IMPLICIT HYBRID BLOCK METHOD FOR NUMERICAL SOLUTION OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATION

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CERTIFICATION

I certify that this project titled **IMPLICIT HYBRID BLOCK METHOD FOR NUMERICAL SOLUTION OF THIRD ORDER ORDINARY DIFFERENTIAL EQUATION** work was carried out by **AKINTADE, OLUWASEYI SAMUEL** with the matriculation number **MTS/15/4164** in the Department of Mathematical Sciences, The Federal University of Technology, Akure, in partial fulfillment of the award of Bachelor of Technology (B.Tech) in Industrial Mathematics.

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DEDICATION

This work is dedicated to Almighty God, my Heavenly Father, who in His infinite mercies granted me the grace, will and strength to start and complete this work. And also to my beloved father for his invaluable support, without him I wouldn't have realized this work. May God let him reap the fruit of his labor (Amen).

ACKNOWLEDGMENTS

I appreciate God for life and all the people He surrounds me with; you have been so awesome. My amazing, loving, caring and understanding family, I do appreciate you, I love you so much.

I indeed appreciate my project supervisor, Dr. M.K Duromola, his love, care, effort and sacrifice ultimately made this project work possible.

Thank you all.

ABSTRACT

Physical processes and real life problems particularly in sciences and engineering can be expressed in differential equations for good understanding of the physical phenomena involved. These differential equations involve the rates of change of and with respect to independent variables. Also there are initial value problems of these ordinary differential equations with conditions added to make the process of obtaining solutions very easy. This research works out a computational method for the solution of a third order ordinary differential equation using the implicit hybrid block. The derivation was based on interpolation-collocation techniques in which a power series acts as the basis function. The system of equations gotten from the basis function was collocated at both the grid and off-grids points while the approximate solution to the problem was interpolated at three off-grid points. The resulting system of equations was solved to obtain the values of the unknown parameters. Substituting the values of these parameters and evaluating the result at different grid and off-grid points to yield the required discrete scheme. The properties such as; the order, consistency, zero-stability, convergence and region of absolute stability of the method will also be determined. The performance of the method, was tested by solving sample problems of initial value problems of general second order ordinary differential equation.

Contents

Chapter 1
INTRODUCTION

1.1 PREAMBLES

In solving a third order Ordinary Differential Equations [ODE], we are not sure there would always be an obtainable analytic result. Some kinds of these equations have a way to solve them analytically or directly irrespective of their seemingly complexity.

The versatility and effectiveness of numerical methods available for obtaining solutions such as the Runge-Kutta and multi-step methods gives us a way out. There are other schemes like the shooting methods, the Euler scheme, Taylor series, Picard method of successive approximation e.t.c. Each of these do the job well but not without a flaw however we try to perfect the flaw by the development of a scheme but not without some explanations.

1.2 ORDINARY DIFFERENTIAL EQUATIONS

In Mathematics, there are different ways to classify or describe equations; an equation could be described based on types of variables it has, the exponent of the variables involved, the presence and number of derivatives in it. Basically an equation with derivatives is called a differential equation.

A differential equation is a relationship between some function and one of its derivatives. Differential equations are among the most important mathematical tools used in producing models in the physical and biological sciences, in technology and engineering as well. In this study, we are concerned with Ordinary Differential Equations, that is, those differential equations that have only one independent variable. What we mean is any equation of the form below is a differential equation.

$$f(x, y(x), y', y'', ..., y^{(n)}(x)) = k$$

$$where\ k\ is\ constant$$

A case of a differential equation where there is just one independent variable [and maybe one dependent variable as in the example above] is an Ordinary Differential Equation.

In general, such equations are encountered in scientific problems in which a statement is made about some rate of change. The goal is usually to identify the function from the given relationship, and at times with a given initial value.

1.2.1 THE ORDER OF AN ODE

The order of an ODE should not be confuse with its degree, the order is the highest power of derivative in the equation [the highest order of the derivative in the equation is the order of the differential equation], while the degree is the power/exponent of the highest order of derivative in the equation. In simple words, the order determines the degree and it does not go the other way. The difference is made clear in the examples below.

$$\begin{array}{ll} a. & y'' + 2x^2y^3 - y^{(4)} = 7 \\ b. & y'^{12} - (xy^{(3)})^7 = 8x \\ c. & y''^{\frac{1}{2}} = \dfrac{1}{x^2y}y'^4 - 2 \end{array}$$

- (a) is an ODE of order 4, (b) is an ODE of order 3, (c) is an ODE of order 2.

- (a) is an ODE of degree 1, (b) is an ODE of degree 7, (c) is an ODE of degree? Can you guess it? REMEMBER: the order determines the degree.

For a first order ODE, the general form is below

$$f(x, y(x), y'(x)) = k$$

and for the third order it is

$$f(x, y(x), y'(x), y''(x), y'''(x)) = k$$

It should be noted that k is a constant.

1.3 INITIAL VALUE PROBLEMS

In trying to solve an ODE we introduce some conditions similar to the behaviour of the function of the solution called initial value. The combination of the ODE and its initial point(s) is called the *Initial Value Problem* otherwise called IVP of the ODE. Usually with these, investigations and analysis on the solution are easy to make.

An example of a third order Order Differential Equation is

$$y'''(x) = f(x, y(x), y'(x), y''(x)), \quad y(x_0) = a_0,\ y'(x_0) = b_0,\ y''(x_0) = c_0$$

1.4 NUMERICAL METHODS

We encounter problems with equations involving derivatives and many of them may not be easily solved analytically; the development of numerical methods becomes inevitable. Conventionally, ordinary differential equations of high orders are solved by reducing the order to a system of first order equations. The approaches have been extensively studied and developed by many researchers over the years. Numerical methods for solving differential equations include the Euler method, Runge-Kutta, linear multi-step method, predictor-corrector method, finite difference method, block method, hybrid method, the midpoint method, secant method, the Simpson and trapezoidal rule.

1.5 THE IMPLICIT HYBRID BLOCK METHOD

Numerical schemes are either explicit or implicit depending on how the computation of the dependent variables are made. If the computation of the dependent variables are made in terms of the known quantities the scheme is explicit. It is implicit if it depends on both known and unknown quantities. An instance is this:

$$y(t + \Delta t) = f(y(t))$$

is explicit while

$$f(y(t), y(t + \Delta t)) = \alpha$$

to find $y(t + \Delta t)$ is implicit.

In terms of time state; explicit methods use the current time state to determine the next while implicit uses the current and subsequent time state to determine the next time state. The knowledge compares to that of implicit and explicit functions in differential calculus.

The methods for solving differential equations such as the Euler's method and the midpoint method are basically written as *explicit* methods for solving ordinary differential equations. However, sometimes an ODE can become *stiff* in which case explicit methods don't do the job of providing solutions well. Whenever possible, it is desirable to change your problem formulation so that you don't have to solve a stiff ODE. Sometimes however that's not possible, and you have just have to be able to solve stiff ODE. If that is the case, you will usually have to use an ODE solution method which is *implicit*.

In a very plain term, an implicit function includes the unknown in its representation.

1.5.1 NUMERICAL BLOCK METHODS

Block methods were proposed firstly by Milne. Method of collocation and interpolation of power series approximate solution to generate a continuous linear multi-step method has been discussed by many authors. These authors developed method which is implemented in predictor-corrector method or block method.

Block methods are one of the numerical methods introduced by several researcher for solving ordinary differential equations. The advantage of block method is that, at each application of a block method, the solution will be approximated in more than one point. The number of points are depend on the structure of the block method.

Block methods generate independent solution at selected grid points without overlapping. It is less expensive in terms of the number of function evaluation compared to the predictor-corrector method, moreover it possess the properties of Runge-Kutta method for being self-starting which means it does not require starting values.

Block methods have the advantages of being more efficient in terms of cost implementation, time of execution and accuracy, and were developed to tackle some of the setbacks of predictor-corrector methods.

A block method is formulated in terms of linear multistep methods. It preserves the traditional advantage of one step methods, of being self-starting and permitting easy change of step length. Their advantage over Runge-Kutta methods lies in the fact that they are less expensive in terms of the number of functions evaluation for a given order. The method generates simultaneous solutions at all grid points. In the use of the block method, we encounter some matrix-values Y_m and F_m be defined as

$$Y_m = (y_n, y_{n+1}, ..., y_{n+r-1})^T$$
$$F_m = (f_n, f_{n+1}, ..., f_{n+r-1})^T$$

Then a general k-block, r-point block method is a matrix of finite difference equation of the form

$$Y_m = \sum_{l=0}^{k} A_l Y_{m+l} + h \sum_{l=0}^{k} B_l F_{m+l}$$

where A_l and B_l are carefully chosen $r \times r$ matrix coefficients and $m = 0, 1, 2, ...$ represents the block number, $n = mr$ is the first step number of the mth block and r is the proposed block size.

1.5.2 One Step Methods

One step methods are methods that use data at a single point, say point n, to advance the solution to point $n+1$. Conventionally, one step numerical integrators for initial value problems are described as

$$y_{n+1} = y_n + h\phi(x_n, y_n; h)$$

where $\phi(x_n, y_n; h)$ is the increment function and h is the step size adopted in the subinterval $[x_n; x_{n+1}]$. The methods can be formulated in explicit form, in which case the increment function is defined as above or in implicit form where the increment function is defined in terms of the independent variable as $\phi(x_n, y_n, y_{n+1}; h)$.

1.5.3 Linear Multistep Methods

Unlike the one step methods considered in the previous section where only a single value y_n was required to compute the next approximation y_{n+1}, LMM need two or more preceding values to be able to calculate y_{n+1}.

A general k-step linear multistep method is defined as

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h^n \sum_{l=0}^{k} \beta_l f_{m+l}$$

where n is the order of the ODE, α_l, β_l are real constants for each

$$l,\ y_{m+l} = y(x_{m+l}) = y(x_m + lh),\ f_{m+l} = f(x_{m+l},\ y(x_{m+l}), ...,\ y^{n-1}(x_{m+l}))$$

and the following suppositions are made that $\alpha_l \neq 0$ and particularly that $\alpha_0 \neq 0, \beta_0 \neq 0$ as well.

1.5.4 HYBRID SCHEMES

Hybrid schemes have been developed since the 1960's though in literature they have not received great attention yet despite their higher accuracy over the single linear multi-step methods of the same step size. This could be because of the need for special predictors to estimate the off-step solutions present in the corrector formulae.

Hybrid from the word hybridization, seeks to combine two or more features that are not necessarily the same. For example, an IVP may be solved by the combination of the Shooting method and the Runge-Kutta method of order 4. That is the idea of the Hybrid scheme, to combine the strength of different schemes and overcoming the weakness of each.

