Q1a (10 marks)

Partial Differential Eq. is a mathematical expression of the relationship between a dependent variable and two or more independent variables together with its partial derivatives. In some cases of PDE (i.e system of PDEs), we may have two or more dependent variables. 3

Four categories of First order PDE

- Linear 1
- Non-linear 1
- Semi-linear 1
- Quasi-linear 1

Three categories of Second order PDE

- Hyperbolic (Wave Eq.) 1
- Parabolic (Heat Eq.) 1
- Eliptic (Laplace Eq.) 1

Q1b (5 marks)

Let $u = x^2 + y^2 + z^2 \qquad v = z^2 - 2xy$ —(2)

$$\phi(x^2+y^2+z^2,\ z^2-2xy) = \phi(u,v) = 0 \quad \text{—(1)}$$

From Eq. (2)

$$\left.\begin{array}{llll} \frac{\partial u}{\partial x} = 2x & \frac{\partial u}{\partial y} = 2y & \frac{\partial u}{\partial z} = 2z & \frac{\partial v}{\partial x} = -2y \\ \frac{\partial v}{\partial y} = -2x & \frac{\partial v}{\partial z} = 2z & \frac{\partial x}{\partial x} = \frac{\partial y}{\partial y} = 1 & \\ & \frac{\partial x}{\partial y}\ \frac{\partial y}{\partial x} = 0 & & \end{array}\right\} \text{—(3)}$$

Starting with the partial diff. of Eq. (1) wrt x

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) +$$

$$\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(2x + 2z\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(-2y + 2z\frac{\partial z}{\partial x}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(2x + 2z\frac{\partial z}{\partial x}\right) = -\frac{\partial \phi}{\partial v}\left(-2y + 2z\frac{\partial z}{\partial x}\right)$$

Hence

$$\frac{\frac{\partial \phi}{\partial u}}{\frac{\partial \phi}{\partial v}} = \frac{y - z\frac{\partial z}{\partial x}}{x + z\frac{\partial z}{\partial x}} \quad \text{—(4)}$$

2

Back to Eq. (1), differentiate wrt y partially

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) +$$

$$\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial y} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\partial \phi}{\partial u}\left(2y + 2z\frac{\partial z}{\partial y}\right) + \frac{\partial \phi}{\partial v}\left(-2x + 2z\frac{\partial z}{\partial y}\right) = 0$$

$$\frac{\dfrac{\partial \phi}{\partial u}}{\dfrac{\partial \phi}{\partial v}} = \frac{x - z\dfrac{\partial z}{\partial y}}{y + z\dfrac{\partial z}{\partial y}} \quad \text{2} \quad \text{——— (5)}$$

Equate Eq. (4) and Eq. (5)

$$\frac{y - z\dfrac{\partial z}{\partial x}}{x + z\dfrac{\partial z}{\partial x}} = \frac{x - z\dfrac{\partial z}{\partial y}}{y + z\dfrac{\partial z}{\partial y}}$$

leads to

$$\left(y + z\frac{\partial z}{\partial y}\right)\left(y - z\frac{\partial z}{\partial x}\right) = \left(x - z\frac{\partial z}{\partial y}\right)\left(x + z\frac{\partial z}{\partial x}\right)$$

$$y^2 - yz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} - z^2\frac{\partial z}{\partial y}\frac{\partial z}{\partial y} =$$

$$x^2 + xz\frac{\partial z}{\partial x} - xz\frac{\partial z}{\partial y} - z^2\frac{\partial z}{\partial y}\frac{\partial z}{\partial x}$$

Simplify to obtain the PDE of the form

$$y^2 - x^2 = (x+y)z\frac{\partial z}{\partial x} - (x+y)z\frac{\partial z}{\partial y} \quad 1$$

$$y^2 - x^2 = xzz_x - xzz_y + yzz_x - yzz_y$$

$$y^2 - x^2 = xz(z_x \cancel{+ z_y}) - \cancel{yz}(-z_{\cancel{x}} + z_y)$$

Q1c (7 marks)

$$\frac{\partial w}{\partial x} + x\frac{\partial w}{\partial y} + yw = y \qquad w(0,y) = y \quad -(1)$$

To find the characteristics, we shall use the change of variables

$$x = x(s,t) \qquad \text{and} \qquad y = y(s,t)$$

$$\frac{\partial v}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} \qquad (2)$$

