# AEC Computing and Applied Mathematics Center

## AEC RESEARCH AND DEVELOPMENT REPORT

Physics and Mathematics

NYO-2536

### THEORY OF CUSPED GEOMETRIES

### II. Particle Losses

Jerome Berkowitz

January 6, 1959

## Institute of Mathematical Sciences

NEW YORK UNIVERSITY
NEW YORK, NEW YORK

UNCLASSIFIED


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Contract No. AT(30-1)-1480





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# PREFACE

This report elaborates and extends some of results of the Theory of Cusped Geometries, I. General Survey, by Harold Grad [4]. The computation of particle losses through a line cusp is performed under the assumption that there is a good approximation to a sharp boundary separating the field-free plasma from the vacuum magnetic field. A rough calculation along the lines of this report was first presented at Sherwood meetings at Los Alamos in June 1955 (no report issued), and at Princeton in October 1955, [2].

Dr. E. Rubin was very helpful in the preparation of early versions of this report.

# THEORY OF CUSPED GEOMETRIES
## II. Particle Losses

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## 1. Introduction

The purpose of this report is to compute the losses of particles through the cusps from a cusped-shaped fully ionized plasma separated from a vacuum magnetic field by a thin boundary layer. (See Figure 1; the thin boundary layer is not shown.) To find the losses one must use a more realistic model than that of the simplified magnetohydrodynamic picture of a sharp boundary between plasma and vacuum. Among other matters, the equilibrium and stability of cusped geometries is treated in [5,6,7].

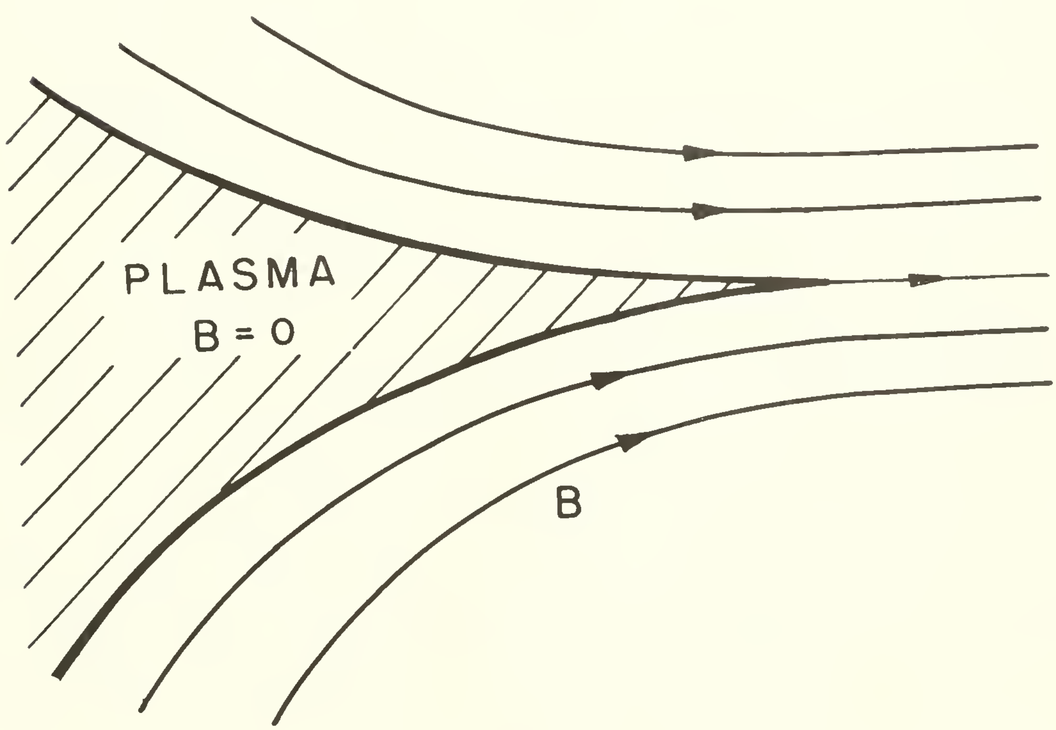


Figure 1

We first consider a two-dimensional problem concerning the motion of particles in cusped domains satisfying a certain unconventional reflection law, "soft-cushion billiard ball" reflection, which simulates the finite thickness boundary layer, cf.[2]. We determine the loss rate of particles from the cusp according to this model, and finally we demonstrate the connection between this model and the actual motion of charged particles in the plasma.

