1 - \frac{1}{2z} + \frac{1}{12z^2} - \cdots\right) \\ &= c\left[z + \ln z - \frac{1}{2} + \mathrm{O}\left(\frac{\ln z}{z}\right)\right].\end{aligned}$$

This establishes the asymptotic form of the second solution.

16.15 *The origin is an ordinary point of the Chebyshev equation,*

$$(1-z^2)y'' - zy' + m^2 y = 0,$$

which therefore has series solutions of the form $z^\sigma \sum_0^\infty a_n z^n$ *for* $\sigma = 0$ *and* $\sigma = 1$.

(a) *Find the recurrence relationships for the* a_n *in the two cases and show that there exist polynomial solutions* $T_m(z)$*:*

(i) *for* $\sigma = 0$*, when* m *is an even integer, the polynomial having* $\frac{1}{2}(m+2)$ *terms;*

(ii) *for* $\sigma = 1$*, when* m *is an odd integer, the polynomial having* $\frac{1}{2}(m+1)$ *terms.*

(b) $T_m(z)$ *is normalised so as to have* $T_m(1) = 1$*. Find explicit forms for* $T_m(z)$ *for* $m = 0, 1, 2, 3$.

(c) *Show that the corresponding non-terminating series solutions* $S_m(z)$ *have as their first few terms*

$$S_0(z) = a_0\left(z + \frac{1}{3!}z^3 + \frac{9}{5!}z^5 + \cdots\right),$$

$$S_1(z) = a_0\left(1 - \frac{1}{2!}z^2 - \frac{3}{4!}z^4 - \cdots\right),$$

$$S_2(z) = a_0\left(z - \frac{3}{3!}z^3 - \frac{15}{5!}z^5 - \cdots\right),$$

$$S_3(z) = a_0\left(1 - \frac{9}{2!}z^2 + \frac{45}{4!}z^4 + \cdots\right).$$

(a)(i) If, for $\sigma = 0$, $y(z) = \sum_{n=0}^\infty a_n z^n$ with $a_0 \neq 0$, the condition for the coefficient of z^r in

$$(1-z^2)\sum_{n=0}^\infty n(n-1)a_n z^{n-2} - z\sum_{n=0}^\infty n a_n z^{n-1} + m^2 \sum_{n=0}^\infty a_n z^n$$

to be zero is that

$$(r+2)(r+1)a_{r+2} - r(r-1)a_r - r a_r + m^2 a_r = 0,$$

$$\Rightarrow \quad a_{r+2} = \frac{r^2 - m^2}{(r+2)(r+1)}\, a_r.$$

This relation relates a_{r+2} to a_r and so to a_0 if r is even. For a_{r+2} to vanish, in this case, requires that $r = m$, which must therefore be an even integer. The non-vanishing coefficients will be $a_0, a_2, \ldots, a_m$, i.e. $\frac{1}{2}(m+2)$ of them in all.

(ii) If, for $\sigma = 1$, $y(z) = \sum_{n=0}^\infty a_n z^{n+1}$ with $a_0 \neq 0$, the condition for the coefficient

of z^{r+1} in

$$(1-z^2)\sum_{n=0}^{\infty}(n+1)na_nz^{n-1} - z\sum_{n=0}^{\infty}(n+1)a_nz^n + m^2\sum_{n=0}^{\infty}a_nz^{n+1}$$

to be zero is that

$$(r+3)(r+2)a_{r+2} - (r+1)ra_r - (r+1)a_r + m^2a_r = 0,$$
$$\Rightarrow \quad a_{r+2} = \frac{(r+1)^2 - m^2}{(r+3)(r+2)}\,a_r.$$

This relation relates a_{r+2} to a_r and so to a_0 if r is even. For a_{r+2} to vanish, in this case, requires that $r+1 = m$, which must therefore be an odd integer. The non-vanishing coefficients will be, as before, $a_0, a_2, \ldots, a_{m-1}$, i.e. $\frac{1}{2}(m+1)$ of them in all.

(b) For $m = 0$, $T_0(z) = a_0$. With the given normalisation, $a_0 = 1$ and $T_0(z) = 1$.

For $m = 1$, $T_1(z) = a_0z$. The required normalisation implies that $a_0 = 1$ and so $T_0(z) = z$.

For $m = 2$, we need the recurrence relation in (a)(i). This shows that

$$a_2 = \frac{0^2 - 2^2}{(2)(1)}\,a_0 = -2a_0 \quad \Rightarrow \quad T_2(z) = a_0(1-2z^2).$$

With the given normalisation, $a_0 = -1$ and $T_2(z) = 2z^2 - 1$.

For $m = 3$, we use the recurrence relation in (a)(ii) and obtain

$$a_2 = \frac{1^2 - 3^2}{(3)(2)}\,a_0 = -\frac{4}{3}a_0 \quad \Rightarrow \quad T_3(z) = a_0(z - \frac{4z^3}{3}).$$

For the required normalisation, we must have $a_0 = -\frac{1}{3}$ and consequently that $T_3(z) = 4z^3 - 3z$.

(c) The non-terminating series solutions $S_m(z)$ arise when $\sigma = 0$ but m is an odd integer and when $\sigma = 1$ with m an even integer. We take each in turn and apply the appropriate recurrence relation to generate the coefficients.

(i) $\sigma = 0$, $m = 1$, using the (a)(i) recurrence relation:

$$a_2 = \frac{0-1}{(2)(1)}\,a_0 = -\frac{1}{2!}\,a_0, \quad a_4 = \frac{4-1}{(4)(3)}\,a_2 = -\frac{3}{4!}\,a_0.$$

Hence,

$$S_1(z) = a_0\left(1 - \frac{1}{2!}z^2 - \frac{3}{4!}z^4 - \cdots\right).$$

(ii) $\sigma = 0$, $m = 3$, using the (a)(i) recurrence relation:

$$a_2 = \frac{0-9}{(2)(1)}\,a_0 = -\frac{9}{2!}\,a_0, \quad a_4 = \frac{4-9}{(4)(3)}\,a_2 = \frac{45}{4!}\,a_0.$$

Hence,

$$S_3(z) = a_0 \left(1 - \frac{9}{2!}z^2 + \frac{45}{4!}z^4 + \cdots\right).$$

(iii) $\sigma = 1$, $m = 0$, using the (a)(ii) recurrence relation:

$$a_2 = \frac{1-0}{(3)(2)}\,a_0 = \frac{1}{3!}\,a_0, \quad a_4 = \frac{9-0}{(5)(4)}\,a_2 = \frac{9}{5!}\,a_0.$$

Hence,

$$S_0(z) = a_0 \left(z + \frac{1}{3!}z^3 + \frac{9}{5!}z^5 + \cdots\right).$$

(iv) $\sigma = 1$, $m = 2$, using the (a)(ii) recurrence relation:

$$a_2 = \frac{1-4}{(3)(2)}\,a_0 = -\frac{3}{3!}\,a_0, \quad a_4 = \frac{9-4}{(5)(4)}\,a_2 = -\frac{15}{5!}\,a_0.$$

Hence,

$$S_2(z) = a_0 \left(z - \frac{3}{3!}z^3 - \frac{15}{5!}z^5 - \cdots\right).$$

17

Eigenfunction methods for differential equations

17.1 *By considering $\langle h|h\rangle$, where $h = f + \lambda g$ with λ real, prove that, for two functions f and g,*

$$\langle f|f\rangle\langle g|g\rangle \geq \tfrac{1}{4}[\langle f|g\rangle + \langle g|f\rangle]^2.$$

The function $y(x)$ is real and positive for all x. Its Fourier cosine transform $\tilde{y}_c(k)$ is defined by

$$\tilde{y}_c(k) = \int_{-\infty}^{\infty} y(x)\cos(kx)\,dx,$$

and it is given that $\tilde{y}_c(0) = 1$. Prove that

$$\tilde{y}_c(2k) \geq 2[\tilde{y}_c(k)]^2 - 1.$$

For any $|h\rangle$ we have that $\langle h|h\rangle \geq 0$, with equality only if $|h\rangle = |0\rangle$. Hence, noting that λ is real, we have

$$0 \leq \langle h|h\rangle = \langle f + \lambda g|f + \lambda g\rangle = \langle f|f\rangle + \lambda\langle g|f\rangle + \lambda\langle f|g\rangle + \lambda^2\langle g|g\rangle.$$

This equation, considered as a quadratic inequality in λ, states that the corresponding quadratic equation has no real roots. The condition for this ('$b^2 < 4ac$') is given by

$$[\,\langle g|f\rangle + \langle f|g\rangle\,]^2 \leq 4\langle f|f\rangle\langle g|g\rangle, \qquad (*)$$

from which the stated result follows immediately. Note that $\langle g|f\rangle + \langle f|g\rangle$ is real and its square is therefore non-negative.

The given datum is equivalent to

$$1 = \tilde{y}_c(0) = \int_{-\infty}^{\infty} y(x)\cos(0x)\,dx = \int_{-\infty}^{\infty} y(x)\,dx.$$

Now consider

$$\begin{aligned}\tilde{y}_c(2k) &= \int_{-\infty}^{\infty} y(x)\cos(2kx)\,dx \\ &= 2\int_{-\infty}^{\infty} y(x)\cos^2 kx - \int_{-\infty}^{\infty} y(x)\,dx, \\ \Rightarrow \quad \tilde{y}_c(2k) + 1 &= 2\int_{-\infty}^{\infty} y(x)\cos^2 kx.\end{aligned}$$

In order to use (*), we need to choose for $f(x)$ and $g(x)$ functions whose product will form the integrand defining $\tilde{y}_c(k)$. With this in mind, we take $f(x) = y^{1/2}(x)\cos kx$ and $g(x) = y^{1/2}(x)$; we may do this since $y(x) > 0$ for all x. Making these choices gives

$$\begin{aligned}\left(\int_{-\infty}^{\infty} y\cos kx\,dx + \int_{-\infty}^{\infty} y\cos kx\,dx\right)^2 &\le 4\int_{-\infty}^{\infty} y\cos^2 kx\,dx \int_{-\infty}^{\infty} y\,dx, \\ \left(\int_{-\infty}^{\infty} 2y\cos kx\,dx\right)^2 &\le 4\int_{-\infty}^{\infty} y\cos^2 kx\,dx \times 1, \\ 4\tilde{y}_c^2(k) &\le 4\int_{-\infty}^{\infty} y\cos^2 kx\,dx.\end{aligned}$$

Thus,

$$\tilde{y}_c(2k) + 1 = 2\int_{-\infty}^{\infty} y(x)\cos^2 kx \ge 2[\tilde{y}_c(k)]^2$$

and hence the stated result.

17.3 *Consider the real eigenfunctions $y_n(x)$ of a Sturm–Liouville equation*

$$(py')' + qy + \lambda\rho y = 0, \qquad a \le x \le b,$$

in which $p(x)$, $q(x)$ and $\rho(x)$ are continuously differentiable real functions and $p(x)$ does not change sign in $a \le x \le b$. Take $p(x)$ as positive throughout the interval, if necessary by changing the signs of all eigenvalues. For $a \le x_1 \le x_2 \le b$, establish the identity

$$(\lambda_n - \lambda_m)\int_{x_1}^{x_2} \rho y_n y_m\,dx = \left[\, y_n\, p\, y_m' - y_m\, p\, y_n' \,\right]_{x_1}^{x_2}.$$

Deduce that if $\lambda_n > \lambda_m$ then $y_n(x)$ must change sign between two successive zeros of $y_m(x)$.

[*The reader may find it helpful to illustrate this result by sketching the first few eigenfunctions of the system $y'' + \lambda y = 0$, with $y(0) = y(\pi) = 0$, and the Legendre polynomials $P_n(z)$ for $n = 2, 3, 4, 5$.*]

The function $p(x)$ does not change sign in the interval $a \le x \le b$; we take it as positive, multiplying the equation all through by -1 if necessary. This means that the weight function ρ can still be taken as positive, but that we must consider all possible functions for $q(x)$ and eigenvalues λ of either sign.

We start with the eigenvalue equation for $y_n(x)$, multiply it through by $y_m(x)$ and then integrate from x_1 to x_2. From this result we subtract the same equation with the roles of n and m reversed, as follows. The integration limits are omitted until the explicit integration by parts is carried through:

$$\int y_m(p\,y_n')'\,dx + \int y_m q\,y_n\,dx + \int y_m \lambda_n \rho y_n\,dx = 0,$$

$$\int y_n(p\,y_m')'\,dx + \int y_n q\,y_m\,dx + \int y_n \lambda_m \rho y_m\,dx = 0,$$

$$\int \left[\, y_m(p\,y_n')' - y_n(p\,y_m')' \,\right] dx + (\lambda_n - \lambda_m) \int y_m \rho y_n\,dx = 0,$$

$$\left[\, y_m p\,y_n' \,\right]_{x_1}^{x_2} - \int y_m' p\,y_n'\,dx - \left[\, y_n p\,y_m' \,\right]_{x_1}^{x_2}$$
$$+ \int y_n' p\,y_m'\,dx + (\lambda_n - \lambda_m) \int y_m \rho y_n\,dx = 0.$$

Hence

$$(\lambda_n - \lambda_m) \int y_m \rho y_n\,dx = \left[\, y_n p\,y_m' - y_m p\,y_n' \,\right]_{x_1}^{x_2}. \qquad (*)$$

Now, in this general result, take x_1 and x_2 as successive zeros of $y_m(x)$, where m is determined by $\lambda_n > \lambda_m$ (after the signs have been changed, if that was necessary). Clearly the sign of $y_m(x)$ does not change in this interval; let it be α. It follows that the sign of $y_m'(x_1)$ is also α, whilst that of $y_m'(x_2)$ is $-\alpha$. In addition, the second term on the RHS of $(*)$ vanishes at both limits, as $y_m(x_1) = y_m(x_2) = 0$.

Let us now *suppose* that the sign of $y_n(x)$ does not change in this same interval and is always β. Then the sign of the expression on the LHS of $(*)$ is $(+1)(\alpha)(+1)\beta = \alpha\beta$. The first $(+1)$ appears because $\lambda_n > \lambda_m$.

The signs of the upper- and lower-limit contributions of the remaining term on the RHS of $(*)$ are $\beta(+1)(-\alpha)$ and $(-1)\beta(+1)\alpha$, respectively, the additional factor of (-1) in the second product arising from the fact that the contribution comes from a lower limit. The contributions at both limits have the same sign, $-\alpha\beta$, and so the sign of the total RHS must also be $-\alpha\beta$.

This contradicts, however, the sign of $+\alpha\beta$ found for the LHS. It follows that it was wrong to suppose that the sign of $y_n(x)$ does not change in the interval; in other words, a zero of $y_n(x)$ does appear between every pair of zeros of $y_m(x)$.

17.5 *Use the properties of Legendre polynomials to carry out the following exercises.*

(a) *Find the solution of* $(1-x^2)y''-2xy'+by=f(x)$ *that is valid in the range* $-1\le x\le 1$ *and finite at* $x=0$, *in terms of Legendre polynomials.*
(b) *Find the explicit solution if* $b=14$ *and* $f(x)=5x^3$. *Verify it by direct substitution.*

[*Explicit forms for the Legendre polynomials can be found in any textbook. In* Mathematical Methods for Physics and Engineering, 3rd edition, *they are given in Subsection 18.1.1.*]

(a) The LHS of the given equation is the same as that of Legendre's equation and so we substitute $y(x)=\sum_{n=0}^{\infty}a_nP_n(x)$ and use the fact that $(1-x^2)P_n''-2xP_n'=-n(n+1)P_n$. This results in

$$\sum_{n=0}^{\infty}a_n[b-n(n+1)]P_n=f(x).$$

Now, using the mutual orthogonality and normalisation of the $P_n(x)$, we multiply both sides by $P_m(x)$ and integrate over x:

$$\sum_{n=0}^{\infty}a_n[b-n(n+1)]\,\delta_{mn}\frac{2}{2m+1}=\int_{-1}^{1}f(z)P_m(z)\,dz,$$

$$\Rightarrow\quad a_m=\frac{2m+1}{2[b-m(m+1)]}\int_{-1}^{1}f(z)P_m(z)\,dz.$$

This gives the coefficients in the solution $y(x)$.

(b) We now express $f(x)$ in terms of Legendre polynomials,

$$f(x)=5x^3=2[\tfrac{1}{2}(5x^3-3x)]+3[x]=2P_3(x)+3P_1(x),$$

and conclude that, because of the mutual orthogonality of the Legendre polynomials, only a_3 and a_1 in the series solution will be non-zero. To find them we need to evaluate

$$\int_{-1}^{1}f(z)P_3(z)\,dz=2\,\frac{2}{2(3)+1}=\frac{4}{7};$$

similarly, $\int_{-1}^{1}f(z)P_1(z)\,dz=3\times(2/3)=2$.

Inserting these values gives

$$a_3=\frac{7}{2(14-12)}\,\frac{4}{7}=1\text{ and }a_1=\frac{3}{2(14-2)}\,2=\frac{1}{4}.$$

Thus the solution is

$$y(x) = \frac{1}{4}P_1(x) + P_3(x) = \frac{1}{4}x + \frac{1}{2}(5x^3 - 3x) = \frac{5(2x^3 - x)}{4}.$$

Check:

$$(1 - x^2)\frac{60x}{4} - 2x\frac{30x^2 - 5}{4} + \frac{140x^3 - 70x}{4} = 5x^3,$$
$$\Rightarrow \quad 60x - 60x^3 - 60x^3 + 10x + 140x^3 - 70x = 20x^3,$$

which is satisfied.

17.7 *Consider the set of functions, $\{f(x)\}$, of the real variable x defined in the interval $-\infty < x < \infty$, that $\to 0$ at least as quickly as x^{-1}, as $x \to \pm\infty$. For unit weight function, determine whether each of the following linear operators is Hermitian when acting upon $\{f(x)\}$:*

$$\text{(a)}\ \frac{d}{dx} + x; \quad \text{(b)}\ -i\frac{d}{dx} + x^2; \quad \text{(c)}\ ix\frac{d}{dx}; \quad \text{(d)}\ i\frac{d^3}{dx^3}.$$

For an operator $\mathcal{L}$ to be Hermitian over the given range with respect to a unit weight function, the equation

$$\int_{-\infty}^{\infty} f^*(x)[\mathcal{L}g(x)]\,dx = \left\{\int_{-\infty}^{\infty} g^*(x)[\mathcal{L}f(x)]\,dx\right\}^* \qquad (*)$$

must be satisfied for general functions f and g.

(a) For $\mathcal{L} = \dfrac{d}{dx} + x$, the LHS of $(*)$ is

$$\int_{-\infty}^{\infty} f^*(x)\left(\frac{dg}{dx} + xg\right)dx = \left[f^*g\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty}\frac{df^*}{dx}g\,dx + \int_{-\infty}^{\infty} f^*xg\,dx$$
$$= 0 - \int_{-\infty}^{\infty}\frac{df^*}{dx}g\,dx + \int_{-\infty}^{\infty} f^*xg\,dx.$$

The RHS of $(*)$ is

$$\int_{-\infty}^{\infty}\left\{g^*(x)\left(\frac{df}{dx} + xf\right)dx\right\}^* = \left\{\int_{-\infty}^{\infty} g^*\frac{df}{dx}\,dx\right\}^* + \left\{\int_{-\infty}^{\infty} g^*xf\,dx\right\}^*$$
$$= \int_{-\infty}^{\infty} g\frac{df^*}{dx}\,dx + \int_{-\infty}^{\infty} gxf^*\,dx.$$

Since the sign of the first term differs in the two expressions, the LHS $\neq$ RHS and $\mathcal{L}$ is *not* Hermitian. It will also be apparent that purely multiplicative terms in the operator, such as x or x^2, will always be Hermitian; thus we can ignore the x^2 term in part (b).

(b) As explained above, we need only consider

$$\int_{-\infty}^{\infty} f^*(x)\left(-i\frac{dg}{dx}\right)\,dx = \left[-if^*g\right]_{-\infty}^{\infty} + i\int_{-\infty}^{\infty}\frac{df^*}{dx}g\,dx$$
$$= 0 + i\int_{-\infty}^{\infty}\frac{df^*}{dx}g\,dx$$

and

$$\int_{-\infty}^{\infty}\left\{g^*(x)\left(-i\frac{df}{dx}\right)\,dx\right\}^* = i\int_{-\infty}^{\infty} g\frac{df^*}{dx}\,dx.$$

These are equal, and so $\mathcal{L} = -i\dfrac{d}{dx}$ is Hermitian, as is $\mathcal{L} = -i\dfrac{d}{dx} + x^2$.

(c) For $\mathcal{L} = ix\dfrac{d}{dx}$, the LHS of $(*)$ is

$$\int_{-\infty}^{\infty} f^*(x)\left(ix\frac{dg}{dx}\right)\,dx = \left[ixf^*g\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty} x\frac{df^*}{dx}g\,dx - i\int_{-\infty}^{\infty} f^*g\,dx$$
$$= 0 - i\int_{-\infty}^{\infty} x\frac{df^*}{dx}g\,dx - i\int_{-\infty}^{\infty} f^*g\,dx.$$

The RHS of $(*)$ is given by

$$\int_{-\infty}^{\infty}\left\{g^*(x)ix\left(\frac{df}{dx}\right)\,dx\right\}^* = -i\int_{-\infty}^{\infty} gx\frac{df^*}{dx}\,dx.$$

Since, in general, $-i\int_{-\infty}^{\infty} fg^*\,dx \neq 0$, the two sides are not equal; therefore $\mathcal{L}$ is not Hermitian.

(d) Since $\mathcal{L} = i\dfrac{d^3}{dx^3}$ is the cube of the operator $-i\dfrac{d}{dx}$, which was shown in part (b) to be Hermitian, it is expected that $\mathcal{L}$ is Hermitian. This can be verified directly as follows.

The LHS of $(*)$ is given by

$$i\int_{-\infty}^{\infty} f^*\frac{d^3g}{dx^3}\,dx = \left[if^*\frac{d^2g}{dx^2}\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\frac{df^*}{dx}\,\frac{d^2g}{dx^2}\,dx$$
$$= 0 - i\left[\frac{df^*}{dx}\,\frac{dg}{dx}\right]_{-\infty}^{\infty} + i\int_{-\infty}^{\infty}\frac{d^2f^*}{dx^2}\,\frac{dg}{dx}\,dx$$
$$= 0 + i\left[\frac{d^2f^*}{dx^2}\,g\right]_{-\infty}^{\infty} - i\int_{-\infty}^{\infty}\frac{d^3f^*}{dx^3}\,g\,dx$$
$$= 0 + \left\{\int_{-\infty}^{\infty} ig^*\frac{d^3f}{dx^3}\,dx\right\}^* = \text{ RHS of } (*).$$

Thus $\mathcal{L}$ is confirmed as Hermitian.

17.9 *Find an eigenfunction expansion for the solution with boundary conditions* $y(0) = y(\pi) = 0$ *of the inhomogeneous equation*

$$\frac{d^2y}{dx^2} + \kappa y = f(x),$$

where κ *is a constant and*

$$f(x) = \begin{cases} x & 0 \le x \le \pi/2, \\ \pi - x & \pi/2 < x \le \pi. \end{cases}$$

The eigenfunctions of the operator $\mathcal{L} = \dfrac{d^2}{dx^2} + \kappa$ are obviously

$$y_n(x) = A_n \sin nx + B_n \cos nx,$$

with corresponding eigenvalues $\lambda_n = n^2 - \kappa$.

The boundary conditions, $y(0) = y(\pi) = 0$, require that n is a positive integer and that $B_n = 0$, i.e.

$$y_n(x) = A_n \sin nx = \sqrt{\frac{2}{\pi}} \sin nx,$$

where A_n (for $n \ge 1$) has been chosen so that the eigenfunctions are normalised over the interval $x = 0$ to $x = \pi$. Since $\mathcal{L}$ is Hermitian on the range $0 \le x \le \pi$, the eigenfunctions are also mutually orthogonal, and so the $y_n(x)$ form an orthonormal set.

If the required solution is $y(x) = \sum_n a_n y_n(x)$, then direct substitution yields the result

$$\sum_{n=1}^{\infty} (\kappa - n^2) a_n y_n(x) = f(x).$$

Following the usual procedure for analysis using sets of orthonormal functions, this implies that

$$a_m = \frac{1}{\kappa - m^2} \int_0^{\pi} f(z) y_m(z)\, dz$$

and, consequently, that

$$y(x) = \sum_{n=1}^{\infty} \sqrt{\frac{2}{\pi}} \frac{\sin nx}{\kappa - n^2} \sqrt{\frac{2}{\pi}} \int_0^{\pi} f(z) \sin(nz)\, dz.$$

It only remains to evaluate

$$\begin{aligned}
I_n &= \int_0^{\pi} \sin(nx) f(x)\, dx \\
&= \int_0^{\pi/2} x \sin nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx\, dx \\
&= \left[\frac{-x\cos nx}{n}\right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos nx}{n}\, dx \\
&\quad + \left[\frac{-(\pi - x)\cos nx}{n}\right]_{\pi/2}^{\pi} + \int_{\pi/2}^{\pi} \frac{(-1)\cos nx}{n}\, dx \\
&= -\frac{\pi}{2}\frac{\cos(n\pi/2)}{n}(1-1) + \left[\frac{\sin nx}{n^2}\right]_0^{\pi/2} - \left[\frac{\sin nx}{n^2}\right]_{\pi/2}^{\pi} \\
&= 0 + \frac{(-1)^{(n-1)/2}}{n^2}(1+1) \text{ for odd } n \text{ and } = 0 \text{ for even } n.
\end{aligned}$$

Thus,

$$y(x) = \frac{4}{\pi} \sum_{n \text{ odd}} \frac{(-1)^{(n-1)/2}}{n^2(\kappa - n^2)} \sin nx$$

is the required solution.

17.11 *The differential operator $\mathcal{L}$ is defined by*

$$\mathcal{L}y = -\frac{d}{dx}\left(e^x \frac{dy}{dx}\right) - \tfrac{1}{4}e^x y.$$

Determine the eigenvalues λ_n of the problem

$$\mathcal{L}y_n = \lambda_n e^x y_n \qquad 0 < x < 1,$$

with boundary conditions

$$y(0) = 0, \qquad \frac{dy}{dx} + \tfrac{1}{2}y = 0 \quad \textit{at} \quad x = 1.$$

(a) *Find the corresponding unnormalised y_n, and also a weight function $\rho(x)$ with respect to which the y_n are orthogonal. Hence, select a suitable normalisation for the y_n.*

(b) *By making an eigenfunction expansion, solve the equation*

$$\mathcal{L}y = -e^{x/2}, \qquad 0 < x < 1,$$

subject to the same boundary conditions as previously.

When written out explicitly, the eigenvalue equation is

$$-\frac{d}{dx}\left(e^x\frac{dy}{dx}\right) - \tfrac{1}{4}e^x y = \lambda e^x y, \qquad (*)$$

or, on differentiating out the product,

$$e^x y'' + e^x y' + (\lambda + \tfrac{1}{4})e^x y = 0.$$

The auxiliary equation is

$$m^2 + m + (\lambda + \tfrac{1}{4}) = 0 \quad \Rightarrow \quad m = -\tfrac{1}{2} \pm i\sqrt{\lambda}.$$

The general solution is thus given by

$$y(x) = Ae^{-x/2}\cos\sqrt{\lambda}x + Be^{-x/2}\sin\sqrt{\lambda}x,$$

with the condition $y(0) = 0$ implying that $A = 0$. The other boundary condition requires that, at $x = 1$,

$$-\tfrac{1}{2}Be^{-x/2}\sin\sqrt{\lambda}x + \sqrt{\lambda}Be^{-x/2}\cos\sqrt{\lambda}x + \tfrac{1}{2}Be^{-x/2}\sin\sqrt{\lambda}x = 0,$$

i.e. that $\cos\sqrt{\lambda} = 0$ and hence that $\lambda = (n + \frac{1}{2})^2\pi^2$ for non-negative integral n.

(a) The unnormalised eigenfunctions are

$$y_n(x) = B_n e^{-x/2}\sin(n + \tfrac{1}{2})\pi x$$

and $(*)$ is in Sturm–Liouville form. However, although $y_n(0) = 0$, the values at the upper limit in $\left[\, y'_m p\, y_n \,\right]_0^1$ are $y_n(1) = B_n e^{-1/2}(-1)^n$, $p(1) = e^1$ and $y'_m(1) = -\frac{1}{2}B_m e^{-1/2}(-1)^m$. Consequently, $\left[\, y'_m p\, y_n \,\right]_0^1 \neq 0$ and S–L theory cannot be applied. We therefore have to find a suitable weight function $\rho(x)$ by inspection. Given the general form of the eigenfunctions, ρ has to be taken as e^x, with the orthonormality integral taking the form

$$\begin{aligned} I_{nm} &= \int_0^1 \rho(x) y_n(x) y_m^*(x)\, dx \\ &= B_n B_m \int_0^1 e^x e^{-x/2}\sin[\,(n + \tfrac{1}{2})\pi x\,]e^{-x/2}\sin[\,(m + \tfrac{1}{2})\pi x\,]\, dx \\ &= \begin{cases} 0 & \text{for } m \neq n, \\ \frac{1}{2}B_n B_m & \text{for } m = n. \end{cases} \end{aligned}$$

It is clear that a suitable normalisation is $B_n = \sqrt{2}$ for all n.

(b) We write the solution as $y(x) = \sum_{n=0}^{\infty} a_n y_n(x)$, giving as the equation to be

solved

$$\begin{aligned}
-e^{x/2} = \mathcal{L}y &= \mathcal{L}\sum_{n=0}^{\infty} a_n y_n(x) \\
&= \sum_{n=0}^{\infty} a_n[\lambda_n \rho(x) y_n(x)] \\
&= \sum_{n=0}^{\infty} a_n (n+\tfrac{1}{2})^2 \pi^2 e^x \sqrt{2} e^{-x/2} \sin[(n+\tfrac{1}{2})\pi x] \\
\Rightarrow \quad -1 &= \sum_{n=0}^{\infty} a_n (n+\tfrac{1}{2})^2 \pi^2 \sqrt{2} \sin[(n+\tfrac{1}{2})\pi x].
\end{aligned}$$

After multiplying both sides of this equation by $\sin(m+\frac{1}{2})\pi x$ and integrating from 0 to 1, we obtain

$$\begin{aligned}
a_m \int_0^1 \sin^2(m+\tfrac{1}{2})\pi x\, dx &= \frac{-1}{(m+\frac{1}{2})^2\pi^2\sqrt{2}} \int_0^1 \sin(m+\tfrac{1}{2})\pi x\, dx, \\
\frac{a_m}{2} &= \frac{-1}{(m+\frac{1}{2})^2\pi^2\sqrt{2}} \int_0^1 \sin(m+\tfrac{1}{2})\pi x\, dx \\
&= \frac{1}{(m+\frac{1}{2})^2\pi^2\sqrt{2}} \left[\frac{\cos(m+\frac{1}{2})\pi x}{(m+\frac{1}{2})\pi}\right]_0^1, \\
a_m &= -\frac{\sqrt{2}}{(m+\frac{1}{2})^3\pi^3}.
\end{aligned}$$

Substituting this result into the assumed expansion, and recalling that $B_n = \sqrt{2}$, gives as the solution

$$y(x) = -\sum_{n=0}^{\infty} \frac{2}{(n+\frac{1}{2})^3\pi^3} e^{-x/2} \sin(n+\tfrac{1}{2})\pi x.$$

17.13 *By substituting $x = \exp t$, find the normalised eigenfunctions $y_n(x)$ and the eigenvalues λ_n of the operator $\mathcal{L}$ defined by*

$$\mathcal{L}y = x^2 y'' + 2xy' + \tfrac{1}{4}y, \qquad 1 \le x \le e,$$

with $y(1) = y(e) = 0$. Find, as a series $\sum a_n y_n(x)$, the solution of $\mathcal{L}y = x^{-1/2}$.

Putting $x = e^t$ and $y(x) = u(t)$ with $u(0) = u(1) = 0$,

$$\frac{dx}{dt} = e^t \quad \Rightarrow \quad \frac{d}{dx} = e^{-t}\frac{d}{dt}$$

and the eigenvalue equation becomes

$$e^{2t}e^{-t}\frac{d}{dt}\left(e^{-t}\frac{du}{dt}\right)+2e^{t}e^{-t}\frac{du}{dt}+\frac{1}{4}u=\lambda u,$$
$$\frac{d^2u}{dt^2}-\frac{du}{dt}+2\frac{du}{dt}+\left(\frac{1}{4}-\lambda\right)=0.$$

The auxiliary equation to this constant-coefficient linear equation for u is

$$m^2+m+(\tfrac{1}{4}-\lambda)=0 \quad\Rightarrow\quad m=-\tfrac{1}{2}\pm\sqrt{\lambda},$$

leading to

$$u(t)=e^{-t/2}\left(Ae^{\sqrt{\lambda}\,t}+Be^{-\sqrt{\lambda}\,t}\right).$$

In view of the requirement that u vanishes at two different values of t (one of which is $t=0$), we need $\lambda<0$ and $u(t)$ to take the form

$$u(t)=Ae^{-t/2}\sin\sqrt{-\lambda}\,t \text{ with } \sqrt{-\lambda}\,1=n\pi, \text{ i.e. } \lambda=-n^2\pi^2,$$

where n is an integer. Thus

$$u_n(t)=A_ne^{-t/2}\sin n\pi t \quad \text{or, in terms of } x, \quad y_n(x)=\frac{A_n}{\sqrt{x}}\sin(n\pi\ln x).$$

Normalisation requires that

$$1=\int_1^e\frac{A_n^2}{x}\sin^2(n\pi\ln x)\,dx=\int_0^1 A_n^2\sin^2(n\pi t)\,dt=\tfrac{1}{2}A_n^2 \quad\Rightarrow\quad A_n=\sqrt{2}.$$

To solve

$$\mathcal{L}y=x^2y''+2xy'+\tfrac{1}{4}y=\frac{1}{\sqrt{x}},$$

we set $y(x)=\sum_{n=0}^{\infty}a_ny_n(x)$. Then the equation becomes

$$\mathcal{L}y=\sum_{n=0}^{\infty}a_n(-n^2\pi^2)y_n(x)=\sum_{n=0}^{\infty}-n^2\pi^2a_n\frac{\sqrt{2}}{\sqrt{x}}\sin(n\pi\ln x)=\frac{1}{\sqrt{x}}.$$

Multiplying through by $y_m(x)$ and integrating, as with ordinary Fourier series,

$$\int_1^e\frac{2a_n}{x}\sin(n\pi\ln x)\,\sin(m\pi\ln x)\,dx=-\frac{1}{n^2\pi^2}\int_1^e\frac{\sqrt{2}\sin(m\pi\ln x)}{x}\,dx.$$

The LHS of this equation is the normalisation integral just considered and has

the value $a_m\delta_{mn}$. Thus

$$\begin{aligned} a_m &= -\frac{\sqrt{2}}{m^2\pi^2}\int_1^e \frac{\sin(m\pi\ln x)}{x}\,dx \\ &= -\frac{\sqrt{2}}{m^2\pi^2}\left[\frac{-\cos(m\pi\ln x)}{m\pi}\right]_1^e \\ &= -\frac{\sqrt{2}}{m^3\pi^3}[\,1-(-1)^m\,] \\ &= \begin{cases} -\dfrac{2\sqrt{2}}{m^3\pi^3} & \text{for } m \text{ odd,} \\ 0 & \text{for } m \text{ even.} \end{cases} \end{aligned}$$

The explicit solution is therefore

$$y(x) = -\frac{4}{\pi^3}\sum_{p=0}^{\infty}\frac{\sin[(2p+1)\pi\ln x]}{(2p+1)^3\sqrt{x}}.$$

17.15 *In the quantum mechanical study of the scattering of a particle by a potential, a Born-approximation solution can be obtained in terms of a function $y(\mathbf{r})$ that satisfies an equation of the form*

$$(-\nabla^2 - K^2)y(\mathbf{r}) = F(\mathbf{r}).$$

Assuming that $y_{\mathbf{k}}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}\cdot\mathbf{r})$ is a suitably normalised eigenfunction of $-\nabla^2$ corresponding to eigenvalue k^2, find a suitable Green's function $G_K(\mathbf{r},\mathbf{r}')$. By taking the direction of the vector $\mathbf{r}-\mathbf{r}'$ as the polar axis for a $\mathbf{k}$-space integration, show that $G_K(\mathbf{r},\mathbf{r}')$ can be reduced to

$$\frac{1}{4\pi^2|\mathbf{r}-\mathbf{r}'|}\int_{-\infty}^{\infty}\frac{w\sin w}{w^2-w_0^2}\,dw,$$

where $w_0 = K|\mathbf{r}-\mathbf{r}'|$.

[*This integral can be evaluated using contour integration and gives the Green's function explicitly as $(4\pi|\mathbf{r}-\mathbf{r}'|)^{-1}\exp(iK|\mathbf{r}-\mathbf{r}'|)$.*]

Given that $y_{\mathbf{k}}(\mathbf{r}) = (2\pi)^{-3/2}\exp(i\mathbf{k}\cdot\mathbf{r})$ satisfies

$$-\nabla^2 y_{\mathbf{k}}(\mathbf{r}) = k^2 y_{\mathbf{k}}(\mathbf{r}),$$

it follows that

$$(-\nabla^2 - K^2)y_{\mathbf{k}}(\mathbf{r}) = (k^2 - K^2)y_{\mathbf{k}}(\mathbf{r}).$$

Thus the same functions are suitable eigenfunctions for the extended operator, but with different eigenvalues.

Its Green's function is therefore (from the general expression for Green's functions in terms of eigenfunctions)

$$\begin{aligned} G_K(\mathbf{r},\mathbf{r}') &= \int \frac{1}{\lambda} y_{\mathbf{k}}(\mathbf{r}) y^*_{\mathbf{k}}(\mathbf{r}')\, d\mathbf{k} \\ &= \frac{1}{(2\pi)^3} \int \frac{\exp(i\mathbf{k}\cdot\mathbf{r})\,\exp(-i\mathbf{k}\cdot\mathbf{r}')}{k^2 - K^2}\, d\mathbf{k}. \end{aligned}$$

We carry out the three-dimensional integration in $\mathbf{k}$-space using the direction $\mathbf{r}-\mathbf{r}'$ as the polar axis (and denote $\mathbf{r}-\mathbf{r}'$ by $\mathbf{R}$). The azimuthal integral is immediate. The remaining two-dimensional integration is as follows:

$$\begin{aligned} G_K(\mathbf{r},\mathbf{r}') &= \frac{1}{(2\pi)^3} \int_0^\infty \int_0^\pi \frac{\exp(i\mathbf{k}\cdot\mathbf{R})}{k^2 - K^2}\, 2\pi k^2 \sin\theta_k\, d\theta_k\, dk \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \int_0^\pi \frac{\exp(ikR\cos\theta_k)}{k^2 - K^2}\, k^2 \sin\theta_k\, d\theta_k\, dk \\ &= \frac{1}{(2\pi)^2} \int_0^\infty \frac{\exp(ikR) - \exp(-ikR)}{ikR(k^2 - K^2)}\, k^2\, dk \\ &= \frac{1}{2\pi^2 R} \int_0^\infty \frac{k \sin kR}{k^2 - K^2}\, dk \\ &= \frac{1}{2\pi^2 R} \int_0^\infty \frac{w \sin w}{w^2 - w_0^2}\, dw, \quad \text{where } w = kR \text{ and } w_0 = kR, \\ &= \frac{1}{4\pi^2 R} \int_{-\infty}^\infty \frac{w \sin w}{w^2 - w_0^2}\, dw. \end{aligned}$$

Here, the final line is justified by noting that the integrand is an even function of the integration variable w.

18

Special functions

18.1 *Use the explicit expressions*

$$Y_0^0 = \sqrt{\tfrac{1}{4\pi}}, \qquad Y_1^0 = \sqrt{\tfrac{3}{4\pi}}\cos\theta,$$

$$Y_1^{\pm 1} = \mp\sqrt{\tfrac{3}{8\pi}}\sin\theta\exp(\pm i\phi), \qquad Y_2^0 = \sqrt{\tfrac{5}{16\pi}}(3\cos^2\theta - 1),$$

$$Y_2^{\pm 1} = \mp\sqrt{\tfrac{15}{8\pi}}\sin\theta\cos\theta\exp(\pm i\phi), \qquad Y_2^{\pm 2} = \sqrt{\tfrac{15}{32\pi}}\sin^2\theta\exp(\pm 2i\phi),$$

to verify for $\ell = 0, 1, 2$ *that*

$$\sum_{m=-\ell}^{\ell} |Y_\ell^m(\theta,\phi)|^2 = \frac{2\ell+1}{4\pi}$$

and so is independent of the values of θ *and* ϕ*. This is true for any* ℓ*, but a general proof is more involved. This result helps to reconcile intuition with the apparently arbitrary choice of polar axis in a general quantum mechanical system.*

We first note that, since every term is the square of a modulus, factors of the form $\exp(\pm mi\phi)$ never appear in the sums. For each value of ℓ, let us denote the sum by S_ℓ. For $\ell = 0$ and $\ell = 1$, we have

$$S_0 = \sum_{m=0}^{0} |Y_0^m(\theta,\phi)|^2 = \frac{1}{4\pi},$$

$$S_1 = \sum_{m=-1}^{1} |Y_1^m(\theta,\phi)|^2 = \frac{3}{4\pi}\cos^2\theta + 2\,\frac{3}{8\pi}\sin^2\theta = \frac{3}{4\pi}.$$

For $\ell = 2$, the summation is more complicated but reads

$$\begin{aligned}
S_2 &= \sum_{m=-2}^{2} |Y_2^m(\theta, \phi)|^2 \\
&= \frac{5}{16\pi}(3\cos^2\theta - 1)^2 + 2\,\frac{15}{8\pi}\sin^2\theta\cos^2\theta + 2\,\frac{15}{32\pi}\sin^4\theta \\
&= \frac{5}{16\pi}(9\cos^4\theta - 6\cos^2\theta + 1 + 12\sin^2\theta\cos^2\theta + 3\sin^4\theta) \\
&= \frac{5}{16\pi}[6\cos^4\theta - 6\cos^2\theta + 1 + 6\sin^2\theta\cos^2\theta + 3(\cos^2\theta + \sin^2\theta)^2] \\
&= \frac{5}{16\pi}[6\cos^2\theta(-\sin^2\theta) + 1 + 6\sin^2\theta\cos^2\theta + 3] = \frac{5}{4\pi}.
\end{aligned}$$

All three sums are independent of θ and ϕ, and are given by the general formula $(2\ell + 1)/4\pi$. It will, no doubt, be noted that $2\ell + 1$ is the number of terms in S_ℓ, i.e. the number of m values, and that 4π is the total solid angle subtended at the origin by all space.

18.3 *Use the generating function for the Legendre polynomials $P_n(x)$ to show that*

$$\int_0^1 P_{2n+1}(x)\,dx = (-1)^n \frac{(2n)!}{2^{2n+1}n!(n+1)!}$$

and that, except for the case $n = 0$,

$$\int_0^1 P_{2n}(x)\,dx = 0.$$

Denote $\int_0^1 P_n(x)\,dx$ by a_n. From the generating function for the Legendre polynomials, we have

$$\frac{1}{(1 - 2xh + h^2)^{1/2}} = \sum_{n=0}^{\infty} P_n(x)h^n.$$

Integrating this definition with respect to x gives

$$\begin{aligned}
\int_0^1 \frac{dx}{(1 - 2xh + h^2)^{1/2}} &= \sum_{n=0}^{\infty}\left(\int_0^1 P_n(x)\,dx\right)h^n, \\
\left[\frac{-(1 - 2xh + h^2)^{1/2}}{h}\right]_0^1 &= \sum_{n=0}^{\infty} a_n h^n, \\
\frac{1}{h}[(1 + h^2)^{1/2} - 1 + h] &= \sum_{n=0}^{\infty} a_n h^n.
\end{aligned}$$

Now expanding $(1+h^2)^{1/2}$ using the binomial theorem yields

$$\sum_{n=0}^{\infty} a_n h^n = \frac{1}{h}\left[1 + \sum_{m=1}^{\infty} {}^{1/2}C_m h^{2m} - 1 + h\right] = 1 + \sum_{m=1}^{\infty} {}^{1/2}C_m h^{2m-1}.$$

Comparison of the coefficients of h^n on the two sides of the equation shows that all a_{2r} are zero except for $a_0 = 1$. For $n = 2r+1$ we need $2m-1 = n = 2r+1$, i.e. $m = r+1$, and the value of a_{2r+1} is ${}^{1/2}C_{r+1}$.

Now, the binomial coefficient ${}^{1/2}C_m$ can be written as

$$\begin{aligned}
{}^{1/2}C_m &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-m+1)}{m!}, \\
&= \frac{1(1-2)(1-4)\cdots(1-2m+2)}{2^m\, m!} \\
&= (-1)^{m-1}\frac{(1)(1)(3)\cdots(2m-3)}{2^m\, m!} \\
&= (-1)^{m-1}\frac{(2m-2)!}{2^m\, m!\, 2^{m-1}\,(m-1)!} \\
&= (-1)^{m-1}\frac{(2m-2)!}{2^{2m-1}\, m!\,(m-1)!}.
\end{aligned}$$

Thus, setting $m = r+1$ gives the value of the integral a_{2r+1} as

$$a_{2r+1} = {}^{1/2}C_{r+1} = (-1)^r \frac{(2r)!}{2^{2r+1}\,(r+1)!\,r!},$$

as stated in the question.

18.5 *The Hermite polynomials $H_n(x)$ may be defined by*

$$\Phi(x,h) = \exp(2xh - h^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) h^n.$$

Show that

$$\frac{\partial^2 \Phi}{\partial x^2} - 2x\frac{\partial \Phi}{\partial x} + 2h\frac{\partial \Phi}{\partial h} = 0,$$

and hence that the $H_n(x)$ satisfy the Hermite equation,

$$y'' - 2xy' + 2ny = 0,$$

where n is an integer ≥ 0.

Use Φ to prove that

(a) $H_n'(x) = 2nH_{n-1}(x)$,
(b) $H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$.

With

$$\Phi(x,h) = \exp(2xh - h^2) = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) h^n,$$

we have

$$\frac{\partial \Phi}{\partial x} = 2h\Phi, \quad \frac{\partial \Phi}{\partial h} = (2x - 2h)\Phi, \quad \frac{\partial^2 \Phi}{\partial x^2} = 4h^2\Phi.$$

It then follows that

$$\frac{\partial^2 \Phi}{\partial x^2} - 2x\frac{\partial \Phi}{\partial x} + 2h\frac{\partial \Phi}{\partial h} = (4h^2 - 4hx + 4hx - 4h^2)\Phi = 0.$$

Substituting the series form into this result gives

$$\sum_{n=0}^{\infty} \left(\frac{1}{n!} H_n'' - \frac{2x}{n!} H_n' + \frac{2n}{n!} \right) h^n = 0,$$
$$\Rightarrow \quad H_n'' - 2xH_n' + 2nH_n = 0.$$

This is the equation satisfied by $H_n(x)$, as stated in the question.

(a) From the first relationship derived above, we have that

$$\frac{\partial \Phi}{\partial x} = 2h\Phi,$$
$$\sum_{n=0}^{\infty} \frac{1}{n!} H_n'(x) h^n = 2h \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x) h^n,$$
$$\Rightarrow \quad \frac{1}{m!} H_m' = \frac{2}{(m-1)!} H_{m-1}, \text{ from the coefficients of } h^m.$$

Hence, $$H_n'(x) = 2nH_{n-1}(x).$$

(b) Differentiating result (a) and then applying it again yields

$$H_n'' = 2nH_{n-1}' = 2n\,2(n-1)H_{n-2}.$$

Using this in the differential equation satisfied by the H_n, we obtain

$$4n(n-1)H_{n-2} - 2x\,2nH_{n-1} + 2nH_n = 0.$$

This gives

$$H_{n+1}(x) - 2xH_n(x) + 2nH_{n-1}(x) = 0$$

after dividing through by $2n$ and changing $n \to n+1$.

18.7 *For the associated Laguerre polynomials, carry through the following exercises.*

(a) *Prove the Rodrigues' formula*

$$L_n^m(x) = \frac{e^x x^{-m}}{n!} \frac{d^n}{dx^n}(x^{n+m} e^{-x}),$$

taking the polynomials to be defined by

$$L_n^m(x) = \sum_{k=0}^{n} (-1)^k \frac{(n+m)!}{k!(n-k)!(k+m)!} x^k.$$

(b) *Prove the recurrence relations*

$$(n+1)L_{n+1}^m(x) = (2n+m+1-x)L_n^m(x) - (n+m)L_{n-1}^m(x),$$

$$x(L_n^m)'(x) = nL_n^m(x) - (n+m)L_{n-1}^m(x),$$

but this time taking the polynomial as defined by

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x)$$

or the generating function.

(a) It is most convenient to evaluate the nth derivative directly, using Leibnitz' theorem. This gives

$$\begin{aligned} L_n^m(x) &= \frac{e^x x^{-m}}{n!} \sum_{r=0}^{n} \frac{n!}{r!(n-r)!} \frac{d^r}{dx^r}(x^{n+m}) \frac{d^{n-r}}{dx^{n-r}}(e^{-x}) \\ &= e^x x^{-m} \sum_{r=0}^{n} \frac{1}{r!(n-r)!} \frac{(n+m)!}{(n+m-r)!} x^{n+m-r} (-1)^{n-r} e^{-x} \\ &= \sum_{r=0}^{n} \frac{(-1)^{n-r}}{r!(n-r)!} \frac{(n+m)!}{(n+m-r)!} x^{n-r}. \end{aligned}$$

Relabelling the summation using the new index $k = n - r$, we immediately obtain

$$L_n^m(x) = \sum_{k=0}^{n} (-1)^k \frac{(n+m)!}{k!(n-k)!(k+m)!} x^k,$$

which is as given in the question.

(b) The first recurrence relation can be proved using the generating function for

the associated Laguerre functions:

$$G(x,h) = \frac{e^{-xh/(1-h)}}{(1-h)^{m+1}} = \sum_{n=0}^{\infty} L_n^m(x)h^n.$$

Differentiating the second equality with respect to h, we obtain

$$\frac{(m+1)(1-h)-x}{(1-h)^{m+3}}\, e^{-xh/(1-h)} = \sum nL_n^m h^{n-1}.$$

Using the generating function for a second time, we may rewrite this as

$$[(m+1)(1-h)-x]\sum L_n^m h^n = (1-h)^2 \sum nL_n^m h^{n-1}.$$

Equating the coefficients of h^n now yields

$$(m+1)L_n^m - (m+1)L_{n-1}^m - xL_n^m = (n+1)L_{n+1}^m - 2nL_n^m + (n-1)L_{n-1}^m,$$

which can be rearranged and simplified to give the first recurrence relation.

The second result is most easily proved by differentiating one of the standard recurrence relations satisfied by the ordinary Laguerre polynomials, but with n replaced by $n+m$. This standard equality reads

$$xL'_{n+m}(x) = (n+m)L_{n+m}(x) - (n+m)L_{n-1+m}(x).$$

We convert this into an equation for the associated polynomials,

$$L_n^m(x) = (-1)^m \frac{d^m}{dx^m} L_{n+m}(x),$$

by differentiating it m times with respect to x and multiplying through by $(-1)^m$. The result is

$$x(L_n^m)' + mL_n^m = (n+m)L_n^m - (n+m)L_{n-1}^m,$$

which immediately simplifies to give the second recurrence relation satisfied by the associated Laguerre polynomials.

18.9 *By initially writing $y(x)$ as $x^{1/2}f(x)$ and then making subsequent changes of variable, reduce Stokes' equation,*

$$\frac{d^2y}{dx^2} + \lambda xy = 0,$$

to Bessel's equation. Hence show that a solution that is finite at $x = 0$ is a multiple of $x^{1/2}J_{1/3}(\frac{2}{3}\sqrt{\lambda x^3})$.

With $y(x) = x^{1/2}f(x)$,

$$y' = \frac{f}{2x^{1/2}} + x^{1/2}f' \text{ and } y'' = -\frac{f}{4x^{3/2}} + \frac{f'}{x^{1/2}} + x^{1/2}f''$$

and the equation becomes

$$-\frac{f}{4x^{3/2}} + \frac{f'}{x^{1/2}} + x^{1/2}f'' + \lambda x^{3/2}f = 0,$$
$$x^2f'' + xf' + (\lambda x^3 - \tfrac{1}{4})f = 0.$$

Now, guided by the known form of Bessel's equation, change the independent variable to $u = x^{3/2}$ with $f(x) = g(u)$ and

$$\frac{du}{dx} = \frac{3}{2}x^{1/2} \quad \Rightarrow \quad \frac{d}{dx} = \frac{3}{2}u^{1/3}\frac{d}{du}.$$

This gives

$$u^{4/3}\frac{3}{2}u^{1/3}\frac{d}{du}\left(\frac{3}{2}u^{1/3}\frac{dg}{du}\right) + u^{2/3}\frac{3}{2}u^{1/3}\frac{dg}{du} + \left(\lambda u^2 - \frac{1}{4}\right)g = 0,$$
$$\frac{3}{2}u^{5/3}\left(\frac{3}{2}u^{1/3}\frac{d^2g}{du^2} + \frac{1}{2}u^{-2/3}\frac{dg}{du}\right) + \frac{3}{2}u\frac{dg}{du} + \left(\lambda u^2 - \frac{1}{4}\right)g = 0,$$
$$\frac{9}{4}u^2\frac{d^2g}{du^2} + \frac{9}{4}u\frac{dg}{du} + \left(\lambda u^2 - \frac{1}{4}\right)g = 0,$$
$$u^2\frac{d^2g}{du^2} + u\frac{dg}{du} + \left(\frac{4}{9}\lambda u^2 - \frac{1}{9}\right)g = 0.$$

This is close to Bessel's equation but still needs a scaling of the variables. So, set $\frac{2}{3}\sqrt{\lambda}u \equiv \mu u = v$ and $g(u) = h(v)$, obtaining

$$\frac{v^2}{\mu^2}\mu^2\frac{d^2h}{dv^2} + \frac{v}{\mu}\mu\frac{dh}{dv} + \left(v^2 - \frac{1}{9}\right)h = 0.$$

This *is* Bessel's equation and has a general solution

$$h(v) = c_1J_{1/3}(v) + c_2J_{-1/3}(v),$$
$$\Rightarrow \quad g(u) = c_1J_{1/3}(\tfrac{2\sqrt{\lambda}}{3}u) + c_2J_{-1/3}(\tfrac{2\sqrt{\lambda}}{3}u),$$
$$\Rightarrow \quad f(x) = c_1J_{1/3}(\tfrac{2\sqrt{\lambda}}{3}x^{3/2}) + c_2J_{-1/3}(\tfrac{2\sqrt{\lambda}}{3}x^{3/2}).$$

For a solution that is finite at $x = 0$, only the Bessel function with a positive subscript can be accepted. Therefore the required solution is

$$y(x) = c_1x^{1/2}J_{1/3}(\tfrac{2\sqrt{\lambda}}{3}x^{3/2}).$$

18.11 *Identify the series for the following hypergeometric functions, writing them in terms of better-known functions.*

(a) $F(a, b, b; z)$,
(b) $F(1, 1, 2; -x)$,
(c) $F(\frac{1}{2}, 1, \frac{3}{2}; -x^2)$,
(d) $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2)$,
(e) $F(-a, a, \frac{1}{2}; \sin^2 x)$*; this is a much more difficult exercise.*

The hypergeometric equation is

$$z(1-z)y'' + [\,c - (a+b+1)z\,]\,y' - aby = 0.$$

The $(n+1)$th term of its series solution, the hypergeometric function $F(a, b, c; z)$, is given by

$$\frac{a(a+1)\cdots(a+n-1)\,b(b+1)\cdots(b+n-1)}{c(c+1)\cdots(c+n-1)}\,\frac{z^n}{n!}$$

for $n \geq 1$ and unity for $n = 0$.

(a) $F(a, b, b; z)$. In each term the equal factors arising from the second and third parameters cancel, as one is in the numerator and the other in the denominator. Thus,

$$\begin{aligned} F(a, b, b; z) &= 1 + az + \frac{a(a+1)}{2!}\,z^2 + \frac{a(a+1)(a+2)}{3!}\,z^3 + \cdots \\ &= (1-z)^{-a}. \end{aligned}$$

(b) $F(1, 1, 2; -x)$. The $n+1$th term is

$$\frac{(n!)\,(n!)}{(n+1)!\,(n!)}\,(-x)^n = \frac{(-1)^n x^n}{n+1}$$

making the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{n+1} = 1 - \frac{x}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \cdots = \frac{1}{x}\ln(1+x).$$

(c) $F(\frac{1}{2}, 1, \frac{3}{2}; -x^2)$. Directly from the series:

$$\begin{aligned} F(\tfrac{1}{2}, 1, \tfrac{3}{2}; -x^2) &= 1 + \frac{(\frac{1}{2})\,(1)}{1!\,(\frac{3}{2})}\,(-x^2) + \frac{(\frac{1}{2})\,(\frac{3}{2})\,(1)\,(2)}{2!\,(\frac{3}{2})\,(\frac{5}{2})}\,(-x^2)^2 + \cdots \\ &= 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} + \cdots. \end{aligned}$$

The coefficients are those of $\tan^{-1} x$, though the powers of x are all too small by one. Thus $F(\frac{1}{2}, 1, \frac{3}{2}; -x^2) = x^{-1}\tan^{-1} x$.

(d) $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; x^2)$. Again, directly from the series:

$$F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}; x^2) = 1 + \frac{(\frac{1}{2})^2}{1!\,(\frac{3}{2})}\,(x^2) + \frac{(\frac{1}{2})^2\,(\frac{3}{2})^2}{2!\,(\frac{3}{2})\,(\frac{5}{2})}\,(x^2)^2 + \cdots$$

$$= 1 + \tfrac{1}{6}\,x^2 + \tfrac{3}{40}\,5x^4 + \frac{15}{336}\,x^6 + \cdots.$$

From the larger standard tables of Maclaurin series it can be seen that, although the successive coefficients are those of $\sin^{-1} x$, the powers of x are all too small by one. Thus $F(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}; -x^2) = x^{-1} \sin^{-1} x$.

(e) $F(-a, a, \frac{1}{2}; \sin^2 x)$. Since we will obtain a series involving terms such as $\sin^{2m} x$, the series may be difficult to identify. The series is

$$1 + \frac{(-a)\,(a)}{(\frac{1}{2})}\,\sin^2 x + \frac{(-a)\,(-a+1)\,(a)\,(a+1)}{2!\,(\frac{1}{2})\,(\frac{3}{2})}\,\sin^4 x + \cdots. \qquad (*)$$

Clearly, this contains only even powers of x, though just the first two terms alone constitute an infinite power series in x. However, a term containing x^{2m} can only arise from the first $m+1$ terms of $(*)$ and a few trials may be helpful.

If $F(-a, a, \frac{1}{2}; \sin^2 x) = \sum_{n=0}^{\infty} b_n x^{2n}$, then $b_0 = 1$ and $b_1 = -2a^2$ since the corresponding powers of x can only arise from the first and second terms of $(*)$, respectively. The coefficient of x^4 is determined by the second and third terms of $(*)$ and is given by

$$b_2 = -2a^2\left(-\frac{2}{3!}\right) + \frac{2a^2(a^2-1)}{3}\,(1) = \frac{2a^4}{3}.$$

The coefficient of x^6, namely b_3, has contributions from the second, third and fourth terms of $(*)$ and is given by

$$\begin{aligned}
&-2a^2\left[\left(\frac{1}{3!}\right)^2 + \frac{2}{5!}\right] + \frac{2a^2(a^2-1)}{3}\left(\frac{-4}{3!}\right) + \frac{4a^2(a^2-1)(4-a^2)}{45}\,(1)\\
&= -2a^2\left(\frac{20+12}{720}\right) - \frac{8a^4}{18} + \frac{8a^2}{18} + \frac{4}{45}(-4a^2 + 5a^4 - a^6)\\
&= \left(-\frac{64}{720} + \frac{8}{18} - \frac{16}{45}\right)a^2 + \left(-\frac{8}{18} + \frac{20}{45}\right)a^4 + \left(-\frac{4}{45}\right)a^6\\
&= -\frac{4}{45}\,a^6.
\end{aligned}$$

Thus, in powers up to x^6,

$$\begin{aligned}
F(-a, a, \tfrac{1}{2}; \sin^2 x) &= 1 - 2a^2x^2 + \tfrac{2}{3}\,a^4x^4 - \tfrac{4}{45}\,a^6x^6\\
&= 1 - \frac{(2ax)^2}{2!} + \frac{(2ax)^4}{4!} - \frac{(2ax)^6}{6!}.
\end{aligned}$$

Though not totally conclusive, this sequence of coefficients strongly suggests that $F(-a, a, \frac{1}{2}; \sin^2 x) = \cos 2ax$. Note that a does not need to be an integer.

This tentative conclusion can be tested by transforming the original hypergeometric equation as follows. With $z = \sin^2 x$, we have that $dz/dx = 2\sin x \cos x = \sin 2x$, implying that $d/dz = (\sin 2x)^{-1}\, d/dx$. The equation becomes

$$\sin^2 x(1-\sin^2 x)\frac{1}{\sin 2x}\frac{d}{dx}\left(\frac{1}{\sin 2x}\frac{dy}{dx}\right) + \left[\frac{1}{2} - (-a+a+1)\sin^2 x\right]\frac{1}{\sin 2x}\frac{dy}{dx} + a^2 y = 0.$$

This can be simplified as follows:

$$\frac{1}{4}\sin 2x\left(\frac{1}{\sin 2x}\frac{d^2y}{dx^2} - \frac{2\cos 2x}{\sin^2 2x}\frac{dy}{dx}\right) + \frac{1-2\sin^2 x}{2\sin 2x}\frac{dy}{dx} + a^2 y = 0,$$

$$\frac{1}{4}\frac{d^2y}{dx^2} - \frac{\cos 2x}{2\sin 2x}\frac{dy}{dx} + \frac{\cos 2x}{2\sin 2x}\frac{dy}{dx} + a^2 y = 0,$$

$$\frac{d^2y}{dx^2} + 4a^2 y = 0.$$

For a solution with $y(0) = 1$, this implies that $y(x) = \cos 2ax$, thus confirming our provisional conclusion.

18.13 *Find a change of variable that will allow the integral*

$$I = \int_1^\infty \frac{\sqrt{u-1}}{(u+1)^2}\, du$$

to be expressed in terms of the beta function and so evaluate it.

The beta function is normally expressed in terms of an integral, over the range 0 to 1, of an integrand of the form $v^m(1-v)^n$, with $m, n > -1$. We therefore need a change of variable $u = f(x)$ such that $u+1$ is an inverse power of x; this being so, we also need $f(0) = \infty$ and $f(1) = 1$.

Consider

$$u + 1 = \frac{A}{x}, \quad \text{i.e.} \quad f(x) = \frac{A}{x} - 1.$$

This satisfies the first two requirements, and also satisfies the third one if we choose $A = 2$.

So, substitute $u = \dfrac{2}{x} - 1 = \dfrac{2-x}{x}$, with $u - 1 = \dfrac{2(1-x)}{x}$ and $du = -\dfrac{2}{x^2}$. The

integral then becomes

$$\begin{aligned} I &= \int_1^0 \frac{2^{1/2}(1-x)^{1/2}\,x^2\,(-2)}{x^{1/2}\,2^2\,x^2}\,dx \\ &= \frac{1}{\sqrt{2}}\int_0^1 (1-x)^{1/2}x^{-1/2}\,dx \\ &= \frac{1}{\sqrt{2}}\,B(\tfrac{1}{2},\tfrac{3}{2}) = \frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}{\sqrt{2}\,\Gamma(\frac{1}{2}+\frac{3}{2})} \\ &= \frac{\sqrt{\pi}\,\frac{1}{2}\sqrt{\pi}}{\sqrt{2}\,1} = \frac{\pi}{2\sqrt{2}}. \end{aligned}$$

18.15 *The complex function $z!$ is defined by*

$$z! = \int_0^\infty u^z e^{-u}\,du \qquad \textit{for } \operatorname{Re} z > -1.$$

For $\operatorname{Re} z \le -1$ *it is defined by*

$$z! = \frac{(z+n)!}{(z+n)(z+n-1)\cdots(z+1)},$$

where n is any (positive) integer $> -\operatorname{Re} z$. Being the ratio of two polynomials, $z!$ is analytic everywhere in the finite complex plane except at the poles that occur when z is a negative integer.

(a) *Show that the definition of $z!$ for* $\operatorname{Re} z \le -1$ *is independent of the value of n chosen.*
(b) *Prove that the residue of $z!$ at the pole $z = -m$, where m is an integer > 0, is $(-1)^{m-1}/(m-1)!$.*

(a) Let m and n be two choices of integer with $m > n > -\operatorname{Re} z$. Denote the corresponding definitions of $z!$ by $(z!)_m$ and $(z!)_n$ and consider the ratio of these two functions:

$$\begin{aligned} \frac{(z!)_m}{(z!)_n} &= \frac{(z+m)!}{(z+m)(z+m-1)\cdots(z+1)}\,\frac{(z+n)(z+n-1)\cdots(z+1)}{(z+n)!} \\ &= \frac{(z+m)!}{(z+m)(z+m-1)\cdots(z+n+1)\times(z+n)!} \\ &= \frac{(z+m)!}{(z+m)!} = 1. \end{aligned}$$

Thus the two functions are identical for all z, i.e the definition of $z!$ is independent of the choice of n, provided that $n > -\operatorname{Re} z$.

(b) From the given definition of $z!$ it is clear that its pole at $z = -m$ is a simple one. The residue R at the pole is therefore given by

$$\begin{aligned}
R &= \lim_{z\to -m} (z+m)z! \\
&= \lim_{z\to -m} \frac{(z+m)\,(z+n)!}{(z+n)(z+n-1)\cdots(z+1)} \qquad \text{(integer } n \text{ is chosen} > m) \\
&= \lim_{z\to -m} \frac{(z+n)!}{(z+n)(z+n-1)\cdots(z+m+1)(z+m-1)\cdots(z+1)} \\
&= \frac{(-m+n)!}{(-m+n)\cdots(-m+m+1)(-m+m-1)\cdots(-m+1)} \\
&= \frac{1}{[-1]\,[-2]\cdots[-(m-1)]} \\
&= (-1)^{m-1}\frac{1}{(m-1)!},
\end{aligned}$$

as stated in the question.

18.17 *The integral*

$$I = \int_{-\infty}^{\infty} \frac{e^{-k^2}}{k^2+a^2}\,dk, \qquad (*)$$

in which $a > 0$, occurs in some statistical mechanics problems. By first considering the integral

$$J = \int_0^{\infty} e^{iu(k+ia)}\,du$$

and a suitable variation of it, show that $I = (\pi/a)\exp(a^2)\,\mathrm{erfc}(a)$, where $\mathrm{erfc}(x)$ is the complementary error function.

The fact that $a > 0$ will ensure that the improper integral J is well defined. It is

$$J = \int_0^{\infty} e^{iu(k+ia)}\,du = \left[\frac{e^{iu(k+ia)}}{i(k+ia)}\right]_0^{\infty} = \frac{i}{k+ia}.$$

We note that this result contains one of the factors that would appear as a denominator in one term of a partial fraction expansion of the integrand in $(*)$. Another term would contain a factor $(k-ia)^{-1}$, and this can be generated by

$$J' = \int_0^{\infty} e^{-iu(k-ia)}\,du = \left[\frac{e^{-iu(k-ia)}}{-i(k-ia)}\right]_0^{\infty} = \frac{-i}{k-ia}.$$

Now, actually expressing the integrand in partial fractions, using the integral

expressions J and J' for the factors, and then reversing the order of integration gives

$$\begin{aligned} I &= \frac{1}{2a}\int_{-\infty}^{\infty}\left(\frac{ie^{-k^2}}{k+ia} - \frac{ie^{-k^2}}{k-ia}\right)dk \\ &= \frac{1}{2a}\int_{-\infty}^{\infty} e^{-k^2}\,dk \int_0^{\infty} e^{iu(k+ia)}\,du + \frac{1}{2a}\int_{-\infty}^{\infty} e^{-k^2}\,dk \int_0^{\infty} e^{-iu(k-ia)}\,du, \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad 2aI &= \int_0^{\infty} du \int_{-\infty}^{\infty} e^{-k^2+iuk-ua}\,dk + \int_0^{\infty} du \int_{-\infty}^{\infty} e^{-k^2-iuk-ua}\,dk \\ &= \int_0^{\infty} du \int_{-\infty}^{\infty} e^{-(k-iu/2)^2-u^2/4-ua}\,dk \\ &\qquad\qquad + \int_0^{\infty} du \int_{-\infty}^{\infty} e^{-(k+iu/2)^2-u^2/4-ua}\,dk \\ &= 2\sqrt{\pi}\int_0^{\infty} e^{-u^2/4-ua}\,du, \end{aligned}$$

using the standard Gaussian result. We now complete the square in the exponent and set $2v = u + 2$, obtaining

$$\begin{aligned} 2aI &= 2\sqrt{\pi}\int_0^{\infty} e^{-(u+2a)^2/4+a^2}\,du, \\ &= 2\sqrt{\pi}\int_a^{\infty} e^{-v^2}\,e^{a^2}\,2dv. \end{aligned}$$

From this it follows that

$$I = \frac{\sqrt{\pi}}{a}\,2e^{a^2}\,\frac{\sqrt{\pi}}{2}\,\mathrm{erfc}(a) = \frac{\pi}{a}\,e^{a^2}\,\mathrm{erfc}(a),$$

as stated in the question.

18.19 *For the functions $M(a,c;z)$ that are the solutions of the confluent hypergeometric equation:*

(a) *use their series representation to prove that*

$$c\,\frac{d}{dz}M(a,c;z) = a\,M(a+1,c+1;z);$$

(b) *use an integral representation to prove that*

$$M(a,c;z) = e^z\,M(c-a,c;-z).$$

(a) Directly differentiating the explicit series term by term gives

$$\begin{aligned}\frac{d}{dz}M(a,c;z) &= \frac{d}{dz}\left[1+\frac{a}{c}z+\frac{a(a+1)}{2!\,c(c+1)}z^2+\cdots\right]\\ &= \frac{a}{c}+\frac{2a(a+1)}{2!\,c(c+1)}z+\frac{3a(a+1)(a+2)}{3!\,c(c+1)(c+2)}z^2+\cdots\\ &= \frac{a}{c}\left[1+\frac{a+1}{c+1}z+\frac{(a+1)(a+2)}{2!\,(c+1)(c+2)}z^2+\cdots\right]\\ &= \frac{a}{c}\,M(a+1,c+1;z).\end{aligned}$$

The quoted result follows immediately.

(b) This will be achieved most simply if we choose a representation in which the parameters can be rearranged without having to perform any actual integration. We therefore take the representation

$$M(a,c;z)=\frac{\Gamma(c)}{\Gamma(c-a)\,\Gamma(a)}\int_0^1 e^{zt}\,t^{a-1}\,(1-t)^{c-a-1}\,dt$$

and change the variable of integration to $s=1-t$ whilst regrouping the parameters (without changing their values, of course). This gives

$$\begin{aligned}&M(a,c;z)\\ &= \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)}\int_1^0 e^{z}\,e^{-zs}\,(1-s)^{a-1}\,s^{c-a-1}\,(-ds)\\ &= \frac{\Gamma(c)\,e^{z}}{\Gamma[\,c-(c-a)\,]\,\Gamma[\,(c-a)\,]}\int_0^1 e^{-zs}\,(1-s)^{c-(c-a)-1}\,s^{(c-a)-1}\,ds\\ &= e^{z}\,M(c-a,c,-z),\end{aligned}$$

thus establishing the identity, in which $a\to c-a$ and $z\to -z$ whilst c remains unchanged.

18.21 *Find the differential equation satisfied by the function $y(x)$ defined by*

$$y(x)=Ax^{-n}\int_0^x e^{-t}t^{n-1}\,dt\equiv Ax^{-n}\gamma(n,x),$$

and, by comparing it with the confluent hypergeometric function, express y as a multiple of the solution $M(a,c;z)$ of that equation. Determine the value of A that makes y equal to M.

As the comparison is to be made with the hypergeometric equation, which is a second-order differential equation, we must calculate the first two derivatives of

$y(x)$. Further, as it is a homogeneous equation, we may omit the multiplicative constant A for the time being:

$$\begin{aligned}
y(x) &= x^{-n}\gamma(n,x),\\
y'(x) &= -nx^{-n-1}\gamma(n,x) + x^{-n}e^{-x}x^{n-1}\\
&= -nx^{-1}y + x^{-1}e^{-x},\\
y''(x) &= nx^{-2}y - nx^{-1}y' - x^{-2}e^{-x} - x^{-1}e^{-x}\\
&= nx^{-2}y - nx^{-1}y' - (x^{-1}+1)(y' + nx^{-1}y),\\
x^2y'' &= (-nx - x - x^2)y' + (n - n - nx)y.
\end{aligned}$$

The second line uses the standard result for differentiating an indefinite integral with respect to its upper limit. In the fifth line we substituted for $x^{-1}e^{-x}$ from the expression obtained for $y'(x)$ in the third line. Thus the equation to be compared with the confluent hypergeometric equation is

$$xy'' + (n+1+x)y' + ny = 0.$$

This has to be compared with

$$zw'' + (c-z)w' - aw = 0.$$

Now xy'' and xy' terms have the same signs (both positive), whereas the zw'' and zw' terms have opposite signs. To deal with this, we must set $z = -x$ in the confluent hypergeometric equation; renaming $w(z) = y(x)$ gives $w' = -y'$ and $w'' = y''$. The equation then becomes (after an additional overall sign change)

$$xy'' + (c+x)y' + ay = 0.$$

The obvious assignments, to go with $z \to -x$, are now $a \to n$ and $c \to n+1$. We therefore conclude that $y(x)$ is a multiple of $M(n, n+1; -x)$.

