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Courant Institute of
Mathematical Sciences

Magneto-Fluid Dynamics Division

The Guiding Center Plasma

Harold Grad

AEC Research and Development Report

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Abstract

A survey is given of the theory of equilibrium, motion, and stability of a plasma in the guiding center limit.

Table of Contents

1. Introduction

Perhaps the single feature which dominates the landscape in any view of plasma physics is the complexity arising from the large number of simultaneous and interfering phenomena [as compared to the relative simplicity of the related classical disciplines of fluid dynamics, kinetic theory, electromagnetic theory]. We cannot expect the growth, even ultimately, of a significant quantitative theory using substantial mathematical tools which will embrace an appreciable part of the physics and at the same time incorporate a non-trivial geometry. One type of model allows the use of significant linear techniques (essentially Fourier analysis enhanced by asymptotics) to describe a relatively large amount of physics by remaining within the realm of trivial geometries such as an infinite homogeneous plasma, possibly with a superposed small linear macroscopic gradient. We shall be concerned with a model of a second type containing somewhat less physics but aimed at the exploitation of more elaborate mathematical tools in more practical geometrical configurations.

The prime example of the latter type of model is given by potential theory as applied in fluid dynamics. There are almost no fluid flows which can be correctly described purely within the scope of the irrotational theory. But the enormous mathematical power that can be brought to bear on the potential equation gives this subject its importance. In effect one develops a foundation for the study of fluid dynamics through a profound mathematical intuition about a nonexistent fluid. Whenever possible, a real fluid flow is thought of in terms of deviations from this ideal and completely understood model. The value of this ideal and physically primitive model depends first on the extent of the mathematical theory and second on a knowledge of exactly where this theory needs radical modification (e.g., at boundary layers, wakes, shocks).

Because of the complexity and scope of plasma physics, we should hope to be able to develop several such mathematically accessible models to give information in various parameter ranges. One such example is the perfectly conducting, scalar pressure macroscopic theory (MH). This is well on its way to achieving a status comparable to classical fluid dynamics [1], except that the second prerequisite, investigation of innumerable boundary layers, is quite incomplete at the present time.

Another candidate for this role as the source of a strong mathematical intuition about plasmas is the partly microscopic and partly macroscopic <u>guiding center plasma</u> (GCP) and the associated fully macroscopic but anisotropic <u>guiding center fluid</u> (GCF). These models are based on the simplified asymptotic description of a charged particle orbit in the limit of small gyro radius. We recall that there is a standard procedure for the formulation of an exact kinetic description (Liouville equation) when given the equations of motion of an individual particle. And there are standard (though heuristic) procedures for reducing a kinetic model to a macroscopic one. The kinetic result, when carried out with the "exact" particle equations is called the Vlasov equations. The simplest procedure in deriving the GCP equations is to accept as given the orbits that are obtained as the leading term in the guiding center asymptotic expansion of the ordinary differential equations of motion. We can then write down the Liouville equation for these artificial orbits. A more intricate procedure is to introduce the guiding center expansion directly into the Vlasov equations. This amounts to repeating the guiding center orbit expansion in the context of the Vlasov equations, but since the expansion is so singular, the calculations present themselves quite differently.

The two methods of obtaining the GCP equations were introduced simultaneously and independently, the simpler by Grad [2] (later by Sagdeev et al. [3]), and the more complex by Goldberger [4]. Extensions of the GCP theory using the second method were made by Chew, Goldberger, and Low [5]; by Brueckner, Chandrasekhar, Kaufmann, and Watson [6], [7], [8]; by Rosenbluth and Longmire [9], and Rosenbluth and Rostoker [10]; and following the simpler procedure by Grad [11] and Kulsrud [12]. In a slightly different direction, variational and quasi-variational formulations of the theory have been made by Kruskal and Oberman [13], Kulsrud [14], [12], Trubnikov [15], and Grad [16]. In perturbations about a uniform magnetic field the GC limit is treated in [3], by Chandrasekhar et al. [17], by Rose [18], and by Kadish [19].

We shall give a brief account of most of the GCP and GCF theory, for which there is a modicum of existence theory (and more of nonexistence!), paying particular attention to recent developments in stability theory. The latter is by far the most subtle and difficult part of the subject, as witnessed by the fact that the literature is in large part contradictory. For further details in the variational stability theory we refer primarily to [16], [29], [30], also to [22], and [24]-[27].

Probably the most interesting feature of the GCP is that it is microscopic in only one dimension, along the field lines. This is an immediate consequence of the singular orbit expansion which yields a constraint (first order system) for the perpendicular motion and a conventional second order equation only for the motion in the parallel direction. A further source of singular behavior in the GC limit is the fact that momentum, mnv, is multiplied by the large expansion parameter e/m to become current, env. As a result, the fluid velocities and electric currents that appear in the GCP are independent concepts, as distinguished from the Vlasov theory.

