Stochastic Processes: Homework 1

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1 σ-field

1.
   - $\{\emptyset, \Omega\}$
   - $\{\emptyset, \{1\}, \{2,3\}, \Omega\}$
   - $\{\emptyset, \{2\}, \{1,3\}, \Omega\}$
   - $\{\emptyset, \{3\}, \{1,2\}, \Omega\}$
   - $\{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{2,3\}, \{1,3\}, \Omega\}$

2. The σ-field generated by $\mathcal{E}$ is
$$\sigma(\mathcal{E}) = \Big\{\emptyset, \{1\}, \{2\}, \{1,2\}, \{3,4\}, \{2,3,4\}, \{1,3,4\}, \Omega\Big\}$$

3. $\mathcal{A}$ is not a σ-field because it is not closed under countable union. Simply consider the countable union of all integers. However the complement of $\mathbb{Z}$ is also infinite, hence $\mathbb{Z} \notin \mathcal{A}$.

4. $\mathcal{A}$ is a σ-field. $\emptyset$ is countable hence $\Omega \in \mathcal{A}$. If $A \in \mathcal{A}$ then either A is countable so $A^c \in \mathcal{A}$ or A^c is countable so $A^c \in \mathcal{A}$. Finally for closure under union, we need only show that if $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$. If both A and B are countable then their union is countable. Likewise, if both A^c and B^c are countable then $A^c \cap B^c$ is countable, thus $(A \cup B)^c = A^c \cap B^c$ is countable. Finally if A is countable and B^c is countable then $(A \cup B)^c = A^c \cap B^c \subset B^c$ is countable. Thus $\mathcal{A}$ is a σ-field. $\square$

5. In this light, $x \in \liminf_{n\to\infty} A_n$ means that x is in all but a finite number of A_n. Both $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$ are $\mathcal{F}$-measurable. To see this, recall that $\mathcal{F}$ is closed under countable union and intersection. Thus for both the inner term is always in $\mathcal{F}$, so we have another countable union/intersection of members of $\mathcal{F}$ so that is in $\mathcal{F}$. $\square$

2 Probability Measure

1. Let $q_1, q_2, \ldots$ be an enumeration of $\mathbb{Q}_{[0,1]} = \mathbb{Q} \cap [0,1]$. Then for any $n > 0$ define $L_i = \left[q_i - \frac{n}{2^{i+1}}, q_i + \frac{n}{2^{i+1}}\right]$. Let $L = \bigcup_{i=1}^{\infty} L_i$. Clearly then $\mathbb{Q}_{[0,1]} \subseteq L$. By the

union bound

$$P(L) \leq \sum_{i=1}^{\infty} P(L_i) = \sum_{i=1}^{\infty} \frac{n}{2^i} = n$$

This is true for any $n > 0$ so $P(\mathbb{Q} \cap [0,1]) < n$ for all $n > 0$ so $P(\mathbb{Q}_{[0,1]}) = 0$ $\square$

2. We have $\mathbb{Q}_{[0,1]} \cup I = [0,1]$ and $\mathbb{Q}_{[0,1]} \cap I = \emptyset$, thus $P(\mathbb{Q}_{[0,1]}) + P(I) = P([0,1]) = 1$, Thus by the previous problem we have $P(I) = 1$. $\square$

3. The calculation is invalid because while it is true that the measure of the countable union of disjoint sets is the sum of the measures of those sets, this is in general only true when we are considering a countable union. For example under this logic $1 = P([0,1]) = \sum_{x \in [0,1]} P(\{x\}) = 0$. $\square$

4. (a) The $\inf_{n \geq k} P(A_n)$ is defined as the largest number which is less than or equal to all of the $P(A_n)$. Since $L = P\left(\bigcap_{n \geq k}^{\infty} A_n\right) \leq P(A_j), \forall j \geq k$, then L is at most the largest value which satisfies this condition, namely the infimum. $\square$

   (b)

$$\begin{aligned} P\left(\lim_{n \to \infty} \inf A_n\right) &= P\left(\bigcup_{k \geq 1} \bigcap_{n \geq k} A_n\right) \\ &= \lim_{k \to \infty} P\left(\bigcap_{n \geq k} A_n\right) \end{aligned}$$

   From the first part we have $P\left(\bigcap_{n \geq k} A_n\right) \leq \inf_{n \geq k} A_n$ for all k. This implies that

$$\lim_{k \to \infty} P\left(\bigcap_{n \geq k} A_n\right) \leq \lim k \to \infty \inf_{n \geq k} A_n$$