To determine the constant A in the given form of $y(x)$ we expand both its definition and $M(n, n+1; -x)$ in powers of x. Strictly, only the first term is necessary, but the second acts as a check.

From the hypergeometric series,

$$M(n, n+1; -x) = 1 + \frac{n(-x)}{n+1} + \cdots.$$

From the definition of $y(x)$,

$$\begin{aligned} y(x) &= Ax^{-n}\int_0^x e^{-t}t^{n-1}\,dt \\ &= Ax^{-n}\int_0^x \left[t^{n-1} - t(t^{n-1}) + \frac{t^2}{2!}(t^{n-1}) + \cdots\right] \\ &= Ax^{-n}\left[\frac{t^n}{n} - \frac{t^{n+1}}{n+1} + \cdots\right]_0^x \\ &= Ax^{-n}\left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1} + \cdots\right] \\ &= \frac{A}{n} - \frac{Ax}{n+1} + \cdots. \end{aligned}$$

This reproduces the first two terms of $M(n, n+1; -x)$ if $A = n$, yielding, finally, that

$$y(x) = nx^{-n}\gamma(n, x) = M(n, n+1; -x).$$

18.23 *Prove two of the properties of the incomplete gamma function $P(a, x^2)$ as follows.*

(a) *By considering its form for a suitable value of a, show that the error function can be expressed as a particular case of the incomplete gamma function.*

(b) *The Fresnel integrals, of importance in the study of the diffraction of light, are given by*

$$C(x) = \int_0^x \cos\left(\frac{\pi}{2}t^2\right)\,dt, \qquad S(x) = \int_0^x \sin\left(\frac{\pi}{2}t^2\right)\,dt.$$

Show that they can be expressed in terms of the error function by

$$C(x) + iS(x) = A\,\mathrm{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right],$$

where A is a (complex) constant, which you should determine. Hence express $C(x) + iS(x)$ in terms of the incomplete gamma function.

(a) From the definition of the incomplete gamma function, we have

$$P(a, x^2) = \frac{1}{\Gamma(a)}\int_0^{x^2} e^{-t}\,t^{a-1}\,dt.$$

Guided by the x^2 in the upper limit, we now change the integration variable to

$y = +\sqrt{t}$, with $2y\,dy = dt$, and obtain

$$P(a, x^2) = \frac{1}{\Gamma(a)} \int_0^x e^{-y^2}\, y^{2(a-1)}\, 2y\, dy.$$

To make the RHS into an error function we need to remove the y-term; to do this we choose a such that $2(a-1)+1 = 0$, i.e. $a = \frac{1}{2}$. With this choice, $\Gamma(a) = \sqrt{\pi}$ and

$$P\left(\tfrac{1}{2}, x^2\right) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-y^2}\, dy,$$

i.e. a correctly normalised error function.

(b) Consider the given expression:

$$z = A \operatorname{erf}\left[\frac{\sqrt{\pi}}{2}(1-i)x\right] = \frac{2A}{\sqrt{\pi}} \int_0^{\sqrt{\pi}(1-i)x/2} e^{-u^2}\, du.$$

Changing the variable of integration to s, given by $u = \frac{1}{2}\sqrt{\pi}(1-i)s$, and recalling that $(1-i)^2 = -2i$, we obtain

$$\begin{aligned} z &= \frac{2A}{\sqrt{\pi}} \int_0^x e^{-s^2\pi(-2i)/4}\, \frac{\sqrt{\pi}}{2}(1-i)\, ds \\ &= A(1-i) \int_0^x e^{i\pi s^2/2}\, ds \\ &= A(1-i) \int_0^x \left[\cos\left(\frac{\pi s^2}{2}\right) + i \sin\left(\frac{\pi s^2}{2}\right)\right] ds \\ &= A(1-i)\,[\,C(x) + iS(x)\,]. \end{aligned}$$

For the correct normalisation we need $A(1-i) = 1$, implying that $A = (1+i)/2$.

Now, from part (a), the error function can be expressed in terms of the incomplete gamma function $P(a, x)$ by

$$\operatorname{erf}(x) = P(\tfrac{1}{2}, x^2).$$

Here the argument of the error function is $\frac{1}{2}\sqrt{\pi}(1-i)x$, whose square is $-\frac{1}{2}\pi i x^2$, and so

$$C(x) + iS(x) = \frac{1+i}{2}\, P\left(\frac{1}{2}, -\frac{i\pi}{2}\, x^2\right).$$

19

Quantum operators

19.1 *Show that the commutator of two operators that correspond to two physical observables cannot itself correspond to another physical observable.*

Let the two operators be A and B, both of which must be Hermitian since they correspond to physical variables, and consider the Hermitian conjugate of their commutator:

$$[A, B]^\dagger = (AB)^\dagger - (BA)^\dagger = B^\dagger A^\dagger - A^\dagger B^\dagger = BA - AB = -[A, B].$$

Thus, the commutator is anti-Hermitian or zero and therefore cannot represent a non-trivial physical variable (as its eigenvalues are imaginary).

19.3 *In quantum mechanics, the time dependence of the state function $|\psi\rangle$ of a system is given, as a further postulate, by the equation*

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle,$$

where H is the Hamiltonian of the system. Use this to find the time dependence of the expectation value $\langle A\rangle$ of an operator A that itself has no explicit time dependence. Hence show that operators that commute with the Hamiltonian correspond to the classical 'constants of the motion'.

For a particle of mass m moving in a one-dimensional potential $V(x)$, prove Ehrenfest's theorem:

$$\frac{d\langle p_x\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle \qquad \textit{and} \qquad \frac{d\langle x\rangle}{dt} = \frac{\langle p_x\rangle}{m}.$$

The expectation value of A at any time is $\langle\psi(x,t)|A|\psi(x,t)\rangle$, where we have explicitly indicated that the state function varies with time. Now

$$\frac{d}{dt}\langle\psi\,|\,A\,|\,\psi\rangle = \left(\frac{\partial}{\partial t}\langle\psi|\right) A\,|\,\psi\rangle + \langle\psi\,|\,\frac{\partial A}{\partial t}\,|\,\psi\rangle + \langle\psi\,|\,A\left(\frac{\partial}{\partial t}|\,\psi\rangle\right).$$

Since A has no explicit time dependence, $\partial A/\partial t = 0$ and the second term drops out.

The given (quantum) equation of motion and its Hermitian conjugate are

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = H|\psi\rangle \quad \text{and} \quad \frac{\partial}{\partial t}\langle\psi\,| = -\frac{1}{i\hbar}\langle\psi\,|H^\dagger = -\frac{1}{i\hbar}\langle\psi\,|H,$$

since H is Hermitian. Thus,

$$\begin{aligned}\frac{d}{dt}\langle\psi\,|\,A\,|\,\psi\rangle &= -\frac{1}{i\hbar}\langle\psi\,|\,HA\,|\,\psi\rangle + \frac{1}{i\hbar}\langle\psi\,|\,AH\,|\,\psi\rangle \\ &= -\frac{1}{i\hbar}\langle\psi\,|\,[H,A]\,|\,\psi\rangle \\ &= \frac{i}{\hbar}\langle\psi\,|\,[H,A]\,|\,\psi\rangle.\end{aligned}$$

This shows that the rate of change of the expectation value of A is proportional to the expectation value of the commutator of A and the Hamiltonian. If A and H commute, the RHS is zero, the expectation value of A has a zero rate of change, and $\langle\psi\,|\,A\,|\,\psi\rangle$ is a constant of the motion.

For the particle moving in the one-dimensional potential $V(x)$,

$$H = \frac{p_x^2}{2m} + V(x).$$

(i) For $\langle p_x\rangle$,

$$\begin{aligned}[H,p_x]\,|\,\psi\rangle &= [V,p_x]\,|\,\psi\rangle, \text{ since } p_x \text{ clearly commutes with } p_x^2, \\ &= -i\hbar V\frac{\partial}{\partial x}|\,\psi\rangle + i\hbar\frac{\partial}{\partial x}\left(V|\,\psi\rangle\right) \\ &= -i\hbar V\frac{\partial}{\partial x}|\,\psi\rangle + i\hbar V\frac{\partial}{\partial x}|\,\psi\rangle + i\hbar\frac{\partial V}{\partial x}\,|\,\psi\rangle \\ &= i\hbar\frac{\partial V}{\partial x}\,|\,\psi\rangle,\end{aligned}$$

implying that

$$\begin{aligned}\frac{d}{dt}\langle p_x\rangle = \frac{d}{dt}\langle\psi\,|\,p_x\,|\,\psi\rangle &= \frac{i}{\hbar}\langle\psi\,|\,[H,p_x]\,|\,\psi\rangle \\ &= \frac{i}{\hbar}\langle\psi\,|\,i\hbar\frac{\partial V}{\partial x}\,|\,\psi\rangle = -\left\langle\frac{\partial V}{\partial x}\right\rangle.\end{aligned}$$

(ii) For $\langle x\rangle$ we will need the general commutator property $[AB,C] = A\,[B,C] +$

$[A, C]\,B$ to evaluate $\left[p_x^2, x\right]$:

$$\begin{aligned}
[H, x]\,|\,\psi\rangle &= \frac{1}{2m}\,\left[p_x^2, x\right]|\,\psi\rangle, \text{ since } x \text{ clearly commutes with } V(x),\\
&= \frac{1}{2m}\,\{p_x\,[p_x, x]\,|\,\psi\rangle + [p_x, x]\,p_x|\,\psi\rangle\}, \quad \text{as above,}\\
&= \frac{1}{2m}\,\{-i\hbar p_x|\,\psi\rangle - i\hbar p_x|\,\psi\rangle\} = -\frac{i\hbar}{m}\,p_x\,|\,\psi\rangle,
\end{aligned}$$

implying that

$$\begin{aligned}
\frac{d}{dt}\langle x\rangle = \frac{d}{dt}\langle\psi\,|\,x\,|\,\psi\rangle &= \frac{i}{\hbar}\langle\psi\,|\;[H, x]\;|\,\psi\rangle\\
&= \frac{i}{\hbar}\langle\psi\,|\,-\frac{i\hbar}{m}p_x\,|\,\psi\rangle = \frac{1}{m}\langle\psi\,|\,p_x\,|\,\psi\rangle = \frac{1}{m}\langle p_x\rangle.
\end{aligned}$$

Ehrenfest's theorem should be compared to the classical statements 'momentum equals mass times velocity', 'the force is given by minus the gradient of the potential' and 'force is equal to the rate of change of momentum'.

19.5 *Find closed-form expressions for* $\cos\mathsf{C}$ *and* $\sin\mathsf{C}$, *where* C *is the matrix*

$$\mathsf{C} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Demonstrate that the 'expected' relationships

$$\cos^2\mathsf{C} + \sin^2\mathsf{C} = \mathsf{I} \quad \textit{and} \quad \sin 2\mathsf{C} = 2\sin\mathsf{C}\cos\mathsf{C}$$

are valid.

Consider the square of C:

$$\mathsf{C}^2 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\mathsf{I}.$$

Now

$$\cos\mathsf{C} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\mathsf{C}^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}\,2^n\,\mathsf{I}^n = (\cos\sqrt{2})\,\mathsf{I}$$

and

$$\sin\mathsf{C} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\mathsf{C}^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}\,2^n\,\mathsf{I}^n\mathsf{C} = \frac{1}{\sqrt{2}}(\sin\sqrt{2})\,\mathsf{C}.$$

To test the analogue of '$\cos^2\theta + \sin^2\theta = 1$':

$$\begin{aligned}
\cos^2\mathsf{C} + \sin^2\mathsf{C} &= (\cos^2\sqrt{2})\mathsf{I} + \tfrac{1}{2}(\sin^2\sqrt{2})\mathsf{C}^2\\
&= (\cos^2\sqrt{2})\mathsf{I} + \tfrac{1}{2}(\sin^2\sqrt{2})\,2\,\mathsf{I} = \mathsf{I},
\end{aligned}$$

as expected.

To test the analogue of '$\sin 2\theta = 2\sin\theta\cos\theta$', we note that $(2\mathsf{C})^{2n} = 2^{2n}(2\mathsf{I})^n = (2^2\,2\,\mathsf{I})^n = 8^n\mathsf{I}^n$ and obtain

$$\begin{aligned}
\sin 2\mathsf{C} &= \sin\begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\, 8^n\,\mathsf{I}^n \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \\
&= \frac{1}{\sqrt{8}}(\sin\sqrt{8}) \begin{pmatrix} 2 & 2 \\ 2 & -2 \end{pmatrix} \\
&= \frac{\sin 2\sqrt{2}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.
\end{aligned}$$

But we also have that

$$\begin{aligned}
2\sin\mathsf{C}\cos\mathsf{C} &= 2\,\frac{1}{\sqrt{2}}(\sin\sqrt{2}) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (\cos\sqrt{2}) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\
&= \frac{2\,\sin\sqrt{2}\,\cos\sqrt{2}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \\
&= \frac{\sin 2\sqrt{2}}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\end{aligned}$$

thus confirming the relationship (at least in this case).

19.7 *Expressed in terms of the annihilation and creation operators A and $A^\dagger$, a system has an unperturbed Hamiltonian $H_0 = \hbar\omega A^\dagger A$. The system is disturbed by the addition of a perturbing Hamiltonian $H_1 = g\hbar\omega(A + A^\dagger)$, where g is real. Show that the effect of the perturbation is to move the whole energy spectrum of the system down by $g^2\hbar\omega$.*

The total Hamiltonian H for the system is $H_0 + H_1$, where

$$H_0 = \hbar\omega A^\dagger A \qquad \text{and} \qquad H_1 = g\hbar\omega(A + A^\dagger).$$

We note that both terms are Hermitian ($H_0^\dagger = H_0$, $H_1^\dagger = H_1$) and that the energy spectrum of the system is given by the eigenvalues μ_i for which

$$(H_0 + H_1)\,|\,\psi_i\rangle = \mu_i\,|\,\psi_i\rangle$$

has solutions.

Now,
$$\begin{aligned}
H &= H_0 + H_1 \\
&= \hbar\omega[\,A^\dagger A + g(A + A^\dagger)\,] \\
&= \hbar\omega[\,(A^\dagger + gI)(A + gI) - g^2 I\,].
\end{aligned}$$

We define B by $B = A + gI$, with $B^\dagger = A^\dagger + gI$, and consider

$$\left[B, B^\dagger\right] = \left[A + gI, A^\dagger + gI\right] = \left[A, A^\dagger\right],$$

since $[C, I] = 0$ for any C and clearly $[I, I] = 0$.

Thus, H is expressible as

$$H = \hbar\omega B^\dagger B - g^2\hbar\omega I$$

with $\left[B, B^\dagger\right] = \left[A, A^\dagger\right]$. That is, H has the same structure with respect to B as H_0 has with respect to A (apart from an additional term proportional to the identity operator) and B and $B^\dagger$ have the same commutation relation as A and $A^\dagger$. This implies that H has the same spectrum of eigenvalues λ_i as H_0, except for a (downward) shift of $-g^2\hbar\omega$, i.e. $\mu_i = \lambda_i - g^2\hbar\omega$ for each value of i. Thus the whole spectrum is lowered by this amount.

19.9 *By considering the function*

$$F(\lambda) = \exp(\lambda A) B \exp(-\lambda A),$$

where A and B are linear operators and λ a parameter, and finding its derivatives with respect to λ, prove that

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots.$$

Use this result to express

$$\exp\left(\frac{iL_x\theta}{\hbar}\right) L_y \exp\left(\frac{-iL_x\theta}{\hbar}\right)$$

as a linear combination of the angular momentum operators L_x, L_y and L_z.

Starting from the definition of $F(\lambda)$, we calculate its first few derivatives with respect to λ, remembering that operator A commutes with any function of A but not necessarily with any function of B:

$$\begin{aligned}
F(\lambda) &= \exp(\lambda A) B \exp(-\lambda A), \\
\frac{dF(\lambda)}{d\lambda} &= A \exp(\lambda A) B \exp(-\lambda A) - \exp(\lambda A) B \exp(-\lambda A) A \\
&= AF(\lambda) - F(\lambda)A = [A, F(\lambda)], \\
\frac{d^2F(\lambda)}{d\lambda^2} &= A\frac{dF}{d\lambda} - \frac{dF}{d\lambda}A = \left[A, \frac{dF(\lambda)}{d\lambda}\right] = [A, [A, F(\lambda)]], \\
\frac{d^3F(\lambda)}{d\lambda^3} &= [A, [A, [A, F(\lambda)]]], \text{ and so on for higher derivatives.}
\end{aligned}$$

Now we use a Taylor series in λ, based on the values of the derivatives at $\lambda = 0$, to evaluate $F(1)$. At $\lambda = 0$, $F(\lambda) = B$, and we obtain

$$\begin{aligned} e^A B e^{-A} &= F(1) \\ &= F(0) + \frac{dF(0)}{d\lambda} + \frac{1}{2!}\frac{d^2F(0)}{d\lambda^2} + \frac{1}{3!}\frac{d^3F(0)}{d\lambda^3} + \cdots \\ &= B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots . \end{aligned}$$

To apply this result to

$$\Theta \equiv \exp\left(\frac{iL_x\theta}{\hbar}\right) L_y \exp\left(\frac{-iL_x\theta}{\hbar}\right),$$

we need to take A as $iL_x\theta/\hbar$ and B as L_y. The corresponding commutator is given by

$$\left[\frac{iL_x\theta}{\hbar}, L_y\right] = \frac{i\theta}{\hbar}[L_x, L_y] = -\theta L_z.$$

Because multiple commutators are involved, we will also need

$$\left[\frac{iL_x\theta}{\hbar}, L_z\right] = \frac{i\theta}{\hbar}[L_x, L_z] = \theta L_y.$$

Substituting in the derived series, we obtain

$$\begin{aligned} \Theta &= L_y + (-\theta L_z) + \frac{1}{2!}\left[\frac{i\theta L_x}{\hbar}, -\theta L_z\right] + \cdots \\ &= L_y - \theta L_z + \frac{1}{2!}(-\theta^2 L_y) + \frac{1}{3!}\left[\frac{i\theta L_x}{\hbar}, -\theta^2 L_y\right] + \cdots \\ &= L_y - \theta L_z - \frac{1}{2!}\theta^2 L_y + \frac{1}{3!}(\theta^3 L_z) + \frac{1}{4!}\left[\frac{i\theta L_x}{\hbar}, \theta^3 L_z\right] + \cdots \\ &= L_y - \theta L_z - \frac{1}{2!}\theta^2 L_y + \frac{1}{3!}\theta^3 L_z + \frac{1}{4!}(\theta^4 L_y) + \frac{1}{5!}\left[\frac{i\theta L_x}{\hbar}, \theta^4 L_y\right] + \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \cdots\right) L_y - \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right) L_z \\ &= \cos\theta\, L_y - \sin\theta\, L_z. \end{aligned}$$

At each stage in the above calculation, the value of the commutator in the nth term of the series has been used to reduce the $(n+1)$th term from a multiple commutator, with n levels of nesting, to a single commutator.

20

Partial differential equations: general and particular solutions

20.1 *Determine whether the following can be written as functions of* $p = x^2 + 2y$ *only, and hence whether they are solutions of*

$$\frac{\partial u}{\partial x} = x\frac{\partial u}{\partial y}.$$

(a) $x^2(x^2 - 4) + 4y(x^2 - 2) + 4(y^2 - 1)$;
(b) $x^4 + 2x^2y + y^2$;
(c) $[x^4 + 4x^2y + 4y^2 + 4]/[2x^4 + x^2(8y + 1) + 8y^2 + 2y]$.

As a first step, we verify that any function of $p = x^2 + 2y$ will satisfy the given equation. Using the chain rule, we have

$$\frac{\partial u}{\partial p}\frac{\partial p}{\partial x} = x\frac{\partial u}{\partial p}\frac{\partial p}{\partial y},$$
$$\Rightarrow \quad \frac{\partial u}{\partial p}2x = x\frac{\partial u}{\partial p}2.$$

This is satisfied for *any* function $u(p)$, thus completing the verification.

To test the given functions we substitute for $y = \frac{1}{2}(p - x^2)$ or for $x^2 = p - 2y$ in each of the $f(x, y)$ and then examine whether the resulting forms are independent of x or y, respectively.

(a) $$\begin{aligned} f(x, y) &= x^2(x^2 - 4) + 4y(x^2 - 2) + 4(y^2 - 1) \\ &= x^2(x^2 - 4) + 2(p - x^2)(x^2 - 2) + p^2 - 2p\,x^2 + x^4 - 4 \\ &= x^4(1 - 2 + 1) + x^2(-4 + 2p + 4 - 2p) - 4p + p^2 - 4 \\ &= p^2 - 4p - 4 = g(p). \end{aligned}$$

This is a function of p only, and therefore the original $f(x, y)$ is a solution of the PDE.

Though not necessary for answering the question, we will repeat the verification, but this time by substituting for x rather than for y:

$$\begin{aligned} f(x, y) &= x^2(x^2 - 4) + 4y(x^2 - 2) + 4(y^2 - 1) \\ &= (p - 2y)(p - 2y - 4) + 4y(p - 2y - 2) + 4(y^2 - 1) \\ &= p^2 - 4py + 4y^2 - 4p + 8y + 4yp - 8y^2 - 8y + 4y^2 - 4 \\ &= p^2 - 4p - 4 = g(p); \end{aligned}$$

i.e. it is the same as before, as it must be, and again this shows that $f(x, y)$ is a solution of the PDE.

(b)
$$\begin{aligned} f(x, y) &= x^4 + 2x^2y + y^2 \\ &= (p - 2y)^2 + 2y(p - 2y) + y^2 \\ &= p^2 - 4p\,y + 4y^2 + 2p\,y - 4y^2 + y^2 \\ &= (p - y)^2 \neq g(p). \end{aligned}$$

As this is a function of both p and y, it is not a solution of the PDE.

(c)
$$\begin{aligned} f(x, y) &= \frac{x^4 + 4x^2y + 4y^2 + 4}{2x^4 + x^2(8y + 1) + 8y^2 + 2y} \\ &= \frac{p^2 - 4p\,y + 4y^2 + 4yp - 8y^2 + 4y^2 + 4}{2p^2 - 8p\,y + 8y^2 + 8yp + p - 16y^2 - 2y + 8y^2 + 2y} \\ &= \frac{p^2 + 4}{2p^2 + p} = g(p). \end{aligned}$$

This is a function of p only and therefore $f(x, y)$ is a solution of the PDE.

20.3 *Solve the following partial differential equations for $u(x, y)$ with the boundary conditions given:*

(a) $x\dfrac{\partial u}{\partial x} + xy = u, \quad u = 2y$ *on the line* $x = 1$;

(b) $1 + x\dfrac{\partial u}{\partial y} = xu, \quad u(x, 0) = x.$

(a) This can be solved as an ODE for u as a function of x, though the 'constant of integration' will be a function of y. In standard form, the equation reads

$$\frac{\partial u}{\partial x} - \frac{u}{x} = -y.$$

By inspection (or formal calculation) the IF for this is x^{-1} and the equation can be rearranged as

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{u}{x}\right) &= -\frac{y}{x}, \\ \Rightarrow \quad \frac{u}{x} &= -y\ln x + f(y), \\ u = 2y \text{ on } x = 1 &\Rightarrow f(y) = 2y, \\ \text{and so } u(x,y) &= xy(2-\ln x). \end{aligned}$$

(b) This equation can be written in standard form, with u as a function of y:

$$\frac{\partial u}{\partial y} - u = -\frac{1}{x},$$

for which the IF is clearly e^{-y}, leading to

$$\begin{aligned} \frac{\partial}{\partial y}\left(e^{-y}u\right) &= -\frac{e^{-y}}{x}, \\ \Rightarrow \quad e^{-y}u &= \frac{e^{-y}}{x} + f(x), \\ u(x,0) = x \Rightarrow f(x) &= x - \frac{1}{x}. \end{aligned}$$

Substituting this result, and multiplying through by e^y, gives $u(x,y)$ as

$$u(x,y) = \frac{1}{x} + \left(x - \frac{1}{x}\right)e^y.$$

20.5 *Find solutions of*

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} = 0$$

for which (a) $u(0,y) = y$ *and* (b) $u(1,1) = 1$.

As usual, we find $p(x,y)$ from

$$\frac{dx}{x^{-1}} = \frac{dy}{y^{-1}} \quad \Rightarrow \quad x^2 - y^2 = p.$$

(a) On $x = 0$, $p = -y^2$ and

$$u(0,y) = y = (-p)^{1/2} \quad \Rightarrow \quad u(x,y) = [-(x^2-y^2)]^{1/2} = (y^2-x^2)^{1/2}.$$

(b) At $(1,1)$, $p = 0$ and

$$u(1,1) = 1 \quad \Rightarrow \quad u(x,y) = 1 + g(x^2-y^2),$$

where g is any function that has $g(0) = 0$.

We note that in part (a) the solution is uniquely determined because the boundary values are given along a line, whereas in part (b), where the value is fixed at only an isolated point, the solution is indeterminate to the extent of a loosely determined function. This is the normal situation, though it is modified if the line in (a) happens to be a characteristic of the PDE.

20.7 *Solve*

$$\sin x \frac{\partial u}{\partial x} + \cos x \frac{\partial u}{\partial y} = \cos x \qquad (*)$$

subject to (a) $u(\pi/2, y) = 0$ *and* (b) $u(\pi/2, y) = y(y+1)$.

As usual, the CF is found from

$$\frac{dx}{\sin x} = \frac{dy}{\cos x} \quad \Rightarrow \quad y - \ln \sin x = p.$$

Since the RHS of $(*)$ is a factor in one of the terms on the LHS, a trivial PI is any function of y only whose derivative (with respect to y) is unity, of which the simplest is $u(x, y) = y$. The general solution is therefore

$$u(x, y) = f(y - \ln \sin x) + y.$$

The actual form of the arbitrary function $f(p)$ is determined by the form that $u(x, y)$ takes on the boundary, here the line $x = \pi/2$.

(a) With $u(\pi/2, y) = 0$:

$$0 = f(y - 0) + y \quad \Rightarrow \quad f(p) = -p$$
$$\Rightarrow \quad u(x, y) = \ln \sin x - y + y = \ln \sin x.$$

(b) With $u(\pi/2, y) = y(y+1)$:

$$y(y+1) = f(y - 0) + y \quad \Rightarrow \quad f(p) = p^2$$
$$\Rightarrow \quad u(x, y) = (y - \ln \sin x)^2 + y.$$

20.9 *If* $u(x, y)$ *satisfies*

$$\frac{\partial^2 u}{\partial x^2} - 3\frac{\partial^2 u}{\partial x \partial y} + 2\frac{\partial^2 u}{\partial y^2} = 0$$

and $u = -x^2$ *and* $\partial u/\partial y = 0$ *for* $y = 0$ *and all* x, *find the value of* $u(0, 1)$.

If we are to find solutions to this homogeneous second-order PDE of the form $u(x, y) = f(x + \lambda y)$, then λ must satisfy

$$1 - 3\lambda + 2\lambda^2 = 0 \quad \Rightarrow \quad \lambda = \tfrac{1}{2},\ 1.$$

Thus $u(x, y) = g(x + \frac{1}{2}y) + f(x + y) \equiv g(p_1) + f(p_2)$.

On $y = 0$, $p_1 = p_2 = x$ and

$$-x^2 = u(x, 0) = g(x) + f(x), \qquad (*)$$

$$0 = \frac{\partial u}{\partial y}(x, 0) = \tfrac{1}{2}g'(x) + f'(x).$$

From $(*)$, $\quad -2x = g'(x) + f'(x).$

Subtracting, $\quad 2x = -\frac{1}{2}g'(x).$

Integrating, $\quad g(x) = -2x^2 + k \quad \Rightarrow \quad f(x) = x^2 - k, \quad$ from $(*)$.

Hence, $\quad u(x, y) = -2(x + \frac{1}{2}y)^2 + k + (x + y)^2 - k$

$$= -x^2 + \tfrac{1}{2}y^2.$$

At the particular point $(0, 1)$ we have $u(0, 1) = -0^2 + \frac{1}{2}(1)^2 = \frac{1}{2}$.

20.11 *In those cases in which it is possible to do so, evaluate $u(2, 2)$, where $u(x, y)$ is the solution of*

$$2y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = xy(2y^2 - x^2)$$

that satisfies the (separate) boundary conditions given below.

(a) $u(x, 1) = x^2$ *for all* x.
(b) $u(x, 1) = x^2$ *for* $x \geq 0$.
(c) $u(x, 1) = x^2$ *for* $0 \leq x \leq 3$.
(d) $u(x, 0) = x$ *for* $x \geq 0$.
(e) $u(x, 0) = x$ *for all* x.
(f) $u(1, \sqrt{10}) = 5$.
(g) $u(\sqrt{10}, 1) = 5$.

To find the CF, $u(x, y) = f(p)$, we set

$$\frac{dx}{2y} = -\frac{dy}{x} \quad \Rightarrow \quad x^2 + 2y^2 = p.$$

This result also defines the characteristic curves, which are right ellipses centred on the origin with semi-axes of lengths $\sqrt{p}$ and $\sqrt{p/2}$. The point $(2, 2)$ lies on the characteristic with $p = 2^2 + 2(2^2) = 12$; we will only be able to determine the value of $u(2, 2)$ if this curve cuts the boundary on which u is specified.

For a PI we try $u(x, y) = Ax^n y^m$:

$$2Anx^{n-1}y^{m+1} - Amx^{n+1}y^{m-1} = 2xy^3 - x^3y,$$

which has a solution, $n = m = 2$ with $A = \frac{1}{2}$. Thus the general solution is

$$u(x, y) = f(x^2 + 2y^2) + \tfrac{1}{2}x^2y^2.$$

(a) With $u(x, 1) = x^2$ for all x:

$$\begin{aligned} x^2 = u(x,1) &= f(x^2+2) + \tfrac{1}{2}x^2 \\ \Rightarrow \qquad f(p) &= \tfrac{1}{2}(p-2) \\ \Rightarrow \quad u(x,y) &= \tfrac{1}{2}(x^2+2y^2-2) + \tfrac{1}{2}x^2y^2 \\ &= \tfrac{1}{2}(x^2+x^2y^2+2y^2-2), \\ u(2,2) &= \tfrac{1}{2}(4+16+8-2) = 13. \end{aligned}$$

The line $y = 1$ cuts each characteristic in zero (for $p < 2$) or two (for $p > 2$) distinct points. Here $p = 12$ (> 2) and the characteristic (ellipse) that passes through $(2, 2)$ cuts the boundary (the line $y = 1$) in two places. In general, a double intersection can lead to inconsistencies and hence to no solution. However, it causes no difficulty with the given boundary conditions since the required values of x^2 at $x = \pm\sqrt{12 - 2(1^2)}$ are equal and u is a even function of x.

(b) With $u(x, 1) = x^2$ for $x \geq 0$.
Since every characteristic ellipse (with $p > 2$) cuts the line $y = 1$ once (and only once in $x > 0$), this gives the same result as in part (a).

(c) With $u(x, 1) = x^2$ for $0 \leq x \leq 3$.
The ellipses that cut the line $y = 1$ with $0 \leq x \leq 3$ have p-values lying between $0^2 + 2(1)^2 = 2$ and $3^2 + 2(1)^2 = 11$. Thus the $p = 12$ curve does not do so and $u(2, 2)$ is undetermined.

(d) With $u(x, 0) = x$ for $x \geq 0$:

$$\begin{aligned} x = u(x,0) &= f(x^2+0) + 0 \\ \Rightarrow \qquad f(p) &= p^{1/2} \\ \Rightarrow \quad u(x,y) &= (x^2+2y^2)^{1/2} + \tfrac{1}{2}x^2y^2 \\ \Rightarrow \quad u(2,2) &= \sqrt{12} + 8. \end{aligned}$$

The characteristic (ellipse) $p = 12$ cuts the positive x-axis (i.e. $y = 0$) in one and only one place ($x = +\sqrt{12}$) and so the solution is well defined and the above evaluation valid.

(e) With $u(x, 0) = x$ for all x.
This is as in part (d) except that now a characteristic ellipse cuts the defining boundary in two places, $x = \pm\sqrt{p}$, and requires both $u(\sqrt{p}, 0) = \sqrt{p}$ and

$u(-\sqrt{p}, 0) = -\sqrt{p}$. Since $\sqrt{z}$ is not differentiable at $z = 0$, this is not possible and no solution exists.

(f) With $u(1, \sqrt{10}) = 5$.
At the point $(1, \sqrt{10})$ the value of p is $1 + 2(10) = 21$. As the 'boundary' consists of just this one point, it is only at the points that lie on the characteristic $p = 21$ that the value of $u(x, y)$ can be known. Since for the point $(2, 2)$ the value of p is 12, the value of $u(2, 2)$ cannot be determined.

(g) With $u(\sqrt{10}, 1) = 5$.
At the point $(\sqrt{10}, 1)$ the value of p is $10 + 2(1) = 12$. Since for $(2, 2)$ it is also 12 the value of $u(2, 2)$ is determined and is given by $f(12) + \frac{1}{2}(4)(4) = 5 + 8 = 13$.

20.13 *The solution to the equation*

$$6\frac{\partial^2 u}{\partial x^2} - 5\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 14$$

that satisfies $u = 2x + 1$ *and* $\partial u/\partial y = 4 - 6x$, *both on the line* $y = 0$, *is*

$$u(x, y) = -8y^2 - 6xy + 2x + 4y + 1.$$

By changing the independent variables in the equation to

$$\xi = x + 2y \qquad \text{and} \qquad \eta = x + 3y,$$

show that it must be possible to write $14(x^2 + 5xy + 6y^2)$ *in the form*

$$f_1(x + 2y) + f_2(x + 3y) - (x^2 + y^2),$$

and determine the forms of $f_1(z)$ *and* $f_2(z)$.

Let $u(x, y) = v(\xi, \eta)$, with $\xi = x + 2y$ and $\eta = x + 3y$. We must first express the differential operators $\partial/\partial x$ and $\partial/\partial y$ in terms of differentiation with respect to ξ and η; to do this we use the chain rule:

$$\frac{\partial}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}; \text{ similarly } \frac{\partial}{\partial y} = 2\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}.$$

The equation becomes

$$6\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial v}{\partial \xi} + \frac{\partial v}{\partial \eta}\right) - 5\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(2\frac{\partial v}{\partial \xi} + 3\frac{\partial v}{\partial \eta}\right)$$
$$+ \left(2\frac{\partial}{\partial \xi} + 3\frac{\partial}{\partial \eta}\right)\left(2\frac{\partial v}{\partial \xi} + 3\frac{\partial v}{\partial \eta}\right) = 14,$$

$$(6 - 10 + 4)\frac{\partial^2 v}{\partial \xi^2} + (12 - 25 + 12)\frac{\partial^2 v}{\partial \xi \partial \eta} + (6 - 15 + 9)\frac{\partial^2 v}{\partial \eta^2} = 14.$$

Collecting similar terms together, we find

$$\frac{\partial^2 v}{\partial\xi\partial\eta} = -14.$$

This equation has a CF of the form $f(\xi) + g(\eta)$ and a PI of $-14\xi\eta$. Thus its general solution is

$$v(\xi, \eta) = f(\xi) + g(\eta) - 14\xi\eta.$$

This must be the same as the given answer, i.e.

$$-8y^2 - 6xy + 2x + 4y + 1 = f(x + 2y) + g(x + 3y) - 14(x + 2y)(x + 3y)$$

for some functions f and g. Thus

$$\begin{aligned} w(x, y) &= 14(x^2 + 5xy + 6y^2) \\ &= 14(x + 2y)(x + 3y) \\ &= f(x + 2y) + g(x + 3y) + 8y^2 + 6xy - 2x - 4y - 1 \\ &= f(x + 2y) + g(x + 3y) - (x^2 + y^2) + h(x, y), \end{aligned}$$

where
$$\begin{aligned} h(x, y) &= x^2 + 9y^2 + 6xy - 2x - 4y - 1 \\ &= (x + 3y)^2 - 2(x + 2y) - 1 \\ &= F(x + 2y) + G(x + 3y). \end{aligned}$$

It follows that

$$w(x, y) = f_1(x + 2y) + f_2(x + 3y) - (x^2 + y^2),$$

where $f_1(z) = f(z) - 2z - 1$ and $f_2(z) = g(z) + z^2$.

After rearrangement this reads

$$15x^2 + 70xy + 85y^2 = f_1(x + 2y) + f_2(x + 3y). \qquad (**)$$

Taking second derivatives with respect to x and y separately,

$$\begin{aligned} 30 &= f_1'' + f_2'', \\ 170 &= 4f_1'' + 9f_2'', \\ \Rightarrow \quad 50 &= 5f_2'' \quad \Rightarrow \quad f_2(z) = 5z^2 + \alpha z + \beta \\ \text{and } 100 &= 5f_1'' \quad \Rightarrow \quad f_1(z) = 10z^2 + \gamma z + \delta. \end{aligned}$$

Equating the coefficients of xy, x and y and the constants in $(**)$ gives $70 = 40+30$, $0 = \alpha + \gamma$, $0 = 3\alpha + 2\gamma$, $0 = \beta + \delta$. These equations have the solution $\alpha = \gamma = 0$ and $\beta = k = -\delta$. Thus

$$f_1(z) = 10z^2 - k \qquad \text{and} \qquad f_2(z) = 5z^2 + k.$$

Clearly, k can take any value without affecting the final form given in the question.

20.15 *Find the most general solution of* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 = x^2 y^2$.

The complementary function for this equation is the solution to the two-dimensional Laplace equation and [either as a general known result or from substituting the trial form $h(x+\lambda y)$ which leads to $\lambda^2 = -1$ and hence to $\lambda = \pm i$] has the form $f(x+iy) + g(x-iy)$ for arbitrary functions f and g.

It therefore remains only to find a suitable PI. As f and g are not specified, there are infinitely many possibilities and which one we finish up with will depend upon the details of the approach adopted. When a solution has been obtained it should be checked by substitution.

As no PI is obvious by inspection, we make a change of variables with the object of obtaining one by means of an explicit integration. To do this, we use as new variables the arguments of the arbitrary functions appearing in the CF.

Setting $\xi = x + iy$ and $\eta = x - iy$, with $u(x,y) = v(\xi,\eta)$, gives

$$\begin{aligned}
&\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\left(\frac{\partial v}{\partial\xi} + \frac{\partial v}{\partial\eta}\right) \\
&\qquad + \left(i\frac{\partial}{\partial\xi} - i\frac{\partial}{\partial\eta}\right)\left(i\frac{\partial v}{\partial\xi} - i\frac{\partial v}{\partial\eta}\right) = \left(\frac{\xi+\eta}{2}\right)^2\left(\frac{\xi-\eta}{2i}\right)^2, \\
&(1-1)\frac{\partial^2 v}{\partial\xi^2} + (2+2)\frac{\partial^2 v}{\partial\xi\partial\eta} + (1-1)\frac{\partial^2 v}{\partial\eta^2} = -\frac{1}{16}(\xi^2-\eta^2)^2, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \frac{\partial^2 v}{\partial\xi\partial\eta} = -\frac{1}{64}(\xi^2-\eta^2)^2.
\end{aligned}$$

When we integrate this we can set all constants of integration and all arbitrary functions equal to zero as *any* solution will suffice:

$$\begin{aligned}
\frac{\partial^2 v}{\partial\xi\partial\eta} &= -\frac{1}{64}(\xi^4 - 2\xi^2\eta^2 + \eta^4), \\
\frac{\partial v}{\partial\eta} &= -\frac{1}{64}\left(\frac{\xi^5}{5} - \frac{2\xi^3\eta^2}{3} + \xi\eta^4\right), \\
v &= -\frac{1}{64}\left(\frac{\xi^5\eta}{5} - \frac{2\xi^3\eta^3}{9} + \frac{\xi\eta^5}{5}\right).
\end{aligned}$$

Re-expressing this solution as a function of x and y (noting that $\xi\eta = x^2 + y^2$)

gives

$$\begin{aligned}
u(x,y) &= \frac{1}{(64)(45)}[\,10\xi^3\eta^3 - 9\xi\eta(\xi^4+\eta^4)\,] \\
&= \frac{1}{(64)(45)}[\,10(x^2+y^2)^3 - 18(x^2+y^2)(x^4-6x^2y^2+y^4)\,] \\
&= \frac{x^2+y^2}{(64)(45)}(10x^4+20x^2y^2+10y^4-18x^4+108x^2y^2-18y^4) \\
&= \frac{x^2+y^2}{(64)(45)}(128x^2y^2-8x^4-8y^4) \\
&= \frac{1}{360}(15x^4y^2-x^6+15x^2y^4-y^6).
\end{aligned}$$

Check

Applying $\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}$ to the final expression yields

$$\frac{1}{360}[\,15(12)x^2y^2-30x^4+30y^4+30x^4+15(12)x^2y^2-30y^4\,] = x^2y^2,$$

as it should.

20.17 *The non-relativistic Schrödinger equation,*

$$-\frac{\hbar^2}{2m}\nabla^2 u + V(\mathbf{r})u = i\hbar\frac{\partial u}{\partial t},$$

is similar to the diffusion equation in having different orders of derivatives in its various terms; this precludes solutions that are arbitrary functions of particular linear combinations of variables. However, since exponential functions do not change their forms under differentiation, solutions in the form of exponential functions of combinations of the variables may still be possible.

Consider the Schrödinger equation for the case of a constant potential, i.e. for a free particle, and show that it has solutions of the form $A\exp(lx+my+nz+\lambda t)$, *where the only requirement is that*

$$-\frac{\hbar^2}{2m}\left(l^2+m^2+n^2\right) = i\hbar\lambda.$$

In particular, identify the equation and wavefunction obtained by taking λ *as* $-iE/\hbar$, *and* l, m *and* n *as* $ip_x/\hbar, ip_y/\hbar$ *and* $ip_z/\hbar$, *respectively, where* E *is the energy and* $\mathbf{p}$ *the momentum of the particle; these identifications are essentially the content of the de Broglie and Einstein relationships.*

For a free particle we may omit the potential term $V(\mathbf{r})$ from the Schrödinger

equation, which then reads (in Cartesian coordinates)

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=i\hbar\frac{du}{dt}.$$

We try $u(x,y,z,t)=A\exp(lx+my+nz+\lambda t)$, i.e. the product of four exponential functions, and obtain

$$-\frac{\hbar^2}{2m}(l^2+m^2+n^2)u=i\hbar\lambda u.$$

This equation is clearly satisfied provided

$$-\frac{\hbar^2}{2m}(l^2+m^2+n^2)=i\hbar\lambda.$$

With λ as $-iE/\hbar$, and l,m and n as $ip_x/\hbar, ip_y/\hbar$ and $ip_z/\hbar$, respectively, where E is the energy and $\mathbf{p}$ is the momentum of the particle, we have

$$-\frac{\hbar^2}{2m}\left(-\frac{p_x^2}{\hbar^2}-\frac{p_y^2}{\hbar^2}-\frac{p_z^2}{\hbar^2}\right)=E,$$

which can be written more compactly as $E=p^2/2m$, the classical non-relativistic relationship between the (kinetic) energy and momentum of a free particle.

The wavefunction obtained is

$$\begin{aligned}u(\mathbf{r},t)&=A\exp\left[\frac{i}{\hbar}(p_x x+p_y y+p_z z-Et)\right]\\&=A\exp\left[\frac{i}{\hbar}(\mathbf{p}\cdot\mathbf{r}-Et)\right],\end{aligned}$$

i.e. a classical plane wave of wave number $\mathbf{k}=\mathbf{p}/\hbar$ and angular frequency $\omega=E/\hbar$ travelling in the direction $\mathbf{p}/p$.

20.19 *An incompressible fluid of density ρ and negligible viscosity flows with velocity v along a thin, straight, perfectly light and flexible tube, of cross-section A which is held under tension T. Assume that small transverse displacements u of the tube are governed by*

$$\frac{\partial^2 u}{\partial t^2}+2v\frac{\partial^2 u}{\partial x\partial t}+\left(v^2-\frac{T}{\rho A}\right)\frac{\partial^2 u}{\partial x^2}=0.$$

(a) *Show that the general solution consists of a superposition of two waveforms travelling with different speeds.*

(b) *The tube initially has a small transverse displacement $u=a\cos kx$ and is suddenly released from rest. Find its subsequent motion.*

(a) This is a second-order equation and will (in general) have two solutions of the form $u(x,t) = f(x + \lambda t)$, where both λ satisfy

$$\lambda^2 + 2v\lambda + \left(v^2 - \frac{T}{\rho A}\right) = 0 \quad \Rightarrow \quad \lambda = -v \pm \sqrt{v^2 - v^2 + \frac{T}{\rho A}} \equiv -v \pm \alpha,$$

and gives (minus) the speed of the corresponding profile. Thus the general displacement consists of a superposition of waveforms travelling with speeds $v \mp \alpha$.

(b) Now $u(x,0) = a\cos kx$ and $\dot{u}(x,0) = 0$, where the dot denotes differentiation with respect to time t. Let the general solution be given by

$$u(x,t) = f[x - (v+\alpha)t] + g[x - (v-\alpha)t],$$

with $\quad a\cos kx = f(x) + g(x)$

and $\quad 0 = -(v+\alpha)f'(x) - (v-\alpha)g'(x).$

We differentiate the first of these with respect to x and then eliminate the function $f'(x)$:

$$\begin{aligned}
-ka\sin kx &= f'(x) + g'(x),\\
-ka(v+\alpha)\sin kx &= (v+\alpha-v+\alpha)g'(x),\\
g'(x) &= -\frac{ka(v+\alpha)}{2\alpha}\sin kx,\\
\Rightarrow \quad g(x) &= \frac{v+\alpha}{2\alpha}a\cos kx + c,\\
\Rightarrow \quad f(x) &= \frac{\alpha-v}{2\alpha}a\cos kx - c.
\end{aligned}$$

Now that the forms of the initially arbitrary functions $f(x)$ and $g(x)$ have been determined, it follows that, for a *general* time t,

$$\begin{aligned}
u(x,t) &= \frac{\alpha-v}{2\alpha}a\cos[kx - k(v+\alpha)t] + \frac{\alpha+v}{2\alpha}a\cos[kx - k(v-\alpha)t]\\
&= \frac{a}{2}2\cos(kx - kvt)\cos k\alpha t + \frac{va}{2\alpha}2\sin(kx - kvt)\sin(-k\alpha t)\\
&= a\cos[k(x-vt)]\cos k\alpha t - \frac{va}{\alpha}\sin[k(x-vt)]\sin k\alpha t.
\end{aligned}$$

20.21 *In an electrical cable of resistance R and capacitance C, each per unit length, voltage signals obey the equation $\partial^2 V/\partial x^2 = RC\partial V/\partial t$. This (diffusion-type) equation has solutions of the form*

$$f(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} \exp(-v^2)\, dv, \text{ where } \zeta = \frac{x(RC)^{1/2}}{2t^{1/2}}.$$

It also has solutions of the form $V = Ax + D$.

(a) *Find a combination of these that represents the situation after a steady voltage V_0 is applied at $x = 0$ at time $t = 0$.*
(b) *Obtain a solution describing the propagation of the voltage signal resulting from the application of the signal $V = V_0$ for $0 < t < T$, $V = 0$ otherwise, to the end $x = 0$ of an infinite cable.*
(c) *Show that for $t \gg T$ the maximum signal occurs at a value of x proportional to $t^{1/2}$ and has a magnitude proportional to t^{-1}.*

(a) Consider the given function

$$f(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} \exp(-v^2)\, dv, \text{ where } \zeta = \frac{x(RC)^{1/2}}{2t^{1/2}}.$$

The requirements to be satisfied by the correct combination of this function and $V(x,t) = Ax + D$ are (i) that, at $t = 0$, V is zero for all x, except $x = 0$ where it is V_0, and (ii) that, as $t \to \infty$, V is V_0 for all x.

(i) At $t = 0$, $\zeta = \infty$ and $f(\zeta) = 1$ for all $x \neq 0$.
(ii) As $t \to \infty$, $\zeta \to 0$ and $f(\zeta) \to 0$ for all finite x.

The required combination is therefore $D = V_0$ and $-V_0 f(\zeta)$, i.e.

$$V(x,t) = V_0 \left[1 - \frac{2}{\sqrt{\pi}} \int_0^{\frac{1}{2}x(CR/t)^{1/2}} \exp(-v^2)\, dv \right].$$

(b) The equation is linear and so we may superpose solutions. The response to the input $V = V_0$ for $0 < t < T$ can be considered as that to V_0 applied at $t = 0$ and continued, together with $-V_0$ applied at $t = T$ and continued. The solution is therefore the difference between two solutions of the form found in part (a):

$$V(x,t) = \frac{2V_0}{\sqrt{\pi}} \int_{\frac{1}{2}x(CR/t)^{1/2}}^{\frac{1}{2}x[CR/(t-T)]^{1/2}} \exp\left(-v^2\right)\, dv.$$

(c) To find the maximum signal we set $\partial V/\partial x$ equal to zero. Remembering that we are differentiating with respect to the limits of an integral (whose integrand

does not contain x explicitly), we obtain

$$\frac{1}{2}\left(\frac{CR}{t-T}\right)^{1/2}\exp\left[-\frac{x^2CR}{4(t-T)}\right]-\frac{1}{2}\left(\frac{CR}{t}\right)^{1/2}\exp\left[-\frac{x^2CR}{4t}\right]=0.$$

This requires

$$\begin{aligned}\left(\frac{t-T}{t}\right)^{1/2}&=\exp\left[-\frac{x^2CR}{4(t-T)}+\frac{x^2CR}{4t}\right]\\&=\exp\left[\frac{x^2CR(-t+t-T)}{4t(t-T)}\right].\end{aligned}$$

For $t \gg T$, we expand both sides:

$$\begin{aligned}&1-\frac{1}{2}\frac{T}{t}+\cdots=1-\frac{Tx^2CR}{4t^2}+\cdots,\\&\Rightarrow\quad x^2\approx\frac{2t}{CR}\quad\Rightarrow\quad v=\frac{1}{2}\sqrt{\frac{2t}{CR}}\left(\frac{CR}{t}\right)^{1/2}=\frac{1}{\sqrt{2}}.\end{aligned}$$

The corresponding value of V is approximately equal to the value of the integrand, evaluated at this value of v, multiplied by the difference between the two limits of the integral. Thus

$$\begin{aligned}V_{\max}&\approx\frac{2V_0}{\sqrt{\pi}}\exp(-v^2)\frac{x\sqrt{CR}}{2}\left[\frac{1}{(t-T)^{1/2}}-\frac{1}{t^{1/2}}\right]\\&\approx\frac{2V_0}{\sqrt{\pi}}e^{-1/2}\frac{x\sqrt{CR}}{2}\frac{1}{2}\frac{T}{t^{3/2}}\\&=\frac{V_0Te^{-1/2}}{\sqrt{2\pi}\,t}.\end{aligned}$$

In summary, for $t \gg T$ the maximum signal occurs at a value of x proportional to $t^{1/2}$ and has a magnitude proportional to t^{-1}.

20.23 *Consider each of the following situations in a qualitative way and determine the equation type, the nature of the boundary curve and the type of boundary conditions involved:*

(a) *a conducting bar given an initial temperature distribution and then thermally isolated;*
(b) *two long conducting concentric cylinders, on each of which the voltage distribution is specified;*
(c) *two long conducting concentric cylinders, on each of which the charge distribution is specified;*
(d) *a semi-infinite string, the end of which is made to move in a prescribed way.*

We use the notation

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = R(x, y)$$

to express the most general type of PDE, and the following table

Equation type	Boundary	Conditions
hyperbolic	open	Cauchy
parabolic	open	Dirichlet or Neumann
elliptic	closed	Dirichlet or Neumann

to determine the appropriate boundary type and hence conditions.

(a) The diffusion equation $\kappa\dfrac{\partial^2 T}{\partial x^2} = \dfrac{\partial T}{\partial t}$ has $A = \kappa$, $B = 0$ and $C = 0$; thus $B^2 = 4AC$ and the equation is parabolic. This needs an open boundary. In the present case, the initial heat distribution (at the $t = 0$ boundary) is a Dirichlet condition and the insulation (no temperature gradient at the external surfaces) is a Neumann condition.

(b) The governing equation in two-dimensional Cartesians (not the natural choice for this situation, but this does not matter for the present purpose) is the Laplace equation, $\dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} = 0$, which has $A = 1$, $B = 0$ and $C = 1$ and therefore $B^2 < 4AC$. The equation is therefore elliptic and requires a closed boundary. Since ϕ is specified on the cylinders, the boundary conditions are Dirichlet in this particular situation.

(c) This is the same as part (b) except that the specified charge distribution σ determines $\partial\phi/\partial n$, through $\partial\phi/\partial n = \sigma/\epsilon_0$, and imposes Neumann boundary conditions.

(d) For the wave equation $\dfrac{\partial^2 u}{\partial x^2} - \dfrac{1}{c^2}\dfrac{\partial^2 u}{\partial t^2} = 0$, we have $A = 1$, $B = 0$ and $C = -c^{-2}$, thus making $B^2 > 4AC$ and the equation hyperbolic. We thus require an open boundary and Cauchy conditions, with the displacement of the end of the string having to be specified at all times — this is equivalent to the displacement and the velocity of the end of the string being specified at all times.

20.25 *The Klein–Gordon equation (which is satisfied by the quantum-mechanical wavefunction $\Phi(\mathbf{r})$ of a relativistic spinless particle of non-zero mass m) is*

$$\nabla^2\Phi - m^2\Phi = 0.$$

Show that the solution for the scalar field $\Phi(\mathbf{r})$ in any volume V bounded by a surface S is unique if either Dirichlet or Neumann boundary conditions are specified on S.

Suppose that, for a given set of boundary conditions ($\Phi = f$ or $\partial\Phi/\partial n = g$ on S), there are two solutions to the Klein–Gordon equation, Φ_1 and Φ_2. Then consider $\Phi_3 = \Phi_1 - \Phi_2$, which satisfies

$$\nabla^2\Phi_3 = \nabla^2\Phi_1 - \nabla^2\Phi_2 = m^2\Phi_1 - m^2\Phi_2 = m^2\Phi_3$$

and

$$\text{either } \Phi_3 = f - f = 0, \text{ or } \frac{\partial\Phi_3}{\partial n} = g - g = 0 \text{ on } S.$$

Now apply Green's first theorem with the scalar functions equal to Φ_3 and Φ_3^*:

$$\int^S \Phi_3^* \frac{\partial\Phi_3}{\partial n}\, dS = \int_V [\,\Phi_3^*\nabla^2\Phi_3 + (\nabla\Phi_3^*)\cdot(\nabla\Phi_3)\,]\, dV,$$

$$\Rightarrow \quad 0 = \int_V (m^2|\Phi_3|^2 + |\nabla\Phi_3|^2)\, dV,$$

whichever set of boundary conditions applies. Since both terms in the integrand on the RHS are non-negative, each must be equal to zero. In particular, $|\Phi_3| = 0$ implies that $\Phi_3 = 0$ everywhere, i.e $\Phi_1 = \Phi_2$ everywhere; the solution is unique.

21

Partial differential equations: separation of variables and other methods

21.1 *Solve the following first-order partial differential equations by separating the variables:*

$$\text{(a)}\ \frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = 0; \qquad \text{(b)}\ x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} = 0.$$

In each case we write $u(x, y) = X(x)Y(y)$, separate the variables into groups that each depend on only one variable, and then assert that each must be equal to a constant, with the several constants satisfying an arithmetic identity.

(a)
$$\begin{aligned}
\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} &= 0,\\
X'Y - xXY' &= 0,\\
\frac{X'}{xX} = \frac{Y'}{Y} = k \quad &\Rightarrow \quad \ln X = \tfrac{1}{2}kx^2 + c_1, \quad \ln Y = ky + c_2,\\
&\Rightarrow \quad X = Ae^{kx^2/2}, \quad Y = Be^{ky},\\
\Rightarrow \quad u(x, y) &= Ce^{\lambda(x^2+2y)}, \text{ where } k = 2\lambda.
\end{aligned}$$

(b)
$$\begin{aligned}
x\frac{\partial u}{\partial x} - 2y\frac{\partial u}{\partial y} &= 0,\\
xX'Y - 2yXY' &= 0,\\
\frac{xX'}{X} = \frac{2yY'}{Y} = k \quad &\Rightarrow \quad \ln X = k\ln x + c_1,\\
&\qquad\quad \ln Y = \tfrac{1}{2}k\ln y + c_2,\\
&\Rightarrow \quad X = Ax^k, \quad Y = By^{k/2},\\
\Rightarrow \quad u(x, y) &= C(x^2y)^{\lambda}, \text{ where } k = 2\lambda.
\end{aligned}$$

21.3 *The wave equation describing the transverse vibrations of a stretched membrane under tension T and having a uniform surface density ρ is*

$$T\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=\rho\frac{\partial^2 u}{\partial t^2}.$$

Find a separable solution appropriate to a membrane stretched on a frame of length a and width b, showing that the natural angular frequencies of such a membrane are given by

$$\omega^2=\frac{\pi^2 T}{\rho}\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right),$$

where n and m are any positive integers.

We seek solutions $u(x,y,t)$ that are periodic in time and have $u(0,y,t)=u(a,y,t)=u(x,0,t)=u(x,b,t)=0$. Write $u(x,y,t)=X(x)Y(y)S(t)$ and substitute, obtaining

$$T(X''YS+XY''S)=\rho XYS'',$$

which, when divided through by XYS, gives

$$\frac{X''}{X}+\frac{Y''}{Y}=\frac{\rho}{T}\frac{S''}{S}=-\frac{\omega^2\rho}{T}.$$

The second equality, obtained by applying the separation of variables principle with separation constant $-\omega^2\rho/T$, gives $S(t)$ as a sinusoidal function of t of frequency ω, i.e. $A\cos(\omega t)+B\sin(\omega t)$.

We then have, on applying the separation of variables principle a second time, that

$$\frac{X''}{X}=\lambda \text{ and } \frac{Y''}{Y}=\mu, \text{ where } \lambda+\mu=-\frac{\omega^2\rho}{T}. \qquad (*)$$

These equations must also have sinusoidal solutions. This is because, since $u(0,y,t)=u(a,y,t)=u(x,0,t)=u(x,b,t)=0$, each solution has to have zeros at two different values of its argument. We are thus led to

$$X=A\sin(p\,x) \text{ and } Y=B\sin(q\,x), \text{ where } p^2=-\lambda \text{ and } q^2=-\mu.$$

Further, since $u(a,y,t)=u(x,b,t)=0$, we must have $p=n\pi/a$ and $q=m\pi/b$, where n and m are integers. Putting these values back into $(*)$ gives

$$-p^2-q^2=-\frac{\omega^2\rho}{T} \quad\Rightarrow\quad \pi^2\left(\frac{n^2}{a^2}+\frac{m^2}{b^2}\right)=\frac{\omega^2\rho}{T}.$$

Hence the quoted result.

21.5 *Denoting the three terms of ∇^2 in spherical polars by ∇_r^2, ∇_θ^2, ∇_ϕ^2 in an obvious way, evaluate $\nabla_r^2 u$, etc. for the two functions given below and verify that, in each case, although the individual terms are not necessarily zero their sum $\nabla^2 u$ is zero. Identify the corresponding values of ℓ and m.*

(a) $u(r,\theta,\phi) = \left(Ar^2 + \dfrac{B}{r^3}\right)\dfrac{3\cos^2\theta - 1}{2}$.

(b) $u(r,\theta,\phi) = \left(Ar + \dfrac{B}{r^2}\right)\sin\theta \exp i\phi$.

In both cases we write $u(r,\theta,\phi)$ as $R(r)\Theta(\theta)\Phi(\phi)$ with

$$\nabla_r^2 = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right),\ \nabla_\theta^2 = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right),\ \nabla_\phi^2 = \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}.$$

(a) $u(r,\theta,\phi) = \left(Ar^2 + \dfrac{B}{r^3}\right)\dfrac{3\cos^2\theta - 1}{2}$.

$$\nabla_r^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(2Ar^3 - \frac{3B}{r^2}\right)\Theta = \left(6A + \frac{6B}{r^5}\right)\Theta = \frac{6u}{r^2},$$

$$\begin{aligned}\nabla_\theta^2 u &= \frac{R}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(-3\sin^2\theta\cos\theta) = \frac{R}{r^2}\left(\frac{-6\sin\theta\cos^2\theta + 3\sin^3\theta}{\sin\theta}\right)\\ &= \frac{R}{r^2}(-9\cos^2\theta + 3) = -\frac{6u}{r^2},\end{aligned}$$

$$\nabla_\phi^2 u = 0.$$

Thus, although $\nabla_r^2 u$ and $\nabla_\theta^2 u$ are not individually zero, their sum is. From $\nabla_r^2 u = \ell(\ell+1)u = 6u$, we deduce that $\ell = 2$ (or -3) and from $\nabla_\phi^2 u = 0$ that $m = 0$.

(b) $u(r,\theta,\phi) = \left(Ar + \dfrac{B}{r^2}\right)\sin\theta\, e^{i\phi}$.

$$\nabla_r^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^2 - \frac{2B}{r}\right)\Theta\Phi = \left(\frac{2A}{r} + \frac{2B}{r^4}\right)\Theta\Phi = \frac{2u}{r^2},$$

$$\begin{aligned}\nabla_\theta^2 u &= \frac{R\Phi}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\cos\theta) = \frac{R\Phi}{r^2}\left(\frac{-\sin^2\theta + \cos^2\theta}{\sin\theta}\right)\\ &= -\frac{u}{r^2} + \frac{\cos^2\theta}{\sin^2\theta}\frac{u}{r^2},\end{aligned}$$

$$\nabla_\phi^2 u = \frac{R\Theta}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}(e^{i\phi}) = -\frac{u}{r^2\sin^2\theta}.$$

Hence,

$$\nabla^2 u = \frac{2u}{r^2} - \frac{u}{r^2} + \frac{\cos^2\theta}{\sin^2\theta}\frac{u}{r^2} - \frac{u}{r^2\sin^2\theta} = \frac{u}{r^2}\left(1 + \frac{\cos^2\theta - 1}{\sin^2\theta}\right) = 0.$$

Here each individual term is non-zero, but their sum *is* zero. Further, $\ell(\ell+1) = 2$ and so $\ell = 1$ (or -2), and from $\nabla^2_\phi u = -u/(r^2\sin\theta)$ it follows that $m^2 = 1$. In fact, from the normal definition of spherical harmonics, $m = +1$.

21.7 *If the stream function $\psi(r,\theta)$ for the flow of a very viscous fluid past a sphere is written as $\psi(r,\theta) = f(r)\sin^2\theta$, then $f(r)$ satisfies the equation*

$$f^{(4)} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0.$$

At the surface of the sphere $r = a$ the velocity field $\mathbf{u} = \mathbf{0}$, whilst far from the sphere $\psi \simeq (Ur^2\sin^2\theta)/2$.

Show that $f(r)$ can be expressed as a superposition of powers of r, and determine which powers give acceptable solutions. Hence show that

$$\psi(r,\theta) = \frac{U}{4}\left(2r^2 - 3ar + \frac{a^3}{r}\right)\sin^2\theta.$$

For solutions of

$$f^{(4)} - \frac{4f''}{r^2} + \frac{8f'}{r^3} - \frac{8f}{r^4} = 0$$

that are powers of r, i.e. have the form Ar^n, n must satisfy the quartic equation

$$\begin{aligned} n(n-1)(n-2)(n-3) - 4n(n-1) + 8n - 8 &= 0, \\ (n-1)[\,n(n-2)(n-3) - 4n + 8\,] &= 0, \\ (n-1)(n-2)[\,n(n-3) - 4\,] &= 0, \\ (n-1)(n-2)(n-4)(n+1) &= 0. \end{aligned}$$

Thus the possible powers are 1, 2, 4 and -1.

Since $\psi \to \frac{1}{2}Ur^2\sin^2\theta$ as $r \to \infty$, the solution can contain no higher (positive) power of r than the second. Thus there is no $n = 4$ term and the solution has the form

$$\psi(r,\theta) = \left(\frac{Ur^2}{2} + Ar + \frac{B}{r}\right)\sin^2\theta.$$

On the surface of the sphere $r = a$ both velocity components, u_r and u_θ, are zero. These components are given in terms of the stream functions, as shown below;

note that u_r is found by differentiating with respect to θ and u_θ by differentiating with respect to r.

$$u_r = 0 \quad \Rightarrow \quad \frac{1}{a^2 \sin\theta}\frac{\partial\psi}{\partial\theta} = 0 \quad \Rightarrow \quad \frac{Ua^2}{2} + Aa + \frac{B}{a} = 0,$$

$$u_\theta = 0 \quad \Rightarrow \quad \frac{-1}{a\sin\theta}\frac{\partial\psi}{\partial r} = 0 \quad \Rightarrow \quad Ua + A - \frac{B}{a^2} = 0,$$

$$\Rightarrow \quad A = -\tfrac{3}{4}Ua \text{ and } B = \tfrac{1}{4}Ua^3.$$

The full solution is thus

$$\psi(r,\theta) = \frac{U}{4}\left(2r^2 - 3ar + \frac{a^3}{r}\right)\sin^2\theta.$$

21.9 *A circular disc of radius a is heated in such a way that its perimeter $\rho = a$ has a steady temperature distribution $A + B\cos^2\phi$, where ρ and ϕ are plane polar coordinates and A and B are constants. Find the temperature $T(\rho,\phi)$ everywhere in the region $\rho < a$.*

This is a steady state problem, for which the (heat) diffusion equation becomes the Laplace equation. The most general single-valued solution to the Lapace equation in plane polar coordinates is given by

$$T(\rho,\phi) = C\ln\rho + D + \sum_{n=1}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)(C_n\rho^n + D_n\rho^{-n}).$$

The region $\rho < a$ contains the point $\rho = 0$; since $\ln\rho$ and all ρ^{-n} become infinite at that point, $C = D_n = 0$ for all n.

On $\rho = a$

$$T(a,\phi) = A + B\cos^2\phi = A + \tfrac{1}{2}B(\cos 2\phi + 1).$$

Equating the coefficients of $\cos n\phi$, including $n = 0$, gives $A + \frac{1}{2}B = D$, $A_2C_2a^2 = \frac{1}{2}B$ and $A_nC_na^n = 0$ for all $n \neq 2$; further, all $B_n = 0$. The solution everywhere (not just on the perimeter) is therefore

$$T(\rho,\phi) = A + \frac{B}{2} + \frac{B\rho^2}{2a^2}\cos 2\phi.$$

It should be noted that 'equating coefficients' to determine unknown constants is justified by the fact that the sinusoidal functions in the sum are mutually orthogonal over the range $0 \le \phi < 2\pi$.

21.11 *The free transverse vibrations of a thick rod satisfy the equation*

$$a^4\frac{\partial^4 u}{\partial x^4}+\frac{\partial^2 u}{\partial t^2}=0.$$

Obtain a solution in separated-variable form and, for a rod clamped at one end, $x=0$, and free at the other, $x=L$, show that the angular frequency of vibration ω satisfies

$$\cosh\left(\frac{\omega^{1/2}L}{a}\right)=-\sec\left(\frac{\omega^{1/2}L}{a}\right).$$

[*At a clamped end both u and $\partial u/\partial x$ vanish, whilst at a free end, where there is no bending moment, $\partial^2 u/\partial x^2$ and $\partial^3 u/\partial x^3$ are both zero.*]

The general solution is written as the product $u(x,t)=X(x)T(t)$, which, on substitution, produces the separated equation

$$a^4\frac{X^{(4)}}{X}=-\frac{T''}{T}=\omega^2.$$

Here the separation constant has been chosen so as to give oscillatory behaviour (in the time variable). The spatial equation then becomes

$$X^{(4)}-\mu^4 X=0,\ \text{where } \mu=\omega^{1/2}/a.$$

The required auxiliary equation is $\lambda^4-\mu^4=0$, leading to the general solution

$$X(x)=A\sin\mu x+B\cos\mu x+C\sinh\mu x+D\cosh\mu x.$$

The constants A, B, C and D are to be determined by requiring $X(0)=X'(0)=0$ and $X''(L)=X'''(L)=0$.

At the clamped end,

$$\begin{aligned}
X(0)=0 \quad &\Rightarrow \quad D=-B,\\
X' \quad &= \quad \mu(A\cos\mu x-B\sin\mu x+C\cosh\mu x-B\sinh\mu x),\\
X'(0)=0 \quad &\Rightarrow \quad C=-A.
\end{aligned}$$

At the free end,

$$\begin{aligned}
X''&=\mu^2(-A\sin\mu x-B\cos\mu x-A\sinh\mu x-B\cosh\mu x),\\
X'''&=\mu^3(-A\cos\mu x+B\sin\mu x-A\cosh\mu x-B\sinh\mu x),\\
X''(L)=0 \quad &\Rightarrow \quad A(\sin\mu L+\sinh\mu L)+B(\cos\mu L+\cosh\mu L)=0,\\
X'''(L)=0 \quad &\Rightarrow \quad A(-\cos\mu L-\cosh\mu L)+B(\sin\mu L-\sinh\mu L)=0.
\end{aligned}$$

Cross-multiplying then gives

$$\begin{aligned} -\sin^2 \mu L + \sinh^2 \mu L &= \cos^2 \mu L + 2\cos \mu L \cosh \mu L + \cosh^2 \mu L, \\ 0 &= 1 + 2\cos \mu L \cosh \mu L + 1, \\ -1 &= \cos \mu L \cosh \mu L, \\ \cosh\left(\frac{\omega^{1/2}L}{a}\right) &= -\sec\left(\frac{\omega^{1/2}L}{a}\right). \end{aligned}$$

Because sinusoidal and hyperbolic functions can all be written in terms of exponential functions, this problem could also be approached by assuming solutions that are (exponential) functions of linear combinations of x and t (as in Chapter 20). However, in practice, eliminating the t-dependent terms leads to involved algebra.

21.13 *A string of length L, fixed at its two ends, is plucked at its mid-point by an amount A and then released. Prove that the subsequent displacement is given by*

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8A}{\pi^2(2n+1)^2} \sin\left[\frac{(2n+1)\pi x}{L}\right] \cos\left[\frac{(2n+1)\pi ct}{L}\right],$$

where, in the usual notation, $c^2 = T/\rho$.

Find the total kinetic energy of the string when it passes through its unplucked position, by calculating it in each mode (each n) and summing, using the result

$$\sum_{0}^{\infty} \frac{1}{(2n+1)^2} = \frac{\pi^2}{8}.$$

Confirm that the total energy is equal to the work done in plucking the string initially.

We start with the wave equation:

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0$$

and assume a separated-variable solution $u(x,t) = X(x)S(t)$. This leads to

$$\frac{X''}{X} = \frac{1}{c^2}\frac{S''}{S} = -k^2.$$

The solution to the spatial equation is given by

$$X(x) = B\cos kx + C\sin kx.$$

Taking the string as anchored at $x = 0$ and $x = L$, we must have $B = 0$ and k constrained by $\sin kL = 0 \Rightarrow k = n\pi/L$ with n an integer.

The solution to the corresponding temporal equation is

$$S(t) = D\cos kct + E\sin kct.$$

Since there is no initial motion, i.e. $\dot{S}(0) = 0$, it follows that $E = 0$.
For any particular value of k, the constants C and D can be amalgamated. The general solution is given by a superposition of the allowed functions, i.e.

$$u(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}.$$

We now have to determine the C_n by making $u(x,0)$ match the given initial configuration, which is

$$u(x,0) = \begin{cases} \dfrac{2Ax}{L} & \text{for } 0 \le x \le \dfrac{L}{2}, \\ \dfrac{2A(L-x)}{L} & \dfrac{L}{2} < x \le L. \end{cases}$$

This is now a Fourier series calculation yielding

$$\begin{aligned} \frac{C_n L}{2} &= \int_0^{L/2} \frac{2Ax}{L}\sin\frac{n\pi x}{L}\,dx + \int_{L/2}^{L} \frac{2A(L-x)}{L}\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2A}{L}J_1 + 2AJ_2 - \frac{2A}{L}J_3, \end{aligned}$$

with

$$\begin{aligned} J_1 &= \left[-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_0^{L/2} + \int_0^{L/2}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx \\ &= -\frac{L^2}{2\pi n}\cos\frac{n\pi}{2} + \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}, \\ J_2 &= \int_{L/2}^{L}\sin\frac{n\pi x}{L}\,dx = -\frac{L}{n\pi}\left[\cos\frac{n\pi x}{L}\right]_{L/2}^{L} \\ &= -\frac{L}{n\pi}\left[(-1)^n - \cos\frac{n\pi}{2}\right], \\ J_3 &= \left[-\frac{xL}{n\pi}\cos\frac{n\pi x}{L}\right]_{L/2}^{L} + \int_{L/2}^{L}\frac{L}{n\pi}\cos\frac{n\pi x}{L}\,dx \\ &= \frac{L^2}{2\pi n}\cos\frac{n\pi}{2} - \frac{L^2}{n\pi}(-1)^n - \frac{L^2}{n^2\pi^2}\sin\frac{n\pi}{2}. \end{aligned}$$

Thus

$$\begin{aligned} J_1 - J_3 &= -\frac{2L^2}{2\pi n}\cos\frac{n\pi}{2} + \frac{L^2}{n\pi}(-1)^n + \frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2} \\ &= -LJ_2 + \frac{2L^2}{n^2\pi^2}\sin\frac{n\pi}{2}, \end{aligned}$$

and so it follows that

$$\frac{C_n L}{2} = \frac{2A}{L}(J_1 - J_3 + LJ_2) = 2A\frac{2L}{n^2\pi^2}\sin\frac{n\pi}{2}.$$

This is zero if n is even and $C_n = 8A(-1)^{(n-1)/2}/(n^2\pi^2)$ if n is odd. Write $n = 2m+1$, $m = 0, 1, 2, \ldots$, with $C_{2m+1} = \dfrac{8A(-1)^m}{(2m+1)^2\pi^2}$.

