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CONTAINMENT IN CUSPED PLASMA SYSTEMS

Harold Grad

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AEC Research and Development Report

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Abstract

In this report we present a brief descriptive survey of the current theoretical picture of plasma containment in cusped magnetic configurations together with -- not a comparison with experiment, which is not feasible at present -- but a mention of the points of contact which exist or might soon be made to exist with experiment. Possibly the most significant recent advance is the development of a theory of containment which is applicable to the whole range of plasma densities from a tenuous plasma in an essentially vacuum magnetic field to a fully developed plasma which completely excludes the magnetic field from its interior. Also presented are cursory accounts of the situation with regard to stability, cyclotron radiation, and methods of creating this type of plasma configuration.

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Containment in Cusped Plasma Systems

1. Stability

The idealized cusped geometry, with a sharp interface separating a vacuum magnetic field from a field-free plasma, Fig. 1, was not only the first plasma "bottle" that was demonstrated to be macroscopically stable, but it is the only one up to the present time that has been shown to be stable with respect to finite amplitude disturbances from equilibrium.[1] To be sure, this model was chosen for its mathematical simplicity rather than its easy physical attainability. We recall that in the case of the "stabilized" pinch, replacement of a sharp interface by a gradual transition layer has led to much more stringent requirements to insure stability.[2]. For a cusped geometry, one can refer to a general theorem which states that an equilibrium state is macroscopically stable provided that the pressure everywhere decreases in the direction of the principal curvature of a magnetic line.[3] Unfortunately this criterion, which seems to give a positive answer, is only suggestive. The only equilibria which satisfy the criterion

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1. J. Berkowitz, H. Grad, and H. Rubin, "Magnetohydrodynamic Stability", in Proceedings of the Second International Conference on the Peaceful Uses of Atomic Energy, Geneva, 1958, vol. 31, p. 177.

2. W. A. Newcomb and A. N. Kaufman, "Hydromagnetic Stability of a Tubular Pinch", Physics of Fluids 4, 314 (1961).

3. K. Hain, R. Lüst and A. Schlüter, "Zur Stabilität eines Plasmas", Z. für Naturforschung 12a, 833 (1957).

have open "cusps" with uncontained plasma extending to infinity, Fig. 2; (within the macroscopic framework of this theory, the pressure is constant along the full length of a magnetic line.)

A more suitable theory would have to consider either an equilibrium state with plasma flowing out through the cusps, or else refer to a more refined microscopic theory of equilibrium which takes into account mirror reflection of particles and therefore permits the existence of a finite contained plasma without flow. The first analysis has not been done. The second possibility, which requires the introduction of an anisotropic stress tensor, has only been examined for plane or cylindrical equilibria and indicates instability when the ratio $p_{\perp}/p_{\parallel}$ (the ratio of the pressure components perpendicular and parallel to the field) is too large or too small. In summary, the questions of possible instability arising from plasma flow or from anistropy of the stress tensor or from electrostatic mechanisms such as relative flow between the electron and ion streams in the cusps are largely unanswered. On the other hand, so long as the transition region remains reasonably thin, the finite amplitude stability theory of the sharp interface (which uses not much more than conservation of energy) shows that there can be no large scale motions of the plasma which carry it to the walls. What is not excluded is that small scale "turbulence" can lead to loss of plasma by enhanced diffusion. Although this would occur slowly compared to the ordinary conception of unstable growth, it might be too rapid for useful plasma containment.

The most striking evidence of the inherent stability of this type of plasma configuration is given by an experiment in which an ordinary pinch is initiated followed by a cusped field which is superposed just before the pinched plasma would become unstable.[4] Although the magnitude of the cusped field is comparable to the pinch field and does not dominate it, the plasma gives the appearance of stability.

2. Containment Formulas

It is possible to give a fairly good quantitative estimate of the rate of loss of particles through a cusp in magnetic fields of some generality.[5] When applied to the case of a sharp boundary cusped geometry for which a precise theory is known[6], this estimate is found to be in error by a factor 3/2.

The computation is based on the hypothesis that there exists a critical flux tube, which separates an outer zone of completely adiabatic orbits from an inner zone where every

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4. J. P. Watteau, "Experiments with Pulsed Magnetic Cusps", Phys. Fluids 4, 607-610 (1961).

