SOLUTIONS OF FIRST AND SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS: DIFFERENTIAL TRANSFORM TECHNIQUE

BY

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MTS/15/4148

A PROJECT SUBMITTED TO

THE DEPARTMENT OF MATHEMATICAL SCIENCES,

FEDERAL UNIVERSITY OF TECHNOLOGY, AKURE, ONDO STATE, NIGERIA.

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE AWARD OF BACHELOR OF TECHNOLOGY (B.TECH.) IN INDUSTRIAL MATHEMATICS.

APRIL, 2021

Certification

I certify that this project work titled SOLUTIONS OF FIRST AND SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS: DIFFERENTIAL TRANSFORM TECHNIQUE was carried out by ADESINA, EMMANUEL ADEDAYO with the matriculation number MTS/15/4148, in the Department of Mathematical Sciences, Federal University of Technology, Akure under the supervision of Dr A J OMOWAYE, in partial fulfillment of the award of Bachelor of Technology (B.Tech) in Industrial Mathematics.

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Dedication

I dedicate this work to God the Almighty, and my beloved parents.

Acknowledgment

All thanks and praises be to the Lord God, by whose strength I had been seen through all my experiences as a student.

A heart deep thanks to my parents **Mr and Dns Adesina** and all of my family for their support, care and love, without which I would not have realized this work.

I am grateful to my supervisor **Dr A J Omowaye** for the unwavering support and how you painstakingly guided me for the success of this project work. Many thanks to the Head of Department of Mathematical Sciences **Prof K S Adegbie**, the entire lecturers and non-teaching staff in the department for their moral and academic assistance in all course of my learning.

Big thanks to every person, friends and colleagues who was at some points or another an help to my academic life.

Abstract

In this work, differential transform technique is used to obtain solutions to first and second-order ordinary differential equation. Differential transform is an alternative iterative method for obtaining Taylor series solutions of differential equations. This method saves the large computational work for problems with large orders compared to Taylor series; Differential Transform Method reduces the size of computational domain and helps solve many problems easily. This method is illustrated by solving some differential equations and the results are compared with the exact solutions. A generalized procedure of the solution is stated and a computer program is written using MATLAB to help automate the iterative procedure.

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Chapter 1

INTRODUCTION

1.1 DIFFERENTIAL EQUATIONS

Differential equations were first proposed in 1676 by G. Leibniz. Differential equations are very important and powerful tool in the study of many problems in the natural sciences and technology; they are extensively employed in mechanics, astronomy, physics, and in many problems of chemistry and biology **?**. In the simplest term, they describe how things change, move, decay, etc. It can be used to analyze many problems in various disciplines **?**. It is important in studies such as the study of growth and decay, study of change in the size of a population, and change in an investment's return over periods, etc.

Denoting t as the independent variable, and unknown functions by x, y, z, etc. For an arbitrary equation $g = f(t)$ its differential is given by:

$$\dot{x} = \frac{dg}{dx} = \frac{df(t)}{dx} = f(t, x) \tag{1.1}$$

which indicates the change in $g = f(x)$ and $\dot{x}, \ddot{x}, etc$ denotes the differentials with respect to x. The base of the problem of finding the primitive function of a given continuous function $f(t)$ amounts to finding an unknown function $x(t)$ which satisfies the equation:

$$\dot{x} = f(t) \tag{1.2}$$

A generalization of equation 1.2 is a first order ordinary differential equation solved with respect to the derivative:

$$\dot{x} = f(t, x) \tag{1.3}$$

Equation 1.3 can be generalized for ordinary differential equations of order **n** with respect to it's leading derivative as:

$$x^{(n)} = f(t, x, \dot{x}, \ddot{x}, ..., x^{(n-1)}) \tag{1.4}$$

1.2 ORDER OF DIFFERENTIAL EQUATIONS

For differential equations that describe physical phenomena, they can be classified based on a variety of cases such as linearity, degree, homogeneity, and order, where the order of differential equations is the highest derivative involved. Among these, the second-order differential equation is the most common and the most important special case as they arise in various fields of mechanics, heat, electricity, aerodynamics, stress analysis, and so on, **?**.
From equation 1.4 we can write:

$$x^{'} = f(t, x) \tag{1.5}$$

$$x^{''} = f(t, x, \dot{x}) \tag{1.6}$$

Where equation 1.5 is said to be a first ordinary differential equation, 1.6 a second ordinary differential equation.

1.3 METHODS OF SOLUTION

The importance of differential equations cannot be over-emphasized as it has many applications in science and engineering, but as beautiful as they are, one can not use them except they are solved so that the solutions can be used in studies, estimations, and decision making. For example, in engineering fields, differential equations are used to model physical problems, these problems then need to be solved and the solutions can be used to study the behavior of the systems concerned, and this applies to all other fields of application of differential equations.
Figure 1.1 illustrates the process of deriving, solving and applying differential equations. When real life problems are encountered, (such as the need to construct solid structures that will be subject to different physical conditions) in engineering, or (the need to determine the spread or growth rate of a culture medium) in biology, etc, these problems are made into mathematical

models that factors in all possible variable that could include physical or environmental properties such as the strength, density and elasticity of the materials to be used in the construction, or the acidity of the culture medium and the number of micro-organisms to be introduced in the culture. Here, many physical laws and relationship between quantities are expressed mathematically as differential equations **?**. These models then needs to be solved using appropriate mathematical tool and methods, and the result of the solution is then interpreted and applied to the real life problem that gave rise to the model