The implicit block methods [by combinig an implicit scheme with the block method] are one of numerical methods adopted for directly approximating the solutions of ODEs. They perform favourably well when obtaining approximations or numerical solutions to initial value problems as it combines the advantages of block method and overcoming the zero stability barrier in linear multi-step method.

The implicit hybrid block method has the advantage of reducing the step number of a method and still remains zero stable. This method while retaining certain characteristics of the continuous linear multi-step method share with the Runge-Kutta method the property of utilizing data at other points other than the step points.

A k-step hybrid formula is defined as

$$\sum_{l=0}^{k} \alpha_l y_{m+l} = h \sum_{l=0}^{k} \beta_l f_{m+l} + h' \beta_v f_{m+v}$$

1.5.5 DERIVATION OF THE IMPLICIT HYBRID BLOCK SCHEME

The implicit hybrid block method preserves the advantage of the Runge-kutta method in that it is self-starting.

We consider approximating the solution of the ODE $y(x)$ by a polynomial $p(x)$ using the Taylor's series expansion of the form

$$y(x) = \sum \frac{(x - x_0)^m}{m!} y^{(m)(x_0)}$$

This polynomial is not necessarily the solution of the problem but a means to obtain the iterative implicit method which is given by an implicit set of equations.

The set of implicit equations we obtain is the block which allows us to obtain a numerical solution on each iteration at the grid points being used. We firstly found the coefficients of this polynomial.

The values of the coefficients do not come out in terms of numeric values but are found in terms of approximate values of y, and f at various grid points, including two intermediate ones, and h, where h is the constant step size taken, that is $h = x_{j+1} - x_j$.

So we have the assumption that the solution $y(x)$ is approximated by the polynomial $p(x)$ as mentioned above, the process to the result is continued in subsequent chapters.

1.6 DEFINITION OF TERMS

There are a number of terms that will be encountered as we go along with the work, so it seems quite good to pre-define them and give succinct explanation to them before hand.

i. Interpolation: a method of constructing new data points from within the range of a discrete set of known data points.

ii. Collocation: a method solving equations by picking (approximate) solutions and points in the domain to be used in the solutions

iii. Step size: steps are the partitioning of the domain of the function to be solved, so the step size is the difference between successive steps; $h = x_n - x_{n-1}$ or $x_{n+1} - x_n$

iv. Grid points:

v. Starting values:

vi. Off-steps:

vii. Convergence: a numerical method is convergent if and only if the sequence of result for each process approaches a fixed point.

viii. Consistency: a numerical scheme is consistent if its discrete operator converges towards it continuous operator, as the mesh size tends to zero the discretization becomes sames as the exact. Consistency is used to indicate the accuracy of the method.

ix. Zero-stability: a numerical method is zero-stable if small changes in the initial conditions do not cause in the solution.

x. Weak-stability:

xi. Truncation error: error caused by approximating a mathematical process, it is the difference between the truncated value and the real value of a process/equation.

xii. Multi-step: The general conformation of multi-step method is

$$y_{n+1} = \sum a_i y_{n-i} + h \sum b_i f(x_{n-i}, y_{n-i})$$

xiii. Error bound:

xiv. Absolute values:

1.7 AIM AND OBJECTIVES

The objective of this research are outlined as follows;

i To develop an Implicit Hybrid Block scheme for the solution of initial value problem of third order ordinary differential equation without reduction to a system of first order differential equations.

ii Examine the properties of the method developed like the order of accuracy, consistency, zero-stability, convergence and region of stability.

iii Test the accuracy of the develop scheme using sample problems and compare the accuracy with the existing methods for third order IVPs.

The aim is to develop a numerical method for solving third-order ordinary differential equation directly without reduction to system of first order differential equations.

1.8 MOTIVATION

Mathematicians develop mathematical models to help them understand physical phenomena in real life problems. These models very frequently lead to equations involving derivatives; these equations are the differential equations of which the ordinary differential equations is a case. Ordinary Differential Equations are very resourceful, applicable and useful fields such as engineering, medicine, economics etc.

There are however a limited number of methods available for solving these equations directly which are called *analytical* methods. There are other methods that look like these but involves reduction to a system of first-order equations. But these won't come without any disadvantage.

Researchers have have proposed solutions to ordinary differential equations and the initial value problems of ordinary differential equations using different approaches. We are motivated in this study therefore to propose an hybrid block method that gives direct solution of third order ordinary differential equations or at least a very accurate analytical solution of it.

1.9 CONTRIBUTION TO KNOWLEDGE

We shall bring to light the absolute usefulness of the Implicit Hybrid Block Method in providing solutions to Ordinary Differential Equations. And also establish the superiority of Implicit Hybrid Block Method over other means of obtaining solutions to ODEs.

Chapter 2
LITERATURE REVIEW

Though the study of Ordinary Differential Equations as regards providing solutions to them is not new in any ways, the bias this work has is that of its focus being on third order equations. We consider what work has been done and discuss some of them with a close look at the concepts they propose.

2.1 A BRIEF SHOT AT RESEARCHES AND WORK ALREADY DONE

Jikantoro, Y.D., Ismail, F., Senu, N., Ibrahim, Z.B., and Aliyu, Y.B. (2019) in the paper *Hybrid Method for Solving Special Fourth Order Ordinary Differential Equations* presented the idea of the approximation of three points of y'_n, y''_n and y'''_n at each step of the integration. This constitutes computational efficiency issue as the approximation of y_n depends on the derivatives. Then derived an integrator that is multistage in nature like the direct Runge-Kutta methods mentioned above, which does not require approximation of y'_n, y''_n and y'''_n at all for special third order ODEs. Although the method is not self staring, it requires approximation of back values to start the integration like most of the linear multi-step methods. The combined properties of multiple stage and multiple step give the method the name hybrid method.

E.A. Areo and R.B. Adeniyi (2014) in *Block Implicit one-step method for the numerical integration of Initial Value Problems in Ordinary Differential Equations* derived a continuous formulation of the proposed block implicit one-step method for the numerical integration of initial value problems in ordinary differential equations is presented and employs it to deduce the discrete ones. The continuous scheme is used to obtain finite difference methods which are combined as simultaneous numerical integrators to constitute conveniently the block method. In order to derive the continuous scheme, the method of Sirisena et al. (2004) is applied where a k-step multi-step collocation method with m collocation points.

Adoghe Lawrence Osa and Omole Ezekiel Olaoluwa (2019) in *A fifth-fourth Continuous Block Implicit Hybrid Method for the Solution of Third Order Initial Value Problems in Ordinary Differential Equations* used Taylor series expansion of exponential function to form the base function for the approximation .

U. Mohammed and R.B Adeniyi (2014) proposed a block hybrid multi-step method for the direct solution of third order initial value problems of ordinary differential equations in the work *A Three Step Implicit Hybrid Linear Multistep Method for the Solution of Third Order Ordinary Differential Equations.*

Kayode S.J and Obarhua F.O (2017) had their work *Symmetric 2-Step 4-Point Hybrid Method for the Solution of General Third Order Differential Equations* considering a symmetric hybrid continuous linear multi-step method for the solution of general third order ordinary differential equations. The method generated by interpolation and collocation approach from a combination of power series and exponential function as basis function. The approximate basis function is interpolated at both grid and off-grid points but the collocation of the differential function is only at the grid points. The derived method was found to be symmetric, consistent, zero stable and of order six with low error constant. Accuracy of the method was confirmed by implementing the method on linear and non-linear test problems. The results show better performance over known existing methods solved with the same third order problems.

Ukpebor L.A and Omole E.O's (2020) *Three-step optimized block backward differentiation formulae for solving stiff ordinary differential equations* focuses on the adoption of polynomial of order 6 and three hybrid points chosen appropriately to optimize the local truncation errors of the main formulas for the block. The method is zero-stable and consistent with sixth algebraic order. Some numerical examples were solved to examine the efficiency and accuracy of the proposed method.

I.A Bakari, Y. Skwame, Kumleng G.M (2018) *An Application of Second Derivative Backward Differentiation Formula Hybrid Block Method on Stiff Ordinary Differential Equations.* A continuous scheme of four and five step with one off-grid point at collocation was developed which provides the approximate solution of both linear and nonlinear stiff ordinary differential equations with constant step size. The continuous scheme is evaluated at both interpolation and collocation where necessary to give continuous hybrid block scheme and high order of accuracy with low error constants.

E. Hasan, Z.A Majid (2018) *One Step Block Method for Solving Third Order Ordinary Differential Equations Directly.* The research discusses a direct two-point one step block method for solving general third-order initial value problems (IVPs) of ordinary differential equations (ODEs) using constant step size. The proposed method will compute the approximation solutions directly without reducing to systems of first order ODEs at two-points in a block simultaneously. Lagrange polynomial was used to derive the block method.

Zanariah Abdul Majid, Mohamed Bin Suleiman and Zurni Omar (2006) *3-Point Implicit Block Method for Solving Ordinary Differential Equations.* In the $3-$point implicit block method, the interval $[a, b]$ is divided into a series of blocks with each block containing 3 points. The following strategy is employed to calculate the solutions at each block. The solution at the point x_n, which is the end point of $(k-1)$ block, is used to calculate the solutions of k blocks. Similarly, the solution at the end point of k block, which is at x_{n+3}, is used to calculate the solutions of $(k+1)$ block. The same process applied for calculating the next blocks until the end point $x = b$ is reached.

We take each of these as guideline and motivation on which we set out our work.

Chapter 3
METHODOLOGY

3.1 Statement of Problem

Therefore this work proposes a more efficient way in the implicit hybrid block method for solving third order ordinary differential equations (IVP).

We use

$$y(x) = \sum_{l=0}^{k+m-1} a_l x^l$$

as the approximation of the solution, k is the number of points of interpolation and m the number of points of collocation as the basis. With it we perform the necessary actions. The polynomial used has a degree $(k+m)-1$ on an interval say, $[b,c]$ with a_l being real. The partitioning of the interval then is

$$b = x_0 < x_1 < ... < x_n = c$$

$$\textit{with a constant step size } h = x_n - x_{n-1}$$

$$\textit{mesh is } \Delta = \{x_n : h = x_n - x_{n-1}, x_n = a + nh, n = 0, 1, 2, ..., N\} \textit{ and } N = \frac{c-b}{h}$$

3.2 Derivation of the Method

For the sake of emphasis, we restate the aim and objectives in two-folds as;

i. We will derive the Implicit Hybrid Block scheme.

ii. Apply the scheme in solving the general third order ODE [IVP].

For the sake of simplicity and ease of computation, we re-write the general third order ODE as

$$y''' = f(x, y, y', y'')$$

rather than

$$f(x, y, y', y'', y''') = k$$

as we had earlier written. The IVP corresponding to the equation is;

$$y''' = f(x, y, y', y'') \quad y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \tag{3.1}$$

The steps involved for the **Implicit Hybrid Block Scheme** derivation is detailed below.

