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ION LOSSES FROM END-STOPPERED MIRROR TRAP*

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ABSTRACT. The loss rate of singly charged ions from an end-stoppered mirror trap is discussed by using a linearized Fokker-Planck equation and the associated variational principle. A comparison is made with the earlier work of Pastukhov and with numerical results.

1. INTRODUCTION

The notion that end losses in magnetic mirror devices might be significantly reduced by imposing electrostatic potentials at the two ends of the mirror has received some attention recently [1–5]. Some schemes for establishing such an "end-stoppering" effect have been proposed and theoretical estimates of the loss rates of particles and energy have been made [1–5]. In this paper, we present an independent calculation of the singly charged ion loss rate due to ion-ion collisions in the steady state using a linearized Fokker-Planck equation. Pastukhov [5] has given a somewhat different analysis of the linearized problem, the essential difference being that he has used a modified differential equation while we have used a modified boundary condition. It should be stated, however, that neither Pastukhov nor we have been able to solve exactly even the linearized problem. The method discussed here though, unlike that of Ref. [5], allows an estimate of the error incurred; this may justify presenting an alternative approach.

Our final results are in qualitative agreement, at least as far as the functional dependence on the various parameters (e.g. potential, ϕ ; mirror ratio, R; etc.) is concerned (except for arguments of some logarithms). There is one dimensionless number, only bounded by our analysis, but occurring only in the argument of a logarithm which could be matched to numerical results. Still, owing, to the sizes of the various errors involved in our approximate calculation, we expect only ~ 50% agreement with experiment with practical parameters.

We also wish to point out what we believe is an error [6] of a factor of 2 in the work of Ref.[5]. When corrected, this brings our results more nearly in agreement.

Let us now state the major assumptions we make in the present calculation:

1. Collisions are adequately treated by the Fokker-Planck equation; that is, small-angle (large impact parameter), two-body encounters are assumed to be most important in calculating the particle loss rate.
2. The "lost" particles are those of large kinetic energy (compared to temperature, T) and these particles are lost owing only to collisions with particles of average kinetic energy ($\approx$ T). This assumption allows the use of a linearized version of the Fokker-Planck equation for the high-velocity part of the distribution function; some such linearization appears to be essential in order to make any progress analytically, because of the complexity of the Fokker-Planck equation.
3. The electrostatic end potential is a square-well potential, zero in the well, finite and positive at the ends so that ion confinement is enhanced.

2. KINETIC EQUATION FOR IONS

The kinetic equation for the ion distribution function $f(\vec{v},t)$ in the central section of the mirror trap (where the magnetic field, $\vec{B}$, is sensibly constant) is

$$\frac{\partial f}{\partial t} + \frac{1}{c}\,\vec{v} \times \vec{B} \cdot \frac{\partial}{\partial \vec{v}} f = \left(\frac{\delta f}{\delta t}\right)_{coll} + \frac{\partial}{\partial \vec{v}} \cdot (\vec{\lambda}(\vec{v},t) f) + S(\vec{v},t) \tag{1}$$

where S is a particle source that we shall take to be non-vanishing only for small $v \equiv |\vec{v}|$ and $(\partial/\partial\vec{v})(\vec{\lambda}(\vec{v},t)f)$ is a term that describes heating due to external sources. In the steady state, $f = f(v,\mu)$ where μ is the cosine of the angle between $\vec{B}$ and $\vec{v}$, and Eq.(1) reduces to

$$\left(\frac{\partial f}{\partial t}\right)_{coll} + \frac{\partial}{\partial \vec{v}} \cdot (\vec{\lambda} f) + S(\vec{v}) = 0 \tag{2}$$

The Fokker-Planck collision term for a plasma has been given by Rosenbluth, MacDonald and Judd [7]. Their result for the case of a single-component (singly charged ion) plasma whose distribution function possesses axial symmetry in velocity space is:

$$\Gamma^{-1}\left(\frac{\delta f}{\delta t}\right)_{coll} = -\frac{1}{v^2}\frac{\partial}{\partial v}\left[f v^2 \frac{\partial h}{\partial v} - \frac{1}{2}\frac{\partial}{\partial v}\left(f v^2 \frac{\partial^2 g}{\partial v^2}\right) + f\frac{\partial g}{\partial v}\right] + \frac{1}{2v^3}\frac{\partial g}{\partial v}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu} \tag{3}$$

where $\Gamma \equiv 4\pi\epsilon^4 \ln\Lambda/m^2$
$\ln\Lambda \equiv$ "Coulomb logarithm"
m = ion mass
ϵ = proton charge

and g and h are potentials defined as

$$g(\vec{v}) = \int d\vec{v}'\, f(v',\mu')\,|\vec{v}-\vec{v}'|$$

$$h(\vec{v}) = 2\int d\vec{v}'\, f(v',\mu')\,|\vec{v}-\vec{v}'|^{-1}$$

Equation (3) is seen to be non-linear and of fourth order. The problem may be considerably simplified, however, by making the assumption that the scatterers (the low-velocity ions) have, owing to their large collision frequency, an essentially Maxwellian distribution. Then, for large v (compared to the thermal velocity) we may evaluate g and h using a Maxwellian distribution in the integrand. We thereby take into account only the scattering *of* high-velocity ions *by* low-velocity ions. The validity of this procedure may be checked a posteriori. Note that is is not a "linearization" in the standard sense: If we write $f = f_{maxwell} + f_1$, where f_1 is appreciable only at large velocities, we are considering here only the contribution to $(\delta f/\delta t)_{coll}$ of the scattering of f_1 by $f_{maxwell}$ and not of $f_{maxwell}$ by f_1 which is also a "linear" process but one which is assumed (and may be verified) to contribute negligibly to $(\delta f/\delta t)_{coll}$ at high energies. Under this assumption, g and h become, for large v:

$$g(v) \sim 4\pi\left[\int_0^v dv'\, f_m(v')\, v'^2 v\left(1 + \frac{1}{3}\frac{v'^2}{v^2}\right) + \int_v^\infty dv'\, f_m(v')\, v'^3\left(1 + \frac{1}{3}\frac{v^2}{v'^2}\right)\right]$$

$$h(v) \sim 8\pi\left[\frac{1}{v}\int_0^v dv'\, v'^2 f_m(v') + \int_v^\infty dv'\, v' f_m(v')\right]$$

where

$$f_m(v) \equiv n(m/2\pi T)^{3/2} \exp(-m\, v^2/2T)$$

n is the ion number density and T is the ion temperature; here Boltzmann's constant has been taken as unity. Equation (3) now becomes, for large v:

$$\Gamma^{-1}\left(\frac{\delta f}{\delta t}\right)_{coll} = \frac{n}{v^2}\frac{\partial}{\partial v}\left(f + \frac{T}{mv}\frac{\partial f}{\partial v}\right) + \frac{n}{2v^3}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial f}{\partial \mu}$$

