COLLISIONAL END LOSS OF ELECTROSTATICALLY CONFINED PARTICLES IN A MAGNETIC MIRROR FIELD

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ABSTRACT. The basic problem of collisional end loss of electrostatically confined particles in a magnetic mirror field with a multi-species background plasma is considered. An analytical procedure similar to, but more accurate than, that of Pastukhov is employed. Using the method of images, in the limit of large mirror ratio and modest confining potential, a class of approximate solutions, $\{F_a\}$, to the corresponding linearized Fokker-Planck equation is found, which is characterized by some adjustable parameters. These parameters are then chosen so that the surface of $F_a = 0$ approximately matches the true loss boundary. An analytical estimate of the error in the loss rate due to this matching procedure is made. A new expression for the loss rate is developed and compared with the results from Fokker-Planck codes, and remarkable agreement over a wide range of mirror ratios and confining potentials is found.

1. INTRODUCTION

The basic problem of collisional end loss of electrostatically confined particles in a magnetic mirror field has been considered by many authors [1–5]. If the confining potential, Φ_m, is larger than the mean energy of confined species 'a', T_a, the bulk of the distribution function, f_a, is Maxwellian and f_a departs significantly from a Maxwellian only in the vicinity of the loss boundary in the velocity space. In this case, the corresponding Fokker-Planck equation can be linearized (Rosenbluth potentials are evaluated using a Maxwellian background distribution function) and the problem is simplified considerably. However, because of the shape of the loss boundary as shown in Fig.1 (hyperbolic in $v_\perp - v_\parallel$ space when the maximum confining potential occurs at the mirror throat), the solution of the linearized Fokker-Planck equation is still very tedious. Indeed, no exact analytical solution has yet been found.

Various approaches to this problem have appeared in the literature. Pastukhov [1] considered the electrostatic confinement of electrons by the ambipolar potential in a conventional mirror machine, in the limit of large mirror ratio ($R \gg 1$) and large confining potential ($x_a^2 = z_a e\Phi_m/T_a \gg 1$, where z_a is the atomic number of species 'a' and e is the electronic charge). Using the method of images, a class of approximate solutions was constructed which were characterized by two adjustable parameters. These parameters were then chosen so that the surface of $F_a = 0$ matched the position and radius of curvature of the true loss boundary at its vertex (the minimum energy point). No estimate of the error due to this matching approximation was given.

Chernin and Rosenbluth [2] considered electrostatic confinement of ions in a tandem mirror device. They replaced the true loss boundary by line segments of constant pitch angle or energy and solved the linearized

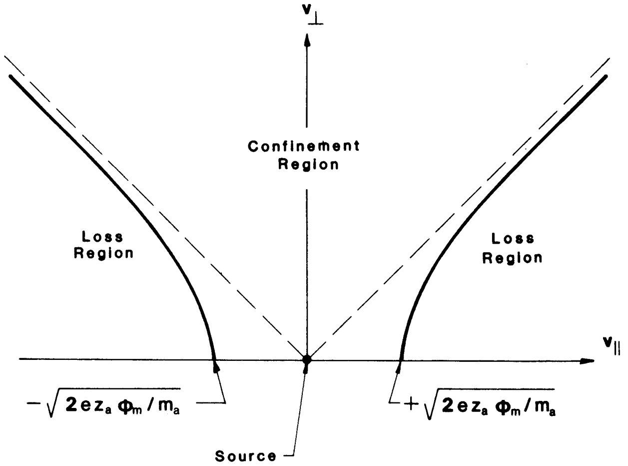

FIG.1. Velocity space loss boundary for electrostatically confined particles in a magnetic mirror field.

Fokker-Planck equation in each region by separation of variables. The solutions in different regions were then approximately matched (for $R \gg 1$ and $x_a^2 \gg 1$). The calculations of both Pastukhov [1] and Chernin and Rosenbluth [2] were done for square-well magnetic fields.

Cohen et al. [3] generalized the two previous analytical treatments to any electrostatically confined species in a multi-species plasma. In addition, Pastukhov's solution was generalized to any arbitrary magnetic field profile. The analytical results for square-well magnetic fields were also compared with those of Fokker-Planck codes, and reasonably good agreement between the generalized Pastukhov expression for the loss rate and the code results was found. However, the expression of Chernin and Rosenbluth [2] did not agree particularly well with the Fokker-Planck codes.

Catto and Bernstein [4] used a variational approach to the general problem of any electrostatically confined species in a multi-species plasma with arbitrary magnetic field profile and constructed the variational integral corresponding to the linearized Fokker-Planck equation. Assuming $R \gg 1$ and $x_a^2 \gg 1$ (but not $x_a^2 \gtrsim 1$, as stated in Ref.[4]), analytical expressions for the loss-rate and well-shape factors were found which are similar to those given in Ref.[3].

Since the Fokker-Planck codes used in Ref.[3] are based on finite-difference approximations, the hyperbolic shape of the loss boundary can only be approximated in these codes. To remove the effect of the complicated loss boundary on the accuracy of the numerical results, Fyfe and Bernstein [5] used a Rayleigh-Ritz finite-element procedure to numerically approximate the minimum of the associated variational integral to the linearized Fokker-Planck equation. However, they found that their results and the numerical results of Ref.[3] were in close agreement. They also examined the effect of well shapes and found reasonably good agreement with the analytical results of Ref.[3].

The previous analytical work has been limited to cases with $x_a^2 \gg 1$. However, present tandem-mirror experiments and reactor designs use a confining potential, x_a^2, of about 1 to 5. In addition, the previous analytical solutions [1–4] which are computed to O(1/R) do not fit asymptotically to numerical results at large mirror ratios. In this paper, we present a new expression for the loss rate which is accurate to O(1/R) and the new solution does correctly fit asymptotically to numerical results at large R.

