

(Homework III) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S , and \mathcal{E}_m denotes the standard basis of \mathbb{R}^m .

Problem I. Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.

- (1) Let S be a linear subspace of \mathbb{R}^n . Prove that $T(S)$ is also a linear subspace of \mathbb{R}^m .
- (2) Let $S \subset \mathbb{R}^m$ be a linear subspace. Define,

$$T^{-1}(S) := \{v \in \mathbb{R}^n : T(v) \in S\}.$$

Show that $T^{-1}(S)$ is a linear subspace of \mathbb{R}^n .

- (3) Write the definition of the kernel of T (aka the null space of T). Using above argue that it is a linear subspace of T .
- (4) Suppose $Q \neq \emptyset$ is a finite subset of \mathbb{R}^m . Show that there exists a map $\phi : \ker T \rightarrow T^{-1}(Q)$ that is injective. (Hint: say, $q \in Q$ and $T(p) = q$. For all $\alpha \in \ker T$, see what is $T(\alpha + p)$.)

Problem II. Let $n > m$, $\mathcal{B} \subset \mathbb{R}^n$ be a basis, and $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map.

- Say, $\mathcal{B} = \{v_1, \dots, v_{n-m}, \dots, v_n\}$, $\ker T = \mathcal{L}(\{v_1, \dots, v_{n-m}\})$. Show that

$$\mathcal{C} := \{T(v_i) : n - m + 1 \leq i \leq n\}$$

is a basis of \mathbb{R}^m .

- Show that the first $n - m$ columns of the matrix $[T]_{\mathcal{B}, \mathcal{E}_m}$ (associated to T with respect to the bases $\mathcal{B} \subset \mathbb{R}^n$, $\mathcal{E}_m \subset \mathbb{R}^m$ on domain and target) are zero.
- Let $T_n : \mathbb{R}^n \rightarrow \mathbb{R}^n$, and $S_m : \mathbb{R}^m \rightarrow \mathbb{R}^m$ be linear maps satisfying $T_n : e_i \mapsto v_i$, and $S_m : T(v_{n-m+i}) \mapsto e_i$ respectively. That is, $T_n : \mathcal{E}_n \mapsto \mathcal{B}$, and $S_m : \mathcal{C} \mapsto \mathcal{E}_m$. Show that the matrix associated to the linear map $S_m \circ T \circ T_n$ with the standard bases on the domain and the target is of the form $[0|I_m] \in M_{m \times n}(\mathbb{R})$.

Problem III. Exercise 1.3.2, 1.3.4, 1.3.10, and 1.3.20 of textbook.