

The Einstein–Vlasov System/Kinetic Theory

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Abstract

The main purpose of this article is to provide a guide to theorems on global properties of solutions to the Einstein–Vlasov system. This system couples Einstein’s equations to a kinetic matter model. Kinetic theory has been an important field of research during several decades in which the main focus has been on nonrelativistic and special relativistic physics, i.e. to model the dynamics of neutral gases, plasmas, and Newtonian self-gravitating systems. In 1990, Rendall and Rein initiated a mathematical study of the Einstein–Vlasov system. Since then many theorems on global properties of solutions to this system have been established. The Vlasov equation describes matter phenomenologically, and it should be stressed that most of the theorems presented in this article are not presently known for other such matter models (i.e. fluid models). This paper gives introductions to kinetic theory in non-curved spacetimes and then the Einstein–Vlasov system is introduced. We believe that a good understanding of kinetic theory in non-curved spacetimes is fundamental to good comprehension of kinetic theory in general relativity.

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1 Introduction to Kinetic Theory

In general relativity, kinetic theory has been used relatively rarely to model phenomenological matter in comparison to fluid models. From a mathematical point of view, however, there are fundamental advantages to using a kinetic description. In non-curved spacetimes kinetic theory has been studied intensively as a mathematical subject during several decades, and it has also played an important role from an engineering point of view. In the first part of this introduction, we will review kinetic theory in non-curved spacetimes and we will consider mainly the special relativistic case, but mathematical results in the nonrelativistic case will also be discussed. We believe that a good understanding of kinetic theory in non-curved spacetimes is fundamental to good comprehension of kinetic theory in general relativity. Moreover, it is often the case that mathematical methods used to treat the Einstein–Vlasov system are carried over from methods developed in the special relativistic or nonrelativistic case.

The purpose of kinetic theory is to model the time evolution of a collection of particles. The particles may be entirely different objects depending on the physical situation. For instance, the particles are atoms and molecules in a neutral gas or electrons and ions in a plasma. In stellar dynamics the particles are stars and in a cosmological case they are galaxies or even clusters of galaxies. Mathematical models of particle systems are most frequently described by kinetic or fluid equations. A characteristic feature of kinetic theory is that its models are statistical and the particle systems are described by distribution functions $f = f(t, x, p)$, which represent the density of particles with given spacetime position $(t, x) \in \mathbb{R} \times \mathbb{R}^3$ and momentum $p \in \mathbb{R}^3$. A distribution function contains a wealth of information, and macroscopical quantities are easily calculated from this function. In a fluid model the quantities that describe the system do not depend on the momentum p but only on the spacetime point (t, x) . A choice of model is usually made with regard to the physical properties of interest for the system or with regard to numerical considerations. It should be mentioned that a fluid model that is too naive may give rise to shell-crossing singularities, which are unphysical. In a kinetic description such phenomena are ruled out.

The time evolution of the system is determined by the interactions between the particles which depend on the physical situation. For instance, the driving mechanism for the time evolution of a neutral gas is the collision between particles (the relativistic Boltzmann equation). For a plasma the interaction is through the electric charges (the Vlasov–Maxwell system), and in the stellar and cosmological cases the interaction is gravitational (the Einstein–Vlasov system). Of course, combinations of interaction processes are also considered but in many situations one of them is strongly dominating and the weaker processes are neglected.

1.1 The relativistic Boltzmann equation

Consider a collection of neutral particles in Minkowski spacetime. Let the signature of the metric be $(-, +, +, +)$, let all the particles have rest mass $m = 1$, and normalize the speed of light c to one. The four-momentum of a particle is denoted by p^a , $a = 0, 1, 2, 3$. Since all particles have equal rest mass, the four-momentum for each particle is restricted to the mass shell $p^a p_a = -m^2 = -1$. Thus, by denoting the three-momentum by $p \in \mathbb{R}^3$, p^a may be written $p^a = (-p^0, p)$, where $|p|$ is the usual Euclidean length of p and $p^0 = \sqrt{1 + |p|^2}$ is the energy of a particle with three-momentum p . The relativistic velocity of a particle with momentum p is denoted by \hat{p} and is given by

$$\hat{p} = \frac{p}{\sqrt{1 + |p|^2}}. \quad (1)$$

Note that $|\hat{p}| < 1 = c$. The relativistic Boltzmann equation models the space-time behaviour of the one-particle distribution function $f = f(t, x, p)$, and it has the form

$$\left(\partial_t + \frac{p}{p^0} \cdot \nabla_x \right) f = Q(f, f), \quad (2)$$

where the relativistic collision operator $Q(f, g)$ is defined by

$$Q(f, g) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} k(p, q, \omega) \times \\ \times [f(p + a(p, q, \omega)\omega) g(q - a(p, q, \omega)\omega) - f(p)g(q)] dp d\omega. \quad (3)$$

(Note that $g = f$ in (2)). Here $d\omega$ is the element of surface area on \mathbb{S}^2 and $k(p, q, \omega)$ is the scattering kernel, which depends on the scattering cross-section in the interaction process. See [23] for a discussion about the scattering kernel. The function $a(p, q, \omega)$ results from the collision mechanics. If two particles, with momentum p and q respectively, collide elastically (no energy loss) with scattering angle $\omega \in \mathbb{S}^2$, their momenta will change, $p \rightarrow p'$ and $q \rightarrow q'$. The relation between p, q and p', q' is

$$p' = p + a(p, q, \omega)\omega, \quad q' = q - a(p, q, \omega)\omega, \quad (4)$$

where

$$a(p, q, \omega) = \frac{2(p^0 + q^0)p^0 q^0 (\omega \cdot (\hat{q} - \hat{p}))}{(p^0 + q^0)^2 - (\omega \cdot (p + q))^2}. \quad (5)$$

This relation is a consequence of four-momentum conservation,

$$p^a + q^a = p^{a'} + q^{a'},$$

or equivalently

$$p^0 + q^0 = p^{0'} + q^{0'}, \quad (6)$$

$$p + q = p' + q'. \quad (7)$$

These are the conservation equations for relativistic particle dynamics. In the classical case these equations read

$$|p|^2 + |q|^2 = |p'|^2 + |q'|^2, \quad (8)$$

$$p + q = p' + q'. \quad (9)$$

The function $a(p, q, \omega)$ is the distance between p and p' (q and q'), and the analogue function in the Newtonian case has the form

$$a_{\text{cl}}(p, q, \omega) = \omega \cdot (q - p). \quad (10)$$

By inserting a_{cl} in place of a in (2) we obtain the classical Boltzmann collision operator (disregarding the scattering kernel, which is also different).

