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ABSTRACT

A formalism is presented for wave reflection for a slowly varying spatially inhomogeneous thermal plasma described by the Vlasov equation. The formalism generalizes a method originated by Bremmer for differential wave equations. In a numerical example we show that the intrinsic thermal properties of the plasma can supply reflection mechanisms that compete with the reflection coefficient predicted for a simple inhomogeneous fluid.

PLASMA WAVE REFLECTION IN SLOWLY VARYING MEDIA

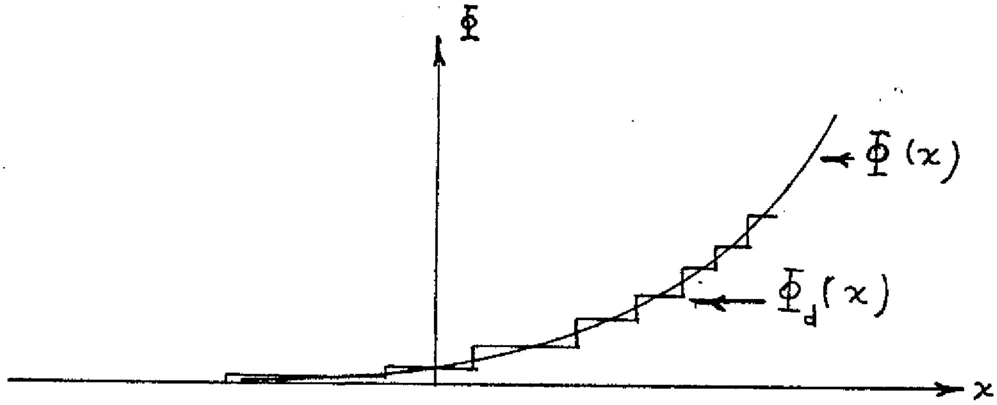
I. INTRODUCTION

Recent investigations of the POST-ROSENBLUTH loss cone instability[1,2,3] have given rise to the question of how reflection of convectively unstable waves in mirror machines can affect stability criteria. In the usual description, one expects that waves generated in the center of the machine are Landau damped at the ends. Hence if the axial wavelength in the device is long, (typically more than 1/10 of the machine length) the wave amplitude does not grow to a level dangerous enough to cause particle loss. However, to these considerations reflection effects due to spatial inhomogeneity should be considered. Although the reflection coefficient can be expected to be exponentially small if the wavelength is much less than the plasma length, the reflected wavelets are themselves exponentiated due to the plasma instability. Thus it may still be possible that the reflection coefficient in the center of the device is of order unity or greater, in which case the system will have a noise level detrimental to particle containment.

AAMODT and BOOK[3] have already treated this problem starting from fluid equations. However, since the effects of reflection might be determined by Landau damping and other non-fluid behavior, we shall attempt to develop here the mathematical formalism for the reflection problem starting from the Vlasov equation. In this paper we shall develop the formalism for a stable plasma. At a later date, application to the Post-Rosenbluth instability will be presented.

Now let us consider the propagation and reflection of plasma waves in a spatially inhomogeneous, but slowly varying, plasma medium in a strong magnetic field. An external potential $\Phi(x)$, which simulates a confining magnetic field plus any static electric potential arising from the charged particle equilibrium, is used to maintain a decreasing electron density along the magnetic field. The ions are considered in the infinite mass limit and effects of their motion are neglected.

In order to calculate the wave propagation, we approximate the continuous potential, $\Phi(x)$, by a discontinuous potential $\Phi_d(x)$ as shown in Fig. 1. The potential, $\Phi(x)$, is taken as zero at $-\infty$ and is monotonically increasing.



Smooth potential $\Phi(x)$ and step potential approximation $\Phi_d(x)$

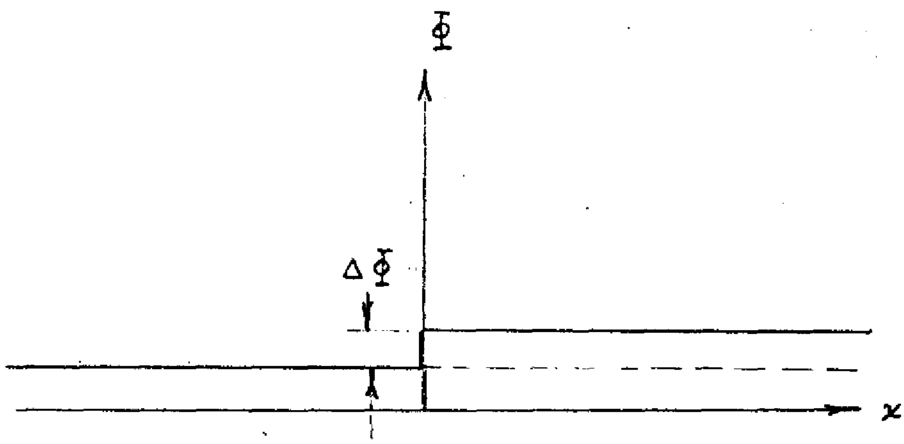
Fig. 1

We assume that in the neighborhood of each discontinuity, we can solve the Vlasov-Poisson equations as if the medium is uniform on each side of the discontinuity (i.e., we neglect the other discontinuities). This enables us to calculate the transmission and reflection coefficients at each separate discontinuity. The overall transmitted and reflected wave is then obtained by evaluating the superposition of waves transmitted through and reflected from each of the discontinuities in the limit $\Phi_d(x) \rightarrow \Phi(x)$.

This method has been used by BREMMER and others[3,4,5] to obtain wave propagation and reflection from a system of differential equations. In these cases, it can be shown that under certain restrictions[5,6,7] the "Bremmer method" produces an exact solution to the differential equation. On the other hand, for the Vlasov equation, it is difficult to demonstrate if the Bremmer method yields in principle an exact solution to the problem. However, we show that the lowest order transmitted wave obtained by our generalized Bremmer method yields the same W.K.B. result obtained from a more direct calculation[8] and we expect from physical intuition and agreement with special cases that the expression we obtain for the reflected wave properly describes the scattering due to local gradients.

II. SINGLE STEP PROBLEM

Let us now compute the transmission and reflection coefficients of a plasma wave, propagating at an angle to the magnetic field, where the external potential $\Phi(x)$ is chosen as a step discontinuity (see Fig. 2). The time-varying perturbed potential, ϕ , is of the form $\phi = \phi_0(x)\, exp(ik_\perp y - i\omega t)$ where y is the spatially uniform direction perpendicular to the magnetic field and $k_\perp$ is the wave number component in the perpendicular direction. We assume that in the x direction the incoming wave propagates to the right and we look for outgoing waves, which propagate to the left for $x < 0$ and to the right for $x > 0$.



Step discontinuity

Fig. 2

The linearized Vlasov equation for the step discontinuity, $\Phi(x) = \Delta\Phi\ \Theta(x)$, in a strong magnetic field is

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\delta(x)\Delta\Phi\frac{\partial f}{\partial v} - \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial F(E)}{\partial v} = 0 \tag{1}$$

where F and f are the equilibrium and perturbed distribution functions averaged over their perpendicular velocities, v is the particle velocity parallel to the magnetic field, $E = \frac{e}{m}\Phi(x) + \frac{v^2}{2}$ is the normalized parallel particle energy, and e and m are the particle charge and mass.

The oscillating potential ϕ satisfies the Poisson equation

$$\frac{\partial^2 \phi}{\partial x^2} - k_\perp^2 \phi = -4\pi e n_0 \int f \, dv \qquad (2)$$

where n_0 is the equilibrium density for $x < 0$.

The equilibrium Vlasov equation requires that the equilibrium distribution depend only on E.

Eqs. (1) and (2) are solved by perturbation theory by considering the external potential $\Delta \Phi$ small. Hence we take

$$f = f^{(0)} + f^{(1)} + \ldots$$

$$\phi = \phi^{(0)} + \phi^{(1)} + \ldots$$

$$F = F^{(0)} + F^{(1)}$$

$$\Phi = 0 + \Delta \Phi \, \theta(x)$$

To lowest order, Eqs. (1) and (2) become the well-known equations for a spatially homogeneous medium.

$$\frac{\partial f^{(0)}}{\partial t} + v \frac{\partial f^{(0)}}{\partial x} - \frac{e}{m} \frac{\partial \phi^{(0)}}{\partial x} \frac{\partial F^{(0)}}{\partial v} = 0 \qquad (3)$$

$$\frac{\partial^2 \phi^{(0)}}{\partial x^2} - k_\perp^2 \phi^{(0)} = -4\pi e n_0 \int f^{(0)} dv \qquad (4)$$

These equations allow the propagation of a wave

$$\phi = \phi_0 \exp(-i\omega t + i k_\perp y + i k_\parallel x)$$

where ω, $k_\perp$, $k_\parallel$ satisfy the dispersion relation

$$\epsilon(\omega, k_\parallel) = 1 - \frac{\omega_p^2}{k_\perp^2 + k_\parallel^2} \int \frac{dv \, \frac{\partial F^{(0)}}{\partial v}}{(v - \omega/k_\parallel)} = 0 \qquad (5)$$

where $\omega_p^2 = \frac{4\pi n_0 e^2}{m}$

We choose $Re\ \omega > 0$ and $Re\ k_\parallel > 0$ so that the wave is moving to the right. The oscillating distribution function is given by

$$f^{(0)} = i\frac{e}{m} \frac{k_\parallel \ \phi^{(0)} \ \frac{\partial F^{(0)}}{\partial v}}{(-i\omega + i k_\parallel v)} \tag{6}$$

In Appendix A, it is indicated that $k_\parallel$ should be taken in the upper half of the complex plane. For a stable plasma, this criterion is automatically satisfied when ω is real. For an unstable plasma, $k_\parallel$ is in the lower half plane, for real ω. The proper scattering behavior is then obtained if ω is first treated in the upper half plane, above the roots of $\epsilon\ (\omega, k_\parallel)$. For this case, $k_\parallel$ is in the upper half plane. The transmission and reflection coefficients are then obtained as a function of complex ω and then analytically continued for ω real.

To first order in $\Delta\Phi$, the equations become

$$\frac{\partial f^{(1)}}{\partial t} + v\frac{\partial f^{(1)}}{\partial x} - \frac{e}{m}\frac{\partial \phi^{(1)}}{\partial x}\frac{\partial F^{(0)}}{\partial v} = \frac{e}{m}\frac{\partial \phi^{(0)}}{\partial x}\frac{\partial F^{(1)}}{\partial v} + \frac{e}{m}\delta(x)\,\Delta\Phi\,\frac{\partial f^{(0)}}{\partial v} \tag{7}$$

$$\frac{\partial^2 \phi^{(1)}}{\partial x^2} - k_\perp^2\,\phi^{(1)} = 4\pi e n_0 \int f^{(1)}\,dv \tag{8}$$

The function $F^{(1)}$, determined from the relation $F^{(0)} + F^{(1)} = F\left(\frac{v^2}{2} + \Phi(x)\right)$ is given by

$$F^{(1)} = \Delta\Phi\ \frac{\theta(x)}{v}\frac{\partial F}{\partial v} \tag{9}$$

In Fourier transform space, Eqs. (7) and (8) are

$$
\begin{aligned}
(-i\omega + ikv) f_k^{(1)} &- \frac{ie}{m} k \phi_k^{(1)} \frac{\partial F^{(0)}}{\partial v} \\
&= \frac{e}{m} C \left[\frac{1}{k-k_{\parallel}} \frac{\partial}{\partial v} \left(\frac{1}{v} \frac{\partial F^{(0)}}{\partial v} \right) - \frac{\partial}{\partial v} \left(\frac{\frac{\partial F^{(0)}}{\partial v}}{\omega - k_{\parallel} v} \right) \right] \\
&= \frac{e}{m} C \frac{\partial}{\partial v} \left[\frac{(\omega - kv) \frac{\partial F}{\partial v}}{v\,(k-k_{\parallel})(\omega - k_{\parallel} v)} \right],
\end{aligned}
\tag{10}
$$

$$
(k^2 + k_{\perp}^2)\, \phi_k = 4\pi e n_0 \int f_k \, dv \tag{11}
$$

where $\quad f_k = \int_{-\infty}^{\infty} dx \; e^{-ikx} \; f(x)$

and $\quad C = \frac{e}{m} k_{\parallel} \phi_0 \Delta \Phi \, e^{ik_{\perp} y} .$

We can readily solve for f_k and ϕ_k, and obtain

$$
\begin{aligned}
f_k^{(1)} = \; & \frac{-\frac{e}{m} k \phi_k^{(1)}}{(\omega - kv)} \frac{\partial F^{(0)}}{\partial v} \\
& + \frac{ie}{m} \frac{C}{(\omega - kv)(k - k_{\parallel})} \frac{\partial}{\partial v} \left[\frac{(\omega - kv)}{v\,(\omega - k_{\parallel} v)} \frac{\partial F^{(0)}}{\partial v} \right],
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\phi_k^{(1)} &= -i\omega_p^2 C \int \frac{dv}{(\omega - kv)} \frac{\partial}{\partial v} \left[\frac{(\omega - kv)}{v(\omega - k_{\parallel} v)} \frac{\partial F^{(0)}}{\partial v} \right] \\
&= \frac{i\omega_p^2 C k}{(k^2 + k_{\perp}^2)(k - k_{\parallel})\, \epsilon(\omega, k)} \int \frac{dv \, \frac{\partial F^{(0)}}{\partial v}}{v\,(\omega - kv)(\omega - k_{\parallel} v)}
\end{aligned}
\tag{13}
$$

When $\phi_k^{(1)}$ is transformed to x-space, we have

$$\phi^{(1)}(x) = \int \frac{dk}{2\pi} \; e^{ikx} \; \phi_k^{(1)}$$
$$= i\omega_p^2 C \int_{-\infty}^{\infty} \frac{dk \; k \; e^{ikx} \displaystyle\int_{-\infty}^{\infty} \frac{dv \; \frac{\partial F^{(0)}}{\partial v}}{v(\omega - kv)(\omega - k_{\parallel} v)}}{2\pi \; (k^2 + k_\perp^2) \; (k - k_{\parallel}) \; \epsilon(\omega, k)} \tag{14}$$

With this integral, we can evaluate the wave response far from the source $(|kx| >> 1)$. The k contour can be distorted into the complex k plane so that $e^{ikx} \to 0$ (we distort into the upper half plane for $x > 0$, and the lower half plane for $x < 0$). The singularities that are first encountered are a double pole at $k = k_{\parallel}$ when $x > 0$ and a single pole at $k = -k_{\parallel}$ for $x < 0$. The residue of these poles is the coherent wave response. In addition to the wave response, it is known that additional singularities in $\epsilon(\omega, k)$ generate fields that decay on a faster scale than the wave decrement $1/\mathrm{Im}\, k_{\parallel}$.[9] However, this field does not decay exponentially, so that at very large distances from the source, these fields dominate the wave field. However, here we shall assume that we can neglect all fields other than the coherent response. Similarly, we neglect the poles at $k = v/\omega$ since these terms produce only rapidly decaying transients.

If now the residue of the double pole at $k = k_{\parallel}$ when $x > 0$ is evaluated, we find

$$\phi^{(1)}(x) = \frac{-\omega_p^2 C \; e^{ik_{\parallel} x}}{\frac{\partial \epsilon}{\partial k}(\omega, k_{\parallel}) \; (k_{\parallel}^2 + k_\perp^2)} \left[\frac{\frac{\partial^2 \epsilon}{\partial k^2}(\omega, k_{\parallel})}{2 \frac{\partial \epsilon}{\partial k}(\omega, k_{\parallel})} \int \frac{dv \; \frac{\partial F^{(0)}}{\partial v}}{v(\omega - k_{\parallel} v)^2} - \int \frac{dv \; \frac{\partial F^{(0)}}{\partial v}}{(\omega - k_{\parallel} v)^3} + ix \int \frac{dv \; \frac{\partial F^{(0)}}{\partial v}}{v(\omega - k_{\parallel} v)^2} \right] \tag{15}$$

The first two terms in the bracket are the change in amplitude of the forward wave, while the coefficient of $ix\phi_0$ in the last term is the wave number shift, Δk, of the incident wave when $x > 0$.

Since first-order perturbation theory only produces terms proportional to $\Delta\Phi$, the wave number shift which to all orders in perturbation theory would appear in the form $e^{i(k+\Delta k)x}$, appears here in the form $e^{ikx}(1+i\Delta kx)$. Equation (15) suggests that Δk is given by

$$\Delta k = -\frac{e}{m}\frac{k_{\parallel}^2\omega_p^2 \quad \Delta\Phi}{(k_{\parallel}^2+k_{\perp}^2)\ \frac{\partial\epsilon}{\partial k_{\parallel}}(\omega,k_{\parallel})} .$$

This indeed is the lowest order Δk that is obtained by setting the local dielectric function to zero,

$$\epsilon(\omega,\ k,\ \Phi(x)) = 1 - \frac{\omega_p^2}{k^2+k_{\perp}^2}\int_{-\infty}^{\infty}\frac{dv\ \frac{\partial F}{\partial v}\left(\frac{v^2}{2}+\frac{e}{m}\Phi(x)\right)}{(v-\omega/k)} \tag{16}$$

The remaining terms in the l.h.s. of Eq. (15) define the transmission coefficient. Combining the solution of zeroth and first order perturbation theory, we see that the transmission coefficient, $1+\delta(x=0)$, at $x=0$, is

$$1+\delta(0) = 1 + \frac{\phi^{(1)}(0)}{\phi_0}$$

$$= 1 - \frac{e}{m}\frac{\omega_p^2 k_{\parallel}^2\Delta\Phi\ e^{ik_{\perp}y}}{(k_{\parallel}^2+k_{\perp}^2)\ \frac{\partial\epsilon}{\partial k_{\parallel}}(\omega,k_{\parallel})} \tag{17}$$

$$\cdot\left[\frac{\frac{\partial^2\epsilon}{\partial k_{\parallel}^2}}{2\frac{\partial\epsilon}{\partial k_{\parallel}}}\int\frac{dv\ \frac{\partial F^{(0)}}{\partial v}}{v(\omega-k_{\parallel}v)^2} - \int\frac{dv\ \frac{\partial F^{(0)}}{\partial v}}{(\omega-k_{\parallel}v)^3}\right].$$

If we now consider $x<0$, the evaluation of the residue at $k=-k_{\parallel}$ yields the following expression for the reflected wave:

$$\phi^{(1)}(x) = \frac{C\omega_p^2\ e^{-ik_{\parallel}x}}{2(k_{\parallel}^2+k_{\perp}^2)\ \frac{\partial\epsilon}{\partial k}(\omega,-k_{\parallel})}\int dv\ \frac{\partial F^{(0)}}{\partial v}\left[v(\omega+k_{\parallel}v)(\omega-k_{\parallel}v)\right]^{-1}$$

$$= \frac{-C\ e^{-ik_{\parallel}x}\ \epsilon(0,k_{\parallel})}{2\omega^2\ \frac{\partial\epsilon}{\partial k}(\omega,-k_{\parallel})} \tag{18}$$

We have used the relation

$$-\omega_p^2 \int \frac{dv \, \frac{\partial F^{(0)}}{\partial v}}{v(\omega + k_{\parallel} v)(\omega - k_{\parallel} v)} = \frac{(k_{\parallel}^2 + k_{\perp}^2)}{\omega^2} \quad \epsilon(0, k_{\parallel})$$

which is obtained if use is made of the relations $\epsilon(\omega, \pm k_{\parallel}) = 0$.