The final solution (in which m is replaced by n, to match the question) is thus

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8A(-1)^n}{\pi^2(2n+1)^2}\sin\left[\frac{(2n+1)\pi x}{L}\right]\cos\left[\frac{(2n+1)\pi ct}{L}\right].$$

The velocity profile derived from this is given by

$$\begin{aligned}\dot{u}(x,t) = \sum_{n=0}^{\infty} &\frac{8A(-1)^n}{\pi^2(2n+1)^2}\left(\frac{-(2n+1)\pi c}{L}\right)\\ &\times \sin\left[\frac{(2n+1)\pi x}{L}\right]\sin\left[\frac{(2n+1)\pi ct}{L}\right],\end{aligned}$$

giving the energy in the $(2n+1)$th mode (evaluated when the time-dependent sine function is maximal) as

$$\begin{aligned}E_{2n+1} &= \int_0^L \tfrac{1}{2}\rho\dot{u}_n^2\,dx\\ &= \int_0^L \frac{\rho}{2}\,\frac{(8A)^2c^2}{L^2(2n+1)^2\pi^2}\sin^2\frac{(2n+1)\pi x}{L}\\ &= \frac{32A^2\rho c^2}{L^2(2n+1)^2\pi^2}\,\frac{L}{2}.\end{aligned}$$

Therefore

$$E = \sum_{n=0}^{\infty} E_{2n+1} = \frac{16A^2\rho c^2}{\pi^2 L}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{2A^2\rho c^2}{L}.$$

When the mid-point of the string has been displaced sideways by y ($\ll L$), the net (resolved) restoring force is $2T[y/(L/2)] = 4Ty/L$. Thus the total work done to produce a displacement of A is

$$W = \int_0^A \frac{4Ty}{L}\,dy = \frac{2TA^2}{L} = \frac{2\rho c^2A^2}{L},$$

i.e. the same as the total energy of the subsequent motion.

21.15 *Prove that the potential for $\rho < a$ associated with a vertical split cylinder of radius a, the two halves of which ($\cos\phi > 0$ and $\cos\phi < 0$) are maintained at equal and opposite potentials $\pm V$, is given by*

$$u(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{\rho}{a}\right)^{2n+1}\cos(2n+1)\phi.$$

The most general solution of the Laplace equation in cylindrical polar coordinates that is independent of z is

$$T(\rho,\phi) = C\ln\rho + D + \sum_{n=1}^{\infty}(A_n\cos n\phi + B_n\sin n\phi)(C_n\rho^n + D_n\rho^{-n}).$$

The required potential must be single-valued and finite in the space inside the cylinder (which includes $\rho = 0$), and on the cylinder it must take the boundary values $u = V$ for $\cos\phi > 0$ and $u = -V$ for $\cos\phi < 0$, i.e the boundary-value function is a square-wave function with average value zero. Although the function is antisymmetric in $\cos\phi$, it is symmetric in ϕ and so the solution will contain only cosine terms (and no sine terms).

These considerations already determine that $C = D = B_n = D_n = 0$, and so have reduced the solution to the form

$$u(\rho,\phi) = \sum_{n=1}^{\infty}A_n\rho^n\cos n\phi.$$

On $\rho = a$ this must match the stated boundary conditions, and so we are faced with a Fourier cosine series calculation. Multiplying through by $\cos m\phi$ and integrating yields

$$\begin{aligned}
A_m a^m\frac{1}{2}2\pi &= 2\int_0^{\pi/2}V\cos m\phi\,d\phi + 2\int_{\pi/2}^{\pi}(-V)\cos m\phi\,d\phi\\
&= 2V\left[\frac{\sin m\phi}{m}\right]_0^{\pi/2} - 2V\left[\frac{\sin m\phi}{m}\right]_{\pi/2}^{\pi}\\
&= \frac{2V}{m}\left(\sin\frac{m\pi}{2} + \sin\frac{m\pi}{2}\right)\\
&= (-1)^{(m-1)/2}\frac{4V}{m}\ \text{for } m \text{ odd}, = 0 \text{ for } m \text{ even}.
\end{aligned}$$

Writing $m = 2n+1$ gives the solution as

$$u(\rho,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left(\frac{\rho}{a}\right)^{2n+1}\cos(2n+1)\phi.$$

21.17 *Two identical copper bars are each of length a. Initially, one is at* $0\,^\circ\mathrm{C}$ *and the other at* $100\,^\circ\mathrm{C}$*; they are then joined together end to end and thermally isolated. Obtain in the form of a Fourier series an expression* $u(x,t)$ *for the temperature at any point a distance x from the join at a later time t. Bear in mind the heat flow conditions at the free ends of the bars.*

Taking $a = 0.5\,\mathrm{m}$ *estimate the time it takes for one of the free endsto attain a temperature of* $55\,^\circ\mathrm{C}$*. The thermal conductivity of copper is* $3.8\times10^2\ \mathrm{J\,m^{-1}\,K^{-1}\,s^{-1}}$*, and its specific heat capacity is* $3.4\times10^6\ \mathrm{J\,m^{-3}\,K^{-1}}$*.*

The equation governing the heat flow is

$$k\frac{\partial^2 u}{\partial x^2} = s\frac{\partial u}{\partial t},$$

which is the diffusion equation with diffusion constant $\kappa = k/s = 3.8\times10^2/3.4\times10^6 = 1.12\times10^{-4}\ \mathrm{m^2\,s^{-1}}$.

Making the usual separation of variables substitution shows that the time variation is of the form $T(t) = T(0)e^{-\kappa\lambda^2 t}$ when the spatial solution is a sinusoidal function of λx. The final common temperature is $50\,^\circ\mathrm{C}$ and we make this explicit by writing the general solution as

$$u(x,t) = 50 + \sum_\lambda (A_\lambda \sin\lambda x + B_\lambda \cos\lambda x)e^{-\kappa\lambda^2 t}.$$

This term having been taken out, the summation must be antisymmetric about $x = 0$ and therefore contain no cosine terms, i.e. $B_\lambda = 0$.

The boundary condition is that there is no heat flow at $x = \pm a$; this means that $\partial u/\partial x = 0$ at these points and requires

$$\lambda A_\lambda \cos\lambda x|_{x=\pm a} = 0 \quad\Rightarrow\quad \lambda a = (n+\tfrac{1}{2})\pi \quad\Rightarrow\quad \lambda = \frac{(2n+1)\pi}{2a},$$

where n is an integer. This corresponds to a fundamental Fourier period of $4a$. The solution thus takes the form

$$u(x,t) = 50 + \sum_{n=0}^{\infty} A_n \sin\frac{(2n+1)\pi x}{2a}\exp\left(-\frac{(2n+1)^2\pi^2\kappa t}{4a^2}\right).$$

At $t = 0$, the sum must take the values $+50$ for $0 < x < 2a$ and -50 for $-2a < x < 0$. This is (yet) another square-wave function — one that is antisymmetric about $x = 0$ and has amplitude 50. The calculation will not be repeated here but gives $A_n = 200/[(2n+1)\pi]$, making the complete solution

$$u(x,t) = 50 + \frac{200}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{(2n+1)\pi x}{2a}\exp\left(-\frac{(2n+1)^2\pi^2\kappa t}{4a^2}\right).$$

For a free end, where $x = a$ and $\sin[(2n+1)\pi x/2a] = (-1)^n$, to attain 55 °C needs

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \exp\left(-\frac{(2n+1)^2\,\pi^2\,1.12\times 10^{-4}}{4\times 0.25}\,t\right) = \frac{5\pi}{200} = 0.0785.$$

In principle this is an insoluble equation but, because the RHS $\ll 1$, the $n = 0$ term alone will give a good approximation to t:

$$\exp(-1.105\times 10^{-3}t) \approx 0.0785 \quad \Rightarrow \quad t \approx 2300 \text{ s}.$$

21.19 *For an infinite metal bar that has an initial ($t = 0$) temperature distribution $f(x)$ along its length, the temperature distribution at a general time t can be shown to be given by*

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] f(\xi)\,d\xi.$$

Find an explicit expression for $u(x,t)$ given that $f(x) = \exp(-x^2/a^2)$.

The given initial distribution is $f(\xi) = \exp(-\xi^2/a^2)$ and so

$$u(x,t) = \frac{1}{\sqrt{4\pi\kappa t}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-\xi)^2}{4\kappa t}\right] \exp\left(-\frac{\xi^2}{a^2}\right) d\xi.$$

Now consider the exponent in the integrand, writing $1 + \dfrac{4\kappa t}{a^2}$ as τ^2 for compactness:

$$\begin{aligned}
\text{exponent} &= -\frac{\xi^2\tau^2 - 2\xi x + x^2}{4\kappa t} \\
&= -\frac{(\xi\tau - x\tau^{-1})^2 - x^2\tau^{-2} + x^2}{4\kappa t} \\
&\equiv -\eta^2 + \frac{x^2\tau^{-2} - x^2}{4\kappa t}, \quad \text{defining } \eta, \\
\text{with } d\eta &= \frac{\tau\,d\xi}{\sqrt{4\kappa t}}.
\end{aligned}$$

With a change of variable from ξ to η, the integral becomes

$$\begin{aligned}
u(x,t) &= \frac{1}{\sqrt{4\pi\kappa t}} \exp\left(\frac{x^2\tau^{-2} - x^2}{4\kappa t}\right) \int_{-\infty}^{\infty} \exp(-\eta^2)\,\frac{\sqrt{4\kappa t}}{\tau}\,d\eta \\
&= \frac{1}{\sqrt{\pi}}\frac{1}{\tau} \exp\left(x^2\frac{1-\tau^2}{4\kappa t\,\tau^2}\right) \sqrt{\pi} \\
&= \frac{a}{\sqrt{a^2 + 4\kappa t}} \exp\left(-\frac{x^2}{a^2 + 4\kappa t}\right).
\end{aligned}$$

In words, although it retains a Gaussian shape, the initial distribution spreads symmetrically about the origin, its variance increasing linearly with time ($a^2 \to a^2 + 4\kappa t$). As is typical with diffusion processes, for large enough times the width varies as $\sqrt{t}$.

21.21 *In the region $-\infty < x,\ y < \infty$ and $-t \le z \le t$, a charge-density wave $\rho(\mathbf{r}) = A\cos qx$, in the x-direction, is represented by*

$$\rho(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{\rho}(\alpha) e^{i\alpha z}\, d\alpha.$$

The resulting potential is represented by

$$V(\mathbf{r}) = \frac{e^{iqx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{V}(\alpha) e^{i\alpha z}\, d\alpha.$$

Determine the relationship between $\tilde{V}(\alpha)$ and $\tilde{\rho}(\alpha)$, and hence show that the potential at the point $(0,0,0)$ is given by

$$\frac{A}{\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\sin kt}{k(k^2+q^2)}\, dk.$$

Poisson's equation,

$$\nabla^2 V(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon_0},$$

provides the link between a charge density and the potential it produces.

Taking $V(\mathbf{r})$ in the form of its Fourier representation gives $\nabla^2 V$ as

$$\frac{\partial^2 V(\mathbf{r})}{\partial x^2} + \frac{\partial^2 V(\mathbf{r})}{\partial y^2} + \frac{\partial^2 V(\mathbf{r})}{\partial z^2} = \frac{e^{iqx}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-q^2 - \alpha^2)\tilde{V}(\alpha) e^{i\alpha z}\, d\alpha,$$

with the $-q^2$ arising from the x-differentiation and the $-\alpha^2$ from the z-differentiation; the $\partial^2 V/\partial y^2$ term contributes nothing.

Comparing this with the integral expression for $-\rho(\mathbf{r})/\epsilon_0$ shows that

$$-\tilde{\rho}(\alpha) = \epsilon_0(-q^2 - \alpha^2)\tilde{V}(\alpha).$$

With the charge-density wave confined in the z-direction to $-t \le z \le t$, the expression for $\rho(\mathbf{r})$ in Cartesian coordinates is (in terms of Heaviside functions)

$$\rho(\mathbf{r}) = Ae^{iqx}[H(z+t) - H(z-t)].$$

The Fourier transform $\tilde{\rho}(\alpha)$ is therefore given by

$$\begin{aligned}\tilde{\rho}(\alpha) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} A[H(z+t)-H(z-t)]\,e^{-i\alpha z}\,dz \\ &= \frac{A}{\sqrt{2\pi}}\int_{-t}^{t} e^{-i\alpha z}\,dz \\ &= \frac{A}{\sqrt{2\pi}}\frac{e^{-i\alpha t}-e^{i\alpha t}}{-i\alpha} \\ &= \frac{A}{\sqrt{2\pi}}\frac{2\sin\alpha t}{\alpha}.\end{aligned}$$

Now,
$$\begin{aligned}V(x,0,z) &= \frac{e^{iqx}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\tilde{\rho}(\alpha)}{\epsilon_0(q^2+\alpha^2)}\,e^{i\alpha z}\,d\alpha \\ &= \frac{e^{iqx}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{i\alpha z}}{\epsilon_0(q^2+\alpha^2)}\,\frac{A}{\sqrt{2\pi}}\,\frac{2\sin\alpha t}{\alpha}\,d\alpha, \\ \Rightarrow\quad V(0,0,0) &= \frac{A}{\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\sin\alpha t}{\alpha(\alpha^2+q^2)}\,d\alpha,\end{aligned}$$

as stated in the question.

21.23 *Find the Green's function $G(\mathbf{r},\mathbf{r}_0)$ in the half-space $z>0$ for the solution of $\nabla^2\Phi=0$ with Φ specified in cylindrical polar coordinates (ρ,ϕ,z) on the plane $z=0$ by*

$$\Phi(\rho,\phi,z)=\begin{cases}1 & \text{for } \rho\le 1,\\ 1/\rho & \text{for } \rho>1.\end{cases}$$

Determine the variation of $\Phi(0,0,z)$ along the z-axis.

For the half-space $z>0$ the bounding surface consists of the plane $z=0$ and the (hemi-spherical) surface at infinity; the Green's function must take zero value on these surfaces. In order to ensure this when a unit point source is introduced at $\mathbf{r}=\mathbf{y}$, we must place a compensating negative unit source at $\mathbf{y}$'s reflection point in the plane. If, in cylindrical polar coordinates, $\mathbf{y}=(\rho,\phi,z_0)$, then the image charge has to be at $\mathbf{y}'=(\rho,\phi,-z_0)$. The resulting Green's function $G(\mathbf{x},\mathbf{y})$ is given by

$$G(\mathbf{x},\mathbf{y})=-\frac{1}{4\pi|\mathbf{x}-\mathbf{y}|}+\frac{1}{4\pi|\mathbf{x}-\mathbf{y}'|}.$$

The solution to the problem with a given potential distribution $f(\rho,\phi)$ on the $z=0$ part of the bounding surface S is given by

$$\Phi(\mathbf{y})=\int_S f(\rho,\phi)\left(-\frac{\partial G}{\partial z}\right)\rho\,d\phi\,d\rho,$$

the minus sign arising because the outward normal to the region is in the negative z-direction. Calculating these functions explicitly gives

$$G(\mathbf{x},\mathbf{y}) = -\frac{1}{4\pi[\,\rho^2+(z-z_0)^2\,]^{1/2}} + \frac{1}{4\pi[\,\rho^2+(z+z_0)^2\,]^{1/2}},$$
$$\frac{\partial G}{\partial z} = \frac{z-z_0}{4\pi[\,\rho^2+(z-z_0)^2\,]^{3/2}} - \frac{(z+z_0)}{4\pi[\,\rho^2+(z+z_0)^2\,]^{3/2}},$$
$$-\left.\frac{\partial G}{\partial z}\right|_{z=0} = -\frac{-2z_0}{4\pi[\,\rho^2+z_0^2\,]^{3/2}}.$$

Substituting the various factors into the general integral gives

$$\begin{aligned}\Phi(0,0,z_0) &= \int_0^\infty f(\rho)\frac{2z_0}{4\pi[\,\rho^2+z_0^2\,]^{3/2}}\,2\pi\,\rho\,d\rho\\ &= \int_0^1 \frac{z_0\,\rho}{(\rho^2+z_0^2)^{3/2}}\,d\rho + \int_1^\infty \frac{z_0}{(\rho^2+z_0^2)^{3/2}}\,d\rho\\ &= -z_0\left[\,(\rho^2+z_0^2)^{-1/2}\,\right]_0^1 + \int_\theta^{\pi/2}\frac{z_0^2\sec^2 u}{z_0^3\sec^3 u}\,du,\end{aligned}$$

where, in the second integral, we have set $\rho = z_0\tan u$ with $d\rho = z_0\sec^2 u\,du$ and $\theta = \tan^{-1}(1/z_0)$. The integral can now be obtained in closed form as

$$\begin{aligned}\Phi(0,0,z_0) &= -\frac{z_0}{(1+z_0^2)^{1/2}} + 1 + \frac{1}{z_0}\,[\,\sin u\,]_\theta^{\pi/2}\\ &= 1 - \frac{z_0}{(1+z_0^2)^{1/2}} + \frac{1}{z_0} - \frac{1}{z_0(1+z_0^2)^{1/2}}.\end{aligned}$$

Thus the variation of Φ along the z-axis is given by

$$\Phi(0,0,z) = \frac{z(1+z^2)^{1/2} - z^2 + (1+z^2)^{1/2} - 1}{z(1+z^2)^{1/2}}.$$

21.25 *Find, in the form of an infinite series, the Green's function of the ∇^2 operator for the Dirichlet problem in the region $-\infty < x < \infty$, $-\infty < y < \infty$, $-c \le z \le c$.*

The fundamental solution in three dimensions of $\nabla^2\psi = \delta(\mathbf{r})$ is $\psi(\mathbf{r}) = -1/(4\pi r)$.

For the given problem, $G(\mathbf{r},\mathbf{r}_0)$ has to take the value zero on $z = \pm c$ and $\to 0$ for $|x| \to \infty$ and $|y| \to \infty$. Image charges have to be added in the regions $z > c$ and $z < -c$ to bring this about after a charge q has been placed at $\mathbf{r}_0 = (x_0, y_0, z_0)$ with $-c < z_0 < c$. Clearly all images will be on the line $x = x_0$, $y = y_0$.

Each image placed at $z = \xi$ in the region $z > c$ will require a further image of the same strength but opposite sign at $z = -c - \xi$ (in the region $z < -c$) so as

to maintain the plane $z = -c$ as an equipotential. Likewise, each image placed at $z = -\chi$ in the region $z < -c$ will require a further image of the same strength but opposite sign at $z = c + \chi$ (in the region $z > c$) so as to maintain the plane $z = c$ as an equipotential. Thus succesive image charges appear as follows:

$$\begin{array}{ccc} -q & 2c - z_0 & -2c - z_0 \\ +q & -3c + z_0 & 3c + z_0 \\ -q & 4c - z_0 & -4c - z_0 \\ +q & \text{etc.} & \text{etc.} \end{array}$$

The terms in the Green's function that are additional to the fundamental solution,

$$-\frac{1}{4\pi}[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{-1/2},$$

are therefore

$$-\frac{(-1)}{4\pi}\sum_{n=2}^{\infty}\left\{\frac{(-1)^n}{[(x - x_0)^2 + (y - y_0)^2 + (z + (-1)^n z_0 - nc)^2]^{1/2}} + \frac{(-1)^n}{[(x - x_0)^2 + (y - y_0)^2 + (z + (-1)^n z_0 + nc)^2]^{1/2}}\right\}.$$

21.27 *Determine the Green's function for the Klein–Gordon equation in a half-space as follows.*

(a) *By applying the divergence theorem to the volume integral*

$$\int_V \left[\phi(\nabla^2 - m^2)\psi - \psi(\nabla^2 - m^2)\phi\right] dV,$$

obtain a Green's function expression, as the sum of a volume integral and a surface integral, for the function $\phi(\mathbf{r}')$ that satisfies

$$\nabla^2\phi - m^2\phi = \rho$$

in V and takes the specified form $\phi = f$ on S, the boundary of V. The Green's function, $G(\mathbf{r}, \mathbf{r}')$, to be used satisfies

$$\nabla^2 G - m^2 G = \delta(\mathbf{r} - \mathbf{r}')$$

and vanishes when $\mathbf{r}$ is on S.

(b) *When V is all space, $G(\mathbf{r}, \mathbf{r}')$ can be written as $G(t) = g(t)/t$, where $t = |\mathbf{r} - \mathbf{r}'|$ and $g(t)$ is bounded as $t \to \infty$. Find the form of $G(t)$.*

(c) *Find $\phi(\mathbf{r})$ in the half-space $x > 0$ if $\rho(\mathbf{r}) = \delta(\mathbf{r} - \mathbf{r}_1)$ and $\phi = 0$ both on $x = 0$ and as $r \to \infty$.*

(a) For general ϕ and ψ we have

$$\begin{aligned}\int_V \left[\phi(\nabla^2 - m^2)\psi - \psi(\nabla^2 - m^2)\phi\right]\, dV &= \int_V \left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]\, dV \\ &= \int_V \nabla\cdot(\phi\nabla\psi - \psi\nabla\phi)\, dV \\ &= \int_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\, dS.\end{aligned}$$

Now take ϕ as ϕ, with $\nabla^2\phi - m^2\phi = \rho$ and $\phi = f$ on the surface S, and ψ as $G(\mathbf{r},\mathbf{r}')$ with $\nabla^2 G - m^2 G = \delta(\mathbf{r}-\mathbf{r}')$ and $G(\mathbf{r},\mathbf{r}') = 0$ on S:

$$\int_V [\,\phi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}') - G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r})\,]\, dV = \int_S [\,f(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}') - \mathbf{0}\,]\cdot\mathbf{n}\, dS,$$

which, on rearrangement, gives

$$\phi(\mathbf{r}') = \int_V G(\mathbf{r},\mathbf{r}')\rho(\mathbf{r})\, dV + \int_S f(\mathbf{r})\nabla G(\mathbf{r},\mathbf{r}')\cdot\mathbf{n}\, dS.$$

(b) In the following calculation we start by formally integrating the defining Green's equation,

$$\nabla^2 G - m^2 G = \delta(\mathbf{r}-\mathbf{r}'),$$

over a sphere of radius t centred on $\mathbf{r}'$. Having replaced the volume integral of $\nabla^2 G$ with the corresponding surface integral given by the divergence theorem, we move the origin to $\mathbf{r}'$, denote $|\mathbf{r}-\mathbf{r}'|$ by t' and integrate both sides of the equation from $t' = 0$ to $t' = t$:

$$\begin{aligned}\int_V \nabla^2 G\, dV - \int_V m^2 G\, dV &= \int_V \delta(\mathbf{r}-\mathbf{r}')\, dV, \\ \int_S \nabla G\cdot\mathbf{n}\, dS - m^2\int_V G\, dV &= 1, \\ 4\pi t^2\frac{dG}{dt} - m^2\int_0^t G(t')4\pi t'^2\, dt' &= 1, \qquad (*) \\ 4\pi t^2 G'' + 8\pi t G' - 4\pi m^2 t^2 G &= 0, \text{ from differentiating w.r.t. } t, \\ tG'' + 2G' - m^2 tG &= 0.\end{aligned}$$

With $G(t) = g(t)/t$,

$$G' = -\frac{g}{t^2} + \frac{g'}{t} \quad\text{and}\quad G'' = \frac{2g}{t^3} - \frac{2g'}{t^2} + \frac{g''}{t},$$

and the equation becomes

$$\begin{aligned}0 &= \frac{2g}{t^2} - \frac{2g'}{t} + g'' - \frac{2g}{t^2} + \frac{2g'}{t} - m^2 g, \\ 0 &= g'' - m^2 g, \\ \Rightarrow\quad g(t) &= Ae^{-mt}, \text{ since } g \text{ is bounded as } t\to\infty.\end{aligned}$$

The value of A is determined by resubstituting into (*), which then reads

$$\begin{aligned} 4\pi t^2\left(-\frac{Ae^{-mt}}{t^2}-\frac{mAe^{-mt}}{t}\right)-m^2\int_0^t\frac{Ae^{-mt'}}{t'}\,4\pi t'^2\,dt' &= 1,\\ -4\pi Ae^{-mt}(1+mt)-4\pi Am^2\left(-\frac{te^{-mt}}{m}+\frac{1-e^{-mt}}{m^2}\right) &= 1,\\ -4\pi A &= 1, \end{aligned}$$

making the solution

$$G(\mathbf{r},\mathbf{r}')=-\frac{e^{-mt}}{4\pi t},\text{ where } t=|\mathbf{r}-\mathbf{r}'|.$$

(c) For the situation in which $\rho(\mathbf{r})=\delta(\mathbf{r}-\mathbf{r}_1)$, i.e. a unit positive charge at $\mathbf{r}_1=(x_1,y_1,z_1)$, and $\phi=0$ on the plane $x=0$, we must have a unit negative image charge at $\mathbf{r}_2=(-x_1,y_1,z_1)$. The solution in the region $x>0$ is then

$$\phi(\mathbf{r})=-\frac{1}{4\pi}\left(\frac{e^{-m|\mathbf{r}-\mathbf{r}_1|}}{|\mathbf{r}-\mathbf{r}_1|}-\frac{e^{-m|\mathbf{r}-\mathbf{r}_2|}}{|\mathbf{r}-\mathbf{r}_2|}\right).$$

22

Calculus of variations

22.1 *A surface of revolution, whose equation in cylindrical polar coordinates is $\rho = \rho(z)$, is bounded by the circles $\rho = a$, $z = \pm c$ $(a > c)$. Show that the function that makes the surface integral $I = \int \rho^{-1/2}\, dS$ stationary with respect to small variations is given by $\rho(z) = k + z^2/(4k)$, where $k = [a \pm (a^2 - c^2)^{1/2}]/2$.*

The surface element lying between z and $z + dz$ is given by

$$dS = 2\pi\rho\,[(d\rho)^2 + (dz)^2]^{1/2} = 2\pi\rho\,(1 + \rho'^2)^{1/2}\, dz$$

and the integral to be made stationary is

$$I = \int \rho^{-1/2}\, dS = 2\pi \int_{-c}^{c} \rho^{-1/2}\rho\,(1 + \rho'^2)^{1/2}\, dz.$$

The integrand $F(\rho', \rho, z)$ does not in fact contain z explicitly, and so a first integral of the E–L equation, symbolically given by $F - \rho'\partial F/\partial\rho' = k$, is

$$\rho^{1/2}(1 + \rho'^2)^{1/2} - \rho'\left[\frac{\rho^{1/2}\rho'}{(1 + \rho'^2)^{1/2}}\right] = A,$$

$$\frac{\rho^{1/2}}{(1 + \rho'^2)^{1/2}} = A.$$

On rearrangement and subsequent integration this gives

$$\frac{d\rho}{dz} = \left(\frac{\rho - A^2}{A^2}\right)^{1/2},$$

$$\int \frac{d\rho}{\sqrt{\rho - A^2}} = \int \frac{dz}{A},$$

$$2\sqrt{\rho - A^2} = \frac{z}{A} + C.$$

Now, $\rho(\pm c) = a$ implies both that $C = 0$ and that $a - A^2 = \dfrac{c^2}{4A^2}$. Thus, writing A^2 as k,

$$4k^2 - 4ka + c^2 = 0 \quad \Rightarrow \quad k = \tfrac{1}{2}[\,a \pm (a^2 - c^2)^{1/2}\,].$$

The two stationary functions are therefore

$$\rho = \frac{z^2}{4k} + k,$$

with k as given above. A simple sketch shows that the positive sign in k corresponds to a smaller value of the integral.

22.3 *The refractive index n of a medium is a function only of the distance r from a fixed point O. Prove that the equation of a light ray, assumed to lie in a plane through O, travelling in the medium satisfies (in plane polar coordinates)*

$$\frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2}{a^2}\frac{n^2(r)}{n^2(a)} - 1,$$

where a is the distance of the ray from O at the point at which $dr/d\phi = 0$.

If $n = [1 + (\alpha^2/r^2)]^{1/2}$ and the ray starts and ends far from O, find its deviation (the angle through which the ray is turned), if its minimum distance from O is a.

An element of path length is $ds = [\,(dr)^2 + (r\,d\phi)^2\,]^{1/2}$ and the time taken for the light to traverse it is $n(r)\,ds/c$, where c is the speed of light *in vacuo*. Fermat's principle then implies that the light follows the curve that minimises

$$T = \int \frac{n(r)\,ds}{c} = \int \frac{n(r'^2 + r^2)^{1/2}}{c}\,d\phi,$$

where $r' = dr/d\phi$. Since the integrand does not contain ϕ explicitly, the E–L equation integrates to (see exercise 22.1)

$$n(r'^2 + r^2)^{1/2} - r'\frac{nr'}{(r'^2 + r^2)^{1/2}} = A,$$

$$\frac{nr^2}{(r'^2 + r^2)^{1/2}} = A.$$

Since $r' = 0$ when $r = a$, $A = n(a)a^2/a$, and the equation is as follows:

$$a^2n^2(a)(r'^2 + r^2) = n^2(r)r^4,$$

$$r'^2 = \frac{n^2(r)r^4}{n^2(a)a^2} - r^2,$$

$$\Rightarrow \quad \frac{1}{r^2}\left(\frac{dr}{d\phi}\right)^2 = \frac{n^2(r)r^2}{n^2(a)a^2} - 1.$$

If $n(r) = [\,1 + (\alpha/r)^2\,]^{1/2}$, the minimising curve satisfies

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^2(r^2+\alpha^2)}{a^2+\alpha^2} - r^2$$
$$= \frac{r^2(r^2-a^2)}{a^2+\alpha^2},$$
$$\Rightarrow \quad \frac{d\phi}{(a^2+\alpha^2)^{1/2}} = \pm\frac{dr}{r\sqrt{r^2-a^2}}.$$

By symmetry,

$$\begin{aligned}
\frac{\Delta\phi}{(a^2+\alpha^2)^{1/2}} &\equiv \frac{\phi_{\text{final}} - \phi_{\text{initial}}}{(a^2+\alpha^2)^{1/2}} \\
&= 2\int_a^\infty \frac{dr}{r\sqrt{r^2-a^2}}, \quad \text{set } r = a\cosh\psi, \\
&= 2\int_0^\infty \frac{a\sinh\psi}{a^2\cosh\psi\sinh\psi}\,d\psi \\
&= \frac{2}{a}\int_0^\infty \operatorname{sech}\,\psi\,d\psi, \quad \text{set } e^\psi = z, \\
&= \frac{2}{a}\int_1^\infty \frac{z^{-1}\,dz}{\frac{1}{2}(z+z^{-1})} \\
&= \frac{2}{a}\int_1^\infty \frac{2\,dz}{z^2+1} \\
&= \frac{4}{a}\left[\tan^{-1} z\right]_1^\infty \\
&= \frac{4}{a}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{a}.
\end{aligned}$$

If the refractive index were everywhere unity ($\alpha = 0$), $\Delta\phi$ would be π (no deviation). Thus the deviation is given by

$$\frac{\pi}{a}(a^2+\alpha^2)^{1/2} - \pi.$$

22.5 *Prove the following results about general systems.*

(a) *For a system described in terms of coordinates q_i and t, show that if t does not appear explicitly in the expressions for x, y and z ($x = x(q_i, t)$, etc.) then the kinetic energy T is a homogeneous quadratic function of the $\dot{q}_i$ (it may also involve the q_i). Deduce that $\sum_i \dot{q}_i(\partial T/\partial \dot{q}_i) = 2T$.*

(b) *Assuming that the forces acting on the system are derivable from a potential V, show, by expressing dT/dt in terms of q_i and $\dot{q}_i$, that $d(T+V)/dt = 0$.*

To save space we will use the summation convention for summing over the index of the q_i.

(a) The space variables x, y and z are not explicit functions of t and the kinetic energy, T, is given by

$$\begin{aligned} T &= \tfrac{1}{2}(\alpha_x \dot{x}^2 + \alpha_y \dot{y}^2 + \alpha_z \dot{z}^2) \\ &= \frac{1}{2}\left[\alpha_x \left(\frac{\partial x}{\partial q_i}\dot{q}_i\right)^2 + \alpha_y \left(\frac{\partial y}{\partial q_j}\dot{q}_j\right)^2 + \alpha_z \left(\frac{\partial z}{\partial q_k}\dot{q}_k\right)^2\right] \\ &= A_{mn}\dot{q}_m\dot{q}_n, \end{aligned}$$

with

$$A_{mn} = \frac{1}{2}\left(\alpha_x \frac{\partial x}{\partial q_m}\frac{\partial x}{\partial q_n} + \alpha_y \frac{\partial y}{\partial q_m}\frac{\partial y}{\partial q_n} + \alpha_z \frac{\partial z}{\partial q_m}\frac{\partial z}{\partial q_n}\right) = A_{nm}.$$

Hence T is a homogeneous quadratic function of the $\dot{q}_i$ (though the A_{mn} may involve the q_i). Further,

$$\begin{aligned} \frac{\partial T}{\partial \dot{q}_i} &= A_{in}\dot{q}_n + A_{mi}\dot{q}_m = 2A_{mi}\dot{q}_m \\ \text{and} \quad \dot{q}_i\frac{\partial T}{\partial \dot{q}_i} &= 2\dot{q}_i A_{mi}\dot{q}_m = 2T. \end{aligned}$$

(b) The Lagrangian is $L = T - V$, with $T = T(q_i, \dot{q}_i)$ and $V = V(q_i)$. Thus

$$\frac{dT}{dt} = \frac{\partial T}{\partial q_i}\dot{q}_i + \frac{dT}{d\dot{q}_i}\ddot{q}_i \quad \text{and} \quad \frac{dV}{dt} = \frac{\partial V}{\partial q_i}\dot{q}_i. \qquad (*)$$

Hamilton's principle requires that

$$\begin{aligned} &\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}, \\ \Rightarrow \quad &\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial T}{\partial q_i} - \frac{\partial V}{\partial q_i}. \qquad (**) \end{aligned}$$

But, from part (a),

$$\begin{aligned} 2T &= \dot{q}_i\frac{\partial T}{\partial \dot{q}_i}, \\ \frac{d}{dt}(2T) &= \ddot{q}_i\frac{\partial T}{\partial \dot{q}_i} + \dot{q}_i\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) \\ &= \ddot{q}_i\frac{\partial T}{\partial \dot{q}_i} + \dot{q}_i\frac{\partial T}{\partial q_i} - \dot{q}_i\frac{\partial V}{\partial q_i}, \text{ using } (**), \\ &= \frac{dT}{dt} - \frac{dV}{dt}, \qquad \text{using } (*). \end{aligned}$$

This can be rearranged as

$$\frac{d}{dt}(T + V) = 0.$$

22.7 *In cylindrical polar coordinates, the curve* $(\rho(\theta), \theta, \alpha\rho(\theta))$ *lies on the surface of the cone* $z = \alpha\rho$. *Show that geodesics (curves of minimum length joining two points) on the cone satisfy*

$$\rho^4 = c^2[\beta^2\rho'^2 + \rho^2],$$

where c *is an arbitrary constant, but* β *has to have a particular value. Determine the form of* $\rho(\theta)$ *and hence find the equation of the shortest path on the cone between the points* $(R, -\theta_0, \alpha R)$ *and* $(R, \theta_0, \alpha R)$.

[*You will find it useful to determine the form of the derivative of* $\cos^{-1}(u^{-1})$.]

In cylindrical polar coordinates the element of length is given by

$$(ds)^2 = (d\rho)^2 + (\rho\, d\theta)^2 + (dz)^2,$$

and the total length of a curve between two points parameterised by θ_0 and θ_1 is

$$\begin{aligned} s &= \int_{\theta_0}^{\theta_1} \sqrt{\left(\frac{d\rho}{d\theta}\right)^2 + \rho^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta \\ &= \int_{\theta_0}^{\theta_1} \sqrt{\rho^2 + (1+\alpha^2)\left(\frac{d\rho}{d\theta}\right)^2}\, d\theta, \text{ since } z = \alpha\rho. \end{aligned}$$

Since the independent variable θ does not occur explicitly in the integrand, a first integral of the E–L equation is

$$\sqrt{\rho^2 + (1+\alpha^2)\rho'^2} - \rho' \frac{(1+\alpha^2)\rho'}{\sqrt{\rho^2 + (1+\alpha^2)\rho'^2}} = c.$$

After being multiplied through by the square root, this can be arranged as follows:

$$\begin{aligned} \rho^2 + (1+\alpha^2)\rho'^2 - (1+\alpha^2)\rho'^2 &= c\sqrt{\rho^2 + (1+\alpha^2)\rho'^2}, \\ \rho^4 &= c^2[\rho^2 + (1+\alpha^2)\rho'^2]. \end{aligned}$$

This is the given equation of the geodesic, in which c is arbitrary but β^2 must have the value $1 + \alpha^2$.

Guided by the hint, we first determine the derivative of $y(u) = \cos^{-1}(u^{-1})$:

$$\frac{dy}{du} = \frac{-1}{\sqrt{1-u^{-2}}}\,\frac{-1}{u^2} = \frac{1}{u\sqrt{u^2-1}}.$$

Now, returning to the geodesic, rewrite it as

$$\begin{aligned} \rho^4 - c^2\rho^2 &= c^2\beta^2\rho'^2, \\ \rho(\rho^2 - c^2)^{1/2} &= c\beta\frac{d\rho}{d\theta}. \end{aligned}$$

Setting $\rho = cu$,

$$uc^2(u^2-1)^{1/2} = c^2\beta\frac{du}{d\theta},$$
$$d\theta = \frac{\beta\, du}{u(u^2-1)^{1/2}},$$

which integrates to

$$\theta = \beta\cos^{-1}\left(\frac{1}{u}\right) + k,$$

using the result from the hint.

Since the geodesic must pass through both $(R, -\theta_0, \alpha R)$ and $(R, \theta_0, \alpha R)$, we must have $k = 0$ and

$$\cos\frac{\theta_0}{\beta} = \frac{c}{R}.$$

Further, at a general point on the geodesic,

$$\cos\frac{\theta}{\beta} = \frac{c}{\rho}.$$

Eliminating c then shows that the geodesic on the cone that joins the two given points is

$$\rho(\theta) = \frac{R\cos(\theta_0/\beta)}{\cos(\theta/\beta)}.$$

22.9 *You are provided with a line of length $\pi a/2$ and negligible mass and some lead shot of total mass M. Use a variational method to determine how the lead shot must be distributed along the line if the loaded line is to hang in a circular arc of radius a when its ends are attached to two points at the same height. Measure the distance s along the line from its centre.*

We first note that the total mass of shot available is merely a scaling factor and not a constraint on the minimisation process.

The length of string is sufficient to form one-quarter of a complete circle of radius a, and so the ends of the string must be fixed to points that are $2a\sin(\pi/4) = \sqrt{2}a$ apart.

We take the distribution of shot as $\rho = \rho(s)$ and have to minimise the integral $\int gy(s)\rho(s)\,ds$, but subject to the requirement $\int dx = a/\sqrt{2}$. Expressed as an integral over s, this requirement can be written

$$\frac{a}{\sqrt{2}} = \int_{s=0}^{s=\pi a/4} dx = \int_0^{\pi a/4} (1-y'^2)^{1/2}\,ds,$$

where the derivative y' of y is with respect to s (not x).

We therefore consider the minimisation of $\int F(y, y', s)\, ds$, where

$$F(y, y', s) = gy\rho + \lambda\sqrt{1 - y'^2}.$$

The E–L equation takes the form

$$\frac{d}{ds}\left(\frac{\partial F}{\partial y'}\right) = \frac{\partial F}{\partial y},$$

$$\lambda\frac{d}{ds}\left(\frac{-y'}{\sqrt{1 - y'^2}}\right) = g\rho(s),$$

$$\frac{-\lambda y'}{\sqrt{1 - y'^2}} = \int_0^s g\rho(s')\, ds' \equiv gP(s),$$

since $y'(0) = 0$ by symmetry.

Now we require $P(s)$ to be such that the solution to this equation takes the form of an arc of a circle, $y(s) = y_0 - a\cos(s/a)$. If this is so, then $y'(s) = \sin(s/a)$ and

$$\frac{-\lambda\sin(s/a)}{\cos(s/a)} = gP(s).$$

When $s = \pi a/4$, $P(s)$ must have the value $M/2$, implying that $\lambda = -Mg/2$ and that, consequently,

$$P(s) = \frac{M}{2}\tan\left(\frac{s}{a}\right).$$

The required distribution $\rho(s)$ is recovered by differentiating this to obtain

$$\rho(s) = \frac{dP}{ds} = \frac{M}{2a}\sec^2\left(\frac{s}{a}\right).$$

22.11 *A general result is that light travels through a variable medium by a path that minimises the travel time (this is an alternative formulation of Fermat's principle). With respect to a particular cylindrical polar coordinate system (ρ, ϕ, z), the speed of light $v(\rho, \phi)$ is independent of z. If the path of the light is parameterised as $\rho = \rho(z), \phi = \phi(z)$, show that*

$$v^2(\rho'^2 + \rho^2\phi'^2 + 1)$$

is constant along the path.

For the particular case when $v = v(\rho) = b(a^2 + \rho^2)^{1/2}$, show that the two Euler–Lagrange equations have a common solution in which the light travels along a helical path given by $\phi = Az + B$, $\rho = C$, provided that A has a particular value.

In cylindrical polar coordinates with $\rho = \rho(z)$ and $\phi = \phi(z)$,

$$ds = \left[1 + \left(\frac{d\rho}{dz}\right)^2 + \rho^2\left(\frac{d\phi}{dz}\right)^2\right]^{1/2} dz.$$

The total travel time of the light is therefore given by

$$\tau = \int \frac{(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}}{v(\rho, \phi)}\, dz.$$

Since z does not appear explicitly in the integrand, we have from the general first integral of the E–L equations for more than one dependent variable that

$$\frac{(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}}{v(\rho, \phi)} - \frac{1}{v}\frac{\rho'^2}{(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}} - \frac{1}{v}\frac{\rho^2\phi'^2}{(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}} = k.$$

Rearranging this gives

$$\begin{aligned} 1 + \rho'^2 + \rho^2\phi'^2 - \rho'^2 - \rho^2\phi'^2 &= kv(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}, \\ 1 &= kv(1 + \rho'^2 + \rho^2\phi'^2)^{1/2}, \\ \Rightarrow \quad v^2(1 + \rho'^2 + \rho^2\phi'^2) &= c, \text{ along the path.} \end{aligned}$$

Denoting $(1 + \rho'^2 + \rho^2\phi'^2)$ by $(**)$ for brevity, the E–L equations for ρ and ϕ are, respectively,

$$\frac{\rho\phi'^2}{v(**)^{1/2}} - \frac{(**)^{1/2}}{v^2}\frac{\partial v}{\partial \rho} = \frac{d}{dz}\left[\frac{\rho'}{v(**)^{1/2}}\right], \quad (1)$$

and

$$-\frac{(**)^{1/2}}{v^2}\frac{\partial v}{\partial \phi} = \frac{d}{dz}\left[\frac{\rho^2\phi'}{v(**)^{1/2}}\right]. \quad (2)$$

Now, if $v = b(a^2 + \rho^2)^{1/2}$, the only dependence on z in a possible solution $\phi = Az + B$ with $\rho = C$ is through the first of these equations. To see this we note that the square brackets on the RHS's of the two E–L equations do not contain any undifferentiated ϕ-terms and so the derivatives (with respect to z) of both are zero. Since $\partial v/\partial\phi$ is also zero, equation (2) is identically satisfied as $0 = 0$. This leaves only (1), which reads

$$\frac{CA^2}{b(a^2 + C^2)^{1/2}(1 + 0 + C^2A^2)^{1/2}} - \frac{(1 + 0 + C^2A^2)^{1/2}bC}{b^2(a^2 + C^2)(a^2 + C^2)^{1/2}} = 0.$$

This is satisfied provided

$$\begin{aligned} A^2(a^2 + C^2) &= 1 + C^2A^2, \\ \text{i.e.} \quad A &= a^{-1}. \end{aligned}$$

Thus, a solution in the form of a helix is possible provided that the helix has a particular pitch, $2\pi a$.

22.13 *A dam of capacity V (less than $\pi b^2 h/2$) is to be constructed on level ground next to a long straight wall which runs from $(-b, 0)$ to $(b, 0)$. This is to be achieved by joining the ends of a new wall, of height h, to those of the existing wall. Show that, in order to minimise the length L of new wall to be built, it should form part of a circle, and that L is then given by*

$$\int_{-b}^{b} \frac{dx}{(1-\lambda^2 x^2)^{1/2}},$$

where λ is found from

$$\frac{V}{hb^2} = \frac{\sin^{-1}\mu}{\mu^2} - \frac{(1-\mu^2)^{1/2}}{\mu}$$

and $\mu = \lambda b$.

The objective is to chose the wall profile, $y = y(x)$, so as to minimise

$$L = \int_{-b}^{b} \sqrt{(dx)^2 + (dy)^2} = \int_{-b}^{b} (1 + y'^2)^{1/2}\, dx$$

subject to the constraint that the capacity of the dam formed is

$$V = h \int_{-b}^{b} y\, dx.$$

For this constrained variation problem we consider the minimisation of

$$K = \int_{-b}^{b} [\,(1 + y'^2)^{1/2} - \lambda y\,]\, dx,$$

where λ is a Lagrange multiplier.

Since x does not appear in the integrand, a first integral of the E–L equation is

$$(1 + y'^2)^{1/2} - \lambda y - y' \frac{y'}{(1 + y'^2)^{1/2}} = k,$$

$$\frac{1}{(1 + y'^2)^{1/2}} = k + \lambda y.$$

Rearranging this and integrating gives

$$\frac{1}{(k + \lambda y)^2} - 1 = y'^2,$$

$$\frac{(k + \lambda y)\, dy}{\sqrt{1 - (k + \lambda y)^2}} = dx,$$

$$\Rightarrow \quad -\frac{\sqrt{1 - (k + \lambda y)^2}}{\lambda} = x + c.$$

This result can be arranged in a more familiar form as

$$\lambda^2(x+c)^2 + (k+\lambda y)^2 = 1.$$

This is the equation of a circle that is centred on $(-c, -k/\lambda)$; from symmetry $c = 0$. Further, since $(\pm b, 0)$ lies on the curve, we must have

$$\lambda^2 b^2 + k^2 = 1, \qquad (*)$$

giving a connection between the Lagrange multiplier and one of the constants of integration. The length of the wall is given by

$$L = \int_{-b}^{b} (1+y'^2)^{1/2}\,dx = \int_{-b}^{b} \frac{1}{k+\lambda y}\,dx = \int_{-b}^{b} \frac{1}{(1-\lambda^2x^2)^{1/2}}\,dx.$$

The remaining constraint determines the value of λ and is that

$$\begin{aligned}
\frac{V}{h} &= \int_{-b}^{b} y\,dx = \frac{1}{\lambda}\int_{-b}^{b}\left(\sqrt{1-\lambda^2x^2} - k\right)\,dx \\
&= \frac{1}{\lambda}\int_{-b}^{b}\left(\sqrt{1-\lambda^2x^2} - \sqrt{1-\lambda^2b^2}\right)\,dx, \text{ using}(*), \\
\frac{\lambda V}{h} &= \left[x\sqrt{1-\lambda^2x^2}\right]_{-b}^{b} - \int_{-b}^{b} \frac{-\lambda^2 x\,x}{\sqrt{1-\lambda^2x^2}}\,dx - \left[x\sqrt{1-\lambda^2b^2}\right]_{-b}^{b} \\
&= \lambda^2 \int_{-b}^{b} \frac{x^2}{\sqrt{1-\lambda^2x^2}}\,dx.
\end{aligned}$$

To evaluate this integral we set $\lambda x = \sin\theta$ and $\mu = \lambda b = \sin\phi$, to give

$$\begin{aligned}
\frac{\lambda V}{h} &= \int_{-\phi}^{\phi} \frac{\sin^2\theta\,\cos\theta}{\lambda\cos\theta}\,d\theta, \\
\frac{\lambda^2 V}{h} &= \int_{-\phi}^{\phi} \frac{1}{2}(1-\cos 2\theta)\,d\theta \\
&= \phi - \frac{2}{4}\sin 2\phi, \\
\frac{\mu^2 V}{hb^2} &= \sin^{-1}\mu - \frac{1}{2}\,2\mu(1-\mu^2)^{1/2}, \\
\frac{V}{hb^2} &= \frac{\sin^{-1}\mu}{\mu^2} - \frac{(1-\mu^2)^{1/2}}{\mu}.
\end{aligned}$$

This equation determines μ and hence λ.

22.15 *The Schwarzchild metric for the static field of a non-rotating spherically symmetric black hole of mass M is given by*

$$(ds)^2 = c^2\left(1 - \frac{2GM}{c^2 r}\right)(dt)^2 - \frac{(dr)^2}{1 - 2GM/(c^2 r)} - r^2\,(d\theta)^2 - r^2\sin^2\theta\,(d\phi)^2.$$

Considering only motion confined to the plane $\theta = \pi/2$, and assuming that the path of a small test particle is such as to make $\int ds$ stationary, find two first integrals of the equations of motion. From their Newtonian limits, in which GM/r, $\dot{r}^2$ and $r^2\dot{\phi}^2$ are all $\ll c^2$, identify the constants of integration.

For motion confined to the plane $\theta = \pi/2$, $d\theta = 0$ and the corresponding term in the metric can be ignored. With this simplification, we can write

$$ds = \left\{ c^2\left(1 - \frac{2GM}{c^2 r}\right) - \frac{\dot{r}^2}{1 - (2GM)/(c^2 r)} - r^2\dot{\phi}^2 \right\}^{1/2} dt.$$

Writing the terms in braces as $\{**\}$, the E–L equation for ϕ reads

$$\frac{d}{dt}\left(\frac{-r^2\dot{\phi}}{\{**\}^{1/2}}\right) - 0 = 0,$$

$$\Rightarrow \quad \frac{r^2\dot{\phi}}{\{**\}^{1/2}} = A.$$

In the Newtonian limit $\{**\} \to c^2$ and the equation becomes $r^2\dot{\phi} = Ac$. Thus, Ac is a measure of the angular momentum of the particle about the origin.

The E–L equation for r is more complicated but, because ds does not contain t explicitly, we can use the general result for the first integral of the E–L equations when there is more than one dependent variable: $F - \sum_i \dot{q}_i \dfrac{\partial F}{\partial \dot{q}_i} = k$. This gives us a second equation as follows:

$$F - \dot{r}\frac{\partial F}{\partial \dot{r}} - \dot{\phi}\frac{\partial F}{\partial \dot{\phi}} = B,$$

$$\{**\}^{1/2} + \frac{\dot{r}}{\{**\}^{1/2}}\,\frac{\dot{r}}{[1 - (2GM)/(c^2 r)]} + \frac{\dot{\phi}}{\{**\}^{1/2}}\,r^2\dot{\phi} = B.$$

Multiplying through by $\{**\}^{1/2}$ and cancelling the terms in $\dot{r}^2$ and $\dot{\phi}^2$ now gives

$$c^2 - \frac{2GM}{r} = B\left\{c^2 - \frac{2GM}{r} - \frac{\dot{r}^2}{[1 - (2GM)/(c^2 r)]} - r^2\dot{\phi}^2\right\}^{1/2}.$$

In the Newtonian limits, in which GM/r, $\dot{r}^2$ and $r^2\dot{\phi}^2$ are all $\ll c^2$, the equation can be rearranged and the braces expanded to first order in small quantities to

give

$$B = \left(c^2 - \frac{2GM}{r}\right)\left\{c^2 - \frac{2GM}{r} - \frac{\dot{r}^2}{[1-(2GM)/(c^2r)]} - r^2\dot{\phi}^2\right\}^{-1/2},$$

$$cB = c^2 - \frac{2GM}{r} + \frac{c^2GM}{c^2r} + \frac{c^2\dot{r}^2}{2c^2} + \frac{c^2r^2\dot{\phi}^2}{2c^2} + \cdots,$$

$$= c^2 - \frac{GM}{r} + \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) + \cdots,$$

which can be read as 'total energy = rest mass energy + gravitational energy + radial and azimuthal kinetic energy'. Thus Bc is a measure of the total energy of the test particle.

22.17 *Determine the minimum value that the integral*

$$J = \int_0^1 [x^4(y'')^2 + 4x^2(y')^2]\,dx$$

can have, given that y is not singular at $x = 0$ and that $y(1) = y'(1) = 1$. Assume that the Euler–Lagrange equation gives the lower limit and verify retrospectively that your solution satisfies the end-point condition

$$\left[\eta\frac{\partial F}{\partial y'}\right]_a^b = 0,$$

where $F = F(y', y, x)$ and $\eta(x)$ is the variation from the minimising curve.

We first set $y'(x) = u(x)$ with $u(1) = y'(1) = 1$. The integral then becomes

$$J = \int_0^1 [x^4(u')^2 + 4x^2u^2]\,dx. \qquad (*)$$

This will be stationary if (using the E–L equation)

$$\frac{d}{dx}(2x^4u') - 8x^2u = 0,$$

$$8x^3u' + 2x^4u'' - 8x^2u = 0,$$

$$x^2u'' + 4xu' - 4u = 0.$$

As this is a homogeneous equation, we try $u(x) = Ax^n$, obtaining

$$n(n-1) + 4n - 4 = 0 \quad \Rightarrow \quad n = -4, \text{ or } n = 1.$$

The form of $y'(x)$ is thus

$$y'(x) = u(x) = \frac{A}{x^4} + Bx \quad \text{with} \quad A + B = 1.$$

Further,

$$y(x) = -\frac{A}{3x^3} + \frac{Bx^2}{2} + C.$$

Since y is not singular at $x = 0$ and $y(1) = 1$, we have that $A = 0$, $B = 1$ and $C = \frac{1}{2}$, yielding $y(x) = \frac{1}{2}(1 + x^2)$. The minimal value of J is thus

$$J_{\min} = \int_0^1 [\,x^4(1)^2 + 4x^2(x)^2\,]\,dx = \int_0^1 5x^4\,dx = \left[\,x^5\,\right]_0^1 = 1.$$

In $(*)$ the integrand is $G(u', u, x)$ and so the end-point condition reads

$$\left[\eta \frac{\partial G}{\partial u'}\right]_0^1 = 0.$$

At the upper limit $\eta(1) = 0$, since $u(1) = y'(1) = 1$ is fixed. At the lower limit,

$$\left.\frac{\partial G}{\partial u'}\right|_{x=0} = \left.2x^4 u'\right|_{x=0} = 0.$$

Thus the contributions at the two limits are individually zero and the boundary condition is satisfied in the simplest way.

22.19 *Find an appropriate but simple trial function and use it to estimate the lowest eigenvalue λ_0 of Stokes' equation*

$$\frac{d^2y}{dx^2} + \lambda xy = 0, \qquad y(0) = y(\pi) = 0.$$

Explain why your estimate must be strictly greater than λ_0.

Stokes' equation is an S–L equation with $p = 1$, $q = 0$ and $\rho = x$. For the given boundary conditions the obvious trial function is $y(x) = \sin x$. The lowest eigenvalue $\lambda_0 \le I/J$, where

$$I = \int_0^\pi p y'^2\,dx = \int_0^\pi \cos^2 x\,dx = \frac{\pi}{2}$$

and

$$\begin{aligned} J &= \int_0^\pi \rho y^2\,dx = \int_0^\pi x \sin^2 x\,dx \\ &= \int_0^\pi \tfrac{1}{2}x(1 - \cos 2x)\,dx \\ &= \left[\frac{x^2}{4}\right]_0^\pi - \left[\frac{x}{2}\,\frac{\sin 2x}{2}\right]_0^\pi + \frac{1}{2}\int_0^\pi \frac{\sin 2x}{2}\,dx \\ &= \frac{\pi^2}{4}. \end{aligned}$$

Thus $\lambda_0 \leq (\frac{1}{2}\pi)/(\frac{1}{4}\pi^2) = 2/\pi$.

However, if we substitute the trial function directly into the equation we obtain

$$-\sin x + \frac{2}{\pi} x \sin x = 0,$$

which is clearly not satisfied. Thus the trial function is not an eigenfunction, and the actual lowest eigenvalue must be strictly less than the estimate of $2/\pi$.

22.21 *A drumskin is stretched across a fixed circular rim of radius a. Small transverse vibrations of the skin have an amplitude $z(\rho, \phi, t)$ that satisfies*

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

in plane polar coordinates. For a normal mode independent of azimuth, in which case $z = Z(\rho)\cos\omega t$, find the differential equation satisfied by $Z(\rho)$. By using a trial function of the form $a^\nu - \rho^\nu$, with adjustable parameter ν, obtain an estimate for the lowest normal mode frequency.

[*The exact answer is* $(5.78)^{1/2}c/a$.]

In cylindrical polar coordinates, (ρ, ϕ), the wave equation,

$$\nabla^2 z = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2},$$

has azimuth-independent solutions (i.e. independent of ϕ) of the form $z(\rho, t) = Z(\rho)\cos\omega t$, and reduces to

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dZ}{d\rho}\right)\cos\omega t = -\frac{Z\omega^2}{c^2}\cos\omega t,$$

$$\frac{d}{d\rho}\left(\rho\frac{dZ}{d\rho}\right) + \frac{\omega^2}{c^2}\rho Z = 0.$$

The boundary conditions require that $Z(a) = 0$ and, so that there is no physical discontinuity in the slope of the drumskin at the origin, $Z'(0) = 0$.

This is an S–L equation with $p = \rho$, $q = 0$ and weight function $w = \rho$. A suitable trial function is $Z(\rho) = a^\nu - \rho^\nu$, which automatically satisfies $Z(a) = 0$ and, provided $\nu > 1$, has $Z'(0) = -\nu\rho^{\nu-1}|_{\rho=0} = 0$.

We recall that the lowest eigenfrequency satisfies the general formula

$$\frac{\omega^2}{c^2} \leq \frac{\displaystyle\int_0^a [(pZ')^2 - qZ^2]\,d\rho}{\displaystyle\int_0^a wZ^2\,d\rho}.$$

In this case

$$\frac{\omega^2}{c^2} \le \frac{\displaystyle\int_0^a \rho\, v^2 \rho^{2v-2}\, d\rho}{\displaystyle\int_0^a \rho(a^v - \rho^v)^2\, d\rho}$$

$$= \frac{\displaystyle\int_0^a v^2 \rho^{2v-1}\, d\rho}{\displaystyle\int_0^a (\rho a^{2v} - 2\rho^{v+1} a^v + \rho^{2v+1})\, d\rho}$$

$$= \frac{(v^2 a^{2v})/2v}{\dfrac{a^{2v+2}}{2} - \dfrac{2a^{2v+2}}{v+2} + \dfrac{a^{2v+2}}{2v+2}}$$

$$= \frac{1}{a^2}\,\frac{v(v+2)(2v+2)}{(v+2)(2v+2) - 4(2v+2) + 2(v+2)}$$

$$= \frac{(v+2)(v+1)}{v a^2}.$$

Since v is an adjustable parameter and we know that, however we choose it, the resulting estimate can *never* be less than the lowest true eigenvalue, we choose the value that minimises the above estimate. Differentiating the estimate with respect to v gives

$$v(2v+3) - (v^2 + 3v + 2) = 0 \quad \Rightarrow \quad v^2 - 2 = 0 \quad \Rightarrow \quad v = \sqrt{2}.$$

Thus the least upper bound to be found with this parameterisation is

$$\omega^2 \le \frac{c^2}{a^2}\,\frac{(\sqrt{2}+2)(\sqrt{2}+1)}{\sqrt{2}} = \frac{c^2}{2a^2}(\sqrt{2}+2)^2 \quad \Rightarrow \quad \omega = (5.83)^{1/2}\,\frac{c}{a}.$$

As noted, the actual lowest eigenfrequency is very little below this.

22.23 *For the boundary conditions given below, obtain a functional $\Lambda(y)$ whose stationary values give the eigenvalues of the equation*

$$(1+x)\frac{d^2y}{dx^2} + (2+x)\frac{dy}{dx} + \lambda y = 0, \qquad y(0) = 0,\ y'(2) = 0.$$

Derive an approximation to the lowest eigenvalue λ_0 using the trial function $y(x) = xe^{-x/2}$. For what value(s) of γ would

$$y(x) = xe^{-x/2} + \beta \sin \gamma x$$

be a suitable trial function for attempting to obtain an improved estimate of λ_0?

Since the derivative of $1 + x$ is not equal to $2 + x$, the given equation is not in self-adjoint form and an integrating factor for the standard form equation,

$$\frac{d^2y}{dx^2} + \frac{2+x}{1+x}\frac{dy}{dx} + \frac{\lambda y}{1+x} = 0,$$

is needed. This will be

$$\exp\left\{\int^x \frac{2+u}{1+u}\,du\right\} = \exp\left\{\int^x \left(1 + \frac{1}{1+u}\right)\,du\right\} = e^x(1+x).$$

Thus, after multiplying through by this IF, the equation takes the S–L form

$$[\,(1+x)e^x y'\,]' + \lambda e^x y = 0,$$

with $p(x) = (1+x)e^x$, $q(x) = 0$ and $\rho(x) = e^x$.

The required functional is therefore

$$\Lambda(y) = \frac{\int_0^2 [\,(1+x)e^x y'^2 + 0\,]\,dx}{\int_0^2 y^2 e^x\,dx},$$

provided that, for the eigenfunctions y_i of the equation, $\left[\,y_i p(x) y_j'(x)\,\right]_0^2 = 0$; this condition is automatically satisfied with the given boundary conditions.

For the trial function $y(x) = xe^{-x/2}$, clearly $y(0) = 0$ and, less obviously, $y'(x) = (1 - \frac{1}{2}x)e^{-x/2}$, making $y'(2) = 0$. The functional takes the following form:

$$\begin{aligned}
\Lambda &= \frac{\int_0^2 (1+x)e^x\,(1-\frac{1}{2}x)^2 e^{-x}\,dx}{\int_0^2 x^2 e^{-x} e^x\,dx} \\
&= \frac{\int_0^2 (1+x)(1-\frac{1}{2}x)^2\,dx}{\int_0^2 x^2\,dx} \\
&= \frac{\int_0^2 (1 - x^2 + \frac{1}{4}x^2 + \frac{1}{4}x^3)\,dx}{8/3} \\
&= \frac{3}{8}\left(2 - \frac{3}{4}\,\frac{8}{3} + \frac{16}{16}\right) = \frac{3}{8}.
\end{aligned}$$

Thus the lowest eigenvalue is $\leq \frac{3}{8}$.

We already know that $xe^{-x/2}$ is a suitable trial function and thus $y_2(x) = \sin\gamma x$ can be considered on its own. It satisfies $y_2(0) = 0$, but must also satisfy $y_2'(2) = \gamma\cos(2\gamma) = 0$. This requires that $\gamma = \frac{1}{2}(n + \frac{1}{2})\pi$ for some integer n; trial functions with γ of this form can be used to try to obtain a better bound on λ_0 by choosing the best value for n and adjusting the parameter β.

22.25 *The unnormalised ground-state (i.e. the lowest-energy) wavefunction of the simple harmonic oscillator of classical frequency ω is $\exp(-\alpha x^2)$, where $\alpha = m\omega/2\hbar$. Take as a trial function the orthogonal wavefunction $x^{2n+1}\exp(-\alpha x^2)$, using the integer n as a variable parameter, and apply either Sturm–Liouville theory or the Rayleigh–Ritz principle to show that the energy of the second lowest state of a quantum harmonic oscillator is $\leq 3\hbar\omega/2$.*

We first note that, for n a non-negative integer,

$$\int_{-\infty}^{\infty} x^{2n+1} e^{-\alpha x^2} e^{-\alpha x^2}\, dx = 0$$

on symmetry grounds and so confirm that the ground-state wavefunction, $\exp(-\alpha x^2)$, and the trial function, $\psi_{2n+1} = x^{2n+1}\exp(-\alpha x^2)$, are orthogonal with respect to a unit weight function.

The Hamiltonian for the quantum harmonic oscillator in one-dimension is given by

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{k}{2}x^2.$$

This means that to prepare the elements required for a Rayleigh–Ritz analysis we will need to find the second derivative of the trial function and evaluate integrals with integrands of the form $x^n\exp(-2\alpha x^2)$. To this end, define

$$I_n = \int_{-\infty}^{\infty} x^n e^{-2\alpha x^2}\, dx, \text{ with recurrence relation } I_n = \frac{n-1}{4\alpha} I_{n-2}.$$

Using Leibnitz' formula shows that

$$\begin{aligned}\frac{d^2\psi_{2n+1}}{dx^2} &= \big[\, 2n(2n+1)x^{2n-1} + 2(2n+1)(-2\alpha)x^{2n+1} \\ &\qquad + (4\alpha^2x^2 - 2\alpha)x^{2n+1}\,\big]\, e^{-\alpha x^2} \\ &= \big[\, 2n(2n+1)x^{2n-1} - 2(4n+3)\alpha x^{2n+1} + 4\alpha^2 x^{2n+3}\,\big]\, e^{-\alpha x^2}.\end{aligned}$$

Hence, we find that $\langle H\rangle$ is given by

$$\begin{aligned}&-\frac{\hbar^2}{2m}\int_{-\infty}^{\infty} x^{2n+1}e^{-\alpha x^2}\frac{d^2\psi_{2n+1}}{dx^2}\, dx + \frac{k}{2}\int_{-\infty}^{\infty} x^2 x^{4n+2} e^{-2\alpha x^2}\, dx \\ &= -\frac{\hbar^2}{2m}\big[\, 2n(2n+1)I_{4n} - 2(4n+3)\alpha I_{4n+2} + 4\alpha^2 I_{4n+4}\,\big] + \frac{k}{2}I_{4n+4} \\ &= I_{4n+2}\left\{-\frac{\hbar^2}{2m}\left[\frac{2n(2n+1)4\alpha}{4n+1} - 2(4n+3)\alpha + \frac{4\alpha^2(4n+3)}{4\alpha}\right] + \frac{k(4n+3)}{8\alpha}\right\},\end{aligned}$$

where we have used the recurrence relation to express all integrals in terms of

I_{4n+2}. This has been done because the denominator of the Rayleigh–Ritz quotient is this (same) normalisation integral, namely

$$\int_{-\infty}^{\infty} \psi_{2n+1}^{*}\psi_{2n+1}\,dx = I_{4n+2}.$$

Thus, the estimate $E_{2n+1} = \langle H\rangle / I_{4n+2}$ is given by

$$\begin{aligned} E_{2n+1} &= -\frac{\hbar^2\alpha}{2m}\left(\frac{16n^2+8n-16n^2-16n-3}{4n+1}\right) + \frac{k(4n+3)}{8\alpha} \\ &= \frac{\hbar^2\alpha}{2m}\,\frac{8n+3}{4n+1} + \frac{k(4n+3)}{8\alpha}. \end{aligned}$$

Using $\omega^2 = \dfrac{k}{m}$ and $\alpha = \dfrac{m\omega}{2\hbar}$ then yields

$$E_{2n+1} = \frac{\hbar\omega}{4}\left(\frac{8n+3}{4n+1} + 4n+3\right) = \frac{\hbar\omega}{2}\,\frac{8n^2+12n+3}{4n+1}.$$

For non-negative integers n (the orthogonality requirement is not satisfied for non-integer values), this has a minimum value of $\frac{3}{2}\hbar\omega$ when $n = 0$. Thus the second lowest energy level is less than or equal to this value. In fact, it is equal to this value, as can be shown by substituting ψ_1 into $H\psi = E\psi$.

22.27 *The upper and lower surfaces of a film of liquid, which has surface energy per unit area (surface tension) γ and density ρ, have equations $z = p(x)$ and $z = q(x)$, respectively. The film has a given volume V (per unit depth in the y-direction) and lies in the region $-L < x < L$, with $p(0) = q(0) = p(L) = q(L) = 0$. The total energy (per unit depth) of the film consists of its surface energy and its gravitational energy, and is expressed by*

$$E = \tfrac{1}{2}\rho g\int_{-L}^{L}(p^2-q^2)\,dx + \gamma\int_{-L}^{L}\left[(1+p'^2)^{1/2} + (1+q'^2)^{1/2}\right]dx.$$

(a) *Express V in terms of p and q.*

(b) *Show that, if the total energy is minimised, p and q must satisfy*

$$\frac{p'^2}{(1+p'^2)^{1/2}} - \frac{q'^2}{(1+q'^2)^{1/2}} = \textit{constant}.$$

(c) *As an approximate solution, consider the equations*

$$p = a(L-|x|), \qquad q = b(L-|x|),$$

where a and b are sufficiently small that a^3 and b^3 can be neglected compared with unity. Find the values of a and b that minimise E.

(a) The total volume constraint is given simply by

$$V = \int_{-L}^{L} [\,p(x) - q(x)\,]\,dx.$$

(b) To take account of the constraint, consider the minimisation of $E - \lambda V$, where λ is an undetermined Lagrange multiplier. The integrand does not contain x explicitly and so we have two first integrals of the E–L equations, one for $p(x)$ and the other for $q(x)$. They are

$$\frac{1}{2}\rho g(p^2 - q^2) + \gamma(1 + p'^2)^{1/2} + \gamma(1 + q'^2)^{1/2} - \lambda(p - q) - p'\frac{\gamma p'}{(1 + p'^2)^{1/2}} = k_1$$

and

$$\frac{1}{2}\rho g(p^2 - q^2) + \gamma(1 + p'^2)^{1/2} + \gamma(1 + q'^2)^{1/2} - \lambda(p - q) - q'\frac{\gamma q'}{(1 + q'^2)^{1/2}} = k_2.$$

Subtracting these two equations gives

$$\frac{p'^2}{(1 + p'^2)^{1/2}} - \frac{q'^2}{(1 + q'^2)^{1/2}} = \text{constant}.$$

(c) If

$$p = a(L - |x|), \qquad q = b(L - |x|),$$

the derivatives of p and q only take the values $\pm a$ and $\pm b$, respectively, and the volume constraint becomes

$$V = \int_{-L}^{L} (a - b)(L - |x|)\,dx = (a - b)L^2 \quad \Rightarrow \quad b = a - \frac{V}{L^2}.$$

The total energy can now be expressed entirely in terms of a and the given parameters, as follows:

$$\begin{aligned}
E &= \frac{1}{2}\rho g \int_{-L}^{L} (a^2 - b^2)(L - |x|)^2\,dx + 2\gamma L(1 + a^2)^{1/2} + 2\gamma L(1 + b^2)^{1/2} \\
&= \frac{1}{2}\rho g(a^2 - b^2)\frac{2L^3}{3} + 2\gamma L(1 + \tfrac{1}{2}a^2 + 1 + \tfrac{1}{2}b^2) + \mathrm{O}(a^4) + \mathrm{O}(b^4) \\
&\approx \frac{\rho g L^3}{3}\left[a^2 - \left(a - \tfrac{V}{L^2}\right)^2\right] + 2\gamma L\left[2 + \tfrac{1}{2}a^2 + \tfrac{1}{2}\left(a - \tfrac{V}{L^2}\right)^2\right] \\
&= \frac{\rho g L^3}{3}\left(\frac{2aV}{L^2} - \frac{V^2}{L^4}\right) + 2\gamma L\left(2 + a^2 - \frac{aV}{L^2} + \frac{V^2}{2L^4}\right).
\end{aligned}$$

This is minimised with respect to a when

$$\frac{2\rho g L^3 V}{3L^2} + 4\gamma L a - \frac{2\gamma L V}{L^2} = 0,$$

$$\Rightarrow \quad a = \frac{V}{2L^2} - \frac{\rho g V}{6\gamma},$$

$$\Rightarrow \quad b = -\frac{V}{2L^2} - \frac{\rho g V}{6\gamma}.$$

As might be expected, $|b| > |a|$ and there is more of the liquid below the $z = 0$ plane than there is above it.

22.29 *The 'stationary value of an integral' approach to finding the eigenvalues of a Sturm–Liouville equation can be extended to two independent variables, x and z, with little modification. In the integral to be minimised, y'^2 is replaced by $(\nabla y)^2$ and the integrals of the various functions of $y(x, z)$ become two-dimensional, i.e. the infinitesimal is $dx\,dz$.*

The vibrations of a trampoline 4 units long and 1 unit wide satisfy the equation

$$\nabla^2 y + k^2 y = 0.$$

By taking the simplest possible permissible polynomial as a trial function, show that the lowest mode of vibration has $k^2 \leq 10.63$ and, by direct solution, that the actual value is 10.49.

Written explicitly, the equation is

$$\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2 y}{\partial z^2} + k^2 y = 0.$$

This is an extended S–L equation with $p(x, z) = 1$, $q(x, z) = 0$, $\rho(x, z) = 1$ and eigenvalue λ. We therefore consider the stationary values of $\Lambda = I/J$, where

$$I = \int\int \left[\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 \right] dx\,dz$$

and J is the normalisation integral $\int y^2(x, z)\,dx\,dz$.

For a trampoline 4 units long and 1 unit wide, the simplest trial function that satisfies $y(0, z) = y(4, z) = y(x, 0) = y(x, 1) = 0$ is

$$y(x, z) = x(4 - x)z(1 - z).$$

For this function,

$$\frac{\partial y}{\partial x} = (4 - 2x)z(1 - z) \quad \text{and} \quad \frac{\partial y}{\partial z} = (1 - 2z)x(4 - x).$$

Thus, I is given by

$$\begin{aligned}
&\int_0^4 (4-2x)^2\,dx \int_0^1 z^2(1-z)^2\,dz + \int_0^4 x^2(4-x)^2\,dx \int_0^1 (1-2z)^2\,dz \\
&= \left[16(4) - 16\left(\frac{16}{2}\right) + 4\left(\frac{64}{3}\right)\right]\left[\left(\frac{1}{3}\right) - 2\left(\frac{1}{4}\right) + \left(\frac{1}{5}\right)\right] \\
&\quad + \left[16\left(\frac{64}{3}\right) - 8\left(\frac{256}{4}\right) + \left(\frac{1024}{5}\right)\right]\left[1(1) - 4\left(\frac{1}{2}\right) + 4\left(\frac{1}{3}\right)\right] \\
&= 64\left(1 - 2 + \frac{4}{3}\right)\frac{1}{30} + \frac{1024}{30}\left(1 - 2 + \frac{4}{3}\right) = \frac{1088}{90}.
\end{aligned}$$

Similarly, J is given by

$$\begin{aligned}
&\int_0^4 x^2(4-x)^2\,dx \int_0^1 z^2(1-z)^2\,dz \\
&= \left[16\left(\frac{64}{3}\right) - 8\left(\frac{256}{4}\right) + \left(\frac{1024}{5}\right)\right]\left[\left(\frac{1}{3}\right) - 2\left(\frac{1}{4}\right) + \left(\frac{1}{5}\right)\right] \\
&= 1024\left(\frac{1}{3} - \frac{2}{4} + \frac{1}{5}\right)^2 = \frac{1024}{900}.
\end{aligned}$$

Thus the lowest eigenvalue $k^2 \leq (1088/90) \div (1024/900) = 10.63$.