If we define

$$f(v,\mu) = f_m(v)\,\bar{g}(v,\mu) \tag{4}$$

Then the kinetic equation for $\bar{g}$ is

$$\frac{T}{m}\frac{\partial}{\partial v}\left(e^{-mv^2/2T} \; \frac{1}{v}\frac{\partial \bar{g}}{\partial v}\right)$$

$$+ \frac{1}{2v} e^{-mv^2/2T} \; \frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial \bar{g}}{\partial \mu}$$

$$= \frac{-Sv^2}{n^2\Gamma}\left(\frac{2\pi T}{m}\right)^{3/2} \tag{5}$$

where we have assumed that the heating term is negligible for the high-velocity ions as follows from the fact that the bulk of the heating goes into the low-energy (Maxwellian) ions. [The ions in the loss-cone region have their energy primarily determined by exchange with the thermal ions rather than by external heating.] If we further assume that S is a low-velocity source, i.e. S is now zero only near where $\bar{g} \approx 1$, then we may calculate the source strength using a variational principle, for which Eq.(5) is the Euler-Lagrange equation:

$$\mathscr{I} \equiv \iint dv d\mu \; \frac{1}{v} e^{-mv^2/2T} \left[\frac{T}{m}\left(\frac{\partial \bar{g}}{\partial v}\right)^2\right.$$

$$\left. + \frac{1}{2}(1-\mu^2)\left(\frac{\partial \bar{g}}{\partial \mu}\right)^2\right]$$

$$= \frac{4\pi \int dv \; v^2 S}{2\pi n^2 \; \Gamma} \cdot \left(\frac{2\pi T}{m}\right)^{3/2} \tag{6}$$

where the double integral over v and μ extends over that region of velocity space in which $\bar{g}$ is non-vanishing. $\bar{g}(v, \mu)$ is assumed to be a continuous function of both variables throughout the region of integration in expression (6).

Equivalently, we may solve Eq.(5) with $S = 0$, apply the matching condition $\bar{g} \to 1$ as $v \to 0$, and use expression (6) to calculate the loss rate:

$$\frac{1}{n\tau} \approx \frac{1}{n^2}\frac{\partial n}{\partial t} = \frac{4\pi}{n^2}\int_0^\infty dv \; v^2 \; S(v)$$

$$= 2\pi\Gamma\mathscr{I}\left[\frac{2\pi T}{m}\right]^{-3/2} \tag{7}$$

Under the change of variables $x = mv^2/2T$, Eq.(5) becomes ($S = 0$):

$$x\frac{\partial^2 g}{\partial x^2} - x\frac{\partial g}{\partial x} + \frac{1}{4}\frac{\partial}{\partial \mu}(1-\mu^2)\frac{\partial}{\partial \mu} g = 0 \tag{8}$$

where $g(x, \mu) \equiv \bar{g}(v, \mu)$ with

$$g(0,\mu) = 1 \tag{9}$$

For a function $g(x, \mu)$ satisfying Eqs (8) and (9) and, additionally,

$$\lim_{x\to\infty} g\, e^{-x} = 0 \tag{10}$$

the variational integral takes the value

$$\mathscr{I} = \int\int dx d\mu \; e^{-x}\left[g_x^2\right.$$

$$\left. + \frac{1}{4x}(1-\mu^2)\; g_\mu^2\right] = -\int d\mu g_x(0,\mu) \tag{11}$$

3. BOUNDARY CONDITIONS

In addition to satisfying Eqs (9) and (10), the function $g(x, \mu)$ must satisfy two more conditions. The first condition follows from a purely kinematic constraint, i.e. if both kinetic energy and magnetic moments of single ions are conserved it follows that no ion having

$$\frac{x}{x_o}\left[1 - (1-\mu^2)R\right] > 1 \tag{12}$$

where $x_0 \equiv \epsilon\phi/T$; ϕ = end potential > 0; R = mirror ratio will be confined. Hence, on the boundary of this region in velocity space

$$\frac{x}{x_o}\left[1 - (1-\mu^2)R\right] = 1 \tag{13}$$

we require $g(x, \mu) = 0$.

Finally, since the force felt by an ion whose velocity vector $\vec{v}$ makes an angle θ with the axis of symmetry, $\vec{B}$, is the same as that experienced by an ion whose velocity vector makes an angle $\pi - \theta$ to $\vec{B}$, we require that $g(x, \mu) = g(x, -\mu)$.

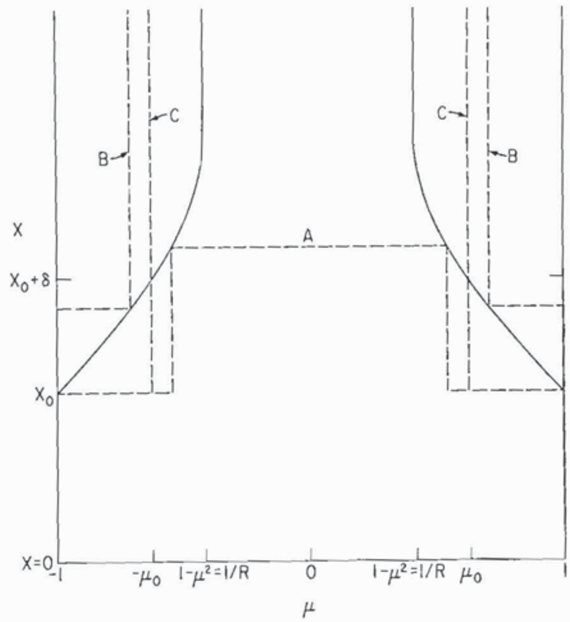

FIG.1. Actual and model loss boundaries:
A = pessimistic model;
B = optimistic model;
C = compromise model.

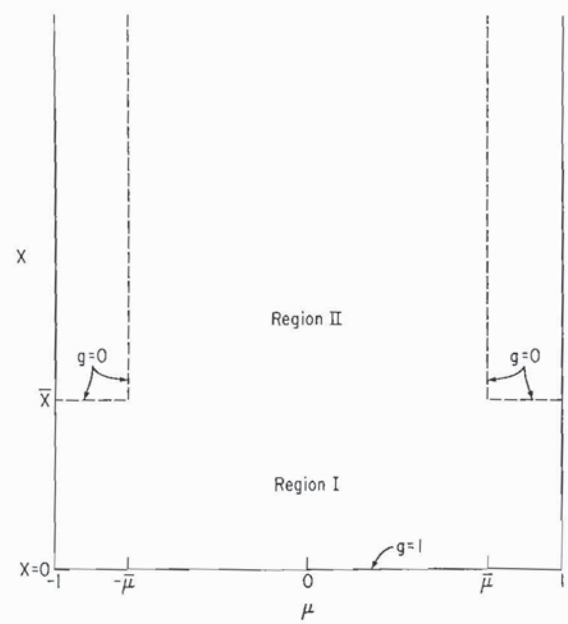

FIG.2. Sample model boundary value problem.

To summarize: The problem is to calculate

$$\mathscr{I} = - \int_{-1}^{1} d\mu \; g_x(o,\mu) \tag{14}$$

for $g(x, \mu)$ satisfying

$$x \frac{\partial^2 g}{\partial x^2} - x \frac{\partial g}{\partial x} + \frac{1}{4} \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial g}{\partial \mu} = 0$$

$$g(0,\mu) = 1; \qquad \lim_{x\to\infty} e^{-x} g(x,\mu) = 0 \tag{15}$$

$$g(x,\mu) = 0 \text{ on } \frac{x}{x_o} [1 - (1-\mu^2)R] = 1$$

$$g(x,\mu) = g(x,-\mu)$$

The region of interest in velocity space appears in Fig.1.

4. ESTIMATE OF LOSS RATE

Owing to the shape of the loss boundary, we have been unable to solve Eq.(15) exactly. Consequently, it would be desirable to use a simple model of the loss boundary such that the problem may be solved by separating x and μ variables. Such a boundary appears as a dotted line (marked A) in Fig.1. Clearly, by its use, one would *over*-estimate the actual loss. It is also possible, by the use of a model boundary inscribed in the actual boundary (dotted line B, Fig.1), to obtain a *lower* bound on the loss rate.

