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Adiabatic Charged-Particle Motion

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Abstract. The adiabatic theory of charged-particle motion is developed systematically in this review. We present the essentials of the theory without giving all the analysis in detail. The general expressions for guiding-center motion and particle energy change are given, with application to the Van Allen radiation and to Fermi acceleration. It is shown that Fermi acceleration and betatron acceleration should not be regarded as distinct processes. Modifications of the nonrelativistic theory that are necessary when the particle is relativistic are discussed. Proofs are given of the invariance to lowest order of the first and second adiabatic invariants for the case of static fields. Finally, applications are made to the theory of plasmas.

INTRODUCTION

The adiabatic approximation to charged-particle motion has been widely used in our attempts to understand the Van Allen radiation and to predict the results of high-altitude nuclear explosions. It has also been used extensively in the theory of plasma confinement and stability in strong magnetic fields. A thorough understanding of the adiabatic predictions is therefore desirable, particularly since deviations from these predictions may be important in explaining what we observe. Our purpose in this review is to present what adiabatic theory says, without presenting all the analysis in the greatest possible generality. Some of the analysis, especially for relativistic particles in time-dependent fields, becomes quite lengthy and will be omitted. (Many of the subjects presented here are amplified in a monograph by *Northrop* [1963].)

THE GUIDING-CENTER MOTION OF NONRELATIVISTIC PARTICLES

In a uniform magnetic field that is constant in time a charged particle moves in a helical path. The motion can be described exactly as motion about a circle whose center is moving along a line of force. If the field is not quite uniform and not quite time-independent, we expect that the motion will not be quite helical; we also expect that something approximating helical motion will still be discernible, and therefore that a good approximation will contain gyration about a center that now may move at right angles to the line of force as well as along it. This expectation is indeed correct, and the equations governing this 'guiding-center' motion can be derived by following physical intuition. To do this let $\mathbf{r} = \mathbf{R} + \boldsymbol{\rho}$, where the vectors are defined in Figure 1. To correspond to the picture of rapid gyration about the guiding center, let $\boldsymbol{\rho} = \rho(\hat{e}_2 \sin \omega t + \hat{e}_3 \cos \omega t)$, where ω is the angular frequency of gyration $eB(\mathbf{R})/mc$, $B(\mathbf{R})$ is the magnetic field at $\mathbf{R}$, and $\hat{e}_2(\mathbf{R})$ and $\hat{e}_3(\mathbf{R})$ are unit vectors perpendicular to $\mathbf{B}(\mathbf{R})$ and to each other. If $\mathbf{R} + \boldsymbol{\rho}$ is now substituted into the equation of motion for the particle

$$m\ddot{\mathbf{r}} = (e\dot{\mathbf{r}}/c) \times \mathbf{B}(\mathbf{r}) + e\mathbf{E}(\mathbf{r}) \tag{1}$$

and an average is taken over a period of the gyration. The result, after a little algebra with the unit vectors, is [*Hellwig*, 1955; *Northrop*, 1961]

$$\ddot{\mathbf{R}} = \frac{e}{m}\left[\mathbf{E}(\mathbf{R}) + \frac{\dot{\mathbf{R}}}{c} \times \mathbf{B}(\mathbf{R})\right] - \frac{M}{m} \nabla B(\mathbf{R}) + \text{terms proportional to } \frac{m}{e} \tag{2}$$

Here M is the well-known magnetic moment $e\rho^2\omega/2c = mv_\perp^2/2B$, where $v_\perp$ is the particle velocity perpendicular to $\mathbf{B}(\mathbf{R})$. In (2) only terms through zero order in m/e have been kept; m/e can be used as the expansion parameter because, if (1) is written in suitable dimensionless form, the dimensionless parameter that appears is the gyration radius divided by the dimensions of the system, and m/e is proportional to this ratio.

The component of $\dot{\mathbf{R}}$ perpendicular to $\mathbf{B}(\mathbf{R})$ in (2) is the guiding-center velocity perpendicular to $\mathbf{B}(\mathbf{R})$. It is the so-called 'drift velocity' and is obtained by taking the vector product of (2) with $\mathbf{B}$. We have

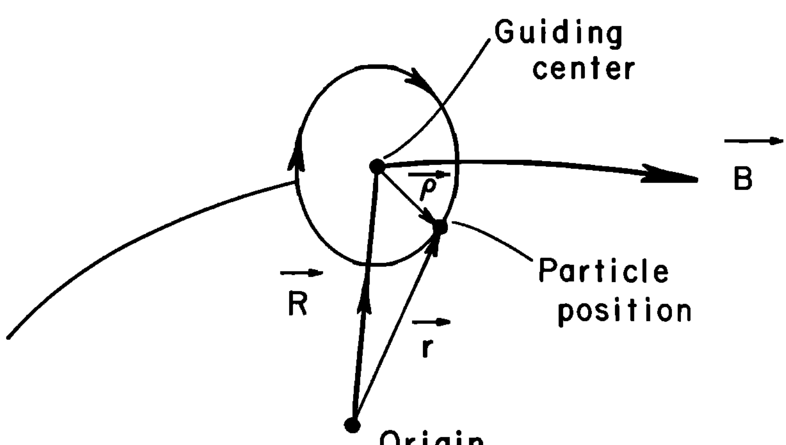

Fig. 1. The charged particle gyrates about its guiding center.

$$\dot{\mathbf{R}}_\perp = \frac{c\mathbf{E} \times \hat{e}_1}{B} + \frac{Mc}{e}\frac{\hat{e}_1 \times \nabla B}{B} + \frac{mc}{e}\frac{\hat{e}_1 \times \ddot{\mathbf{R}}}{B} + 0(\epsilon^2) \tag{3}$$

where ϵ is m/e, $\hat{e}_1$ is $\mathbf{B}/B$, and all field quantities are evaluated at $\mathbf{R}$. There are three drift terms here. The first is the well-known '$\mathbf{E} \times \mathbf{B}$' drift, and the second is the 'gradient $\mathbf{B}$' drift. The third term contains the 'line curvature' drift, but it also contains quite a few other drifts, as will be developed below. All the drifts occur because the curvature of the particle trajectory is alternately larger and smaller as the particle goes around its 'circle' of gyration; the gyration 'circle' is not really quite a circle. This variation in the curvature produces a gradual drift to one side as illustrated in Figure 2. The cause of the alternately large and small curvature is different for each of the drifts. The '$\mathbf{E} \times \mathbf{B}$' and '$\nabla B$' drifts have been frequently described before [*Alfvén*, 1950; *Spitzer*, 1952]. The six drifts that are contained in the last term of (3) also can be given geometric interpretations. That term can be expanded by writing $\ddot{\mathbf{R}}$ as

$$\frac{d}{dt}(\dot{\mathbf{R}}_\perp + \hat{e}_1\dot{\mathbf{R}}\cdot\hat{e}_1) = \hat{e}_1\frac{dv_\parallel}{dt} + v_\parallel\frac{d\hat{e}_1}{dt} + \frac{d\dot{\mathbf{R}}_\perp}{dt}$$

where $v_\parallel$ is $\dot{\mathbf{R}}\cdot\hat{e}_1(\mathbf{R})$, the component of guiding-center velocity parallel to the line of force at $\mathbf{R}$. We only need $d\dot{\mathbf{R}}_\perp/dt$ to zero order in ϵ, since the entire term is multiplied by ϵ in (3). By iteration of (3), we obtain $d\dot{\mathbf{R}}_\perp/dt = d\mathbf{u}_E(\mathbf{R})/dt + 0(\epsilon)$, where $\mathbf{u}_E$ is $c\mathbf{E} \times \hat{e}_1/B$. Also, $d\hat{e}_1(\mathbf{R})/dt$ is needed. It is the rate of change of the unit vector as we follow the guiding center. This unit vector changes direction in a time-dependent magnetic field even in the absence of guiding-center motion. In addition the guiding center sees a change in $\hat{e}_1$ as it moves in a field whose direction in space is not constant. Consequently, the total derivative $d\hat{e}_1/dt$ equals

$$\partial\hat{e}_1/\partial t + v_\parallel\, \partial\hat{e}_1/\partial s + \mathbf{u}_E\cdot\nabla\hat{e}_1 + 0(\epsilon)$$

where s is distance along the line of force. Similarly $d\mathbf{u}_E/dt$ equals $\partial\mathbf{u}_E/\partial t + v_\parallel\, \partial\mathbf{u}_E/\partial s + \mathbf{u}_E\cdot\nabla\mathbf{u}_E$. With these substitutions, the total drift velocity becomes

