Brillouin Limit and Beyond: A Route to Inertial-Electrostatic Confinement of a Single-Species Plasma

Leaf Turner and D. C. Barnes
Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545
(Received 15 September 1992)

The physics limiting the density of a cold non-neutral plasma is presented. It is shown that when a flow has no strain, the instantaneous maximum number of charges that can be stored within a fixed boundary is equal to the total magnetic field energy within divided by the relativistic rest energy of a single charge. A higher limit can be supported by the presence of a deviatoric strain in the flow. Brillouin flow equilibria with arbitrarily high values of the Brillouin ratio, leading to the possibility of pure inertial-electrostatic confinement of non-neutral plasmas, are demonstrated.

PACS numbers: 52.25.Wz, 52.55.Dy, 52.55.Lf, 52.55.Mg

It is well known that one-species plasmas can be confined virtually indefinitely by a uniform magnetic field in a state of uniform density corresponding to a cylindrically symmetric thermodynamic equilibrium [1]. We shall refer to the ratio of relativistic rest energy density to magnetic energy density as the Brillouin ratio. The limit on this ratio, the Brillouin limit, has received scrutiny over the years [2]. The interest in such plasmas extends to other fields, in particular, that of astrophysics [3].

In the first portion of this Letter, we show that the nine components of the velocity strain tensor, $\partial_i v_j$, divide naturally into three separate classes according to their effect on the stored plasma density; namely, the compressive strain, $\nabla\cdot\mathbf{v}$, the three components of the vorticity, $\nabla\times\mathbf{v}$, and the five components of the deviatoric strain, $\partial_i v_j + \partial_j v_i - \frac{2}{3}\delta_{ij}\nabla\cdot\mathbf{v}$ [4]. (We use the convention that $\partial_k = \partial/\partial x_k$.) With this insight, in the concluding portion of this Letter, we construct Brillouin flow equilibria that possess arbitrarily high Brillouin ratios, leading to the possibility of pure inertial-electrostatic confinement (IEC).

The fundamental equation we employ is the force-balance equation for the cold single-species plasma:

$$\frac{\partial \mathbf{v}(\mathbf{r},t)}{\partial t} = -\frac{1}{2}\nabla v^2(\mathbf{r},t) + \frac{q\mathbf{E}(\mathbf{r},t)}{m} + \mathbf{v}(\mathbf{r},t)\times[\boldsymbol{\omega}(\mathbf{r},t)+\boldsymbol{\Omega}_B(\mathbf{r},t)]\,, \tag{1}$$

in which $\boldsymbol{\omega}(\mathbf{r},t)$ is the vorticity, $\nabla\times\mathbf{v}(\mathbf{r},t)$, and $\boldsymbol{\Omega}_B(\mathbf{r},t)$ equals $q\mathbf{B}(\mathbf{r},t)/m$ [with $\mathbf{B}(\mathbf{r},t)$ the magnetic field], and the charge and mass of the confined species are specified by q and m, respectively. The electric field is represented by $\mathbf{E}(\mathbf{r},t)$. Defining the square of the local plasma frequency by $\omega_p^2(\mathbf{r},t)\equiv q^2 n(\mathbf{r},t)/\epsilon_0 m$, where $n(\mathbf{r},t)$ is the number density and ϵ_0 is the electric permittivity of free space, we take the divergence of Eq. (1):

$$\omega_p^2 = \frac{\partial(\nabla\cdot\mathbf{v})}{\partial t} - \omega^2 - \boldsymbol{\omega}\cdot\boldsymbol{\Omega}_B + \nabla^2\left(\frac{\mathbf{v}^2}{2}\right) + \mathbf{v}\cdot[\nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}]\,.$$

(In nonrelativistic, non-neutral plasmas, the Coulomb force dominates any Lorentz force due to self-generated magnetic fields by a factor of order c^2/v^2. Therefore, in marked contrast to the physics of quasineutral plasmas, plasma diamagnetic effects are negligible; i.e., the magnetic field may be taken to be the externally imposed one.) We assume that the plasma remains confined within a fixed, stationary boundary at all times, so that on the boundary $\mathbf{v}\cdot\hat{\mathbf{n}}=0$. Utilizing this condition, we find

$$\int \omega_p^2 d^3r = \int\left\{-\omega^2 - \boldsymbol{\omega}\cdot\boldsymbol{\Omega}_B + \nabla^2\left(\frac{v^2}{2}\right) - \mathbf{v}\cdot\nabla^2\mathbf{v} - (\nabla\cdot\mathbf{v})^2\right\}d^3r\,.$$