Let us consider first a single step boundary like B of Fig.1. We proceed by dividing that part of velocity space in which g is non-vanishing into two regions as shown in Fig.2. Region I (low-velocity region) is defined by

$$x < \bar{x}$$

$$|\mu| \leq 1 \tag{16a}$$

and Region II by

$$x > \bar{x}$$

$$|\mu| \leq \bar{\mu} \tag{16b}$$

and on the common boundary we require

$$g_I(\bar{x},\mu) = g_{II}(\bar{x},\mu) \quad |\mu| \leq 1 \qquad (17a)$$

$$\frac{\partial}{\partial x} g_I(\bar{x},\mu) = \frac{\partial}{\partial x} g_{II}(\bar{x},\mu) \; |\mu| \leq \bar{\mu} \qquad (17b)$$

where it will be convenient to define $g_{II}(x, \mu) \equiv 0$ for $|\mu| \geqslant \bar{\mu}$.

If it were possible to perform this matching (17) exactly and to obtain the value of $\mathscr{I}$ even for these simplified boundary conditions, the use of a circumscribed boundary would, as stated, give an upper bound on the loss rate and the use of an inscribed boundary, a lower bound. However, at least for the case $R \gg \bar{x}$, $x_0 \gg 1$, which we shall assume to hold in the remainder of this paper, these results appear to differ only slightly owing to the exponential dependence of the Maxwellian on energy, as shown later in Appendix C.

In any case, as is typical of matching problems, this one cannot be exactly solved. It is only transformed into a different problem involving inversion of an infinite matrix. To make any further progress we are forced to evaluate matrix elements and to do this inversion in an approximate manner and in such a way that we cannot say with certainty whether our result is too high or too low. Since the optimistic and pessimistic models give very similar results we shall for simplicity choose an intermediate model as shown in Fig. 1, curve C. Here we have chosen $1 - \bar{\mu}^2 \equiv 1 - \mu_0^2$ such that the lines $\mu = \pm \mu_0$ intersect the actual loss boundary at $x_0 + \delta$, with δ a number of order $1 (\delta \ll x_0)$, i.e. the lines $\mu = \mu_0$ intersect the loss-boundary δ "thermal units" of energy above the point at which the distribution function peaks:

$$1 - \mu_o^2 \approx \delta/Rx_o << 1 \qquad (18)$$

Since most of the loss occurs near $x = x_0$, a value for δ of ~ 1 should give good results in what follows. In fact, the final result will depend very weakly (logarithmically) on the exact choice of δ. As a practical matter, δ could be matched to numerical results.

We now write, for our model problem

$$g_I(x,\mu) = \sum_{n=0}^{\infty} a_{2n}(x) P_{2n}(\mu) \qquad (19)$$

$$g_{II}(x,\mu) = \sum_{n=0}^{\infty} b_{2n}(x) H_{2n}(\mu;\mu_o)$$

where the P_{2n}'s are Legendre polynomials satisfying

$$\frac{d}{d\mu}(1-\mu^2)\frac{d}{d\mu}P_{2n} + \alpha_{2n}P_{2n} = 0$$

$\alpha_{2n} = 2n(2n + 1)$ and the H_{2n}'s are functions satisfying

$$\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu}H_{2n}(\mu;\mu_o)$$

$$+ \beta_{2n}H_{2n}(\mu;\mu_o) = 0 \qquad (20a)$$

with

$$H_{2n}(\mu_o;\mu_o) = 0, \; H_{2n}(-\mu;\mu_o)$$

$$= H_{2n}(\mu;\mu_o)$$

These functions may be written

$$H_{2n}(\mu;\mu_o) = h_{2n}(\mu_o)\left[\frac{P_\nu(\mu)}{P_\nu(\mu_o)}\right.$$

$$\left. - \frac{Q_\nu(\mu)}{Q_\nu(\mu_o)}\right] \quad |\mu| \leq \mu_o \qquad (20b)$$

where $\beta_{2n} = \nu(\nu + 1)$ may be shown, using standard identities, to satisfy

$$\frac{\pi}{2}\frac{P_\nu(\mu_o)}{Q_\nu(\mu_o)} = \tan\frac{\nu\pi}{2} \qquad (21)$$

We choose to define

$$H_{2n}(\mu;\mu_o) = 0 \text{ for } |\mu| > \mu_o$$

and define the normalization $h_{2n}(\mu_0)$ such that

$$\int_{-\mu_o}^{\mu_o} d\mu \; H^2_{2n}(\mu;\mu_o) = 1$$

The expansions (19), when substituted in the kinetic equation, will be a solution to Eq.(8) if and only if

$$x \frac{d^2 a_{2n}}{dx^2} - x \cdot \frac{da_{2n}}{dx} - \frac{1}{4}\alpha_{2n} a_{2n} = 0$$

$$x \frac{d^2 b_{2n}}{dx^2} - x \frac{db_{2n}}{dx} - \frac{1}{4}\beta_{2n} b_{2n} = 0$$

These equations are standard forms of the confluent hypergeometric (Kummer's) equation. Their solution gives the following general solution for the distribution function $g(x, \mu)$ satisfying Eqs (9) and (10):

$$g_I(x,\mu) = 1 + \sum_{n=0}^{\infty} A'_{2n} \left[\frac{x\Phi(1 + \frac{1}{4}\alpha_{2n}, 2; x)}{x_o\Phi(1 + \frac{1}{4}\alpha_{2n}, 2; x_o)} \right] P_{2n}(\mu) \qquad (22)$$

$$g_{II}(x,\mu) = \sum_{n=0}^{\infty} B'_{2n} \left[\frac{x\Psi(1 + \frac{1}{4}\beta_{2n}, 2; x)}{x_o\Psi(1 + \frac{1}{4}\beta_{2n}, 2; x_o)} \right] H_{2n}(\mu;\mu_o)$$

where $\Phi(a,b;x)$ and $\Psi(a,b;x)$ are the confluent hypergeometric functions in the notation of Erdélyi et al. [8] and where the A'_{2n} and B'_{2n} are constants. From expression (14) we have

$$\mathscr{I} = \frac{-2A'_o}{x_o\Phi(1,2;x_o)} = \frac{-e^{-x_o/2}}{\sinh(x_o/2)} A'_o \qquad (23)$$

The matching conditions (17) become

$$A'_{2n} = (2n + \tfrac{1}{2}) \sum_{m=o}^{\infty} B'_{2m} J_{nm} - \delta_{n,o} \qquad (24)$$

$$B'_{2n} = \frac{1}{E_{2n}} \sum_{m=o}^{\infty} A'_{2m} D_{2m} J_{mn}$$

where

$$J_{nm} \equiv \int_{-\mu_o}^{\mu_o} d\mu P_{2n}(\mu) H_{2m}(\mu;\mu_o)$$

$$D_{2n} \equiv \frac{\partial}{\partial x_o} \ln\left[x_o\Phi(1 + \tfrac{1}{4}\alpha_{2n}, 2; x_o)\right]$$

$$E_{2n} \equiv \frac{\partial}{\partial x_o} \ln\left[x_o\Psi(1 + \tfrac{1}{4}\beta_{2n}, 2; x_o)\right] \qquad (25)$$