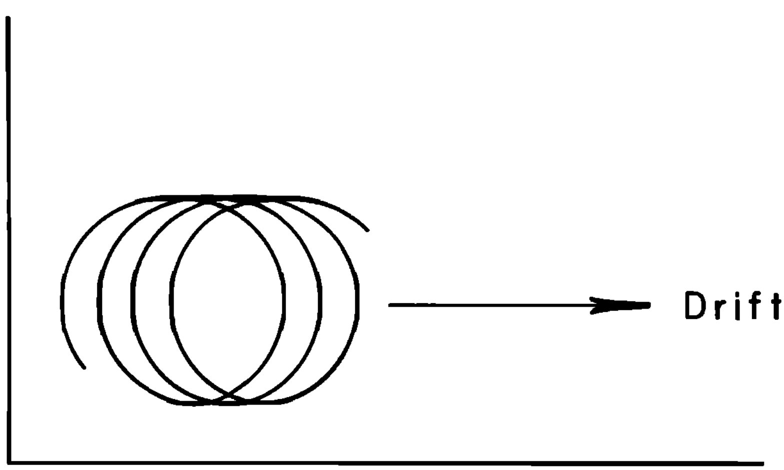

Fig. 2. A drift.

$$\dot{\mathbf{R}}_\perp = \frac{\hat{e}_1}{B} \times \left(-c\mathbf{E} + \frac{Mc}{e}\nabla B + \frac{mc}{e} v_\parallel \frac{d\hat{e}_1}{dt} + \frac{mc}{e}\frac{d\mathbf{u}_E}{dt}\right)$$

$$= \frac{\hat{e}_1}{B} \times \left\{-c\mathbf{E} + \frac{Mc}{e}\nabla B + \frac{mc}{e}\right.$$

$$\left.\cdot\left[v_\parallel \frac{\partial \hat{e}_1}{\partial t} + v_\parallel^2 \frac{\partial \hat{e}_1}{\partial s} + v_\parallel \mathbf{u}_E \cdot \nabla \hat{e}_1 + \frac{\partial \mathbf{u}_E}{\partial t} + v_\parallel \frac{\partial \mathbf{u}_E}{\partial s} + \mathbf{u}_E \cdot \nabla \mathbf{u}_E\right]\right\} + 0(\epsilon^2) \qquad (4)$$

where all quantities are evaluated at **R**. The term proportional to $\partial\hat{e}_1/\partial s$ is the well-known line curvature drift. However, the other five terms in the square bracket, although possibly less familiar, should not be overlooked. In practical cases the electric fields are often so small that the four terms containing $\mathbf{u}_E$ are negligible, and the field lines may change direction so slowly that the $\partial\hat{e}_1/\partial t$ drift is small. But these five terms in the bracket are not necessarily small, and situations in which each is of primary importance are known in plasma physics. For example, the term proportional to $\mathbf{u}_E \cdot \nabla \mathbf{u}_E$ is responsible for the shear, or Helmholtz, instability of a plasma [*Northrop*, 1956, 1961]. Shears occur at the solar wind-geomagnetic field interface, where the solar plasma slides over the geomagnetic field.

The $\partial\hat{e}_1/\partial t$ drift is an easy one to understand geometrically. If the direction of the magnetic field changes without a change in the particle velocity, then some of what was 'parallel' velocity will become 'perpendicular,' and vice versa. In other words, if there is a change in the reference direction, with respect to which we define parallel and perpendicular, then the respective components of velocity will change. It is easy to work out the details and see that there is a periodic variation (at the gyration frequency) in the curvature of the particle trajectory while the line of force changes direction. This leads to a drift, just as in the more familiar case of the $\mathbf{E} \times \mathbf{B}$ and ∇B drifts.

The component of (2) parallel to the magnetic field gives the parallel acceleration of the guiding center. The scalar product of (2) with $\hat{e}_1(\mathbf{R})$ is

$$\ddot{\mathbf{R}} \cdot \hat{e}_1 = (e/m)E_\parallel - (M/m)\hat{e}_1 \cdot \nabla B + 0(\epsilon) \qquad (5)$$

where $E_\parallel$ is $\mathbf{E}(\mathbf{R}) \cdot \hat{e}_1(\mathbf{R})$. The parallel acceleration $dv_\parallel/dt$ is $(d/dt)(\dot{\mathbf{R}} \cdot \hat{e}_1)$, which differs from $\ddot{\mathbf{R}} \cdot \hat{e}_1$ by $\dot{\mathbf{R}} \cdot d\hat{e}_1/dt$, and since the latter equals $(\hat{e}_1 v_\parallel + \mathbf{u}_E) \cdot (d\hat{e}_1/dt) + 0(\epsilon)$, then

$$\frac{dv_\parallel}{dt} = \frac{e}{m}E_\parallel - \frac{M}{m}\frac{\partial B}{\partial s} + \mathbf{u}_E \cdot \frac{d\hat{e}_1}{dt} + 0(\epsilon) \qquad (6)$$

The term $v_\parallel \hat{e}_1 \cdot d\hat{e}_1/dt$ vanished because $\hat{e}_1$ is a unit vector. The term $-(M/m)(\partial B/\partial s)$ is the usual mirror effect that produces reflection of particles and makes them oscillate north and south in the geomagnetic field, thus trapping them. The total time derivative $d\hat{e}_1/dt$ may be expanded to $(\partial\hat{e}_1/\partial t) + (v_\parallel \partial\hat{e}_1/\partial s) + \mathbf{u}_E \cdot \nabla \hat{e}_1$, just as in the drift equation. This $\mathbf{u}_E \cdot d\hat{e}_1/dt$ term is another example of an effect caused by a change in the reference direction. If the electric field is small, the term may be negligible.

ENERGY CHANGES

The kinetic energy W of a particle, averaged over a gyration, is $(mv_{\parallel}^2/2) + (mu_E^2/2) + MB$. This can be demonstrated, but it is really obvious: the first two terms are the energy of the guiding-center motion, and MB is the energy of rotation about the guiding center. The parallel energy $W_{\parallel}$ is $mv_{\parallel}^2/2$, and the average perpendicular energy $W_{\perp}$ is $(mu_E^2/2) + MB$. The rate of change dW/dt of total kinetic energy, averaged over a gyration, can be deduced in a formal fashion, but the result is so intuitively correct that the procedure will be omitted here. The result is

$$\frac{1}{e}\frac{dW}{dt} = \dot{\mathbf{R}}\cdot\mathbf{E}(\mathbf{R}, t) + \frac{M}{e}\frac{\partial B}{\partial t}(\mathbf{R}, t) + 0(\epsilon^2) \tag{7}$$

where $\dot{\mathbf{R}}$ is $\hat{e}_1 v_{\parallel} + \dot{\mathbf{R}}_{\perp}$. The first term on the right side is energy increase resulting from the average particle motion in the electric field; the second term is the induction effect, or 'betatron acceleration,' caused by the curl of $\mathbf{E}$ acting about the circle of gyration. Part of the energy increase given by (7) is fed into the parallel energy, and the rest into perpendicular energy. Simultaneously, energy is exchanged between parallel and perpendicular components by the mirror effect, the exchange occurring without a change in total kinetic energy. The process can be visualized as in Figure 3, where the partition of dW/dt between $dW_{\perp}/dt$ and $dW_{\parallel}/dt$ comes from the formal analysis. Note that $M\ \partial B/\partial t$ is only part of the perpendicular energy increase; $e\dot{\mathbf{R}}\cdot\mathbf{E}$ contains the rest of the perpendicular energy increase plus the entire rate of increase of parallel energy.

FERMI ACCELERATION

Fermi acceleration [*Fermi*, 1949, 1954; *Teller*, 1954; *Davis*, 1956; *Parker*, 1958] is a special case of the adiabatic energy change of the preceding section. Fermi

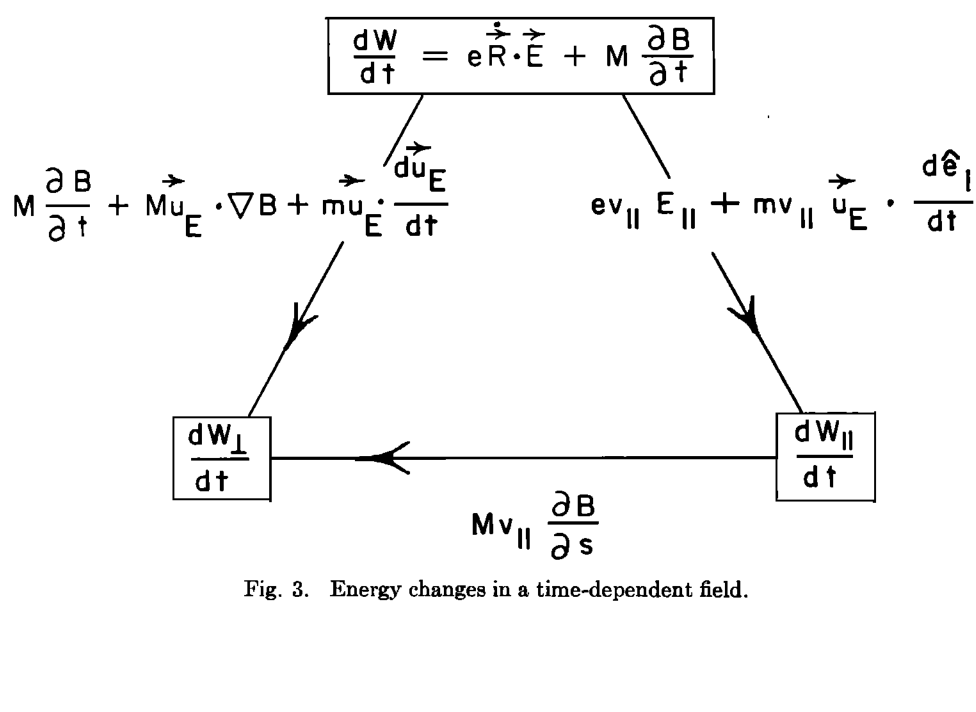

Fig. 3. Energy changes in a time-dependent field.

suggested that repeated collisions between a charged particle and moving clumps of magnetized plasma in space would accelerate a few particles to extreme energies. In effect, the clumps act as massive particles with which the high-energy particles attempt to establish kinetic equilibrium. The many particles in a clump, although of low energy, give it a very large mass. Thus at thermal equilibrium the high-energy particles will have very high energies indeed. The statistics of these collisions will not be discussed here [see *Teller*, 1954]; instead, details of a single Fermi-type collision will be interpreted in light of the preceding section.