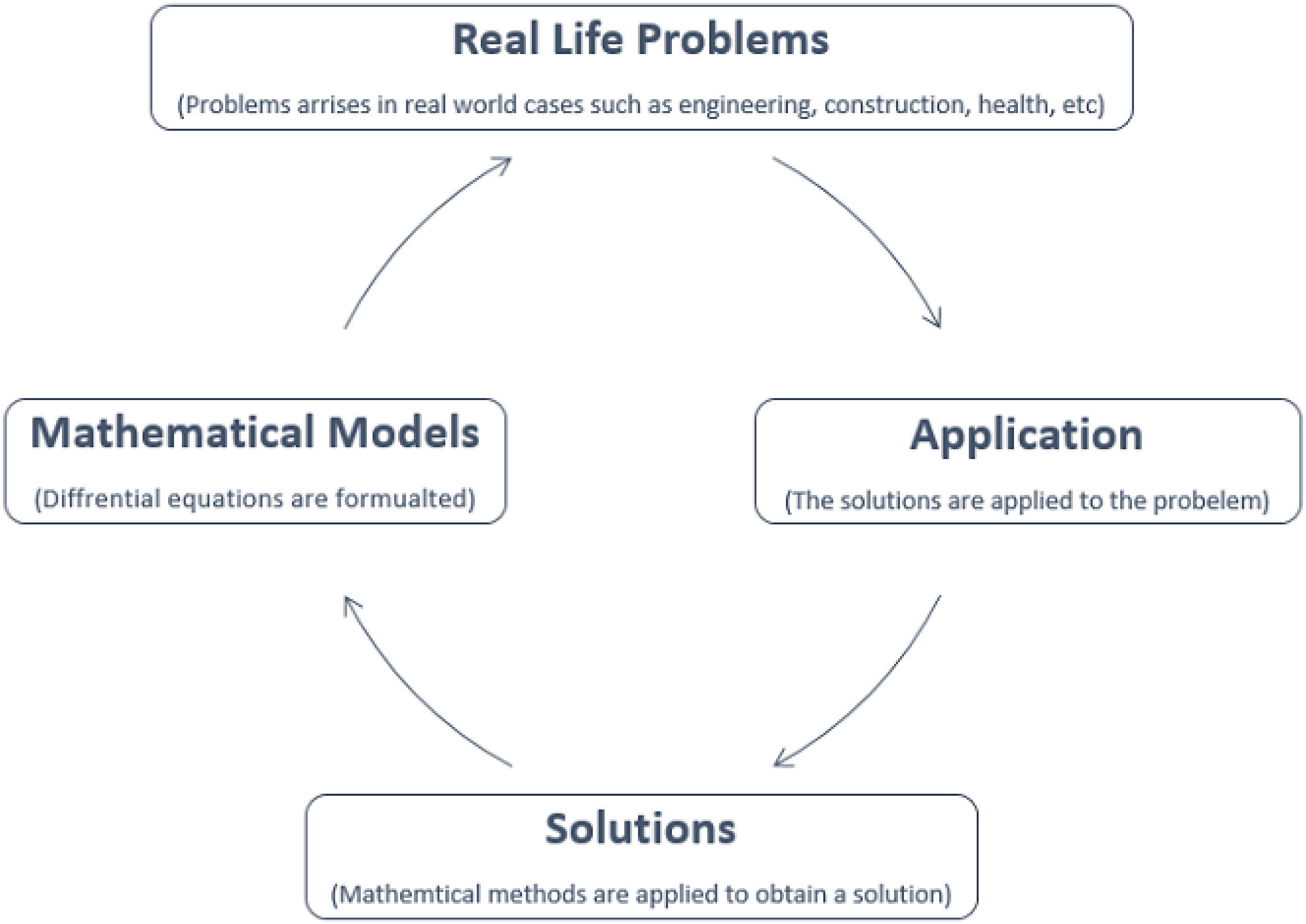

Figure 1.1: Differential Equations Solutions Chart

Thus, since its origination, many methods of solution have been developed, each targeting different types and classes of differential equations and aiming at providing simplicity, with a large variety of numerical and analytical methods been developed to obtain accurate approximate and analytic solutions **??** and **?**. One of the earliest analytic techniques to many problems has been the Taylor's series method, especially for ordinary differential equations, however, it requires a lot of symbolic calculation and takes a large computational time for derivatives of a higher order.

Among many other methods, the more commonly used once includes, but not limited to:

Analytical Methods such as the direct integration method, variable separable, integrating factor, and method of undetermined coefficient

Numerical Methods where we try to find values y_n for $\phi(x_n)$ at a sequence of points $x_0, x_1 = x_0 + h, x_2 = x_0 + 2h, x_3 = x_0 + 3h, ...$ where h is called a step size (**?**). Examples of such methods includes the Eular Method, Runge-Kuta Method which obtains the solution of an initial value problem $y' = f(x, y), y(x_0) = y_0$ by taking approximations of $y_(n+1)$ on points of the interval $[x_0, x_n]$, Picard iterations.

Here, we introduce the Differential Transform Technique (DTM) which **???** regards as an updated version of the Taylor series. The Differential Transform Technique solves the induced recursive equation from the given differential equations to determine the coefficients of the Taylor series. We introduce notations of the DTM and illustrates its operation.

1.4 AIM AND OBJECTIVES OF THE PROJECT

This project illustrates how to obtain a solution to first and second-order ordinary differential equations using the Differential Transform Technique. The objectives are to:

- illustrate the method and demonstrate it by solving some differential equations, and
- write a MATLAB computer program to automate the method.

1.5 ORGANIZATION OF THE PROJECT

This project outlines an introduction which details the concept of differential equations and why obtaining solutions to them is important and the various existing methods of solution. The differential transform technique is described and the steps to using it to obtain solutions is outlined giving a clear understanding of the technique.

Differential Transform Technique is then used to solve some differential equations and the solution is compare to their exact solutions. With MATLAB program written to solve the problems. The outcome of these are then discussed.

Chapter 2

LITERATURE REVIEW

Due to the complexity of different types and classes of differential equations with the different conditions that surrounds them, one needs to carefully experiment with a method of solution with a large variation of differential equations and compare the result with other existing methods to be sure of which works best and give the best accurate solution. Compare to other methods, the differential transform technique has been studied by many authors in attempt to establish it as a suitable method in solving differential equations and investigate its simplicity and easy of use.

? employed the differential transform technique for obtaining solutions to initial value problems and obtained series solutions that can be easily converted to exact solutions

? illustrated that linear and nonlinear systems of ordinary differential equations, such that if the system was considered as a solution in terms of the series expansion of known functions, the method catches up with the exact solution. **?** solved linear and non-linear Goursat problem using the two-dimensional Differential Transform Technique.

In **?** Volterra integral equations with separable kernels was solved using the Differential Transform Technique to obtain an approximate solution of this equation. The results of using the Differential Transform Technique were compared with the LaPlace transform by **?** showing the accuracy and efficiency and less computational effort and time requirement in comparison to the other methods

In **?** Lane-Emden equations as singular initial value problems were solved using the Differential Transform Technique and also presented some numerical examples to illustrate the method.

Amidst vast variation of literature, **?** and **?** gave a detailed look into different types and classifications of differential equations and examined some common analytical and exact methods of solving them in the scope of undergraduate studies.

This project is devoted to solutions of first and second order ordinary differential equation using differential transform technique

Chapter 3

METHODOLOGY

3.1 DIFFERENTIAL TRANSFORM TECHNIQUE

The Differential Transform Method was first introduced by **?**, with it's application in Electrical Circuits Analysis. The Method applies to both Linear and Non-linear initial or boundary value problems to obtain both numerical solutions which can be presented as a polynomial.

