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Fusion cross sections for fusion energy☆

Xing Z. Li [a,*], Bin Liu [a], Qing M. Wei [a], Shu X. Zheng [b], Dong X. Cao [b]

[a] *Department of Physics, Tsinghua University, Beijing 100084, China*
[b] *Department of Engineering Physics, Tsinghua University, Beijing 100084, China*

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Abstract

A new formula for fusion cross sections is proposed with less parameters and better understanding of fusion physics. It is based on the complex nuclear potential in connection with the Coulomb repulsion field. It reproduces the ENDF/B-VI data from 200 eV to 280 keV for D + T fusion reaction reasonably well. The widely used formula in *NRL's Handbook* is discussed while some mistakes there are corrected. This formalism is particularly useful in searching the low energy resonance between two charged nuclei.

Keywords: Fusion cross section; D + T fusion; Gamow factor; *S*-factor; Matching damping

1. Introduction

The fusion cross section is one of the most important physical quantities in the fusion engineering and design. Early in 1970s an empirical formula was proposed [1] with five parameters. This formula has been widely used in fusion research since then. It gave correctly the Gamow factor and the geometric factor, and the remaining part was approximated by a fraction of polynomials. It was cited in Naval Research Laboratory (NRL) Plasma Formulary, which stemmed from D.L. Book NRL Memorandum Report No. 3332 (1977). This 5-parameter formula for fusion cross sections was cited first from reference 27 of NRL Plasma Formulary (G.H. Miley, H. Towner and N. Ivich, *Fusion Cross Section and Reactivities*, Rept. COO-2218-17, University of Illinois, Urbana, IL, 1974). Later, after 1983, in NRL Plasma Formulary (revised) the same 5-parameter formula was cited from reference 28, i.e. B.H. Duane, *Fusion Cross Section Theory*, Rept. BNWL-1685 (Brookhaven National Laboratory, 1972) [2]. This report attributed mistakenly to Brookhaven National Laboratory instead of Battelle Pacific Northwest Laboratory, contained no theory, but an empirical fitting of the fusion cross sections. In a word, the theory about fusion cross sections was just the Gamow factor and the geometric factor. In 1992, H.-S. Bosch and G.M. Hale published "Improved Formulas for Fusion

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* Corresponding author. Tel.: +86 10 6278 4343; fax: +86 10 6278 4343 or 86 10 6278 1604.
E-mail address: lxz-dmp@tsinghua.edu.cn (X.Z. Li).

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Cross Sections and Thermal Reactivities" [3]. Based on R-matrix theory and on new and more accurate measurements of fusion cross sections, it pointed out that this 5-parameter formula was not adequate, in particular at low energy. A 10-parameter formula was proposed there to replace this 5-parameter formula. In the present paper, we propose a 2-parameter formula which has a clearer indication of the physics than the 5-parameter.

2. Astrophysical factor (*S*-factor)

Due to historical reasons, the fusion cross section, $\sigma(E)$, is separated into three factors: Gamow factor, geometric factor and astrophysical factor (*S*-factor) as:

$$\sigma(E) = \frac{2\pi}{\exp[2\pi/(ka_c)] - 1}\frac{\pi}{k^2}S(E) \tag{1}$$

Here, E is the energy of the relative motion of the fusion particles; k the corresponding wave number; a_c is the Coulomb unit of length:

$$k = \sqrt{\frac{2\mu E}{\hbar^2}}; \qquad a_c = \frac{\hbar^2}{Z_1 Z_2 \mu e^2}. \tag{2}$$

Here, μ is the reduced mass of two fusion nuclei; $(Z_1 e)$ and $(Z_2 e)$ are their electrical charges, respectively; $\hbar$ is the Planck constant divided by 2π. In the *NRL Handbook*, this *S*-factor was written as:

$$S(E) = A_5 + \frac{A_2}{(A_4 - A_3 E)^2 + 1} \tag{3}$$

Here, A_2, A_3, A_4 and A_5 are four fitting parameters; in the *NRL Handbook* the Gamow factor is approximated by a variant form of that in Eq. (1) and includes the fifth parameter. In the present paper, a new formula for $S(E)$ is proposed as:

$$S^*(E) = \frac{-4w_i}{w_r^2 + \left(w_i - (1/\theta^2)\right)^2} \tag{4}$$

Here, θ^2 is related to the Gamow factor as:

$$\theta^2 = \frac{\exp[2\pi/(ka_c)] - 1}{2\pi} \tag{5}$$

w_r and w_i are the new parameters related to the phase shift of the Coulomb wave function defined in the following section.

