(Practice Midterm I) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S, and $\mathcal{E}_m$ denotes the standard basis of $\mathbb{R}^m$.

Problem I. Sets, Relations, and Cardinality

(1) Write the power set of the set $A = \{1, 2, a\}$.
(2) Recall the definition of equivalence relation on a non-empty set.
(3) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Define a binary relation R on $\mathbb{R}^n$ as vRw iff $v - w \in kerT$. Show that R is an equivalence relation on $\mathbb{R}^n$.
(4) Define, the equivalence class $[v]$ of v as $[v] := \{w \in \mathbb{R}^n : vRw\}$. Prove that $[v] = \{T^{-1}(T(v))\}$, the set of pre-images of $T(v)$.
(5) Recall, the Schroder-Bernstein theorem. Using it or otherwise show that $\mathbb{N} \times \mathbb{N}$ is countable.
(6) Recall Cantor's diagonal trick. Say $A := \{a \in (0, 1) : a = 0.a_1a_2 \dots, a_i \in \{2, 3\}\}$. That is, A is all the real numbers whose decimal expansion consists only of 2 and 3. Show that A is uncountable.

Problem II. Subspace

(1) Check whether $\{(x, y, z) \in \mathbb{R}^2 : x + 2y + 3z = 1\}$ is a linear subspace.
(2) Check whether $\{(x, y, z) : x + 2y + z = 0\}$ is a subspace. If yes, write a basis for this.
(3) Let $p(x, y) = x^2 + xy + x$. Determine if $\{(x, y) : p(x, y) = 0\}$ is a subspace.
(4) Let S_1, S_2 be subspaces. Is $S_1 \cup S_2$ also a subspace? Define $S_1 + S_2 := \{s_1 + s_2 : s_i \in S_i\}$. Is $S_1 + S_2$ a subspace?

Problem III. Span and Linear Independence

(1) Let $S = \{(x, y) : x^2 + y^2 = 1\}$, the unit circle. What is $\mathcal{L}(S)$?
(2) Let S be a linear subspace of $\mathbb{R}^n$. What is $\mathcal{L}(S)$?
(3) Show that $S_1 + S_2 = \mathcal{L}(S_1 \cup S_2)$.
(4) Check linear independence of $\{(1, 0, 0), (1, 2, 3), (0, -2, -3)\}$. Drop elements if necessary to make it linearly independent.
(5) Prove linear independence of $\{(1, 0, 0), (0, -2, -3)\}$. Extend this to a basis of $\mathbb{R}^3$.
(6) Write the definition of a coordinate of a vector w.r.t. a basis. Choose two bases of $\mathbb{R}^3$ and compute the coordinates w.r.t. these bases.

Problem IV. Linear Transformation, Kernel, and Rank-nullity theorem

(1) Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be linear, and $kerT$ be trivial. Is $n > m$?
(2) Let $W_1, W_2 \subset T(\mathbb{R}^n)$ be linear subspaces with $W_1 \cap W_2 = \{0\} \subset \mathbb{R}^m$. Show that $T^{-1}(W_i)$ are linear subspaces of $\mathbb{R}^n$. Further show that $T^{-1}(W_1) \cap T^{-1}(W_2) = \{0\} \subset \mathbb{R}^n$. Moreover if $W_1 + W_2 = T(\mathbb{R}^n)$, show any $v \in \mathbb{R}^n$ can be be uniquely written as $v = w_1 + w_2$, where $w_i \in T^{-1}(W_i)$. Thus,
$$T^{-1}(W_1) + T^{-1}(W_2) = \mathbb{R}^n.$$
(3) Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, and S^{n-1} is the unit sphere in $\mathbb{R}^n$. Moreover, let, $T(S^{n-1}) = \{v\} \subset \mathbb{R}^m$. That is, T sends the entire sphere to one single element of $\mathbb{R}^m$. Compute the dimension of the kernel of T. Describe $T(\mathbb{R}^n)$.

Problem V. Matrix Representation, row reduction

(1) Find the matrix representation of the rotation, inclusion, projection linear maps.
(2) Do last two problems of Homework 4.

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Notation: $<,>$ denotes the dot product in $\mathbb{R}^n$. $\{e_i\}_1^n$ is the standard basis of $\mathbb{R}^n$.