Compare Eq.(1) and Eq.(2)

$$\left.\begin{aligned} &\frac{\partial x}{\partial t} = 1 && x(s,0) = 0 \\ &\frac{\partial y}{\partial t} = x && y(s,0) = s \\ &\frac{\partial v}{\partial t} = -yw + y && v(0,y) = y \\ & && v(0,s) = s \end{aligned}\right\} - (3)$$

1

Starting with Eq. (3a)

$$\partial x = \partial t$$

$$x = t + c$$

Using $x(s,0) = 0$ to find c

$$0 = 0 + c \qquad \therefore c = 0$$

$$x = t$$

1

Next is Eq. (3b)

$$\frac{\partial y}{\partial t} = x$$

Sub $x = t$

$$\frac{dy}{dt} = t$$

Integrate to obtain

$$y = \frac{t^2}{2} + c$$

Using $y(s,0) = s$ to obtain $c = s$, then

$$y = \frac{t^2}{2} + s$$

1

Whereas

$$s = y - \frac{1}{2}t^2$$

Back to ~~Eq.~~(3c)

$$\frac{\partial v}{\partial t} + yu = y$$

Sub $y = \frac{t^2}{2} + s$

$$\frac{\partial v}{\partial t} + \left(\frac{t^2}{2} + s\right)v = \left(s + \frac{1}{2}t^2\right)$$

Subject to $v(0,s) = s$

$$\frac{\partial v}{\partial t} = (1-v)\left\{s + \frac{1}{2}t^2\right\}$$

$$\frac{dv}{1-v} = \left(s + \frac{1}{2}t^2\right) dt$$

Integrate

$$\int \frac{dv}{1-v} = \int \left(s + \frac{1}{2}t^2\right) dt$$

$$-\ln(1-v) = \frac{t^3}{2*3} + st + c$$

Using $v(s,0) = s$

$$-\ln(1-s) = 0 + 0 + c$$

$$c = -\ln(1-s)$$

$$-\ln(1-v) = \frac{t^3}{3!} + st - \ln(1-s)$$

$$e^{\ln(1-v)^{-1}} = e^{\frac{t^3}{3!} + st - \ln(1-s)}$$

$$\ln(1-v)^{-1} = e^{\frac{t^3}{3!} + st} * e^{-\ln(1-s)}$$

$$\frac{1}{1-v} = e^{\frac{t^3}{3!} + st} * e^{\ln(1-s)^{-1}}$$

$$1 - v = \frac{1}{e^{\frac{t^3}{3!} + st} * e^{-\ln(1-s)}}$$

$$1-v = e^{\frac{-t^3}{3!} - st} * e^{\ln(1-s)}$$

$$1-v = e^{\frac{-t^3}{6} - st} * (1-s)$$

$$-v = -1 + (1-s)e^{\frac{-t^3}{6} - st}$$

$$v = 1-(1-s)e^{\frac{-t^3}{6} - st}$$

$$v = 1+(s-1)e^{\frac{-t^3}{6} - st}$$ 2

sub $s = y - \frac{x^2}{2}$ $\$$ $t = x$

$$v(s,t) = W(x,y)$$

Hence, the final solution (the mathematical expression to quantify the weight) is

$$W(x,y) = 1 + \left[y - \frac{1}{2}x^2 - 1\right] e^{-xy + \frac{x^3}{3!}}$$ 2

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Q 1d (3 marks)

Theorem: If the Newmann problem of Laplace Equation (2nd order PDE) for a bounded region has a solution, then it is either Unique or it differs from one another by a constant only.

Proof

Let ϕ_1 and ϕ_2 be two distinct solutions of the Neumann Problem. Then, we have

$$\nabla^2\phi_1 \text{ in } \mathbb{R} \quad \therefore \quad \frac{\partial \phi_1}{\partial n} = f \text{ on } \partial\mathbb{R}$$ 1

$$\nabla^2\phi_2 \text{ in } \mathbb{R} \quad \therefore \quad \frac{\partial \phi_2}{\partial n} = f \text{ on } \partial\mathbb{R}$$

Let $\psi = \phi_1 - \phi_2$, then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \quad \text{in } \mathbb{R}$$

Likewise the boundary condition gives

$$\frac{\partial \psi}{\partial n} = \frac{\partial \phi_1}{\partial n} - \frac{\partial \phi_2}{\partial n} = 0 \quad \text{on } \partial\mathbb{R}$$ 1

Meanwhile, if ψ is a harmonic function in $\mathbb{R}$ and $\frac{\partial \psi}{\partial n} = 0$ on $\partial\mathbb{R}$, then ψ is a constant in $\mathbb{R}$. Based on this,

$$\phi_1 - \phi_2 = \text{constant}$$ 1

Therefore, the solution of the Newmann Problem is not unique. Thus, the solutions of a certain Newmann problem of Laplace Eq. can differ from one another by a constant only.