## 2. Orbits in the plasma

To fix our notation, see Fig. 2, we take a rectangular x,y coordinate system in which the axis of the cusp is the x-axis, with the cusp at the origin; the domain of interest extends symmetrically about the negative x-axis. We denote by $\psi$ the angle between the positive x-axis and the direction of motion of a particle. When a particle crosses the cusp axis, if it is advancing toward the cusp, we have $|\psi| < \frac{\pi}{2}$ , whereas if it is moving away, $|\psi| > \frac{\pi}{2}$.

We make the following general assumptions:

In the interior of the domain a particle moves on a straight line. When it flies out of the domain into the boundary layer, it is returned shortly via some trajectory which need not be specified in detail now. Then it continues inside the domain on a straight line in the direction it had at the instant of its return until it emerges again, is returned again, and so forth. Our analysis is made close to the tip of the cusp, where the boundary curves of the cusp are very nearly parallel straight lines.

Upon reentry into the plasma domain a particle moves in that direction which it would have if it had undergone specular reflection at the magnetic wall.

A particle in two successive flights outside of the cusp domain advances (in the x-direction) a distance $r_1 + r_2 = 2r(\psi)$; $r(\psi)$ is called the <u>range</u>, (see Figure 2b). The range depends continuously on $\psi$ in such a way that:

(a) $r(\psi) \geq 0 \qquad 0 \leq \psi \leq \frac{\pi}{2}$ ,

(b) $r(\pi-\psi) = -r(\psi) \qquad 0 \leq \psi \leq \pi$ ,

(c) $r(-\psi) = r(\psi) \qquad 0 \leq \psi < \pi$ .