For a function $f(x)$ that can be expressed by Taylor series about a point $x = 0$ as:

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \{\frac{d^k f}{dx^k}\}_{x=0} \tag{3.1}$$

The one dimensional transform of the kth derivative is defined as:

$$V(k) = \frac{1}{k!} \{\frac{d^k v(x)}{dx^k}\} \tag{3.2}$$

and the inverse transformation is defined by,

$$f(x) = \sum_{k=0}^{\infty} x^k V(k) \tag{3.3}$$

From 3.2 and 3.3 , if $Y(k)$ and $G(k)$ are transformed functions of $y(x) \pm G(k)$ respectively, then we can define the properties of the Differential Transform Techniqueas in Table 3.1

The procedure for using the Differential Transform Technique can be itemized simply as follows:

- Transform the terms of the differential equations such that $V(k) = \frac{1}{k!}\{\frac{d^k v(x)}{dx^k}\}$
- Make unknown $V(k+n)$ term the subject

- Solve iteratively for $k = 0, 1, 2, 3, ..., n$ using the initial value to get values of $V(k)$
- Compute the approximate solution in the form of $\sum V(k)t^k$
- Substitute the values of $V(k)$ into the Inverse Transforms of the form $\sum \frac{(t-t_i)}{k!}V(k)$

Original function $f(x)$	**The transform** $F(k)$
$\alpha y(x)$	$\alpha Y(k)$
$y(x) \pm g(x)$	$Y(k) \pm G(k)$
$y(x)g(x)$	$\sum_{l=0}^{k} Y(l)G(l)$
$\frac{dy(x)}{dx}$	$(k+1)Y(k)$
$\frac{d^m y(x)}{dx^m}$	$(k+1)(k+2)...(k+m)Y(k+m)$
$\int_{x_0}^{x} y(t)$	$\frac{U(k-1)}{k}, k \geq 1, F(0) = 0$
x^m	$\delta(k-m)$
$\exp(\lambda x)$	$\frac{\lambda^k}{k!}$
$\sin(\omega x + \alpha)$	$\frac{w^k}{k!}\sin(\pi k/2 + \alpha)$
$\cos(\omega x + \alpha)$	$\frac{w^k}{k!}\cos(\pi k/2 + \alpha)$

Table 3.1: Properties of the DTM

3.2 MATRIX LABORATORY (MATLAB)

Being a numerical method of solution, the Differential Transform Technique requires quite many repetitive steps which can be time-consuming if done by hand. Hence, a MATLAB computer program will be developed to automate the repetitive steps.

The solutions will be compare with the exact solution to the differential equations to ascertain its correctness

Chapter 4

ILLUSTRATIVE EXAMPLES AND APPLICATIONS

In this section, the Differential Transform Technique will be used to obtain solution of some differential equations, which is then evaluated for values of x and compare with the evaluations of the exact solution.

Problem 1

Consider the linear second order non-homogeneous (boundary value) differential equation

$$y'' + 7y' + 10y = 4e^{-3t}; y(0) = 0, y'(0) = -1 \tag{4.1}$$

With exact solution $e^{-2x} + e^{-5x} - 2e^{-3x}$.

Transforming each term of the differential equations, we have:

$$(k+1)(k+2)Y(k+2) + 7(k+1)Y(k+1) + 10Y(k) = \frac{4(-3)^k}{k!}$$

$$Y(k+2) = \frac{\frac{4(-3)^k}{k!} - 7(k+1)Y(k+1) - 10Y(k)}{(k+1)(k+2)}$$

Solving for $Y(k+2)$, $k = 0, 1, 2, ..., n$, we have:

$\mathbf{k = 0}$

$$Y(2) = \frac{\frac{4(-3)^0}{0!} - 7(1)Y(1) - 10Y(0)}{(1)(2)}$$

$$= \frac{\frac{4(-3)^0}{0!} - 7(-1) - 10(0)}{2} = \frac{4+7-0}{2}$$

$$Y(2) = 5.5$$

k = 1

$$Y(3) = \frac{\frac{4(-3)^1}{1!} - 7(2)Y(2) - 10Y(1)}{(2)(3)}$$

$$= \frac{\frac{4(-3)^1}{1!} - 14(5.5) - 10(-1)}{6} = \frac{-12 - 77 + 10}{6}$$

$$Y(2) = -13.1667$$

k = 2

$$Y(3) = \frac{\frac{4(-3)^2}{2!} - 7(3)Y(3) - 10Y(2)}{(3)(4)}$$

$$= \frac{\frac{4(-3)^2}{2!} - 14(-13.1667) - 10(5.5)}{6} = \frac{36 + 184.3338 - 55}{12}$$

$$Y(3) = 19.9583$$

k = 3

$$Y(3) = \frac{\frac{4(-3)^3}{3!} - 7(3)Y(3) - 10Y(3)}{(4)(5)}$$

$$= \frac{\frac{4(-3)^3}{3!} - 21(19.9583) - 10(5.5)}{20} = \frac{36 - 419.1243 - 55}{12}$$

$$Y(3) = -22.2583$$

The results is shown for different values of n solved by using a Matlab computer program and compared with results of the exact solution. The solution is given as:
$-t + 5.5t^2 - 13.1667t^3 + 19.9583t^4 - 22.2583t^5 + 19.7653t^6 - 14.6585t^7 + 9.369t^8 - 5.2752t^9 + 2.6589t^10 - 1.2144t^11 + 0.50748t^12 - 0.19552t^13 + ...$

x	Exact Solution	DTM Solution	Error
0.0	0.00000000	0.00000000	0.00000000
0.1	-0.05637503	-0.05637515	0.00000012
0.2	-0.05942378	-0.05944563	0.00002184
0.3	-0.04119752	-0.04172325	0.00052573
0.4	-0.01772418	-0.02271913	0.00499495
0.5	0.00370412	-0.02470234	0.02840646
0.6	0.02038350	-0.09639128	0.11677478
0.7	0.03188149	-0.35196548	0.38384697
0.8	0.03877625	-1.03278592	1.07156217
0.9	0.04199686	-2.59921155	2.64120841
1.0	0.04249909	-5.85990000	5.90239909

Table 4.1: Solutions to Problem 1 at N = 5

x	Exact Solution	DTM Solution	Error
0.0	0.00000000	0.00000000	0.00000000
0.1	-0.05637503	-0.05637506	0.00000004
0.2	-0.05942378	-0.05942409	0.00000031
0.3	-0.04119752	-0.04119856	0.00000104
0.4	-0.01772418	-0.01772554	0.00000136
0.5	0.00370412	0.00371970	-0.00001558
0.6	0.02038350	0.02058630	-0.00020280
0.7	0.03188149	0.03338424	-0.00150275
0.8	0.03877625	0.04710090	-0.00832465
0.9	0.04199686	0.07945204	-0.03745518
1.0	0.04249909	0.18588000	-0.14338091

Table 4.2: Solutions to Problem 1 at N = 10

x	Exact Solution	DTM Solution	Error
0.0	0.00000000	0.00000000	0.00000000
0.1	-0.05637503	-0.05637506	0.00000004
0.2	-0.05942378	-0.05942409	0.00000031
0.3	-0.04119752	-0.04119859	0.00000107
0.4	-0.01772418	-0.01772669	0.00000251
0.5	0.00370412	0.00369948	0.00000464
0.6	0.02038350	0.02037645	0.00000706
0.7	0.03188149	0.03187251	0.00000898
0.8	0.03877625	0.03876295	0.00001330
0.9	0.04199686	0.04193750	0.00005936
1.0	0.04249909	0.04209160	0.00040749