Defining the tensor $d_{ij}(\mathbf{r},t)$ by $d_{ij}(\mathbf{r},t)\equiv\partial_i v_j(\mathbf{r},t)$, this can be reexpressed as

$$\int \omega_p^2(\mathbf{r},t)d^3r = \int[d_{jk}(\mathbf{r},t)d_{kj}(\mathbf{r},t) - \epsilon_{ijk}d_{jk}(\mathbf{r},t)\Omega_{Bi}(\mathbf{r},t) - d_{kk}(\mathbf{r},t)d_{jj}(\mathbf{r},t)]d^3r\,. \tag{2}$$

Throughout this Letter, we use the Einstein summation convention that repeated indices are summed on. The third-rank ϵ tensor is the totally antisymmetric Levi-Cività tensor. We employ a representation obtained as the Kronecker product of the gradient vector operating on the velocity vector. In this representation, the basic vector $\tilde{\mathbf{d}}$, satisfying

$$\tilde{\mathbf{d}} = \left(\frac{\nabla\cdot\mathbf{v}}{\sqrt{3}}, \frac{\nabla\times\mathbf{v}}{\sqrt{2}}, \frac{2\partial_x v_x - \partial_y v_y - \partial_z v_z}{\sqrt{6}}, \frac{\partial_y v_y - \partial_z v_z}{\sqrt{2}}, \frac{\partial_y v_z + \partial_z v_y}{\sqrt{2}}, \frac{\partial_z v_x + \partial_x v_z}{\sqrt{2}}, \frac{\partial_x v_y + \partial_y v_x}{\sqrt{2}}\right), \tag{3}$$

has nine linearly independent components related to the nine tensor elements, d_{ij}, by an orthogonal transformation. In this representation, the nine-component tensor $-\epsilon_{ijk}\Omega_{Bi}$ is specified by the vector $\tilde{\boldsymbol{\Omega}}_B = (0, -\sqrt{2}\boldsymbol{\Omega}_B, 0,0,0,0,0)$. Hence, Eq. (2) can be expressed equivalently in the more readily physically interpretable version,

$$\int \omega_p^2(\mathbf{r},t)d^3r = \int [\tilde{\mathbf{d}}(\mathbf{r},t)\cdot \mathbf{M}\cdot \tilde{\mathbf{d}}(\mathbf{r},t) + \tilde{\mathbf{d}}(\mathbf{r},t)\cdot \tilde{\mathbf{\Omega}}_B(\mathbf{r},t)]d^3r\,, \tag{4}$$

where the matrix $\mathbf{M}$ is diagonal in the chosen representation with the form

$$M_{ij} = \delta_{ij}\tilde{\lambda}_j, \quad \tilde{\lambda} = (-2,-1,-1,-1,1,1,1,1,1)\,. \tag{5}$$

We have thus converted Eq. (2) to a simple diagonal quadratic form.

Let us first suppose that the deviatoric strain components, i.e., the last five components of $\tilde{\mathbf{d}}$, vanish. For such a flow, the number density is maximized when

$$\delta\int \omega_p^2(\mathbf{r},t)d^3r = \int \delta\tilde{\mathbf{d}}(\mathbf{r},t)\cdot[2\mathbf{M}\cdot\tilde{\mathbf{d}}(\mathbf{r},t)+\tilde{\mathbf{\Omega}}_B(\mathbf{r},t)]d^3r = 0\,.$$

This condition must hold for arbitrary variations and thus, for all points $\mathbf{r}$ within the plasma volume, the following compressibility and vorticity conditions must be met:

$$\sqrt{3}\tilde{d}_1(\mathbf{r},t) = \nabla\cdot\mathbf{v}(\mathbf{r},t) = 0\,,$$
$$\sqrt{2}\tilde{d}_2(\mathbf{r},t) = \omega_1(\mathbf{r},t) = -\Omega_{B1}(\mathbf{r},t)\,,$$
$$\sqrt{2}\tilde{d}_3(\mathbf{r},t) = \omega_2(\mathbf{r},t) = -\Omega_{B2}(\mathbf{r},t)\,,$$
$$\sqrt{2}\tilde{d}_4(\mathbf{r},t) = \omega_3(\mathbf{r},t) = -\Omega_{B3}(\mathbf{r},t)\,.$$