The obvious direct solution satisfying the boundary conditions is

$$y(x,z) = A \sin\frac{\pi x}{4} \sin \pi z.$$

Substituting this into the original equation gives

$$-\frac{\pi^2}{16} y(x,z) - \pi^2 y(x,z) + k^2 y(x,z) = 0,$$

which is clearly satisfied if

$$k^2 = \frac{17\pi^2}{16} = 10.49.$$

23

Integral equations

23.1 *Solve the integral equation*

$$\int_0^\infty \cos(xv)y(v)\,dv = \exp(-x^2/2)$$

for the function $y = y(x)$ *for* $x > 0$. *Note that for* $x < 0$, $y(x)$ *can be chosen as is most convenient.*

Since $\cos uv$ is an even function of v, we will make $y(-v) = y(v)$ so that the complete integrand is also an even function of v. The integral I on the LHS can then be written as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \cos(xv)y(v)\,dv = \frac{1}{2}\mathrm{Re}\int_{-\infty}^{\infty} e^{ixv}y(v)\,dv = \frac{1}{2}\int_{-\infty}^{\infty} e^{ixv}y(v)\,dv,$$

the last step following because $y(v)$ is symmetric in v. The integral is now $\sqrt{2\pi} \times$ a Fourier transform, and it follows from the inversion theorem for Fourier transforms applied to

$$\frac{1}{2}\int_{-\infty}^{\infty} e^{ixv}y(v)\,dv = \exp(-x^2/2)$$

that

$$\begin{aligned} y(x) &= \frac{2}{2\pi}\int_{-\infty}^{\infty} e^{-u^2/2}e^{-iux}\,du \\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} e^{-(u+ix)^2/2}e^{-x^2/2}\,dx \\ &= \frac{1}{\pi}\sqrt{2\pi}\,e^{-x^2/2} \\ &= \sqrt{\frac{2}{\pi}}\,e^{-x^2/2}. \end{aligned}$$

Although, as noted in the question, $y(x)$ is arbitrary for $x < 0$, because its form in this range does not affect the value of the integral, for $x > 0$ it *must* have the form given. This is tricky to prove formally, but any second solution $w(x)$ has to satisfy

$$\int_0^\infty \cos(xv)[y(v) - w(v)]\,dv = 0$$

for all $x > 0$. Intuitively, this implies that $y(x)$ and $w(x)$ are identical functions.

23.3 *Convert*

$$f(x) = \exp x + \int_0^x (x - y)f(y)\,dy$$

into a differential equation, and hence show that its solution is

$$(\alpha + \beta x)\exp x + \gamma\ \exp(-x),$$

where α, β, γ are constants that should be determined.

We differentiate the integral equation twice and obtain

$$f'(x) = e^x + (x - x)f(x) + \int_0^x f(y)\,dy,$$
$$f''(x) = e^x + f(x).$$

Expressed in the usual differential equation form, this last equation is

$$f''(x) - f(x) = e^x, \text{ for which the CF is } f(x) = Ae^x + Be^{-x}.$$

Since the complementary function contains the RHS of the equation, we try as a PI $f(x) = Cxe^x$:

$$Cxe^x + 2Ce^x - Cxe^x = e^x \quad \Rightarrow \quad \beta = C = \tfrac{1}{2}.$$

The general solution is therefore $f(x) = Ae^x + Be^{-x} + \frac{1}{2}xe^x$.

The boundary conditions needed to evaluate A and B are constructed by considering the integral equation and its derivative(s) at $x = 0$, because with $x = 0$ the integral on the RHS contributes nothing. We have

$$f(0) = e^0 + 0 = 1 \quad \Rightarrow \quad A + B = 1$$

and $$f'(0) = e^0 + 0 = 1 \quad \Rightarrow \quad A - B + \tfrac{1}{2} = 1.$$

Solving these yields $\alpha = A = \frac{3}{4}$ and $\gamma = B = \frac{1}{4}$ and makes the complete solution

$$f(x) = \tfrac{3}{4}e^x + \tfrac{1}{4}e^{-x} + \tfrac{1}{2}xe^x.$$

23.5 *Solve for $\phi(x)$ the integral equation*

$$\phi(x) = f(x) + \lambda \int_0^1 \left[\left(\frac{x}{y}\right)^n + \left(\frac{y}{x}\right)^n \right] \phi(y)\, dy,$$

where $f(x)$ is bounded for $0 < x < 1$ and $-\frac{1}{2} < n < \frac{1}{2}$, expressing your answer in terms of the quantities $F_m = \int_0^1 f(y) y^m\, dy$.

(a) *Give the explicit solution when $\lambda = 1$.*
(b) *For what values of λ are there no solutions unless $F_{\pm n}$ are in a particular ratio? What is this ratio?*

This equation has a symmetric degenerate kernel, and so we set

$$\phi(x) = f(x) + a_1 x^n + a_2 x^{-n},$$

giving

$$\begin{aligned}
\frac{\phi(x) - f(x)}{\lambda} &= \int_0^1 \left(\frac{x^n}{y^n} + \frac{y^n}{x^n}\right) [f(y) + a_1 y^n + a_2 y^{-n}]\, dy \\
&= x^n \int_0^1 \frac{f(y)}{y^n}\, dy + x^{-n} \int_0^1 y^n f(y)\, dy + a_1 x^n \\
&\qquad + a_2 x^{-n} + a_1 x^{-n} \int_0^1 y^{2n}\, dy + a_2 x^n \int_0^1 y^{-2n}\, dy \\
&= x^n \left(F_{-n} + a_1 + \frac{a_2}{1-2n}\right) + x^{-n} \left(F_n + a_2 + \frac{a_1}{2n+1}\right).
\end{aligned}$$

This is consistent with the assumed form of $\phi(x)$, provided

$$a_1 = \lambda \left(F_{-n} + a_1 + \frac{a_2}{1-2n}\right) \quad \text{and} \quad a_2 = \lambda \left(F_n + a_2 + \frac{a_1}{2n+1}\right).$$

These two simultaneous linear equations can now be solved for a_1 and a_2.

(a) For $\lambda = 1$, the equations simplify and decouple to yield

$$a_2 = -(1-2n)F_{-n} \quad \text{and} \quad a_1 = -(1+2n)F_n,$$

respectively, giving as the explicit solution

$$\phi(x) = f(x) - (1+2n)F_n x^n - (1-2n)F_{-n} x^{-n}.$$

(b) For a general value of λ,

$$\begin{aligned}
(1-\lambda)a_1 - \frac{\lambda}{1-2n}\, a_2 &= \lambda F_{-n}, \\
-\frac{\lambda}{1+2n}\, a_1 + (1-\lambda)a_2 &= \lambda F_n.
\end{aligned}$$

The case $\lambda = 0$ is trivial, with $\phi(x) = f(x)$, and so suppose that $\lambda \neq 0$. Then, after being divided through by λ, the equations can be written in the matrix and vector form $\mathsf{Aa} = \mathsf{F}$:

$$\begin{pmatrix} \dfrac{1}{\lambda} - 1 & -\dfrac{1}{1-2n} \\ -\dfrac{1}{1+2n} & \dfrac{1}{\lambda} - 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} F_{-n} \\ F_n \end{pmatrix}.$$

In general, this matrix equation will have no solution if $|\mathsf{A}| = 0$. This will be the case if

$$\left(\frac{1}{\lambda} - 1\right)^2 - \frac{1}{1-4n^2} = 0,$$

which, on rearrangement, shows that λ would have to be given by

$$\frac{1}{\lambda} = 1 \pm \frac{1}{\sqrt{1-4n^2}}.$$

We note that this value for λ is real because n lies in the range $-\frac{1}{2} < n < \frac{1}{2}$. In fact $-\infty < \lambda < \frac{1}{2}$. Even for these two values of λ, however, if *either* $F_n = F_{-n} = 0$ *or* the matrix equation

$$\begin{pmatrix} \pm\dfrac{1}{\sqrt{1-4n^2}} & -\dfrac{1}{1-2n} \\ -\dfrac{1}{1+2n} & \pm\dfrac{1}{\sqrt{1-4n^2}} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} F_{-n} \\ F_n \end{pmatrix}$$

is equivalent to two linear equations that are multiples of each other, there will still be a solution. In this latter case, we must have

$$\frac{F_n}{F_{-n}} = \mp\sqrt{\frac{1-2n}{1+2n}}.$$

Again we note that, because of the range in which n lies, this ratio is real; this condition can, however, require any value in the range $-\infty$ to ∞ for F_n/F_{-n}.

23.7 *The kernel of the integral equation*

$$\psi(x) = \lambda \int_a^b K(x, y)\psi(y)\, dy$$

has the form

$$K(x, y) = \sum_{n=0}^{\infty} h_n(x) g_n(y),$$

where the $h_n(x)$ *form a complete orthonormal set of functions over the interval* $[a, b]$.

(a) *Show that the eigenvalues* λ_i *are given by*

$$|\mathsf{M} - \lambda^{-1}\mathsf{I}| = 0,$$

where M *is the matrix with elements*

$$M_{kj} = \int_a^b g_k(u) h_j(u)\, du.$$

If the corresponding solutions are $\psi^{(i)}(x) = \sum_{n=0}^{\infty} a_n^{(i)} h_n(x)$, *find an expression for* $a_n^{(i)}$.

(b) *Obtain the eigenvalues and eigenfunctions over the interval* $[0, 2\pi]$ *if*

$$K(x, y) = \sum_{n=1}^{\infty} \frac{1}{n} \cos nx \cos ny.$$

(a) We write the ith eigenfunction as

$$\psi^{(i)}(x) = \sum_{n=0}^{\infty} a_n^{(i)} h_n(x).$$

From the orthonormality of the $h_n(x)$, it follows immediately that

$$a_m^{(i)} = \int_a^b h_m(x) \psi^{(i)}(x)\, dx.$$

However, the coefficients $a_m^{(i)}$ have to be found as the components of the eigenvectors $\mathsf{a}^{(i)}$ defined below, since the $\psi^{(i)}$ are not initially known.

Substituting this assumed form of solution, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty} a_m^{(i)} h_m(x) &= \lambda_i \int_a^b \sum_{n=0}^{\infty} h_n(x) g_n(y) \sum_{l=0}^{\infty} a_l^{(i)} h_l(y)\, dy \\ &= \lambda_i \sum_{n,l} a_l^{(i)} M_{nl} h_n(x).\end{aligned}$$

Since the $\{h_n\}$ are an orthonormal set, it follows that

$$a_m^{(i)} = \lambda_i \sum_{n,l} a_l^{(i)} M_{nl}\delta_{mn} = \lambda_i \sum_{l=0}^{\infty} M_{ml} a_l^{(i)},$$

$$\text{i.e.} \quad (\mathsf{M} - \lambda_i^{-1}\mathsf{I})\mathsf{a}^{(i)} = 0.$$

Thus, the allowed values of λ_i are given by $|\mathsf{M} - \lambda^{-1}\mathsf{I}| = 0$, and the expansion coefficients $a_m^{(i)}$ by the components of the corresponding eigenvectors.

(b) To make the set $\{h_n(x) = \cos nx\}$ into a complete orthonormal set we need to add the set of functions $\{\eta_\nu(x) = \sin \nu x\}$ and then normalise all the functions by multiplying them by $1/\sqrt{\pi}$. For this particular kernel the general functions $g_n(x)$ are given by $g_n(x) = n^{-1}\sqrt{\pi}\cos nx$.

The matrix elements are then

$$M_{kj} = \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \cos ju \frac{\sqrt{\pi}}{k} \cos ku\, du = \frac{\pi}{k}\delta_{kj},$$
$$M_{k\nu} = \int_0^{2\pi} \frac{1}{\sqrt{\pi}} \sin \nu u \frac{\sqrt{\pi}}{k} \cos ku\, du = 0.$$

Thus the matrix M is diagonal and particularly simple. The eigenvalue equation reads

$$\sum_{j=0}^{\infty} \left(\frac{\pi}{k}\delta_{kj} - \lambda_i^{-1}\delta_{kj} \right) a_j^{(i)} = 0,$$

giving the immediate result that $\lambda_k = k/\pi$ with $a_k^{(k)} = 1$ and all other $a_j^{(k)} = a_\nu^{(k)} = 0$. The eigenfunction corresponding to eigenvalue k/π is therefore

$$\psi^{(k)}(x) = h_k(x) = \frac{1}{\sqrt{\pi}} \cos kx.$$

23.9 *For $f(t) = \exp(-t^2/2)$, use the relationships of the Fourier transforms of $f'(t)$ and $tf(t)$ to that of $f(t)$ itself to find a simple differential equation satisfied by $\tilde{f}(\omega)$, the Fourier transform of $f(t)$, and hence determine $\tilde{f}(\omega)$ to within a constant. Use this result to solve for $h(t)$ the integral equation*

$$\int_{-\infty}^{\infty} e^{-t(t-2x)/2} h(t)\, dt = e^{3x^2/8}.$$

As a standard result,

$$\mathscr{F}\left[f'(t) \right] = i\omega \tilde{f}(\omega),$$

though we will not need this relationship in the following solution.

From its definition,

$$\begin{aligned}\mathscr{F}\,[\,tf(t)\,] &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} tf(t)\,e^{-i\omega t}\,dt \\ &= \frac{1}{-i}\frac{d}{d\omega}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,e^{-i\omega t}\,dt\right) = i\frac{d\tilde{f}}{d\omega}.\end{aligned}$$

Now, for the particular given function,

$$\begin{aligned}\tilde{f}(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-t^2/2}\,e^{-i\omega t}\,dt \\ &= \frac{1}{\sqrt{2\pi}}\left[\frac{e^{-t^2/2}\,e^{-i\omega t}}{-i\omega}\right]_{-\infty}^{\infty} + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{te^{-t^2/2}\,e^{-i\omega t}}{-i\omega}\,dt \\ &= 0 - \frac{1}{i\omega}\,i\frac{d\tilde{f}}{d\omega}.\end{aligned}$$

Hence,

$$\frac{d\tilde{f}}{d\omega} = -\omega\tilde{f} \quad\Rightarrow\quad \ln\tilde{f} = -\tfrac{1}{2}\omega^2 + k \quad\Rightarrow\quad \tilde{f} = Ae^{-\omega^2/2}, \qquad (*)$$

giving $\tilde{f}(\omega)$ to within a multiplicative constant.

Now, we are given

$$\begin{aligned}&\int_{-\infty}^{\infty} e^{-t(t-2x)/2}\,h(t)\,dt = e^{3x^2/8}, \\ \Rightarrow\quad &\int_{-\infty}^{\infty} e^{-(t-x)^2/2}\,e^{x^2/2}\,h(t)\,dt = e^{3x^2/8}, \\ \Rightarrow\quad &\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,h(t)\,dt = e^{-x^2/8}. \qquad (**)\end{aligned}$$

The LHS of (**) is a convolution integral, and so applying the convolution theorem for Fourier transforms and result (*), used twice, yields

$$\begin{aligned}&\sqrt{2\pi}Ae^{-\omega^2/2}\tilde{h}(\omega) = \mathscr{F}\left[e^{-(x/2)^2/2}\right] = Ae^{-(2\omega)^2/2}, \\ \Rightarrow\quad &\sqrt{2\pi}\,\tilde{h}(\omega) = e^{-3\omega^2/2} = e^{-(\sqrt{3}\omega)^2/2}, \\ \Rightarrow\quad &h(t) = \frac{1}{\sqrt{2\pi}A}\,e^{-(t/\sqrt{3})^2/2} = \frac{1}{\sqrt{2\pi}A}\,e^{-t^2/6}.\end{aligned}$$

We now substitute in (**) to determine A:

$$\int_{-\infty}^{\infty} e^{-(x-t)^2/2}\,\frac{1}{\sqrt{2\pi}A}\,e^{-t^2/6}\,dt = e^{-x^2/8},$$

$$\frac{1}{\sqrt{2\pi}A}\int_{-\infty}^{\infty} e^{-2t^2/3}\,e^{xt}\,e^{-x^2/2}\,e^{x^2/8}\,dt = 1,$$

$$\frac{1}{\sqrt{2\pi}A}\int_{-\infty}^{\infty} \exp\left[-\frac{2}{3}\left(t-\frac{3x}{4}\right)^2\right]\,dt = 1.$$

From the normalisation of the Gaussian integral, this implies that

$$\frac{1}{\sqrt{2\pi}A} = \frac{2}{\sqrt{2\pi}\,\sqrt{3}},$$

which in turn means $A = \sqrt{3}/2$, giving finally that

$$h(t) = \sqrt{\frac{2}{3\pi}}\,e^{-t^2/6}.$$

This solution can be checked by resubstitution.

23.11 *At an international 'peace' conference a large number of delegates are seated around a circular table with each delegation sitting near its allies and diametrically opposite the delegation most bitterly opposed to it. The position of a delegate is denoted by θ, with $0 \le \theta \le 2\pi$. The fury $f(\theta)$ felt by the delegate at θ is the sum of his own natural hostility $h(\theta)$ and the influences on him of each of the other delegates; a delegate at position ϕ contributes an amount $K(\theta-\phi)f(\phi)$. Thus*

$$f(\theta) = h(\theta) + \int_0^{2\pi} K(\theta-\phi)f(\phi)\,d\phi.$$

Show that if $K(\psi)$ takes the form $K(\psi) = k_0 + k_1\cos\psi$ then

$$f(\theta) = h(\theta) + p + q\cos\theta + r\sin\theta$$

and evaluate p, q and r. A positive value for k_1 implies that delegates tend to placate their opponents but upset their allies, whilst negative values imply that they calm their allies but infuriate their opponents. A walkout will occur if $f(\theta)$ exceeds a certain threshold value for some θ. Is this more likely to happen for positive or for negative values of k_1?

Given that $K(\psi) = k_0 + k_1\cos\psi$, we try a solution $f(\theta) = h(\theta) + p + q\cos\theta + r\sin\theta$,

reducing the equation to

$$\begin{aligned} & p + q\cos\theta + r\sin\theta \\ &= \int_0^{2\pi} [k_0 + k_1(\cos\theta\cos\phi + \sin\theta\sin\phi)] \\ &\qquad \times\ [h(\phi) + p + q\cos\phi + r\sin\phi]\ d\phi \\ &= k_0(H + 2\pi p) + k_1(H_c\cos\theta + H_s\sin\theta + \pi q\cos\theta + \pi r\sin\theta), \end{aligned}$$

where $H = \int_0^{2\pi} h(z)\,dz$, $H_c = \int_0^{2\pi} h(z)\cos z\,dz$ and $H_s = \int_0^{2\pi} h(z)\sin z\,dz$.

Thus, on equating the constant terms and the coefficients of $\cos\theta$ and $\sin\theta$, we have

$$\begin{aligned} p &= k_0H + 2\pi k_0 p \quad\Rightarrow\quad p = \frac{k_0H}{1 - 2\pi k_0}, \\ q &= k_1H_c + k_1\pi q \quad\Rightarrow\quad q = \frac{k_1H_c}{1 - k_1\pi}, \\ r &= k_1H_s + k_1\pi r \quad\Rightarrow\quad r = \frac{k_1H_s}{1 - k_1\pi}. \end{aligned}$$

And so the full solution for $f(\theta)$ is given by

$$\begin{aligned} f(\theta) &= h(\theta) + \frac{k_0H}{1 - 2\pi k_0} + \frac{k_1H_c}{1 - k_1\pi}\cos\theta + \frac{k_1H_s}{1 - k_1\pi}\sin\theta \\ &= h(\theta) + \frac{k_0H}{1 - 2\pi k_0} + \frac{k_1}{1 - k_1\pi}(H_c^2 + H_s^2)^{1/2}\cos(\theta - \alpha), \end{aligned}$$

where $\tan\alpha = H_s/H_c$.

Clearly, the maximum value of $f(\theta)$ will depend upon $h(\theta)$ and its various integrals, but it is most likely to exceed any particular value if k_1 is positive and $\approx \pi^{-1}$. Stick with your friends!

23.13 *The operator $\mathscr{M}$ is defined by*

$$\mathscr{M}f(x) \equiv \int_{-\infty}^{\infty} K(x, y)f(y)\,dy,$$

where $K(x, y) = 1$ inside the square $|x| < a, |y| < a$ and $K(x, y) = 0$ elsewhere. Consider the possible eigenvalues of $\mathscr{M}$ and the eigenfunctions that correspond to them; show that the only possible eigenvalues are 0 and $2a$ and determine the corresponding eigenfunctions. Hence find the general solution of

$$f(x) = g(x) + \lambda\int_{-\infty}^{\infty} K(x, y)f(y)\,dy.$$

From the given properties of $K(x, y)$ we can assert the following.

(i) No matter what the form of $f(x)$, $\mathscr{M}f(x) = 0$ if $|x| > a$.

(ii) All functions for which both $\int_{-a}^{a} f(y)\,dy = 0$ and $f(x) = 0$ for $|x| > a$ are eigenfunctions corresponding to eigenvalue 0.

(iii) For any function $f(x)$, the integral $\int_{-a}^{a} f(y)\,dy$ is equal to a constant whose value is independent of x; thus $f(x)$ can only be an eigenfunction if it is equal to a constant, μ, for $-a \le x \le a$ and is zero otherwise. For this case $\int_{-a}^{a} f(y)\,dy = 2a\mu$ and the eigenvalue is $2a$.

Point (iii) gives the only possible non-zero eigenvalue, whilst point (ii) shows that eigenfunctions corresponding to zero eigenvalues do exist.

Denote by $S(x, a)$ the function that has unit value for $|x| \le a$ and zero value otherwise; $K(x, y)$ could be expressed as $K(x, y) = S(x, a)S(y, a)$. Substitute the trial solution $f(x) = g(x) + kS(x, a)$ into

$$f(x) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y) f(y)\,dy.$$

This gives

$$g(x) + kS(x, a) = g(x) + \lambda \int_{-\infty}^{\infty} K(x, y)[g(y) + kS(y, a)]\,dy,$$
$$kS(x, a) = \lambda S(x, a) \int_{-a}^{a} g(y)\,dy + \lambda k\, 2aS(x, a).$$

Here, having replaced $K(x, y)$ by $S(x, a)S(y, a)$, we use the factor $S(y, a)$ to reduce the limits of the y-integration from $\pm\infty$ to $\pm a$. As this result is to hold for all x we must have

$$k = \frac{\lambda G}{1 - 2a\lambda}, \quad \text{where } G = \int_{-a}^{a} g(y)\,dy.$$

The general solution is thus

$$f(x) = \begin{cases} g(x) + \dfrac{\lambda G}{1 - 2a\lambda} & \text{for } |x| \le a, \\ g(x) & \text{for } |x| > a. \end{cases}$$

23.15 *Use Fredholm theory to show that, for the kernel*

$$K(x,z) = (x+z)\exp(x-z)$$

over the interval $[0,1]$, *the resolvent kernel is*

$$R(x,z;\lambda) = \frac{\exp(x-z)[(x+z) - \lambda(\frac{1}{2}x + \frac{1}{2}z - xz - \frac{1}{3})]}{1 - \lambda - \frac{1}{12}\lambda^2},$$

and hence solve

$$y(x) = x^2 + 2\int_0^1 (x+z)\exp(x-z)\, y(z)\, dz,$$

expressing your answer in terms of I_n, *where* $I_n = \int_0^1 u^n \exp(-u)\, du$.

We calculate successive values of d_n and $D_n(x,z)$ using the Fredholm recurrence relations:

$$d_n = \int_a^b D_{n-1}(x,x)\, dx,$$

$$D_n(x,z) = K(x,z)d_n - n\int_a^b K(x,z_1)D_{n-1}(z_1,z)\, dz_1,$$

starting from $d_0 = 1$ and $D_0(x,z) = (x+z)e^{x-z}$. In the first iteration we obtain

$$d_1 = \int_0^1 (u+u)e^{u-u}\, du = 1,$$

$$\begin{aligned} D_1(x,z) &= (x+z)e^{x-z}(1) - 1\int_0^1 (x+u)e^{x-u}(u+z)e^{u-z}\, du \\ &= (x+z)e^{x-z} - e^{x-z}\int_0^1 [\,xz + (x+z)u + u^2\,]\, du \\ &= e^{x-z}[\,\tfrac{1}{2}(x+z) - xz - \tfrac{1}{3}\,]. \end{aligned}$$

Performing the second iteration gives

$$d_2 = \int_0^1 e^{u-u}(u - u^2 - \tfrac{1}{3})\, du = \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{3} = -\tfrac{1}{6},$$

$$\begin{aligned} D_2(x,z) &= (x+z)e^{x-z}(-\tfrac{1}{6}) \\ &\quad - 2\int_0^1 (x+u)e^{x-u}e^{u-z}\left[\,\tfrac{1}{2}(u+z) - uz - \tfrac{1}{3}\,\right] du \\ &= e^{x-z}\left\{-\tfrac{1}{6}(x+z) - 2\left[\,x\left(\frac{1}{4} + \frac{z}{2} - \frac{z}{2} - \frac{1}{3}\right) + \left(\frac{1}{6} + \frac{z}{4} - \frac{z}{3} - \frac{1}{6}\right)\right]\right\} \\ &= e^{x-z}\left\{-\tfrac{1}{6}(x+z) - 2\left[-\frac{x}{12} - \frac{z}{12}\right]\right\} = 0. \end{aligned}$$

Since $D_2(x,z) = 0$, $d_3 = 0$, $D_3(x,z) = 0$, etc. Consequently both $D(x,z;\lambda)$ and $d(\lambda)$ are finite, rather than infinite, series:

$$D(x,z;\lambda) = (x+z)e^{x-z} - \lambda\,[\,\tfrac{1}{2}(x+z) - xz - \tfrac{1}{3}\,]\,e^{x-z},$$
$$d(\lambda) = 1 - \lambda + \left(-\tfrac{1}{6}\right)\frac{\lambda^2}{2!} = 1 - \lambda - \tfrac{1}{12}\,\lambda^2.$$

The resolvent kernel $R(x,z;\lambda)$, given by the ratio $D(x,z;\lambda)/d(\lambda)$, is therefore as stated in the question.

For the particular integral equation, $\lambda = 2$ and $f(x) = x^2$. It follows that

$$d(\lambda) = 1 - 2 - \tfrac{4}{12} = -\tfrac{4}{3} \quad \text{and} \quad D(x,z:\lambda) = (2xz + \tfrac{2}{3})e^{x-z}.$$

The solution is therefore given by

$$\begin{aligned} y(x) &= f(x) + \lambda \int_0^1 R(x,z;\lambda) f(z)\,dz \\ &= x^2 + 2\int_0^1 \frac{(2xz + \frac{2}{3})z^2 e^{x-z}}{-\frac{4}{3}}\,dz \\ &= x^2 - \int_0^1 (3xz^3 + z^2)e^{x-z}\,dz \\ &= x^2 - (3xI_3 + I_2)e^x. \end{aligned}$$

24

Complex variables

24.1 *Find an analytic function of $z = x + iy$ whose imaginary part is*

$$(y\cos y + x\sin y)\exp x.$$

If the required function is $f(z) = u + iv$, with $v = (y\cos y + x\sin y)\exp x$, then, from the Cauchy–Riemann equations,

$$\frac{\partial v}{\partial x} = e^x(y\cos y + x\sin y + \sin y) = -\frac{\partial u}{\partial y}.$$

Integrating with respect to y gives

$$\begin{aligned} u &= -e^x \int (y\cos y + x\sin y + \sin y)\,dy + f(x) \\ &= -e^x\left(y\sin y - \int \sin y\,dy - x\cos y - \cos y\right) + f(x) \\ &= -e^x(y\sin y + \cos y - x\cos y - \cos y) + f(x) \\ &= e^x(x\cos y - y\sin y) + f(x). \end{aligned}$$

We determine $f(x)$ by applying the second Cauchy–Riemann equation, which equates $\partial u/\partial x$ with $\partial v/\partial y$:

$$\begin{aligned} \frac{\partial u}{\partial x} &= e^x(x\cos y - y\sin y + \cos y) + f'(x), \\ \frac{\partial v}{\partial y} &= e^x(\cos y - y\sin y + x\cos y). \end{aligned}$$

By comparison, $\qquad f'(x) = 0 \quad \Rightarrow \quad f(x) = k,$

where k is a real constant that can be taken as zero. Hence, the analytic function is given by

$$\begin{aligned} f(z) = u + iv &= e^x(x\cos y - y\sin y + iy\cos y + ix\sin y) \\ &= e^x[(\cos y + i\sin y)(x+iy)] \\ &= e^x e^{iy}(x+iy) \\ &= ze^z. \end{aligned}$$

The final line confirms explicitly that this is a function of z alone (as opposed to a function of both z and z^*).

24.3 *Find the radii of convergence of the following Taylor series:*

$$\text{(a)} \sum_{n=2}^{\infty} \frac{z^n}{\ln n}, \qquad \text{(b)} \sum_{n=1}^{\infty} \frac{n!z^n}{n^n},$$

$$\text{(c)} \sum_{n=1}^{\infty} z^n n^{\ln n}, \quad \text{(d)} \sum_{n=1}^{\infty} \left(\frac{n+p}{n}\right)^{n^2} z^n, \text{ with } p \text{ real.}$$

In each case we consider the series as $\sum_n a_n z^n$ and apply the formula

$$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{1/n}$$

derived from considering the Cauchy root test for absolute convergence.

(a) $$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{1}{\ln n}\right)^{1/n} = 1, \text{ since } -n^{-1}\ln\ln n \to 0 \text{ as } n\to\infty.$$

Thus $R = 1$. For interest, we also note that at the point $z = 1$ the series is

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} > \sum_{n=2}^{\infty} \frac{1}{n},$$

which diverges. This shows that the given series diverges at this point on its circle of convergence.

(b) $$\frac{1}{R} = \lim_{n\to\infty}\left(\frac{n!}{n^n}\right)^{1/n}.$$

Since the nth root of $n!$ tends to n as $n\to\infty$, the limit of this ratio is that of n/n, namely unity. Thus $R = 1$ and the series converges inside the unit circle.

(c) $$\begin{aligned} \frac{1}{R} &= \lim_{n\to\infty}\left(n^{\ln n}\right)^{1/n} = \lim_{n\to\infty} n^{(\ln n)/n} \\ &= \lim_{n\to\infty} \exp\left[\frac{\ln n}{n}\ln n\right] = \exp(0) = 1. \end{aligned}$$

Thus $R = 1$ and the series converges inside the unit circle. It is obvious that the series diverges at the point $z = 1$.

(d)
$$\frac{1}{R} = \lim_{n\to\infty}\left[\left(\frac{n+p}{n}\right)^{n^2}\right]^{1/n} = \lim_{n\to\infty}\left(\frac{n+p}{n}\right)^{n}$$
$$= \lim_{n\to\infty}\left(1+\frac{p}{n}\right)^{n} = e^{p}.$$

Thus $R = e^{-p}$ and the series converges inside a circle of this radius centred on the origin $z = 0$.

24.5 *Determine the types of singularities (if any) possessed by the following functions at $z = 0$ and $z = \infty$:*

(a) $(z-2)^{-1}$, (b) $(1+z^3)/z^2$, (c) $\sinh(1/z)$,
(d) e^z/z^3, (e) $z^{1/2}/(1+z^2)^{1/2}$.

(a) Although $(z-2)^{-1}$ has a simple pole at $z = 2$, at both $z = 0$ and $z = \infty$ it is well behaved and analytic.

(b) Near $z = 0$, $f(z) = (1+z^3)/z^2$ behaves like $1/z^2$ and so has a double pole there. It is clear that as $z \to \infty$ $f(z)$ behaves as z and so has a simple pole there; this can be made more formal by setting $z = 1/\xi$ to obtain $g(\xi) = \xi^2 + \xi^{-1}$ and considering $\xi \to 0$. This leads to the same conclusion.

(c) As $z \to \infty$, $f(z) = \sinh(1/z)$ behaves like $\sinh\xi$ as $\xi \to 0$, i.e. analytically. However, the definition of the sinh function involves an infinite series — in this case an infinite series of inverse powers of z. Thus, no finite n for which

$$\lim_{z\to 0}[z^n f(z)] \text{ is finite}$$

can be found, and $f(z)$ has an essential singularity at $z = 0$.

(d) Near $z = 0$, $f(z) = e^z/z^3$ behaves as $1/z^3$ and has a pole of order 3 at the origin. At $z = \infty$ it has an obvious essential singularity; formally, the series expansion of $e^{1/\xi}$ about $\xi = 0$ contains arbitrarily high inverse powers of ξ.

(e) Near $z = 0$, $f(z) = z^{1/2}/(1+z^2)^{1/2}$ behaves as $z^{1/2}$ and therefore has a branch point there. To investigate its behaviour as $z \to \infty$, we set $z = 1/\xi$ and obtain

$$f(z) = g(\xi) = \left(\frac{\xi^{-1}}{1+\xi^{-2}}\right)^{1/2} = \left(\frac{\xi}{\xi^2+1}\right)^{1/2} \sim \xi^{1/2} \text{ as } \xi \to 0.$$

Hence $f(z)$ also has a branch point at $z = \infty$.

24.7 *Find the real and imaginary parts of the functions* (i) z^2, (ii) e^z, *and* (iii) $\cosh \pi z$. *By considering the values taken by these parts on the boundaries of the region* $x \geq 0$, $y \leq 1$, *determine the solution of Laplace's equation in that region that satisfies the boundary conditions*

$$\phi(x,0) = 0, \qquad \phi(0,y) = 0,$$
$$\phi(x,1) = x, \qquad \phi(1,y) = y + \sin \pi y.$$

Writing $f_k(z) = u_k(x,y) + iv_k(x,y)$, we have

$$\begin{aligned}
&\text{(i)} \quad f_1(z) &&= z^2 = (x+iy)^2 \\
&&&\Rightarrow u_1 = x^2 - y^2 \text{ and } v_1 = 2xy, \\
&\text{(ii)} \quad f_2(z) &&= e^z = e^{x+iy} = e^x(\cos y + i \sin y) \\
&&&\Rightarrow u_2 = e^x \cos y \text{ and } v_2 = e^x \sin y, \\
&\text{(iii)} \quad f_3(z) &&= \cosh \pi z = \cosh \pi x \cos \pi y + i \sinh \pi x \sin \pi y \\
&&&\Rightarrow u_3 = \cosh \pi x \cos \pi y \text{ and } v_3 = \sinh \pi x \sin \pi y.
\end{aligned}$$

All of these u and v are necessarily solutions of Laplace's equation (this follows from the Cauchy–Riemann equations), and, since Laplace's equation is linear, we can form any linear combination of them and it will also be a solution. We need to choose the combination that matches the given boundary conditions.

Since the third and fourth conditions involve x and $\sin \pi y$, and these appear only in v_1 and v_3, respectively, let us try a linear combination of them:

$$\phi(x,y) = A(2xy) + B(\sinh \pi x \sin \pi y).$$

The requirement $\phi(x,0) = 0$ is clearly satisfied, as is $\phi(0,y) = 0$. The condition $\phi(x,1) = x$ becomes $2Ax + 0 = x$, requiring $A = \frac{1}{2}$, and the ,remaining condition, $\phi(1,y) = y + \sin \pi y$, takes the form $y + B \sinh \pi \sin \pi y = y + \sin \pi y$, thus determining B as $1/\sinh \pi$.

With ϕ a solution of Laplace's equation and all of the boundary conditions satisfied, the uniqueness theorem guarantees that

$$\phi(x,y) = xy + \frac{\sinh \pi x \sin \pi y}{\sinh \pi}$$

is the correct solution.

24.9 *The* fundamental theorem of algebra *states that, for a complex polynomial $p_n(z)$ of degree n, the equation $p_n(z) = 0$ has precisely n complex roots. By applying Liouville's theorem, which reads*

If $f(z)$ is analytic and bounded for all z then f is a constant,

to $f(z) = 1/p_n(z)$, prove that $p_n(z) = 0$ has at least one complex root. Factor out that root to obtain $p_{n-1}(z)$ and, by repeating the process, prove the fundamental theorem.

We prove this result by the method of contradiction. Suppose $p_n(z) = 0$ has no roots in the complex plane, then $f_n(z) = 1/p_n(z)$ is bounded for all z and, by Liouville's theorem, is therefore a constant. It follows that $p_n(z)$ is also a constant and that $n = 0$. However, if $n > 0$ we have a contradiction and it was wrong to suppose that $p_n(z) = 0$ has no roots; it must have at least one. Let one of them be $z = z_1$; i.e. $p_n(z)$, being a polynomial, can be written $p_n(z) = (z - z_1)p_{n-1}(z)$.

Now, by considering $f_{n-1}(z) = 1/p_{n-1}(z)$ in just the same way, we can conclude that either $n - 1 = 0$ or a further reduction is possible. It is clear that n such reductions are needed to make f_0 a constant, thus establishing that $p_n(z) = 0$ has precisely n (complex) roots.

Many of the remaining exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar contours that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we show in figure 24.1 a number of typical contour types to which reference can be made.

24.11 *The function*

$$f(z) = (1 - z^2)^{1/2}$$

of the complex variable z is defined to be real and positive on the real axis for $-1 < x < 1$. Using cuts running along the real axis for $1 < x < +\infty$ and $-\infty < x < -1$, show how $f(z)$ is made single-valued and evaluate it on the upper and lower sides of both cuts.

Use these results and a suitable contour in the complex z-plane to evaluate the integral

$$I = \int_1^\infty \frac{dx}{x(x^2 - 1)^{1/2}}.$$

Confirm your answer by making the substitution $x = \sec\theta$.

Figure 24.1 Typical contours for use in contour integration.

As usual when dealing with branch cuts aimed at making a multi-valued function into a single-valued one, we introduce polar coordinates centred on the branch points. For $f(z)$ the branch points are at $z = \pm 1$, and so we define r_1 as the distance of z from the point 1 and θ_1 as the angle the line joining 1 to z makes with the part of the x-axis for which $1 < x < +\infty$, with $0 \le \theta_1 \le 2\pi$. Similarly, r_2 and θ_2 are centred on the point -1, but θ_2 lies in the range $-\pi \le \theta_2 \le \pi$.

With these definitions,

$$\begin{aligned} f(z) = (1-z^2)^{1/2} &= (1-z)^{1/2}(1+z)^{1/2} \\ &= \left[(-r_1 e^{i\theta_1})(r_2 e^{i\theta_2})\right]^{1/2} \\ &= (r_1 r_2)^{1/2} e^{i(\theta_1+\theta_2-\pi)/2}. \end{aligned}$$

In the final line the choice between $\exp(+i\pi)$ and $\exp(-i\pi)$ for dealing with the

minus sign appearing before r_1 in the second line was resolved by the requirement that $f(z)$ is real and positive when $-1 < x < 1$ with $y = 0$. For these values of z, $r_1 = 1 - x$, $r_2 = 1 + x$, $\theta_1 = \pi$ and $\theta_2 = 0$. Thus,

$$f(z) = [(1-x)(1+x)]^{1/2} e^{(\pi+0-\pi)/2} = (1-x^2)^{1/2} e^{i0} = +(1-x^2)^{1/2},$$

as required.

Now applying the same prescription to points lying just above and just below each of the cuts, we have

$$\begin{aligned}
x > 1, y = 0_+ \qquad & r_1 = x - 1 \quad r_2 = x + 1 \quad \theta_1 = 0 \quad \theta_2 = 0 \\
\Rightarrow \quad & f(z) = (x^2-1)^{1/2} e^{i(0+0-\pi)/2} = -i(x^2-1)^{1/2}, \\
x > 1, y = 0_- \qquad & r_1 = x - 1 \quad r_2 = x + 1 \quad \theta_1 = 2\pi \quad \theta_2 = 0 \\
\Rightarrow \quad & f(z) = (x^2-1)^{1/2} e^{i(2\pi+0-\pi)/2} = i(x^2-1)^{1/2}, \\
x < -1, y = 0_+ \qquad & r_1 = 1 - x \quad r_2 = -x - 1 \quad \theta_1 = \pi \quad \theta_2 = \pi \\
\Rightarrow \quad & f(z) = (x^2-1)^{1/2} e^{i(\pi+\pi-\pi)/2} = i(x^2-1)^{1/2}, \\
x < -1, y = 0_- \qquad & r_1 = 1 - x \quad r_2 = -x - 1 \quad \theta_1 = \pi \quad \theta_2 = -\pi \\
\Rightarrow \quad & f(z) = (x^2-1)^{1/2} e^{i(\pi-\pi-\pi)/2} = -i(x^2-1)^{1/2}.
\end{aligned}$$

To use these results to evaluate the given integral I, consider the contour integral

$$J = \int_C \frac{dz}{z(1-z^2)^{1/2}} = \int_c \frac{dz}{zf(z)}.$$

Here C is a large circle (consisting of arcs Γ_1 and Γ_2 in the upper and lower half-planes, respectively) of radius R centred on the origin but indented along the positive and negative x-axes by the cuts considered earlier. At the ends of the cuts are two small circles γ_1 and γ_2 that enclose the branch points $z = 1$ and $z = -1$, respectively. Thus the complete closed contour, starting from γ_1 and moving along the positive real axis, consists of, in order, circle γ_1, cut C_1, arc Γ_1, cut C_2, circle γ_2, cut C_3, arc Γ_2 and cut C_4, leading back to γ_1.

On the arcs Γ_1 and Γ_2 the integrand is $O(R^{-2})$ and the contributions to the contour integral $\to 0$ as $R \to \infty$. For the small circle γ_1, where we can set $z = 1 + \rho e^{i\phi}$ with $dz = i\rho e^{i\phi}\, d\phi$, we have

$$\int_{\gamma_1} \frac{dz}{z(1+z)^{1/2}(1-z)^{1/2}} = \int_0^{2\pi} \frac{i\rho e^{i\phi}}{(1+\rho e^{i\phi})(2+\rho e^{i\phi})^{1/2}(-\rho e^{i\phi})^{1/2}}\, d\phi,$$

and this $\to 0$ as $\rho \to 0$. Similarly, the small circle γ_2 contributes nothing to the contour integral. This leaves only the contributions from the four arms of the

branch cuts. To relate these to I we use our previous results about the value of $f(z)$ on the various arms:

$$\text{on } C_1,\ z = x \text{ and } \int_{C_1} = \int_1^\infty \frac{dx}{x[-i(x^2-1)^{1/2}]} = iI\,;$$

$$\text{on } C_2,\ z = -x \text{ and } \int_{C_2} = \int_\infty^1 \frac{-dx}{-x[\,i(x^2-1)^{1/2}]} = iI\,;$$

$$\text{on } C_3,\ z = -x \text{ and } \int_{C_3} = \int_1^\infty \frac{-dx}{-x[-i(x^2-1)^{1/2}]} = iI\,;$$

$$\text{on } C_4,\ z = x \text{ and } \int_{C_1} = \int_\infty^1 \frac{dx}{x[\,i(x^2-1)^{1/2}]} = iI.$$

So the full contour integral around C has the value $4iI$. But, this must be the same as $2\pi i$ times the residue of $z^{-1}(1-z^2)^{-1/2}$ at $z = 0$, which is the only pole of the integrand inside the contour. The residue is clearly unity, and so we deduce that $I = \pi/2$.

This particular integral can be evaluated much more simply using elementary methods. Setting $x = \sec\theta$ with $dx = \sec\theta\tan\theta\,d\theta$ gives

$$\begin{aligned} I &= \int_1^\infty \frac{dx}{x(x^2-1)^{1/2}} \\ &= \int_0^{\pi/2} \frac{\sec\theta\tan\theta\,d\theta}{\sec\theta\,(\sec^2\theta-1)^{1/2}} = \int_0^{\pi/2} d\theta = \frac{\pi}{2}, \end{aligned}$$

and so verifies the result obtained by contour integration.

24.13 *Prove that if $f(z)$ has a simple zero at z_0 then $1/f(z)$ has residue $1/f'(z_0)$ there. Hence evaluate*

$$\int_{-\pi}^{\pi} \frac{\sin\theta}{a-\sin\theta}\,d\theta,$$

where a is real and > 1.

If $f(z)$ is analytic and has a simple zero at $z = z_0$ then it can be written as

$$f(z) = \sum_{n=1}^{\infty} a_n(z-z_0)^n, \qquad \text{with } a_1 \neq 0.$$

Using a binomial expansion,

$$\frac{1}{f(z)} = \frac{1}{a_1(z-z_0)\left(1+\sum_{n=2}^{\infty}\frac{a_n}{a_1}(z-z_0)^{n-1}\right)}$$
$$= \frac{1}{a_1(z-z_0)}(1+b_1(z-z_0)+b_2(z-z_0)^2+\cdots),$$

for some coefficients, b_i. The residue at $z = z_0$ is clearly a_1^{-1}.

But, from differentiating the Taylor expansion,

$$f'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1},$$
$$\Rightarrow \quad f'(z_0) = a_1 + 0 + 0 + \cdots = a_1,$$

i.e. the residue $= \dfrac{1}{a_1}$ can also be expressed as $\dfrac{1}{f'(z_0)}$.

Denote the required integral by I and consider the contour integral

$$J = \int_C \frac{dz}{a - \frac{1}{2i}(z - z^{-1})} = \int_C \frac{2iz\,dz}{2aiz - z^2 + 1},$$

where C is the unit circle, i.e. contour (c) of figure 24.1 with $R = 1$. The denominator has simple zeros at $z = ai \pm \sqrt{-a^2+1} = i(a \pm \sqrt{a^2-1})$. Since a is strictly greater than 1, $\alpha = i(a - \sqrt{a^2-1})$ lies strictly inside the unit circle, whilst $\beta = i(a + \sqrt{a^2-1})$ lies strictly outside it (and need not be considered further).

Extending the previous result to the case of $h(z) = g(z)/f(z)$, where $g(z)$ is analytic at z_0, the residue of $h(z)$ at $z = z_0$ can be seen to be $g(z_0)/f'(z_0)$. Applying this, we find that the residue of the integrand at $z = \alpha$ is given by

$$\left|\frac{2iz}{2ai - 2z}\right|_{z=\alpha} = \frac{i\alpha}{ai - ai + i\sqrt{a^2-1}} = \frac{\alpha}{\sqrt{a^2-1}}.$$

Now on the unit circle, $z = e^{i\theta}$ with $dz = i\,e^{i\theta}\,d\theta$, and J can be written as

$$J = \int_{-\pi}^{\pi} \frac{i\,e^{i\theta}\,d\theta}{a - \frac{1}{2i}(e^{i\theta} - e^{-i\theta})} = \int_{-\pi}^{\pi} \frac{i(\cos\theta + i\sin\theta)\,d\theta}{a - \sin\theta}.$$

Hence,

$$I = -\mathrm{Re}\,J = -\mathrm{Re}\;2\pi i\frac{i(a-\sqrt{a^2-1})}{\sqrt{a^2-1}}$$
$$= 2\pi\left(\frac{a}{\sqrt{a^2-1}} - 1\right).$$

Although it is not asked for, we can also deduce from the fact that the residue at $z = \alpha$ is purely imaginary that

$$\int_{-\pi}^{\pi} \frac{\cos\theta}{a - \sin\theta}\, d\theta = 0,$$

a result that can also be obtained by more elementary means, when it is noted that the numerator of the integrand is the derivative of the denominator.

24.15 *Prove that*

$$\int_0^{\infty} \frac{\cos mx}{4x^4 + 5x^2 + 1}\, dx = \frac{\pi}{6}\left(4e^{-m/2} - e^{-m}\right) \qquad \textit{for } m > 0.$$

Since, when z is on the real axis, the integrand is equal to

$$\operatorname{Re} \frac{e^{imz}}{(z^2+1)(4z^2+1)} = \operatorname{Re} \frac{e^{imz}}{(z+i)(z-i)(2z+i)(2z-i)},$$

we consider the integral of $f(z) = \dfrac{e^{imz}}{(z+i)(z-i)(2z+i)(2z-i)}$ around contour (d) in figure 24.1.

As $|f(z)| \sim |z|^{-4}$ as $z \to \infty$ and $m > 0$, all the conditions for Jordan's lemma to hold are satisfied and the integral around the large semicircle contributes nothing. For this integrand there are two poles inside the contour, at $z = i$ and at $z = \frac{1}{2}i$. The respective residues are

$$\frac{e^{-m}}{2i\,3i\,i} = \frac{ie^{-m}}{6} \quad \text{and} \quad \frac{e^{-m/2}}{\frac{3i}{2}\left(-\frac{i}{2}\right)2i} = \frac{-2ie^{-m/2}}{3}.$$

The residue theorem therefore reads

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{4x^4 + 5x^2 + 1}\, dx + 0 = 2\pi i\left(\frac{ie^{-m}}{6} - \frac{2ie^{-m/2}}{3}\right),$$

and the stated result follows from equating real parts and changing the lower integration limit, recognising that the integrand is symmetric about $x = 0$ and so the integral from 0 to ∞ is equal to half of that from $-\infty$ to ∞.

Figure 24.2 The contour used in exercise 24.17.

24.17 *The following is an alternative (and roundabout!) way of evaluating the Gaussian integral.*

(a) *Prove that the integral of* $[\exp(i\pi z^2)]\operatorname{cosec}\pi z$ *around the parallelogram with corners* $\pm 1/2 \pm R\exp(i\pi/4)$ *has the value* $2i$.

(b) *Show that the parts of the contour parallel to the real axis give no contribution when* $R \to \infty$.

(c) *Evaluate the integrals along the other two sides by putting* $z' = r\exp(i\pi/4)$ *and working in terms of* $z' + \frac{1}{2}$ *and* $z' - \frac{1}{2}$. *Hence by letting* $R \to \infty$ *show that*

$$\int_{-\infty}^{\infty} e^{-\pi r^2}\, dr = 1.$$

The integral is

$$\int_C e^{i\pi z^2} \operatorname{cosec} \pi z\, dz = \int_C \frac{e^{i\pi z^2}}{\sin \pi z}\, dz$$

and the suggested contour C is shown in figure 24.2.

(a) The integrand has (simple) poles only on the real axis at $z = n$, where n is an integer. The only such pole enclosed by C is at $z = 0$. The residue there is

$$a_{-1} = \lim_{z\to 0} \frac{z e^{i\pi z^2}}{\sin \pi z} = \frac{1}{\pi}.$$

The value of the integral around C is therefore $2\pi i \times (\pi^{-1}) = 2i$.

(b) On the parts of C parallel to the real axis, $z = \pm Re^{i\pi/4} + x'$, where $-\frac{1}{2} \le x' \le \frac{1}{2}$. The integrand is thus given by

$$\begin{aligned} f(z) &= \frac{1}{\sin \pi z} \exp\left[i\pi(\pm Re^{i\pi/4} + x')^2 \right] \\ &= \frac{1}{\sin \pi z} \exp\left[i\pi \left(R^2 e^{i\pi/2} \pm 2Rx' e^{i\pi/4} + x'^2 \right) \right] \\ &= \frac{1}{\sin \pi z} \exp\left[-\pi R^2 \pm \frac{2\pi i R x'}{\sqrt{2}}(1+i) + i\pi x'^2 \right] \\ &= \mathrm{O}\left(\exp[-\pi R^2 \mp \sqrt{2}\pi R x'] \right) \\ &\to 0 \text{ as } R \to \infty. \end{aligned}$$

Since the integration range is finite ($-\frac{1}{2} \le x' \le \frac{1}{2}$), the integrals $\to 0$ as $R \to \infty$.

(c) On the first of the other two sides, let us set $z = \frac{1}{2} + re^{i\pi/4}$ with $-R \le r \le R$. The corresponding integral I_1 is

$$\begin{aligned} I_1 &= \int_{L_1} e^{i\pi z^2} \operatorname{cosec} \pi z \, dz \\ &= \int_{-R}^{R} \frac{\exp\left[i\pi(\frac{1}{2} + re^{i\pi/4})^2 \right]}{\sin\left[\pi(\frac{1}{2} + re^{i\pi/4}) \right]} e^{i\pi/4} \, dr \\ &= \int_{-R}^{R} \frac{e^{i\pi/4} \exp(i\pi r e^{i\pi/4}) \exp(i\pi r^2 i) e^{i\pi/4}}{\cos(\pi r e^{i\pi/4})} \, dr \\ &= \int_{-R}^{R} \frac{i \exp(i\pi r e^{i\pi/4}) e^{-\pi r^2}}{\cos(\pi r e^{i\pi/4})} \, dr. \end{aligned}$$

Similarly (remembering the sense of integration), the remaining side contributes

$$I_2 = -\int_{-R}^{R} \frac{i \exp(-i\pi r e^{i\pi/4}) e^{-\pi r^2}}{-\cos(\pi r e^{i\pi/4})} \, dr.$$

Adding together all four contributions gives

$$0 + 0 + \int_{-R}^{R} \frac{i[\exp(i\pi r e^{i\pi/4}) + \exp(-i\pi r e^{i\pi/4})] e^{-\pi r^2}}{\cos(\pi r e^{i\pi/4})} \, dr,$$

which simplifies to

$$\int_{-R}^{R} 2ie^{-\pi r^2} \, dr.$$

From part (a), this must be equal to $2i$ as $R \to \infty$, and so $\displaystyle\int_{-\infty}^{\infty} e^{-\pi r^2} \, dr = 1$.

24.19 *Using a suitable cut plane, prove that if α is real and $0 < \alpha < 1$ then*

$$\int_0^\infty \frac{x^{-\alpha}}{1+x}\,dx$$

has the value $\pi \operatorname{cosec} \pi\alpha$.

As α is not an integer, the complex form of the integrand $f(z) = \dfrac{z^{-\alpha}}{1+z}$ is not single-valued. We therefore need to perform the contour integration in a cut plane; contour (f) of figure 24.1 is a suitable contour. We will be making use of the fact that, because the integrand takes different values on γ_1 and γ_2, the contributions coming from these two parts of the complete contour, although related, do *not* cancel.

The contributions from γ and Γ are both zero because:

(i) around γ, $|zf(z)| \sim \dfrac{z\,z^{-\alpha}}{1} = z^{1-\alpha} \to 0$ as $|z| \to 0$, since $\alpha < 1$;

(ii) around Γ, $|zf(z)| \sim \dfrac{z\,z^{-\alpha}}{z} = z^{-\alpha} \to 0$ as $|z| \to \infty$, since $\alpha > 0$.

Therefore, the only contributions come from the cut; on γ_1, $z = xe^{0i}$, whilst on γ_2, $z = xe^{2\pi i}$. The only pole inside the contour is a simple one at $z = -1 = e^{i\pi}$, where the residue is $e^{-i\pi\alpha}$. The residue theorem now reads

$$0 + \int_0^\infty \frac{x^{-\alpha}}{1+x}\,dx + 0 - \int_0^\infty \frac{x^{-\alpha}e^{-2\pi i\alpha}}{1+xe^{2\pi i}}\,dx = 2\pi i\,e^{-i\pi\alpha},$$

$$\Rightarrow \quad (1 - e^{-2\pi i\alpha}) \int_0^\infty \frac{x^{-\alpha}}{1+x}\,dx = 2\pi i\,e^{-i\pi\alpha}.$$

This can be rearranged to read

$$\int_0^\infty \frac{x^{-\alpha}}{1+x}\,dx = \frac{2\pi i\,e^{-i\pi\alpha}}{(1-e^{-2\pi i\alpha})} = \frac{2\pi i}{e^{i\pi\alpha} - e^{-i\pi\alpha}} = \frac{\pi}{\sin \pi\alpha},$$

thus establishing the stated result.

24.21 *By integrating a suitable function around a large semicircle in the upper half plane and a small semicircle centred on the origin, determine the value of*

$$I = \int_0^\infty \frac{(\ln x)^2}{1+x^2}\,dx$$

and deduce, as a by-product of your calculation, that

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0.$$

The suggested contour is that shown in figure 24.1(e), but with only one indentation γ on the real axis (at $z = 0$) and with $R = \infty$. The appropriate complex function is

$$f(z) = \frac{(\ln z)^2}{1 + z^2}.$$

The only pole inside the contour is at $z = i$, and the residue there is given by

$$\frac{(\ln i)^2}{i + i} = \frac{(\ln 1 + i(\pi/2))^2}{2i} = -\frac{\pi^2}{8i}.$$

To evaluate the integral around γ, we set $z = \rho\, e^{i\theta}$ with $\ln z = \ln \rho + i\theta$ and $dz = i\rho\, e^{i\theta}\, d\theta$; the integral becomes

$$\int_{\pi}^{0} \frac{\ln^2 \rho + 2i\theta \ln \rho - \theta^2}{1 + \rho^2 e^{2i\theta}}\, i\rho\, e^{i\theta}\, d\theta, \text{ which} \to 0 \text{ as } \rho \to 0.$$

Thus γ contributes nothing. Even more obviously, on Γ, $|zf(z)| \sim z^{-1}$ and tends to zero as $|z| \to \infty$, showing that Γ also contributes nothing.

On γ_+, $z = xe^{i0}$ and the contribution is equal to I.

On γ_-, $z = xe^{i\pi}$ and the contribution is (remembering that the contour actually runs from $x = \infty$ to $x = 0$) given by

$$\begin{aligned} I_- &= -\int_0^\infty \frac{(\ln x + i\pi)^2}{1 + x^2}\, e^{i\pi}\, dx \\ &= I + 2i\pi \int_0^\infty \frac{\ln x}{1 + x^2}\, dx - \pi^2 \int_0^\infty \frac{1}{1 + x^2}\, dx. \end{aligned}$$

The residue theorem for the complete closed contour thus reads

$$0 + I + 0 + I + 2i\pi \int_0^\infty \frac{\ln x}{1 + x^2}\, dx - \pi^2 \left[\tan^{-1} x \right]_0^\infty = 2\pi i \left(\frac{-\pi^2}{8i} \right).$$

Equating the real parts $\quad \Rightarrow \quad 2I - \frac{1}{2}\pi^3 = -\frac{1}{4}\pi^3 \quad \Rightarrow \quad I = \frac{1}{8}\pi^3.$

Equating the imaginary parts gives the stated by-product.

25

Applications of complex variables

Many of the exercises in this chapter involve contour integration and the choice of a suitable contour. In order to save the space taken by drawing several broadly similar contours that differ only in notation, the positions of poles, the values of lengths or angles, or other minor details, we make reference to figure 24.1 which shows a number of typical contour types.

25.1 *In the method of complex impedances for a.c. circuits, an inductance L is represented by a complex impedance $Z_L = i\omega L$ and a capacitance C by $Z_C = 1/(i\omega C)$. Kirchhoff's circuit laws,*

$$\sum_i I_i = 0 \text{ at a node and } \sum_i Z_i I_i = \sum_j V_j \text{ around any closed loop,}$$

are then applied as if the circuit were a d.c. one.

Apply this method to the a.c. bridge connected as in figure 25.1 to show that if the resistance R is chosen as $R = (L/C)^{1/2}$ then the amplitude of the current I_R through it is independent of the angular frequency ω of the applied a.c. voltage $V_0\, e^{i\omega t}$.

Determine how the phase of I_R, relative to that of the voltage source, varies with the angular frequency ω.

Omitting the common factor $e^{i\omega t}$ from all currents and voltages, let the current drawn from the voltage source be (the complex quantity) I and the current flowing from A to D be I_1. Then the currents in the remaining branches are $AE : I - I_1$, $DB : I_1 - I_R$ and $EB : I - I_1 + I_R$.

Figure 25.1 The inductor–capacitor–resistor network for exercise 25.1.

Applying $\sum_i Z_i I_i = \sum_j V_j$ to three separate loops yields

loop $ADBA$
$$i\omega L I_1 + \frac{1}{i\omega C}(I_1 - I_R) = V_0,$$

loop $ADEA$
$$i\omega L I_1 + R I_R - \frac{1}{i\omega C}(I - I_1) = 0,$$

loop $DBED$
$$\frac{1}{i\omega C}(I_1 - I_R) - i\omega L(I - I_1 + I_R) - R I_R = 0.$$

Now, denoting $(LC)^{-1}$ by ω_0^2 and choosing R as $(L/C)^{1/2} = (\omega_0 C)^{-1}$, we can write these equations as follows:

$$\left(1 - \frac{\omega^2}{\omega_0^2}\right) I_1 - I_R = i\omega C V_0,$$

$$-I + \left(1 - \frac{\omega^2}{\omega_0^2}\right) I_1 + i\frac{\omega}{\omega_0} I_R = 0,$$

$$\frac{\omega^2}{\omega_0^2} I + \left(1 - \frac{\omega^2}{\omega_0^2}\right) I_1 + \left(-1 + \frac{\omega^2}{\omega_0^2} - i\frac{\omega}{\omega_0}\right) I_R = 0.$$

Eliminating I from the last two of these yields

$$\left(1 + \frac{\omega^2}{\omega_0^2}\right)\left(1 - \frac{\omega^2}{\omega_0^2}\right) I_1 - \left(\frac{i\omega}{\omega_0} + 1\right)\left(1 - \frac{\omega^2}{\omega_0^2}\right) I_R = 0.$$

Thus,

$$I_R = \frac{1 + \dfrac{\omega^2}{\omega_0^2}}{1 + i\dfrac{\omega}{\omega_0}} I_1 = \frac{\omega_0^2 + \omega^2}{\omega_0(\omega_0 + i\omega)} \frac{\omega_0^2(i\omega C V_0 + I_R)}{\omega_0^2 - \omega^2}.$$

After some cancellation and rearrangement,

$$(\omega_0^2 - \omega^2) I_R = \omega_0(\omega_0 - i\omega)(i\omega C V_0 + I_R),$$
$$(i\omega\omega_0 - \omega^2) I_R = \omega_0\omega(i\omega_0 + \omega) C V_0,$$

and so

$$I_R = \omega_0 C V_0 \frac{i\omega_0 + \omega}{i\omega_0 - \omega} = \omega_0 C V_0 \frac{(i\omega_0 + \omega)(-i\omega_0 - \omega)}{(i\omega_0 - \omega)(-i\omega_0 - \omega)}$$
$$= \omega_0 C V_0 \frac{\omega_0^2 - \omega^2 - 2i\omega\omega_0}{\omega_0^2 + \omega^2}.$$

From this we can read off

$$|I_R| = \omega_0 C V_0 \frac{\left[(\omega^2 - \omega_0^2)^2 + 4\omega^2\omega_0^2\right]^{1/2}}{\omega_0^2 + \omega^2} = \omega_0 C V_0, \text{ i.e. independent of } \omega,$$

and

$$\phi = \text{phase of } I_R = \tan^{-1} \frac{-2\omega\omega_0}{\omega_0^2 - \omega^2}.$$

Thus I_R (which was arbitrarily and notionally defined as flowing from D to E in the equivalent d.c. circuit) has an imaginary part that is always negative but a real part that changes sign as ω passes through ω_0. Its phase ϕ, relative to that of the voltage source, therefore varies from 0 when ω is small to $-\pi$ when ω is large.

25.3 *For the function*

$$f(z) = \ln\left(\frac{z+c}{z-c}\right),$$

where c is real, show that the real part u of f is constant on a circle of radius $c\,\mathrm{cosech}\, u$ centred on the point $z = c\coth u$. Use this result to show that the electrical capacitance per unit length of two parallel cylinders of radii a, placed with their axes $2d$ apart, is proportional to $[\cosh^{-1}(d/a)]^{-1}$.

From

$$f(z) = \ln\left(\frac{z+c}{z-c}\right) = \ln\left|\frac{z+c}{z-c}\right| + i\arg\left(\frac{z+c}{z-c}\right),$$

we have that

$$u = \ln\left|\frac{z+c}{z-c}\right| = \frac{1}{2}\ln\frac{(x+c)^2 + y^2}{(x-c)^2 + y^2} \quad \Rightarrow \quad e^{2u} = \frac{(x+c)^2 + y^2}{(x-c)^2 + y^2}.$$

The curve upon which $u(x, y)$ is constant is therefore given by

$$(x^2 - 2cx + c^2 + y^2)e^{2u} = x^2 + 2xc + c^2 + y^2.$$

This can be rewritten as

$$x^2(e^{2u}-1)-2xc(e^{2u}+1)+y^2(e^{2u}-1)+c^2(e^{2u}-1)=0,$$
$$x^2-2xc\frac{e^{2u}+1}{e^{2u}-1}+y^2+c^2=0,$$
$$x^2-2xc\coth u+y^2+c^2=0,$$

which, in conic-section form, becomes

$$(x-c\coth u)^2+y^2=c^2\coth^2 u-c^2=c^2\mathrm{cosech}^2 u.$$

This is a circle with centre $(c\coth u, 0)$ and radius $|c\,\mathrm{cosech}\,u|$.

Now consider two such circles with the same value of $|c\mathrm{cosech}\ u|$, equal to a, but different values of u satisfying $c\coth u_1=-d$ and $c\coth u_2=+d$. These two equations imply that $u_1=-u_2$, corresponding physically to equal but opposite charges $-Q$ and $+Q$ placed on identical cylindrical conductors that coincide with the circles; the conductors are raised to potentials u_1 and u_2.

We have already established that we need $c\coth u_2=d$ and $c\,\mathrm{cosech}\ u_2=a$. Dividing these two equations gives $\cosh u_2=d/a$.

The capacitance (per unit length) of the arrangement is given by the magnitude of the charge on one conductor divided by the potential difference between the conductors that results from the presence of that charge, i.e.

$$C=\frac{Q}{u_2-u_1}\propto\frac{1}{2u_2}=\frac{1}{2\cosh^{-1}(d/a)},$$

as stated in the question.

25.5 *By considering in turn the transformations*

$$z=\tfrac{1}{2}c(w+w^{-1})\quad\textit{and}\quad w=\exp\zeta,$$

where $z=x+iy$, $w=r\exp i\theta$, $\zeta=\xi+i\eta$ *and* c *is a real positive constant, show that* $z=c\cosh\zeta$ *maps the strip* $\xi\geq 0$, $0\leq\eta\leq 2\pi$, *onto the whole* z-*plane. Which curves in the* z-*plane correspond to the lines* $\xi=$ *constant and* $\eta=$ *constant? Identify those corresponding to* $\xi=0$, $\eta=0$ *and* $\eta=2\pi$.

The electric potential ϕ *of a charged conducting strip* $-c\leq x\leq c$, $y=0$, *satisfies*

$$\phi\sim -k\ln(x^2+y^2)^{1/2}\textit{for large values of }(x^2+y^2)^{1/2},$$

with ϕ *constant on the strip. Show that* $\phi=\mathrm{Re}\ [-k\cosh^{-1}(z/c)]$ *and that the magnitude of the electric field near the strip is* $k(c^2-x^2)^{-1/2}$.

We first note that the combined transformation is given by

$$z = \frac{c}{2}(e^{\zeta} + e^{-\zeta}) = c\cosh\zeta \quad \Rightarrow \quad \zeta = \cosh^{-1}\frac{z}{c}.$$

The successive connections linking the strip in the ζ-plane and its image in the z-plane are

$$\begin{aligned} z &= c\cosh\zeta = c\cosh(\xi + i\eta) \\ &= c\cosh\xi\cos\eta + ic\sinh\xi\sin\eta, \text{ with } \xi > 0,\ 0 \le \eta \le 2\pi, \end{aligned}$$

$$re^{i\theta} = w = e^{\zeta} = e^{\xi}e^{i\eta}, \text{ with the strip as } 1 < r < \infty,\ 0 \le \theta \le 2\pi,$$

$$\begin{aligned} x + iy = z &= \frac{c}{2}(w + w^{-1}) \\ &= \frac{c}{2}[r(\cos\theta + i\sin\theta) + r^{-1}(\cos\theta - i\sin\theta)] \\ &= \frac{c}{2}\left(r + \frac{1}{r}\right)\cos\theta + i\frac{c}{2}\left(r - \frac{1}{r}\right)\sin\theta. \end{aligned}$$

This last expression for z and the previous specification of the strip in terms of r and θ show that both x and y can take all values, i.e. that the original strip in the ζ-plane is mapped onto the whole of the z-plane. From the two expressions for z we also see that $x = c\cosh\xi\cos\eta$ and $y = c\sinh\xi\sin\eta$.

For ξ constant, the contour in the xy-plane, obtained by eliminating η, is

$$\frac{x^2}{c^2\cosh^2\xi} + \frac{y^2}{c^2\sinh^2\xi} = 1, \quad \text{i.e. an ellipse.}$$

The eccentricity of the ellipse is given by

$$e = \left(\frac{c^2\cosh^2\xi - c^2\sinh^2\xi}{c^2\cosh^2\xi}\right)^{1/2} = \frac{1}{\cosh\xi}.$$

The foci of the ellipse are at $\pm e\times$ the major semi-axis, i.e. $\pm 1/\cosh\xi \times c\cosh\xi = \pm c$. This is independent of ξ and so all the ellipses are confocal.

Similarly, for η constant, the contour is

$$\frac{x^2}{c^2\cos^2\eta} - \frac{y^2}{c^2\sin^2\eta} = 1.$$

This is one of a set of confocal hyperbolae.

(i) $\xi = 0 \quad \Rightarrow \quad y = 0,\ x = c\cos\eta.$
This is the finite line (degenerate ellipse) on the x-axis, $-c \le x \le c$.

(ii) $\eta = 0 \quad \Rightarrow \quad y = 0,\ x = c\cosh\xi.$
This is a part of the x-axis not covered in (i), $c < x < \infty$. The other part, $-\infty < x < -c$, corresponds to $\eta = \pi$.

(iii) This is the same as (the first case) in (ii).

Now, in the ζ-plane, consider the real part of the function $F(\zeta) = -k\zeta$, with k real. On $\xi = 0$ [case (i) above] it reduces to Re $\{-ik\eta\}$, which is zero for all η, i.e. a constant. This implies that the real part of the transformed function will be a constant (actually zero) on $-c \le x \le c$ in the z-plane.

Further,

$$\begin{aligned}(x^2 + y^2)^{1/2} &= (c^2 \cosh^2 \xi \cos^2 \eta + c^2 \sinh^2 \xi \sin^2 \eta)^{1/2} \\ &\approx \tfrac{1}{2} c e^{\xi} \text{ for large } \xi, \\ \Rightarrow \quad \xi &\approx \ln(x^2 + y^2)^{1/2} + \text{fixed constant.}\end{aligned}$$

Hence,

$$\text{Re}\,\{-k\zeta\} = -k\xi \approx -k \ln(x^2 + y^2)^{1/2} \text{ for large } (x^2 + y^2)^{1/2}.$$

Thus, the transformation

$$F(\zeta) = -k\zeta \quad \rightarrow \quad f(z) = -k \cosh^{-1} \frac{z}{c}$$

produces a function in the z-plane that satisfies the stated boundary conditions (as well as satisfying Laplace's equation). It is therefore the required solution.

The electric field near the conducting strip, where $y = 0$ and $z^2 = x^2$, can have no component in the x-direction (except at the points $x = \pm c$), but its magnitude is still given by

$$E = |f'(z)| = \left| -\frac{k}{\sqrt{z^2 - c^2}} \right| = \frac{k}{(c^2 - x^2)^{1/2}}.$$

25.7 *Use contour integration to answer the following questions about the complex zeros of a polynomial equation.*

(a) *Prove that* $z^8 + 3z^3 + 7z + 5$ *has two zeros in the first quadrant.*
(b) *Find in which quadrants the zeros of* $2z^3 + 7z^2 + 10z + 6$ *lie. Try to locate them.*

(a) Consider the principle of the argument applied to the integral of $f(z) = z^8 + 3z^3 + 7z + 5$ around contour (b) in figure 24.1.

On OA $f(z)$ is always real and $\Delta_{AB} \arg(f) = 0$.

On AB the argument of f increases by $8 \times \frac{1}{2}\pi = 4\pi$.

On BO $z = iy$ and $f(z) = h(y) = y^8 - 3iy^3 + 7iy + 5$. The argument of $h(y)$ is therefore

$$\tan^{-1} \frac{-3y^3 + 7y}{y^8 + 5}.$$

The appropriate choice at $y = \infty$ for this multi-valued function is 4π, as we have just shown. As y decreases from ∞ the argument initially decreases, but passes through 4π again when $y = \sqrt{7/3}$. After that it remains greater than 4π until returning to that value at $y = 0$. Further, since $y^8 + 5$ has no zeros for real y, $\arg(h)$ can reach neither $\frac{7}{2}\pi$ nor $\frac{9}{2}\pi$. Consequently, we deduce that $\Delta_{BO} \arg(f) = 0$.

In summary, $\Delta \arg(f)$ around the closed contour is 4π, and it follows from the principle of the argument that the first quadrant must contain 2 zeros of $f(z)$.

(b) For $f(z) = 2z^3 + 7z^2 + 10z + 6$ we initially follow the same procedure as in part (a), although, as it is a cubic with all of its coefficients positive, we know that it must have at least one negative real zero.