Table 4.3: Solutions to Problem 1 at N = 15

x	Exact Solution	DTM Solution	Error
0.0	0.00000000	0.00000000	0.00000000
0.1	-0.05637503	-0.05637506	0.00000004
0.2	-0.05942378	-0.05942409	0.00000031
0.3	-0.04119752	-0.04119859	0.00000107
0.4	-0.01772418	-0.01772669	0.00000251
0.5	0.00370412	0.00369948	0.00000464
0.6	0.02038350	0.02037650	0.00000700
0.7	0.03188149	0.03187333	0.00000817
0.8	0.03877625	0.03877180	0.00000445
0.9	0.04199686	0.04200968	-0.00001282
1.0	0.04249909	0.04256250	-0.00006341

Table 4.4: Solutions to Problem 1 at N = 20

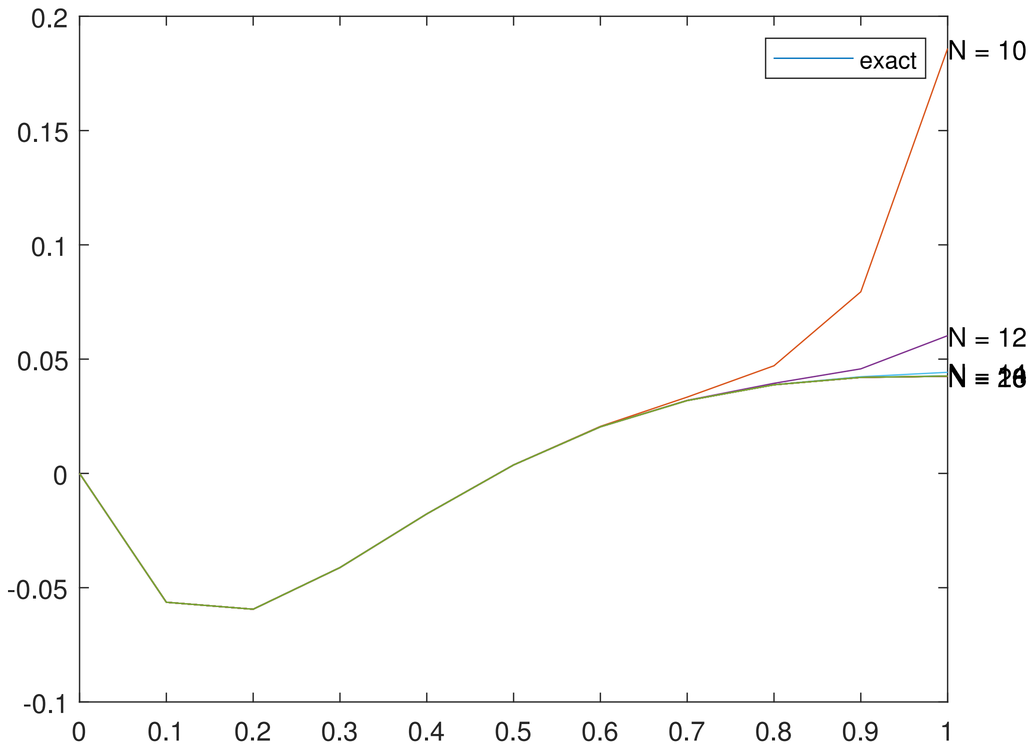

Figure 4.1: Solutions of Problem 1

Problem 2

Consider the first order non-homogeneous differential equation

$$y' + 2x = e^{3t}; y(0) = \frac{6}{5} \tag{4.2}$$

With exact solution $\frac{1}{5}e^{3x} + e^{-2x}$.

Transforming each term of the differential equations, we have:

$$(k+1)Y(k+1) + 2Y(k) = \frac{3^k}{k!}$$

$$Y(k+1) = \frac{\frac{3^k}{k!} - 2Y(k)}{(k+1)}$$

Solving for $Y(k+1)$, $k = 0, 1, 2, ..., n$, we have:

k = 0

$$Y(1) = \frac{\frac{3^0}{0!} - 2Y(0)}{(1)}$$

$$= \frac{\frac{3^0}{0!} - 2(Y(0))}{(1)} = \frac{1 - 2(6/5)}{(1)}$$

$$Y(1) = -1.4$$

k = 1

$$Y(2) = \frac{\frac{3^1}{1!} - 2Y(1)}{(2)}$$
$$= \frac{\frac{3^1}{1!} - 2(Y(1))}{(2)} = \frac{3 - 2(-1.4)}{(2)}$$
$$Y(2) = 2.9$$

k = 2

$$Y(3) = \frac{\frac{3^2}{2!} - 2Y(2)}{(3)}$$
$$= \frac{\frac{3^2}{2!} - 2(Y(3))}{(3)} = \frac{\frac{9}{2} - 2(2.9)}{(3)}$$
$$Y(3) = -0.433$$

The results is shown for different values of n solved by using a Matlab computer program and compared with results of the exact solution.

The solution is given as:

$1.2 - 1.4t + 2.9t^2 - 0.43333t^3 + 1.3417t^4 + 0.13833t^5 + 0.29139t^6 + 0.061389t^7 + 0.038894t^8 + ...$

x	Exact Solution	DTM Solution	Error
0.0	1.20000000	1.20000000	0.00000000
0.1	1.08870251	1.08870222	0.00000029
0.2	1.03474381	1.03472435	0.00001946
0.3	1.04073226	1.04050400	0.00022826
0.4	1.11335235	1.11203090	0.00132145
0.5	1.26421726	1.25901281	0.00520444
0.6	1.51112370	1.49504158	0.01608212
0.7	1.87983095	1.83775910	0.04207184
0.8	2.40653179	2.30902333	0.09750846
0.9	3.14124523	2.93507428	0.20617095
1.0	4.15244267	3.74670000	0.40574267

Table 4.5: Solutions to Problem 2 at N = 4

x	Exact Solution	DTM Solution	Error
0.0	1.20000000	1.20000000	0.00000000
0.1	1.08870251	1.08870252	-0.00000001
0.2	1.03474381	1.03474388	-0.00000007
0.3	1.04073226	1.04073240	-0.00000014
0.4	1.11335235	1.11335050	0.00000185
0.5	1.26421726	1.26419731	0.00001994
0.6	1.51112370	1.51100844	0.00011526
0.7	1.87983095	1.87933866	0.00049229
0.8	2.40653179	2.40480901	0.00172278
0.9	3.14124523	3.13603563	0.00520960
1.0	4.15244267	4.13837300	0.01406967