In Section 2 we derive the linearized Fokker-Planck equation in the limit of $x_a^2 \gtrsim 1$ and show that the corresponding equations in Refs [1–3] are accurate only for $x_a^2 \gg 1$. A new set of variables is introduced in Section 3 which transforms the linearized Fokker-Planck equation into Laplacian form. Then, using the method of images, we replace the loss boundary by an imaginary sink inside the loss region and construct a class of solutions, $\{F_a\}$, which are characterized by the parameters of the imaginary sink. These parameters are then adjusted so that the surface of $F_a = 0$ matches the true loss boundary via some specified criteria. An analytical estimate of the error in the loss rate due to this matching technique is given in Section 4. In Section 5 we investigate various criteria for matching of the loss boundaries and compare the resulting expressions for the loss rate with the numerical results. A summary and conclusions are given in Section 6.

2. THE LINEARIZED FOKKER-PLANCK EQUATION

Using dimensionless notation similar to that of Refs [1, 3] and assuming gyro-symmetry and isotropic Rosenbluth potentials, the Fokker-Planck equation for the gyration-centre distribution function of species 'a', f_a, of a multi-species plasma may be written as

$$\frac{Df_a}{Dt} = \sum_b \Gamma_{ab} \frac{n_a n_b}{v_a^6} \left\{ \frac{1}{x^2} \frac{\partial}{\partial x} \left[\frac{x^2 y_b''}{2} \frac{\partial F_a}{\partial x} - \frac{m_a}{m_b} \left(\frac{x^2 y_b'''}{2} + x y_b'' - y_b' \right) F_a \right] \right.$$

$$+\frac{y_b'}{2x^3}\frac{\partial}{\partial\mu}\left[(1-\mu^2)\frac{\partial F_a}{\partial\mu}\right]\Bigg]\Bigg\} + S_a(x,\mu) \qquad (1)$$

where the prime denotes d/dx, $\mu \equiv \cos\theta = \vec{v}\cdot\vec{B}/(vB)$, and the summation is performed over all species. The parameters m, n and v_a are, respectively, mass, number density and mean velocity of species 'a', $v_a \equiv (2kT_a/m_a)^{1/2}$. Also,

$$\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v}\cdot\frac{\partial}{\partial\vec{r}} + \frac{ez_a}{m_a}(\vec{E} + \vec{v}\times\vec{B})\cdot\frac{\partial}{\partial\vec{v}}$$

$$\Gamma_{ab} = \frac{4\pi}{m_a^2}\left[\frac{z_a z_b e^2}{4\pi\epsilon_0}\right]^2 \lambda_{ab}$$

where λ_{ab} is the Coulomb logarithm, G_b is the Rosenbluth potential,

$$G_b \equiv \int d^3\vec{v}'|\vec{v}-\vec{v}'|f_b(\vec{v}') \qquad (2)$$

and $x \equiv v/v_b$, $F_a \equiv v_a^3 f_a/n_a$ and $y_b \equiv G_b/(n_b v_a)$.

Assuming that the confining potential is larger than the mean energy of the confined species, $x_a^2 \gtrsim 1$, the bulk of the distribution function f_a is Maxwellian. In addition, we assume that the distribution functions of other species are also close to Maxwellians. One can then linearize Eq.(1) by taking $f_b(\vec{v}')$ to be a Maxwellian in the expression for G_b. We then have

$$y_b = \frac{v_b}{v_a}\left(\frac{1}{\sqrt{\pi}}\exp(-x_b^2) + \left(x_b + \frac{1}{2x_b}\right)\operatorname{erf}(x_b)\right) \qquad (3)$$

where $x_b \equiv v/v_b$ and $v_b \equiv (2kT_b/m_b)^{1/2}$. For the limiting cases of $x_b^2 \gtrsim 1$ or $x_b^2 \ll 1$, one can expand Eq.(3) to get

$$y_b = \frac{v_b}{v_a}\left(x_b + \frac{1}{2x_b}\right)\left(1 + O(\exp(-x_b^2))\right) \qquad x_b \gtrsim 1 \qquad (4a)$$

$$y_b = \frac{2}{\sqrt{\pi}}\frac{v_b}{v_a}\left(1 + x_b^2/3 + O(x_b^4)\right) \qquad x_b \ll 1 \qquad (4b)$$

If species 'a' are electrons, $v_a = v_e$. Then, $x_e \equiv v/v_e = x \gtrsim 1$ and $x_i \equiv v/v_i = (v_e/v_i)\,x \gg 1$, and one can use the high-velocity limit, Eq.(4a), for both electron and ion Rosenbluth potentials. When species 'a' are ions, $v_a = v_i$. It follows that $x_i \equiv v/v_i = x \gtrsim 1$ while $x_e \equiv v/v_e = (v_i/v_e)x \ll 1$. We then use Eq.(4a) for ion and Eq.(4b) for electron Rosenbluth potentials. We also assume that the bounce frequency is much larger than any frequency of interest. Using these approximations, Eq.(1) can be simplified to

$$\frac{\tau_a v_a^3}{n_a}\frac{Df_a}{Dt} = \mathcal{L}(F_a)(1+\Delta_a) + \frac{\tau_a v_a^3}{n_a}S_a(x,\mu) \qquad (5a)$$

$$\mathcal{L}(F_a) \equiv \frac{1}{x^2}\frac{\partial}{\partial x}\left(F_a + \frac{1}{2x}\frac{\partial F_a}{\partial x}\right) + \frac{1}{x^3}\left(Z_a - \frac{1}{4x^2}\right)\frac{\partial}{\partial\mu}\left((1-\mu^2)\frac{\partial F_a}{\partial\mu}\right) \qquad (5b)$$

The constants τ_a and Z_a for electrons (a = e) and ions (a = i) are

$$\tau_e^{-1} \equiv \frac{4\pi}{m_e^2 v_e^3}\left(\frac{e^2}{4\pi\epsilon_0}\right)^2 n_e\lambda_{ee} \qquad (6a)$$

$$Z_e \equiv \frac{1}{2}\left(1 + \frac{\sum_j n_j z_j^2\lambda_{ej}}{n_e\lambda_{ee}}\right) \qquad (6b)$$

$$\tau_i^{-1} \equiv 4\pi\left(\frac{e^2}{4\pi\epsilon_0}\right)^2\sum_j \frac{n_j z_i^2 z_j^2\lambda_{ij}}{m_i m_j v_i^3}\frac{T_j}{T_i} \qquad (7a)$$

$$Z_i \equiv \frac{1}{2}\,\frac{\sum_j n_j z_j^2\lambda_{ij}}{\sum_j n_j z_j^2\lambda_{ij}(T_j/T_i)(m_i/m_j)} \qquad (7b)$$

where $\sum_j$ is a summation over ions only and the error terms, Δ_a, for electrons and ions are

$$\Delta_e = O(\exp(-x_a^2)) + O((m_e/m_i)^{1/2}) \qquad (8a)$$

$$\Delta_i = O(\exp(-x_a^2)) + O((m_e/m_i)^{1/2}x_a^3) + O(1 - T_i/T_j) \qquad (8b)$$