Equation 7 applies to any adiabatic situation, but Fermi had in mind special ones—namely, those where there is a frame of reference (that of the clump) in which the magnetic field is static and there is no electric field. In the frame of the clump there is therefore no energy gain or loss by the particle. The collision is elastic, and its net effect is to alter the velocity of the guiding center. In the earth's frame, with respect to which the clump is in motion, there may be an energy change, somewhat in analogy to a ball struck by a baseball bat. A particle will lose energy if the clump is overtaken by the particle, and it will gain if the clump overtakes the particle.

Suppose the earth is fixed at 0 in Figure 4 and that the clump is fixed in a frame 0* moving at velocity **u** with respect to the earth. The rate of energy gain is, from (7) and (4)

$$\begin{aligned}\frac{dW}{dt} &= ev_{\parallel}\mathbf{E}_{\parallel} + e\dot{\mathbf{R}}_{\perp}\cdot\mathbf{E} + M\,\partial B/\partial t \\ &= ev_{\parallel}E_{\parallel} + M\mathbf{u}_E\cdot\nabla B + mv_{\parallel}\mathbf{u}_E\cdot\frac{d\hat{\mathbf{e}}_1}{dt} + m\mathbf{u}_E\cdot\frac{d\mathbf{u}_E}{dt} + M\,\frac{\partial B}{\partial t}\end{aligned} \tag{8}$$

Quantities in (8) must now be expressed in terms of **u**. For example, the electric field seen in 0 is

$$\mathbf{E} - (\mathbf{u}/c) \times \mathbf{B}^* \cong -(\mathbf{u}/c) \times \mathbf{B}$$

The magnetic fields **B** and **B*** are equal through order $\mathbf{u}/c$, i.e., nonrelativistically. The actual cosmic problem may have relativistic clump velocities, and relativistic energy for the colliding particle. Relativistic adiabatic motion will be reviewed in the next section, but the nonrelativistic case is adequate here for illustrative purposes.

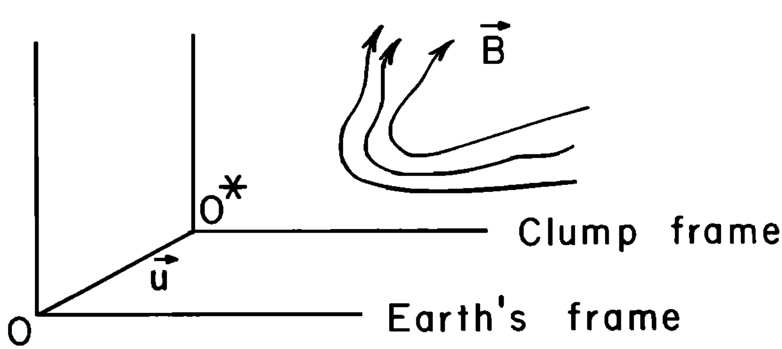

Fig. 4. Fermi acceleration.

The following relations also hold, as seen from the earth's frame of reference:

$$\partial B/\partial t = -\mathbf{u}\cdot\nabla B$$

$$d\hat{e}_1/dt = (v_\parallel - u_\parallel)\,\partial\hat{e}_1/\partial s$$

$$\mathbf{u}_E\cdot d\mathbf{u}_E/dt = -(v_\parallel - u_\parallel)u_\parallel\mathbf{u}_\perp\cdot\partial\hat{e}_1/\partial s$$

Substitution into (8) gives

$$\frac{1}{e}\frac{dW}{dt} = -\frac{M}{e}u_\parallel\frac{\partial B}{\partial s} + \frac{m}{e}(v_\parallel - u_\parallel)^2\mathbf{u}_\perp\cdot\frac{\partial\hat{e}_1}{\partial s} + 0(\epsilon^2) \tag{9}$$

If the magnetic field in the clump is such that the guiding center moves along a straight line of force, the last term in (9) is zero, and we then have what Fermi named 'type *a*' acceleration. As seen from the clump frame, the particle moves into an increasing magnetic field (magnetic mirror) along a straight line of force, and reflects with no energy change. As viewed from the earth's frame, it will show an energy change.

On the other hand, if the field line along which the guiding center moves is curved, and if the magnitude of the field is constant along the line, the first term on the right side of (9) vanishes. The last term is then Fermi's 'type *b*' acceleration. In either case, (9) can be integrated with respect to time to give the total energy change produced by the particle's collision with the clump. Types *a* and *b* really differ only in the mechanism whereby the guiding-center velocity is reversed in the clump frame. In either case the energy change seen by the observer on the earth is $2mu_\parallel(v_\parallel - u_\parallel)$, where $v_\parallel$ is the component of guiding-center velocity parallel to the magnetic field after the collision (i.e., far from the clump), and $u_\parallel$ is the component of $\mathbf{u}$ parallel to that field. This energy change is naturally more easily obtained from the fact that the velocity in the static frame is merely reversed by the collision. But our purpose here has been to apply (7) in the frame of reference in which there is an energy change. Equation 9 can also be integrated over a collision without breaking it up into the special cases *a* and *b*.

Fermi acceleration and betatron acceleration are sometimes invoked as distinct processes whereby a particle gains energy. However, they are not distinct. If we follow the fate of the $(M/e)\,\partial B/\partial t$ term in the transition from (8) to (9), we find the term goes into forming $-(M/e)u_\parallel\,(\partial B/\partial s)$, which is the type *a* acceleration. Consequently, betatron acceleration should not be viewed as a process distinct from Fermi acceleration, since it is part of type *a*. It *is* correct to distinguish between betatron acceleration and acceleration resulting from guiding-center motion in the electric field, since these appear as distinct terms in (9).

Pure betatron acceleration in space is improbable, since, if there is a $\partial B/\partial t$, there will usually be an electric field at the guiding center, and the $\dot{\mathbf{R}}\cdot\mathbf{E}$ term in (7) will be nonvanishing.

RELATIVISTIC ADIABATIC MOTION

If the particle has relativistic energy, (1) is replaced by

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\frac{m_0\dot{\mathbf{r}}}{(1-\beta^2)^{1/2}} = \frac{e}{c}\dot{\mathbf{r}}\times\mathbf{B}(\mathbf{r}) + e\mathbf{E}(\mathbf{r}) \tag{10}$$

where $\mathbf{p}$ is the momentum, $\beta = v/c$, and m_0 is the rest mass. Three cases can be distinguished: when the electric field is zero, when its component $\mathbf{E}_\perp$ perpendicular to $\mathbf{B}$ is small, and when $\mathbf{E}_\perp$ is large.

If there is no electric field, the force on the particle is always at right angles to the velocity, with the result that the energy is constant. Then $m_0/(1 - \beta^2)^{1/2}$ can be removed from under the d/dt in (10), and the equation is identical with the nonrelativistic one for a particle of mass $m_0/(1 - \beta^2)^{1/2}$. All the preceding nonrelativistic theory, with $\mathbf{E}$ set equal to zero, now applies. In the following two equations the nonrelativistic guiding-center equations are rewritten with $m_0/(1 - \beta^2)^{1/2}$ replacing m. The drift velocity is

$$\dot{\mathbf{R}}_\perp = \frac{1}{(1 - \beta^2)^{1/2}} \frac{m_0 v_\perp^2}{2B} \frac{c}{e} \frac{\hat{e}_1 \times \nabla B}{B} + \frac{1}{(1 - \beta^2)^{1/2}} \frac{m_0 c}{e} v_\parallel^2 \frac{\hat{e}_1}{B} \times \frac{\partial \hat{e}_1}{\partial s} \tag{11}$$

and the parallel force is

$$\frac{m_0}{(1 - \beta^2)^{1/2}} \frac{dv_\parallel}{dt} = -\frac{1}{(1 - \beta^2)^{1/2}} \frac{m_0 v_\perp^2}{2B} \frac{\partial B}{\partial s} \tag{12}$$

Nonrelativistically, the magnetic moment is $M = mv_\perp^2/2B$. Relativistically, the corresponding invariant is

$$M_r = m_0 v_\perp^2/2B(1 - \beta^2) = p_\perp^2/2m_0 B$$

It is not obvious that this is the correct generalization of M for relativistic energy. It is easy enough to verify for the simple case of a particle in a uniform, azimuthally symmetric field that changes with time. The general case is not so easy to prove. The adiabatic invariants will be studied more in the next section.