Table 4.6: Solutions to Problem 2 at N = 7

x	Exact Solution	DTM Solution	Error
0.0	1.20000000	1.20000000	0.00000000
0.1	1.08870251	1.08870252	-0.00000001
0.2	1.03474381	1.03474389	-0.00000008
0.3	1.04073226	1.04073261	-0.00000035
0.4	1.11335235	1.11335338	-0.00000103
0.5	1.26421726	1.26421961	-0.00000235
0.6	1.51112370	1.51112797	-0.00000426
0.7	1.87983095	1.87983592	-0.00000498
0.8	2.40653179	2.40652725	0.00000454
0.9	3.14124523	3.14118744	0.00005779
1.0	4.15244267	4.15218327	0.00025940

Table 4.7: Solutions to Problem 2 at N = 10

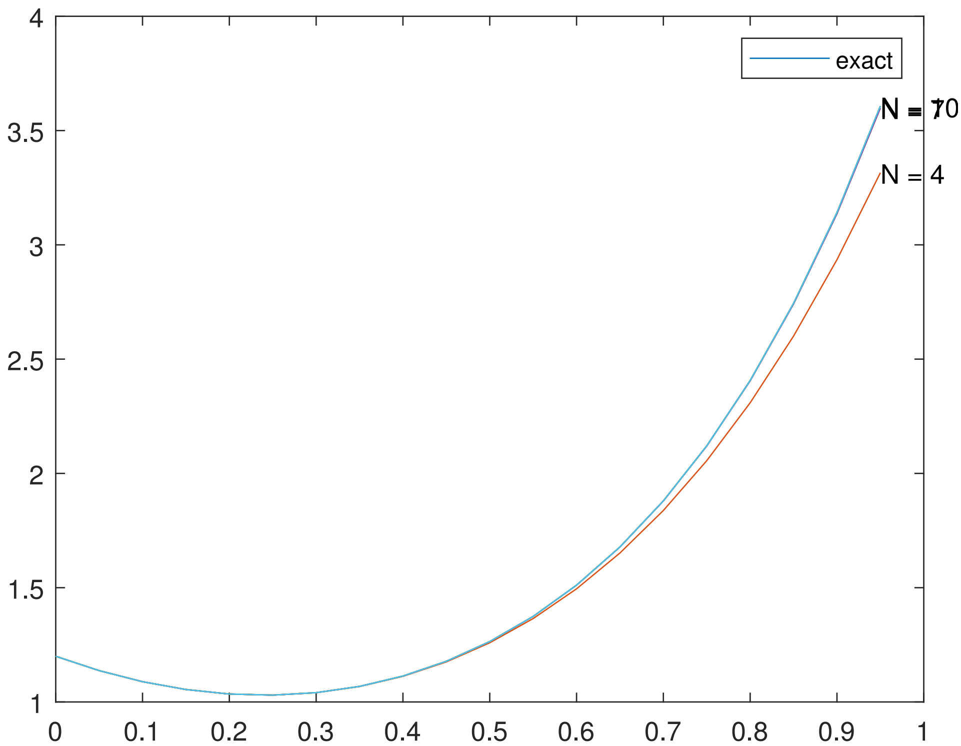

Figure 4.2: Solutions of Problem 2

Problem 3

Consider the system of 1st order differential equation:

$$y_1^{'}(x) = y_1(x) + y_2(x) \tag{4.3}$$

$$y_2^{'}(x) = -y_1(x) + y_2(x) \tag{4.4}$$

$$y_1(0) = 0, \;\; y_2(0) = 1 \tag{4.5}$$

Transforming each term of the differential equations, we have:

$$(k+1)Y1(k) = a - b \tag{4.6}$$

$$(k+1)Y2(k) = b - a \tag{4.7}$$

$$y1(k+1) = \frac{a-b}{k+1} \tag{4.8}$$

$$y2(k+1) = \frac{b-a}{k+1} \tag{4.9}$$

Solving for $Y(k+1)$ and $Y(k+1)$, $k = 0, 1, 2, 3, 4,$ using a Matlab computer program.

n: 0	n: 1	n: 3	n: 4
y1: 0	y1: -1	y1: -0.66667	y1: -0.33333
y2: 1	y2: 1	y2: 0.66667	y2: 0.33333

Table 4.8: Solutions to Problem 3

Problem 4 - Landing a Plane to the Race Track

Here we consider a real life situation of an air-place has it is about to land in an airport. With a velocity, V at height y above the ground, the Plane tires of the plane (located at the center part of the plane) must touch the horizontal ground at $y = 0$ assuming that the velocity remains constant as the plane moves and tilts to land. We can see the representation in Figure 4.3

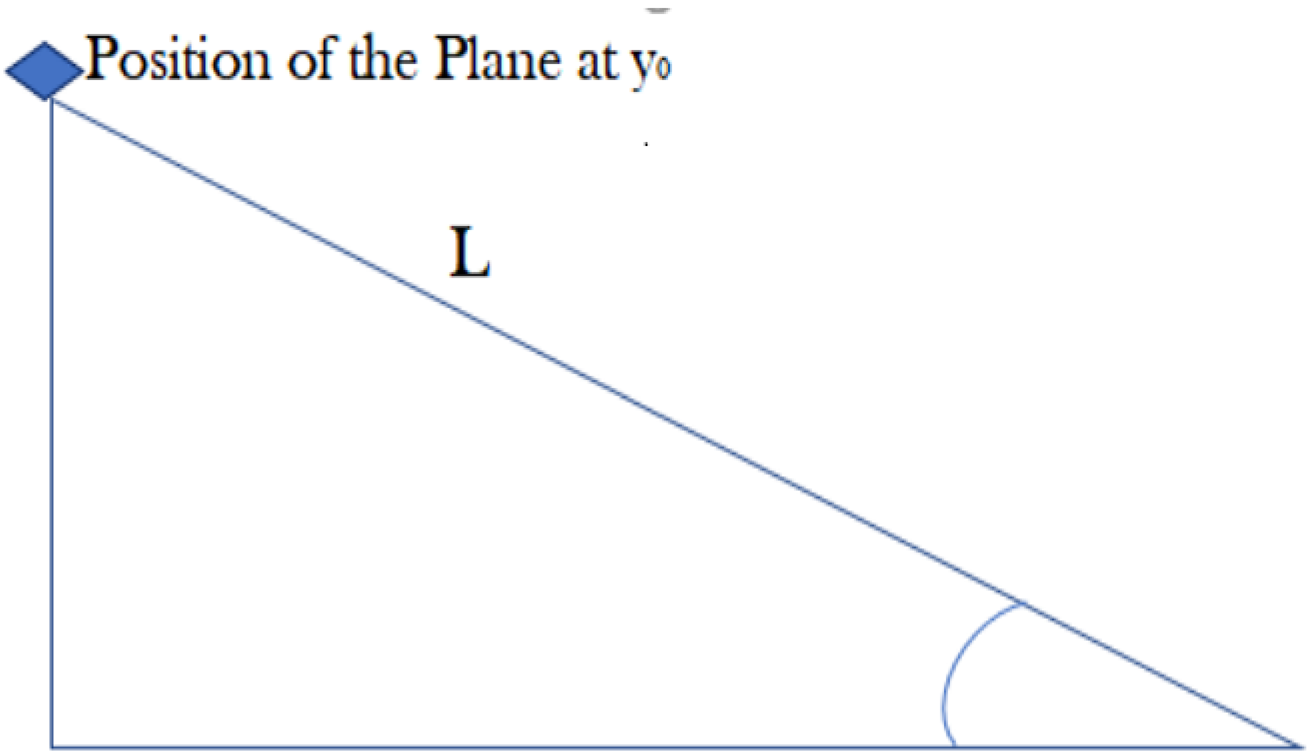

Figure 4.3: Motion of a plane as it descends to the landing track

Generating a differential equations to represent the change in the height of the plane above the ground with change in time, and taking $V = 1$ and $L = 6$, we have:

$$\frac{dy}{dt} = -\frac{y}{6} \tag{4.10}$$

With exact solution $y = 10e^{\frac{-t}{6}}$.

Transforming each term of the differential equations, we have:

$$(k+1)Y(k+1) = \tfrac{-Y(K)}{6}$$
$$Y(k+1) = \tfrac{-Y(K)}{6(K+1)}$$

Solving for $Y(k+1)$, $k = 0, 1, 2, ..., n$, we have:

k = 0

$$Y(1) = \tfrac{-Y(0)}{6(1)}$$
$$Y(1) = \tfrac{-10}{6}$$
$$Y(1) = -1.6667$$

k = 1

$$Y(2) = \tfrac{-Y(1)}{6(2)}$$
$$Y(2) = \tfrac{-(-1.6667)}{12}$$
$$Y(2) = 0.1389$$

k = 2

$$Y(3) = \tfrac{-Y(2)}{6(3)}$$
$$Y(3) = \tfrac{-(0.1389)}{18}$$
$$Y(3) = -0.007716$$

The results is shown for different values of n solved by using a Matlab computer program and compared with results of the exact solution.

The solution is therefore given as:

$$10 - 1.6667t + 0.13889t^2 - 0.007716t^3 + 0.0003215t^4...$$

time, t	Exact Solution	DTM Solution	Error
0.0	10.00000000	10.00000000	0.00000000
0.1	9.83471454	9.83471122	0.00000332
0.2	9.67216100	9.67215438	0.00000662
0.3	9.51229425	9.51228435	0.00000990
0.4	9.35506985	9.35505670	0.00001315
0.5	9.20044415	9.20042776	0.00001639
0.6	9.04837418	9.04835458	0.00001960
0.7	8.89881771	8.89879490	0.00002281
0.8	8.75173319	8.75170718	0.00002601
0.9	8.60707976	8.60705054	0.00002922
1.0	8.46481725	8.46478478	0.00003247

Table 4.9: Solutions to Problem 4 at N = 4

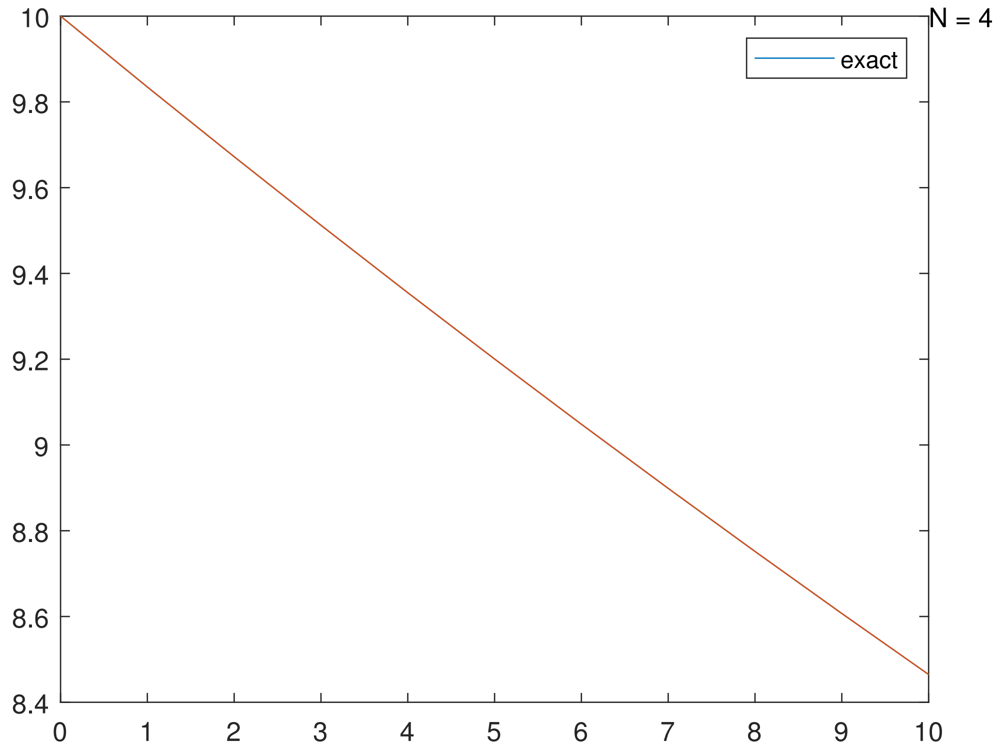

Figure 4.4: Solutions of Problem 4

Problem 5 - Changing Weight with Height

In the case of objects such as rockets, planes and other air bound vehicles, when the are start to move away from the ground, their weight changes as they move away due to changes in their height above the ground and the change in the volume of fuel they carry as it gets consumed in the flight. Considering a dumb-bell with a suspended chain being lifted during athletes exercise. The weight of the dumb-bell changes as the height is being varied, the length of the chain above the ground adds to the weight of the dumb bell with gravity, similar to the change in fuel volume of a lunched rocket.

A model of these results in the second order ordinary differential equation:

$$x\frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^2 + x = 5 \tag{4.11}$$

With exact solution .

Transforming each term of the differential equations, we have:

$$X(k+2) = \frac{5 - ((k+1)\ X(k+1))^2 - X(k)}{X(k)\ (k+2)\ (k+1)}$$

Solving for $Y(k+2)$, $k = 0, 1, 2, ..., n$, we have:

k = 0

k = 1

k = 2

The results is shown for different values of n solved by using a Matlab computer program and compared with results of the exact solution.

