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Walton L. Howes

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Effect of Initial Velocity on One-Dimensional, Bipolar, Space-Charge Currents

WALTON L. HOWES

Lewis Research Center, National Aeronautics and Space Administration, Cleveland, Ohio

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Oppositely directed flows of positive and negative charges constitute a bipolar current. A general case is analyzed in which the charges traverse the evacuated space between plane-parallel boundaries and all particles of a given charge species are monoenergetic. Moreover, it is assumed that the charges may possess nonvanishing initial kinetic energy. This represents an extension of analyses by Langmuir and Müller-Lübeck for vanishing initial kinetic energy.

Sample calculations of dimensionless current densities, electric and potential fields, and charge-density distributions are exhibited for cases where the electric field is assumed to vanish at one boundary but the species initial velocities may not. It is shown that resulting ion currents may be several times the Child's law limit if initial kinetic energies are of the order of the potential energy.

INTRODUCTION

THE analysis of one-dimensional, space-charge flows determined by the action of an electric field on charges of a given sign originated with studies by Child¹ and Langmuir² of unipolar currents in an otherwise evacuated space between plane electrodes. The recognition of space-charge-limited current, that is, of the existence of a maximum current for a given electrode separation and potential difference, resulted from these studies. Vanishing initial velocity of all particles was assumed. Ultimately this assumption was dropped. In particular, Salzberg and Haef³ and Fay, Samuel, and Shockley⁴ coincidentally and independently derived all possible nonrelativistic solutions for one-dimensional, unipolar, space-charge current with unique initial particle velocity.

One possible method of overcoming the unipolar space-charge-current limitation is to neutralize the space charge by injecting opposite charges at the final boundary plane traversed by the original charge species. Langmuir⁵ derived the theory for the resulting bipolar currents. A required second integration of Poisson's equation, yielding the potential distribution, was performed by numerical integration. Müller-Lübeck⁶ aesthetically improved on Langmuir's analysis by showing that the second integration involved elliptic integrals.

For the bipolar-current case, Langmuir and Müller-Lübeck both assumed vanishing initial velocity of each charge species; however, as in the unipolar-current case, this assumption may be discarded. The resulting analysis, presented herein, is essentially the bipolar equivalent of the unipolar theories in Refs. 3 and 4; however, the results presented are concerned only with monotonic potential fields. The electric and

potential fields and charge-density distributions are calculated. The possibility of overcoming the Child's law limit on ion current by means of bipolar currents with nonvanishing initial velocities is emphasized.

All of the cited analyses represent special cases of that which follows, and may be derived from it.

The present problem was encountered in studying possible ion propulsion schemes.

ANALYSIS

Consider two infinite parallel planes (Fig. 1) separated by a distance l and associated with different electric potentials. (Only an association is intended. The planes are not to be regarded necessarily as physical entities, say electrodes, which might mechanically limit the motion of charges at the plane boundaries.) Let positive ions be introduced normal to the plane associated with the higher potential, and let negatively charged particles (electrons or negative ions) be introduced normal to the plane associated with the lower potential. Assume that the rate of charge injection is steady. Then, presumably, a steady positive-negative ion counterflow, that is, a bipolar current, exists between the boundary planes. The currents are assumed

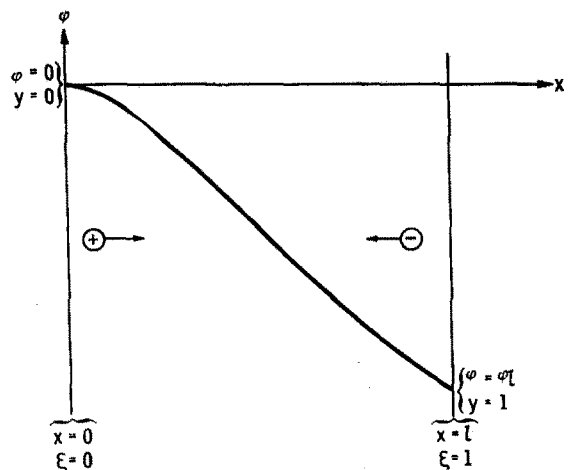


FIG. 1. Geometry and coordinate scheme.

¹ C. D. Child, Phys. Rev. **32**, 492 (1911).

² I. Langmuir, Phys. Rev. **2**, 450 (1913); or *The Collected Works of Irving Langmuir*, edited by C. G. Suits (Pergamon Press, New York, 1961), Vol. 3, pp. 3-37.

³ B. Salzberg and A. V. Haef, RCA Rev. **2**, 336 (1938).

⁴ C. E. Fay, A. L. Samuel, and W. Shockley, Bell System Tech. J. **17**, 49 (1938).

⁵ I. Langmuir, Phys. Rev. **33**, 954 (1929); or *The Collected Works of Irving Langmuir*, edited by C. G. Suits; (Pergamon Press, New York, 1961), Vol. 5, pp. 140-175.