The main result concerning the existence of solutions to the classical Boltzmann equation is a theorem by DiPerna and Lions [25] that proves existence, but not uniqueness, of renormalized solutions (i.e. solutions in a weak sense, which are even more general than distributional solutions). An analogous result holds in the relativistic case, as was shown by Dudyński and Ekiel-Jezewska [26]. Regarding classical solutions, Illner and Shinbrot [46] have shown global existence of solutions to the nonrelativistic Boltzmann equation for small initial data (close to vacuum). At present there is no analogous result for the relativistic Boltzmann equation and this must be regarded as an interesting open problem. When the data are close to equilibrium (see below), global existence of classical solutions has been proved by Glassey and Strauss [36] in the relativistic case and by Ukai [87] in the nonrelativistic case (see also [84]).

The collision operator $Q(f, g)$ may be written in an obvious way as

$$Q(f, g) = Q^+(f, g) - Q^-(f, g),$$

where Q^+ and Q^- are called the gain and loss term respectively. In [2] it is proved that given $f \in L^2(\mathbb{R}^3)$ and $g \in L^1(\mathbb{R}^3)$ with $f, g \geq 0$, then

$$\|Q^+(f, g)\|_{H^1(\mathbb{R}^3)} \leq C \|f\|_{L^2(\mathbb{R}^3)} \|g\|_{L^1(\mathbb{R}^3)}, \quad (11)$$

under some technical requirements on the scattering kernel. Here H^s is the usual Sobolev space. This regularizing result was first proved by P.L. Lions [49] in the classical situation. The proof relies on the theory of Fourier integral operators and on the method of stationary phase, and requires a careful analysis of the collision geometry, which is very different in the relativistic case.

The regularizing theorem has many applications. An important application is to prove that solutions tend to equilibrium for large times. More precisely, Lions used the regularizing theorem to prove that solutions to the (classical) Boltzmann equation, with periodic boundary conditions, converge in L^1 to a global Maxwellian,

$$M = e^{-\alpha|p|^2 + \beta \cdot p + \gamma} \quad \text{with } \alpha, \gamma \in \mathbb{R}, \quad \alpha > 0, \quad \beta \in \mathbb{R}^3,$$

as time goes to infinity. This result had first been obtained by Arkeryd [8] by using non-standard analysis. It should be pointed out that the convergence

takes place through a sequence of times tending to infinity and it is not known whether the limit is unique or depends on the sequence. In the relativistic situation, the analogous question of convergence to a relativistic Maxwellian, or a Jüttner equilibrium solution,

$$J = e^{-\alpha\sqrt{1+|p|^2}+\beta\cdot p+\gamma}, \quad \alpha, \beta \text{ and } \gamma \text{ as above, with } \alpha > |\beta|,$$

had been studied by Glassey and Strauss [36, 37]. In the periodic case they proved convergence in a variety of function spaces for initial data close to a Jüttner solution. Having obtained the regularizing theorem for the relativistic gain term, it is a straightforward task to follow the method of Lions and prove convergence to a *local* Jüttner solution for arbitrary data (satisfying the natural bounds of finite energy and entropy) that is periodic in the space variables. In [2] it is next proved that the local Jüttner solution must be a global one, due to the periodicity of the solution.

For more information on the relativistic Boltzmann equation on Minkowski space we refer to [29, 23, 86] and in the nonrelativistic case we refer to the excellent review paper by Villani [88] and the books [29, 16].

1.2 The Vlasov–Maxwell and Vlasov–Poisson systems

Let us consider a collisionless plasma, which is a collection of particles for which collisions are relatively rare and the interaction is through their charges. We assume below that the plasma consists only of one type of particle, whereas the results below also hold for plasmas with several particle species. The particle rest mass is normalized to one. In the kinetic framework, the most general set of equations for modelling a collisionless plasma is the relativistic Vlasov–Maxwell system:

$$\partial_t f + \hat{v} \cdot \nabla_x f + (E(t, x) + \hat{v} \times B(t, x)) \cdot \nabla_v f = 0, \quad (12)$$

$$\partial_t E + j = c \nabla \times B, \quad \nabla \cdot E = \rho, \quad (13)$$

$$\partial_t B = -c \nabla \times E, \quad \nabla \cdot B = 0. \quad (14)$$

The notation follows the one already introduced with the exception that the momenta are now denoted by v instead of p . This has become a standard notation in this field. E and B are the electric and magnetic fields, and \hat{v} is the relativistic velocity

$$\hat{v} = \frac{v}{\sqrt{1 + |v|^2/c^2}}, \quad (15)$$

where c is the speed of light. The charge density ρ and current j are given by

$$\rho = \int_{\mathbb{R}^3} f dv, \quad j = \int_{\mathbb{R}^3} \hat{v} f dv. \quad (16)$$

Equation (12) is the relativistic Vlasov equation and (13, 14) are the Maxwell equations.

A special case in three dimensions is obtained by considering spherically symmetric initial data. For such data it can be shown that the solution will also be spherically symmetric, and that the magnetic field has to be constant. The Maxwell equation $\nabla \times E = -\partial_t B$ then implies that the electric field is the gradient of a potential ϕ . Hence, in the spherically symmetric case the relativistic Vlasov–Maxwell system takes the form

$$\partial_t f + \hat{v} \cdot \nabla_x f + \beta E(t, x) \cdot \nabla_v f = 0, \quad (17)$$

$$E = \nabla \phi, \quad \Delta \phi = \rho. \quad (18)$$

Here $\beta = 1$, and the constant magnetic field has been set to zero, since a constant field has no significance in this discussion. This system makes sense for any initial data, without symmetry constraints, and is called the relativistic Vlasov–Poisson equation. Another special case of interest is the classical limit, obtained by letting $c \rightarrow \infty$ in (12, 13, 14), yielding:

$$\partial_t f + v \cdot \nabla_x f + \beta E(t, x) \cdot \nabla_v f = 0, \quad (19)$$

$$E = \nabla \phi, \quad \Delta \phi = \rho, \quad (20)$$

where $\beta = 1$. See Schaeffer [80] for a rigorous derivation of this result. This is the (nonrelativistic) Vlasov–Poisson equation, and $\beta = 1$ corresponds to repulsive forces (the plasma case). Taking $\beta = -1$ means attractive forces and the Vlasov–Poisson equation is then a model for a Newtonian self-gravitating system.

One of the fundamental problems in kinetic theory is to find out whether or not spontaneous shock formations will develop in a collisionless gas, i.e. whether solutions to any of the equations above will remain smooth for all time, given smooth initial data.

If the initial data are small this problem has an affirmative solution in all cases considered above (see [30, 35, 10, 11]). For initial data unrestricted in size the picture is more involved. In order to obtain smooth solutions globally in time, the main issue is to control the support of the momenta

$$Q(t) := \sup\{|v| : \exists (s, x) \in [0, t] \times \mathbb{R}^3 \text{ such that } f(s, x, v) \neq 0\}, \quad (21)$$

i.e. to bound $Q(t)$ by a continuous function so that $Q(t)$ will not blow up in finite time. That such a control is sufficient for obtaining global existence of smooth solutions follows from well-known results in the different cases (see [34, 42, 12, 30]). For the full three-dimensional relativistic Vlasov–Maxwell system, this important problem of establishing whether or not solutions will remain smooth for all time is open. In two space and three momentum dimensions, Glassey and Schaeffer [31] have shown that $Q(t)$ can be controlled, which thus yields global existence of smooth solutions in that case (see also [32]).