Q5a (4 marks)

Theorem: If the Dirichlet problem of Laplace Eq (second order PDE) for a bounded region has a solution, then such a solution is unique.

Proof: If ϕ_1 and ϕ_2 are two solutions of the interior Dirichlet problem, then

$$\nabla^2\phi_1 = 0 \quad \text{in } R \qquad \phi_1 = f \quad \text{on } \partial R$$
$$\nabla^2\phi_2 = 0 \quad \text{in } R \qquad \phi_2 = f \quad \text{on } \partial R \qquad (1)$$

Let $\psi = \phi_1 - \phi_2$. Then

$$\nabla^2\psi = \nabla^2\phi_1 - \nabla^2\phi_2 = 0 \quad \text{in } R$$
$$\psi = \phi_1 - \phi_2 = f - f = 0 \quad \text{on } \partial R \qquad (1)$$

Therefore,

$$\nabla^2\psi = 0 \quad \text{in } R \qquad \psi = 0 \quad \text{on } \partial R$$

Using the fact that if a harmonic function vanishes everywhere on the boundary, then it is identically zero everywhere, we obtain (1)

$$\psi = 0 \quad \text{on } \overline{R}$$

This implies that $\phi_1 = \phi_2$. Hence, the solution of the Dirichlet problem is unique. (1)

Q5b

$$\frac{\partial^2 u}{\partial t^2} = h^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{——— (1)}$$

Subject to

$$u(x,0) = f(x) \qquad \frac{\partial u}{\partial x} = g(x) \qquad \text{——— (2)}$$
$$-\infty < x < \infty$$

Q5b (11 Marks)

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad \text{———— (1)}$$

Subject to

$$u(x,0) = \sin x$$

$$\frac{\partial u}{\partial t} = \cos x \qquad -\infty < x < \infty \quad \text{———— (2)}$$

We shall ~~sup~~ assume that

$$V = x + 2t \qquad J = \frac{\partial(V,W)}{\partial(x,t)} = \begin{vmatrix} V_x & V_t \\ W_x & W_t \end{vmatrix} \neq 0$$

$$W = x - 2t \qquad \text{—— (3)}$$

Starting with $V_x = 1$, $V_t = 2$, $W_x = 1$, and $W_t = -2$.

$$U_x = \frac{\partial u}{\partial x} = U_V V_x + U_W W_x$$

$$U_x = U_V + U_W \quad \text{———— (4)}$$

And

$$\frac{\partial^2 u}{\partial x^2} = U_{xx} = \frac{\partial}{\partial x}(U_x) = \frac{\partial}{\partial x}(U_V + U_W)$$

$$= \frac{\partial}{\partial x}(U_V) + \frac{\partial}{\partial x}(U_W)$$

$$\frac{\partial^2 u}{\partial x^2} = U_{VV} + 2U_{VW} + U_{WW} \quad 2 \text{ ———— (5)}$$

Also,

$$\frac{\partial u}{\partial t} = U_t = U_V V_t + V_W W_t$$

$$U_t = 2U_V - 2U_W$$

$$\frac{\partial^2 u}{\partial t^2} = U_{tt} = 2\frac{\partial}{\partial t}\left[U_V - U_W\right]$$

$$= 2\left[\frac{\partial}{\partial t}(U_V) - \frac{\partial}{\partial t}(U_W)\right]$$

$$= 2\left[2U_{VV} - 2U_{WV} - 2U_{WV} + 2U_{WW}\right]$$

$$\frac{\partial^2 u}{\partial t^2} = 4U_{VV} - 8U_{WV} + 4U_{WW} \quad 2 \text{ ———— (6)}$$