Note that changes in the reference potential of the single step problem, i.e., $\Delta\Phi\,\Theta(x) \to \Phi_0 + \Delta\Phi\,\Theta(x)$, and the position of the discontinuity from $x = 0$ to $x = x_i$, alters the preceding results given by Eqs. (15) and (18) only in that $F^{(0)} \to F^{(0)}(\frac{v^2}{2} + \frac{e\Phi_0}{m})$ and $x \to x - x_i$. The dielectric function $\epsilon(\omega, k)$ appearing in the solutions is then changed to $\epsilon(\omega, k, \Phi_0)$, defined by Eq. (16).

III. CONTINUOUS PROBLEM

We are now in a position to apply the results of the single step problem to the problem of a continuously varying static potential. Now imagine that the incident wave propagates through a potential $\Phi_d(x)$ consisting of N steps, and approximating a continuous potential $\Phi(x)$ as shown in Fig. 1. We assume, as in the work of Bremmer, that the incident wave at each point in space is determined to first approximation only by the transmission at each singularity. Obviously, this assumption neglects the additional effects of multiple reflections. With this assumption, the incident wave $\phi(x)$ between the jth and (j + 1)th discontinuities is given by

$$\phi(x) = \phi_0 \, e^{i k_{\perp} y} \left[\prod_{i=1}^{j} (1 + \delta(x_i)) \right] \cdot \exp\left[i \sum_{i=1}^{j} k_i (x_i - x_{i-1}) + i k_j x + i k_0 x_0 + i \beta \right] \tag{19}$$

Here, $\phi_0 \, e^{i k_0 x + i\beta}$ is the incident field when $x = x_0$

(the position of the first discontinuity) x_i $(i=0, 1, \ldots N-1)$ is the position of each discontinuity, k_i is the local wave number determined by the relation $\epsilon(\omega, k_i, \Phi_d(x)) = 0$, $1 + \delta(x_i)$ is the transmission coefficient at x_i , and β is an arbitrary phase factor which, for convenience, is chosen here so that the phase of the incident wave is ultimately zero at $x=0$.

Proceeding to the limit $\Phi_d(x) \rightarrow \Phi(x)$, where each discontinuity becomes arbitrarily small, and an infinite number of them arise, we see that the phase factor becomes the integral $\int_0^x k(x')\,dx'$

and the product $\prod_{i=1}^{j} (1+\delta(x_i))$ becomes

$$\lim \left(\prod_{i=1}^{j} (1+\delta(x_i)) \right) = \exp\left[\int_{-\infty}^{x} dx'\, \sigma(x') \right], \tag{20}$$

where $\sigma(x)$ is the limit of $\dfrac{\delta(x')}{\Delta x}$ when $\Phi_d(x) \rightarrow \Phi(x)$.

At the point x , the transmitted wave is then given by

$$\phi(x) = \phi_0 \exp\left[i \int_0^x k(x)\,dx + \int_{-\infty}^{x} dx\, \sigma(x) \right]. \tag{21}$$

From Eq. (15) and the limit relation $\Delta\Phi = \dfrac{\partial \Phi}{\partial x}\,dx$ we have

$$\delta(x_i) \rightarrow \sigma(x_i)\,dx_i = \frac{dx_i\, \Phi'(x_i)\, \frac{e}{m}\, \omega_p^2\, k_\parallel^2}{(k_\parallel^2 + k_\perp^2)\, \frac{\partial \epsilon}{\partial k}(\omega, k_\parallel)} \cdot \left[\frac{\frac{\partial^2 \epsilon(\omega, k_\parallel)}{\partial k^2}}{2\, \frac{\partial \epsilon}{\partial k}(\omega, k_\parallel)} \int \frac{dv\, \frac{\partial F}{\partial v}\left(\frac{v^2}{2} + \frac{e}{m}\Phi(x)\right)}{v\,(\omega - k_\parallel v)^2} - \int \frac{dv\, \frac{\partial F}{\partial v}}{(\omega - k_\parallel v)^3} \right] \tag{22}$$

We shall establish below that

$$\sigma(x) = \frac{d}{dx} \ln \left(\frac{\partial}{\partial k_\parallel} \epsilon(\omega, k_\parallel, \Phi(x)) \right)^{1/2} . \qquad (23)$$

Upon substituting this equation into Eq. (21), we find that the transmitted wave is given by

$$\phi(x) = \phi_0 \left(\frac{\frac{\partial \epsilon}{\partial k}(\omega, k_\parallel(-\infty), 0)}{\frac{\partial \epsilon}{\partial k}(\omega, k_\parallel(x), \Phi(x))} \right)^{1/2} e^{i \int_0^x k_\parallel x + i k_\perp y} . \qquad (24)$$

This is the same answer that one obtains from a direct W.K.B. calculation of the Vlasov equation[8].

In order to establish Eq. (23), we note that $\frac{\partial}{\partial k} \epsilon(\omega, k_\parallel, \Phi)$ depends on x through $k_\parallel$ and Φ. If we then perform the derivative operation shown in Eq. (23), we obtain

$$\sigma(x) = \frac{-\left[\frac{dk_\parallel}{dx} \frac{\partial^2 \epsilon}{\partial k_\parallel^2} + \frac{d\Phi}{dx} \frac{\partial^2 \epsilon(\omega, k_\parallel, \Phi)}{\partial \Phi \partial k} \right]}{2 \frac{\partial \epsilon(\omega, k_\parallel, \Phi)}{\partial k}} \qquad (25)$$

Since $\epsilon(\omega, k_\parallel, \Phi) = 0$, the total derivative $\frac{d}{dx} \epsilon(\omega, k_\parallel, \Phi) = 0$ and therefore

$$\frac{dk_\parallel}{dx} = - \frac{d\Phi}{dx} \frac{\frac{\partial \epsilon}{\partial \Phi}}{\frac{\partial \epsilon}{\partial k}} . \qquad (26)$$

Upon substituting this result into Eq. (25), we obtain

$$\sigma(x) = - \frac{\Phi'(x)}{2 \frac{\partial \epsilon}{\partial k}(\omega, k_\parallel, \Phi)} \left[\frac{\partial \epsilon}{\partial \Phi} \frac{\partial^2 \epsilon}{\partial k_\parallel^2} - \frac{\partial^2 \epsilon}{\partial \Phi \partial k_\parallel} \right] . \qquad (27)$$

We confirm that $\sigma(x)$ is our desired expression, given by Eq. (22), when we substitute the relations,

$$\frac{\partial \epsilon}{\partial \Phi} = -\frac{e}{m}\,\omega_p^2 \int_{-\infty}^{\infty} \frac{dv\,\frac{\partial F}{\partial v}\left(\frac{v^2}{2} + \frac{e}{m}\Phi(x)\right)}{v\,(\omega - kv)^2} , \tag{28}$$

$$\frac{\partial^2 \epsilon}{\partial k\,\partial \Phi} = -2\frac{e}{m}\,\omega_p^2 \int_{-\infty}^{\infty} \frac{dv\,\frac{\partial F}{\partial v}}{(\omega - kv)^3} . \tag{29}$$

If a wave is initially travelling to the left with wave number $-k_{\parallel}$, and has an amplitude ϕ_1 at the point x_1, a similar analysis would yield the result

$$\phi(x) = \phi_1\, e^{i k_\perp y} \left(\frac{\frac{\partial \epsilon}{\partial k}(\omega, k_{\parallel}, \Phi(x_1))}{\frac{\partial \epsilon}{\partial k}(\omega, k_{\parallel}, \Phi(x))} \right)^{1/2} e^{-i\int_{x_1}^{x} k_{\parallel}\, dx} \tag{30}$$

where we have used the relation $\frac{\partial \epsilon}{\partial k}(\omega, k_{\parallel}) = -\frac{\partial \epsilon}{\partial k}(\omega, -k_{\parallel})$.

In order to calculate the reflected wave, we observe that the total field moving to the left at a point x arises from a superposition of each of the wavelets generated at each discontinuity to the right of x. In the neighborhood of each discontinuity x_i we see from Eq. (18) that the reflected wavelet is given by

$$\phi^{(1)}(x) = -dx_i\,\frac{e}{m}\, \frac{e^{i k_\perp y}\, \Phi'(x_i)\, k_{\parallel}\, \phi(x_i)\, \epsilon(0, k_{\parallel})\, e^{-i k_{\parallel} x}}{2\omega^2\, \frac{\partial \epsilon}{\partial k}(\omega, -k_{\parallel}, \Phi(x_i))} \tag{31}$$

Now each wavelet, once formed, is assumed to propagate without further reflection. Hence if one accounts for the alteration of the phase and amplitude of the wavelet, due to transmission effects, we see from the previous discussion concerning wave transmission that we should replace the factor $e^{-i k_{\parallel}(x - x_i)}$ by the factor

$$\left(\frac{\frac{\partial\epsilon}{\partial k}(\omega, k_{\parallel}, \Phi(x_i))}{\frac{\partial\epsilon}{\partial k}(\omega, k_{\parallel}, \Phi(x))}\right)^{1/2} e^{-i\int_{x_i}^{x} k_{\parallel}(x'')\,dx''} .$$

The field $\phi(x_i)$ is given by Eq. (24). If we now sum all the wavelets reaching the point x, we see that as $x \to -\infty$ the total reflected wave $\phi^{(1)}(x)$ is given by

$$\phi^{(1)}(x) = \frac{e\,\phi_0}{2m\omega^2} \exp\left(ik_\perp y - i\int_0^x dx'\,k_{\parallel}(x')\right) \cdot \int_{-\infty}^{\infty} \frac{dx'\; k_{\parallel}\; \Phi'(x')\; \epsilon(0, k_{\parallel})\; e^{2i\int_0^{x'} k_{\parallel} dx}}{\frac{\partial\epsilon}{\partial k}(\omega, k_{\parallel}, \Phi(x'))}$$

$$= -\frac{e}{2m\omega^2}\,\phi_0 \exp\left(ik_\perp y - i\int_0^x k_{\parallel}\,dx'\right) \cdot \int_{-\infty}^{\infty} \frac{dx\; k_{\parallel}}{\frac{\partial\epsilon}{\partial\Phi}} \frac{dk_{\parallel}}{dx}\,\epsilon(0, k_{\parallel})\, e^{2i\int_0^{x'} k\,dx} \tag{32}$$

where we have used Eq. (26).

The derivation of the reflected and transmitted waves has been heuristic and we shall not attempt a rigorous justification of the method. We have, however, several consistency checks for our method. We note that as in the Bremmer method for differential equations, the transmitted wave alone is the same as the lowest order W.K.B. solution. It can be shown that as $k \to 0$ and $k_\perp$, much less than the Debye wave number, the expression for the reflected wave, Eq. (32), approaches the lowest order Bremmer solution

$$\frac{\phi^{(1)}(x)}{\phi_0} \xrightarrow[k\to 0]{} \frac{e}{2}^{-i\int_0^x k_{\parallel} dx + ik_\perp y} \int_{-\infty}^{\infty} \frac{dx}{k_{\parallel}} \frac{dk_{\parallel}}{dx}\, e^{2i\int_0^x k_{\parallel} dx} \tag{33}$$

It is shown further in Appendix B that if a distribution function

$$F^{(0)} = \frac{1}{2u} \quad \Theta\left(\frac{u^2}{2} - E\right),$$

is used, an exact differential equation is obtained,

$$\frac{d}{dx}\left(v(x)\frac{d\mathcal{E}}{dx}\right) + \left(\frac{\omega^2}{v(x)} - \frac{\omega_p^2}{u}\right)\mathcal{E} = 0$$

where $\mathcal{E}$ is the electric field, $v(x) = \left[2\left(E - \frac{e}{m}\Phi(x)\right)\right]^{1/2}$, and $k_\perp = 0$.

The reflected wave obtained by the Bremmer method applied to this differential equation is found to be

$$\phi^{(1)}(x) = \phi_0 \, e^{-i\int^x k\,dx} \int_{-\infty}^{\infty} dx' \frac{\frac{d}{dx}\left(k(x)\,v(x)\right)}{2\,k(x)\,v(x)} \, e^{2i\int^{x'} k\,dx'} \tag{34}$$

An identical result is obtained from Eq. (32) when the dielectric function, $\epsilon(\omega, k) = 1 + \frac{\omega_p^2 v(x)}{v_0\left(k^2 v^2(x) - \omega^2\right)}$, is used.

Finally, it can be shown that in the case in which perturbation theory is applicable to a slowly varying potential, (i.e., when the following conditions are satisfied: $e\Phi << m v_{th}^2$ where $\Delta\Phi$ is the total change in the external potential, $\frac{1}{k^2}\frac{dk}{dx} << 1$, and $\Delta k L << 1$ where L is the range in which Φ changes) the resulting transmitted and reflected waves agree with the expressions derived here.

However, whereas for differential equations the Bremmer method can be continually iterated to produce an exact solution, for the Vlasov equation an exact solution cannot be obtained from such an iteration. There are two obvious reasons for the limitations of the Bremmer method in our application. The first is that the transmitted and reflected fields on either side of the discontinuity are not exactly wave fields as in differential equations, but only approach wave fields some distance from the discontinuity. Hence, to higher orders one might expect residual reflection and transmission due to incoherent components interacting with additional discontinuities.

Another limitation is that our perturbation theory does not properly describe the complete history of a particle in an actual system and additional non-local phenomena can perhaps affect the scattering. For example, how does the past history of a reflected particle prior to its reflection modify the wave reflection coefficient? An approach to answer this query is proposed in Appendix C.

IV. CALCULATION OF REFLECTION COEFFICIENT

We will now compute the reflection coefficient given by Eq. (32) for the case of a plasma with a Maxwellian distribution of electrons along the lines of force.

For a Maxwellian distribution function

$$F = (2\pi v_{th}^2)^{-1/2} \exp\left(\frac{-e\Phi}{m v_{th}^2} - \frac{1}{2}\frac{v^2}{v_{th}^2}\right), \tag{35}$$

the dielectric function given by Eq. (16) becomes

$$\epsilon = 1 - \frac{\omega_p^2\, e^{-\frac{e\Phi}{m v_{th}^2}}}{2(k_\perp^2 + k_\parallel^2)\, v_{th}^2}\, Z'\left(\frac{\omega}{k_\parallel v_{th}\sqrt{2}}\right) \tag{36}$$

where Z and $Z' = \frac{dZ(\zeta)}{d\zeta}$ are the functions tabulated by FRIED and CONTE[10]. Using Equations (26) and (32), the reflection coefficient can be written as

$$R = \frac{v_{th}^2}{2\omega^2}\int_C dk_\parallel\, k_\parallel\, \epsilon(0, k_\parallel) \exp\left[2i\int_{k_\parallel(x=-\infty)}^{k_\parallel} k_\parallel' \frac{dx}{dk_\parallel'}\, dk'\right] \tag{37}$$

where C is the contour determined by $\epsilon(k_\parallel, \Phi) = 0$ as the density goes from its initial value to zero. The contour is independent of the particular potential and is shown in Figure 3.

Since the phase of the integral is rapidly oscillating, the major contributions to the integral come from the points of stationary phase and from the end points. The saddle points are given by

$$k_\parallel \frac{dx}{dk_\parallel} = 0$$

which implies either

$$k_\parallel = 0 \quad \text{or} \quad \frac{dx}{dk_\parallel} \propto \frac{\partial \epsilon}{\partial k_\parallel} = 0.$$

The point $k_\parallel = 0$ is an essential singularity of ϵ, and its value at this point depends on the direction of approach. The saddle point at $k_\parallel = 0$ associated with that part of the contour C going down the imaginary axis gives zero contribution to the integral because the integrand vanishes. The condition $\frac{\partial \epsilon}{\partial k_\parallel} = 0$ has a denumerable infinity of roots.

Assuming $k_\perp^2 \gg k_\parallel^2$, $\frac{\partial \epsilon}{\partial k_\parallel} = 0$ at the roots of $Z''(\zeta) = 0$. These roots and the directions of steepest descent are located in the $k_\parallel$-plane by using the tables of the Z' function. The paths of steepest descent or stationary phase are found by following

$$\operatorname{Re} \int_{k_{\parallel, j}}^{k_\parallel} k_\parallel' \frac{dx}{dk_\parallel'} \, dk_\parallel' = 0$$

numerically from the jth saddle point $k_{\parallel, j}$. The paths do not cross the original contour C but instead start and end in the part of the plane where Z' is divergent. Portions of the paths from neighboring saddle points cancel, leaving the sum of the paths of steepest descent equivalent to a contour C_{SP} as shown in Fig. 3.

Thus the scheme for evaluating the integral is to deform the original path of integration C to a path going from $k_\parallel(x = -\infty)$ to $k_\parallel = 0$ and then through the saddle points on the paths of stationary phase. On the deformed path of integration, Eq. (37) becomes

$$R = \frac{V_{th}^2}{2\omega^2} \left\{ \int_{k_\parallel(x=-\infty)}^{0} dk_\parallel \, k_\parallel \, \epsilon(0, k_\parallel) \exp\left[2i \int_{k_\parallel(x=-\infty)}^{k_\parallel} k_\parallel' \frac{dx}{dk_\parallel'} \, dk_\parallel' \right] \right. +$$

$$\sum_{j=1}^{\infty} \int_{C_j} dk_\parallel \, k_\parallel \epsilon(0,k_\parallel) \exp\left[2i \int_{k_\parallel(x=-\infty)}^{k_\parallel} k_\parallel' \frac{dx}{dk_\parallel'} dk_\parallel'\right] \Bigg\} \qquad (38)$$

The integrals through the saddle points are easily evaluated. The contribution from the infinite sum is evaluated by converting the sum from $j = J$ to $j = \infty$, where J is sufficiently large, into an integral. This gives an algebraic quantity multiplied by the phase factor

$$\exp\left[2i \int_{k_\parallel(x=-\infty)}^{0} k_\parallel' \frac{dx}{dk_\parallel'} dk_\parallel'\right] .$$

The evaluation of the integral from $k_\parallel(x=-\infty)$ to $k_\parallel = 0$ depends in more detail on the potential.