5. For details see H. Grad, "Plasma Containment in Cusped Systems", to appear.

6. J. Berkowitz, K. O. Friedrichs, H. Goertzel, H. Grad, J. Killeen, and E. Rubin, "Cusped Geometries", in Proc. Second Internat'l Conf. on the Peaceful Uses of Atomic Energy, Geneva, 1958, vol. 31, p. 171.

magnetic line passes through a distinctly nonadiabatic region[7], Fig. 3. An orbit which is guided by a magnetic line of the latter type will consist of successive adiabatic stretches (near the turning points) separated by a nonadiabatic portion which serves to randomize the magnetic moment. The existence of a critical flux tube with approximately these properties has been verified by numerical computation.[8]

A more precise description includes a third or transition zone in which the change in magnetic moment, although small for a given transit, is so sensitive to very slight changes of phase as to appear completely uncorrelated from one transit to the next, thereby producing diffusion of velocities.[9] We choose to ignore this refinement since the volume occupied by particles suffering small nonadiabatic changes is not large.

On the basis of this hypothesis, we postulate an isotropic distribution of particles leaving the central region and aimed at a cusp. Those with a sufficiently small magnetic moment (determined by the mirror ratio) emerge through the loss cone, while the others return for another try. An easy computation yields

$$Q_2 = \frac{1}{4} n_o V \delta \tag{1}$$

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7. H. Grad, "Thermonuclear Plasma Containment in Open-Ended Systems", NYO-9355, Institute of Mathematical Sciences, New York Univ. (September 1960).

8. R. van Norton, "The Motion of a Charged Particle Near the Zero Field Point", NYO-9495, Inst. Math. Sci., New York Univ., to appear.

9. H. Grad and R. van Norton, to appear.

for the rate of loss of particles per unit length of the cusp in a two-dimensional problem and

$$Q_3 = \frac{1}{4} n_o V A \tag{2}$$

for the total loss rate in a three-dimensional problem. Here n_o is the density in the central region, V is the speed, and δ and A are the geometrical sizes of the openings provided by the critical flux tube, Fig. 4.

In the configuration of Fig. 4a, the total flux entering at the circular rim is equal to that leaving by the two "point" cusps. If the maximum magnetic field strength is the same at the line and point cusps, then the areas must be equal. Thus the loss rate through the rim is exactly equal to the sum for the two point cusps. This result is also verified explicitly by the exact formulas in the sharp boundary case.[10]

For the specific axially symmetric vacuum field defined by the flux tubes

$$\psi = r^2 z = \text{constant}, \tag{3}$$

the critical value of ψ, obtained numerically[11], is

$$\psi_c = (a \lambda_o)^3, \quad a \sim 8.8 \tag{4}$$

where

$$\lambda_o = mV/eB_o \tag{5}$$

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10. H. Grad, "Plasma Containment in Cusped Systems", to appear.

11. R. van Norton, NYO-9495, loc. cit.

is a Larmor radius constructed from the speed V (not its perpendicular component) and the minimum value, B_o, of the magnetic field on the line ψ_c. There are many alternative expressions for the area, A. For the line cusp one can write

$$(6) \qquad A_1 = (2\pi R)\left(\frac{2}{3}\frac{a}{M}\lambda\right)$$

where R is the radius of the plasma at the cusp, λ is the Larmor radius at the cusp, and M is the mirror ratio

$$(7) \qquad M = B/B_o \ ;$$

B is the field at the cusp. Similarly, for one of the point cusps,

$$(8) \qquad A_2 = (\pi\lambda^2)(a^2 M/\sqrt{3}\,).$$

We see that the equivalent opening at the line cusp can be made considerably smaller than a Larmor radius while the opening at the point cusp is much larger than a circle of radius λ. This contradicts certain widely held opinions.

In a more general magnetic field, one would estimate the critical flux tube by comparing the local Larmor radius with the logarithmic gradient of the field; a simple recipe, equivalent to (4), is that the minimum radius of curvature of the critical field line be about $10\lambda_o$. For the simple two-dimensional vacuum field defined by the flux tube

$$(9) \qquad \psi = xy = \text{constant}$$

this yields the criterion

(10) $$\psi_c = (a\lambda_o)^2, \quad a \sim 7.0.$$

The cusp opening δ can be estimated as

(11) $$\delta \sim a\sqrt{2}\,\lambda \sim 10\lambda$$

In this case, the width of the opening is directly related to the Larmor radius at the cusp, but it is considerably larger.