The parallel force in (12) is now larger by $(1 - \beta^2)^{-1/2}$ than would be predicted by the nonrelativistic equation for the same rest mass. Similarly, the drifts in (11) are faster by the same factor. These effects are caused by the increased gyration radius resulting from the relativistic mass increase. For example, the increased gyration radius increases the amount of field inhomogeneity sampled by the particle, hence increases the ∇B drift. Similarly, the parallel force increases because the larger gyration radius subjects the particle to a greater convergence of the field lines, and it is this convergence that produces the mirror effect. As we can see in Figure 5, it is the product of $v_\perp$ and the radial component of $\mathbf{B}$ that results in a parallel force.

If the electric field is sufficiently small (formally, of order ϵ), the four terms containing $\mathbf{u}_E$ in (4) become of the order of ϵ^2 and can be dropped. The drift proportional to $\partial \hat{e}_1/\partial t$ will also probably be negligible, since $\nabla \times \mathbf{E}$ and $\partial \mathbf{B}/\partial t$ are related by the Maxwell equation. Then only the three familiar drifts remain. We can surmise that the correct relativistic modification is obtained by adding $c\mathbf{E} \times \mathbf{B}/B^2$ to (11) and $eE_\parallel$ to the parallel force in (12). This does in fact turn out to be the correct procedure, but it is not a deductive one, since (11) and (12) were derived by assuming no electric field. The relativistic case has been studied by *Hellwig* [1955] and by *Vandervoort* [1960] for $\mathbf{E}_\perp$ large (i.e., of the order of 1), and the small $\mathbf{E}_\perp$ results are a special case.

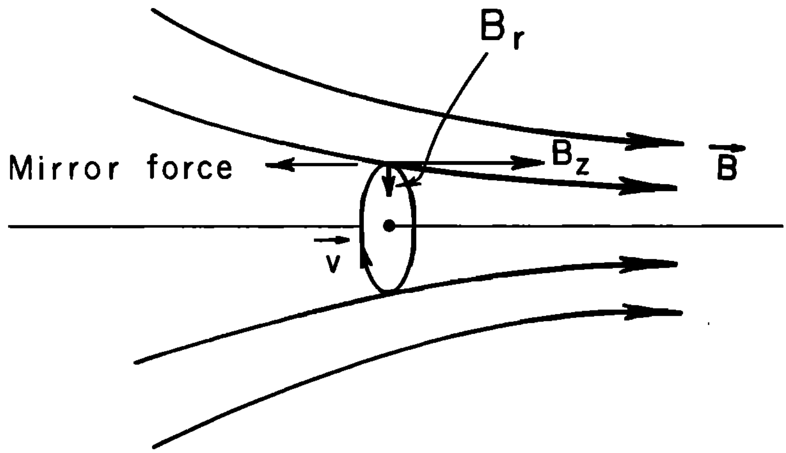

Fig. 5. The mirror effect.

The relativistic rate of energy change for $\mathbf{E}_\perp$ small is

$$dW/dt = e\dot{\mathbf{R}}\cdot\mathbf{E} + M_r(1 - \beta^2)^{1/2}\,\partial B/dt \tag{13}$$

Only the betatron term has been altered, a comparison with (7) shows.

The complete guiding-center equations for large $\mathbf{E}_\perp$ are rather long and will not be repeated here [see *Vandervoort,* 1960; *Northrop,* 1963]. Their principal features are corrections to existing terms of the small $\mathbf{E}_\perp$ relativistic expressions above. Additionally, two new drift terms that are in the *direction* of $\mathbf{E}_\perp$ appear. They are pure relativistic effects that have no analog in the small $\mathbf{E}_\perp$ relativistic case. One of these two drifts can be explained by the change in direction of **B** when a Lorentz transformation is made in the presence of an electric field. Basically, the drift is a result of the change in the reference direction with respect to which parallel and perpendicular are defined. Some of what was parallel velocity is converted to perpendicular velocity.

THE ADIABATIC INVARIANTS

The magnetic moment. The emphasis so far has been on the guiding-center motion and on energy changes. Not only are the guiding-center equations useful, but also valuable are quantities that are constant over long periods of guiding-center motion, i.e., any invariants of the adiabatic motion, or 'adiabatic invariants.' They are not exact invariants of the particle motion, any more than the guiding-center equations are exact equations for the particle motion. Formal analysis [*Kruskal,* 1960; *Northrop and Teller,* 1960] shows that there are at the most three adiabatic invariants for the charged particle. Each one is really an asymptotic series in a smallness parameter ϵ, a series of the form: constant $= a_0 + \epsilon a_1 + \epsilon^2 a_2 + \cdots$. Systematic analysis [*Gardner,* 1959; *Kruskal,* 1962] is essential for obtaining higher-order terms in the series. Historically, however, the forms of the lowest-order invariants (i.e., the a_0's) were deduced by physical insight and by consideration of special cases [*Alfvén,* 1950; *Chew et al.,* 1955; *Northrop and Teller,* 1960]. The connection with more formal theory was made later. Such an evolutionary history is common in physical science. In this paper only invariance to lowest order (the a_0's) will be proven.

The formal theories also show that the adiabatic invariant series are not the action integrals of the form $\oint p\,dq$, where p and q are canonical variables, but are instead Poincaré integral invariants of the form

$$\sum_i \oint p_i\, dq_i$$

where the number of terms in the sum is the number of degrees of freedom of the canonical system. However, the number of adiabatic invariants may vary from one to three, depending on the field geometry, as will become apparent shortly. The number of invariants is less than or equal to the number of degrees of freedom of the system [*Kruskal*, 1962].

The first invariant is the magnetic moment, defined previously as $mv_\perp^2/2B$ for the nonrelativistic case; $mv_\perp^2/2B$ is really M_0 of the magnetic moment series: constant $= M_0 + \epsilon M_1 + \epsilon^2 M_2 + \cdots$. The definition of $v_\perp$ was glossed over slightly in the beginning of this review. If the component of **E** perpendicular to **B** is small, the $\mathbf{E} \times \mathbf{B}$ drift is much less than the particle velocity, and the particle trajectory will be as in Figure 2. The motion is almost circular, and the $v_\perp$ to be used in the magnetic moment is the velocity about the circle. When $\mathbf{E}_\perp$ is this small, the last four drifts in (4) will probably be negligible. Suppose that $\mathbf{E}_\perp$ is now increased. Eventually the trajectory will resemble a prolate cycloid as in Figure 6. There is no resemblance to circular motion in the laboratory frame, but in the frame moving at $\mathbf{u}_E$ the motion is approximately circular again as in Figure 2. It is the $v_\perp$ in this drifting frame that should be used in $mv_\perp^2/2B$. Adiabatic theory therefore can still apply even when the perpendicular electric field is so large that the particle trajectory in the observer's frame shows no looping or resemblance to circular motion. We must only be careful to use the complete expressions in (4) and (6), and to define $v_\perp$ properly.

The invariance of M is easy to demonstrate for simple cases, like a time-dependent magnetic field with azimuthal symmetry and straight lines of force. A proof for the most general situation (general time-dependent magnetic field and

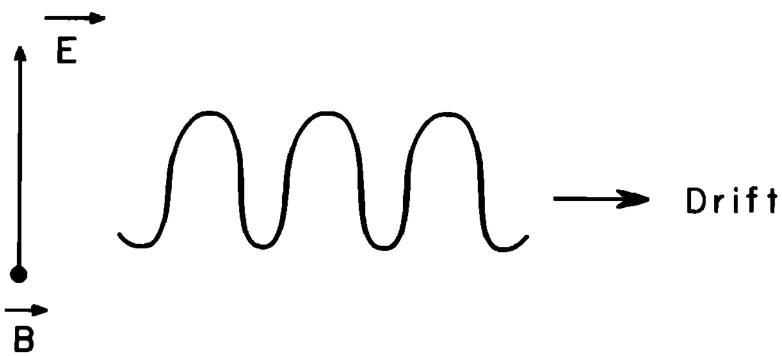

Fig. 6. Particle trajectory when $\mathbf{E}_\perp$ is large.

large electric field) seems to be rather long [*Kruskal*, 1958; *Gardner*, 1959; *Northrop*, 1963]. The most general case for which a simple proof seems to exist is the static one, where the energy is constant; a large curl-free electric field may be present. By conservation of energy,

$$\frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \frac{mu_E^2}{2} + MB + e\phi\right) = 0 \tag{14}$$

where ϕ is the electrostatic potential. Recall that the invariance of M was not invoked in deriving the guiding-center equations. Thus the value of $dv_\parallel/dt$ from (6) can be used to convert (14) to

$$\frac{d(MB)}{dt} = -\frac{ed\phi}{dt} - m\mathbf{u}_E\cdot\frac{d\mathbf{u}_E}{dt} - mv_\parallel\left(\frac{e}{m}E_\parallel - \frac{M}{m}\frac{\partial B}{\partial s} + \mathbf{u}_E\cdot\frac{d\hat{\mathbf{e}}_1}{dt}\right) \tag{15}$$

The total derivative $d\phi/dt$ equals $v_\parallel(\partial\phi/\partial s) + \dot{\mathbf{R}}_\perp\cdot\nabla\phi$, where $\dot{\mathbf{R}}_\perp$ is given by (4). Putting it all together and doing a little vector algebra gives

$$\frac{d(MB)}{dt} = M\mathbf{u}_E\cdot\nabla B + Mv_\parallel\frac{\partial B}{\partial s} \equiv M\frac{dB}{dt} \tag{16}$$

or

$$dM/dt = 0$$

The next two higher terms in the magnetic moment series have also been derived [*Kruskal*, 1958] (C. S. Gardner, private communication, 1962; Gardner's work is quoted in *Northrop* [1963]). They are rather complicated.

