MTS 522

STOCHASTIC PROCESS

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RANDOM VARIABLE

Consider a random experiment having sample space $\boldsymbol{S}$. A random variable $\boldsymbol{X}$ is a function that assigns a real value to each outcome in $\boldsymbol{S}$. For any set of real numbers A, the probability that X will assume a value that is contained in the set A is equal to the probability that the outcome of the experiment is contained in$\boldsymbol{X^{-1}(A)}$. That

$$P\{X \in A\} = P\{X^{-1}(A)\}$$

Where $\boldsymbol{X^{-1}(A)}$ is the event consisting of all points $s \in S \text{ such that } X(s) \in A$.

The distribution function *F* of the random Variable *X* is defined for any real number ***x*** by

$$F(x) = P\{X \leq x\} = P\{X \in (-\infty, x)\}$$

We shall denote $1 - F(x)$ $by\ \hat{F}(x), and\ so\ \hat{F}(x) = P\{X > x\}$

STOCHASTIC PROCESSES

- Random Walk
- Simple and General Random Walk with Absorbing and Reflecting Barriers
- Markovian Processes with Finite Chains
- Limit Theory
- Passion, Branching, Birth and Death Processes
- Queuing Processes
- Queues and their Waiting Time Distribution
- Relevant Applications

1) A stochastic process $X = \{X(t), t \in T\}$ is a collection of random variables. That is, for each $\boldsymbol{t}$ in the index set T, $X(t)$ is a random variable. We often interpret $\boldsymbol{t}$ as time and call $\boldsymbol{X}$ the state of the process at time $\boldsymbol{t}$. if the index set $\boldsymbol{T}$ is a countable set, we call $\boldsymbol{X}$ a discrete time stochastic process, and if $\boldsymbol{T}$ is a continuum, we call it a continuous time process.

   Definition: A stochastic process is any system that develops in time or space in accordance with probabilistic laws.

   Mathematically, a stochastic process is collection of random variables $\{(X_n)\}$ or $\{X(t)\}$ defined for all relevant values of

   $n = 1, 2, 3, \ldots$

   $n = 0 + 1 + 2 \ldots$ discrete time or $t \geq 0$ $or - \infty < t < \infty$ continuous time respectively. Thus, there are two basic time points (discrete or continuous) on which the stochastic process is defined.

2) A stochastic process is a family $\{X(t) | t\, \varepsilon\, T\}$ of random variable X(t), all defined on the same sample space S, where domain T of the parameter is a subset of $\Re$

3) A stochastic process is said to be of discrete time if the system is defined as a finite or enumerably infinite set of time points. e.g. if you are to calculate the waiting time of bus at the bus stop

   X_1 can be the waiting time for the first cab and X_2 can be the waiting time for the second cab e.g $12{:}00, 12{:}12, 12{:}16 \ldots (X_1, X_2, \ldots, X_n)$

   That is a stochastic process is of discrete time if the time $t(t \geq 0)$ has been converted to $n = 0, \pm 1, \pm 2$ where n is an integer

4) A stochastic process is said to be of continuous time if the system defined in an interval of time $t > 0 \; or -\infty < t < \infty$

The state space of the stochastic process is the set of possible values of an individual $X_n \;\; or \; X(t)$ which can be one dimensional or multidimensional. (We consider only the one dimensional)

Definition: Random (or stochastic) process refers to any quantity that evolves randomly in time or space. It is usually a dynamic object of same kind which varies in an unpredictable fashion.

Mathematically, a random or stochastic process is defined as a collection of random variables. The various members of the family are distinguished by different values of a parameter, $\propto$, say the entire set of values of $\propto$ which we shall denote by A, is called an index set or parameter set. A random process is then a collection of such as

$$\{X_k, \propto \in A\} \; of \; a \; random \; variable$$

The index set A may be discrete (finite or countably infinite) or continuous.

The space in which the values of the random variable X lie is called the $\underline{\boldsymbol{State\ Space}}$. Usually there is some connection which unites in some sense, the individual numbers of the process. Suppose a coin is tossed 3 times.

Let X_k with possible values $0 \; and \; 1$ be the number of heads on the K^{th} toss. Then the collection $\{X_1, X_2, X_3\}$ fits our definition of stochastic process but as such is of no more interest then its individual members since each of these random variables is independent of the others.

If however we introduce

$$Y_1 = X_1$$

$$Y_2 = X_1 + X_2$$

$$Y_3 = X_1 + X_2 + X_3$$

So that Y_k records the number of heads up to and including the K^{th} toss, then the collection $\{Y_k, k \in \{1,2,3\}\}$ is a stochastic process which fits in the physical concept outlined earlier. In this example the index set is $A = \{1,2,3\}$ (we have used k rather than $\propto$ for the index) and the state space is the set $(0,1,2,3,)$

The following two physical examples illustrate some of the possibilities for index sets and state space.

Examples

I. *Discrete Time Parameter*
Let X_k be the amount of rainfall in a day $\boldsymbol{K}$, with $\boldsymbol{K} \; = \; 0, 1, 2, 3, \boldsymbol{\ldots}$ The collection of random variable $\boldsymbol{X} = \{ \; \boldsymbol{X_k} \,, \;\; \boldsymbol{K} = 0, 1, \; 2 \ldots \}$ stochastic process in discrete time. Since the amount of rainfall can be any non-negative number X_k have a continuous range. Hence, **X** is said to have a continuous state space.

II. *Continuous Time Parameter*
Let $X(t)$ be the number of vehicles on a certain roadway at time $\boldsymbol{t}$ where $\boldsymbol{t} \; \geq \boldsymbol{0}$ is measured relative to some reference time. Then the collection of random variable $\boldsymbol{X} = \{\boldsymbol{X_{(t)}} \,, \; \boldsymbol{t} \geq \boldsymbol{0}\}$ is a stochastic process in a continuous time. Here the state space is discrete since the number of vehicles is a member of discrete set $\{\boldsymbol{0}, \boldsymbol{1}, \boldsymbol{2}, \ldots, \boldsymbol{N}\}$ where N is the minimum number of vehicles that may be on the roadway.

SAMPLE PATHS OF A STOCHASTIC PROCESS

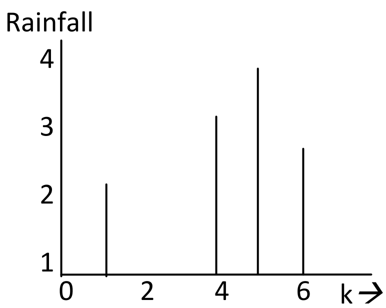
The sequence of possible values of the family of random variables constituting a stochastic process, taken in increasing order of time, say are called sample paths (or trajectories or realization).

The various sample paths corresponding to elementary outcomes in the case of observations on a single random variable. It is often convinient to draw graphs of these and examples are shown in the figure below for the cases;

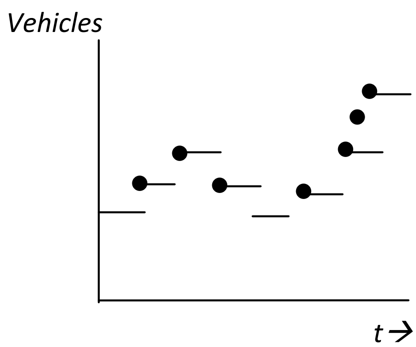
a. 1) Discrete time – discrete state space, e.g the number of death in a city due to automobile accident on day K;
   2) The number of customers in front of the n arriving automobile
b. Discrete time – continuous state space e.g the rainfall on day K;
c. Continuous time – discrete space state e.g the no of vehincles on the roadway at time t.
d. Continuous time- continuous state space e.g the temperature at a given location at time t.

a.

b.



c.



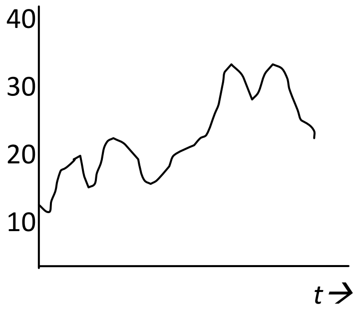
N.B: **STATE SPACES**

The set of all possible values assumed by is called a state space. A state space may be integer value or real value.

PARAMETER SPACE

The indexing parameter takes a range of values. Thus, the set of all possible values of this parameter is called a parameter space. A parameter space may be integer value or real value.

d.



The figures (a)------(d) are sketches of representatives sample paths for the various kinds of stochastic processes.

PROBABILISTIC DESCRIPTION OF RANDOM PROCESSES

Any random variable X, may be characterised by its distribution function

$$F(X) \leq Pr\{X \leq x\}, -\infty \leq x \leq \infty$$

A discrete-parameter stochastic process$\{X_{k,}\ k = 0, 1, 2, \dots, n\}$ may be characterised by the joint distribution function of all the random variables involved;

$F(\ x_0, x_1, \dots, x_n) = \Pr\{X_0\ \leq\ x_0\ , X_1 \leq\ x_1, \dots, X_n\ \ \leq\ x_n\ \}, x_n\ \in (-\infty, \infty),$

$k = 1,2, \dots, n$ and by the joint distributions of all distinct subsets of (X_k). Similar, but more complicated descriptions apply to continous time stochastic processes. The probability structure of some processes, however, enables them to be characterized much more simply. One important of such class of processes is called ***Markov Processes.***

A Markov Process is a discrete stochastic process of values in N_o for which also,

$$\Pr\{X_{(n)} = S_{(kn)} \mid X_{(n-1)} = S_{(kn-1)} \wedge \dots \wedge X_{(1)} = S_{(1)}\}$$

$$\Pr\{X_{(n)} = S_{(kn)} \mid X_{(n-1)} = S_{(kn-1)}\}$$

Markov Processes: Let $X = \{X_{k,}\ k = 0, 1, 2, \dots\}$ be a stochastic process with a discrete index set and a discrete state space $S\ =\ \{S_1\ , S_2, S_3\ \dots\}$

if

$\Pr\{X_n =\ S_{kn} \mid X_{n-1} =\ S_{kn-1},\ X_{n-2} =\ S_{kn-2}, \dots, X_1 = S_{k1},\ X_0 = S_{k0}\}$

$= \Pr\{X_n =\ S_{kn} \mid X_{n-1} =\ S_{kn-1},\ \} \dots \dots \dots \dots \dots \dots \dots \dots \dots ..$***

for any $n > 1$ and any collection of $S_k \in S, j = 0,1, \dots, n$ then X is called a ***Markov Process.***

A Markov Process is a process that has no memory extending beyond the previous instance. This implies that $X_n's$ are not independent. i.e

$$P(X_{1,} \dots, X_n) = P(X_1).P(X_2\backslash X_1).P(X_3\backslash X_1X_2)\dots P(X_n\backslash X_1 \dots X_{n-1})$$

$$= P(X_1).P(X_2\backslash X_1).P(X_3\backslash X_2)\dots P(X_n\backslash X_{n-1})$$

$$P(X_{1,} \dots, X_n) = P(X_i)\prod_{i=2}^{n} P(X_i\backslash X_{i-1})$$

so that the marked process is defined by the conditional distribution $P(X_i\backslash X_i)$ for any i and the initial distribution $P(X_i)$ when the state space X_n is discrete, the process is called the ***Markov Chain.***

From (***) it can be deduced that the values of X at all times prior to $n-1$ have no effect whatsoever in the conditional probability distribution of $X_n\ given\ X_{n-1}$. Thus, a $markov\ process$ has memory of its past values but only to a limited extent.

The collection of quantities

$$\Pr\{X_n = S_{kn}\backslash X_{n-1} = S_{kn-1}\}$$

for various n, S_{kn} and S_{kn-1}; is called the set of one-time-step transition probabilities.

MARKOV CHAIN

A *MARKOV CHAIN (M.C)* is a sequence of random variable such that X_n is a random walk such that the probability of occurence depends on its immediate past.

Let X_n => state space

And X_{n-j} => The state space X_n is at state j at time n.

The stochastic process

$X = \{X_n, n \in N\}$ is called a *Markov Chain* provided that

$P(X_{n+1}$= j|$X_1, \dots, X_n)$ = $P(X_{n+1}$= j| $X_n)$

MARKOV PROPERTY

A simple random walk is clearly a Markov Process

For example

$$Pr\{X_4 = 2|\ X_3 = 3\ X_2\ = 2\ X_1 = 1\ X_0 = 0\}$$

$$= Pr\{X_4 = 2|\ X_3 = 3\} = Pr\{Z_4 =\ \pm 1\} = q$$

That is, the probability is q that X_4 has the value 2 given that $X_3 = 3$, regardless of the values of the process at epochs 0, 1, 2.

The one- time step transition probabilities are:

$$P_{jk} = Pr\{X_n = K, X_{n-1} = j\}\begin{Bmatrix} p. & if\ k = & j+1 \\ q. & if\ k = & j-1 \\ o & olw & \end{Bmatrix}$$

And in this case these does not depend on n.

MEAN AND VARIANCE

We first observe that

$$X_1 = X_0 + Z_1$$

$$X_2 = X_1 + Z_2 = X_0 + Z_1 + Z_2$$

$$\vdots$$

$$\vdots$$

$$X_n = X_0 + Z_1 + Z_2 + \cdots + Z_n$$

Then, because the Z_n are identically distributed and independent random variables and

$X_0 = 0$ With probability one,

$$E(X_n) = E\left(\sum_{k=1}^{n} Z_k\right) = n\,E(Z_1)$$

$$And\ Var(X_n) = Var\left(\sum_{k=1}^{n} Z_k\right) = nVar(Z_1)$$

Now

$$E(Z_1) = 1.p + (-1)q = p - q$$

And

$$E(Z_1^2) = 1.p + q = p + q = 1$$

Thus

$$Var(Z_1) = E(Z_1^2) - E^2(Z_1)$$

$$= 1 - (p-q)^2$$

$$= 1 - (p^2 + q^2 - 2pq)$$

$$= 1 - (p^2 + q^2 - 2pq) + 4pq = 4pq$$

Since

$$p^2 + q^2 - 2pq = (p+q)^2 = 1$$

Hence we arrive at the following expressions for the mean and variance of the process at epoch n.

$$E(X_n) = n(p - q)$$

$$Var(X_n) = 4npq$$

We see that the mean and variance grow linearly with time.