We now consider the following potential,

$$\frac{e}{m v_{th}^2} \Phi(x) = \ln\left(1 + \delta e^{x/L}\right) , \qquad (39)$$

with the associated dispersion relation

$$2 k_\perp^2 \lambda_D^2 \left(1 + \delta e^{x/L}\right) = Z'\left(\frac{\omega}{k_\parallel v_{th} \sqrt{2}}\right) . \qquad (40)$$

The reflection coefficient is of the form

$$R = \sum_{j=1}^{\infty} a_j \exp\left[2i \int_{k_\parallel(x=-\infty)}^{k_{\parallel,j}} k_\parallel' \frac{dx}{dk_\parallel'} dk_\parallel'\right] \qquad (41)$$

where a_j is the residue of the jth saddle point. For this potential the integral from $k_\parallel(x=-\infty)$ to $k_\parallel = 0$ yields a term of the form

$$\exp\left[2i \int_{k_\parallel(x=-\infty)}^{0} k_\parallel' \frac{dx}{dk'} dk'\right]$$

which is identical to the result from the differential equation derived from fluid equations,

$$\frac{d^2\phi}{dx^2} + k_\perp^2 \frac{\omega^2}{\omega_p^2}\left(1+\delta e^{x/L}\right)\phi = 0 . \qquad (42)$$

We now introduce the approximation, $Z'(\zeta) = \zeta^{-2}$, which is best for small $k_\parallel$ (i.e., $k_\parallel \ll \omega/v_{th}$). At the largest saddle point, $K_{\parallel,1}$, and at $x = -\infty$ for $2k_\perp^2\lambda_D^2 \leq .10$, the error in the approximation is about 10%. Then, absorbing the real parts of the phase integrals in the coefficients a_j; we have

$$R = \sum_{j=1}^{\infty} a_j \exp\left\{-\frac{\sqrt{2}\,\omega L}{v_{th}}\left[2\,\mathrm{Im}(\kappa_j) + \sqrt{2k_\perp^2\lambda_D^2}\cdot \arg\left(\frac{\kappa_j - \sqrt{2k_\perp^2\lambda_D^2}}{\kappa_j + \sqrt{2k_\perp^2\lambda_D^2}}\right)\right]\right\} \qquad (43)$$

where κ is the dimensionless $k_\parallel$ variable defined by $\kappa = \frac{k_\parallel v_{th}\sqrt{2}}{\omega}$.

For very small $k_\perp^2\lambda_D^2$ the term arising from the saddle point at $k_\parallel = 0$ dominates the sum.

$$R \approx e^{-\sqrt{2}\frac{\omega L}{v_{th}}\sqrt{2k_\perp^2\lambda_D^2}\,\pi} \approx e^{-2\pi k_\parallel(x=-\infty)L} \qquad (44)$$

This is the reflection coefficient given by the fluid equations for this density function. For $2k_\perp^2\lambda_D^2 \geqslant .005$ the first term in the series dominates, and we obtain

$$R \approx \exp -\frac{\sqrt{2}\,\omega L}{v_{th}}\left[.2 + \sqrt{2k_\perp^2\lambda_D^2}\tan^{-1}\left(\frac{.2\sqrt{2k_\perp^2\lambda_D^2}}{.1 - 2k_\perp^2\lambda_D^2}\right)\right] \qquad (45)$$

using $\kappa_1 \simeq (.3, .1)$. For values of $k_\perp^2\lambda_D^2$ for which the approximation $Z'(\zeta) = \zeta^{-2}$ is valid this term varies as

$$R \approx e^{-.3\frac{\omega L}{V_{th}}} \tag{46}$$

For larger values of $K_\perp^2 \lambda_D^2$ the $\mathrm{Im}\int_{K_\parallel(x=-\infty)}^{K_{\parallel,1}} K_\parallel' \, dx/dK_\parallel' \, dK_\parallel'$ must be computed using the exact dispersion relation.

Thus, in this example, the formalism establishes the transition between the reflection due to the fluid behavior which dominates at very long wavelengths compared to V_{th}/ω and the thermal behavior which becomes important and even dominates at shorter wavelengths. For the unstable plasma, which will be analyzed in a later paper, it is the short wavelength regime, $K_\parallel^{-1} \approx V_{th}/\omega$, that gives the largest reflection coefficient and can cause a non-convective instability.

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APPENDIX A. COMMENT ON ANALYTIC CONTINUATION PRESCRIPTION

In the text we found that the response to a step function perturbation is of the form

$$\phi^{(1)}(x,t) = e^{-i\omega_0 t}\int_{-\infty}^{+\infty}\frac{dk_{\parallel}}{2\pi i}\cdot\frac{e^{ik_{\parallel}x}\,G(k_{\parallel},\omega_0)\,S_{\omega_0}}{\epsilon(\omega_0,k_{\parallel})\,(k_{\parallel}-k_{\parallel}(\omega_0))} \tag{A.1}$$

where ω_0 is a real frequency, $k_{\parallel}(\omega_0)$ is the wave number for the forward wave determined by the equation $\epsilon(\omega_0,k_{\parallel})=0$, and S_{ω_0} is proportional to the amplitude of the incident wave.

It has been observed that if the system is unstable, $k_{\parallel}(\omega_0)$ is in the lower half plane. If we believe Eq. (A.1) in its present form we see that for an unstable system we have a reflected wave to the right of the discontinuity and a transmitted wave to the left; a result that violates our boundary conditions.

In order to obtain the correct results it must be remembered that a problem must be posed with initial conditions present. If, for example, we assume we have a dipole source at the point $x_0 < 0$ whose time behavior is of the form $Q \propto \theta(t)\,e^{-i\omega_0 t}$, then after transients have died, waves with wave number $k_{\parallel}(\omega_0)$ propagate to the right and left of the source. The wave propagating to the right can be taken as $\phi = S_{\omega_0}\,e^{ik_{\parallel}(\omega_0)x-i\omega_0 t}$. It can then be shown that the perturbed field due to the step function at $x=0$ has the form

$$\phi^{(1)}(x,t) = \int_{C_\omega}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega t}}{\omega-\omega_0}\int_{-\infty}^{+\infty}\frac{dk_{\parallel}}{2\pi}\,\frac{e^{ik_{\parallel}x}\,G(k_{\parallel},\omega)\,S_{\omega}}{\epsilon(\omega,k_{\parallel})\,(k_{\parallel}-k_{\parallel}(\omega))}$$

where C_ω is a contour in the upper half plane above any roots $\omega(k_{\parallel})$ determined by $\epsilon(\omega,k_{\parallel})=0$ for real $k_{\parallel}$. Since the C_ω contour is chosen above the zeros of $\epsilon(\omega,k_{\parallel})$, it follows that the roots $k_{\parallel}(\omega)$ for ω on C_ω can be chosen in the upper half plane. Thus for $x>0$ where we can enclose the $k_{\parallel}$ contour in the upper $k_{\parallel}$ plane, we find that the poles $k_{\parallel}=k_{\parallel}(\omega)$ are encircled, while for $x<0$ we can enclose the $k_\parallel$ contour in the lower $k_\parallel$ plane where the poles $k_\parallel = -k_\parallel(\omega)$ are encircled. Now if $\frac{\partial \epsilon(\omega, k_\parallel(\omega))}{\partial k_\parallel} \neq 0$ in the upper half ω plane, the only contribution that persists after a long time in the final ω integral is from the pole at $\omega = \omega_0$. Hence we may now replace ω by ω_0 treating $k_\parallel(\omega_0)$ in the upper half plane, as prescribed in the text. The condition $\frac{\partial \epsilon(\omega, k_\parallel(\omega))}{\partial k_\parallel} \neq 0$ in the upper half plane guarantees that any instability present is convective,[11] i.e., disturbances propagate away from its source.

APPENDIX B. SOLUTION FOR "θ" DISTRIBUTION FUNCTION

For the special case in which the equilibrium distribution function is a Θ function, ($F = \frac{1}{2u}\theta(E - u^2/2)$ where $E = \frac{1}{2}v^2 + \Phi(x)$) the Vlasov-Poisson equations can be reduced to a differential equation. Here we consider only a one-dimensional system or, equivalently, $k_\perp = 0$. The solution to this problem enables us to test the general expression derived in the text.

Although one can proceed directly from the Vlasov-Poisson equations, the equations of the system are more quickly derived by observing that if the initial state is a θ -function, the distribution function can only change at its points of discontinuity, $E = E_i$, since

$$\frac{dF}{dt} = -\frac{e}{m}\mathcal{E} v \frac{\partial F}{\partial E} \propto \sum_i' \delta(E - E_i) \quad . \qquad \text{(B.1)}$$

Hence, only the width of the θ -function changes in time and space. The density of particles for all time is then given by $n(x,t) = C\left(v^+(x,t) - v^-(x,t)\right)$ where C is determined from the equilibrium to be $C = \frac{n_0}{2u}$, n_0 is the particle density at $x = -\infty$, and v^+ and v^- are the points of discontinuity. Now $n(x,t)$ can be related to the electric field by Poisson's equation and we need only solve the linearized equations of motion for $v^{\pm}(x,t)$.

Thus we have,

$$\frac{\partial v^{\pm}(x,t)}{\partial t} = -v^{\pm}\frac{\partial v^{\pm}}{\partial x} - \frac{\partial \Phi}{\partial x} + \frac{e}{m}\frac{\partial \mathcal{E}}{\partial x} \qquad (B.2)$$

$$\frac{\partial \mathcal{E}}{\partial x} = 4\pi e\left[\frac{n_0}{2u}\left(v^{+}-v^{-}\right) - N(x)\right] \qquad (B.3)$$

where $\mathcal{E}$ is the perturbed electric field and $N(x) = \frac{n_0}{u}\sqrt{2(E-\Phi(x))}$ is the density of the rigid ion background.

The equilibrium solution is $v_0^{+} = -v_0^{-} = \sqrt{2(E-\Phi)} = v_0(x)$.

If now we add and subtract the equations for perturbed velocities, $v_1^{\pm}$, we find

$$\frac{\partial}{\partial t}\left(v_1^{+} - v_1^{-}\right) = -\frac{\partial}{\partial x}\left[v_0(x)\left(v_1^{+} + v_1^{-}\right)\right] \qquad (B.4)$$

$$\frac{\partial}{\partial t}\left(v_1^{+} + v_1^{-}\right) = -\frac{\partial}{\partial x}\left[v_0(x)\left(v_1^{+} - v_1^{-}\right)\right] + \frac{e}{m}\mathcal{E} \qquad (B.5)$$

Combining (B.4) and (B.5), we have

$$\frac{\partial^2}{\partial t^2}\left(v_1^{+} - v_1^{-}\right) = \frac{\partial}{\partial x} v_0(x)\frac{\partial}{\partial x}\left[v_0(x)\left(v^{+} - v^{-}\right)\right]$$

$$- \frac{e}{m}\frac{\partial}{\partial x}\left[v_0(x)\,\mathcal{E}(x,t)\right] \qquad (B.6)$$

Using (B.3), integrating with respect to x with the boundary condition $\mathcal{E} = 0$ at $x = \infty$, and substituting $\partial^2/\partial t^2 = -\omega^2$, we find

$$\frac{\partial}{\partial x}\left(v_0(x)\frac{\partial \mathcal{E}}{\partial x}\right) - \left(\frac{\omega_p^2}{v_0(x)} - \frac{\omega^2}{u}\right)\mathcal{E} = 0 \qquad (B.7)$$

This is our desired result and we can apply the Bremmer method directly to this equation. If we consider $\Phi(x)$ to be a step function, we obtain the boundary conditions by integrating across the discontinuous jump. These yield the conditions $V(x)\,\mathcal{E}'(x)$, and $\mathcal{E}(x)$ are continuous across the jump. These boundary conditions determine the transmission (t) and reflection (r) coefficients to be

$$t = \frac{2k_1 v_1}{k_1 v_1 + k_2 v_2}$$
$$r = \frac{k_1 v_1 - k_2 v_2}{k_1 v_1 + k_2 v_2} \qquad \text{(B.8)}$$

where the subscripts 1 and 2 refer to quantities to the left and right of the jumps.

Using the same procedure as in the text, we can construct the incident and reflected waves for a continuous potential and find that they are respectively given by

$$\mathcal{E}_i(x) = \mathcal{E}_0 \left(\frac{k(-\infty)\,V(-\infty)}{k(x)\,V(x)}\right)^{1/2} e^{i\int_0^x k(x')\,dx'} \qquad \text{(B.9)}$$

$$\mathcal{E}_r(x) = -\mathcal{E}_0\, e^{-i\int_0^x k\,dx'} \int_{-\infty}^{+\infty} \frac{dx\,\frac{d}{dx}(kV)}{2kV}\, e^{2i\int_0^x k\,dx} \qquad \text{(B.10)}$$

where $\mathcal{E}_0$ is the initial amplitude of the incident wave, $\mathcal{E}_i(x)$ is the transmittal wave, $\mathcal{E}_r(x)$ is the reflected wave for large negative x and $k^2(x) = \omega^2/v^2(x) - \omega_p^2/u\,V(x)$. Note that if the oscillating potential, ϕ, is used instead of the electric field in Eq. (B.10) (as in the text), the sign of Eq. (B.10) reverses.

APPENDIX C. NON-LOCAL REFLECTION

Here we indicate how a more detailed history of the particle orbits can perhaps be taken into account in our formalism and isolate what other terms might be important for wave scattering.

The equation for the response of the distribution function to a step is given by Eq. (7). If we view Eq. (7) as the response of a system with a smoothly varying potential to a step discontinuity, we can improve the accuracy of the right-hand side by substituting for $\frac{\partial \phi^{(0)}}{\partial x}$ and $f^{(0)}$ a more accurate approximation than the solution for the spatially homogeneous system. Instead we shall substitute the best available solution to the original equations. For convenience we restrict ourselves to the case $K_\perp = 0$. Then it can be found from orbit integrations that f is exactly given by

$$f^+ = e^{i\int_0^x k\,dx} \frac{e}{m}\frac{1}{v}\frac{\partial F}{\partial v}\left[\phi(x) + i\omega\int_{-\infty}^{x}\frac{dx'}{v(x')}\,\phi(x')\,e^{i\int_x^{x'}\left(k-\frac{\omega}{v}\right)dx''}\right] \tag{C.1a}$$

$$f^- = e^{i\int_0^x k\,dx} \frac{e}{m}\frac{1}{v}\frac{\partial F}{\partial v}\left[\phi(x) + i\omega\int_{x}^{x_0}\frac{dx'}{v(x')}\,\phi(x')\,e^{i\int_x^{x'}\left(k+\frac{\omega}{v}\right)dx''}\right.$$
$$\left. + i\omega\, e^{i\int_x^{x_0}\left(k+\frac{\omega}{v}\right)dx'}\int_{-\infty}^{x_0}\frac{dx'}{v(x')}\,\phi(x')\,e^{i\int_{x_0}^{x'}\left(k-\frac{\omega}{v}\right)dx''}\right] \tag{C.1b}$$

where the signs $\pm$ refer to positive and negative velocity particles and $v(x) = \sqrt{2\left(E - \frac{e}{m}\Phi(x)\right)}$.

For the oscillating field we can substitute in a W.K.B. solution given by Eq. (30). For such choices of $f^{(0)}$ and $\frac{\partial \phi^{(0)}}{\partial x}$, Eq. (6) can be solved for the response to the step. We can then use the Bremmer method of superposition to find the response of many steps which form the potential $\Phi(x)$. In this way we find that the reflection coefficient is given by

$$R = \frac{-\omega_p^2}{\left(\frac{\partial\epsilon(K(-\infty))}{\partial K}\right)^{1/2}} \int_{-\infty}^{+\infty} \frac{dx \;\; \Phi'(x)}{K^2\left(\frac{\partial\epsilon(K(x))}{\partial K}\right)^{1/2}} \int_{-\infty}^{+\infty} \frac{dv}{\omega + k(x)v} \cdot$$

$$\left[\frac{\partial f^{(0)}(x)}{\partial v} e^{i\int_0^x k\,dx'} - \frac{e}{2m}\frac{\partial}{\partial v}\left(\frac{1}{v}\frac{\partial F}{\partial v}\right)\phi(x)\, e^{i\int_0^x k\,dx'}\right] \quad (C.2)$$

where

$$\phi(x) = \left(\frac{\frac{\partial\epsilon}{\partial K}(K(-\infty))}{\frac{\partial\epsilon}{\partial K}(K(x))}\right)^{1/2} e^{i\int_0^x k\,dx'}$$

This expression contains more information as to the history of the particle orbits than Eq. (32). Equation (32) is recovered if we substitute for $f^{(0)}$ that part of Eq. (C.1) that is obtained by integrating by parts once and neglecting the integral remainders. Now the last term on the right-hand side of Eq. (C.1b) describes how particles arriving with negative velocities at the point (X,t) affect the field because they have interacted with the field at (X',t') where the same particles had a positive velocity. Certainly, this term is in no way described in the formulation in the text. It is therefore of interest to look at this term in more detail.

If we substitute into (C.2) just that part of f^- due to particle motion prior to reflection, we obtain ΔR, the additional wave reflection coefficient,

$$\Delta R = \frac{-i\omega\omega_p^2\, e/m}{\left(\frac{\partial\epsilon}{\partial k}(K(-\infty))\right)^{1/2}} \int_0^{\infty} dE\,\frac{\partial F}{\partial E} \int_{-\infty}^{x_0} \frac{dx\;\Phi'(x)\; e^{2i\int_0^x k\,dx' + i\int_x^{x_0}\left(k+\frac{\omega}{v}\right)dx'}}{K\left(\frac{\partial\epsilon}{\partial k}(K(x))\right)^{1/2}\; v\,(\omega - kv)^2}$$

$$\cdot \int_{-\infty}^{x_0} \frac{dx'}{v(x')}\,\phi(x')\, e^{i\int_{x_0}^{x'}\left(k-\frac{\omega}{v}\right)dx''}$$

(C.3)

This multiple integral is quite difficult to evaluate in general. However, it can be reduced somewhat if we extract only the contribution from those particles whose velocity was resonant at some point with the phase velocity of the wave, i.e., mathematically speaking, we evaluate the integrals at the stationary points $v(x) = \omega / k(x)$. This enables us to perform two of the integrals and reduce (C.3) to the form

$$\Delta R = \int_0^\infty dx_s \frac{dE}{dx_s} \frac{\partial F}{\partial E} C(x_s) \, e^{2i\int_0^{x_0} \frac{\omega}{v} dx' + 2i\int_0^{x_s} \left(k - \frac{\omega}{v}\right) dx'} \tag{C.4}$$

where x_s is defined by $\omega / k(x_s) = \sqrt{2(E - e/m\,\Phi(x_s))}$, and $C(x_s)$ is a slowly varying function. To evaluate this integral, we again have to seek the points of stationary phase and possible end point contributions.

Our analysis here is incomplete, but tentative results indicate a much smaller scattering coefficient than previously calculated for $\frac{\omega L}{v_{th}} \gg 1$.

