(Homework IV) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S, and $\mathcal{E}_m$ denotes the standard basis of $\mathbb{R}^m$.

Problem I. Let $T : \mathbb{R}^n \to \mathbb{R}^2$ be a linear map.

(1) Suppose $T(v) = (T_1(v), T_2(v)) \in \mathbb{R}^2$, for all $v \in \mathbb{R}^n$. Show that $T_1, T_2 : \mathbb{R}^n \to \mathbb{R}$ are linear maps.

(2) Prove that $ker(T_1) \cap ker(T_2) = ker(T)$.

(3) Recall that, the product of an $1 \times n$ matrix (a 'row') with an $n \times 1$ matrix (a 'column') is a real number (that is, an 1×1 matrix). This phenomena is also referred as *dot product* or *scalar product* of elements in $\mathbb{R}^n$. We shall denote the dot product of vectors v, w by $v.w$. Show that there exists $A_1 \in \mathbb{R}^n$ such that $T_1(v) = A_1.v$, for all $v \in \mathbb{R}^n$.

(4) Suppose the corresponding vector for T_2 is $A_2 \in \mathbb{R}^n$; that is, $T_2 : v \mapsto A_2.v$. Show that the matrix $[T]$ associated to the linear map T (with respect to the standard bases on domain and target) is $[T] = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \in M_{2\times n}(\mathbb{R})$.

I overheard some students talking about something like this near the board after class last Thursday. Thanks to you for reminding me about this!

Problem II. Use row reduction to do the following.

- Solve the system for x, y, z whenever possible and write the complete set of solution.

$$\begin{aligned} x + y &= a, \\ y + z &= b, \\ x + z &= c. \end{aligned}$$

- For what values of a, b is $\begin{bmatrix} 0 & a & 1 \\ -a & 0 & b \\ -1 & -b & 0 \end{bmatrix}$ invertible?

Problem III. Exercise 2.2.5 of textbook.