The relativistic and nonrelativistic Vlasov–Poisson equations are very similar in form. In particular, the equation for the field is identical in the two cases. However, the mathematical results concerning the two systems are very different. In the nonrelativistic case Batt [12] gave an affirmative solution 1977 in the case

of spherically symmetric data. Pfaffelmoser [55] (see also Schaeffer [82]) was the first one to give a proof for general smooth data. He obtained the bound

$$Q(t) \leq C(1+t)^{(51+\delta)/11},$$

where $\delta > 0$ could be taken arbitrarily small. This bound was later improved by different authors. The sharpest bound valid for $\beta = 1$ and $\beta = -1$ has been given by Horst [43] and reads

$$Q(t) \leq C(1+t) \log(2+t).$$

In the case of repulsive forces ($\beta = 1$) Rein [59] has found the sharpest estimate by using a new identity for the Vlasov–Poisson equation, discovered independently by Illner and Rein [45] and by Perthame [54]. Rein’s estimate reads

$$Q(t) \leq C(1+t)^{2/3}.$$

Independently and about the same time as Pfaffelmoser gave his proof, Lions and Perthame [50] used a different method for proving global existence. To some extent their method seems to be more generally applicable to attack problems similar to the Vlasov–Poisson equation but which are still quite different (see [3], [48]). On the other hand, their method does not give such strong growth estimates on $Q(t)$ as described above. For the relativistic Vlasov–Poisson equation (with $\beta = 1$), Glassey and Schaeffer [30] showed that if the data are spherically symmetric, $Q(t)$ can be controlled, which is analogous to the result by Batt mentioned above (we also mention that the cylindrical case has been considered in [33]). If $\beta = -1$, it was also shown in [30] that blow-up occurs in finite time for spherically symmetric data with negative total energy. This system, however, is unphysical in the sense that it is not a special case of the Einstein–Vlasov system. Quite surprisingly, for general smooth initial data none of the techniques discussed above for the nonrelativistic Vlasov–Poisson equation apply in the relativistic case. This fact is annoying since it has been suggested that an understanding of this equation may be necessary for understanding the three-dimensional relativistic Vlasov–Maxwell equation. However, the relativistic Vlasov–Poisson equation lacks the Lorentz invariance; it is a hybrid of a classical Galilei invariant field equation and a relativistic transport equation (17). Only for spherical symmetric data is the equation a fundamental physical equation. The classical Vlasov–Poisson equation is on the other hand Galilean invariant. In [1] a different equation for the field is introduced that is observer independent among Lorentz observers. By coupling this equation for the field to the relativistic Vlasov equation, the function $Q(t)$ may be controlled as shown in [1]. This is an indication that the transformation properties are important in studying existence of smooth solutions (the situation is less subtle for weak solutions, where energy estimates and averaging are the main tools, see [44] and [24]). Hence, it is unclear whether or not the relativistic Vlasov–Poisson equation will play a central role in the understanding of the Lorentz invariant relativistic Vlasov–Maxwell equation.

We refer to the book by Glassey [29] for more information on the relativistic Vlasov–Maxwell system and the Vlasov–Poisson equation.

1.3 The Einstein–Vlasov system

In this section we will consider a self-gravitating collisionless gas and we will write down the Einstein–Vlasov system and describe its general mathematical features. Our presentation follows to a large extent the one by Rendall in [76]. We also refer to Ehlers [27] and Stewart [85] for more background on kinetic theory in general relativity.

Let M be a four-dimensional manifold and let g_{ab} be a metric with Lorentz signature $(-, +, +, +)$ so that (M, g_{ab}) is a spacetime. We use the abstract index notation, which means that g_{ab} is a geometric object and not the components of a tensor. See [89] for a discussion on this notation. The metric is assumed to be time-orientable so that there is a distinction between future and past directed vectors. The worldline of a particle with non-zero rest mass m is a timelike curve and the unit future-directed tangent vector v^a to this curve is the four-velocity of the particle. The four-momentum p^a is given by mv^a . We assume that all particles have equal rest mass m and we normalize so that $m = 1$. One can also consider massless particles but we will rarely discuss this case. The possible values of the four-momentum are all future-directed unit timelike vectors and they constitute a hypersurface P in the tangent bundle TM , which is called the mass shell. The distribution function f that we introduced in the previous sections is a non-negative function on P . Since we are considering a collisionless gas, the particles travel along geodesics in spacetime. The Vlasov equation is an equation for f that exactly expresses this fact. To get an explicit expression for this equation we introduce local coordinates on the mass shell. We choose local coordinates on M such that the hypersurfaces $t = x^0 = \text{constant}$ are spacelike so that t is a time coordinate and x^j , $j = 1, 2, 3$, are spatial coordinates (letters in the beginning of the alphabet always take values $0, 1, 2, 3$ and letters in the middle take $1, 2, 3$). A timelike vector is future directed if and only if its zero component is positive. Local coordinates on P can then be taken as x^a together with the spatial components of the four-momentum p^a in these coordinates. The Vlasov equation then reads

$$\partial_t f + \frac{p^j}{p^0} \partial_{x^j} f - \frac{1}{p^0} \Gamma_{ab}^j p^a p^b \partial_{p^j} f = 0. \quad (22)$$

Here $a, b = 0, 1, 2, 3$ and $j = 1, 2, 3$, and Γ_{ab}^j are the Christoffel symbols. It is understood that p^0 is expressed in terms of p^j and the metric g_{ab} using the relation $g_{ab} p^a p^b = -1$ (recall that $m = 1$).

In a fixed spacetime the Vlasov equation is a linear hyperbolic equation for f and we can solve it by solving the characteristic system,

$$dX^i/ds = \frac{P^i}{P^0}, \quad (23)$$

$$dP^i/ds = -\Gamma_{ab}^i \frac{P^a P^b}{P^0}. \quad (24)$$

In terms of initial data f_0 the solution to the Vlasov equation can be written as

$$f(x^a, p^i) = f_0(X^i(0, x^a, p^i), P^i(0, x^a, p^i)), \quad (25)$$

where $X^i(s, x^a, p^i)$ and $P^i(s, x^a, p^i)$ solve (23) and (24), and where $X^i(t, x^a, p^i) = x^i$ and $P^i(t, x^a, p^i) = p^i$.