Inserting these conditions back into Eq. (4) and using Eq. (5), we obtain the final result that

$$\frac{2\int\omega_p^2(\mathbf{r},t)d^3r|_{\max}}{\int\Omega_B^2(\mathbf{r},t)d^3r} = 1\,.$$

We have thus proved that when there is no deviatoric strain, as in the case of thermodynamic equilibrium—a state of uniform translation and rigid rotation, the maximum number of charges that can be stored at any instant of time in the fixed bounding volume is equal to the total magnetic field energy contained in that volume divided by the relativistic rest energy of a single charge. For a particular geometry, the physical upper limit actually may be lower.

With this heightened understanding, we now develop hydrodynamic flow equilibria that exceed the Brillouin limit. We consider a flow that (a) satisfies the Brillouin flow criterion, i.e., $\boldsymbol{\omega} = -\mathbf{\Omega}_B$, so that the flow is incompressible and satisfies $\nabla\times\boldsymbol{\omega}=0$; and (b) possesses a nonvanishing deviatoric strain tensor. Not only does the first condition, along with the second condition, according to Eqs. (3), (4), and (5), not lead to a diminution of the density confined, but it prevents the nonvanishing deviatoric strain of the second condition from producing any viscous force, $\nu\nabla^2\mathbf{v}$, within the plasma. [Recall that $\nabla^2\mathbf{v} = \nabla(\nabla\cdot\mathbf{v}) - \nabla\times\boldsymbol{\omega}$.] When finite viscosity effects are considered, only a viscous force imbalance at the boundary prevents the trapped non-neutral plasma from remaining serenely in hydrodynamic equilibrium.

We construct such equilibria for a non-neutral toroidally symmetric plasma whose confining vacuum magnetic field $\mathbf{B}(r,z)$ is purely poloidal and satisfies

$$\mathbf{B}(r,z) = \nabla\times A(r,z)\hat{\boldsymbol{\theta}}, \quad \nabla^2 A(r,z) = A(r,z)/r^2\,. \tag{6}$$

The first condition is met by setting

$$\mathbf{v}(r,z) = -(q/m)A(r,z)\hat{\boldsymbol{\theta}}\,. \tag{7}$$

Inserting Eq. (7) into the equilibrium case of Eq. (1), we easily observe that

$$\mathbf{E}(r,z) = \frac{q}{2m}\nabla A^2(r,z), \quad \omega_p^2(r,z) = \frac{q^2}{2m^2}\nabla^2 A^2(r,z)\,. \tag{8}$$

Now we can evaluate the Brillouin ratio in terms of the vector potential using Eqs. (6) and (8):

$$\frac{2\mu_0 mc^2 n(r,z)}{B^2(r,z)} = \frac{2\omega_p^2(r,z)}{\Omega_B^2(r,z)} = \frac{\nabla^2 A^2(r,z)}{B^2(r,z)} = 1 + \frac{[\partial A(r,z)/\partial r - A(r,z)/r]^2 + [\partial A(r,z)/\partial z]^2}{[\partial A(r,z)/\partial r + A(r,z)/r]^2 + [\partial A(r,z)/\partial z]^2}\,. \tag{9}$$

One immediately notes that this ratio will exceed the value of 2 wherever

$$\frac{1}{r}\frac{\partial A^2(r,z)}{\partial r} < 0\,. \tag{10}$$

The $1/r$ is retained to emphasize that exceeding the value of 2 requires a true toroidal geometry, excluding $r=0$. We can relate the Brillouin ratio directly to the deviatoric strain tensor $\mathbf{s}$. We first define s_{ij} to equal $\partial_i v_j + \partial_j v_i$, or, equivalently, $-(q/m)(\partial_i A_j + \partial_j A_i)$. We next observe that for a vacuum magnetic field,

$$\tfrac{1}{2}\nabla^2 A^2 = (\partial_i A_j)(\partial_i A_j)\,. \tag{11}$$

Using the Brillouin flow criterion and expressing the magnetic field in a Cartesian coordinate system, we obtain

$$\frac{m^2}{2q^2}\mathrm{Tr}(s^2) = (\partial_i A_j)^2 + (\partial_i A_j)(\partial_j A_i)\,,$$
$$B^2 = (\partial_i A_j)^2 - (\partial_i A_j)(\partial_j A_i)\,. \tag{12}$$