The probability distribution of X_n

Let us derive an expression for the probability distribution of the random variable X_n, the value of the process or $x -$ coordinate of the particle at time n ≥1. That is, we seek

$$P(k,n) = \Pr\{X_n = k\}, where\ k\ is\ an\ integer$$

We first note that $P(k,n) = 0\ if\ n < |k|$ because the process cannot get to level k in less than $|k|$ steps. Henceforth, therefore $n \geq |k|$

Of the n-steps, let the number of magnitude

$+1\ be\ N_n^+\ and\ the\ number\ of\ magnitude$

$-1\ be\ N_n^-, where\ N_n^+\ and\ N_n^-$ are random variables

We must have

$$X_n = N_n^+ - N_n^- ---------------(i)$$

And

$$n = N_n^+ + N_n^- ---------------(ii)$$

Adding these two equations to eliminate N_n^- yields

$$N_n^+ = \frac{1}{2}(n + X_n) ------ (iii)$$

Thus $X_n = k$ if and only if $N_n^+ = \frac{1}{2}(n+k)'$

We note that $N_n^+$is a binomial random variable with parameters $n\ and\ p$. Also from (iii)

$2N_n^+ = n + X_n$ is necessary even, $X_n$must be odd if n is odd

Thus we arrive at

$$P(k,n) = \binom{n}{(k+n)/2} p^{(k+n)/2} q^{(n-k)/2}$$

$n \geq |k|\ and\ n\ either\ both\ are\ even\ or\ both\ odd$

For example, the probability that the particle is at $k = -2$ after $n = 4$ steps is

$$P(-2,4) = \binom{4}{1} pq^3 = 4pq^3 -----------(iv)$$

$$\lambda(t) = E(t^X) = \sum_{j=0}^{\infty} t^x p_j$$

$$Q(t) = \sum_{j=1}^{\infty} t^x q_j$$

Theorem*:* Define $\lambda(t)$ and $Q(t)$ as above

$(1-t)Q(t) = 1 - \lambda(t)$

Proof: Given that

$$\lambda(t) = \sum_{j=0}^{\infty} t^j p_j$$

$$1 - \lambda(t) = 1 - \sum_{j=0}^{\infty} t^j p_j$$

$$= 1 - p_0 - p_1 t - p_2 t^2 - p_3 t^3 - \cdots . - p_j t^j \ldots$$

$$Q(t) = \sum_{j=1}^{\infty} t^j q_j$$

$$= q_0 + q_1 t + q_2 t^2 + q_3 t^3 + \cdots . + q_j t^j \ldots$$

$$(1 - t)Q(t) = Q(t) - tQ(t)$$

$$= (q_0 + q_1 t + q_2 t^2 + q_3 t^3 + \cdots . + q_j t^j \ldots) - (q_0 t + q_1 t^2 + q_2 t^3 + \cdots . + q_{j-1} t^j \ldots)$$

Now,

$$q_j - q_{j-1} = P(X > j) - P(X > j - 1)$$

$$= \left(p_{j+1} + p_{j+2} + \cdots\right) - \left(p_j + p_{j+1} + p_{j+2} + \cdots\right)$$

$$= -p_j$$

Recall $q_j = P(X \leq j)$

$$= p_{j+1} + p_{j+2}$$

And $q_j = P(X > 0)$

$$= p_1 + p_2 + \cdots$$

$$= 1 - p_0$$

$$Q(t) - tQ(t) = q_0 - p_1 t - p_2 t^2 - p_3 t^3 - \cdots$$

$$= 1 - \sum_{j=0}^{\infty} t^j p_j$$

$$= 1 - \lambda(t)$$

The above can be used to generate the moments via factorial moments.

Note therefore,

$\lambda'(1) = E(X)$

$\lambda''(1) = E(X(X-1))$

$V(X) = \lambda''(1) + \lambda'(1) - [\lambda'(1)]^2$

Theorem: The pgf $\lambda(t)$ of X_n is given by $\left(E(t^{Z_n})\right)^n$

Proof:

$$X_n = \sum_{i=0}^{\infty} Z_n$$

$$E(t^{X_n}) = E(t^{\sum Z_n})$$

$$= \left(E\left(t^{Z_j}\right)\right)^n$$

Consider a symmetric random walk

$$Z_n = \begin{cases} 1 & p \\ -1 & q \end{cases} \quad where\ p + q = 1$$

PGF of Z_n , $\lambda(t)$ is given by

$$\lambda(t) = \ E(t^{Z_n}) = \sum t^{Z_n} P(Z_n) = t^1 p + t^{-1} q$$

$$= tp + \frac{q}{t}$$

$$\lambda'(t) = p - \frac{q}{t^2}$$

$$\lambda'(t) = p - \frac{q}{t^2}$$

$$\lambda'(1) = p - q = E(Z_n)$$

$$\lambda''(t) = \frac{2q}{t^3}$$

$$\lambda''(1) = 2q$$

$$V(Z_n) = 2q + (p - q) - (p - q)^2$$

$$= 2q + (p - q)(1 - p + q)$$

$$= 2q + (p - q)(q + q)$$

$$= 2q + (p - q)2q$$

$$= 2q(1 + p - q)$$

$$= 2q(1 - q + p) = 2q(p + p) = 2q(2p) = \boldsymbol{4pq}$$

To maximize thus, the highest value of $\boldsymbol{pq} = {}^{\mathbf{1}}\!/_{\mathbf{4}}$ which makes it

But

$$PGF(X_n) = \left(E(t^{Z_n})\right)^n$$

$$= \left(tp - \frac{q}{t}\right)^n$$

$$\lambda'(t) = n\left(p + \frac{q}{t^2}\right)^{n-1}\left(p - \frac{q}{t^2}\right)$$

$$\lambda'(1) = n(p+q)^{n-1}(p-q)$$

Recall that $\boldsymbol{p + q = 1}$

$$= n(p-q)$$

GENERATING FUNCTION

Consider a sequence of real number $a_0, a_1, a_2, \dots, a_n$

Let

$$\lambda(t) = E(t^X) = \begin{cases} \int_{-\infty}^{\infty} t^x f(x)dx \\ \sum t^x P(X = x) \end{cases}$$

$$\lambda(t) = E(t^X)$$

$$\lambda'(t)\Big|_{t\,=\,1} = E[Xt^{X-1}]_{t=1} = E[X]$$

$$\lambda''(t)\Big|_{t\,=\,1} = E[X(X-1)t^{X-2}]_{t=1} = E[X(X-1)]$$

.

.

.

$$\lambda^j(t)\Big|_{t\,=\,1} = E[X(X-1)(X-2)\dots(X-j)]$$

If $\sum_{j=0}^{4} a_j = 1$

$\lambda(t) = E(t^X)$ *is called the* ***generating function*** of the Random variable X

$$\lambda(t) = E(t^X) = \sum_{j=0}^{\infty} t^x a_j$$

$\lambda(t)$ is Probability generating function (PGF) $if\ \sum_{j=0}^{\infty} a_j = 1\ holds$

$if\ P(X = j) = p_j$

Suppose $P(X \geq j) = q_j$; so that $P(X \leq j) = 1 - q_j$

$\lambda(t) = E(t^X) = \sum_{j=0}^{\infty} t^x p_j$

Consider $Q(t) = \sum_{j=0}^{\infty} t^x q_j$, Note that $Q(t)$is not a probability function, since $\sum_{j=0}^{\infty} t^x q_j$

$$i.e. q_0 = P(X > 0) = 1 - p_0 \quad , \quad q_1 = P(X > 1) = 1 - p_0 - p_1$$

RANDOM WALK

Random Walk: - Consider a sequence (X_t) of mutually independent identically distributed random variables, where the distribution can be expressed as

$P(X_t = 1) = p$ and $P(X_t = -1) = q$

p, q > 0 and p + q = 1 $\quad$ t ε N.

We define another sequence of random variables (Z_n) by

Z_o = 0 and $Z_n = Z_0 + \sum_{t=1}^{n} X_t$ $\quad$ for n ε N

If $p = q = \frac{1}{2}$, this process is called the **drunkard's Walk.**

The Ballot Theorem: At an election, a candidate A obtains in total a votes, while another candidate B obtains b votes, where b<a. The probability that A is leading during the whole of the counting is equal to $\frac{a-b}{a+b}$.

UNRESTRICTED SIMPLE RANDOM WALK

Suppose a particle is initially at the point $x = 0$ on the $x - axis$. At each subsequent time unit it moves a unit distance to the right with probability q,where $p + q = 1$

At time unit and let the position of the particle be X_n the above assumption yield $X_0 = 0$, with probability and in general,

$$X_n = X_{n-1} + Z_n \qquad n = 1,2,3, \ldots$$

Where the Z_n are identically distributed with

$$Pr\{Z_1 = +1\} = p$$

$$Pr\{Z_1 = -1\} = q$$

It is further assumed that the step taken by the particles are mutually independent random variable.

Definition: The collection of random variable

$X = \{X_0, X_1, X_2, \dots\}$ is called a simple random walk in one dimension. It is simple because the step takes only the value ±1 in distinction to cases where for example the Z_n are continuous random variables.

The simple random walk is a stochastic process indexed by a discrete time parameter $(n = 1, 2, 3, \dots)$ and has a discrete state space because its possible values are$\{0, \pm1, \pm2, \dots\}$. Furthermore, because there are no bounds on the possible values of x, the random walk is said to be unrestricted.

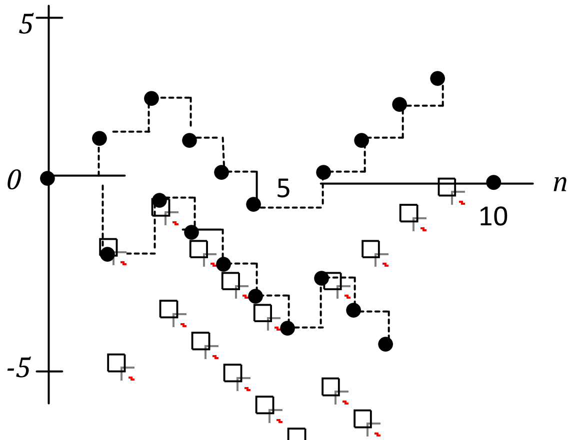
SAMPLE PATHS

Two possible beginning of sequence of values X are;

$$\{0, +1, +2, +1, 0, -1, 0, +1, +2, +3, \dots\}$$

$$\{0, -1, 0, -1, 2, -3, -4, -3, -4, -5, \dots\}$$

The sketches for two possible sample paths for the simple random walk



RANDOM WALK AND DISCRETE TIME MARKOV CHAIN

Definition 1: A Random Walk (R.W) is any stochastic process of the form $X_n = X_{n-1} + Z_n$in which $\{Z_1, Z_2, \dots, Z_n\}$ are independent and identically distributed random variables

Some values of Z_n

If

$$Z_n = \begin{cases} 1 & with\ Prob\ p \\ 0 & with\ Prob\ 1-p-q \\ -1 & with\ Prob\ q \end{cases}$$

The R.W here is simple if $p = q$ then the R.W is symmetric.

A R.W can be restricted if X_n is bounded or unrestricted if o/w

If two guys are gambling

$$Z_i = \begin{cases} 1 & with\ Prob\ p \\ 0 & with\ Prob\ p \\ -1 & \ \ with\ Prob\ \ q \end{cases}$$

$$X_n = X_{n-1} + Z_n$$

RANDOM WALK WITH ABSORBING STATES

The paths of the process considered in the previously increase or decrease at random, indefinitely. In many important applications this is not the case as particular values have special significance. This is illustrated in the following classical example.

A SIMPLE GAMBLING GAME

Let two gamblers, A and B, initially have $S_a$and S_b respectively, where a and b are positive integers. Suppose that at each round of their game, player A wins $1 from B with probability p and loses $1 to B with probability $q = 1 - p$.

The total capital of the two players at all times is $c = a + b$

Let X_n be players $A's$ capital at round n where $n = 0,1,2, \ldots\ and\ X_0 = a$.

Let Z_n be the amount A wins on trial n. The Z_n are assumed to be independent. It is clear that as both players have money left,

$$X_n = X_{n-1} + Z_n\ , n = 1,2, \ldots$$

Where the Z_n are iid. Thus $\{X_n, n = 0,1,2, \ldots\}$ is a simple random walk but there are now some restrictions or boundary conditions on the values it takes.

ABSORBING STATES

Let us assume that Ade and Bola play until one of them has no money left i.e. 'gone broke'. This may occur in two ways. Ade's capital may reach zero or Ade's capital may reach c, in which case Bola has gone broke. The process $X = \{X_n, n = 0,1,2, \ldots\}$ is thus restricted to the set of integers $\{0,1,2, \ldots . c\}$ and terminates when either the value 0 or c is attained called ***absorbing state,*** or we say there are ***absorbing barriers*** at 0 and c

THE PROBABILITIES OF ABSORBTION AT 0

Let P_a, $a = 0,1,2, \ldots, C$ denote the probability that player A goes broke when his initial capital is S_a equivalently, $P_a$is the probability that X is absorbed at 0 when $X_0 = a$.

The calculation of $P_a$is referred to as a gambler's ruin problem.

We will obtain a difference equation for P_a. First however, we observe that the following boundary conditions must apply; $P_0 = 1$

$$P_c = 0$$

Since if $a = 0$, the probability of absorption at 0 is one whereas if $a = c$, absorption at c has already occurred and absorption at 0 is impossible.

Now when $\boldsymbol{a}$ is not equal to either 0 or c, all games can be divided into two mutually exclusive categories.

i) A wins the first round

ii) A loses the first round

Thus the event $\{A\ goes\ broke\ from\ a\}\ is\ the$ union of two mutually exclusive events

$\{A\ goes\ broke\ from\ a\}$

$$= \{\ A\ wins\ the\ first\ round\ and\ goes\ broke\ from\ a+1\} \\ \cup \{A\ loses\ the\ first\ round\ and\ goes\ broke\ from\ a-1\}$$

Also, since going broke after winning the first round and winning the first round are independent

$$\Pr\{A\ wins\ the\ 1st\ round\ and\ goes\ broke\ from\ a+1\}$$

$$= \Pr\{A\ wins\ the\ 1st\ round\ \} * \Pr\{A\ goes\ broke\ from\ a+1\} = p.P_{a+1}$$

Similarly,

$$\Pr\{A\ loses\ the\ 1st\ round\ and\ goes\ broke\ from\ a-1\} = q.P_{a-1}$$

Since the probability of the union of two mutually exclusive events is the sum of their individual probabilities, we obtain $\boldsymbol{P_a = p.P_{a+1} + q.P_{a-1}}$ ———————————— $(\mathbf{1})$

This is a difference equation for $\boldsymbol{P_a}$which we will solve subject to the above boundary conditions

Solving equation (1) involves three main steps;

(1) The 1st step is to rearrange the equations since $p+q=1, we\ have$

$$(p+q)P_a = pP_{a+1} + q.P_{a-1} \text{ ———————————— } (2)$$

$$or \qquad p(P_{a+1}-P_a) = q(P_a - P_{a-1}) \text{ ———————————— } (3)$$

Dividing by p and letting $r = \frac{q}{p}$ ————————————(4) gives

$$P_{a+1} - P_a = r(P_a - P_{a-1}) \text{ —————————————— } (5)$$

(2) The 2nd step is to find P_1

To do this, we write out the system of equations and utilize the boundary condition $P_0 = 1$

$$a = 1\ : P_2 - P_1 = r(P_1 - P_0) = r(P_1 - 1)$$

$$a = 2\ : P_3 - P_2 = r(P_2 - P_1) = r^2(P_1 - 1)$$

:

.

$$a = c-2\ : P_{c-1} - P_{c-2} = r(P_{c-2} - P_{c-3}) = r^{c-2}(P_1 - 1)$$

$$a = c-1\ : P_c - P_{c-1} = r(P_{c-1} - P_{c-2}) = r^{c-1}(P_1 - 1)$$

Adding all these and cancelling gives

$$P_c - P_1 = -P_1 = (P_1 - 1)(r + r^2 + \cdots + r^{c-1}) - - - - - - - - - - (7)$$

Where we have used the fact that $P_c = 0$ special case:

$$p = q = {}^1/_2 \ if \ p = q = {}^1/_2 \ then \ r = 1$$

So

$$(r + r^2 + \cdots + r^{c-1}) = c - 1$$

Hence

$$-P_1 = (P_1 - 1)(c - 1)$$

Solving this gives

$$P_1 = 1 - {}^1/_c - - - - - - - - - - - - (8), r = 1$$

General case: $p \neq q$, equation (7) can be rearranged to give

$$(P_1 - 1)(1 + r + r^2 + \cdots + r^{c-1}) + 1 = 0$$

So

$$P_1 = 1 - \frac{1}{(1 + r + r^2 + \cdots + r^{c-1})}$$

For $r \neq 1$, we utilize the following formula for the sum of a finite number of terms of a geometric series

$$1 + r + r^2 + \cdots + r^{c-1} = \frac{1 - r^c}{1 - r} - - - - - - - - - - - - - - (9)$$

Hence, after a little algebra,

$$P_1 = \frac{r - r^c}{1 - r^c}, \qquad r \neq 1 - - - - - - - - - - - - - - - - - - - (10)$$

equation (8) and (9) gives the probabilities that the random walk is absorbed at zero when $x_0 = 1$, or the chances that player A goes broke when starting with one unit of capital.