As boundary condition we use our assumption of large bounce frequency to demand that F_a vanish on the loss boundary. If the maximum confining potential occurs at the mirror throat, the loss boundary is

$$1 - \mu^2 = \frac{1}{R}\left(1 - \frac{x_a^2}{x^2}\right) \qquad (9)$$

In addition, F_a should exponentially approach zero for large velocities ($x \to \infty$) and F_a should approach a

Maxwellian for low velocities, $x \to 0$, or, far from the loss boundary, $\mu \to 0$. (Note that we assume S_a to be a low-velocity source.)

For a square-well magnetic field, Eq.(5a) can be trivially bounce-averaged. The result is similar to Eq.(5a), except that D/Dt is simply replaced by $\partial/\partial t$. For arbitrarily shaped magnetic fields, one can follow Cohen et al. [3] to bounce-average Eq.(5a). The result is similar to Eq.(5a), with some modifications [3]. However, such bounce-averaging is accurate only to $O(1/x_a^2)$.

Finally, we assume that S_a is a low-velocity source, which either maintains a steady state or, in a dynamical problem, entails a loss rate small enough so that $f_a^{-1}\,\partial f_a/\partial t$ is much smaller than the collision frequency. In this case, the linearized Fokker-Planck equation (5a) will reduce to a steady-state equation, $\mathcal{L}(F_a) = 0$.

The term $1/4x^2$ in the coefficient of the scattering term in Eq.(5b) is missing in previous analytical work [1–3]. Therefore, the starting equations given in these references are accurate only to $O(1/x_a^2)$. This term has been included in Ref.[4].

3. APPROXIMATE SOLUTION

In this section we construct a class of solutions to the linearized Fokker-Planck equation. Following Pastukhov [1], we extend the domain of F_a into the loss region where an imaginary sink of particles, $Q(x, \mu)$, is assumed to exist.

$$\frac{v_a^3 \tau_a}{n_a}\frac{\partial f_a}{\partial t} \equiv \mathcal{L}(F_a) + Q(x,\mu) = 0 \tag{10}$$

One can attempt to solve Eq.(10) and to find the form of Q (the image of the low-velocity source S_a) such that $F_a = 0$ on the true loss boundary. But such an approach is at least as hard as directly solving Eq.(5). We therefore limit ourselves to a subset of image sources which are physically reasonable and simplify the problem:

$$Q(x,\mu) = -\frac{\delta(1-\mu^2)}{4\pi}\, q(x)\, \eta(x-a) \tag{11}$$

where η is the step function. Note that, because of this simplification, even with the best choice of q(x) the surface of $F_a = 0$ may only approximately match the true loss boundary.

Even with this simple sink form (11), no exact solution to Eq.(10) has yet been found. To construct an approximate solution, we assume a large mirror ratio and use our assumptions to introduce the following orderings:

(1) We note that F_a significantly departs from a Maxwellian in a region near the loss boundary; the flux of particles across the loss boundary is maximum at the vertex ($x = x_a$ and $\mu = 1$); and both the flux and F_a scale as $\exp(-x^2)$. Therefore, we look for solutions in a region which extends about one e-folding away from the vertex, i.e. in the region where

$$|x^2 - x_a^2| \lesssim 1 \tag{12a}$$

Using Eq.(9), the corresponding range for θ can be found,

$$\theta^2 \sim \sin^2\theta \sim \frac{1}{R x_a^2} \tag{12b}$$

(2) In the region of interest near the loss boundary, the change in F_a is mostly in the direction perpendicular to the loss boundary rather than parallel to it; the contours of F_a = constant run approximately parallel to the loss boundary. In other words,

$$\frac{\partial F_a/\partial x}{\partial F_a/\partial \mu} \sim -\left.\frac{d\mu}{dx}\right|_{L.B.} \tag{13}$$

In addition, we note that F_a is symmetric about the $v_\perp$ axis, $F_a(x,\mu) = F_a(x,-\mu)$ and F_a is assumed to be a Maxwellian far away from the loss boundary ($\mu \to 0$). Therefore, one can consider only the right half-portion of the $v_\perp - v_\parallel$ space in Fig.1 ($0 \leqslant \theta \leqslant \pi/2$ or $\mu \geqslant 0$).

We now introduce a set of new variables, ζ and ρ, defined by

$$\zeta \equiv \exp(x^2) \tag{14a}$$

$$\rho \equiv \left(\frac{2x^2}{Z_a - 1/4x^2}\right)^{1/2} \exp(x^2)\tan\theta \tag{14b}$$

Using the ordering (12), it is easy to show that $\rho^2/\zeta^2 \sim x^2\theta^2 \sim O(1/RZ_a) \ll 1$. Defining g as $g \equiv F_a/F_{max} = \pi^{3/2}\exp(x^2)\, F_a$, the operator $\mathcal{L}(F_a)$ in Eq.(10) can be written in terms of the new variables,

$$\mathcal{L}(F_a) = 2\pi^{-3/2}\,\frac{\zeta}{x} \times\left(\frac{\partial^2 g}{\partial \zeta^2} + \Delta_x + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) + \Delta_\theta\right) \tag{15}$$

We will show that the remainder terms Δ_x and Δ_θ are small and can be ignored to $O(1/RZ_a)$. For simplicity of notation, we denote

$$L \equiv \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g}{\partial\rho}\right) = \frac{1}{\rho}\frac{\partial g}{\partial\rho} + \frac{\partial^2 g}{\partial\rho^2} \equiv L_1 + L_2 \quad (16)$$