STEP 1

First we write out the power series approximation

$$y(x) = \sum_{l=0}^{k+m-1} a_l x^l \tag{3.2}$$

where k is the number of points of interpolation and m the number of points of collocation.

Then we find the derivatives of $y(x)$

$$y'(x) = \sum_{l=1}^{k+m-1} l a_l x^{(l-1)} \tag{3.3}$$

$$y''(x) = \sum_{l=2}^{k+m-1} l(l-1) a_l x^{(l-2)} \tag{3.4}$$

$$y'''(x) = \sum_{l=3}^{k+m-1} l(l-1)(l-2) a_l x^{(l-3)} \tag{3.5}$$

<u>REMARK</u>: y and y''' are very important, y is used for interpolation and and y''' for collocation.

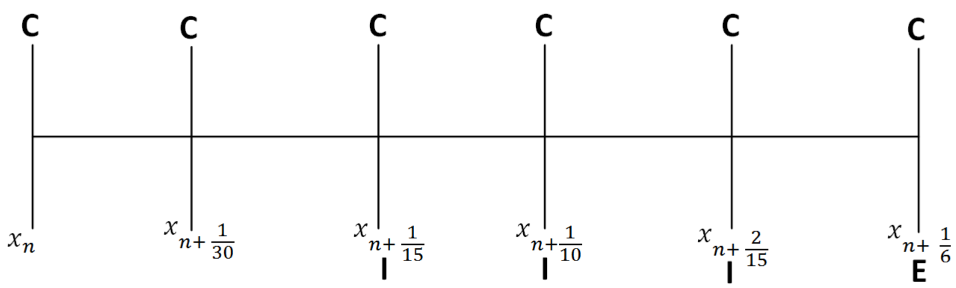
Substituting (3.5) for y''' in (3.1) the IVP transforms to

$$\sum_{l=3}^{k+m-1} l(l-1)(l-2) a_l x^{(l-3)} = f(x, y, y', y'') \quad y(x_0) = y_0, y'(x_0) = y'_0, y''(x_0) = y''_0 \tag{3.6}$$

STEP 2

We pick the points to be used for collocation and interpolation, however we need to raise a very vital point.

3.3 Specification of the Scheme



In the diagram above, a sketch of how the points we are to use are picked is seen.

From the mesh, the collocation points are picked as $x = x_{n+m}$ $m = 0(\frac{1}{30})\frac{1}{6}$.

We interpolate at $x = x_{n+k}$ $k = \frac{1}{15}(\frac{1}{30})\frac{2}{15}$

This brings us to $m + k - 1 = 6 + 3 - 1 = 8$, then we derive the system of equations.

COLLOCATION

$$f_{n+m} = \sum_{l=3}^{8} l(l-1)(l-2)a_l x_{n+m}^{(l-3)} \quad m = 0(\frac{1}{30})\frac{1}{6} \tag{3.7}$$

$$f_{n+m} = 6a_3 + 24a_4 x_{n+m} + 60a_5 x_{n+m}^2 + 120a_6 x_{n+m}^3 + 210a_7 x_{n+m}^4 + 336a_8 x_{n+m}^5 \; m = 0(\frac{1}{30})\frac{1}{6} \tag{3.8}$$

The above yields a system of equations written below

$f_n = 6a_3 + 24a_4 x_n + 60a_5 x_n^2 + 120a_6 x_n^3 + 210a_7 x_n^4 + 336a_8 x_n^5$ m $= 0$

$f_{n+\frac{1}{30}} = 6a_3 + 24a_4 x_{n+\frac{1}{30}} + 60a_5 x_{n+\frac{1}{30}}^2 + 120a_6 x_{n+\frac{1}{30}}^3 + 210a_7 x_{n+\frac{1}{30}}^4 + 336a_8 x_{n+\frac{1}{30}}^5 \; m = \frac{1}{30}$

$f_{n+\frac{1}{15}} = 6a_3 + 24a_4 x_{n+\frac{1}{15}} + 60a_5 x_{n+\frac{1}{15}}^2 + 120a_6 x_{n+\frac{1}{15}}^3 + 210a_7 x_{n+\frac{1}{15}}^4 + 336a_8 x_{n+\frac{1}{15}}^5 \; m = \frac{1}{15}$

$f_{n+\frac{1}{10}} = 6a_3 + 24a_4 x_{n+\frac{1}{10}} + 60a_5 x_{n+\frac{1}{10}}^2 + 120a_6 x_{n+\frac{1}{10}}^3 + 210a_7 x_{n+\frac{1}{10}}^4 + 336a_8 x_{n+\frac{1}{10}}^5 \; m = \frac{1}{10}$

$f_{n+\frac{2}{15}} = 6a_3 + 24a_4 x_{n+\frac{2}{15}} + 60a_5 x_{n+\frac{2}{15}}^2 + 120a_6 x_{n+\frac{2}{15}}^3 + 210a_7 x_{n+\frac{2}{15}}^4 + 336a_8 x_{n+\frac{2}{15}}^5 \; m = \frac{2}{15}$

$f_{n+\frac{1}{6}} = 6a_3 + 24a_4 x_{n+\frac{1}{6}} + 60a_5 x_{n+\frac{1}{6}}^2 + 120a_6 x_{n+\frac{1}{6}}^3 + 210a_7 x_{n+\frac{1}{6}}^4 + 336a_8 x_{n+\frac{1}{6}}^5 \; m = \frac{1}{6}$

(3.9)

INTERPOLATION

$$y_{n+k}(x) = \sum_{l=0}^{8} a_l x_{n+k}^l \quad k = \frac{1}{15}\left(\frac{1}{30}\right)\frac{2}{15} \tag{3.10}$$

$$y_{n+k}(x) = a_0 + a_1 x_{n+k} + a_2 x_{n+k}^2 + a_3 x_{n+k}^3 + a_4 x_{n+k}^4 + a_5 x_{n+k}^5 + a_6 x_{n+k}^6 + a_7 x_{n+k}^7 + a_8 x_{n+k}^8$$

It also yields a system of equations which is

at $k = \frac{1}{15}$

$y_{n+\frac{1}{15}}(x) = a_0 + a_1 x_{n+\frac{1}{15}} + a_2 x_{n+\frac{1}{15}}^2 + a_3 x_{n+\frac{1}{15}}^3 + a_4 x_{n+\frac{1}{15}}^4 + a_5 x_{n+\frac{1}{15}}^5 + a_6 x_{n+\frac{1}{15}}^6 + a_7 x_{n+\frac{1}{15}}^7 + a_8 x_{n+\frac{1}{15}}^8$

at $k = \frac{1}{10}$

$y_{n+\frac{1}{10}}(x) = a_0 + a_1 x_{n+\frac{1}{10}} + a_2 x_{n+\frac{1}{10}}^2 + a_3 x_{n+\frac{1}{10}}^3 + a_4 x_{n+\frac{1}{10}}^4 + a_5 x_{n+\frac{1}{10}}^5 + a_6 x_{n+\frac{1}{10}}^6 + a_7 x_{n+\frac{1}{10}}^7 + a_8 x_{n+\frac{1}{10}}^8$

at $k = \frac{2}{15}$

$y_{n+\frac{2}{15}}(x) = a_0 + a_1 x_{n+\frac{2}{15}} + a_2 x_{n+\frac{2}{15}}^2 + a_3 x_{n+\frac{2}{15}}^3 + a_4 x_{n+\frac{2}{15}}^4 + a_5 x_{n+\frac{2}{15}}^5 + a_6 x_{n+\frac{2}{15}}^6 + a_7 x_{n+\frac{2}{15}}^7 + a_8 x_{n+\frac{2}{15}}^8$

By combining the results obtained from using the collocation and interpolation points, we come again to another system of equations, which will be put in a matrix equation of the form $AX = B$.

$$y_{n+\frac{1}{15}}(x) = a_0 + a_1x_{n+\frac{1}{15}} + a_2x^2_{n+\frac{1}{15}} + a_3x^3_{n+\frac{1}{15}} + a_4x^4_{n+\frac{1}{15}} + a_5x^5_{n+\frac{1}{15}} + a_6x^6_{n+\frac{1}{15}} + a_7x^7_{n+\frac{1}{15}} + a_8x^8_{n+\frac{1}{15}}$$

$$y_{n+\frac{1}{10}}(x) = a_0 + a_1x_{n+\frac{1}{10}} + a_2x^2_{n+\frac{1}{10}} + a_3x^3_{n+\frac{1}{10}} + a_4x^4_{n+\frac{1}{10}} + a_5x^5_{n+\frac{1}{10}} + a_6x^6_{n+\frac{1}{10}} + a_7x^7_{n+\frac{1}{10}} + a_8x^8_{n+\frac{1}{10}}$$

$$y_{n+\frac{2}{15}}(x) = a_0 + a_1x_{n+\frac{2}{15}} + a_2x^2_{n+\frac{2}{15}} + a_3x^3_{n+\frac{2}{15}} + a_4x^4_{n+\frac{2}{15}} + a_5x^5_{n+\frac{2}{15}} + a_6x^6_{n+\frac{2}{15}} + a_7x^7_{n+\frac{2}{15}} + a_8x^8_{n+\frac{2}{15}}$$

$$f_n = 6a_3 + 24a_4x_n + 60a_5x_n^2 + 120a_6x_n^3 + 210a_7x_n^4 + 336a_8x_n^5$$

$$f_{n+\frac{1}{30}} = 6a_3 + 24a_4x_{n+\frac{1}{30}} + 60a_5x^2_{n+\frac{1}{30}} + 120a_6x^3_{n+\frac{1}{30}} + 210a_7x^4_{n+\frac{1}{30}} + 336a_8x^5_{n+\frac{1}{30}}$$

$$f_{n+\frac{1}{15}} = 6a_3 + 24a_4x_{n+\frac{1}{15}} + 60a_5x^2_{n+\frac{1}{15}} + 120a_6x^3_{n+\frac{1}{15}} + 210a_7x^4_{n+\frac{1}{15}} + 336a_8x^5_{n+\frac{1}{15}}$$

$$f_{n+\frac{1}{10}} = 6a_3 + 24a_4x_{n+\frac{1}{10}} + 60a_5x^2_{n+\frac{1}{10}} + 120a_6x^3_{n+\frac{1}{10}} + 210a_7x^4_{n+\frac{1}{10}} + 336a_8x^5_{n+\frac{1}{10}}$$

$$f_{n+\frac{2}{15}} = 6a_3 + 24a_4x_{n+\frac{2}{15}} + 60a_5x^2_{n+\frac{2}{15}} + 120a_6x^3_{n+\frac{2}{15}} + 210a_7x^4_{n+\frac{2}{15}} + 336a_8x^5_{n+\frac{2}{15}}$$