(3) The zero and final step is to solve for $P_a, a \neq 1$

From the system of equation (6)

$$P_2 = P_1 + r(P_1 - 1)$$

$$P_3 = P_2 + r^2(P_1 - 1) = P_1 + (P_1 - 1)(r + r^2)$$

$$\vdots$$

$$\vdots$$

$$\vdots$$

$$P_a = P_{a-1} + r^{a-1}(P_1 - 1) = P_1 + (P_1 - 1)(r + r^2 + \cdots + r^{a-1})$$

Adding and subtracting one gives

$$P_a = (P_1 - 1)(1 + r + r^2 + \cdots + r^{a-1}) + 1 - - - - - - - - - - - - - (11)$$

Special case:$p = q = {}^1/_2$when $r = 1$, we have

$$1 + r + r^2 + \cdots + r^{a-1} = a,$$

so using (8) gives;

$$P_a = 1 - \frac{a}{c},\ \ p = q - - - - - - - - (12)$$

General case:$p \neq q\ \ from\ (10)$, we find

$$P_1 - 1 = \frac{r-1}{1-rc}$$

substituting this in (11)and utilizing (9)for the sum of the geometric series

$$P_a = \left(\frac{r-1}{1-rc}\right)\left(\frac{1-r^c}{1-r}\right) + 1$$

which rearranges to

$$P_a = \left(\frac{r^a - r^c}{1-r^c}\right), \qquad r \neq 1$$

Thus, in terms of $p\ and\ q$ we finally obtain the following results.

Then the probability that the R.W is absorbed at 0 when it starts at $X_0 = a$,

{or the chances that player A goes broke from $a\}$ is

$$P_a = \frac{\left({}^{q}/_{p}\right)^a - \left({}^{q}/_{p}\right)^c}{1 - \left({}^{q}/_{p}\right)^c} p \neq q - - - - - - (13)$$

Values of P_a for various p

a	**P=0.25**	**P=0.4**	**P=0.5**
0	1.00000	1.00000	1.0
1	0.99997	0.99118	0.9
2	0.99986	0.97794	0.8
3	0.99956	0.95809	0.7
4	0.99865	0.92831	0.6
5	0.99590	0.88364	0.5
6	0.98767	0.81663	0.4
7	0.96298	0.71612	0.3
8	0.88890	0.56536	0.2
9	0.66667	0.33922	0.1
10	0	0	0

A USE OF RANDOM WALK

Gambler's Ruin is found in game theory where two individuals gamble between each other. Let them be gamblers A and B, for simplicity, let the gambler A have an initial value of 2 and B has initial value of 1. Thus, the combined value is $2 + 1 = 3$.

The game proceeds in stages. Assume that B plays A 1 unit of a wins and the reverse o/w. suppose further that A has a probability **p** of winning.

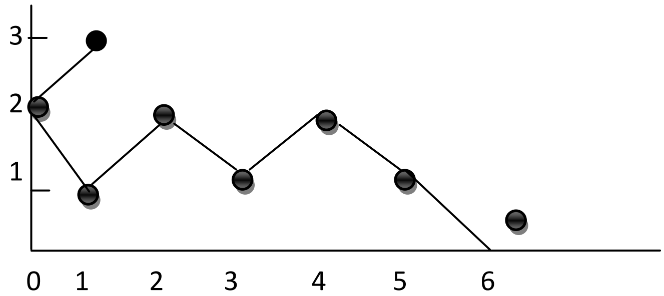
Clearly, it is correct to assume that the chance of winning in any game is independent i.e. Z_n *is independent*

The state of the capital can be represented by

A	B
3	0
2	1
1	2
0	3

The series of the game stops when either A or B has 0. The absorbing barriers are $X_n = 0 \text{ or } 3$. In this situation absorption is being interpreted as the ruin of one of the gamblers.

Let us consider the movement of A's capital



Let's examine the probability of Ruin for B

Games	1	3	5	7	……	2n+1…
Probability	q	pq^2	P^2q^3	P^3q^4	……	P^nq^{n+1}

$$P(B \text{ is ruined}) = q + pq^2 + p^2q^3 + \cdots + p^nq^{n+1}$$

$$= \sum_{j=0}^{\infty} p^j q^{j+1} = q \sum_{j=0}^{\infty} (pq)^j = q\left(\frac{1}{1 - pq}\right)$$

Recall that $S_\infty = \frac{a}{1-r} \quad when\ a = 1, r = pq$

$$if\ q = 1\ then\ P(B\ is\ ruined) = 1$$

$$if\ q = 0\ then\ P(B\ is\ ruined) = 0$$

Another way of obtaining the probability of Ruin is as follows;

Let

$$U_1 = prob(B\ is\ Ruined\ eventually)$$

After the 1st game B's capital is 2 or 0 according as he wins or loses.

Hence

$$U_1 = pU_2 + qU_0$$

$$= pU_2 + q \qquad as \quad U_0 = 1$$

$$p(qU_1) + q \qquad as\ U_2 = qU_1$$

Solving for U_1

$$U_1 = \frac{q}{1 - pq}$$

$since\ U_2 = qU_1 = \frac{q^2}{1-pq}$

Let us examine the probability of ruin for A

Games	2	4	6	8	.....	2(n+1)
Probability	q^2	pq^3	p^2q^4		....	p^nq^{n+2}

$Prob(A\ is\ ruined)\ =\ q^2 + pq^3 + p^2q^4 + p^nq^{n+2}$

$$= \sum_{j=0}^{\infty} p^j q^{j+2} = q^2 \sum_{j=0}^{\infty} (pq)^j = \frac{q^2}{1 - pq}$$

In general, let a be the total capital such that A's initial capital is k and B's is $a - k$

Let $U_k = p\ (A\ is\ ruined\ eventual)$

$$U_k = pU_{k+1} + qU_{k-1}; \quad 1 \le k \le a - 1$$

$provided: U_0 = 1, U_a = 0$

$let\ \ \propto^k = U_k\ \ (\propto \neq 0)$

Then

$$U_k = pU_{k+1} + qU_{k-1};$$

$$\propto^{k} = p \propto^{k+1} + q \propto^{k-1}$$

or

$$p \propto^{k+1} - \propto^{k} + q \propto^{k-1} = 0$$

$$\propto^{k-1} [p \propto^{2} - \propto + q] = 0$$

$$p \propto^{2} - \propto + q = (\propto - 1)(p \propto - q) = 0$$

$$\propto = 1 \; or \; \propto = {}^{q}/_{p}, \quad p \neq q$$

$$U_k = C(1) + D({}^{q}/_{p})^{k}$$

Where C and D are constants which satisfy the boundary conditions $U_0 = 1 \; and \; U_a = 0$

If $P \neq \frac{1}{2}$

$$U_k = \frac{\left(\frac{q}{p}\right)^k - \left(\frac{q}{p}\right)^N}{1 - \left(\frac{q}{p}\right)^N} \qquad \text{for k = 0, 1, 2, . . ., N.}$$

But if $P = \frac{1}{2} = q$

the $$U_k = \frac{N-k}{N} \qquad \text{for k = 0, 1, 2, . . ., N.}$$

When $k = 0$,

$$U_0 = C(1) + D({}^{q}/_{p})^{0} => C + D \;\; = 1$$

When $k = a$

$$U_a = C(1) + D({}^{q}/_{p})^{a} => C + D({}^{q}/_{p})^{a} \;\; = 0$$

$$C + D \;\; = 1$$

$$C + D({}^{q}/_{p})^{a} \;\; = 0$$

$$D = \frac{-1}{({}^{q}/_{p})^{a} - 1} \quad C = \frac{({}^{q}/_{p})^{a}}{({}^{q}/_{p})^{a} - 1}$$

$$U_k = \frac{({}^{q}/_{p})^{a} - ({}^{q}/_{p})^{k}}{({}^{q}/_{p})^{a} - 1} \qquad for \;\; p \neq q$$

V_k *is probability that A wins, then*

$$V_k = 1 - U_k$$

$$= {\left(q/p\right)^k - 1} \Big/ {\left(q/p\right)^a - 1}$$

Then

$$\left(q/p\right)^a \to 0$$

$$==> \frac{0-1}{0-1} = V_k = 1$$

Then probability of ruined is close to 1. The chance of A's winning is close to 1

Now if $p < q$

$\left(q/p\right)^k$ *becomes very large and* $\left(q/p\right)^k - 1$ *becomes less important*

$\left(q/p\right)^a$ *is likewise very large*

$$\frac{\left(q/p\right)^k}{\left(q/p\right)^a} \to 0$$

So the probability of A winning is close to 0

Summary: Once you have a fine resources in playing a game and $p \neq q$and then we expect the game to come to an end with a gambler been ruined

There is expected duration of such pattern or time

EXPECTED DURATION OF A GAME

(Average Time of Ruin)

Let d_s be the duration to ruin of Gambler A

$$d_s = p(1 + d_{s+1}) + q(1 + d_{s-1})$$

$$= p + pd_{s+1} + q + q\ d_{s-1}$$

$$= p + q + pd_{s+1} + q\ d_{s-1}$$

$1 \le x \le a - 1$, *recall that* $p + q = 1$

Provided $d_0 = 0$ *and* $d_a = 0$

Let the difference equation above be solved by letting $d_s = \lambda s$

$$\lambda s = 1 + p\lambda(s + 1) + q\lambda(s - 1)$$

$$= 1 + \lambda x + \lambda(p - q)$$

➔ $\lambda = {}^{1}/_{q-p}$ where $p \neq q$

Thus if $p \neq {}^{1}/_{2}$, the general solution of the difference equation is

$d_s = \frac{s}{q-p} + C + D\left(\frac{q}{p}\right)^s$

where $C\ and\ \Delta\ are\ constants\ satisfied\ by\ d_0 = d_a = 0$

$$C + D = 0, when\ d_0 = 0$$

$$\frac{a}{q-p} + C + D\left(\frac{q}{p}\right)^a = 0$$

$$\therefore C\left[1 - \left(\frac{q}{p}\right)^a\right] = \frac{-a}{q-p}$$

$$C = \frac{{}^{-a}/_{q-p}}{1 - \left(\frac{q}{p}\right)^a}$$

$$D = \frac{{}^{a}/_{q-p}}{1 - \left(\frac{q}{p}\right)^a}$$

When $p > q$, the chance of your winning continuously is going through steps a. If otherwise$q > p$, the chance of ruining is X because you're going s −downward

i.e

$$\frac{s}{q-p} - \frac{a}{q-p}\left[\frac{1 - \left(\frac{q}{p}\right)^s}{1 - \left(\frac{q}{p}\right)^a}\right] \Rightarrow \quad \frac{s}{q-p} - \frac{a}{q-p}$$

$$\Rightarrow \frac{s-a}{q-p}\ as\ q - p \to -1$$

$$\Rightarrow a - s$$

Exercise: Two players A & B had an initial capitals of 2 and 1 units respectively suppose that the probability that A wins a unit from B is 0.25. Find

i. The probability that A was eventually ruined
ii. The probability that A eventually succeeded
iii. The average time for A to be ruined

Solution: Let $A = 2 = k\ \ B = 1\ p = 0.25,\ \ q = 0.75,\ \ p + q = 1$

$a = 2 + 1$ (total capital)

Let $U_x$be the probability that A was eventually ruined

Then

$$U_k = \frac{\left(\frac{q}{p}\right)^a - \left(\frac{q}{p}\right)^k}{\left(\frac{q}{p}\right)^a - 1} = \frac{\left(\frac{0.75}{0.25}\right)^3 - \left(\frac{0.75}{0.25}\right)^2}{\left(\frac{0.75}{0.25}\right)^2 - 1}$$

$$= \frac{3^3 - 3^2}{3^2 - 1} = \frac{9}{13}$$

Let $V_n =$ probability that A wins eventually

$$V_n = 1 - U_k$$

$$V_n = 1 - \frac{18}{26} = \frac{4}{13}$$

Let d_s be the duration to the ruin of A

$$d_s = \frac{s}{q-p} - \frac{a}{q-p}\left[\frac{1 - \left(\frac{q}{p}\right)^s}{1 - \left(\frac{q}{p}\right)^a}\right]$$

$$= \frac{12}{\frac{1}{2}} - \frac{3}{\frac{1}{2}}\left(\frac{1-(3)^2}{1-(3)^3}\right) = 2\frac{2}{13} \approx 2$$

TRANSITION PROBABILITY

Consider the process$\{X_n, n \geq 1\}$

Define

$$P[X_n = j | X_{n-1} = i] = P_{ij}(n-1, n)$$

Thus the one –step transition probability similarly

$$P[X_{m+n} = j | X_n = i] = P_{ij}(m, m+n)$$

is the n −step transition probability

TIME HOMOGENEOUS PROCESS

A process is time Homogeneous if the conditional probabilities stated above depend on the difference in steps i.e.$n-(n-1)=1$ and $(m+n)-m=n$

$$P_{ij}(n-1,n)=P_{ij}(0)=P_{ij}$$

and

$$P_{ij}(m,m+n)=P_{ij}(n)$$

MARKOV CHAIN

Definition

Vector: A vector is an ordered $n-tupple.$

Probability Vector: The vector $Z=\{z_1,z_2,\dots z_n\}$ is a probability vector if

(i)$Z_i\geq 0\ \ \forall\, i$ (ii) $\sum_{i=1}^{n}Z_i=1$

STOCHASTIC MATRIX

A square matrix P_{ij} is stochastic if every row in the matrix is a probability vector.

Lemma: If $A=(a_{ij})$ is a stochastic matrix and $Z=\{z_1,z_2,\dots z_n\}$ is a probability vector then, $Z(A)$ is also a probability vector.

$P_{ij}\geq 0$ for every i and j

$\sum_{j}P_{ij}=1$ for every i, thus the sum of the rows are 1.

A probability vector Z is called invariant with respect to the stochastic matrix P, or a stationary distribution of the Markov chain if ZP = Z

Proof:

$$Z(A)=\{z_1,z_2,\dots z_n\}\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & & a_{2n}\\ \vdots & & \ddots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$$

$$=\begin{bmatrix} z_1a_{11}+z_2a_{21}+\cdots+z_na_{n1}\\ \vdots\\ \vdots\\ z_1a_{1n}+z_2a_{2n}+\cdots+z_na_{nn}\end{bmatrix}$$

We are required to show that the sum entries is equal to

$$z_1(a_{11}+a_{12}+\cdots+a_{1n})+z_2(a_{21}+a_{22}+\cdots+a_{2n})+\cdots+z_n(a_{n1}+a_{n2}+\cdots+a_{nn})$$

$$=z_1+z_2+\cdots+z_n=1\ \ since\ \sum_{i=1}^{n}z_i=1$$

REGULAR STOCHASTIC MATRIX

A stochastic matrix P is regular if P^n has only non-zero entires for some n.

e.g

$$P_1 = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$$

$$P_1^2 = \begin{pmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{pmatrix}$$

i.e P_1 is a regular Stochastic Matrix

e.g

$$P_2 = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}, \quad P_2^2 = \begin{pmatrix} 1 & 0 \\ 3/4 & 1/4 \end{pmatrix}$$

P_2 is not regular since for any n, all entire of P^n is non-zero

PROPERTIES OF A REGULAR MATRIX

1. Let P be a regular stochastic matric, then $P, P^2, P^3, \dots P^n$ will tend to a matrix T with identical rows.
2. Suppose t is a row of T then if P is a probability Vector, $PP^1, PP^2, PP^3, \dots, PP^n$ will also tend to t.
3. The matrix T is determined once a row is determined.
4. The row t of T is obtain by solving the following equation $tP = t$.