⁶ K. Müller-Lübeck, Z. Angew. Phys. **3**, 409 (1951).

collisionless. (This implies that the currents are one-dimensional.) All particles of a given species are assumed to be injected at the same velocity. A coordinate is defined so that its origin is at the plane of positive-ion injection, parallel to and increasing in value in the direction of the ion flow. The nonrelativistic equations (in Gaussian units) describing the particle motions and field distributions are, for two charge species,

$$dj_{\pm}/dx=0 \text{ (charge conservation),} \quad (1)$$

$$d(m_{\pm}v_{\pm}^2/2)/dx=e_{\pm}\mathcal{E} \text{ (equation of motion),} \quad (2)$$

$$d^2\varphi/dx^2=-4\pi(\rho_++\rho_-) \text{ (Poisson's equation),} \quad (3)$$

where

$$\rho=\rho_++\rho_-, \quad (4)$$

$$\rho_{\pm}=j_{\pm}/v_{\pm}, \quad (5)$$

j is the current density of the positive ions, or electrons (denoted by the subscript + or -, respectively), m is the particle mass, v the particle velocity, e the charge, ρ the charge density, \mathcal{E} the electric field, and φ the electric potential.

By introducing φ in Eq. (2) and integrating, the particle velocity of both species may be represented by

$$v_{\pm}=\pm[-\eta(\varphi-\varphi_{0,l})+v_{\pm 0,l}^2]^{\frac{1}{2}}, \quad (6)$$

where the subscripts 0 associated with the positive ions (+) and l associated with the electrons (-) refer here to initial conditions and

$$\eta_{\pm}=2e_{\pm}/m_{\pm}. \quad (7)$$

Substitute the right sides of Eqs. (5) and (6) in Eq. (3) and define

$$y=(\varphi-\varphi_0)/(\varphi_l-\varphi_0), \quad (8)$$

where $\varphi=\varphi_l$ at the final boundary ($x=l$) traversed by the positive ions. Let

$$x\equiv\xi l \quad (9)$$

and

$$a\equiv(j_-/j_+)(-\eta_+/\eta_-)^{\frac{1}{2}}\geq 0. \quad (10)$$

The quantity a^2 can be shown to be the ratio of the product of charge density and total kinetic energy of the electron current to that of the positive ion current at the same value of ξ . Next, define

$$w_{+0}\equiv-v_{+0}^2/(\eta_+\varphi_l), \quad (11)$$

where v_{+0} is the initial velocity (at $x, \xi=0$) of the positive ions, and

$$w_{-l}\equiv v_{-l}^2/(\eta_-\varphi_l), \quad (12)$$

where v_{-l} is the initial velocity (at $x=l$, or $\xi=1$) of the electrons. Finally, let

$$J_+=4\pi l^2 j_+/(-\eta_+\varphi_l^{\frac{3}{2}}). \quad (13)$$

Then, Eq. (3) may be rewritten in the dimensionless

form

$$y''=J_+[(y+w_{+0})^{-\frac{1}{2}}-a(1-y+w_{-l})^{-\frac{1}{2}}], \quad (14)$$

where primes denote differentiation with respect to ξ , and, without loss of generality, $\varphi=0$ at $x, \xi=0$. The quantities w_{+0} and w_{-l} are initial kinetic- to potential-energy ratios for ions and electrons, respectively, whereas the terms $y+w_{+0}$ and $1-y+w_{-l}$, respectively, represent the local kinetic energies of ions and electrons relative to the initial potential energy of both species. The quantity J_+ is proportional to the ratio between the ion current density j_+ and the unipolar, space-charge-limited ion current density j_{m+} for vanishing initial velocity. In particular,

$$J_+=(4/9)(j_+/j_{m+}), \quad (15)$$

where

$$j_{m+}=(-\eta_+\varphi_l^{\frac{3}{2}})/(9\pi l^2) \quad (16)$$

results from the solution of Eq. (14) for unipolar, space-charge-limited ion current.

In order to determine the charge-density distribution, Eq. (3) may also be written in the form

$$\rho=-\varphi_l y''/(4\pi l^2). \quad (17)$$

In Eq. (17), y'' may be replaced by the right side of Eq. (14). In the absence of electrons, this result yields, for $y=1$ (i.e., $\xi=1$),

$$\rho_{m+l}=-\varphi_l J_+/(4\pi l^2), \quad (18)$$

where ρ_{m+l} is the ion charge density at $x=l$ for unipolar, space-charge-limited current. For bipolar currents, Eqs. (14), (17), and (18) result in

$$\rho/\rho_{m+l}=(y+w_{+0})^{-\frac{1}{2}}-a(1-y+w_{-l})^{-\frac{1}{2}}, \quad (19)$$

which expresses the net charge-density distribution. Eq. (19) consists of the sum of the two components

$$\rho_+/\rho_{m+l}=(y+w_{+0})^{-\frac{1}{2}}, \quad (20)$$

$$\rho_-/\rho_{m+l}=-a(1-y+w_{-l})^{-\frac{1}{2}}, \quad (21)$$

determining the charge-density distribution for each species.