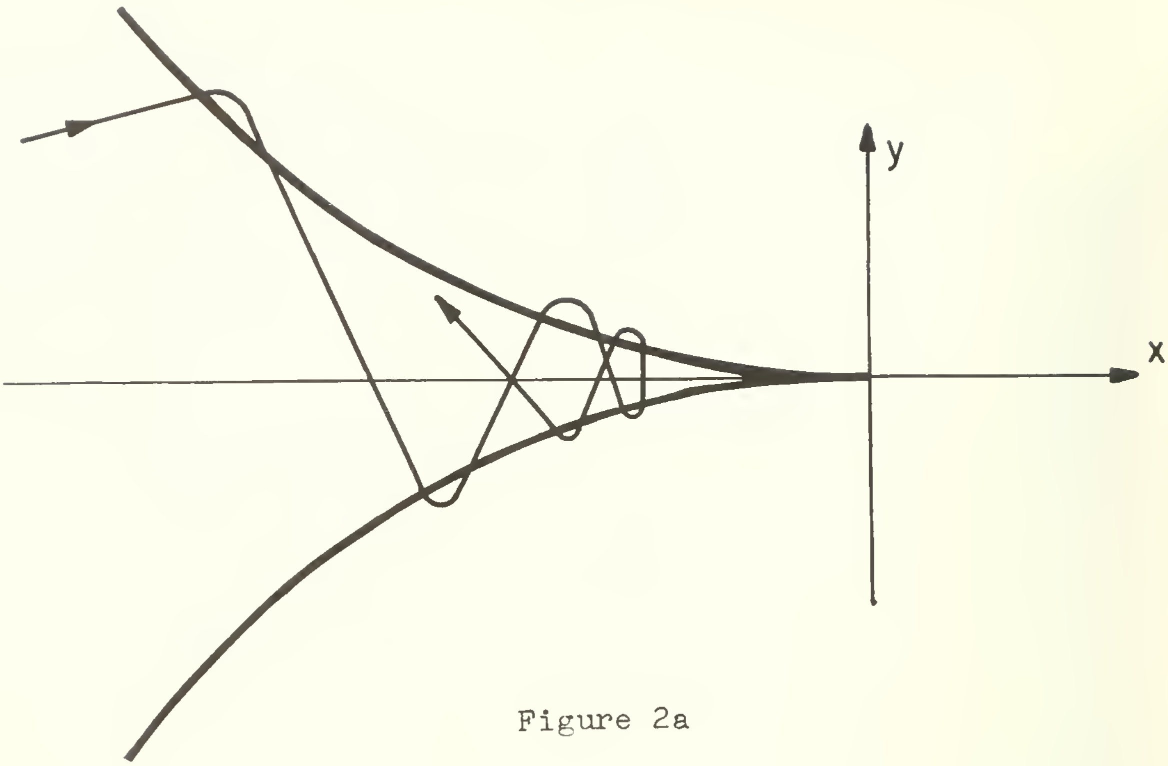


Figure 2a

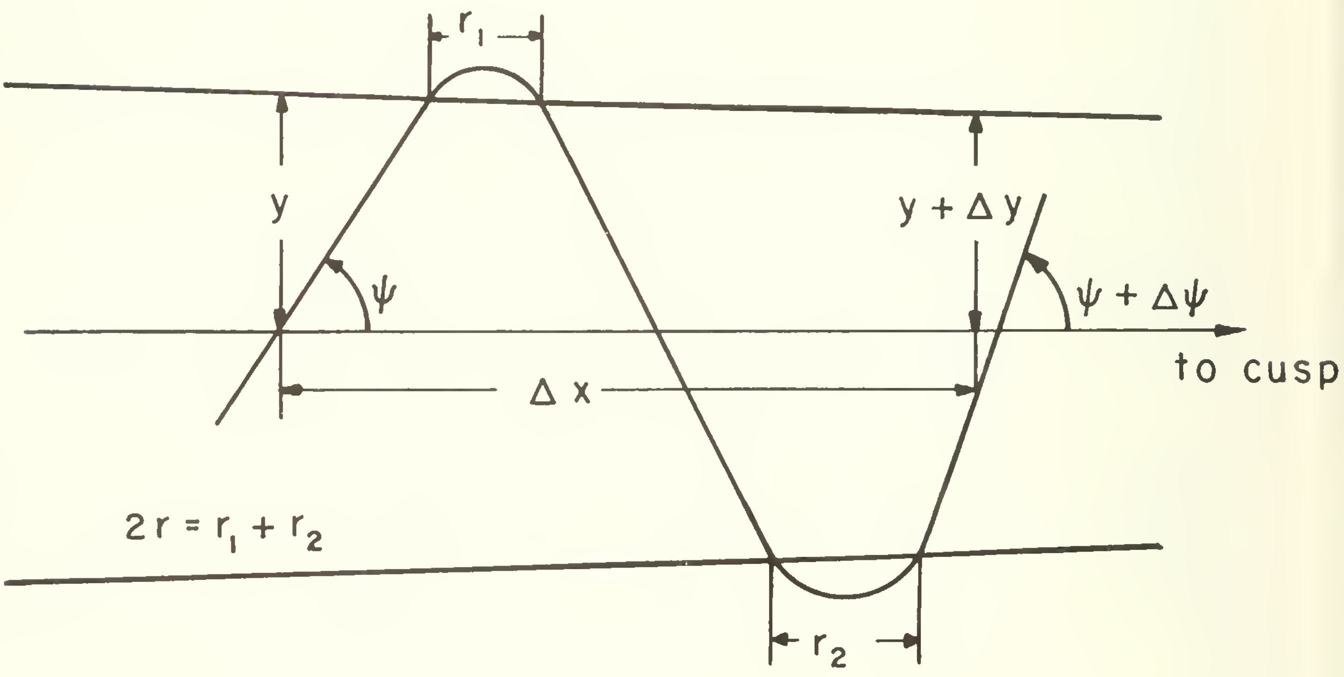


Figure 2b

This prescription completely determines the motion of a particle, provided the particle path never extends beyond $x = 0$. When the latter situation arises, we say the particle is lost through the cusp. Thus we regard as lost all particles not turned back before reaching the tip of the cusp. What happens to particles which are lost--whether they return to the interior of the plasma, or to the boundary layer--is a complicated problem, which is not considered here. However, see [4].

In the vicinity of the cusp, the boundary curves of the cusp are nearly parallel straight lines, and the range $r(\psi)$ is small compared to the dimensions of the cusp domain. If we denote by $\Delta\psi$ the change in $\psi$ after two successive reflections from the exterior, then it follows that $|\Delta\psi| \ll |\psi|$. Thus we have to good approximation (cf. Figure 2b), for $\psi$ positive,

$$\Delta x = 4y \cot \psi + 2r(\psi)$$

and

$$\Delta\psi = -4 \frac{\Delta y}{\Delta x} ;$$

and we find

$$\frac{\Delta y}{\Delta\psi} = -y \cot \psi - \frac{1}{2} r(\psi) .$$

For small $\Delta\psi$, we can replace this difference equation by the corresponding differential equation*

$$\frac{dy}{dx} = -y \cot \psi - \frac{1}{2} r(\psi),$$

which can be integrated immediately to yield

*This analysis is approximately equivalent to using an adiabatic invariant which generalizes the magnetic moment; cf. [3] and the appendix to [5].

$$y = \frac{1}{\sin\psi} \quad \frac{1}{2}\int_{\psi}^{\pi/2} r(\psi)\,\sin\psi\,d\psi + y(\tfrac{\pi}{2}) \quad .$$

We sketch some of the integral curves in Figure 3.