Using this notation, Δ_θ has the form

$$\Delta_\theta = O(\theta^2)\,L + O(\theta^2)\,L_1 \quad (17a)$$

The other remainder term, Δ_x, includes the terms of the form $\partial^2 g/\partial\rho\partial\zeta$. To find an estimate for these terms, we rewrite the ordering (13) in terms of ρ and ζ to find an estimate for $\partial g/\partial\zeta$. Substituting the results in $\partial^2 g/\partial\rho\partial\zeta = \partial/\partial\rho\,(\partial g/\partial\zeta)$ and evaluating Δ_x, one has

$$\Delta_x = O(\rho^2/\zeta^2)\,L_1 + O(\rho^2/\zeta^2)\,L_2 + O(\rho^2/\zeta^2)\,L \quad (17b)$$

We assume that the magnitudes of L_1, L_2 and L are of the same order. Then, relations (17) result in $\Delta_x \sim O(1/RZ_a)$ L and $\Delta_\theta \sim O(1/Rx_a^2)$ L, and, therefore, these remainder terms can be ignored compared with L in Eq.(15).

For simplicity, we redefine the sink term (11) as

$$Q(x,\mu) = -\frac{\delta(1-\mu^2)}{4\pi} \times \left[\frac{8}{\sqrt{\pi}}\left(Z_a - \frac{1}{4x^2}\right)\frac{\exp(-x^2)}{x^3}\,\bar{q}(x)\right]\eta(x-a) \quad (18)$$

where the bracket is replacing q(x) in definition (11). Next, we write the sink expression (18) in terms of ρ and ζ, noting that we are only interested in the right half-portion of $v_\perp - v_\parallel$ space ($\mu \geq 0$). Using the result and substituting for $\mathcal{L}(F_a)$ from relation (15), the linearized Fokker-Planck equation (10) can be written as

$$\nabla^2 g - \frac{\delta(\rho)}{2\pi\rho}\,4\pi\bar{q}(x)\,\eta(\zeta-\zeta_a) = 0 \quad (19)$$

where $\zeta_a \equiv \exp(a^2)$, $x \equiv (\ln\zeta)^{1/2}$, and ∇^2 is the Laplacian operator in cylindrical co-ordinates ρ and ζ. Note that Eq.(19) is correct to $O(\exp(-x_a^2))$ used in deriving the linearized Fokker-Planck equation and to $O(1/RZ_a)$ in dropping the remainder terms in Eq.(15). The boundary conditions for Eq.(19) are simply g = 1 for ζ approaching zero ($\zeta = 0$ plane) or infinity.

Equation (19) is similar to Poisson's equation, $\nabla^2 g = -4\pi\sigma$, with a Dirichlet boundary condition (g = 1) on the surface $\mathcal{S}$ (plane of $\zeta = 0$ and infinity). We denote G_D as the Green's function, satisfying

$$\nabla_r^2 G_D(\vec{r},\vec{r}_0) = -4\pi\delta(\vec{r}-\vec{r}_0) \quad (20)$$

with $G_D = 0$ for $\vec{r}$ on $\mathcal{S}$. Then, using Green's theorems, the solution to Eq.(19) can be written as [6]

$$g(\vec{r}) = \int_{\mathcal{V}} \sigma(\vec{r}_0)\,G_D(\vec{r},\vec{r}_0)\,d^3\vec{r}_0 - \frac{1}{4\pi}\int_{\mathcal{S}} g(\vec{r}_0)\,\frac{\partial G_D}{\partial\vec{n}_0}\,da_0 \quad (21)$$

where $\mathcal{V}$ is the volume enclosed by $\mathcal{S}$, $\partial G_D/\partial\vec{n}_0 \equiv \vec{n}_0\cdot\nabla_{r_0}G_D$, and $\vec{n}_0$ is the outward normal to $\mathcal{S}$. Evaluating G_D by the method of images and substituting the result in Eq.(21), one finds that the surface integral is exactly equal to unity and the expression for g is

$$g = 1 - \int_{\zeta_a}^{\infty} d\zeta_0\,\bar{q}(x_0)\left[\left(\rho^2 + (\zeta-\zeta_0)^2\right)^{-1/2} - \left(\rho^2 + (\zeta+\zeta_0)^2\right)^{-1/2}\right] \quad (22)$$

where $x_0 = (\ln\zeta_0)^{1/2}$. For $\bar{q}(x) = q_0$, a constant, Eq.(22) can be integrated to obtain

$$g = 1 - q_0 \ln\left(\frac{\zeta_a + \zeta + \sqrt{\rho^2 + (\zeta_a+\zeta)^2}}{\zeta_a - \zeta + \sqrt{\rho^2 + (\zeta_a-\zeta)^2}}\right) \quad (23)$$

The adjustable parameters 'q_0' and 'a' should then be chosen such that the surface of $F_a = 0$ (or g = 0) matches the true loss boundary (Section 6).

If one replaces $\tan\theta$ with $\sin\theta$ or θ itself in the definition (14b) of ρ, the above conclusions are still correct, since we are interested in the region near the vertex where $\theta^2 \ll 1$. We have chosen the form (14b) so that the right half-portion of $v_\perp - v_\parallel$ space ($0 \leq \theta \leq \pi/2$) transforms into the entire top half of $\rho\zeta$ space ($0 \leq \rho$ and $0 \leq \zeta$). This choice simplifies the evaluation of the surface integral in Eq.(21). In addition, the lower limit for ζ, $\zeta = 1$ for $x \to 0$, can be easily extended to $\zeta = 0$ since $\zeta \equiv \exp(x^2) \gg 1$ (or set $\zeta' \equiv \exp(x^2) - 1 = \zeta(1 + O(\exp(-x^2)))$.