Finally, we can solve Eqs. (11) and (12) for $\nabla^2 A^2$ in terms of the magnetic field and the deviatoric strain. Inserting the result back into Eq. (9) yields the effect of the deviatoric strain on the Brillouin ratio:

$$\frac{2\mu_0 mc^2 n(r,z)}{B^2(r,z)} = 1 + \frac{\mathrm{Tr}[\mathbf{s}^2(r,z)]}{2\Omega_B^2(r,z)} \geq 1\,.$$

Since $\mathbf{s}$ is a symmetric matrix, this ratio will equal unity only when all elements of $\mathbf{s}$ vanish, which transpires only for flows consisting of nothing other than uniform translation and rigid rotation, the case of thermodynamic equilibrium. Using Eqs. (11) and (12), it is also useful to express the square of the deviatoric strain tensor in terms of the magnetic field and the vector potential:

$$\frac{m^2}{2q^2}\mathrm{Tr}[\mathbf{s}^2(r,z)] = \nabla^2 A^2(r,z) - B^2(r,z) = r^2\left\{\nabla\left[\frac{A(r,z)}{r}\right]\right\}^2 . \tag{13}$$

To maximize the Brillouin ratio, we wish to minimize the magnetic field strength. For a vacuum magnetic field, the minimization is accomplished by constructing a field possessing an x point. We shall do so in two distinct ways.

In case I, we examine the neighborhood of a magnetic x point located at $r=r_0$, $z=0$. To do so, we set

$$A(r,z) = [(r^2-r_0^2)z/r]B_0/r_0 , \tag{14}$$

where B_0 is some constant relating to the magnetic field strength. Equations (6), (8), (13), and (14) then yield the following expressions for $\mathbf{B}$, $\mathbf{E}$, and the Brillouin ratio within the plasma,

$$\mathbf{B}(r,z) = \left[\left(\frac{r_0}{r}-\frac{r}{r_0}\right)\hat{\mathbf{r}} + 2\frac{z}{r_0}\hat{\mathbf{z}}\right]B_0 ,$$

$$\mathbf{E}(r,z) = \frac{qB_0}{m}A(r,z)\left[\frac{z}{r_0}\left(1+\frac{r_0^2}{r^2}\right)\hat{\mathbf{r}} + \frac{r}{r_0}\left(1-\frac{r_0^2}{r^2}\right)\hat{\mathbf{z}}\right] ,$$

$$\frac{2\mu_0 mc^2 n(r,z)}{B^2(r,z)} = \frac{2\omega_p^2(r,z)}{\Omega_p^2(r,z)} = 2\left[\frac{2[(r^4+r_0^4)/r^4]z^2 + (r^2-r_0^2)^2/r^2}{4z^2+(r^2-r_0^2)^2/r^2}\right] .$$

A numerical evaluation of the Brillouin ratio near the x point for this case is depicted in Fig. 1(a). The value of this ratio at the x point is 2, just that value which was first observed by Walker and by Pöschl and Veith [5] as the maximum value for the ratio in their studies of ellipsoidal Brillouin flow. In our example, one notes from the figure that higher values than 2 can be attained. Indeed, inserting Eq. (14) into Eq. (10), we see that such values will be obtained whenever $r<r_0$ and $z\neq 0$.

Another representative example is given in case II where

$$A(r,z) = \frac{1}{2r}\left[r^2\ln\left(\frac{r}{r_0}\right) + \frac{r_0^2-r^2}{2} - z^2\right]B_0 ,$$