It is straightforward to conclude that $\Delta_{OA} \arg(f) = 0$ and that around the curve AB the change of argument is $\Delta_{AB} \arg(f) = \frac{3}{2}\pi$. On BO,

$$\arg(f) = \tan^{-1} \frac{10y - 2y^3}{6 - 7y^2}.$$

At $y = \infty$ this is $\frac{3}{2}\pi$ (as we have just established) and, as y decreases towards 0, it passes through π at $y = \sqrt{5}$ and $\frac{1}{2}\pi$ at $y = \sqrt{6/7}$, and finally becomes zero at $y = 0$. Thus the net change around the whole closed contour is zero, and we conclude that there are no zeros in the first quadrant. Since the zeros of polynomials with real coefficients occur in complex conjugate pairs, it follows that the fourth quadrant also contains no zeros. This shows that the complex conjugate zeros of $f(z)$ are located in the second and third quadrants.

We start our search for the negative real zero by tabulating some easy-to-calculate values of $f(x)$, the choice of successive values of x being guided by previous results:

z	0	-1	-2	-1.5
$f(z)$	6	1	-2	0

By chance, we have hit upon an exact zero, $z = -\frac{3}{2}$. It follows that $(2z + 3)$ is a factor of $f(z)$, which can be written

$$f(z) = (2z + 3)(z^2 + 2z + 2).$$

The other two zeros are therefore

$$z = -1 \pm \sqrt{1 - 2} = -1 \pm i.$$

As expected, these are in the second and third quadrants.

25.9 *Prove that*

$$\sum_{-\infty}^{\infty} \frac{1}{n^2 + \frac{3}{4}n + \frac{1}{8}} = 4\pi.$$

Carry out the summation numerically, say between -4 *and* 4*, and note how much of the sum comes from values near the poles of the contour integration.*

In order to evaluate this sum, we must first find a function of z that takes the value of the corresponding term in the sum whenever z is an integer. Clearly this is

$$\frac{1}{z^2 + \frac{3}{4}z + \frac{1}{8}}.$$

Further, too make use of the properties of contour integrals, we need to multiply this function by one that has simple poles at the same points, each with unit residue. An appropriate choice of integrand is therefore

$$f(z) = \frac{\pi \cot \pi z}{z^2 + \frac{3}{4}z + \frac{1}{8}} = \frac{\pi \cot \pi z}{(z + \frac{1}{2})(z + \frac{1}{4})}.$$

The contour to be used must enclose all integer values of z, both positive and negative and, in practical terms, must give zero contribution for $|z| \to \infty$, except possibly on the real axis. A large circle C, centred on the origin (see contour (c) in figure 24.1) suggests itself.

As $|zf(z)| \to 0$ on C, the contour integral has value zero. This implies that the residues at the enclosed poles add up to zero. The residues are

$$\begin{aligned}
&\frac{\pi \cot(-\frac{1}{2}\pi)}{-\frac{1}{2} + \frac{1}{4}} = 0 && \text{at } z = -\tfrac{1}{2},\\
&\frac{\pi \cot(-\frac{1}{4}\pi)}{-\frac{1}{4} + \frac{1}{2}} = -4\pi && \text{at } z = -\tfrac{1}{4},\\
&\sum_{n=-\infty}^{\infty} \frac{1}{(n + \frac{1}{2})(n + \frac{1}{4})} && \text{at } z = n,\ -\infty < n < \infty.
\end{aligned}$$

The quoted result follows immediately.

For the rough numerical summation we tabulate n, $D(n) = n^2 + \frac{3}{4}n + \frac{1}{8}$ and the

nth term of the series, $1/D(n)$:

n	$D(n)$	$1/D(n)$
-4	13.125	0.076
-3	6.875	0.146
-2	2.625	0.381
-1	0.375	2.667
0	0.125	8.000
1	1.875	0.533
2	5.625	0.178
3	11.375	0.088
4	19.125	0.052

The total of these nine terms is 12.121; this is to be compared with the total for the entire infinite series (of positive terms), which is $4\pi = 12.566$. It will be seen that the sum is dominated by the terms for $n = 0$ and $n = -1$. These two values bracket the positions on the real axis of the poles at $z = -\frac{1}{2}$ and $z = -\frac{1}{4}$.

25.11 *By considering the integral of*

$$\left(\frac{\sin \alpha z}{\alpha z}\right)^2 \frac{\pi}{\sin \pi z}, \qquad \alpha < \frac{\pi}{2},$$

around a circle of large radius, prove that

$$\sum_{m=1}^{\infty} (-1)^{m-1} \frac{\sin^2 m\alpha}{(m\alpha)^2} = \frac{1}{2}.$$

Denote the given function by $f(z)$ and consider its integral around contour (c) in figure 24.1.

As $|z| \to \infty$, $\sin \alpha z \sim e^{\alpha|z|}$, and so $f(z) \sim |z|^{-2} e^{2\alpha|z|} e^{-\pi|z|} = z^{-2} e^{(2\alpha-\pi)|z|}$, and, since $\alpha < \frac{1}{2}\pi$, $|z f(z)\, dz| \to 0$ as $|z| \to \infty$ and the integral around the contour has value zero for $R = \infty$.

The function $f(z)$ has simple poles at $z = n$, where n is an integer, $-\infty < n < \infty$. The pole at $z = 0$ is only a first-order pole as the term in parentheses $\to 1$ as $z \to 0$ and has no singularity there. It follows that the sum of the residues of $f(z)$ at all of its poles is zero. For $n \neq 0$, that residue is

$$\pi \left(\frac{\sin n\alpha}{n\alpha}\right)^2 \left(\left.\frac{d(\sin \pi z)}{dz}\right|_{z=n}\right)^{-1} = \left(\frac{\sin n\alpha}{n\alpha}\right)^2 \frac{1}{\cos \pi n}$$
$$= (-1)^n \left(\frac{\sin n\alpha}{n\alpha}\right)^2.$$

For $n = 0$ the residue is 1.

Since the general residue is an even function of n, the sum for $-\infty < n \leq -1$ is equal to that for $1 \leq n < \infty$, and the zero sum of the residues can be written

$$1 + 2\sum_{n=1}^{\infty}(-1)^n \left(\frac{\sin n\alpha}{n\alpha}\right)^2 = 0,$$

leading immediately to the stated result.

25.13 *Find the function $f(t)$ whose Laplace transform is*

$$\bar{f}(s) = \frac{e^{-s} - 1 + s}{s^2}.$$

Consider first the Taylor series expansion of $\bar{f}(s)$ about $s = 0$:

$$\bar{f}(s) = \frac{e^{-s} - 1 + s}{s^2} = \frac{(1 - s + \frac{1}{2}s^2 + \cdots) - 1 + s}{s^2} \sim \frac{1}{2} + \mathrm{O}(s).$$

Thus $\bar{f}$ has no pole at $s = 0$, and λ in the Bromwich integral can be as small as we wish (but > 0). When the integration line is made part of a closed contour C, the inversion integral becomes

$$f(t) = \int_C \frac{e^{-s}e^{st} - e^{st} + se^{st}}{s^2}\, ds.$$

For $t < 0$, all the terms $\to 0$ as $\mathrm{Re}\, s \to \infty$, and so we close the contour in the right half-plane, as in contour (h) of figure 24.1. On Γ, s times the integrand $\to 0$, and, as the contour encloses no poles, it follows that the integral along L is zero. Thus $f(t) = 0$ for $t < 0$.

For $t > 1$, all terms $\to 0$ as $\mathrm{Re}\, s \to -\infty$, and so we close the contour in the left half-plane, as in contour (g) of figure 24.1. On Γ, s times the integrand again $\to 0$, and, as this contour also encloses no poles, it again follows that the integral along L is zero. Thus $f(t) = 0$ for $t > 1$, as well as for $t < 0$.

For $0 < t < 1$, we need to separate the Bromwich integral into two parts (guided by the different ways in which the parts behave as $|s| \to \infty$):

$$f(t) = \int_L \frac{e^{-s}e^{st}}{s^2}\, ds + \int_L \frac{(s-1)e^{st}}{s^2}\, ds \equiv I_1 + I_2.$$

For I_1 the exponent is $s(t-1)$; $t - 1$ is negative and so, as in the case $t < 0$, we close the contour in the right half-plane [contour (h)]. No poles are included in this contour, and we conclude that $I_1 = 0$.

For I_2 the exponent is st, indicating that (g) is the appropriate contour. However,

$(s-1)/s^2$ does have a pole at $s=0$ and that is inside the contour. The integral around Γ contributes nothing (that is why it was chosen), and the integral along L must be equal to the residue of $(s-1)e^{st}/s^2$ at $s=0$. Now,

$$\frac{(s-1)e^{st}}{s^2} = \left(\frac{1}{s} - \frac{1}{s^2}\right)\left(1 + st + \frac{s^2t^2}{2!} + \cdots\right) = -\frac{1}{s^2} + \frac{1}{s}(1-t) + \cdots.$$

The residue, and hence the value of I_2, is therefore $1-t$. Since I_1 has been shown to have value 0, $1-t$ is also the expression for $f(t)$ for $0 < t < 1$.

25.15 *Use contour* (i) *in figure 24.1 to show that the function with Laplace transform* $s^{-1/2}$ *is* $(\pi x)^{-1/2}$.

[*For an integrand of the form* $r^{-1/2}\exp(-rx)$, *change variable to* $t = r^{1/2}$.]

With the suggested contour no poles of $s^{-1/2}e^{sx}$ are enclosed and so the integral of $(2\pi i)^{-1}s^{-1/2}e^{sx}$ around the closed curve must have the value zero. It is also clear that the integral along Γ will be zero since $\operatorname{Re} s < 0$ on Γ.

For the small circle γ enclosing the origin, set $s = \rho\, e^{i\theta}$, with $ds = i\rho\, e^{i\theta}\, d\theta$, and consider

$$\lim_{\rho\to 0}\int_0^{2\pi} \rho^{-1/2}e^{-i\theta/2}\exp(x\rho\, e^{i\theta})i\rho\, e^{i\theta}\, d\theta.$$

This $\to 0$ as $\rho \to 0$ (as $\rho^{1/2}$).

On the upper cut, γ_1, $s = re^{i\pi}$ and the contribution to the integral is

$$\frac{1}{2\pi i}\int_\infty^0 \frac{e^{-i\pi/2}}{r^{1/2}}\exp(rxe^{i\pi})e^{i\pi}\, dr,$$

whilst, on the lower cut, γ_2, $s = re^{-i\pi}$, and its contribution to the integral is

$$\frac{1}{2\pi i}\int_0^\infty \frac{e^{i\pi/2}}{r^{1/2}}\exp(rxe^{-i\pi})e^{-i\pi}\, dr.$$

Combining the two (and making both integrals run over the same range) gives

$$-\frac{1}{2\pi i}\int_0^\infty \frac{2i}{r^{1/2}}\, e^{-rx}\, dr = -\frac{1}{\pi}\int_0^\infty \frac{1}{t}\, e^{-t^2x}2t\, dt, \text{ after setting } r = t^2,$$
$$= -\frac{2}{\pi}\frac{\sqrt{\pi}}{2\sqrt{x}}.$$

Since this must add to the Bromwich integral along L to make zero, it follows that the function with Laplace transform $s^{-1/2}$ is $(\pi x)^{-1/2}$.

25.17 *The equation*

$$\frac{d^2y}{dz^2} + \left(\nu + \frac{1}{2} - \frac{1}{4}z^2\right) y = 0,$$

sometimes called the Weber–Hermite equation, has solutions known as parabolic cylinder functions. Find, to within (possibly complex) multiplicative constants, the two W.K.B. solutions of this equation that are valid for large $|z|$. *In each case, determine the leading term and show that the multiplicative correction factor is of the form* $1 + \mathrm{O}(\nu^2/z^2)$.

Identify the Stokes and anti-Stokes lines for the equation. On which of the Stokes lines is the W.K.B. solution that tends to zero for z *large, real and negative, the dominant solution*?

If we consider the equation to be of the generic form

$$\frac{d^2y}{dz^2} + f(z)y = 0,$$

then the W.K.B. solutions are, to within a constant multiplier,

$$y_\pm(z) = \frac{1}{[f(z)]^{1/4}} \exp\left\{\pm i \int^z \sqrt{f(u)}\, du\right\}.$$

In this particular case, writing $\nu + \frac{1}{2}$ as μ for the time being, these solutions are

$$y_\pm(z) = \frac{1}{(\mu - \frac{1}{4}z^2)^{1/4}} \exp\left\{\pm i \int^z \sqrt{\mu - \frac{u^2}{4}}\, du\right\}.$$

Now we seek solutions for large z, and, in this spirit, make binomial expansions of both roots in inverse powers of the relevant variable, z or u. This enables us to write, for a succession of multiplicative complex constants and working to $\mathrm{O}(z^{-2})$,

$$\begin{aligned} y_\pm(z) &= \frac{A}{(\frac{1}{4}z^2 - \mu)^{1/4}} \exp\left\{\pm i^2 \int^z \sqrt{\frac{u^2}{4} - \mu}\, du\right\} \\ &= \frac{B}{\sqrt{z}} \left(1 + \frac{\mu}{z^2} + \cdots\right) \exp\left\{\pm i^2 \int^z \frac{u}{2}\left(1 - \frac{4\mu}{u^2}\right)^{1/2} du\right\} \\ &= \frac{B}{\sqrt{z}} \left(1 + \frac{\mu}{z^2}\right) \exp\left\{\mp \int^z \left(\frac{u}{2} - \frac{\mu}{u} - \frac{\mu^2}{u^3} + \cdots\right) du\right\}. \end{aligned}$$

Performing the indefinite integral in the exponent yields

$$\begin{aligned} y_{\pm}(z) &= \frac{B}{\sqrt{z}}\left(1+\frac{\mu}{z^2}\right)\exp\left\{\mp\left(\frac{z^2}{4}-\mu\ln z+\frac{\mu^2}{2z^2}+\cdots\right)\right\} \\ &= \frac{B}{\sqrt{z}}\left(1+\frac{\mu}{z^2}\right)e^{\mp z^2/4}z^{\pm\mu}\left(1\mp\frac{\mu^2}{2z^2}+\cdots\right) \\ &= \frac{B}{\sqrt{z}}\,e^{\mp z^2/4}z^{\pm\mu}\left(1+\frac{2\mu\mp\mu^2}{2z^2}+\cdots\right). \end{aligned}$$

Replacing μ by $\nu+\frac{1}{2}$ and writing the two solutions separately, we have

$$y_1(z) = C\,e^{-z^2/4}\,z^{\nu}\left[1+\mathrm{O}\left(\frac{\nu^2}{z^2}\right)\right], \qquad y_2(z) = D\,e^{z^2/4}\,z^{-(\nu+1)}\left[1+\mathrm{O}\left(\frac{\nu^2}{z^2}\right)\right].$$

The Stokes lines are determined by the argument(s) of z that make the exponent in the solutions purely real, resulting in one solution being very large (dominant) and one very small (subdominant). As the exponent is proportional to z^2, the Stokes lines are given by $\arg z$ equals 0, $\pi/2$, π or $3\pi/2$. For z large, real and negative, the solution that tends to zero is $y_1(z)\propto e^{-z^2/4}$. This is dominant when z^2 is real and negative, i.e. when z lies on either $\arg z=\pi/2$ or $\arg z=3\pi/2$.

The anti-Stokes lines, on which the exponent is purely imaginary and consequently the two solutions are comparable in magnitude, are clearly given by the four lines $\arg z=(2n+1)\pi/4$ for $n=0,1,2,3$.

25.19 *The function $h(z)$ of the complex variable z is defined by the integral*

$$h(z)=\int_{-i\infty}^{i\infty}\exp(t^2-2zt)\,dt.$$

(a) *Make a change of integration variable, $t=iu$, and evaluate $h(z)$ using a standard integral. Is your answer valid for all finite z?*

(b) *Evaluate the integral using the method of steepest descents, considering in particular the cases* (i) *z is real and positive,* (ii) *z is real and negative and* (iii) *z is purely imaginary and equal to $i\beta$, where β is real. In each case sketch the corresponding contour in the complex t-plane.*

(c) *Evaluate the integral for the same three cases as specified in part* (b) *using the method of stationary phases. To determine an appropriate contour that passes through a saddle point $t=t_0$, write $t=t_0+(u+iv)$ and apply the criterion for determining a level line. Sketch the relevant contour in each case, indicating what freedom there is to distort it.*

Comment on the accuracy of the results obtained using the approximate methods adopted in (b) *and* (c).

Before we consider the three different methods of evaluating the integral, we note that its limits lie one in each of the $\pi/2$ sectors of the complex t-plane that are centred on the negative and positive parts of the imaginary axis. All contours that we employ must do the same, though it will not matter exactly where in these sectors they formally end, as, within them, the integrand, which behaves like $\exp(-|t|^2)$, goes (rapidly) to zero as $|t| \to \infty$.

(a) Making the change of integration variable $t = iu$ with $dt = i\,du$ gives $h(z)$ as

$$\begin{aligned} h(z) &= \int_{-\infty}^{\infty} \exp(-u^2 - 2izu)\, i\,du \\ &= \int_{-\infty}^{\infty} \exp[-(u+iz)^2]\, \exp(-z^2)\, i\,du \\ &= i\sqrt{\pi}\, e^{-z^2}. \end{aligned}$$

It is the behaviour of the dominant term in the exponent that determines the convergence or otherwise of the integral. In this case, the t^2 term dominates the term containing z, and, since, as discussed above, it produces convergence, the result is valid for all (finite) values of z.

(b) We first identify the saddle point(s) t_0 of the integrand by setting the derivative of the exponent equal to zero:

$$0 = \frac{d}{dt}(t^2 - 2zt) = 2t - 2z \quad \Rightarrow \quad t_0 = z\text{; only one saddle point.}$$

The second derivative of the exponent is 2 (independent of the value of z in this case), and so, in the standard notation $f''(t_0) = Ae^{i\alpha}$, we have $A = 2$ and $\alpha = 0$. The value of $f_0 \equiv f(t_0)$ is $t_0^2 - 2zt_0 = -z^2$.

The remaining task is to determine the orientation and direction of traversal of the saddle point. With $t - t_0 = se^{i\theta}$, the possible lines of steepest descents (l.s.d.) are given by $2\theta + \alpha = 0$, $\pm\pi$ or 2π. Of these, the need for $\frac{1}{2}As^2\cos(2\theta + \alpha)$ to be negative picks out $\theta = \pm\frac{1}{2}\pi$. Thus the l.s.d. through the saddle point is parallel to the imaginary axis and the direction of traversal is $+\frac{1}{2}\pi$. Since this lies (just) in the range $-\frac{1}{2}\pi < \theta \le \frac{1}{2}\pi$, we take the positive sign from the general formula

$$\pm\left(\frac{2\pi}{A}\right)^{1/2} \exp(f_0)\exp[\tfrac{1}{2}i(\pi - \alpha)]$$

and obtain

$$\begin{aligned} h(z) &= +\left(\frac{2\pi}{2}\right)^{1/2} \exp(-z^2)\exp[\tfrac{1}{2}i(\pi - 0)] \\ &= i\sqrt{\pi}\, e^{-z^2}. \end{aligned}$$

The conclusion about the orientation and sense of traversal of the saddle point did not depend upon the value of z (because $f''(t_0)$ did not). Consequently the

(b)(i) $z > 0$ (b)(ii) $z < 0$ (b)(iii) $z = i\beta$

(c)(i) $z > 0$ (c)(ii) $z < 0$ (c)(iii) $z = i\beta$

Figure 25.2 The contours following (b) the lines of steepest descents and (c) the lines of stationary phase for the integral in exercise 25.19.

value of the integral is the same for all three cases, though the path in the complex t-plane is determined by z, as is shown in the upper row of sketches in figure 25.2.

(c) We know from general theory that the directions of the level lines at a saddle point make an angle of $\pi/4$ with the l.s.d. through the point. From this and the results of part (b) we can say that the level lines at $t_0 = z$ have directions $\theta = \pm\pi/4$ and $\pm 3\pi/4$.

The same conclusion can be reached, and an indication of suitable contours obtained, by writing $t = t_0 + u + iv$ and requiring that the resulting integrand has a constant magnitude for all u and v. That magnitude must be the same as it is at the saddle point, i.e. when $u = v = 0$.

We consider first cases (i) and (ii) in which $t_0 = z$ is real and $t = (z + u) + iv$. The integrand is then

$$g(u, v) = \exp(t^2 - 2zt) = \exp[\,(z + u)^2 - v^2 + 2iv(z + u) - 2z(z + u + iv)\,],$$

with $g(0, 0) = \exp(-z^2)$. For the integrand to have a constant magnitude, the real part of the exponent must not depend upon u and v. The u- and v-dependent part of the real part of the exponent is $2zu + u^2 - v^2 - 2zu$, and this must therefore

have the value 0 for all u and v, i.e. $v = \pm u$. These are the same lines as $\theta = \pm\pi/4$ and $\theta = \pm 3\pi/4$.

Now, although the saddle point at $t_0 = z$ lies outside both of the regions in which the contour must begin and end, the contour must go through it. It is therefore necessary for the contour to turn through a right angle at the saddle point; it transfers from one of the level lines that pass through the saddle to the other one. As will be seen from sketches (c)(i) and (c)(ii), the contour in case (i), $z > 0$, turns to the left by $\pi/2$ as it passes through the saddle; that for $z < 0$ turns to the right through $\pi/2$.

The formula for the total contribution to the integral from integrating through the saddle point along a level line is the same as that for an l.s.d. evaluation, though the former is a Fresnel integral and the latter is an error integral. The stationary phase calculation therefore also yields the value $i\sqrt{\pi}\,\exp(-z^2)$ for $h(z)$.

In both of the present cases, the sharp turn through a right angle at the saddle point means that the vector diagram for the integral consists of one-half from each of two Cornu spirals that are mirror images of each other. Each is broken at its centre point where the phase of the integrand is stationary. The two half spirals join at right angles at the point that is midway between their 'winding points'.

We now turn to case (iii), in which $z = i\beta$ is imaginary. In this case the saddle point lies within one of the two regions that each contain one end of the contour. However, a parallel analysis to that for cases (i) and (ii), setting $t = u + i(\beta + v)$, yields the same conclusion, namely that $v = \pm u$ are appropriate level lines through the saddle.

It is a matter of choice whether the solid line shown in sketch (c)(iii), or its mirror image in the imaginary axis, is chosen; the calculated value for the integral will be the same. The result for $h(z)$ will also be the same as for cases (i) and (ii), i.e. $i\sqrt{\pi}\,\exp(-z^2)$, or, more explicitly in this case, $i\sqrt{\pi}\,\exp(\beta^2)$. Since the contour does not have to go through any particular point other than $t = i\beta$ (and does not need to take a right-angled turn there) and the integrand is analytic, the contour in the end-region not containing z can follow almost any path. One variation from two intersecting straight lines is shown dashed in figure 25.2(c)(iii).

Finally, we note that the fact that all methods give the same answer for $h(z)$, even though the l.s.d. and stationary phase calculations are, in general, approximations, can be put down to the particular form of the integrand. The exponent, $t^2 - 2zt$, is a quadratic function, and so its Taylor series terminates after three terms (of which the second vanishes at the saddle point). Consequently, the l.s.d. and stationary phase approaches which ignore the cubic and higher terms in the Taylor series are *not* approximations. This, together with the fact that there is only one saddle point in the whole t-plane, means that the methods produce exact results for this form of integrand.

25.21 *The stationary phase approximation to an integral of the form*

$$F(v) = \int_a^b g(t)e^{ivf(t)}\,dt, \qquad |v| \gg 1,$$

where $f(t)$ is a real function of t and $g(t)$ is a slowly varying function (when compared with the argument of the exponential), can be written as

$$F(v) \sim \left(\frac{2\pi}{|v|}\right)^{1/2} \sum_{n=1}^{N} \frac{g(t_n)}{\sqrt{A_n}} \exp\left\{ i\left[vf(t_n) + \frac{\pi}{4}\,\mathrm{sgn}\Big(vf''(t_n)\Big)\right]\right\},$$

where the t_n are the N stationary points of $f(t)$ that lie in $a < t_1 < t_2 < \cdots < t_N < b$, $A_n = |f''(t_n)|$, and sgn(x) is the sign of x.

Use this result to find an approximation, valid for large positive values of v, to the integral

$$F(v, z) = \int_{-\infty}^{\infty} \frac{1}{1+t^2} \cos[(2t^3 - 3zt^2 - 12z^2t)v]\,dt,$$

where z is a real positive parameter.

Since the argument of the cosine function is everywhere real, we can consider the required integral as the real part of

$$\int_{-\infty}^{\infty} \frac{1}{1+t^2} \exp\{ i[(2t^3 - 3zt^2 - 12z^2t)v]\}\,dt,$$

to which we can apply the stated approximation directly. To do so, we need to calculate values for all of the terms appearing in the quoted 'omnibus' formula.

We start by determining the stationary points involved, given by

$$\begin{aligned} 0 = f'(t) = 6t^2 - 6zt - 12z^2 \quad &\Rightarrow \quad (t+z)(t-2z) = 0 \\ &\Rightarrow \quad t_1 = -z \text{ and } t_2 = 2z. \end{aligned}$$

Thus $N = 2$ and the required second derivatives, $f''(t) = 12t - 6z$, and values, $f_n = f(t_n)$, are given by

$$\begin{gathered} f_1 = -2z^3 - 3z^3 + 12z^3 = 7z^3, \;\; f_2 = 16z^3 - 12z^3 - 24z^3 = -20z^3, \\ f''(t_1) = -12z - 6z = -18z, \;\; f''(t_2) = 24z - 6z = 18z. \end{gathered}$$

The two corresponding values of the multiplicative function $g(t) = (1+t^2)^{-1}$ are

$$g(t_1) = (1+z^2)^{-1} \quad \text{and} \quad g(t_2) = (1+4z^2)^{-1}.$$

Substituting all of these gives

$$F(v,z) \sim \operatorname{Re}\left(\frac{2\pi}{v}\right)^{1/2}\left\{\frac{\exp[\,i(7vz^3-\frac{1}{4}\pi)\,]}{\sqrt{18z}\,(1+z^2)}+\frac{\exp[\,i(-20vz^3+\frac{1}{4}\pi)\,]}{\sqrt{18z}\,(1+4z^2)}\right\}$$

$$=\left(\frac{\pi}{9zv}\right)^{1/2}\left[\frac{\cos(7vz^3-\frac{1}{4}\pi)}{1+z^2}+\frac{\cos(20vz^3-\frac{1}{4}\pi)}{1+4z^2}\right]$$

as the stationary phase approximation to the integral.

25.23 *Use the method of steepest descents to find an asymptotic approximation, valid for z large, real and positive, to the function defined by*

$$F_v(z)=\int_C \exp(-iz\sin t+ivt)\,dt,$$

where v is real and non-negative and C is a contour that starts at $t=-\pi+i\infty$ and ends at $t=-i\infty$.

Let us denote the integrand by $g(t)$ and the exponent by $f(t)$; thus $g(t)=\exp[\,f(t)\,]$.

We first check that the integrand $\to 0$ at the two end-points; if it did not, the method could not be even approximately correct. As the end-points involve $\pm\infty$, we should formally consider a limiting process, but in practice we need only identify the dominant term in each expression and determine its behaviour as $t\to\infty$.

At $t=-\pi+i\infty$,

$$\sin t=\frac{1}{2i}\,[\,e^{i(-\pi+i\infty)}-e^{-i(-\pi+i\infty)}\,]=\frac{1}{2i}\,(-0+e^{\infty})=\frac{1}{2i}\,e^{\infty}.$$

Thus

$$g(-\pi+i\infty)=\exp[\,-iz(e^{\infty}/2i)+iv(-\pi+i\infty)\,]=0,$$

for z real and >0 and for all v. Similarly, at $t=-i\infty$,

$$\sin(-i\infty)=\frac{1}{2i}\,(e^{-i^2\infty}-e^{i^2\infty})=\frac{1}{2i}\,e^{\infty}$$

and

$$g(-i\infty)=\exp[\,-iz(e^{\infty}/2i)+iv(-i\infty)\,]=0.$$

In each case, the behaviour of $f(t)$ is dominated by the exponentiation appearing in the sine term; as this produces a negative exponent for the exponential function determining $g(t)$, the latter $\to 0$ at both end-points.

We next determine the position(s) and properties of the saddle points. These are given by

$$\begin{aligned}
0 = \frac{df}{dt} &= -iz\cos t + iv \\
&\Rightarrow \quad t_0 = \cos^{-1}\frac{v}{z} \text{ (real) with } -\pi < t_0 < 0 \text{ and } z > v, \\
\frac{d^2 f}{dt^2} &= iz\sin t, \\
f''(t_0) &= iz\sin\left(\cos^{-1}\frac{v}{z}\right) = \frac{-iz\sqrt{z^2 - v^2}}{z} \\
&= -i\sqrt{z^2 - v^2} \\
&\equiv Ae^{i\alpha} \text{ with } A = \sqrt{z^2 - v^2} \text{ and } \alpha = \frac{3\pi}{2}, \\
f_0 \equiv f(t_0) &= -iz\sin\left(\cos^{-1}\frac{v}{z}\right) + iv\left(\cos^{-1}\frac{v}{z}\right) \\
&= i\sqrt{z^2 - v^2} + iv\left(\cos^{-1}\frac{v}{z}\right).
\end{aligned}$$

Thus the only saddle point is at $t_0 = \cos^{-1}(v/z)$, and the values of $f(t_0)$ and $f''(t_0)$ are given above.

The final step before evaluating the approximate expression for the integral is to determine the direction of the contour through the saddle point. A line of steepest descents (l.s.d.), on which the phase of $f(t)$ is constant, is given by $\sin(2\theta + \alpha) = 0$, where $t - t_0 = se^{i\theta}$ and α is as determined above by $f''(t_0)$. Thus $2\theta + 3\pi/2 = 0$, $\pm\pi$, 2π are possible lines, and, of the resulting possible values of $\pm\pi/4$ and $\pm 3\pi/4$ for θ, it is clear that approaching from the direction $\theta = 3\pi/4$ and leaving in the direction $\theta = -\pi/4$ is appropriate. This can be verified by considering the first non-constant, non-vanishing term in the Taylor expansion of $f(t)$, namely

$$\frac{1}{2!}(t - t_0)^2 f''(t_0) = \frac{1}{2!}s^2 e^{-2i\pi/4}(-i\sqrt{z^2 - v^2}) = -\frac{s^2\sqrt{z^2 - v^2}}{2}.$$

This is real and negative (in both cases, since $e^{-2i\pi/4} = e^{6i\pi/4}$), thus confirming that the standard result for integrating over the saddle point can be used. This is

$$I \approx \pm\left(\frac{2\pi}{A}\right)^{1/2}\exp(f_0)\exp[\tfrac{1}{2}i(\pi - \alpha)],$$

with the $\pm$ choice being resolved by the direction in which the l.s.d. passes through the saddle point; it is positive if $|\theta| < \pi/2$ and negative otherwise. In this particular case, the l.s.d. is traversed in the direction $-\pi/4$ through the saddle point and the plus sign is appropriate.

Finally, inserting all of the specific data for this case into the general formula, we

find that

$$\begin{aligned} F_\nu(z) &\approx \left[\frac{2\pi}{\sqrt{z^2-\nu^2}}\right]^{1/2} \exp\left[i\sqrt{z^2-\nu^2} + i\nu\cos^{-1}\frac{\nu}{z}\right] \exp[\tfrac{1}{2}i(\pi - \tfrac{3}{2}\pi)] \\ &= \left[\frac{2\pi}{\sqrt{z^2-\nu^2}}\right]^{1/2} \exp\left[i\left(\sqrt{z^2-\nu^2} + \nu\cos^{-1}\frac{\nu}{z} - \frac{\pi}{4}\right)\right] \\ &\approx \left[\frac{2\pi}{z}\right]^{1/2} \exp\left[i\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)\right], \text{ for } z \gg \nu. \end{aligned}$$

This last approximation enables us to identify the function $F_\nu(z)$ as probably being a multiple of the Hankel function (Bessel function of the third kind) $H_\nu^{(1)}(z)$, though, as different functions can have the same asymptotic form, this cannot be certain.

26

Tensors

26.1 *Use the basic definition of a Cartesian tensor to show the following.*

(a) *That for any general, but fixed, ϕ,*

$$(u_1, u_2) = (x_1 \cos\phi - x_2 \sin\phi,\ x_1 \sin\phi + x_2 \cos\phi)$$

are the components of a first-order tensor in two dimensions.

(b) *That*

$$\begin{pmatrix} x_2^2 & x_1 x_2 \\ x_1 x_2 & x_1^2 \end{pmatrix}$$

is not a tensor of order 2. To establish that a single element does not transform correctly is sufficient.

Consider a rotation of the (unprimed) coordinate axes through an angle θ to give the new (primed) axes. Under this rotation,

$$\begin{aligned} x_1 &\to x_1' = x_1 \cos\theta + x_2 \sin\theta, \\ x_2 &\to x_2' = -x_1 \sin\theta + x_2 \cos\theta, \\ x_3 &\to x_3' = x_3, \end{aligned}$$

and the transformation matrix L_{ij} is given by

$$\mathsf{L} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Denoting $\cos\theta$ and $\sin\theta$ by c and s, respectively, we compare

$$u_1' = x_1' \cos\phi - x_2' \sin\phi = cx_1 \cos\phi + sx_2 \cos\phi + sx_1 \sin\phi - cx_2 \sin\phi$$

with

$$u_1' = cu_1 + su_2 = cx_1 \cos\phi - cx_2 \sin\phi + sx_1 \sin\phi + sx_2 \cos\phi.$$

These two are equal, showing that the first component transforms correctly. However, this alone is not sufficient; for (u_1, u_2) to be the components of a first-order tensor, *all* components must transform correctly. We therefore also compare the remaining transformed component:

$$u_2' = x_1' \sin\phi + x_2' \cos\phi = cx_1 \sin\phi + sx_2 \sin\phi - sx_1 \cos\phi + cx_2 \cos\phi$$

is to be compared with

$$u_2' = -su_1 + cu_2 = -sx_1 \cos\phi + sx_2 \sin\phi + cx_1 \sin\phi + cx_2 \cos\phi.$$

These two are also equal, showing that both components do transform correctly and that (u_1, u_2) are indeed the components of a first-order tensor.

We note, in passing, that $u_1 + iu_2$ is the complex vector obtained by rotating the 'base vector' $x_1 + ix_2$ through an angle ϕ in the complex plane:

$$\begin{aligned} u_1 + iu_2 &= e^{i\phi}(x_1 + ix_2) \\ &= (\cos\phi + i\sin\phi)(x_1 + ix_2) \\ &= (x_1 \cos\phi - x_2 \sin\phi) + i(x_1 \sin\phi + x_2 \cos\phi). \end{aligned}$$

In view of this observation, and of the definition of a first-order tensor as a set of objects 'that transform in the same way as a position vector', it is perhaps not surprising to find that the given expressions form the components of a tensor.

(b) Consider the transform of the first element $u_{11} = x_2^2$. This becomes

$$u_{11}' = (x_2')^2 = (-sx_1 + cx_2)^2 = s^2x_1^2 - 2scx_1x_2 + c^2x_2^2.$$

If it transforms as a component of a tensor, then it must also be the case that

$$u_{11}' = L_{1k}L_{1l}u_{kl} = c^2x_2^2 + csx_1x_2 + scx_1x_2 + s^2x_1^2.$$

But, these two RHSs are not equal, and it follows that the given set of expressions cannot form the components of a tensor of order 2. It is not necessary to consider any more u_{ij}; failure of any one element to transform correctly rules out the possibility of the set being a tensor.

26.3 *In the usual approach to the study of Cartesian tensors the system is considered fixed and the coordinate axes are rotated. The transformation matrix used is therefore that for components relative to rotated coordinate axes. An alternative view is that of taking the coordinate axes as fixed and rotating the components of the system; this is equivalent to reversing the signs of all rotation angles.*

Using this alternative view, determine the matrices representing (a) *a positive rotation of* $\pi/4$ *about the x-axis and* (b) *a rotation of* $-\pi/4$ *about the y-axis. Determine the initial vector* $\mathbf{r}$ *which, when subjected to* (a) *followed by* (b), *finishes at* $(3, 2, 1)$.

The normal notation for the two rotation matrices would be

$$\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \text{ and } \mathsf{B} = \begin{pmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{pmatrix},$$

with $\theta = \phi = \pi/4$.

In the alternative view (denoted by ") they would have the same forms but with $\theta = \phi = -\pi/4$, namely

$$\mathsf{A}" = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \text{ and } \mathsf{B}" = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

The matrix representing (a) followed by (b) in this alternative view is thus

$$\begin{aligned} \mathsf{B}"\mathsf{A}" &= \frac{1}{2} \begin{pmatrix} 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix}. \end{aligned}$$

The required point is the solution of

$$\frac{1}{2} \begin{pmatrix} \sqrt{2} & -1 & -1 \\ 0 & \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}.$$

Using the fact that $\mathsf{B}"\mathsf{A}"$ is orthogonal, and therefore its inverse is simply its transpose, this can be solved directly as

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & \sqrt{2} \\ -1 & \sqrt{2} & 1 \\ -1 & -\sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2\sqrt{2} \\ -1+\sqrt{2} \\ -1-\sqrt{2} \end{pmatrix}.$$

As a partial check, we compute $|\mathbf{r}_{\text{initial}}|^2 = 8 + (3 - 2\sqrt{2}) + (3 + 2\sqrt{2}) = 14 = 3^2 + 2^2 + 1^2 = |\mathbf{r}_{\text{final}}|^2$, i.e. the length of the vector is unchanged by the rotations, as it should be.

26.5 *Use the quotient law for tensors to show that the array*

$$\begin{pmatrix} y^2 + z^2 - x^2 & -2xy & -2xz \\ -2yx & x^2 + z^2 - y^2 & -2yz \\ -2zx & -2zy & x^2 + y^2 - z^2 \end{pmatrix}$$

forms a second-order tensor.

To test whether the array is a second-order tensor we need to contract it with an arbitrary known second-order tensor. By 'arbitrary tensor' we mean a tensor in which any one component can be made to be the only non-zero component.

Since any second-order tensor can always be written as the sum of a symmetric and an anti-symmetric tensor, and all operations are linear, it will be sufficient to prove the result for one known symmetric tensor and one known antisymmetric tensor. The simplest symmetric second-order tensor S_{ij} is the (symmetric) outer product of the (by definition) first-order tensor (x, y, z) with itself, i.e $S_{ij} = x_i x_j$. Denoting the given array by B_{ij}, we consider

$$\begin{aligned} B_{ij}S_{ij} &= B_{ij}x_i x_j \\ &= x^2(y^2 + z^2 - x^2) + y^2(x^2 + z^2 - y^2) + z^2(x^2 + y^2 - z^2) \\ &\qquad + 2[\,xy(-2xy) + xz(-2xz) + yz(-2yz)\,] \\ &= -2x^2y^2 - 2x^2z^2 - 2y^2z^2 - x^4 - y^4 - z^4 \\ &= -(x^2 + y^2 + z^2)^2 = -|\mathbf{x}|^4. \end{aligned}$$

The term in parentheses in the last line is formally $x_i x_i$, i.e the contracted product of a first-order tensor with itself, and therefore an invariant zero-order tensor. Squaring an invariant or multiplying it by a constant (-1) leaves it as an invariant, leading to the conclusion that $B_{ij}S_{ij}$ is a zero-order tensor.

We now turn to an antisymmetric tensor, where a suitable second-order tensor A_{ij} is the contraction of the third-order tensor ϵ_{ijk} with the first-order tensor x_i. Thus A_{ij} has the form

$$A = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

and the contracted tensor is

$$B_{ij}A_{ij} = 0 - 2xyz + 2yxz + 2zyx + 0 - 2xyz - 2yzx + 2xzy + 0 = 0.$$

Now, 0 is an even more obvious invariant than $|\mathbf{x}|^2$ and so $B_{ij}A_{ij}$ is also a zero-order tensor.

Taking the results of the last two paragraphs together, it follows from the quotient law that B_{ij} is a second-order tensor.

26.7 *Use tensor methods to establish that*

$$grad\, \tfrac{1}{2}(\mathbf{u}\cdot\mathbf{u}) = \mathbf{u}\times curl\,\mathbf{u} + (\mathbf{u}\cdot grad)\mathbf{u}.$$

Now use this result and the general divergence theorem for tensors to show that, for a vector field $\mathbf{A}$,

$$\int_S \left[\mathbf{A}(\mathbf{A}\cdot d\mathbf{S}) - \tfrac{1}{2}A^2 d\mathbf{S}\right] = \int_V \left[\mathbf{A}\, div\mathbf{A} - \mathbf{A}\times curl\,\mathbf{A}\right]\, dV,$$

where S *is the surface enclosing the volume* V.

We start with the most complicated of the terms in the identity:

$$\begin{aligned}[\,\mathbf{u}\times(\nabla\times\mathbf{u})\,]_i &= \epsilon_{ijk}u_j(\nabla\times\mathbf{u})_k = \epsilon_{ijk}u_j\,\epsilon_{klm}\frac{\partial u_m}{\partial x_l}\\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})u_j\frac{\partial u_m}{\partial x_l} = u_j\frac{\partial u_j}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j}\\ &= \frac{1}{2}\frac{\partial}{\partial x_i}(u_ju_j) - (\mathbf{u}\cdot\nabla)u_i = [\,\tfrac{1}{2}\nabla(\mathbf{u}\cdot\mathbf{u}) - (\mathbf{u}\cdot\nabla)\mathbf{u}\,]_i,\end{aligned}$$

which establishes the first result. To establish the second result we first note that

$$\frac{\partial}{\partial x_j}(A_iA_j) = A_i\frac{\partial A_j}{\partial x_j} + A_j\frac{\partial A_i}{\partial x_j} = [\,\mathbf{A}\nabla\cdot\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{A}\,]_i. \qquad (*)$$

Next we consider the ith component of the integrand on the RHS of the putative equation and use the first result to replace $\mathbf{A}\times(\nabla\times\mathbf{A})$.

$$\begin{aligned}[\,\mathrm{RHS}\,]_i &= A_i(\nabla\cdot\mathbf{A}) - [\,\mathbf{A}\times(\nabla\times\mathbf{A})\,]_i\\ &= A_i(\nabla\cdot\mathbf{A}) - \tfrac{1}{2}\nabla_iA^2 + (\mathbf{A}\cdot\nabla)\mathbf{A}_i = \frac{\partial}{\partial x_j}(A_iA_j) - \frac{1}{2}\frac{\partial(A^2)}{\partial x_i}, \qquad \text{using } (*).\end{aligned}$$

We can now integrate this equation over the volume V and apply the divergence theorem for tensors to both terms individually:

$$\begin{aligned}\int_V [\,\mathrm{RHS}\,]_i\, dV &= \int_V \frac{\partial}{\partial x_j}(A_iA_j)\, dV - \frac{1}{2}\int_V \frac{\partial(A^2)}{\partial x_i}\, dV\\ &= \int_S A_iA_j\, dS_j - \frac{1}{2}\int_S A^2\, dS_i = \int_S \left[\mathbf{A}(\mathbf{A}\cdot d\mathbf{S}) - \tfrac{1}{2}A^2 d\mathbf{S}\right]_i.\end{aligned}$$

This concludes the proof.

26.9 *The equation*

$$|\mathsf{A}|\epsilon_{lmn} = A_{li}A_{mj}A_{nk}\epsilon_{ijk} \qquad (*)$$

is a more general form of the expression for the determinant of a 3×3 *matrix* A. *This would normally be written as*

$$|\mathsf{A}| = \epsilon_{ijk}A_{i1}A_{j2}A_{k3},$$

but the form $(*)$ *removes the explicit mention of* $1, 2, 3$ *at the expense of an additional Levi–Civita symbol. The* $(*)$ *form of expression for a determinant can be readily extended to cover a general* $N \times N$ *matrix.*

The following is a list of some of the common properties of determinants.

(a) *Determinant of the transpose.* The transpose matrix A^{T} has the same determinant as A itself, i.e.

$$|\mathsf{A}^{\mathrm{T}}| = |\mathsf{A}|.$$

(b) *Interchanging two rows or two columns.* If two rows (or columns) of A are interchanged, its determinant changes sign but is unaltered in magnitude.

(c) *Identical rows or columns.* If any two rows (or columns) of A are identical or are multiples of one another, then it can be shown that $|\mathsf{A}| = 0$.

(d) *Adding a constant multiple of one row (or column) to another.* The determinant of a matrix is unchanged in value by adding to the elements of one row (or column) any fixed multiple of the elements of another row (or column).

(e) *Determinant of a product.* If A and B are square matrices of the same order then

$$|\mathsf{AB}| = |\mathsf{A}||\mathsf{B}| = |\mathsf{BA}|.$$

A simple extension of this property gives, for example,

$$|\mathsf{AB}\cdots\mathsf{G}| = |\mathsf{A}||\mathsf{B}|\cdots|\mathsf{G}| = |\mathsf{A}||\mathsf{G}|\cdots|\mathsf{B}| = |\mathsf{A}\cdots\mathsf{GB}|,$$

which shows that the determinant is invariant to permutations of the matrices in a multiple product.

Use the form given in $(*)$ *to prove the above properties. For definiteness take* $N = 3$, *but convince yourself that your methods of proof would be valid for any positive integer* $N > 1$.

(a) We write the expression for $|\mathsf{A}^{\mathrm{T}}|$ using the given formalism, recalling that $(\mathsf{A}^{\mathrm{T}})_{ij} = (\mathsf{A})_{ji}$. We then contract both sides with ϵ_{lmn}:

$$\begin{aligned}
|\mathsf{A}^{\mathrm{T}}|\epsilon_{lmn} &= A_{il}A_{jm}A_{kn}\epsilon_{ijk},\\
|\mathsf{A}^{\mathrm{T}}|\epsilon_{lmn}\epsilon_{lmn} &= A_{il}A_{jm}A_{kn}\epsilon_{lmn}\epsilon_{ijk},\\
&= |\mathsf{A}|\epsilon_{ijk}\epsilon_{ijk},\\
|\mathsf{A}^{\mathrm{T}}| &= |\mathsf{A}|.
\end{aligned}$$

In the third line we have used the definition of $|\mathsf{A}|$ (with the roles of the sets of

dummy variables $\{i, j, k\}$ and $\{l, m, n\}$ interchanged), and in the fourth line, we have cancelled the scalar quantity $\epsilon_{lmn}\epsilon_{lmn} = \epsilon_{ijk}\epsilon_{ijk}$; the value of this scalar is $N(N-1)$, but that is irrelevant here.

(b) Every non-zero term on the RHS of (*) contains any particular row index once and only once. The same can be said for the Levi–Civita symbol on the LHS. Thus interchanging two rows is equivalent to interchanging two of the subscripts of ϵ_{lmn} and thereby reversing its sign. Consequently, the whole RHS changes sign and the magnitude of $|\mathsf{A}|$ remains the same, though its sign is changed.

(c) If, say, $A_{pi} = \lambda A_{pj}$, for some particular pair of values i and j and all p, then in the (multiple) summation on the RHS of (*) each A_{nk} appears multiplied by (with no summation over i and j)

$$\epsilon_{ijk}A_{li}A_{mj} + \epsilon_{jik}A_{lj}A_{mi} = \epsilon_{ijk}\lambda A_{lj}A_{mj} + \epsilon_{jik}A_{lj}\lambda A_{mj} = 0,$$

since $\epsilon_{ijk} = -\epsilon_{jik}$. Consequently, grouped in this way, all pairs of terms contribute nothing to the sum and $|\mathsf{A}| = 0$.

(d) Consider the matrix B whose m, jth element is defined by $B_{mj} = A_{mj} + \lambda A_{pj}$, where $p \neq m$. The only case that needs detailed analysis is when l, m and n are all different. Since $p \neq m$ it must be the same as either l or n; suppose that $p = l$. The determinant of B is given by

$$\begin{aligned}|\mathsf{B}|\epsilon_{lmn} &= A_{li}(A_{mj} + \lambda A_{lj})A_{nk}\epsilon_{ijk}\\ &= A_{li}A_{mj}A_{nk}\epsilon_{ijk} + \lambda A_{li}A_{lj}A_{nk}\epsilon_{ijk}\\ &= |\mathsf{A}|\epsilon_{lmn} + \lambda 0,\end{aligned}$$

where we have used the row equivalent of the intermediate result obtained for columns in (c). Thus we conclude that $|\mathsf{B}| = |\mathsf{A}|$.

(e) If $\mathsf{X} = \mathsf{AB}$, then

$$|\,\mathsf{X}\,|\epsilon_{lmn} = A_{lx}B_{xi}A_{my}B_{yj}A_{nz}B_{zk}\epsilon_{ijk}.$$

Contract both sides with ϵ_{lmn}:

$$\begin{aligned}|\,\mathsf{X}\,|\epsilon_{lmn}\epsilon_{lmn} &= \epsilon_{lmn}A_{lx}A_{my}A_{nz}\,\epsilon_{ijk}B_{xi}B_{yj}B_{zk}\\ &= \epsilon_{xyz}|\mathsf{A}^{\mathrm{T}}|\epsilon_{xyz}|\mathsf{B}|,\\ \Rightarrow\quad |\,\mathsf{X}\,| &= |\mathsf{A}^{\mathrm{T}}|\,|\mathsf{B}| = |\mathsf{A}|\,|\mathsf{B}|, \quad \text{using result (a).}\end{aligned}$$

To obtain the last line we have cancelled the non-zero scalar $\epsilon_{lmn}\epsilon_{lmn} = \epsilon_{xyz}\epsilon_{xyz}$ from both sides, as we did in the proof of result (a).

The extension to the product of any number of matrices is obvious. Replacing B by CD or by DC and applying the result just proved extends it to a product of three matrices. Extension to any higher number is done in the same way.

26.11 *Given a non-zero vector* $\mathbf{v}$, *find the value that should be assigned to* α *to make*

$$P_{ij} = \alpha v_i v_j \qquad \text{and} \qquad Q_{ij} = \delta_{ij} - \alpha v_i v_j$$

into parallel and orthogonal projection tensors, respectively, i.e. tensors that satisfy, respectively, $P_{ij}v_j = v_i$, $P_{ij}u_j = 0$ *and* $Q_{ij}v_j = 0$, $Q_{ij}u_j = u_i$, *for any vector* $\mathbf{u}$ *that is orthogonal to* $\mathbf{v}$.

Show, in particular, that Q_{ij} *is unique, i.e. that if another tensor* T_{ij} *has the same properties as* Q_{ij} *then* $(Q_{ij} - T_{ij})w_j = 0$ *for* any *vector* $\mathbf{w}$.

Consider

$$P_{ij}v_j = \alpha v_i v_j v_j = \alpha|\mathbf{v}|^2 v_i, \text{ and}$$
$$P_{ij}u_j = \alpha v_i v_j u_j = \alpha v_i(v_j u_j) = 0, \text{ as } \mathbf{u} \text{ is orthogonal to } \mathbf{v}.$$

For $P_{ij}v_j = v_i$ it is clearly necessary that $\alpha = |\mathbf{v}|^{-2}$.

With this choice,

$$Q_{ij}v_j = (\delta_{ij} - \alpha v_i v_j)v_j = v_i - \alpha(v_j v_j)v_i = v_i - |\mathbf{v}|^{-2}(v_j v_j)v_i = 0, \text{ and}$$
$$Q_{ij}u_j = (\delta_{ij} - \alpha v_i v_j)u_j = u_i - \alpha(v_j u_j)v_i = u_i - 0v_i = u_i.$$

Thus the one assigned value for α gives both P_{ij} and Q_{ij} the required properties.

Let $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ be any two linearly independent non-zero vectors orthogonal to $\mathbf{v}$. Then *any* vector $\mathbf{w}$ can be expressed as $\lambda\mathbf{v} + \mu\mathbf{u}^{(1)} + \nu\mathbf{u}^{(2)}$.
Now suppose that T_{ij} has the same properties as Q_{ij} and consider

$$\begin{aligned}(Q_{ij} - T_{ij})w_j &= (Q_{ij} - T_{ij})(\lambda v_j + \mu u_j^{(1)} + \nu u_j^{(2)}) \\ &= \lambda 0 + \mu u_j^{(1)} + \nu u_j^{(2)} - \lambda T_{ij}v_j - \mu T_{ij}u_j^{(1)} - \nu T_{ij}u_j^{(2)} \\ &= 0 + \mu u_j^{(1)} + \nu u_j^{(2)} - 0 - \mu u_j^{(1)} - \nu u_j^{(2)} = 0.\end{aligned}$$

In this sense, Q_{ij} is unique.

26.13 *In a certain crystal the unit cell can be taken as six identical atoms lying at the corners of a regular octahedron. Convince yourself that these atoms can also be considered as lying at the centres of the faces of a cube and hence that the crystal has cubic symmetry. Use this result to* prove *that the conductivity tensor for the crystal,* σ_{ij}, *must be isotropic.*

It is easiest to start with a cube and then join the centre points of any pair of

faces that have a common edge. The network of 12 lines so formed are the edges of a regular octahedron.

The crystal has cubic symmetry and must therefore be invariant under rotations that leave a cube unchanged (apart from the labelling of its corners). One such symmetry operation is rotation (by $2\pi/3$) about a body diagonal; this relabels the axes $O123$ as axes $O3'1'2'$ in the rotated system. The (orthogonal) base-vector transformation matrix S has as its i, jth component the ith component of $\mathbf{e}'_j$ with respect to the basis $\{\mathbf{e}_k\}$. The coordinate transformation matrix L is the transpose of this. For the rotation under consideration,

$$\mathsf{S} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathsf{L} = \mathsf{S}^{\mathrm{T}} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The conductivity tensor is a second-order tensor and so $\sigma'_{ij} = L_{ik}L_{jm}\sigma_{km}$ or, in matrix form,

$$\begin{aligned} \sigma' &= \mathsf{L}\sigma\mathsf{L}^{\mathrm{T}} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{21} \\ \sigma_{32} & \sigma_{33} & \sigma_{31} \\ \sigma_{12} & \sigma_{13} & \sigma_{11} \end{pmatrix}. \end{aligned}$$

This must be the same tensor as σ and so requires that

$$\sigma_{11} = \sigma_{22} = \sigma_{33}; \quad \sigma_{12} = \sigma_{23} = \sigma_{31}; \quad \sigma_{21} = \sigma_{32} = \sigma_{13}.$$

We also note that the transformed tensor is the original one, but with $1 \to 2$, $2 \to 3$ and $3 \to 1$.

Now, restarting from the original situation, consider a rotation of $\pi/2$ about the 3-axis. This clearly carries the $O1$-axis onto the original $O2$-axis and the $O2$-axis onto the original negative $O1$-axis. Therefore, by the substitutions $1 \to 2$, $2 \to -1$ and $3 \to 3$ (where a component changes sign for each minus sign on its subscripts) or by a matrix calculation similar to the previous one, the new transformed conductivity tensor is

$$\sigma' = \begin{pmatrix} \sigma_{22} & -\sigma_{21} & \sigma_{23} \\ -\sigma_{12} & \sigma_{11} & -\sigma_{13} \\ \sigma_{32} & -\sigma_{31} & \sigma_{33} \end{pmatrix}.$$

Again the invariance of σ imposes requirements. In this case,

$$\sigma_{11} = \sigma_{22}; \quad \sigma_{13} = \sigma_{23} = -\sigma_{13}.$$

The last set of equalities requires that $\sigma_{13} = \sigma_{23} = 0$ and hence, by the previous result, that $\sigma_{ij} = 0$ whenever $i \neq j$. Since $\sigma_{11} = \sigma_{22} = \sigma_{33}$, σ is a multiple of the unit matrix, and it follows that σ_{ij} is an isotropic tensor.

Either by direct calculation or by noting that any rotational symmetry of a cube can be represented as an ordered sequence of the two rotations already used, it can be shown that other symmetries do not impose any further constraint on the remaining non-zero elements of the conductivity tensor. Intuitively this must be so, since σ now contains only one free parameter, the common value of σ_{11}, σ_{22} and σ_{33}, and this is required to describe the level of conductivity, which must vary from one crystal to another, and certainly between crystals of different elements.

26.15 *In a certain system of units the electromagnetic stress tensor M_{ij} is given by*

$$M_{ij} = E_iE_j + B_iB_j - \tfrac{1}{2}\delta_{ij}(E_kE_k + B_kB_k),$$

where the electric and magnetic fields, **E** *and* **B**, *are first-order tensors. Show that M_{ij} is a second-order tensor.*

Consider a situation in which $|\mathbf{E}| = |\mathbf{B}|$ but the directions of **E** *and* **B** *are not parallel. Show that* $\mathbf{E} \pm \mathbf{B}$ *are principal axes of the stress tensor and find the corresponding principal values. Determine the third principal axis and its corresponding principal value.*

In the calculation of the transformed RHS, **E** and **B** transform with a single 'L-matrix', but δ_{ij}, being a second-order tensor, requires two. It may simply be noticed that E_kE_k and B_kB_k are scalars and therefore unaltered in the transformation; but, if not, then the orthogonal properties of L, $L_{ik}L_{jk} = \delta_{ij}$ and $L_{ki}L_{kj} = \delta_{ij}$, are needed:

$$\begin{aligned}
M_{ij} &= E_iE_j + B_iB_j - \tfrac{1}{2}\delta_{ij}(E_kE_k + B_kB_k),\\
M'_{ij} &= L_{im}E_mL_{jn}E_n + L_{im}B_mL_{jn}B_n\\
&\qquad - \tfrac{1}{2}L_{ip}L_{jq}\delta_{pq}(L_{kr}E_rL_{ks}E_s + L_{kr}B_rL_{ks}B_s)\\
&= L_{im}L_{jn}(E_mE_n + B_mB_n) - \tfrac{1}{2}L_{ip}L_{jq}\delta_{pq}(\delta_{rs}E_rE_s + \delta_{rs}B_rB_s)\\
&= L_{im}L_{jn}[\,E_mE_n + B_mB_n - \tfrac{1}{2}\delta_{mn}(E_rE_r + B_rB_r)\,]\\
&= L_{im}L_{jn}M_{mn}.
\end{aligned}$$

To obtain the penultimate line we relabelled the dummy suffices p and q as m and n. Thus M_{ij} transforms as a second-order tensor; it is real and symmetric and will therefore have orthogonal eigenvectors.

For the case $|\mathbf{E}| = |\mathbf{B}|$, i.e. $E^2 = B^2$, denote $E_i \pm B_i$ by v_i and consider

$$\begin{aligned}
M_{ij}v_j &= M_{ij}(E_j \pm B_j) \\
&= E_iE_j(E_j \pm B_j) + B_iB_j(E_j \pm B_j) - \tfrac{1}{2}\delta_{ij}(E^2 + B^2)(E_j \pm B_j) \\
&= E_iE^2 \pm E_i(\mathbf{E}\cdot\mathbf{B}) + B_i(\mathbf{B}\cdot\mathbf{E}) \pm B_iB^2 \\
&\qquad - \tfrac{1}{2}(E^2 + B^2)(E_i \pm B_i) \\
&= (E_i \pm B_i)[\,E^2 \pm (\mathbf{E}\cdot\mathbf{B}) - \tfrac{1}{2}\,2E^2\,], \text{ using } E^2 = B^2, \\
&= \pm(\mathbf{E}\cdot\mathbf{B})(E_i \pm B_i) \\
&= \pm(\mathbf{E}\cdot\mathbf{B})v_i.
\end{aligned}$$

This shows that $\mathbf{E}\pm\mathbf{B}$ are eigenvectors of M_{ij} (i.e. its principal axes) with principal values $\pm(\mathbf{E}\cdot\mathbf{B})$.

The third principal axis is orthogonal to both of these and is therefore in the direction

$$(\mathbf{E}+\mathbf{B})\times(\mathbf{E}-\mathbf{B}) = \mathbf{0} + (\mathbf{B}\times\mathbf{E}) - (\mathbf{E}\times\mathbf{B}) - \mathbf{0} = 2(\mathbf{B}\times\mathbf{E}).$$

To determine its principal value, consider

$$\begin{aligned}
M_{ij}(\mathbf{B}\times\mathbf{E})_j &= M_{ij}\epsilon_{jlm}B_lE_m \\
&= E_iE_j\epsilon_{jlm}B_lE_m + B_iB_j\epsilon_{jlm}B_lE_m - \tfrac{1}{2}\delta_{ij}\,2E^2\epsilon_{jlm}B_lE_m \\
&= 0 + 0 - E^2(\mathbf{B}\times\mathbf{E})_i, \text{ since } \epsilon_{jlm}X_lX_j = 0.
\end{aligned}$$

Thus, the third principal value is $-E^2$ (or $-B^2$). This value could have been deduced from the trace of $M_{ij} = E^2 + B^2 - \frac{3}{2}(E^2 + B^2) = -E^2$, since the two eigenvalues found previously are $\pm\mathbf{E}\cdot\mathbf{B}$, which sum to zero. The three eigenvalues together must add up to the trace; hence, the third one is $-E^2$.

26.17 *A rigid body consists of eight particles, each of mass m, held together by light rods. In a certain coordinate frame the particles are at positions*

$$\pm a(3,1,-1), \qquad \pm a(1,-1,3), \qquad \pm a(1,3,-1), \qquad \pm a(-1,1,3).$$

Show that, when the body rotates about an axis through the origin, if the angular velocity and angular momentum vectors are parallel then their ratio must be $40ma^2$, $64ma^2$ *or* $72ma^2$.

Because the particles are symmetrically placed in pairs with respect to the origin, the inertia tensor, given by

$$I_{ij} = \sum_{\text{particles}} m(r^2\delta_{ij} - x_ix_j),$$

will be twice that calulated for the + signs alone. It components are therefore

$$\begin{aligned}
I_{11} &= 2ma^2(2+10+10+10) = 64ma^2,\\
I_{12} = I_{21} &= -2ma^2(3-1+3-1) = -8ma^2,\\
I_{13} = I_{31} &= -2ma^2(-3+3-1-3) = 8ma^2,\\
I_{22} &= 2ma^2(10+10+2+10) = 64ma^2,\\
I_{23} = I_{32} &= -2ma^2(-1-3-3+3) = 8ma^2,\\
I_{33} &= 2ma^2(10+2+10+2) = 48ma^2.
\end{aligned}$$

The resulting tensor is

$$8ma^2 \begin{pmatrix} 8 & -1 & 1 \\ -1 & 8 & 1 \\ 1 & 1 & 6 \end{pmatrix}$$

and its principal moments are $8ma^2\lambda$, where

$$\begin{aligned}
0 &= \begin{vmatrix} 8-\lambda & -1 & 1 \\ -1 & 8-\lambda & 1 \\ 1 & 1 & 6-\lambda \end{vmatrix}\\
&= (8-\lambda)(\lambda^2 - 14\lambda + 47) + (-7+\lambda) + (-9+\lambda)\\
&= (8-\lambda)(\lambda^2 - 14\lambda + 47 - 2)\\
&= (8-\lambda)(\lambda-9)(\lambda-5).
\end{aligned}$$

Thus the principal moments are $40ma^2$, $64ma^2$ and $72ma^2$. As a partial check: $40 + 64 + 72 = 8(8 + 8 + 6)$.

If the angular velocity ω and the angular momentum $\mathbf{J} = I\omega$ are parallel, then the body is rotating about one of its principal axes (the eigenvectors of I); their ratio is the principal moment about that axis and is thus one of the three values calculated above.

26.19 *A block of wood contains a number of thin soft-iron nails (of constant permeability). A unit magnetic field directed eastwards induces a magnetic moment in the block having components* $(3, 1, -2)$*, and similar fields directed northwards and vertically upwards induce moments* $(1, 3, -2)$ *and* $(-2, -2, 2)$ *respectively. Show that all the nails lie in parallel planes.*

The magnetic moment M, the permeability μ and the magnetic field H for iron of constant pemeability are connected by $M_i = \mu_{ij}H_j$. Taking the 1-, 2- and

3-directions as East, North and vertical, μ has the form

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 2 \end{pmatrix}.$$

By adding the first two columns to twice the third one, it can be seen that this matrix has zero determinant. The matrix therefore has at least one zero eigenvalue. (The same conclusion can be reached using the routine method for finding eigenvalues; they are 0, 2 and 6.)

Thus, a field parallel to the eigenvector corresponding to this zero eigenvalue will induce no moment in the block. Physically this means that all the nails lie in planes to which this direction is a normal. To find the direction we solve

$$\begin{pmatrix} 3 & 1 & -2 \\ 1 & 3 & -2 \\ -2 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}.$$

We conclude that all the nails lie at right angles to this direction.

26.21 *For a general isotropic medium, the stress tensor p_{ij} and strain tensors e_{ij} are related by*

$$p_{ij} = \frac{\sigma E}{(1+\sigma)(1-2\sigma)} e_{kk}\delta_{ij} + \frac{E}{1+\sigma} e_{ij},$$

where E is Young's modulus and σ is Poisson's ratio.

By considering an isotropic body subjected to a uniform hydrostatic pressure (no shearing stress), show that the bulk modulus k, defined by the ratio of the pressure to the fractional decrease in volume, is given by $k = E/[3(1-2\sigma)]$.

Consider a small rectangular parallelepiped, with one corner at the origin and the opposite one at (a_1, a_2, a_3), subjected to a uniform hydrostatic pressure. The isotropy of the pressure means that all forces are normal to the surfaces on which they act and that the stress and strain tensor components $p_{ij} = e_{ij} = 0$ for $i \neq j$. Furthermore, because of the symmetry of the situation, when $i \neq j$, not only is e_{ij} zero, but so are the individual $\partial u_i/\partial x_j$ that are its constituents.

In the current situation, $p_{11} = p_{22} = p_{33} = -p$ and so, writing $\sum_k e_{kk}$ as θ, we have, for each i ($i = 1, 2, 3$) with no summation over i implied, that

$$-p = p_{ii} = \frac{E}{(1+\sigma)(1-2\sigma)} [\sigma\theta + (1-2\sigma)e_{ii}].$$

Adding the three equations together gives

$$-3p = \frac{E}{(1+\sigma)(1-2\sigma)}\,[\,3\sigma\theta + (1-2\sigma)\theta\,] = \frac{E\theta}{1-2\sigma}.$$

Now the fractional increase f in the volume of the parallelepiped is given by

$$\frac{1}{a_1 a_2 a_3}\left(a_1 + \frac{\partial u_1}{\partial x_i}a_i + \cdots\right)\left(a_2 + \frac{\partial u_2}{\partial x_i}a_i + \cdots\right)\left(a_3 + \frac{\partial u_3}{\partial x_i}a_i + \cdots\right) - 1.$$

Since $\partial u_i/\partial x_j = 0$ for $i \neq j$, the only three non-zero first-order terms are

$$f = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{11} + e_{22} + e_{33} = \theta.$$

We conclude that the bulk modulus, k, is given by

$$k = \frac{p}{-f} = \frac{-E\theta}{3(1-2\sigma)}\,\frac{1}{(-\theta)} = \frac{E}{3(1-2\sigma)}.$$

26.23 *A fourth-order tensor T_{ijkl} has the properties*

$$T_{jikl} = -T_{ijkl}, \qquad T_{ijlk} = -T_{ijkl}.$$

Prove that for any such tensor there exists a second-order tensor K_{mn} such that

$$T_{ijkl} = \epsilon_{ijm}\epsilon_{kln}K_{mn}$$

and give an explicit expression for K_{mn}. Consider two (separate) special cases, as follows.

(a) *Given that T_{ijkl} is isotropic and $T_{ijji} = 1$, show that T_{ijkl} is uniquely determined and express it in terms of Kronecker deltas.*

(b) *If now T_{ijkl} has the additional property*

$$T_{klij} = -T_{ijkl},$$

show that T_{ijkl} has only three linearly independent components and find an expression for T_{ijkl} in terms of the vector

$$V_i = -\tfrac{1}{4}\epsilon_{jkl}T_{ijkl}.$$

As K_{mn} is to be a second-order tensor, we need to construct such a tensor from T_{ijkl}. Since the latter is of fourth order, it needs to be contracted n times with a tensor of order $2n-2$ for some positive integer n. In view of the final stated

expression for T_{ijkl}, involving $\epsilon_{ijm}\epsilon_{kln}$, i.e. a sixth-order tensor, we try $n = 4$ and, starting from $T_{ijkl} = \epsilon_{ijm}\epsilon_{kln}K_{mn}$, consider

$$\begin{aligned}\epsilon_{pij}\epsilon_{qkl}T_{ijkl} &= \epsilon_{pij}\epsilon_{qkl}\epsilon_{ijm}\epsilon_{kln}K_{mn}\\ &= (\delta_{jj}\delta_{pm} - \delta_{jm}\delta_{pj})(\delta_{ll}\delta_{qn} - \delta_{ln}\delta_{ql})K_{mn}\\ &= (3\delta_{pm} - \delta_{pm})(3\delta_{qn} - \delta_{qn})K_{mn}\\ &= 4K_{pq}.\end{aligned}$$

Clearly, $K_{mn} = \frac{1}{4}\epsilon_{mij}\epsilon_{nkl}T_{ijkl}$ has the required property.

(a) Given that T_{ijkl} is isotropic, and noting that ϵ_{mij} and ϵ_{nkl} are also isotropic, we conclude that K_{mn} must itself be isotropic. It must therefore be some multiple of δ_{mn} (as this is the most general isotropic second-order tensor), i.e. $K_{mn} = \lambda\delta_{mn}$ for one or more values of λ. Thus,

$$\begin{aligned}T_{ijkl} &= \epsilon_{ijm}\,\epsilon_{kln}\lambda\delta_{mn}\\ &= \lambda\epsilon_{ijm}\,\epsilon_{klm}\\ &= \lambda(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}).\end{aligned}$$

Now, since $T_{ijji} = 1$,

$$\begin{aligned}1 &= \lambda(\delta_{ij}\delta_{ji} - \delta_{ii}\delta_{jj})\\ &= \lambda[\,\delta_{ii} - (\delta_{ii})^2\,] = \lambda(3-9) \quad \Rightarrow \quad \lambda = -\tfrac{1}{6}.\end{aligned}$$

We conclude that λ, and therefore also T_{ijkl}, is unique with $T_{ijkl} = \frac{1}{6}(\delta_{il}\delta_{jk} - \delta_{ik}\delta_{jl})$.

(b) To examine the implications of the antisymmetry indicated by $T_{klij} = -T_{ijkl}$, we interchange the pair of dummy suffices $\{i, j\}$ with the pair $\{k, l\}$ to obtain the third line below — and then switch them back again in the fourth line using the antisymmetry:

$$\begin{aligned}K_{mn} &= \tfrac{1}{4}\epsilon_{mij}\,\epsilon_{nkl}\,T_{ijkl},\\ K_{nm} &= \tfrac{1}{4}\epsilon_{nij}\,\epsilon_{mkl}\,T_{ijkl}\\ &= \tfrac{1}{4}\epsilon_{nkl}\,\epsilon_{mij}\,T_{klij}\\ &= -\tfrac{1}{4}\epsilon_{nkl}\,\epsilon_{mij}\,T_{ijkl}\\ &= -K_{mn}.\end{aligned}$$

Thus K_{mn} is antisymmetric. It therefore has zeros on its leading diagonal and only three linearly independent components as non-diagonal elements. Since T_{ijkl} is uniquely defined in terms of K_{mn}, it too has only three linearly independent components.

Now consider

$$\begin{aligned}\epsilon_{jkl}T_{ijkl} &= \epsilon_{jkl}\,\epsilon_{ijm}\,\epsilon_{kln}K_{mn},\\ -4V_i &= (\delta_{km}\delta_{li} - \delta_{ki}\delta_{lm})\epsilon_{kln}K_{mn}\\ &= (\epsilon_{min} - \epsilon_{imn})K_{mn}\\ &= 2\epsilon_{min}K_{mn}.\end{aligned}$$

To 'invert' this relationship, consider

$$\begin{aligned}\epsilon_{irs}V_i &= -\tfrac{1}{2}\epsilon_{irs}\,\epsilon_{min}K_{mn}\\ &= -\tfrac{1}{2}(\delta_{rn}\delta_{sm} - \delta_{rm}\delta_{sn})K_{mn}\\ &= -\tfrac{1}{2}(K_{sr} - K_{rs})\\ &= K_{rs} \quad \Rightarrow \quad K_{mn} = \epsilon_{pmn}V_p.\end{aligned}$$

Finally, expressing T_{ijkl}, as given in the question, explicitly in terms of the vector V_i, using the result obtained above, we have

$$\begin{aligned}T_{ijkl} &= \epsilon_{ijm}\,\epsilon_{kln}\,\epsilon_{pmn}V_p\\ &= (\delta_{in}\delta_{jp} - \delta_{ip}\delta_{jn})\epsilon_{kln}V_p\\ &= \epsilon_{kli}V_j - \epsilon_{klj}V_i.\end{aligned}$$

26.25 *In a general coordinate system u^i, $i = 1, 2, 3$, in three-dimensional Euclidean space, a volume element is given by*

$$dV = |\mathbf{e}_1\,du^1 \cdot (\mathbf{e}_2\,du^2 \times \mathbf{e}_3\,du^3)|.$$

Show that an alternative form for this expression, written in terms of the determinant g of the metric tensor, is given by

$$dV = \sqrt{g}\,du^1\,du^2\,du^3.$$

Show that under a general coordinate transformation to a new coordinate system u'^i, the volume element dV remains unchanged, i.e. show that it is a scalar quantity.