Problem I. Dot Product, Cross Product, Determinants

(1) Define cross product $v \times w$ between two vectors v, w in $\mathbb{R}^3$.
(2) Prove that $v \times w$ is orthogonal to v, w.
(3) Define dot product in $\mathbb{R}^n$.
(4) Suppose $a, b, c, v, w \in \mathbb{R}^3$. Prove that $\det[abc] = \langle a, (b \times c)\rangle$. Argue $\det[vw(v \times w)] = 0$.
(5) See problems from "supplementary exercises (determinants)" file.
(6) Compute the determinant of a 4×4 matrix of your choice.

Problem II. Eigenvalues, Eigenvectors

(1) Let $A \in M_n(\mathbb{R})$. Write the definition of characteristic polynomial of A. Define eigenvalues of A using characteristic polynomial of A.
(2) Practice computing eigenvalues, eigenvectors of given 3×3 matrices.
(3) $A \in M_n(\mathbb{R})$ is called *similar* to $B \in M_n(\mathbb{R})$ if there exists an invertible matrix Q such that $QAQ^{-1} = B$. Show that A, B have the same characteristic polynomials. Thus, they have the same eigenvalues.
(4) (Continuation of the above problem) Suppose λ is an eigenvalue of B and v is an associated eigenvector. Prove that λ is an eigenvalue of A with $Q^{-1}v$ as an associated eigenvector.

Problem III. Orthogonality

(1) Let $S \subset \mathbb{R}^n$ be non-empty. Define the orthogonal complement $S^\perp$ of S.
(2) Show that $S^\perp$ is a linear subspace of $\mathbb{R}^n$.
(3) Prove that $\mathcal{L}(S) \cap S^\perp = \{0\}$.
(4) Prove that $\mathcal{L}(S) + S^\perp = \mathbb{R}^n$. In other words, $\mathbb{R}^n$ is a direct sum of $\mathcal{L}(S)$ and $S^\perp$; denoted as
$$\mathcal{L}(S) \oplus S^\perp.$$
(5) Let $S_1 \subset \mathbb{R}^n$ be a non-empty set of mutually orthogonal vectors. Show that S_1 is linearly independent.
(6) Let $v, w \in \mathbb{R}^n$. The *projection* of v on w is defined as $\frac{<v,w>}{\|w\|^2}w$. As the name indicates, erasing this *projection* from v should produce a vector that is orthogonal to w. Confirm this intuition. That is, show $v - \frac{<v,w>}{\|w\|^2}w$ is orthogonal to v.
(7) Write the definition of an *orthonormal set.*
(8) Let $A \in M_n(\mathbb{R})$ be such that the column vectors of A constitute an orthonormal basis of $\mathbb{R}^n$. Prove that $A^T A = Id_n$. Such matrices are called *orthogonal matrices.*
(9) Suppose $V_1, V_2 \subset \mathbb{R}^n$ be k-dimensional linear subspaces of $\mathbb{R}^n$, and $T : V_1 \to V_2$ be a linear map. Further suppose $< T(v), T(w) >=< v, w >$ for all $v, w \in V_1$. That is, T *preserves* the dot product.
    - Prove that T maps any orthogonal subset of V_1 to another orthogonal set in V_2.
    - Show that the matrix associated with T (w.r.t. any basis of V_1) is an orthogonal matrix.
    - Show that any eigenvalue λ of T satisfies $\lambda^2 = 1$.
(10) Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map (just a linear map, orthogonality is not assumed) and $T_1, \ldots, T_n$ are the component functions. That is, $T(v) = (T_1(v), T_2(v), \ldots, T_n(v)) \in \mathbb{R}^n$.
    - Show that $\ker T_i = \{(T_i(e_1), \ldots, T_i(e_n))\}^\perp$.

- Since $\ker T = \cap_1^n \ker T_i$ see that $\ker T = \{(T_i(e_1), \ldots, T_i(e_n)) : 1 \leq i \leq n\}^{\perp}$. Therefore every linear map T decomposes $\mathbb{R}^n$ as
$$\mathbb{R}^n = \ker T \oplus \mathcal{L}(\{(T_i(e_1), \ldots, T_i(e_n)) : 1 \leq i \leq n\}).$$

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Problem I. Orthogonality

(1) Suppose $S \subset \mathbb{R}^n$ is finite and $\mathcal{L}(S) \subsetneq \mathbb{R}^n$. Construct a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ such that $kerT \oplus S^{\perp} = \mathbb{R}^n$.