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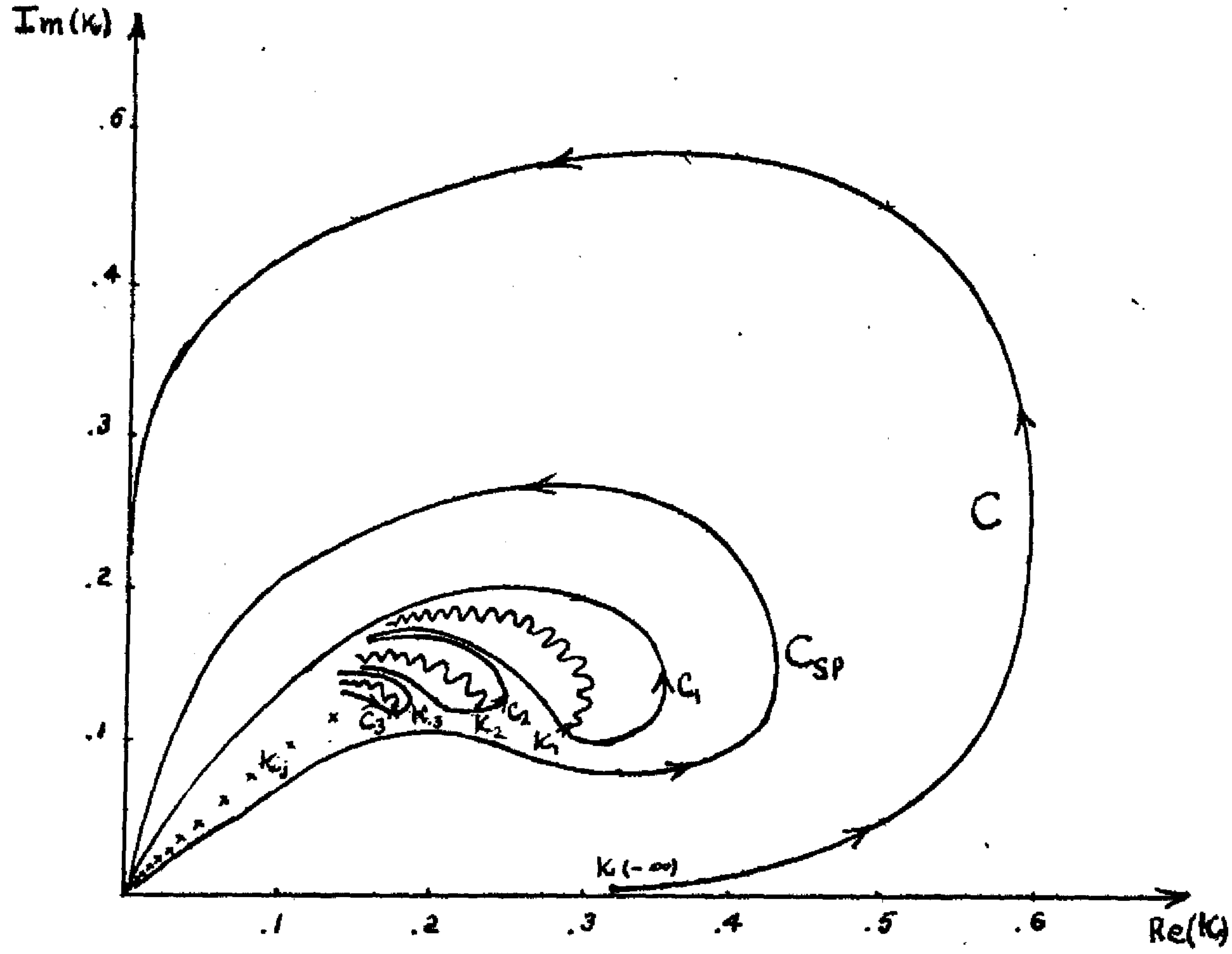

The contour C denotes the path of integration in the K plane for x on the real axis. The path C_{sp} is equivalent to the sum of the paths C_j, which are the paths of steepest descent through the stationary points K_j.

Figure 3

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PLASMA WAVE REFLECTION IN SLOWLY VARYING MEDIA

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ABSTRACT

Two mathematical formalisms are presented to describe wave reflection in a slowly varying spatially inhomogeneous thermal plasma described by the Vlasov equation. We find that the transmitted wave, which is the W.K.B. solution, and the reflected wave, can be expressed in terms of the local dielectric properties of the medium. In a numerical example, we show that the intrinsic thermal properties of the plasma can supply reflection mechanisms that compete with the reflection coefficient predicted when the plasma is described by fluid equations.

I. Introduction

Recent investigations of the POST-ROSENBLUTH loss cone instability[1,2,3] have given rise to the question of how reflection of convectively unstable waves in mirror machines can affect stability criteria. In the usual description, one expects that waves generated in the center of the machine are Landau damped at the ends. Hence if the axial wavelength in the device is sufficiently long, (typically more than 1/10 of the machine length) the wave amplitude does not grow to a level dangerous enough to cause particle loss. However, to these considerations reflection effects due to spatial inhomogeneity should be considered. Although the reflection coefficient can be expected to be exponentially small if the wavelength is much less than the plasma length, the reflected wavelets are themselves exponentiated due to the plasma instability. Thus it may still be possible that the reflection coefficient in the center of the device is of order unity or greater, in which case the system will have a noise level detrimental to particle containment.

AAMODT and BOOK[3] have already treated this problem starting from fluid equations. However, since the effects of reflection might be determined by Landau damping and other non-fluid behavior, we shall attempt to develop here the mathematical formalism for the reflection problem starting from the Vlasov equation. In this paper we shall develop the formalism for a stable plasma. At a later date, application to the Post-Rosenbluth instability will be presented.

Since we are ultimately interested in the application to the Post-Rosenbluth loss cone instability, we shall use approximations associated with this mode in our formalism. If the ion motion is neglected, then the mode of interest is a stable electrostatic oscillation of electrons where $k_\perp$, the wave number perpendicular to the magnetic field is much greater than $k_{\parallel}$, the wave number parallel to the field and the oscillation frequency ω is much less than the electron gyrofrequency ω_{ce}. Further, the group velocity of this mode propagates mostly along the field. Hence, we can consider a plasma model which is uniform perpendicular to the field but spatially inhomogeneous along the field. The inhomogeneity is provided by an external potential $\Phi(x)$, which simulates a confining magnetic field plus any static electric field arising from the charged particle equilibrium. The potential illustrated in Figure 1, is taken as zero at $x = -\infty$ and monotonically increasing to either a constant or infinity as $x \to \infty$. The ions will be considered as rigid and are present only to provide a neutralizing background. For this model, the basic equations for the linearized system take the form,

$$\frac{\partial f(x,y,v,t)}{\partial t} + v\frac{\partial f}{\partial x} - \frac{e}{m}\frac{\partial \Phi}{\partial x}\frac{\partial f}{\partial v} = \frac{e}{m}\frac{\partial \phi}{\partial x}\frac{\partial F(E)}{\partial v} \qquad (1)$$

$$\nabla^2 \phi(x,y,t) = -4\pi n_o e \int_{-\infty}^{+\infty} f\, dv \qquad (2)$$

(Hereafter we take $\nabla^2 = \frac{\partial^2}{\partial y^2}$ since it is assumed that $k_\perp \gg k_\parallel$ throughout).

Here F(E) is the equilibrium distribution function which depends only on the energy $E = \frac{v^2}{2} + \frac{e}{m}\Phi(x)$, f and ϕ are the oscillating distribution function and potential, x and v are the position and velocity coordinates along the field, e and m the charge and mass and n_o the electron density at $x = -\infty$. The distribution function has been averaged over its perpendicular velocities. We shall treat the case of a source situated far to the left of the inhomogeneous region and providing a disturbance proportional to $\exp(-i\omega t + i k_\perp y)$. Hence, a wave impinges on the inhomogeneous part of the plasma and we are required to find the reflected and transmitted waves. We see that the y,t dependence enters only through the factor $\exp(-i\omega t + i k_\perp y)$ which shall be subsequently suppressed.

We have developed two mathematical formalisms in an attempt to describe plasma waves in media slowly varying in space. Both methods are generalizations of techniques for obtaining the reflection coefficient for waves governed by differential equations.

The first method generalizes the one used by Bremmer[4] and others[3,5] and will be referred to as the "Bremmer method." For this method, the continuous potential $\Phi(x)$ is approximated by a discontinuous potential $\Phi_d(x)$ as shown in Fig. 1. We assume that in the neighborhood of each discontinuity we can solve the Vlasov-Poisson equation as if the medium is uniform on each side of the discontinuity (i.e. we neglect the other discontinuities). This enables us to calculate the transmission and reflection coefficients at each separate discontinuity.

The overall transmitted and reflected wave is then obtained from the coherent superposition of wavelets transmitted through and reflected from each discontinuity in the limit $\Phi_d(x) \to \Phi(x)$.

For differential equations this summation technique produces under certain restrictions an exact solution[5,6,7]. On the other hand, it will be clear from our construction that the Bremmer method cannot produce an exact solution of the Vlasov equation. Nevertheless, this method seems physically realistic and yields the correct W.K.B. solution as well as the correct answers for special distribution functions that can be treated exactly.

The second method generalizes an approach of Ginzberg[8] and is more direct than the Bremmer method. Here we begin with the integral equation for the oscillating field that is obtained by integrating the Vlasov equation over its unperturbed orbits. The integral equation is then solved by an iteration scheme in which the lowest order solution is the W.K.B. solution originally obtained by Berk, Rosenbluth and Sudan[9]. The neglected terms then serve as sources for reflected waves.

Neither of the above methods is rigorous although the second method can perhaps ultimately be made rigorous. However, the two methods complement one another in that after some approximation the methods yield the same result but the most obvious neglected correction terms come from different sources.

In Sections II and III we shall derive the reflection and transmission coefficients for the above two methods, while Section IV is devoted to a discussion of our derivations. In Section V the reflection coefficient is computed for a non-trivial choice of distribution function and $\xi(x)$. In this example the thermal reflection coefficient is found to be $r \sim \exp\left(-\frac{\omega L}{V_{th}}\right)$ which dominates the fluid result, $r \sim \exp\left(-\frac{2\pi}{\omega}\omega_p k_\perp L\right)$, if $2\left(\frac{k_\perp V_{th}}{\omega_p}\right)^2 > .005$. Here ω_p is the typical electron plasma frequency of the system and $L = n_0 / dn_0/dx$.

Although we have limited ourselves to a special mode of oscillation, our method has general application to problems where spatial inhomogeneity exists in one dimension.

II. The Bremmer Method

A. Step Problem

In order to apply the Bremmer method, we first calculate the elementary transmission and reflection coefficient of a wave incident on a single step at $x = x_i$, shown in Figure 2. We assume that the incoming wave propagates to the right and we look for outgoing waves for $x > x_i$ and $x < x_i$. These elementary wavelets will then ultimately be superimposed to calculate the fields propagating throughout the plasma.

The basic equations for the single step problem are,

$$-i\omega f + v\frac{\partial f}{\partial x} - \frac{e}{m}\,\delta(x-x_i)\,\Delta\Phi(x_i)\,\frac{\partial f}{\partial v} \qquad (3)$$

$$- \frac{e}{m}\,\frac{\partial \phi}{\partial x}\,\frac{\partial F^{(E)}}{\partial v} = 0$$

$$k_\perp^2\,\phi = 4\pi n_o e \int_{-\infty}^{+\infty} f\,dv \qquad (4)$$

where $\Delta\Phi(x_i)$ is the discontinuous jump in Φ .

This system can be solved by perturbation theory in the parameter $\Delta\Phi$. Hence we take,

$$f = f^{(0)} + f^{(1)} + \cdots$$

$$\phi = \phi^{(0)} + \phi^{(1)} + \cdots$$

$$F = F^{(0)} + F^{(1)} + \cdots$$

$$\Phi = 0 + \Delta\Phi\,\theta(x-x_i)$$

Then to lowest order the system describes a spatially homogeneous medium. The solution with an incoming wave boundary condition is,

$$\phi^{(0)} = \phi_o\,e^{ik_\parallel x} \qquad (5)$$

$$f^{(0)} = \frac{e}{m}\cdot\frac{k_\parallel\,\phi^{(0)}\,\frac{\partial F^{(0)}}{\partial v}}{-\omega + k_\parallel v} \qquad (6)$$

where $k_\parallel$ is determined by the dispersion relation

$$\epsilon(\omega, k_{\parallel}, \Phi_0) = 1 - \frac{\omega_p^2}{k_\perp^2} \int_{-\infty}^{+\infty} dv \, \frac{\frac{\partial F^{(0)}}{\partial v}(\frac{1}{2}v^2 + \frac{e}{m}\Phi_0)}{v - \omega/k_{\parallel}} = 0 \qquad (7)$$

where $\omega_p^2 = \dfrac{4\pi n_0 e^2}{m}$.

We choose $Re(\omega) > 0$ and $Re(k_{\parallel}) > 0$ so that the wave moves to the right. In Appendix A it is indicated that $k_{\parallel}$ should be treated as if it is in the upper half complex plane. For a stable plasma this criterion is automatically satisfied when ω is real. For an unstable plasma, $k_{\parallel}$ is in the lower half plane for real ω. The proper reflection behavior is obtained only if ω is first treated in the upper half plane so that the root of $\epsilon(\omega, k_{\parallel}) = 0$ occurs for $k_{\parallel}$ in the upper half plane. The transmission and reflection coefficients can then be obtained as a function of complex ω and analytically continued for ω real.

Now to first order in $\Delta\Phi$, equations (3) and (4) become,

$$-i\omega f^{(1)} + v\frac{\partial f^{(1)}}{\partial x} - \frac{e}{m}\frac{\partial \phi^{(1)}}{\partial x}\cdot\frac{\partial F^{(0)}}{\partial v} = \frac{e}{m}\frac{\partial \phi^{(0)}}{\partial x}\cdot\frac{\partial F^{(1)}}{\partial v} + \frac{e}{m}\delta(x-x_i)\Delta\Phi\frac{\partial f^{(0)}}{\partial v} \qquad (8)$$

$$k_\perp^2 \phi^{(1)} = 4\pi n_0 e \int_{-\infty}^{+\infty} f^{(1)} dv \qquad (9)$$

where $F^{(1)} = \frac{e}{m}\Delta\Phi \, \dfrac{\Theta(x-x_i)}{v} \, \dfrac{\partial F}{\partial v}(\frac{1}{2}v^2 + \frac{e}{m}\Phi_0)$ since

$F(E) = F(\frac{1}{2}v^2 + \frac{e}{m}\Phi(x))$

We can readily solve this system of equations in terms of their Fourier transforms defined by $\left(f_k^{(1)}, \phi_k^{(1)}\right) = \int_{-\infty}^{+\infty} dx\, e^{-ikx}\left(f^{(1)}(x), \phi^{(1)}(x)\right)$.

We then find that the solutions for $f_k^{(1)}$ and $\phi_k^{(1)}$ are given by,

$$f_k^{(1)} = -\frac{e}{m}\cdot\frac{k\,\phi_k^{(1)}\,\frac{\partial F^{(0)}}{\partial v}}{\omega - kv} + i\,\frac{e^2}{m^2}\,\frac{\Delta\Phi\;\phi_0\,k_{\parallel}\,e^{i(k_{\parallel}-k)x_i}}{(\omega - kv)(k - k_{\parallel})}\cdot\frac{\partial}{\partial v}\left[\frac{\omega - kv}{v(\omega - k_{\parallel}v)}\,\frac{\partial F^{(0)}}{\partial v}\right] \tag{10}$$

$$\phi_k^{(1)} = -i\,\omega_p^2\,\frac{e}{m}\,\frac{\Delta\Phi\;\phi_0\,k_{\parallel}k\,e^{i(k_{\parallel}-k)x_i}}{k_{\perp}^2\,\epsilon(\omega,k)\,(k - k_{\parallel})}\int_{-\infty}^{+\infty}\frac{dv\,\frac{\partial F^{(0)}}{\partial v}}{v(\omega - kv)(\omega - k_{\parallel}v)} \tag{11}$$

When $\phi_k^{(1)}$ is transformed to x space, we have,

$$\phi^{(1)}(x) = -i\,\omega_p^2\,\frac{e}{m}\,\frac{\Delta\Phi\;\phi_0\,k_{\parallel}}{k_{\perp}^2}\,e^{ik_{\parallel}x_i}\cdot\int_{-\infty}^{+\infty}\frac{dk_{\parallel}\,k\,e^{ik(x-x_i)}}{2\pi\,(k-k_{\parallel})\,\epsilon(\omega,k)}\int_{-\infty}^{+\infty}\frac{dv\,\frac{\partial F^{(0)}}{\partial v}}{v(\omega - kv)(\omega - k_{\parallel}v)} \tag{12}$$

From this integral we can evaluate the wave response far from the source $|kx| \gg 1$. The k contour can be distorted into the complex k plane so that $e^{ik(x-x_i)} \rightarrow 0$ (we distort into the upper half plane for $x > x_i$ and the lower half plane for $x < x_i$). The singularities that are first encountered are a double pole at $k = k_{\parallel}$ when $x > x_i$ and a single pole at $k = -k_{\parallel}$ for $x < x_i$. The residue of these poles is the coherent wave response. In addition to this coherent wave response, it is known that the additional singularities in $\epsilon(\omega, k)$ generate "stray" fields that decay on a faster scale than the wave decrement $[\mathrm{Im}(k_{\parallel})]^{-1}$.[10]

The stray fields do not decay exponentially, so that at very large distances from the source, these fields dominate the wave field. However, here we assume that we can neglect all fields other than the coherent response. Similarly, we neglect the poles $k = v/\omega$ since these terms also produce only rapidly decaying transients for a smooth distribution function.

If we now evaluate Eq. (12) for $x > x_i$ by extracting the residue of the double pole at $k = k_{\parallel}$ we find,

$$\phi^{(1)}(x) \equiv \phi_0 e^{ik_{\parallel}x} \left[\delta(x_i) + i \Delta k (x - x_i) \right]$$

$$= \frac{\omega_p^2}{k_\perp^2} \cdot \frac{e}{m} \cdot \frac{\Delta\Phi \, \phi_0 \, k_{\parallel}^2 \, e^{ik_{\parallel}x}}{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel})} \cdot$$

$$\left[\left(\frac{1}{k_{\parallel}} - \frac{\frac{\partial^2 \epsilon}{\partial k_{\parallel}^2}}{2 \frac{\partial \epsilon}{\partial k_{\parallel}}} \right) \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}}{v(\omega - k_{\parallel} v)^2} + \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}}{(\omega - k_{\parallel} v)^3} \right. \tag{13}$$

$$\left. + i(x - x_i) \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}}{v(\omega - k_{\parallel} v)^2} \right]$$

The first two terms in the bracket are the relative change in the amplitude of the forward wave from unity, so that the transmission coefficient at x_i is given by $t(x_i) = 1 + \tau(x_i)$. The coefficient of $i(x - x_i) e^{ik_{\parallel}x}$ in the last term represents the wave number shift Δk for $x > x_i$ (to all orders in perturbation theory the wave number shifts appear as $e^{ik_{\parallel}x_i + i(k_{\parallel} + \Delta k)(x-x_i)}$, but to first order in Δk this exponent has the form $e^{ik_{\parallel}x}(1 + i\Delta k(x - x_i))$. This shift can be verified by expanding $\epsilon(\omega, k_{\parallel}, \Phi + \Delta\Phi) = 0$ for small $\Delta\Phi$.