By defining J_+ proportional to an ion-current ratio, the emphasis in Eq. (14) is on ion current characteristics; however, by redefining J proportional to an electron-current ratio, Eq. (14) may be derived in an alternative form, namely,

$$y''=J_-[(1/a)(y+w_{+0})^{-\frac{1}{2}}-(1-y+w_{-l})^{-\frac{1}{2}}], \quad (22)$$

where

$$J_-=4\pi l^2 j_-/(\eta_-\varphi_l^{\frac{3}{2}})=(4/9)(j_-/j_{m-}) \quad (23)$$

and

$$j_{m-}=(\eta_-\varphi_l^{\frac{3}{2}})/(9\pi l^2) \quad (24)$$

is the unipolar, space-charge-limited, electron-current density. Eq. (22) may be treated in a manner analogous

to the preceding treatment of Eq. (14), and thus alternative forms for the charge-density distributions may be obtained. Specifically,

$$\rho/\rho_{m-0} = (1/a)(y+w_{+0})^{-\frac{1}{2}} - (1-y+w_{-1})^{-\frac{1}{2}}, \quad (25)$$

and when Eqs. (21) and (20), respectively, are considered,

$$\rho_-/\rho_{m-0} = -\rho_-/(a\rho_{m+1}), \quad (26)$$

$$\rho_+/\rho_{m-0} = -\rho_+/(a\rho_{m+1}), \quad (27)$$

where ρ_{m-0} is the electron charge density at $x=0$ for unipolar, space-charge-limited current.

Eq. (26) or (27), allows a new interpretation of the coefficient a . In particular,

$$a = -\rho_{m-0}/\rho_{m+1}; \quad (28)$$

that is, " a " is the absolute value of the electron- to ion-charge-density ratio associated with the final boundaries traversed by the respective species under the conditions of unipolar, space-charge-limited currents. The coefficient a also relates the relative electron current and ion current according to

$$j_-/j_{m-} = aj_+/j_{m+}, \quad (29)$$

which follows from definitions (10), (16), and (24).

According to Eqs. (19)–(21), the charge-density distributions may be calculated as functions of the dimensionless potential y for assumed values of the dimensionless energy parameters w_{+0} and w_{-1} , and the coefficient a . However, the variation of y , itself, as a function of the dimensionless distance ξ is determined by Eq. (14).

Integrating Eq. (14) twice yields, consecutively, the electric field and the potential distribution. The first integration is elementary and results in

$$y' = -l\mathcal{E}/\varphi_l = [4J_+ \{ (y+w_{+0})^{\frac{1}{2}} - w_{+0}^{\frac{1}{2}} + a[(1-y+w_{-1})^{\frac{1}{2}} - (1+w_{-1})^{\frac{1}{2}}] + (l\mathcal{E}_0/\varphi_l)^2 \}^{\frac{1}{2}}, \quad (30)$$

where $\mathcal{E} = \mathcal{E}_0$ at $\xi = 0$. Eq. (30) is not integrable in terms of elementary functions. When the boundary condition $y=0$ at $\xi=0$ is used, the solution has the form

$$\xi = (j_+/j_{m+})^{-\frac{1}{2}} h_0[y, a, w_{+0}, w_{-1}, (y_0')^2/J_+], \quad (31)$$

where

$$h_0[y, a, w_{+0}, w_{-1}, (y_0')^2/J_+] = \left(\frac{3}{4}\right) \int_0^y [(y+w_{+0})^{\frac{1}{2}} + a(1-y+w_{-1})^{\frac{1}{2}} - g_0]^{-\frac{1}{2}} dy \quad (32)$$

and

$$g_0 = w_{+0}^{\frac{1}{2}} + a(1+w_{-1})^{\frac{1}{2}} - (y_0')^2/4J_+. \quad (33)$$

Eq. (14) can be integrated alternatively by using $y=1$ at $\xi=1$ as the lower limit to obtain, consecutively,

$$y' = -l\mathcal{E}/\varphi_l = [4J_+ \{ (y+w_{+0})^{\frac{1}{2}} - (1+w_{+0})^{\frac{1}{2}} + a[(1-y+w_{-1})^{\frac{1}{2}} - w_{-1}^{\frac{1}{2}}] + (l\mathcal{E}_1/\varphi_l)^2 \}^{\frac{1}{2}}, \quad (34)$$

and

$$1-\xi = (j_+/j_{m+})^{-\frac{1}{2}} h_1[y, a, w_{+0}, w_{-1}, (y_1')^2/J_+], \quad (35)$$

where $\mathcal{E} = \mathcal{E}_1$ at $\xi=1$,

$$h_1[y, a, w_{+0}, w_{-1}, (y_1')^2/J_+] = \left(\frac{3}{4}\right) \int_y^1 [(y+w_{+0})^{\frac{1}{2}} + a(1-y+w_{-1})^{\frac{1}{2}} - g_1]^{-\frac{1}{2}} dy, \quad (36)$$

and

$$g_1 = (1+w_{+0})^{\frac{1}{2}} + aw_{-1}^{\frac{1}{2}} - (y_1')^2/(4J_+). \quad (37)$$