The expression 'nonadiabatic behavior' as applied to the magnetic moment has by custom come to mean any deviation of $mv_\perp^2/2B$ from constancy. However, it is actually the series $M_0 + \epsilon M_1 + \epsilon^2 M_2 + \cdots$ that is the invariant of the particle motion, and not just M_0. Therefore M_0 can vary according to adiabatic theory. It seems preferable to define as nonadiabatic any behavior not predicted by the series. Since the series is asymptotic [*Berkowitz and Gardner*, 1959], and not convergent, it would not be surprising to see particle behavior that completely ignores the adiabatic predictions, even in low order, and this would be genuine nonadiabatic behavior. Examples of such motion are known [*Garren et al.*, 1958; *Northrop*, 1963] for the magnetic moment.

The second, or longitudinal, invariant. Another invariant of the particle motion, or really of the guiding-center motion, is

$$J = \oint p_\parallel \, ds \tag{17}$$

where $p_\parallel$ is $mv_\parallel$, the guiding-center momentum parallel to the line of force. The invariant J exists if there is a mirror-type geometry such that the guiding center oscillates back and forth along the lines of force while drifting slowly at right angles to them, as illustrated in Figure 7. For J to be constant, it is necessary that the drift be slow compared to $v_\parallel$, i.e., that $\mathbf{E}_\perp$ be of order ϵ. The integral is taken over a complete oscillation, the deviation of the guiding center from a line due to the drift during one oscillation being negligible if $\mathbf{E}_\perp$ is small.

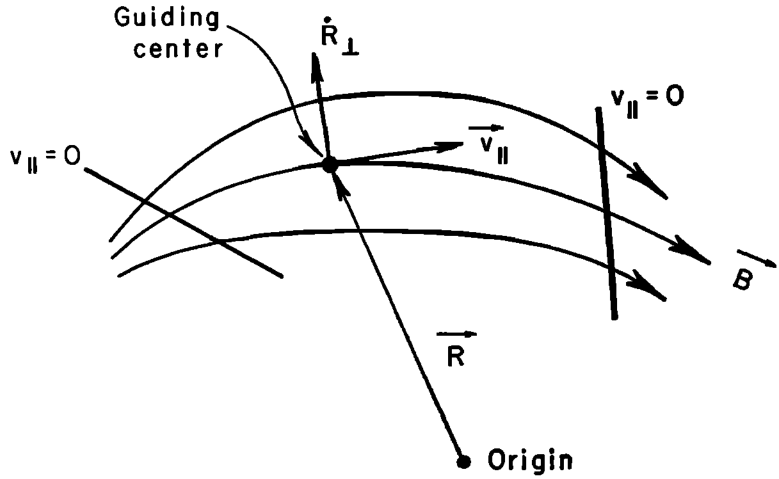

Fig. 7. Mirror geometry needed for existence of second adiabatic invariant.

The earliest suggestion that J is an invariant appears to have come from Rosenbluth [*Chew et al.*, 1955]. A proof of the invariance of J and some applications to laboratory magnetic field configurations were given by *Kadomtsev* [1958] for a nonrelativistic particle in a static magnetic field. A proof that remains valid at relativistic energies, and that includes time-dependent fields, has been given by *Northrop and Teller* [1960] along with applications to the Van Allen radiation. The proof of the invariance of J given below is for a nonrelativistic particle in a static field with no electric field; inclusion of nonstatic fields greatly increases the length of the proof. Therefore, only the results will be given for the time-dependent case. The time-dependent results will be needed for discussion of the third invariant. Relativistic modifications do not seem to materially complicate the proofs.

To begin the proof of J, a curvilinear coordinate system will now be introduced. The three coordinates will be denoted by α, β, and s, where α and β are two parameters specifying the line of force, and s denotes position along the line. (Distinguish this β from v/c in a previous section.) A system of nonintersecting lines can be generated as the intersections of two families of surfaces $\alpha(\mathbf{r})$ = constant, and $\beta(\mathbf{r})$ = constant, where $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ are two different functions of position. It is apparent that for a given system of lines the functions $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ are not unique. Consider the simple example of straight lines of force. They can be generated by the intersections of two families of planes, by a family of planes and one of cylinders, etc. Among the many possible pairs of functions $\alpha(\mathbf{r})$ and $\beta(\mathbf{r})$ for a given magnetic field, there is a subclass for which the vector potential $\mathbf{A}$ is $\alpha\nabla\beta$ and $\mathbf{B}$ therefore is $\nabla\alpha \times \nabla\beta$. That such a subclass exists is not quite obvious, but it is not difficult to prove. The utility of the subclass is that for it $|\nabla\alpha \times \nabla\beta|/B$ is constant everywhere, being unity, and this fact reduces the algebra involved.

In the absence of electric fields, the energy W equals $mv_\parallel^2/2 + MB$, so that

$$J(\alpha, \beta, M, W) = \oint \{2m[W - MB(\alpha, \beta, s)]^{1/2}\, ds\} \tag{18}$$

The instantaneous rate of change of J due to the particle drift $\dot{\mathbf{R}}_\perp$ in Figure 7 is

$$\frac{dJ}{dt} = \frac{\partial J}{\partial \alpha}\frac{d\alpha}{dt} + \frac{\partial J}{\partial \beta}\frac{d\beta}{dt} \tag{19}$$

Differentiation of the integral in (18) gives

$$\frac{\partial J}{\partial \alpha} = -mM \oint \frac{ds}{[2m(W - MB)]^{1/2}} \frac{\partial B(\alpha, \beta, s)}{\partial \alpha}$$

and

$$\frac{\partial J}{\partial \beta} = -mM \oint \frac{ds}{[2m(W - MB)]^{1/2}} \frac{\partial B(\alpha, \beta, s)}{\partial \beta} \tag{20}$$

Because α and β are constant on a line of force, they are changed only by the drift velocity and not by the parallel velocity. Therefore $d\alpha/dt = \dot{\mathbf{R}}\cdot\nabla\alpha(\mathbf{R})$ and $d\beta/dt = \dot{\mathbf{R}}_\perp\cdot\nabla\beta(\mathbf{R})$. Substituting $\dot{\mathbf{R}}_\perp$ from (4), with the electric field zero, gives

$$\frac{d\alpha}{dt} = \frac{\hat{e}_1}{B} \times \left(\frac{Mc}{e}\nabla B + \frac{mc}{e}v_\parallel{}^2\frac{\partial \hat{e}_1}{\partial s}\right)\cdot\nabla\alpha \tag{21}$$

Consider now the quantity $(\partial\mathbf{R}/\partial\beta) \times \mathbf{B}$, where the guiding-center position $\mathbf{R}$ is a function of (α, β, s):

$$\frac{\partial \mathbf{R}}{\partial \beta} \times \mathbf{B} = \frac{\partial \mathbf{R}}{\partial \beta} \times (\nabla\alpha \times \nabla\beta) = -\left(\nabla\alpha\cdot\frac{\partial \mathbf{R}}{\partial \beta}\right)\nabla\beta + \left(\nabla\beta\cdot\frac{\partial \mathbf{R}}{\partial \beta}\right)\nabla\alpha \tag{22}$$

By implicit differentiation of $\alpha = \alpha[\mathbf{R}(\alpha, \beta, s)]$ we find that

$$\nabla\alpha\cdot\partial\mathbf{R}/\partial\beta = 0 \qquad \text{and} \qquad \nabla\beta\cdot\partial\mathbf{R}/\partial\beta = 1$$

Thus, $(\partial\mathbf{R}/\partial\beta) \times \mathbf{B} = \nabla\alpha$, and (21) becomes

$$\frac{d\alpha}{dt} = \frac{\hat{e}_1}{B} \times \left(\frac{Mc}{e}\nabla B + \frac{mc}{e}v_\parallel{}^2\frac{\partial \hat{e}_1}{\partial s}\right)\cdot\left(\frac{\partial \mathbf{R}}{\partial \beta} \times \mathbf{B}\right) \tag{23}$$