Sub Eq. (6) and Eq. (5) into Eq. (1) and simplify to obtain

$$16 U_{wv} = 0$$

Therefore

$$\frac{\partial^2 u}{\partial w \partial v} = 0 \qquad 2$$

Integrate twice wrt. w & wrt v partially

$$u(x,t) = h(w) + g(v)$$

where $h(w) = \int f(w)\, dw$. Hence

$$u(x,t) = h(x-2t) + g(x+2t) \quad \text{———} (7)$$

Next is to use the given conditions Eq. (2)

$$u(x,0) = h(x-2(0)) + g(x+2(0)) = \sin x \qquad 1$$

$$h(x) + g(x) = \sin x \quad \text{———} 8$$

Using the second boundary conditions

$$\frac{\partial u}{\partial t} = -2h'(x-2t) + 2g'(x+2t)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = -2h'(x) + 2g'(x) = \cos x \qquad 1 \quad \text{———} (9)$$

Integrate both sides of Eq. (9) from 0 to x since the 2nd condition holds for $-\infty < x < \infty$

$$2\int_0^x (g' - h')\, dx = \int_0^x \cos x\, dx$$

Integrating dummy variable $x = s$

$$g(s) - h(s)\Big|_0^x = \frac{1}{2}\left[\sin x\right]_0^x$$

$$\left[g(x) - h(x)\right] - \left[g(0) - h(0)\right] = \frac{1}{2}\sin x$$

Let $c = g(0) - h(0)$

$$g(x) - h(x) = \frac{1}{2}\sin x + c \quad \text{———} (10)$$

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Next is to Consider Eq. (8) and Eq. (10)

$$h(x) + g(x) = \sin x$$

$$g(x) - h(x) = \frac{1}{2}\sin x + c$$

upon solving simultaneously, we shall obtain

$$h(x) = \frac{\sin x}{4} - \frac{c}{2}$$ (1)

$$g(x) = \frac{3\sin x}{4} + \frac{c}{2}$$

Where as

$$\left.\begin{aligned} h(x-2t) &= \frac{\sin(x-2t)}{4} - \frac{c}{2} \\ g(x+2t) &= \frac{3\sin(x+2t)}{4} + \frac{c}{2} \end{aligned}\right\} \text{— (11)}$$ (1)

Sub. Eq. (11) into Eq. (7)

$$u(x,t) = \frac{\sin(x-2t)}{4} - \frac{c}{2} + \frac{3\sin(x+2t)}{4} + \frac{c}{2}$$

$$u(x,t) = \frac{\sin(x-2t)}{4} + \frac{3\sin(x+2t)}{4}$$

Finally

$$u(x,t) = \frac{1}{4}\left[\sin(x-2t) + 3\sin(x+2t)\right]$$ (1)

Q3a (11 marks)

$$u_{xx} - 2\sin(x)u_{xy} - \cos^2(x)u_{yy} - \cos(x)u_y = 0 \quad (1)$$

In order to classify, we shall use the discriminant

$$B^2 - 4AC = 0$$

where

$$A = 1 \qquad B = -2\sin x \qquad C = -\cos^2 x$$

$$(-2\sin x)^2 - 4(1)(-\cos^2 x) = 4\sin^2 x + 4\cos^2 x$$
$$= 4 > 0 \quad 2$$

Hence, it is worthy to conclude that the second order PDE Eq-(1) is hyperbolic and the relevant characteristic equations are

$$\frac{dy}{dx} = \frac{B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{(-2\sin x) \pm \sqrt{4\sin^2 x + 4\cos^2 x}}{2}$$

$$\frac{dy}{dx} = \frac{-2\sin x \pm \sqrt{4}}{2} = \frac{-2\sin x \pm 2}{2}$$

$$\frac{dy}{dx} = -\sin x - 1 \quad 1 \qquad \frac{dy}{dx} = 1 - \sin x \quad 1 \quad (2)$$

Integrate both equations to obtain

$$dy = (-\sin x - 1)\,dx \qquad dy = (1 - \sin x)\,dx$$

$$y = \cos x - x + c_1 \qquad y = \cos x + x + c_2 \quad (3)$$

Thus, if we choose the characteristic lines are

$$\xi = x + y - \cos x = c_1 \quad (4)$$
$$\eta = -x + y - \cos x = c_2$$

Note that the functions ξ and η are continuously differentiable and the Jacobian

$$J = \frac{\partial(\xi, \eta)}{\partial(x, y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x \neq 0 \quad 1$$