It is not surprising, in view of the very singular GC limit, that the GCP and GCF equations turn out to be mathematically unacceptable in some parameter ranges. However, the nonexistence of solutions under some circumstances can be turned to advantage. We interpret this phenomenon as heralding the appearance of instability. This is not a logical consequence but a physical interpretation. Moreover, if we credit the equations with complete honesty, they are telling us that the growth rate of these instabilities is too rapid to be described by the theory and must be governed by parameters (such as gyro frequency and plasma frequency) which have been set equal to infinity. Such an instability has been traditionally called a micro-instability.

Another distinguishing feature of the micro-instabilities within the GCP theory is that they are localized physically. If unstable, they can be excited by arbitrarily localized disturbances. If micro-stable, then any remaining instability is necessarily global. The GCP therefore has the pleasant feature of distinguishing categorically between micro- and macro-instabilities, both in localization and in growth rate.

Micro-instability makes its appearance in three distinct ways: in both the initial value problem and in the boundary value problem for static equilibrium by violating well-posedness or continuous dependence criteria; and in non-existence of solutions of the integral equation which governs the charge neutralizing potential.

There are several sets of auxiliary assumptions and approximations that can be made within the framework of the GC theory. For example, displacement current can be kept or dropped. The latter yields a more elegant Galilean invariant system, is simpler, and turns out to be not more restrictive. In some treatments higher order terms in the GC expansion are kept. This can, in principle, extend the range to cover new phenomena (in the realm of micro-stability). But it also restricts the range of validity in other respects (both micro- and macro-stability). If done without care, a partial contribution from the higher order GC terms can narrow the domain of validity of the equations in one direction without any compensating advantages in others (e.g., in the analyses [21], [23] of interchange stability). We shall distinguish between a two-fluid treatment and a one-fluid treatment which does not differentiate electrons from ions. The distinction is subtle, and combining the two fluids can only be considered a marriage of convenience. All of the references cited except for Grad [11], [16], [29] and Andreoletti [28] are single fluid (or incorrectly formulated as two fluids).* An adiabatic system of equations is introduced in Sec. 5 based on an entirely different set of state variables than the normal GCP. It is macroscopic in the sense that magnetic field displacements and motions alone enter; but it is distinct from the GCP. These equations are related to the second adiabatic invariant and they can give information about some classes of instabilities if interpreted carefully.

While our main purpose is to present a coherent account of various results which have already appeared in the literature (but in conflicting and inconsistent expositions), we also

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*In [3] and [17] the specialization to a two-temperature Maxwellian sidesteps this distinction.

present several new results. These are principally the theory of the neutralizing potential (Secs. 10 and 12), and most of the equilibrium theory (Secs. 9 and 10); also the Galilean representation of the Vlasov equations (Sec. 4), the treatment of interchange stability at the plasma edge (Sec. 14) and the adiabatic equations (Sec. 5).

2. Plasma Stability

The subject of plasma stability is very rich in the choice of alternative procedures and even equations of motion. Within the GC theory there are many definitions of stability, some explicit and some only implicit, frequently noncomparable and often contradictory. To some extent this proliferation is necessary because of the difficulty of the subject. If, for example, boundedness of solutions of the nonlinear equations of motion is an unattainable goal, some other definition of stability must be introduced. Even this "best" definition is not at all capable of answering the questions which would be asked by the designer of a plasma containment experiment. But totally unnecessary difficulties arise when the formulations and even the equations of motion are not specified.

As we have mentioned, there are essentially no results known with regard to boundedness of solutions in the non-linear GCP theory. Linear boundedness is not comparable with nonlinear, so it must be taken as an independent definition of stability. But this also is frequently too difficult, and we retire to the still simpler concept of exponential stability, the presence or absence of exponentially growing normal modes. Explicit counterexamples can be given for the GCP of the existence of growing solutions when there are no eigenvalues in the right half plane.

A still more special concept is that of marginal stability. This requires that ω^2 have a real spectrum, that no modes $te^{i\omega t}$ can arise from multiple poles, and that $\omega^2 > 0$ transfer to $\omega^2 < 0$ only by passing through the origin. In other words, the motional operator should be self adjoint, the continuous spectrum, if any, should not approach the origin, and the spectrum should be bounded and depend continuously on parameters. Each of these requirements is violated within the GCP theory.