Interchanging the dot and cross, and expanding the triple vector product $\hat{e}_1 \times (\partial\mathbf{R}/\partial\beta \times \mathbf{B})$, gives

$$\begin{aligned}
\frac{d\alpha}{dt} &= -\left(\frac{Mc}{e}\nabla B + \frac{mc}{e}v_\parallel{}^2\frac{\partial \hat{e}_1}{\partial s}\right)\cdot\left(\frac{\partial \mathbf{R}}{\partial \beta} - \hat{e}_1\hat{e}_1\cdot\frac{\partial \mathbf{R}}{\partial \beta}\right) \\
&= -\frac{Mc}{e}\frac{\partial \mathbf{R}}{\partial \beta}\cdot\nabla B - \frac{mc}{e}v_\parallel{}^2\frac{\partial \mathbf{R}}{\partial \beta}\cdot\frac{\partial \hat{e}_1}{\partial s} + \frac{Mc}{e}\hat{e}_1\cdot\frac{\partial \mathbf{R}}{\partial \beta}\hat{e}_1\cdot\nabla B \\
&= -\frac{Mc}{e}\frac{\partial B(\alpha, \beta, s)}{\partial \beta} - \frac{mc}{e}v_\parallel{}^2\frac{\partial \mathbf{R}(\alpha, \beta, s)}{\partial \beta}\cdot\frac{\partial \hat{e}_1(\alpha, \beta, s)}{\partial s} + \frac{Mc}{e}\hat{e}_1\cdot\frac{\partial \mathbf{R}}{\partial \beta}\frac{\partial B(\alpha, \beta, s)}{\partial s}
\end{aligned} \tag{24}$$

In the second term on the right side of (24), we have

$$\frac{\partial \mathbf{R}}{\partial \beta}\cdot\frac{\partial \hat{e}_1}{\partial s} = \frac{\partial}{\partial s}\left(\hat{e}_1\cdot\frac{\partial \mathbf{R}}{\partial \beta}\right)$$

since

$$\hat{e}_1\cdot\frac{\partial}{\partial s}\frac{\partial \mathbf{R}(\alpha, \beta, s)}{\partial \beta} = \hat{e}_1\cdot\frac{\partial}{\partial \beta}\frac{\partial \mathbf{R}}{\partial s}$$

and

$$\hat{e}_1 \cdot \frac{\partial}{\partial \beta} \frac{\partial \mathbf{R}}{\partial s} = \hat{e}_1 \cdot \frac{\partial \hat{e}_1}{\partial \beta}$$

which is zero. Therefore, the second term becomes

$$-\frac{mc}{e} v_\parallel{}^2 \frac{\partial}{\partial s}\left(\hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \beta}\right) = -\frac{mc}{e} v_\parallel \frac{d}{dt}\left(\hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \beta}\right)$$

From (6),

$$\partial B/\partial s = -(m/M)\, dv_\parallel/dt$$

so that the last two terms in (24) combine to

$$-\frac{mc}{e} \frac{d}{dt}\left(v_\parallel \hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \beta}\right)$$

The instantaneous rate of change of α finally is

$$\frac{d\alpha}{dt} = -\frac{Mc}{e} \frac{\partial B(\alpha, \beta, s)}{\partial \beta} - \frac{mc}{e} \frac{d}{dt}\left(v_\parallel \hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \beta}\right) \tag{25}$$

By a similar analysis, we have

$$\frac{d\beta}{dt} = \frac{Mc}{e} \frac{\partial B}{\partial \alpha} + \frac{mc}{e} \frac{d}{dt}\left(v_\parallel \hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \alpha}\right) \tag{26}$$

If $\partial B/\partial \beta$ from (25) is substituted into $\partial J/\partial \beta$ from (20), the result is

$$\begin{aligned} \frac{\partial J}{\partial \beta} &= \oint \frac{m\,ds}{[2m(W - MB)]^{1/2}} \frac{e}{c}\left[\frac{d\alpha}{dt} + \frac{mc}{e} \frac{d}{dt}\left(v_\parallel \hat{e}_1 \cdot \frac{\partial \mathbf{R}}{\partial \beta}\right)\right] \\ &= \frac{e}{c} \oint \frac{ds}{v_\parallel} \frac{d\alpha}{dt} \end{aligned} \tag{27}$$

The integral of $m(d/dt)[v_\parallel \hat{e}_1 \cdot (\partial \mathbf{R}/\partial \beta)]$ has vanished because $ds/v_\parallel$ is dt, and $v_\parallel$ is zero at the reflection points. Equation 27 can be written as

$$\partial J/\partial \beta = (e/c) T \langle \dot{\alpha} \rangle \tag{28}$$

where T is the time for a longitudinal oscillation, and the brackets denote the time average over an oscillation. Similarly,

$$\partial J/\partial \alpha = -(e/c) T \langle \dot{\beta} \rangle \tag{29}$$

Equation 19 can then be written as

$$dJ/dt = (e/c) T [\langle \dot{\alpha} \rangle \dot{\beta} - \langle \dot{\beta} \rangle \dot{\alpha}] \tag{30}$$

Now this quantity is not zero except under very special circumstances, so that J is not instantaneously being conserved by the guiding-center motion. However the rate of change of J averaged over a longitudinal oscillation is

$$\begin{aligned} \oint \frac{ds}{v_\parallel} \frac{dJ}{dt} &= \frac{eT}{c}\left[\langle \dot{\alpha} \rangle \oint \frac{ds}{v_\parallel} \dot{\beta} - \langle \dot{\beta} \rangle \oint \frac{ds}{v_\parallel} \dot{\alpha}\right] \\ &= \frac{eT}{c} [\langle \dot{\alpha} \rangle \langle \dot{\beta} \rangle - \langle \dot{\beta} \rangle \langle \dot{\alpha} \rangle] \end{aligned} \tag{31}$$

which is identically zero, and this is the important fact for the long-term motion.

Equations 28 and 29 are new equations of motion with the guiding-center oscillation averaged out; they are the analog of the guiding-center equations of motion, which are the particle equations of motion with the particle gyration averaged out.

When (28) and (29) are solved for $\langle\dot{\alpha}\rangle$ and $\langle\dot{\beta}\rangle$, they are at first sight suggestively canonical in form, with $J(\alpha, \beta, M, W)$ playing the role of Hamiltonian. But they are not quite canonical. In the first place, the time of oscillation T is also a function of (α, β, M, W). Furthermore, there are the time averages of $\dot{\alpha}$ and $\dot{\beta}$, rather than the instantaneous values. The first difficulty can be overcome by differentiating $J = J(\alpha, \beta, M, W)$ implicitly with respect to α and β to yield $\partial J(\alpha, \beta, M, W)/\partial\beta = -(\partial J/\partial W)\ \partial W(\alpha, \beta, M, J)/\partial\beta$, etc., for $\partial J/\partial\alpha$. The factor $\partial J/\partial W$ is simply T, as can be verified from (18). Then

$$\langle\dot{\alpha}\rangle = -\frac{c}{e}\frac{\partial W(\alpha, \beta, M, J)}{\partial\beta} \qquad \langle\dot{\beta}\rangle = \frac{c}{e}\frac{\partial W}{\partial\alpha} \tag{32}$$

Except for the time averages, these are now canonical. It would seem that the matter of the time averages could be overlooked if we are interested only in the average guiding-center position, and therefore that the equations of motion can be regarded as canonical. If this is the case, any theorems in classical mechanics that come from the canonical equations should have an analog in the (α, β) space. Liouville's theorem comes to mind immediately, and it is possible to derive it [*Northrop and Teller*, 1960] for the density in (α, β) space by disregarding the problem with time averages. To dispel doubts about the time averages, a more direct derivation can be made by using the expressions for the instantaneous values of $\dot{\alpha}$ and $\dot{\beta}$. The consequences of the Liouville theorem will be described shortly.

The third adiabatic invariant. As a guiding center oscillates between mirror points, it gradually changes lines of force. During its motion along a line, it instantaneously drifts toward a variety of adjacent lines, but there is one line toward which it moves on the average; this line is specified by (32). Thus a surface composed of lines on which J is constant is gradually traversed by the guiding center. Now it may happen that this surface is closed, so that the particle eventually returns to a line it traversed earlier. If so, there is a third periodicity, and a third adiabatic invariant is to be expected. The surfaces seem to be closed for particles in the inner Van Allen belt. Such a surface (idealized) is sketched in Figure 8.

Note that if the particle is not trapped between mirrors, the longitudinal motion is not periodic and there is not even a second adiabatic invariant, nor is there a third. Only the magnetic moment exists. This illustrates the fact that the number of adiabatic invariants depends on the geometry and is less than or equal to the number of degrees of freedom.

To return to the Liouville theorem: it says that, in the steady state in the absence of electric fields, contours of constant magnetic field are also constant guiding-center density contours on a longitudinal invariant surface (Figure 8).

The third adiabatic invariant is the flux Φ of $\mathbf{B}$ enclosed by the surface of Figure 8. That this flux should be constant in a static situation is a trivial statement, much as the invariance of the magnetic moment in a uniform field is trivially true.