But subtracting Eq. (30) from Eq. (34) leads to

$$l^2(\mathcal{E}_1^2 - \mathcal{E}_0^2)/(4J_+\varphi_l^2) = (1+w_{+0})^{\frac{1}{2}} - w_{+0}^{\frac{1}{2}} - a[(1+w_{-1})^{\frac{1}{2}} - w_{-1}^{\frac{1}{2}}], \quad (38)$$

or $g_0 = g_1$. Thus, for preassigned values of \mathcal{E}_0 and \mathcal{E}_1 , values of the coefficient a are not completely arbitrary. In particular, if $\mathcal{E}_1 > \mathcal{E}_0$, the left side of Eq. (38) is positive, so that, necessarily,

$$0 \leq a \leq [(1+w_{+0})^{\frac{1}{2}} - w_{+0}^{\frac{1}{2}}]/[(1+w_{-1})^{\frac{1}{2}} - w_{-1}^{\frac{1}{2}}] \equiv r, \quad (39a)$$

which also defines r . Similarly, if $\mathcal{E}_1 < \mathcal{E}_0$,

$$a \geq r, \quad (39b)$$

where the equality applies if $\mathcal{E}_1 = \pm \mathcal{E}_0$. However, an explicit expression for the ion-current ratio j_+/j_{m+} is derivable from Eq. (31) or (35) only if \mathcal{E} vanishes at one or both boundaries.

If $\mathcal{E}_0 = 0$ or $\mathcal{E}_1 = 0$, then J_+ vanishes from the integrand appearing in the respective Eq. (32) or (36). Then an explicit expression for j_+/j_{m+} is derivable from (31) or (35). (If $\mathcal{E}_0 \neq 0$ and $\mathcal{E}_1 \neq 0$, the integrations can still be expressed in closed form.) Because of charge conservation the ion-current ratio is unique for specified boundary conditions and may be evaluated therefrom. Hence, for $\mathcal{E}_0 = 0$,

$$j_+/j_{m+} = h_0^2(y=1, a, w_{+0}, w_{-1}, \mathcal{E}_0=0), \quad (40)$$

whereas for $\mathcal{E}_1 = 0$,

$$j_+/j_{m+} = h_1^2(y=0, a, w_{+0}, w_{-1}, \mathcal{E}_1=0) \quad (41)$$

are expressions for the ion-current ratio. The first expression is obtained by setting $\xi=1$ in Eq. (31); and the second, by setting $\xi=0$ in Eq. (35). Then, Eqs. (31) and (35) become, respectively,

$$\xi = h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0=0)/h_0(y=1, a, w_{+0}, w_{-1}, \mathcal{E}_0=0) \quad (42)$$

and

$$1-\xi = h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1=0)/h_1(y=0, a, w_{+0}, w_{-1}, \mathcal{E}_1=0). \quad (43)$$

The final problem is that of evaluating the integral represented by h_0 or h_1 in Eq. (32) or (36), respectively.

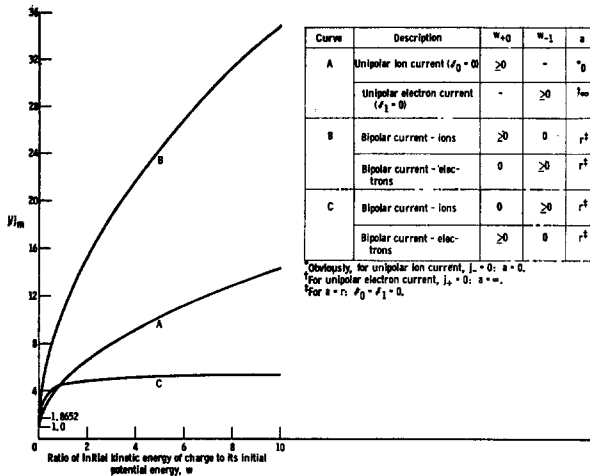


FIG. 2. Relative current as function of initial relative energy ($\mathcal{E}_0 = 0$).