Working in terms of the Cartesian bases vectors $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$, let

$$\mathbf{e}_m = \lambda_{mx}\,\mathbf{i} + \lambda_{my}\,\mathbf{j} + \lambda_{mz}\,\mathbf{k}, \text{ for } m = 1, 2, 3.$$

Then,

$$\begin{aligned}\mathbf{e}_2\,du^2 \times \mathbf{e}_3\,du^3 = du^2\,du^3(&\lambda_{2x}\lambda_{3y}\,\mathbf{k} - \lambda_{2x}\lambda_{3z}\,\mathbf{j} - \lambda_{2y}\lambda_{3x}\,\mathbf{k}\\ &+ \lambda_{2y}\lambda_{3z}\,\mathbf{i} + \lambda_{2z}\lambda_{3x}\,\mathbf{j} - \lambda_{2z}\lambda_{3y}\,\mathbf{i}),\end{aligned}$$

and it follows that

$$\begin{aligned}
dV &= \mathbf{e}_1\, du^1 \cdot (\mathbf{e}_2\, du^2 \times \mathbf{e}_3\, du^3) \\
&= du^1\, du^2\, du^3 [\, \lambda_{1x}(\lambda_{2y}\lambda_{3z} - \lambda_{2z}\lambda_{3y}) + \lambda_{1y}(\lambda_{2z}\lambda_{3x} - \lambda_{2x}\lambda_{3z}) \\
&\qquad\qquad + \lambda_{1z}(\lambda_{2x}\lambda_{3y} - \lambda_{2y}\lambda_{3x})\,] \\
&= du^1\, du^2\, du^3 \begin{vmatrix} \lambda_{1x} & \lambda_{1y} & \lambda_{1z} \\ \lambda_{2x} & \lambda_{2y} & \lambda_{2z} \\ \lambda_{3x} & \lambda_{3y} & \lambda_{3z} \end{vmatrix} \\
&\equiv du^1\, du^2\, du^3 |\mathsf{A}|, \text{ thus defining } \mathsf{A}.
\end{aligned}$$

Now consider an element of the matrix $\mathsf{A}\mathsf{A}^{\mathrm{T}}$:

$$(\mathsf{A}\mathsf{A}^{\mathrm{T}})_{mn} = \sum_r \mathsf{A}_{mr}\mathsf{A}_{nr} = \lambda_{mx}\lambda_{nx} + \lambda_{my}\lambda_{ny} + \lambda_{mz}\lambda_{nz}.$$

But the elements of the metric tensor are given by

$$\mathsf{g}_{mn} = \mathbf{e}_m \cdot \mathbf{e}_n = \lambda_{mx}\lambda_{nx} + \lambda_{my}\lambda_{ny} + \lambda_{mz}\lambda_{nz}.$$

Hence $\mathsf{A}\mathsf{A}^{\mathrm{T}} = \mathsf{g}$ and, in particular, $|\mathsf{A}|\,|\mathsf{A}^{\mathrm{T}}| = |\,\mathsf{g}\,|$. Since $|\mathsf{A}| = |\mathsf{A}^{\mathrm{T}}|$, it follows that $|\mathsf{A}| = |\,\mathsf{g}\,|^{1/2} = \sqrt{g}$ and

$$dV = du^1\, du^2\, du^3\, |\mathsf{A}| = \sqrt{g}\, du^1\, du^2\, du^3.$$

For a transformation $u'^i = u'^i(u^1, u^2, u^3)$,

$$du'^1\, du'^2\, du'^3 = \left| \frac{\partial u'}{\partial u} \right| du^1\, du^2\, du^3,$$

and the covariant components of the second-order tensor g_{ij} transform as

$$\begin{aligned}
g'_{ij} &= \frac{\partial u^k}{\partial u'^i} \frac{\partial u^l}{\partial u'^j}\, g_{kl}, \\
\Rightarrow \quad g' &= \left| \frac{\partial u}{\partial u'} \right| \left| \frac{\partial u}{\partial u'} \right| g \text{ (on taking determinants)}, \\
\Rightarrow \quad \sqrt{g'} &= \left| \frac{\partial u}{\partial u'} \right| \sqrt{g}.
\end{aligned}$$

Thus, the new volume element is

$$\begin{aligned}
dV' &= \sqrt{g'}\, du'^1\, du'^2\, du'^3 \\
&= \left| \frac{\partial u}{\partial u'} \right| \sqrt{g} \left| \frac{\partial u'}{\partial u} \right| du^1\, du^2\, du^3 \\
&= \sqrt{g}\, du^1\, du^2\, du^3 = dV.
\end{aligned}$$

This shows that dV is a scalar quantity.

26.27 *Find an expression for the second covariant derivative, written in semicolon notation as* $v_{i;jk} \equiv (v_{i;j})_{;k}$, *of a vector* v_i. *By interchanging the order of differentiation and then subtracting the two expressions, we define the components* R^l_{ijk} *of the* Riemann tensor *as*

$$v_{i;jk} - v_{i;kj} \equiv R^l_{ijk} v_l.$$

Show that in a general coordinate system u^i *these components are given by*

$$R^l_{ijk} = \frac{\partial \Gamma^l{}_{ik}}{\partial u^j} - \frac{\partial \Gamma^l{}_{ij}}{\partial u^k} + \Gamma^m{}_{ik}\Gamma^l{}_{mj} - \Gamma^m{}_{ij}\Gamma^l{}_{mk}.$$

By first considering Cartesian coordinates, show that all the components $R^l_{ijk} \equiv 0$ *for* any *coordinate system in three-dimensional Euclidean space.*
In such a space, therefore, we may change the order of the covariant derivatives without changing the resulting expression.

For the covariant derivative of the covariant components of a vector, we have

$$v_{i;j} = \frac{\partial v_i}{\partial u^j} - \Gamma^k_{ij} v_k,$$

where Γ^k_{ij} is a Christoffel symbol of the second kind. Hence,

$$\begin{aligned} v_{i;jk} &\equiv (v_{i;j})_{;k} \\ &= \left(\frac{\partial v_i}{\partial u^j} - \Gamma^l{}_{ij} v_l\right)_{;k} \\ &= \frac{\partial}{\partial u^k}\left(\frac{\partial v_i}{\partial u^j} - \Gamma^l{}_{ij} v_l\right) - \Gamma^m_{ik}\left(\frac{\partial v_m}{\partial u^j} - \Gamma^l{}_{mj} v_l\right) \\ &= \frac{\partial^2 v_i}{\partial u^k \partial u^j} - \Gamma^l{}_{ij}\frac{\partial v_l}{\partial u^k} - v_l \frac{\partial \Gamma^l{}_{ij}}{\partial u^k} - \Gamma^m_{ik}\frac{\partial v_m}{\partial u^j} + \Gamma^m_{ik}\Gamma^l{}_{mj} v_l. \end{aligned}$$

Interchanging subscripts j and k,

$$v_{i;kj} = \frac{\partial^2 v_i}{\partial u^j \partial u^k} - \Gamma^l_{ik}\frac{\partial v_l}{\partial u^j} - v_l \frac{\partial \Gamma^l_{ik}}{\partial u^j} - \Gamma^m_{ij}\frac{\partial v_m}{\partial u^k} + \Gamma^m_{ij}\Gamma^l{}_{mk} v_l.$$

When these two expressions are subtracted to define the Riemann tensor, the first, second and fourth terms (the second of one with the fourth of the other and vice versa) on the two RHSs cancel in pairs to yield

$$R^l_{ijk} v_l \equiv v_{i;jk} - v_{i;kj} = \left(\frac{\partial \Gamma^l_{ik}}{\partial u^j} - \frac{\partial \Gamma^l_{ij}}{\partial u^k} + \Gamma^m_{ik}\Gamma^l{}_{mj} - \Gamma^m_{ij}\Gamma^l{}_{mk}\right) v_l.$$

Now, in three-dimensional Euclidean space, one possible coordinate system is the Cartesian one. In this system $g = 1$ and all of its derivatives are zero. Thus all Christoffel symbols and their derivatives are zero, as are all components of

the Riemann tensor. As *all* the components vanish in this Cartesian coordinate system, they must do so in *any* coordinate system in this space.

26.29 *We may define Christoffel symbols of the first kind by*

$$\Gamma_{ijk} = g_{il}\Gamma^{l}{}_{jk}.$$

Show that these are given by

$$\Gamma_{ijk} = \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right).$$

By permuting indices, verify that

$$\frac{\partial g_{ij}}{\partial u^k} = \Gamma_{ijk} + \Gamma_{jik}.$$

Using the fact that $\Gamma^{l}{}_{jk} = \Gamma^{l}{}_{kj}$, *show that*

$$g_{ij;k} \equiv 0,$$

i.e. that the covariant derivative of the metric tensor is identically zero in all coordinate systems.

Starting from Christoffel symbols of the second kind, we have

$$\begin{aligned}
\Gamma_{ijk} &= g_{il}\Gamma^{l}{}_{jk} \\
&= \frac{1}{2}g_{il}g^{ln}\left(\frac{\partial g_{kn}}{\partial u^j} + \frac{\partial g_{nj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^n}\right) \\
&= \frac{1}{2}\delta_i^n\left(\frac{\partial g_{kn}}{\partial u^j} + \frac{\partial g_{nj}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^n}\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right).
\end{aligned}$$

Next, forming the symmetric sum of two Christoffel symbols:

$$\begin{aligned}
\Gamma_{ijk} + \Gamma_{jik} &= \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} + \frac{\partial g_{ij}}{\partial u^k} - \frac{\partial g_{jk}}{\partial u^i}\right) + \frac{1}{2}\left(\frac{\partial g_{kj}}{\partial u^i} + \frac{\partial g_{ji}}{\partial u^k} - \frac{\partial g_{ik}}{\partial u^j}\right) \\
&= \frac{1}{2}\left(\frac{\partial g_{ij}}{\partial u^k} + \frac{\partial g_{ji}}{\partial u^k}\right) + \frac{1}{2}\left(\frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ik}}{\partial u^j}\right) \\
&\quad + \frac{1}{2}\left(-\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{kj}}{\partial u^i}\right) \\
&= \frac{\partial g_{ij}}{\partial u^k} + 0 + 0.
\end{aligned}$$

To obtain the last line we have used the fact that the metric tensor is symmetric, $g_{ij} = g_{ji}$.

Further, since $\Gamma^l{}_{jk} = \Gamma^l{}_{kj}$, and therefore $g_{il}\Gamma^l{}_{jk} = g_{il}\Gamma^l{}_{kj}$, we have that $\Gamma_{ijk} = \Gamma_{ikj}$, i.e Christoffel symbols of the first kind are symmetric under the interchange of the last two indices.

Finally, forming the covariant derivative of g_{ij}:

$$\begin{aligned}
g_{ij;k} &= \frac{\partial}{\partial u^k}(g_{ij}\,\mathbf{e}^i \otimes \mathbf{e}^j) \\
&= \frac{\partial g_{ij}}{\partial u^k}\,\mathbf{e}^i \otimes \mathbf{e}^j + g_{ij}\,\frac{\partial \mathbf{e}^i}{\partial u^k} \otimes \mathbf{e}^j + g_{ij}\,\mathbf{e}^i \otimes \frac{\partial \mathbf{e}^j}{\partial u^k} \\
&= \frac{\partial g_{ij}}{\partial u^k}\,\mathbf{e}^i \otimes \mathbf{e}^j + g_{ij}\,(-\Gamma^i{}_{lk}\mathbf{e}^l) \otimes \mathbf{e}^j + g_{ij}\,\mathbf{e}^i \otimes (-\Gamma^j{}_{mk}\mathbf{e}^m) \\
&= \frac{\partial g_{ij}}{\partial u^k}\,\mathbf{e}^i \otimes \mathbf{e}^j - \Gamma_{jlk}\mathbf{e}^l \otimes \mathbf{e}^j - \Gamma_{imk}\mathbf{e}^i \otimes \mathbf{e}^m, \text{ since } g_{ij} = g_{ji}, \\
&= \left(\frac{\partial g_{ij}}{\partial u^k} - \Gamma_{jik} - \Gamma_{ijk}\right)\mathbf{e}^i \otimes \mathbf{e}^j, \text{ renaming dummy suffices,} \\
&= 0\,(\mathbf{e}^i \otimes \mathbf{e}^j), \text{ from the previous result.}
\end{aligned}$$

Thus the covariant derivative of the metric tensor is identically zero in all coordinate systems.

27

Numerical methods

27.1 *Use an iteration procedure to find the root of the equation* $40x = \exp x$ *to four significant figures.*

To provide a satisfactory iteration scheme, the equation must be rearranged in the form $x = f(x)$, where $f(x)$ is a slowly varying function of x; we then use $x_{n+1} = f(x_n)$ as the iteration scheme.

In the present case the rearrangement is straightforward, as, by taking logarithms, we can write the equation as $x = \ln 40x$. Since $\ln z$ is a slowly varying function of z, we can take $x_{n+1} = \ln 40x_n$ as the iteration scheme.

We start with the (poor) guess that $x = 1$. The successive values generated by the scheme are (to 5 s.f.)

$$1,\ 3.6889,\ 4.9942,\ 5.2972,\ 5.3560,\ 5.3671,\ 5.3691,\ 5.3696,\ 5.3696,\ \ldots .$$

Thus to 4 s.f. we give the answer as $x = 5.370$. In fact, after 15 iterations the calculated value is stable to 10 s.f. at 5.369640395.

27.3 *Show the following results about rearrangement schemes for polynomial equations.*

(a) *That if a polynomial equation* $g(x) \equiv x^m - f(x) = 0$, *where* $f(x)$ *is a polynomial of degree less than* m *and for which* $f(0) \neq 0$, *is solved using a rearrangement iteration scheme* $x_{n+1} = [f(x_n)]^{1/m}$, *then, in general, the scheme will have only first-order convergence.*

(b) *By considering the cubic equation*

$$x^3 - ax^2 + 2abx - (b^3 + ab^2) = 0$$

for arbitrary non-zero values of a *and* b, *demonstrate that, in special cases, the same rearrangement scheme can give second- (or higher-) order convergence.*

(a) If we represent the iteration scheme as $x_{n+1} = F(x_n)$ then the scheme will have only first-order convergence unless $F'(\xi) = 0$, where ξ is the solution to the original equation satisfying $\xi^m = f(\xi)$ or, equivalently, $\xi = F(\xi)$.

In this case $F(x) = [f(x)]^{1/m}$ and

$$F'(\xi) = \frac{1}{m}[f(\xi)]^{(1-m)/m} f'(\xi).$$

Since $f(0) \neq 0$, $x = 0$ cannot be one of the solutions ξ of the original equation. Now, $f(\xi) = \xi^m$ and so the first two factors in the expression for $F'(\xi)$ have the value $m^{-1}(\xi^m)^{(1-m)/m} = m^{-1}\xi^{1-m}$. This is neither zero nor infinite and so $F'(\xi)$ can only be zero if $f'(\xi) = 0$; in general this will not be the case and the convergence will be only of first order.

(b) For the given equation $m = 3$ and $f(x) = ax^2 - 2abx + (b^3 + ab^2)$. It follows that $f'(x) = 2ax - 2ab$ and that $f'(x) = 0$ when $x = b$. However, $x = b$, also satisfies the original equation

$$b^3 - ab^2 + 2ab^2 - b^3 - ab^2 = 0,$$

and therefore, in the terminology used in part (a), $\xi = b$ and $F'(\xi) = F'(b) = 0$. This shows that the convergence will be of second (or higher) order.

In fact, further differentiation shows that $F''(b) = 2a/3b^2$ and, as this is non-zero, the convergence is only of second order.

27.5 *Solve the following set of simultaneous equations using Gaussian elimination (including interchange where it is formally desirable):*

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 2x_4 &= 0, \\ 2x_1 + 10x_2 - 5x_3 + x_4 &= 6, \\ 4x_2 + 3x_3 + 3x_4 &= 20, \\ -3x_1 + 6x_2 + 12x_3 - 4x_4 &= 16. \end{aligned}$$

Since the largest (in magnitude) coefficient of x_1 appears in the final equation, we reorder them to make it first (labelled I) and divide through by -1 to make the coefficient of x_1 positive:

$$3x_1 - 6x_2 - 12x_3 + 4x_4 = -16. \qquad \text{(I)}$$

The first and second equations now have $\frac{1}{3}$ and $\frac{2}{3}$ (respectively) of (I) subtracted from them to eliminate x_1. The third equation does not contain x_1 and so is left unchanged:

$$\begin{aligned} 5x_2 + 8x_3 + \tfrac{2}{3}x_4 &= \tfrac{16}{3}, \qquad \text{(a)} \\ 14x_2 + 3x_3 - \tfrac{5}{3}x_4 &= \tfrac{50}{3}, \qquad \text{(b)} \\ 4x_2 + 3x_3 + 3x_4 &= 20. \qquad \text{(c)} \end{aligned}$$

Equation (b) is now the one with the largest coefficient of x_2, and so we take as the second finalised equation

$$14x_2 + 3x_3 - \tfrac{5}{3}x_4 = \tfrac{50}{3}, \qquad \text{(II)}$$

and subtract the needed fractions of this from (a) and (c) to eliminate x_2 from them:

$$\begin{aligned} \tfrac{97}{14}x_3 + (\tfrac{2}{3} + \tfrac{25}{42})x_4 &= \tfrac{16}{3} - \tfrac{250}{42}, \qquad \text{(d)} \\ (3 - \tfrac{12}{14})x_3 + (3 + \tfrac{20}{42})x_4 &= 20 - \tfrac{200}{42}. \qquad \text{(e)} \end{aligned}$$

Rationalising these two equations we have

$$\begin{aligned} 291x_3 + 53x_4 &= -26, \qquad \text{(d}'\text{)} \equiv \text{(III)} \\ 90x_3 + 146x_4 &= 640. \qquad \text{(e}'\text{)} \end{aligned}$$

Finally, eliminating x_3 from (e′) gives

$$\begin{aligned} (146 - \tfrac{90}{291}\,53)x_4 &= 640 - \tfrac{90}{291}\,(-26), \\ 37716x_4 &= 188580, \\ x_4 &= 5. \qquad \text{(IV)} \end{aligned}$$

Resubstitution then gives

$$\text{from (III)},\ x_3 = \tfrac{-26-(53\times 5)}{291} = -1,$$

$$\text{from (II)},\ x_2 = \tfrac{1}{14}(\tfrac{50}{3} + \tfrac{5\times 5}{3} - 3(-1)) = 2,$$

$$\text{from (I)},\ x_1 = \tfrac{1}{3}(-16 - (4\times 5) + 12(-1) + 6(2)) = -12,$$

making the solution $x_1 = -12$, $x_2 = 2$, $x_3 = -1$ and $x_4 = 5$.

27.7 *Simultaneous linear equations that result in tridiagonal matrices can sometimes be solved in the same way as three-term recurrence relations. Consider the tridiagonal simultaneous equations*

$$x_{i-1} + 4x_i + x_{i+1} = 3(\delta_{i+1,0} - \delta_{i-1,0}), \quad i = 0, \pm 1, \pm 2, \ldots .$$

Prove that for $i > 0$ the equations have a general solution of the form $x_i = \alpha p^i + \beta q^i$, where p and q are the roots of a certain quadratic equation. Show that a similar result holds for $i < 0$. In each case express x_0 in terms of the arbitrary constants $\alpha, \beta, \ldots$.

Now impose the condition that x_i is bounded as $i \to \pm\infty$ and obtain a unique solution.

We substitute the trial solution $x_i = \alpha p^i + \beta q^i$ into the given equation for $i \geq 2$ and obtain

$$\alpha(p^{i-1} + 4p^i + p^{i+1}) + \beta(q^{i-1} + 4q^i + q^{i+1}) = 3(0-0) = 0;$$

this is satisfied for arbitrary α and β if p and q are the two roots of the quadratic equation $1 + 4r + r^2 = 0$.

Using the same form for $i = 1$, but with these specific values for p and q, we have

$$\begin{aligned} x_0 + 4x_1 + x_2 &= 3(0-1), \\ x_0 + 4(\alpha p + \beta q) + \alpha p^2 + \beta q^2 &= -3, \\ x_0 + \alpha(4p + p^2) + \beta(4q + q^2) &= -3, \\ x_0 + \alpha(-1) + \beta(-1) &= -3. \end{aligned}$$

To obtain the final line we used the fact that both p and q satisfy $4r + r^2 = -1$. Similarly, for $i \leq -1$ the solution is $x_i = \alpha' p^i + \beta' q^i$, with $x_0 - \alpha' - \beta' = +3$.

In addition, for $i = 0$, we have from the original equation that

$$\frac{\alpha'}{p} + \frac{\beta'}{q} + 4x_0 + \alpha p + \beta q = 0.$$

The values of p and q are $-2 \pm \sqrt{4-1}$, with, say, $p = -2 + \sqrt{3}$ and $|p| < 1$, and $q = -2 - \sqrt{3}$ and $|q| > 1$.

Now, the solution is to be bounded as $i \to \pm\infty$. The fact that $|q| > 1$ and the condition at $+\infty$ together require that $\beta = 0$, whilst $|p|^{-1} > 1$ and the condition at $-\infty$ imply that $\alpha' = 0$. We are left with three equations for three unknowns:

$$\begin{aligned} x_0 - \alpha + 3 &= 0, \\ x_0 - \beta' - 3 &= 0, \\ \frac{\beta'}{-2-\sqrt{3}} + 4x_0 + \alpha(-2+\sqrt{3}) &= 0. \end{aligned}$$

We now rearrange the last of these and substitute from the first two:

$$\begin{aligned} \beta' + 4(-2-\sqrt{3})x_0 + \alpha &= 0, \\ \Rightarrow \quad (x_0 - 3) - (8 + 4\sqrt{3})x_0 + (x_0 + 3) &= 0, \end{aligned}$$

and $x_0 = 0$, $\alpha = 3$, $\beta' = -3$. The solution is thus

$$x_i = \begin{cases} 3(-2+\sqrt{3})^i & i \geq 1, \\ 0 & i = 0, \\ -3(-2-\sqrt{3})^i & i \leq -1. \end{cases}$$

The final entry could be written as $-3(-2+\sqrt{3})^{-i}$.

27.9 *Although it can easily be shown, by direct calculation, that*

$$\int_0^\infty e^{-x}\cos(kx)\,dx = \frac{1}{1+k^2},$$

the form of the integrand is also appropriate for a Gauss–Laguerre numerical integration. Using a 5-point formula, investigate the range of values of k for which the formula gives accurate results. At about what value of k do the results become inaccurate at the 1% level?

The integrand is an even function of k and so only positive k need be considered. The points and weights for the 5-point Gauss–Laguerre integration are

x_i	w_i
0.26356 03197	0.52175 56106
1.41340 30591	0.39866 68111
3.59642 57710	0.07594 24497
7.08581 00059	0.00361 17587
12.6408 00844	0.00002 33700

The table below gives the exact and calculated results to four places of decimals, as well as the percentage error in the calculated result. It shows that the error is not more than 1% for $|k|$ less than about 1.1.

k	Exact	Calculated	% error
0.0	1.0000	1.0000	0.0
0.5	0.8000	0.8000	0.0
0.8	0.6098	0.6097	0.0
1.0	0.5000	0.5005	0.1
1.1	0.4525	0.4545	0.4
1.2	0.4098	0.4145	1.1
1.3	0.3717	0.3800	2.2
1.5	0.3077	0.3200	4.0
1.7	0.2571	0.2535	−1.4
2.0	0.2000	0.1184	−40.8
3.0	0.1000	0.1674	67.4

27.11 *Consider the integrals I_p defined by*

$$I_p = \int_{-1}^{1} \frac{x^{2p}}{\sqrt{1-x^2}}\,dx.$$

(a) *By setting $x = \sin\theta$ and using the recurrence relation quoted below, show that I_p has the value*

$$I_p = 2\,\frac{2p-1}{2p}\,\frac{2p-3}{2p-2}\cdots\frac{1}{2}\,\frac{\pi}{2}.$$

Recurrence relation: *If $J(n)$ is defined for a non-negative integer n by*

$$J(n) = \int_0^{\pi/2} \sin^n\theta\,d\theta,$$

then, for $n > 2$,

$$J(n) = \frac{n-1}{n}J(n-2).$$

(b) *Evaluate I_p for $p = 1, 2, \ldots, 6$ using 5- and 6-point Gauss–Chebyshev integration (conveniently run on a spreadsheet such as* Excel*) and compare the results with those in* (a). *In particular, show that, as expected, the 5-point scheme first fails to be accurate when the order of the polynomial numerator $(2p)$ exceeds $(2 \times 5) - 1 = 9$. Likewise, verify that the 6-point scheme evaluates I_5 accurately but is in error for I_6.*

(a) Setting $x = \sin\theta$ with $dx = \cos\theta$ converts I_p to

$$I_p = \int_{-\pi/2}^{\pi/2} \frac{\sin^{2p}\theta}{\cos\theta}\cos\theta\, d\theta = 2\int_0^{\pi/2} \sin^{2p}\theta\, d\theta = 2J(2p),$$

using the given definition of $J(n)$. Applying the reduction formula then gives

$$I_p = 2\,\frac{2p-1}{2p}\,\frac{2p-3}{2p-2}\cdots\frac{1}{2}\,\frac{\pi}{2},$$

where we have used the obvious result $J(0) = \pi/2$.

(b) The points and weights needed for a Gauss–Chebyshev integration are given analytically by

$$x_i = \cos\frac{(i-\frac{1}{2})\pi}{n}, \qquad w_i = \frac{\pi}{n}, \qquad \text{for } i = 1, \ldots, n.$$

Here we have to take the cases $n = 5$ and $n = 6$. The following table gives the exact result calculated in (a) and the values obtained using the n-point Gauss–Chebyshev formula.

p	Exact	$n = 5$	$n = 6$
1	1.570796	1.570796	1.570796
2	1.178097	1.178097	1.178097
3	0.981748	0.981748	0.981748
4	0.859029	0.859029	0.859029
5	0.773126	0.766990	0.773126
6	0.708699	0.690291	0.707165

It will be seen that, as stated in the question, the $p = 5$, $n = 5$ and both the $p = 6$ values diverge from the exact result. The discrepancy is of the order of 1% when $p = n$, i.e. when the order of the polynomial in the numerator of I_p first exceeds $2n - 1$.

27.13 *Given a random number η uniformly distributed on $(0, 1)$, determine the function $\xi = \xi(\eta)$ that would generate a random number ξ distributed as*

(a) 2ξ *on* $0 \le \xi < 1$,
(b) $\frac{3}{2}\sqrt{\xi}$ *on* $0 \le \xi < 1$,
(c) $\dfrac{\pi}{4a}\cos\dfrac{\pi\xi}{2a}$ *on* $-a \le \xi < a$,
(d) $\frac{1}{2}\exp(-|\,\xi\,|)$ *on* $-\infty < \xi < \infty$.

For each required distribution $f(t)$ in the range (a, b) we need to determine the cumulative distribution function $F(y) = \int_a^y dt$ and then take $F(y)$ as uniformly distributed on $(0, 1)$. A correctly normalised distribution has $F(b) = 1$. For any given random number η, the corresponding variable, distributed as $f(\xi)$, is $\xi = F^{-1}(\eta)$.

(a) For $f(t) = 2t$,

$$F(y) = \int_0^y 2t\,dt = y^2 \quad \Rightarrow \quad \eta = \xi^2 \quad \Rightarrow \quad \xi = \sqrt{\eta}.$$

(b) For $f(t) = \frac{3}{2}\sqrt{t}$,

$$F(y) = \int_0^y \tfrac{3}{2}\sqrt{t}\,dt = y^{3/2} \quad \Rightarrow \quad \eta = \xi^{3/2} \quad \Rightarrow \quad \xi = \eta^{2/3}.$$

(c) For $f(t) = \dfrac{\pi}{4a}\cos\dfrac{\pi t}{2a}$,

$$F(y) = \frac{\pi}{4a}\int_{-a}^y \cos\left(\frac{\pi t}{2a}\right)\,dt = \frac{1}{2}\left[\sin\left(\frac{\pi y}{2a}\right) + 1\right],$$

$$\Rightarrow \quad \eta = \frac{1}{2}\left[\sin\left(\frac{\pi\xi}{2a}\right) + 1\right] \quad \Rightarrow \quad \xi = \frac{2a}{\pi}\sin^{-1}(2\eta - 1).$$

(d) For $f(t) = \frac{1}{2}\exp(-|\,t\,|)$,

for $y < 0$, $\quad F(y) = \displaystyle\int_{-\infty}^y \frac{e^t}{2}\,dt = \frac{e^y}{2}$,

for $y > 0$, $\quad F(y) = \displaystyle\int_{-\infty}^0 \frac{e^t}{2}\,dt + \int_0^y \frac{e^{-t}}{2}\,dt$

$$= \frac{1}{2} + \frac{1 - e^{-y}}{2} = \frac{1}{2}(2 - e^{-y}).$$

It follows that

$$\eta = \begin{cases} \frac{1}{2}e^{\xi} & \xi \le 0, \\ 1 - \frac{1}{2}e^{-\xi} & \xi > 0, \end{cases} \quad \text{and } \xi = \begin{cases} \ln 2\eta & \eta \le 0.5, \\ -\ln(2 - 2\eta) & 0.5 < \eta < 1. \end{cases}$$

27.15 *Use a Taylor series to solve the equation*

$$\frac{dy}{dx} + xy = 0, \qquad y(0) = 1,$$

evaluating $y(x)$ *for* $x = 0.0$ *to* 0.5 *in steps of* 0.1.

In order to construct the Taylor series we need to find the derivatives $y^{(n)} \equiv$

$d^{(n)}y/dx^n$ up to, say, $n = 6$ and evaluate them at $x = 0$. We will also need $y(0) = 1$. The derivatives are

$$\begin{aligned}
y' &= -xy \quad \Rightarrow \quad y^{(1)}(0) = 0,\\
y^{(2)} &= = -y - xy'(x) = -y + x^2 y \quad \Rightarrow \quad y^{(2)}(0) = -1,\\
y^{(3)} &= 2xy + (-1 + x^2)y'(x) = 3xy - x^3 y \quad \Rightarrow \quad y^{(3)}(0) = 0,\\
y^{(4)} &= 3y - 3x^2 y + (3x - x^3)y'(x)\\
&= 3y - 6x^2 y + x^4 y \quad \Rightarrow \quad y^{(4)}(0) = 3,\\
y^{(5)} &= -12xy + 4x^3 y + (3 - 6x^2 + x^4)y'(x)\\
&= -15xy + 10x^3 y - x^5 y \quad \Rightarrow \quad y^{(5)}(0) = 0,\\
y^{(6)} &= -15y + 30x^2 y - 5x^4 y + (-15x + 10x^3 - x^5)y'(x)\\
&= -15y + 45x^2 y - 15x^4 y + x^6 y \quad \Rightarrow \quad y^{(6)}(0) = -15.
\end{aligned}$$

Thus, the Taylor series for an expansion about $x = 0$ is given by

$$\begin{aligned}
y(x) &= 1 - \frac{x^2}{2!} + \frac{3x^4}{4!} - \frac{15x^6}{6!} + \mathrm{O}(x^8)\\
&= 1 - \frac{x^2}{2} + \frac{x^4}{8} - \frac{x^6}{48} + \mathrm{O}(x^8).
\end{aligned}$$

To four significant figures the values of $y(x)$ calculated using this Taylor series are $y(0.1) = 0.9950$, $y(0.2) = 0.9802$, $y(0.3) = 0.9560$, $y(0.4) = 0.9231$ and $y(0.5) = 0.8825$.

For interest, we note that the exact solution of the differential equation, which is separable, is given by

$$\begin{aligned}
\frac{dy}{y} = -x\,dx \quad &\Rightarrow \quad \ln y = -\frac{x^2}{2} + c\\
y(0) = 1 \quad &\Rightarrow \quad c = 0 \quad \Rightarrow \quad y(x) = e^{-x^2/2},
\end{aligned}$$

which has the Taylor series

$$y(x) = 1 - \frac{x^2}{2^1\,1!} + \frac{x^4}{2^2\,2!} - \frac{x^6}{2^3\,3!} + \cdots.$$

As expected, this is the same as that found directly from the differential equation, up to the last term calculated; clearly the next term is $O(x^8)$.

To four significant figures the exact solution and the Taylor expansion give the same values over the given range of x; for $x = 0.6$ they differ by 1 in the fourth decimal place.

27.17 *A more refined form of the Adams predictor–corrector method for solving the first-order differential equation*

$$\frac{dy}{dx} = f(x, y)$$

is known as the Adams–Moulton–Bashforth scheme. At any stage (say the nth) in an Nth-order scheme, the values of x and y at the previous N solution points are first used to predict the value of y_{n+1}. *This approximate value of y at the next solution point,* x_{n+1}, *denoted by* $\bar{y}_{n+1}$, *is then used together with those at the previous* $N-1$ *solution points to make a more refined (corrected) estimation of* $y(x_{n+1})$. *The calculational procedure for a third-order scheme is summarised by the following two equations:*

$$\overline{y_{n+1}} = y_n + h(a_1 f_n + a_2 f_{n-1} + a_3 f_{n-2}) \qquad \textit{(predictor)},$$
$$y_{n+1} = y_n + h(b_1 f(x_{n+1}, \overline{y_{n+1}}) + b_2 f_n + b_3 f_{n-1}) \qquad \textit{(corrector)}.$$

(a) *Find Taylor series expansions for* f_{n-1} *and* f_{n-2} *in terms of the function* $f_n = f(x_n, y_n)$ *and its derivatives at* x_n.

(b) *Substitute them into the predictor equation and, by making that expression for* $\bar{y}_{n+1}$ *coincide with the true Taylor series for* y_{n+1} *up to order* h^3, *establish simultaneous equations that determine the values of* a_1, a_2 *and* a_3.

(c) *Find the Taylor series for* f_{n+1} *and substitute it and that for* f_{n-1} *into the corrector equation. Make the corrected prediction for* y_{n+1} *coincide with the true Taylor series by choosing the weights* b_1, b_2 *and* b_3 *appropriately.*

(d) *The values of the numerical solution of the differential equation*

$$\frac{dy}{dx} = \frac{2(1+x)y + x^{3/2}}{2x(1+x)}$$

at three values of x are given in the following table.

x	0.1	0.2	0.3
$y(x)$	0.030628	0.084107	0.150328

Use the above predictor–corrector scheme to find the value of $y(0.4)$ *and compare your answer with the accurate value,* 0.225577.

(a) 'Taylor series' expansions, using increments in x of $-h$ and $-2h$, give

$$f_{n-1} = f_n - hf'_n + \tfrac{1}{2}h^2 f''_n - \tfrac{1}{6}h^3 f^{(3)}_n + \cdots,$$
$$f_{n-2} = f_n - 2hf'_n + \tfrac{4}{2}h^2 f''_n - \tfrac{8}{6}h^3 f^{(3)}_n + \cdots.$$

These expansions are not true Taylor series as the only derivatives used are those with respect to x; however, the same is true of all subsequent expansions.

(b) Substitution in the predictor equation gives

$$\begin{aligned}\overline{y_{n+1}} &= y_n + h(a_1 f_n + a_2 f_{n-1} + a_3 f_{n-2}) \\ &= y_n + h[(a_1 + a_2 + a_3) f_n + h(-a_2 - 2a_3) f_n' \\ &\qquad + h^2(\tfrac{1}{2}a_2 + 2a_3) f_n'' + \cdots].\end{aligned}$$

Now, the accurate Taylor series for y_{n+1} is

$$y_{n+1} = y_n + h f_n + \tfrac{1}{2}h^2 f_n' + \tfrac{1}{6}h^3 f_n'' + \cdots .$$

To make these two expressions coincide up to order h^3, we need

$$\left.\begin{aligned} a_1 + a_2 + a_3 &= 1 \\ -a_2 - 2a_3 &= \tfrac{1}{2} \\ \tfrac{1}{2}a_2 + 2a_3 &= \tfrac{1}{6} \end{aligned}\right\} a_2 = -\tfrac{4}{3},\ a_3 = \tfrac{5}{12},\ a_1 = \tfrac{23}{12}.$$

(c) In the same way as in part (a),

$$f_{n+1} = f_n + h f_n' + \tfrac{1}{2}h^2 f_n'' + \tfrac{1}{6}h^3 f_n^{(3)} + \cdots ,$$

and substitution in the corrector equation gives

$$\begin{aligned} y_{n+1} &= y_n + h(b_1 f(x_{n+1}, \overline{y_{n+1}}) + b_2 f_n + b_3 f_{n-1}) \\ &= y_n + h[b_1 f(x_{n+1}, y_{n+1}) + b_2 f_n + b_3 f_{n-1}], \text{ to order } h^3 \\ &\equiv y_n + h(b_1 f_{n+1} + b_2 f_n + b_3 f_{n-1}), \text{ to order } h^3 \\ &= y_n + h[(b_1 + b_2 + b_3) f_n + h(b_1 - b_3) f_n' \\ &\qquad + h^2(\tfrac{1}{2}b_1 + \tfrac{1}{2}b_3) f_n'' + \cdots]. \end{aligned}$$

To make this coincide with the accurate Taylor series up to order h^3, we need

$$\left.\begin{aligned} b_1 + b_2 + b_3 &= 1 \\ b_1 - b_3 &= \tfrac{1}{2} \\ \tfrac{1}{2}b_1 + \tfrac{1}{2}b_3 &= \tfrac{1}{6} \end{aligned}\right\} b_1 = \tfrac{5}{12},\ b_3 = -\tfrac{1}{12},\ b_2 = \tfrac{2}{3}.$$

(d) We repeat the given table, indexing it and adding a line giving the values of $f(x, y)$.

n	1	2	3
x_n	0.1	0.2	0.3
$y_n(x_n)$	0.030628	0.084107	0.150328
$f_n(x_n, y_n)$	0.450020	0.606874	0.711756

Now, taking $n = 3$, we apply the predictor formula with the calculated values for the a_i and find $\overline{y_4} = 0.224582$. This allows us to calculate $f(x_4, \overline{y_4})$ as 0.787332. Finally, applying the corrector formula, using the calculated values for the b_i, we

find the corrected value $y_4 = 0.225527$. This is to be compared with the accurate value of 0.225577 (and the predicted, but uncorrected, value of 0.224582).

27.19 *To solve the ordinary differential equation*

$$\frac{du}{dt} = f(u, t)$$

for $f = f(t)$, the explicit two-step finite difference scheme

$$u_{n+1} = \alpha u_n + \beta u_{n-1} + h(\mu f_n + \nu f_{n-1})$$

may be used. Here, in the usual notation, h is the time step, $t_n = nh$, $u_n = u(t_n)$ and $f_n = f(u_n, t_n)$; α, β, μ, and ν are constants.

(a) *A particular scheme has $\alpha = 1$, $\beta = 0, \mu = 3/2$ and $\nu = -1/2$. By considering Taylor expansions about $t = t_n$ for both u_{n+j} and f_{n+j}, show that this scheme gives errors of order h^3.*

(b) *Find the values of α, β, μ and ν that will give the greatest accuracy.*

We will need the Taylor expansions of $u_{n\pm1}$ and f_{n-1}. They are given by

$$u_{n\pm1} = u_n \pm hu'_n + \frac{1}{2!}h^2u''_n \pm \frac{1}{3!}h^3u^{(3)}_n + \cdots,$$

$$f_{n-1} = u'_{n-1} = u'_n - hu''_n + \frac{1}{2!}h^2u^{(3)}_n - \frac{1}{3!}h^3u^{(4)}_n + \cdots.$$

(a) This scheme calculates u_{n+1} as

$$\begin{aligned} u_{n+1} &= u_n + h\left(\frac{3}{2}f_n - \frac{1}{2}f_{n-1}\right) \\ &= u_n + h\left[\frac{3}{2}u'_n - \frac{1}{2}\left(u'_n - hu''_n + \frac{1}{2!}h^2u^{(3)}_n - \frac{1}{3!}h^3u^{(4)}_n + \cdots\right)\right]. \end{aligned}$$

This is to be compared with

$$u_{n+1} = u_n + hu'_n + \frac{1}{2!}h^2u''_n + \frac{1}{3!}h^3u^{(3)}_n + \cdots.$$

Omitting terms that appear in both expressions, we have

$$\frac{1}{3!}h^3u^{(3)}_n + \cdots \approx -\frac{1}{4}h^3u^{(3)}_n + \cdots,$$

showing that the error is

$$\left(\frac{1}{3!} + \frac{1}{4}\right)h^3u^{(3)}_n = \frac{5}{12}h^3u^{(3)}_n + \mathrm{O}(h^4).$$

(b) For the best accuracy we require that

$$u_{n+1} = u_n + hu'_n + \frac{1}{2!}h^2u''_n + \frac{1}{3!}h^3u^{(3)}_n + \cdots$$

and

$$\alpha u_n + \beta\left(u_n - hu'_n + \frac{1}{2!}h^2u''_n - \frac{1}{3!}h^3u^{(3)}_n + \cdots\right) + h\mu u'_n$$
$$+ h\nu\left(u'_n - hu''_n + \frac{1}{2!}h^2u^{(3)}_n - \frac{1}{3!}h^3u^{(4)}_n + \cdots\right)$$

should match up to as high a positive power of h as possible.

With four parameters available, we can expect to match terms in h^n up to $n = 3$:

$$\begin{aligned} h^0 &: \ 1 = \alpha + \beta, \\ h^1 &: \ 1 = -\beta + \mu + \nu, \\ h^2 &: \ \tfrac{1}{2} = \tfrac{1}{2}\beta - \nu, \\ h^3 &: \ \tfrac{1}{6} = -\tfrac{1}{6}\beta + \tfrac{1}{2}\nu. \end{aligned}$$

The final two equations are equivalent to $\beta = 1 + 2\nu$ and $1 + \beta = 3\nu$, yielding $\nu = 2$ and $\beta = 5$; it then follows that $\mu = 4$ and $\alpha = -4$. With this set of values, the finite difference scheme,

$$u_{n+1} = -4u_n + 5u_{n-1} + h(4f_n + 2f_{n-1}),$$

has errors of order h^4.

27.21 *Write a computer program that would solve, for a range of values of λ, the differential equation*

$$\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + \lambda y^2}}, \qquad y(0) = 1,$$

using a third-order Runge–Kutta scheme. Consider the difficulties that might arise when $\lambda < 0$.

The relevant equations for a third-order Runge–Kutta scheme are

$$y_{i+1} = y_i + \tfrac{1}{6}(b_1 + 4b_2 + b_3),$$

where

$$\begin{aligned} b_1 &= hf(x_i, y_i), \\ b_2 &= hf(x_i + \tfrac{1}{2}h,\ y_i + \tfrac{1}{2}b_1), \\ b_3 &= hf(x_i + h,\ y_i + 2b_2 - b_1). \end{aligned}$$

The function $f(x, y)$, in this case, is $(x^2 + \lambda y^2)^{-1/2}$.

This calculation can be set up easily on a spreadsheet such as *Excel*, and it is immediately apparent that, with the given boundary value $y(0) = 1$, no significant finesse is needed. For positive values of λ the solution y is a monotonically (and boringly!) increasing function of x with values lying between 1 and ∞, the latter being approached rapidly only when λ is very small. Even with λ as small as 0.01, a step size Δx of 0.1 is adequate unless great precision is needed.

The difficulties that might arise for $\lambda < 0$ do not need much consideration; there is no real solution for *any* negative value of λ. The reason for this is easy to see. At the initial point, $x = 0$, $y = 1$ and λy^2 is negative and so the square root does not yield a real value for the derivative dy/dx.

More interesting results arise if the initial value is given elsewhere than at $x = 0$. For example, if $f(1) = 1$ then a solution can be calculated for negative values of λ greater than about -0.582 and if $f(1) = 2$ then a solution exists for $\lambda > -0.2057$.

27.23 *For some problems, numerical or algebraic experimentation may suggest the form of the complete solution. Consider the problem of numerically integrating the first-order wave equation*

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0,$$

in which A is a positive constant. A finite difference scheme for this partial differential equation is

$$\frac{u(p, n+1) - u(p, n)}{\Delta t} + A\frac{u(p, n) - u(p-1, n)}{\Delta x} = 0,$$

where $x = p\Delta x$ and $t = n\Delta t$, with p any integer and n a non-negative integer. The initial values are $u(0, 0) = 1$ and $u(p, 0) = 0$ for $p \neq 0$.

(a) *Carry the difference equation forward in time for two or three steps and attempt to identify the pattern of solution. Establish the criterion for the method to be numerically stable.*

(b) *Suggest a general form for $u(p, n)$, expressing it in generator function form, i.e. as '$u(p, n)$ is the coefficient of s^p in the expansion of $G(n, s)$'.*

(c) *Using your form of solution (or that given in the answers!), obtain an explicit general expression for $u(p, n)$ and verify it by direct substitution into the difference equation.*

(d) *An analytic solution of the original PDE indicates that an initial disturbance propagates undistorted. Under what circumstances would the difference scheme reproduce that behaviour?*

If we write $A\Delta t/\Delta x$ as c, the equation becomes

$$u(p,n+1) - u(p,n) + c[\,u(p,n) - u(p-1,n)\,] = 0,$$

with $u(0,0) = 1$ and $u(p,0) = 0$ for $p \neq 0$.

(a) For calculational purposes we rearrange the equation and then substitute trial values:

$$\begin{aligned}
u(p,n+1) &= (1-c)u(p,n) + cu(p-1,n), \qquad (*)\\
u(0,1) &= (1-c)u(0,0) + cu(-1,0) = 1-c,\\
u(1,1) &= (1-c)u(1,0) + cu(0,0) = c,\\
u(m,1) &= (1-c)u(m,0) + cu(m-1,0) = 0 \text{ for } m > 1,\\
u(0,2) &= (1-c)u(0,1) + cu(-1,1) = (1-c)^2,\\
u(1,2) &= (1-c)u(1,1) + cu(0,1) = 2c(1-c),\\
u(2,2) &= (1-c)u(2,1) + cu(1,1) = c^2,\\
u(m,2) &= (1-c)u(m,1) + cu(m-1,1) = 0 \text{ for } m > 2.
\end{aligned}$$

By now the pattern is clear, as is the condition for numerical stability, namely $c < 1$.

(b) For the nth time-step, the $n+1$ values of $u(p,n)$, $p = 0,1,\ldots,n$ appear to be given by the terms in the binomial expansion of $[\,(1-c) + cs\,]^n$. Using the language of generating functions, we would say that '$u(p,n)$ is the coefficient of s^p in the expansion of $[\,(1-c) + cs\,]^n$'.

(c) If this conjecture is correct, then

$$u(p,n) = \frac{n!\,(1-c)^{n-p}\,c^p}{p\,!\,(n-p)!}.$$

Substituting this form into the difference equation $(*)$ yields

$$\frac{(n+1)!\,(1-c)^{n+1-p}\,c^p}{p\,!\,(n+1-p)!} = \frac{(1-c)\,n!\,(1-c)^{n-p}\,c^p}{p\,!\,(n-p)!} + \frac{c\,n!\,(1-c)^{n+1-p}\,c^{p-1}}{(p-1)!\,(n+1-p)!}.$$

Multiplying through by $p\,!\,(n+1-p)!$ and dividing by $n!\,(1-c)^{n+1-p}\,c^p$ gives

$$(n+1) = (n-p+1) + p.$$

This is satisfied for all n and p, showing that the proposed solution satisfies the equation. It also gives $u(0,0) = 1$, confirming that it is the required solution.

(d) For the special case $c = 1$, the recurrence relation reduces to

$$u(p,n+1) = u(p-1,n),$$

i.e. the disturbance u at the point $p\Delta x$ at time $(n+1)\Delta t$ is exactly the same as that at position $(p-1)\Delta x$ one time-step earlier. In other words, the disturbance propagates undistorted at speed A.

From the point of view of the numerical integration, this situation ($c = 1$ exactly) is both on the edge of instability and unlikely to be realised in practice.

27.25 *Laplace's equation,*

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0,$$

is to be solved for the region and boundary conditions shown in figure 27.1.

Starting from the given initial guess for the potential values V and using the simplest possible form of relaxation, obtain a better approximation to the actual solution. Do not aim to be more accurate than ± 0.5 units and so terminate the process when subsequent changes would be no greater than this.

We start by imposing a coordinate grid symmetrically on the region, so that the initial guess is

$$V(0,1) = V(\pm 1, 1) = 20,$$
$$V(i,2) = 40 \text{ for all } i,$$

and the fixed boundary conditions are

$$V(i,0) = 0 \text{ for } |i| < 2,$$
$$V(i,1) = 0 \text{ for all } |i| \geq 2,$$
$$V(i,3) = 80 \text{ for all } i.$$

On symmetry grounds, we need consider only non-negative values of i.

We now apply the simplest relaxation scheme,

$$V_{i,j} \to \tfrac{1}{4}(V_{i+1,j} + V_{i-1,j} + V_{i,j+1} + V_{i,j-1}),$$

Figure 27.1 Region, boundary values and initial guessed solution values.

Figure 27.2 The solution to exercise 27.25.

for each point (i, j) that does not lie on the boundaries, where V_{ij} is prescribed and cannot be changed. The very simplest scheme would use only values from the previous iteration, but there is no additional labour involved in using previously calculated values from the current iteration when evaluating the RHS of the relationship. For this scheme the first few iterations produce the following results (to 3 s.f.):

$V_{0,1}$	$V_{1,1}$	$V_{0,2}$	$V_{1,2}$	$V_{2,2}$	$V_{3,2}$
20.0	20.0	40.0	40.0	40.0	40.0
20.0	15.0	45.0	45.0	41.3	40.3
18.8	15.9	47.2	46.1	41.6	40.4
19.8	16.5	48.0	46.5	41.7	40.4
20.2	16.7	48.3	46.7	41.8	40.4
20.4	16.8	48.4	46.8	41.8	40.4

The value at $(0, 1)$ is the one most likely to show the largest change at each iteration, as it is the one 'furthest from the fixed boundaries'. As the most recent changes have been 0.4 and 0.2, the process can be halted at this point, although the monotonic behaviour of values after the second iteration makes it harder to be sure that the differences between the final values and the current ones are within any given range.

The correct self-consistent solution (again to 3 s.f.) has corresponding values 20.6, 16.8, 48.5, 46.8, 41.8 and 40.5. This set of values is reached after nine iterations and is shown in figure 27.2. If the values from the previous iteration (rather than the most recently calculated ones) are used, the same ultimate result is reached (as expected), but about 17 iterations are needed to achieve the same self-consistency.

27.27 *The Schrödinger equation for a quantum mechanical particle of mass m moving in a one-dimensional harmonic oscillator potential $V(x) = kx^2/2$ is*

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{kx^2\psi}{2} = E\psi.$$

For physically acceptable solutions the wavefunction $\psi(x)$ must be finite at $x = 0$, tend to zero as $x \to \pm\infty$ and be normalised, so that $\int |\psi|^2\,dx = 1$. In practice, these constraints mean that only certain (quantised) values of E, the energy of the particle, are allowed. The allowed values fall into two groups, those for which $\psi(0) = 0$ and those for which $\psi(0) \neq 0$.

Show that if the unit of length is taken as $[\hbar^2/(mk)]^{1/4}$ and the unit of energy as $\hbar(k/m)^{1/2}$ then the Schrödinger equation takes the form

$$\frac{d^2\psi}{dy^2} + (2E' - y^2)\psi = 0.$$

Devise an outline computerised scheme, using Runge–Kutta integration, that will enable you to:

- *determine the three lowest allowed values of E;*
- *tabulate the normalised wavefunction corresponding to the lowest allowed energy.*

You should consider explicitly:

- *the variables to use in the numerical integration;*
- *how starting values near $y = 0$ are to be chosen;*
- *how the condition on ψ as $y \to \pm\infty$ is to be implemented;*
- *how the required values of E are to be extracted from the results of the integration;*
- *how the normalisation is to be carried out.*

We start by setting $x = \alpha y$, where α is the new unit of length; then $d/dx = \alpha^{-1}d/dy$ and

$$-\frac{\hbar^2}{2m}\alpha^{-2}\frac{d^2\psi}{dy^2} + \frac{k\alpha^2 y^2}{2} = E\psi,$$

$$\frac{d^2\psi}{dy^2} - \alpha^4\frac{mk}{\hbar^2}y^2\psi + \alpha^2\frac{2mE}{\hbar^2}\psi = 0.$$

Although, strictly, it should be given a new symbol, we continue to denote the required solution by ψ, now taken as a function of y rather than of x.

Now if α is chosen as $(\hbar^2/mk)^{1/4}$ and E is written as $E = \beta E'$, where $\beta = \hbar(k/m)^{1/2}$,

this equation becomes

$$\frac{d^2\psi}{dy^2} + (2E' - y^2)\psi = 0.$$

We note that this is a Sturm–Liouville equation with $p = 1$, $q = -y^2$, unit weight function and eigenvalue $2E'$; we therefore expect its solutions for different values of E' to be orthogonal.

To keep the notation the same as that normally used when describing numerical integration, we rewrite the equation as

$$\frac{d^2y}{dx^2} + (\lambda - x^2)y = 0. \qquad (*)$$

So that this second-order equation can be handled using an R–K routine, it has to be written as two first-order equations using an auxiliary variable. We make the simplest choice of $z \equiv dy/dx$, thus making a (two-component) 'vector' of dependent variables $(y,\ z)^{\mathrm{T}}$ with governing equations

$$\frac{dy}{dx} = z \qquad \text{and} \qquad \frac{dz}{dx} = (x^2 - \lambda)y.$$

The computer program will need to contain a subroutine that, given an input vector $(x,\ y,\ z)^{\mathrm{T}}$, returns an output vector $(dy/dx,\ dz/dx)^{\mathrm{T}}$ calculated as $(z,\ (x^2 - \lambda)y)^{\mathrm{T}}$. This is used to calculate the function $f(x_i,\ u_i)$ that appears on the (four) RHSs of, say, a fourth-order RK routine:

$$u_{i+1} = u_i + \tfrac{1}{6}(c_1 + 2c_2 + 2c_3 + c_4),$$

where

$$\begin{aligned} c_1 &= hf(x_i, u_i), \\ c_2 &= hf(x_i + \tfrac{1}{2}h,\ u_i + \tfrac{1}{2}c_1), \\ c_3 &= hf(x_i + \tfrac{1}{2}h,\ u_i + \tfrac{1}{2}c_2), \\ c_4 &= hf(x_i + h,\ u_i + c_3). \end{aligned}$$

Here, at each stage in the calculation of a one-step advance, u_i stands for y_i and z_i in turn.

Since equation $(*)$ is unchanged under the substitution $x \to -x$, and the boundary conditions, $y \to 0$ at $\pm\infty$, can be considered as both symmetric and antisymmetric, we can expect to find solutions that are either purely symmetric or purely antisymmetric. Consequently, we need only consider positive values of x, starting with $y(0) = 1$ for symmetric solutions and $y(0) = 0$ for antisymmetric ones. What will distinguish one potential solution from another is the value assigned to the initial slope $z(0)$. Clearly, one combination to be avoided is $y(0) = z(0) = 0$; such a computation will 'never get off the ground'. Intuition suggests that the initial slope should be zero if $y(0) = 1$ and non-zero if $y(0) = 0$.

As the formal range of x is infinite, we need to investigate the likely behaviour of a computed solution for large x; we want it to tend to zero for acceptable solutions. For large x equation (*) approximates to

$$\frac{d^2y}{dx^2} = x^2 y,$$

and if we substitute a trial function $y = e^{\gamma x^n}$ with $n > 0$ we find that

$$\frac{d^2y}{dx^2} = n(n-1)\gamma x^{n-2} e^{\gamma x^n} + n^2\gamma^2 x^{2n-2} e^{\gamma x^n}.$$

The dominant term in this expression is the second one; if this is to match $x^2 y$ then $n = 2$ and $\gamma = \pm\frac{1}{2}$. The case $\gamma = -\frac{1}{2}$ is clearly the one that is required, but, inevitably, even if the appropriate eigenvalue could be hit upon exactly, rounding errors are bound to introduce some of the $\gamma = +\frac{1}{2}$ solution. Thus we could never expect a computed solution actually to tend to zero and remain close to it however many steps are taken.

A more practical way to implement the boundary condition is to require y (and hence necessarily z) to remain within some specified narrow (but empirical) band about zero over, say, the interval $5 < x < 6$ — chosen because $5^n \exp(-5^2/2)$ is less than $\sim 10^{-3}$ for any moderate value of n and we cannot hope to achieve better accuracy than one part in a thousand without using more sophisticated techniques. Thus, in practice, the integration has to be over a finite range.

A crude technique is therefore to run the integration routine from $x = 0$ up to $x = 5$ for a mesh of values for λ (≥ 0) and $z(0)$ (in the ranges discussed above) and so evaluate the solution $v(\lambda, z(0)) = y(5)$. If all v have the same sign and vary smoothly with λ and $z(0)$, then a larger range of λ is indicated. However, if the v have mixed signs, interpolated values of λ and $z(0)$ should be tried, aiming to produce $v(\lambda, z(0)) \approx 0$. When this has been achieved, the test in the previous paragraph should be implemented to give further refinement. A graphical screen display of the calculated solution would be a considerable advantage in following what is happening.

Once a value of λ that results in a solution that approaches and stays near zero over the test range has been found, the corresponding values of y need to be divided by the square root of the value of the integral $\int_{-\infty}^{\infty} y^2\, dx$, so as to normalise the solution; they can then be tabulated. The integral can be evaluated well enough using the trapezium or Simpson's rule formulae over the finite range $0 < x < 5$ and doubling the result.

In order to be reasonably certain of finding the three lowest allowed values of E, the search should start from λ (i.e. $2E$) equal to zero and incremented in amounts $\Delta\lambda$ less than, but not negligible compared with, the average values of x^2 to be covered. The latter are of order unity, and so $\Delta\lambda = 0.1$ is reasonable. The step

length h in the x-variable might be chosen in the range 0.01 to 0.1 with the smaller values used when near a potential solution in the $(\lambda, z(0))$ grid.

[As has been indicated in several exercises in previous chapters, the actual eigenvalues λ are 1, 3, 5, $\ldots$, $2n+1$, $\ldots$ and the corresponding solutions are $\exp(-x^2/2)$ multiplied by a Hermite polynomial $H_n(x)$.]

28

Group theory

28.1 *For each of the following sets, determine whether they form a group under the operation indicated (where it is relevant you may assume that matrix multiplication is associative):*

(a) *the integers (mod* 10*) under addition;*
(b) *the integers (mod* 10*) under multiplication;*
(c) *the integers* $1, 2, 3, 4, 5, 6$ *under multiplication (mod* 7*);*
(d) *the integers* $1, 2, 3, 4, 5$ *under multiplication (mod* 6*);*
(e) *all matrices of the form*

$$\begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix},$$

where a and b are integers (mod 5*) and* $a \neq 0 \neq b$*, under matrix multiplication;*
(f) *those elements of the set in* (e) *that are of order* 1 *or* 2 *(taken together);*
(g) *all matrices of the form*

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix},$$

where a, b, c are integers, under matrix multiplication.

In all cases we need to establish whether the prescribed combination law is associative and whether, under it, the set possesses the properties of (i) closure, (ii) having an identity element and (iii) containing an inverse for every element present. If any one of these conditions fails, the set cannot form a group under the given law.

(a) Addition is associative and the set $\{0, 1, 2, \ldots, 9\}$ is closed under addition (mod 10), e.g. $7 + 6 = 3$. The identity is 0 and every element has an inverse, e.g. $(7)^{-1} = 3$. The set does form a group.

(b) For the set $\{0, 1, 2, \ldots, 9\}$ under multiplication the identity can only be 1. However, for any element $X \neq 1$ the set does *not* contain an inverse Y such that $XY = 1$. As a specific example, if $X = 2$ then 0.5 would need to be in the set – but it is not. The set does *not* form a group under multiplication.

(c) Multiplication is associative and the group table would be as below. The entries are calculated by expressing each product modulo 7. For example, $4 \times 5 = 20 = (2 \times 7) + 6 = 6 \pmod 7$

	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

This demonstrates (i) closure, (ii) the existence of an identity element (1) and (iii) an inverse for each element (1 appears in every row). The set does form a group.

(d) The set is not closed under multiplication (mod 6) and *cannot* form a group. For example, $2 \times 3 = 0 \pmod 6$ and 0 is not in the given set.

(e) With the associativity of matrix multiplication assumed and $a = b = 1$ yielding a unit element, consider

$$\begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix} \begin{pmatrix} c & c-d \\ 0 & d \end{pmatrix} = \begin{pmatrix} ac & ac - ad + ad - bd \\ 0 & bd \end{pmatrix}$$
$$= \begin{pmatrix} ac & ac - bd \\ 0 & bd \end{pmatrix},$$

implying closure. We also note that interchanging a with c and b with d shows that any two matrices in the set commute.

Since neither a nor b is 0, the determinant of a general matrix in the set is non-zero and its inverse can be constructed as

$$\frac{1}{ab} \begin{pmatrix} b & b-a \\ 0 & a \end{pmatrix} = \begin{pmatrix} a^{-1} & a^{-1} - b^{-1} \\ 0 & b^{-1} \end{pmatrix}.$$

The question then arises as to whether a^{-1} is an integer; in multiplication mod 5 it is. For example, if $a = 3$ then $a^{-1} = 2$ since $3 \times 2 = 6 = 1 \pmod 5$. The full set of values is: $1^{-1} = 1$, $2^{-1} = 3$, $3^{-1} = 2$ and $4^{-1} = 4$.

Thus each inverse is of the required form and a general one can be verified:

$$\frac{1}{ab}\begin{pmatrix} b & b-a \\ 0 & a \end{pmatrix}\begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix} = \frac{1}{ab}\begin{pmatrix} ab & ab-b^2+b^2-ab \\ 0 & ab \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

All four requirements are satisfied and the set of matrices is, in fact, a group.

(f) As always, the only element of order 1 is the unit element. Elements of order 2 must satisfy

$$\begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix}\begin{pmatrix} a & a-b \\ 0 & b \end{pmatrix} = \begin{pmatrix} a^2 & a^2-b^2 \\ 0 & b^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus a and b must both be elements whose squares are unity (mod 5); each must be either 1 or 4 [since $4^2 = 16 = 1 \pmod 5$]. The four matrices to consider are thus

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \quad \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}.$$

The identity element is present and, from the way they were defined, each is its own inverse. Only closure remains to be tested. As all matrices in the set commute [see (e) above], we need test only

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix},$$
$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix},$$
$$\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}.$$

Each product is one of the set of four. So closure is established and the set does form a group – a subgroup, of order 4, of the group in (e).

(g) The product of two such matrices is

$$\begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ a+x & 1 & 0 \\ b+cx+y & c+z & 1 \end{pmatrix}.$$

Since all elements of the original two matrices are integers, so are all elements of the product and closure is established.

Clearly, $a = b = c = 0$ provides the identity element and, since the determinant of each matrix is 1, inverses can be constructed in the usual way, typically

$$\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix}.$$

This is of the correct form as can be verified as follows:

$$\begin{pmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ ac-b & -c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus, assuming associativity, the group property of the set is established.

28.3 *Define a binary operation $\bullet$ on the set of real numbers by*

$$x \bullet y = x + y + rxy,$$

where r is a non-zero real number. Show that the operation $\bullet$ is associative.

Prove that $x \bullet y = -r^{-1}$ if, and only if, $x = -r^{-1}$ or $y = -r^{-1}$. Hence prove that the set of all real numbers excluding $-r^{-1}$ forms a group under the operation $\bullet$.

To demonstrate the associativity we need to show that $x \bullet (y \bullet z)$ is the same thing as $(x \bullet y) \bullet z$. So consider

$$\begin{aligned} x \bullet (y \bullet z) &= x + (y \bullet z) + rx(y \bullet z) \\ &= x + y + z + ryz + rx(y + z + ryz) \\ &= x + y + z + r(yz + xy + xz) + r^2xyz \end{aligned}$$

and

$$\begin{aligned} (x \bullet y) \bullet z &= (x \bullet y) + z + r(x \bullet y)z \\ &= x + y + rxy + z + r(x + y + rxy)z \\ &= x + y + z + r(xy + xz + yz) + r^2xyz. \end{aligned}$$

The two RHSs are equal, showing that the operation $\bullet$ is associative.

Firstly, suppose that $x = -r^{-1}$. Then

$$x \bullet y = -\frac{1}{r} + y + r\left(-\frac{1}{r}\right)y = -\frac{1}{r} + y - y = -\frac{1}{r}.$$

Similarly $y = -r^{-1} \quad \Rightarrow \quad x \bullet y = -r^{-1}$.

Secondly, suppose that $x \bullet y = -r^{-1}$. Then

$$\begin{aligned} x + y + rxy &= -r^{-1}, \\ rx + ry + r^2xy + 1 &= 0, \\ (rx + 1)(ry + 1) &= 0, \\ \Rightarrow \quad \text{either } x = -r^{-1} &\text{ or } y = -r^{-1}. \end{aligned}$$

Thus $x \bullet y = -r^{-1} \iff (x = -r^{-1} \text{ or } y = -r^{-1})$.

If $\mathcal{S} = \{\text{real numbers } \neq -r^{-1}\}$, then

(i) Associativity under $\bullet$ has already been shown.
(ii) If x and y belong to $\mathcal{S}$, then $x \bullet y$ is a real number and, in view of the second result above, is $\neq -r^{-1}$. Thus $x \bullet y$ belongs to $\mathcal{S}$ and the set is closed under the operation $\bullet$.
(iii) For any x belonging to $\mathcal{S}$, $x \bullet 0 = x + 0 + rx0 = x$. Thus 0 is an identity element.
(iv) An inverse x^{-1} of x must satisfy $x \bullet x^{-1} = 0$, i.e.

$$x + x^{-1} + rxx^{-1} = 0 \quad \Rightarrow \quad x^{-1} = -\frac{x}{1 + rx}.$$

This is a real (finite) number since $x \neq -r^{-1}$ and, further, $x^{-1} \neq -r^{-1}$, since if it were we could deduce that $1 = 0$. Thus the set $\mathcal{S}$ contains an inverse for each of its elements.

These four results together show that $\mathcal{S}$ is a group under the operation $\bullet$.

28.5 *The following is a 'proof' that reflexivity is an unnecessary axiom for an equivalence relation.*

Because of symmetry $X \sim Y$ implies $Y \sim X$. Then by transitivity $X \sim Y$ and $Y \sim X$ imply $X \sim X$. Thus symmetry and transitivity imply reflexivity, which therefore need not be separately required.

Demonstrate the flaw in this proof using the set consisting of all real numbers plus the number i. Show by investigating the following specific cases that, whether or not reflexivity actually holds, it cannot be deduced from symmetry and transitivity alone.

(a) $X \sim Y$ *if* $X + Y$ *is real.*
(b) $X \sim Y$ *if* XY *is real.*

Let elements X and Y be drawn from the set $\mathcal{S}$ consisting of the real numbers together with i.

(a) For the definition $X \sim Y$ if $X + Y$ is real, we have

(i) that

$$X \sim Y \quad \Rightarrow \quad X + Y \text{ is real} \quad \Rightarrow \quad Y + X \text{ is real} \quad \Rightarrow \quad Y \sim X,$$

i.e symmetry holds;

(ii) that if $X \sim Y$ then neither X nor Y can be i and, equally, if $Y \sim Z$ then neither Y nor Z can be i. It then follows that $X + Z$ is real and $X \sim Z$, i.e. transitivity holds.

Thus both symmetry and transitivity hold, though it is obvious that $X \not\sim X$ if X is i. Thus symmetry and transitivity do *not* necessarily imply reflexivity, showing

that the 'proof' is flawed – in this case the proof fails when X is i because there is no distinct 'Y' available, something assumed in the proof.

(b) For the definition $X \sim Y$ if XY is real, we have

(i) that

$$X \sim Y \quad \Rightarrow \quad XY \text{ is real} \quad \Rightarrow \quad YX \text{ is real} \quad \Rightarrow \quad Y \sim X,$$

i.e symmetry holds;

(ii) that if $X \sim Y$ then neither X nor Y is i. Equally, if $Y \sim Z$ then neither Y nor Z is i. It then follows that XZ is real and $X \sim Z$, i.e. transitivity holds.

Thus both symmetry and transitivity hold and, setting Z equal to X, they do imply the reflexivity property for the real elements of $\mathcal{S}$. However, they cannot establish it for the element i – though it happens to be true in this case as i^2 is real.

28.7 *$\mathcal{S}$ is the set of all 2×2 matrices of the form*

$$A = \begin{pmatrix} w & x \\ y & z \end{pmatrix}, \qquad \textit{where } wz - xy = 1.$$

Show that $\mathcal{S}$ is a group under matrix multiplication. Which element(s) have order 2? Prove that an element A has order 3 if $w + z + 1 = 0$.

The condition $wz - xy = 1$ is the same as $|\mathsf{A}| = 1$; it follows that the set contains an identity element (with $w = z = 1$ and $x = y = 0$). Moreover, each matrix in $\mathcal{S}$ has an inverse and, since $|\mathsf{A}^{-1}|\,|\mathsf{A}| = |\,\mathsf{I}\,| = 1$ implies that $|\mathsf{A}^{-1}| = 1$, the inverses also belong to the set.

If A and B belong to $\mathcal{S}$ then, since $|\mathsf{AB}| = |\mathsf{A}|\,|\mathsf{B}| = 1 \times 1 = 1$, their product also belongs to $\mathcal{S}$, i.e. the set is closed.

These observations, together with the associativity of matrix multiplication establish that the set $\mathcal{S}$ is, in fact, a group under this operation.

If A is to have order 2 then

$$\begin{pmatrix} w & x \\ y & z \end{pmatrix}\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e. $w^2 + xy = 1, \quad x(w+z) = 0, \quad y(w+z) = 0, \quad xy + z^2 = 1.$

These imply that $w^2 = z^2$ and that either $z = -w$ or $x = y = 0$. If $z = -w$, then both

$$\begin{aligned} & w^2 + xy = 1, && \text{from the above condition,} \\ \text{and} \quad & -w^2 - xy = 1, && \text{from } wz - xy = 1. \end{aligned}$$

This is not possible and so we must have $x = y = 0$, implying that w and z are either both $+1$ or both -1. The former gives the identity (of order 1), and so the matrix given by the latter, $\mathsf{A} = -\mathsf{I}$, is the only element in $\mathcal{S}$ of order 2.

If $w + z + 1 = 0$ (as well as $xy = wz - 1$), A^2 can be written as

$$\begin{aligned}\mathsf{A}^2 &= \begin{pmatrix} w^2 + xy & x(w+z) \\ y(w+z) & xy + z^2 \end{pmatrix} \\ &= \begin{pmatrix} w^2 + wz - 1 & -x \\ -y & wz - 1 + z^2 \end{pmatrix} \\ &= \begin{pmatrix} -w - 1 & -x \\ -y & -z - 1 \end{pmatrix}.\end{aligned}$$

Multiplying again by A gives

$$\begin{aligned}\mathsf{A}^3 &= \begin{pmatrix} -w - 1 & -x \\ -y & -z - 1 \end{pmatrix} \begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ &= -\begin{pmatrix} w(w+1) + xy & (w+1)x + xz \\ wy + y(z+1) & xy + z(z+1) \end{pmatrix} \\ &= -\begin{pmatrix} w(w+1) + wz - 1 & x \times 0 \\ y \times 0 & wz - 1 + z(z+1) \end{pmatrix} \\ &= -\begin{pmatrix} (w \times 0) - 1 & 0 \\ 0 & (z \times 0) - 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

Thus A has order 3.

28.9 *If $\mathcal{A}$ is a group in which every element other than the identity, I, has order 2, prove that $\mathcal{A}$ is Abelian. Hence show that if X and Y are distinct elements of $\mathcal{A}$, neither being equal to the identity, then the set $\{I, X, Y, XY\}$ forms a subgroup of $\mathcal{A}$.*

Deduce that if $\mathcal{B}$ is a group of order $2p$, with p a prime greater than 2, then $\mathcal{B}$ must contain an element of order p.

If every element of $\mathcal{A}$, apart from I, has order 2, then, for any element X, $X^2 = I$. Consider two elements X and Y and let $XY = Z$. Then

$$\begin{aligned} X^2 Y &= XZ \quad \Rightarrow \quad Y = XZ, \\ XY^2 &= ZY \quad \Rightarrow \quad X = ZY. \end{aligned}$$

It follows that $YX = XZZY$. But, since $XY = Z$, Z must belong to $\mathcal{A}$ and

therefore $Z^2 = I$. Substituting this gives $YX = XY$, proving that the group is Abelian.

Consider the set $\mathcal{S} = \{I, X, Y, XY\}$, for which

(i) Associativity holds, since it does for $\mathcal{A}$.
(ii) It is closed, the only products needing non-trivial examinations being $XYX = XXY = X^2Y = Y$ and $YXY = XYY = XY^2 = X$ (here we have twice used the fact that $\mathcal{A}$, and hence $\mathcal{S}$, is Abelian).
(iii) The identity I is contained in the set.
(iv) Since all elements are of order 2 (or 1), each is its own inverse.

Thus the set is a subgroup of $\mathcal{A}$ of order 4.

Now consider the group $\mathcal{B}$ of order $2p$, where p is prime. Since the order of an element must divide the order of the group, all elements in $\mathcal{B}$ must have order 1 (I only) or 2 or p.

Suppose *all* elements, other than I, have order 2. Then, as shown above, $\mathcal{B}$ must be Abelian and have a subgroup of order 4. However, the order of any subgroup must divide the order of the group and 4 cannot divide $2p$ since p is prime. It follows that the supposition that all elements can be of order 2 is false, and consequently at least one must have order p.

28.11 *Identify the eight symmetry operations on a square. Show that they form a group $\mathcal{D}_4$ (known to crystallographers as 4mm and to chemists as $\mathcal{C}_{4v}$) having one element of order 1, five of order 2 and two of order 4. Find its proper subgroups and the corresponding cosets.*

The operation of leaving the square alone is a trivial symmetry operation, but an important one, as it is the identity I of the group; it has order 1.

Rotations about an axis perpendicular to the plane of the square by $\pi/4$, $\pi/2$ and $3\pi/2$ each take the square into itself. The first and last of these have to be repeated four times to reproduce the effect of I, and so they have order 4. The rotation by $\pi/2$ clearly has order 2.

Reflections in the two axes parallel to the sides of the square and passing through its centre are also symmetry operations, as are reflections in the two principal diagonals of the square; all of these reflections have order 2.

Using the notation indicated in figure 28.1, R being a rotation of $\pi/2$ about an axis perpendicular to the square, we have: I has order 1; R^2, m_1, m_2, m_3, m_4 have order 2; R, R^3 have order 4.

Figure 28.1 The notation for exercise 28.11.