In order to write down the Einstein–Vlasov system we need to define the energy-momentum tensor T_{ab} in terms of f and g_{ab} . In the coordinates (x^a, p^a) on P we define

$$T_{ab} = - \int_{\mathbb{R}^3} f p_a p_b |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0},$$

where as usual $p_a = g_{ab} p^b$ and $|g|$ denotes the absolute value of the determinant of g . Equation (22) together with Einstein's equations

$$G_{ab} := R_{ab} - \frac{1}{2} R g_{ab} = 8\pi T_{ab}$$

then form the Einstein–Vlasov system. Here G_{ab} is the Einstein tensor, R_{ab} the Ricci tensor and R is the scalar curvature. We also define the particle current density

$$N^a = - \int_{\mathbb{R}^3} f p^a |g|^{1/2} \frac{dp^1 dp^2 dp^3}{p_0}.$$

Using normal coordinates based at a given point and assuming that f is compactly supported it is not hard to see that T_{ab} is divergence-free which is a necessary compatibility condition since G_{ab} is divergence-free by the Bianchi identities. A computation in normal coordinates also shows that N^a is divergence-free, which expresses the fact that the number of particles is conserved. The definitions of T_{ab} and N^a immediately give us a number of inequalities. If V^a is a future directed timelike or null vector then we have $N_a V^a \leq 0$ with equality if and only if $f = 0$ at the given point. Hence N^a is always future directed timelike if there are particles at that point. Moreover, if V^a and W^a are future directed timelike vectors then $T_{ab} V^a W^b \geq 0$, which is the dominant energy condition. If X^a is a spacelike vector then $T_{ab} X^a X^b \geq 0$. This is called the non-negative pressure condition. These last two conditions together with the Einstein equations imply that $R_{ab} V^a V^b \geq 0$ for any timelike vector V^a , which is the strong energy condition. That the energy conditions hold for Vlasov matter is one reason that the Vlasov equation defines a well-behaved matter model in general relativity. Another reason is the well-posedness theorem by Choquet-Bruhat for the Einstein–Vlasov system that we will state below. Before stating that theorem we will first discuss the initial conditions imposed.

The data in the Cauchy problem for the Einstein–Vlasov system consist of the induced Riemannian metric g_{ij} on the initial hypersurface S , the second fundamental form k_{ij} of S and matter data f_0 . The relations between a given initial data set (g_{ij}, k_{ij}) on a three-dimensional manifold S and the metric g_{ij} on the spacetime manifold is that there exists an embedding ψ of S into the spacetime such that the induced metric and second fundamental form of $\psi(S)$ coincide with the result of transporting (g_{ij}, k_{ij}) with ψ . For the relation of the distribution functions f and f_0 we have to note that f is defined on the mass shell. The initial condition imposed is that the restriction of f to the part of the mass shell over $\psi(S)$ should be equal to $f_0 \circ (\psi^{-1}, d(\psi)^{-1}) \circ \phi$, where ϕ

sends each point of the mass shell over $\psi(S)$ to its orthogonal projection onto the tangent space to $\psi(S)$. An initial data set for the Einstein–Vlasov system must satisfy the constraint equations, which read

$$R - k_{ij}k^{ij} + (\text{tr } k)^2 = 16\pi\rho, \quad (26)$$

$$\nabla_i k_l^i - \nabla_l(\text{tr } k) = 8\pi j_l. \quad (27)$$

Here $\rho = T_{ab}n^a n^b$ and $j^a = -h^{ab}T_{bc}n^c$, where n^a is the future directed unit normal vector to the initial hypersurface and $h^{ab} = g^{ab} + n^a n^b$ is the orthogonal projection onto the tangent space to the initial hypersurface. In terms of f_0 we can express ρ and j^l by (j^a satisfies $n_a j^a = 0$ so it can naturally be identified with a vector intrinsic to S)

$$\rho = \int_{\mathbb{R}^3} f p^a p_a |^{(3)}g|^{1/2} \frac{dp^1 dp^2 dp^3}{1 + p_j p^j}, \quad (28)$$

$$j_l = \int_{\mathbb{R}^3} f p_l |^{(3)}g|^{1/2} dp^1 dp^2 dp^3. \quad (29)$$

Here $|^{(3)}g|$ is the determinant of the induced Riemannian metric on S . We can now state the local existence theorem by Choquet-Bruhat [17] for the Einstein–Vlasov system.

Theorem 1 *Let S be a 3-dimensional manifold, g_{ij} a smooth Riemannian metric on S , k_{ij} a smooth symmetric tensor on S and f_0 a smooth non-negative function of compact support on the tangent bundle TS of S . Suppose that these objects satisfy the constraint equations (26) and (27). Then there exists a smooth spacetime (M, g_{ab}) , a smooth distribution function f on the mass shell of this spacetime, and a smooth embedding ψ of S into M which induces the given initial data on S such that g_{ab} and f satisfy the Einstein–Vlasov system and $\psi(S)$ is a Cauchy surface. Moreover, given any other spacetime (M', g'_{ab}) , distribution function f' and embedding ψ' satisfying these conditions, there exists a diffeomorphism χ from an open neighbourhood of $\psi(S)$ in M to an open neighbourhood of $\psi'(S)$ in M' which satisfies $\chi \circ \psi = \psi'$ and carries g_{ab} and f to g'_{ab} and f' , respectively.*

In this context we also mention that local existence has been proved for the Einstein–Maxwell–Boltzmann system [9] and for the Yang–Mills–Vlasov system [18].

A main theme in the following sections is to discuss special cases for which the local existence theorem can be extended to a global one. There are interesting situations when this can be achieved, and such global existence theorems are not known for Einstein’s equations coupled to other forms of phenomenological matter models, i.e. fluid models (see, however, [21]). In this context it should be stressed that the results in the previous sections show that the mathematical understanding of kinetic equations on a flat background space is well-developed. On the other hand the mathematical understanding of fluid equations on a flat

background space (also in the absence of a Newtonian gravitational field) is not that well-understood. It would be desirable to have a better mathematical understanding of these equations in the absence of gravity before coupling them to Einstein's equations. This suggests that the Vlasov equation is natural as matter model in mathematical general relativity.

2 Global Existence Theorems for the Einstein–Vlasov System

In general relativity two classes of initial data are distinguished. If an isolated body is studied, the data are called asymptotically flat. The initial hypersurface is topologically \mathbb{R}^3 and (since far away from the body one expects spacetime to be approximately flat) appropriate fall off conditions are imposed. Roughly, a smooth data set (g_{ij}, k_{ij}, f_0) on \mathbb{R}^3 is said to be asymptotically flat if there exist global coordinates x^i such that as $|x|$ tends to infinity the components g_{ij} in these coordinates tend to δ_{ij} , the components k_{ij} tend to zero, f_0 has compact support and certain weighted Sobolev norms of $g_{ij} - \delta_{ij}$ and k_{ij} are finite (see [76]). The symmetry classes that admit asymptotical flatness are few. The important ones are spherically symmetric and axially symmetric spacetimes. One can also consider a case in which spacetime is asymptotically flat except in one direction, namely cylindrical spacetimes. Regarding global existence questions, only spherically symmetric spacetimes have been considered for the Einstein–Vlasov system in the asymptotically flat case.