Example: Given the matrix $P = \begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix}$ obtain the fixed point t.

Solution

Let $t = (t_1, t_2)$, where t_1, t_2 is a probability vector and the sum is one i.e $t_1 + t_2 = 1$

$$\Rightarrow t_2 = 1 - t_1$$

Thus,

$$t = (t_1, 1 - t_1)$$

Therefore,

$$(t_1, 1 - t_1)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (t_1, 1 - t_1)$$

$$\Rightarrow \left(\frac{1}{2}(1 - t_1), t_1 + \frac{1}{2}(1 - t_1)\right) = (t_1, 1 - t_1)$$

We now set up the equation

(i) $t_1 = \frac{1}{2}(1 - t_1)$

(ii) $1 - t_1 = t_1 + \frac{1}{2}(1 - t_1)$

From (ii)

$$1 - t_1 = t_1 + \frac{1}{2}(1 - t_1)$$

$$1 - 2t_1 = \frac{1}{2}(1 - t_1)$$

$$2 - 4t_1 = 1 - t_1$$

$$t_1 = {}^{1}\!/_{3}$$

Recall that $t_1 + t_2 = 1$

$$t_2 = 1 - t_1 = 1 - {}^{1}\!/_{3} = {}^{2}\!/_{3}$$

Hence $t = \left({}^{1}\!/_{3}, {}^{2}\!/_{3}\right)$

$$T = \begin{pmatrix} {}^{1}\!/_{3} & {}^{2}\!/_{3} \\ {}^{1}\!/_{3} & {}^{2}\!/_{3} \end{pmatrix}$$

$$\lim_{n \to \infty} P^n = T = \begin{pmatrix} {}^{1}\!/_{3} & {}^{2}\!/_{3} \\ {}^{1}\!/_{3} & {}^{2}\!/_{3} \end{pmatrix}$$

CHAPMAN KOLMOGOROV EQUATION

Suppose we have a Markov Chain with r-steps let $P_{ij}^{(n)}$ be the nth-step transition probability matrix, then for any number of step C;

$$P_{ij}^{(n)} = \sum_{k=1}^{r} P_{ik}^{c} P_{kj}^{(n-c)}$$

Proof: Consider the process after successive steps

Case I: After one step, the process must be somewhere

i.e

$$P_{ij}^{(n)} = P_{i1}^{(1)} P_{1j}^{(n-1)} + P_{i2}^{(1)} P_{2j}^{(n-1)} + \cdots + P_{ir}^{(1)} P_{rj}^{(n-1)}$$

Case II: After 2 steps

$$P_{ij}^{(n)} = P_{i1}^{(2)} P_{1j}^{(n-2)} + P_{i2}^{(2)} P_{2j}^{(n-2)} + \cdots + P_{ir}^{(2)} P_{rj}^{(n-2)}$$

Case III: After C steps

$$P_{ij}^{(n)} = P_{i1}^{(c)} P_{1j}^{(n-c)} + P_{i2}^{(c)} P_{2j}^{(n-c)} + \cdots + P_{ir}^{(c)} P_{rj}^{(n-c)}$$

In general

$$P_{ij}^{(n)} = \sum_{k=1}^{r} P_{ik}^{(c)} P_{kj}^{(n-c)}$$

ABSOLUTE PROBABILITY DISTRIBUTION

The absolute Probability distribution $\left\{P_j^{(n)}, j = 1 \dots r\right\}$, is the probability of finding the process in any of the r-steps. Indeed, $P_{ij}^{(n)}$ is an unconditional probability distribution.

e.g Given a two-state M.C with a one-step transition probability matrix $\underline{P}$ given by

$$\underline{P} = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}\ ,\ \ 0 \le a, b \le 1, \qquad |1-a-b| \le 1$$

Then the n-step transition matrix is given by

$$P_{ij}^{(n)} = \begin{bmatrix} \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b} & , & \frac{a}{a+b} - \frac{a(1-a-b)^n}{a+b} \\ \frac{b}{a+b} - \frac{b(1-a-b)^n}{a+b} & , & \frac{a}{a+b} + \frac{b(1-a-b)^n}{a+b} \end{bmatrix}$$

Proof: We are required to show that

$$P_{00}^{(n)} = \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b}$$

Observe that:

(i) $P_{00}^{(n)} = 1 - a$

(ii) $P_{00}^{(n)} = \sum_{i=0} P_{i0}^{(1)} P_{0i}^{(n-1)}$

If $i = 1$

$$\Rightarrow P_{00}^{(1)} P_{00}^{(n-1)} + P_{10}^{(1)} P_{01}^{(n-1)}$$

$$= (1-a) P_{00}^{(n-1)} + b P_{01}^{(n-1)}$$

$$= (1-a) P_{00}^{(n-1)} + b P_{01}^{(n-1)}$$

NOTICE THAT:

$$P_{00}^{(n-1)} + P_{01}^{(n-1)} = 1$$

$$P_{01}^{(n-1)} = 1 - P_{00}^{(n-1)}$$

$$\Rightarrow (1-a) P_{00}^{(n-1)} + b\left(1 - P_{00}^{(n-1)}\right)$$

$$\Rightarrow P_{00}^{(n-1)} - a P_{00}^{(n-1)} + b - b P_{00}^{(n-1)}$$

Thus we can write

$$P_{00}^{(1)} = 1 - a$$

$$P_{00}^{(2)} = b + (1-a-b)(1-a)$$

$$P_{00}^{(3)} = b + (1-a-b) P_{00}^{(2)} = b + b(1-a-b) + (1-a-b)^2 (1-a)$$

In general,

$$P_{00}^{(n)} = b + b(1-a-b) + b(1-a-b)^2 + \cdots + (1-a-b)^{n-1}(1-a)$$

$$= b\sum_{r=0}^{n=\cdots}(1-a-b)^r + (1-a)(1-a-b)^{n-1}$$

$$= b\left[\frac{1-(1-a-b)^{n-1}}{1-(1-a-b)}\right] + (1-a(1-a-b))^{n-1}$$

$$= b\left[\frac{1-(1-a-b)^{n-1}}{a+b}\right] + (1-a)(1-a-b)^{n-1}$$

$$= \frac{b}{a+b} - \frac{b(1-a-b)^{n-1}}{a+b} + \frac{(1-a)(1-a-b)^{n-1}(a+b)}{a+b}$$

$$= \frac{b}{a+b} + \frac{(1-a-b)^{n-1}}{a+b}.a(1-a-b)$$

$$= \frac{b}{a+b} + \frac{(1-a-b)^{n-1}}{a+b}$$

We note that having obtained $P_{00}^{(n)}$ we can obtain $P_{01}^{(n)}$ by subtraction i.e $P_{00}^{(n)} + P_{01}^{(n)} = 1$

$$\Rightarrow P_{01}^{(n)} = 1 - P_{00}^{(n)}$$

Theorem: In the result above and for $|1-a-b| \leq 1$

$$\underset{n\to\infty}{Lim} P^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

e.g suppose we have two political parties A and B and an electorate vote for either of these. To study the voting habit or support pattern, a sample of electorates is studied over two time periods.

The following transition probability matrix was obtained

$$P = \begin{array}{c} A \\ B \end{array}\begin{pmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{pmatrix}$$

(1) Find the pattern of support after 2 times period

(2) After very many elections, obtain the constant matrix which describes the election process for the country

Solution:

Recall

$$P_n = \begin{bmatrix} \frac{b}{a+b} + \frac{a(1-a-b)^n}{a+b} & , & \frac{a}{a+b} - \frac{a(1-a-b)^n}{a+b} \\ \frac{b}{a+b} - \frac{b(1-a-b)^n}{a+b} & , & \frac{a}{a+b} + \frac{b(1-a-b)^n}{a+b} \end{bmatrix}$$

We are given that $a = 0.3, b = 0.4$

$$P_2 = \begin{bmatrix} \frac{0.4}{0.7} + \frac{0.3(1-0.7)^2}{0.7} & , & \frac{0.3}{0.7} - \frac{0.3(1-0.7)^2}{0.7} \\ \frac{0.4}{0.7}\frac{0.4(1-0.7)^2}{0.7} & , & \frac{0.3}{0.7} + \frac{0.4(1-0.7)^2}{0.7} \end{bmatrix}$$

$$= \begin{bmatrix} 0.61 & 0.39 \\ 0.52 & 0.48 \end{bmatrix}$$

(b)

$$\underset{n \to \infty}{Lim} P_n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix} = \begin{pmatrix} \frac{0.4}{0.7} & \frac{0.3}{0.7} \\ \frac{0.4}{0.7} & \frac{0.3}{0.7} \end{pmatrix}$$

CLOSURE AND CLOSED SETS

We normally say that ε_i can be reached from ε_j if there exist some ≥ 0 such that $P_{ji} > 0$ (i.e if there is a positive probability of reaching ε_i from ε_i including the case $\varepsilon_i = \varepsilon_j$)

e.g In an unrestricted Random Walk each state can be reached from every other state, but from an absorbing barrier, no other state can be reached.

Definition: A set C of a state is closed if no state outside C can be reached from any state ε_j in C i.e ($\varepsilon_j \in C$). for any ordinary set C of state, the smallest closed set contains C is called chasine of C.

A single state ε_i forming a closed set will called an absorbing states.

A state j is recurrent if $P_j(N_j < \infty) = 0$

VISITS TO FIXED STATE

Define X to be Markov Chain with the state space E and transition probability matrix P.

Also, let N_j be the total number of visits to state j by the process X.

If $N_j < \infty$ then X eventually leaving state j never to return to it again that is there is an integer n such that $X_n = j$ and $X_m \neq j$ for $m > n$.

On the other hand, if $N_j = \infty$ then X visits again and again (and there is no n for which $X_m \neq j$ for all $m > n$.

Let t_1, t_2 ... be the successive indices $n \geq 1$ for which $X_n = j$ as long as there exists such n. (If there is no such m then $t_1 = t_2 = \cdots = \infty$.

If j appears only a finite number of times, let $t_1, t_2 \ldots t_m$ be the successive n for which $X_n = j$ $t_{m+1} = t_{m+2} = \cdots = \infty$ for any event $\in N$, $t_m \leq n$ if appears in $\{x_1 \ldots x_n\}$ at least m times.

Thus t_m is a stopping time and the Markov property holds at t_m.

$$P_i\{t_{m-1} - t_m = k | t_1 \ldots t_m\} = \begin{cases} 0 \quad if\ t_m = \infty \\ \\ P_j(t_i < \infty)\ if\ t_m < \infty \end{cases}$$

Computation of $P_i(t_i = k)$

Let $F_k(i,j) = P_i(t_i = k)\ i \in E, k = 1,2, \ldots.$

(i is the probability that you are leaving and at the kth step you get to j)

For $k = 1$

$$F_k(i,j) = P_i(t_1 = 1) = P_i(X_i = j) = P(i,j)$$

For $k \geq 2$

$$F_k(i,j) = P_i(X_1 \neq j, X_2 \neq j, \ldots X_{k-1} \neq j, X_k = j)$$

$$\sum_{\substack{b \in E \\ b \neq j}} P_i(X_1 = b, X_2 \neq j \ldots X_{k-1} \neq j, X_k = j)$$

$$\sum_{b \in E-[j]} P_i(X_1 = b, X_2 \neq j \ldots X_{k-1} \neq j, X_k = j)$$

$$\sum_{b \in E[j]} P_i(X_1 = b) P(X_2 \neq j \ldots X_{k-1} \neq j, X_k = j)$$

$$F_k(i,j) = \begin{cases} P(i,j)\ if\ k = 1 \\ \sum_{b \in E-[j]} P(i,b) F_{k-1}(b_j) \end{cases}$$

Which is a recursive equation when $k > 1$

Theorem: The Arcus sinus law for the latest visit.

The probability that the process up to time 2n last time is in state E_0 to time 2s, is expressed as β2s,2n = $U_{2s} \cdot U_{2n-2s}$

Let $U_{2n} = P\{Z_{2n} = 0\}$

The probability β2s,2n of the given distribution at the point 2s, where $0 \leq \delta \leq n,$ is best described as discrete Arcus sinus distribution of order n. this name can be justified if $\alpha\ and\ \tau$ are given numbers, where $0 < \alpha < \tau < 1$

$$P\{last\, visit\, to\, E_0\ lies\ between\ 2\alpha n\ \ and\ \ 2\tau n\} = \sum_{\alpha n \leq s \leq \tau n} U_{2s} . U_{2n-2s}$$

$$\approx \sum_{\alpha \le \frac{s}{n} \le \tau} \frac{1}{n} \cdot \frac{1}{\pi \sqrt{\frac{s}{n}\left(1-\frac{s}{n}\right)}} \approx \int_{\alpha}^{\tau} \frac{1}{\pi\sqrt{x(1-x)}}\, dx$$

$$= \frac{2}{\pi} Ar\sin\sqrt{\tau} - \frac{2}{\pi} Ar\sin\sqrt{\alpha}$$

Let us assume that the sum as mean sum of integral of Arcsin.

This implies that

$$P(last\, visit\, of\ E\, before\, 2nx) \approx \frac{2}{\pi} Arc\sin\sqrt{x}, \qquad for\ \ x \in \left]0,1\right[$$

It can be explained that for every player A and B, there exist a large probability that one is almost at all the time in the winning side and the other one is almost all the time on the losing side.

Example

Consider a Markov Chain defined by the random variable X with state space $E = \{1,2,3\}$ with transition probability matrix given by

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/6 & 1/3 \\ 1/3 & 3/5 & 1/15 \end{pmatrix}$$

(a) Does there exist absorbing states
(b) Is it possible to visit state 3 form state 2
(c) What is the chance of revisit state 3

Solution:

(a) for absorbing state $P_{ij} = \begin{cases} 1 & if\ i = j \\ 0 & if\ i \neq j \end{cases}$

There exist an absorbing states in state 1

(b) $P(2,3) = 1/3$ Since it is not equal to zero, it means we can revisit 3 from 2.