These results are more illuminating when expressed in terms of the mean containment time, which we define as the plasma volume (or area in two dimensions) divided by the flow rate. We find

(12) $$\tau_2 \sim \frac{L}{V}\,[1.69 + 2 \log M]$$

in two dimensions, and

(13) $$\tau_3 \sim \frac{R}{V}\,[1.96 + 3 \log M]$$

in three dimensions; L is the distance between cusps and R is the radius at the line cusp. The mean containment time is seen to be somewhat but not very much larger than the transit time. The advantage of the high β plasma is now clearly seen. For a given coil arrangement, if we imagine that plasma is introduced into the central region, we observe a slow increase of the opening A as the plasma spreads the field but a much more rapid increase in the volume occupied by the plasma. Thus the containment time increases. For the case of complete exclusion of the plasma, we have[12]

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12. J. Berkowitz, et al., "Cusped Geometries", loc. cit.

$$(14) \qquad \begin{cases} \tau_2 \sim \frac{L}{V}\,[0.19\,\frac{L}{\lambda}\,] \\ \\ \tau_3 \sim \frac{R}{V}\,[0.24\,\frac{R}{\lambda}\,]\ , \end{cases}$$

which can give values much larger than the transit time.

A word of caution is perhaps appropriate here. The formulas (1) and (2) are the ones to be used in a given case together with the proper opening size; the formulas (4), (6), (8), (10) and (11) are applicable only for the specific fields taken as examples.

In a problem in which the mean-free-path is comparable to or smaller than the plasma dimensions, all orbits are effectively nonadiabatic no matter what is the size of the Larmor radius. The loss formulas (1) and (2) hold for the full plasma opening, δ or A, without reference to a critical flux tube.[13] For a <u>small</u> mean-free-path one can also use a conventional gas-dynamics analysis with a result that differs from (1) or (2) by less than a factor 2.

The formulas given above are modified in the presence of electric fields. It is convenient to consider separately electric fields which are perpendicular and parallel to the magnetic field.

For an electric field oriented along the magnetic field, assuming that the potential is monotone, we need only know its value at the cusp (we take it to be zero in the interior) to compute the modification in the loss rate. An elementary

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13. H. Grad, NYO-9355, loc. cit.

computation shows that for an accelerating field the Maxwell-averaged loss-rate is multiplied by a factor

(15) $$1 + |e\phi|/kT_o \;;$$

whereas a decelerating field reduces the loss rate by a factor

(16) $$\exp(-|e\phi|/kT_o) \;;$$

T_o is the temperature in the central region. The significance of these formulas lies in the conclusion that a large disparity between the electron and ion loss rates will yield a compromise much closer to the smaller of the two values.

The effect of a transverse electric field on the loss rate is significant only when the potential difference $e\phi$ is on the order of kT over a distance comparable to the Larmor radius. The reason is that electric fields of such magnitude modify the adiabatic invariant which defines the loss cone; it is no longer permissible to use the magnetic moment.[14] The most important effect can be expected in the vicinity of a three-dimensional circular cusp where the flux tubes converge and the opening can be much thinner than a Larmor radius. We recall that this opening refers to the guiding magnetic lines; the guiding centers have a narrow spread, but the actual orbits extend over a wider region which is at least comparable to the Larmor radius, Fig. 5. Taking that value of the electric field

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14. H. Grad, "General Adiabatic Theory", Conference on Controlled Thermonuclear Reactions, June 1956, TID-7520, Sept. 1956.

which is required to produce approximate neutrality of the beam (the electrons spread by a small factor while the ions are pulled in to meet the electrons), yields a reduction of the ion loss rate on the order of the larger of the two quantities $2/M$ or $(m_-/m_+)^{1/2}$ together with a small increase in the electron loss rate (less than a factor two).

This estimate could be made more precise, but the exact value would not be significant because of the idealization which is implicit in the model.

3. Examples

The results given in the previous section are to be considered as building blocks from which a theory is to be constructed. They apply only to a certain class of particles, viz., the strongly nonadiabatic ones. This will include a certain part of the ion population, another part of the electrons. The formulas must be supplemented by appropriate mirror loss estimates for the adiabatic particles taking into account Coulomb encounters.[14]

The speed, V, enters into the loss formula in two ways. It determines the volume within which this theory applies, and

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14. R. F. Post, "Summary of UCRL Pyrotron (Mirror Machine) Program", Proc. of the Second Internat'l Conf. on the Peaceful Uses of Atomic Energy, Geneva, 1958, vol. 32, p. 245.

if fixes the relevant hole size. For rough estimates, the critical flux tube can be prescribed in terms of a mean thermal speed. If the field is excluded from the plasma, then the entire plasma volume is enclosed by the critical flux tube, but the opening size is still energy-dependent.