Spacetimes that possess a compact Cauchy hypersurface are called cosmological spacetimes, and data are accordingly given on a compact 3-manifold. In this case the whole universe is modelled and not only an isolated body. In contrast to the asymptotically flat case, cosmological spacetimes admit a large number of symmetry classes. This gives one the possibility to study interesting special cases for which the difficulties of the full Einstein equations are strongly reduced. We will discuss below cases for which the spacetime is characterized by the dimension of its isometry group together with the dimension of the orbit of the isometry group.

2.1 Spherically symmetric spacetimes

The study of the global properties of solutions to the spherically symmetric Einstein–Vlasov system was initiated by Rein and Rendall in 1990. They chose to work in coordinates where the metric takes the form

$$ds^2 = -e^{2\mu(t,r)} dt^2 + e^{2\lambda(t,r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $t \in \mathbb{R}$, $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. These are called Schwarzschild coordinates. Asymptotic flatness is expressed by the boundary conditions

$$\lim_{r \rightarrow \infty} \lambda(t, r) = \lim_{r \rightarrow \infty} \mu(t, r) = 0, \quad \forall t \geq 0.$$

A regular centre is also required and is guaranteed by the boundary condition

$$\lambda(t, 0) = 0.$$

With

$$x = (r \sin \phi \cos \theta, r \sin \phi \sin \theta, r \cos \phi)$$

as spatial coordinate and

$$v^j = p^j + (e^\lambda - 1) \frac{x \cdot p}{r} \frac{x^j}{r}$$

as momentum coordinates the, Einstein–Vlasov system reads

$$\partial_t f + e^{\mu-\lambda} \frac{v}{\sqrt{1+v^2}} \cdot \partial_x f - (\lambda_t \frac{x \cdot v}{r} + e^{\mu-\lambda} \mu_r \sqrt{1+v^2}) \frac{x}{r} \cdot \partial_v f = 0, \quad (30)$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho, \quad (31)$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p. \quad (32)$$

The matter quantities are defined by

$$\rho(t, x) = \int_{\mathbb{R}^3} \sqrt{1+|v|^2} f(t, x, v) dv, \quad (33)$$

$$p(t, x) = \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r} \right)^2 f(t, x, v) \frac{dv}{\sqrt{1+|v|^2}}. \quad (34)$$

Let us point out that this system is not the full Einstein–Vlasov system. The remaining field equations, however, can be derived from these equations. See [62] and the erratum [65] for more details. Let the square of the angular momentum be denoted by L , i.e.

$$L := |x|^2 |v|^2 - (x \cdot v)^2.$$

A consequence of spherical symmetry is that angular momentum is conserved along the characteristics of (30). Introducing the variable

$$w = \frac{x \cdot v}{r},$$

the Vlasov equation for $f = f(t, r, w, L)$ becomes

$$\partial_t f + e^{\mu-\lambda} \frac{w}{E} \partial_r f - \left(\lambda_t w + e^{\mu-\lambda} \mu_r E - e^{\mu-\lambda} \frac{F}{r^3 E} \right) \partial_w f = 0, \quad (35)$$

where

$$E = E(r, w, L) = \sqrt{1 + w^2 + L/r^2}.$$

The matter terms take the form

$$\rho(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} E f(t, r, w, L) dw dL, \quad (36)$$

$$p(t, r) = \frac{\pi}{r^2} \int_{-\infty}^{\infty} \int_0^{\infty} \frac{w^2}{E} f(t, r, w, L) dw dL. \quad (37)$$

Let us write down a couple of known facts about the system (31, 32, 35, 36, 37). A solution to the Vlasov equation can be written as

$$f(t, r, w, L) = f_0(R(0, t, r, w, L), W(0, t, r, w, L), L), \quad (38)$$

where R and W are solutions to the characteristic system

$$\frac{dR}{ds} = e^{(\mu-\lambda)(s,R)} \frac{W}{E(R,W,L)}, \quad (39)$$

$$\begin{aligned} \frac{dW}{ds} = & -\lambda_t(s,R)W - e^{(\mu-\lambda)(s,R)}\mu_r(s,R)E(R,W,L) \\ & + e^{(\mu-\lambda)(s,R)} \frac{L}{R^3 E(R,W,L)}, \end{aligned} \quad (40)$$

such that the trajectory $(R(s,t,r,w,L), W(s,t,r,w,L), L)$ goes through the point (r,w,L) when $s = t$. This representation shows that f is nonnegative for all $t \geq 0$ and that $f \leq \|f_0\|_\infty$. There are two known conservation laws for the Einstein–Vlasov system: conservation of the number of particles,

$$4\pi^2 \int_0^\infty e^{\lambda(t,r)} \left(\int_{-\infty}^\infty \int_0^\infty f(t,r,w,L) dL dw \right) dr,$$

and conservation of the ADM mass

$$M := 4\pi \int_0^\infty r^2 \rho(t,r) dr. \quad (41)$$

Let us now review the global results concerning the Cauchy problem that have been proved for the spherically symmetric Einstein–Vlasov system. As initial data we take a spherically symmetric, nonnegative, and continuously differentiable function f_0 with compact support that satisfies

$$4\pi^2 \int_0^r \int_{-\infty}^\infty \int_0^\infty E f_0(r,w,L) dw dL dr < \frac{r}{2}.$$

This condition guarantees that no trapped surfaces are present initially. In [62] it is shown that for such an initial datum there exists a unique, continuously differentiable solution f with $f(0) = f_0$ on some right maximal interval $[0, T)$. If the solution blows up in finite time, i.e. if $T < \infty$, then $\rho(t)$ becomes unbounded as $t \rightarrow T$. Moreover, a continuation criterion is shown that says that a local solution can be extended to a global one provided the v -support of f can be bounded on $[0, T)$. (In [62] they chose to work in the momentum variable v rather than w, L .) This is analogous to the situation for the Vlasov–Maxwell system where the function $Q(t)$ was introduced for the v -support. A control of the v -support immediately implies that ρ and p are bounded in view of (33) and (34). In the Vlasov–Maxwell case the field equations have a regularizing effect in the sense that derivatives can be expressed through spatial integrals, and it follows [34] that the derivatives of f also can be bounded if the v -support is bounded. For the Einstein–Vlasov system such a regularization is less clear, since μ_r depends on ρ in a pointwise manner. However, certain combinations of second and first order derivatives of the metric components can be expressed in terms of matter components only, without derivatives (a consequence of the