(c) $F_k(j,i)$ When $j = 3$

$$f_k = F_k(i,j)$$

$$f_k = Qf_{k-1}$$

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 1/2 & 1/6 & 0 \\ 1/3 & 3/5 & 0 \end{pmatrix}$$

$$f_1 = \begin{bmatrix} 0 \\ 1/3 \\ 1/15 \end{bmatrix}$$

$$f_2 = Qf_1 = F_3(i,j)$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ {}^{1}/_{2} & {}^{1}/_{6} & 0 \\ {}^{1}/_{3} & {}^{3}/_{5} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{1}/_{3} \\ {}^{1}/_{15} \end{bmatrix} = \begin{bmatrix} 0 \\ {}^{1}/_{6} \cdot {}^{1}/_{3} \\ {}^{3}/_{5} \cdot {}^{1}/_{3} \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ {}^{1}/_{8} \\ {}^{1}/_{5} \end{bmatrix}$$

$$f_3 = Qf_2 = F_3(i,j) = \begin{bmatrix} 1 & 0 & 0 \\ {}^{1}/_{2} & {}^{1}/_{6} & 0 \\ {}^{1}/_{3} & {}^{3}/_{5} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ {}^{1}/_{6} \cdot {}^{1}/_{3} \\ {}^{3}/_{5} \cdot {}^{1}/_{3} \end{bmatrix} = \begin{bmatrix} 0 \\ {}^{1}/_{6} \cdot {}^{1}/_{18} \\ {}^{3}/_{5} \cdot {}^{1}/_{18} \end{bmatrix}$$

$$F_k(1,3) = 0$$

$$F_k(2,3) = \begin{cases} {}^{1}/_{3} & k = 1 \\ \left(\frac{1}{6}\right)\left(\frac{1}{3}\right) & k = 2 \\ \left(\frac{1}{6}\right)^2 \left(\frac{1}{3}\right) & k = 3 \end{cases}$$

$$= \left(\frac{1}{3}\right)\left(\frac{1}{6}\right)^{k-1}$$

Probability of ever visiting 3 from 2 is given by

$$\sum_{k=1}^{\infty} F_k(2,3) = \sum_{k=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{6}\right)^{k-1} = \frac{1}{3}\sum_{k=1}^{\infty}\left(\frac{1}{6}\right)^{k-1} = \frac{1}{3}\left(\frac{1}{1-\frac{1}{6}}\right) = \frac{2}{5}$$

$$F_k(3,3) = \begin{cases} \frac{1}{15} & K = 1 \\ \left(\frac{3}{5}\right)\left(\frac{1}{3}\right) & K = 2 \\ \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right) & K = 3 \\ \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)^2 & K = 4 \end{cases}$$

$$= \sum_{k=2}^{\infty} \left(\frac{3}{5}\right)\left(\frac{1}{3}\right)\left(\frac{1}{6}\right)^{k-2}$$

$F_4\ (3,3) = Qf_3$

$$= \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & \frac{1}{6} & 0 \\ \frac{1}{3} & \frac{3}{5} & 0 \end{bmatrix} \begin{bmatrix} 0 & & \\ \left(\frac{1}{6}\right)^2 & \frac{1}{3} & \\ \left(\frac{1}{6}\right) & \left(\frac{1}{3}\right) & \left(\frac{3}{5}\right) \end{bmatrix} = \begin{bmatrix} 0 & & \\ \left(\frac{1}{6}\right)^3 & \frac{1}{3} & \\ \left(\frac{1}{6}\right)^2 & \left(\frac{1}{3}\right) & \left(\frac{3}{5}\right) \end{bmatrix}$$

$$= \sum_{k=2}^{\infty} \left(\frac{1}{6}\right)^{k-2}$$

$$= \frac{1}{5} \sum_{k=2}^{\infty} \left(\frac{1}{6}\right)^{k-2}$$

$$= \frac{1}{5} \frac{1}{1 - \frac{1}{6}} = \frac{1}{5} \frac{6}{5} = \frac{6}{25}$$

For each pair (i, j) we define

$$F\ (i,j)\ =\ Pi\ (ti\ <\ N)\ =\ \sum_{k=1}^{N} Fk(i,j)$$

Thus $F\ (i, j)$ represent the proof that starting from i the Markov chain X ever visit j.

$$F(i,j) =\ P\ (i,j) + \sum\ P\ (i,b)\ \ F\ (b,j) i \in E\ \ b \in E - [j]$$

Consider how N_j as the no of visit to state j.

If $Nj\ =\ m$ iff $T_i < \infty, T_2 < \infty\ \ \dots\dots . T_m < \infty, T_{m+1} < \infty$

And $\{T_2 - T_1 < \infty\} = \cdots\ \{Tm - Tm + < \infty\}$ are independent and their associates probability starting on i are $F(i,j), F(j,j) \dots\ F(j,j)(i - F(j,j))$

$$\therefore P_i\ (N_j = m)\ =\ F\ (i,j).(F(j,j))^{m-1}\ (i - F(j,j) P_i\ (N_j = m)\ = (F(j,j))^{m-1}\ (i - F(j,j)$$

$$P_i\left(N_j = \infty\right) = \begin{cases} 1 & if\ (j,j) < 1\ from\ 8 \\ & 0\ if\ F(j,j) = 1 \end{cases}$$

Probability that $N_j = \begin{cases} 1 \\ 0 \end{cases}$

Thus if $F(j,j)\ <\ 1$ then the state j is **TRANSIENT** and if $F(j,j)\ =\ 1$, then, the state j is **RECURRENT**

If $F\ (j,j)\ =\ 1$, then $N_j\ =\ \infty$ so that $E_j\ (N_j)\ =\ \infty$ (recurrent property)

But if $F(j,j)\ <\ 1$, N_j has a geometric distribution with probability of success $P\ = 1 - F(j,j)$

Therefore

$$E_j\ (Nj)\ = \frac{1}{P} = \frac{1}{1 - F(j,j)}$$

Definitions

(1) A state j is recurrent if $P_j\ (N_j\ <\ \infty)\ =\ 0 \Rightarrow\ F\ (j,j)\ =\ 1$

(2) A recurrent state is called null $if\ E_j\ \{T\}\ =\ \infty$, otherwise it is not-null.
(3) A recurrent state is said to be periodic with probability δ, if

$$P_j\ \{T\ =\ n\sigma\ for\ some\ n \geq 1\}\ =\ 1$$

Otherwise, it is aperiodic.

Conversely, a state is transient if $P_j\left(N_j\ <\ \infty\right) = 1$

and

$$F\ (j,j) < 1$$
$$E_j,\left(N_j\right) = \frac{1}{1 - F(j,j).}$$

A set of states is said to be closed if no state outside it can be reached from any state in it.

A state forming a closed set by itself is called an absorbing state $[P(j,j)\ = 1]$

A closed set is irreducible if no proper subset of it is closed.

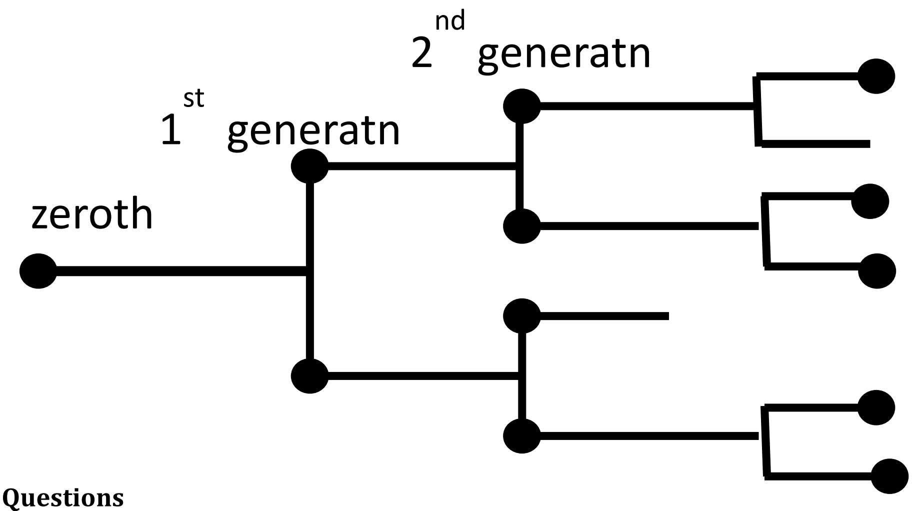
A Markov chain is called irreducible if its only closed set is the set of all states

BRANCHING PROCESSES

This is concerned with the multiplication pattern of population, propagation of infectious disease or epidemics on the possible continuation of a family name. We consider this within the simplest context assuming starting as an individual or a single man starting a family name.

Consider a population of individuals that reproduce in discrete generations at constant intervals. For simplicity assume that only one individual exists at the initial (or zeroth) generation and no immigration of individuals allowed into the population.

Assume further that at each stage each individual has probability Pj of giving rise to new individuals for the next generation. Note j=0 implies the death of individual and $j = 1$ continued existence of individual and $j > 1$ implies the appearance of new individuals.



Typical Questions

1. What is the distribution of the total size after n generations?
2. What is the probability that the process become extinct?

$$N_0, N_1, \dots, Nk \qquad \exists n : j > n \qquad N_j <\ Nn$$

$$\lim_{\substack{k \to 0 \\ n \to \infty}} P(N_n = k) = 1$$

The above implies that there is extinction

GENERAL MODEL

Let P_j be the probability of the single individual $j = 0,1,2, \ldots$ ....having j in the new generation. The Probability Generating Function (PGF) $P(x)$ is given by

$$P(t) = E(t^x) = \sum_{j=0}^{N} P_j t^j$$

Now the second generation consists of the descendants of the generation where each of the j descendants produces j offspring or infectious independently of every other individual. The random variable X_2 for the size of the second (2nd) generation is thus the sum of X_1 mutually independent random variables each having the probability generating function $P(t)$. Thus the random variable X_2 therefore has a compound distribution and its PGF $P2(t) = P\{P(t)\}$

similarly.

$$P_3 = P\{P_2(t)\}$$

In general, the PGF

$$P_n(t) = P\{P_n - 1(t)\}, \qquad n > 1$$

So that for

$$n = 0, \qquad P_0(t) = t$$

$$n = 1\, define\, P_1(t) = Pt$$

$$\forall n$$

$$P_n(t) = P\{P_{n-1}(t)\}$$

$$= P_{n-1}(P(t))$$

If we started with m individual

$$P_m n(t) = (P_n(t))^m$$

PROBABILITY OF EXTINCTION

The probability of extinction is the probability that the system dies out sometime. Consider a process starting with a single individual. Let P_j be the probability that this individual gives rise to j individuals in the next generation.

Suppose the random variable X_1 denotes the population size at the 1st generation. This has the following PDF

j	$0 \quad 1 \quad 2$ - - -	J
Probability	$P_0 \quad P_1 \quad P_2$ - - - $P \quad P^2 \quad P^3$ - - -	P_j P^{j+1}

$$P_1\,(t) = \; P(t) = \sum_{j=o}^{\infty} P_j\; t$$

$$= \sum_{j=o}^{\infty} P^{j+1}\, t^j \; = \; P \sum_{j=o}^{\infty} (Pt)^j = P\left(\frac{1}{1-Pt}\right) \quad if\ P < 1$$

$$= \frac{1}{2-t} \qquad if\ P = \frac{1}{2}$$

$$P_1(t) \; = \frac{1}{2-t.}$$

$$P_2(t) = P\,\{P(t)\} = P\left(\frac{1}{2-t}\right)$$

$$= \frac{1}{2-\frac{1}{2-t}} = \frac{2-t}{3-2t}$$

$$P_3\,(t) = P\left(P_2\,(t)\right)$$

$$= \frac{1}{2-\frac{2-t}{3-2t}} = \frac{3-2t}{4-3t}$$

In general,

$$P_n(t) \;\; = \;\; P\,(P_{n-1}\,(t))$$

$$= \frac{n-(n-1)t}{(n+1)-nt}$$

Therefore, the probability of extinction at the nth generation is obtained as

$$P_n(t = 0)$$

$$P_n(0) = \; \frac{n}{n+1}$$

$$\lim_{n \to \infty} P_n(0) \; = \; 1$$

THE POISSON PROCESS

A stochastic process $\{N(t), t \geq 0\}$ is said to be a counting process if $N(t)$ respresents the total number of 'events' that have occurred up to time t. Hence, a counting process $N(t)$ must satisfy :

1. $N(t) \geq 0$
2. $N(t)$ is integer valued.
3. If $s < t$, then $N(s) \leq \; N(t)$
4. For $s < t$, $N(t) - N(s)$ equals the number of events that have occurred in the interval $(s, t]$

A counting process is said to possess independent increments if the number of events that occur in disjoint time intervals are independent. For example, this means that the number of events that have occurred by time t $[that\ is\ N(t)]$ must be independent of the number of events occurring between time t and $t+s$ $[that\ is\ N(t+s)-N(t)]$.

A counting process is said to possess stationary increments if the distribution of the number of events occur in any interval of time depends only on the length of the time interval. In other words, the process has stationary increments if the number of events in the interval $(t_1+s, t_2+s]$ (that is, $N(t_2+s-N(t_1+s))$ has the same distribution as the number of events in the interval $(t_1, t_2]$ (that is $N(t_2)-N(t_1)$) for all $t_1 < t_2$, and $s > 0$. One of the most important counting process is the Poisson process which is defined as follows:

Definition 1(a).

The Poisson process $\{N(t), t \geq 0\}$is said to be a Poisson process having rate λ , $\lambda > 0$ if :

I. $N(0) = 0$
II. The process has independent increments
III. The number of events in any interval of length t is Poisson distributed with mean λt. i.e for all $s, t \geq 0$;

$$P\{N(t+s)-N(s)=n\} = e^{\lambda t}\frac{(\lambda t)^n}{n!} \quad ; n = 0,1, \ldots.$$

NOTE: from (III), it means Poisson process has stationary increments and also that $E[N(t)] = \lambda t$. Which explains why λ is called the rate of the process.

Definition 1(b).

The function f is said to be $0(h)$ if

$$\lim_{h\to 0}\frac{f(h)}{h} = 0$$

Definition 1(c).

By implication of 1(b) above we can define a Poisson process alternatively thus :
The counting process $\{N(t), t \geq 0\}$ is said to be a Poisson process which rate λ , $\lambda > 0$,if

I. $N(0) = 0$
II. The process has stationary and independent increments
III. $P\{N(h) = 1\} = \lambda h + 0(h)$
IV. $P\{N(h) \geq 2\} = 0(h)$

PROOF: DEFINITION 1(A) = DEFINITION 1(C)

Let $P_n(t) = P\{N(t) = n\}$

We derive a differential equation $P_0(t)$ in the following names

$$\begin{aligned} P_0(t+h) &= P\{N(t+h) = 0\} \\ &= P\{N(t) = 0, N(t+h) - N(t) = 0\} \\ &= P\{N(t) = 0\}\, P\{N(t+h) - N(t) = 0\} \\ &= P_0(t)[1 - \lambda h + 0(h)] \end{aligned}$$

Where the final two equations follows from assumption (ii) and the fact that (iii) and (iv) imply that $P\{N(h) = 0\} = 1 - \lambda h + 0(h)$. Hence

$$\frac{P_0(t+h)-P_0(t)}{h} = -\lambda P_0(t) + \frac{0(h)}{h}$$

Letting $h \to 0$ yields

$$P_0'(t) = -P_0'(t)P_0(t)$$

Or

$$\frac{P_0'(t)}{P_0(t)} = -\lambda$$

Which implies, by integration:

$$\log P_0\,(t) = -\lambda t + c$$

Or

$$P_0(t) = Ke^{-\lambda t}$$

Since $P_0 = P\{N(0) = 0\} = 1$, we arrive at

$$P_0(t) = e^{-\lambda t}$$

Similarly, for $n \geq 1$

$$P_n(t+h) = P\{N(t+h) = n\} = P\{N(t) = n, N(t+h) - N(t) = 0\} + P\{N(t) = n-1, N(t+h) - N(t) = 1\} + P\{N(t+h) = n, N(t+h) - N(t) \geq 2\}$$

However by (iv), the last term in the above is $0(h)$; hence , by using (ii), we obtain

$$P_n(t+h) = P_n(t)P_0(h) + P_{n-1}(t)P_1(h) + 0(h)$$

$$= (1-\lambda h)P_n(t) + \lambda h P_{n-1}(t) + 0(h)$$

Thus

$$\frac{P_n(t+h)-P_n(t)}{n} = -\lambda P_n(t) + \lambda P_{n-1}(t) + 0(h)$$

Letting $h \to 0$

$$P_n'(t) = -\lambda P_n(t) + \lambda P_{n-1}(t)$$

Or equivalently

$$e^{\lambda t}[P_n'(t) + \lambda P_n(t)] = \lambda e^{\lambda t} P_{n-1}(t)$$

$$\frac{d[e^{\lambda t}P_n(t)]}{dt} = \lambda e^{\lambda t} P_{n-1}(t)$$

Now by (i), we have when $n = 1$

$$\frac{d[e^{\lambda t}P_1(t)]}{dt} = \lambda$$

or

$$P_1(t) = (\lambda t + c)e^{-\lambda t}$$

which since $P_1(0) = 0,$ yields

$$P_1(t) = \lambda t e^{-\lambda t}$$

Definition 1 (d):- Given the sequence of independent events, each of them indicating the time when they occur, we assume the following:

(1) The probability of an event occurs in a time interval between 0 and +∞ does not depend on the position or location of the time interval but only on the interval length.

