CHAPTER 2

OPERATIONS ON FUZZY SETS

❖. *Fuzzy Complement*

❖. *Fuzzy Union*

❖. *Fuzzy Intersection*

❖. *Combinations of Operations*

❖. *General Aggregation Operations*

❖. *Fuzzy Complement*

Def: A ***complement*** of a fuzzy set A is specified by a function

$$c : [0,1] \rightarrow [0,1],$$

which assigns a value $c(\mu_A(x))$ to each membership grade $\mu_A(x)$.

Axiom:

The function of a fuzzy complement must satisfy at least the following two requirements:

c1.: $c(0) = 1$ and $c(1) = 0$ (boundary conditions).

c2.: For all $a, b \in [0,1]$, if $a < b$, then $c(a) \geq c(b)$ (c is monotonic nonincreasing).

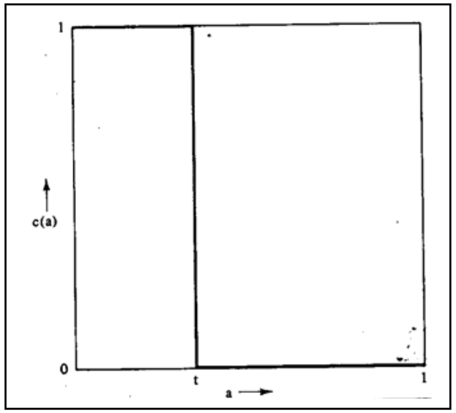
Remark:

1. All functions that satisfy axioms **c1** and **c2** form the most general class of fuzzy complements.
2. Axioms **c1** and **c2** are called the *axiomatic skeleton for fuzzy complements*.
3. Axiom **c1** is the ordinary complement for crisp sets.

Example: The threshold-type complement

$$c(a) = \begin{cases} 1 & \text{for } a \leq t, \\ 0 & \text{for } a > t, \end{cases}$$

where $a \in [0,1]$ and $t \in [0,1)$; t is called the threshold of c.



Axiom:

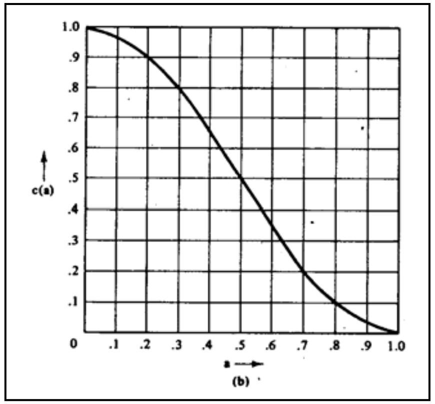
In practical significance, it is desirable to consider various additional requirements such as:

c3. c is a *continuous* function.

c4. c *involutive, i.e.* $c(c(a)) = a$ for all $a \in [0,1]$.

Example: An example that satisfies axioms **c1** to **c3**

$$c(a) = \frac{1}{2}(1 + \cos \pi a)$$

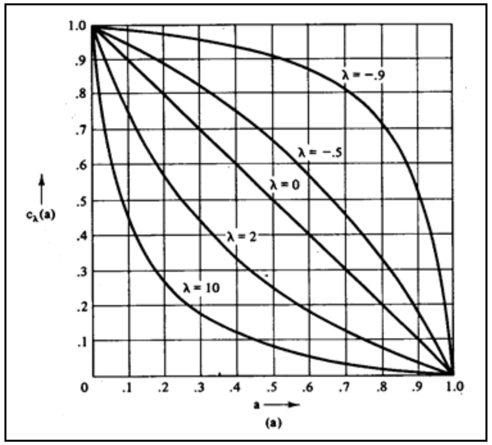


Examples of involutive fuzzy complement:

1. The *sugeno class*:

$$c_\lambda(a) = \frac{1-a}{1+\lambda a},$$

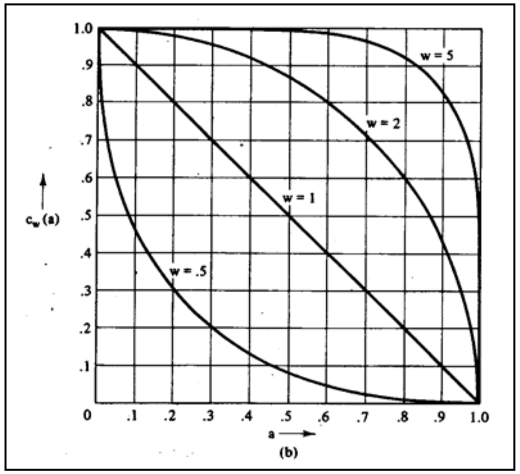
where $\lambda \in (-1, \infty)$.



2. The *yager class*:

$$c_w(a) = (1 - a^w)^{1/w},$$

where $w \in (0, \infty)$.



Note:

1) For $\lambda = 0, c_\lambda(a) = 1 - a$.
2) For $w = 1, c_w(a) = 1 - a$.

❖. *Fuzzy Union*

Def: The ***union*** of two fuzzy sets A and B is specified by a function of the form:

$$\mu : [0,1] \times [0,1] \to [0,1].$$

Formally,

$$\mu_{A \cup B}(x) = \mu[\mu_A(x), \mu_B(x)].$$

Axioms:

The function of a fuzzy union must satisfy at least the following axioms (*the axiomatic skeleton for fuzzy set unions*):

u1. $u(0,0) = 0$; $u(0,1) = u(1,0) = u(1,1) = 1$ (*boundary conditions*).

u2. $u(a,b) = u(b,a)$ (*commutative*).

u3. If $a \le a'$ and $b \le b'$, then $u(a,b) \le u(a',b')$ (*monotonic*).

u4. $u(u(a,b),c) = u(a,u(b,c))$ (*associative*).

Axiom:

It is often desirable to restrict the class of fuzzy union by considering various additional such as:

u5. u is a *continuous* function.

u6. $u(a,a) = a$, *i.e.* u is *idempotent*.

Example: The *Yager class union* satisfies **u1** to **u5**.

$$u_w(a,b) = \min\left[1, \left(a^w + b^w\right)^{1/w}\right]$$

where $w \in (0, \infty)$.

Property:

$w = 1$, $u_1(a,b) = \min(1, a+b)$

$w = 2$, $u_2(a,b) = \min\left(1, \sqrt{a^2 + b^2}\right)$

$w = \infty$, $u_\infty(a,b) = \max(a,b)$

❖. *Fuzzy Intersection*

<u>*Def*</u>: The ***<u>intersection</u>*** of two fuzzy sets A and B is specific by a function:

$$i : [0,1] \times [0,1] \to [0,1] .$$

Formally,

$$\mu_{A \cap B}(x) = \mu[\mu_A(x), \mu_B(x)].$$

<u>*Axioms*</u>:

The function of fuzzy intersection must satisfy the following axioms (*the axiomatic skeleton for fuzzy set intersections*):

i1. $i(1,1) = 1; \quad i(0,1) = i(1,0) = i(0,0) = 0$; (*boundary conditions*).

i2. $i(a,b) = i(b,a)$; (*commutative*)

i3. If $a \le a'$ and $b \le b'$, then $i(a,b) \le i(a',b')$ (*monotonic*).

i4. $i(i(a,b),c) = i(a,i(b,c))$ (*associative*).

▪. The most important additional requirements:

i5. i is a *continuous* function.

i6. $i(a,a) = a$, *i.e.* i is *idempotent*.

<u>*Example*</u>: The *Yager class union* satisfies **i1** to **i4:**

$$i_w(a,b) = 1 - \min\left[1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right]$$

where $w \in (0, \infty)$.