4. ESTIMATE OF ERROR DUE TO MODIFICATION OF LOSS BOUNDARY

Both the Pastukhov solution [1] and ours result in distribution functions which vanish on a surface that approximates, but is not identical to, the true loss boundary (9). Before introducing some criteria for

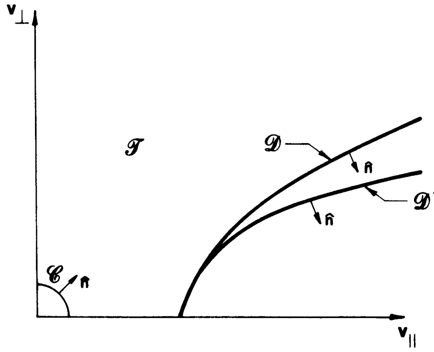

FIG.2. True loss boundary, $\mathscr{D}$ $(g = 0)$, approximate loss boundary, $\mathscr{D}'$ $(g' = 0)$, and low-energy (source) boundary, $\mathscr{C}$ $(g = g' = 1)$, in velocity space.

this matching, we consider the error introduced in the loss rate by this change in the loss boundary. As in Refs [4, 7], we rewrite the linearized Fokker-Planck equation (5) in the form

$$\frac{\partial f_a}{\partial t} = \nabla \cdot \overleftrightarrow{D} \cdot \nabla g = 0 \tag{24}$$

where $\nabla \equiv \partial/\partial \vec{v}$. The diffusion tensor, $\overleftrightarrow{D}$, is symmetric and is given by

$$\overleftrightarrow{D} \equiv \frac{v_a^3}{\tau_a} \left\{ \left(Z_a - \frac{v_a^2}{4v^2} \right) \frac{v^2 \overleftrightarrow{I} - \vec{v}\vec{v}}{v^3} + \frac{\vec{v}\vec{v}}{v^4} \frac{T}{mv} \right\} f_{max} \tag{25}$$

where f_{max} is a Maxwellian distribution. As the boundary conditions for Eq.(24), we have $g = 0$ on the true loss boundary $\mathscr{D}$, and $g = 1$ on the low-energy boundary $\mathscr{C}$ (Fig.2).

The loss rate, $-dn/dt$, is the flux of particles across $\mathscr{C}$ or $\mathscr{D}$. This particle flux, Γ, can be found by integrating Eq.(24) over the velocity space,

$$\Gamma = -\int_{\mathscr{S}} da\, \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g \tag{26}$$

where $\mathscr{S}$ denotes $\mathscr{C}$ or $\mathscr{D}$, da is the surface element on $\mathscr{S}$, and the direction of the unit normal, $\vec{n}$, is indicated in Fig.2.

We denote our approximate solution by g': g' satisfies Eq.(24) and $g' = 1$ on $\mathscr{C}$, but g' vanishes on the modified loss boundary $\mathscr{D}'$ (Fig.2). The approximate particle loss rate due to g', Γ', can be found similar to Eq.(26)

$$\Gamma' = -\int_{\mathscr{S}'} da\, \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g' \tag{27}$$

where $\mathscr{S}'$ denotes $\mathscr{C}$ or $\mathscr{D}$ or $\mathscr{D}'$.

Now, in the spirit of Ref.[7], consider the functional I,

$$I \equiv \int_{\mathscr{T}} d^3\vec{v}\, \nabla g' \cdot \overleftrightarrow{D} \cdot \nabla g \tag{28}$$

where $\mathscr{T}$ is the volume enclosed between $\mathscr{C}$ and $\mathscr{D}$, and $d^3\vec{v}$ is the differential volume element in the velocity space. The integral (28) can be evaluated by integration by parts, namely, using $\nabla \cdot (g' \cdot \overleftrightarrow{D} \cdot \nabla g) = \nabla g' \cdot \overleftrightarrow{D} \cdot \nabla g + g' \nabla \cdot \overleftrightarrow{D} \cdot \nabla g$, the divergence theorem, and noting that $\nabla \cdot \overleftrightarrow{D} \cdot \nabla g = 0$ in $\mathscr{T}$. Thus,

$$I = \int_{\mathscr{D}} da\, g' \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g - \int_{\mathscr{C}} da\, g' \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g \tag{29}$$

Since $g' = 1$ on $\mathscr{C}$, the surface integral on $\mathscr{C}$ in Eq.(29) is exactly the true flux, Γ. Therefore,

$$I = \Gamma + \int_{\mathscr{D}} da\, g' \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g \equiv \Gamma - \delta\Gamma \tag{30}$$

In addition, we have $\nabla g' \cdot \overleftrightarrow{D} \cdot \nabla g = \nabla g \cdot \overleftrightarrow{D} \cdot \nabla g'$ ($\overleftrightarrow{D}$ is symmetric). Rewriting I through this new variation and integrating by parts, we have

$$I = \int_{\mathscr{D}} da\, g \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g' - \int_{\mathscr{C}} da\, g \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g' \tag{31}$$

Now, $g = 0$ on $\mathscr{D}$ and $g = 1$ on $\mathscr{C}$. Therefore, the surface integral on $\mathscr{D}$ in Eq.(31) vanishes while the surface integral on $\mathscr{C}$ is exactly Γ'; $I = \Gamma'$. Using Eq.(30), we find that $\delta\Gamma$, defined by relation (30), is the difference between the true flux and the approximate flux, $\delta\Gamma = \Gamma - \Gamma'$. However, expression (30) for $\delta\Gamma$ is not useful since it contains the unknown distribution function g. We therefore write $\delta\Gamma$ as $\delta\Gamma \equiv \Delta\Gamma + I_1$, where

$$\Delta\Gamma \equiv -\int_{\mathscr{D}} da\, g' \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla g' \tag{32}$$

$$I_1 \equiv -\int_{\mathscr{D}} da\, g' \vec{n} \cdot \overleftrightarrow{D} \cdot \nabla (g - g') \tag{33}$$

If $\mathscr{D}'$ is a good match for $\mathscr{D}$, g' will be small on $\mathscr{D}$, $g' \sim O(\epsilon)$. Comparing the definition (32) of $\Delta\Gamma$ with the expression for Γ, one can write $\Delta\Gamma \sim O(\epsilon)\,\Gamma$. On the other hand, if $\mathscr{D}'$ is a good match for $\mathscr{D}$, the difference between Γ' and Γ should be small, namely

$$\Gamma - \Gamma' = -\int_{\mathscr{D}} da\, \vec{n}\cdot\vec{\vec{D}}\cdot\nabla(g-g') = O(\epsilon)\,\Gamma \tag{34}$$

Comparing Eqs (33) and (34), one has $I_1 \sim O(\epsilon)\, g'\Gamma \sim O(\epsilon^2)\,\Gamma$. Therefore, to first order in ϵ, I_1 can be ignored, $\delta\Gamma \approx \Delta\Gamma$, or the true flux, Γ, is

$$\Gamma \equiv \Gamma' + \delta\Gamma = \Gamma' + \Delta\Gamma + I_1 = \bar{\Gamma} + I_1 \approx \bar{\Gamma} \tag{35}$$

where $\bar{\Gamma} \equiv \Gamma' + \Delta\Gamma$ is the corrected flux.