Defining the additional quantities

$$A_{2n} = \sqrt{\frac{D_{2n}}{2n + 1/2}} \; A'_{2n}$$

$$B_{2n} = \sqrt{-E_{2n}} \; B_{2n}$$

The matching conditions, in matrix form, are written

$$A = \mathcal{M} B + v$$

$$B = -\mathcal{M}^T A \qquad (26)$$

where

$$\mathcal{M}_{mn} = \left[\frac{(2m + 1/2) D_{2m}}{-E_{2n}}\right]^{1/2} J_{mn}$$

$$v_n = -\sqrt{2D_o} \; \delta_{n,o} \qquad (27)$$

$$(\mathcal{M}^T)_{nm} = \mathcal{M}_{mn}$$

Formally, then, it follows that the variational integr[al] is given by

$$\mathscr{I} = \frac{e^{-x_o/2}}{\sinh(x_o/2)} (I + \mathcal{M}\mathcal{M}^T)^{-1}_{oo} \qquad (28)$$

where I is the unit matrix and the matrix element o[n] the right-hand side of expression (28) is the "zero-zero" element of the indicated inverse. It is to be stressed that expression (28) is strictly a formal result since it is in general unknown how to invert an infinite matrix. The problem, in a nutshell, is to do an effective job approximating this inverse for the case of large x_0 and R. This we have found to be extremely difficult to do. After an extended effort we are able only to obtain the value of

$$(I + \mathcal{M}\mathcal{M}^T)^{-1}_{oo}$$

to within a relative error of order

$$[\ln x_o/(\ln R + \ln x_o)]^2$$

Hence, our result is good only in the extreme limit $R \gg x_0 \gg 1$. However, we feel that this is still worth reporting since the paper of Pastukhov [5] includes no error analysis; in fact, it is far from clear how to discuss error at all for his method.

We proceed by defining the variational quantity

$$Q[A] = \frac{A^T[I + \mathcal{M}\mathcal{M}^T]A}{(A^T v)^2} \tag{29}$$

Its minimum value is

$$Q_{min} = \left[v^T(I + \mathcal{M}\mathcal{M}^T)^{-1} v\right]^{-1}$$

$$= \left[v_o^2 \; (I + \mathcal{M}\mathcal{M}^T)^{-1}_{oo}\right]^{-1} \tag{30}$$

Therefore

$$\mathcal{I} = \frac{e^{-x_o/2}}{\sinh(x_o/2)} \quad \frac{1}{v_o^2 \; Q_{min}} \tag{31}$$

We propose to minimize Q[A] approximately as follows: Writing out Q[A], we have

$$A_o^2 v_o^2 Q = \sum_{n=0}^{\infty} A_{2n}^2 + \sum_{m=0}^{\infty} \left[\sum_{n=0}^{\infty} \mathcal{M}_{nm} A_{2n}\right]^2 \tag{32}$$

In the second (double) sum we propose to separate out what appear (and may later be shown) to be the largest terms. These turn out to be the m = 0 and m = n terms. Roughly speaking, we may say that we include exactly the effects of the lowest mode in Region II and include only approximately the effects of higher modes. The essential difficulty of this problem stems from the fact that the lowest mode is *not* strongly dominant over the totality of higher modes though it is more important (i.e. contributes more to Q_{min}, hence to $\mathcal{I}$) than any other single mode.

We now have

$$A_o^2 v_o^2 Q = \sum_{n=0}^{\infty} A_{2n}^2 + \left[\sum_{n=0}^{\infty} \mathcal{M}_{no} A_{2n}\right]^2 + \sum_{m=1}^{\infty} \mathcal{M}_{mm}^2 A_{2m}^2 + \mathcal{R} \tag{33}$$

with the remainder

$$\mathcal{R} = \sum_{m=1}^{\infty} \left\{\left[\sum_{n=0}^{\infty} \mathcal{M}_{nm} A_{2n}\right]^2 - \mathcal{M}_{mm}^2 A_{2m}^2\right\} \tag{34}$$

which will be estimated in Appendix A.

Choosing $A_0 = 1$ (Q is independent of the normalization of the vector A) we minimize v_0^2Q-R to find

$$A_{2n} = \frac{-\mathcal{M}_{oo}\mathcal{M}_{no}}{S(1 + \mathcal{M}_{nn}^2)} ; n = 1,2,\ldots \tag{35}$$

where

$$S = 1 + \sum_{m=1}^{\infty} \frac{\mathcal{M}_{mo}^2}{1 + \mathcal{M}_{mm}^2} \tag{36}$$

and

$$\left[v_o^2 Q - \mathcal{R}\right]_{min} = 1 + \mathcal{M}_{oo}^2 / S \tag{37}$$

In Appendix A it is shown that with A_n as determined above

$$\frac{\mathcal{R}}{\mathcal{M}_{oo}^2} \sim 0\ [\beta_o \ln x_o]^2$$

$$S \sim 0(1)$$

We conclude

$$\mathcal{I} = \frac{2e^{-x_o}}{1 + \mathcal{M}_{oo}^2\ S^{-1}} \left[1 + 0\ [\beta_o \ln x_o]^2\right] \tag{38}$$

For the matrix element $\mathcal{M}_{00}$ we have

$$\mathcal{M}_{oo} \equiv \left[\frac{D_o}{-2E_o}\right]^{1/2} \int_{-\mu_o}^{\mu_o} d\mu\ H_o(\mu;\mu_o)$$

Using the results of Appendix B for D_0 and E_0 gives

$$D_o \sim 1 + 0(1/x_o)$$

$$-E_o \sim \beta_o/4x_o\ [1 + 0(1/x_o)]$$

Using (20a) it is possible to estimate $H_0(\mu;\mu_0)$ for $|\mu| \leqslant \mu_0$

$$H_o(\mu;\mu_o)$$

$$= h_o(\mu_o)\left[1 + \frac{1}{2}\beta_o \ln(1-\mu^2) + 0(\beta_o^2)\right]$$

from which it follows,

$$\int_{-\mu_o}^{\mu_o} d\mu H_o(\mu;\mu_o) = \sqrt{2}\ \left[1 + 0(\beta_o^2)\right]$$

(The positive root has been chosen arbitrarily for $h_0(\mu_0)$.) so that

$$\mathcal{M}_{oo}^2 = \frac{4x_o}{\beta_o}\ [1 + 0(1/x_o)] \tag{39}$$

β_0 may be obtained from the eigenvalue condition (21). It is

$$\beta_o = -2\Big/[\ln\left((1-\mu_o^2)/4\right) + 2]$$

$$+ 0[1/\ln^3(1-\mu_o^2)] \tag*{(40}$$

Finally, from Appendix A

$$S = 1 + \frac{1}{2}\ \beta_o\ [\ln x_o$$

$$+ .54 + 0(1/x_o)]$$

Collecting our results gives the source strength

$$\mathcal{I} = \frac{e^{-x_o}}{x_o}\ \frac{\beta_o}{2}\left[1 + \frac{1}{2}\beta_o \ln 1.7x_o\right.$$

$$\left. + 0[\beta_o \ln x_o]^2\right] \tag*{(41}$$

[We may now verify our claim that use of the inscribed and circumscribed boundaries shown in Fig.1 will affect the result (41) only slightly. In order not to interrupt the discussion, this calculation is placed in Appendix C].