$$f_{n+\frac{1}{6}} = 6a_3 + 24a_4x_{n+\frac{1}{6}} + 60a_5x^2_{n+\frac{1}{6}} + 120a_6x^3_{n+\frac{1}{6}} + 210a_7x^4_{n+\frac{1}{6}} + 336a_8x^5_{n+\frac{1}{6}} \tag{3.11}$$

$$\begin{bmatrix} 1 & x_{n+\frac{1}{15}} & x^2_{n+\frac{1}{15}} & x^3_{n+\frac{1}{15}} & x^4_{n+\frac{1}{15}} & x^5_{n+\frac{1}{15}} & x^6_{n+\frac{1}{15}} & x^7_{n+\frac{1}{15}} & x^8_{n+\frac{1}{15}} \\ 1 & x_{n+\frac{1}{10}} & x^2_{n+\frac{1}{10}} & x^3_{n+\frac{1}{10}} & x^4_{n+\frac{1}{10}} & x^5_{n+\frac{1}{10}} & x^6_{n+\frac{1}{10}} & x^7_{n+\frac{1}{10}} & x^8_{n+\frac{1}{10}} \\ 1 & x_{n+\frac{2}{15}} & x^2_{n+\frac{2}{15}} & x^3_{n+\frac{2}{15}} & x^4_{n+\frac{2}{15}} & x^5_{n+\frac{2}{15}} & x^6_{n+\frac{2}{15}} & x^7_{n+\frac{2}{15}} & x^8_{n+\frac{2}{15}} \\ 0 & 0 & 0 & 6 & 24x_n & 60x_n^2 & 120x_n^3 & 210x_n^4 & 336x_n^5 \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{30}} & 60x^2_{n+\frac{1}{30}} & 120x^3_{n+\frac{1}{30}} & 210x^4_{n+\frac{1}{30}} & 336x^5_{n+\frac{1}{30}} \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{15}} & 60x^2_{n+\frac{1}{15}} & 120x^3_{n+\frac{1}{15}} & 210x^4_{n+\frac{1}{15}} & 336x^5_{n+\frac{1}{15}} \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{10}} & 60x^2_{n+\frac{1}{10}} & 120x^3_{n+\frac{1}{10}} & 210x^4_{n+\frac{1}{10}} & 336x^5_{n+\frac{1}{10}} \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{2}{15}} & 60x^2_{n+\frac{2}{15}} & 120x^3_{n+\frac{2}{15}} & 210x^4_{n+\frac{2}{15}} & 336x^5_{n+\frac{2}{15}} \\ 0 & 0 & 0 & 6 & 24x_{n+\frac{1}{6}} & 60x^2_{n+\frac{1}{6}} & 120x^3_{n+\frac{1}{6}} & 210x^4_{n+\frac{1}{6}} & 336x^5_{n+\frac{1}{6}} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} = \begin{bmatrix} y_{n+\frac{1}{15}} \\ y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{15}} \\ f_n \\ f_{n+\frac{1}{30}} \\ f_{n+\frac{1}{15}} \\ f_{n+\frac{1}{10}} \\ f_{n+\frac{2}{15}} \\ f_{n+\frac{1}{6}} \end{bmatrix} \tag{3.12}$$

In the above matrix equation, A, B and X can be easily identified since we have it to be $AX = B$. The result for X can be gotten by the Gaussian elimination method, by the means of inversion [i.e $X = A^{-1}B$ so that the values of each a_l could be gotten] or any other suitable means. Solving for these unknowns (a_l) gives us the required coefficients of the polynomial $p(x)$ in terms of $y_{n+\frac{1}{15}}, y_{n+\frac{1}{10}}, y_{n+\frac{2}{15}}, f_n, f_{n+\frac{1}{30}}, f_{n+\frac{1}{15}}, f_{n+\frac{1}{10}}, f_{n+\frac{2}{15}}, f_{n+\frac{1}{6}}$.

3.4 The derivation of the Block Method

Here by a Mathematical Software [The Scientific WorkPlace], we evaluate the continuous scheme at $x = x_n,\ x_{n+\frac{1}{30}},\ x_{n+\frac{1}{6}}$ to obtain the following discrete values.

$$\text{at } t = \frac{1}{6},$$

$$y_{n+\frac{1}{30}} - y_{n+\frac{1}{15}} + 3y_{n+\frac{1}{10}} - 3y_{n+\frac{2}{15}} = -\frac{h^3}{12960000}\left[f_n - 7f_{n+\frac{1}{30}} + 18f_{n+\frac{1}{15}} - 262f_{n+\frac{1}{10}} - 227f_{n+\frac{2}{15}} - 3f_{n+\frac{1}{6}}\right] \tag{3.13}$$

$$\text{at } t = 0,$$

$$y_{n+\frac{1}{30}} - y_{n+\frac{1}{15}} + 3y_{n+\frac{1}{10}} - 3y_{n+\frac{2}{15}} = -\frac{h^3}{6480000}\left[3f_n + 109f_{n+\frac{1}{30}} + 494f_{n+\frac{1}{15}} + 354f_{n+\frac{1}{10}} - f_{n+\frac{2}{15}} + f_{n+\frac{1}{6}}\right] \tag{3.14}$$

$$\text{at } t = \frac{1}{30},$$

$$y_{n+\frac{1}{30}} - y_{n+\frac{1}{15}} + 3y_{n+\frac{1}{10}} - 3y_{n+\frac{2}{15}} = -\frac{h^3}{12960000}\left[f_n - 3f_{n+\frac{1}{30}} + 242f_{n+\frac{1}{15}} + 242f_{n+\frac{1}{10}} - 3f_{n+\frac{2}{15}} + f_{n+\frac{1}{6}}\right] \tag{3.15}$$

Differentiating the continuous scheme twice and evaluating at points

$$x = x_n,\ x_{n+\frac{1}{30}},\ x_{n+\frac{1}{15}},\ x_{n+\frac{1}{10}},\ x_{n+\frac{2}{15}},\ x_{n+\frac{1}{6}}$$

gives the following derivatives.

First Order Derivatives

when $T = 0$

$$hy'_n + 105y_{n+\frac{1}{15}} - 180y_{n+\frac{1}{10}} + 75y_{n+\frac{2}{15}} = \frac{h^3}{18144000}\Big[1437f_n + 20755f_{n+\frac{1}{30}} + 39698f_{n+\frac{1}{15}} + 25758f_{n+\frac{1}{10}} - 415f_{n+\frac{2}{15}} + 127f_{n+\frac{1}{6}}\Big]$$

when $T = \frac{1}{30}$

$$hy'_{n+\frac{1}{30}} + 75y_{n+\frac{1}{15}} - 120y_{n+\frac{1}{10}} + 45y_{n+\frac{2}{15}} = -\frac{h^3}{9072000}\Big[11f_n - 732f_{n+\frac{1}{30}} - 10370f_{n+\frac{1}{15}} + 7364f_{n+\frac{1}{10}} - 9f_{n+\frac{2}{15}} - 16f_{n+\frac{1}{6}}\Big]$$

when $T = \frac{1}{15}$

$$hy'_{n+\frac{1}{15}} + 45y_{n+\frac{1}{15}} - 60y_{n+\frac{1}{10}} + 15y_{n+\frac{2}{15}} = \frac{h^3}{18144000}\Big[31f_n - 271f_{n+\frac{1}{30}} + 2118f_{n+\frac{1}{15}} + 4874f_{n+\frac{1}{10}} - 53f_{n+\frac{2}{15}} + 21f_{n+\frac{1}{6}}\Big]$$

when $T = \frac{1}{10}$

$$hy'_{n+\frac{1}{10}} + 15y_{n+\frac{1}{15}} - 0y_{n+\frac{1}{10}} - 15y_{n+\frac{2}{15}} = \frac{h^3}{9072000}\Big[0f_n + 5f_{n+\frac{1}{30}} - 104f_{n+\frac{1}{15}} - 1482f_{n+\frac{1}{10}} - 104f_{n+\frac{2}{15}} + 5f_{n+\frac{1}{6}}\Big]$$

when $T = \frac{2}{15}$

$$hy'_{n+\frac{2}{15}} - 15y_{n+\frac{1}{15}} + 60y_{n+\frac{1}{10}} - 45y_{n+\frac{2}{15}} = -\frac{h^3}{18144000}\Big[31f_n - 207f_{n+\frac{1}{30}} + 518f_{n+\frac{1}{15}} - 549f_{n+\frac{1}{10}} - 1653f_{n+\frac{2}{15}} + 85f_{n+\frac{1}{6}}\Big]$$

when $T = \frac{1}{6}$

$$hy'_{n+\frac{1}{6}} - 45y_{n+\frac{1}{15}} + 120y_{n+\frac{1}{10}} - 75y_{n+\frac{2}{15}} = \frac{h^3}{9072000}\Big[11f_n - 50f_{n+\frac{1}{30}} + 174f_{n+\frac{1}{15}} + 714f_{n+\frac{1}{10}} + 10535f_{n+\frac{2}{15}} + 660f_{n+\frac{1}{6}}\Big] \tag{3.16}$$

Second Order Derivatives

when $T = 0$

$h^2y''_n - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = -\frac{h^3}{604800}\Big[6383f_n + 2792f_{n+\frac{1}{30}} + 12870f_{n+\frac{1}{15}} + 14799f_{n+\frac{1}{10}} - 1794f_{n+\frac{2}{15}} + 309f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{30}$
$h^2y''_{n+\frac{1}{30}} - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = -\frac{h^3}{604800}\Big[267f_n - 7943f_{n+\frac{1}{30}} - 2404f_{n+\frac{1}{15}} - 8046f_{n+\frac{1}{10}} - 625f_{n+\frac{2}{15}} + 69f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{15}$
$h^2y''_{n+\frac{1}{15}} - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = \frac{h^3}{604800}\Big[111f_n - 975f_{n+\frac{1}{30}} + 9734f_{n+\frac{1}{15}} + 11658f_{n+\frac{1}{10}} - 453f_{n+\frac{2}{15}} + 85f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{10}$
$h^2y''_{n+\frac{1}{10}} - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = \frac{h^3}{604800}\Big[43f_n - 327f_{n+\frac{1}{30}} + 1494f_{n+\frac{1}{15}} - 430f_{n+\frac{1}{10}} - 849f_{n+\frac{2}{15}} + 69f_{n+\frac{1}{6}}\Big]$

when $T = \frac{2}{15}$
$h^2y''_{n+\frac{2}{15}} - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = -\frac{h^3}{604800}\Big[111f_n - 751f_{n+\frac{1}{30}} + 211f_{n+\frac{1}{15}} - 13878f_{n+\frac{1}{10}} - 8069f_{n+\frac{2}{15}} + 309f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{6}$
$h^2y''_{n+\frac{1}{6}} - 900y_{n+\frac{1}{15}} + 1800y_{n+\frac{1}{10}} - 900y_{n+\frac{2}{15}} = \frac{h^3}{604800}\Big[267f_n - 1671f_{n+\frac{1}{30}} + 4630f_{n+\frac{1}{15}} + 2706f_{n+\frac{1}{10}} + 28047f_{n+\frac{2}{15}} + 6341f_{n+\frac{1}{6}}\Big]$