The form (33) for I_1 suggests some interesting features. First, since the differential flux, $da\, \vec{n}\cdot\vec{\vec{D}}\cdot\nabla g$, has exponential behaviour, definition (33) shows that a good way to minimize I_1 is to ensure that g' is small near the vertex, or $\mathscr{D}'$ match $\mathscr{D}$ near the vertex. Also, since $g = 0$ on $\mathscr{D}$, one can replace g' in the integrand of I_1 by $-(g-g')$. Then, using the divergence theorem and noting that $g = g' = 1$ on $\mathscr{C}$, I_1 can be written as

$$I_1 = +\int_{\mathscr{T}} d^3\vec{v}\, \nabla(g-g')\cdot\vec{\vec{D}}\cdot\nabla(g-g') \tag{36}$$

where the integrand is a quadratic form. We note that the eigenvalues of $\vec{\vec{D}}$ are $\lambda_1 = \lambda_2 = d(Z_a - 1/2x^2)$ and $\lambda_3 = d/2x^2$, where $d \equiv (\pi^{-3/2} n_a/v_a\tau_{aa}) \exp(-x^2)/x$. Therefore, the quadratic form in the integrand of I_1 is positive definite if $x^2 \geqslant 1/2Z_a$. Since g and g' both depart from a Maxwellian ($g = g' = 1$) only near the loss boundary, the integrand of Eq.(36) should be positive over $\mathscr{T}$ if $x_a^2 \gtrsim 1/2Z_a$, resulting in $I_1 > 0$. Therefore, using Eq.(34), the corrected flux is smaller than the true flux for $x_a^2 \gtrsim 1/2Z_a$ (or the corrected confinement time is larger than the true one).

One should note that, in addition to ignoring I_1, the corrected flux is still an approximation to the true flux since we have assumed that g' exactly satisfies Eq.(24). The error terms in finding g' ($O(1/RZ_a)$ and $O(\exp(-x_a^2))$ for the new solution, and $O(1/RZ_a)$ and $O(1/x_a^2)$ for the Pastukhov solution) are still present in the expression for Γ.

Last, we note that expression (32) for $\Delta\Gamma$ can be explicitly written as

$$\Delta\Gamma = \frac{2}{\sqrt{\pi}}\frac{n_a}{\tau_a}\int_{\epsilon_a}^{\infty} d\epsilon\, \exp(-\epsilon)\, g'(\epsilon,\mu_0)$$
$$\times\left[-2\frac{\partial g'}{\partial\epsilon}\frac{d\mu}{d\epsilon} + \left(Z_a - \frac{1}{4\epsilon}\right)\left(\frac{1-\mu^2}{\epsilon}\right)\frac{\partial g'}{\partial\mu}\right]_{\mu=\mu_0} \tag{37}$$

where $\epsilon \equiv x^2$, $\epsilon_a \equiv x_a^2$, and μ_0 is given by Eq.(9): $1-\mu_0^2 = (\epsilon-\epsilon_a)/R\epsilon$ and $d\mu/d\epsilon = -\epsilon_a/2R\mu\epsilon^2$.

5. LOSS RATE

Our discussion of the error estimate in Section 4 showed that the corrected flux, $\bar{\Gamma}$, is a good approximation to the true flux, Γ, when g' is small on $\mathscr{D}$, especially near the vertex. Therefore, as our first criteria for matching $\mathscr{D}$ and $\mathscr{D}'$, we demand that the modified loss boundary, $\mathscr{D}'$, match the position and the radius of curvature of the true loss boundary at its vertex. In other words,

$$F_a\,(x = x_a, \mu = 1) = 0 \tag{38a}$$

$$\left.\frac{\partial F_a/\partial x}{\partial F_a/\partial\mu}\right|_{x_a,1} = -\left(\frac{d\mu}{dx}\right)_{\mathscr{D};x_a,1} = \frac{1}{Rx_a} \tag{38b}$$

For $q(x_0) = q_0 =$ constant, the distribution function, F_a, is given by Eq.(23). Then, the matching conditions (38) result in

$$q_0 = \left(\ln\left(\frac{w+1}{w-1}\right)\right)^{-1} \tag{39a}$$

$$w^2 = 1 + 1/R(Z_a - 1/4\,x_a^2) \tag{39b}$$

where $w \equiv \exp(a^2 - x_a^2)$.

Even in this simple case, the error term, $\Delta\Gamma$, should be calculated by numerical integration of Eq.(37). To obtain an approximate analytical estimate for $\Delta\Gamma$, we examine the integrand of Eq.(32) (or Eq.(37)) using the distribution function (23) together with the sink parameters (39). We find from numerical examples that g' varies nearly linearly with $\epsilon - \epsilon_a$, while the differential flux, $(da/d\epsilon)\, \vec{n}\cdot\vec{\vec{D}}\cdot\nabla g'$, falls slightly more rapidly than $\exp(-\epsilon)$. Thus, a reasonable estimate for $\Delta\Gamma$ is

$$\Delta\Gamma \approx g'(\epsilon^*, \mu^*)\int_{\mathscr{D}} da\, \vec{n}\cdot\vec{\vec{D}}\cdot\nabla g' \tag{40}$$

where $\epsilon^* = \epsilon_a + 1$ and $\mu^* = \mu_0(\epsilon^*)$. Using the distribution function (23) in the limit of large mirror ratio, $O(1/RZ_a)$, $g'(\epsilon^*, \mu^*)$ can be estimated. Then,

$$\frac{\Delta\Gamma}{\Gamma'} \approx g'(\epsilon^*, \mu^*) \approx 0.84\, q_0 \tag{41a}$$

where q_0 is given by Eq.(39a). It is interesting to note that if one assumes that the differential flux $(da/d\epsilon)\,\vec{n}\cdot\vec{\vec{D}}\cdot\nabla g$ has exponential behaviour (i.e. scales as $\exp(-\epsilon)$), then the above argument can be applied to definition (30) for $\delta\Gamma$ to obtain

$$\frac{\delta\Gamma}{\Gamma} \approx g'(\epsilon^*, \mu^*) \approx 0.84\, q_0 \tag{41b}$$

Substituting for q_0 from Eq.(39a) and in the limit of large mirror ratio, we find that $\Delta\Gamma/\Gamma' \approx \delta\Gamma/\Gamma \approx O(1/\ln(4RZ_a))$.