These integrals may be transformed into a sum of elementary and elliptic integrals. The substitution

$$y + w_{+0} \equiv \epsilon^2 \sin^2 \alpha, \quad (44)$$

where

$$\epsilon^2 \equiv 1 + w_{+0} + w_{-1}, \quad (45)$$

transforms the integrals into a known elliptic form. However, a more tractable form, in which unipolar and bipolar contributions to the integrals are essentially separated, results by subsequently employing the trigonometric identity

$$A \cos(\alpha - \delta) = A \sin \delta \sin \alpha + A \cos \delta \cos \alpha,$$

where

$$\tan \delta \equiv 1/a \quad (46)$$

and

$$A^2 \equiv 1 + a^2. \quad (47)$$

In addition, let

$$\alpha - \delta \equiv 2\beta. \quad (48)$$

Then, the integrals may be written in a form involving the square root $(1 - n^2 \sin^2 \beta)^{-1/2}$, where

$$n^2 \equiv 2A\epsilon / \left(A\epsilon - \begin{Bmatrix} g_0 \\ g_1 \end{Bmatrix} \right) \equiv k^{-2} \quad (49)$$

also defines a new modulus k , and the upper quantity within the braces corresponds to the representation of Eq. (32), whereas the lower quantity corresponds to the representation of Eq. (36). Usually, $n^2 > 1$. Then, in order to obtain a value of the modulus less than unity, the substitution

$$n \sin \beta \equiv \sin \gamma \quad (50)$$

may be employed to transform Eqs. (32) and (36)

into a final form

$$\begin{aligned} & \left\{ \begin{aligned} h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0) \end{aligned} \right\} \\ &= 3(\epsilon^3/2A)^{1/2} \int_{\gamma_0}^{\gamma_1} [2k \cos(2\delta)(\sin \gamma - 2k^2 \sin^3 \gamma) \\ &+ \frac{1}{2} \sin(2\delta)(1 - 8k^2 \sin^2 \gamma + 8k^4 \sin^4 \gamma) \\ &\times (1 - k^2 \sin^2 \gamma)^{-1/2}] d\gamma, \quad (51) \end{aligned}$$

where

$$\sin \gamma = n \sin \left(\frac{1}{2} \left\{ \sin^{-1}[(y + w_{+0})\epsilon^{-2}]^2 - \delta \right\} \right), \quad (52)$$

with $\gamma = \gamma_0$ for $y = 0$, and $\gamma = \gamma_1$ for $y = 1$. Real solutions of Eq. (51) are obtained only if $|\sin \gamma| \leq 1$. In Eq. (51), the elliptic integrals vanish for unipolar ion currents, whereas the elementary integrals vanish for bipolar currents if $a = 1$. Although Eq. (51) has been written for $\mathcal{E}_0 = 0$ and $\mathcal{E}_1 = 0$, these assumptions are not necessary for evaluating the integrals. However, they are necessary for expressing j_+/j_{m+} explicitly, as mentioned previously.

The elliptic integrals in Eq. (51) correspond to integrals 280.00 and 281.01 in Ref. 7. When these are used the alternate solutions for h_0 and h_1 are

$$\begin{aligned} & \left\{ \begin{aligned} h_0(y, a, w_{+0}, w_{-1}, \mathcal{E}_0 = 0) \\ h_1(y, a, w_{+0}, w_{-1}, \mathcal{E}_1 = 0) \end{aligned} \right\} \\ &= (\epsilon^3/2A)^{1/2} \{ 2k \cos(2\delta) [3(2k^2 - 1) \cos \gamma - 2k^2 \cos^3 \gamma] \\ &+ \frac{1}{2} \sin(2\delta) [(8k^2 - 5)F(\gamma, k) + 8(1 - 2k^2)E(\gamma, k) \\ &+ 8k^2 \sin \gamma \cos \gamma (1 - k^2 \sin^2 \gamma)^{1/2}] \} \Big|_{\gamma_0}^{\gamma_1}, \quad (53) \end{aligned}$$

where the former set of limits is associated with h_0 and the latter set with h_1 . The quantities $F(\gamma, k)$ and $E(\gamma, k)$ are elliptic integrals of the first and second kinds, respectively, defined by

$$F(\gamma, k) \equiv \int_0^\gamma (1 - k^2 \sin^2 \gamma)^{-1/2} d\gamma \quad (54)$$

$$E(\gamma, k) \equiv \int_0^\gamma (1 - k^2 \sin^2 \gamma)^{1/2} d\gamma. \quad (55)$$

Eq. (53) in conjunction with Eqs. (42) and (43), respectively, determine implicit solutions for the potential distribution.