Finally the loss rate is

$$\left(\frac{1}{n^2}\frac{dn}{dt}\right)_{inj} = 2\left(\frac{2\pi}{mT}\right)^{1/2}\frac{\varepsilon^3}{\phi}\ \frac{\ln\Lambda}{\ln y_o}\ e^{-x_o}$$

$$\times\left[1 + \frac{\ln 1.7x_o}{\ln y_o} + 0\left(\frac{\ln x_o}{\ln y_o}\right)^2\right] \tag*{(4}$$

where $x_0 = \epsilon\phi/T$; $y_0 = x_0(4R/e^2\delta)$ with $\delta \sim O(1)$.

In Appendix C, we show that an upper bound for δ is $\ln x_0$, while a lower bound is $[\ln Rx_0]^{-1}$. A reasonable compromise then is to put

$$\delta = \left[\frac{\ln x_o}{\ln Rx_o}\right]^{1/2} \tag*{(42}$$

This is similar to the result of Ref.[5] as calculated there for electron loss from *two*-component plasma in which ion-electron collisions involving pitch angle scattering are included. Here, though, the potential ϕ appears in the argument of the logarithm. For a single component plasma, Pastukhov's result for the loss rate is modified; apart from small factors in the limit of large R, Pastukhov's result would be *halved*. Therefore, Eq.(42) predicts roughly *twice* the loss rate as would be given by the suitably modified version of Pastukhov's result in the limit of large R. We believe there is an error of a factor of two in Eq.(17) [9] of Ref.[5] which, when corrected, gives the following result for the one component plasma:

$$\left(\frac{1}{n^2}\frac{dn}{dt}\right)_{inj} = 2\left(\frac{2\pi}{mT}\right)^{1/2} \frac{\varepsilon^3}{\phi} \ln\Lambda \quad \frac{R}{R+1} \frac{e^{-\varepsilon\phi/T}}{\ln(2R+2)} \times \left[1 + \frac{T}{2\varepsilon\phi} - \left(\frac{T}{2\varepsilon\phi}\right)^2 + \ldots\right] \tag{43}$$

and renders our results in agreement, at least for very large values of R.

5. NUMERICAL RESULTS

Results of a Fokker-Planck code for ion-ion (D-D) scattering have been reported to us by R. Cohen and M.E. Rensink [10]. For the parameters $x_0 = 3.61$, $R = 10$, the code gives $(n\tau)_{ion} = 4.27 \times 10^{14}\ cm^{-3} \cdot s$ while Eq.(43) gives $(n\tau)_{ion} = 4.2 \times 10^{14}\ cm^{-3} \cdot s$ which is remarkably good agreement.

Our result, Eq.(42), using (42a) ($\delta = 0.6$) is $(n\tau)_{ion} = 3.2 \times 10^{14}\ cm^{-3} \cdot s$ which is more pessimistic than either Eq.(43) or the code results. The best upper bound (C-14) [for $(n\tau)^{-1}$] gives $\delta \sim \ln x_0 = 1.28$ and $(n\tau)_{ion,\,min} = 2.3 \times 10^{14}\ cm^{-3} \cdot s$ and the best lower bound (C-17) gives $\delta = 0.22$ and $(n\tau)_{ion,\,max} = 4.4 \times 10^{14}\ cm^{-3} \cdot s$. A value of $\delta = 0.24$ matches the numerical result of Cohen and Rensink. This is smaller than we would have expected for good agreement. It is possible, however, that the error term in Eq.(42) is larger than we would like.

[After this work was completed, we were informed of some earlier numerical work of McHarg and Oakes [12] who solve the initial value problem for the Fokker-Planck equation. For such a problem, they define a confinement time as the time required for the initial plasma to decay to half the initial density. By empirically fitting their data they postulate a formula for $(n\tau)^{-1}$ which closely resembles Eq.(42) to leading order with $\delta = 0.36$.

We wish to thank Dr. Paul J. Channell for bringing this paper to our attention.]

6. CONCLUSION

An independent calculation has been given of the singly charged ion loss rate from an end-stoppered mirror in which both the magnetic field strength and electrostatic potential are "square wells." Our result for the linearized problem agrees to about 30% with both numerical results and the earlier work of Pastukhov, after correction of a factor of two in this latter work.

ACKNOWLEDGEMENTS

We would like to thank Dr. P.J. Channell and Dr. J.A. Krommes for many useful discussions, and Dr. R. Cohen and Dr. M.E. Rensink for correspondence and discussions regarding this problem and especially for providing us with a summary of their numerical work.

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APPENDIX A

In this Appendix we evaluate certain matrix elements needed in the text and calculate the remainder, $\mathscr{R}$, in Eq.(33). We require the following quantities: $\mathscr{M}_{n0}(n = 0,1,\ldots.)$, $\mathscr{M}_{nn}(n = 1,2,\ldots)$, and

$$S = 1 + \sum_{n=1}^{\infty} \mathcal{M}_{no}^2 / (1 + \mathcal{M}_{nn}^2) \tag{A.1}$$

where (see expressions (27))

$$\mathcal{M}_{mn} = \left[\frac{(2m + 1/2) D_{2m}}{-E_{2n}}\right]^{1/2} \times \int_{-\mu_o}^{\mu_o} d\mu P_{2m}(\mu) H_{2n}(\mu;\mu_o) \tag{A.2}$$

We are interested in the behaviour of these quantities in the case

$$x_o \to \infty\ ;\ 1-\mu_o^2 \to 0;\ x_o(1-\mu_o^2) \to 0$$

The first result we record is that of Appendix B:

$$D_{2m} = \frac{1}{2}\left[1 + (1+\alpha_{2m}/x_o)^{1/2}\right] \times \left[1 + 0(1/x_o)\right]$$

$$E_{2n} = \frac{1}{2}\left[1 - (1+\beta_{2n}/x_o)^{1/2}\right] \times \left[1 + 0(1/x_o)\right] \tag{A.3}$$

Next we consider the integrals [see expressions (25)]:

$$J_{mn} = \int_{-\mu_o}^{\mu_o} d\mu\, P_{2m}(\mu)\, H_{2n}(\mu;\mu_o) \tag{A.4}$$

Referring to the explicit expression of the $H_{2n}(\mu;\mu_0)$ in terms of Legendre functions (20b), we find, using standard integrals

$$J_{mn} \approx \frac{\frac{4}{\pi}(-1)^{n+1}\sin\frac{\nu_n\pi}{2}\sqrt{\nu_n+1/2}}{\left[1-\frac{4}{\pi^2}\sin^2\frac{\nu_n\pi}{2}\,\psi'(1+\nu_n)\right]^{1/2}(2m-\nu_n)(2m+\nu_n+1)} \tag{A.5}$$

We then find that

$$J_{oo} = \sqrt{2} + 0[\beta_o^2]$$

$$J_{mo} = -\frac{1}{4}\sqrt{2}\,\frac{1}{m(m+1/2)}\,\beta_o + 0[\beta_o^2]$$

where (see (21), (40))

$$\beta_o = -2/\left[\ln\left((1-\mu_o^2)/4\right) + 2\right] + 0[\beta_o^3]$$

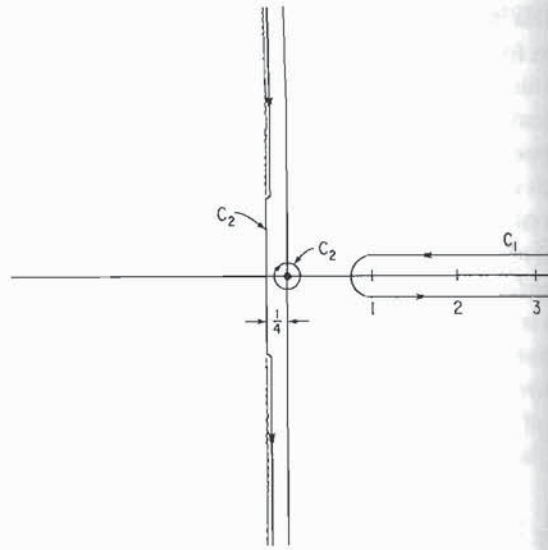

FIG.A-1. Contours for (A-7): C_1 = original contour; C_2 = deformed contour. The branch points are at $1/4\,[-1 \pm i(4x_0 - 1)^{1/2}]$.