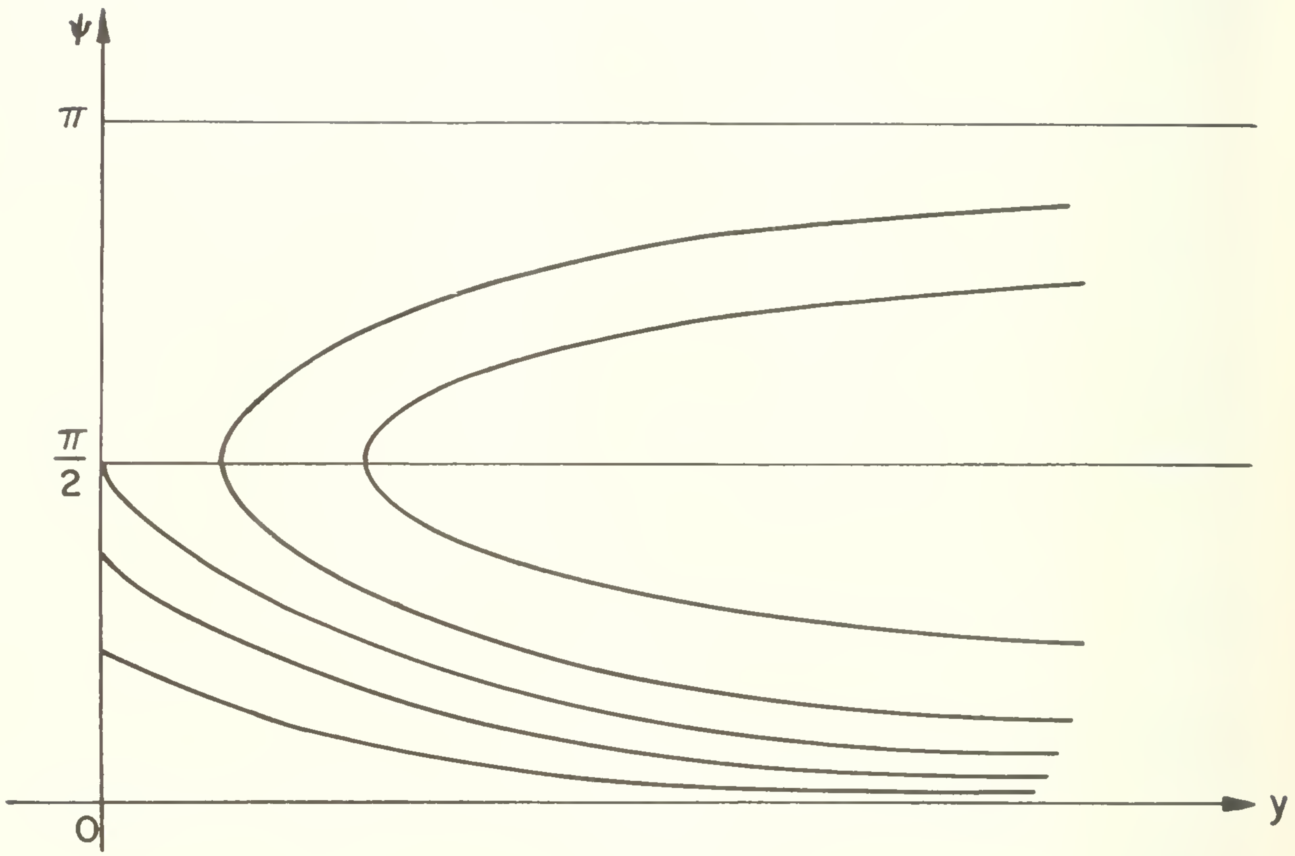


Figure 3

Note that there is a critical integral curve for which $y(\frac{\pi}{2}) = 0$. We denote this curve by $\psi = \psi^*(y)$; the inverse of this function is given by

$$y = \frac{1}{2\sin\psi^*} \int_{\psi^*}^{\pi/2} r(\psi)\,\sin\psi\,d\psi \ .$$

This critical curve $\psi = \psi^*(y)$ has the property that, for a given value of y, (the half-width of the cusp), all particles starting with $|\psi| > \psi^*(y)$ will be returned before reaching the cusp, whereas those starting with $|\psi| < \psi^*(y)$ will be lost through the cusp. We remark, for future use, that $2y\psi^*$ tends to a constant as $\psi^*$ tends to zero, and y tends to $\infty$; we denote this constant value by $r^*$, where

$$(1) \qquad r^* = \int_0^{\pi/2} r(\psi) \sin \psi \, d\psi.$$

Incidentally, in the special case of specular reflection, i.e. $r(\psi) \equiv 0$, the integral curves are

$$y = y(\tfrac{\pi}{2})/\sin \psi ,$$

and the critical integral curve is $y = 0$. Thus in this case our theory yields the (correct) result that the only particles lost are those aimed exactly along the cusp axis towards the tip.