Re-writing the above set of equations

when $T = 0$
$6480000y_n - 38880000y_{n+\frac{1}{15}} + 51840000y_{n+\frac{1}{10}} - 19440000y_{n+\frac{2}{15}} = -h^3\Big[3f_n + 109f_{n+\frac{1}{30}} + 494f_{n+\frac{1}{15}} + 354f_{n+\frac{1}{10}} - f_{n+\frac{2}{15}} + f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{30}$
$12960000y_{n+\frac{1}{30}} - 38880000y_{n+\frac{1}{15}} + 38880000y_{n+\frac{1}{10}} - 12960000y_{n+\frac{2}{15}} = -h^3\Big[f_n - 3f_{n+\frac{1}{30}} + 242f_{n+\frac{1}{15}} + 242f_{n+\frac{1}{10}} - 3f_{n+\frac{2}{15}} + f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{6}$

$12960000y_{n+\frac{1}{10}} - 12960000y_{n+\frac{1}{15}} + 38880000y_{n+\frac{1}{10}} - 38880000y_{n+\frac{2}{15}} = -h^3\Big[f_n - 7f_{n+\frac{1}{30}} + 18f_{n+\frac{1}{15}} - 262f_{n+\frac{1}{10}} - 272f_{n+\frac{2}{15}} - 3f_{n+\frac{1}{6}}\Big]$

and

when $T = 0$

$18144000hy'_n + 1905120000y_{n+\frac{1}{15}} - 3265920000y_{n+\frac{1}{10}} + 136080000y_{n+\frac{2}{15}} = h^3\Big[1437f_n + 20755f_{n+\frac{1}{30}} + 39698f_{n+\frac{1}{15}} + 25758f_{n+\frac{1}{10}} - 415f_{n+\frac{2}{15}} + 127f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{30}$

$9072000hy'_{n+\frac{1}{30}} + 680400000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} + 408240000y_{n+\frac{2}{15}} = -h^3\Big[11f_n - 732f_{n+\frac{1}{30}} - 10370f_{n+\frac{1}{15}} + 7364f_{n+\frac{1}{10}} - 9f_{n+\frac{2}{15}} - 16f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{15}$

$18144000hy'_{n+\frac{1}{15}} + 816480000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} + 272160000y_{n+\frac{2}{15}} = h^3\Big[31f_n - 271f_{n+\frac{1}{30}} + 2118f_{n+\frac{1}{15}} + 4874f_{n+\frac{1}{10}} - 53f_{n+\frac{2}{15}} + 21f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{10}$

$9072000hy'_{n+\frac{1}{10}} + 136080000y_{n+\frac{1}{15}} - 0y_{n+\frac{1}{10}} - 136080000y_{n+\frac{2}{15}} = h^3\Big[0f_n + 5f_{n+\frac{1}{30}} - 104f_{n+\frac{1}{15}} - 1482f_{n+\frac{1}{10}} - 104f_{n+\frac{2}{15}} + 5f_{n+\frac{1}{6}}\Big]$

when $T = \frac{2}{15}$

$18144000hy'_{n+\frac{2}{15}} - 272160000y_{n+\frac{1}{15}} + 1088640000y_{n+\frac{1}{10}} - 816480000y_{n+\frac{2}{15}} = -h^3\Big[31f_n - 207f_{n+\frac{1}{30}} + 518f_{n+\frac{1}{15}} - 549f_{n+\frac{1}{10}} - 1653f_{n+\frac{2}{15}} + 85f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{6}$

$9072000hy'_{n+\frac{1}{6}} - 408240000y_{n+\frac{1}{15}} + 1088640000y_{n+\frac{1}{10}} - 68040000y_{n+\frac{2}{15}} = h^3\Big[11f_n - 50f_{n+\frac{1}{30}} + 174f_{n+\frac{1}{15}} + 7144f_{n+\frac{1}{10}} + 10535f_{n+\frac{2}{15}} + 666f_{n+\frac{1}{6}}\Big]$

when $T = 0$

$604800h^2y''_n - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = -h^3\Big[6383f_n + 27921f_{n+\frac{1}{30}} + 12870f_{n+\frac{1}{15}} +$

$14794f_{n+\frac{1}{10}} - 1794f_{n+\frac{2}{15}} + 309f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{30}$

$604800h^2y''_{n+\frac{1}{30}} - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = h^3\Big[267f_n - 7943f_{n+\frac{1}{30}} - 2404f_{n+\frac{1}{15}} -$
$8046f_{n+\frac{1}{10}} - 625f_{n+\frac{2}{15}} + 69f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{15}$

$604800h^2y''_{n+\frac{1}{15}} - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = -h^3\Big[11f_n - 975f_{n+\frac{1}{30}} + 9734f_{n+\frac{1}{15}} +$
$11658f_{n+\frac{1}{10}} - 453f_{n+\frac{2}{15}} + 85f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{10}$

$604800h^2y''_{n+\frac{1}{10}} - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = h^3\Big[43f_n - 377f_{n+\frac{1}{30}} + 1494f_{n+\frac{1}{15}} -$
$430f_{n+\frac{1}{10}} - 849f_{n+\frac{2}{15}} + 69f_{n+\frac{1}{6}}\Big]$

when $T = \frac{2}{15}$

$604800h^2y''_{n+\frac{2}{15}} - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = -h^3\Big[11f_n - 757f_{n+\frac{1}{30}} + 211f_{n+\frac{1}{15}} -$
$13878f_{n+\frac{1}{10}} - 8069f_{n+\frac{2}{15}} + 309f_{n+\frac{1}{6}}\Big]$

when $T = \frac{1}{6}$

$604800h^2y''_{n+\frac{1}{6}} - 544320000y_{n+\frac{1}{15}} - 1088640000y_{n+\frac{1}{10}} - 54430000y_{n+\frac{2}{15}} = h^3\Big[267f_n - 1671f_{n+\frac{1}{30}} + 4630f_{n+\frac{1}{15}} +$
$2706f_{n+\frac{1}{10}} + 28047f_{n+\frac{2}{15}} + 6341f_{n+\frac{1}{6}}\Big]$

Combining (3.27) to (3.41) and by the modified block method formula to obtain values for

$$y_{n+\frac{1}{15}},\ y_{n+\frac{1}{10}},\ y_{n+\frac{2}{15}},\ y_{n+\frac{1}{6}},\ y_{n+\frac{1}{30}},\ y'_{n+\frac{1}{15}},\ y'_{n+\frac{1}{10}},\ y'_{n+\frac{2}{15}},\ y'_{n+\frac{1}{6}},\ y'_{n+\frac{1}{30}},\ y''_{n+\frac{1}{15}},\ y''_{n+\frac{1}{10}},\ y''_{n+\frac{2}{15}},\ y''_{n+\frac{1}{6}},\ y''_{n+\frac{1}{30}}$$

$$
\begin{bmatrix}
-12960000 & 38880000 & -38880000 & 12960000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-38880000 & 51840000 & -19440000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-38880000 & 38880000 & -12960000 & 0 & 12960000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1905120000 & -326590000 & 136080000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
68040000 & -108840000 & 408240000 & 0 & 0 & 0 & 0 & 0 & 0 & 12960000 & 0 & 0 & 0 & 0 & 0 \\
816480000 & -10888640000 & 272160000 & 0 & 0 & 1814400 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
136080000 & 0 & -136080000 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-272160000 & 10888640000 & -816480000 & 0 & 0 & 0 & 0 & 18141000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-408240000 & 10888640000 & -68040000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 0 & 0 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 0 & 0 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 0 & 604800 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 604800 & 0 & 0 & 0 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 604800 & 0 & 0 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 604800 & 0 \\
-544320000 & 10888640000 & -544320000 & 0 & 0 & 0 & 0 & 0 & 9072000 & 0 & 0 & 0 & 0 & 0 & 604800
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{15}} \\ y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{30}} \\
y'_{n+\frac{1}{15}} \\ y'_{n+\frac{1}{10}} \\ y'_{n+\frac{2}{15}} \\ y'_{n+\frac{1}{6}} \\ y'_{n+\frac{1}{30}} \\
y''_{n+\frac{1}{15}} \\ y''_{n+\frac{1}{10}} \\ y''_{n+\frac{2}{15}} \\ y''_{n+\frac{1}{6}} \\ y''_{n+\frac{1}{30}}
\end{bmatrix}
$$

$$
= \begin{bmatrix}
0 & 0 & 0 \\
-6480000 & 0 & 0 \\
0 & 0 & 0 \\
0 & -18144000 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & -604800 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix} y_n \\ hy'_n \\ h^2y''_n \end{bmatrix}
+
\begin{bmatrix}
-1 \\ -3 \\ -1 \\ 1437 \\ -11 \\ 31 \\ 0 \\ -31 \\ 11 \\ -6383 \\ 267 \\ -111 \\ 43 \\ -111 \\ 267
\end{bmatrix}
h^3 f_n +
\begin{bmatrix}
7 & -18 & 262 & 227 & 3 \\
-109 & -494 & -354 & 1 & -1 \\
3 & -242 & -242 & 3 & -1 \\
20755 & 39698 & 25758 & -415 & 127 \\
732 & 10370 & -7364 & 9 & 16 \\
-271 & 2118 & 4874 & -53 & 21 \\
5 & -104 & -1482 & -104 & 5 \\
207 & -518 & 549 & 1653 & -85 \\
-50 & 174 & 7144 & 10535 & 660 \\
-27921 & -12870 & -14794 & 1794 & -309 \\
7943 & -2404 & -8046 & -625 & 69 \\
-975 & -9734 & -11658 & 453 & -85 \\
-377 & 1494 & -430 & -849 & 69 \\
751 & -211 & 13878 & 8069 & -309 \\
1671 & 6430 & 2706 & 28047 & 6341
\end{bmatrix}
\begin{bmatrix}
h^3 f_{n+\frac{1}{30}} \\ h^3 f_{n+\frac{1}{15}} \\ h^3 f_{n+\frac{1}{10}} \\ h^3 f_{n+\frac{2}{15}} \\ h^3 f_{n+\frac{1}{6}}
\end{bmatrix}
$$