The group multiplication table takes the form

	I	R	R^2	R^3	m_1	m_2	m_3	m_4
I	I	R	R^2	R^3	m_1	m_2	m_3	m_4
R	R	R^2	R^3	I	m_4	m_3	m_1	m_2
R^2	R^2	R^3	I	R	m_2	m_1	m_4	m_3
R^3	R^3	I	R	R^2	m_3	m_4	m_2	m_1
m_1	m_1	m_3	m_2	m_4	I	R^2	R	R^3
m_2	m_2	m_4	m_1	m_3	R^2	I	R^3	R
m_3	m_3	m_2	m_4	m_1	R^3	R	I	R^2
m_4	m_4	m_1	m_3	m_2	R	R^3	R^2	I

Inspection of this table shows the existence of the non-trivial subgroups listed below, and tedious but straightforward evaluation of the products of selected elements of the group with all the elements of any one subgroup provides the cosets of that subgroup. The results are as follows:

subgroup $\{I, R, R^2, R^3\}$ has cosets $\{I, R, R^2, R^3\}$, $\{m_1, m_2, m_3, m_4\}$;

subgroup $\{I, R^2, m_1, m_2\}$ has cosets $\{I, R^2, m_1, m_2\}$, $\{I, R^2, m_3, m_4\}$;

subgroup $\{I, R^2, m_3, m_4\}$ has cosets $\{I, R^2, m_3, m_4\}$, $\{I, R^2, m_1, m_2\}$;

subgroup $\{I, R^2\}$ has cosets $\{I, R^2\}$, $\{R, R^3\}$, $\{m_1, m_2\}$, $\{m_3, m_4\}$;

subgroup $\{I, m_1\}$ has cosets $\{I, m_1\}$, $\{R, m_3\}$, $\{R^2, m_2\}$, $\{R^3, m_4\}$;

subgroup $\{I, m_2\}$ has cosets $\{I, m_2\}$, $\{R, m_4\}$, $\{R^2, m_1\}$, $\{R^3, m_3\}$;

subgroup $\{I, m_3\}$ has cosets $\{I, m_3\}$, $\{R, m_2\}$, $\{R^2, m_4\}$, $\{R^3, m_1\}$;

subgroup $\{I, m_4\}$ has cosets $\{I, m_4\}$, $\{R, m_1\}$, $\{R^2, m_3\}$, $\{R^3, m_2\}$.

28.13 *Find the group $\mathcal{G}$ generated under matrix multiplication by the matrices*

$$\mathsf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Determine its proper subgroups, and verify for each of them that its cosets exhaust $\mathcal{G}$.

Before we can draw up a group multiplication table to search for subgroups, we must determine the multiple products of A and B with themselves and with each other:

$$\mathsf{A}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{I}.$$

Since $\mathsf{B} = i\mathsf{A}$, it follows that $\mathsf{B}^2 = -I$, that $\mathsf{AB} = i\mathsf{I} = \mathsf{BA}$, and that $\mathsf{B}^3 = -\mathsf{B}$. In brief, A is of order 2, B is of order 4, and A and B commute. The eight distinct elements of the group are therefore: $\mathsf{I}, \mathsf{A}, \mathsf{B}, \mathsf{B}^2, \mathsf{B}^3, \mathsf{AB}, \mathsf{AB}^2$ and AB^3.

The group multiplication table is

	I	A	B	B^2	B^3	AB	AB^2	AB^3
I	I	A	B	B^2	B^3	AB	AB^2	AB^3
A	A	I	AB	AB^2	AB^3	B	B^2	B^3
B	B	AB	B^2	B^3	I	AB^2	AB^3	A
B^2	B^2	AB^2	B^3	I	B	AB^3	A	AB
B^3	B^3	AB^3	I	B	B^2	A	AB	AB^2
AB	AB	B	AB^2	AB^3	A	B^2	B^3	I
AB^2	AB^2	B^2	AB^3	A	B	B^3	I	B
AB^3	AB^3	B^3	A	AB	AB^2	I	B	B^2

By inspection, the subgroups and their cosets are as follows:

$$\begin{aligned}
\{\mathsf{I}, \mathsf{A}\} &: \{\mathsf{I}, \mathsf{A}\}, \{\mathsf{B}, \mathsf{AB}\}, \{\mathsf{B}^2, \mathsf{AB}^2\}, \{\mathsf{B}^3, \mathsf{AB}^3\}; \\
\{\mathsf{I}, \mathsf{B}^2\} &: \{\mathsf{I}, \mathsf{B}^2\}, \{\mathsf{A}, \mathsf{AB}^2\}, \{\mathsf{B}, \mathsf{B}^3\}, \{\mathsf{AB}, \mathsf{AB}^3\}; \\
\{\mathsf{I}, \mathsf{AB}^2\} &: \{\mathsf{I}, \mathsf{AB}^2\}, \{\mathsf{A}, \mathsf{B}^2\}, \{\mathsf{B}, \mathsf{AB}^3\}, \{\mathsf{B}^3, \mathsf{AB}\}; \\
\{\mathsf{I}, \mathsf{B}, \mathsf{B}^2, \mathsf{B}^3\} &: \{\mathsf{I}, \mathsf{B}, \mathsf{B}^2, \mathsf{B}^3\}, \{\mathsf{A}, \mathsf{AB}, \mathsf{AB}^2, \mathsf{AB}^3\}; \\
\{\mathsf{I}, \mathsf{AB}, \mathsf{B}^2, \mathsf{AB}^3\} &: \{\mathsf{I}, \mathsf{AB}, \mathsf{B}^2, \mathsf{AB}^3\}, \{\mathsf{A}, \mathsf{B}, \mathsf{AB}^2, \mathsf{B}^3\}.
\end{aligned}$$

As expected, in each case the cosets exhaust the group, with each element in one and only one coset.

28.15 *Consider the following mappings between a permutation group and a cyclic group.*

(a) *Denote by A_n the subset of the permutation group S_n that contains all the even permutations. Show that A_n is a subgroup of S_n.*
(b) *List the elements of S_3 in cycle notation and identify the subgroup A_3.*
(c) *For each element X of S_3, let $p(X) = 1$ if X belongs to A_3 and $p(X) = -1$ if it does not. Denote by $\mathcal{C}_2$ the multiplicative cyclic group of order 2. Determine the images of each of the elements of S_3 for the following four mappings:*

$$\begin{aligned} \Phi_1 &: S_3 \to \mathcal{C}_2 & X &\to p(X), \\ \Phi_2 &: S_3 \to \mathcal{C}_2 & X &\to -p(X), \\ \Phi_3 &: S_3 \to A_3 & X &\to X^2, \\ \Phi_4 &: S_3 \to S_3 & X &\to X^3. \end{aligned}$$

(d) *For each mapping, determine whether the kernel $\mathcal{K}$ is a subgroup of S_3 and, if so, whether the mapping is a homomorphism.*

(a) With A_n as the subset of S_n that contains all the even permutations, we need to demonstrate that it has the four properties that characterise a group:

(i) If X and Y belong to A_n so does XY, as the product of two even permutations is even. This establishes closure.
(ii) From the definition of an even permutation, the identity I belongs to A_n.
(iii) If X belongs to A_n so does X^{-1}, as the number of pair interchanges needed to change from X to I is the same as the number needed to go in the opposite direction. This establishes the existence, within the subset, of an inverse for each member of the subset.
(iv) Associativity follows from that of the group S_n.

Thus A_n does possess the four properties and is a subgroup of S_n.

(b) (1), (123) and (132) belong to A_3. The permutations (12), (13) and (23) do not belong, as each involves only one pair interchange.

(c) With the given definition of $p(X)$,

$$p(X) = 1 \text{ for } X = (1),\ (123),\ (132),$$
$$p(X) = -1 \text{ for } X = (12),\ (13),\ (23).$$

$\mathcal{C}_2$ consists of the two elements $+1$ and -1.

For $\Phi_1 : S_3 \to \mathcal{C}_2,\ X \to p(X)$, elements in A_3 have image $+1$; those not in A_3 have image -1.

For $\Phi_2 : S_3 \to \mathcal{C}_2,\ X \to -p(X)$, elements in A_3 have image -1; those not in A_3 have image $+1$.

For $\Phi_3 : S_3 \to A_3,\ X \to X^2$

$$\begin{aligned}(1) &\to (1)(1) = (1),\\ (123) &\to (123)(123) = (132),\\ (132) &\to (132)(132) = (123),\\ (12) &\to (12)(12) = (1), \quad \text{similarly, (13) and (23).}\end{aligned}$$

For $\Phi_4 : S_3 \to S_3,\ X \to X^3$

$$\begin{aligned}(1) &\to (1)(1) = (1),\\ (123) &\to (123)(123)(123) = (132)(123) = (1),\\ (132) &\to (132)(132)(132) = (123)(132) = (1),\\ (12) &\to (12)(12)(12) = (1)(12) = (12), \quad \text{similarly, (13) and (23).}\end{aligned}$$

(d) For Φ_1, the kernel is the set of elements belonging to A_3 and, as already shown, this is a subgroup of S_3.

We note that the product of two even or two odd permutations is an even permutation, whilst the product of an odd and an even permutation is an odd permutation. We also note that $+1 \times +1$ and -1×-1 are both equal to $+1$, whilst $+1 \times -1 = -1$. Since Φ_1 maps even permutations onto $+1$ and odd permutations onto -1, the preceding observations imply that Φ_1 is a homomorphism.

For Φ_2, the kernel is the set of elements *not* belonging to A_3. Since this set does not contain the identity (1), it cannot be a subgroup of S_3.

For Φ_3 the kernel is the set $\{(1), (12), (13), (23)\}$. Since, for example, $(12)(13) = (132)$, the set is not closed and so cannot form a group. It cannot, therefore, be a subgroup of S_3.

For Φ_4 the kernel is the set $\{(1), (123), (132)\}$, i.e. the subgroup A_3. However, for example, $[(12)(13)]' = (132)' = (1)$, whilst $(12)'(13)' = (12)(13) = (132)$; these two results are not equal, showing that the mapping *cannot* be a homomorphism.

28.17 *The group of all non-singular $n \times n$ matrices is known as the general linear group $GL(n)$ and that with only real elements as $GL(n, \mathbf{R})$. If $\mathbf{R}^*$ denotes the multiplicative group of non-zero real numbers, prove that the mapping $\Phi : GL(n, \mathbf{R}) \to \mathbf{R}^*$, defined by $\Phi(\mathsf{M}) = \det \mathsf{M}$, is a homomorphism.*

Show that the kernel $\mathcal{K}$ of Φ is a subgroup of $GL(n, \mathbf{R})$. Determine its cosets and show that they themselves form a group.

If P and Q are two matrices belonging to $GL(n, \mathbf{R})$ then, under Φ,

$$(\mathsf{PQ})' = |\mathsf{PQ}| = |\mathsf{P}|\,|\mathsf{Q}| = \mathsf{P}'\mathsf{Q}'.$$

Thus Φ is a homomorphism.

The kernel $\mathcal{K}$ of the mapping consists of all matrices in $GL(n, \mathbf{R})$ that map onto the identity in $\mathbf{R}^*$, i.e all matrices whose determinant is 1.

To determine whether $\mathcal{K}$ is a subgroup of the general linear group, let X and Y belong to $\mathcal{K}$. Then, testing $\mathcal{K}$ for the four group-defining properties, we have

(i) $(XY)' = X'Y' = 1 \times 1 = 1$, i.e. XY also belongs to $\mathcal{K}$, showing the closure of the kernel.
(ii) The associative law holds for the elements of $\mathcal{K}$ since it does so for all elements of $GL(n, \mathbf{R})$.
(iii) $|\mathsf{I}| = 1$ and so I belongs to $\mathcal{K}$.
(iv) Since $\mathsf{X}^{-1}\mathsf{X} = \mathsf{I}$, it follows that $|\mathsf{X}^{-1}|\,|\mathsf{X}| = |\mathsf{I}|$ and $|\mathsf{X}^{-1}| \times 1 = 1$. Hence $|\mathsf{X}^{-1}| = 1$ and so X^{-1} also belongs to $\mathcal{K}$.

This completes the proof that $\mathcal{K}$ is a subgroup of $GL(n, \mathbf{R})$.

Two matrices P and Q in $GL(n, \mathbf{R})$ belong to the same coset of $\mathcal{K}$ if

$$\mathsf{Q} = \mathsf{PX}, \text{ where } \mathsf{X} \text{ is some element in } \mathcal{K}.$$

It then follows that

$$|\mathsf{Q}| = |\mathsf{P}|\,|\mathsf{X}| = |\mathsf{P}| \times 1.$$

Thus the requirement for two matrices to be in the same coset is that they have equal determinants.

Let us denote by $\mathcal{C}_i$ the (infinite) set of all matrices whose determinant has the value i; the label i will itself take on an infinite continuum of values, excluding 0. Then,

(i) For *any* $\mathsf{M}_i \in \mathcal{C}_i$ and *any* $\mathsf{M}_j \in \mathcal{C}_j$ we have

$$|\mathsf{M}_i\mathsf{M}_j| = |\mathsf{M}_i|\,|\mathsf{M}_j| = i \times j,$$

implying that we always have $\mathsf{M}_i\mathsf{M}_j \in \mathcal{C}_{(i\times j)}$. Thus the set of cosets is closed, with $\mathcal{C}_i \times \mathcal{C}_j = \mathcal{C}_{(i\times j)}$.
(ii) The associative law holds, since it does so for matrix multiplication in general, and the product of three matrices, and hence its determinant, is independent of the order in which the individual multiplications are carried out.
(iii) Since

$$|\mathsf{M}_i\mathsf{M}_1| = |\mathsf{M}_i|\,|\mathsf{M}_1| = i,$$

$\mathcal{C}_i \times \mathcal{C}_1 = \mathcal{C}_i$, showing that $\mathcal{C}_1$ is an identity element in the set.
(iv) Since

$$|\mathsf{M}_i\mathsf{M}_{1/i}| = |\mathsf{M}_i|\,|\mathsf{M}_{1/i}| = i \times (1/i) = 1,$$

$\mathcal{C}_i \times \mathcal{C}_{1/i} = \mathcal{C}_1$, showing that the set of cosets contains an inverse for each coset.

This completes the proof that the cosets themselves form a group under coset multiplication (and also that $\mathcal{K}$ is a normal subgroup).

28.19 *Given that matrix* M *is a member of the multiplicative group* $GL(3, \mathbf{R})$*, determine, for each of the following additional constraints on* M *(applied separately), whether the subset satisfying the constraint is a subgroup of* $GL(3, \mathbf{R})$*:*

(a) $\mathsf{M}^T = \mathsf{M}$;
(b) $\mathsf{M}^T\mathsf{M} = \mathsf{I}$;
(c) $|\mathsf{M}| = 1$;
(d) $M_{ij} = 0$ *for* $j > i$ *and* $M_{ii} \neq 0$.

The matrices belonging to $GL(3, \mathbf{R})$ have the general properties that they are non-singular, possess inverses and have real elements. The operation of matrix multiplication is associative, and this will be assumed in the rest of the exercise, in which A and B are general matrices satisfying the various defining constraints.

(a) $\mathsf{M}^T = \mathsf{M}$: the set of symmetric matrices.
Now, for two symmetric matrices A and B,

$$(\mathsf{AB})^{\mathrm{T}} = \mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}} = \mathsf{BA}$$

and this is not equal to AB in general, as matrix multiplication is not necessarily commutative. The set is therefore not closed and cannot form a group; equally it cannot be a subgroup of $GL(3, \mathbf{R})$.

(b) $\mathsf{M}^T\mathsf{M} = \mathsf{I}$: the set of orthogonal matrices.
Clearly, the identity I belongs to the set, and furthermore

$$(\mathsf{M}^{-1})^{\mathrm{T}}\mathsf{M}^{-1} = (\mathsf{M}^{\mathrm{T}})^{-1}\mathsf{M}^{-1} = (\mathsf{M}^{-1})^{-1}\mathsf{M}^{-1} = \mathsf{MM}^{-1} = \mathsf{I},$$

i.e. if M belongs to the set then so does M^{-1}. Finally,

$$(\mathsf{AB})^{\mathrm{T}}\mathsf{AB} = \mathsf{B}^{\mathrm{T}}\mathsf{A}^{\mathrm{T}}\mathsf{AB} = \mathsf{B}^{\mathrm{T}}\mathsf{I}\,\mathsf{B} = \mathsf{I},$$

showing that the set is closed. This completes the proof that the orthogonal matrices form a subgroup of $GL(3, \mathbf{R})$.

(c) $|\mathsf{M}| = 1$: the set of unimodular matrices.

$$\begin{aligned} |\mathsf{AB}| = |\mathsf{A}|\,|\mathsf{B}| = 1 \times 1 = 1 &\quad \text{closure,} \\ |\,\mathsf{I}\,| = 1 &\quad \text{identity,} \\ \mathsf{M}^{-1}\mathsf{M} = \mathsf{I} \quad \Rightarrow \quad |\mathsf{M}^{-1}|\,|\mathsf{M}| = 1 \quad \Rightarrow \quad |\mathsf{M}^{-1}| = 1 &\quad \text{inverse.} \end{aligned}$$

These three results (and associativity) show that the unimodular matrices do form a subgroup of $GL(3, \mathbf{R})$.

(d) $M_{ij} = 0$ for $j > i$ and $M_{ii} \neq 0$: the set of lower diagonal matrices with non-zero diagonal elements.
Taking first the question of closure, consider the matrix product $\mathsf{C} = \mathsf{AB}$. A typical element of C above the leading diagonal is

$$\begin{aligned} C_{12} &= A_{11}B_{12} + A_{12}B_{22} + A_{13}B_{32} \\ &= A_{11}\,0 + 0\,B_{22} + 0\,A_{32} = 0, \end{aligned}$$

and a typical element on the leading diagonal is

$$\begin{aligned} C_{11} &= A_{11}B_{11} + A_{12}B_{21} + A_{13}B_{31} \\ &= A_{11}B_{11} + 0\,B_{21} + 0\,A_{31} = A_{11}B_{11} \neq 0. \end{aligned}$$

That C_{11} is not equal to zero follows from the fact that neither A_{11} nor B_{11} is zero. Similarly, C_{13} and C_{23} are zero, whilst C_{22} and C_{33} are non-zero. Thus C has all the properties defining the set, and so belongs to it. The set is therefore closed.

Clearly the matrix I_3 belongs to the set, which therefore contains an identity element.

Since no diagonal element is zero, the determinant (which is the product of the diagonal elements for lower triangular matrices) of any member of the set cannot be zero. All members must therefore have inverses. For the matrix

$$\mathsf{A} = \begin{pmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{pmatrix},$$

the inverse is given by

$$\mathsf{A}^{-1} = \frac{1}{A_{11}A_{22}A_{33}} \begin{pmatrix} A_{22}A_{33} & 0 & 0 \\ -A_{21}A_{33} & A_{11}A_{33} & 0 \\ A_{21}A_{32} - A_{22}A_{31} & -A_{11}A_{32} & A_{11}A_{22} \end{pmatrix}.$$

We note that each of the diagonal elements of A^{-1} is the product of two non-zero terms, and is therefore itself non-zero. Thus A^{-1} has the correct form for a member of the set – lower diagonal with non-zero diagonal elements – and so belongs to the set, which has now been shown to have all the properties needed to make it a group and hence a subgroup of $GL(3, \mathbf{R})$.

28.21 *Show that $\mathcal{D}_4$, the group of symmetries of a square, has two isomorphic subgroups of order* 4.

The quaternion group $\mathcal{Q}$ is the set of elements

$$\{1, -1, i, -i, j, -j, k, -k\},$$

with $i^2 = j^2 = k^2 = -1$, $ij = k$ and its cyclic permutations, and $ji = -k$ and its cyclic permutations. Its multiplication table reads as follows:

	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

Show that there exists a two-to-one homomorphism from the quaternion group $\mathcal{Q}$ onto one (and hence either) of the two subgroups of $\mathcal{D}_4$, and determine its kernel.

We first reproduce the multiplication table for $\mathcal{D}_4$:

	I	R	R^2	R^3	m_1	m_2	m_3	m_4
I	I	R	R^2	R^3	m_1	m_2	m_3	m_4
R	R	R^2	R^3	I	m_4	m_3	m_1	m_2
R^2	R^2	R^3	I	R	m_2	m_1	m_4	m_3
R^3	R^3	I	R	R^2	m_3	m_4	m_2	m_1
m_1	m_1	m_3	m_2	m_4	I	R^2	R	R^3
m_2	m_2	m_4	m_1	m_3	R^2	I	R^3	R
m_3	m_3	m_2	m_4	m_1	R^3	R	I	R^2
m_4	m_4	m_1	m_3	m_2	R	R^3	R^2	I

Here R is a rotation by $\pi/2$ in the plane of the square, m_1 and m_2 are reflections in the axes parallel to the sides of the square, and m_3 and m_4 are reflections in the square's diagonals.

As shown in exercise 28.11, $\mathcal{D}_4$ has three proper subgroups of order 4. They are $\{I, R, R^2, R^3\}$, $\mathcal{H}_1 = \{I, R^2, m_1, m_2\}$ and $\mathcal{H}_2 = \{I, R^2, m_3, m_4\}$. The first of these is a cyclic subgroup but the other two are not. The group tables for the latter two,

extracted from the one above, are as follows

$\mathcal{H}_1$	I	R^2	m_1	m_2
I	I	R^2	m_1	m_2
R^2	R^2	I	m_2	m_1
m_1	m_1	m_2	I	R^2
m_2	m_2	m_1	R^2	I

$\mathcal{H}_2$	I	R^2	m_3	m_4
I	I	R^2	m_3	m_4
R^2	R^2	I	m_4	m_3
m_3	m_3	m_4	I	R^2
m_4	m_4	m_3	R^2	I

Clearly, these two subgroups are isomorphic with $m_1 \leftrightarrow m_3$ and $m_2 \leftrightarrow m_4$ and the other elements unchanged.

Next, we reproduce the group table for the quaternion group, but with the columns and rows reordered (this does not alter the information it carries):

	1	i	j	k	-1	$-i$	$-j$	$-k$
1	1	i	j	k	-1	$-i$	$-j$	$-k$
i	i	-1	k	$-j$	$-i$	1	$-k$	j
j	j	$-k$	-1	i	$-j$	k	1	$-i$
k	k	j	$-i$	-1	$-k$	$-j$	i	1
-1	-1	$-i$	$-j$	$-k$	1	i	j	k
$-i$	$-i$	1	$-k$	j	i	-1	k	$-j$
$-j$	$-j$	k	1	$-i$	j	$-k$	-1	i
$-k$	$-k$	$-j$	i	1	k	j	$-i$	-1

If we now make the two-to-one mapping

$$\Phi \quad : \quad \pm 1 \to I, \quad \pm i \to R^2, \quad \pm j \to m_1, \quad \pm k \to m_2,$$

each quadrant of the table for $\mathcal{Q}$ becomes identical to that for $\mathcal{H}_1$, showing that Φ is a homomorphism of $\mathcal{Q}$ onto $\mathcal{H}_1$. As $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic there exists a similar homomorphism onto $\mathcal{H}_2$.

Finally, the kernel of either mapping contains those elements of $\mathcal{Q}$ that map onto I, namely 1 and -1.

28.23 *Find* (a) *all the proper subgroups and* (b) *all the conjugacy classes of the symmetry group of a regular pentagon.*

A regular pentagon (see figure 28.2) has rotational symmetries and reflection symmetries about lines that join a vertex to the centre-point of the opposite side. Clearly there are five of the latter, m_i ($i = 1, 2, \ldots, 5$). If R represents a rotation by $2\pi/5$, then the rotational symmetries are R, R^2, R^3 and R^4. To these must be added the 'do nothing' identity I. The symmetry group of the regular pentagon

Figure 28.2 The regular pentagon of exercise 28.23.

($\mathcal{C}_{5v}$ in chemical notation) therefore consists of the following ten elements (with their orders):

element	:	I	R	R^2	R^3	R^4	m_1	m_2	m_3	m_4	m_5
order	:	1	5	5	5	5	2	2	2	2	2

(a) As the order of the group is 10, the order of any proper subgroup can only be 2 or 5. As I must be in every subgroup and the order of any element in it must divide the order of the subgroup, it is clear that there is only one subgroup of order 5 and that is $\{I, R, R^2, R^3, R^4\}$. Similarly, there are five subgroups of order 2, namely $\{I, m_i\}$ for m_i $(i = 1, 2, \ldots, 5)$.

(b) As always, I is in a class by itself.

We now prove a useful general result about elements in the same conjugacy class: namely, that they have the same order. Let X and Y be in the same class ($Y = g_i X g_i^{-1}$ for some g_i belong to the group) and let X have order m, i.e. $X^m = I$. Then

$$Y^m = g_i X g_i^{-1}\, g_i X g_i^{-1} \cdots g_i X g_i^{-1} = g_i X^m g_i^{-1} = g_i g_i^{-1} = I.$$

This implies that the order of Y divides the order of X. Similarly the order of X divides the order of Y. Therefore X and Y have the same order. Applying this result to the given group, we see that a conjugacy class cannot contain a mixture of rotations and reflections.

We first note the obvious result that $R^p R^q (R^p)^{-1} = R^q$ for any valid p and q. Next, by considering the effects of various combinations of symmetries on a general point x of the pentagon (as marked in the figure), we find that for any i and j, $(i, j = 1, 2 \ldots, 5)$,

$$m_i R m_i = R^4 \quad \text{and} \quad m_j R^4 m_j = R.$$

These results, together with that noted above, imply that R and R^4 constitute a class. Similarly R^2 and R^3 make up a (different) class.

Turning to the reflections, we see that the following chain of results, for example, shows that all five reflections must be in the same class (recall that each reflection is its own inverse):

$$m_1 m_2 m_1 = m_5, \quad m_3 m_5 m_3 = m_1, \quad m_2 m_1 m_2 = m_3, \quad m_1 m_3 m_1 = m_4.$$

In summary, there are four conjugacy classes and they are I, (R, R^4), (R^2, R^3) and $(m_1, m_2, m_3, m_4, m_5)$.

29

Representation theory

29.1 *A group $\mathcal{G}$ has four elements I, X, Y and Z, which satisfy $X^2 = Y^2 = Z^2 = XYZ = I$. Show that $\mathcal{G}$ is Abelian and hence deduce the form of its character table.*

Show that the matrices

$$\mathsf{D}(I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{D}(X) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\mathsf{D}(Y) = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}, \qquad \mathsf{D}(Z) = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix},$$

where p is a real number, form a representation D of $\mathcal{G}$. Find its characters and decompose it into irreps.

Since I necessarily commutes with all other elements, we need only consider products such as XY. Now,

$$\begin{aligned} XYZ = I \quad &\Rightarrow \quad X^2YZ = X \quad \Rightarrow \quad YZ = X, \\ XYZ = I \quad &\Rightarrow \quad XYZ^2 = Z \quad \Rightarrow \quad XY = Z \\ &\Rightarrow \quad XY^2 = ZY \quad \Rightarrow \quad X = ZY. \end{aligned}$$

Thus, $YZ = X = ZY$, showing that Y and Z commute. Similarly, $XY = Z = YX$ and $XZ = Y = ZX$. We conclude that the group is Abelian.

As the group is Abelian, each element is in a class of its own and there are therefore four classes and consequently four irreps $\mathrm{D}^{(\lambda)}$. Since

$$\sum_{\lambda=1}^{4} n_\lambda^2 = g = 4,$$

where n_λ is the dimension of representation λ, the only possibility is that $n_\lambda = 1$ for each λ, i.e the group has four one-dimensional irreps.

As for all sets of irreps, the identity irrep $\mathrm{D}^{(1)} = \mathrm{A}_1$ must be present and the characters of the others must be orthogonal to the (1,1,1,1) characters of A_1. Further, for each one-dimensional irrep, the identity I must have the character $+1$. The character table must therefore take the form

χ	I	X	Y	Z
A_1	1	1	1	1
$\mathrm{D}^{(2)}$	1	1	-1	-1
$\mathrm{D}^{(3)}$	1	-1	1	-1
$\mathrm{D}^{(4)}$	1	-1	-1	1

For the proposed representation we first need to verify the multiplication properties. Those for $\mathsf{D}(I)$ are immediate. For the others:

$$\mathsf{D}(X)\mathsf{D}(X) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{D}(I),$$

$$\mathsf{D}(Y)\mathsf{D}(Y) = \begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{D}(I),$$

$$\mathsf{D}(Z)\mathsf{D}(Z) = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \mathsf{D}(I),$$

$$\begin{aligned}\mathsf{D}(X)\mathsf{D}(Y)\mathsf{D}(Z) &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & -p \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = \mathsf{D}(I),\end{aligned}$$

Since the defining relationships for I, X, Y and Z are $X^2 = Y^2 = Z^2 = XYZ = I$, these results show that the matrices form a representation of $\mathcal{G}$, whatever the value of p.

Now $\chi(g_i)$ is equal to the trace of $\mathsf{D}(g_i)$ and so the character set for this representation (in the order I, X, Y, Z) is $(2, -2, 0, 0)$. The only rows in the character table that can be added to produce the correct totals for all four elements are the third and fourth. This shows that

$$\mathsf{D} = \mathrm{D}^{(3)} \oplus \mathrm{D}^{(4)}.$$

29.3 *The quaternion group $\mathcal{Q}$ (see exercise 28.21) has eight elements $\{\pm 1, \pm i, \pm j, \pm k\}$ obeying the relations*

$$i^2 = j^2 = k^2 = -1, \quad ij = k = -ji.$$

Determine the conjugacy classes of $\mathcal{Q}$ and deduce the dimensions of its irreps. Show that $\mathcal{Q}$ is homomorphic to the four-element group $\mathcal{V}$, which is generated by two distinct elements a and b with $a^2 = b^2 = (ab)^2 = I$. Find the one-dimensional irreps of $\mathcal{V}$ and use these to help determine the full character table for $\mathcal{Q}$.

As always, the identity, $+1$, is in a class by itself and, since it commutes with every other element in the group, so is -1.

Now consider all products of the form $X^{-1}iX$:

$$\begin{aligned} 1\,i\,1 &= i, & (-1)\,i\,(-1) &= i, \\ (-i)\,i\,i &= i, & i\,i\,(-i) &= i, \\ (-j)\,i\,j = (-j)\,k &= -i, & j\,i\,(-j) = (-k)\,(-j) &= -i, \\ (-k)\,i\,k = (-k)\,(-j) &= -i, & k\,i\,(-k) = j\,(-k) &= -i. \end{aligned}$$

These show that i and $-i$ are in the same class. Similarly $\{j, -j\}$ and $\{k, -k\}$ are two other classes. This exhausts the group.

In summary there are five classes, the elements in any one class all having the same order. They are

class :	$\{1\}$	$\{-1\}$	$\{\pm i\}$	$\{\pm j\}$	$\{\pm k\}$
order :	1	2	4	4	4

It follows that there are five irreps and, since $\sum_{\lambda=1}^{5} n_\lambda^2 = 8$, they can only be one two-dimensional and four one-dimensional irreps.

Turning to the group $\mathcal{V}$,

$$\begin{aligned} (ab)^2 = I \quad &\Rightarrow \quad abab^2 = b \quad \Rightarrow \quad aba = b \quad \Rightarrow \quad aba^2 = ba \\ &\Rightarrow \quad ab = ba, \end{aligned}$$

i.e. a and b commute. Also, it follows that

$$a(ab) = a(ba) = (ab)a \quad \text{and} \quad b(ab) = (ba)b = (ab)b.$$

Thus, all four elements commute, the group $\mathcal{V}$ is Abelian, and each of its elements is in a class of its own. As in exercise 29.1, an Abelian group of order 4 must

have four irreps and the character table

$\chi_{\mathcal{V}}$	I	a	b	ab
A_1	1	1	1	1
$D^{(2)}$	1	1	-1	-1
$D^{(3)}$	1	-1	1	-1
$D^{(4)}$	1	-1	-1	1

The multiplication table for the quaternion group is, as given in exercise 21 of chapter 28 of the form

$\mathcal{Q}$	1	-1	i	$-i$	j	$-j$	k	$-k$
1	1	-1	i	$-i$	j	$-j$	k	$-k$
-1	-1	1	$-i$	i	$-j$	j	$-k$	k
i	i	$-i$	-1	1	k	$-k$	$-j$	j
$-i$	$-i$	i	1	-1	$-k$	k	j	$-j$
j	j	$-j$	$-k$	k	-1	1	i	$-i$
$-j$	$-j$	j	k	$-k$	1	-1	$-i$	i
k	k	$-k$	j	$-j$	$-i$	i	-1	1
$-k$	$-k$	k	$-j$	j	i	$-i$	1	-1

If each entry ± 1 is replaced by I, each entry $\pm i$ by a, each entry $\pm j$ by b and each entry $\pm k$ by ab, then this table reduces to four copies of the table

$\mathcal{V}$	I	a	b	ab
I	I	a	b	ab
a	a	I	ab	b
b	b	ab	I	a
ab	ab	b	a	I

which is the group multiplication table for group $\mathcal{V}$. The same conclusion can be reached by replacing each 2×2 block containing only ± 1 by I, each 2×2 block containing only $\pm i$ by a, etc.; this results in a single copy. Both approaches lead to the conclusion that there is a two-to-one homomorphism from $\mathcal{Q}$ onto $\mathcal{V}$.

Since the homomorphism maps all elements of any one conjugacy class of $\mathcal{Q}$ onto the same element of $\mathcal{V}$, the one-dimensional irreps of $\mathcal{Q}$ must be the same as those of $\mathcal{V}$. Further, since both of the classes $\{1\}$ and $\{-1\}$ map onto I in $\mathcal{V}$ they will have common characters in each one-dimensional irrep (1 in every case, as it happens). As shown earlier, there will also be a two-dimensional irrep. Its character for I must be 2 (the dimension of the irrep); the other characters can be determined from the requirement of orthogonality to the characters of the other (one-dimensional) irreps.

The character table for $\mathcal{Q}$ therefore has the form

$\chi_{\mathcal{Q}}$	1	-1	$\pm i$	$\pm j$	$\pm k$
A_1	1	1	1	1	1
$D^{(2)}$	1	1	1	-1	-1
$D^{(3)}$	1	1	-1	1	-1
$D^{(4)}$	1	1	-1	-1	1
$D^{(5)}$	2	w	x	y	z

We require that

$$\begin{aligned}
1(1)(2) + 1(1)(w) + 2(1)(x) + 2(1)y + 2(1)z &= 0,\\
1(1)(2) + 1(1)(w) + 2(1)(x) + 2(-1)y + 2(-1)z &= 0,\\
1(1)(2) + 1(1)(w) + 2(-1)(x) + 2(1)y + 2(-1)z &= 0,\\
1(1)(2) + 1(1)(w) + 2(-1)(x) + 2(-1)y + 2(1)z &= 0.
\end{aligned}$$

These equations have the solution $w = -2$, $x = y = z = 0$, thus completing the full character table for $\mathcal{Q}$.

29.5 *The group of pure rotations (excluding reflections and inversions) that take a cube into itself has 24 elements. The group is isomorphic to the permutation group S_4 and hence has the same character table, once corresponding classes have been established. By counting the number of elements in each class, make the correspondences below (the final two cannot be decided purely by counting, and should be taken as given).*

Permutation class type	*Symbol (physics)*	*Action*
(1)	I	*none*
(123)	3	*rotations about a body diagonal*
(12)(34)	2_z	*rotation of π about the normal to a face*
(1234)	4_z	*rotations of $\pm\pi/2$ about the normal to a face*
(12)	2_d	*rotation of π about an axis through the centres of opposite edges*

Given in table 29.1 is the character table for S_4. Reformulate it in terms of the elements of the rotation symmetry group (432 or O) of a cube and use it when answering exercise 29.7.

The rotational symmetries of the cube are as follows.

(i) 'Do nothing'. There is only one such operation; it is the identity and so corresponds to $(1) \equiv (1)(2)(3)(4)$.

Irrep	Typical element and class size				
	(1)	(12)	(123)	(1234)	(12)(34)
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

Table 29.1 The character table for the permutation group S_4.

(ii) Rotations about a body diagonal. There are four body diagonals and rotations of $2\pi/3$ and $4\pi/3$ are possible about each. Thus there are eight elements and this must correspond to $(123) \equiv (123)(4)$.

(iii) Rotations by π about a face normal. Although there are six faces to the cube, they define only three distinct face normals and hence there are three elements in this set. They therefore correspond to (12)(34).

(iv) Rotations of $\pi/2$ and $3\pi/2$ about a face normal. With three distinct face normals and two possible rotation angles for each, the set contains six elements. These could correspond to $(12) \equiv (12)(3)(4)$ or to (1234).

(v) Rotations of π about axes that join the centres of opposite edges. There are six such axes and hence six elements. As in (iv), these could correspond to $(12) \equiv (12)(3)(4)$ or to (1234).

Taking the identification given in the question for (iv) and (v), the reformulated table (in which only the headings have changed) is given in table 29.2.

Irrep	Typical element and class size				
	I	2_d	3	4_z	2_z
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	−1	1	−1	1
E	2	0	−1	0	2
T_1	3	1	0	−1	−1
T_2	3	−1	0	1	−1

Table 29.2 The character table for the symmetry group 432 or O.

We note that the rotational symmetries of a cube can, alternatively, be characterised by the effects they have on the orientations in space of its four body diagonals. For example, a rotation of π about a face normal interchanges them in pairs – represented in cycle notation by the form (12)(34). Using this formulation of the symmetry group, the assignments for (iv) and (v) are unambiguous.

29.7 *In a certain crystalline compound, a thorium atom lies at the centre of a regular octahedron of six sulphur atoms at positions* $(\pm a, 0, 0)$, $(0, \pm a, 0)$, $(0, 0, \pm a)$. *These can be considered as being positioned at the centres of the faces of a cube of side* $2a$. *The sulphur atoms produce at the site of the thorium atom an electric field that has the same symmetry group as a cube (*432 *or* O*).*

The five degenerate d-electron orbitals of the thorium atom can be expressed, relative to any arbitrary polar axis, as

$$(3\cos^2\theta - 1)f(r), \qquad e^{\pm i\phi}\sin\theta\cos\theta f(r), \qquad e^{\pm 2i\phi}\sin^2\theta f(r).$$

A rotation about that polar axis through an angle ϕ' *in effect changes* ϕ *to* $\phi - \phi'$. *Use this to show that the character of the rotation in a representation based on the orbital wavefunctions is given by*

$$1 + 2\cos\phi' + 2\cos 2\phi'$$

and hence that the characters of the representation, in the order of the symbols given in exercise 29.5, is 5, −1, 1, −1, 1. *Deduce that the five-fold degenerate level is split into two levels, a doublet and a triplet.*

The electric field at the thorium atom has the symmetries of group 432 and the d-electron orbitals are

$$\begin{aligned}\psi_1 &= (3\cos^2\theta - 1)f(r),\\ \psi_{2,3} &= e^{\pm i\phi}\sin\theta\cos\theta f(r),\\ \psi_{4,5} &= e^{\pm 2i\phi}\sin^2\theta f(r).\end{aligned}$$

Taking the ψ_i $(i = 1, 2, \ldots, 5)$ as a basis, the representation of a rotation by ϕ' is a 5×5 matrix whose diagonal elements are equal to the factor by which each basis function is multiplied when that function is subjected to the rotation.

ψ_1 does not depend upon ϕ and so is unaltered; its entry is 1.

$\psi_{2,3}$ become $\sin\theta\cos\theta f(r)\, e^{\pm i\phi \mp i\phi'}$; their entries are $e^{-i\phi'}$ and $e^{i\phi'}$.

$\psi_{4,5}$ become $\sin^2\theta f(r)\, e^{\pm 2i\phi \mp 2i\phi'}$; their entries are $e^{-2i\phi'}$ and $e^{i2\phi'}$.

The trace of the representative matrix, and therefore the character of the rotation, is thus

$$\chi = 1 + e^{-i\phi'} + e^{i\phi'} + e^{-2i\phi'} + e^{2i\phi'} = 1 + 2\cos\phi' + 2\cos 2\phi'.$$

For the symmetry elements in the group 432, the corresponding rotation angles

and characters are as follows:

Symmetry	ϕ'	χ
I	0	$1+2+2=5$
3	$\pm 2\pi/3$	$1+2(-\frac{1}{2})+2(-\frac{1}{2})=-1$
2_z	π	$1+2(-1)+2(1)=1$
4_z	$\pm\pi/2$	$1+2(0)+2(-1)=-1$
2_d	π	$1+2(-1)+2(1)=1$

Rewriting these results in a form similar to that in which the character table of 432 has been previously presented, we have

Symmetry	I	2_d	3	$4z$	$2z$
Character, χ	5	1	-1	-1	1

We now compare this with table 29.2, compiled in exercise 29.5, and see, or calculate using the equation

$$m_\mu = \frac{1}{g}\sum_X \left[\chi^{(\mu)}(X)\right]^* \chi(X) = \frac{1}{g}\sum_i c_i \left[\chi^{(\mu)}(X_i)\right]^* \chi(X_i),$$

that this character set is the direct sum of those for the two dimensional irrep E and the three-dimensional irrep T_1, as given in that table:

	I	2_d	3	4_z	2_z
E	2	0	-1	0	2
T_1	3	1	0	-1	-1
χ	5	1	-1	-1	1

The n (mixed) orbitals transforming according to any particular n-dimensional irrep will all have the same energy, but, barring accidental coincidences, it will be a different energy from that corresponding to a different irrep. Thus the five-fold degenerate level is split into a doublet (E) and a triplet (T_1).

29.9 *The hydrogen atoms in a methane molecule* CH_4 *form a perfect tetrahedron with the carbon atom at its centre. The molecule is most conveniently described mathematically by placing the hydrogen atoms at the points* $(1,1,1)$, $(1,-1,-1)$, $(-1,1,-1)$ *and* $(-1,-1,1)$. *The symmetry group to which it belongs, the tetrahedral group (*$\bar{4}3m$ *or* T_d*), has classes typified by* I, 3, 2_z, m_d *and* $\bar{4}_z$, *where the first three are as in exercise 29.5,* m_d *is a reflection in the mirror plane* $x-y=0$ *and* $\bar{4}_z$ *is a rotation of* $\pi/2$ *about the* z*-axis followed by an inversion in the origin. A reflection in a mirror plane can be considered as a rotation of* π *about an axis perpendicular to the plane, followed by an inversion in the origin.*

The character table for the group $\bar{4}3m$ *is very similar to that for the group* 432, *and has the form shown in table 29.3. By following the steps given below, determine how many different internal vibration frequencies the* CH_4 *molecule has.*

(a) *Consider a representation based on the twelve coordinates* x_i, y_i, z_i *for* $i = 1,2,3,4$. *For those hydrogen atoms that transform into themselves, a rotation through an angle* θ *about an axis parallel to one of the coordinate axes gives rise in this natural representation to the diagonal elements* 1 *for the corresponding coordinate and* $2\cos\theta$ *for the two orthogonal coordinates. If the rotation is followed by an inversion then these entries are multiplied by* -1. *Atoms not transforming into themselves give a zero diagonal contribution. Show that the characters of the natural representation are* 12, 0, 0, 0, 2 *and hence that its expression in terms of irreps is*

$$A_1 \oplus E \oplus T_1 \oplus 2T_2.$$

(b) *The irreps of the bodily translational and rotational motions are included in this expression and need to be identified and removed. Show that when this is done it can be concluded that there are three different internal vibration frequencies in the* CH_4 *molecule. State their degeneracies and check that they are consistent with the expected number of normal coordinates needed to describe the internal motions of the molecule.*

(a) We consider each type of rotation in turn and determine how many of the hydrogen atoms are transformed into themselves, i.e. do not change position.

Under I all twelve atoms retain their original positions and so $\chi(I) = 12$.

For the symmetry 3 the rotation angle is $\pm 2\pi/3$ and for each such rotation one atom retains its original place. However, it contributes $1 + 2\cos(2\pi/3) = 1 + 2(-\frac{1}{2}) = 0$ and so $\chi(3) = 0$.

For the symmetries 2_z and $\bar{4}_z$ no atoms retain their original places and the corresponding characters are both 0.

Irreps	Typical element and class size I 1	3 8	2_z 3	$\bar{4}_z$ 6	m_d 6	Functions transforming according to irrep
A_1	1	1	1	1	1	$x^2+y^2+z^2$
A_2	1	1	1	-1	-1	
E	2	-1	2	0	0	$(x^2-y^2, 3z^2-r^2)$
T_1	3	0	-1	1	-1	(R_x, R_y, R_z)
T_2	3	0	-1	-1	1	(x, y, z); (xy, yz, zx)

Table 29.3 The character table for the symmetry group $\bar{4}3m$.

Finally, for m_d, a reflection in one of the six mirror planes, there are two atoms that lie in any one of the planes (with the other two atoms placed symmetrically, one on either side of it). Thus two atoms are unchanged. As explained in the question, a reflection in a mirror plane can be considered as a rotation of π about an axis perpendicular to the plane, followed by an inversion in the origin; the latter gives rise to an additional factor of -1. As a result, each of the two atoms contributes $(-1)(1+2\cos\pi)=1$ to the character of m_d. Thus $\chi(m_d)=2$ and the full character set of the natural representation is $(12, 0, 0, 0, 2)$.

It then follows that

$$m_{A_1} = \frac{1(1)(12)+8(1)(0)+3(1)(0)+6(1)(0)+6(1)(2)}{24} = 1,$$

$$m_{A_2} = \frac{1(1)(12)+8(1)(0)+3(1)(0)+6(-1)(0)+6(-1)(2)}{24} = 0,$$

$$m_{E} = \frac{1(2)(12)+8(-1)(0)+3(2)(0)+6(0)(0)+6(0)(2)}{24} = 1,$$

$$m_{T_1} = \frac{1(3)(12)+8(0)(0)+3(-1)(0)+6(1)(0)+6(-1)(2)}{24} = 1,$$

$$m_{T_2} = \frac{1(3)(12)+8(0)(0)+3(-1)(0)+6(-1)(0)+6(1)(2)}{24} = 2.$$

Thus the irreps present in this representation are

$$A_1 \oplus E \oplus T_1 \oplus 2T_2.$$

(b) Bodily translation of the centre of mass of the molecule is included in this representation since the representation allows all coordinates to vary independently. From table 29.3, the set (x, y, z) transforms as T_2 and so this motion corresponds to one of the two T_2 irreps found above.

Equally a rigid-body rotation of the molecule *about* its centre of mass is included; from the table (R_x, R_y, R_z) transform as T_1 and so this rotation, which contains no internal vibrations, accounts for the T_1 irrep.

After these two irreps are removed we are left with the irreps of the internal vibrations, which are A_1, E, and T_2. They are, respectively, one-, two- and three-dimensional irreps and therefore the corresponding vibration frequencies have degeneracies of 1, 2 and 3. This gives a total of six internal coordinates, in accordance with the twelve original ones, less the three translational coordinates of the centre of mass and the three coordinates needed to specify the direction of the axis of a rigid-body rotation.

29.11 *Use the results of exercise 28.23 to find the character table for the dihedral group $\mathcal{D}_5$, the symmetry group of a regular pentagon.*

As shown in exercise 28.23, the group $\mathcal{D}_5$ has ten elements and four classes: $\{I\}$, $\{R, R^4\}$, $\{R^2, R^3\}$ and $\{m_1, m_2, m_3, m_4, m_5\}$. Here R is a rotation through $2\pi/5$.

Since there are ten elements and four classes, and hence four irreps, we must have that the dimensionalities of the irreps satisfy

$$n_1^2 + n_2^2 + n_3^2 + n_4^2 = 10.$$

This has only one (non-zero) integral solution, $n_1 = n_2 = 1$ and $n_3 = n_4 = 2$. The identity irrep, A_1, must be one of the one-dimensional irreps, and the character table must have the form

Irrep	I	R, R^4	R^2, R^3	m_i $(i = 1, 5)$
A_1	1	1	1	1
A_2	1	a	b	c
E_1	2	d	e	f
E_2	2	g	h	j

For A_2 we must have both $1 + 2|a|^2 + 2|b|^2 + 5|c|^2 = 10$ (summation rule) and $1 + 2a + 2b + 5c = 0$ (orthogonality with A_1). Since the m_i have order 2 and A_2 is one-dimensional, c can only be a second root of unity, i.e. either of 1 or -1. The only solution to these simultaneous equations, even allowing a and b to be complex (but restricted to each being a fifth root of unity), is $a = b = 1$ and $c = -1$.

For E_1 (and similarly for E_2),

$$4 + 2|d|^2 + 2|e|^2 + 5|f|^2 = 10.$$

Arguing as previously, we conclude that, because E is two-dimensional, f can only be the sum of two values which are either $+1$ or -1. Hence, only 0 and ± 2 are possible, and the values ± 2 are impossible in this case. This conclusion can

be confirmed by noting that the E_1 character set has to be orthogonal to those for both A_1 and A_2. So both

$$1(1)2 + 2(1)d + 2(1)e + 5(1)f = 0$$

and

$$1(1)2 + 2(1)d + 2(1)e + 5(-1)f = 0,$$

implying that $f = 0$.

We are left with

$$|d|^2 + |e|^2 = 3 \text{ and } 1 + d + e = 0.$$

This clearly has no integer solutions, but we attempt to find real solutions before considering complex ones. If d is real then e must also be real. Substituting $e = -1 - d$ into the first equation gives the quadratic equation

$$d^2 + (-1-d)^2 = 3 \quad \Rightarrow \quad d^2 + d - 1 = 0 \quad \Rightarrow \quad d = \frac{-1 \pm \sqrt{5}}{2}.$$

If d is taken as $\frac{1}{2}(-1+\sqrt{5})$ (the golden mean!) then $e = \frac{1}{2}(-1-\sqrt{5})$, the other root of the quadratic. This completes the character set for E_1. That for E_2 is obtained by setting $j = 0$ and assigning $\frac{1}{2}(-1+\sqrt{5})$ to h and $\frac{1}{2}(-1-\sqrt{5})$ to g; this can be confirmed by checking the orthogonality relation

$$\begin{aligned} &1(2)(2) + 2[\tfrac{1}{2}(-1+\sqrt{5})\,\tfrac{1}{2}(-1-\sqrt{5})] \\ &\qquad + 2[\tfrac{1}{2}(-1-\sqrt{5})\,\tfrac{1}{2}(-1+\sqrt{5})] + 5(0)(0) = 4 - 2 - 2 + 0 = 0. \end{aligned}$$

We also note that, for example,

$$\frac{-1+\sqrt{5}}{2} = \exp\left(\frac{2\pi i}{5}\right) + \exp\left(\frac{4 \times 2\pi i}{5}\right) = 2\cos\frac{2\pi}{5} = 0.6180,$$

i.e. d and h are each the sum of two fifth roots of unity. The same applies to e and g. The final character table reads

Irrep	I	R, R^4	R^2, R^3	m_i $(i = 1, 5)$
A_1	1	1	1	1
A_2	1	1	1	-1
E_1	2	$\frac{1}{2}(-1+\sqrt{5})$	$-\frac{1}{2}(1+\sqrt{5})$	0
E_2	2	$-\frac{1}{2}(1+\sqrt{5})$	$\frac{1}{2}(-1+\sqrt{5})$	0

29.13 *Further investigation of the crystalline compound considered in exercise 29.7 shows that the octahedron is not quite perfect but is elongated along the* $(1,1,1)$ *direction with the sulphur atoms at positions* $\pm(a+\delta,\delta,\delta)$, $\pm(\delta,a+\delta,\delta)$, $\pm(\delta,\delta,a+\delta)$, *where* $\delta \ll a$. *This structure is invariant under the (crystallographic) symmetry group* 32 *with three two-fold axes along directions typified by* $(1,-1,0)$. *The latter axes, which are perpendicular to the* $(1,1,1)$ *direction, are axes of two-fold symmetry for the perfect octahedron. The group* 32 *is really the three-dimensional version of the group* 3*m* *and has the same character table. That for* 3*m* *is*

$3m$	I	A,B	C,D,E
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Use this to show that, when the distortion of the octahedron is included, the doublet found in exercise 29.7 is unsplit but the triplet breaks up into a singlet and a doublet.

The perfect octahedron is invariant under the operations of group 432, whose character table is as follows:

Irrep	Typical element and class size				
	I	2_d	3	4_z	2_z
	1	6	8	6	3
A_1	1	1	1	1	1
A_2	1	-1	1	-1	1
E	2	0	-1	0	2
T_1	3	1	0	-1	-1
T_2	3	-1	0	1	-1

The distorted octahedron is invariant only under the operations of the smaller group 32, whose character table is

$3m$	I	R,R^2	m_i
A_1	1	1	1
A_2	1	1	-1
E	2	-1	0

Here R is a rotation through $2\pi/3$ and its class corresponds to the class denoted by '3' in group 432. The reflection symmetries correspond to rotations by π when considered as operations in three dimensions (as opposed to in a plane); thus they correspond to the class 2_d. We are thus concerned with the first three classes

in the 432 table, but with the second and third interchanged as compared with the 32 table.

Using the order in the 32 table, E has the characters $(2, -1, 0)$. This two-dimensional irrep also appears in the 432 table, and so the corresponding doublet level in the thorium atom is not split as a result of the distortion of the sulphur octahedron. However, the triplet level, whose components transform as T_1, will be affected by the distortion. The irrep T_1 does not appear in the 32 table but has to be made up from E and A_1; in terms of character sets

$$(3, 0, 1) = (2, -1, 0) + (1, 1, 1).$$

In physical terms, the triplet state in thorium is split by the distorted electric field due to the sulphur atoms into a doublet and a singlet.

30

Probability

30.1 *By shading or numbering Venn diagrams, determine which of the following are valid relationships between events. For those that are, prove the relationship using de Morgan's laws.*

(a) $\overline{(\bar{X} \cup Y)} = X \cap \bar{Y}$.
(b) $\bar{X} \cup \bar{Y} = \overline{(X \cup Y)}$.
(c) $(X \cup Y) \cap Z = (X \cup Z) \cap Y$.
(d) $X \cup \overline{(Y \cap Z)} = (X \cup \bar{Y}) \cap \bar{Z}$.
(e) $X \cup \overline{(Y \cap Z)} = (X \cup \bar{Y}) \cup \bar{Z}$.

For each part of this question we refer to the corresponding part of figure 30.1.

(a) This relationship is correct as both expressions define the shaded region that is both inside X and outside Y.

(b) This relationship is *not* valid. The LHS specifies the whole sample space *apart from* the region marked with the heavy shading. The RHS defines the region that is lightly shaded. The unmarked regions of X and Y are included in the former but not in the latter.

(c) This relationship is *not* valid. The LHS specifies the sum of the regions marked 2, 3 and 4 in the figure, whilst the RHS defines the sum of the regions marked 1, 3 and 4.

(d) This relationship is *not* valid. On the LHS, $\overline{Y \cap Z}$ is the whole sample space apart from regions 3 and 4. So $X \cup \overline{(Y \cap Z)}$ consists of all regions except for region 3. On the RHS, $X \cup \bar{Y}$ contains all regions except 3 and 7. The events $\bar{Z}$ contain regions 1, 6, 7 and 8 and so $(X \cup \bar{Y}) \cap \bar{Z}$ consists of regions 1, 6 and 8. Thus regions 2, 4, 5 and 7 are in one specification but not in the other.

Figure 30.1 The Venn diagrams used in exercise 30.1.

(e) This relationship is valid. The LHS is as found in (d), namely all regions except for region 3. The RHS consists of the union (as opposed to the intersection) of the two subregions found in (d) and thus contains those regions found in either or both of $X \cup \bar{Y}$ (1, 2, 4, 5, 6 and 8) and $\bar{Z}$ (1, 6, 7 and 8). This covers all regions except region 3 – in agreement with those found for the LHS.

For the two valid relationships, their proofs using de Morgan's laws are:

(a) $\overline{(\bar{X} \cup Y)} = \bar{\bar{X}} \cap \bar{Y} = X \cap \bar{Y}$,

(e) $X \cup \overline{(Y \cap Z)} = X \cup (\bar{Y} \cup \bar{Z}) = (X \cup \bar{Y}) \cup \bar{Z}$.

30.3 *A and B each have two unbiased four-faced dice, the four faces being numbered* 1, 2, 3 *and* 4. *Without looking, B tries to guess the sum x of the numbers on the bottom faces of A's two dice after they have been thrown onto a table. If the guess is correct B receives* x^2 *euros, but if not he loses x euros.*

Determine B's expected gain per throw of A's dice when he adopts each of the following strategies:

(a) *he selects x at random in the range* $2 \le x \le 8$;
(b) *he throws his own two dice and guesses x to be whatever they indicate;*
(c) *he takes your advice and always chooses the same value for x. Which number would you advise?*

We first calculate the probabilities $p(x)$ and the corresponding gains $g(x) = p(x)x^2 - [1 - p(x)]x$ for each value of the total x. Expressing both in units of $1/16$, they are as follows:

x	2	3	4	5	6	7	8
$p(x)$	1	2	3	4	3	2	1
$g(x)$	−26	−24	−4	40	30	0	−56

(a) If B's guess is random in the range $2 \le x \le 8$ then his expected return is

$$\frac{1}{16}\frac{1}{7}(-26 - 24 - 4 + 40 + 30 + 0 - 56) = -\frac{40}{112} = -0.36 \text{ euros.}$$

(b) If he picks by throwing his own dice then his distribution of guesses is the same as that of $p(x)$ and his expected return is

$$\frac{1}{16}\frac{1}{16}[1(-26) + 2(-24) + 3(-4) + 4(40) + 3(30) + 2(0) + 1(-56)]$$
$$= \frac{108}{256} = 0.42 \text{ euros.}$$

(c) As is clear from the tabulation, the best return of $40/16 = 2.5$ euros is expected if B always chooses '5' as his guess. Of course, you should not advise him but offer to take his place!

30.5 *Two duellists, A and B, take alternate shots at each other, and the duel is over when a shot (fatal or otherwise!) hits its target. Each shot fired by A has a probability α of hitting B, and each shot fired by B has a probability β of hitting A. Calculate the probabilities P_1 and P_2, defined as follows, that A will win such a duel: P_1, A fires the first shot; P_2, B fires the first shot.*

If they agree to fire simultaneously, rather than alternately, what is the probability P_3 that A will win, i.e. hit B without being hit himself?

Each shot has only two possible outcomes, a hit or a miss. P_1 is the probability that A will win when it is his turn to fire the next shot, and he is still able to do so (event W). There are three possible outcomes of the first two shots: C_1, A hits with his shot; C_2, A misses but B hits; C_3, both miss. Thus

$$P_1 = \sum_i \Pr(C_i)\Pr(W|C_i)$$
$$= [\alpha \times 1] + [(1-\alpha)\beta \times 0] + [(1-\alpha)(1-\beta) \times P_1]$$
$$\Rightarrow \quad P_1 = \frac{\alpha}{\alpha + \beta - \alpha\beta}.$$

When B fires first but misses, the situation is the one just considered. But if B

hits with his first shot then clearly A's chances of winning are zero. Since these are the only two possible outcomes of B's first shot, we can write

$$P_2 = [\beta \times 0] + [(1-\beta) \times P_1] \quad \Rightarrow \quad P_2 = \frac{(1-\beta)\alpha}{\alpha + \beta - \alpha\beta}.$$

When both fire at the same time there are four possible outcomes D_i to the first round: D_1, A hits and B misses; D_2, B hits but A misses; D_3, they both hit; D_4, they both miss. If getting hit, even if you manage to hit your opponent, does not count as a win, then

$$\begin{aligned} P_3 &= \sum_i \Pr(D_i)\Pr(W|D_i) \\ &= [\alpha(1-\beta) \times 1] + [(1-\alpha)\beta \times 0] + [\alpha\beta \times 0] + [(1-\alpha)(1-\beta) \times P_3]. \end{aligned}$$

This can be rearranged as

$$P_3 = \frac{\alpha(1-\beta)}{\alpha + \beta - \alpha\beta} = P_2.$$

Thus the result is the same as if B had fired first. However, we also note that if all that matters to A is that B is hit, whether or not he is hit himself, then the third bracket takes the value $\alpha\beta \times 1$ and P_3 takes the same value as P_1.

30.7 *A tennis tournament is arranged on a straight knockout basis for 2^n players, and for each round, except the final, opponents for those still in the competition are drawn at random. The quality of the field is so even that in any match it is equally likely that either player will win. Two of the players have surnames that begin with 'Q'. Find the probabilities that they play each other*

(a) *in the final,*
(b) *at some stage in the tournament.*

Let p_r be the probability that *before* the rth round the two players are both still in the tournament (and, by implication, have not met each other). Clearly, $p_1 = 1$.

Before the rth round there are 2^{n+1-r} players left in. For both 'Q' players to still be in before the $(r+1)$th round, Q_1 must avoid Q_2 in the draw and both must win their matches. Thus

$$p_{r+1} = \frac{2^{n+1-r} - 2}{2^{n+1-r} - 1}\left(\frac{1}{2}\right)^2 p_r.$$

(a) The probability that they meet in the final is p_n, given by

$$\begin{aligned}
p_n &= 1\,\frac{2^n-2}{2^n-1}\,\frac{1}{4}\,\frac{2^{n-1}-2}{2^{n-1}-1}\,\frac{1}{4}\cdots\frac{2^2-2}{2^2-1}\,\frac{1}{4}\\
&= \left(\frac{1}{4}\right)^{n-1} 2^{n-1}\left[\frac{(2^{n-1}-1)(2^{n-2}-1)\cdots(2^1-1)}{(2^n-1)(2^{n-1}-1)\cdots(2^2-1)}\right]\\
&= \left(\frac{1}{4}\right)^{n-1} 2^{n-1}\,\frac{1}{2^n-1}\\
&= \frac{1}{2^{n-1}(2^n-1)}.
\end{aligned}$$

(b) The more general solution to the recurrence relation derived above is

$$\begin{aligned}
p_r &= 1\,\frac{2^n-2}{2^n-1}\,\frac{1}{4}\,\frac{2^{n-1}-2}{2^{n-1}-1}\,\frac{1}{4}\cdots\frac{2^{n+2-r}-2}{2^{n+2-r}-1}\,\frac{1}{4}\\
&= \left(\frac{1}{4}\right)^{r-1} 2^{r-1}\left[\frac{(2^{n-1}-1)(2^{n-2}-1)\cdots(2^{n+1-r}-1)}{(2^n-1)(2^{n-1}-1)\cdots(2^{n+2-r}-1)}\right]\\
&= \left(\frac{1}{2}\right)^{r-1}\frac{2^{n+1-r}-1}{2^n-1}.
\end{aligned}$$

Before the rth round, if they are both still in the tournament, the probability that they will be drawn against each other is $(2^{n-r+1}-1)^{-1}$. Consequently, the chance that they will meet at *some* stage is

$$\begin{aligned}
\sum_{r=1}^{n} p_r\,\frac{1}{2^{n-r+1}-1} &= \sum_{r=1}^{n}\left(\frac{1}{2}\right)^{r-1}\frac{2^{n+1-r}-1}{2^n-1}\,\frac{1}{2^{n-r+1}-1}\\
&= \frac{1}{2^n-1}\sum_{r=1}^{n}\left(\frac{1}{2}\right)^{r-1}\\
&= \frac{1}{2^n-1}\,\frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}} = \frac{1}{2^{n-1}}.
\end{aligned}$$

This same conclusion can also be reached in the folowing way. The probability that Q_1 *is not* put out of (i.e. wins) the tournament is $(\frac{1}{2})^n$. It follows that the probability that Q_1 *is* put out is $1-(\frac{1}{2})^n$ and that the player responsible is Q_2 with probability $[1-(\frac{1}{2})^n]/(2^n-1) = 2^{-n}$. Similarly, the probability that Q_2 is put out and that the player responsible is Q_1 is also 2^{-n}. These are exclusive events but cover all cases in which Q_1 and Q_2 meet during the tournament, the probability of which is therefore $2\times 2^{-n} = 2^{n-1}$.

30.9 *An electronics assembly firm buys its microchips from three different suppliers; half of them are bought from firm X, whilst firms Y and Z supply 30% and 20%, respectively. The suppliers use different quality-control procedures and the percentages of defective chips are 2%, 4% and 4% for X, Y and Z, respectively. The probabilities that a defective chip will fail two or more assembly-line tests are 40%, 60% and 80%, respectively, whilst all defective chips have a 10% chance of escaping detection. An assembler finds a chip that fails only one test. What is the probability that it came from supplier X?*

Since the number of tests failed by a defective chip are mutually exclusive outcomes (0, 1 or ≥ 2), a chip supplied by X has a probability of failing just one test given by $0.02(1-0.1-0.4) = 0.010$. The corresponding probabilities for chips supplied by Y and Z are $0.04(1-0.1-0.6) = 0.012$ and $0.04(1-0.1-0.8) = 0.004$, respectively.

Using '1' to denote failing a single test, Bayes' theorem gives the probability that the chip was supplied by X as

$$\begin{aligned}\Pr(X|1) &= \frac{\Pr(1|X)\ \Pr(X)}{\Pr(1|X)\ \Pr(X) + \Pr(1|Y)\ \Pr(Y) + \Pr(1|Z)\ \Pr(Z)} \\ &= \frac{0.010 \times 0.5}{0.010 \times 0.5 + 0.012 \times 0.3 + 0.004 \times 0.2} = \frac{50}{94}.\end{aligned}$$

30.11 *A boy is selected at random from amongst the children belonging to families with n children. It is known that he has at least two sisters. Show that the probability that he has $k-1$ brothers is*

$$\frac{(n-1)!}{(2^{n-1}-n)(k-1)!(n-k)!},$$

for $1 \leq k \leq n-2$ and zero for other values of k. Assume that boys and girls are equally likely.

The boy has $n-1$ siblings. Let A_j be the event that $j-1$ of them are brothers, i.e. his family contains j boys and $n-j$ girls. The probability of event A_j is

$$\Pr(A_j) = \frac{{}^{n-1}C_{j-1}\left(\frac{1}{2}\right)^{n-1}}{\sum_{j=1}^{n} {}^{n-1}C_{j-1}\left(\frac{1}{2}\right)^{n-1}} = \frac{(n-1)!}{2^{n-1}(j-1)!(n-j)!}.$$

If B is the event that the boy has at least two sisters, then

$$\Pr(B|A_j) = \begin{cases} 1 & 1 \leq j \leq n-2, \\ 0 & n-1 \leq j \leq n. \end{cases}$$

Now we apply Bayes' theorem to give the probability that he has $k-1$ brothers:

$$\Pr(A_k|B) = \frac{1\ \Pr(A_k)}{\sum_{j=1}^{n-2} 1\ \Pr(A_j)},$$

for $1 \le k \le n-2$. The denominator of this expression is the sum $1 = (\frac{1}{2}+\frac{1}{2})^{n-1} = \sum_{j=1}^{n} {}^{n-1}C_{j-1} \left(\frac{1}{2}\right)^{n-1}$, but omitting the $j = n-1$ and the $j = n$ terms, and so is equal to

$$1 - \frac{(n-1)!}{2^{n-1}\,(n-2)!\,1!} - \frac{(n-1)!}{2^{n-1}\,(n-1)!\,0!} = \frac{1}{2^{n-1}}\left[2^{n-1} - (n-1) - 1\right].$$

Thus,

$$\Pr(A_k|B) = \frac{(n-1)!}{2^{n-1}(k-1)!(n-k)!}\,\frac{2^{n-1}}{2^{n-1}-n} = \frac{(n-1)!}{(2^{n-1}-n)(k-1)!(n-k)!},$$

as given in the question.

30.13 *A set of $2N+1$ rods consists of one of each integer length $1, 2, \ldots, 2N, 2N+1$. Three, of lengths a, b and c, are selected, of which a is the longest. By considering the possible values of b and c, determine the number of ways in which a non-degenerate triangle (i.e. one of non-zero area) can be formed (i) if a is even, and (ii) if a is odd. Combine these results appropriately to determine the total number of non-degenerate triangles that can be formed with the $2N+1$ rods, and hence show that the probability that such a triangle can be formed from a random selection (without replacement) of three rods is*

$$\frac{(N-1)(4N+1)}{2(4N^2-1)}.$$

Rod a is the longest of the three rods. As no two are the same length, let $a > b > c$. To form a non-degenerate triangle we require that $b + c > a$, and, in consequence, $4 \le a \le 2N+1$.

(i) With a even. Consider each b ($< a$) in turn and determine how many values of c allow a triangle to be made:

b	Values of c	Number of c values
$a-1$	$2, 3, \cdots, a-2$	$a-3$
$a-2$	$3, 4, \cdots, a-3$	$a-5$
$\cdots$	$\cdots$	$\cdots$
$\frac{1}{2}a+1$	$\frac{1}{2}a$	1

Thus, there are $1 + 3 + 5 + \cdots + (a-3)$ possible triangles when a is even.

(ii) A table for odd a is similar, except that the last line will read $b = \frac{1}{2}(a+3)$, $c = \frac{1}{2}(a-1)$ or $\frac{1}{2}(a+1)$, and the number of c values $= 2$. Thus there are $2+4+6+\cdots+(a-3)$ possible triangles when a is odd.

To find the total number $n(N)$ of possible triangles, we group together the cases $a = 2m$ and $a = 2m+1$, where $m = 1, 2, \ldots, N$. Then,

$$\begin{aligned}
n(N) &= \sum_{m=2}^{N} [1+3+\cdots+(2m-3)] + [2+4+\cdots+(2m+1-3)] \\
&= \sum_{m=2}^{N} \sum_{k=1}^{2m-2} k = \sum_{m=2}^{N} \tfrac{1}{2}(2m-2)(2m-1) = \textstyle\sum_{m=2}^{N} 2m^2 - 3m + 1 \\
&= 2\left[\tfrac{1}{6}N(N+1)(2N+1) - 1\right] - 3\left[\tfrac{1}{2}N(N+1) - 1\right] + N - 1 \\
&= \frac{N}{6}[2(N+1)(2N+1) - 9(N+1) + 6] \\
&= \frac{N}{6}(4N^2 - 3N - 1) = \frac{N}{6}(4N+1)(N-1).
\end{aligned}$$

The number of ways that three rods can be drawn at random (without replacement) is $(2N+1)(2N)(2N-1)/3!$ and so the probability that they can form a triangle is

$$\frac{N(4N+1)(N-1)}{6} \frac{3!}{(2N+1)(2N)(2N-1)} = \frac{(N-1)(4N+1)}{2(4N^2-1)},$$

as stated in the question.

30.15 *The duration (in minutes) of a telephone call made from a public call-box is a random variable T. The probability density function of T is*

$$f(t) = \begin{cases} 0 & t < 0, \\ \frac{1}{2} & 0 \le t < 1, \\ ke^{-2t} & t \ge 1, \end{cases}$$

where k is a constant. To pay for the call, 20 pence has to be inserted at the beginning, and a further 20 pence after each subsequent half-minute. Determine by how much the average cost of a call exceeds the cost of a call of average length charged at 40 pence per minute.

From the normalisation of the PDF, we must have

$$1 = \int_0^\infty f(t)\,dt = \frac{1}{2} + \int_1^\infty ke^{-2t}\,dt = \frac{1}{2} + \frac{ke^{-2}}{2} \quad \Rightarrow \quad k = e^2.$$

The average length of a call is given by

$$\begin{aligned}\bar{t} &= \int_0^1 t\,\frac{1}{2}\,dt + \int_1^\infty t\,e^2 e^{-2t}\,dt \\ &= \frac{1}{2}\,\frac{1}{2} + \left[\frac{te^2e^{-2t}}{-2}\right]_1^\infty + \int_1^\infty \frac{e^2\,e^{-2t}}{2}\,dt = \frac{1}{4} + \frac{1}{2} + \frac{e^2}{2}\left[\frac{e^{-2t}}{-2}\right]_1^\infty = \frac{3}{4} + \frac{1}{4} = 1.\end{aligned}$$

Let $p_n = \Pr\{\frac{1}{2}(n-1) < t < \frac{1}{2}n\}$. The corresponding cost is $c_n = 20n$. Clearly, $p_1 = p_2 = \frac{1}{4}$ and, for $n > 2$,

$$p_n = e^2 \int_{(n-1)/2}^{n/2} e^{-2t}\,dt = e^2\left[\frac{e^{-2t}}{-2}\right]_{(n-1)/2}^{n/2} = \frac{1}{2}e^2(e-1)e^{-n}.$$

The average cost of a call is therefore

$$\bar{c} = 20\left[\frac{1}{4} + 2\,\frac{1}{4} + \sum_{n=3}^{\infty} n\frac{1}{2}e^2(e-1)e^{-n}\right] = 15 + 10e^2(e-1)\sum_{n=3}^{\infty} ne^{-n}.$$

Now, the final summation might be recognised as part of an arithmetico-geometric series whose sum can be found from the standard formula

$$S = \frac{a}{1-r} + \frac{rd}{(1-r)^2},$$

with $a = 0$, $d = 1$ and $r = e^{-1}$, or could be evaluated directly by noting that as a geometric series,

$$\sum_{n=0}^{\infty} e^{-nx} = \frac{1}{1-e^{-x}}.$$

Differentiating this with respect to x and then setting $x = 1$ gives

$$-\sum_{n=0}^{\infty} ne^{-nx} = -\frac{e^{-x}}{(1-e^{-x})^2} \quad \Rightarrow \quad \sum_{n=0}^{\infty} ne^{-n} = \frac{e^{-1}}{(1-e^{-1})^2}.$$

From either method it follows that

$$\begin{aligned}\sum_{n=3}^{\infty} ne^{-n} &= \frac{e}{(e-1)^2} - e^{-1} - 2e^{-2} \\ &= \frac{e - e + 2 - e^{-1} - 2 + 4e^{-1} - 2e^{-2}}{(e-1)^2} = \frac{3e^{-1} - 2e^{-2}}{(e-1)^2}.\end{aligned}$$

The total charge therefore exceeds that of a call of average length (1 minute) charged at 40 pence per minute by the amount (in pence)

$$15 + 10e^2(e-1)\frac{3e^{-1} - 2e^{-2}}{(e-1)^2} - 40 = \frac{10(3e-2) - 25e + 25}{e-1} = \frac{5e+5}{e-1} = 10.82.$$

30.17 *If the scores in a cup football match are equal at the end of the normal period of play, a 'penalty shoot-out' is held in which each side takes up to five shots (from the penalty spot) alternately, the shoot-out being stopped if one side acquires an unassailable lead (i.e. has a lead greater than its opponents have shots remaining). If the scores are still level after the shoot-out a 'sudden death' competition takes place.*

In sudden death each side takes one shot and the competition is over if one side scores and the other does not; if both score, or both fail to score, a further shot is taken by each side, and so on. Team 1, which takes the first penalty, has a probability p_1, which is independent of the player involved, of scoring and a probability q_1 $(= 1 - p_1)$ of missing; p_2 and q_2 are defined likewise.