If we now consider $x < x_i$, the evaluation of the residue at $k = -k_{\parallel}$

yields the following expression for the reflected wave,

$$\phi^{(r)}(x) \equiv \Delta\rho(x_i)\, e^{-i k_\parallel (x-x_i)}$$

$$= -\frac{e}{2m} \frac{\omega_p^2 \,\Delta\Phi\, k_\parallel\, \phi_0\, e^{-i k_\parallel (x-2x_i)}}{k_\perp^2 \frac{\partial\epsilon}{\partial k_\parallel}(\omega,-k_\parallel)} \int_{-\infty}^{+\infty} \frac{dv \, \frac{\partial F}{\partial v}}{v\,(\omega+k_\parallel v)(\omega-k_\parallel v)} \tag{14}$$

$$= -\frac{e}{2m} \frac{\Delta\Phi\, k_\parallel\, \phi_0\, e^{-i k_\parallel (x-2x_i)}\quad \epsilon(0,k_\parallel)}{\omega^2 \frac{\partial\epsilon}{\partial k_\parallel}(\omega,k_\parallel)}$$

We have used the relation

$$-\frac{\omega_p^2}{k_\perp^2}\, \omega^2 \int_{-\infty}^{+\infty} \frac{dv\, \frac{\partial F}{\partial v}}{v\,(\omega+k_\parallel v)(\omega-k_\parallel v)}$$

$$= \epsilon(0,k_\parallel) - \frac{1}{2}\left[\epsilon(\omega,k_\parallel) + \epsilon(\omega,-k_\parallel)\right]$$

$$= \epsilon(0,k_\parallel)$$

and

$$\frac{\partial\epsilon(\omega,-k_\parallel)}{\partial k_\parallel} = -\frac{\partial\epsilon(\omega,k_\parallel)}{\partial k_\parallel}.$$

B. Continuous Problem

We can now apply the results of the step problem to the continuously varying potential. First we imagine that the incident wave propagates through a potential $\Phi_d(x)$ consisting of N steps which approximate the continuous potential $\Phi(x)$ as shown in Fig. 1. As a first approximation, we neglect multiple wave reflections so that the field at X is determined by the transmissions coefficients $t(x_n) = 1 + \tau(x_n)$ of the steps to the left of X. Hence the incident wave $\phi_t(X)$ is given by

$$\phi_t(x) = \phi_0 \prod_{n=1}^{m} (1 + \tau(x_n)) \exp\Big[i \sum_{n=1}^{m-1} k_n (x_n - x_{n-1}) + i k_m (X - X_{m-1}) + i\beta \Big] \tag{15}$$

$$X_{m-1} < X < X_m$$

Here $\phi_0 e^{i k_0 x + i\beta}$ is the incident field for $X < X_0$ (the first step), β is an arbitrary phase factor which, for convenience, is chosen so that the phase is ultimately zero at $X = 0$ and k_i is the local wave number determined by the relation $\epsilon(\omega, k_i, \Phi_d(x_i)) = 0$

Proceeding to the limit $\Phi_d(x) \to \Phi(x)$ where each discontinuity becomes arbitrarily small and an infinite number arise, we find that Eq. (15) becomes,

$$\phi_t(x) = \phi_0 \exp\Big[i \int_0^x k(x)\,dx + \int_{-\infty}^{x} \sigma(x)\,dx \Big] \tag{16}$$

where $\sigma(X) = \lim_{\Delta x \to 0} \frac{\tau(x)}{\Delta X}$. From Eq. (13) we see that $\sigma(x)$ is given by,

$$\sigma(x) = \frac{e}{m} \frac{\omega_p^2 \Phi'(x) k_{\parallel}^2}{k_{\perp}^2 \frac{\partial \epsilon(\omega, k_{\parallel})}{\partial k_{\parallel}}} \Bigg[\Bigg(\frac{1}{k_{\parallel}} - \frac{\frac{\partial^2 \epsilon(\omega, k_{\parallel})}{\partial k_{\parallel}}}{2 \frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel})} \Bigg) \cdot \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}}{v(\omega - k_{\parallel} v)^2} + \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}}{(\omega - k_{\parallel} v)^3} \Bigg] \tag{17}$$

Now having obtained an expression for the incident wave at each point in space, we can find the magnitude of the reflected wavelet at a step X_i. If we assume it propagates to the left without further reflection, we then find that the reflected wave field $\phi_r(x)$ in the limit of a continuous potential is given by

$$\phi_r(x) = \int_x^{\infty} dx' \, \rho(x') \, \phi_t(x') \, e^{\int_x^{x'} [i k_{\parallel}(x'') - \sigma(x'')] dx''}$$

$$\xrightarrow[x \to -\infty]{} \phi_0 e^{-i\int_0^x k_{\parallel}(x') dx'} \int_{-\infty}^{+\infty} dx' \, \rho(x') \, e^{2i\int_0^x k_{\parallel}(x') dx'} \tag{18}$$

where $\rho(x) = \lim_{\Delta x \to 0} \frac{\Delta \rho}{\Delta x}$ is determined from Equation (14). In this way, we see we have constructed a reflection coefficient which neglects multiply reflected wavelets.

Now we need only substitute for $\sigma(x)$ and $\rho(x)$ found from Eqs. (13) and (14). It can be shown that $\sigma(x)$ given by Eq. (17) can be written as,

$$\sigma(x) = \frac{d}{dx} \ln \left(\frac{\partial \epsilon(\omega, k_{\parallel}, \Phi(x))}{\partial k_{\parallel}} \right)^{-1/2} \tag{19}$$

after the following identities are used:

$$\frac{dk_{\parallel}}{dx} = - \frac{d\Phi}{dx} \frac{\frac{\partial \epsilon}{\partial \Phi}}{\frac{\partial \epsilon}{\partial k_{\parallel}}} \tag{20}$$

$$\frac{\partial \epsilon}{\partial \Phi} = - \frac{e}{m} \omega_p^2 \frac{k_{\parallel}^2}{k_{\perp}^2} \int_{-\infty}^{+\infty} \frac{dv \frac{\partial F}{\partial v}(\frac{1}{2}v^2 + \frac{e}{m}\Phi)}{v(\omega - k_{\parallel} v)^2} \tag{21}$$

and

$$\frac{\partial^2 \epsilon}{\partial k_\parallel \partial \Phi} = -\frac{2e}{m}\frac{\omega_p^2}{k_\perp^2}\left[k_\parallel^2 \int_{-\infty}^{+\infty} \frac{dv\,\frac{\partial F}{\partial v}}{(\omega - k_\parallel v)^3} + k_\parallel \int_{-\infty}^{+\infty} \frac{dv\,\frac{\partial F}{\partial v}}{v\,(\omega - k_\parallel v)^2}\right] \tag{22}$$

Hence the incident wave is given by

$$\phi_t(x) = \phi_0 \left[\frac{\frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel(-\infty), 0)}{\frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel(x), \Phi(x))}\right]^{1/2} e^{i\int_0^x k_\parallel(x')dx'} \tag{23}$$

which is similar to the result of ref. 9[11].

For the reflected wave we find from $\rho(x)$ determined from Eq. (14) and Eqs. (18) and (19) that as $x \rightarrow -\infty$ the reflected wave is given by

$$\begin{aligned}\phi_r(x) &\equiv r\,\phi_0\, e^{-i\int_0^x k_\parallel(x')dx'} \\ &= -\frac{e}{2m\omega^2}\phi_0\, e^{-i\int_0^x k_\parallel dx'} \int_{-\infty}^{+\infty} \frac{dx'\, k_\parallel\, \Phi'(x')\, \epsilon(0, k_\parallel)}{\frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel)}\, e^{2i\int_0^{x'} k_\parallel dx''} \\ &= \frac{e}{2m\omega^2}\phi_0\, e^{-i\int_0^x k_\parallel dx'} \int_{-\infty}^{+\infty} \frac{dx'\, k_\parallel}{\frac{\partial \epsilon}{\partial \Phi}}\, \frac{dk}{dx}\, \epsilon(0, k_\parallel)\, e^{2i\int_0^{x'} k_\parallel dx''}\end{aligned} \tag{24}$$

Notice that $\phi_r(x)$ is exponentially small since the phase of the exponential is rapidly varying in space while the rest of the integrand varies slowly in space.

III. Alternate Method

Instead of following the indirect route of the Bremmer method in deriving the transmission and reflection coefficients, we can obtain them more directly from the integral equation determining $\phi(x)$. If Eq. (1) is solved by integrating over the characteristic orbits and then substituted into Eq. (2), the following integral equation is obtained,

$$k_\perp^2 \phi(x) = \omega_p^2 \int_{\frac{e}{m}\Phi(x)}^{\infty} \frac{dE}{v(x,E)} \frac{\partial F(E)}{\partial E} \Big\{ 2\phi(x) + i\omega \int_{-\infty}^{x} \frac{dx'}{v(x',E)} \phi(x') e^{-i\omega \int_x^{x'} \frac{dx''}{v}} + i\omega \int_x^{x_0(E)} \frac{dx'}{v(x',E)} \phi(x') e^{+i\omega\int_x^{x'} \frac{dx}{v}} + i\omega e^{i\omega \int_x^{x_0} \frac{dx}{v}} \int_{-\infty}^{x_0(E)} \frac{dx'}{v(x',E)} \phi(x') e^{-i\int_{x_0}^{x'} \frac{dx}{v}} \Big\}, \tag{25}$$

where $v(x,E) = \sqrt{2(E - \frac{e}{m}\Phi(x))}$ and $\frac{e}{m}\Phi(x_0) = E$

We now assume that $\phi(x)$ is given by,

$$\phi(x) = \phi_1(x) e^{i\int_0^x k_\parallel dx'} + \phi_2(x) e^{-i\int_0^x k_\parallel dx'}$$

where $\phi_1(x)$ is the amplitude of an incident wave propagating from $x = -\infty$ and $\phi_2(x)$ is the amplitude of the reflected wave propagating to $x = -\infty$ and $k_\parallel(x)$ is determined by the relation $\epsilon(\omega, k_\parallel(x), \Phi(x)) = 0$

If we substitute this form of $\phi(x)$ into Eq. (25) and integrate by parts with respect to x' twice (in the same manner as Ref. 9), we obtain exactly the following expressions,

$$e^{i\int_0^x k_\parallel dx}\left[\epsilon(\omega, k_\parallel, \Phi(x))\,\phi_1(x) + \frac{d\phi_1}{dx}\,\frac{\partial\epsilon}{\partial k_\parallel}(\omega, k_\parallel, \Phi) + \frac{1}{2}\,\phi_1(x)\frac{d}{dx}\left(\frac{\partial\epsilon}{\partial k_\parallel}(\omega, k_\parallel, \Phi)\right)\right]$$

$$+\ e^{-i\int_0^x k_\parallel dx}\left[\epsilon(\omega, -k_\parallel, \Phi)\,\phi_2(x) + \frac{d\phi_2}{dx}\,\frac{\partial\epsilon}{\partial k_\parallel}(\omega, -k_\parallel, \Phi) + \frac{1}{2}\,\phi_2(x)\frac{d}{dx}\left(\frac{\partial\epsilon}{\partial k_\parallel}(\omega, -k_\parallel, \Phi)\right)\right]$$

$$= e^{i\int_0^x k_\parallel dx}\, S_+(x, \phi_1(x)) + e^{-i\int_0^x k_\parallel dx}\, S_-(x, \phi_2(x)) \qquad (26)$$

Here $S_\pm(x,\phi)$ are the integral remainder terms and are given by

$$S_\pm = \frac{\omega\omega_p^2}{k_\perp^2}\int_{\frac{e\Phi}{m}}^{\infty}\frac{dE}{v(x,E)}\,\frac{\partial F}{\partial E}\left\{\int_{-\infty}^{x} dx'\, e^{i\int_x^{x'}(\pm k_\parallel - \frac{\omega}{v})\,dx''}\right.$$

$$\frac{d}{dx'}\left[\frac{v}{\pm k_\parallel v - \omega}\,\frac{d}{dx'}\left(\frac{\phi_{1,2}(x)}{\pm k_\parallel v - \omega}\right)\right] \qquad (27)$$

$$+ \int_x^{x_0} dx'\, e^{i\int_x^{x'}(\pm k_\parallel + \frac{\omega}{v})\,dx''}\,\frac{d}{dx'}\left[\frac{v}{\pm k_\parallel v + \omega}\,\frac{d}{dx'}\left(\frac{\phi_{1,2}(x')}{\pm k_\parallel v + \omega}\right)\right]$$

$$+\ e^{i\int_x^{x_0}(\pm k_\parallel + \frac{\omega}{v})\,dx''}\int_{-\infty}^{x_0} dx'\, e^{i\int_{x_0}^{x}(\pm k_\parallel - \frac{\omega}{v})\,dx''}\,\frac{d}{dx'}\left[\frac{v}{\pm k_\parallel v - \omega}\,\frac{d}{dx'}\left(\frac{\phi_{1,2}}{\pm k_\parallel v - \omega}\right)\right.$$

The terms on the left side of Eq. (26) are the WKB operators for waves of wave number $k_{\parallel}$ and $-k_{\parallel}$ and are similar to the WKB operator found in Ref. (9). Equation (26) is still exact but now the right hand side is smaller in a WKB sense since each term involves two derivatives of a slowly varying quantity.

We proceed further if we assume that $\phi_2(x)$, the reflected wave is much less than $\phi_1(x)$, the incident wave. Then, since $\phi_2(x)$ is small, we neglect the expression containing $\phi_1(x)$ on the right hand side. If, for $\phi_1(x)$ we take the WKB solution,

$$\phi_1(x) = \phi_0 \left[\frac{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, -k_{\parallel}(-\infty), 0)}{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel}(x), \Phi(x))} \right]^{1/2} \tag{28}$$

then Eq. (26) becomes

$$\frac{d}{dx}\left[\phi_2(x) \left(\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, -k_{\parallel}, \Phi) \right)^{1/2} \right]$$

$$= \left[\frac{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, -k_{\parallel}(-\infty), 0)}{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel}, \Phi)} \right]^{1/2} \phi_0 \, e^{2i\int_0^x k_{\parallel} dx} S_+\left(x, \left(\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel}, \Phi) \right)^{-1/2} \right) \tag{29}$$

We can now easily solve for $\phi_2(x)$. With the boundary condition that $\phi_2 = 0$ at $x = +\infty$, we find,

$$\phi_2(x) = -\phi_0 \left(\frac{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, k_{\parallel}(-\infty))}{\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, -k_{\parallel})} \right)^{1/2} \int_x^{\infty} dy \, \frac{e^{2i\int_0^y k_{\parallel}(y')dy'} S_+(y)}{\left[\frac{\partial \epsilon}{\partial k_{\parallel}}(\omega, -k_{\parallel}) \right]^{1/2}} \tag{30}$$

The reflection coefficient, $r = \phi_2(-\infty)/\phi_0$ is obtained by taking $x \to -\infty$. If we then reverse the order of the integration and use $\frac{\partial \epsilon}{\partial k_\parallel}(\omega, -k_\parallel) = -\frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel)$, we find that r can be written as,

$$
\begin{aligned}
r = \frac{\omega\,\omega_p^2}{k_\perp^2} \int_{-\infty}^{+\infty} dx'\, e^{2i\int_0^{x'} k_\parallel dy} \int_{\frac{e}{m}\Phi(x')}^{\infty} dE\, \frac{\partial F}{\partial E} \Bigg\{ & \frac{d}{dx'}\left[\frac{v}{k_\parallel v - \omega} \cdot \frac{d}{dx'} \frac{1}{(k_\parallel v - \omega)\left(\frac{\partial \epsilon}{\partial k_\parallel}\right)^{1/2}} \right] \cdot \Bigg[\int_{x'}^{x_0} \frac{dx}{\left(\frac{\partial \epsilon}{\partial k_\parallel}\right)^{1/2} v(x,E)}\, e^{i\int_{x'}^{x}\left(k_\parallel + \frac{\omega}{v}\right) dx''} \\
& + e^{i\int_{x'}^{x}\left(k_\parallel + \frac{\omega}{v}\right) dx} \int_{-\infty}^{x_0} \frac{dx}{v\left(\frac{\partial \epsilon}{\partial k_\parallel}\right)^{1/2}}\, e^{i\int_{x_0}^{x}\left(k_\parallel - \frac{\omega}{v}\right) dx''} \Bigg] \\
& + \frac{d}{dx'}\left[\frac{v}{k_\parallel v + \omega}\, \frac{d}{dx} \frac{1}{(k_\parallel v + \omega)\left(\frac{\partial \epsilon}{\partial k_\parallel}\right)^{1/2}} \right] \int_{-\infty}^{x'} \frac{dx}{v(x,E)\left(\frac{\partial \epsilon}{\partial k_\parallel}\right)^{1/2}}\, e^{i\int_{x'}^{x}\left(k_\parallel - \frac{\omega}{v}\right) dx''} \Bigg\}
\end{aligned}
\qquad (31)
$$

As it stands, the expression derived here contains more information than the reflection coefficient derived from the Bremmer method, Eq. (24). We obtain Eq. (24) from Eq. (31) after we perform an approximate phase integration on the x-integral in Eq. (31), and then neglect double derivatives of slowly varying quantities. The procedure is shown in detail in Appendix B.

IV. Discussion

Our derivations of the reflected and transmitted waves have been heuristic and we shall not attempt any rigorous justification. Instead we shall point out several short comings, consistency checks and further information that can be gleaned from our results.

The principal criticism of our methods is that there is no guarantee that the terms that have been neglected produce small corrections. This criticism even applies to differential equations when the Bremmer method is used. The reason is that although higher order terms are smaller in a WKB sense, the reflection is exponentially small. There is then no guarantee that the higher order terms arising from multiple reflections are annihilated as efficiently as the lower order terms and thus they can conceivably be important. For differential equations it can be shown that the Bremmer integral gives at least the correct exponential behavior. To check the accuracy of the Bremmer integral more precisely we have also evaluated numerically the Bremmer reflection integral arising from the Helmholtz equation

$$\frac{d^2\psi}{dx^2} + k^2(x)\,\psi = 0 \qquad \text{where} \qquad k^2(x) = k_0^2\left[1 + \frac{\delta}{1+e^{x/L}}\right]$$

and compared the result with the exact solution.[12] The results shown in Fig. 4 show excellent agreement as long as the WKB criteria are obeyed, or $\frac{\Delta k}{k} \ll 1$.

For our problem additional corrections arise from fields that propagate at wave numbers determined from other zeros of the dispersion relation, $\epsilon(\omega, k_\parallel) = 0$. If in the Bremmer method only first order perturbation theory is employed, these fields are unimportant since they damp quickly, i.e., on the order of a Debye length,[10] and the magnitude of the reflected coherent wave is unaffected. However, if two interactions of the wave with the inhomogeneties are considered, it is possible that the stray fields will rescatter back into a coherent mode. We attempted to estimate the order of magnitude of this effect by formulating a two step problem but unfortunately this procedure lead us to an unexplained divergence.

