(Homework V) Math 31AH Fall 2025

Assume $\mathbb{N}, \mathbb{Q}, \mathbb{R}$ are used in the usual sense. We are using same notations as in class. $\mathcal{L}(S)$ denotes the linear span of S, and $\mathcal{E}_m$ denotes the standard basis of $\mathbb{R}^m$.

Problem I. Exercise 2.1.2 and 2.1.3 from the textbook.

Problem II. Exercise 1.4.2, 1.4.5, 1.4.8, 1.4.12 (a), 1.4.24, 1.4.27 from the textbook.

Problem III. Let, A be a real 3×3 matrix.

- Write the definition of an eigenvalue and an associated eigenvector of A.
- Argue that A must have at least one real eigenvalue.
- Let, λ be an eigenvalue and v be an associated eigenvector of A. Suppose T is a linear map such that $\mathcal{L}(\{v\}) \subset ker(T - \lambda Id)$. Let $p(x)$ be some polynomial with real coefficients such that $p(T) = 0 \in M_3(\mathbb{R})$. That is, the 3×3 matrix $p(T)$ is just the zero matrix. Prove that $(x - \lambda)$ is a factor of $p(x)$. (Caution: This fact is true for all square matrices, do not get distracted by the $M_3(\mathbb{R})$.)