(Practice Midterm I) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S, and $\mathcal{E}_m$ denotes the standard basis of $\mathbb{R}^m$.

Problem I. Sets, Relations, and Cardinality

(1) Write the power set of the set $A = \{1, 2, a\}$.
(2) Recall the definition of equivalence relation on a non-empty set.
(3) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Define a binary relation R on $\mathbb{R}^n$ as vRw iff $v - w \in kerT$. Show that R is an equivalence relation on $\mathbb{R}^n$.
(4) Define, the equivalence class $[v]$ of v as $[v] := \{w \in \mathbb{R}^n : vRw\}$. Prove that $[v] = \{T^{-1}(T(v))\}$, the set of pre-images of $T(v)$.
(5) Recall, the Schroder-Bernstein theorem. Using it or otherwise show that $\mathbb{N} \times \mathbb{N}$ is countable.
(6) Recall Cantor's diagonal trick. Say $A := \{a \in (0, 1) : a = 0.a_1a_2 \ldots, a_i \in \{2, 3\}\}$. That is, A is all the real numbers whose decimal expansion consists only of 2 and 3. Show that A is uncountable.

Problem II. Subspace

(1) Check whether $\{(x, y, z) \in \mathbb{R}^2 : x + 2y + 3z = 1\}$ is a linear subspace.
(2) Check whether $\{(x, y, z) : x + 2y + z = 0\}$ is a subspace. If yes, write a basis for this.
(3) Let $p(x, y) = x^2 + xy + x$. Determine if $\{(x, y) : p(x, y) = 0\}$ is a subspace.
(4) Let S_1, S_2 be subspaces. Is $S_1 \cup S_2$ also a subspace? Define $S_1 + S_2 := \{s_1 + s_2 : s_i \in S_i\}$. Is $S_1 + S_2$ a subspace?

Problem III. Span and Linear Independence

(1) Let $S = \{(x, y) : x^2 + y^2 = 1\}$, the unit circle. What is $\mathcal{L}(S)$?
(2) Let S be a linear subspace of $\mathbb{R}^n$. What is $\mathcal{L}(S)$?
(3) Show that $S_1 + S_2 = \mathcal{L}(S_1 \cup S_2)$.
(4) Check linear independence of $\{(1, 0, 0), (1, 2, 3), (0, -2, -3)\}$. Drop elements if necessary to make it linearly independent.
(5) Prove linear independence of $\{(1, 0, 0), (0, -2, -3)\}$. Extend this to a basis of $\mathbb{R}^3$.
(6) Write the definition of a coordinate of a vector w.r.t. a basis. Choose two bases of $\mathbb{R}^3$ and compute the coordinates w.r.t. these bases.

Problem IV. Linear Transformation, Kernel, and Rank-nullity theorem

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and $kerT$ be trivial. Is $n > m$?
(2) Let $W_1, W_2 \subset T(\mathbb{R}^n)$ be linear subspaces with $W_1 \cap W_2 = \{0\} \subset \mathbb{R}^m$. Show that $T^{-1}(W_i)$ are linear subspaces of $\mathbb{R}^n$. Further show that $T^{-1}(W_1) \cap T^{-1}(W_2) = \{0\} \subset \mathbb{R}^n$. Moreover if $W_1 + W_2 = T(\mathbb{R}^n)$, show any $v \in \mathbb{R}^n$ can be be uniquely written as $v = w_1 + w_2$, where $w_i \in T^{-1}(W_i)$. Thus,
$$T^{-1}(W_1) + T^{-1}(W_2) = \mathbb{R}^n.$$
(3) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. Moreover, let, $T(S^{n-1}) = \{v\} \subset \mathbb{R}^m$. That is, T sends the entire sphere to one single element of $\mathbb{R}^m$. Compute the dimension of the kernel of T. Describe $T(\mathbb{R}^n)$.

Problem V. Matrix Representation, row reduction

(1) Find the matrix representation of the rotation, inclusion, projection linear maps.
(2) Do last two problems of Homework 4.

(Practice Midterm II) Math 31AH Fall 2025

Notation: $<,>$ denotes the dot product in $\mathbb{R}^n$. $\{e_i\}_1^n$ is the standard basis of $\mathbb{R}^n$.