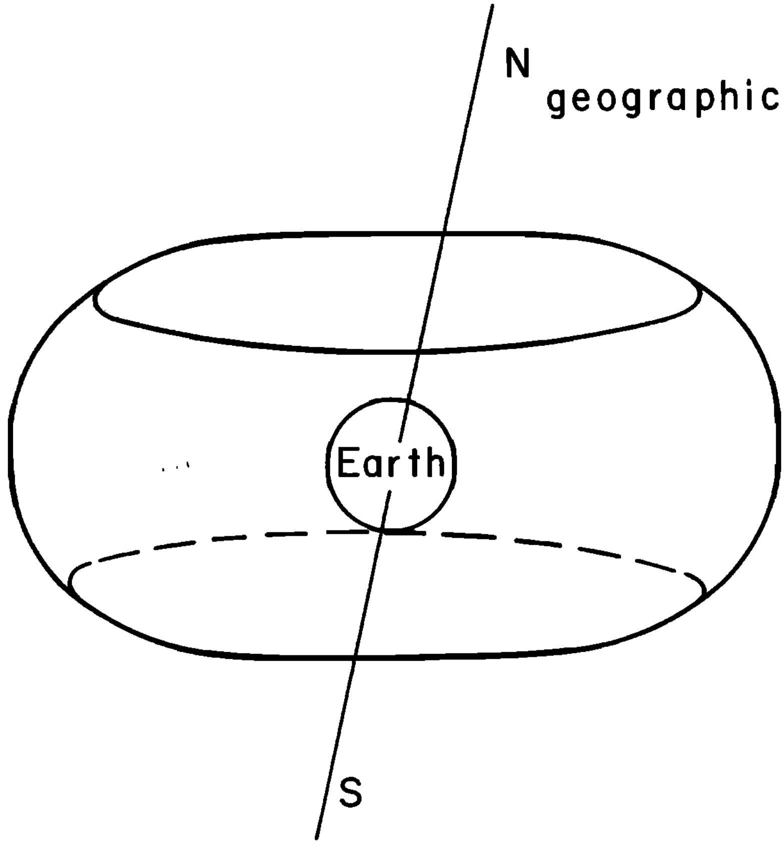

Fig. 8. An invariant surface for a particle trapped in the earth's field.

But the flux is also invariant if the field is time-dependent, and this is the significant fact. The surface about which the particle precesses is not even well defined unless the particle traverses it in a time small compared to the time scale for fields to change. It is not surprising therefore that this rapid precession assumption is necessary to prove the invariance of Φ. From a practical standpoint, the time scale of field fluctuations must be slowest to conserve Φ; they can be faster and still conserve J, and fastest of all without disturbing M, since the time scale then need only be long compared to the gyration period.

Proof of the invariance of Φ is reminiscent of the proof for J. It is necessary to extend (32) to include time-dependent fields. When the fields are time-dependent, it is appropriate to generalize the quantity W used previously to a quantity K, defined by

$$K = \frac{mv_{\parallel}^2}{2} + MB + e\left(\phi + \frac{\alpha}{c}\frac{\partial\beta}{\partial t}\right) \tag{33}$$

where ϕ is the scalar potential for the electric field, so that

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial(\alpha\nabla\beta)}{\partial t}$$

In a time-dependent field α and β are functions of both time and position. The second invariant is now defined by

$$J(\alpha, \beta, M, K, t) = \oint \left\{ 2m \left[K - e\left(\frac{\alpha}{c} \frac{\partial \beta}{\partial t} + \phi \right) - MB \right]^{1/2} \right\} ds \tag{34}$$

where $\partial\beta/\partial t$ is to be expressed as a function of (α, β, s, t). The generalizations of (32) turn out to be

$$\begin{aligned} \langle \dot{\alpha} \rangle &= -(c/e)(\partial K(\alpha, \beta, J, M, t)/\partial \beta) \\ \langle \dot{\beta} \rangle &= (c/e)(\partial K/\partial \alpha) \\ \langle \dot{K} \rangle &= (\partial K/\partial t) \\ 1 &= T(\partial K/\partial J) \end{aligned} \tag{35}$$

The quantity $\langle \dot{K} \rangle$ is related to the gain in energy averaged over a longitudinal oscillation.

The details of the proof that Φ is invariant will not be given here [see *Northrop and Teller*, 1960]. We find that $d\Phi/dt$ is not zero as the particle drifts around the surface defined by the invariance of J (i.e., as it precesses around the earth); the average motion from line to line as given by (35) does not conserve Φ. But if $d\Phi/dt$ is averaged over a complete precession, the time average is zero. This is analogous to the situation with dJ/dt. The instantaneous rate of change of Φ is

$$\frac{d\Phi}{dt} = \frac{cT_p}{e} [\langle \dot{K} \rangle - \langle\langle \dot{K} \rangle\rangle] \neq 0 \tag{36}$$

where $\langle\langle \dot{K} \rangle\rangle$ means $\langle \dot{K} \rangle$ averaged over a precession, and T_p is the time for the particle to precess once around the surface. The right side of (36) obviously vanishes when averaged over the period T_p.

Before we leave the subject of the third invariant, several points should be discussed about motion of lines of force and the average (over a longitudinal oscillation) guiding-center drift. The 'velocity' of a line of force in a time-dependent field is not physically observable. We cannot see lines of force. We are therefore free to define line velocity, and it should be defined so as to enhance our visualization of how the magnetic field pattern changes with time. Usually a picture is used in which a magnetic field has an intensity proportional to the line density drawn. As the field changes with time this picture remains valid if the lines are moved around at a 'flux-preserving velocity.' To define this velocity, suppose an arbitrary closed curve is drawn in space; now let each element of the curve move at a velocity $\mathbf{U}(\mathbf{r}, t)$. If the flux through the curve remains constant as the curve distorts, $\mathbf{U}$ is said to be flux-preserving. As was shown by *Newcomb* [1958], $\mathbf{U}$ must satisfy $\nabla \times (\mathbf{E} + \mathbf{U} \times \mathbf{B}/c) = 0$. This limits $\mathbf{U}$ but does not determine it uniquely. We often choose $\mathbf{U}$ as $c\mathbf{E} \times \mathbf{B}/B^2$, which is acceptable if $\nabla \times \mathbf{E}_\parallel$ is zero.

A more general definition of line velocity that is always acceptable (but not unique) is

$$\mathbf{U}(\mathbf{r}, t) = \left(\frac{\partial \beta}{\partial t} \nabla \alpha - \frac{\partial \alpha}{\partial t} \nabla \beta \right) \times \frac{\hat{e}_1}{cB} \tag{37}$$

It is not difficult to show that $\nabla \times (\mathbf{E} + \mathbf{U} \times \mathbf{B}/c)$ is zero for this choice of $\mathbf{U}$. Moreover, this choice has the advantage that $(\partial\alpha/\partial t) + \mathbf{U}\cdot\nabla\alpha$ is zero, as it is for β. The significance of this is that as an observer moves at the line velocity the (α, β) label on the line he is following remains unchanging with time.

A convenient space in which to visualize the invariant surfaces is a cartesian (α, β, s) space in which the field lines are straight and parallel to the s axis, as in Figure 9. The choice of $\mathbf{U}$ in (37) makes the lines of force fixed in this space; by contrast, a particle for which J (but not necessarily Φ) is invariant moves in (α, β) space in accord with (35), and consequently does not remain attached to a line of force.

The picture developed so far of line motion is very appealing, but is not unique. To illustrate, suppose $\mathbf{U}$ is defined by

$$\mathbf{U}(\mathbf{r}, t) = \frac{c}{eB}\,\hat{e}_1 \times \nabla K(\mathbf{r}, M, J, t) + \left(\frac{\partial\beta}{\partial t}\,\nabla\alpha - \frac{\partial\alpha}{\partial t}\,\nabla\beta\right) \times \frac{\hat{e}_1}{B} \tag{38}$$

where K is to be regarded as a function of the specified variables via (34). This velocity can also be proved flux-preserving. However, for it we have

$$\begin{aligned}\frac{\partial\alpha}{\partial t} + \mathbf{U}\cdot\nabla\alpha &= \frac{c}{eB}\,(\hat{e}_1 \times \nabla K)\cdot\nabla\alpha \\ &= -\frac{c}{e}\frac{\partial K}{\partial\beta}\,(\alpha, \beta, M, J, t) = \langle\dot{\alpha}\rangle\end{aligned} \tag{39}$$

and similarly for $\langle\dot{\beta}\rangle$. With this definition of line velocity, the line of force consequently moves at exactly the average particle drift velocity, and the particle remains attached to the line.