(2) The probability that there exist at every time interval t at least one event is equal to $\lambda t + t\delta(t)$, where $\lambda > 0$ is a given positive constant.

(3) The probability that there exist at least two event in any time interval of length t is $t\delta(t)$. It follows that,

(4) The probability that there exist no event within a time interval of length t is $1-\lambda t + t\delta(t)$

(5) The probability that there exist exactly one event within a time interval of length t is $\lambda t + t\delta(t)$.

The notation $\delta(t)$ denotes some unspecified functions, which approaches zero as the time (t) tends to zero.

$$\therefore \; P_s(t) = P\{X(t)=s\}, \qquad for \; s \in N_0$$

Then X(t) is a Poisson distributed random variable of parameter λt. The process $\{X(t) \setminus t \in (0+\infty)\}$ is called a Poisson Process, and the parameter λ is called the intensity of the Poisson Process.

The following results are satisfied by Poisson processes:

1. If t = 0, (i.e. X(0) = 0), then

$$P_s = \begin{cases} 1, & for \;\; s = 0 \\ 0, & for \;\; s \in N \end{cases}$$

2. If t > 0, the $P_s(t)$ is a differentiable function, and

$$P_s^1(t) = \begin{cases} \lambda\{P_{s-1}(t) - P_s(t)\}, & for \; s \in N \quad and \; t > 0 \\ -\lambda P_0(t), & for \;\; s = 0 \;\; and \;\; t > 0 \end{cases}$$

When, we solve these differential equations, we obtain

$$P_s(t) = \frac{(\lambda t)^s \, e^{-\lambda t}}{s!}, \quad for \; s \in N_0$$

Proving that X(t) is Poisson distributed with parameter λt.

Given a Poisson Process X(t) as expressed above, then X(v+t) –X(v) has the same distribution as X(t), thus

$$P\{X(v+t) - X(v)\} \; = \; \frac{(\lambda t)^s \, e^{-\lambda t}}{s!}, \quad for \; s \in N_0$$

If $0 \le t_1 \le t_2 \le t_3 \le t_4$, then the two random variables $X(t_4)$ - $X(t_3)$ and $X(t_2)$ - $X(t_1)$ are independent. The Poisson Processes are said to possess an independent and stationary growth.

The mean value function of a Poisson Process is expressed as

$$\min(t) = E\{X(t)\} = \lambda t$$

The auto-covariance [Covariance function] is given by

$$C(v,t) = \operatorname{cov}(X(v), X(t))$$

$$= \lambda \min\{v,t\}$$

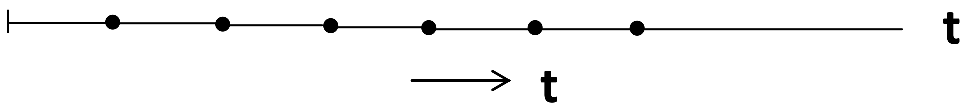
The auto correlation is given by

$$R(v,t) = E\{X(v), X(t)\}$$

$$= \lambda \min(v,t) + \lambda^2 vt$$

DEFINITION

A point process is a realization of a number of occurrences in a given interval of time



Poisson process is an example of the point process. Consider events occurring in time on the interval $(0, \infty)$. For $t > 0$, Let $n(t)$ be the stochastic process representing the number of events that has occurred in $(0, t]$ For a small change δt in time i.e

$$\delta y > 0$$

$$n(t + \delta t) - n(t)$$

Assumes only non-negative integer values i.e

$$n(t + \delta t) - n(t) = 0,1,2,3 \ldots \ldots \ldots.$$

Such process which are Poisson process are

1. Number of accidents per week on a busy highway
2. Number of vehicles passing through a toll gate at a specified period of time
3. Number of telephone conversations I a small interval of time
4. The number of letters lost in a big post office per day
5. The number of typographical error per page

EXAMPLES

1. Let X_n be a M.C such that $E = \{a, b, c\}$ having transition matrix P^n defined by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix}$$

Find $P(X_{n+2} = c|X_n = b)$
Solution:

$$P^2(b,c) = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ \frac{2}{3} & 0 & \frac{1}{3} \\ \frac{2}{5} & \frac{2}{5} & 0 \end{bmatrix} = \frac{1}{6}$$

2. Let X denote the number of success in n Bernoulli trials where the probability of a success in any one trial is p.
   Let $P(i,j)$ be the probability transition matrix P where
   $$P(i,j) = P(X_{n+1=j}|X_{n-1} = i)$$
   Determine the probability transition matrix.

   Solution
   $$P(i,j) = P(X_{n=j}|X_{n-1} = i)$$
   $$= \begin{cases} p & if\ j = i+1 \\ q & if\ j = i \\ 0 & otherwise \end{cases}$$

   $$P_{ij} = \begin{bmatrix} q & p & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & q & p & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & q & p & 0 & \dots & 0 & 0 \\ & \vdots & & \dots & \dots & & \vdots & \vdots \\ & \vdots & & & & & & \\ 0 & 0 & 0 & \dots & \dots & & q & p \end{bmatrix}$$

3. Consider this inventory control. A commodity is being stocked to satisfy a continuing demand. The inventory policy is described by specifying two non-negative critical values A and B with $A < B$. The inventory level is checked periodically at fixed points $t_0, t_1, \dots$ …. If the stock on hand at t_n is less than or equal to A, then by immediate procurement the stock is brought to level B. On the other hand, if the stock level is greater than A, then no procurement is undertaken. Can this Stock control be examined using stochastic process? if yes, formulate the control mathematically

   Solution
   t_0 t_1 t_2 $t_3 \dots \dots$ Step 1: Replenishment

   Step 2: No Replenishment

   Let Z_n be sales between t_n and t_{n+1}

   X_n be stock between t_n and t_{n+1}

   $$Z_0 = \begin{cases} 1 \\ 2 \end{cases}$$

   $$X_{n+1} = \begin{cases} X_n - Z_{n+1} & if\ X_n > A, if\ 0 < Z_n < X_n & 1st\ state \\ B - Z_{n+1} & if\ \ X_n < A & 2nd\ state \\ 0 & otherwise & \end{cases}$$

$$X_{n+2} = \begin{cases} X_{n+1} - Z_{n+2} = X_n - Z_{n+1} - Z_{n+0} = X_n - (Z_{n+1} + Z_n) \\ \end{cases}$$

Therefore

$$X_i = -\sum_{j=0}^{1} Z_j$$

$$X_{n+1} = X_n + Z_n$$

X_{n+1} is a r.w and Z_n is a step jump (discrete).
Therefore, X_{n+1} is Markov process and a Markov chain.

Since X_{n+1} is discrete and it undergoes a step jump leading to a Markov process and it is a Markov Chain and it means it undergoes probability transition matrix with dimension 2 by 2.

Recall: $\sum Z_i = -X_i$ implies that the sum of sales leads to decrease in stock.

Now

$$X_i = -\sum_{j=0}^{1} Z_j$$

$$= P(X_{n+1} = j | X_n \dots X_n)$$

$$= P(X_{n+1} = j | X_n)$$

Examples on Poisson Process

Let N be the arrival number which has a Poisson process with rate $\lambda = 8$. Compute

1. $P(N_{2.5} = 17)$
2. $P(N_{2.5} = 17, N_{3.5} = 22, N_{4.3} = 36)$

Solution

1. $P(N_{2.5} = 17)$

Because

$$P(N_t = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

$$\lambda t = \lambda(2.5)$$

$$= 8(2.5)$$

$$= 20$$

Hence

$$P(N_{2.5} = 17) = \frac{e^{-20} 20^{17}}{17!} = 0.07595$$

2. $P(N_{2.5} = 17, N_{3.5} = 22, N_{4.3} = 36)$

Recall

$$P(N_t = n_1, N_{t+s} = n_2, N_{t+s+k} = n_3)$$

$$= P(N_t = n_1).P(N_s = n_2 - n_1).P(\ N_k = n_3 - n_2)$$

BIRTH AND DEATH PROCESSES

A continuous – time M.C with states $0, 1, \ldots$ for which $q_{ij} = 0$ whenever $|i - j| > 1$ is called a birth and death process. Thus a birth and death process is a continuous – time Markov chain with states $0, 1, \ldots$ for which transitions from state i can only go to either state $i - 1$ or state $i + 1$.

The state of the process is usually thought of as representing the size of some population, and when the state increases by 1 we can say that a birth occurs, and when it decreases by 1 we say that a death occurs.

Let $\{X(t) \mid t \in (0,\infty)\}$ be a Stochastic process, which can be found in states E_0, E_1, E_2, …,E_k. The process can one state to the next state, either to the left or to the right.

Let E_s be the starting state, then, if the process moves to the right, it attracts a positive signal E_{s+1} and if it moves to the left, it attracts a negative signal E_{s-1}.

We assume that there exist non-negative constants λ_s *and* μ_s, such that $s \in N$.

Let λ_i and μ_i be given by

$$\lambda_i - q_i, \quad i + 1,$$

$$\mu_i = q_i, \quad i - 1$$

The value $\{ \lambda_i, i \geq 0 \}$ and $\{ dw, i \geq 1\}$ are called the birth rates and the death rates.

Since

$$\sum_j q_{ij} = \mathrm{V}_i$$

we see that

$$\mathrm{V}_i = \lambda_i + \mu_i$$

$$P_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i} = 1 - P_{i,i-1}$$

Hence, we can think of a birth and death process by supposing that whenever there are i people in the system the time until.

The next birth is exponential with rate λ_i and is independent of the time until the next death which is exponential with rate μ_i.

Two Birth and Death Processes

1) The M/M/S queue
   - (i) Arrival time is exponential
   - (ii) Service time is exponential
   - (iii) Numbers of servers

Suppose that customers arrive at an S server service station in accordance with a poisson process having rate λ. That is the time between successive arrivals are independent exponential random variables having mean $1/_{\lambda}$. Each customer, upon arrival , goes directly into service if any of the servers are free, and if not , then the customers joins the queue (that is, waits in line) when a server finishes serving a customer, and the next customer in line, if there are any waiting time, enter the service. The successive service time are assumed to be independent exponential random variables having mean $1/_{\mu}$.

If we let X(t) denotes the number in the system at time t, then $\{\lambda\,(t), t \geq 0\}$ is a birth and death process with

$$\mu_n = \begin{cases} n\mu & 1 \leq n \leq S \\ S\mu & n > S \end{cases}$$

$$\lambda_n = \lambda \qquad n \geq 0$$

ii) A Linear Growth Model with immigration

A model in which

$$\mu_n = n\,\mu \qquad n \geq 1$$

$$\lambda_n = n\,\lambda + \theta \ , n \geq \theta$$

is called a linear growth process with immigration such processes occur naturally in the study of biological reproduction and population growth. Each individual in the population is assumed to give birth exponential rate , in addition , there is an exponential rate of increase θ of the population due to the external source such as immigration. Hence, the total birth rate where there are n persons in the system is $n\lambda + \theta$. Death are assume to occur at an exponential rate μ for each member of the population and hence $\mu_n = n\,\mu$.

A birth and death process is said to be a pure process if $\mu_n = 0$ for all n (that is, if death is impossible).

The simplest example of a pure birth process is the poisson process, which as a constant birth rate $\lambda_n = \lambda$, $n \geq 0$

A second example of pure birth process results from a population in which each member acts independently and gives birth at an exponential rate. If we suppose that no one ever dies, then if $X(t)$ represent the population size at time t, $\{X(t), t \geq 0\}$ is a pure birth process with

$$\lambda_n = n\,\lambda \ , \ n \geq 0$$

The pure birth process is a called Yule process.

Consider a Yule process starting with a single individual at time and let T_i, $i \geq 1$, denote the time between $(i-1)st$ and the ith birth. That is T_i is the time that it takes for the population size to go from i to $i+1$. It easily follows from the definition of a Yule process that the T_i, $i > 1$, is independent and T_i is exponential with rate $i\,\lambda$.

Now

$$P\{ T_i \leq t \} = 1 - e^{-\lambda t}$$

$$P\{T_1 + T_2 \leq t \} = \int_0^t P\{T_1 + T_2 \leq t / T_1 = x \} \lambda e^{-\lambda x} \, dx$$

$$= \int_0^t \left(1 - e^{-2\lambda(t-x)}\right) \lambda e^{-\lambda x} dx$$

$$= (1 - e^{-\lambda t})^2$$

$$P\{T_1 + T_2 + T_3 \leq t \} = \int_0^t P\,(T_1 + T_2 + T_3 \leq t / T_1 + T_2 = x\} \, dx$$

$$= \int_0^t (1 - e^{-3\lambda(t-x)} 2\lambda e^{-\lambda x}(1 - e^{-\lambda x}) \, dx = (1 - e^{-\lambda t})^3$$

And in general, we can show by induction that

$$P\{ T_1 + \cdots . + T_j \leq t \} = (1 - e^{-\lambda t})^j$$

Hence, as $P\{ X(t) \geq j + 1 / X(0) = 1\}$, we see that for a Yule process,

$$P_{ij}\,(t) = (1 - e^{-\lambda t})\, j - 1 - (1 - e^{-\lambda t})\, j$$

$$= e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}, j \geq 1$$

Thus, we see from the above that, starting with a single individual, the population size at t will have a geometric distribution with mean $e^{\lambda t}$, Hence if the population starts with i individual, it follows that its size at t will be the sum of i independent and identically distributed and geometric r.v's, and will thus have a negative binomial distribution that is, for the Yule process,

$$P_{ij}\,(t) = \binom{j-1}{i-1} e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}, \qquad j \geq i \geq 1$$

Another interesting result about the Yule process, starting with a single individual, concerns the conditional distribution of the times of birth given the population time at t. Since the ith birth occurs at time $= T_1 + \cdots + T_j$. Let us complete the condition joint distribution of $S_1, \ldots\ldots, S_n$ given that $X(t) = n + 1$

Reasoning heuristically and treating densities as if they were probabilities yields that for $0 \leq S_1 \leq S_2 \leq \cdots .. \leq S_n \leq t$

$$P\,\{ S_1 = s_1, S_2 = s_2, \ldots., S_n = s_n \,/\, X(t) = n + 1 \}$$

$$= \frac{P\,\{ T_1 = s_1,\ T_2 = s_2 - s_1, \ldots\ldots.,\ T_n = s_n - s_n - 1,\ T_{n+1} > t - s_n \}}{P\{ \lambda(t) = n + 1 \}}$$

$$= \frac{\lambda\, e^{\lambda s_1}\ 2\lambda\, e^{-2\lambda(s_2 - s_1)} \ldots\ldots\ n\,\lambda\, e^{-n\lambda(s_n - s_n - 1)}\, e^{-(n+1)\lambda(t - s_n)}}{P\{X(t) = n + 1\}}$$

$$= C\, e^{-\lambda t(t-S1)} e^{-\lambda(t-S2)} \ldots\ldots\, e^{-\lambda(t-Sn)}$$

Where C is some constants that does not depend on $S_1, \ldots\ldots, S_n$. Hence we see that the conditional density of $S_1, \ldots\ldots, S_n$ given that $X(t) = n + 1$ is given by

$$f(S_1, \ldots\ldots, S_n \,/\, n+1) \;=\; \prod_{i=1}^{n} f(S_i) \quad 0 \le S_1 \;\le\; S_n \;\le\; t,$$

Where f is the density function

$$f \;=\; \begin{cases} \dfrac{\lambda e^{-\lambda\,(\,\mu - x)}}{1 - e^{-\lambda t}} & 0 \;\le\; x \;\le\; t \\ 0 \quad O/W \end{cases}$$

E.g Consider a Yule process with $X(0) \;= 1$ let us complete the expected sum of the ages of the members of the population at time t.