To obtain an answer in a given problem, a separation must be made into adiabatic (mirror loss) and nonadiabatic (cusp loss) components for the electrons and for the ions. Then the combined losses are modified by the inclusion of electric field effects to equalize the ion and electron losses. We shall demonstrate this procedure with a few representative examples.

First consider a low density plasma in an essentially unperturbed vacuum magnetic field and entirely contained within the critical ion flux tube. The basic ion loss rate, Q_+^o, corresponds to a containment time which is several times the ion transit time. The critical electron flux tube is much smaller (assuming comparable ion and electron temperatures), and it is easily verified that the nonadiabatic electrons form an unimportant minority. The basic electron containment time must therefore be computed as in a mirror machine and is closely related to the mean electron-electron collision time. This can bear any relation to the ion transit time, but, except for very hot or very rare plasmas, this basic electron loss rate, Q_-^o, will be considerably larger than Q_+^o ; we adopt this alternative. The transverse electric field at the circular line cusp cuts down this half of Q_+ by a large factor, while leaving Q_- essentially unaffected. A potential gradient along the field lines sufficient to equalize the ion and electron loss rates will increase the ion loss rate on the order of a factor two; thus the final result is a loss rate Q, approximately equal to the original, basic ion rate, Q_+^0. It is clear that the value of Q is very insensitive to the value of Q_-^0, provided that the latter is larger than Q_+^0.

Next we turn to the case of a high density plasma which excludes most of the magnetic field from its interior. If the exclusion is enough to yield a large electron Larmor radius compared to the plasma dimensions, the sharp boundary theory applies.[15] A more practical eventuality is that, although the ion Larmor radius is large, the electron Larmor radius is not. This is qualitatively similar to the situation in the first example; the electrons are adiabatic and the ions are not. There are several differences, however. The basic ion containment time can be much larger than the transit time. Also, the electrons are in a mirror machine with a very large mirror ratio. The improvement in containment is more strongly felt by the ions than by the electrons, and the former effect dominates. The final result, again, is that $Q \sim Q_+^0$.

Finally, to try to optimize the effect of the transverse electric field acting at the line cusp, we may take a series of plasmas, joining them at the point cusps, or even eliminate the point cusps entirely by forming a torus, Fig. 6. Comparing a torus of, say, six sections with a single section having the

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15. J. Berkowitz, et al., "Cusped Geometries", loc. cit.

same total volume, we compute a maximum advantage for the torus of a factor close to ten in containment. For the torus itself we find $Q \sim \frac{1}{30} Q_+^o$, and for the single section $\bar{Q} \sim \bar{Q}_+^o$; the volume scaling yields $\bar{Q}_+^o \sim 0.3\, Q_+^o$.

4. Methods of Obtaining a Cusped Plasma

We distinguish three general methods of creating a cusped plasma configuration experimentally. One is to establish the cusped field and inject an externally created plasma. Or the plasma can be created *in situ* by extraneous means and the cusped field then applied. Finally, the ionization and heating can be effected by the rising cusped field itself. The first two methods have the advantage of allowing the use of more efficient heating devices than the cusped geometry, while the third has the possible advantage of simplicity.

The most active of these procedures at the present time is injection.[16] It is very difficult at the present time to separate out incidental from basic difficulties, and the plasma measurements with regard to containment are fairly rudimentary.

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16. a) D. C. Hagerman and J. E. Osher, "Injection and Trapping of a β=1 Plasma into a Cusped Magnetic Field", to appear.

    b) F. Coensgen, A. Sherman, W. Nexsen and W. Cummins, "Plasma Injection into a Magnetic Field of Cusped Geometry", Phys. of Fluids 3, 764 (1960).

    c) F. R. Scott and H. G. Voorhies, "Plasma Injection into a Vacuum Magnetic Field", Phys. Flu. 4, 600-606 (1961).

Nevertheless, it seems clear that

1) substantial plasmas can be created in and fired from guns,

2) they can be made to enter a cusped region by penetrating through a cusp,

3) at least some portion of the plasma can be trapped under certain circumstances.