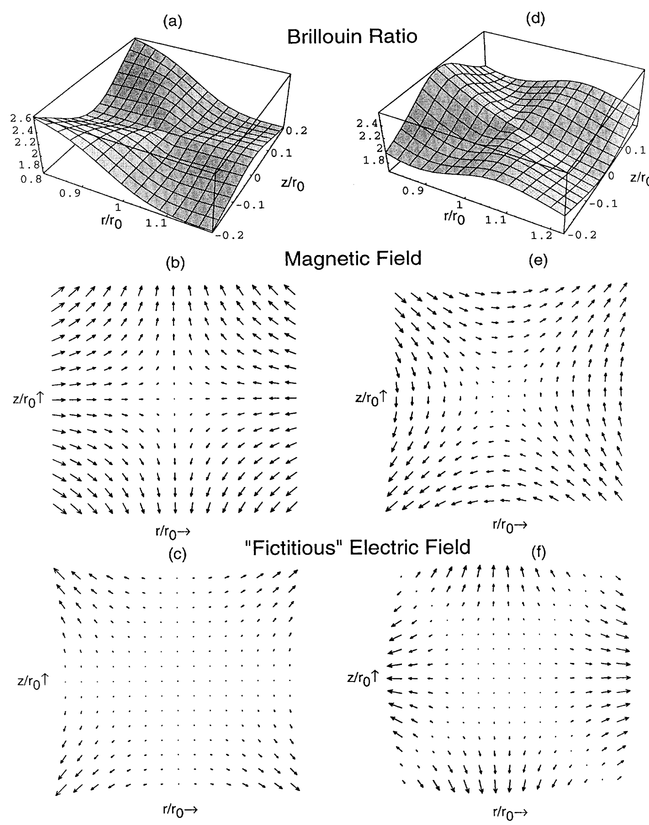

FIG. 1. For case I: (a) The Brillouin ratio as a function of r and z near the magnetic x point at $r=r_0$ and $z=0$; (b) the magnetic field near the x point; (c) the "fictitious" electric field, defined in text, near the x point. For case II: (d) The Brillouin ratio as a function of r and z near the magnetic x point at $r=r_0$ and $z=0$; (e) the magnetic field near the x point; (f) the "fictitious" electric field near the x point. (The magnetic and electric fields are shown in arbitrary units.)

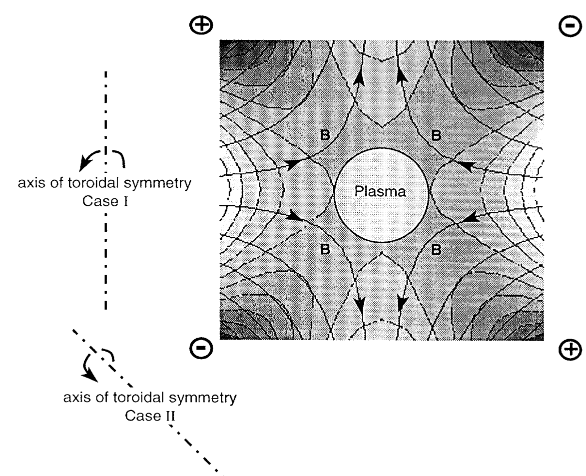

FIG. 2. A representative cross section of toroidal plasma located at the x point of a magnetic field $\mathbf{B}$, whose direction is shown in this sketch by the lines with arrows. The shaded regions are bounded by constant values of "fictitious" electric potential. For the two cases of this paper, the two axes shown demonstrate the orientation of the symmetry axis of the torus with respect to the magnetic and electric field configuration of the plasma confinement. In the corners, notice the quadrupolar current source.

where again B_0 is some constant relating to the magnetic field strength. This case also has a magnetic x point located at $r = r_0$, $z = 0$. However, the x point is rotated 45° clockwise with respect to case I [cf. Figs. 1(b) and 1(e)]. Proceeding as we did in case I, we find within the plasma

$$\mathbf{B}(r,z) = \left[\frac{z}{r}\hat{\mathbf{r}} + \ln\left(\frac{r}{r_0}\right)\hat{\mathbf{z}}\right]B_0 ,$$

$$\mathbf{E}(r,z) = \frac{qB_0}{m}A(r,z)\left\{\frac{1}{2}\left[\ln\left(\frac{r}{r_0}\right) + \left(\frac{1}{2} + \frac{z^2}{r^2} - \frac{r_0^2}{2r^2}\right)\right]\hat{\mathbf{r}} - \frac{z}{r}\hat{\mathbf{z}}\right\},$$

$$\frac{2\mu_0 mc^2 n(r,z)}{B^2(r,z)} = \frac{2\omega_p^2(r,z)}{\Omega_p^2(r,z)} = \left\{\frac{[\ln(r/r_0)]^2 + (\frac{1}{2} + z^2/r^2 - r_0^2/2r^2)^2 + 2z^2/r^2}{[\ln(r/r_0)]^2 + z^2/r^2}\right\}.$$

A numerical evaluation of the Brillouin ratio in the neighborhood of the x point for this case is shown in Fig. 1(d). As in case I, the value of this ratio at the x point is 2. Again one notes that higher values than 2 can be attained. Of course, any other orientation of the x point can be obtained from a linear superposition of the vector potentials of cases I and II. (A lattice of these x points can be created by considering sums of products of first-order Bessel functions with exponentials of real or imaginary argument, as appropriate.)