<u>*Property*</u>:

$w = 1$, $\quad i_1(a,b) = 1 - \min(1, 2 - a - b)$

$w = 2$, $\quad i_2(a,b) = 1 - \min\left(1, \sqrt{(1-a)^2 + (1-b)^2}\right)$

$w = \infty$, $\quad i_\infty(a,b) = \min(a,b)$

TABLE 2.2. SOME CLASSES OF FUZZY SET UNIONS AND INTERSECTIONS.

Reference	Fuzzy Unions	Fuzzy Intersections	Range of Parameter
Schweizer & Sklar [1961]	$1 - \max[0, (1-a)^{-p} + (1-b)^{-p} - 1)]^{1/p}$	$\max(0, a^{-p} + b^{-p} - 1)^{-1/p}$	$p \in (-\infty, \infty)$
Hamacher [1978]	$\dfrac{a + b - (2-\gamma)ab}{1 - (1-\gamma)ab}$	$\dfrac{ab}{\gamma + (1-\gamma)(a + b - ab)}$	$\gamma \in (0, \infty)$
Frank [1979]	$1 - \log_s\left[1 + \dfrac{(s^{1-a} - 1)(s^{1-b} - 1)}{s - 1}\right]$	$\log_s\left[1 + \dfrac{(s^a - 1)(s^b - 1)}{s - 1}\right]$	$s \in (0, \infty)$
Yager [1980]	$\min[1, (a^w + b^w)^{1/w}]$	$1 - \min[1, (1-a)^w + (1-b)^w)^{1/w}]$	$w \in (0, \infty)$
Dubois & Prade [1980]	$\dfrac{a + b - ab - \min(a, b, 1-\alpha)}{\max(1-a, 1-b, \alpha)}$	$\dfrac{ab}{\max(a,b,\alpha)}$	$\alpha \in (0, 1)$
Dombi [1982]	$\dfrac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{-\lambda} + \left(\frac{1}{b} - 1\right)^{-\lambda}\right]^{-1/\lambda}}$	$\dfrac{1}{1 + \left[\left(\frac{1}{a} - 1\right)^{\lambda} + \left(\frac{1}{b} - 1\right)^{\lambda}\right]^{1/\lambda}}$	$\lambda \in (0, \infty)$

❖. *COMBINATION OF OPERATORS*:

Theorem:

$$\max(a,b) \le u(a,b) \le u_{\max}(a,b),$$

where

$$u_{\max}(a,b) = \begin{cases} a & \text{when } b = 0, \\ b & \text{when } a = 0, \\ 1 & \text{otherwise.} \end{cases}$$

proof:

1) Using associativity (**u4**)

$$u\big(a, u(0,0)\big) = u\big(u(a,0),0\big)$$

By applying the boundary conditions (**u1**)

$$u(a,0) = u\big(u(a,0),0\big)$$

so

$$u(a,0) = a$$

By monotonicity of u (**u3**)

$$u(a,b) \ge u(a,0) = a$$

By employing commutativity (**u2**)

$$u(a,b) = u(b,a) \ge u(b,0) = b$$

Hence $u(a,b) \ge \max(a,b)$.

2) When

1. $b = 0$, then $u(a,b) = a = u_{\max}(a,b)$
2. $a = 0$, then $u(a,b) = b = u_{\max}(a,b)$
3. $a,b \in (0,1]$, then

$$u(a,b) \le u(a,1) = u(1,b) = 1 = u_{\max}(a,b)$$

From 1, 2, 3, $u_{\max}(a,b) \ge u(a,b)$.

Theorem:

$$i_{\min}(a,b) \le i(a,b) \le \min(a,b)$$

where

$$i_{\min}(a,b) = \begin{cases} a & \text{when } b = 1, \\ b & \text{when } a = 1, \\ 0 & \text{otherwise.} \end{cases}$$

proof: Similar to the previous theorem

1) Basing on the associativity (**u4**)

$$i\big(a, i(1,1)\big) = i\big(i(a,1),1\big)$$

Using the boundary condition (**u1**)

$$i(a,1) = i\big(i(a,1),1\big)$$

so

$$i(a,1) = a$$

By monotonicity (**u3**)

$$i(a,b) \le i(a,1) = a$$

With the commutativity (**u2**)

$$i(a,b) = i(b,a) \le i(b,1) = b$$

2) Left as an exercise!