Let $\Pr(i : x, y)$ *be the probability that team i has scored x goals after y attempts, and $f(M)$ be the probability that the shoot-out terminates after a* total *of M shots.*

(a) *Prove that the probability that 'sudden death' will be needed is*

$$f(11+) = \sum_{r=0}^{5} ({}^5C_r)^2 (p_1 p_2)^r (q_1 q_2)^{5-r}.$$

(b) *Give reasoned arguments (preferably without first looking at the expressions involved) which show that, for $N = 3, 4, 5$,*

$$f(M = 2N) = \sum_{r=0}^{2N-6} \left\{ \begin{array}{l} p_2 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N - 1) \\ \quad + q_2 \Pr(1 : 6 - N + r, N) \Pr(2 : r, N - 1) \end{array} \right\}$$

and, for $N = 3, 4$,

$$f(M = 2N + 1) = \sum_{r=0}^{2N-5} \left\{ \begin{array}{l} p_1 \Pr(1 : 5 - N + r, N) \Pr(2 : r, N) \\ \quad + q_1 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N). \end{array} \right\}$$

(c) *Give an explicit expression for* $\Pr(i : x, y)$ *and hence show that if the teams are so well matched that $p_1 = p_2 = 1/2$ then*

$$f(2N) = \sum_{r=0}^{2N-6} \left(\frac{1}{2^{2N}} \right) \frac{N!(N-1)!6}{r!(N-r)!(6-N+r)!(2N-6-r)!},$$

$$f(2N+1) = \sum_{r=0}^{2N-5} \left(\frac{1}{2^{2N}} \right) \frac{(N!)^2}{r!(N-r)!(5-N+r)!(2N-5-r)!}.$$

(d) *Evaluate these expressions to show that, expressing $f(M)$ in units of 2^{-8},*

M	6	7	8	9	10	11+
$f(M)$	8	24	42	56	63	63

Give a simple explanation of why $f(10) = f(11+)$.

(a) For 'sudden death' to be needed the scores must be equal after ten shots, five from each side. A score of r goals each has a probability

$$\left({}^5C_r\, p_1^r\, q_1^{5-r} \right) \times \left({}^5C_r\, p_2^r\, q_2^{5-r} \right),$$

and the total probability that the scores are equal after ten shots is obtained by summing this over all possible values of r ($r = 0, 1, \ldots, 5$). Thus

$$f(11+) = \sum_{r=0}^{5} ({}^5C_r)^2 (p_1 p_2)^r (q_1 q_2)^{5-r}.$$

(b) For the shoot-out to terminate after $2N$ shots (≤ 10 shots), one team must be $6 - N$ goals ahead and team 2 must just have taken the last shot.

(i) If team 1 won, it was because team 2 failed with their Nth shot and team 1 must have been $6 - N$ goals ahead before the final shot was taken. The probability for this is $q_2 \Pr(1 : 6 - N + r, N) \Pr(2 : r, N - 1)$.
(ii) If team 2 won, it must have been successful with its last shot and, before it, must have been $5 - N$ goals ahead. The probability for this is $p_2 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N - 1)$.

This type of finish can only arise if $N > 5 - N$, i.e. $N = 3$, 4 or 5. Further, since in $\Pr(i : x, y)$ we must have $x \leq y$, the range for r is determined, from (i), by $6 - N + r \leq N$ and, from (ii), by $5 - N + r \leq N - 1$; these both give $0 \leq r \leq 2N - 6$. Thus

$$f(M = 2N) = \sum_{r=0}^{2N-6} \left\{ \begin{array}{l} p_2 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N - 1) \\ \quad + q_2 \Pr(1 : 6 - N + r, N) \Pr(2 : r, N - 1) \end{array} \right\}.$$

For $M = 2N + 1$, the shoot-out terminates after team 1's $(N + 1)$th shot, which must have been successful if it wins, or unsuccessful if team 2 wins.

(i) If team 1 wins, it must now be $6 - N$ goals ahead, i.e. it was $5 - N$ goals ahead before its successful $(N + 1)$th shot. This has probability $p_1 \Pr(1 : 5 - N + r, N) \Pr(2 : r, N)$.
(i) If team 2 wins, it must have been $5 - N$ goals ahead, before team 1's unsuccessful $(N + 1)$th shot. The probability for this is $q_1 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N)$.

This type of ending can only occur if $N > 5 - N$ and $2N + 1 \leq 10$, i.e. $N = 3$ or 4. Arguing as before, we see that both (i) and (ii) require $5 - N + r \leq N$, i.e. $0 \leq r \leq 2N - 5$. Thus

$$f(M = 2N + 1) = \sum_{r=0}^{2N-5} \left\{ \begin{array}{l} p_1 \Pr(1 : 5 - N + r, N) \Pr(2 : r, N) \\ \quad + q_1 \Pr(1 : r, N) \Pr(2 : 5 - N + r, N) \end{array} \right\}.$$

(c) As in part (a), $\Pr(i : x, y)$ is given by the binomial distribution as

$$\Pr(i : x, y) = {}^yC_x\, p_i^x\, q_i^{y-x}.$$

We now set $p_1 = p_2 = q_1 = q_2 = \frac{1}{2}$ and calculate

$$\begin{aligned} f(2N) &= \sum_{r=0}^{2N-6} \left[\frac{1}{2}\, {}^{N}C_r\ {}^{N-1}C_{5-N+r} \left(\frac{1}{2}\right)^{N} \left(\frac{1}{2}\right)^{N-1} \right. \\ &\qquad \left. + \frac{1}{2}\, {}^{N}C_{6-N+r}\ {}^{N-1}C_r \left(\frac{1}{2}\right)^{N} \left(\frac{1}{2}\right)^{N-1} \right] \\ &= \sum_{r=0}^{2N-6} \left(\frac{1}{2^{2N}}\right) \left[\frac{N!\,(N-1)!}{r!\,(N-r)!\,(5-N+r)!\,(2N-6-r)!} \right. \\ &\qquad \left. + \frac{N!\,(N-1)!}{(6-N+r)!\,(2N-6-r)!\,r!\,(N-1-r)!} \right] \\ &= \sum_{r=0}^{2N-6} \left(\frac{1}{2^{2N}}\right) \frac{N!\,(N-1)!\,[6-N+r+N-r]}{r!\,(N-r)!\,(6-N+r)!\,(2N-6-r)!} \\ &= \sum_{r=0}^{2N-6} \left(\frac{1}{2^{2N}}\right) \frac{N!\,(N-1)!\,6}{r!\,(N-r)!\,(6-N+r)!\,(2N-6-r)!}. \end{aligned}$$

The value of $f(2N+1)$ is found in a similar way. But, since $p_1 = p_2 = q_1 = q_2 = \frac{1}{2}$, the two terms contributing to it for any particular value of r are equal and each has the value

$$\frac{1}{2}\, {}^{N}C_{5-N+r} \left(\frac{1}{2}\right)^{N} {}^{N}C_r \left(\frac{1}{2}\right)^{N}.$$

When these terms are added and then summed over r we obtain

$$f(2N+1) = \sum_{r=0}^{2N-5} \left(\frac{1}{2^{2N}}\right) \frac{(N!)^2}{r!(N-r)!(5-N+r)!(2N-5-r)!}.$$

(d) Evaluating these expressions for the allowed values of N, that is 3, 4 and 5 for $f(2N)$, and 3 and 4 for $f(2N+1)$, is straightforward but somewhat tedious. The results, as given in the question, are

M	6	7	8	9	10	11+
$f(M)$	8	24	42	56	63	63

Here $f(M)$ is expressed in units of 2^{-8}. As expected, these probabilities add up to unity, and it can be seen that sudden death is needed in about one-quarter of such shoot-outs.

The equality of $f(10)$ and f(11+) is simply explained by the fact that, if the shoot-out has not been settled by then, team 2 is just as likely ($p_2 = \frac{1}{2}$) to take it into sudden death by scoring with its fifth shot as it is to lose it ($q_2 = \frac{1}{2}$) by missing.

30.19 *A continuous random variable X has a probability density function $f(x)$; the corresponding cumulative probability function is $F(x)$. Show that the random variable $Y = F(X)$ is uniformly distributed between* 0 *and* 1.

We first note that, as $F(x)$ is a cumulative probability density function, it has values $F(-\infty) = 0$ and $F(\infty) = 1$ and that $y = F(x)$ has a single-valued inverse $x = x(y)$.

With $Y = F(X)$, we have from the standard result for the distribution of single-valued inverse functions that

$$g(Y) = f(X(Y))\left|\frac{dX}{dY}\right|.$$

However, in this particular case of Y being the cumulative probability function of X, we can evaluate $|dX/dY|$ more explicitly. This is because

$$\frac{dY}{dX} = \frac{d}{dX}F(X) = \frac{d}{dX}\int_{-\infty}^{X} f(u)\,du = f(X),$$

and is non-negative. So,

$$g(Y) = f(X(Y))\left|\frac{dX}{dY}\right| = \frac{dY}{dX}\left|\frac{dX}{dY}\right| = 1.$$

This shows that Y is uniformly distributed on $(0, 1)$.

30.21 *This exercise is about interrelated binomial trials.*

(a) *In two sets of binomial trials T and t, the probabilities that a trial has a successful outcome are P and p, respectively, with corresponding probabilites of failure of $Q = 1 - P$ and $q = 1 - p$. One 'game' consists of a trial T, followed, if T is successful, by a trial t and then a further trial T. The two trials continue to alternate until one of the T-trials fails, at which point the game ends. The score S for the game is the total number of successes in the t-trials. Find the PGF for S and use it to show that*

$$E[S] = \frac{Pp}{Q}, \qquad V[S] = \frac{Pp(1 - Pq)}{Q^2}.$$

(b) *Two normal unbiased six-faced dice A and B are rolled alternately starting with A; if A shows a* 6 *the experiment ends. If B shows an odd number no points are scored, one point is scored for a* 2 *or a* 4, *and two points are awarded for a* 6. *Find the average and standard deviation of the score for the experiment and show that the latter is the greater.*

This is a situation in which the score for the game is a variable length sum, the length N being determined by the outcome of the T-trials. The probability that $N = n$ is given by $h_n = P^n Q$, since n T-trials must succeed and then be followed by a failing T-trial. Thus the PGF for the length of each 'game' is given by

$$\chi_N(t) \equiv \sum_{n=0}^{\infty} h_n t^n = \sum_{n=0}^{\infty} P^n Q t^n = \frac{Q}{1 - Pt}.$$

For each permitted Bernoulli t-trial, $X_i = 1$ with probability p and $X_i = 0$ with probability q; its PGF is thus $\Phi_X(t) = q + pt$. The score for the game is $S = \sum_{i=1}^{N} X_i$ and its PGF is given by the compound function

$$\begin{aligned}\Xi_S(t) &= \chi_N(\Phi_X(t)) \\ &= \frac{Q}{1 - P(q + pt)},\end{aligned}$$

in which the PGF for a single t-trial forms the argument of the PGF for the length of each 'game'.

It follows that the mean of S is found from

$$\Xi_S'(t) = \frac{QPp}{(1 - Pq - Ppt)^2} \quad \Rightarrow \quad E[S] = \Xi_S'(1) = \frac{QPp}{(1 - P)^2} = \frac{Pp}{Q}.$$

To calculate the variance of S we need to find $\Xi_S''(1)$. This second derivative is

$$\Xi_S''(t) = \frac{2QP^2p^2}{(1 - Pq - Ppt)^3} \quad \Rightarrow \quad \Xi_S''(1) = \frac{2P^2p^2}{Q^2}.$$

The variance is therefore

$$\begin{aligned}V[S] &= \Xi_S''(1) + \Xi_S'(1) - [\Xi_S'(1)]^2 \\ &= \frac{2P^2p^2}{Q^2} + \frac{Pp}{Q} - \frac{P^2p^2}{Q^2} \\ &= \frac{Pp(Pp + Q)}{Q^2} = \frac{Pp(P - Pq + Q)}{Q^2} = \frac{Pp(1 - Pq)}{Q^2}.\end{aligned}$$

(b) For die A: $P = \frac{5}{6}$ and $Q = \frac{1}{6}$ giving $\chi_N(t) = 1/(6 - 5t)$.

For die B: $\Pr(X = 0) = \frac{3}{6}$, $\Pr(X = 1) = \frac{2}{6}$ and $\Pr(X = 2) = \frac{1}{6}$ giving $\Phi_X(t) = (3 + 2t + t^2)/6$.

The PGF for the game score S is thus

$$\Xi_S(t) = \frac{1}{6 - \frac{5}{6}(3 + 2t + t^2)} = \frac{6}{21 - 10t - 5t^2}.$$

We need to evaluate the first two derivatives of $\Xi_S(t)$ at $t = 1$, as follows:

$$\Xi'_S(t) = \frac{-6(-10-10t)}{(21-10t-5t^2)^2} = \frac{60+60t}{(21-10t-5t^2)^2}$$

$$\Rightarrow \quad E[S] = \Xi'_S(1) = \frac{120}{6^2} = \frac{10}{3} = 3.33,$$

$$\Xi''_S(t) = \frac{60}{(21-10t-5t^2)^2} - \frac{2(60+60t)(-10-10t)}{(21-10t-5t^2)^3}$$

$$\Rightarrow \quad \Xi''_S(1) = \frac{60}{36} - \frac{2(120)(-20)}{(6)^3} = \frac{215}{9}.$$

Substituting the calculated values gives $V[S]$ as

$$V[S] = \frac{215}{9} + \frac{10}{3} - \left(\frac{10}{3}\right)^2 = \frac{145}{9},$$

from which it follows that

$$\sigma_S = \sqrt{V[S]} = 4.01, \text{ i.e. greater than the mean.}$$

30.23 *A point P is chosen at random on the circle $x^2 + y^2 = 1$. The random variable X denotes the distance of P from $(1,0)$. Find the mean and variance of X and the probability that X is greater than its mean.*

With O as the centre of the unit circle and Q as the point $(1,0)$, let OP make an angle θ with the x-axis OQ. The random variable X then has the value $2\sin(\theta/2)$ with θ uniformly distributed on $(0, 2\pi)$, i.e.

$$f(x)\,dx = \frac{1}{2\pi}\,d\theta.$$

The mean of X is given straightforwardly by

$$\langle X\rangle = \int_0^2 X f(x)\,dx = \int_0^{2\pi} 2\sin\left(\frac{\theta}{2}\right)\frac{1}{2\pi}\,d\theta = \frac{1}{\pi}\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} = \frac{4}{\pi}.$$

For the variance we have

$$\sigma_X^2 = \langle X^2\rangle - \langle X\rangle^2 = \int_0^{2\pi} 4\sin^2\left(\frac{\theta}{2}\right)\frac{1}{2\pi}\,d\theta - \frac{16}{\pi^2} = \frac{4}{2\pi}\,\frac{1}{2}\,2\pi - \frac{16}{\pi^2} = 2 - \frac{16}{\pi^2}.$$

When $X = \langle X\rangle = 4/\pi$, the angle $\theta = 2\sin^{-1}(2/\pi)$ and so

$$\Pr(X > \langle X\rangle) = \frac{2\pi - 4\sin^{-1}\dfrac{2}{\pi}}{2\pi} = 0.561.$$

30.25 *The number of errors needing correction on each page of a set of proofs follows a Poisson distribution of mean μ. The cost of the first correction on any page is α and that of each subsequent correction on the same page is β. Prove that the average cost of correcting a page is*

$$\alpha + \beta(\mu - 1) - (\alpha - \beta)e^{-\mu}.$$

Since the number of errors on a page is Poisson distributed, the probability of n errors on any particular page is

$$\Pr(n \text{ errors}) = p_n = e^{-\mu}\frac{\mu^n}{n!}.$$

The average cost per page, found by averaging the corresponding cost over all values of n, is

$$\begin{aligned} c &= 0\,p_0 + \alpha p_1 + \sum_{n=2}^{\infty}[\,\alpha + (n-1)\beta\,]p_n \\ &= \alpha\mu e^{-\mu} + (\alpha - \beta)\sum_{n=2}^{\infty} p_n + \beta\sum_{n=2}^{\infty} np_n. \end{aligned}$$

Now, $\sum_{n=0}^{\infty} p_n = 1$ and, for a Poisson distribution, $\sum_{n=0}^{\infty} np_n = \mu$. These can be used to evaluate the above, once the $n = 0$ and $n = 1$ terms have been removed. Thus

$$\begin{aligned} c &= \alpha\mu e^{-\mu} + (\alpha - \beta)(1 - e^{-\mu} - \mu e^{-\mu}) + \beta(\mu - 0 - \mu e^{-\mu}) \\ &= \alpha + \beta(\mu - 1) + e^{-\mu}(\alpha\mu - \alpha + \beta - \mu\alpha + \mu\beta - \mu\beta) \\ &= \alpha + \beta(\mu - 1) + e^{-\mu}(\beta - \alpha), \end{aligned}$$

as given in the question.

30.27 *Show that for large r the value at the maximum of the PDF for the gamma distribution of order r with parameter λ is approximately $\lambda/\sqrt{2\pi(r-1)}$.*

The gamma distribution takes the form

$$f(x) = \frac{\lambda}{\Gamma(r)}(\lambda x)^{r-1}e^{-\lambda x}$$

and its maximum will occur when $y(x) = x^{(r-1)}e^{-\lambda x}$ is maximal. This requires

$$0 = \frac{dy}{dx} = (r-1)x^{(r-2)}e^{-\lambda x} - \lambda x^{(r-1)}e^{-\lambda x} \quad \Rightarrow \quad \lambda x = r - 1.$$

The maximum value is thus

$$\gamma_{\max}(r) = \frac{\lambda}{\Gamma(r)}\,(r-1)^{(r-1)}\,e^{-(r-1)}.$$

Now, using Stirling's approximation,

$$\Gamma(n+1) = n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{for large } n,$$

we obtain

$$\begin{aligned}\gamma_{\max}(r) &\approx \frac{\lambda}{\sqrt{2\pi(r-1)}}\,\frac{e^{(r-1)}}{(r-1)^{(r-1)}}\,(r-1)^{(r-1)}\,e^{-(r-1)}\\ &= \frac{\lambda}{\sqrt{2\pi(r-1)}}.\end{aligned}$$

30.29 *The probability distribution for the number of eggs in a clutch is $Po(\lambda)$, and the probability that each egg will hatch is p (independently of the size of the clutch). Show by direct calculation that the probability distribution for the number of chicks that hatch is $Po(\lambda p)$.*

Clearly, to determine the probability that a clutch produces k chicks, we must consider clutches of size n, for all $n \geq k$, and for each such clutch find the probability that exactly k of the n chicks do hatch. We then average over all n, weighting the results according to the distribution of n.

The probability that k chicks hatch from a clutch of size n is ${}^nC_k p^k q^{n-k}$, where $q = 1-p$. The probability that the clutch is of size n is $e^{-\lambda}\lambda^n/n!$. Consequently, the overall probability of k chicks hatching from a clutch is

$$\begin{aligned}\Pr(k \text{ chicks}) &= \sum_{n=k}^{\infty} e^{-\lambda}\,\frac{\lambda^n}{n!}\;{}^nC_k\,p^k\,q^{n-k}\\ &= e^{-\lambda}p^k\lambda^k\sum_{n=k}^{\infty}\frac{(\lambda q)^{n-k}}{n!}\,\frac{n!}{k!\,(n-k)!}, \quad \text{set } n-k=m,\\ &= e^{-\lambda}\frac{(\lambda p)^k}{k!}\sum_{m=0}^{\infty}\frac{(\lambda q)^m}{m!}\\ &= e^{-\lambda}\,\frac{(\lambda p)^k}{k!}\,e^{\lambda q}\\ &= \frac{e^{-\lambda p}(\lambda p)^k}{k!},\end{aligned}$$

since $q = 1-p$. Thus $\Pr(k \text{ chicks})$ is distributed as a Poisson distribution with parameter $\mu = \lambda p$.

30.31 *Under EU legislation on harmonisation, all kippers are to weigh* 0.2000 kg *and vendors who sell underweight kippers must be fined by their government. The weight of a kipper is normally distributed with a mean of* 0.2000 kg *and a standard deviation of* 0.0100 kg. *They are packed in cartons of* 100 *and large quantities of them are sold.*

Every day a carton is to be selected at random from each vendor and tested according to one of the following schemes, which have been approved for the purpose.

(a) *The entire carton is weighed and the vendor is fined* 2500 *euros if the average weight of a kipper is less than* 0.1975 kg.
(b) *Twenty-five kippers are selected at random from the carton; the vendor is fined* 100 *euros if the average weight of a kipper is less than* 0.1980 kg.
(c) *Kippers are removed one at a time, at random, until one has been found that weighs* more *than* 0.2000 kg*; the vendor is fined* $4n(n-1)$ *euros, where* n *is the number of kippers removed.*

Which scheme should the Chancellor of the Exchequer be urging his government to adopt?

For these calculations we measure weights in grammes.

(a) For this scheme we have a normal distribution with mean $\mu = 200$ and s.d. $\sigma = 10$. The s.d. for a carton is $\sqrt{100}\,\sigma = 100$ and the mean weight is 20000. There is a penalty if the weight of a carton is less than 19750. This critical value represents a standard variable of

$$Z = \frac{19750 - 20000}{100} = -2.5.$$

The probability that $Z < -2.5 = 1 - \Phi(2.5) = 1 - 0.9938 = 0.0062$. Thus the average fine per carton tested on this scheme is $0.0062 \times 2500 = 15.5$ euros.

(b) For this scheme the general parameters are the same but the mean weight of the sample measured is 5000 and its s.d is $\sqrt{25}\,(10) = 50$. The Z-value at which a fine is imposed is

$$Z = \frac{(198 \times 25) - 5000}{50} = -1.$$

The probability that $Z < -1.0 = 1 - \Phi(1.0) = 1 - 0.8413 = 0.1587$. Thus the average fine per carton tested on this scheme is $0.1587 \times 100 = 15.9$ euros.

(c) This scheme is a series of Bernoulli trials in which the probability of success is $\frac{1}{2}$ (since half of all kippers weigh more than 200 and the distribution is normal). The probability that it will take n kippers to find one that passes the test is

$q^{n-1}p = (\frac{1}{2})^n$. The expected fine is therefore

$$f = \sum_{n=2}^{\infty} 4n(n-1)\left(\frac{1}{2}\right)^n = 4\,\frac{2\,(\frac{1}{4})}{(\frac{1}{2})^3} = 16 \text{ euros.}$$

The expression for the sum was found by twice differentiating the sum of the geometric series $\sum r^n$ with respect to r, as follows:

$$\begin{aligned}
\sum_{n=0}^{\infty} r^n = \frac{1}{1-r} \quad &\Rightarrow \quad \sum_{n=1}^{\infty} nr^{n-1} = \frac{1}{(1-r)^2} \\
&\Rightarrow \quad \sum_{n=2}^{\infty} n(n-1)r^{n-2} = \frac{2}{(1-r)^3} \\
&\Rightarrow \quad \sum_{n=2}^{\infty} n(n-1)r^{n} = \frac{2r^2}{(1-r)^3}.
\end{aligned}$$

There is, in fact, little to choose between the schemes on monetary grounds; no doubt political considerations, such as the current unemployment rate, will decide!

30.33 *A practical-class demonstrator sends his twelve students to the storeroom to collect apparatus for an experiment, but forgets to tell each which type of component to bring. There are three types, A, B and C, held in the stores (in large numbers) in the proportions* 20%, 30% *and* 50%, *respectively, and each student picks a component at random. In order to set up one experiment, one unit each of A and B and two units of C are needed. Let* $\Pr(N)$ *be the probability that at least N experiments can be setup.*

(a) *Evaluate* $\Pr(3)$.

(b) *Find an expression for* $\Pr(N)$ *in terms of* k_1 *and* k_2, *the numbers of components of types A and B, respectively, selected by the students. Show that* $\Pr(2)$ *can be written in the form*

$$\Pr(2) = (0.5)^{12} \sum_{i=2}^{6} {}^{12}C_i\ (0.4)^i \sum_{j=2}^{8-i} {}^{12-i}C_j\ (0.6)^j.$$

(c) *By considering the conditions under which no experiments can be set up, show that* $\Pr(1) = 0.9145$.

(a) To make three experiments possible the twelve components picked must be

three each of A and B and six of C. The probability of this is given by the multinomial distribution as

$$\Pr(3) = \frac{(12)!}{3!\,3!\,6!}(0.2)^3(0.3)^3(0.5)^6 = 0.06237.$$

(b) Let the numbers of A, B and C selected be k_1, k_2 and k_3, respectively, and consider when *at least* N experiments can be set up. We have the obvious inequalities $k_1 \geq N$, $k_2 \geq N$ and $k_3 \geq 2N$. In addition $k_3 = 12 - k_1 - k_2$, implying that $k_2 \leq 12 - 2N - k_1$. Further, k_1 cannot be greater than $12 - 3N$ if at least N experiments are to be set up, as each requires three other components that are not of type A. These inequalities set the limits on the acceptable values of k_1 and k_2 (k_3 is not a third independent variable). Thus $\Pr(N)$ is given by

$$\sum_{k_1 \geq N}^{12-3N} \sum_{k_2 \geq N}^{12-2N-k_1} \frac{(12)!}{k_1!\,k_2!\,(12-k_1-k_2)!}(0.2)^{k_1}(0.3)^{k_2}(0.5)^{12-k_1-k_2}.$$

The answer to part (a) is a particular case of this with $N = 3$, when each summation reduces to a single term.

For $N = 2$ the expression becomes

$$\begin{aligned}\Pr(2) &= \sum_{k_1 \geq 2}^{6} \sum_{k_2 \geq 2}^{8-k_1} \frac{(12)!}{k_1!\,k_2!\,(12-k_1-k_2)!}(0.2)^{k_1}(0.3)^{k_2}(0.5)^{12-k_1-k_2} \\ &= (0.5)^{12} \sum_{i=2}^{6} \sum_{j=2}^{8-i} \frac{(12)!\,(0.2/0.5)^i}{i!\,(12-i)!} \frac{(12-i)!\,(0.3/0.5)^j}{j!\,(12-i-j)!} \\ &= (0.5)^{12} \sum_{i=2}^{6} {}^{12}C_i\,(0.4)^i \sum_{j=2}^{8-i} {}^{12-i}C_j\,(0.6)^j.\end{aligned}$$

(c) No experiment can be set up if any one of the following four events occurs: $A_1 = (k_1 = 0)$, $A_2 = (k_2 = 0)$, $A_3 = (k_3 = 0)$ and $A_4 = (k_3 = 1)$. The probability for the union of these four events is given by

$$\Pr(A_1 \cup A_2 \cup A_3 \cup A_4) = \sum_{i=1}^{4} \Pr(A_i) - \sum_{i,j} \Pr(A_i \cap A_j) + \cdots.$$

The probabilities $\Pr(A_i)$ are straightforward to calculate as follows:

$$\begin{aligned}\Pr(A_1) &= (1-0.2)^{12}, \qquad \Pr(A_2) = (1-0.3)^{12}, \\ \Pr(A_3) &= (1-0.5)^{12}, \qquad \Pr(A_4) = {}^{12}C_1(1-0.5)^{12}(0.5).\end{aligned}$$

The calculation of the probability for the intersection of two events is typified by

$$\begin{aligned}\Pr(A_1 \cap A_2) &= [1-(0.2+0.3)]^{12} \\ \text{and } \Pr(A_1 \cap A_4) &= {}^{12}C_1[1-(0.2+0.5)]^{11}(0.5)^1.\end{aligned}$$

A few trial evaluations show that these are of order 10^{-4} and can be ignored by comparison with the larger terms in the first sum, which are (after rounding)

$$\begin{aligned}\sum_{i=1}^{4} \Pr(A_i) &= (0.8)^{12} + (0.7)^{12} + (0.5)^{12} + 12(0.5)^{11}(0.5) \\ &= 0.0687 + 0.0138 + 0.0002 + 0.0029 = 0.0856.\end{aligned}$$

Since the probability of no experiments being possible is 0.0856, it follows that $\Pr(1) = 0.9144$.

30.35 *The continuous random variables X and Y have a joint PDF proportional to $xy(x-y)^2$ with $0 \le x \le 1$ and $0 \le y \le 1$. Find the marginal distributions for X and Y and show that they are negatively correlated with correlation coefficient $-\frac{2}{3}$.*

This PDF is clearly symmetric between x and y. We start by finding its normalisation constant c:

$$\int_0^1 \int_0^1 c(x^3y - 2x^2y^2 + xy^3)\,dx\,dy = c\left(\frac{1}{4}\frac{1}{2} - 2\frac{1}{3}\frac{1}{3} + \frac{1}{2}\frac{1}{4}\right) = \frac{c}{36}.$$

Thus, we must have that $c = 36$.

The marginal distribution for x is given by

$$\begin{aligned} f(x) &= 36\int_0^1 (x^3y - 2x^2y^2 + xy^3)\,dy \\ &= 36(\tfrac{1}{2}x^3 - \tfrac{2}{3}x^2 + \tfrac{1}{4}x) \\ &= 18x^3 - 24x^2 + 9x, \end{aligned}$$

and the mean of x by

$$\mu_X = \bar{x} = \int_0^1 (18x^4 - 24x^3 + 9x^2)\,dx = \frac{18}{5} - \frac{24}{4} + \frac{9}{3} = \frac{3}{5}.$$

By symmetry, the marginal distribution and the mean for y are $18y^3 - 24y^2 + 9y$ and $\frac{3}{5}$, repectively.

To calculate the correlation coefficient we also need the variances of x and y and their covariance. The variances, obviously equal, are given by

$$\begin{aligned} \sigma_X^2 &= \int_0^1 x^2(18x^3 - 24x^2 + 9x)\,dx - (\tfrac{3}{5})^2 \\ &= \frac{18}{6} - \frac{24}{5} + \frac{9}{4} - \frac{9}{25} \\ &= \frac{900 - 1440 + 675 - 108}{300} = \frac{9}{100}. \end{aligned}$$

The standard deviations σ_X and σ_Y are therefore both equal to 3/10.

The covariance is calculated next; it is given by

$$\begin{aligned}
\mathrm{Cov}[X,Y] &= \langle XY\rangle - \mu_X\mu_Y \\
&= 36\int_0^1\int_0^1 (x^4y^2 - 2x^3y^3 + x^2y^4)\,dx\,dy - \frac{3}{5}\,\frac{3}{5} \\
&= \frac{36}{5\times 3} - \frac{72}{4\times 4} + \frac{36}{3\times 5} - \frac{9}{25} \\
&= \frac{12}{5} - \frac{9}{2} + \frac{12}{5} - \frac{9}{25} \\
&= \frac{120 - 225 + 120 - 18}{50} = -\frac{3}{50}.
\end{aligned}$$

Finally,

$$\mathrm{Corr}[X,Y] = \frac{\mathrm{Cov}[X,Y]}{\sigma_X\,\sigma_Y} = \frac{-\frac{3}{50}}{\frac{3}{10}\,\frac{3}{10}} = -\frac{2}{3}.$$

30.37 *Two continuous random variables X and Y have a joint probability distribution*

$$f(x,y) = A(x^2 + y^2),$$

where A is a constant and $0 \le x \le a$, $0 \le y \le a$. Show that X and Y are negatively correlated with correlation coefficient $-15/73$. By sketching a rough contour map of $f(x,y)$ and marking off the regions of positive and negative correlation, convince yourself that this (perhaps counter-intuitive) result is plausible.

The calculations of the various parameters of the distribution are straightforward (see exercise 30.35). The parameter A is determined by the normalisation condition:

$$1 = \int_0^a\int_0^a A(x^2+y^2)\,dx\,dy = A\left(\frac{a^4}{3} + \frac{a^4}{3}\right) \quad\Rightarrow\quad A = \frac{3}{2a^4}.$$

The two expectation values required are given by

$$\begin{aligned}
E[X] &= \int_0^a\int_0^a Ax(x^2+y^2)\,dx\,dy \\
&= \frac{3}{2a^4}\left(\frac{a^5}{4\times 1} + \frac{a^5}{2\times 3}\right) = \frac{5a}{8}, \qquad (E[Y] = E[X]), \\
E[X^2] &= \int_0^a\int_0^a Ax^2(x^2+y^2)\,dx\,dy \\
&= \frac{3}{2a^4}\left(\frac{a^6}{5\times 1} + \frac{a^6}{3\times 3}\right) = \frac{7a^2}{15}.
\end{aligned}$$

Hence the variance, calculated from the general result $V[X] = E[X^2] - (E[X])^2$, is

$$V[X] = \frac{7a^2}{15} - \left(\frac{5a}{8}\right)^2 = \frac{73}{960}a^2,$$

and the standard deviations are given by

$$\sigma_X = \sigma_Y = \sqrt{\frac{73}{960}}\,a.$$

To obtain the correlation coefficient we need also to calculate the following:

$$\begin{aligned} E[XY] &= \int_0^a \int_0^a Axy(x^2 + y^2)\,dx\,dy \\ &= \frac{3}{2a^4}\left(\frac{a^6}{4 \times 2} + \frac{a^6}{2 \times 4}\right) = \frac{3a^2}{8}. \end{aligned}$$

Then the covariance, given by $\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$, is evaluated as

$$\mathrm{Cov}[X, Y] = \frac{3}{8}a^2 - \frac{5a}{8}\,\frac{5a}{8} = -\frac{a^2}{64}.$$

Combining this last result with the standard deviations calculated above, we then obtain

$$\mathrm{Corr}[X, Y] = \frac{-(a^2/64)}{\sqrt{\frac{73}{960}}\,a\,\sqrt{\frac{73}{960}}\,a} = -\frac{15}{73}.$$

As the means of both X and Y are $\frac{5}{8}a = 0.62a$, the areas of the square of side a for which $X - \mu_X$ and $Y - \mu_Y$ have the same sign (i.e. regions of positive correlation) are about $(0.62)^2 \approx 39\%$ and $(0.38)^2 \approx 14\%$ of the total area of the square. The regions of negative correlation occupy some 47% of the square.

However, $f(x, y) = A(x^2 + y^2)$ favours the regions where one or both of x and y are large and close to unity. Broadly speaking, this gives little weight to the region in which both X and Y are less than their means, and so, although it is the largest region in area, it contributes relatively little to the overall correlation. The two (equal area) regions of negative correlation together outweigh the smaller high probability region of positive correlation in the top right-hand corner of the square; the overall result is a net negative correlation coefficient.

30.39 *Show that, as the number of trials n becomes large but $np_i = \lambda_i$, $i = 1, 2, \ldots, k-1$, remains finite, the multinomial probability distribution,*

$$M_n(x_1, x_2, \ldots, x_k) = \frac{n!}{x_1!x_2!\cdots x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k},$$

can be approximated by a multiple Poisson distribution with $k-1$ factors:

$$M'_n(x_1, x_2, \ldots, x_{k-1}) = \prod_{i=1}^{k-1} \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!}.$$

(Write $\sum_i^{k-1} p_i = \delta$ and express all terms involving subscript k in terms of n and δ, either exactly or approximately. You will need to use $n! \approx n^\epsilon[(n-\epsilon)!]$ and $(1 - a/n)^n \approx e^{-a}$ for large n.)

(a) *Verify that the terms of M'_n add up to unity when summed over all possible values of the random variables $x_1, x_2, \ldots, x_{k-1}$.*
(b) *If $k = 7$ and $\lambda_i = 9$ for all $i = 1, 2, \ldots, 6$, estimate, using the appropriate Gaussian approximation, the chance that at least three of $x_1, x_2, \ldots, x_6$ will be 15 or greater.*

The probabilities p_i are not all independent, and $p_k = 1 - \sum' p_i$, where, for compactness and typographical clarity, we denote $\sum_{i=1}^{k-1}$ by $\sum'$. We further write $\sum' p_i$ as δ. In the same way, we denote $\sum' x_i$ by ϵ and can write $x_k = n - \epsilon$.

Now, as $n \to \infty$ with $p_i \to 0$, whilst the product np_i remains finite and equal to λ_i, we will have that $\delta \to 0$, $n\delta \to \sum' \lambda_i$ and $(n-\epsilon)/n \to 1$. Making these replacements in the factors that contain subscript k gives

$$\begin{aligned}
&M_n(x_1, x_2, \ldots, x_k) \\
&\quad = \quad \frac{n!}{x_1!\,x_2!\cdots x_{k-1}!\,(n-\epsilon)!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} (1-\delta)^{n-\epsilon} \\
&\quad \approx \quad \frac{n^\epsilon (n-\epsilon)!}{x_1!\,x_2!\cdots x_{k-1}!\,(n-\epsilon)!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} \left(1 - \frac{n\delta}{n}\right)^{n-\epsilon} \\
&\quad = \quad \frac{n^{x_1+x_2+\cdots+x_{k-1}}}{x_1!\,x_2!\cdots x_{k-1}!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} \left(1 - \frac{n\delta}{n}\right)^{n-\epsilon} \\
&\quad \to \quad \frac{\lambda_1^{x_1} \lambda_2^{x_2} \cdots \lambda_{k-1}^{x_{k-1}}}{x_1!\,x_2!\cdots x_{k-1}!} e^{-(\lambda_1+\lambda_2+\cdots+\lambda_{k-1})} \\
&\quad = \quad \prod_{i=1}^{k-1} \frac{e^{-\lambda_i} \lambda_i^{x_i}}{x_i!},
\end{aligned}$$

i.e. as $n \to \infty$ $M_n(x_1, x_2, \ldots, x_k)$ can be approximated by the direct product of $k-1$ separate Poisson distributions.

(a) Since the modified expression $M'_n(x_1, x_2, \ldots, x_{k-1})$ consists of this multiple product of factors, the summation between 0 and ∞ over any particular variable, x_j say, can be carried out separately, with the factors not involving x_j treated as constant multipliers. A typical sum is

$$\sum_{x_j=0}^{\infty} \frac{e^{-\lambda_j} \lambda_j^{x_j}}{x_j!} = e^{-\lambda_j} e^{\lambda_j} = 1.$$

When all the summations have been carried out,

$$\sum_{\text{all } x_i} M'_n(x_1, x_2, \ldots, x_{k-1}) = (1)^{k-1} = 1.$$

(b) The Gaussian approximation to each Poisson distribution Po(9) is $N(9, 9)$, for which the standard variable is given by

$$Z = \frac{X - 9}{\sqrt{9}}.$$

Thus the probability that one of the x_i will exceed 15 (after including a continuity correction) is

$$\Pr(x_i \geq 15) = \Pr\left(Z > \frac{14.5 - 9}{3}\right) = 1 - \Phi(1.833) = 1 - 0.966 = 0.0334.$$

That (any) three of them should do so has probability

$${}^6C_3(0.0334)^3 = 20 \times 3.726\ 10^{-5} = 7.5 \times 10^{-4}.$$

The probabilities that 4, 5 or 6 of the x_i will exceed 15 make negligible additions to this, which is already an approximation in any case.

31

Statistics

31.1 *A group of students uses a pendulum experiment to measure g, the acceleration of free fall, and obtains the following values (in* $\mathrm{m\,s^{-2}}$*):* 9.80, 9.84, 9.72, 9.74, 9.87, 9.77, 9.28, 9.86, 9.81, 9.79, 9.82. *What would you give as the best value and standard error for g as measured by the group*?

We first note that the reading of 9.28 m s^{-2} is so far from the others that it is almost certainly in error and should not be used in the calculation. The mean of the ten remaining values is 9.802 and the standard deviation of the sample about its mean is 0.04643. After including Bessel's correction factor, the estimate of the population s.d. is $\sigma = 0.0489$, leading to a s.d. in the measured value of the mean of $0.0489/\sqrt{10} = 0.0155$. We therefore give the best value and standard error for g as 9.80 ± 0.02 m s^{-2}.

31.3 *The following are the values obtained by a class of* 14 *students when measuring a physical quantity x:* 53.8, 53.1, 56.9, 54.7, 58.2, 54.1, 56.4, 54.8, 57.3, 51.0, 55.1, 55.0, 54.2, 56.6.

(a) *Display these results as a histogram and state what you would give as the best value for x.*
(b) *Without calculation, estimate how much reliance could be placed upon your answer to* (a).
(c) *Databooks give the value of x as* 53.6 *with negligible error. Are the data obtained by the students in conflict with this*?

Figure 31.1 Histogram of the data in exercise 31.3.

(a) The histogram in figure 31.1 shows no reading that is an obvious mistake and there is no reason to suppose other than a Gaussian distribution. The best value for x is the arithmetic mean of the fourteen values given, i.e. 55.1.

(b) We note that eleven values, i.e. approximately two-thirds of the fourteen readings, lie within ± 2 bins of the mean. This estimates the s.d for the population as 2.0 and gives a standard error in the mean of $\approx 2.0/\sqrt{14} \approx 0.6$.

(c) Within the accuracy we are likely to achieve by estimating σ for the sample by eye, the value of Student's t is $(55.1 - 53.6)/0.6$, i.e. about 2.5. With fourteen readings there are 13 degrees of freedom. From standard tables for the Student's t-test, $C_{13}(2.5) \approx 0.985$. It is therefore likely at the $2 \times 0.015 = 3\%$ significance level that the data are in conflict with the accepted value.

[Numerical analysis of the data, rather than a visual estimate, gives the lower value 0.51 for the standard error in the mean and implies that there is a conflict between the data and the accepted value at the 1.0% significance level.]

31.5 *Measured quantities x and y are known to be connected by the formula*

$$y = \frac{ax}{x^2 + b},$$

where a and b are constants. Pairs of values obtained experimentally are

x:	2.0	3.0	4.0	5.0	6.0
y:	0.32	0.29	0.25	0.21	0.18.

Use these data to make best estimates of the values of y that would be obtained for (a) $x = 7.0$, *and* (b) $x = -3.5$. *As measured by fractional error, which estimate is likely to be the more accurate?*

In order to use this limited data to best advantage when estimating a and b graphically, the equation needs to be arranged in the linear form $v = mu + c$, since a straight-line graph is much the easiest form from which to extract parameters. The given equation can be arranged as

$$\frac{x}{y} = \frac{x^2}{a} + \frac{b}{a},$$

which is represented by a line with slope a^{-1} and intercept b/a when x^2 is used as the independent variable and x/y as the dependent one. The required tabulation is:

x	2.0	3.0	4.0	5.0	6.0
y	0.32	0.29	0.25	0.21	0.18
x^2	4.0	9.0	16.0	25.0	36.0
x/y	6.25	10.34	16.00	23.81	33.33

Plotting these data as a graph for $0 \leq x^2 \leq 40$ produces a straight line (within normal plotting accuracy). The line has a slope

$$\frac{1}{a} = \frac{28.1 - 2.7}{30.0 - 0.0} = 0.847 \quad \Rightarrow \quad a = 1.18.$$

The intercept is at $x/y = 2.7$, and, as this is equal to b/a, it follows that $b = 2.7 \times 1.18 = 3.2$. In fractional terms this is not likely to be very accurate as $b \ll x^2$ for all but two of the x-values used.

(a) For $x = 7.0$, the estimated value of y is

$$y = \frac{1.18 \times 7.0}{49.0 + 3.2} = 0.158.$$

(b) For $x = -3.5$, the estimated value of y is

$$y = \frac{1.18 \times (-3.5)}{12.25 + 3.2} = -0.267.$$

Although as a graphical extrapolation estimate (b) is further removed from the measured values, it is likely to be the more accurate because, using the fact that $y(-x) = -y(x)$, it is effectively obtained by (visual) *inter*polation amongst measured data rather than by *extra*polation from it.

31.7 *A population contains individuals of k types in equal proportions. A quantity X has mean μ_i amongst individuals of type i and variance σ^2, which has the same value for all types. In order to estimate the mean of X over the whole population, two schemes are considered; each involves a total sample size of nk. In the first the sample is drawn randomly from the whole population, whilst in the second* (stratified sampling) *n individuals are randomly selected from each of the k types.*

Show that in both cases the estimate has expectation

$$\mu = \frac{1}{k}\sum_{i=1}^{k}\mu_i,$$

but that the variance of the first scheme exceeds that of the second by an amount

$$\frac{1}{k^2 n}\sum_{i=1}^{k}(\mu_i - \mu)^2.$$

(i) For the first scheme the estimator $\hat{\mu}$ has expectation

$$\langle\hat{\mu}\rangle = \frac{1}{nk}\sum_{j=1}^{nk}\langle x_j\rangle,$$

where

$$\langle x_j\rangle = \frac{1}{k}\sum_{i=1}^{k}\mu_i \text{ for all } j,$$

since the k types are in equal proportions in the population. Thus,

$$\langle\hat{\mu}\rangle = \frac{1}{nk}\sum_{j=1}^{nk}\frac{1}{k}\sum_{i=1}^{k}\mu_i = \frac{1}{k}\sum_{i=1}^{k}\mu_i = \mu.$$

The variance of $\hat{\mu}$ is given by

$$\begin{aligned} V[\hat{\mu}] &= \frac{1}{n^2k^2}\, nk\, V[x] \\ &= \frac{1}{nk}\left(\langle x^2\rangle - \mu^2\right) \\ &= \frac{1}{nk}\left(\frac{1}{k}\sum_{i=1}^{k}\langle x_i^2\rangle - \mu^2\right), \end{aligned}$$

again since the k types are in equal proportions in the population.

Now we use the relationship $\sigma^2 = \langle x_i^2\rangle - \mu_i^2$ to replace $\langle x_i^2\rangle$ for each type, noting

that σ^2 has the same value in each case. The expression for the variance becomes

$$\begin{aligned}
V[\hat{\mu}] &= \frac{1}{nk}\left[\frac{1}{k}\sum_{i=1}^{k}(\mu_i^2+\sigma^2)-\mu^2\right] \\
&= \frac{\sigma^2-\mu^2}{nk}+\frac{1}{nk^2}\sum_{i=1}^{k}(\mu_i-\mu+\mu)^2 \\
&= \frac{\sigma^2-\mu^2}{nk}+\frac{1}{nk^2}\sum_{i=1}^{k}\left[(\mu_i-\mu)^2+2\mu(\mu_i-\mu)+\mu^2\right] \\
&= \frac{\sigma^2-\mu^2}{nk}+\frac{1}{nk^2}\sum_{i=1}^{k}(\mu_i-\mu)^2+0+\frac{k\mu^2}{nk^2} \\
&= \frac{\sigma^2}{nk}+\frac{1}{nk^2}\sum_{i=1}^{k}(\mu_i-\mu)^2.
\end{aligned}$$

(ii) For the second scheme the calculations are more straightforward. The expectation value of the estimator $\hat{\mu}=(nk)^{-1}\sum_{i=1}^{k}\langle x_i\rangle$ is

$$\langle\hat{\mu}\rangle=\frac{1}{nk}\sum_{i=1}^{k}n\mu_i=\frac{1}{k}\sum_{i=1}^{k}\mu_i=\mu,$$

whilst the variance is given by

$$V[\hat{\mu}]=\frac{1}{n^2k^2}\sum_{i=1}^{k}V[\langle x_i\rangle]=\frac{1}{n^2k^2}\sum_{i=1}^{k}n\sigma_i^2=\frac{1}{k^2}\frac{k\sigma^2}{n}=\frac{\sigma^2}{kn},$$

since $\sigma_i^2=\sigma^2$ for all i.

Comparing the results from (i) and (ii), we see that the variance of the estimator in the first scheme is larger by

$$\frac{1}{nk^2}\sum_{i=1}^{k}(\mu_i-\mu)^2.$$

31.9 *Each of a series of experiments consists of a large, but unknown, number n ($\gg 1$) of trials, in each of which the probability of success p is the same, but also unknown. In the ith experiment, $i = 1, 2, \ldots, N$, the total number of successes is x_i ($\gg 1$). Determine the log-likelihood function.*

Using Stirling's approximation to $\ln(n - x)$, *show that*

$$\frac{d\,\ln(n-x)}{dn} \approx \frac{1}{2(n-x)} + \ln(n-x),$$

and hence evaluate $\partial({}^nC_x)/\partial n$.

By finding the (coupled) equations determining the ML estimators $\hat{p}$ *and* $\hat{n}$, *show that, to order* n^{-1}, *they must satisfy the simultaneous 'arithmetic' and 'geometric' mean constraints*

$$\hat{n}\hat{p} = \frac{1}{N}\sum_{i=1}^{N} x_i \quad \text{and} \quad (1-\hat{p})^N = \prod_{i=1}^{N}\left(1 - \frac{x_i}{\hat{n}}\right).$$

The likelihood function for these N Bernoulli trials is given by

$$L(\mathbf{x}; n, p) = \prod_{i=1}^{N} {}^nC_{x_i}\, p^{x_i}\,(1-p)^{n-x_i}$$

and the corresponding log-likelihood function is

$$\ln L = \sum_{i=1}^{N} \ln\, {}^nC_{x_i} + \ln p \sum_{i=1}^{N} x_i + \ln(1-p)\left[Nn - \sum_{i=1}^{N} x_i\right].$$

The binomial coefficient depends upon n and so we need to determine $\partial({}^nC_x)/\partial n$. To do so, we first consider the derivative of $n!$. Stirling's approximation to $n!$ is

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, \qquad \text{for large } n.$$

The derivative of n^n is found by setting $y = n^n$ and proceeding as follows:

$$\ln y = n\ln n \quad \Rightarrow \quad \frac{1}{y}\frac{dy}{dn} = \ln n + \frac{n}{n} \quad \Rightarrow \quad \frac{dy}{dn} = n^n(1 + \ln n).$$

It follows that

$$\begin{aligned}
\frac{d(n!)}{dn} &= \sqrt{2\pi}\left[\frac{1}{2\sqrt{n}}\left(\frac{n}{e}\right)^n + \frac{\sqrt{n}}{e^n}\,n^n(1+\ln n) - \sqrt{n}\,n^n\,e^{-n}\right] \\
&= \sqrt{2\pi}\left[\frac{1}{2\sqrt{n}}\left(\frac{n}{e}\right)^n + \frac{\sqrt{n}}{e^n}\,n^n\ln n\right] \\
&= \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left[\frac{1}{2n} + \ln n\right] = n!\left(\frac{1}{2n} + \ln n\right).
\end{aligned}$$

An immediate consequence of this is

$$\frac{d(\ln n!)}{dn} = \frac{1}{n!}\frac{d(n!)}{dn} = \frac{1}{2n} + \ln n.$$

We now return to the log-likelihood function, the first term of which is

$$\sum_{i=1}^{N} \ln\ {}^{n}C_{x_i} = \sum_{i=1}^{N} [\ln n! - \ln x_i! - \ln(n - x_i)!],$$

with, for large n, a partial derivative with respect to n of

$$\begin{aligned}&\sum_{i=1}^{N}\left[\frac{1}{2n} + \ln n - 0 - \frac{1}{2(n-x_i)} - \ln(n-x_i)\right]\\ &= \sum_{i=1}^{N}\left[\ln\frac{n}{n-x_i} - \frac{x_i}{2n(n-x_i)}\right].\end{aligned}$$

We are now in a position to find the partial derivatives of the log-likelihood function with respect to p and n and equate each of them to zero, thus yielding the equations $\hat{p}$ and $\hat{n}$ must satisfy.

Firstly, differentiating with respect to p gives

$$\begin{aligned}\frac{\partial(\ln L)}{\partial p} &= \frac{1}{p}\sum_{i=1}^{N} x_i - \frac{1}{(1-p)}\left[Nn - \sum_{i=1}^{N} x_i\right] = 0,\\ \frac{N\hat{n}}{1-\hat{p}} &= \left(\frac{1}{\hat{p}} + \frac{1}{1-\hat{p}}\right)\sum_{i=1}^{N} x_i,\\ \frac{1}{\hat{p}}\sum_{i=1}^{N} x_i = N\hat{n} \quad &\Rightarrow \quad \hat{n}\hat{p} = \frac{1}{N}\sum_{i=1}^{N} x_i.\end{aligned}$$

Secondly, differentiation with respect to n yields

$$\frac{\partial(\ln L)}{\partial n} = \sum_{i=1}^{N}\left[\ln\frac{n}{n-x_i} - \frac{x_i}{2n(n-x_i)}\right] + N\ln(1-p) = 0.$$

For large n (and, consequently, large x_i), the first term in the square brackets is of zero-order in n whilst the second is of order n^{-1}. Ignoring the second term and recalling that $\ln 1 = 0$, the equation is equivalent to

$$(1-\hat{p})^N \prod_{i=1}^{N} \frac{\hat{n}}{\hat{n} - x_i} = 1 \quad \Rightarrow \quad (1-\hat{p})^N = \prod_{i=1}^{N}\left(1 - \frac{x_i}{\hat{n}}\right).$$

31.11 *According to a particular theory, two dimensionless quantities X and Y have equal values. Nine measurements of X gave values of* 22, 11, 19, 19, 14, 27, 8, 24 *and* 18, *whilst seven measured values of Y were* 11, 14, 17, 14, 19, 16 *and* 14. *Assuming that the measurements of both quantities are Gaussian distributed with a common variance, are they consistent with the theory? An alternative theory predicts that $Y^2 = \pi^2 X$; are the data consistent with this proposal?*

On the hypothesis that $X = Y$ and both quantities have Gaussian distributions with a common variance, we need to calculate the value of t given by

$$t = \frac{\bar{w} - \omega}{\hat{\sigma}} \left(\frac{N_1 N_2}{N_1 + N_2} \right)^{1/2},$$

where $\bar{w} = \bar{x}_1 - \bar{x}_2$, $\omega = \mu_1 - \mu_2 = 0$ and

$$\hat{\sigma} = \left[\frac{N_1 s_1^2 + N_2 s_2^2}{N_1 + N_2 - 2} \right]^{1/2}.$$

The nine measurements of X have a mean of 18.0 and a value for s^2 of 33.33. The corresponding values for the seven measurements of Y are 15.0 and 5.71. Substituting these values gives

$$\hat{\sigma} = \left[\frac{9 \times 33.33 + 7 \times 5.71}{9 + 7 - 2} \right]^{1/2} = 4.93,$$

$$t = \frac{18.0 - 15.0 - 0}{4.93} \left(\frac{9 \times 7}{9 + 7} \right)^{1/2} = 1.21.$$

This variable follows a Student's t-distribution for $9 + 7 - 2 = 14$ degrees of freedom. Interpolation in standard tables gives $C_{14}(1.21) \approx 0.874$, showing that a larger value of t could be expected in about $2 \times (1 - 0.874) = 25\%$ of cases. Thus no inconsistency between the data and the first theory has been established.

For the second theory we are testing Y^2 against $\pi^2 X$; the former will not be Gaussian distributed and the two distributions will not have a common variance. Thus the best we can do is to compare the difference between the two expressions, evaluated with the mean values of X and Y, against the estimated error in that difference.

The difference in the expressions is $(15.0)^2 - 18.0\pi^2 = 47.3$. The error in the difference between the functions of Y and X is given approximately by

$$\begin{aligned} V(Y^2 - \pi^2 X) &= (2Y)^2 V[Y] + (\pi^2)^2 V[X] \\ &= (30.0)^2 \frac{5.71}{7-1} + (\pi^2)^2 \frac{33.33}{9-1} \\ &= 1262 \quad \Rightarrow \quad \sigma \approx 35.5. \end{aligned}$$

The difference is thus about $47.3/35.5 = 1.33$ standard deviations away from the theoretical value of 0. The distribution will not be truly Gaussian but, if it were, this figure would have a probability of being exceeded in magnitude some $2 \times (1 - 0.908) = 18\%$ of the time. Again no inconsistency between the data and theory has been established.

31.13 *The χ^2 distribution can be used to test for correlations between characteristics of sampled data. To illustrate this consider the following problem.*

During an investigation into possible links between mathematics and classical music, pupils at a school were asked whether they had preferences (a) *between mathematics and english, and* (b) *between classical and pop music. The results are given below.*

	Classical	*None*	*Pop*
Mathematics	23	13	14
None	17	17	36
English	30	10	40

By computing tables of expected numbers, based on the assumption that no correlations exist, and calculating the relevant values of χ^2, determine whether there is any evidence for

(a) *a link between academic and musical tastes, and*
(b) *a claim that pupils either had preferences in both areas or had no preference.*

You will need to consider the appropriate value for the number of degrees of freedom to use when applying the χ^2 test.

We first note that there were 200 pupils taking part in the survey. Denoting no academic preference between mathematics and english by NA and no musical preference by NM, we draw up an enhanced table of the actual numbers m_{XY} of preferences for the various combinations that also shows the overall probabilities p_X and p_Y of the three choices in each selection.

	C	NM	P	Total	p_X
M	23	13	14	50	0.25
NA	17	17	36	70	0.35
E	30	10	40	80	0.40
Total	70	40	90	200	
p_Y	0.35	0.20	0.45		

(a) If we now assume the (null) hypothesis that there are no correlations in the

data and that any apparent correlations are the result of statistical fluctuations, then the expected number of pupils opting for the combination X and Y is $n_{XY} = 200 \times p_X \times p_Y$. A table of n_{XY} is as follows:

	C	NM	P	Total
M	17.5	10	22.5	50
NA	24.5	14	31.5	70
E	28	16	36	80
Total	70	40	90	200

Taking the standard deviation as the square root of the expected number of votes for each particular combination, the value of χ^2 is given by

$$\chi^2 = \sum_{\text{all XY combinations}} \left(\frac{n_i - m_i}{\sqrt{n_i}} \right)^2 = 12.3.$$

For an $n \times n$ correlation table (here $n = 3$), the $(n-1) \times (n-1)$ block of entries in the upper left-hand can be filled in arbitrarily. But, as the totals for each row and column are predetermined, the remaining $2n - 1$ entries are not arbitrary. Thus the number of degrees of freedom (d.o.f.) for such a table is $(n-1)^2$, here 4 d.o.f. From tables, a χ^2 of 12.3 for 4 d.o.f. makes the assumed hypothesis less than 2% likely, and so it is almost certain that a correlation between academic and musical tastes does exist.

(b) To investigate a claim that pupils either had preferences in both areas or had no preference, we must combine expressed preferences for classical or pop into one set labelled PM meaning 'expressed a musical preference'; similarly for academic subjects. The correlation table is now a 2×2 one and will have only one degree of freedom. The actual and expected ($n_{XY} = 200 p_X p_Y$) data tables are

	PM	NM	Total	p_X
PA	107	23	130	0.65
NA	53	17	70	0.35
Total	160	40	200	
p_Y	0.80	0.20		

	PM	NM	Total
PA	104	26	130
NA	56	14	70
Total	160	40	200

The value of χ^2 is

$$\chi^2 = \frac{(-3)^2}{104} + \frac{(3)^2}{26} + \frac{(3)^2}{56} + \frac{(-3)^2}{14} = 1.24.$$

This is close to the expected value (1) of χ^2 for 1 d.o.f. and is neither too big nor too small. Thus there is no evidence for the claim (or for any tampering with the data!).

31.15 *A particle detector consisting of a shielded scintillator is being tested by placing it near a particle source whose intensity can be controlled by the use of absorbers. It might register counts even in the absence of particles from the source because of the cosmic ray background.*

The number of counts n registered in a fixed time interval as a function of the source strength s is given as:

source strength s:	0	1	2	3	4	5	6
counts n:	6	11	20	42	44	62	61

At any given source strength, the number of counts is expected to be Poisson distributed with mean

$$n = a + bs,$$

where a and b are constants. Analyse the data for a fit to this relationship and obtain the best values for a and b together with their standard errors.

(a) *How well is the cosmic ray background determined?*
(b) *What is the value of the correlation coefficient between a and b? Is this consistent with what would happen if the cosmic ray background were imagined to be negligible?*
(c) *Do the data fit the expected relationship well? Is there any evidence that the reported data 'are too good a fit'?*

Because in this exercise the independent variable s takes only consecutive integer values, we will use it as a label i and denote the number of counts corresponding to $s = i$ by n_i. As the data are expected to be Poisson distributed, the best estimate of the variance of each reading is equal to the best estimate of the reading itself, namely the actual measured value. Thus each reading n_i has an error of $\sqrt{n_i}$, and the covariance matrix N takes the form $N = \mathrm{diag}(n_0, n_1, \ldots, n_6)$, i.e. it is diagonal, but not a multiple of the unit matrix.

The expression for χ^2 is

$$\chi^2(a,b) = \sum_{i=0}^{6} \left(\frac{n_i - a - bi}{\sqrt{n_i}} \right)^2 \qquad (*).$$

Minimisation with respect to a and b gives the simultaneous equations

$$0 = \frac{\partial \chi^2}{\partial a} = -2 \sum_{i=0}^{6} \frac{n_i - a - bi}{n_i},$$

$$0 = \frac{\partial \chi^2}{\partial b} = -2 \sum_{i=0}^{6} \frac{i(n_i - a - bi)}{n_i}.$$

As is shown more generally in textbooks on numerical computing (e.g. William H. Press *et al.*, *Numerical Recipes in C*, 2nd edn (Cambridge: Cambridge University Press, 1996), Sect. 15.2), these equations are most conveniently solved by defining the quantities

$$S \equiv \sum_{i=0}^{6} \frac{1}{n_i}, \quad S_x \equiv \sum_{i=0}^{6} \frac{i}{n_i}, \quad S_y \equiv \sum_{i=0}^{6} \frac{n_i}{n_i},$$

$$S_{xx} \equiv \sum_{i=0}^{6} \frac{i^2}{n_i}, \quad S_{xy} \equiv \sum_{i=0}^{6} \frac{in_i}{n_i}, \quad \Delta \equiv SS_{xx} - (S_x)^2.$$

With these definitions (which correspond to the quantities calculated and accessibly stored in most calculators programmed to perform least-squares fitting), the solutions for the best estimators of a and b are

$$\hat{a} = \frac{S_{xx}S_y - S_xS_{xy}}{\Delta},$$
$$\hat{b} = \frac{S_{xy}S - S_xS_y}{\Delta},$$

with variances and covariance given by

$$\sigma_a^2 = \frac{S_{xx}}{\Delta}, \quad \sigma_b^2 = \frac{S}{\Delta}, \quad \mathrm{Cov}(a,b) = -\frac{S_x}{\Delta}.$$

The computed values of these quantities are: $S = 0.38664$; $S_x = 0.53225$; $S_y = 7$; $S_{xx} = 1.86221$; $S_{xy} = 21$; $\Delta = 0.43671$.

From these values, the best estimates of $\hat{a}$, $\hat{b}$ and the variances σ_a^2 and σ_b^2 are

$$\hat{a} = 4.2552, \quad \hat{b} = 10.061, \quad \sigma_a^2 = 4.264, \quad \sigma_b^2 = 0.8853.$$

The covariance is $\mathrm{Cov}(a,b) = -1.2187$, giving estimates for a and b of

$$a = 4.3 \pm 2.1 \qquad \text{and} \qquad b = 10.06 \pm 0.94,$$

with a correlation coefficient $r_{ab} = -0.63$.

(a) The cosmic ray background must be present, since $n(0) \neq 0$, but its value of about 4 is uncertain to within a factor of 2.

(b) The correlation between a and b is negative and quite strong. This is as expected since, if the cosmic ray background represented by a were reduced towards zero, then b would have to be increased to compensate when fitting to the measured data for non-zero source strengths.

(c) A measure of the goodness-of-fit is the value of χ^2 achieved using the best-fit values for a and b. Direct resubstitution of the values found into (*) gives $\chi^2 = 4.9$. If the weight of a particular reading is taken as the square root of the predicted (rather than the measured) value, then χ^2 rises slightly to 5.1. In either case the result is almost exactly that 'expected' for 5 d.o.f. – neither too good nor too bad.

There are five degrees of freedom because there are seven data points and two parameters have been chosen to give a best fit.

31.17 *The following are the values and standard errors of a physical quantity $f(\theta)$ measured at various values of θ (in which there is negligible error):*

θ	0	$\pi/6$	$\pi/4$	$\pi/3$
$f(\theta)$	3.72 ± 0.2	1.98 ± 0.1	-0.06 ± 0.1	-2.05 ± 0.1

θ	$\pi/2$	$2\pi/3$	$3\pi/4$	π
$f(\theta)$	-2.83 ± 0.2	1.15 ± 0.1	3.99 ± 0.2	9.71 ± 0.4

Theory suggests that f should be of the form $a_1 + a_2 \cos\theta + a_3 \cos 2\theta$. Show that the normal equations for the coefficients a_i are

$$\begin{aligned} 481.3a_1 + 158.4a_2 - 43.8a_3 &= 284.7, \\ 158.4a_1 + 218.8a_2 + 62.1a_3 &= -31.1, \\ -43.8a_1 + 62.1a_2 + 131.3a_3 &= 368.4. \end{aligned}$$

(a) *If you have matrix inversion routines available on a computer, determine the best values and variances for the coefficients a_i and the correlation between the coefficients a_1 and a_2.*

(b) *If you have only a calculator available, solve for the values using a Gauss–Seidel iteration and start from the approximate solution $a_1 = 2$, $a_2 = -2$, $a_3 = 4$.*

Assume that the measured data have uncorrelated errors. The quoted errors are not all equal and so the covariance matrix N, whilst being diagonal, will not be a multiple of the unit matrix; it will be

$$\mathsf{N} = \mathrm{diag}(0.04, 0.01, 0.01, 0.01, 0.04, 0.01, 0.04, 0.16).$$

Using as base functions the three functions $h_1(\theta) = 1$, $h_2(\theta) = \cos\theta$ and $h_3(\theta) = \cos 2\theta$, we calculate the elements of the 8×3 response matrix $\mathsf{R}_{ij} = h_j(\theta_i)$. To save space we display its 3×8 transpose:

$$\mathsf{R}^{\mathrm{T}} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0.866 & 0.707 & 0.500 & 0 & -0.500 & -0.707 & -1 \\ 1 & 0.500 & 0 & -0.500 & -1 & -0.500 & 0 & 1 \end{pmatrix}$$

Then

$$\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1} = \begin{pmatrix} 25 & 100 & 100 & 100 & 25 & 100 & 25 & 6.25 \\ 25 & 86.6 & 70.7 & 50 & 0 & -50 & -17.7 & -6.25 \\ 25 & 50.0 & 0 & -50.0 & -25 & -50 & 0 & 6.25 \end{pmatrix}$$

and

$$\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R} = \mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0.866 & 0.500 \\ 1 & 0.707 & 0 \\ 1 & 0.500 & -0.500 \\ 1 & 0 & -1 \\ 1 & -0.500 & -0.500 \\ 1 & -0.707 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 481.25 & 158.35 & -43.75 \\ 158.35 & 218.76 & 62.05 \\ -43.75 & 62.05 & 131.25 \end{pmatrix}.$$

From the measured values,

$$\mathsf{f} = (3.72,\ 1.98,\ -0.06,\ -2.05,\ -2.83,\ 1.15,\ 3.99,\ 9.71)^{\mathrm{T}},$$

we need to calculate $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{f}$, which is given by

$$\begin{pmatrix} 25 & 100 & 100 & 100 & 25 & 100 & 25 & 6.25 \\ 25 & 86.6 & 70.7 & 50 & 0 & -50 & -17.7 & -6.25 \\ 25 & 50.0 & 0 & -50 & -25 & -50 & 0 & 6.25 \end{pmatrix}\begin{pmatrix} 3.72 \\ 1.98 \\ -0.06 \\ -2.05 \\ -2.83 \\ 1.15 \\ 3.99 \\ 9.71 \end{pmatrix},$$

i.e. $(284.7,\ -31.08,\ 368.44)^{\mathrm{T}}$.

The vector of LS estimators of a_i satisfies $\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R}\hat{\mathsf{a}} = \mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{f}$. Substituting the forms calculated above into the two sides of the equality gives the set of equations stated in the question.

(a) Machine (or manual!) inversion gives

$$(\mathsf{R}^{\mathrm{T}}\mathsf{N}^{-1}\mathsf{R})^{-1} = 10^{-3}\begin{pmatrix} 3.362 & -3.177 & 2.623 \\ -3.177 & 8.282 & -4.975 \\ 2.623 & -4.975 & 10.845 \end{pmatrix}.$$

From this (covariance matrix) we can calculate the standard errors on the a_i from the square roots of the terms on the leading diagonal as ± 0.058, ± 0.091 and ± 0.104. We can further calculate the correlation coefficient r_{12} between a_1 and a_2 as

$$r_{12} = \frac{-3.177 \times 10^{-3}}{0.058 \times 0.091} = -0.60.$$

The best values for the a_i are given by the result of multiplying the column matrix $(284.7,\ -31.08,\ 368.44)^{\mathrm{T}}$ by the above inverted matrix. This yields $(2.022,\ -2.944,\ 4.897)^{\mathrm{T}}$ to give the best estimates of the a_i as

$$a_1 = 2.02 \pm 0.06, \quad a_2 = -2.99 \pm 0.09, \quad a_3 = 4.90 \pm 0.10.$$

(b) Denote the given set of equations by $\mathsf{Aa} = \mathsf{b}$ and start by dividing each equation by the quantity needed to make the diagonal elements of A each equal to unity; this produces $\mathsf{Ca} = \mathsf{d}$. Then, writing $\mathsf{C} = \mathsf{I} - \mathsf{F}$ yields the basis of the iteration scheme,

$$\mathsf{a}_{n+1} = \mathsf{Fa}_n + \mathsf{d}.$$

We use only the simplest form of Gauss–Seidel iteration (with no separation into upper and lower diagonal matrices).

The explicit form of $\mathsf{Ca} = \mathsf{d}$ is

$$\begin{pmatrix} 1 & 0.3290 & -0.0909 \\ 0.7239 & 1 & 0.2836 \\ -0.3333 & 0.4728 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0.5916 \\ -0.1421 \\ 2.8072 \end{pmatrix}$$

and

$$\mathsf{F} = \begin{pmatrix} 0 & -0.3290 & 0.0909 \\ -0.7239 & 0 & -0.2836 \\ 0.3333 & -0.4728 & 0 \end{pmatrix}.$$

Starting with the approximate solution $a_1 = 2$, $a_2 = -2$, $a_3 = 4$ gives as the result of the first ten iterations

a_1	a_2	a_3
2.000	−2.000	4.000
1.613	−2.724	4.419
1.890	−2.563	4.633
1.856	−2.824	4.649
1.943	−2.804	4.761
1.947	−2.899	4.781
1.980	−2.907	4.827
1.987	−2.944	4.842
2.000	−2.953	4.861
2.005	−2.969	4.870
2.011	−2.975	4.879

This final set of values is in close agreement with that obtained by direct inversion; in fact, after eighteen iterations the values agree exactly to three significant figures. Of course, using this method makes it difficult to estimate the errors in the derived values.

31.19 *The F-distribution $h(F)$ for the ratio F of the variances of two samples of sizes N_1 and N_2 drawn from populations with a common variance is*

$$\left(\frac{n_1}{n_2}\right)^{n_1/2} \frac{F^{(n_1-2)/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(1 + \frac{n_1}{n_2} F\right)^{-(n_1+n_2)/2},$$

where, to save space, we have written $N_1 - 1$ as n_1 and $N_2 - 1$ as n_2.

Verify that the F-distribution $P(F)$ is symmetric between the two data samples, i.e. that it retains the same form but with N_1 and N_2 interchanged, if F is replaced by $F' = F^{-1}$. Symbolically, if $P'(F')$ is the distribution of F' and $P(F) = \eta(F, N_1, N_2)$, then $P'(F') = \eta(F', N_2, N_1)$.

We first write $F^{-1} = F'$ with $|dF| = |dF'|/F'^2$ and rewrite $h(F)$ as

$$\left(\frac{n_1}{n_2}\right)^{n_1/2} \frac{(F')^{-(n_1-2)/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(1 + \frac{n_1}{n_2 F'}\right)^{-(n_1+n_2)/2}$$

$$= \left(\frac{n_1}{n_2}\right)^{n_1/2} \frac{(F')^{-(n_1-2)/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} (F')^{(n_1+n_2)/2} \left(\frac{n_2}{n_1}\right)^{(n_1+n_2)/2} \left(\frac{F' n_2}{n_1} + 1\right)^{-(n_1+n_2)/2}$$

$$= \left(\frac{n_2}{n_1}\right)^{n_2/2} \frac{(F')^{(n_2+2)/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(1 + \frac{n_2 F'}{n_1}\right)^{-(n_1+n_2)/2}.$$

Further,

$$h(F)\,|dF| = \left(\frac{n_2}{n_1}\right)^{n_2/2} \frac{(F')^{(n_2+2)/2}}{B\left(\frac{n_1}{2}, \frac{n_2}{2}\right)} \left(1 + \frac{n_2 F'}{n_1}\right)^{-(n_1+n_2)/2} \frac{1}{F'^2}\,|dF'|$$

$$= \left(\frac{n_2}{n_1}\right)^{n_2/2} \frac{(F')^{(n_2-2)/2}}{B\left(\frac{n_2}{2}, \frac{n_1}{2}\right)} \left(1 + \frac{n_2 F'}{n_1}\right)^{-(n_1+n_2)/2} |dF'|.$$

In the last step we have made use of the symmetry of the beta function $B(x, y)$ with respect to its arguments. To express the final result in the usual F-distribution form, we need to restore n_1 to $N_1 - 1$ and n_2 to $N_2 - 1$, but the symmetry between the data samples has already been demonstrated.