(Practice Midterm I) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S, and $\mathcal{E}_m$ denotes the standard basis of $\mathbb{R}^m$.

Problem I. Sets, Relations, and Cardinality

(1) Write the power set of the set $A = \{1, 2, a\}$.
(2) Recall the definition of equivalence relation on a non-empty set.
(3) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Define a binary relation R on $\mathbb{R}^n$ as vRw iff $v - w \in kerT$. Show that R is an equivalence relation on $\mathbb{R}^n$.
(4) Define, the equivalence class $[v]$ of v as $[v] := \{w \in \mathbb{R}^n : vRw\}$. Prove that $[v] = \{T^{-1}(T(v))\}$, the set of pre-images of $T(v)$.
(5) Recall, the Schroder-Bernstein theorem. Using it or otherwise show that $\mathbb{N} \times \mathbb{N}$ is countable.
(6) Recall Cantor's diagonal trick. Say $A := \{a \in (0, 1) : a = 0.a_1a_2 \ldots, a_i \in \{2, 3\}\}$. That is, A is all the real numbers whose decimal expansion consists only of 2 and 3. Show that A is uncountable.

Problem II. Subspace

(1) Check whether $\{(x, y, z) \in \mathbb{R}^2 : x + 2y + 3z = 1\}$ is a linear subspace.
(2) Check whether $\{(x, y, z) : x + 2y + z = 0\}$ is a subspace. If yes, write a basis for this.
(3) Let $p(x, y) = x^2 + xy + x$. Determine if $\{(x, y) : p(x, y) = 0\}$ is a subspace.
(4) Let S_1, S_2 be subspaces. Is $S_1 \cup S_2$ also a subspace? Define $S_1 + S_2 := \{s_1 + s_2 : s_i \in S_i\}$. Is $S_1 + S_2$ a subspace?

Problem III. Span and Linear Independence

(1) Let $S = \{(x, y) : x^2 + y^2 = 1\}$, the unit circle. What is $\mathcal{L}(S)$?
(2) Let S be a linear subspace of $\mathbb{R}^n$. What is $\mathcal{L}(S)$?
(3) Show that $S_1 + S_2 = \mathcal{L}(S_1 \cup S_2)$.
(4) Check linear independence of $\{(1, 0, 0), (1, 2, 3), (0, -2, -3)\}$. Drop elements if necessary to make it linearly independent.
(5) Prove linear independence of $\{(1, 0, 0), (0, -2, -3)\}$. Extend this to a basis of $\mathbb{R}^3$.
(6) Write the definition of a coordinate of a vector w.r.t. a basis. Choose two bases of $\mathbb{R}^3$ and compute the coordinates w.r.t. these bases.

Problem IV. Linear Transformation, Kernel, and Rank-nullity theorem

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and $kerT$ be trivial. Is $n > m$?
(2) Let $W_1, W_2 \subset T(\mathbb{R}^n)$ be linear subspaces with $W_1 \cap W_2 = \{0\} \subset \mathbb{R}^m$. Show that $T^{-1}(W_i)$ are linear subspaces of $\mathbb{R}^n$. Further show that $T^{-1}(W_1) \cap T^{-1}(W_2) = \{0\} \subset \mathbb{R}^n$. Moreover if $W_1 + W_2 = T(\mathbb{R}^n)$, show any $v \in \mathbb{R}^n$ can be be uniquely written as $v = w_1 + w_2$, where $w_i \in T^{-1}(W_i)$. Thus,
$$T^{-1}(W_1) + T^{-1}(W_2) = \mathbb{R}^n.$$
(3) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. Moreover, let, $T(S^{n-1}) = \{v\} \subset \mathbb{R}^m$. That is, T sends the entire sphere to one single element of $\mathbb{R}^m$. Compute the dimension of the kernel of T. Describe $T(\mathbb{R}^n)$.

Problem V. Matrix Representation, row reduction

(1) Find the matrix representation of the rotation, inclusion, projection linear maps.
(2) Do last two problems of Homework 4.