It is important to note that our conclusions are independent of the shape of the cusp. The equation $y = y(x)$ of the plasma boundary does not matter; one only needs to know the range $r(\psi)$.

Having determined the path of a particle moving in a plane cusp-shaped domain in accordance with the laws of motion specified above, we now consider a general three-dimensional motion in a cusp-shaped wedge in x,y,z-space. Now Figure 2 represents a section of the wedge for z constant. (Note that we are now dealing with a line cusp.) We suppose that a gas consisting of particles is contained in the cusp-shaped domain. We shall compute the rate of loss of particles through the cusp.

In the physical model described below it will be seen that the motion of a particle is such that the projection of its trajectory on the x,y-plane satisfies the general assumptions made above. The components of velocity in the x,y,z directions are denoted by u,v,w respectively. We also write $u = U \cos \psi$ and $v = U \sin \psi$. In the interior of the cusp-shaped domain we assume that the speeds U and w remain constant for a particular particle in the course of its motion. The range r now depends on U and w as well as on $\psi$; i.e. $r = r(\psi,U,w)$. We shall assume that the distribution of particle velocities is isotropic far enough from the cusp.

We remark first that the distribution of particle velocities does not remain isotropic as the cusp is approached. If we calculate the rate of flow through a section of width 2y at some given x, of those particles which have $|\psi| < \psi^*(y)$, we find that this quantity is independent of y. Hence it is easiest to perform the calculation at a relatively large value of y where the velocity distribution is isotropic. This loss rate will now be computed in terms of $r(\psi,U,w)$ or $r^*(U,w)$.

The isotropic velocity distribution function $f(u,v,w)$ is given. The flow rate per unit area in the positive x-direction is

$$\int_{u>0} u\, f\, du\, dv\, dw = \int_{|\psi|<\frac{\pi}{2}} u\, f\, U\, dU\, d\psi\, dw\ ;$$

and the loss rate per unit area, determined by $|\psi| < \psi^*$, is easily seen to be, for $\psi^*$ small,

$$\int 2\psi^*(U,w)\, U^2 f\, dU\, dw.$$

Therefore the loss rate per unit length of line cusp is given by

$$Q = 2 \int r^* U \, f \, U \, dU \, dw$$

or

$$(2) \qquad Q = \int_{u > 0} |rv| \, f \, du \, dv \, dw \; .$$

Within a numerical factor of order unity, the mean value of $r^*$ represents the effective size of the "hole" at the cusp.

3. Range function

We shall now calculate the range r, which was found to be basic in computing particle losses through cusps. To do this we need to choose some model for the boundary layer between the plasma and the vacuum. We adopt the simple model of a magnetic field B, which jumps discontinuously from zero to its vacuum value together with a constant electric field E, caused by charge separation. We can either compute a roughly self-consistent value of E, e.g. by equating the average penetration of ions and electrons into the boundary layer, or else leave open the value of E. The latter course is appropriate in case there are stray electron orbits near the plasma, but not entering it.*
The number of such external electrons, which influence the value of E, would have to be determined by an analysis of the whole system. Order of magnitude estimates have been made, cf. [4].

In the boundary layer we introduce (x,y,z) coordinates as indicated in Figure 4. The x-axis is directed towards the cusp, and the y-axis is in the outward normal direction to the plasma. Thus we use a different coordinate system in the other boundary layer. The components of velocity are (u,v,w), as before. It is convenient to introduce

$$V = \sqrt{u^2 + v^2 + w^2} \qquad \text{and } V' = E/B .$$

Now suppose a particle enters the boundary layer at x = y = x = 0 with initial velocity (u,v,w). It is a simple matter to find the

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*cf. M. Rosenbluth [1], who gives a more general self-consistent analysis, but treats a special velocity distribution. However, a more precise self-consistent treatment of the boundary layer is not warranted here because of the uncertainty as to the exact value of E.