The last (and probably most important) requirement is, unfortunately, the vaguest. Precise conditions under which effective trapping can be accomplished are exceedingly difficult to analyze theoretically[17] and have yet to be established experimentally.

With regard to the second general method, one interesting experiment is the cusped pinch.[18] A conventional linear pinch discharge is used to create and heat the plasma, and the cusped field is then applied to provide stable confinement. Essentially two experimental points have been established, viz., stability over the duration of the experiment, and confinement of the line cusp losses to a narrow region.

In the third category is an experiment in which the two coils that combine to produce the cusped field are long thin cylinders.[19] The early stages of compression look like separate mirror (or "orthogonal" pinch) compressions which then fire into

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17. a) J. L. Tuck, "Plasma Jet Piercing of Magnetic Fields and Entropy Trapping Into a Conservative System", Phys. Rev. Letters 3, 313 (1959).

    b) H. Grad, "Plasma Trapping in Cusped Geometries", Phys. Rev. Letters 4, 222 (1960).

18. J. P. Watteau, loc. cit.

19. D. Finkelstein, S. Koslov, K. C. Rogers, G. Schmidt, "Combinations of Axial Magnetic Field Heating and Cusp Confinement", Bull. APS 5, 310 (1960).

one another.

To this list can be added a number of suggestions which have not yet been attempted. The dominant consideration is the inefficiency of a cusped field with regard to compression; the field strength decreases away from the coils. A theoretical solution to this difficulty is to apply the cusped field extremely quickly, viz., such that it reaches its maximum by the time a shock wave can penetrate to the center. The irreversible heating becomes as efficient as compression in a longitudinal field (mirror compression) if done very quickly. This possibility would seem to require a substantial but not extraordinary advance in capacitor technology.

Possibly the simplest procedure which has not yet been attempted is to create a plasma by conventional shock heating and then apply the cusped field. An elaboration of this idea would be to make use of reversed fields as in a longitudinal (mirror) compression.[20] One would apply a relatively weak cusped field to a plasma followed by a stronger reversed cusped field. The compression of the combination of plasma and trapped field could give much more effective heating than plasma compression alone.

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20. A. C. Kolb, H. R. Greim, W. R. Faust, "Dense Plasmas Confined by External Fields", Proc. of Fourth International Conf. on Ionization Phenomena, Uppsala, 1959.

5. Cyclotron Radiation

One of the features of the cusped geometry is that it remains stable and has relatively low particle losses in the case of a sharp boundary with magnetic field completely excluded from the interior of the plasma. This is also the optimum configuration with regard to cyclotron radiation. We do not try to estimate the radiation in any generality, but only for this optimum situation. More precisely, one first computes the exact self-consistent magnetic field profile in the sheath joining the plasma to the vacuum[21], and then computes the radiation for the exact electron orbits. The result is[22]

$$W = 1.0 \times 10^{-32}\, T_o^2\, n_o^{3/2}\ , \tag{17}$$

where W is in watts/cm.^{2}, T_o is in electron volts, and n_o is the number of electrons per cm.^{3}. For example, a plasma of thermonuclear interest might have $n_o = 10^{16}$, $T_o = 10^4$ which corresponds to $H = 50{,}000$ gauss. The radiation in this case is one watt/cm.^{2} which is quite tolerable.

This result should not be interpreted as a fair estimate of the radiation in an average plasma, but as an ideal value which can probably be approached if necessary by taking special pains. In other words, radiation may be troublesome but this problem can probably be overcome in a cusped geometry.

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21. H. Grad, "The Boundary Layer Between a Plasma and a Magnetic Field. I", NYO-9491, Institute of Mathematical Sciences, New York Univ., (December 1960).

22. H. Burkhardt, private communication.

6. Conclusions

One of the basic difficulties in the subject is the difficulty in correlating theory with experiment. Because of the competition between ions and electrons, also between adiabatic (collisional) losses and nonadiabatic losses, it is almost impossible to interpret a bare "containment time" observed in a given experiment. Because of the different scaling with regard to both density and temperature of the two loss mechanisms, it would even be difficult to interpret containment results from a series of experiments under varied conditions unless they were quite elaborate. Finally, the loss rates can be extremely sensitive to electrostatic potentials which might very well be altered almost unknowingly by slight changes in experimental procedures.