The sum of the ages at time t, call it A(t), can be expressed as

$$A(t) \;=\; a_0 \;+\; t \;+\; \sum_{i=1}^{X(t)-1} (\,t - Si),$$

Where a_0 is the age at $t \;=\; 0$ of the initial individual

To compute $E[A(T)]$ condition on $X(t)$

$$E[\,A(t)\,/\,X(t) \;=\; n \;+\; 1] \;=\; a_0 \;+\; t \;+\; E\left[\sum_{t=1}^{x(t)-1} (t - S1) \;\; |X(t) = n+1\right]$$

$$= \; a_0 \;+\; t \;+\; n \int_0^t (t-x) \frac{\lambda e^{-\lambda(t-x)}}{1 - \; e^{-\lambda t}} \; dx$$

Or

$$E\,[\,A(t)\,/\,X(t)] \;=\; a_0 \;+\; t \;+\; (\,X(t) \;-1) \;\; \frac{1 \;-\; e^{-\lambda t} - \; \lambda t\, e^{-\lambda t}}{\lambda\,(\,1 - e^{-\lambda t}\,)}$$

Taking the expectations and using the fact that X(t) has mean $e^{\lambda t}$ yields

$$E\,[\,A(t)\,] \;=\; a_0 \;+\; t \;+\; \frac{e^{\lambda t} - 1 - \; \lambda t}{\lambda}$$

$$= \; a_0 \;+\; \frac{e^{\lambda t} - \; 1}{\lambda}$$

Alternatively

Recall

$$A(t) \;=\; a_0 \;+\; \int_0^t X(s)\; ds$$

Give,

$$E[A(t)] = a_0 + E\left[\int_0^t X(s)\,ds\right]$$

$$= a_0 + \int_0^t E\,[X(s)]\;ds \qquad since\; X(s) \geq 0$$

$$= a0 + \int_0^t e^{\lambda s}\,ds$$

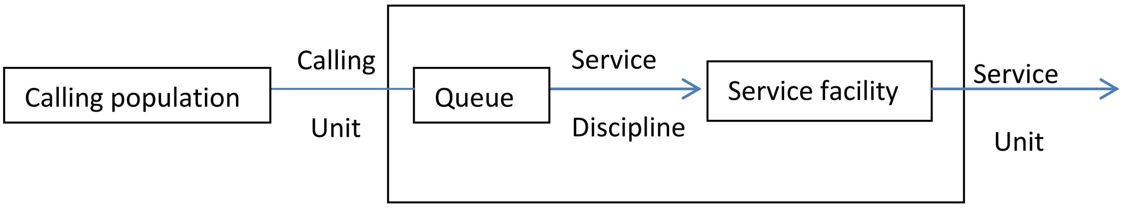
$$= a0 + \frac{e^{\lambda t} - 1}{\lambda}$$

QUEUEING THEORY

When a demand for a service exceeds a supply, lines or queues form.

Queues occur everywhere imaginable, for example, motorist waits for traffic light. Commuters wait for buses. In industry, airplanes wait for runaway, computer, jobs wait computer time and product wait for materials. Investors wait for capitals, workers wait for work orders and pay cheques.

A queuing system contains several basic components



Unit enters the system from an input source or calling population. The members of the calling population arrives the service facility for service. This calling unit produces the demand for service whenever the service facility is not available for service; the calling unit form a queue.

Members of the queue are served according to the prescribed rules of a queue discipline or service discipline.

Service is performed by service mechanism or service facility and finally, service unit leaves the system.

Associated with queuing theory are the following.

(1) **Input process** i.e how unit join the queue.

(a) Regular arrival i.e regularity may be time to time.
(b) Purely – random arrival i.e the inter arrival time are random.
(c) General independence arrival { arrival times are independence} but not necessarily exponentially distributed}
(d) Regular arrival with impunctuality we note that if the variance of regular arrival with impunctuality large, this impunctuality arrival becomes random.
(e) Aggregate arrival.

(2) **Queuing Discipline:** How Unit are Selected for Service.

(a) First come first served or FIFO

(b) First come first last

(c) Priority service e.g when service depends on the urgency of the matter.

(d) Erratic service: In this case there is no order in service or the service is regular.

(3) **Service Mechanism:** Here we consider

(a) A capacity of the system i.e number of servers available.

(b) Length if service is with respect to time

(c) Service availability i.e server is always on seat or server is not always on seat. Server on seat means incomplete availability.

Characteristics of Queues

(1) The distribution of the length of the queue at time t, X(t).
We shall be interested in

1. $E\ (X(t))$

2. $Var\ (X(t))$

(2) The distribution of the waiting time W. Here again we shall obtain

(i) $E(W)$ (ii) $Var\ (W)$. It should be noted that the waiting time involves the time spent in the queue and the time taken to serve the customer.

(3) The distributed of the server busy / idle times.

Queues are normally represent

F_A / Fs/ S where F_A is the arrival time distribution
Fs is the service time distribution
S is the capacity of service

These, associated with every queue are

(i) The arrival Process

(ii) The service process

(iii) The capacity of the system

The following forms the queues shall be considered

(1) M / M / I , when speak of M / M / 1 we mean that the

(i) Arrival time is Exponential

(ii) The distribution of service time is e xponential

(iii) The number of server is 1

M \ G \ I

(i) The arrival time is exponential

(ii) The service time is unspecified

(iii) The number of server is 1

M \ M \ S

(i) Arrival time is exponential

(ii) The service time is exponential

(iii) The number of server is s

G \ M \ I

(i) Arrival time is unspecified
(ii) Service time is exponential
(iii) The number of server is s

G \ G \ I

(i) Arrival time is unspecified
(ii) Service time is unspecified
(iii) The number of server is 1

GENERAL

The queuing process is equivalent to birth and death process. The basic model of queuing process is derived from that of birth and process.
In particular arrival is equivalent to the event of birth while departure is equivalent to the event of death.
Recall: The difference equation for birth and death process is

$$P_n'^{(t)} = -(\lambda_n + \mu_n) P_n^{(t)} + \lambda_{n-1} P_{n-1}^{(t)} + \mu_{n+1} P_{n+1}^{(t)}$$

In the queuing process we consider a stationary process (one that has lasted for a long time and the distributed is independent of time).
We then have,

(i) $P_n^{(t)} = P_n$ (ii) $P_n'^{(t)} = 0$

Thus the generalized model becomes

$$(\lambda_n + \mu_n) P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}, \quad n \geq 0$$

$$= \lambda_n P_n + \mu_n P_n = \lambda_{n-1} P_{n-1} + \mu_{n+1} P_{n+1}, \quad n \geq 0$$

$$\lambda_1 P_1 + 0 = 0 + \mu_1 P_1$$

$$P_1 = \frac{\lambda_0 P_0}{\mu_1}$$

To obtain the distribution of P_n we proceed as follows: when $n = 1$

$$(\lambda_1 + \mu_1) P_1 = \lambda_0 P_0 + \mu_2 P_2$$

$$(\lambda_1 + \mu_1) \frac{\lambda_0 P_0}{\mu_1} P0 = \lambda_0 P_0 + \mu_2 P_2$$

$$P_2 = \frac{\lambda_0 \lambda_1}{\mu_1 \mu_2} P_0$$

When $n = 2$,

$$(\lambda_2 + \mu_2) P_2 = \lambda_1 P_1 + \mu_3 P_3$$

Substitution f_{10}, P_1 and P_2 we have

$$P_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\,\mu_2\,\mu_3}P_0$$

In general ,

$$Pn = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}P_0}{\mu_1\mu_2 \cdots \mu_n}$$

Observed that the unknown is given as a function of another unknown P_0. We need to determine P_0 so as to completely specify the distribution

Since $[P_n, n \geq 0]$ is a distribution,

then

$$P_0 + P_1 + P_2 + \dots.. = 1$$

$$= \left[P_0 + \frac{\lambda_0\,P_0}{\mu_1} + \frac{\lambda_0\lambda_1\,P_0}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2P_0}{\mu_1\mu_2\mu_3} + \dots\dots\dots\dots.\right] = 1$$

$$= P_0\left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \dots\dots\dots..\right] = 1$$

$$P_0 = \frac{I}{L}$$

Where

$$L = [1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \dots\dots\dots.]$$

Therefore the distribution P_n is

$$P_n = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2\mu_3 \cdots \mu_n}.\frac{1}{L} \quad , n \geq 0$$

which has general model for the queuing process.

PARTICULAR CASES

1. Queue of type M/M /I : In this case $\lambda_n = \lambda$ and $\mu_n = \mu$, which are independent of n . we seek the distribution of the function

$$P_n = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2\mu_3 \dots \mu_n} P_0$$

$$= \frac{\lambda\lambda\lambda \dots \lambda}{\mu\mu\mu \dots \mu}$$

$$= \left(\frac{\lambda}{\mu}\right)^n P_0$$

But

$$P_0 = \frac{1}{L_0} \quad L_0 = 1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \cdots.$$

$$= 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \left(\frac{\lambda}{\mu}\right)^3 + \cdots$$

$$= \frac{1}{1 - \frac{\lambda}{\mu}}$$

So that

$$P_0 = \frac{1}{L_0} = 1 - \frac{\lambda}{\mu}$$

and

$$P_n = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) = P^n (1 - P) P = \frac{\lambda}{\mu}$$

and P is the called the traffic intensity.

Let us note that

$$P = \frac{\lambda}{\mu} = \frac{\frac{1}{\mu}}{\frac{1}{\lambda}}$$

$$= \frac{mean\ service\ time}{mean\ rrival\ time}$$

$$\boldsymbol{E[X(t)]}\ \ \boldsymbol{and\ V}\ [(\boldsymbol{X(t)}]$$

$$E[(t)] = \sum_{n=1}^{\infty} nP_n = \sum_{n=1}^{\infty} n\, P^n(1 - P)$$

$$= (1 - P) \sum_{n=1}^{\infty} n\, P^n$$

$$= 1 - P \cdot \frac{P}{(1 - P)^2}$$

$$= \frac{P}{1 - P}$$

EXERCISE

Obtain the variances of the $X(t\ )$ .i .e $var\ [X(t)]$

$$E[X(t)] = \sum_{n=1}^{\infty} n\, Pn = \sum_{n=1}^{\infty} n\, P^n(1 - P)$$

$$= (1 - P)P \sum_{n=1}^{N} n\, P^{n-1}$$

$$= (1-P)\,P\,[\,1 + 2P + 3P^2 + 4P^3 + \ldots$$

$$= (1-P)P[1 + P + P^2 + P^3 + \cdots$$

$$+P + P^2 + P^3 + \cdots$$

$$+P^3 + \cdots$$

$$= (1-P)P\left(\frac{1}{1-P} + \frac{P}{1-P} + \frac{P^2}{1-P} + \cdots\right)$$

$$= (1-P)P\left[\frac{1}{1-P}(1 + P + P^2 + \cdots)\right]$$

$$= (1-P)P\left[\left(\frac{1}{1-P}\right)\left(\frac{1}{1-P}\right)\right]$$

$$= \frac{(1-P)P}{(1-P)^2}$$

=

$$\frac{P}{1-P}$$

$Var\,(X_t) = E[X(t)^2] = \left[E\big(X(t)\big)\right]^2$

$$E\big(X^2(t)\big) = \sum_{n=1}^{\infty} n^2\,(P_n)$$

$$= \sum\nolimits_{n=1}^{\infty} n^2\,P^n(1-P)$$

$$Var(Xt) = \sum_{n=1}^{\infty} n^2 p^n (1-p) - \left[\frac{p}{1-p}\right]^2$$

$$= (1-P)\sum\nolimits_{n=1}^{\infty} n^2 P^n - \left[\frac{P}{1-P}\right]^2$$

DISTRIBUTION OF WAITING TIME

Suppose there n-persons in the system of type M/M/I, we are interested in the waiting time of the $(n+1)$ customers. Clearly, this is the sum of time taken to serve him and the n-person befor him.

Each service time is exponentially distributed so that we are actually interested in distribution of the sum of $(n+1)$ exponentially distributed random variables.

Let X_k be the service time of the kth customer before him in the queue. Thus,

$$Y = X_1 + X_2 + \cdots + X_{n+1}$$

Where Y is the time he wasted in the queue (w) plus the time taken to serve him (v) i.e

$$Y = w + v$$

We note that $X_1, X_2, \ldots, X_{n+1}$ are independently and identically distributed with exponential distribution.

Y then is the sum of $(n+1)$ independently and identically distributed exponential distribution i.e Y has a Gamma distribution $G(n+1, \mu)$

$$P[Y \leq t | N = n] = \int_0^t \frac{\mu(\mu y)^n}{P(n+1)} e^{-\mu y}\, dy$$

i.e the probability that his waiting time equals t given that there are $N = n$ persons before him.

Our interest is in the unconditional distribution $P[Y \leq t]$.

$$P[Y \leq t] = \sum_{n \geq 0} P[Y \leq t | N = n] P[N = n]$$

$$= \sum_{n\geq 0} \int_0^t \frac{\mu(\mu y)^n}{n!} e^{-\mu y} p^n (1-p) dy$$

Recall that

$$= \int_0^t \sum_{n \geq 0} \left(\frac{(\mu p y)^n}{n!}\right) \mu e^{-\mu y} p^n (1-p) dy$$

$$\mu p y - \mu y = \mu y(1-p)$$

$$= \int_0^t \mu(1-p) e^{\mu p y} e^{-\mu y} dy$$

$$= \int_0^t e^{-\mu y(1-p)} \mu(1-p) dy$$

$$= \mu(1-p) \left[\frac{-e^{-\mu y(1-p)}}{\mu(1-p)}\right]_0^t$$

$$= -e^{-\mu y(1-p)} \Big|_0^t$$

$$= 1 - e^{-\mu t(1-p)}$$

Therefore the PDF of Y is given by

$$P(Y) = \mu(1-p) e^{-\mu y(1-p)}$$

Obtain the mean of the distribution

$$E(Y) = E(w + v)$$

Suppose customers wait for $w + v$ units of time, within this time, X customers arrived. The arrival rate is λ. Thus we have

$$\lambda(w + v) = X$$

i.e

$$E(X) = \lambda E(w+v)$$

$$E(w+v) = \frac{E(X)}{\lambda}$$

$$= \frac{p}{(1-p)}\frac{1}{\lambda}$$

$$= \frac{p}{\lambda(1-p)}$$

$$= \frac{\lambda}{\mu}\frac{1}{\lambda(1-p)}$$

$$= \frac{1}{\mu(1-p)}$$

Suppose arrival at a telephone booth follow a prison process with mean of 15 persons per hour. The curves follow a negative exponential distribution with mean 2 minutes. It is the company's policy to install a new telephone booth if on the average customer want for at least 4minutes in the booth. By how much must arrival increase in order to justify a second booth?

