COS 598: Adv. Topics in CS – Image Processing and Analysis
Spring 2022: Section 2
Instructor: Prof. Terry Yoo

Homework Assignment 2: Images and SimpleITK

In class, we have described images as digital sampling from the real world, a function, or other continuous domain. This exercise is intended to familiarize you with working with SimpleITK and to deepen your understanding of the relationship between continuous functions and discrete images.

A continuous function in 2D: a Gaussian function and a little more on linear algebra and its notation

Translating the Gaussian function – One of the most important functions that we use in image processing is a Gaussian. The 1D formulation of a Gaussian function with a scale or standard deviation σ where the output is f (abbreviated from $f = f(x)$) is:

$$f = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\frac{(x-x_0)^2}{\sigma^2}} \tag{1}$$

Where x_0 is the offset along the x-axis shifting the center of the away from the origin. In 2D where the center of the Gaussian function is (x_0, y_0), this becomes:

$$f = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{(x-x_0)^2+(y-y_0)^2}{\sigma^2}} \tag{1a}$$

Introducing the notation of linear algebra – For an isotropic (circular) Gaussian function in 2D, let f = output ($f = f(x, y)$) and let $\bar{x} = (x, y)$ and $\bar{\mu} = (x_0, y_0)$, the offset from the origin:

$$f = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{\|\bar{x}-\bar{\mu}\|^2}{\sigma^2}} \tag{2}$$

Remembering that given a vector $\bar{v}$, the magnitude of that vector is the square root of the inner product of that vector. Specifically,

$$\|\bar{v}\| = \sqrt{\bar{v}\cdot\bar{v}} = \sqrt{\bar{v}\,\bar{v}^{\mathrm{T}}} \tag{3}$$

Applying these terms, equation 2 can be rewritten:

$$f = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\frac{(\bar{x}-\bar{\mu})(\bar{x}-\bar{\mu})^{\mathrm{T}}}{\sigma^2}} \tag{4}$$

The 2 x 2 identity matrix is:

$$\mathrm{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

Given an arbitrary matrix A one may derive an *inverse* matrix A^{-1} such that:

$$\mathrm{A}\,\mathrm{A}^{-1} = \mathrm{I} \tag{6}$$

Not every matrix has an inverse. However, consider the variance term, σ^2. Let Σ be a matrix (in our case a 2 x 2 matrix) and consider its inverse Σ^{-1}:

$$\Sigma = \begin{bmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{bmatrix} \quad , \quad \Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma^2} & 0 \\ 0 & \frac{1}{\sigma^2} \end{bmatrix} \tag{7}$$

You might recognize this simply as an isotropic scaling matrix with a scale factor of σ^2 and its inverse. Applying these terms, equation 4 can be rewritten using the notation of linear algebra:

$$f = \frac{1}{2\pi\sigma^2} e^{-\frac{1}{2}\,(\bar{x}-\bar{\mu})\Sigma^{-1}(\bar{x}-\bar{\mu})^{\mathrm{T}}} \tag{8}$$

Adding elongation – If instead of a circular, isotropic Gaussian function, we wish to create an ellipsoidal function with different scaling along the x and y axes, in 2 dimensions the general function without using linear algebra is:

$$f = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\frac{(x-x_0)^2}{\sigma_x^2}+\frac{(y-y_0)^2}{\sigma_y^2}\right)} \tag{9}$$

Please note that if we move to 3D or higher dimensions, this rudimentary algebraic notation will become cumbersome. Let's revisit this using linear algebra. Given an arbitrary 2 x 2 matrix A, the *determinant* of the matrix $|\mathrm{A}|$ is:

$$\mathrm{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad , \quad determinant\ \mathrm{A} = \det(\mathrm{A}) = |\mathrm{A}| = ad - bc \tag{10}$$

If instead we use the notation of linear algebra, the non-isotropic Gaussian function is merely a recapitulation of equation 8 with a different Σ matrix:

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \quad , \quad f = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{1}{2}\,(\bar{x}-\bar{\mu})\Sigma^{-1}(\bar{x}-\bar{\mu})^{\mathrm{T}}} \tag{11}$$

Adding rotation – Finally, not all Gaussian functions are aligned along the x- or y- (or other basis vector) axes. The general form for a non-isotropic Gaussian function with an arbitrary rotation angle of θ in 2 dimensions is:

$$f = \frac{1}{2\pi\sigma_x\sigma_y} e^{-\frac{1}{2}\left(\left(\frac{\cos^2\theta}{\sigma_x^2}+\frac{\sin^2\theta}{\sigma_y^2}\right)(x-x_0)^2+2\left(-\frac{\sin^2\theta}{\sigma_x^2}+\frac{\sin^2\theta}{\sigma_y^2}\right)(x-x_0)(y-y_0)+\left(\frac{\sin^2\theta}{\sigma_x^2}+\frac{\cos^2\theta}{\sigma_y^2}\right)(y-y_0)^2\right)} \tag{12}$$

Consider the fact that when we are rotating the Gaussian function f, we are actually rotating the matrix Σ. Recall, that in two dimensions, a rotation matrix R is:

$$\mathrm{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{13}$$

Equation 12 becomes:

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \ , \ \mathrm{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \ , \ f = \frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{1}{2}\,(\bar{x}-\bar{\mu})\,\mathrm{R}\,\Sigma^{-1}\,\mathrm{R}^{\mathrm{T}}\,(\bar{x}-\bar{\mu})^{\mathrm{T}}} \tag{14}$$

Again, please observe that this notation will adapt to higher dimensions.

The actual homework:

Write a program (using Python and SimpleITK… you may adapt a Jupyter notebook) to generate a 256 x 256 image of a background b on which is superimposed a Gaussian spot of intensity f centered at location $\bar{\mu} = (x_0, y_0)$. The function will be amplified by a factor a.

$$\Sigma = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \quad , \quad \mathrm{R} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \quad , \quad f = a\left(\frac{1}{2\pi\sqrt{|\Sigma|}} e^{-\frac{1}{2}(\bar{x}-\bar{\mu})\,\mathrm{R}\,\Sigma^{-1}\,\mathrm{R}^{\mathrm{T}}\,(\bar{x}-\bar{\mu})^{\mathrm{T}}}\right) + b \qquad (15)$$

Your variables or input parameters should be $a, b, x_0, y_0, \sigma_x, \sigma_y$, and θ.

Submit your source code (or notebook) and generate three images. You may choose any convenient values for a, b, x_0, and y_0. You should use the following specific values for your three images:

a. $\sigma_x = \sigma_y = 20$ and $\theta = 0$.
b. $\sigma_x = 5, \sigma_y = 20$, and $\theta = {}^{\pi}/_{4}$.
c. $\sigma_x = 10, \sigma_y = 30$, and $\theta = -{}^{\pi}/_{6}$.

Please note, there are versions of the necessary code online. Please do not copy work from the internet, though I will allow that you should be able to refer to such resources to check your work.

Submit your answers on Brightspace.
Due 21-March-2022