In our alternate method, the fields arising from the additional roots of $\epsilon(\omega, k_{\parallel}) = 0$ were explicitly neglected. It would seem that these fields should be included in the WKB propagator if one is to set up an iteration procedure that ultimately converges to an exact solution.

The alternate method can be cast as a formal iteration procedure, where we equate ϕ_1 with $S_-(\phi_2)$ and ϕ_2 with $S_+(\phi_1)$. If to lowest order we choose $\phi_2^{(0)} = 0$ and $\phi_1^{(0)}$ as the WKB solution, a formal iteration procedure leading to an infinite series is defined. The first order term produces our result for the reflection coefficient but we have not been able to exhibit the convergence of the series. Thus there is still some question as to the formal basis of our procedures and whether we have neglected any important sources of reflection arising in higher order perturbation theory.

Our alternate method does have the virtue of exhibiting how the detailed history of a particle contributes to the wave reflection. Remember that in order to reduce Eq. (31) to Eq. (24), only the end point contribution of the x-integral in Eq. (31) was taken into account. However, other processes may be important to wave reflection. One such process that can be isolated is the effect on wave reflection by particles that have already been turned around but were at one time resonating with the wave. To analyze this, we isolate that term in Δr given by Eq. (31) in which the x-integral varies from $-\infty$ to x_0. After some algebra, in which spatial derivatives of k and v are omitted, this contribution to Δr can be written as,

$$\Delta r = -\frac{\omega\omega_p^2}{k_\perp^2}\int_0^\infty dE\ \frac{\partial F}{\partial E}\ \exp\left(2i\int_0^{x_0}\frac{\omega}{v}\,dx\right)\cdot\left[\int_{-\infty}^{x_0}\frac{dx\ \exp\left[i\int_0^x\left(k-\frac{\omega}{v}\right)dx\right]}{V(E,x)\ \left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\right]^2. \tag{32}$$

From this integral we can extract the resonant contribution by evaluating the integrals at the stationary phase points where $v(x,E) = \frac{\omega}{k_\parallel(x)}$. This enables us to do two of the integrals and reduce (32) to the form,

$$\Delta r = \int_0^\infty dx_s \, \frac{\partial E}{\partial x_s} \frac{\partial F}{\partial E} C(x_s) \, e^{2i\int_0^{x_0} \frac{\omega}{v} dx' + 2i\int_0^{x_s} (k_\parallel - \frac{\omega}{v}) \, dx'} \tag{33}$$

where x_s is defined by $\omega / k_\parallel(x_s) = \sqrt{2(E - \frac{e}{m}\Phi(x_s))}$ and $C(x_s)$ is a slowly varying function. To evaluate this integral we again have to seek points of stationary phase and possible end point contributions. Our analysis here is incomplete, but tentative results indicate a much smaller reflection coefficient than is obtained from Eq. (24) when applied to the example given in the next section.

We have several consistency checks for our method. We note that as in the Bremmer method for differential equations, the transmitted wave is the same as the lowest order WKB solution. It can be shown that as $k_\parallel \to 0$ and $k_\perp$ much less than the Debye wave number, the expression for the reflected wave, Eq. (24), approaches the result of Aamodt and Book,[3]

$$\phi_r(x)/\phi_0 \underset{k_\parallel \to 0}{\longrightarrow} \frac{e^{-i\int_0^x k_\parallel dx}}{2} \int_{-\infty}^{+\infty} dx \, \frac{1}{k_\parallel} \frac{dk_\parallel}{dx} \, e^{2i\int_0^x k_\parallel dx} \tag{34}$$

It is shown further in Appendix C that if a distribution function $F = \frac{1}{2u}\Theta\left(\frac{v^2}{2} - E\right)$ where $\Theta(x) = \begin{cases} 0, & x<0 \\ 1, & x>0, \end{cases}$ is used, an exact differential equation is obtained,

$$\frac{d}{dx}\left(V(x)\,\frac{dV\,\phi(x)}{dx}\right) + \frac{\omega_p^2}{k_\perp^2 u}\,V(x)\,\frac{d\phi(x)}{dx} + \omega^2\phi(x) = 0 \quad (35)$$

where $V(x) = \left[2\left(E - \frac{e}{m}\Phi(x)\right)\right]^{1/2}$

The transmitted and reflected waves obtained when the Bremmer method is applied directly to this equation is

$$\phi_t(x) = \phi_0\left[\frac{k(-\infty)\,V(-\infty)\,\left(V(-\infty) + \omega_p^2/k_\perp^2 u\right)}{k(x)\,V(x)\,\left(V(x) + \omega_p^2/k_\perp^2 u\right)}\right]^{1/2} e^{i\int_0^x k(x')dx'} \quad (36a)$$

$$\phi_r(x) = \frac{1}{4}\phi_0\,e^{-i\int_0^x k\,dx'}\int_{-\infty}^{+\infty} dx\,\frac{dV}{dx}\,\frac{e^{2i\int_0^x k\,dx'}}{\left(k_\perp^2 u V^2 + \omega_p^2 V\right)} \quad (36b)$$

where $k^2(x) = \omega^2/\left(V^2(x) + \omega_p^2 V(x)/k_\perp^2 u\right)$.

Identical results are obtained from Eq. (23) and (24) if one uses the dielectric function, $\epsilon(\omega, k) = 1 + \frac{\omega_p^2 k^2 V}{k_\perp^2 u\,(k^2V^2 - \omega^2)}$

V. Calculation of the Reflection Coefficient

We now compute the reflection coefficient for a Maxwellian distribution function and for a particular potential,

$$F(E) = (2\pi v_{th}^2)^{-1/2} \exp\left[-\frac{e\Phi}{m v_{th}^2} - \frac{1}{2}\frac{v^2}{v_{th}^2}\right] \tag{37}$$

$$\frac{e\Phi(x)}{m v_{th}^2} = \ln\left(1 + \delta e^{x/L}\right). \tag{38}$$

For this distribution function the local dielectric constant is

$$\epsilon(\omega, k_{\parallel}, \Phi) = 1 - \frac{\omega_p^2 e^{-\frac{e\Phi}{m v_{th}^2}}}{2 k_{\perp}^2 v_{th}^2} Z'\left(\frac{\omega}{k_{\parallel} v_{th}\sqrt{2}}\right) \tag{39}$$

where $Z'(\phi)$ is the plasma dispersion function tabulated by Fried and Conte,[13] which for $\mathrm{Im}(\phi) > 0$, is given by

$$Z'(\phi) = \frac{1}{\pi^{1/2}} \int_{-\infty}^{+\infty} \frac{-2x\, e^{-x^2}\, dx}{x - \phi} \tag{40}$$

It is convenient to transform the variable of integration in the reflection coefficient, Eq. (24), from x to $k_{\parallel}$ as related by $\epsilon(\omega, k_{\parallel}, \Phi(x)) = 0$ and to use dimensionless variables defined by

$$x = X/L$$

$$\chi = k_{\parallel} v_{th} \sqrt{2}/\omega$$

$$\lambda = \sqrt{2}\, L\omega / v_{th}$$

$$\alpha = 2 k_{\perp}^2 v_{th}^2/\omega_p^2 = 2 k_{\perp}^2 \lambda_D^2 .$$

Making these transformations we find that the reflection coefficient becomes

$$r = \frac{1}{4}\int_C dx\ x\ \frac{2+Z'(x^{-1})}{Z'(x^{-1})}\ e^{i\lambda\Psi(x)} \tag{41}$$

where

$$\Psi(x) = \int_{x(0)}^{x} \frac{Z''(x^{-1})}{\alpha - Z'(x^{-1})}\frac{dx}{x} \tag{42}$$

and $Z'' = \frac{d}{d(x^{-1})} Z'$

The contour of integration C is determined by $\mathcal{E}(\omega, k_{\parallel}, \Phi(x)) = 0$, and is shown schematically in Fig. (4). For Maxwellian, the contour is given by $\mathrm{Im}\left(Z'(x^{-1})\right) = 0$ and hence is independent of the potential.

Since λ is typically a large parameter, the method of stationary phase integration is appropriate for the approximate evaluation of the integral in Eq. (41). This method involves deforming the original integration contour to a path which passes through the saddle points of the integral in such a way that Re(Ψ) is constant and that Im(Ψ) has a minimum on the path.

We shall now list some properties of the phase function $\Psi(x)$ that are needed in the evaluation of the integral. The points of stationary phase are the roots of $\frac{d\Psi}{dx} \propto Z''(x^{-1}) = 0$. The roots of $Z'(x^{-1}) - \alpha = 0$ are branch points of the function $\Psi(x)$, and the roots of $Z'(x^{-1}) = 0$ are simple poles of the integrand in Eq. (41). These equations have a denumerable infinity of roots which lie just beneath a line making a 45° angle with the real axis. The roots converge rapidly on this line to a limit point at the origin. The first four roots of these equations are listed in table (1) for several values of α To determine the behavior of Ψ in the neighborhood of these critical points, we use a Taylor's series expansion of Z' and find that

$$\Psi \simeq \frac{2 Z'_{s,\ell}}{(\alpha - Z'_{s,\ell}) x^3_{s,\ell}} (x - x_{s,\ell})^2 \tag{43}$$

for $|x - x_{s,\ell}| < |x_{s,\ell}|^3$ where $x_{s,\ell}$ is a saddle point, and that

$$\Psi \simeq x_{b,\ell} \ln (x - x_{b,\ell}) \tag{44}$$

for $|x - x_{b,\ell}| < |x_{b,\ell}|^3$ where $x_{b,\ell}$ is a branch point. The derivatives in the Taylor series expansion have been rewritten using the identity $Z' = -2(1 + \phi Z)$. The path of stationary phase, Re(Ψ) = const. is determined from Eq. (43) in the neighborhood of $x_{s,\ell}$. Away from the saddle point the path was followed numerically and was found to loop around the $x_{b,\ell}$ branch point as shown schematically in Fig. (4). The stationary phase curves obtained by numerical integration are shown more precisely in Fig. (5), for α = .01, .1 and .5. The topology of the curves of steepest descent changes for $\alpha \gtrsim .15$ and thus the method of evaluation described below fails for this case. However, we shall restrict ourselves to the more important case where the waves are not too strongly damped in the main part of the plasma so that $\alpha < .15$. The paths approach the origin above the 45° line where the phase varies as $\Psi \simeq -\frac{2}{x}$ for small x; therefore, the integrand vanishes on these paths as $x \to 0$. In determining the paths of stationary phase the branch lines are chosen as shown in Fig. (4). The discontinuity in Ψ across the branch point $x_{b,\ell}$ is $2\pi i \, x_{b,\ell}$

Returning to the evaluation of the integral in Eq. (41), we deform the path of integration to the paths of stationary phase plus the paths that run from

$x(-\infty)$ to a distance ϵ from the origin and a 45^o circular arc of radius ϵ that begins on the real axis and terminates on the L^{th} steepest descent curve. The distorted contour is schematically shown in Fig. (4). Explicitly, we have

$$r = \left[\int_{C_o} + \int_{C_o'} + \sum_{\ell=1}^{L} \int_{C_\ell} \right] \frac{x}{4} \left(\frac{2+z'}{z'} \right) e^{i\lambda \Psi(x)} dx \tag{45}$$

where C_ℓ is a steepest descent contour, C_o goes from $x = x(-\infty)$ to the origin and C_o' is the 45^o arc connecting C_o and C_L.

The integrals on the path C_ℓ are evaluated by the method of stationary phase if $\lambda |x_{s,\ell}|^3 > 1$. If this inequality is not satisfied, then the stationary phase method fails since the phase is not yet large when higher order terms of the Taylor expansion about $x_{s,\ell}$ compete with the quadratic term. First we will evaluate the contribution on the paths C_ℓ when $\lambda |x_{s,\ell}|^3 > 1$ and later we will obtain the contribution from the region $\lambda |x_{s,\ell}|^3 < 1$

Factoring out the phase evaluated at $x_{s,\ell}$, the sum of terms in Eq. (45) has the form

$$\sum_{\ell=1}^{\infty} a_\ell \, e^{i\lambda \Psi(x_{s,\ell})}$$

where a_ℓ represents the residual integral. Note that the exponential order of magnitude is now given even without a_ℓ evaluated. For. $\lambda |x_{s,\ell}|^3 > 1$ we find from the stationary phase method that a_ℓ is given by

$$a_\ell = \frac{1}{4} x_{s,\ell} \frac{2+z'_{s,\ell}}{z'_{s,\ell}} \sqrt{\frac{\pi x^3_{s,\ell} (\alpha - z'_{s,\ell})}{2 i \lambda z'_{s,\ell}}}$$

For the first two terms in Eq. (45) we find that the phase on C_o is given by

$$\Psi(x) = i\pi\, x(-\infty) + P\int_{x(0)}^{x} \frac{z''}{\alpha - z'}\, \frac{dx}{x}$$

The increment in the phase on the small arc around the origin vanishes for $\arg(x) < \pi/4$. Thus, the contribution from the first two terms in Eq. (45) becomes

$$\Delta r_0 = e^{-\lambda\pi x(-\infty)} \left[\frac{1}{4}\int_{x(-\infty)}^{0} dx\, x \left(\frac{2+z'}{z'}\right) e^{i\lambda\Delta\Psi(x)} + \frac{i\pi}{8}\, e^{i\lambda\Delta\Psi(0)} \right] \tag{46}$$

where $\Delta\Psi(x) = P\int_{x(0)}^{x} \frac{z''}{\alpha - z'}\, \frac{dx}{x}$. Since $\lambda\Delta\Psi$ is rapidly varying along its contour of integration, (except near the origin) the first term tends to annihilate itself and thus can be neglected compared to the second term, which is the contribution from the circular arc. However, as $x \to 0$, $z' \to x^2$ and hence the first term has a logarithmic divergence that can only be cancelled from the contribution of the stationary phase integrals near the origin. Note that $\Delta\Psi(x)$ can be considered real if $\alpha \ll 1$, since $z'(x^{-1}) \cong x^2$ and $x(-\infty) \simeq \sqrt{\alpha}$.

We now discuss the convergence of the infinite sum in Eq. (45) and the logarithmic divergence at the upper limit in Eq. (46). The infinite sum diverges and cancels the divergence in Eq. (46) as we know it must, since the original integral is finite. To show the cancellation, we must calculate that part of a_ℓ which gives the singularity. We note that in the limit $x_\ell \to 0$ the discontinuity in the phase vanishes. This suggests that for small x_ℓ we shrink the contour C_ℓ to encircle the pole and the branch line. The integral around the branch point vanishes for $\lambda\, \mathrm{Im}(x_{b,\ell}) < 1$, and the integral along the branch line is easily bounded and found to be less than the integral around the pole by a factor $\lambda \mathcal{X}_{b,\ell}$ which is taken to be small. The residue from the pole is obtained from

$$Z'(x^{-1}) = 0 + Z''_{p,\ell}\left(-x_{p,\ell}^{2}\right)\left(x - x_{p,\ell}\right) + \cdots$$

$$\simeq -\frac{2}{x_{p,\ell}}\left(x - x_{p,\ell}\right)$$

which gives for the infinite sum from the Lth term

$$\Delta r_\infty = \sum_{\ell=L}^{\infty} (2\pi i)\left(\frac{-1}{4} x_{p,\ell}^{2}\right) e^{i\Psi(x_{p,\ell})}$$

where $x_{p,\ell}$ is the root of $Z'(x^{-1}) = 0$ and L is such that $|\lambda x_{p,L}| \ll 1$. The spacing of the infinite roots can be found from the asymptotic form of Z' and gives

$$\frac{1}{i\pi}\,\frac{d x_{p,\ell}}{x_{p,\ell}^{3}} = d\ell = 1 \tag{47}$$

for $|x_{p,\ell}^{3}| \ll 1$. Using Eq. (47) to convert the infinite sum into an integral, we find that

$$\Delta r_\infty = \int_{x_{p,L}}^{0} \frac{1}{i\pi}\,\frac{d x_{p,\ell}}{x_{p,\ell}^{3}}\left(\frac{-2\pi i}{4}\right) x_{p,\ell}^{2}\, e^{i\lambda\Psi(x_{p,\ell})}$$

$$= \frac{1}{2}\int_{0}^{x_{p,L}} \frac{dx}{x}\, e^{i\lambda\Psi(x)} \tag{48}$$

Thus, the sum $\Delta r_0 + \Delta r_\infty$ is finite; it is equal to the integral

$\frac{1}{2}\int \frac{dx}{x}\, e^{i\lambda\Psi}$ on a path like C_0 around the origin at a finite distance from the origin.

To recapitulate, the reflection coefficient reduces to a form where the exponential dependence of the terms is explicit; that is

$$r \simeq \frac{i\pi}{8}\, e^{-\pi\lambda\chi(-\infty) + i\lambda\Delta\Psi(0)} + \sum_{\ell=1}^{L'} a_\ell\, e^{i\lambda\Psi(x_{s,\ell})} \tag{49}$$

$\Delta\Psi(0)$ is real in the approximation that $\chi(0)$ is real, a_ℓ has an explicit algebraic dependence on the parameters λ and α for ℓ not too large, and $\lambda|x^3_{s,\ell'}| \gtrless 1$. In the following, we assume that the magnitude of the terms in r is dominated by their exponential dependence $e^{-\lambda \mathrm{Im}(\Psi)}$, and that r is characterized by the largest term in the series.

The exponential dependence of the first term in Eq. (49) has the form

$$r \simeq e^{-2\pi k_\parallel(-\infty) L} \tag{50}$$

This is the reflection coefficient obtained by solving the problem in the fluid approximation, that is, by solving the differential equation

$$\frac{d^2}{dx^2}\phi + k_\perp^2 \frac{\omega^2}{\omega_p^2}\left(1 + \delta e^{x/L}\right)\phi = 0$$

where $k_\parallel(-\infty) = k_\perp \frac{\omega}{\omega_p}$. The evaluation of the phase $\Psi(x_{s,\ell})$ for finite $x_{s,\ell}$ is, in general, a numerical problem and the results for α = .01, and .1 are given in table 2. However, after determining the location of the saddle points numerically, an estimate of $\Psi(x_{s,\ell})$ is obtained by using the approximations $Z'(\zeta) \cong \zeta^{-2}$ and $Z'''(\zeta) \cong -2\zeta^{3}$. The error made in using this approximation appears to be less than 10% for $\alpha < .10$. Using the approximation the integral for the phase can be performed and the reflection coefficient becomes

$$r = \sum_{\ell=1}^{L'} a_\ell \exp\left\{ -\frac{\sqrt{2}\,L\omega}{v_{th}} \left[2\,\mathrm{Im}(x_{s,\ell}) + \sqrt{2k_\perp^2\lambda_D^2}\,\arg\left(\frac{x_{s,\ell} - \sqrt{2k_\perp^2\lambda_D^2}}{x_{s,\ell} + \sqrt{2k_\perp^2\lambda_D^2}} \right) \right] \right\} \tag{51}$$

where the real part of the phase has been absorbed in a_ℓ .