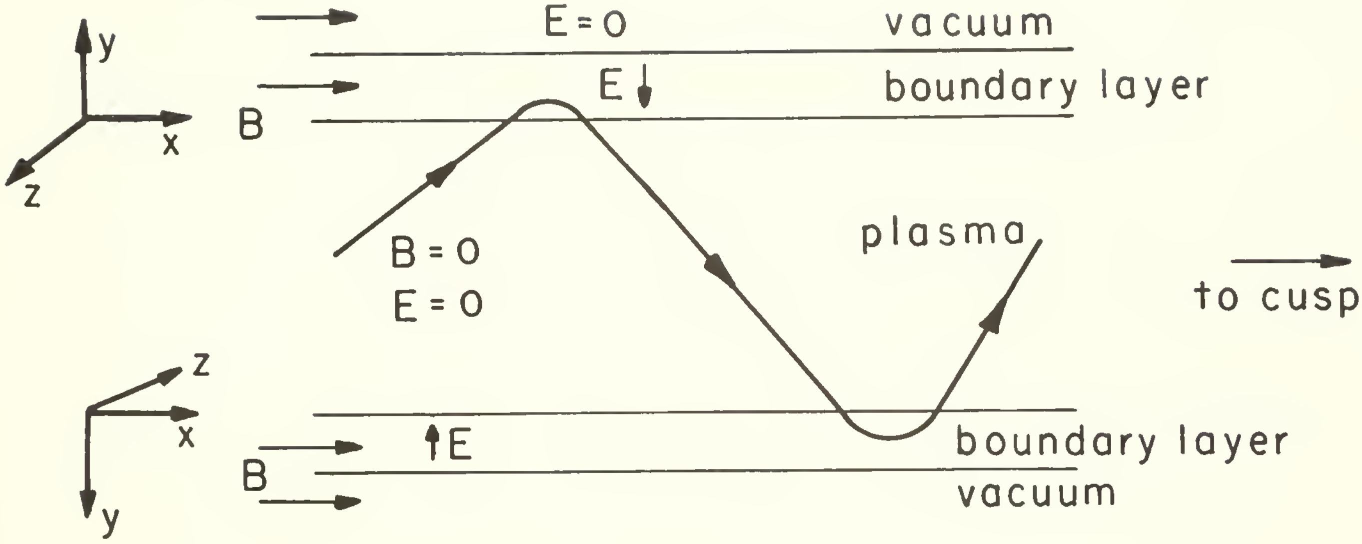


Figure 4

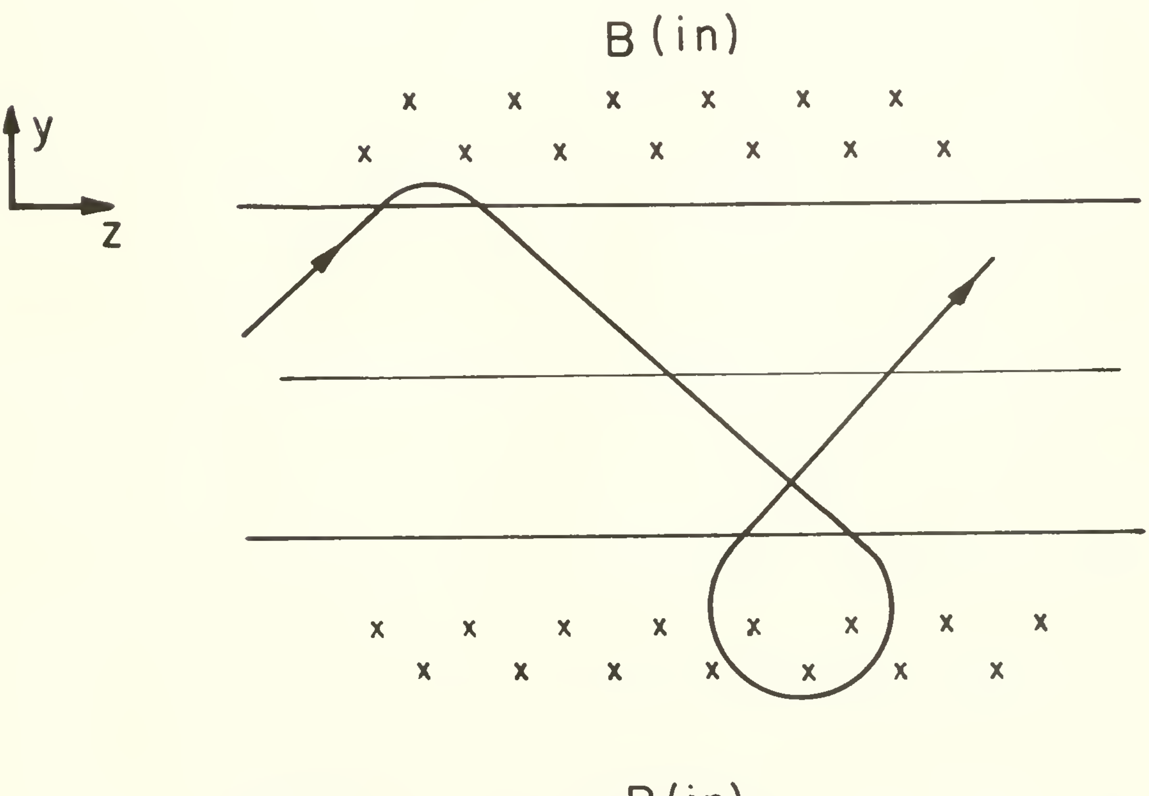


Figure 5

trajectory of a particle with mass m and charge e in a constant electromagnetic field. We find

$$
(3) \qquad
\begin{aligned}
x &= ut , \\
y &= \frac{v}{\omega} \sin \omega t - \frac{1}{\omega} ( w - V')( \cos \omega t - 1) , \\
z &= \frac{v}{\omega} (\cos \omega t - 1) + \frac{1}{\omega} (w - V')\sin \omega t + V't ,
\end{aligned}
$$

where $\omega = eB/m$.

The time of flight of the charged particle in the boundary layer is immediately found to be

$$\frac{2}{|\omega|} \text{arc tan} \frac{\pm v}{V' - w} ,$$

where the $\pm$ sign is chosen to be the same as the sign of the charge e on the particle, and the value of the arc tangent is chosen between 0 and $\pi$. We note that at the point of reentry into the plasma, the velocity is $(u,-v,w)$. At the particle's subsequent emergence into the other boundary layer, its initial velocity (in the appropriate coordinate system there) is $(u,v,-w)$. Consequently the time of flight there is

$$\frac{2}{|\omega|} \text{arc tan} \frac{\pm v}{V' + w} .$$

Therefore the range r is given by

$$(4) \qquad r = \frac{u}{|\omega|} \left(\text{arc tan} \frac{\pm v}{V' - w} + \text{arc tan} \frac{\pm v}{V' + w} \right) .$$

After this result is inserted into the formula (2) for the loss rate Q, it is just a matter of integration to determine the rate at which particles are lost through the cusp. In general, this requires numerical computation. We shall be able to evaluate Q explicitly in a few special cases.

Of course the answer depends on the value of E, or, equivalently, of $V'$. Ultimately the value of E must be determined by an analysis of the entire space charge field which surrounds the plasma in order to equalize the rates of loss of ions and electrons. In the remainder of this paper, we only evaluate the loss rate Q for certain assigned values of E. We note that the loss rates for electrons and ions will turn out to differ, as should be expected.