Problem I. Dot Product, Cross Product, Determinants

(1) Define cross product $v \times w$ between two vectors v, w in $\mathbb{R}^3$.
(2) Prove that $v \times w$ is orthogonal to v, w.
(3) Define dot product in $\mathbb{R}^n$.
(4) Suppose $a, b, c, v, w \in \mathbb{R}^3$. Prove that $\det[abc] = \langle a, (b \times c)\rangle$. Argue $\det[vw(v \times w)] = 0$.
(5) See problems from "supplementary exercises (determinants)" file.
(6) Compute the determinant of a 4×4 matrix of your choice.

Problem II. Eigenvalues, Eigenvectors

(1) Let $A \in M_n(\mathbb{R})$. Write the definition of characteristic polynomial of A. Define eigenvalues of A using characteristic polynomial of A.
(2) Practice computing eigenvalues, eigenvectors of given 3×3 matrices.
(3) $A \in M_n(\mathbb{R})$ is called *similar* to $B \in M_n(\mathbb{R})$ if there exists an invertible matrix Q such that $QAQ^{-1} = B$. Show that A, B have the same characteristic polynomials. Thus, they have the same eigenvalues.
(4) (Continuation of the above problem) Suppose λ is an eigenvalue of B and v is an associated eigenvector. Prove that λ is an eigenvalue of A with $Q^{-1}v$ as an associated eigenvector.

Problem III. Orthogonality

(1) Let $S \subset \mathbb{R}^n$ be non-empty. Define the orthogonal complement $S^\perp$ of S.
(2) Show that $S^\perp$ is a linear subspace of $\mathbb{R}^n$.
(3) Prove that $\mathcal{L}(S) \cap S^\perp = \{0\}$.
(4) Prove that $\mathcal{L}(S) + S^\perp = \mathbb{R}^n$. In other words, $\mathbb{R}^n$ is a direct sum of $\mathcal{L}(S)$ and $S^\perp$; denoted as
$$\mathcal{L}(S) \oplus S^\perp.$$
(5) Let $S_1 \subset \mathbb{R}^n$ be a non-empty set of mutually orthogonal vectors. Show that S_1 is linearly independent.
(6) Let $v, w \in \mathbb{R}^n$. The *projection* of v on w is defined as $\frac{<v,w>}{\|w\|^2}w$. As the name indicates, erasing this *projection* from v should produce a vector that is orthogonal to w. Confirm this intuition. That is, show $v - \frac{<v,w>}{\|w\|^2}w$ is orthogonal to v.
(7) Write the definition of an *orthonormal set.*
(8) Let $A \in M_n(\mathbb{R})$ be such that the column vectors of A constitute an orthonormal basis of $\mathbb{R}^n$. Prove that $A^T A = Id_n$. Such matrices are called *orthogonal matrices.*
(9) Suppose $V_1, V_2 \subset \mathbb{R}^n$ be k-dimensional linear subspaces of $\mathbb{R}^n$, and $T : V_1 \to V_2$ be a linear map. Further suppose $< T(v), T(w) >=< v, w >$ for all $v, w \in V_1$. That is, T *preserves* the dot product.
   - Prove that T maps any orthogonal subset of V_1 to another orthogonal set in V_2.
   - Show that the matrix associated with T (w.r.t. any basis of V_1) is an orthogonal matrix.
   - Show that any eigenvalue λ of T satisfies $\lambda^2 = 1$.

(10) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map (just a linear map, orthogonality is not assumed) and $T_1, \dots, T_n$ are the component functions. That is, $T(v) = (T_1(v), T_2(v), \dots, T_n(v)) \in \mathbb{R}^n$.
   - Show that $\ker T_i = \{(T_i(e_1), \dots, T_i(e_n))\}^\perp$.

- Since $\ker T = \cap_1^n \ker T_i$ see that $\ker T = \{(T_i(e_1), \ldots, T_i(e_n)) : 1 \leq i \leq n\}^\perp$. Therefore every linear map T decomposes $\mathbb{R}^n$ as
$$\mathbb{R}^n = \ker T \oplus \mathcal{L}(\{(T_i(e_1), \ldots, T_i(e_n)) : 1 \leq i \leq n\}).$$