3. Phase shift of Coulomb wave function

Fusion reactions happen between two charged nuclei. When they are far away beyond the nuclear interaction range, the Coulomb repulsion force dominates. The relative motion of two charged nuclei is well described by the Coulomb wave function: the linear combination of the regular Coulomb wave function F_o, and the irregular Coulomb wave function $G_{o.}$ The fusion reaction in the nuclear interaction well will have two effects on this linear combination: (1) the nuclear attractive force will change the phase of the Coulomb wave function (i.e. the coefficients in the linear combination); (2) the absorption of the wave function inside the nuclear well will make this phase shift complex. This complex phase shift, δ_o, may be written as:

$$\operatorname{Cot}\delta_o = \theta^2(w_r + iw_i) \tag{6}$$

The simplest model to calculate this phase shift is the square-well model with a complex nuclear potential $U_r + iU_i$. Using the connection of the wave function at the boundary of the nuclear well, the phase shift of the Coulomb wave function may be expressed by these two parameters as:

$$\operatorname{Cot}\delta_o = \theta^2\left\{k_1 a_c \operatorname{Cot}(k_1 a) - 2\left[\ln\left(\frac{2a}{a_c}\right) + 2C + \frac{(ka_c)^2}{12}\right]\right\} \tag{7}$$

Here, k_1 is the wave number inside the nuclear well; a the radius of the nuclear well; $C = 0.57721\ldots$ is the Euler constant:

$$k_1 = \sqrt{\frac{2\mu(E - U_r - iU_i)}{\hbar^2}} \tag{8}$$

Two adjustable parameters U_r and U_i are involved in k_1. They are the only two parameters in this model. The nuclear radius, a, is fixed by the radius of deuteron (4.4×10^{-15} m [4]) and the relation between mass number and the nuclear radius, i.e. $a = 1.746 \times (2^{1/3} + 3^{1/3}) \times 10^{-15}$ m. It is found that $U_r = -47.6712$ MeV and $U_i = -118.660$ keV would give a good fit with the ENDF/B-VI data [5] as seen in the Table 1.This 2-parameter formula deviates from the ENDF/B-VI data less than 9% in general. Moreover, it

Table 1
Comparison for D + T fusion cross section

Deuteron energy in lab. (eV)	Fusion cross section evaluated by Eq. (9) (barns)	Cross section evaluated by ENDF/B-VI (barns)	Relative error with respect to ENDF/B-VI	Cross section evaluated by Hale–Bosch formula (barns)
200	7.422E−39	7.433E−39	0.001	6.563E−39
300	4.051E−31	4.056E−31	0.001	3.724E−31
400	1.585E−26	1.586E−26	0.001	1.486E−26
700	2.514E−19	2.514E−19	0.000	2.420E−19
1000	1.028E−15	1.000E−15	0.027	9.993E−16
4000	1.171E−06	1.160E−06	0.009	1.155E−06
7000	1.570E−04	1.544E−04	0.016	1.546E−04
10000	1.772E−03	1.730E−03	0.024	1.742E−03
14000	1.174E−02	1.140E−02	0.029	1.151E−02
20000	6.188E−02	5.968E−02	0.036	6.049E−02
26000	1.747E−01	1.683E−01	0.037	1.708E−01
35000	4.854E−01	4.700E−01	0.032	4.769E−01
50000	1.376	1.370	0.004	1.378
65000	2.595	2.656	0.024	2.660
80000	3.833	3.957	0.032	3.970
95000	4.731	4.801	0.015	4.839
104000	5.018	4.996	0.004	5.045
110000	5.103	5.010	0.018	5.062
120000	5.082	4.877	0.040	4.924
135000	4.784	4.465	0.067	4.493
150000	4.325	3.970	0.082	3.975
180000	3.360	3.072	0.086	3.050
220000	2.368	2.233	0.057	2.208
280000	1.481	1.505	0.016	1.496

gives a good indication on the feature of low energy resonance (if any). For comparison, we calculated the cross sections using 10-parameter formula in reference [3] as listed in fifth column of Table 1. At low energy, the present formulation is a little better; however, at higher energy the 10-parameter formula is better. The most useful quantity for fusion applications is the fusion Maxwellian reactivity. We are trying to obtain the fusion Maxwellian reactivity at low temperature using the present formulation.

4. Discussion

The early theory on nuclear fusion was proposed by H.A. Bethe [6] in order to explain the origin of solar energy. It contained a square nuclear well and a Coulomb repulsion field connected to the nuclear well; however, it did not consider the effect of absorption on the Coulomb wave function. In 1950s, B.H. Flowers proposed a similar model with complex nuclear reaction length [7]. It fitted roughly the experimental data from 25 to 200 keV available at that time. In 1960s, D.D. Clayton applied a Breit–Wigner type formula on the resonant fusion reactions [8], although the Breit–Wigner formula was proposed mainly for the heavy nuclei where compound nucleus model was valid. In 1990s, a new formula for fusion cross section was proposed in terms of complex Cot δ_o, where δ_o is the complex phase shift for the s-wave [9–12]. This new formula clearly showed the resonance effect on penetration of Coulomb barrier:

$$\sigma(E) = \underbrace{\frac{2\pi}{\mathrm{Exp}[2\pi/(ka_c)] - 1}}_{\text{Gamow factor}} \underbrace{\frac{\pi}{k^2}}_{\text{Geometric factor}} \times \underbrace{\frac{-4w_i}{w_r^2 + \left(w_i - (1/\theta^2)\right)^2}}_{S\text{-factor}} \tag{9}$$