Normalizing the system above in (3.42), I got the Block Solution of the ODE.

$$
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_{n+\frac{1}{15}} \\ y_{n+\frac{1}{10}} \\ y_{n+\frac{2}{15}} \\ y_{n+\frac{1}{6}} \\ y_{n+\frac{1}{30}} \\
y'_{n+\frac{1}{15}} \\ y'_{n+\frac{1}{10}} \\ y'_{n+\frac{2}{15}} \\ y'_{n+\frac{1}{6}} \\ y'_{n+\frac{1}{30}} \\
y''_{n+\frac{1}{15}} \\ y''_{n+\frac{1}{10}} \\ y''_{n+\frac{2}{15}} \\ y''_{n+\frac{1}{6}} \\ y''_{n+\frac{1}{30}}
\end{bmatrix}
=
$$

$$
\begin{bmatrix}
1 & \frac{1}{15} & \frac{1}{450} \\
1 & \frac{1}{10} & \frac{1}{200} \\
1 & \frac{2}{15} & \frac{1}{255} \\
1 & \frac{1}{6} & \frac{1}{72} \\
1 & \frac{1}{30} & \frac{1}{1800} \\
1 & 1 & \frac{1}{15} \\
1 & 1 & \frac{1}{10} \\
1 & 1 & \frac{2}{15} \\
1 & 1 & \frac{1}{6} \\
1 & 1 & \frac{1}{30} \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
y_n \\ hy'_n \\ h^2 y''_n
\end{bmatrix}
+
\begin{bmatrix}
\frac{317}{17010000} \\ \frac{29}{640000} \\ \frac{89}{1063125} \\ \frac{233}{1741824} \\ \frac{3929}{1088640000} \\
\frac{71}{113400} \\ \frac{41}{42000} \\ \frac{94}{70875} \\ \frac{61}{36288} \\ \frac{1231}{4536000} \\
\frac{7}{675} \\ \frac{17}{1600} \\ \frac{7}{675} \\ \frac{19}{1728} \\ \frac{19}{1728}
\end{bmatrix}
h^3 f_n +
\begin{bmatrix}
\frac{367}{8505000} & \frac{19}{8505000} & \frac{61}{4252500} & \frac{203}{38880000} & \frac{1}{1215000} \\
\frac{597}{4480000} & \frac{81}{2240000} & \frac{47}{1344000} & \frac{1}{78750} & \frac{9}{4480000} \\
\frac{292}{1063125} & \frac{4}{151875} & \frac{88}{1063125} & \frac{1597}{6804000} & \frac{4}{1063125} \\
\frac{815}{1741824} & \frac{5}{870912} & \frac{155}{870912} & \frac{109}{5443200} & \frac{11}{1741824} \\
\frac{199}{4354600} & \frac{1931}{541320000} & \frac{173}{77760000} & \frac{11}{13608000} & \frac{139}{1088640000} \\
\frac{136}{70875} & \frac{37}{56700} & \frac{34}{70875} & \frac{1613}{9072000} & \frac{2}{70875} \\
\frac{389}{112000} & \frac{1}{14000} & \frac{29}{33600} & \frac{23}{86640} & \frac{1}{22400} \\
\frac{356}{70875} & \frac{44}{70875} & \frac{33967}{18144000} & \frac{1277}{4536000} & \frac{4}{70875} \\
\frac{475}{72576} & \frac{25}{18144} & \frac{125}{36288} & \frac{125}{3628800} & \frac{1369}{9072000} \\
\frac{863}{1814400} & \frac{761}{2268000} & \frac{2141}{1512000} & \frac{1361}{18144000} & \frac{107}{9072000} \\
\frac{499}{11200} & \frac{7}{1350} & \frac{7}{1350} & \frac{149}{67200} & \frac{1}{2700} \\
\frac{3443}{75600} & \frac{19}{800} & \frac{19}{800} & \frac{881}{201600} & \frac{1}{1600} \\
\frac{32}{675} & \frac{12659}{604800} & \frac{32}{675} & \frac{251}{24192} & 0 \\
\frac{137}{2800} & \frac{193}{6048} & \frac{25}{675} & \frac{2917}{67200} & \frac{19}{1728} \\
\frac{4483}{75600} & \frac{5233}{302400} & \frac{241}{26100} & \frac{2419}{604800} & \frac{1}{1600}
\end{bmatrix}
\begin{bmatrix}
h^3 f_{n+\frac{1}{30}} \\ h^3 f_{n+\frac{1}{15}} \\ h^3 f_{n+\frac{1}{10}} \\ h^3 f_{n+\frac{2}{15}} \\ h^3 f_{n+\frac{1}{6}}
\end{bmatrix}
\tag{3.17}
$$

which is the normal matrix

Writing (3.43) explicitly, we have

$$y_{n+\frac{1}{15}} = y_n+\frac{1}{15}hy'_n+\frac{1}{450}h^2y''_n+\frac{317}{17010000}h^3f_n+\frac{367}{8505000}h^3f_{n+\frac{1}{30}}+\frac{19}{8505000}h^3f_{n+\frac{1}{15}}+\frac{61}{4252500}h^3f_{n+\frac{1}{10}}+\frac{203}{3888000}h^3f_{n+\frac{2}{15}}+\frac{1}{1215000}h^3f_{n+\frac{1}{6}}$$

$$y_{n+\frac{1}{10}} = y_n+\frac{1}{10}hy'_n+\frac{1}{200}h^2y''_n+\frac{29}{640000}h^3f_n+\frac{597}{4480000}h^3f_{n+\frac{1}{30}}+\frac{81}{850500}h^3f_{n+\frac{1}{15}}+\frac{47}{1344000}h^3f_{n+\frac{1}{10}}+\frac{1}{78750}h^3f_{n+\frac{2}{15}}+\frac{9}{4480000}h^3f_{n+\frac{1}{6}}$$

$$y_{n+\frac{2}{15}} = y_n+\frac{2}{15}hy'_n+\frac{2}{255}h^2y''_n+\frac{89}{1063125}h^3f_n+\frac{292}{1063125}h^3f_{n+\frac{1}{30}}+\frac{4}{151875}h^3f_{n+\frac{1}{15}}+\frac{88}{1063125}h^3f_{n+\frac{1}{10}}+\frac{1597}{68040000}h^3f_{n+\frac{2}{15}}+\frac{4}{1063125}h^3f_{n+\frac{1}{6}}$$

$$y_{n+\frac{1}{6}} = y_n+\frac{1}{6}hy'_n+\frac{1}{72}h^2y''_n+\frac{233}{1741824}h^3f_n+\frac{815}{1741824}h^3f_{n+\frac{1}{30}}+\frac{5}{870912}h^3f_{n+\frac{1}{15}}+\frac{155}{870912}h^3f_{n+\frac{1}{10}}+\frac{109}{5443200}h^3f_{n+\frac{2}{15}}+\frac{11}{1741824}h^3f_{n+\frac{1}{6}}$$

$$y_{n+\frac{1}{30}} = y_n+\frac{1}{30}hy'_n+\frac{1}{1800}h^2y''_n+\frac{3929}{1088640000}h^3f_n+\frac{199}{4354600}h^3f_{n+\frac{1}{30}}+\frac{1931}{544320000}h^3f_{n+\frac{1}{15}}+\frac{173}{77760000}h^3f_{n+\frac{1}{10}}+\frac{11}{13608000}h^3f_{n+\frac{2}{15}}+\frac{139}{1088640000}h^3f_{n+\frac{1}{6}}$$

$$y'_{n+\frac{1}{15}} = hy'_n+\frac{1}{15}h^2y''_n+\frac{71}{113400}h^3f_n+\frac{136}{70875}h^3f_{n+\frac{1}{30}}+\frac{37}{56700}f_{n+\frac{1}{15}}+\frac{34}{70875}f_{n+\frac{1}{10}}+\frac{1613}{9072000}f_{n+\frac{2}{15}}+\frac{2}{70875}h^3f_{n+\frac{1}{6}}$$

$$y'_{n+\frac{1}{10}} = hy'_n+\frac{1}{10}h^2y''_n+\frac{41}{42000}h^3f_n+\frac{389}{112000}h^3f_{n+\frac{1}{30}}+\frac{1}{14000}h^3f_{n+\frac{1}{15}}+\frac{29}{33600}h^3f_{n+\frac{1}{10}}+\frac{23}{80640}h^3f_{n+\frac{2}{15}}+\frac{1}{22400}h^3f_{n+\frac{1}{6}}$$

$$y'_{n+\frac{2}{15}} = hy'_n+\frac{2}{15}h^2y''_n+\frac{94}{70875}h^3f_n+\frac{356}{70875}h^3f_{n+\frac{1}{30}}+\frac{44}{70875}h^3f_{n+\frac{1}{15}}+\frac{33967}{18144000}h^3f_{n+\frac{1}{10}}+\frac{1277}{4536000}h^3f_{n+\frac{2}{15}}+\frac{4}{70875}h^3f_{n+\frac{1}{6}}$$

$$y'_{n+\frac{1}{6}} = hy'_n+\frac{1}{6}h^2y''_n+\frac{61}{36288}h^3f_n+\frac{475}{72576}h^3f_{n+\frac{1}{30}}+\frac{25}{18144}h^3f_{n+\frac{1}{15}}+\frac{125}{36288}h^3f_{n+\frac{1}{10}}+\frac{2503}{3628800}h^3f_{n+\frac{2}{15}}+\frac{1369}{9072000}h^3f_{n+\frac{1}{6}}$$

$$y'_{n+\frac{1}{30}} = hy'_n+\frac{1}{30}h^2y''_n+\frac{1231}{4536000}h^3f_n+\frac{863}{1814400}h^3f_{n+\frac{1}{30}}+\frac{761}{2268000}f_{n+\frac{1}{15}}+\frac{2141}{15120000}h^3f_{n+\frac{1}{10}}+\frac{1361}{18144000}h^3f_{n+\frac{2}{15}}+\frac{107}{9072000}h^3f_{n+\frac{1}{6}}$$

$$y''_{n+\frac{1}{15}} = h^2y''_n+\frac{7}{675}h^3f_n+\frac{499}{11200}h^3f_{n+\frac{1}{30}}+\frac{7}{1350}h^3f_{n+\frac{1}{15}}+\frac{7}{1350}h^3f_{n+\frac{1}{10}}+\frac{149}{67200}h^3f_{n+\frac{2}{15}}+\frac{1}{2700}h^3f_{n+\frac{1}{6}}$$

$$y''_{n+\frac{1}{10}} = h^2y''_n + \frac{17}{1600}h^3f_n + \frac{3443}{75600}h^3f_{n+\frac{1}{30}} + \frac{19}{800}h^3f_{n+\frac{1}{15}} + \frac{19}{800}h^3f_{n+\frac{1}{10}} + \frac{881}{201600}f_{n+\frac{2}{15}} + \frac{1}{1600}h^3f_{n+\frac{1}{6}}$$

$$y''_{n+\frac{2}{15}} = h^2y''_n + \frac{7}{675}h^3f_n + \frac{32}{675}h^3f_{n+\frac{1}{30}} + \frac{12659}{604800}h^3f_{n+\frac{1}{15}} + \frac{32}{675}h^3f_{n+\frac{1}{10}} + \frac{251}{24192}f_{n+\frac{2}{15}}$$

$$y''_{n+\frac{1}{6}} = h^2y''_n + \frac{19}{1728}h^3f_n + \frac{137}{2800}h^3f_{n+\frac{1}{30}} + \frac{193}{6048}h^3f_{n+\frac{1}{15}} + \frac{25}{675}h^3f_{n+\frac{1}{10}} + \frac{2917}{672000}h^3f_{n+\frac{2}{15}} + \frac{19}{1728}h^3f_{n+\frac{1}{6}}$$

$$y''_{n+\frac{1}{30}} = h^2y''_n + \frac{19}{1728}h^3f_n + \frac{4483}{75600}h^3f_{n+\frac{1}{30}} + \frac{5233}{302400}h^3f_{n+\frac{1}{15}} + \frac{241}{21600}h^3f_{n+\frac{1}{10}} + \frac{2419}{604800}h^2f_{n+\frac{2}{15}} + \frac{1}{1600}h^3f_{n+\frac{1}{6}}$$