## 4. Loss rates $Q_{\pm}$

For simplicity we now take a distribution of particles that is monoenergetic as well as isotropic.

First consider the case $E = 0$. The range r, given by (4), reduces to $r = \pi u/|\omega|$. To evaluate the loss rate Q, which is given by (2), we introduce space polar coordinates:

$$u = V \sin \theta \cos \psi, \quad v = V \sin \theta \sin \psi, \quad w = V \cos \theta.$$

The angle $\psi$ is the same as that introduced in section 2; see Figure 2b. The speed V is constant. The distribution function f becomes, in these variables,

$$f(u,v,w)\, du\, dv\, dw = \frac{n}{4\pi} \sin \theta\, d\theta\, d\psi,$$

where n denotes the number density. Then the loss rate Q is found, by a simple integration, to be

$$(5) \qquad Q = \frac{1}{3} n V \lambda ,$$

where $\lambda = V/|\omega|$ is the radius of gyration. Electrons and ions are lost at the same rate if they have the same temperature, $m_+ V_+^2 = m_- V_-^2$. Although the electrons move faster, the size of the "hole", through which they leak is proportionately smaller. If we define the "hole size" $\delta$ by $Q = n \bar{u} \delta$, where $\bar{u}$ is the mean speed

$$\bar{u} = \int_{u>0} u f\, du\, dv\, dw \Big/ \int_{u>0} f\, du\, dv\, dw,$$

then we obtain in this case

$$\delta = \frac{2}{3} \lambda .$$

The next case we treat is more important in that it provides a partially self-consistent picture of the boundary layer. We adjust the constant value of the electric field E to make the mean penetration of ions and electrons equal. (The Debye length is assumed to be small compared to the thickness of the boundary layer.)