Solution

Arrival rate $= 15\ persons/hour$

Mean service time $= 2\ minutes$

$$E(w+v) = \frac{1}{\mu(1-p)}$$

If $\frac{1}{\mu(1-p)} \geq 4\ mins$, a new booth is installed.

i.e

$$\frac{1}{\mu(1-p)} \geq 4$$

$$\Rightarrow \quad \frac{1}{\mu\left(1-\frac{\lambda}{\mu}\right)} \geq 4$$

$$\frac{1}{\mu\left(1-\frac{\lambda}{\mu}\right)} = \frac{1}{\mu\left(\frac{\mu-\lambda}{\mu}\right)} = \frac{1}{\mu-\lambda} \geq 4$$

Mean service is $2\ minutes$ i.e

$$\mu = \frac{1}{2} = 0.5$$

$$\frac{1}{\mu-\lambda} \geq 4$$

$$\frac{1}{\frac{1}{2}-\lambda} \geq 4$$

$$2 - 4\lambda \leq 1$$

So that

$$\lambda \geq \frac{1}{4}$$

i.e

$$\frac{1}{\lambda} = 4\ persons\ per\ minutes$$

$$= 240\ persons\ per\ hour$$

Hence, the arrivals must increase by $225\ persons/hour$ in order to justify the installation of the second booth

$$E(w) = \frac{p}{\mu(1-p)} \quad (show)$$

OTHER TYPES OF QUEUES

1. Queue with linearly dependent service rate. This type of queue has a constant arrival rate: $\lambda_n = \lambda$ Service rate is given by:

$$\mu_n = n\mu$$

The distribution of P_n is given as:

$$P_n = \frac{\lambda_1\lambda_2\lambda_3 \dots \lambda_{n-1}}{\mu_1\mu_2\mu_3 \dots n\mu} P_0$$
$$= \frac{\lambda\lambda\lambda \dots \lambda}{1\mu 2\mu 3\mu \dots n\mu} P_0$$

$$= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{n!} P^n P_0$$

To obtain P_0, we make use of the relation:

$$P_0 + P_1 + P_2 + \cdots + = 1$$

$$P_0 = \frac{1}{S_0}$$

$$S_0 = \left[1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \cdots\right]$$

$$= \left[1 + \left(\frac{\lambda}{\mu}\right) + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \frac{1}{3!}\left(\frac{\lambda}{\mu}\right)^3 + \cdots\right]$$

$$= \left[1 + P + \frac{1}{2!}P^2 + \frac{1}{3!}P^3 + \cdots\right]$$

$$= e^P$$

$$P_0 e^P = 1$$

$$P_0 = e^{-P}$$

So that:

$$P_0 = \frac{1}{n!} P_n e^{-P}$$

2. QUEUES WITH DISCOURAGEMENT

Generally, long queues discourage customers. In this case, the service rate is a constant μ, and the arrival rate is given by:

$$\lambda_n = \frac{\lambda}{n+1}$$

Hence,

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{n-1}}{\mu_1 \mu_2 \dots \mu_n} P_0$$

$$= \frac{\frac{\lambda}{1}\frac{\lambda}{2}\frac{\lambda}{3}\dots\frac{\lambda}{n}}{\mu.\mu.\mu\dots\mu} P_0$$

$$= \frac{1}{n!} P^n P_0$$

As in case (1),

$$P_0 = e^{-P}$$

So that:

$$P_n = \frac{1}{n!} P^n e^{-P}$$

$$= \frac{P^n e^{-P}}{n!}, \quad n \geq 0$$

which is a Poisson distribution.

3. NEGATIVE BINOMIAL DISTRIBUTION

Here,

$$\mu_n = \mu$$

And

$$\lambda_n = \frac{N+n}{N(n+1)}, n \geq 0$$

$$P_n = \frac{\lambda_0 \lambda_1 \lambda_2 \dots \lambda_{n-1}}{\mu_1 \mu_2 \mu_3 \dots \mu_n} P_0$$

$$= \frac{\frac{N}{N}.\frac{N+1}{2N}.\frac{N+2}{3N}.\dots\frac{N+n-1}{nN}}{\mu.\mu.\mu\dots\mu} P_0$$

$$= \frac{N.(N+1).(N+2)\dots(N+n-1)}{N\mu.2N\mu.3N\mu\dots nN\mu} P_0$$

$$= \left(\frac{1}{N\mu}\right)^n . \frac{N(N+1)(N+2) \dots (N+n-1)}{1.2.3 \dots n} P_0$$

$$= \left(\frac{1}{N\mu}\right)^n \frac{1}{n!} N(N+1)(N+2) \dots (N+n-1) P_0$$

$$= \left(\frac{1}{N\mu}\right)^n \frac{1}{n!} (-1)^n - N(-N-1)(-N-2) \dots (-N-n+1) P_0$$

$$= \left(-\frac{1}{N\mu}\right)^n \frac{1}{n!} \frac{(-N)!}{(N-1)!}$$

$$= \binom{-N}{n} \left(-\frac{1}{N\mu}\right)^n P_0$$

It can be shown that:

(SHOW)

$$P_0 = \left(1 - \frac{1}{N\mu}\right)^N$$

And

$$P_n = \left(-\frac{N}{n}\right) \left(-\frac{1}{N\mu}\right)^n \left(1 - \frac{1}{N\mu}\right)^N$$

4. BINOMIAL QUEUES

Here,

$$\lambda_n = \frac{N-n}{N(n+1)}$$

and

$$\mu_n = n$$

Show that:

$$P_n = \binom{N}{n} \left(\frac{1}{1+N\mu}\right) \left(\frac{N\mu}{1+N\mu}\right)^{N-n}$$

Where

$$P_0 = \left(1 + \frac{1}{N\mu}\right)^{-N}$$

QUEUES WITH LIMITED ROOM

Consider a system which has space of N customers. The customers will keep coming for as long as there is space e.g a telephone system. The service rate remains the same i.e

$$\mu_n = \mu \;\; \forall \, n$$

And

$$\lambda_n = \begin{cases} \lambda \; for \; n \leq N \\ 0 \; for \; n > N \end{cases}$$

Show that:

$$P_n = \frac{P^n(1-P)}{1-P^{N+1}}, \qquad n \geq 1$$

Obtain $E(N)$ and the probability when the call lost

(A call is lost when there are no customers)

QUEUES OF TYPE M/M/S

In this case, a customer gets immediate service if the number of customers before is greater than S. If there are S servers and the number of persons is $n \leq S$, then the service rate will be

$$\mu_n = \begin{cases} n\mu \, ; n \leq S \\ S\mu, n > S \end{cases}$$

And

$$\lambda_n = \lambda \quad for\ all\ n$$

CASE I $\quad n \leq S$

$$P_n = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1.\mu_2.\mu_3 \dots \mu_n} P_0$$

$$= \frac{\lambda\lambda\lambda \dots \lambda}{\mu.2\mu.3\mu \dots n\mu} P_0$$

$$= \frac{1}{n!}\left(\frac{\lambda}{\mu}\right)^n P_0$$

$$= \frac{1}{n!} P^n P_0$$

CASE II $\quad n > S$

$$P_n = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1}}{\mu_1\mu_2\mu_3 \dots \mu_s\mu_{s+1}\mu_{s+2} \dots \mu_n} P_0$$

$$= \frac{\lambda\lambda\lambda \dots \lambda_s \dots \lambda_{n-1}}{\mu.2\mu.3\mu \dots s\mu.(s+1)\mu.(s+2)\mu \dots n\mu} P_0$$

$$= \frac{\lambda^n}{s!\, s^{n-s}\mu^n} P_0$$

$$= \frac{s^s}{s!}\left(\frac{\lambda}{s\mu}\right)^n P_0$$

To obtain P_0, we recall that:

$$\sum_{n \geq 0} P_n = 1$$

i.e

$$P_0 + P_1 + P_2 + \cdots + P_s + P_{s+1} + P_{s+2} + \cdots = 1$$

i.e

$$P_0\left[1 + \frac{\lambda}{\mu} + \frac{1}{2!}\left(\frac{\lambda}{\mu}\right)^2 + \cdots + \frac{1}{s!}\left(\frac{\lambda}{\mu}\right)^s + \cdots + \sum_{n\geq s+1}\frac{s^s}{s!}\left(\frac{\lambda}{\mu}\right)^n\right]$$

if we say:

$$P = \frac{\lambda}{s\mu}$$

then

$$P_s = \frac{\lambda}{\mu}$$

$$= P_0\left[1 + P_s + \frac{1}{2!}(P_s)^2 + \cdots + \frac{1}{s!}(P_s)^s + \cdots + \sum_{n\geq s+1}\frac{s^s}{s!}(P_s)^n\right] = 1$$

$$= P_0\left[\sum_{j=1}^{s}\frac{(P_s)^j}{j!} + \frac{(P_s)^j}{s!}\frac{P}{1-P}\right] = 1$$

Note $\sum\frac{S^S}{S!}P^n = \frac{S^S}{S!}\sum P^n = \frac{S^S}{S!}[P^{s+1} + P^{s+2} + \cdots]$

$$= \frac{S^s}{S!}P^s[P + P^2 + P^3 \ldots]$$

$$= \frac{P}{1-P}$$

so that

$$P_0 = \frac{1}{M_0}$$

Where $M_0 = \left[\sum_{j=1}^{S}\frac{(P_S)^j}{j!} + \frac{(P_S)^S}{s!}\frac{P}{1-P}\right]$

That

$$P_n = \frac{S^s}{S!}P^n P_0$$

where $P_0 = \frac{1}{M_0}$

Recall, $M_0 = \left[\sum_{j=0}^{S}\frac{(P_S)^j}{j!} + \frac{(P_S)^S}{s!}\frac{P}{1-P_1}\right]$

Obtain the value of M_0 for $s = 2$

i.e (there are two services)

$$M_0 = \left[\sum_{j=0} \frac{(2P)^j}{j!} + \frac{(2P)^2}{2!}\frac{P}{1-P}\right]$$

$$= 1 + 2P + \frac{(2P)^2}{2!} + \frac{(2P)^2}{2!}\frac{P}{1-P} = 1 + 2P + \frac{4P^2}{2} + \frac{4P^3}{2}\frac{1}{1-P} = \ 1 + 2P + 2P^2 + \ \frac{2P^3}{1-P}$$

$$= \frac{(1-P) + 2P(1-P) + 2P^2(1-P) + 2P^3}{1-P}$$

$$= \frac{1+P}{1-P}$$

Therefore,

$$P_n = \frac{S^s}{S!}P^n\frac{1}{M_0}, for\ n > 0$$

$$= \frac{2^2}{2!}P^n \times \frac{1-P}{1+P}$$

$$= 2P^n\frac{(1-P)}{1+P},\ \ n \geq 2$$

and

$$P^n = \frac{1}{n!}(2P)^n\frac{1-P}{1+P}\ \ , \quad\ \ n < 2$$

DISTRIBUTION OF WAITING TIME IN M|M|S

Let W be the time spent in the queue. We are interested in $P\ [\ W > Y\ ]$.

Lemma: $P\ [\ W\ >\ 0\ ] = P_s(1-P)^{-1}$

Now $P_s\ =\ P\ [\,X = s\,]$

PROOF: $P\ [\ W > 0\ ]\ =\ P\ [\,X\ \geq s\,]$

i.e Probability that there are at least s person in the queue

$$= \sum_{j=s}^{N} P\ [X = j]\ =\ P_s\ \ +\ P_{S+1}\ +\ P_{S+2}\ \ + \cdots\ P_{S+n}$$

$$P_{S+1} =\ \ P[X\ =\ S+1]$$

$$=\ \frac{\lambda_0\lambda_1 \dots \lambda_s}{\mu_1\mu_2 \dots \mu_{s+1}}P_0$$

$$=\ \frac{\lambda_0\lambda_1 \dots \lambda_{s-1}}{\mu_1\mu_2 \dots\ \mu_s}P_0\ \ \frac{\lambda_s}{\mu_{s+1}}$$

$$=\ \ P_S\frac{\lambda_s}{\mu_{s+1}}\ =\ \ P_s\frac{\lambda}{S\mu} =$$

$$P_S\, P$$

Similarly

$$P_{S+2} \;=\; P_S\, P^2$$

Therefore,

$$P\,[\;W > 0\;] \;=\; P_s \;+\; P_S\, P \;+\; P_S\, P^2 \;+\; \ldots..+ P_S\, P^n$$

$$=\; P_S\,[\;1 + \;P \;+\; P^2 \;+\; P^3 \;+ \cdots \ldots. + P^n]$$

$$= P_s \;\left(\frac{1}{1-p}\right) \;=\; P_s(\;1 - P\quad)^{-1}$$

THEOREM: $P\;[\;W \geq y\,] \;=\; P_s(\;1-P\quad)^{-1}\, e\, -^{(\;1-P\quad)s\mu y}$

PROOF: Let V be the number of customers in the queue

A customer wants to for when $X \;=\; s+n\;,\; n > 0$

The time this customer waits is the time taken to serve $(n+1)$ customers.

Let

$$W \;\;= Y_1 \;+\; Y_2 \;\;+\; \ldots\ldots \;+\; Y_{n+1}$$

Since each of the r.vs $Y_1 \;+\; Y_2 \;\;+\; \ldots\ldots \;+\; Y_{n+1}$ is an exponential distribution r.v, we seek the distribution of the sum of $n+1$ exponentially distributed r.vs. Therefore the distribution of W is $Gamma(n+1, s\mu)$

$$P\;\;[\;W \geq y\,\big|\,X = n+s] \;=\; \int \mu_S \frac{(\mu_S x)^n}{n!} e^{-\mu s x}\, dx$$

Our interest is on $P[\;W \geq y\;\;]$

$$P[W \geq y] = \sum_{n\geq 0} P\big[W \geq y\,\big|\,X = n+s\big] P[X = n+s]$$

$$= \sum_{n\geq 0} \int_y^\infty \mu_s \frac{(\mu_s x)^n}{n!} e^{-\mu_s} P_s P^n dx$$

$$= \quad P_s \int_y^\infty \frac{\sum(\mu_s x)^n}{n!} \mu e^{-\mu_s x}\, dx$$

$$= \;\mu_s P_s \int_y^\infty e^{\mu_s x p}\; e^{-\mu_s x} dx$$

$$= \;\mu_s P_s \int_y^\infty e^{-\mu_s x(1-P)} dx$$

$$= \mu_s P_s \frac{1}{\mu_s(1-P)} e^{-\mu_s x(1-P)} \Big|_y^{\infty}$$

$$= P_s(1-P)^{-1} e^{-\mu_s y(1-P)} \qquad , y \geq 0$$

Note that the last lemma is contained in the theorem and for $y = 0$

$$P(W \geq y) = P_s(1-P)^{-1}$$

to obtain the PDF of y ,we differentiate the the above result with respect to y, i.e

$$\frac{\partial}{\partial y}[W \geq y] = \frac{\partial}{\partial y}[1 - P[W \geq y]$$

i.e

$$P(y) = P_s \mu_s e^{-\mu_s y(1-P)}$$

it can be shown that

$$E(W) = \frac{P_s}{\mu_s(1-P)^2}$$

SHOW

PROBLEM 1

A shop keepers has 3 servers .the service time for each server is exponentially distributed with mean 5 minutes .people are in poisson at the rate 30/hr

(a) What is the probability that all server are busy
(b) What is the expected number of customer to be serve
(c) Obtain the expected length of time a customer waits in the system
Suppose customer wait for W unit of time, within this time X is served. The service time(rate) is μ. Thus we have

$$\mu(W) = X$$
$$E\mu(W) = E(X)$$
$$\mu E(W) = E(X)E(W) = E\frac{(X)}{\mu}$$
$$= \left(\frac{P}{1-P}\right) \div \mu$$
$$= \left(\frac{P}{1-P}\right) \times \frac{1}{\mu}$$
$$= \frac{P}{\mu(1-P)}$$

⇨ $E(time\ in\ line) = t(time\ in\ service)\ minus\ \ t(service\ time)$

PROBLEM 2

At a service station ,the rate of service is μ cars/hours and the arrival of cars per service is λ/hours .Assume arrivals are poisson and service time are negative exponential the cost incurred by the service station due to waiting Cars is ₦C 1 per car per hour and the operating and service cost are ₦ μC2 PER hour when the service rate is μ . Determine the service rate that result in the least expected cost.