From Eq. (4)

$$\xi_x = \frac{\partial \xi}{\partial x} = 1 + \sin x \qquad \xi_{xx} = \frac{\partial^2 \xi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) = \cos x$$

$$\xi_y = \frac{\partial \xi}{\partial y} = 1 \qquad \xi_{yy} = \frac{\partial^2 \xi}{\partial y^2} = \xi_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial \xi}{\partial y}\right) = 0$$

$$\eta_x = \frac{\partial \eta}{\partial x} = -1 + \sin x \qquad \eta_y = \frac{\partial \eta}{\partial y} = 1$$

$$\eta_{xx} = \frac{\partial^2 \eta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) = \cos x \qquad \eta_{xy} = \frac{\partial}{\partial y}(\eta_x) = \frac{\partial}{\partial y}(-1+\sin x) = 0$$

$$\eta_{yy} = \frac{\partial^2 \eta}{\partial y^2} = 0 \qquad \text{——— (5)}$$

Next is to consider each term.

$$\frac{\partial u}{\partial x} = u_x = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \qquad \text{——— (6)}$$

$$\frac{\partial u}{\partial y} = u_y = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} \qquad \text{——— (7)}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}\right)$$

$$= \frac{\partial u}{\partial \xi}\frac{\partial}{\partial x}\left(\frac{\partial \xi}{\partial x}\right) + \underline{\frac{\partial \xi}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi}\right)} + \frac{\partial u}{\partial \eta}\frac{\partial}{\partial x}\left(\frac{\partial \eta}{\partial x}\right) + \underline{\frac{\partial \eta}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \eta}\right)}$$

$$u_{xx} = \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \xi}{\partial x}\left[\frac{\partial}{\partial \xi}(u_\xi)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\xi)\frac{\partial \eta}{\partial x}\right] + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial}{\partial x}\left[\frac{\partial}{\partial \xi}(u_\eta)\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}(u_\eta)\frac{\partial \eta}{\partial x}\right]$$

$$u_{xx} = \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \xi}{\partial x}\left[\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right] + \frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \eta}{\partial x}\left[\frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right]$$

Expand

$$U_{xx} = \frac{\partial u}{\partial \xi}\frac{\partial^2 \xi}{\partial x^2} + \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial x}\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 u}{\partial \xi \partial \eta} +$$

$$\frac{\partial u}{\partial \eta}\frac{\partial^2 \eta}{\partial x^2} + \frac{\partial \eta}{\partial x}\frac{\partial \xi}{\partial x}\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 u}{\partial \eta^2}$$

Hence

$$U_{xx} = U_\xi \xi_{xx} + (\xi_x)^2 U_{\xi\xi} + 2\xi_x \eta_x U_{\xi\eta} + U_\eta \eta_{xx} + (\eta_x)^2 U_{\eta\eta} \quad \text{——} (8)$$

1

Likewise

$$U_{yy} = U_{\xi\xi}(\xi_y)^2 + 2U_{\xi\eta}\xi_y \eta_y + U_{\eta\eta}(\eta_y)^2 + U_\xi \xi_{yy} + U_\eta \eta_{yy} \quad \text{——} (9)$$

1

$$U_{xy} = U_{\xi\xi}\xi_x \xi_y + U_{\xi\eta}(\xi_x \eta_y + \xi_y \eta_x) + U_{\eta\eta}\eta_x \eta_y + U_\xi \xi_{xy} + U_\eta \eta_{xy} \quad \text{——} (10)$$

1

Next is to sub Eq.(6) — Eq.(10) into Eq. (1) to obtain

$$U_\xi \xi_{xx} + (\xi_x)^2 U_{\xi\xi} + 2\xi_x \eta_x U_{\xi\eta} + U_\eta \eta_{xx} + (\eta_x)^2 U_{\eta\eta}$$

$$-2\sin x\left[U_{\xi\xi}\xi_x\xi_y + U_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + U_{\eta\eta}\eta_x\eta_y + U_\xi\xi_{xy} + U_\eta\eta_{xy}\right]$$

$$-\cos^2(x)\left[U_{\xi\xi}(\xi_y)^2 + 2U_{\xi\eta}\xi_y\eta_y + U_{\eta\eta}(\eta_y)^2 + U_\xi\xi_{yy} + U_\eta\eta_{yy}\right]$$