We make the tentative hypothesis that $V' = E/B$ lies midway between $V_+$ and $V_-$, i.e. that

$$V_+ << V' << V_- .$$

At the end of our analysis, we shall be able to check on the validity of this assumption. For, the ions the range r, given by (4), in this case reduces approximately to

$$r = \frac{2u}{\omega_+} \text{arc tan} \frac{v}{V'} . \tag{6}$$

Then, using the trajectory equations (3) it is easy to compute $d_+$, the mean penetration of a positive ion into the boundary layer; and we obtain

$$d_+ = \frac{1}{3} \frac{v^2}{\omega_+ V'} .$$

The average mean penetration $\bar{d}_+$, averaged over the positive ion distribution, is then immediately found to be

$$\bar{d}_+ = \frac{1}{9} \frac{V_+^2}{\omega_+ V'} .$$

On the other hand, for electrons we ignore $V' = E/B$ in comparison to $V_-$. Then we find the mean penetration $d_-$ of an electron into the boundary layer to be simply

$$d_- = \frac{\sqrt{v^2 + w^2}}{|\omega_-|} .$$

The average mean penetration of the electrons is consequently

$$\bar{d}_- = \frac{\pi}{4} \frac{V_-}{|\omega_-|} .$$

Now we set $\bar{d}_+ = \bar{d}_-$ -- and assuming that the electron and ion temperatures are equal, i.e. $m_+ V_+^2 = m_- V_-^2$ -- we find

$$V' = \frac{4}{9\pi} V_- .$$

For deuterium, we approximately have

$$V_+ : V' : V_- = \frac{1}{10} : 1 : 7,$$

confirming our hypothesis about the relative magnitudes of $V_+$, $V'$, and $V_-$.

This analysis results in a boundary layer thickness close to an electron gyro-radius, and the value

$$E = \frac{4}{9\pi} V_- B$$

for the electric field. We are now able to apply our general loss rate formula (2). In fact since E hardly alters the electron orbits, the electron loss rate is given as before,

(7) $$Q_- = \frac{1}{3} n V_- \lambda_- .$$

For ions we use for the range $r_+$, the approximate value

$$r_+ = \frac{2u}{\omega_+} \frac{v}{V'} ,$$

obtained from (6). Inserting this expression for $r_+$ into (2), we find for the ion loss rate

$$Q_+ = \frac{9\pi}{32} \frac{nV_+^2}{V_-} \lambda_+ .$$

The "hole size" for ions is then

$$\delta_+ = \frac{9\pi}{16} \sqrt{\frac{m_-}{m_+}} \lambda_+ ,$$

whereas for electrons it is

$$\delta_- = \frac{2}{3} \lambda_- .$$

Therefore we have

$$\delta_+ : \delta_- = 2.7 .$$

For deuterium, we obtain

$$Q_+ : Q_- = 1 : 35 .$$

The difference in loss rates must be equalized by external space charge effects. For a qualitative resolution of this difficulty, see [4]. The equalized loss rate is intimately connected with the thickness of the boundary layer, and there is a factor of at least 35 at stake. In the picture presented here, the electron orbits are hardly affected by the electric field in the boundary layer, and the ion orbits are pulled in almost to the electron orbits.

## REFERENCES

[1] Rosenbluth, M. Infinite Conductivity Theory of the Pinch, LA 1850, September 14, 1954, especially pp. 23-35.

[2] Grad, H. Theory of the Cuspidor, TID-7503, Princeton, October 1955, p. 319.

[3] Grad, H. General Adiabatic Theory, TID-7520, Gatlinburg, June 1956, p. 402.

[4] Grad, H. Theory of Cusped Geometries, I. General Survey, NYO-7969, Inst. Math. Sci., N.Y.U., December 1, 1957.

[5] Berkowitz, J., K.O. Friedrichs, H. Goertzel, H. Grad, J. Killeen, and E. Rubin. Cusped Geometries, Proc. of 2nd U.N. International Conference on the Peaceful Uses of Atomic Energy (September 1958), Vol. 31 (to appear).

[6] Berkowitz, J., H. Grad and H. Rubin. Problems in Magneto-hydrodynamic Stability, Proc. of 2nd U.N. International Conference on the Peaceful Uses of Atomic Energy (September 1958) (to appear).

[7] Berkowitz, J., et al. Theory of Cusped Geometries, III. Computation of Cusped Surfaces, (to appear).

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