The approximate loss rate due to g', Γ', can be found by integrating Eq.(10) over the velocity space, using expression (18) for $Q(x, \mu)$. Then, adding the estimate (41b) for $\delta\Gamma$, the new expression for the confinement time is

$$n\tau = n_a \tau_{aa} \frac{\sqrt{\pi}}{4} \frac{a^2 \exp(a^2)}{\bar{I}(a^2)} \left(\ln\left(\frac{w+1}{w-1}\right) - 0.84\right) \tag{42}$$

where $w \equiv \exp(a^2 - x_a^2) = 1 + 1/R(Z_a - 1/4\, x_a^2)$ and

$$\bar{I}(a^2) \equiv \left(Z_a + \frac{1}{4}\right)\bar{E}_1(a^2) - \frac{1}{4} \tag{43}$$

Here $\bar{E}_1(y) \equiv y \exp(y)\, E_1(y)$ and $E_1(y)$ is the exponential integral. Simple and accurate approximations to $\bar{E}_1(y)$ are given in Refs [8, 9].

We have also calculated $\Delta\Gamma$ by numerical integration of expression (37). The resulting corrected confinement time agrees with the analytical estimate (42) within a few per cent over a wide range of mirror ratio and confining potential. In general, the numerical results for the confinement time are slightly larger than expression (42). This is not surprising since expression (42) includes an estimate for $\delta\Gamma$ while the numerical calculation uses $\Gamma \approx \Gamma' + \Delta\Gamma$, and we have shown in Section 4 that $I_1 = \delta\Gamma - \Delta\Gamma > 0$ for $x_a^2 \gtrsim 1/2Z_a$.

We have also used different expressions for the sink $\bar{q}(x_0)$ (besides $\bar{q}(x_0) = q_0$ = constant). We have considered sinks of the form $\bar{q}(x_0) = q_0(x/x_a)^m$, where m is given. The adjustable parameters, 'q_0' and 'a', are numerically calculated using the integral expression (22) for the distribution function and matching conditions (38). We have found that for m ranging from −2 to 2, the change in corrected flux, Γ, is small (a few per cent). Thus, probably no major improvement in Γ can be found by optimizing the sink form $\bar{q}(x_0)$.

A comparison of the new expression (42) for the confinement time, the Fokker-Planck code results [3]

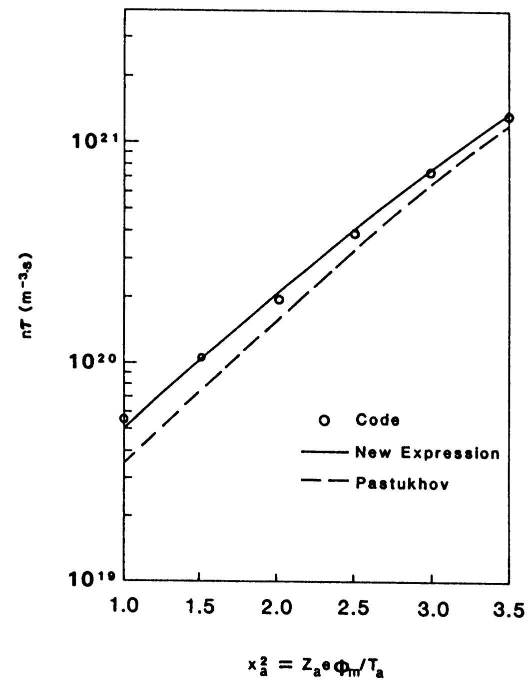

FIG.3. Confinement time of electrostatically confined ions as a function of the confining potential for $R = 10$, $z_i = 1$ ($Z_a = 1/2$), $m_i = 2.5$ amu, $T_i = 30$ keV and $T_e = 45$ keV.

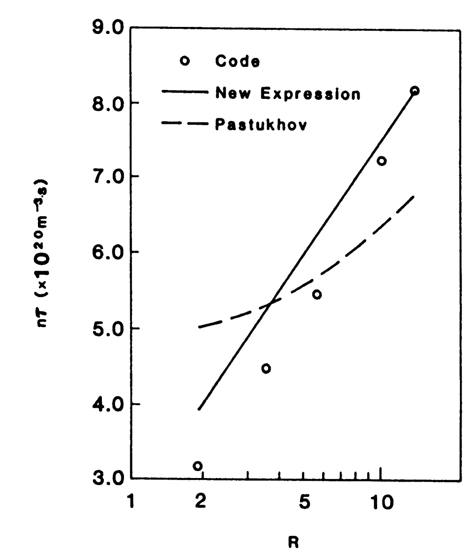

FIG.4. Confinement time of electrostatically confined ions as a function of the mirror ratio for $x_a^2 = 3$, $z_i = 1$ ($Z_a = 1/2$), $m_i = 2.5$ amu, $T_i = 30$ keV and $T_e = 45$ keV.