For the integrals J_{mm} we need an estimate of the eigenvalues ν_{2m}. It follows from the eigenvalue condition (21) that, for large m, $\nu_{2m} \sim 2m$. Since ν_0 is very small, we estimate J_{mm} for $m \geqslant 1$ by using the limit of (A.5) as $\nu \to 2m$ or

$$J_{mm} \approx (2m + 1/2)^{-1/2}$$

It now follows that the sum S is approximately

$$S \approx 1 + \frac{1}{2}\beta_o \sum_{n=1}^{\infty} \frac{2n+1/2}{n(n+1/2)} \times \left[1 + 4n(n+1/2)/x_o\right]^{-1/2} \tag{A.6}$$

The sum appearing in expression (A.6) has asymptotic behaviour:

$$\sum_{n=1}^{\infty} \frac{2n+1/2}{n(n+1/2)} \times \left[1 + 4n(n+1/2)/x_o\right]^{-1/2} = \ln x_o + C + 0(1/x_o)$$

with $C = 2[\ln 2 - 1 + \gamma] = 0.540$ as may be found by considering the corresponding integral

$$\frac{1}{2\pi i} \int_{C_1} dz \, \pi \cot(\pi z) \, \frac{2z+1/2}{z(z+1/2)} \times \left[1+4z(z+1/2)/x_0\right]^{-1/2} \tag{A.7}$$

around the contours of Fig.A-1. The contribution to expression (A.7) from along the branch cut is exponentially small as $x_0 \to \infty$, from the pole at 0 is $-2 + 1/x_0$, and from the gap between the branch points is $\ln x_0 + 2(\ln 2 + \gamma) + O(1/x_0)$ where the integral

$$\int_0^\infty dt \, \frac{1}{e^{4\pi t}+1} \, \frac{t}{t^2+a^2} = \frac{1}{2}\left[\psi(2a+1/2) - \ln(2a)\right]$$

is useful in evaluating this last contribution.

Finally we discuss the remainder $\mathcal{R}$ in the expression for Q (33), (34). The issue is how accurately, for any $m \geq 1$,

$$\sum_n \mathcal{M}_{nm} A_{2n} \approx \mathcal{M}_{mm} A_{2m}$$

We claim

$$\sum_n \mathcal{M}_{nm} A_{2n} = \mathcal{M}_{mm} A_{2m} \left[1 + O[\beta_0 \ln x_0]\right] \tag{A.8}$$

and further

$$\frac{1}{\mathcal{M}_{00}^2} \sum_{m=1}^{\infty} \mathcal{M}_{mm}^2 A_{2m}^2 \lesssim O[\beta_0 \ln x_0] \tag{A.9}$$

It follows that

$$\frac{\mathcal{R}}{\mathcal{M}_{00}^2} \lesssim O[\beta_0 \ln x_0]^2 \tag{A.10}$$

The result (A.8) follows from consideration of the sum

$$\sum_{n\neq m} \frac{\mathcal{M}_{nm} A_{2n}}{\mathcal{M}_{mm} A_{2m}} \approx \sqrt{1+\alpha_{2m}/x_0} \, \frac{\beta_0}{2} \sum_{\substack{n=1 \\ n\neq m}}^{\infty} \times \frac{2n+1/2}{(m-n)(m+n+1/2)} \left[1+4n(n+1/2)/x_0\right]^{-1/2} \tag{A.11}$$

The sum on the right-hand side of expression (A.11) may be done for large x_0 in a way quite similar to that used in expressions (A.6), (A.7). The result is that for $x_0 \gg m^2$ the right-hand side of (A.11) behaves as

$$-\frac{\beta_0}{2}\left[\ln x_0 + 2\ln 2 - \frac{1}{2m+1/2} + \frac{1}{2m(m+1/2)} - 2\psi(1+2m) + O(m^2/x_0) + O(1/x_0)\right]$$

For $m^2 \gg x_0 \gg 1$ the righthand side is

$$O[\beta_0 \sqrt{x_0}/m] + O[\beta_0/m] + O[\beta_0/x_0]$$

and for $m^2 \sim x_0 \gg 1$. The right-hand side is $O[\beta_0]$. For all m, then, the right-hand side of expression (A.11) is at most $O[\beta_0 \ln x_0]$.

The result (A.9) follows from

$$\frac{1}{\mathcal{M}_{00}^2} \sum_{m=1}^{\infty} \mathcal{M}_{mm}^2 A_{2m}^2 < \frac{1}{\mathcal{M}_{00}^2} \sum_{m=1}^{\infty} (\mathcal{M}_{mm}^2+1) A_{2m}^2 = \frac{S-1}{S^2} \sim O[\beta_0 \ln x_0]$$

and (A.6).

APPENDIX B

We require representations for the functions

$$D(x) = \frac{\partial}{\partial x} \ln[x\Phi(1+\tfrac{1}{4}\alpha, 2; x)] \tag{B.1}$$

$$E(x) = \frac{\partial}{\partial x} \ln[x\Psi(1+\tfrac{1}{4}\beta, 2; x)] \tag{B.2}$$

valid when x is large compared to 1 but not necessarily compared to α or β.

Consider first D(x). Using standard identities (e.g. Erdélyi, loc. cit.)

$$D(x) = 1 + \frac{\Phi(\tfrac{1}{4}\alpha, 1; x)}{x\Phi(1+\tfrac{1}{4}\alpha, 2; x)}$$

The following integral representation is given by Magnus, Oberhettinger, and Soni [11]:

$$\Phi(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)} \int_0^\infty dt\ t^{a-1} (xt)^{\frac{1-c}{2}}$$

$$\times\ e^{-t}\ I_{c-1}(2\sqrt{xt})$$

$$= 2\frac{\Gamma(c)}{\Gamma(a)}\ x^{-a} \int_0^\infty ds\ s^{2a-c}\ e^{-s^2/x}\ I_{c-1}(2s) \tag{B.3}$$

If x is large (compared to 1) the major contribution to expression (B-3) will occur for large s. Using

$$I_{c-1}(2s) \sim \frac{e^{2s}}{2\pi^{\frac{1}{2}} s^{\frac{1}{2}}}\ [1+O(1/s)]$$

we obtain for (B-3)

$$\frac{1}{\pi^{\frac{1}{2}}} \frac{\Gamma(c)}{\Gamma(a)}\ x^{-a} \int ds\ s^{-c-\frac{1}{2}}\ e^{f(s)},$$

$$\times\ [1+O(1/s)] \tag{B.4}$$

where $f(s) = -\frac{s^2}{x} + 2s + 2a\ln s$

The point of steepest descent is

$$s^* = \frac{x}{2}[1+\sqrt{1+4a/x}\ ] \tag{B.5}$$

A standard calculation then gives

$$\Phi(a,c;x) \sim$$

$$\frac{\Gamma(c)}{\Gamma(a)}\ x^{a-c}\ \frac{[\tfrac{1}{2}(1+\sqrt{1+4a/x})]^{2a-c}}{[1+4a/x]^{\frac{1}{4}}}$$

$$\times \exp[\tfrac{x}{2}(1-\frac{2a}{x} + \sqrt{1+4a/x})]\ [1+O(1/x)] \tag{B.6}$$

For $x \gg a$, expression (B.6) reduces to

$$\Phi(a,c;x) \sim \frac{\Gamma(c)}{\Gamma(a)}\ x^{a-c}\ e^x\ [1+O(1/x)]$$

which is the usual result. For $a \gg x \gg 1$ we find

$$\Phi(a,c;x) \sim \frac{\Gamma(c)}{2\pi^{\frac{1}{2}}}\ (ax)^{\frac{1}{4}-c/2}$$

$$\times\ e^{x/2}\ e^{2\sqrt{ax}}\ [1+O(1/a)]$$

which also agrees with the usual result to leading order.