The preceding analysis permits the currents, fields, and charge distributions to be calculated for cases in which the electric field vanishes at one boundary, at least, and $\mathcal{E} > 0$ in the interval between the boundaries. Included among these possibilities are all of the corre-

⁷ P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer-Verlag, Berlin, 1954), p. 162.

sponding cases in the references cited. There also exists another series of cases for which the electric field vanishes at one or two locations between the boundaries. These cases, with at least one exception, may be treated by juxtaposing fields so that the boundary condition $\mathcal{E}=0$ (for $\xi=0$, say) in the present analysis coincides with the condition $\mathcal{E}=0$ for the type of distribution being considered. A simple juxtaposition is unsatisfactory if the current of one or both charge species is space-charge limited and the charge transmission coefficient at the juxtaposition plane ($\mathcal{E}=0$) is not unity. Then, reflected currents or instabilities must be considered.

PROCEDURE

With w_{+0} and w_{-1} given and a value of a selected, subject to inequalities (39), the quantity h_0 or h_1 may be computed from Eq. (53) as a function of the dimensionless potential y for the interval $0 \leq y \leq 1$ and subject to the additional limitation $|\sin \gamma| \leq 1$. The value of h_0 or h_1 , associated with the respective boundary value $y=1$ or $y=0$, permits the calculation of the ion-current ratio j_+/j_{m+} explicitly according to Eq. (40), or (41) depending on whether $\mathcal{E}_0=0$ or $\mathcal{E}_1=0$, respectively. Coincidentally, alternative Eqs. (42) and (43) allow computation of the dimensionless distance ξ as a function of the dimensionless potential y . The electric field is given by Eq. (30) if $\mathcal{E}_0=0$ or Eq. (34) if $\mathcal{E}_1=0$. The charge-density distribution and the component distributions are determined by Eqs. (19) to (21). The electric field and charge-density distributions are obtained as functions of y , but y has been uniquely related to the dimensionless distance ξ by Eqs. (42) or (43).

RESULTS AND DISCUSSION

Figures 2 to 6 illustrate results of a few sample calculations of the currents, fields, and charge distributions, all associated with the assumed condition $\mathcal{E}_0=0$ and demonstrating the effects of nonvanishing initial particle velocities.

Current ratios as functions of appropriate energy ratios are shown in Fig. 2. Curve A, applying to a unipolar current [which is space-charge-limited ($\mathcal{E}_0=0$) for $w_{+0}=0$], might just as readily represent the initial energy dependence of a unipolar electron current, where the electron current is space-charge-limited ($\mathcal{E}_1=0$) for $w_{-1}=0$. In the former case the ordinate and the abscissa are j_+/j_{m+} and w_{+0} , respectively, whereas in the latter case, the coordinates become j_-/j_{m-} and w_{-1} . According to curve A, the ion (electron) current density becomes considerably greater than the space-charge-limited value if the ions (electrons) are supplied with initial kinetic energy which is comparable to or greater than the applied electric potential. (Note that, for $w=1$, the initial kinetic and potential energies are equal.) If electrons are introduced with vanishing

Curve	Description	w_{+0}	w_{-1}	a
A	Unipolar space-charge-limited ion current	0	-	∞
B	Unipolar ion current ($\mathcal{E}_0=0$)	1	-	∞
C	Bipolar. Both currents space-charge limited	0	0	r^\dagger
D	Bipolar current	1	0	r^\dagger
E	Bipolar current	0	1	r^\dagger

*Obviously, for unipolar ion current, $j_- = 0$: $a = 0$.

†For $a = r$: $\mathcal{E}_0 = \mathcal{E}_1 = 0$.

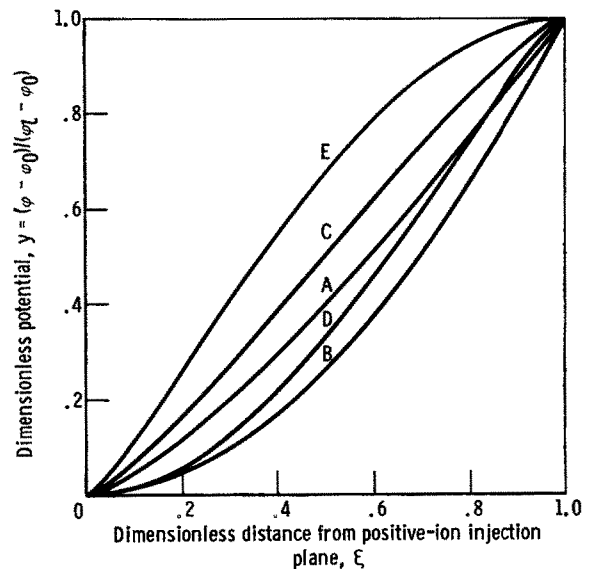


FIG. 3. Dimensionless potential distribution ($\mathcal{E}_0=0$).

initial kinetic energy at the boundary $\xi=1$, then for $\mathcal{E}_0=\mathcal{E}_1=0$ ($a=r$), curve B indicates the ion-current ratio. Curve C depicts the corresponding electron-current ratio for this case. Comparing curves A and B shows that introducing electrons increases the ion-current ratio by a factor 1.86 (for $w_{+0}=0$) to 2.39 (for $w_{+0}=10$) times the corresponding unipolar ion current ratio. More exactly, the present value $j_+/j_{m+} = 1.8652$ for $w_{+0}=0$, agrees more closely with Langmuir's value, $j_-/j_{m-} = 1.8605$ (Ref. 5), than with Müller-Lübeck's value, $j_-/j_{m-} = 1.8532$ (Ref. 6).