Either of the two pictures is acceptable, though definition 37 seems preferable since it does not depend on any particle parameters, whereas definition 38 depends on J and M. It is a little unappealing to use a definition of line velocity that de-

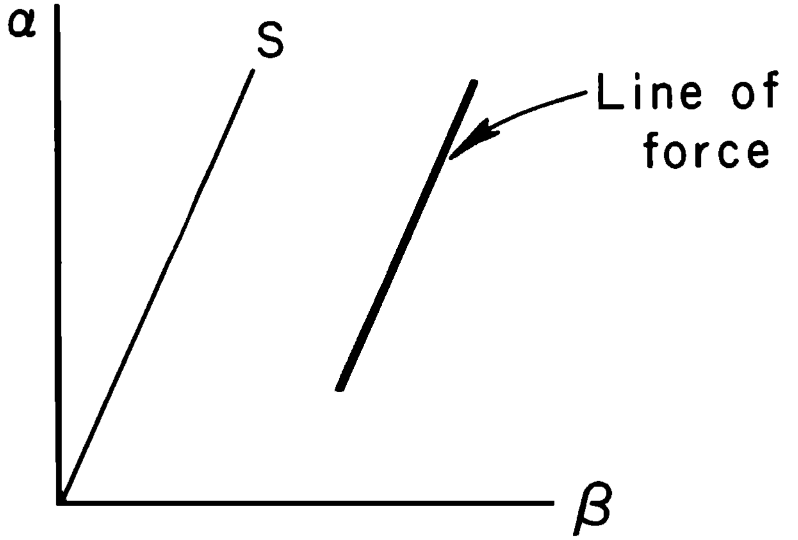

Fig. 9. A line of force in (α, β, s) space.

pends on the particle under observation. We prefer to visualize the motion of field lines as being intrinsic to the field and not dependent on particles. Furthermore, if two particles with different J and M are on the same line of force, there will be an ambivalence in the line velocity. Finally, if the electric fields are so large that all the drift terms in (4) must be retained, J is not conserved. Definition 38 is still flux-preserving, but J is now a time-dependent parameter. And because the guiding center no longer shows a slow average drift, governed by (35), it is not possible to say that the particle follows the line of force on the average. The guiding center follows a trajectory in Figure 9 determined by (4) and (6). Under these circumstances, definition 37 for the line velocity certainly is superior to 38.

APPLICATION OF ADIABATIC THEORY TO PLASMAS

In the preceding sections the motion of a single particle in a prescribed field has been studied. The adiabatic model may also apply to a plasma, where the density of positively and negatively charged particles is so large that their interactions are important in determining their motions. The field each particle moves in is the sum of (*a*) any 'external' field and (*b*) those fields due to the motions and positions of all other particles. For the particle motion to be adiabatic, close collisions between charged particles must be infrequent (high plasma temperature and low density) so that a particle at no time feels a sudden force. Such self-consistent calculations are necessary to analyze the stability of plasma confinement in a given field configuration.

Newcomb [1963] has developed a method for using the first two adiabatic invariants in studying plasma stability. The change in energy of an equilibrium plasma under a prescribed displacement $\boldsymbol{\xi}(\mathbf{r})$ of the element of plasma at $\mathbf{r}$ can be obtained from invariance of the magnetic moment and longitudinal invariant. If this energy change is positive for all possible $\boldsymbol{\xi}(\mathbf{r})$, the plasma is stable. If the change is negative for any $\boldsymbol{\xi}(\mathbf{r})$, it is unstable. It is plausible that the change in particle energies should be derivable from the first two invariants. The magnetic moment is associated with perpendicular energy, whereas the longitudinal invariant is associated with parallel velocity and energy. Changes in field energy under the perturbation must also be accounted for in obtaining the total change in energy.

The mechanism of these instabilities can be explained in terms of the adiabatic particle drifts. In the presence of the perturbation the drifts lead to charge accumulations whose electric fields drive the perturbation further in a typically regenerative fashion [see *Rosenbluth and Longmire*, 1957, and *Northrop*, 1961, for examples].

We can also apply adiabatic motion to the current density in a collisionless plasma. Each component (i.e., ions or electrons) of the plasma obeys the macroscopic momentum conservation equation

$$nm\frac{d\mathbf{V}}{dt} = -\nabla\cdot\mathbf{P} + ne\frac{\mathbf{V}}{c}\times\mathbf{B} + ne\mathbf{E} \tag{40}$$

where $\mathbf{V}$ is the average (over the velocity distribution) of the particle velocity $\mathbf{v}$, and $\mathbf{P}$ is the pressure tensor defined as $\langle nm(\mathbf{v} - \mathbf{V})(\mathbf{v} - \mathbf{V})\rangle$, where the brackets mean an average over the particle velocity distribution. The current density $\mathbf{J}$ of that component is $ne\mathbf{V}$, where n is the particle density. Solving (40) for $\mathbf{V}$ we obtain

$$\mathbf{V} = \mathbf{V}_\parallel + \frac{c\hat{e}_1 \times \nabla \cdot \mathbf{P}}{neB} + \frac{c\mathbf{E} \times \hat{e}_1}{B} + \frac{mc}{eB^2}\hat{e}_1 \times \frac{d\mathbf{V}}{dt} \tag{41}$$

Consider now a steady situation in which there is no electric field. Then $\mathbf{V}_\perp$ is $c\hat{e}_1 \times \nabla \cdot \mathbf{P}/neB$. This is just the east-west asymmetry effect of mirroring protons observed by *Heckman and Nakano* [1963]. They observed that at the inner edge of the inner Van Allen belt more high-energy protons are moving east than west; there is an average proton velocity $\mathbf{V}$ toward the east. The pressure gradient is caused by the atmospheric density gradient, there being fewer particles at lower altitudes because of the greater loss to the atmosphere. At the outer edge of a radiation belt, where the density decreases with increasing radius (for whatever reason), the reverse asymmetry should appear, with more particles moving west than east.

The divergence of the pressure tensor can be expanded in the adiabatic case as [*Chew et al.*, 1956]

$$\nabla \cdot \mathbf{P} = \hat{e}_1\left[\frac{\partial P_\parallel}{\partial s} - \frac{P_\parallel - P_\perp}{B}\frac{\partial B}{\partial s}\right] + \left[(P_\parallel - P_\perp)\frac{\partial \hat{e}_1}{\partial s} + \nabla P_\perp\right] \tag{42}$$

where $P_\parallel$ is $nm\langle(v_\parallel - V_\parallel)^2\rangle$, and $P_\perp$ is $\frac{1}{2}nm\langle v_\perp{}^2\rangle$. In the east-west asymmetry experiment there would be a small contribution from the line curvature $\partial\hat{e}_1/\partial s$ in addition to the one from the pressure gradient $\nabla P_\perp$.

It is possible to prove from the Vlasov (collisionless Boltzmann) equation that

$$\mathbf{J}(\mathbf{r}, t) \equiv ne\mathbf{V} = Ne\langle\dot{\mathbf{R}}_\perp + \hat{e}_1 v_\parallel\rangle + c\nabla \times \mathbf{M} \tag{43}$$

where N is the number of guiding centers per unit volume at $(\mathbf{r}, t)$, and $\mathbf{M}$ is the total magnetic moment per unit volume of particles with guiding centers at $\mathbf{r}$. The brackets mean the average over particles with guiding centers at $\mathbf{r}$. The perpendicular component of (43) is easily derived from (41) and the guiding-center equations. However, the parallel component is rather difficult to prove formally [see *Northrop*, 1963], even though the entire expression 43 is intuitively correct. It says that the total current density in a plasma is the sum of the guiding-center current and the current that results from the curl of the magnetic moment per unit volume.

NONADIABATIC EFFECTS

The application of adiabatic theory and the lowest-order invariants to the Van Allen radiation has been outlined in preceding sections. According to the theory, in the absence of collisions particles would remain indefinitely in the geomagnetic field and repeatedly precess about their invariant surfaces. In practice all three invariants may not hold sufficiently well for this permanent trapping to occur. There is low-temperature plasma permeating the magnetosphere about the earth, and the solar wind may produce disturbances that are propagated through this plasma. These disturbances in turn may be sufficiently fast to affect one or more of the lowest-order invariants. Even if the deviation of one of them from a constant is very small, this very small effect can operate over very long times in the geophysical case. The question becomes whether these effects are cumulative, or whether they are oscillatory and self-canceling over a long period. If the motion is truly nonadiabatic, in the sense defined following (16), the effects may be cumulative and the particle may become lost from the geomagnetic field. For example, if the magnetic moment decreases continuously, the particle will eventually become lost in the atmosphere. However, if the motion is adiabatic, in the sense of being predicted by the first few terms of the invariant series, then the particle may still be permanently trapped, with the guiding center following a slightly different path from that predicted by the lowest-order invariant. The distinction between these two possibilities, cumulative and oscillatory, may not always be sharp, though in one geometry it seemed to be quite sharp for the magnetic moment [see *Garren et al.*, 1958].

There would certainly be value in computing at least one higher term for the longitudinal and flux invariants. The consequences of the earth's rotation, coupled with the azimuthal asymmetry of its field, do not seem to be known except in the limit when Φ is invariant. In this limit a particle precesses rapidly about its invariant surface, and the surface rotates slowly and rigidly with a 24-hour period. The next terms of the longitudinal and flux invariant series ought to describe the lowest-order modification to this simple picture.

To conclude, it does not seem possible at present to make any general statements about nonadiabatic effects other than that numerical computation is probably needed to study them. However, these effects may be important in the dynamics of the trapped radiation and therefore merit attention.

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