For large x, then, we find, for *all* α,

$$D(x) \sim \tfrac{1}{2}[1+\sqrt{1+\alpha/x}\ ]\ [1+O(1/x)] \tag{B.7}$$

A similar calculation may be done for E(x) using [8

$$\Psi(a,c;x)$$

$$= \frac{1}{\Gamma(a)} \int_0^\infty dt\ e^{-xt}\ (1+t)^{c-2} \left[\frac{t}{1+t}\right]^{a-1} \tag{B.8}$$

The point of steepest descent, t^*, is a solution to

$$-x + (a-1)\left[\frac{1}{t} - \frac{1}{1+t}\right] = 0$$

hence

$$t^* = \tfrac{1}{2}\left[-1+\sqrt{1+4(a-1)/x}\,\right] \tag{B.9}$$

It follows that

$$\Psi(a,c;x) \sim \frac{1}{x\Gamma(a)}\sqrt{\frac{\pi(a-1)}{t^*+\frac{1}{2}}}\,[t^*]^{a-1}$$

$$\times\,[1+t^*]^{c-a-1}\,e^{-xt^*}[1+O(1/x)] \tag{B.10}$$

For the case $x \gg a$, expression (B.10) gives

$$\Psi(a,c;x) \sim x^{-a}\,[1+O(1/x)]$$

and for $a \gg x$ we find

$$\Psi(a,c;x) \sim \tfrac{1}{2}\sqrt{2}\,(a/x)^{c/2-1/4}$$

$$\times\, a^{-a} e^{a-2\sqrt{ax}+x/2}[1+O(a^{-1/2})]$$

which agree with standard results.

Using expression (B.10) and an identity for Ψ we now write, for large x and all β:

$$E(x) \equiv -\tfrac{1}{4}\beta\,\frac{\Psi(1+\frac{1}{4}\beta,1;x)}{x\Psi(1+\frac{1}{4}\beta,2;x)}$$

$$\sim \tfrac{1}{2}\left[1-\sqrt{1+\beta/x}\right]\,[1+O(1/x)] \tag{B.11}$$

APPENDIX C

In this Appendix we consider the "step" boundary of the form shown in Fig.1, curve A, as well as the inscribed (optimistic) boundary of curve B. We argue

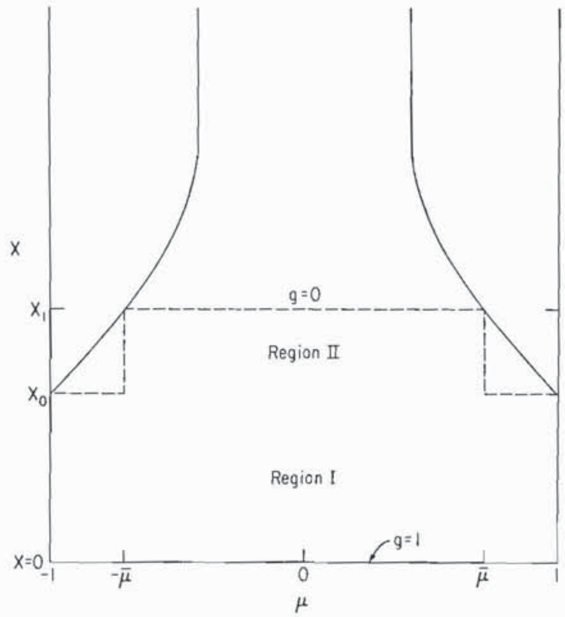

FIG.C-1. A "pessimistic"-model loss boundary.

that their use would modify expression (41) by quantities smaller than the error already present in this expression for the source strength.

Consider first Fig.1, curve A. The boundary value problem of interest is displayed in Fig. C.1. We propose to choose x_1 such that $\mathscr{I}$ is minimized and, using this "best" step size, show that $\mathscr{I}$ is essentially that given in expression (41).

We expand $g(x,\mu)$ as follows:

Region I:

$$g(x,\mu) = 1 + \sum_{n=0}^{\infty} A'_{2n}\left[\frac{x\Phi(1+\frac{1}{4}\alpha_{2n},2;x)}{x_0\Phi(1+\frac{1}{4}\alpha_{2n},2;x_0)}\right]P_{2n}(\mu) \tag{C.1a}$$

Region II:

$$g(x,\mu) = \sum_{n=0}^{\infty} B'_{2n}\left[\frac{x_1\Psi(1+\frac{1}{4}\beta_{2n},2;x_1)}{x_0\Psi(1+\frac{1}{4}\beta_{2n},2;x_0)}\right]$$

$$\times\left[\frac{x\Psi(1+\frac{1}{4}\beta_{2n},2;x)}{x_1\Psi(1+\frac{1}{4}\beta_{2n},2;x_1)} - \frac{x\Phi(1+\frac{1}{4}\beta_{2n},2;x)}{x_1\Phi(1+\frac{1}{4}\beta_{2n},2;x_1)}\right]H_{2n}(\mu;\bar{\mu}) \tag{C.1b}$$

After some algebra exactly analogous to that of (24)–(27) the matching conditions at $x = x_0$ may be written

$$A = \mathcal{M}B + V$$

$$B = -\mathcal{M}^T A \qquad \text{(C.2)}$$

where

$$A_{2n} = A'_{2n}\left[\frac{2n+\frac{1}{2}}{D^{\alpha}_{2n}}\right]^{-\frac{1}{2}}$$

$$B_{2n} = B'_{2n}\left[(-E_{2n}+\rho_{2n}D^{\beta}_{2n})(1-\rho_{2n})\right]^{\frac{1}{2}}$$

$$D^{\xi}_{2n} = \frac{\partial}{\partial x_0}\ln\left[x_0\Phi(1+\tfrac{1}{4}\xi_{2n},2;x_0)\right]$$

$$\xi = \alpha,\beta$$

$$E_{2n} = \frac{\partial}{\partial x_0}\ln\left[x_0\Psi(1+\tfrac{1}{4}\beta_{2n},2;x_0)\right]$$

$$\rho_{2n} = \frac{\Psi(1+\frac{1}{4}\beta_{2n},2;x_1)}{\Psi(1+\frac{1}{4}\beta_{2n},2;x_0)}\,\frac{\Phi(1+\frac{1}{4}\beta_{2n},2;x_0)}{\Phi(1+\frac{1}{4}\beta_{2n},2;x_1)}$$

$$\mathcal{M}_{nm} = \left[\frac{(2n+\frac{1}{2})D^{\alpha}_{2n}(1-\rho_{2m})}{-E_{2m}+\rho_{2m}D^{\beta}_{2m}}\right]^{\frac{1}{2}} \times \int_{-\bar{\mu}}^{\bar{\mu}} d\mu\; P_{2n}(\mu)H_{2m}(\mu;\bar{\mu})$$

$$V_n = -\sqrt{2D^{\alpha}_0}\,\delta_{n,0} \qquad \text{(C.3)}$$

Note that if $x_1 \to \infty$ with x_0 fixed, $\rho_{2n} \to 0$ and the original problem is recovered.