$$-\cos x\left[U_\xi + U_\eta\right] = 0$$

Simplify and sub Eq.(5) to obtain the required cannonical form

$$U_{\xi\eta} = 0$$

1

Integrate wrt ξ to obtain

$$U_\eta = f(\eta)$$

where f is arbitrary. Integrating once again wrt η

$$u(x,y) = \int f(\eta)\, d\eta + g(\xi)$$

or, let $\int f(\eta)\, d\eta = \psi(\eta)$ and $g(\xi)$ is another arbitrary functi

$$u(x,y) = \psi(-x+y-\cos x) + g(x+y-\cos x).$$

1

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Q3b (4 marks)

Cauchy's problems for the existence of solutions of first order PDEs.

Consider an interval I on the real line. If $x_0(s)$, $y_0(s)$, and $z_0(s)$ are three arbitrary functions of a single variable $s \in I$ such that they are continuous in the interval I with their first derivatives. Then, the Cauchy problem for a first order PDE of the form 1

$$f\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0 \quad \text{——— (1)}$$

is to find a region $\mathbb{R}$ in (x, y) [i.e the space containing $[x_0(s), y_0(s)]$ for all $s \in I$, and a solution $z = \phi(x, y)$ of the PDE Eq. (1) such that

$$z[x_0(s), y_0(s)] = z_0(s) \qquad 1$$

and $\phi(x, y)$ together with its partial derivatives with respect to x and y are continous functions of x and y in the region $\mathbb{R}$. Geometrically, there exists a surface $z = \phi(x, y)$ which 1 passess through the curve Γ whose parametric equations are

$$x = x_0(s), \qquad y = y_0(s), \qquad z = z_0(s)$$

and at every point of which the direction $\left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1\right)$ of the normal is such that 1

$$F\left(x, y, z, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0.$$

Q4a (10 marks)

Solution of

$$\frac{\partial^3 z}{\partial x^3} - \frac{\partial^3 z}{\partial y^3} = x^3 y^3 \quad \text{——— (1)}$$

In order to first obtain the complementary function of the third order homogeneous PDE, let seek for the solution

$$z = \phi(y+mx) \quad \text{——— (2)}$$

$$\frac{\partial z}{\partial x} = m\phi(y+mx) \qquad \frac{\partial^2 z}{\partial x^2} = m^2\phi(y+mx) \qquad \frac{\partial^3 z}{\partial x^3} = m^3\phi(y+mx)$$

Also $\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^3 z}{\partial y^3} = \phi(y+mx)$

Sub Eq. (2) into Eq. (1)

$$m^3\phi(y+mx) - \phi(y+mx) = 0$$

$$(m^3 - 1)\phi(y+mx) = 0$$

2

Meanwhile $\phi(y+mx) \neq 0$ but

$$m^3 - 1 = 0$$

In such a case,

$$m_1 = 1 \qquad m_2 = w \qquad m_3 = w^2$$

Where w and w^2 are complex roots of a unit. Hence,

$$C_F = \phi_1(y+x) + \phi_2(y+wx) + \phi_3(y+w^2x)$$ 2

Next is to find P.I using the concept of D-operator for PDE where

$$\frac{\partial^3}{\partial x^3} = D^3 \quad \text{and} \quad \frac{\partial^3}{\partial y^3} = (D')^3$$

Hence, Eq. (1) becomes

$$\left(D^3 - (D')^3\right) z_p = x^3 y^3$$

$$Z_p = \frac{x^3y^3}{D^3-(D')^3} = \frac{x^3y^3}{D^3\left[1-\frac{(D')^3}{D^3}\right]}$$

$$Z_p = \frac{x^3y^3}{D^3}\left[1-\frac{(D')^3}{D^3}\right]^{-1}$$

2

Recall ~~the~~ binomial expansion

$$\left(1-\frac{(D')^3}{D^3}\right)^{-1} = 1+\frac{(D')^3}{D^3}+\cdots+$$

Hence

$$Z_p = \frac{1}{D^3}\left[x^3y^3+\frac{(D')^3}{D^3}x^3y^3+\cdots+\right]$$

Recall that $D' = \frac{\partial}{\partial y}$ $\quad (D')^3 = \frac{\partial^3}{\partial y^3}$

