(Homework I) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense.

Problem I. Let S be a non-empty set.

(1) When is S called countably, and uncountably infinite?

(2) Suppose each element of S can be assigned to at most five distinct k-tuples in $\mathbb{N}^k$. Also, any two distinct element of S has at least one distinct k-tuple assigned to them. Determine the cardinality of S with complete justification. **Hint:** First prove that $\mathbb{N}^k$ is countable. Define a map from S to $\left(\mathbb{N}^k\right)^5$.

(3) Let $S \subset \mathcal{P}(\mathbb{N})$ be the set of all infinite subsets of $\mathbb{N}$. Prove that S is uncountable. Write a complete argument. **Hint:** Use Cantor's diagonal trick from class.

Problem II. Show that the set of algebraic numbers is countable. **Hint:** For an algebraic number a, assume $p_a(x)$ is a monic polynomial of minimal degree. Define, height of the polynomial p_a as $h(p_a)$ as the sum of the absolute values of the co-efficients of p_a, and realize this is a rational number.

Problem III. The set of all formal power series $\mathcal{P}_\infty$ is defined as

$$\mathcal{P}_\infty := \left\{ p(x) = \sum_0^\infty a_n x^n : a_n \in \mathbb{Q} \right\},$$

where x is a variable.

(1) Prove that $\mathcal{P}_\infty$ is a vector space over $\mathbb{Q}$.

(2) Suppose by $\mathcal{P}_k$ we denote the space of all polynomials with rational coefficients of degree at most $k-1$. Prove that $\dim \mathcal{P}_k = k$ by writing down a basis. Therefore, argue that $\mathcal{P}_\infty$ is infinite dimensional. Write down a basis of $\mathcal{P}_\infty$.

(3) Denote by $\mathcal{Q}_\infty$ the set of all sequences (or ordered infinite tuples) of rational numbers,

$$\mathcal{Q} := \{(a_0, a_1, \dots) : a_i \in \mathbb{Q}\}.$$

Prove that $\mathcal{Q}$ is isomorphic to $\mathcal{P}_\infty$. Write down a basis of $\mathcal{Q}$.

Problem IV. Let V be a vector space over $\mathbb{R}$, and L be a subspace of V. Define a relation ρ on V as

$$a\rho b \iff a - b \in L.$$

(1) Show that ρ defines an equivalence relation on V.

(2) Show that the set of equivalence classes of ρ defines a vector space. It is called the quotient space and denoted by V/L.

(3) Assuming $\{e_1, \dots, e_{k+\ell}\}$ is a basis for V, and $\{e_1, \dots, e_\ell\}$ is a basis for L, write down a basis for V/L.