geodesic deviation equation). This fact turns out to be sufficient for obtaining bounds also on the derivatives of f (see [62] for details). By considering initial data sufficiently close to zero, Rein and Rendall show that the v -support is bounded on $[0, T)$, and the continuation criterion then implies that $T = \infty$. It should be stressed that even for small data no global existence result like this one is known for any other phenomenological matter model coupled to Einstein's equations. The resulting spacetime in [62] is geodesically complete, and the components of the energy momentum tensor as well as the metric quantities decay with certain algebraic rates in t . The mathematical method used by Rein and Rendall is inspired by the analogous small data result for the Vlasov–Poisson equation by Bardos and Degond [10]. This should not be too surprising since for small data the gravitational fields are expected to be small and a Newtonian spacetime should be a fair approximation. In this context we point out that in [63] it is proved that the Vlasov–Poisson system is indeed the nonrelativistic limit of the spherically symmetric Einstein–Vlasov system, i.e. the limit when the speed of light $c \rightarrow \infty$. (In [71] this issue is studied in the asymptotically flat case without symmetry assumptions.) Finally, we mention that there is an analogous small data result using a maximal time coordinate [76] instead of a Schwarzschild time coordinate.

The case with general data is more subtle. Rendall has shown [70] that there exist data leading to singular spacetimes as a consequence of Penrose's singularity theorem. This raises the question of what we mean by global existence for such data. The Schwarzschild time coordinate is expected to avoid the singularity, and by global existence we mean that solutions remain smooth as Schwarzschild time tends to infinity. Even though spacetime might be only partially covered in Schwarzschild coordinates, a global existence theorem for general data would nevertheless be very important since weak cosmic censorship would follow from it. A partial investigation for general data was done in [67], where it is shown that if singularities form in finite Schwarzschild time the first one must be at the centre. More precisely, if $f(t, r, w, L) = 0$ when $r > \epsilon$ for some $\epsilon > 0$, and for all t, w and L , then the solution remains smooth for all time. This rules out singularities of the shell-crossing type, which can be an annoying problem for other matter models (e.g. dust). The main observation in [67] is a cancellation property in the term

$$\lambda_t w + e^{\mu-\lambda} \mu_r E$$

in the characteristic equation (40). We refer to the original paper for details. In [68] a numerical study was undertaken. A numerical scheme originally used for the Vlasov–Poisson system was modified to the spherically symmetric Einstein–Vlasov system. It has been shown by Rodewis [79] that the numerical scheme has the desirable convergence properties. (In the Vlasov–Poisson case convergence was proved in [81]. See also [28].) The numerical experiments support the conjecture that solutions are singularity-free. This can be seen as evidence that weak cosmic censorship holds for collisionless matter. It may even hold in a stronger sense than in the case of a massless scalar field (see [20, 22]). There may be no naked singularities formed for any regular initial data rather

than just for generic data. This speculation is based on the fact that the naked singularities that occur in scalar field collapse appear to be associated with the existence of type II critical collapse, while Vlasov matter is of type I. This is indeed the primary goal of their numerical investigation: to analyze critical collapse and decide whether Vlasov matter is type I or type II.

These different types of matter are defined as follows. Given small initial data no black holes are expected to form and matter will disperse (which has been proved for a scalar field [19] and for Vlasov matter [62]). For large data, black holes will form and consequently there is a transition regime separating dispersion of matter and formation of black holes. If we introduce a parameter A on the initial data such that for small A dispersion occurs and for large A a black hole is formed, we get a critical value A_c separating these regions. If we take $A > A_c$ and denote by $m_B(A)$ the mass of the black hole, then if $m_B(A) \rightarrow 0$ as $A \rightarrow A_c$, we have type II matter, whereas for type I matter this limit is positive and there is a mass gap. For more information on critical collapse we refer to the review paper by Gundlach [38].

For Vlasov matter there is an independent numerical simulation by Olabarrieta and Choptuik [53] (using a maximal time coordinate) and their conclusion agrees with the one in [68]. Critical collapse is related to self similar solutions; Martin-Garcia and Gundlach [52] have presented a construction of such solutions for the massless Einstein–Vlasov system by using a method based partially on numerics. Since such solutions often are related to naked singularities, it is important to note that their result is for the massless case (in which case there is no known analogous result to the small data theorem in [62]) and their initial data are not in the class that we have described above.

2.2 Cosmological solutions

In cosmology the whole universe is modelled, and the “particles” in the kinetic description are galaxies or even clusters of galaxies. The main goal again is to determine the global properties of the solutions to the Einstein–Vlasov system. In order to do so, a global time coordinate t must be found (global existence) and the asymptotic behaviour of the solutions when t tends to its limiting values has to be analyzed. This might correspond to approaching a singularity (e.g. the big bang singularity) or to a phase of unending expansion. Since the general case is beyond the range of current mathematical techniques, all known results are for cases with symmetry (see however [7] where to some extent global properties are established in the case without symmetry).

There are several existing results on global time coordinates for solutions of the Einstein–Vlasov system. In the spatially homogeneous case it is natural to choose a Gaussian time coordinate based on a homogeneous hypersurface. The maximal range of a Gaussian time coordinate in a solution of the Einstein–Vlasov system evolving from homogenous data on a compact manifold was determined in [73]. The range is finite for models of Bianchi IX and Kantowski–Sachs types. It is finite in one time direction and infinite in the other for the other Bianchi types. The asymptotic behaviour of solutions in the

spatially homogeneous case has been analyzed in [77] and [78]. In [77], the case of massless particles is considered, whereas the massive case is studied in [78]. Both the nature of the initial singularity and the phase of unlimited expansion are analyzed. The main concern is the behaviour of Bianchi models I, II, and III. The authors compare their solutions with the solutions to the corresponding perfect fluid models. A general conclusion is that the choice of matter model is very important since for all symmetry classes studied there are differences between the collisionless model and a perfect fluid model, both regarding the initial singularity and the expanding phase. The most striking example is for the Bianchi II models, where they find persistent oscillatory behaviour near the singularity, which is quite different from the known behaviour of Bianchi type II perfect fluid models. In [78] it is also shown that solutions for massive particles are asymptotic to solutions with massless particles near the initial singularity. For Bianchi I and II it is also proved that solutions with massive particles are asymptotic to dust solutions at late times. It is conjectured that the same holds true also for Bianchi III. This problem is then settled by Rendall in [69].