CHAPTER 4
ANALYSIS OF METHOD

4.1 INTRODUCTION

This chapter considers the examination of the basic properties of the derived method. These properties include order and error constant, consistency, zero-stability, convergence and region of absolute stability.

4.2 Order and Error Constant

Consider the linear multi-step defined

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h^m \sum_{j=0}^{k} \beta_j f_{n+j} \tag{4.1}$$

and the associated difference operator $\Theta\Big[\gamma(x);h\Big]$ defined by

$$\Theta\Big[\gamma(x);h\Big] = \sum_{j=0}^{k}[\alpha_j\gamma(x+jh) - h\beta_j\gamma'(x+jh)] \tag{4.2}$$

where $\gamma(x) \in C[a,b]$ is an arbitrary function.

If $\gamma(x)$ is differentiable then by expanding $\gamma(x+jh)$ and $\gamma'(x+jh)$ about x, (4.2) becomes

$$\Theta\Big[\gamma(x);h\Big] = C_0\gamma(x) + C_1 h\gamma'(x) + C_2h^2\gamma''(x) + C_3h^3\gamma'''(x) + ... + C_nh^n\gamma'''(x) + ... \tag{4.3}$$

where $C_n's$ are constants.

Thus, the linear multi-step method (4.1) is said to be of order p if in (4.2), the constants $C_0 = C_1 = ... = C_p = 0,\ C_{p+1} \neq 0$ and the corresponding error constant is C_{p+1}.

From the derived method, we obtain the following by re-arranging the terms

$$y_{n+\frac{1}{6}} - 3y_{n+\frac{2}{15}} + 3y_{n+\frac{1}{10}} - y_{n+\frac{1}{15}} + \frac{h^3}{12960000}\Big[f_n - 7f_{n+\frac{1}{30}} + 18f_{n+\frac{1}{15}} - 262f_{n+\frac{1}{10}} - 227f_{n+\frac{2}{15}} - 3f_{n+\frac{1}{6}}\Big] = 0 \tag{4.4}$$

Expanding (4.4) with the Taylor's series in the form

$$\sum_{j=0}^{\infty}\frac{(\frac{1}{6})^jh^j}{j!}y_n^j - 3\sum_{j=0}^{\infty}\frac{(\frac{2}{15})^j}{j!}y_n^j + 3\sum_{j=0}^{\infty}\frac{(\frac{1}{10})^j}{j!}y_n^j - \sum_{j=0}^{\infty}\frac{(\frac{1}{15})^j}{j!}y_n^j + \sum_{j=0}^{\infty}\frac{h^{j+3}}{j!}y_n^j$$

$$\Big[-\tfrac{7}{12960000}(\tfrac{1}{30})^j + \tfrac{18}{12960000}(\tfrac{1}{15})^j - \tfrac{262}{12960000}(\tfrac{1}{10})^j - \tfrac{227}{12960000}(\tfrac{2}{15})^j - \tfrac{3}{12960000}(\tfrac{1}{6})^j + \tfrac{1}{12960000}h^3y_n^3\Big] = 0$$

Collecting terms in h and y heads to the following

$$C_0 = 1 - 3 + 3 - 1$$

$$C_1 = 1(\tfrac{1}{6}) - 3(\tfrac{2}{15}) + 3(\tfrac{1}{10}) - 1(\tfrac{1}{15}) = \tfrac{1}{6} - \tfrac{2}{5} + \tfrac{3}{10} - \tfrac{1}{15}$$

$$C_2 = \frac{(\frac{1}{6})^2}{2!} - 3\frac{(\frac{2}{15})^2}{2!} + 3\frac{(\frac{1}{10})^2}{2!} - \frac{(\frac{1}{15})^2}{2!} = \frac{1}{72} - \frac{2}{75} + \frac{3}{200} - \frac{1}{450} = 0$$

$$C_3 = \frac{(\frac{1}{6})^3}{3!} - 3\frac{(\frac{2}{15})^3}{3!} + 3\frac{(\frac{1}{10})^3}{3!} - \frac{(\frac{1}{15})^3}{3!} + \left[-\frac{7}{12960000} + \frac{18}{12960000} - \frac{262}{12960000} - \frac{227}{12960000} - \frac{3}{12960000} + \frac{1}{12960000}\right] = 0$$

$$C_4 = \frac{(\frac{1}{6})^4}{4!} - 3\frac{(\frac{2}{15})^4}{4!} + 3\frac{(\frac{1}{10})^4}{4!} - \frac{(\frac{1}{15})^4}{4!} + \left[-\frac{7}{12960000}(\tfrac{1}{30}) + \frac{18}{12960000}(\tfrac{1}{15}) - \frac{262}{12960000}(\tfrac{1}{10}) - \frac{227}{12960000}(\tfrac{2}{15}) - \frac{3}{12960000}(\tfrac{1}{6}) + \frac{1}{12960000}(0)\right] = 0$$

$$C_5 = \frac{(\frac{1}{6})^5}{5!} - 3\frac{(\frac{2}{15})^5}{5!} + 3\frac{(\frac{1}{10})^5}{5!} - \frac{(\frac{1}{15})^5}{5!} + \left[-\frac{7}{12960000}\left(\frac{(\frac{1}{30})^2}{2!}\right) + \frac{18}{12960000}\left(\frac{(\frac{1}{15})^2}{2!}\right) - \frac{262}{12960000}\left(\frac{(\frac{1}{10})^2}{2!}\right) - \frac{227}{12960000}\left(\frac{(\frac{2}{15})^2}{2!}\right) - \frac{3}{12960000}\left(\frac{(\frac{1}{6})^2}{2!}\right) + \frac{1}{12960000}\left(\frac{(0)^2}{2!}\right)\right] = 0$$

$$C_6 = \frac{(\frac{1}{6})^6}{6!} - 3\frac{(\frac{2}{15})^6}{6!} + 3\frac{(\frac{1}{10})^6}{6!} - \frac{(\frac{1}{15})^6}{6!} + \left[-\frac{7}{12960000}\left(\frac{(\frac{1}{30})^3}{3!}\right) + \frac{18}{12960000}\left(\frac{(\frac{1}{15})^3}{3!}\right) - \frac{262}{12960000}\left(\frac{(\frac{1}{10})^3}{3!}\right) - \frac{227}{12960000}\left(\frac{(\frac{2}{15})^3}{3!}\right) - \frac{3}{12960000}\left(\frac{(\frac{1}{6})^3}{3!}\right) + \frac{1}{12960000}\left(\frac{(0)^3}{3!}\right)\right] = 0$$

$$C_7 = \frac{(\frac{1}{6})^7}{7!} - 3\frac{(\frac{2}{15})^7}{7!} + 3\frac{(\frac{1}{10})^7}{7!} - \frac{(\frac{1}{15})^7}{7!} + \left[-\frac{7}{12960000}\left(\frac{(\frac{1}{30})^4}{4!}\right) + \frac{18}{12960000}\left(\frac{(\frac{1}{15})^4}{4!}\right) - \frac{262}{12960000}\left(\frac{(\frac{1}{10})^4}{4!}\right) - \frac{227}{12960000}\left(\frac{(\frac{2}{15})^4}{4!}\right) - \frac{3}{12960000}\left(\frac{(\frac{1}{6})^4}{4!}\right) + \frac{1}{12960000}\left(\frac{(0)^4}{4!}\right)\right] = 0$$

$$C_8 = \frac{(\frac{1}{6})^8}{8!} - 3\frac{(\frac{2}{15})^8}{8!} + 3\frac{(\frac{1}{10})^8}{8!} - \frac{(\frac{1}{15})^8}{8!} + \left[-\frac{7}{12960000}\left(\frac{(\frac{1}{30})^5}{5!}\right) + \frac{18}{12960000}\left(\frac{(\frac{1}{15})^5}{5!}\right) - \frac{262}{12960000}\left(\frac{(\frac{1}{10})^5}{5!}\right) - \frac{227}{12960000}\left(\frac{(\frac{2}{15})^5}{5!}\right) - \frac{3}{12960000}\left(\frac{(\frac{1}{6})^5}{5!}\right) + \frac{1}{12960000}\left(\frac{(0)^5}{5!}\right)\right] = 0$$

$$C_9 = \frac{(\frac{1}{6})^9}{9!} - 3\frac{(\frac{2}{15})^9}{9!} + 3\frac{(\frac{1}{10})^9}{9!} - \frac{(\frac{1}{15})^9}{9!} + \left[-\frac{7}{12960000}\left(\frac{(\frac{1}{30})^5}{5!}\right) + \frac{18}{12960000}\left(\frac{(\frac{1}{15})^5}{5!}\right) - \frac{262}{12960000}\left(\frac{(\frac{1}{10})^5}{5!}\right) - \frac{227}{12960000}\left(\frac{(\frac{2}{15})^5}{5!}\right) - \frac{3}{12960000}\left(\frac{(\frac{1}{6})^5}{5!}\right) + \frac{1}{12960000}\left(\frac{(0)^5}{5!}\right)\right] = \frac{1}{18600435000000000}$$

$\Rightarrow$ $C_9 \neq 0$, that is $C_0 = C_1 = ... = C_{n+2} = 0$ and $C_{n+3} \neq 0$.