If we consider a sequence of $x_{s,\ell}$ on the 45° ray, we find that the function in the exponent has a minimum for $|x_{s,\ell}| = \sqrt{2k_\perp^2\lambda_D^2}$ and gives a term of the form $e^{-2k_{\parallel}(-\infty)L(\pi/2+\sqrt{2})}$ which is similar to the fluid term but is slightly larger. For $\alpha \gtrsim .005$ the saddle points which lie appreciably below the 45° ray begin to dominate, and rapidly the saddle point $x_{s,\ell=1}$, which has a smaller imaginary part than the neighboring points, dominates. The contribution from this saddle point gives

$$r \simeq e^{-.3\frac{L\omega}{v_{th}}} \tag{52}$$

This is a new type of dependence which arises from the thermal properties of $\epsilon(\omega, k_\parallel)$.

Thus, in this example, the formalism establishes the transition between the reflection due to the fluid behavior given by Eq. (49) which dominates at long wavelengths and the thermal behavior given by Eq. (51) which dominates at short wavelengths. For the unstable plasma, which will be analyzed in a later paper, it is the short wavelength regime, $k_\parallel \lesssim \omega/v_{th}$ that appears to give the largest reflection.

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POSITION OF SADDLE POINTS AND ITS PHASES

$\alpha = .01$		$\alpha = .10$		$\alpha = .5$
$x(-\infty) = (.0992,\ .0000)$		$x(-\infty) = (.2923,\ .001)$		$x(-\infty) = (.4937,\ .0535)$
$x_{b,1} = (.3172,\ .1526)$	$\mathrm{Im}\,\Psi(x_{s,1}) = .250$	$x_{b,1} = (.2997,\ .1530)$	$\mathrm{Im}\,\Psi(x_{s,1}) = .668$	$x_{b,1} = (.2805,\ .1793)$
$x_{b,2} = (.2230,\ .1430)$	$\mathrm{Im}\,\Psi(x_{s,2}) = .279$	$x_{b,2} = (.2127,\ .1445)$	$\mathrm{Im}\,\Psi(x_{s,2}) = .818$	$x_{b,2} = (.2040,\ .1555)$
$x_{b,3} = (.1793,\ .1291)$	$\mathrm{Im}\,\Psi(x_{s,3}) = .281$	$x_{b,3} = (.1723,\ .1306)$	$\mathrm{Im}\,\Psi(x_{s,3}) = .864$	$x_{b,3} = (.1670,\ .1370)$
$x_{b,4} = (.1533,\ .1178)$	$\mathrm{Im}\,\Psi(x_{s,4}) = .280$	$x_{b,4} = (.1481,\ .1192)$	$\mathrm{Im}\,\Psi(x_{s,4}) = .885$	$x_{b,4} = (.1445,\ .1235)$

Roots of $Z'(x^{-1}) = 0$
$x_{p,1} = (.3188,\ .1534)$
$x_{p,2} = (.2257,\ .1437)$
$x_{p,3} = (.1803,\ .1296)$
$x_{p,4} = (.1517,\ .1183)$

Roots of $Z''(x^{-1}) = 0$
$x_{s,1} = (.2920,\ .1131)$
$x_{s,2} = (.2175,\ .1185)$
$x_{s,3} = (.1786,\ .1126)$
$x_{s,4} = (.1542,\ .1057)$

TABLE I.

APPENDIX A. Comment on Analytic Continuation Prescription

In the text we found that the response to a step function perturbation is of the form

$$\phi^{(1)}(x,t) = e^{-i\omega_0 t}\int_{-\infty}^{+\infty}\frac{dk}{2\pi i}\,\frac{e^{ikx}\,G(k,\omega)\,S_{\omega_0}}{\epsilon(\omega_0,k)\,(k-k(\omega_0))} \tag{A.1}$$

where ω_0 is a real frequency, $k(\omega_0)$ is the wave number for the forward wave determined by the equation, $\epsilon(\omega_0,k)=0$ and S_{ω_0} is proportional to the amplitude of the incident wave.

It has been observed that if the system is unstable, $k(\omega_0)$ is in the lower half plane. If we believe Eq. (A.1) in its present form we see that for an unstable system we have a reflected wave to the right of the discontinuity and a transmitted wave to the left; a result that violates our boundary conditions.

In order to obtain the correct results, we should remember that a problem must be posed with initial conditions present. If, for example, we assume we have a dipole source at the point $X_0 < 0$ whose time behavior is of the form $\Theta(t)\,e^{-i\omega_0 t}$, then after transients have died, waves with wave number $k(\omega_0)$ propagate to the right and left of the source. The wave propagating to the right can be taken as $\phi = S_{\omega_0}\,e^{ik(\omega_0)x - i\omega_0 t}$

It can then be shown that the perturbed field due to the step function at x = 0 has the form

$$\phi^{(1)}(x,t) = \int_{C_\omega}\frac{d\omega}{2\pi}\,\frac{e^{-i\omega t}}{\omega-\omega_0}\int_{-\infty}^{+\infty}\frac{dk}{2\pi}\,\frac{e^{ikx}\,G(k,\omega)\,S_\omega}{\epsilon(\omega,k)\,(k-k(\omega))}$$

where C_ω is a contour in the upper half plane above any roots $\omega(k)$ determined by $\epsilon(\omega, k) = 0$ for real k. Since the C_ω contour is chosen above the zeros of $\epsilon(\omega, k)$, it follows that the roots $k(\omega)$ for ω on C_ω can be chosen in the upper half plane. Thus, for $x > 0$ where we can enclose the k contour in the upper k-plane, we find that the poles $k = k(\omega)$ are encircled, while for $x < 0$ we can enclose the k contour in the lower k plane where the poles $k = -k(\omega)$ are encircled. Now, if $\frac{\partial \epsilon}{\partial k}(\omega, k(\omega)) \neq 0$ in the upper half ω plane, the only contribution that persists after a long time in the final ω integral is from the pole at $\omega = \omega_0$. Hence we may now replace ω by ω_0 treating $k(\omega_0)$ in the upper half plane, as prescribed in the text. The condition $\frac{\partial \epsilon}{\partial k}(\omega, k(\omega)) \neq 0$ in the upper half plane guarantees that any instability present is convective[14] i.e., disturbances propagate away from its source.

APPENDIX B. Reduction of Reflection Coefficient

We would like to show how Eq. (31) can be reduced to Eq. (24). We first evaluate the x-integral in Eq. (31) approximately by integrating by parts and neglecting the remainder term since it is higher order in the parameter $[(k_\parallel - \omega/v)L]^{-1}$. Thus only the end point contributions are important and it turns out that the end points at $x = x_0$ vanish. We then find that r is given by,

$$r = \frac{\omega \omega_p^2}{k_\perp^2} \int_{-\infty}^{+\infty} dx' \, e^{2i\int_0^{x'} k\,dx''} \int_{\frac{e}{m}\Phi(x)}^{\infty} dE \, \frac{\partial F}{\partial E} \left[- \frac{1}{i(kv+\omega)} \frac{1}{\left(\frac{\partial \epsilon}{\partial k}\right)^{1/2}} \right. \tag{B.1}$$

$$\left. \frac{d}{dx'} \left[\frac{v}{(kv-\omega)} \frac{d}{dx'} \frac{1}{(kv-\omega)} \frac{1}{\left(\frac{\partial \epsilon}{\partial k}\right)^{1/2}} \right] + \frac{1}{i(kv-\omega)} \frac{1}{\left(\frac{\partial \epsilon}{\partial k}\right)^{1/2}} \right.$$

$$\frac{d}{dx'}\left[\frac{v}{kv-\omega}\;\frac{d}{dx'}\,\frac{1}{(kv+\omega)\;\left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\right]\Bigg]$$

We now integrate the x' integral by parts and neglect terms of $\vartheta\left(\frac{1}{(k-\frac{\omega}{v})^2 L^2}\right)$ We then obtain,

$$r = 2\,\frac{\omega\omega_p^2}{k_\perp^2}\int_{-\infty}^{+\infty}\frac{dx\;k}{\left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\;e^{2i\int_0^x k\,dx}\int_{\frac{e}{m}\Phi(x)}^{\infty} dE\;\frac{\partial F}{\partial E}$$

$$\left[\frac{v}{(kv-\omega)(kv+\omega)}\;\frac{d}{dx}\;\frac{1}{(kv-\omega)\;\left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\right. \tag{B.2}$$

$$\left. -\;\frac{v}{(kv-\omega)(kv+\omega)}\;\frac{d}{dx}\;\frac{1}{(kv+\omega)\left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\right]$$

It is now shown that this expression reduces to our desired result,

$$r = \frac{-e}{2m\omega^2}\int_{-\infty}^{+\infty}\frac{dx'}{\frac{\partial\epsilon}{\partial k}}\;k\;\Phi'(x)\;\epsilon(0,k)\;e^{2i\int_0^x k\,dx'} \tag{B.3}$$

The reduction of Eq. (B.3) to Eq. (B.2) requires a fair amount of algebraic manipulation. For compactness we suppress the last term in Eq. (B.2). We then focus our attention on the quantity

$$I = \left(\frac{\partial\epsilon}{\partial k}\right)^{-1/2}\frac{2\omega_p^2\omega}{k_\perp^2}\int_{\frac{e}{m}\Phi(x)}^{\infty} dE\;\frac{\partial F}{\partial E}\left\{\frac{v}{(kv+\omega)(kv-\omega)}\frac{d}{dx}\;\frac{1}{(kv-\omega)\left(\frac{\partial\epsilon}{\partial k}\right)^{1/2}}\right. \tag{B.4}$$

$$\left. -\;\left[\omega\to-\omega\right]\right\}$$

If we show that I is given by $\quad I = \frac{-e}{2m\omega^2}\,\Phi'(x)\;\epsilon(0,k)$

we have our desired result. In the work below, it is convenient to rewrite the dielectric function in the form,

$$\epsilon(\omega,k) = 1 - \frac{\omega_p^2}{k_\perp^2}\,k \int_{\frac{e}{m}\Phi(x)}^{\infty} dE\, \frac{\partial F}{\partial E}\left[\frac{1}{kv-\omega} + \frac{1}{kv+\omega}\right] \tag{B.5a}$$

$$= 1 - \frac{2\omega_p^2}{k_\perp^2}\,k \int_{\frac{e\Phi}{m}}^{\infty} dE\, \frac{\partial F}{\partial E}\left[\frac{v}{(kv-\omega)(kv+\omega)}\right] \tag{B.5b}$$

$$= 1 - \frac{\omega_p^2}{k_\perp^2} \int_{\frac{e\Phi}{m}}^{\infty} dE\, \frac{\partial F}{\partial E}\left[\frac{\omega}{kv-\omega} - \frac{\omega}{kv+\omega} + 2\right] \tag{B.5c}$$

Now we note that I can be rewritten as

$$I = -\frac{\omega_p^2\,\omega}{k\,k_\perp^2} \int_{\frac{e}{m}\Phi(x)}^{\infty} dE\, \frac{\partial F}{\partial E}\left\{\left[\frac{1}{kv-\omega} + \frac{1}{kv+\omega}\right]\left[\frac{\frac{d}{dx}(kv)}{(kv-\omega)^2} + \frac{\frac{1}{2}\frac{d}{dx}\left(\frac{\partial\epsilon}{\partial k}\right)}{(kv-\omega)\frac{\partial\epsilon}{\partial k}}\right] - \left[\omega \to -\omega\right]\right\}$$

$$= -\frac{\omega_p^2\,\omega}{k_\perp^2\,k} \int_{\frac{e}{m}\Phi(x)}^{\infty} dE\, \frac{\partial F}{\partial E}\left\{\frac{\frac{1}{2}\frac{d}{dx}\left(\frac{\partial\epsilon}{\partial k}\right)}{(kv-\omega)(kv+\omega)\left(\frac{\partial\epsilon}{\partial k}\right)} + \frac{1}{2}\,\frac{\frac{d}{dx}\left(\frac{\partial\epsilon}{\partial k}\right)}{(kv-\omega)\left(\frac{\partial\epsilon}{\partial k}\right)} - \frac{1}{2}\,\frac{d}{dx}\,\frac{1}{(kv-\omega)^2} + \frac{1}{(kv+\omega)}\,\frac{d}{dx}\,\frac{1}{kv-\omega} - \left[\omega \to -\omega\right]\right\} \tag{B.6}$$

Here the first term vanishes because it is even in ω and thus is cancelled by one of the suppressed terms. The second and third terms can be shown to cancel with use of the relation

$$\frac{\partial \epsilon}{\partial k} = \frac{\omega_p^2 \omega}{k_\perp^2} \int_{\frac{e}{m}\Phi(x)}^{\infty} dE \, \frac{\partial F}{\partial E} \left[\frac{1}{(kv-\omega)^2} - \frac{1}{(kv+\omega)^2} \right] \quad \text{(B.7)}$$

Thus, only the last term remains. This term can be reduced to the form,

$$I = \frac{\omega_p^2}{2k_\perp^2 k} \int_{\frac{e}{m}\Phi(x)}^{\infty} dE \, \frac{\partial F}{\partial E} \, \frac{d}{dx}(kv) \left[\frac{1}{(kv-\omega)^2} + \frac{1}{(kv+\omega)^2} - \frac{2}{(kv-\omega)(kv+\omega)} \right] \quad \text{(B.8)}$$

Since $\epsilon(\omega, k(x), \Phi(x)) = \frac{d}{dx}\epsilon(\omega, k(x), \Phi(x)) = 0$, we have from Eq. (B.5a)

$$\begin{aligned} &\frac{\omega_p^2}{k_\perp^2} \int_{\frac{e}{m}\Phi}^{\infty} dE \, \frac{\partial F}{\partial E} \, \frac{d}{dx}(kv) \left[\frac{1}{(kv-\omega)^2} + \frac{1}{(kv+\omega)^2} \right] \\ &= \frac{\omega_p^2}{k_\perp^2 k} \frac{dk}{dx} \int_{\frac{e\Phi}{m}}^{\infty} dE \, \frac{\partial F}{\partial E} \left[\frac{1}{kv-\omega} + \frac{1}{kv+\omega} \right] \\ &= \frac{1}{k^2} \frac{dk}{dx} \end{aligned} \quad \text{(B.9)}$$

Hence I can be written as

$$\begin{aligned} I &= \frac{1}{2k^3} \frac{dk}{dx} - \frac{\omega_p^2}{k_\perp^2 k} \int_{\frac{e}{m}\Phi}^{\infty} dE \, \frac{\partial F}{\partial E} \, \frac{\frac{d}{dx}(kv)}{(kv-\omega)(kv+\omega)} \\ &= \frac{1}{2k^3} \frac{dk}{dx} - \frac{\omega_p^2}{k_\perp^2 k} \frac{dk}{dx} \int_{\frac{e\Phi}{m}}^{\infty} dE \, \frac{\partial F}{\partial E} \, \frac{v}{(kv-\omega)(kv+\omega)} \\ &\quad + \frac{e}{m} \frac{\omega_p^2}{k_\perp^2} \Phi'(x) \int_{\frac{e}{m}\Phi}^{\infty} dE \, \frac{\partial F}{\partial E} \, \frac{1}{v} \, \frac{1}{(kv-\omega)(kv+\omega)} \end{aligned} \quad \text{(B.10)}$$

where we have used $v \frac{dv}{dx} = -\frac{e}{m}\Phi'(x)$ Using $\epsilon(\omega, k, \Phi) = 0$ and Eq. (B.5b), we see that the first two terms cancel, and we have from the last term,

$$I = \frac{e}{2m} \frac{\omega_p^2}{k_\perp^2} \frac{\Phi'(x)}{\omega} \int_{\frac{e}{m}\Phi}^{\infty} \frac{dE}{v} \frac{\partial F}{\partial E} \left[\frac{1}{kv-\omega} - \frac{1}{kv+\omega} \right] \tag{B.11}$$

Finally, using Eq. (B.5c) and $\epsilon(\omega, k) = 0$, we have

$$I = -\frac{e}{2m\omega^2} \Phi'(x) \, \epsilon(0, k) \tag{B.12}$$

Q. E. D.

APPENDIX C. Solution for "θ" Distribution Function

For the special case in which the equilibrium distribution function is a function, $F = \frac{1}{2u} \Theta(\frac{v^2}{2} - E)$ where $E = \frac{1}{2}v^2 + \frac{e}{m}\Phi(x)$ and $\Theta(x) = \begin{cases} 0, & x<0 \\ 1, & x>1 \end{cases}$ the Vlasov-Poisson equations can be reduced to a differential equation. The solution to this problem enables us to test the general expression derived in the text.

Although one can proceed directly from the Vlasov-Poisson equations, the equations of the system are more quickly derived by observing that if the initial state is a θ -function, the distribution function can only change at its points of discontinuity $V = V_i$, since

$$\frac{dF^{(1)}}{dt} = -\frac{e}{m} E \cdot \frac{\partial F^{(0)}}{\partial v} \propto \sum_i \delta(v - v_i) \tag{C.1}$$

Hence, only the width of the Θ-function changes in time and space. The density of particles is then given by $n(x,t) = C\,(V^+(x,t) - V^-(x,t))$ where C is determined from the equilibrium to be C = $\frac{n_o}{2u}$, n_o is the particle density at $x = -\infty$, and V^+ and V^- are the points of discontinuity. Now n(x,t) can be related to the electric field by Poisson's equation and we need only solve the linearized equations for $V^\pm(x,t)$. Thus we have,

$$\frac{\partial V^\pm(x,t)}{\partial t} = -V^\pm \frac{\partial V^\pm}{\partial x} - \frac{e}{m}\frac{\partial \Phi}{\partial x} - \frac{e}{m}\frac{\partial \phi}{\partial x} \tag{C.2}$$

$$k_\perp^2 \phi = 4\pi e\left[\frac{n_o}{2u}(V^+ - V^-) - N(x)\right] \tag{C.3}$$

where $N(x) = \frac{n_o}{u}\left[2\left(E - \frac{e}{m}\Phi(x)\right)\right]^{1/2}$ is the density of neutralizing background.

The equilibrium solution is $V_o^+ = -V_o^- = \left[2\left(E - \frac{e}{m}\Phi(x)\right)\right]^{1/2} = V(x)$.

If we now add and subtract the equations for the perturbed velocities, $V_1^\pm e^{-i\omega t}$, we find,

$$-i\omega\,(V_1^+ - V_1^-) = -\frac{d}{dx}\left[V(V_1^+ + V_1^-)\right] \tag{C.4}$$

$$-i\omega\,(V_1^+ + V_1^-) = -\frac{d}{dx}\left[V(V_1^+ - V_1^-)\right] - \frac{e}{m}\frac{d\phi}{dx} \tag{C.5}$$

Combining (C.4) and (C.5) and (C.3), we find the differential equation governing ϕ,

$$\frac{d}{dx}\left[V \frac{d}{dx}(V\phi)\right] + \frac{\omega_p^2}{k_\perp^2 u} \frac{d}{dx}\left(V \frac{d\phi}{dx}\right) + \omega^2 \phi = 0 . \qquad (C.6)$$

The reflection and transmission coefficients for this equation can be obtained by the same type of perturbation theory used in the text, and the results are given in Eq. (36).