For $w > 0$, the indicated current ratios are for a monotonic potential distribution with $\mathcal{E}_0=0$; however, the current can be increased further until the ultimate space-charge limit is reached. The resulting potential distribution is then nonmonotonic but can generally be treated by using the present analysis by the juxtaposition method previously mentioned.

Curve	Description	w_{+0}	w_{-1}	a
A	Unipolar space-charge-limited ion current	0	-	∞
B	Unipolar ion current ($\epsilon_0 = 0$)	1	-	∞
C	Bipolar. Both currents space-charge limited	0	0	r^\dagger
D	Bipolar current	1	0	r^\dagger
E	Bipolar current	0	1	r^\dagger

*Obviously, for unipolar ion current, $j_- = 0$ $a = 0$.

†For $a = r$: $\epsilon_0 = \epsilon_1 = 0$.

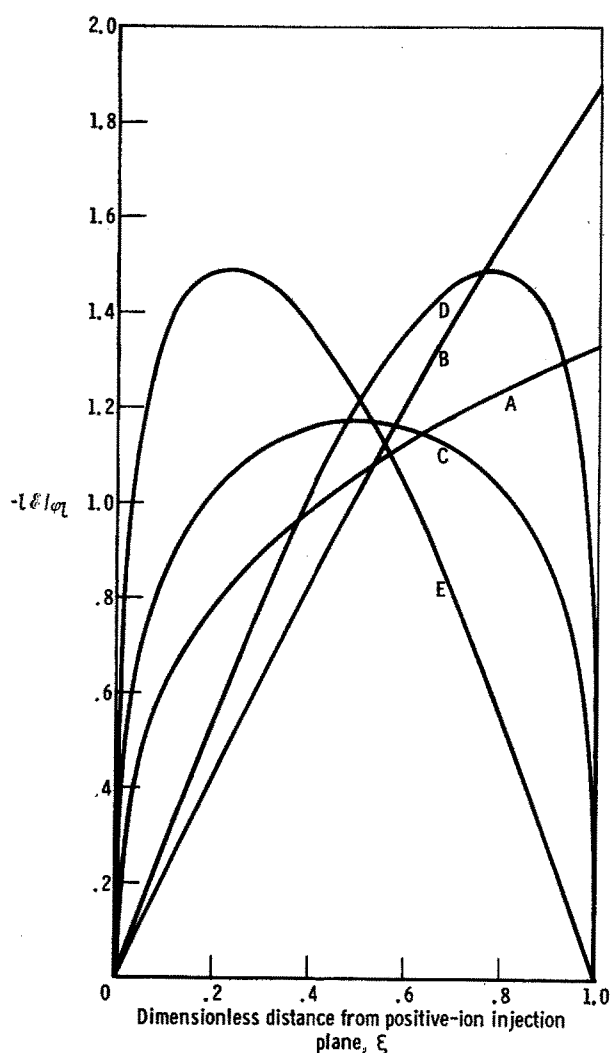


FIG. 4. Dimensionless electric field ($\epsilon_0 = 0$).

If the initial kinetic energy is supplied to the electrons, rather than the ions, then again for $a=r$, curve B is associated with the electron-currents ratio, whereas

curve C is associated with the ion-currents ratio; that is, the association of the curves is reversed. For $w \leq 0.85$, introducing negative ions with $m_- = m_+$, $|e_-| = e_+$, and with or without initial kinetic energy results in a larger positive ion-current density than would exist if the same initial kinetic energy were supplied to a unipolar flow of ions.

Figures 3 and 4 show, respectively, dimensionless potential distributions and dimensionless electric fields corresponding to specific values of the independent quantities w and a included in Fig. 2. (Curves C are the same as those for $a=1$ in Figs. 4 and 6 of Ref. 6.) Figure 5 depicts the dimensionless charge-density distributions corresponding to curves A, B, C, and D in Figs. 3 and 4. Charge-density distributions for bipolar currents with both current components space-charge-limited (curves C) are compared in Fig. 5(a) with the corresponding distribution for unipolar, space-charge-limited ion current. A similar comparison is exhibited in Fig. 5(b), except that $w_{+0}=1$ for both the bipolar and unipolar cases.