$$Z_p = \frac{1}{D^3}\left[x^3y^3+\frac{1}{D^3}\frac{\partial^3}{\partial y^3}(x^3y^3)+\cdots+\right]$$

$$Z_p = \frac{1}{D^3}\left[x^3y^3+\frac{1}{D^3}(6x^3)+\cdots+\right]$$

$$Z_p = \frac{1}{D^3}\left[x^3y^3+D^{-3}(6x^3)+\cdots+\right]$$

Meanwhile $D^{-3}(6x^3) = \iiint 6x^3\,\partial x\,\partial x\,\partial x$

$$= \frac{x^6}{120}$$

$$Z_p = D^{-3}(x^3y^3)+D^{-3}\left(\frac{x^6}{120}\right)+\cdots+$$

$$Z_p = \frac{1}{120}x^6y^3+\frac{1}{10080}x^9$$

2

Finally

$$Z(x,y) = \phi_1(y+x)+\phi_2(y+wx)+\phi_3(y+w^2x)+\frac{1}{120}x^6y^3+\frac{1}{10080}x^9.$$

2

Q4b (5 marks)

Theorem: If a harmonic function vanishes everywhere on the boundary, then it is Identically Zero everywhere.

Proof: If ϕ is a harmonic function, then

$$\nabla^2 \phi = 0 \text{ in the bounded region } \mathbb{R} \text{ in } \mathbb{R}^3$$

Also, if $\phi = 0$ on the boundary $\partial\mathbb{R}$ of the domain, we shall show that $\phi = 0$ in 1

$$\bar{\mathbb{R}} = \mathbb{R} \cup \partial\mathbb{R}$$

Recall Green's Identity

$$\iiint_{\mathbb{R}} (\nabla \phi)^2 \, dV = \iint_{\mathbb{R}} \phi \frac{\partial \phi}{\partial n} \, ds - \iiint_{\mathbb{R}} \phi \nabla^2 \phi \, dV$$ 1

and using the above facts we have, at once, the relation

$$\iiint_{\mathbb{R}} (\nabla \phi)^2 \, dV = 0$$ 1

Since $(\nabla \phi)^2$ is positive, it follows that the Integral will be satisfied only if $\nabla \phi = 0$. This implies that ϕ is a constant in $\mathbb{R}$. 2

Since ϕ is continous in $\bar{\mathbb{R}}$ and ϕ is zero on $\partial\mathbb{R}$, it follows that $\phi = 0$ in $\mathbb{R}$.

Q5a (5 marks)

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u \qquad u(x,0) = 6e^{-3x} \quad \text{---(1)}$$

Let seek the solution of the form

$$u = X(x)\,T(t)$$

2

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}T \quad \& \quad \frac{\partial u}{\partial t} = X\frac{dT}{dt} \quad \text{———} (2)$$

Sub Eq.(2) into Eq.(1)

$$\frac{dX}{dx}T = 2X\frac{dT}{dt} + XT$$

Next is to separate the variables.

$$\frac{1}{X}\frac{dX}{dx} - 1 = \frac{2}{T}\frac{dT}{dt} \quad \text{————} \quad (3)$$

Next stage is to Equate Eq.(3) to a constant of separation since LHS depend strictly on X and RHS depend strictly on t

$$\frac{1}{X}\frac{dX}{dx} - 1 = \frac{2}{T}\frac{dT}{dt} = K \qquad 2$$

Lead to

$$\frac{1}{X}\frac{dX}{dx} - 1 = K \qquad \frac{2}{T}\frac{dT}{dt} = K$$

$$X(x) = P_2 e^{(K+1)x} \quad 1 \qquad T(t) = P_1 e^{\frac{Kt}{2}} \quad 1$$

Hence, the required solution is of the form

$$u(x,t) = P_2 e^{(K+1)x} P_1 e^{Kt/2}$$

Let $A = P_1P_2$

$$u(x,t) = Ae^{(K+1)x} e^{\frac{Kt}{2}} \qquad 1$$

Using the condition $u(x,0) = 6e^{-3x}$ to find A

$$6e^{-3x} = Ae^{(K+1)x}e^{0} \qquad \therefore A = 6 \text{ for } K = -4$$

$$u(x,t) = 6e^{-3x}e^{-2t} = 6e^{-(3x+2t)} \qquad 2$$

Q5b (6 marks)