In Figs. 1(c) and 1(f), respectively, we have displayed the "fictitious" electric field in the neighborhood of the magnetic x point. By fictitious, we mean the field $\mathbf{E} + (mv_\theta^2/qr)\hat{\mathbf{r}}$, which takes into account the fictitious force on the flow. One observes that confinement of a torus of plasma containing an x point requires the application of a toroidally symmetric electric field having an octupolar nature. This applied field, when superimposed on the field emanating from the confined plasma, results in the fictitious field depicted. The octupolar character prevents plasma flow along the magnetic field. The x point magnetic field configuration can be created by using a toroidal quadrupolar current configuration outside of the plasma. See Fig. 2.

To obtain an infinite value of the Brillouin ratio at the magnetic x points, we must add a $1/r$ term to the toroidal flow velocity, which vanishes at these points in the two cases above. This addition will not compromise our two conditions. But, it will make the trace of the square of the deviatoric strain tensor nonzero at the x point and therefore yield the desired infinite Brillouin ratio.

To best understand the infinite Brillouin ratio, we shall take the extreme case in which the magnetic field vanishes everywhere. This is a case of inertial-electrostatic confinement (IEC). Using the analysis of the first two examples, we find within the plasma that

$$\mathbf{v}(r,z) = \frac{c_0\hat{\theta}}{r}, \quad \mathbf{E}(r,z) = -\frac{c_0^2 m}{q}\frac{\mathbf{r}}{r^3},$$
$$\omega_p^2(r,z) = \frac{2c_0^2}{r^4}, \quad \frac{m^2}{2q^2}\mathrm{Tr}[\mathbf{s}^2(r,z)] = \frac{c_0^2 m^2}{q^2}\left(\frac{4}{r^4}\right), \tag{15}$$

where c_0 is a constant. In order to confine a torus of plasma by this means, one must impose an external electric field that both opposes the motion of the torus along the axis of toroidal symmetry, as well as opposes any outward radial expansion along the major axis of the torus. The superposition of such a field with the field induced by the toroidal plasma is that which is given within the plasma by Eq. (15). It is both the multiply connected (toroidal) as well as the non-neutral character of the plasma which makes inertial-electrostatic confinement tractable.

In contrast to the well-studied equilibria of Penning-trap-confined non-neutral plasmas, the hydrodynamic equilibria analyzed in this Letter are not thermodynamic equilibria due to their nonvanishing deviatoric strain. This lack of thermodynamic equilibrium is shared by all magnetohydrodynamic equilibria of neutral plasmas and is largely responsible for their intricate physics. The novelty of the proposed confinement geometry (with the concomitant removal of the Brillouin limit condition on the confined density) and the intricacy of the tapestry of physics of neutral plasmas suggest that research to develop methods to form these non-neutral equilibria, to study their stability, and the effects of finite temperature on transport should prove amply rewarding in increased understanding of the physics of confinement of non-neutral plasmas—with the possibility of derivative benefits for the confinement of neutral plasmas.

This work was supported by the U.S. Department of Energy.

[1] D. H. E. Dubin and T. M. O'Neil, Phys. Fluids **29**, 11 (1985).

[2] R. C. Davidson, *Physics of Nonneutral Plasmas* (Addison-Wesley, Redwood City, CA, 1990), p. 42; L. Turner, Phys. Fluids B **3**, 1355 (1991).

[3] F. C. Michel, Rev. Mod. Phys. **54**, 1 (1982).

[4] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge Univ. Press, Cambridge, 1967), p. 144.

[5] L. R. Walker, J. Appl. Phys. **26**, 780 (1955); K. Pöschl and W. Veith, J. Appl. Phys. **33**, 1013 (1962). See also, K. Avinash and S. N. Bhattacharyya, Phys. Fluids B **4**, 3863 (1992).