   This follows straight from the definition of limits. If the LHS were greater than the RHS in the limit, then for sufficiently large k the LHS would have to be larger the RHS. However we know this to never be the case, thus the above statement holds. $\square$

3 Poisson Process

1. We have $\Pr[N(1) = 0] = 0.2 = e^{-\lambda}$ thus we get our rate $\lambda = -\ln(0.2)$. Thus we have $\Pr[N(2) = 0] = 0.2^2 = 0.04$ and $\Pr[N(2) = 1] = \frac{-\ln(0.2) \cdot 2}{1!} 0.2^2 = 0.129$, thus

$$\Pr[N(2) > 1] = 1 - 0.04 - 0.129 = 0.831$$

2.

$$\begin{aligned} \mathbb{E}[(N(4) - N(2))(N(3) - N(1))] &= \mathbb{E}[N(1)N(2) + N(3)N(4) - N(2)N(3) - N(1)N(4)] \\ &= \mathbb{E}[N(1)N(2)] + \mathbb{E}[N(3)N(4)] - \mathbb{E}[N(2)N(3)] - \mathbb{E}[N(1)N(4)] \end{aligned}$$

For $i < j$

$$\begin{aligned}
\mathbb{E}[N(i)N(j)] &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} mn \cdot \Pr[N(i) = n]\Pr[N(j) = m|N(i) = n] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(m+n)n \cdot \Pr[N(i) = n]\Pr[N(j) = m+n|N(i) = n] \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(m+n)n \cdot \Pr[N(i) = n]\Pr[N(j-i) = m] \\
&= \sum_{n=0}^{\infty} n \cdot \Pr[N(i) = n]\sum_{m=0}^{\infty}(m+n)\Pr[N(j-i) = m] \\
&= \sum_{n=0}^{\infty} n \cdot \Pr[N(i) = n]\left(\sum_{m=0}^{\infty} m\Pr[N(j-i) = m] + n\sum_{m=0}^{\infty}\Pr[N(j-i) = m]\right) \\
&= \sum_{n=0}^{\infty} n \cdot \Pr[N(i) = n]\left(\mathbb{E}[N(j-i)] + n \cdot 1\right) \\
&= \mathbb{E}[N(i)]\mathbb{E}[N(j-i)] + \sum_{n=0}^{\infty} n^2 \cdot \Pr[N(i) = n] \\
&= \mathbb{E}[N(i)]\mathbb{E}[N(j-i)] + \mathbb{E}[N(i)^2] \\
&= \lambda^2 i(j-i) + \lambda^2 i^2 + \lambda i \\
&= \lambda^2 ij + \lambda i
\end{aligned}$$

Plugging this in to our above expression we get

$$\mathbb{E}[(N(4) - N(2))(N(3) - N(1))] = 36 + 204 - 104 - 68 = 68$$

4 Convergence of Random Variables

1.

$$\begin{aligned}
\mathbb{E}[\lim_{n\to\infty} X_n] &= \int_{-\infty}^{\infty} x \cdot \Pr[\lim_{n\to\infty} X_n = x]dx \\
&= 0 \\
\lim_{n\to\infty}[X_n] &= \lim_{n\to\infty}\int_{-\infty}^{\infty} x \cdot \Pr[X_n = x]dx \\
&= \lim_{n\to\infty} n \cdot \frac{1}{n} + 0 \cdot (1 - 1/n) \\
&= 1
\end{aligned}$$

They are not equal because the random variable described does to satisfy the conditions for the dominated convergence theorem, thus there is not reason to believe that the limit of its expectation should be the expectation of its limit.

2. We have

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{nU\log U}{1+n^2U^2}\right] = \lim_{n\to\infty}\int_0^1 \frac{nx\log x}{1+n^2x^2}dx$$

because U is uniform on $[0,1]$. Let $f_n(x) = \frac{nx\log x}{1+n^2x^2}$ Then we can compute

$$\lim_{n\to\infty} \frac{nx\log x}{1+n^2x^2} = 0$$

To see this, note first that as $x \to 0$ the function goes to 0. Otherwise the top and bottom are just bounded constants. Thus the n and n^2 terms dominate, meaning clearly the whole thing goes to 0. Thus there exists a function such that $|f_n(x)| \leq g(x)$ for all $x \in [0,1]$. Thus by the dominated convergence theorem we have

$$\lim_{n\to\infty}\int_0^1 \frac{nx\log x}{1+n^2x^2}dx = \int_0^1 \lim_{n\to\infty} \frac{nx\log x}{1+n^2x^2}dx = 0$$

Thus the expectation is 0. □