

**(Homework III) Math 31AH Fall 2025**

Assume  $\mathbb{N}, \mathbb{Q}, \mathbb{R}$  are used in the usual sense. We are using same notations as in class.  $\mathcal{L}(S)$  denotes the linear span of  $S$ , and  $\mathcal{E}_m$  denotes the standard basis of  $\mathbb{R}^m$ .

**Problem I.** Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map.

- (1) Let  $S$  be a linear subspace of  $\mathbb{R}^n$ . Prove that  $T(S)$  is also a linear subspace of  $\mathbb{R}^m$ .
- (2) Let  $S \subset \mathbb{R}^m$  be a linear subspace. Define,

$$T^{-1}(S) := \{v \in \mathbb{R}^n : T(v) \in S\}.$$

Show that  $T^{-1}(S)$  is a linear subspace of  $\mathbb{R}^n$ .

- (3) Write the definition of the kernel of  $T$  (aka the null space of  $T$ ). Using above argue that it is a linear subspace of  $T$ .
- (4) Suppose  $Q \neq \emptyset$  is a finite subset of  $\mathbb{R}^m$ . Show that there exists a map  $\phi : \ker T \rightarrow T^{-1}(Q)$  that is injective. (Hint: say,  $q \in Q$  and  $T(p) = q$ . For all  $\alpha \in \ker T$ , see what is  $T(\alpha + p)$ .)

**Problem II.** Let  $n > m$ ,  $\mathcal{B} \subset \mathbb{R}^n$  be a basis, and  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map.

- Say,  $\mathcal{B} = \{v_1, \dots, v_{n-m}, \dots, v_n\}$ ,  $\ker T = \mathcal{L}(\{v_1, \dots, v_{n-m}\})$ . Show that

$$\mathcal{C} := \{T(v_i) : n - m + 1 \leq i \leq n\}$$

is a basis of  $\mathbb{R}^m$ .

- Show that the first  $n - m$  columns of the matrix  $[T]_{\mathcal{B}, \mathcal{E}_m}$  (associated to  $T$  with respect to the bases  $\mathcal{B} \subset \mathbb{R}^n$ ,  $\mathcal{E}_m \subset \mathbb{R}^m$  on domain and target) are zero.
- Let  $T_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , and  $S_m : \mathbb{R}^m \rightarrow \mathbb{R}^m$  be linear maps satisfying  $T_n : e_i \mapsto v_i$ , and  $S_m : T(v_{n-m+i}) \mapsto e_i$  respectively. That is,  $T_n : \mathcal{E}_n \mapsto \mathcal{B}$ , and  $S_m : \mathcal{C} \mapsto \mathcal{E}_m$ . Show that the matrix associated to the linear map  $S_m \circ T \circ T_n$  with the standard bases on the domain and the target is of the form  $[0 | I_m] \in M_{m \times n}(\mathbb{R})$ .

**Problem III.** Exercise 1.3.2, 1.3.4, 1.3.10, and 1.3.20 of textbook.