(2) Suppose $<,>$ denotes the dot product in $\mathbb{R}^n$, and A is an orthogonal matrix. Show that $< Av, Aw >=< v, w >$ for all $v, w \in \mathbb{R}^n$.

(3) Suppose $A \in M_n(\mathbb{R})$ is such that $< Av, Aw >=< v, w >$ for all $v, w \in \mathbb{R}^n$. Prove that $A^T = A^{-1}$.

(4) Define $\hat{e}_k = \sum_{j=1}^k je_j$, where $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$. Show that $\{\hat{e}_j : 1 \leq j \leq n\}$ is a basis of $\mathbb{R}^n$.
Find an orthonormal basis of $\mathbb{R}^3$ that contains a vector of the form $c\hat{e_3}$ (using Gram-Schimdt). Here $c \in \mathbb{R}$.

(5) Recall the Taylor expansion of the exponential function. The exponential of a matrix A is defined similarly:
$$\exp(A) = \sum_0^{\infty} \frac{1}{k!} A^k.$$
Suppose $A = (a_{ij})_{n \times n} \in M_n(\mathbb{R})$ is diagonal with $a_{ii} = \lambda_i$. Prove that $\exp(A)$ is diagonal with diagonal entries e^{λ_i}. (Hint: Recall A^k is diagonal with diagonal entries λ_i^k.)

(6) Check that $\exp(0) = Id_n$, $\exp(A^T) = \exp(A)^T$, $\exp(YAY^{-1}) = Y \exp(A) Y^{-1}$.

(7) Note therefore, if A is diagonalizable then so is $\exp(A)$.

(8) Assume the fact that $XY = YX$ implies $\exp(X)\exp(Y) = \exp(X+Y)$. Suppose $A \in M_n(\mathbb{R})$ is orthogonal. Prove that $\exp(A)\exp(A^T) = \exp(A + A^T)$. This means orthogonality of A does not imply orthogonality of $\exp(A)$.

Problem II.

(1) Decide true or false with justifications.
- The product of two symmetric matrices is symmetric.
- The sum of two symmetric matrices is symmetric.
- The sum of two anti-symmetric matrices is anti-symmetric.
- The inverse of an invertible symmetric matrix is symmetric.
- If B is an arbitrary $n \times m$ matrix, then $A = B^T B$ is symmetric.
- If A is similar to B and A is symmetric, then B is symmetric.
- If $A = SBS^{-1}$ with $S^T S = I_n$ and A is symmetric, then B is symmetric.
- Every symmetric matrix is diagonalizable.
- Only the zero matrix is both anti-symmetric and symmetric.

(2) (a) Express the characteristic polynomial of the partitioned matrix
$$\det \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$$
in terms of the characteristic polynomials of A and B.
(b) What is the relation between $\chi(A^T)$ and $\chi(A)$?

(3) Find the characteristic polynomial $\chi(A)$ for the magic square

$$A = \begin{bmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{bmatrix}$$

and factor it.

(4) (a) Given the eigenvalues $\lambda_1 = 3$, $\lambda_2 = 6$, $\lambda_3 = 4$, $\lambda_4 = 9$, find a non-triangular matrix which has these eigenvalues.

(b) Given the eigenvalues $\lambda_1 = 1 + i$, $\lambda_2 = 3 + 4i$, $\lambda_3 = 2 - 2i$, is there a real 3×3 matrix which has these eigenvalues? If not, why not?

(c) Is it true that if λ is an eigenvalue of A and μ is an eigenvalue of B, then $\lambda\mu$ is an eigenvalue of AB?

(d) Is it true that if λ is an eigenvalue of A, then λ^2 is an eigenvalue of A^2?

(e) True or False: if λ is a non-zero eigenvalue of A, then $1/\lambda$ is an eigenvalue of A^{-1}.

(f) True or False: if λ is an eigenvalue of A, then λ is an eigenvalue of A^T.

(5) A matrix with non-negative entries for which the sum of the entries in each column is equal to 1 is called a *stochastic matrix* or *Markov matrix.*

- Any Markov matrix has eigenvalue 1.
- Any Markov matrix with positive entries then the maximum eigenvalue is 1 and has multiplicity 1. This is also known as the *Perron-Frobenius theorem.* Prove this for 2×2 matrices.