KBA-2 - Maximum Likelihood Estimate (MLE)

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1 Quick Summary

The following sections discuss the maximum likelihood estimates for Poisson and Binomial distributions, the results are summarized here for quick reference:

Poisson Distribution

$$\lambda_{MLE} = \frac{x}{t}$$

Where x represents the number of events (failures) over the plant-specific period of time t.

Binomial Distribution

$$p_{MLE} = \frac{x}{n}$$

Where x represents the number of events (failures) for the number of demands n.

Exponential Distribution

$$\lambda_{MLE} = \frac{n}{\sum_{i=1}^{n} t_i}$$

Where t_i is the exposure time for plant i, and n is the total number of plants.

2 Maximum Likelihood Estimate - Discrete Distributions

The most commonly used frequentist estimate is the maximum likelihood estimate (MLE). It is found by determining which value maximizes the likelihood equation. The cases for demand-type failures and run-type failures are derived in the sections below. Note that the sections below assumed that the number of failures x was generated from a Poission distribution, or a Binomial distribution, and is conditional on the plant-specific parameter, namely, λ or p. However, if λ or p was generated from another distribution, say *gamma*(α,β), the equations for the maximum likelihood estimators (also termed MLEs) become more complex. These formulations are not discussed here.

2.1 Poission Distribution (Run-Type Failures)

The MLE for a Poisson distribution can be obtained by treating the Poisson likelihood as a function of λ and using calculus. The likelihood equation for a Poisson distribution is as follows:

$$Poisson(x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}$$

Taking the derivative of the likelihood equation with respect to λ, and finding the maximum results in the following:

$$\frac{d}{d\lambda}e^{-\lambda t}\frac{(\lambda t)^x}{x!}$$

$$\frac{d}{d\lambda}e^{-\lambda t}\frac{(\lambda t)^x}{x!} + e^{-\lambda t}\frac{d}{d\lambda}\frac{\lambda^x t^x}{x!}$$

$$-te^{-\lambda t}\frac{\lambda^x t^x}{x!} + e^{-\lambda t}\frac{t^x x\lambda^{x-1}}{x!} = 0$$

$$te^{-\lambda t}\frac{\lambda^x t^x}{x!} = e^{-\lambda t}\frac{t^x x\lambda^{x-1}}{x!}$$

$$t = x\lambda^{-1}$$

$$\lambda = \frac{x}{t}$$

2.2 Binomial Distribution (Demand-Type Failures)

The MLE for a Binomial distribution can be obtained by treating the Binomial likelihood as a function of p and using calculus. The likelihood equation for a Binomial distribution is as follows:

$$Binomial(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Taking the derivative of the likelihood equation with respect to p, and finding the maximum results in the following:

$$\frac{d}{dp} C p^x (1-p)^{n-x}$$

$$\frac{d}{dp} C p^x (1-p)^{n-x} + C p^x \frac{d}{dp} (1-p)^{n-x}$$

$$C x p^{x-p} (1-p)^{n-x} = C p^x (n-x)(1-p)^{n-x-1}$$

$$\frac{x}{p} = \frac{n-x}{1-p}$$

$$\frac{x}{p}(1-p) + x = n$$

$$n = x(\frac{1-p}{p} + 1)$$

$$\frac{n}{x} = \frac{1-p+p}{p}$$

$$p = \frac{x}{n}$$

3 Maximum Likelihood Estimate - Continous Distributions

In the case of continuous distribution, likelihood refers to the joint probability density of your data.

3.1 Exponential Distribution

The probability density function (p.d.f.) of an exponential distribution is as follows:

$$f(t) = \lambda e^{-\lambda t}$$

Given a set of data $t_1, t_2, ..., t_n$, the joint probability density is as follows:

$$L = f(\lambda | t_1, t_2, ..., t_n) = \prod_{i=1}^{n} \lambda e^{-\lambda t_i}$$

$$L = \lambda^n e^{-\lambda \sum_{i=1}^{n} t_i}$$

The MLE can be determined by finding the value of λ that maximizes the likelihood equation. More easily, we can find the value of λ that maximizes the likelihood equation by finding the value of λ that maximizes the log-likelihood. This generally makes the derivatives easier.

$$ln(L) = ln(\lambda^n e^{-\lambda \sum_{i=1}^{n} t_i})$$

$$ln(L) = ln(\lambda^n) + ln(e^{-\lambda \sum_{i=1}^{n} t_i})$$

$$ln(L) = n\ ln(\lambda) - \lambda \sum_{i=1}^{n} t_i ln(e)$$

$$ln(L) = n\ ln(\lambda) - \lambda \sum_{i=1}^{n} t_i$$

Taking the derivative of the equation above with respect to λ, and setting this to zero results in the following:

$$\frac{d\ ln(L)}{d\lambda} = \frac{d\ ln(L)}{d\lambda}(n\ ln(\lambda) - \lambda \sum_{i=1}^{n} t_i) = 0$$

$$\frac{d\ ln(L)}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} t_i = 0$$

$$\frac{n}{\lambda} = \sum_{i=1}^{n} t_i$$

$$\lambda_{MLE} = \frac{n}{\sum_{i=1}^{n} t_i}$$

4 References

1. NUREG/CR-6823, *Handbook of Parameter Estimation for Probabilistic Risk Assessment*, U.S. Nuclear Regulatory Commission, September 2003.