Property:

1. The ***max*** operation is the *lower bound* of function u, *i.e.* the largest union.

2. The ***min*** operation is the *upper bound* of function i, *i.e.* the weakest intersection.

3. Of all possible pairs of fuzzy sets unions and intersections, the ***max*** and ***min*** functions are closest to each other (an extreme pair), *i.e.*

$$\max(a,b) - \min(a,b) = |a-b| \le u(a,b) - i(a,b)$$

Proposition:

The max and min functions together with the $c(a) = 1 - a$ complement satisfy the DeMorgan's laws, *i.e.*

$$\max(a,b) = 1 - \min(1-a, 1-b)$$

$$\min(a,b) = 1 - \max(1-a, 1-b)$$

Proof: See blackboard!

Proposition:

The operations $u_{\max}$ and $u_{\min}$ is another extreme pair in the sense that

$$u_{\max}(a,b) - i_{\min}(a,b) \ge u(a,b) - i(a,b)$$

The DeMorgan laws are also satisfied under the complement operation $c(a) = 1 - a$.

$$u_{\max}(a,b) = 1 - i_{\min}(1-a, 1-b)$$

$$i_{\min}(a,b) = 1 - u_{\max}(1-a, 1-b)$$

Proof: An exercise!

▪. **The full scope of fuzzy aggregation operation**:

Intersection		Union
Dombi $0 \rightarrow \lambda \rightarrow \infty$		Dombi $\infty \leftarrow \lambda \leftarrow 0$
Schweizer/Sklar $-\infty \rightarrow p \rightarrow \infty$		Schweizer/Sklar $\infty \leftarrow p \leftarrow -\infty$
Yager $0 \rightarrow w \rightarrow \infty$		Yager $\infty \leftarrow w \leftarrow 0$
	Generalized means $-\infty \rightarrow \alpha \rightarrow \infty$	

$i_{\min}$ $\quad i \quad$ min $\quad$ max $\quad u \quad$ $u_{\max}$

$\rightarrow$ Intersection $\leftarrow\rightarrow$ Average $\leftarrow\rightarrow$ Union $\leftarrow$

Theorem:

$u(a,b)=\max(a,b)$ and $i(a,b)=\min(a,b)$ are the only continuous and idempotent fuzzy set union and fuzzy set intersection.

Proof:

a) The fuzzy set union:

By associativity

$$u(a,u(a,b))=u(u(a,a),b)$$

The idempotency leads to(**u6**)

$$u(a,u(a,b))=u(a,b)$$

Similarly,

$$u(u(a,b),b)=u(a,u(b,b))=u(a,b)$$

Hence

$$u(a,u(a,b))=u(u(a,b),b) \qquad (*)$$

When $a = b$, (*) is trival.

W.l.o.g., Let $a < b$.

1) Assume $u(a,b)=\alpha$, where $\alpha \neq a$ and $\alpha \neq b$.
Then (*) becomes

$$u(a,\alpha)=u(b,\alpha) \qquad (**)$$

Since u is continuous (**u5**) and monotonic nondecreasing (**u3**) with $u(0,\alpha)=\alpha$ and $u(1,\alpha)=1$

$$\exists a,b \in [0,1] \ni u(a,\alpha)<u(b,\alpha)$$

It contradicts to (**) $\rightarrow$ the assumption is not warrant.

2) Assume $u(a,b)=a=\min(a,b)$, it violates the boundary condition (**u1**) when a = 0 and b = 1.

3) Assume $u(a,b)=b=\max(a,b)$, it satisfies **u1** and (*) becomes $u(a,b)=u(b,b)$ is also satisfied for all $a < b$.