However, there is a possibility of separating out the two component losses in a given experiment. For mirror losses, a small difference in mirror ratio at the two ends of a flux tube will throw almost all the lost particles toward the side with the smaller value. On the other hand, cusp losses show a continuous dependence on mirror ratio. If the two types of losses could be separated in this way by a sequence of experiments with varying mirror ratio, then the behavior of each of the separate components would be much more amenable to theoretical inspection. It should be remarked that diffusional effects due to small successive nonadiabatic changes (cf. Section 2) would be grouped with mirror losses, while the small part of collisional losses resulting from large angle deflections would be associated with cusp losses.

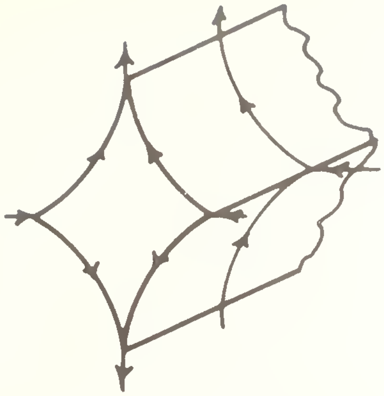
A further caution with regard to the interpretation of experimental containment times is that, even though the loss rate Q is proportional to the density n_o, the observed decay will not be exponential in time. Specifically, in the case of a completely excluded field, Q is proportional to V^2. Thus

$$\begin{aligned} f(V,t) &= f(V,0)\exp(-\alpha V^2 t) \\ &= c\ \exp[-V^2(\alpha t + 1/2RT_o)]. \end{aligned}$$

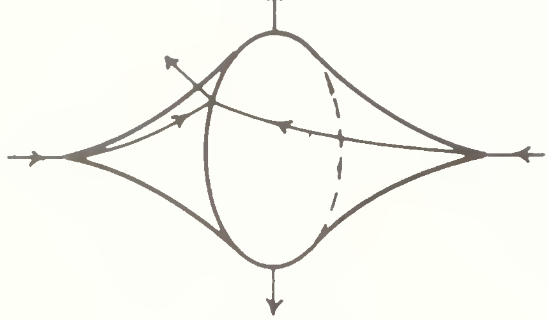
We notice that an initially Maxwellian distribution remains so, and the density and temperature decrease algebraically with time,

$$(18) \qquad \left\{ \begin{aligned} T(t) &= T(0)/(1 + t/\tau) \\ n(t) &= n(0)/(1 + t/\tau)^{3/2} \end{aligned} \right.$$

Theoretical predictions with regard to stability, containment, and radiation indicate that cusped containment should be very effective for dense, moderately hot (less than 1 kev) plasmas. In the thermonuclear range, cusped devices are competitive and could become dominant depending on the position with regard to stability and radiation in other devices. Experimental confirmation of the effectiveness of cusped containment is not yet at hand.



(a)



(b)

