Bayesian Updating (Beta and Gamma Distributions)

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1 Quick Summary

The following sections discuss the derivation of the Bayesian update formulas for Beta and Gamma distributions in more detail. A summary is provided here for quick reference.

Gamma Distribution

$$\alpha_{posterior} = x + \alpha_{prior}$$
$$\beta_{posterior} = t + \beta_{prior}$$
$$Mean_{posterior} = \frac{\alpha_{posterior}}{\beta_{posterior}}$$
$$Variance_{posterior} = \frac{\alpha_{posterior}}{(\beta_{posterior})^2}$$

Where x represents the number of events (failures) over the plant-specific period of time t.

Beta Distribution

$$\alpha_{posterior} = x + \alpha_{prior}$$
$$\beta_{posterior} = \beta_{prior} + n - x$$
$$Mean_{posterior} = \frac{\alpha_{posterior}}{(\alpha_{posterior} + \beta_{posterior})}$$
$$Variance_{posterior} = \frac{\alpha_{posterior}\beta_{posterior}}{(\alpha_{posterior} + \beta_{posterior})^2(\alpha_{posterior} + \beta_{posterior} + 1)}$$

Where x represents the number of events (failures) for the number of demands n.

2 Bayesian Estimation

As discussed in NUREG/CR-6823 (Section B.5.1), Bayes' Theorem can be seen to transform the prior distribution by the effect of the sample data distribution to produce a posterior distribution. The sample data distribution can be viewed as a function of the unknown parameter, instead of the observed data, producing a likelihood function. Using the likelihood function, the fundamental relationship expressed by Bayes' Theorem is given as follows:

$$PosteriorDistribution = \frac{PriorDistribution * Likelihood}{MarginalDistribution}$$

In Bayesian updating, the sampling distribution of the data provides new information about the parameter value. Bayes' Theorem provides a mathematical framework for processing new sample data as they become sequentially available over time.With the new information, the uncertainty of the parameter value has been reduced, but not eliminated.Bayes' Theorem is used to combine the prior and sampling distributions to form the posterior distribution, which then describes the updated state of knowledge (Still in terms of subjective probability) about the parameter. Point and interval estimates of the parameter can then be obtained directly from the posterior distribution, which is viewed as containing the current knowledge about the parameter.This posterior distribution can then be used as the prior distribution when the next set of data becomes available. Thus, Bayesian updating is successively implemented using additional data in conjunction with Bayes' Theorem to obtain successively better posterior distribution that model plant-specific parameters.

The following sections discuss Bayesian updating of Beta and Gamma distributions which are typically encountered in PRA models.

3 Gamma Distribution

The conjugate family of prior distribution for Poisson data is the family of gamma distributions. For Bayesian updating, the following gamma distribution parameterization is the convenient form:

$$f(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\lambda\beta}$$

Here, λ has units $\frac{1}{time}$ and β has units of time, so the product of $\lambda\beta$ is unitless. For example, if λ is the frequency of events per critical-year, β has units of critical-years. The parameter β is a kind of scale parameter. That is, β corresponds to the scale of λ. If we convert λ from events per hour to events per year by multiplying by 8760, we correspondingly divide β by 8760, converting it from hours to years. The other parameter, α, is unitless, and is called the shape parameter.

Probability density functions generally have normalizing constants to make them integrate to 1.0. These constants can be complicated, but using the proportionality instead of equality allows the analyst to neglect the normalizing constants. Stripped of all the normalizing constants, the Gamma distribution probability density function (PDF) is:

$$f(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda\beta}$$

The Poisson likelihood is defined as follows:

$$Poisson(x) = e^{-\lambda t} \frac{(\lambda t)^x}{x!}$$

The Gamma distribution and the Poisson likelihood combined in a convenient way:

$$PosteriorDistribution \propto e^{-\lambda t} \frac{(\lambda t)^x}{x!} \lambda^{\alpha-1} e^{-\lambda\beta}$$

$$PosteriorDistribution \propto \lambda^{x+\alpha-1} e^{-\lambda(t+\beta)}$$

It can be seen that the posterior distribution is also a Gamma distribution. This is the meaning of conjugate: if the prior distribution is a member of the family (in this case, the Gamma family), the posterior distribution is a member of the same family. The convenient form lends itself to show that a posterior distribution can be determined by modifying the prior α and β parameters. From the convenient form we can see that the prior α and β parameters are replaced in the posterior distribution by the following:

$$\alpha_{posterior} = x + \alpha_{prior}$$

$$\beta_{posterior} = t + \beta_{prior}$$

Where x represents the number of events (failures) over the plant-specific period of time t. Therefore, the posterior mean and variance can be calculated as follows:

$$Mean_{posterior} = \frac{\alpha_{posterior}}{\beta_{posterior}}$$

$$Variance_{posterior} = \frac{\alpha_{posterior}}{(\beta_{posterior})^2}$$

4 Beta Distribution

The conjugate family of prior distributions for Binomial data is the family of beta distributions. For Bayesian updating, the following Beta distribution parameterization is the convenient form:

$$f(y) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} y^{\alpha-1}(1-y)^{\beta-1}$$

As noted in the discussion for the Gamma distribution, PDFs generally have normalizing constants to make them integrate to 1.0. These constants can be complicated, but using the proportionality instead of equality allows the analyst to neglect the normalizing constants. Stripped of all the normalizing constants, the Beta distribution PDF is:

$$f(y) \propto y^{\alpha-1}(1-y)^{\beta-1}$$

The Binomial likelihood is defined as follows:

$$Binomial(x) = \binom{n}{x} y^{x}(1-y)^{n-x}$$

The Beta distribution and the Binomial likelihood combine in a convenient way:

$$PosteriorDistribution \propto y^{\alpha-1}(1-y)^{\beta-1} y^{x}(1-y)^{n-x}$$

$$PosteriorDistribution \propto y^{x+\alpha-1}(1-y)^{\beta+n-x-1}$$

In the final expression, it can be seen that the distribution is also a beta distribution. This is the meaning of conjugate: if the prior distribution is a member of the family (in this case, the beta family), the posterior distribution is a member of the same family. The convenient form lends itself to show that a posterior distribution can be determined by modifying the prior α and β parameters. From the convenient form we can see that the prior α and β parameters are replaced in the posterior distribution by the following:

$$\alpha_{posterior} = x + \alpha_{prior}$$

$$\beta_{posterior} = \beta_{prior} + n - x$$

Where x represents the number of events (failures) for the number of demands n. Therefore, the posterior mean and variance can be calculated as follows:

$$Mean_{posterior} = \frac{\alpha_{posterior}}{(\alpha_{posterior} + \beta_{posterior})}$$

$$Variance_{posterior} = \frac{\alpha_{posterior}\beta_{posterior}}{(\alpha_{posterior} + \beta_{posterior})^2(\alpha_{posterior} + \beta_{posterior} + 1)}$$

5 References

1. NUREG/CR-6823, *Handbook of Parameter Estimation for Probabilistic Risk Assessment*, U.S. Nuclear Regulatory Commission, September 2003.