(Remark: The *Yager class* satisfies continuity)

b) The intersection:

An exercise!

❖. About the fuzzy set operations max and min

The structure of $\{\tilde{P}(X),\cup,\cap,-\}$ satisfies

1. Involution $\bar{\bar{A}} = A$
2. Commutativity $A\cup B = B\cup A$, $A\cap B = B\cap A$
3. Associativity $(A\cup B)\cup C = A\cup(B\cup C)$, $(A\cap B)\cap C = A\cap(B\cap C)$
4. Distributivity
5. Idempotence
6. Absorption
7. Absorption by X and ∅.
8. Identity
9. DeMorgan laws

But violates

a) Law of contradiction

$$A\cap\bar{A}\neq\varnothing$$

b) Law of excluded middle

$$A\cup\bar{A}\neq X$$

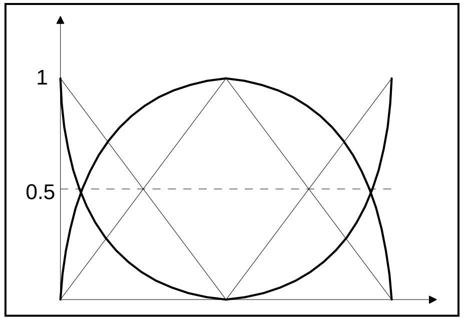
Remark on violations:

1. Since a fuzzy set A has no sharp boundary, and neither has $\bar{A}$, A and $\bar{A}$ would overlap, so $A\cap\bar{A}\neq\varnothing$. However, the overlap is always limited, since

$$u_{A\cap\bar{A}}(x) = \min\left(u_A(x), u_{\bar{A}}(x)\right) \le \frac{1}{2}$$

2. Also, $A\cup\bar{A}$ does not cover X, but

$$u_{A\cup\bar{A}}(x) = \max\left(u_A(x), u_{\bar{A}}(x)\right) \ge \frac{1}{2}$$



▪. Alternative Operators on $\tilde{P}(X)$

***1**. Probabilistic operator*

Def: ***Intersection***

$$A\cdot B = \{X, u_{A\cdot B}\},$$

$$u_{A\cdot B}(x) = u_A(x)\cdot u_B(x),\quad \forall x\in X$$

Union

$$A\hat{+}B = \{X, u_{A\hat{+}B}\},$$

$$u_{A\hat{+}B}(x) = u_A(x) + u_B(x) - u_A(x)\cdot u_B(x),\quad \forall x\in X$$

Remark:

1) Probabilistic union and intersection can be deduced by the Dubois and Prade class of fuzzy set operation, since as α = 1,

a) $$\frac{a+b-ab-\min(a,b,1-\alpha)}{\max(1-a,1-b,\alpha)} = \frac{a+b-ab-0}{1} = a+b-ab$$

b) $$\frac{a\cdot b}{\max(a,b,\alpha)} = \frac{a.b}{1}$$

2) $\{\tilde{P}(X),\hat{+},\cdot,-\}$ satisfies only
Commutativity
associativity
identity
DeMorgan's law
$A\cdot\varnothing = \varnothing$, $A\hat{+}X = X$ (absorption by ∅ and X)

2. *Bold operator*

<u>*Def*</u>: *Intersection*

$$A \cap B = \{X, \mu_{A \cap B}\}$$

$$\mu_{A \cap B}(x) = \max\left(0, \mu_A(x) + \mu_B(x) - 1\right), \quad \forall x \in X$$

Union

$$A \cup B = \{X, \mu_{A \cup B}\}$$

$$\mu_{A \cup B}(x) = \min\left(1, \mu_A(x) + \mu_B(x)\right), \quad \forall x \in X$$

<u>*Remark*</u>:

1) Bold union and intersection can be deduced by the Yager class operator, since as $w = 1$,

   a). $\min\left(1, (a^w + b^w)^{1/w}\right) = \min(1, a + b)$

   b). $1 - \min\left[1, \left((1-a)^w + (1-b)^w\right)^{1/w}\right]$
   $= 1 - \min\left(1, (1-a) + (1-b)\right)$
   $= \max(0, a + b - 1)$

2) $\{\tilde{P}(X), \cup, \cap, -\}$ satisfies only
   Commutativity
   associativity
   identity
   DeMorgan's law
   $A \cap \varnothing = \varnothing$, $A \cup X = X$ (absorption by $\varnothing$ and X)
   $A \cup \overline{A} = X$, $A \cap \overline{A} = \varnothing$ (law of exclude middle)

▪. Comparison on $\{\tilde{P}(X), \cup, \cap, -\}$, $\{\tilde{P}(X), \hat{+}, \cdot, -\}$, $\{\tilde{P}(X), \cup, \cap, -\}$

1. In general,

$$A \cap B \subseteq A \cdot B \subseteq A \cap B$$

$$A \cup B \subseteq A \hat{+} B \subseteq A \cup B$$

2. $\max\left(0, \mu_A(x) + \mu_B(x) - 1\right) \le \mu_A(x) \cdot \mu_B(x)$
   $\le \min\left(\mu_A(x), \mu_B(x)\right) \le \max\left(\mu_A(x), \mu_B(x)\right)$
   $\le \mu_A(x) + \mu_B(x) - \mu_A(x) \cdot \mu_B(x)$
   $\le \min\left(1, \mu_A(x) + \mu_B(x)\right)$

▪. About the standard operators

1> ***<u>Union</u>***

$$A \cup B = \{X, \mu_{A \cup B}\}$$

$$\mu_{A \cup B}(x) = \max_{x \in X}\{\mu_A(x), \mu_B(x)\}$$

2> ***<u>Intersection</u>***

$$A \cap B = \{X, \mu_{A \cap B}\}$$

$$\mu_{A \cap B}(x) = \min_{x \in X}\{\mu_A(x), \mu_B(x)\}$$

3> ***<u>Complement</u>***

$$\overline{A} = \{X, \mu_{\overline{A}}\}$$

$$\mu_{\overline{A}}(x) = 1 - \mu_A(x), \ \forall x \in X$$

Example:

Ages	Infant	Adult	Young	Old
5	0	0	1	0
10	0	0	1	0
20	0	.8	.8	.1
30	0	1	.5	.2
40	0	1	.2	.4
50	0	1	.1	.6
60	0	1	0	.8
70	0	1	0	1
80	0	1	0	1

$$\begin{aligned}\text{Young} \cup \text{Old} &= 1/5 + 1/10 + .8/20 + .5/30 + .4/40 \\ &\quad + .6/50 + .8/60 + 1/70 + 1/80\end{aligned}$$

$$\text{Young} \cap \text{Old} = .1/20 + .2/30 + .2/40 + .1/50$$

Note: When we take {0,1} instead of [0,1], then

1. $A \cup B = \left\{X, \mu_{A\cup B}(x) = \max_{x\in X}\{\mu_A(x), \mu_B(x)\}\right\}$
$= \left\{x \middle| x \in A \text{ or } x \in B\right\}$

2. $A \cap B = \left\{X, \mu_{A\cap B}(x) = \min_{x\in X}\{\mu_A(x), \mu_B(x)\}\right\}$
$= \left\{x \middle| x \in A \text{ and } x \in B\right\}$

3. $\overline{A} = \left\{x \middle| x \in X \text{ and } x \notin A\right\}$

Comment:

1) From the note, they perform identically to the corresponding crisp set operators when {0,1} is instead of [0,1].

2) The max and min operations are additional significant in that they constitute the only fuzzy union and intersection operators that are *continuous* and *idempotent*.

3) Both of these operations are very simple.

4) They are good generalizations of the classical crisp set operators.

5) A lot of applications are baseing on this pair of operations.