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11. If the term $\frac{\partial^2 \phi}{\partial x^2}$ we included in the Poisson equation the result for the transmitted wave is

$$\phi_t(x) = \phi_0 \left[\frac{\left(k_\perp^2 + k_\parallel^2(-\infty)\right) \frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel(-\infty), 0)}{\left(k_\perp^2 + k_\parallel^2(x)\right) \frac{\partial \epsilon}{\partial k_\parallel}(\omega, k_\parallel(x), \Phi(x))} \right]^{1/2} e^{i\int_0^x k_\parallel dx'}$$

The reflection coefficient given in terms of the dielectric function remains the same as Eq. (24).

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FIGURE CAPTIONS

Fig. 1. Smooth Potential $\Phi(x)$ and Step Potential Approximation $\Phi_d(x)$

Fig. 2. Step Discontinuity

Fig. 3. Comparison of Bremmer Integral with Exact Solution. The exact reflection coefficient to the equation $\frac{d^2\psi}{dx^2} + k_0^2\left(1 + \frac{\delta}{1+e^{x/L}}\right)\psi = 0$ is given by

$$r = \frac{\sinh\left[\pi k_0 L\left(\sqrt{1+\delta} - 1\right)\right]}{\sinh\left[\pi k_0 L\left(\sqrt{1+\delta} + 1\right)\right]}$$

This exact answer is compared with the Bremmer integral

$$r = \frac{1}{2}\left| \int_{-\infty}^{+\infty} \frac{dx}{k(x)} \frac{dk}{dx} e^{2i\int_0^x k(x')dx'} \right| .$$

For the parameters in the area below the solid curve the agreement between the two expressions is better than 5%. On the solid curve r varies from $r = .1$ to $r = .4$. In the cross hatched areas the asymptotic form of r is indicated.

Fig. 4. Schematic Drawing of Contour of Integration in $\mathcal{X}$ Plane. The solid line indicates the contour of integration C in the $\mathcal{X}$ plane. The dotted line indicates how this contour is distorted. The contours C_1, C_2,...C_i are the curves of steepest descent passing through the saddle points $\mathcal{X}_{s1}$, $\mathcal{X}_{s2}$ ..... $\mathcal{X}_{si}$.

Fig. 5. Numerically Evaluated Curves of Steepest Descent. Steepest descent curves for various parameters of $\alpha = 2 k_\perp^2 \lambda_D^2$

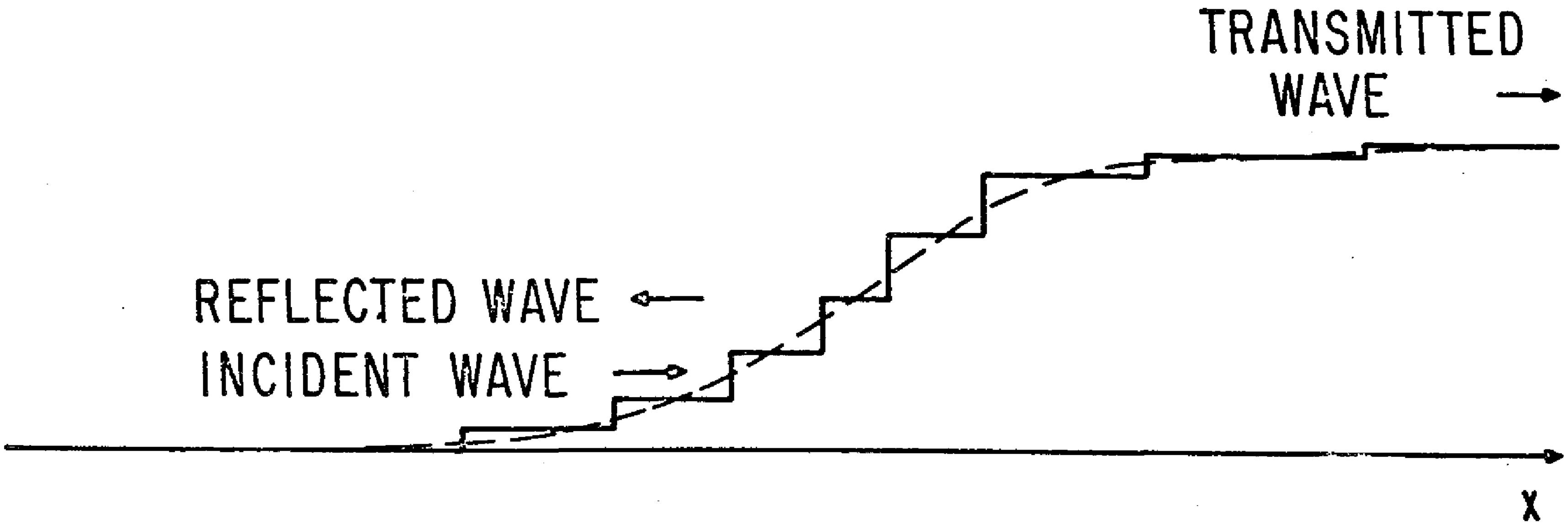


Figure 1

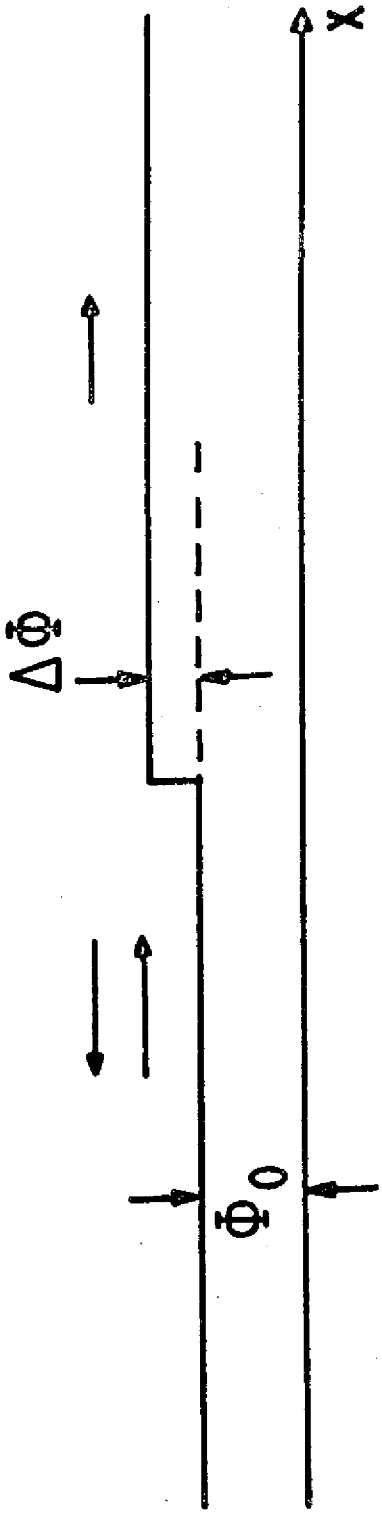
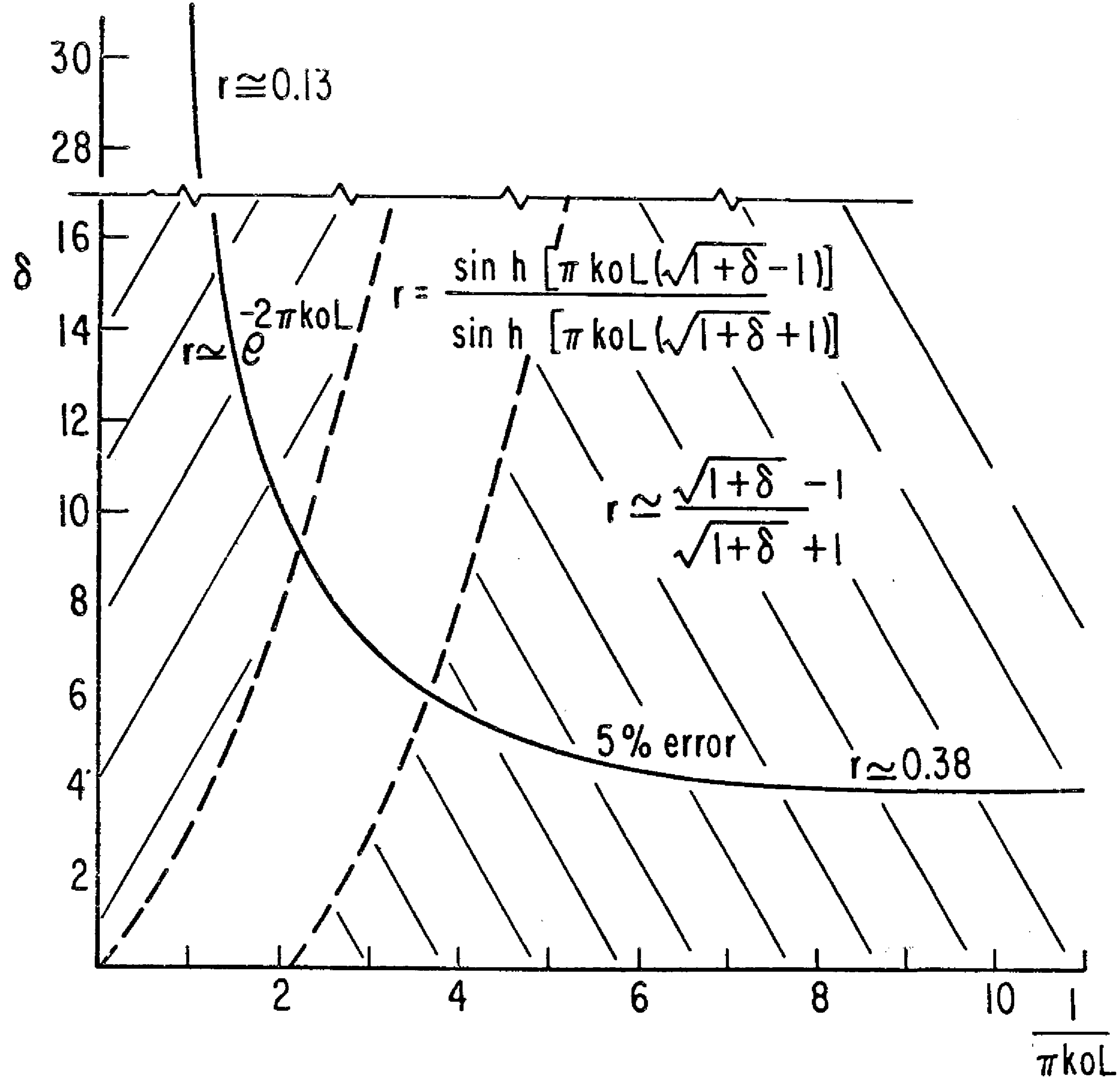


Fig 2



30
28
16
14
12
10
8
6
4
2
δ
r ≅ 0.13
r ≃ e^{-2πkoL}
r = sin h [π koL(√1+δ −1)] / sin h [π koL(√1+δ +1)]
r ≃ (√1+δ −1)/(√1+δ +1)
5 % error
r ≃ 0.38
2
4
6
8
10
1/πkoL

Fig. 3

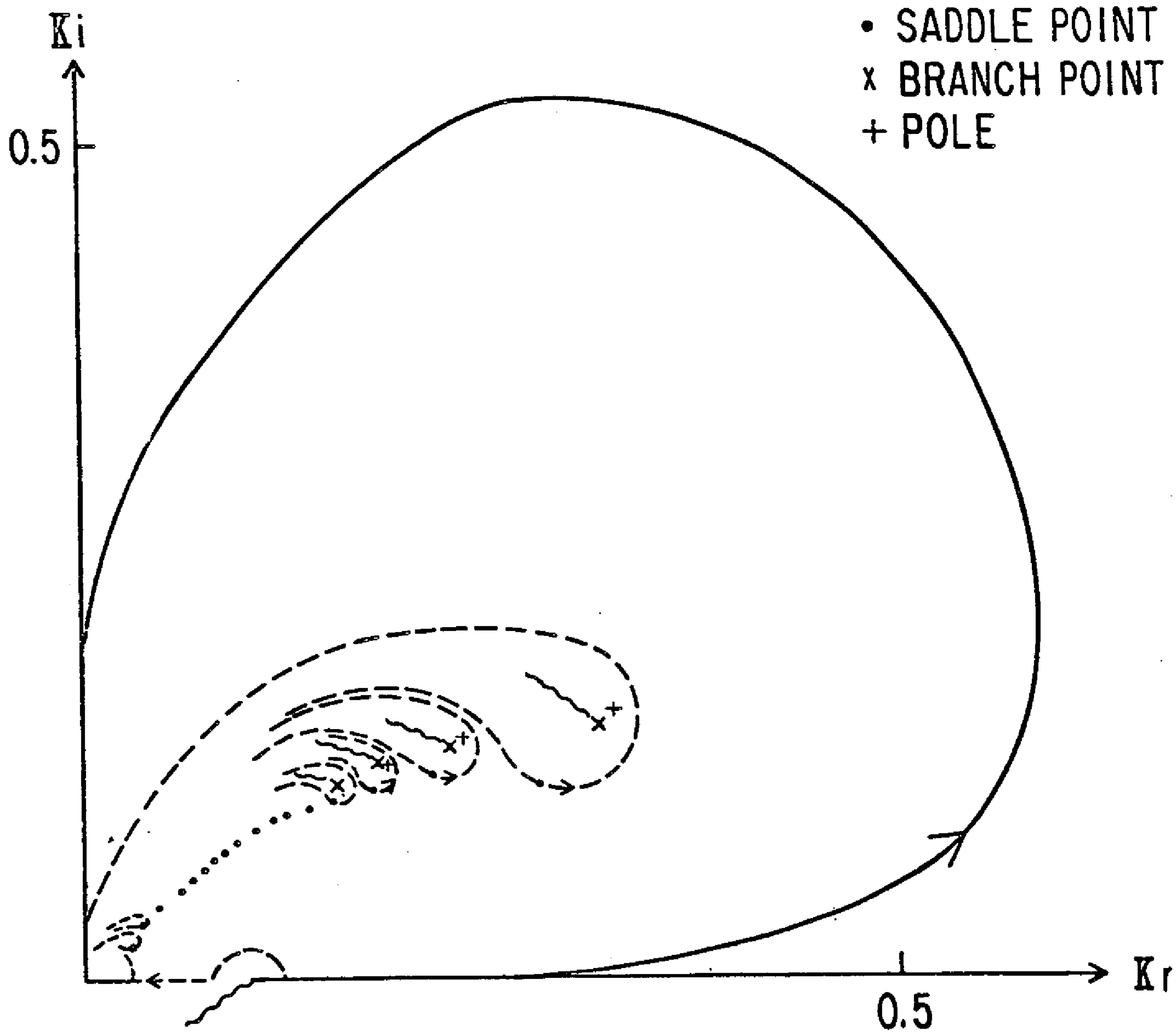


Fig. 4

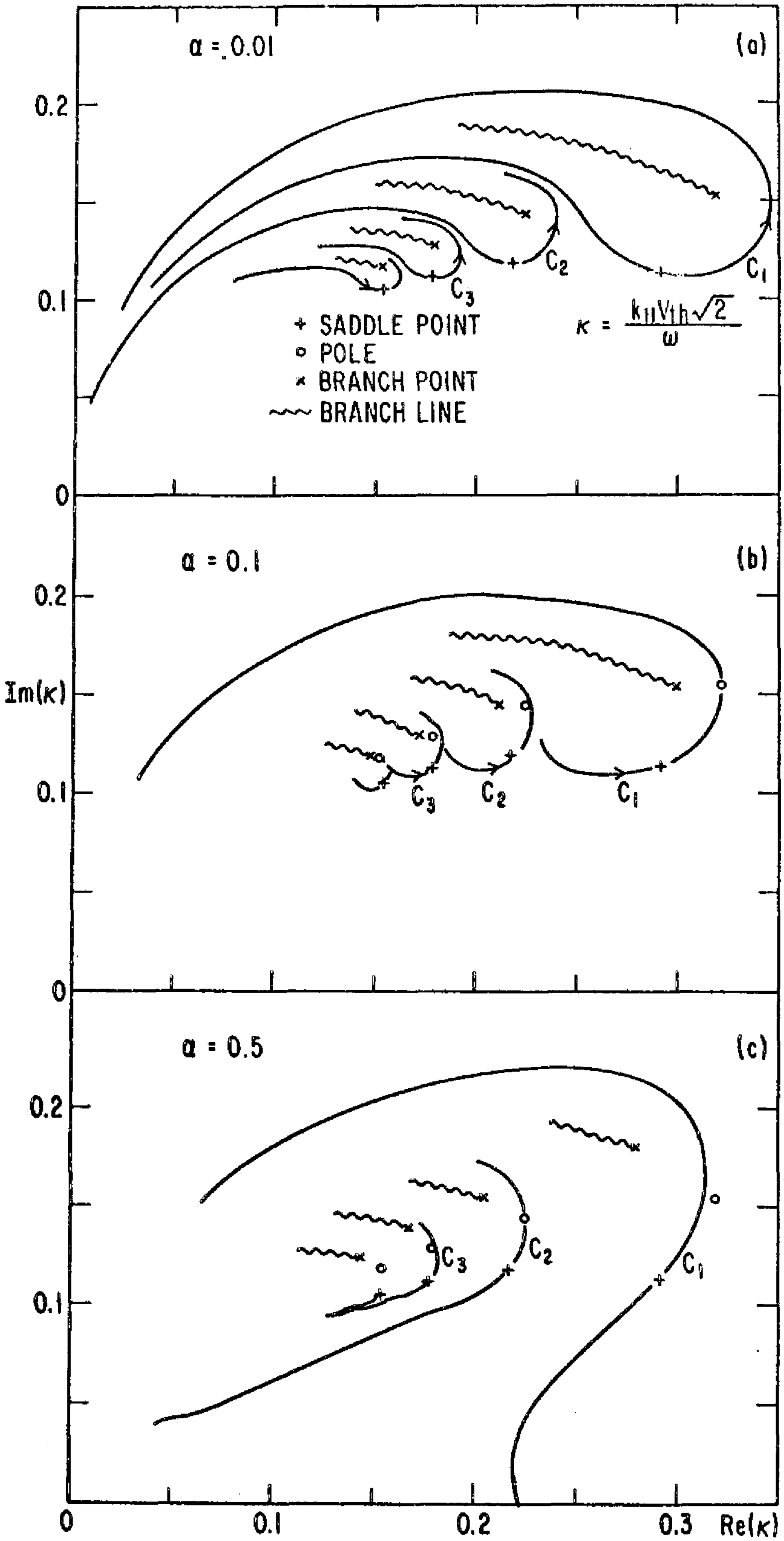

α = 0.01
(a)
0.2
0.1
0
C1
C2
C3
+ SADDLE POINT
∘ POLE
× BRANCH POINT
〰 BRANCH LINE
$\kappa = \frac{k_{\parallel} V_{th}\sqrt{2}}{\omega}$
α = 0.1
(b)
Im(κ)
α = 0.5
(c)
0.1
0.2
0.3
Re(κ)

Fig. 5