The main scheme is of order 6 and the error constant is $\frac{1}{18600435000000000}$

4.3 Consistency

A numerical scheme of order p is said to be consistent if the following conditions are satisfied

i The order $p \geq 1$

ii $\sum_{j=0}^{k} = 0$

iii $\rho(1) - \rho'(1) = 0$

iii $\rho'''(1) = 3'(1)$ for the principal root $r = 1$

- The order of the method is $p \geq 1$ which obviously satisfies (i) above

- For the method, $\alpha_0 = 1,\ \alpha_1 = -1,\ \alpha_2 = 3,\ \alpha_3 = -3.$

Thus

$$\sum_{j=1}^{3} \alpha_j = 1 - 1 + 3 - 3 = 0$$

$$\rho(r) = r^{\frac{1}{6}} - r^{\frac{1}{15}} + 3r^{\frac{1}{10}} - 3r^{\frac{2}{15}}$$
$$\rho(1) = 1^{\frac{1}{6}} - 1^{\frac{1}{15}} + 3.1^{\frac{1}{10}} - 3.1^{\frac{2}{15}} = 1 - 1 + 3 - 3 = 0$$

$$\rho'(r) = \frac{1}{6}r^{-\frac{5}{6}} - \frac{1}{15}r^{-\frac{14}{15}} + \frac{3}{10}r^{-\frac{9}{10}} - \frac{6}{15}r^{-\frac{13}{15}}$$
$$\rho'(1) = \frac{1}{6}1^{-\frac{5}{6}} - \frac{1}{15}1^{-\frac{14}{15}} + \frac{3}{10}1^{-\frac{9}{10}} - \frac{6}{15}1^{-\frac{13}{15}} = \frac{1}{6} - \frac{1}{15} + \frac{3}{10} - \frac{6}{15} = 0$$
$$\rho(1) = \rho'(1) = 0$$

$$\rho(r) = r^{\frac{1}{6}} - r^{\frac{1}{15}} + 3r^{\frac{1}{10}} - 3r^{\frac{2}{15}}$$
$$\delta(r) = -\frac{1}{12960000}r^{0} + \frac{7}{12960000}r^{\frac{1}{30}} - \frac{18}{12960000}r^{\frac{1}{5}} + \frac{262}{12960000}r^{\frac{1}{10}} + \frac{227}{12960000}r^{\frac{2}{15}} + \frac{3}{12960000}r^{\frac{1}{6}}$$
$$\rho'(r) = \frac{1}{6}r^{-\frac{5}{6}} - \frac{1}{15}r^{-\frac{14}{15}} + \frac{3}{10}r^{-\frac{9}{10}} - \frac{6}{15}r^{-\frac{13}{15}}$$
$$\rho''(r) = -\frac{5}{36}r^{-\frac{11}{6}} + \frac{14}{225}r^{-\frac{29}{15}} - \frac{27}{100}r^{-\frac{19}{10}} - \frac{48}{225}r^{-\frac{28}{15}}$$
$$\rho'''(r) = \frac{55}{216}r^{-\frac{17}{6}} + \frac{406}{3375}r^{-\frac{44}{15}} - \frac{513}{1000}r^{-\frac{29}{10}} - \frac{2184}{3375}r^{-\frac{43}{15}}$$
$$\rho'''(1) = \frac{6875-3248+13851-1742}{27000} = \frac{6}{27000}$$

$$\delta(1) = -\frac{1}{12960000} + \frac{7}{12960000} - \frac{18}{12960000} + \frac{262}{12960000} + \frac{227}{12960000} + \frac{3}{12960000} = \frac{1}{27000}$$
$$\delta(1) = 3!p'''(r) = 3!\delta(1) = p$$
$$p'''(1) = 3!\delta(1)$$

$\frac{6}{27000} = 3!\frac{1}{27000}$
PROVED

It follows that the scheme satisfies the fourth condition. Since all the conditions are satisfied, it follows that the derived scheme is consistent.

4.4 Zero Stability of the method

By definition the first characteristic polynomial

$\rho(r) = r^{\frac{1}{6}} - 3r^{\frac{2}{15}} + 3r^{\frac{1}{10}} - r^{\frac{1}{15}}$

$p(1) = 1 - 3 + 3 - 1 = 0$

$\rho(-1) = (-1)^{\frac{1}{6}} - 3(-1)^{\frac{2}{15}} + 3(-1)^{\frac{1}{10}} - (-1)^{\frac{1}{15}} \neq 0$

Showing the first characteristics polynomial of the scheme has only one root $r = 1$, which is not greater than 1. Hence, the condition for the scheme to be zero-stable is satisfied; thus the scheme is zero-stable.

4.5 Convergence of the method

The necessary and sufficient condition for any method to be convergent is for it to be consistent and zero-stable. Thus since it has been successfully shown in sections 4.3 and 4.4 thus the method is convergent.

4.6 Region of Absolute Stability of the Method

We consider the stability polynomial written in general form

$$\pi(r, \bar{h}) = \rho(r) - \bar{h}\,\sigma(r) = 0$$

$$\text{where } \bar{h} = h^2\lambda \text{ and } \lambda = \frac{\partial F}{\partial y} \text{ is assumed constant.}$$

From the main scheme, the first and second characteristic polynomial are as follows

$\rho(r) = r^{\frac{1}{6}} - 3r^{\frac{2}{15}} + 3r^{\frac{1}{10}} - r^{\frac{1}{15}}$

$\sigma(r) = -\frac{1}{12960000} + \frac{7}{12960000}r^{\frac{1}{30}} - \frac{8}{12960000}r^{\frac{1}{15}} + \frac{262}{12960000}r^{\frac{1}{10}} + \frac{227}{12960000}r^{\frac{2}{15}} + \frac{3}{12960000}r^{\frac{1}{6}}$

So that the boundaries of the region of absolute stability is

$$\bar{h} = \frac{\rho(r)}{\sigma(r)} = \frac{12960000\left[r^{\frac{1}{6}} - 3r^{\frac{2}{15}} + 3r^{\frac{1}{10}} - r^{\frac{1}{15}}\right]}{-1 + 7r^{\frac{1}{30}} - 18r^{\frac{1}{15}} + 262r^{\frac{1}{10}} + 227r^{\frac{2}{15}} + 3r^{\frac{1}{6}}}$$

setting $r = e^{i\theta}$, then we have

$$\frac{12960000\left[e^{\frac{1}{6}i\theta}-3e^{\frac{2}{15}i\theta}+3e^{\frac{1}{10}i\theta}-e^{\frac{1}{15}i\theta}\right]}{-1+7e^{\frac{1}{30}i\theta}-18e^{\frac{1}{15}i\theta}+262e^{\frac{1}{10}i\theta}+227e^{\frac{2}{15}i\theta}+3e^{\frac{1}{6}i\theta}}$$

$$=\frac{12960000\left[\cos(\frac{1}{6})\theta+i\sin(\frac{1}{6})\theta-3\cos(\frac{2}{15})\theta-3i\sin(\frac{2}{15})\theta+3\cos(\frac{1}{10})\theta+3i\sin(\frac{1}{10})\theta-\cos(\frac{1}{15})\theta-i\sin(\frac{1}{15})\theta\right]}{-1+7\cos(\frac{1}{30})\theta+7i\sin(\frac{1}{30})\theta-18\cos(\frac{1}{15})\theta-18i\sin(\frac{1}{15})\theta+262\cos(\frac{1}{10})\theta+262i\sin(\frac{1}{10})\theta+227\cos(\frac{2}{15})\theta+227i\sin(\frac{2}{15})\theta+3\cos(\frac{1}{6})\theta+3i\sin(\frac{1}{6})\theta}$$

$$\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{ac+bd}{c^2+d^2}$$

$$=\frac{12960000\left[126-\cos(\frac{1}{6})\theta+10\cos(\frac{2}{15})\theta-45\cos(\frac{1}{10})\theta+120\cos(\frac{1}{15})\theta-210\cos(\frac{1}{30})\theta\right]}{120556-6\cos(\frac{1}{6})\theta-412\cos(\frac{2}{15})\theta+2546\cos(\frac{1}{10})\theta-2896\cos(\frac{1}{15})\theta-110612\cos(\frac{1}{30})\theta}$$

$$=\frac{1632960000-12960000\cos(\frac{1}{6})\theta+129600000\cos(\frac{2}{15})\theta-583200000\cos(\frac{1}{10})\theta+1555200000\cos(\frac{1}{15})\theta-2721600000\cos(\frac{1}{30})\theta}{120556-6\cos(\frac{1}{6})\theta-412\cos(\frac{2}{15})\theta+2546\cos(\frac{1}{10})\theta-2896\cos(\frac{1}{15})\theta-110612\cos(\frac{1}{30})\theta}$$

Equating the above at interval of 30^o gives the following results, the boundaries of the region of the absolute stability of the method

θ	0^o	30^o	60^o	90^o	120^o	150^o	180^o
$\pi(r)$	0.000	71.251	6907.1	28810	13094	437	1.5498×10^{-3}

From here, it could be seen that the region of the absolute stability of the method is given by $x(\theta)=1.5498\times10^{-3}$ which satisfies the condition for $p-$stability, similarly, this interval of periodicity lies in the interval $x(0),\ (0,+\infty)$ showing that the method is $p-$stable.

Chapter 5
SUMMARY, CONCLUSION AND RECOMMENDATION

5.1 INTRODUCTION

In this work, I proposed a self starting continuous one-step hybrid block method of a third order differential equation.

5.2 SUMMARY AND CONCLUSION

One-ninth step hybrid linear multi-step method for the direct solution of the third order differential equation has been developed in this project. The techniques of interpolations and collocations are adopted in the derivation of the method, using power series as the basis function. This yields an order six method, which is as well consistent, convergent and zero-stable with a characteristic low error constant. In addition, the derived method is implemented using the block method, and MATLAB codes are written using the results of the block method. The new method can handle both linear and non-linear third order differential equations with initial conditions given. The results of this scheme in comparison with other methods in terms of error compares favourably in that it is very close to the analytic solution.

5.3 RECOMMENDATION

I learnt quite a lot when carrying out this research, from some of the knowledge a list of recommendations are made

- the hybrid block method for the direct solution of third order differential equations with or without initial and boundary conditions given has been seen to be of a very high accuracy levels.
- the use of MATLAB for the development of programs for numerical schemes should be made utilized more. In its use, it is very flexible, easy yo use and very efficient.

The techniques of interpolations and collocations are adopted in the derivation of the method, using power series as the basis function. This yields an order six method, which is as well consistent, convergent and zero-stable with a characteristic low error constant. The results of this scheme in comparison with other methods in terms of error compares favourably in that it is very close to the analytic solution.

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