All other results presently available on the subject concern spacetimes that admit a group of isometries acting on two-dimensional spacelike orbits, at least after passing to a covering manifold. The group may be two-dimensional (local $U(1) \times U(1)$ or T^2 symmetry) or three-dimensional (spherical, plane, or hyperbolic symmetry). In all these cases, the quotient of spacetime by the symmetry group has the structure of a two-dimensional Lorentzian manifold Q . The orbits of the group action (or appropriate quotients in the case of a local symmetry) are called surfaces of symmetry. Thus, there is a one-to-one correspondence between surfaces of symmetry and points of Q . There is a major difference between the cases where the symmetry group is two- or three-dimensional. In the three-dimensional case no gravitational waves are admitted, in contrast to the two-dimensional case. In the former case, the field equations reduce to ODEs while in the latter their evolution part consists of nonlinear wave equations. Three types of time coordinates that have been studied in the inhomogeneous case are CMC, areal, and conformal coordinates. A CMC time coordinate t is one where each hypersurface of constant time has constant mean curvature (CMC) and on each hypersurface of this kind the value of t is the mean curvature of that slice. In the case of areal coordinates, the time coordinate is a function of the area of the surfaces of symmetry (e.g. proportional to the area or proportional to the square root of the area). In the case of conformal coordinates, the metric on the quotient manifold Q is conformally flat.

Let us first consider spacetimes (M, g) admitting a three-dimensional group of isometries. The topology of M is assumed to be $\mathbb{R} \times S^1 \times F$, with F a compact two-dimensional manifold. The universal covering \hat{F} of F induces a spacetime (\hat{M}, \hat{g}) by $\hat{M} = \mathbb{R} \times S^1 \times \hat{F}$ and $\hat{g} = p^*g$, where $p : \hat{M} \rightarrow M$ is the canonical projection. A three-dimensional group G of isometries is assumed to act on (\hat{M}, \hat{g}) . If $F = S^2$ and $G = SO(3)$, then (M, g) is called spherically symmetric; if $F = T^2$ and $G = E_2$ (Euclidean group), then (M, g) is called plane symmetric; and if F has genus greater than one and the connected component of the symmetry group G of the hyperbolic plane H^2 acts isometrically on $\hat{F} = H^2$,

then (M, g) is said to have hyperbolic symmetry.

In the case of spherical symmetry the existence of one compact CMC hypersurface implies that the whole spacetime can be covered by a CMC time coordinate that takes all real values [72, 15]. The existence of one compact CMC hypersurface in this case was proved later by Henkel [40] using the concept of prescribed mean curvature (PMC) foliation. Accordingly this gives a complete picture in the spherically symmetric case regarding CMC foliations. In the case of areal coordinates, Rein [58] has shown, under a size restriction on the initial data, that the past of an initial hypersurface can be covered. In the future direction it is shown that areal coordinates break down in finite time.

In the case of plane and hyperbolic symmetry, Rendall and Rein showed in [72] and [58], respectively, that the existence results (for CMC time and areal time) in the past direction for spherical symmetry also hold for these symmetry classes. The global CMC foliation results to the past imply that the past singularity is a crushing singularity, since the mean curvature blows up at the singularity. In addition, Rein also proved in his special case with small initial data that the Kretschmann curvature scalar blows up when the singularity is approached. Hence, the singularity is both a crushing and a curvature singularity in this case. In both of these works the question of global existence to the future was left open. This gap was closed by the author, Rein and Rendall in [5], and global existence to the future was established in both CMC and areal coordinates. The global existence result for CMC time is partly a consequence of the global existence theorem in areal coordinates, together with a theorem by Henkel [40] that shows that there exists at least one hypersurface with (negative) constant mean curvature. Also, the past direction was analyzed in areal coordinates and global existence was established without any smallness condition on the data. It is, however, not concluded if the past singularity in this more general case without the smallness condition on the data is a curvature singularity as well. The question whether the areal time coordinate, which is positive by definition, takes all values in the range $(0, \infty)$ or only in (R_0, ∞) for some positive R_0 is also left open. In the special case in [58], it is indeed shown that $R_0 = 0$, but there is an example for vacuum spacetimes in the more general case of $U(1) \times U(1)$ symmetry where $R_0 > 0$.

For spacetimes admitting a two-dimensional isometry group, the first study was done by Rendall [75] in the case of local $U(1) \times U(1)$ symmetry (or local T^2 symmetry). For a discussion of the topologies of these spacetimes we refer to the original paper. In the model case the spacetime is topologically of the form $\mathbb{R} \times T^3$, and to simplify our discussion later on we write down the metric in areal coordinates for this type of spacetime:

$$g = e^{2(\eta-U)}(-\alpha dt^2 + d\theta^2) + e^{-2U}t^2[dy + Hd\theta + Mdt]^2 + e^{2U}[dx + Ady + (G + AH)d\theta + (L + AM)dt]^2. \quad (42)$$

Here the metric coefficients η , U , α , A , H , L , and M depend on t and θ and $\theta, x, y \in S^1$. In [75] CMC coordinates are considered rather than areal coordinates. The CMC and the areal coordinate foliations are both geometrically

based time foliations. The advantage with a CMC approach is that the definition of a CMC hypersurface does not depend on any symmetry assumptions and it is possible that CMC foliations will exist for rather general spacetimes. The areal coordinate foliation, the other hand, is adapted to the symmetry of spacetime but it has analytical advantages that we will see below.

Under the hypothesis that there exists at least one CMC hypersurface, Rendall proves, without any smallness condition on the data, that the past of the given CMC hypersurface can be globally foliated by CMC hypersurfaces and that the mean curvature of these hypersurfaces blows up at the past singularity. Again, the future direction was left open. The result in [75] holds for Vlasov matter and for matter described by a wave map (which is not a phenomenological matter model). That the choice of matter model is important was shown by Rendall [74] who gives a non-global existence result for dust, which leads to examples of spacetimes [47] that are not covered by a CMC foliation.

There are several possible subcases to the $U(1) \times U(1)$ symmetry class. The plane case where the symmetry group is three-dimensional is one subcase and the form of the metric in areal coordinates is obtained by letting $A = G = H = L = M = 0$ and $U = \log t/2$ in (42). Another subcase, which still admits only two Killing fields (and which includes plane symmetry as a special case), is Gowdy symmetry. It is obtained by letting $G = H = L = M = 0$ in (42). In [4], the author considers Gowdy symmetric spacetimes with Vlasov matter. It is proved that the entire maximal globally hyperbolic spacetime can be foliated by constant areal time slices for arbitrary (in size) initial data. The areal coordinates are used in a direct way for showing global existence to the future, whereas the analysis for the past direction is carried out in conformal coordinates. These coordinates are not fixed to the geometry of spacetime and it is not clear that the entire past has been covered. A chain of geometrical arguments then shows that areal coordinates indeed cover the entire spacetime. This method was applied to the problem on hyperbolic and plane symmetry in [5]. The method in [4] was in turn inspired by the work [14] for vacuum spacetimes where the idea of using conformal coordinates in the past direction was introduced. As pointed out in [5], the result by Henkel [41] guarantees the existence of one CMC hypersurface in the Gowdy case and, together with the global areal foliation in [4], it follows that Gowdy spacetimes with Vlasov matter can be globally covered by CMC hypersurfaces as well (also to the future). So, in a sense the areal coordinate approach seems to be analytically favourable to the CMC approach for these spacetimes. It remains to be proved that the general case with local $U(1) \times U(1)$ symmetry and Vlasov matter can be foliated by CMC and by constant areal time hypersurfaces. This project is in progress by Rendall, Weaver, and the author [6]. Regarding global foliations (with respect to a CMC and an areal time coordinate) of spacetimes admitting a two-dimensional isometry group, this result (if affirmative) will complete the picture. As mentioned above, there are, however, a number of important questions open regarding the nature of the initial singularity, the range of the areal coordinate, and the question of the asymptotics in the future direction. Recently, Rein [56] has shown geodesic completeness to the future for solutions to

the Einstein–Vlasov system with hyperbolic symmetry (cf. [5]) under a certain size restriction on the initial data.