When $w_r \to 0$, S-factor approaches a resonance; however, this is not enough to overcome the Gamow factor if w_i does not approaches $\left(-\frac{1}{\theta^2}\right)$. w_i is related to the imaginary part of the nuclear potential U_i as follows:

$$w_i = \mathrm{Im}[k_1 a_\mathrm{c}\,\mathrm{Cot}(k_1 a)] = \left.\frac{\partial[k_1 a_\mathrm{c}\,\mathrm{Cot}(k_1 a)]}{\partial(k_1 a_\mathrm{c})}\right|_{k_1=k_{1r}} (k_{1i} a_\mathrm{c}) \tag{10}$$

$$k_{1i} a_\mathrm{c} \propto k_{1i} a = -\frac{\mu a}{k_{1r}\hbar^2} U_i = \left|\frac{a/(k_{1r}\hbar/\mu)}{\hbar/U_i}\right| \approx \frac{\tau_\mathrm{flight}}{\tau_\mathrm{life}} \tag{11}$$

Here, τ_flight is the flight time of the projectile inside the nuclear well; τ_life is the life-time of the projectile inside the nuclear well.

$w_i \to -\frac{1}{\theta^2}$ would require $\frac{\tau_\mathrm{flight}}{\tau_\mathrm{life}} \to \mathrm{O}\left(\frac{1}{\theta^2}\right)$. It simply means a very long life-time, i.e.

$$\tau_\mathrm{life} \approx \theta^2 \tau_\mathrm{flight} \tag{12}$$

Eqs. (12) and (9) show that if we are looking for a resonance between two charged particles at low energy, then, we have to look for the nuclear products of certain weak-interaction (i.e. long life-time nuclear reaction). This is particularly important when we are searching for the unknown nuclear resonance between two charged particles at low energy. This life-time in Eq. (12) may be considered as a matching damping to this resonance.

In burning plasmas, the ion temperature might be inferred from the yield of fusion neutrons. This better formula for fusion cross section is important for a good diagnostics mainly at low plasma temperatures (the 5-parameter formula gave much lower D–T fusion cross section by a factor of 100 at 200 eV).

Fusion researchers, including plasma physicists and fusion engineers, tend to ignore the nuclear physics aspects of fusion. The citation of the fusion cross section was mistaken early in 1980s and such mistake persisted after many revisions over twenty years. More attention to the nuclear physics of nuclear fusion should be paid.

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References

[1] D.L. Book, NRL Plasma Formulary, Revised 1980, p. 44.
[2] B.H. Duane, "Fusion cross-section theory", in Annual Report on CTR Technology 1972, in: W.C. Wolkenhauer (Ed.), Rep. BNWL-1685, Battelle Pacific Northwest Laboratory, Richland, WA, 1972.
[3] H.-S. Bosch, G.M. Hale, Improved formulas for fusion cross-sections and thermal reactivities, Nucl. Fusion 32 (1992) 611–631.
[4] H. Feshbach, Theoretical Nuclear Physics–Nuclear Reactions, John Wiley & Sons Inc., New York, 1992, p. 488.
[5] V. Pronyaev, Private Communication on December 21, 2000 from Physics Section, IAEA, see also "d–t reaction evaluation, based on ^{5}He system R-matrix analysis by G. Hale (7/94), and T(d,n) Legendre coefficients evaluated by M. Drosg (12/94): (http://t2.lanl.gov/data/data/ENDF-deuteron/H/dt) "; or from National Nuclear Data Center (http://www.nndc.bnl.gov).
[6] H.A. Bethe, C.L. Critchfield, The formation of deuterons by proton combination, Phys. Rev. 54 (1938) 248–254.
[7] B.H. Flowers, The theory of D + T reaction, Proc. R. Soc. A 204 (1951) 503–513.
[8] D.D. Clayton, The Principles of Stellar Evolution and Nucleosynthesis, McGraw-Hill, New York, 1990, pp. 342–350; (originally published 1968), University of Chicago Press edition, reprinted in USA.
[9] X.Z. Li, J. Tian, M.Y. Mei, C.X. Li, Sub-barrier fusion and selective resonant tunneling, Phys. Rev. C 61 (2000), 024610-1–024610-6.
[10] X.Z. Li, C.X. Li, H.F. Huang, Maximum value of the resonant tunneling current through the coulomb barrier, Fusion Technol. 36 (1999) 324–330.
[11] X.Z. Li, B. Liu, X.Z. Ren, J. Tian, W.Z. Yu, D.X. Cao, Study of nuclear physics for nuclear fusion, J. Fusion Energy 19 (2002) 163–168.
[12] X.Z. Li, Nuclear physics for nuclear fusion, Fusion Sci. Technol. 41 (2002) 63–68.