The solution is given as:

x	Exact Solution	DTM Solution	Error

Table 4.10: Solutions to Problem 5

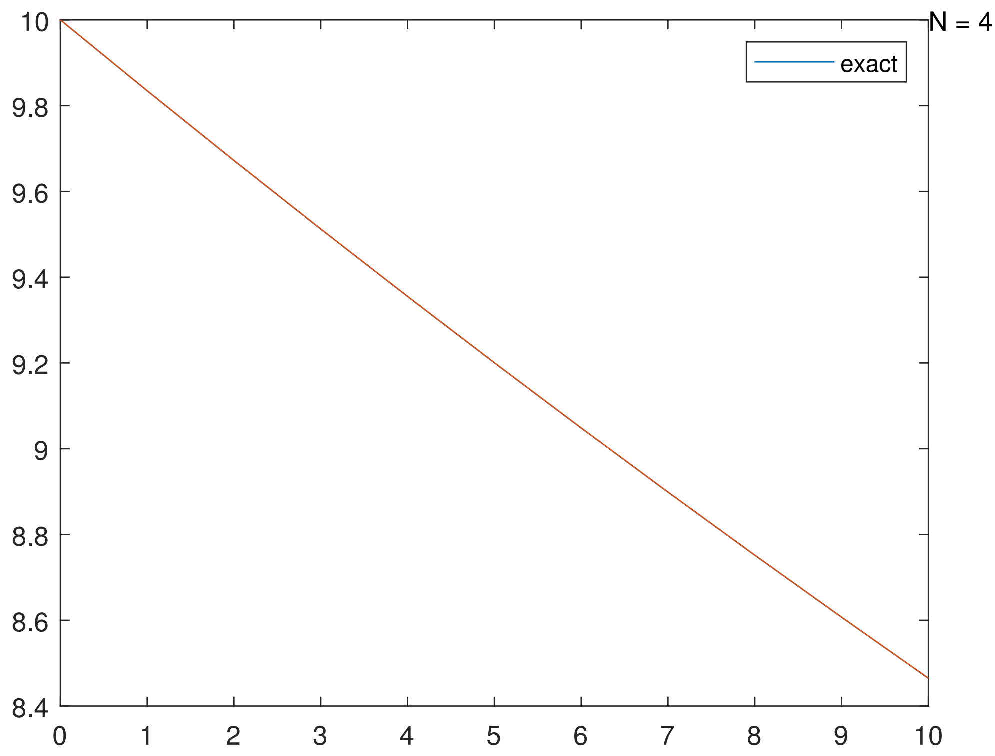

Figure 4.5: Solutions of Problem 5

Problem 6 - Spring Displacement

The change in displacement of a spring with constant k as an object of mass m, with velocity v is modeled as:

$$m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = F$$

Where c is the damping coefficient and F is the external force. For simplicity, we consider a free oscillating system such that, there is no damping effect, and taking oscillating frequency ($\epsilon_0^2 = mg$) as 64, and taking initial values in foot units, the model then results in the second order ordinary differential equation:

$$\frac{d^2y}{dt^2} + 64y = 0 \tag{4.12}$$

$$y(0) = \tfrac{3}{2} \quad y'(0) = 3$$

With exact solution: $y = \frac{3}{2}\cos 8t + \frac{3}{8}\sin 8t$.

Transforming each term of the differential equations, we have:

$$X(k+2) = \frac{5\ -\ (\ (k+1)\ X(k+1)\ )^2\ -\ X(k)}{X(k)\ (k+2)\ (k+1)}$$

Solving for $Y(k+2)$, $k = 0, 1, 2, ..., n$, we have:

$\mathbf{k = 0}$

$\mathbf{k = 1}$

$\mathbf{k = 2}$

The results is shown for different values of n solved by using a Matlab computer program and compared with results of the exact solution.

The solution is given as:

x	Exact Solution	DTM Solution	Error

Table 4.11: Solutions to Problem 6

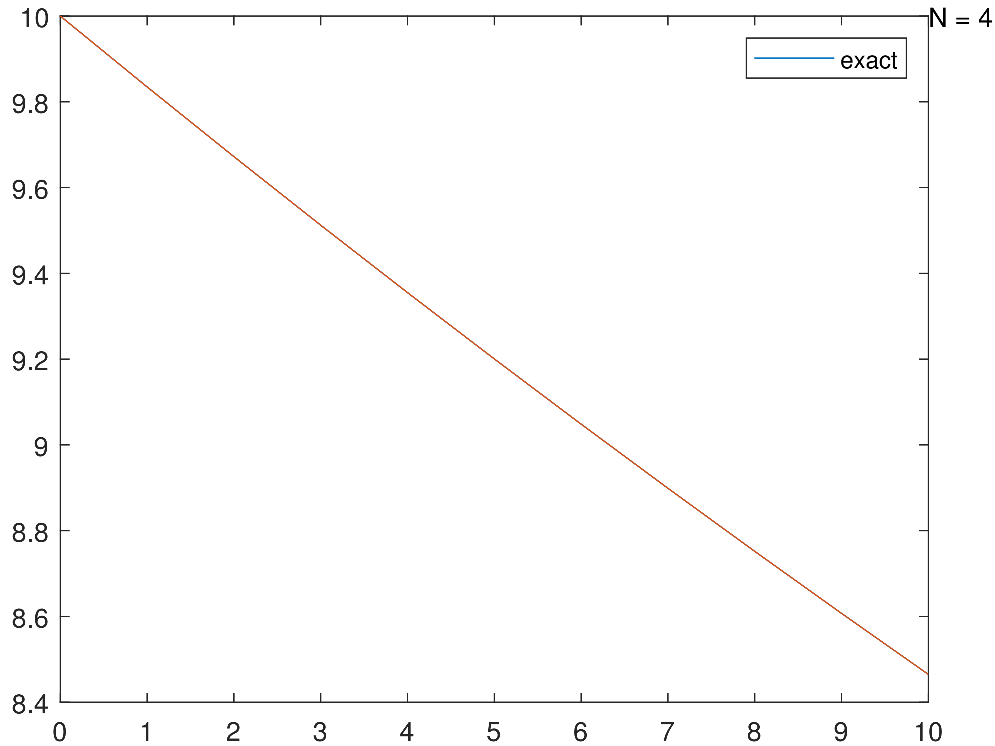

Figure 4.6: Solutions of Problem 6

Chapter 5

SUMMARY AND CONCLUSION

5.1 SUMMARY

In this work, theDifferential Transform Techniquehas been successfully applied to find solutions to linear and nonlinear differential equations. The results of each of the problems are in presented tables and shows that the Differential Transform Technique converges to the exact solution with few iterations and reduces the computational difficulties unlike the other traditional methods.

In figure 4.1, the graph shows how the solution obtained with the Differential Transform Technique converges to the exact solution from small values of $N \geq 10$ for Problem 1, this is shown as the plots for the each values of N overlaps with the plot for the exact solution. For the Problem 2, figure 4.2 show the overlap from values $N = 7, 10...$ For a system of equations as in Problem 3 that would require a lot of computation in obtaining the exact solution, the technique provides a polynomial that can be presented as a numerical solution seen in (4.8). In problem 4, the technique is applied to a real life problem and the result with just 4 iterations is just as the exact one.