3 Stationary Solutions to the Einstein–Vlasov System

Equilibrium states in galactic dynamics can be described as stationary solutions of the Einstein–Vlasov system, or of the Vlasov–Poisson system in the Newtonian case. Here we will consider the former case for which only static, spherically symmetric solutions have been constructed, but we mention that in the latter case also, stationary axially symmetric solutions have been found by Rein [61].

In the static, spherically symmetric case, the problem can be formulated as follows. Let the spacetime metric have the form

$$ds^2 = -e^{2\mu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2),$$

where $r \geq 0$, $\theta \in [0, \pi]$, $\varphi \in [0, 2\pi]$. As before, asymptotic flatness is expressed by the boundary conditions

$$\lim_{r \rightarrow \infty} \lambda(r) = \lim_{r \rightarrow \infty} \mu(r) = 0, \quad \forall t \geq 0,$$

and a regular centre requires

$$\lambda(t, 0) = 0.$$

Following the notation in section 2.1, the time-independent Einstein–Vlasov system reads

$$e^{\mu-\lambda} \frac{v}{\sqrt{1+|v|^2}} \cdot \partial_x f - \sqrt{1+|v|^2} e^{\mu-\lambda} \mu_r \frac{x}{r} \cdot \partial_v f = 0, \quad (43)$$

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 \rho, \quad (44)$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 p. \quad (45)$$

The matter quantities are defined as before:

$$\rho(x) = \int_{\mathbb{R}^3} \sqrt{1+v^2} f(t, x, v) dv, \quad (46)$$

$$p(x) = \int_{\mathbb{R}^3} \left(\frac{x \cdot v}{r} \right)^2 f(t, x, v) \frac{dv}{\sqrt{1+v^2}}. \quad (47)$$

The quantities

$$E := e^\mu(r) \sqrt{1+v^2}, \quad L = x^2 v^2 - (x \cdot v)^2 = |x \times v|^2,$$

are conserved along characteristics. E is the particle energy and L is the angular momentum squared. If we let

$$f(x, v) = \Phi(E, L)$$

for some function Φ , the Vlasov equation is automatically satisfied. The form of Φ is usually restricted to

$$\Phi(E, L) = \phi(E)(L - L_0)^l, \quad (48)$$

where $l > -1/2$ and $L_0 \geq 0$. If $\phi(E) = (E - E_0)_+^k$, $k > -1$, for some positive constant E_0 , this is called the polytropic ansatz. The case of isotropic pressure is obtained by letting $l = 0$ so that Φ only depends on E . We refer to [57] for information on the role of L_0 .

In passing, we mention that for the Vlasov–Poisson system it has been shown [13] that every static spherically symmetric solution must have the form $f = \Phi(E, L)$. This is referred to as Jeans’ theorem. It was an open question for some time to decide whether or not this was also true for the Einstein–Vlasov system. This was settled in 1999 by Schaeffer [83], who found solutions that do not have this particular form globally on phase space, and consequently, Jeans’ theorem is not valid in the relativistic case. However, almost all results in this field rest on this ansatz. By inserting the ansatz for f in the matter quantities ρ and p , a nonlinear system for λ and μ is obtained, in which

$$e^{-2\lambda}(2r\lambda_r - 1) + 1 = 8\pi r^2 G_\Phi(r, \mu), \quad (49)$$

$$e^{-2\lambda}(2r\mu_r + 1) - 1 = 8\pi r^2 H_\Phi(r, \mu), \quad (50)$$

where

$$G_\Phi(r, \mu) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\epsilon^2-1)} \Phi(e^{\mu(r)}\epsilon, L) \frac{\epsilon^2}{\sqrt{\epsilon^2 - 1 - L/r^2}} dL d\epsilon, \quad (51)$$

$$H_\Phi(r, \mu) = \frac{2\pi}{r^2} \int_1^\infty \int_0^{r^2(\epsilon^2-1)} \Phi(e^{\mu(r)}\epsilon, L) \sqrt{\epsilon^2 - 1 - L/r^2} dL d\epsilon. \quad (52)$$

Existence of solutions to this system was first proved in the case of isotropic pressure in [64] and then extended to the general case in [57]. The main problem is then to show that the resulting solutions have finite (ADM) mass and compact support. This is accomplished in [64] for a polytropic ansatz with isotropic pressure and in [57] for a polytropic ansatz with possible anisotropic pressure. They use a perturbation argument based on the fact that the Vlasov–Poisson system is the limit of the Einstein–Vlasov system as the speed of light tends to infinity [63]. Two types of solutions are constructed, those with a regular centre [64, 57], and those with a Schwarzschild singularity in the centre [57]. In [66] Rendall and Rein go beyond the polytropic ansatz and assume that Φ satisfies

$$\Phi(E, L) = c(E - E_0)_+^k L^l + O((E_0 - E)_+^{\delta+k}) L^l \quad \text{as } E \rightarrow E_0,$$

where $k > -1$, $l > -1/2$, $k + l + 1/2 > 0$, $k < l + 3/2$. They show that this assumption is sufficient for obtaining steady states with finite mass and compact support. The result is obtained in a more direct way and is not based on the perturbation argument mentioned above. Their method is inspired by a work on stellar models by Makino [51], in which he considers steady states of the Euler–Einstein system. In [66] there is also an interesting discussion about steady states that appear in the astrophysics literature. They show that their result applies to most of these steady states, which proves that they have the desirable property of finite mass and compact support.

All solutions described so far have the property that the support of ρ contains a ball about the centre. In [60] Rein shows that there exist steady states whose support is a finite, spherically symmetric shell, so that they have a vacuum region in the centre.

At present, there are almost no known results concerning the stability properties of the steady states to the Einstein–Vlasov system. In the Vlasov–Poisson case, however, the nonlinear stability of stationary solutions has been investigated by Guo and Rein [39] using the energy–Casimir method. In the Einstein–Vlasov case, Wolansky [90] has applied the energy–Casimir method and obtained some insights, but the theory in this case is much less developed than in the Vlasov–Poisson case and the stability problem is essentially open.

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