In Fig. 5(a) the partial neutralization of the positive ion charge density when electrons are introduced is evident. This is also characterized by the increased slope (near $\xi=0$) of the potential curve C with respect to curve A in Fig. 3 or by the increased absolute magnitude (near $\xi=0$) associated with the electric-field curve C with respect to curve A in Fig. 4. More importantly, the ion charge densities at $\xi=1$ are identical, so that the increased ion current density (1.86 times) must be associated with increased velocity of ions at $\xi=1$ when electrons are introduced.

If now, the ions possess initial kinetic energy, the ion charge-density distributions for unipolar currents tend to equal that for bipolar currents. For the case $w_{+1}=1$, $a=r$ shown in Fig. 5(b), the two ion charge-density distributions (curves B and D for ρ_+/ρ_{m+i}) are

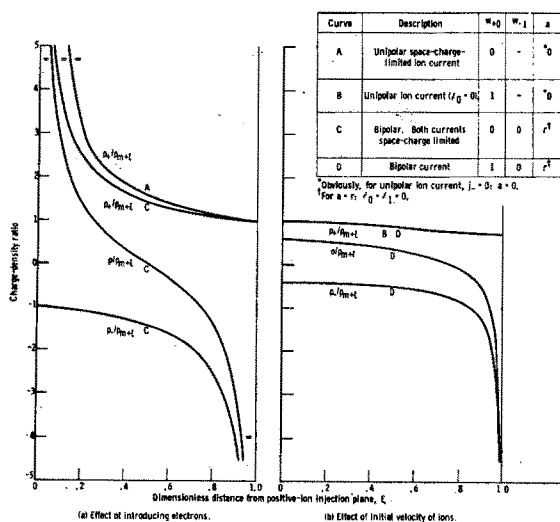


FIG. 5. Dimensionless charge-density distributions ($\epsilon_0 = 0$).

effectively identical. Moreover, the ion charge density at $\xi=1$ is less than the unipolar, space-charge-limited value. Hence, the increased ion current results from a combined effect of increased ion velocity due to the initial velocity of the ions and a further increase of ion velocity caused by the introduction of electrons. In the present instance, the former effect is more significant.

The sample results provide a very limited illustration

of the variety of solutions contained in the theory. Particular cases of interest will, of course, depend on the application.

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Thermoelectric Properties of Niobium in the Temperature Range 300°–1200°K*

V. RAAG AND H. V. KOWGER

Electronic Components and Devices, Radio Corporation of America, Harrison, New Jersey

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The electrical resistivity, thermal diffusivity, and Seebeck coefficient of niobium have been measured from 300°–1200°K. Using measured values of thermal diffusivity and literature values of specific heat, the thermal conductivity has been calculated for the same temperature range. The Lorenz number is 2.75×10^{-8} W- Ω -deg $^{-2}$ and is independent of temperature. The absolute Seebeck coefficient is very close to zero, its largest magnitude being $-2.6 \mu\text{V}/^\circ\text{K}$ and occurring at 450°K. Data are also presented on the Seebeck coefficient of tungsten up to 1400°K.

INTRODUCTION

IN recent years considerable interest has arisen in the thermal and electrical properties of niobium at high temperatures. The impetus for this interest is partially derived from the use of niobium as a canning material for the fuel elements of high-temperature reactors and from its use in aerospace technology. Additional interest has been generated by the advent of high-temperature refractory thermocouples. Although the emphasis has been placed on tungsten/tungsten-rhenium thermocouples, niobium-tungsten thermocouples^{1,2} have also been used on several occasions, although relatively little information has been published on the thermoelectric properties of niobium. Tottle³ and Heal⁴ have determined the thermal conductivity of niobium to 600°C and Fieldhouse *et al.*,⁵ to 1600°C. Its electrical resistivity values to 1450°C have been reported by Tye⁶ and to 900°C by Ames and

McQuillan.⁷ Miller⁸ has calibrated a niobium-platinum thermocouple in the temperature range of 200°–1450°C.

The investigation described in this paper uses concurrent thermal diffusivity and electrical resistivity determinations, along with literature values of the specific heat, to extend the range of calculated Lorenz numbers of niobium to 1200°K. In addition, it more precisely establishes the temperature dependence of the absolute Seebeck coefficient in the temperature range of 300°–1200°K. New values are given for the Seebeck coefficient of tungsten from 500°–1400°K.

EXPERIMENTAL TECHNIQUES

The thermal diffusivity, electrical resistivity, and Seebeck coefficient (referenced to Alumel) of niobium were measured in an RCA model H-1950-E thermal diffusivity apparatus. The apparatus and techniques employed, yielding thermal diffusivity values with an accuracy of about 2% and a repeatability of 1%, are essentially similar to those reported by Abeles *et al.*⁹ The niobium specimen ($\frac{1}{4}$ -in. diam and 2-in. length) measured in the thermal diffusivity apparatus was machined from a 0.5-in.-diam rod purchased from Kawecki Chemical Corporation. The niobium had been refined by electron-beam melting and had been

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