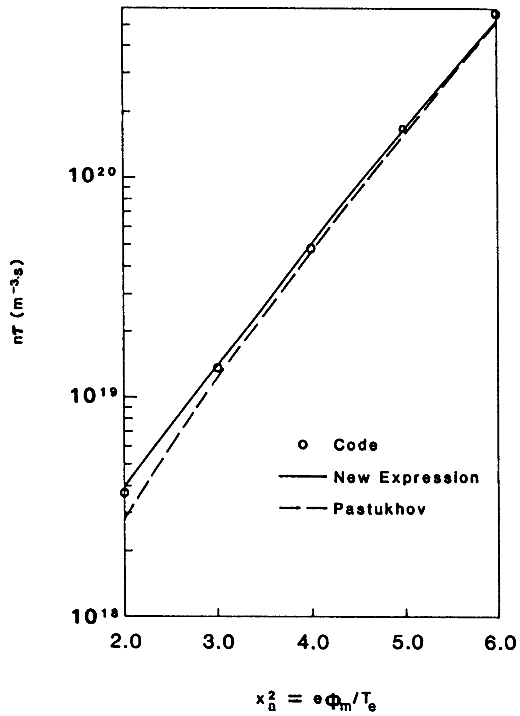

FIG.5. Confinement time of electrostatically confined electrons as a function of the confining potential for $R = 10$, $Z_a = 1$ and $T_e = 45$ keV.

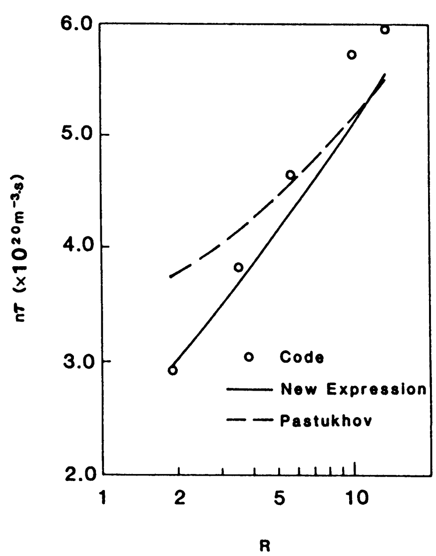

FIG.6. Confinement time of electrostatically confined electrons as a function of the mirror ratio for $x_a^2 = 6$, $Z_a = 1$ and $T_e = 45$ keV.

and the generalized Pastukhov expression [3] is given in Figs 3–6. There, the ion and electron confinement times are plotted versus a wide range of mirror ratio and confining potential. The new expression shows a remarkable agreement with the code results within the accuracy of the calculations and the code results (±7%) [3]. The new expression also asymptotically fits to the code results at large mirror ratio and is accurate where the generalized Pastukhov expression fails (small or large mirror ratios and modest confining potential).

The average energy of the lost particles, E_a, can be found by multiplying Eq.(10) by $m_a v^2/2$ and integrating the result over the velocity space. Using the sink parameters (39), we have

$$\frac{E_a}{T_a} = \frac{a^2 Z_a - \frac{1}{4}\bar{E}_1(a^2)}{\bar{I}(a^2)} \tag{44}$$

The accuracy of this expression is also checked. Choosing $T_i = 14$ keV, $\Phi_m = 50$ keV, $R = 10$ and $Z_a = 1/2$, we find $E_a = 61.6$ keV from the Fokker-Planck codes [3], $E_a = 63.6$ keV from expression (44), and $E_a = 65.3$ keV from Pastukhov's results [1, 3].

As another criterium for matching $\mathscr{D}$ and $\mathscr{D}'$, we demanded that: (1) $\mathscr{D}$ and $\mathscr{D}'$ have a common vertex (i.e. Eq.(39a) is satisfied), and (2) $\Delta\Gamma = 0$. The second condition, $\Delta\Gamma = 0$, has been satisfied both by numerically setting Eq.(32) equal to zero or using the analytical estimate (40) and by setting $g'(\epsilon^*, \mu^*) = 0$. The comparison among the calculated confinement times from these two criteria, expression (42) and the Fokker-Planck codes shows that expression (42) is more accurate. This is due to the fact that for $\Delta\Gamma = 0$ g' switches sign along $\mathscr{D}$ (see Eq.(32)) rather than becoming small. In this case, the argument in Section 4 leading to $I_1 \ll \Delta\Gamma$ is not valid and the correction term I_1 cannot be ignored.

We have also applied our error estimate to the Pastukhov solution. Using an argument similar to the one leading to Eqs (40) and (41), we find

$$\left(\frac{\delta\Gamma}{\Gamma}\right)_p \approx \left(\frac{\Delta\Gamma}{\Gamma}\right)_p \approx g_p(\epsilon^*, \mu^*) \approx 0.84\, q_0 - 1/2x_a^2 \tag{45}$$

where q_0 is given by Eq.(39a), while $w^2 = 1 + 1/RZ_a$. Addition of this error term to the Pastukhov expression results in some improvement at small mirror ratio and in deterioration or no improvement at large mirror ratio.

6. SUMMARY

We have considered a more accurate determination of the loss rate for an electrostatically confined species in a magnetic mirror field. Our method is similar to that of Pastukhov [1, 3]. Using the method of images, a class of approximate distribution functions, $\{F_a\}$, has been found which vanish on a modified loss boundary, $\mathscr{D}'$. Various criteria for matching $\mathscr{D}'$ with the true loss boundary, $\mathscr{D}$, are considered. An estimate of the error in the loss rate due to this modification in the loss boundary is also derived.

The new solution is accurate to $O(1/RZ_a)$ and $O(\exp(-x_a^2))$, unlike previous analytical solutions which were accurate to $O(1/RZ_a)$ and $O(1/x_a^2)$. In particular for the Pastukhov solution, the approximation of $O(1/x_a^2)$ has been used in many places: in deriving the linearized Fokker-Planck equation, during the transformation of variables, in evaluating the distribution function from the Green's function, and in matching procedures (Pastukhov's solution only approximately matches $\mathscr{D}$ and $\mathscr{D}'$ at the vertex). Furthermore, the Pastukhov expression [1] and its generalization [3] are subject to a relative error of $O(1/\ln(4RZ_a))$ and $O(1/x_a^2)$, estimated in Section 5, because of the modification of the loss boundary. In addition, the previous solutions do not correctly fit asymptotically to the Fokker-Planck code results in the limit of large mirror ratio, while the new expression for the loss rate does correctly fit asymptotically to the code results at large mirror ratio.

The new expression for the loss rate is compared with the Fokker-Planck code results, and remarkable agreement, within the accuracy of the codes and the calculations, is found over a wide range of mirror ratio and confining potential. The simplicity of the new expression plus its accuracy make it well suited for use in power balance and rate codes for tandem mirror devices.

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