The variational integral is given by

$$\mathcal{J} = \frac{-2A'_0}{x_0\Phi(1,2;x_0)} = \frac{e^{-x_0/2}}{\sinh(x_0/2)}\,(I+\mathcal{M}\mathcal{M}^T)^{-1}_{00} \qquad \text{(C.4)}$$

as in (28) but with $\mathcal{M}$ now given in expression (C.3).

Proceeding now through steps exactly parallel to (29)–(38) we write down our approximation to $\mathcal{J}$ (see expression (38)):

$$\mathcal{J} \approx \frac{2e^{-x_0}}{1+\mathcal{M}^2_{00}S^{-1}} \qquad \text{(C.5)}$$

where S is defined in (36):

$$S = 1 + \sum_{n=1}^{\infty}\frac{\mathcal{M}^2_{n0}}{1+\mathcal{M}^2_{nn}} \qquad \text{(C.6)}$$

Since the sum S is O(1) we first study

$$\mathcal{M}^2_{00} = \frac{\frac{1}{2}D^{\alpha}_0(1-\rho_0)}{-E_0+\rho_0D^{\beta}_0}\left[\int_{-\bar{\mu}}^{\bar{\mu}} d\mu\; H_0(\mu;\bar{\mu})\right]^2$$

$$= \frac{D^{\alpha}_0(1-\rho_0)}{-E_0+\rho_0D^{\beta}_0}\left[1+O(\beta_0^2)\right] \qquad \text{(C.7)}$$

where we have used our previous evaluation of the integral. We may also use the results of Appendix B to write

$$\mathcal{M}^2_{00} \approx \frac{1-\rho_0}{\frac{\beta_0}{4x_0}+\rho_0} \qquad \text{(C.8)}$$

where

$$\beta_0 \approx \frac{-2}{\ln c(1-\bar{\mu}^2)} = \frac{-2}{\ln\left[c\,\frac{x_1-x_0}{Rx_1}\right]} \qquad \text{(C.9)}$$

with

$$c = e^2/4$$

and where we have used the equation for the loss boundary, Eq.(13).

Using asymptotic forms for Φ and Ψ we find

$$\rho_0 = e^{(x_0-x_1)}\left[1+O(\beta_0)\right] \qquad \text{(C.10)}$$

Next, defining $\delta = x_1 - x_0$, we look for a maximum of $\mathcal{M}_{00}^2$ for $\delta/x_0 \ll 1$ for large x_0. Hence, we replace Rx_1 by Rx_0 in the argument of the logarithm in (C.9). Setting

$$\frac{\partial}{\partial\delta}\mathcal{M}_{00}^{2} = 0 \tag{C.11}$$

we find

$$\frac{1-\rho_0}{\rho_0}\frac{\partial}{\partial\delta}\ln\beta_0 - \frac{4x_0}{\beta_0} = 1 \tag{C.12}$$

Solving for $\rho_0 = e^{-\delta}$ and using $\beta_0' = (1/2\delta)\beta_0^2$ gives

$$\delta = \ln\left[1+\frac{2\delta}{\beta_0}\left(1+4x_0/\beta_0\right)\right] \tag{C.13}$$

which may be iterated for large x_0 to give

$$\delta \simeq \ln x_0 + \ln\ln x_0 + 2\ln\ln Rx_0 + O(1) \tag{C.14}$$

so that

$$\rho_0 \sim \frac{(\mathrm{const})}{x_0\ln x_0[\ln Rx_0]^2}$$

and finally

$$\frac{\mathcal{M}_{00}^{2}}{4x_0} \sim \frac{1}{\beta_0}\left[1 + O\left(\frac{1}{\beta_0\ln x_0[\ln Rx_0]^2}\right)\right]$$

$$\sim \frac{1}{\beta_0}\left[1 + O\left(\frac{1}{\ln x_0\ln Rx_0}\right)\right] \tag{C.15}$$

where the error in (C.15) is much smaller than that incurred in our original minimization procedure [see expression (33) ff.]. To leading order, then, (C.15) reproduces (39), the result for boundary C, Fig.1 (but with $\delta \sim \ln x_0$ instead of $\delta \sim 1$). Further calculation shows that (C.14) is indeed a maximum.

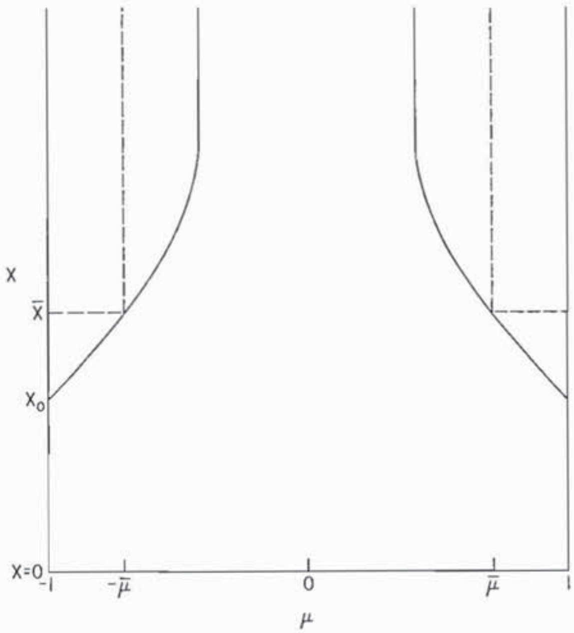

FIG.C-2. Lower bound on loss: "optimistic"-model boundary.

Next we consider the "optimistic" boundary of Fig.C.2. We intend to choose $\bar{x} - x_0$ so as to find the best (greatest) lower bound of $\mathcal{J}$, the variational integral. Using the result (41) to lowest order

$$\mathcal{J} \approx \frac{e^{-\bar{x}}}{\bar{x}}\frac{\beta_0}{2} \tag{C.16}$$

where

$$2/\beta_0 = -\ln c(1-\bar{\mu}^2) = \ln[R\bar{x}/c(\bar{x}-x_0)]$$

and we have again used the equation for the loss boundary, (13).

We anticipate a solution with $\bar{x} - x_0 = \delta \ll x_0$. Maximization of $\mathcal{J}$ then gives an equation for this best value of δ to use for a lower bound:

$$\delta \ln[Rx_0/c\delta] = 1 \tag{C.17}$$

Our conclusion is that the use of any one of the three model boundaries of Fig.1 for the purpose of estimating the ion loss rate gives a result of the form (42), the only difference being in the value used for the parameter δ, which affects the result only weakly.

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[9] Pastukhov's Equation (17) should read

$$F = \pi^{-3/2} e^{-x^2} + \pi \int_a^\infty q e^{-\xi^2} \Psi \xi^2 d\xi$$

He has a factor 2π instead of π in front of the integral. This change has the effect of changing his "sink" strength q, to

$$q = \frac{4}{\pi^{3/2} x_o^3 \ln(4R+2)}$$

or just doubling the value given in Eq.(18) of his paper.

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