Maximum – Minimum Principle for Laplace Eq. (2nd order PDE)

Let $\mathbb{R}$ be a region bounded by $\partial\mathbb{R}$. Also, let u be a function which is continuous in a closed region $\overline{\mathbb{R}} = \mathbb{R} \cup \partial\mathbb{R}$ and satisfies the Laplace equation $\nabla^2 u = 0$ in the interior of $\mathbb{R}$. Further, if u is not constant everywhere on $\overline{\mathbb{R}}$, then the maximum and minimum values of u must occur only on the boundary $\partial\mathbb{R}$. (2) (2) (2)

OR

~~DEF~~: A function $u = u(x)$ is harmonic in an open region (Ω) if u is twice continuously differentiable in Ω and satisfies Laplace's equation in Ω. (i.e. u is the solution of Laplace Eq. $\nabla^2 u = 0$) (2)

Let Ω be a bounded region with boundary S and let u be harmonic in Ω. If M and m are respectively the maximum and minimum values of $u(x)$ for x on S, then Weak Maximum – Minimum Principle says that (2)

$$m \le u(x) \le M \quad \text{for all } x \text{ in } \overline{\Omega}$$

Where $\overline{\Omega}$ is known as the closure of Ω

(i.e. u is harmonic in Ω and continuous in $\overline{\Omega}$).

Strong Maximum – Minimum Principle says that (2)

Either $m < u(x) < M$ for all x in Ω

or Else $m = u(x) = M$ for all x in $\overline{\Omega}$

Q6

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r} = 0 \qquad \text{———} *$$

Let seek the product solution

$$u(r,\theta) = R(r)\,G(\theta) \qquad \text{———} (1)$$

To obtain

$$r^2 G \frac{d^2R}{dr^2} + R\frac{d^2G}{d\theta^2} + rG\frac{dR}{dr} = 0 \qquad 1 \text{———} (2)$$

Equating to separation constant to obtain

$$r^2R'' + rR' - \lambda R = 0 \qquad 1 \text{———} (3)$$

$$G'' + \lambda G = 0$$

Firstly, when $\lambda = 0$

$$R(r) = A_0 + B_0 \ln r \qquad 1 \text{———} (4)$$

Ability to solve the variable coefficient second order ODE using

$$R(r) = Pr^q \qquad 1 \quad \text{when } q = \pm n \text{ ——} (5)$$

$n = 1, 2, \dots$

To obtain

$$R_n(r) = A_n r^n + B_n r^{-n} \qquad 1 \quad \text{Provided } n \neq 0 \text{ ——} (6)$$

Also, ability to obtain

$$G_n(r) = C_n \cos n\theta + D_n \sin n\theta \qquad 1 \text{———} (7)$$

Then

$$u(r,\theta) = U_n(r,\theta) = R_n(r)\,G_n(\theta) \qquad 1 \text{———} (8)$$

$$u_n(r,\theta) = (A_0 + B_0\ln r) + (A_n r^n + B_n r^{-n}) * (C_n\cos n\theta + D_n \sin n\theta) \qquad 1 \text{———} (9)$$

Ability to reduce the solution to

$$U_n(r,\theta) = A_0 + r^n[A_nC_n\cos n\theta + A_nD_n \sin n\theta] \qquad 1 \text{———} (10)$$

Let

$$\alpha_n = A_nC_n \qquad \beta_n = A_nD_n$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left[\alpha_n \cos n\theta + B_n \sin n\theta \right] \quad (11)$$

Using the condition $u(r=a,\theta) = f(\theta)$ and Fourier Series expansion to obtain

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n \left(\frac{a_n}{a^n} \cos n\theta + \frac{b_n}{a^n} \sin n\theta \right) \quad (12)$$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} \frac{r^n}{a^n} \left(a_n \cos n\theta + b_n \sin n\theta \right)$$

Where

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos n\theta \, d\theta \quad (13a)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta \, d\theta . \quad (13b)$$

Remark 1: Note that Eq. (9) for $n = 0, 1, 2, \cdots, r > 0$ $U_n(r,\theta)$ are Periodic of Period 2π. This satisfies Laplace's equation (m^*) in polar coordinate.

Remark 2: The functions $U_n(r,\theta)$ is called the circular harmonics.