5.2 CONCLUSION

This project has verified the effectiveness of Differential Transform Technique in solving differential equations. This method can be applied to solve many differential equations without the need for linearization. With solutions from the Differential Transform Technique converging to the exact solution with just a few iterations, it makes it a suitable method to save computational time and effort to obtain accurate results

APPENDIX

Matlab computer programs used to obtain the solutions

MATLAB Program for Problem 1

```
function ques1()

n = 5; x = (0.1:0.1:1); l = 1;

while n <= 20 %how many iterations to perform

errors = zeros();        exact = zeros(); % arrays

dtm = zeros();        k = 0;        kvalues = zeros(); %array

kvalues(1) = 0;  % value y1

kvalues(2) = -1; % value y2

theequation = '';

while k <= n % the loop to solve the k values

a = kvalues(k + 1); b = kvalues(k + 2);

kvalues(k + 3) = ( ( ((4*((-3)^k))/(factorial(k))) + (-7*(k+1)*(b)) + (-10*(a))) / ((k+1

% disp(kvalues(k + 1)); k = k+1;

end

i = 1; while i <= n+3 % loop to generate the polynomial solutionn of the DTM

p = num2str(i - 1); if kvalues(i) < 0

kval = num2str( -1 *  kvalues(i) );

formed = strcat( ' -(',kval,'*(t','^', p ,'))' );

else

kval = num2str(kvalues(i));

formed = strcat( ' +(',kval,'*(t','^', p ,'))' );

end
```

```
theequation = strcat( theequation, formed);            i = i+1;
end
f = str2func(strcat('@(t)',theequation));
g = 'exp(-2*x) + exp(-5*x) - (2 * exp(-3*x))';
gx = str2func(strcat('@(x)',g));
ti = 0.1;            m = 1; disp ( strcat('N =  ', num2str(n)) );
while ti <= 1
exact(m) = gx(ti);                dtm(m) = f(ti);
errors(m) = exact(m) - dtm(m);
str = ' %2.1f & %4.8f & %4.8f & %4.8f \\\\ \\hline \n ';
fprintf (str, ti , exact(m), dtm(m), errors(m) );
m = m + 1;                ti = ti + 0.1;
end
plot(x,exact, x,dtm); dd = strcat("N = ", num2str(n)); legend("exact")
if max(dtm) > 0 text(max(x), max(dtm), dd)
end
if max(dtm) < 0 text(max(x), min(dtm), dd) hold on
end
l = l + 1;            n = n + 5;
end
```

MATLAB Program for Problem 2

```
function ques2()
n = 4;    x = (0.1:0.1:1);    l = 1;
while n <= 10 %how many iterations to perform
errors = zeros(10,1);         exact = zeros(10,1);
dtm = zeros(10,1);        solutions = strings(); % arrays
k = 0;   %Initial value of k is 0
kvalues = zeros(); % the array to save the values of k
kvalues(1) = (6/5);  % value y1        %kvalues(2) = -1; % value y2
theequation = '';
```

```
while k <= n % the loop to solve the k values
a = kvalues(k + 1);            %b = kvalues(k + 2);
kvalues(k + 2) = ( ( ((3^k)/(factorial(k))) + (-2*a)) / (k+1)); k = k+1;
end
i = 1;
while i <= n+2 %generate the polynomial solution of the DTM
p = num2str(i - 1); if kvalues(i) < 0
kval = num2str( -1 *  kvalues(i) ); formed = strcat( ' -(',kval,'*(t','^', p ,'))' );
else
kval = num2str(kvalues(i)); formed = strcat( ' +(',kval,'*(t','^', p ,'))' );
end
theequation = strcat( theequation, formed); i = i+1;
end
solutions(l) = theequation;
f = str2func(strcat('@(t)',theequation));
g = '((1/5) * exp(3*x)) + exp(-2*x)';
gx = str2func(strcat('@(x)',g));
ti = 0.1;            m = 1;
disp ( strcat('N =  ', num2str(n)) ) disp ( solutions(l) )
while ti <= 1
exact(m) = gx(ti); dtm(m) = f(ti); errors(m) = exact(m) - dtm(m);
str = ' %2.1f & %4.8f & %4.8f & %4.8f \\\\ \\hline \n ';
fprintf (str, ti , exact(m), dtm(m), errors(m) );
m = m + 1; ti = ti + 0.1;
end
plot(x,exact, x,dtm) dd = strcat("N = ", num2str(n));
legend("exact") text(max(x), max(dtm), dd)
hold on
l = l + 1; n = n + 3;
end
```

Problem 3

```
function ques3()

y1 = zeros(); y2 = zeros();

y1(1) = 0; y2(1) = 1;

k = 0; n = 5;

while k < n

a = y1(k+1); b = y2(k+1);

y1(k+2) = (a-b)/(k+1); y2(k+2) = (b-a)/(k+1);

disp ('........'); disp ( strcat('n:', num2str(k)) );

disp ( strcat('y1:',num2str(y1(k+1))) ); disp ( strcat('y2:',num2str(y2(k+1))) );

k = k + 1;

end
```

Problem 4

```
function ques4()

n = 4;    x = (1:1:10);    l = 1;

while n == 4 %how many iterations to perform

errors = zeros(10,1);         exact = zeros(10,1);

dtm = zeros(10,1);        solutions = strings(); % arrays

k = 0;   %Initial value of k is 0

kvalues = zeros(); % the array to save the values of k

kvalues(1) = (10);  % value y1

theequation = '';

while k <= n % the loop to solve the k values

a = kvalues(k + 1);

kvalues(k + 2) = (-a / (6 * (k + 1))); k = k+1;

end

i = 1;

while i <= n+2 %generate the polynomial solution of the DTM

p = num2str(i - 1);
```

```
if kvalues(i) < 0
kval = num2str( -1 *  kvalues(i) );
formed = strcat( ' -(',kval,'*(t','^', p ,'))' );
else
kval = num2str(kvalues(i));
formed = strcat( ' +(',kval,'*(t','^', p ,'))' );
end
theequation = strcat( theequation, formed); i = i+1;
end
solutions(l) = theequation;
f = str2func(strcat('@(t)',theequation)); g = '10 * exp((-x)/6)';
gx = str2func(strcat('@(x)',g)); ti = 0.1;       m = 1;
disp ( strcat('N =  ', num2str(n)) ) disp ( solutions(l) )
while ti <= 10
    exact(m) = gx(ti); dtm(m) = f(ti);
  errors(m) = exact(m) - dtm(m);
str = ' %2.1f & %4.8f & %4.8f & %4.8f \\\\ \\hline \n ';
fprintf (str, ti , exact(m), dtm(m), errors(m) );
m = m + 1; ti = ti + 1;
end
plot(x,exact, x,dtm) dd = strcat("N = ", num2str(n));
legend("exact") text(max(x), max(dtm), dd)
hold on l = l + 1; n = n + 3;
end
```