FIG. 1

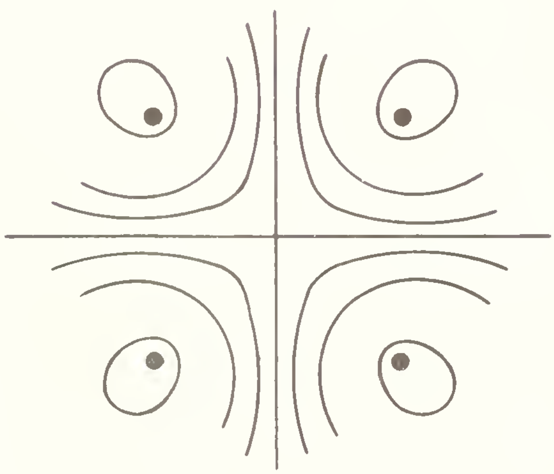


FIG. 2

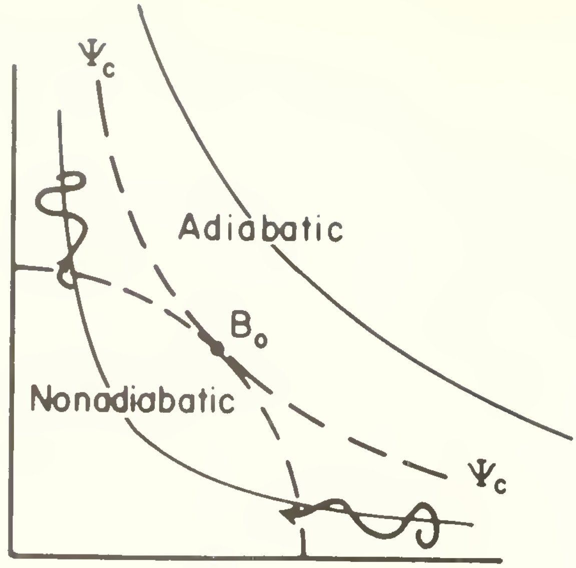


FIG 3

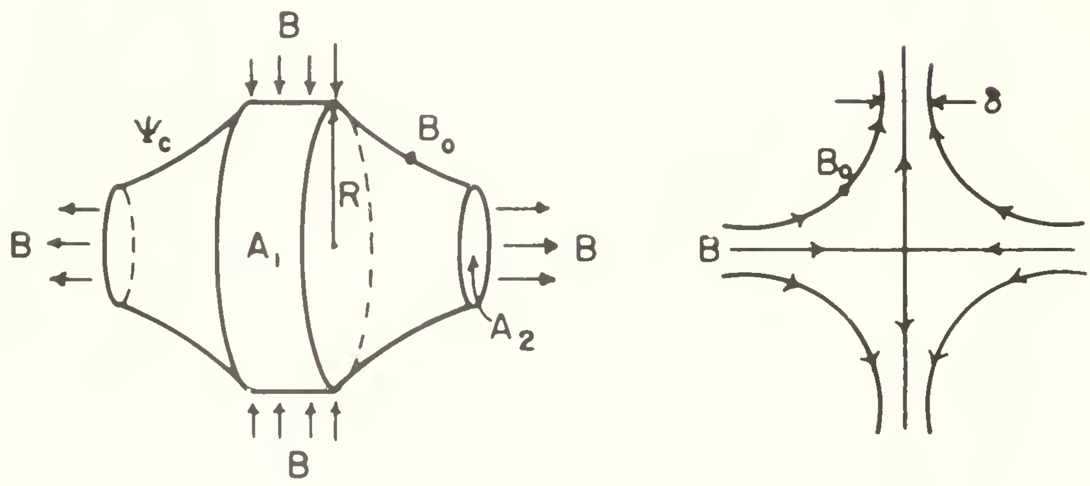


FIG. 4

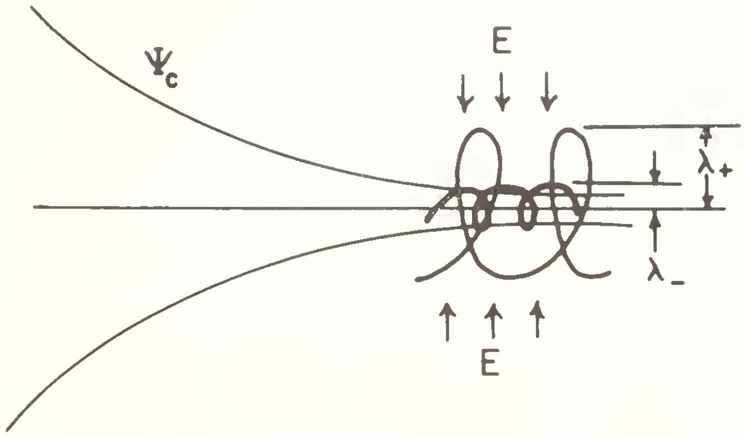


FIG. 5

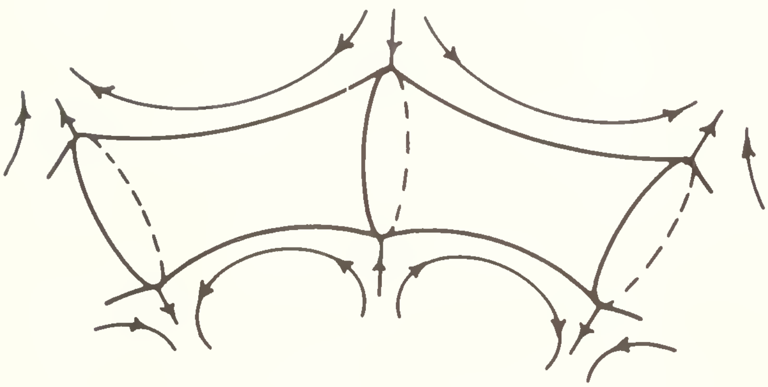


FIG. 6