Almost all of plasma physics is blessed with the presence of continuous spectra. The common streaming operator, $\xi \partial f/\partial x$, has the entire imaginary axis as a continuum. The Poisson bracket [H,f] which describes the GC motion along a magnetic line has as a continuum all transit frequencies ω of particles between turning points. This continuum covers the entire ω-axis (in particular including the origin) since, in the restoring potential μB, we have $0 < \mu < \infty$. Although the complete GC operator couples the Poisson bracket $[H,f]$ with other operators, this will not disturb the continuum.

As to continuous dependence of the spectrum, consider the anisotropic wave equation (the right side is self adjoint)

$$u_{tt} = c_1^2 u_{xx} + c_2^2 u_{yy} \tag{2.1}$$

in a rectangular domain with $u = 0$ at the boundary. We shall see (Sec. 5) that this example is directly related to stability of a uniform GCP. Suppose that c_2 approaches zero; $c_2 = 0$ is marginal. The lowest eigenvalue is

$$\omega^2 = \pi^2(c_1^2/L_1^2 + c_2^2/L_2^2) > \pi^2 c_1^2/L_1^2 \tag{2.2}$$

and it remains bounded away from zero. It is the high order eigenvalues that change sign with c_2^2 by passing through infinity rather than the origin.

We should distinguish the marginal criterion $\omega \to 0$ (which is required, e.g., to justify use of the second adiabatic invariant) from the distinct condition $\omega/k \to 0$ which occurs in various complex analyses of stability and is more frequently satisfied in a transition from stability to instability. In the uniform plasma [as in (2.1)] the dispersion relation involves only the ratio ω/k. Therefore a transition to instability which occurs for large k can be discovered by setting $\omega = 0$, even though the transitional mode is not slow. We shall find that some calculations based on the concept of marginal stability give correct answers and others give incorrect answers. To distinguish one from the other it is necessary to abandon this concept and examine the question from a less precarious position.

An entirely different class of stability analyses is concerned with variational methods instead of boundedness of solutions. The basic argument is drawn from the existence of an energy integral, $K + \Phi$ = const. where $K > 0$ is kinetic and Φ is potential. If Φ_0 is a minimum value (subject to appropriate constraints), then any perturbed motion which has an energy constant $\Phi_0 + \delta$ will satisfy the inequalities

$$0 < K < \delta \ , \quad \Phi_0 < \Phi < \Phi_0 + \delta \ . \tag{2 3}$$

This we can interpret as stability. The precise consequences depend on what departures from equilibrium are allowed by a small value of K and a prescribed excursion of Φ from its minimum. We take minimum Φ_0 (with a suitable choice of neighborhood) as the <u>definition</u> of variational stability. To correspond to the classical case of finite degrees of freedom, stationary Φ should give an acceptable definition of equilibrium. The connection with boundedness of solutions is not simple (in either direction, necessary or sufficient), and the results depend crucially on completing the definition by a specification of the class of admissible functions.

As in the case of boundedness, there is no direct connection between the linear and nonlinear formulations, here between existence of a local minimum and positivity of the second variation. We adopt $\delta^2\Phi > 0$ as the definition of linear variational stability. We shall give several examples of situations which are linearly stable but are unstable with respect to finite displacements of arbitrarily small amplitude (Secs. 11 and 14).

There is a direct connection between the sign of the second variation and linear boundedness of solutions if the system is self-adjoint and the spectrum is discrete. In this case the normal modes are obtained by minimizing $\delta^2\Phi$ with respect to the independent variation subject to suitable constraints (Rayleigh's principle). This connection is lost, even with a self-adjoint system, if there is a continuum which extends to the origin. One can have unbounded solutions, in this event, even with a strictly positive definite $\delta^2\Phi$.* On the other hand, $\delta^2\Phi > 0$ is equivalent to <u>exponential</u> boundedness when the system is self-adjoint, provided that we define boundedness to be in L_2. It is not clear whether there is any definite connection between $\delta^2\Phi > 0$ and exponential boundedness when the system is not self-adjoint; and one can show by explicit counterexamples that $\delta^2\Phi > 0$ is neither necessary nor sufficient for simple boundedness of solutions.

Although self-adjointness has not been established, a

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*There is reason to believe that such degenerate behavior at the origin may hide within the GCP theory certain types of instabilities (e.g., "drift") which might seem to be excluded from a lowest order GC theory.

closely related formal symmetry of the single fluid GCP equations of motion has been established by Kulsrud for an unperturbed equilibrium which is a monotone function of energy, $\partial f^{o}/\partial \varepsilon < 0$ [12]. From the associated variational principle, it can be seen that the continuous spectrum extends to the origin under most circumstances. And when this monotonicity condition is violated (as it often is experimentally), explicit examples can be given of normal modes with complex exponents.*

The variational approach gives an improved tool to replace the discredited notion of marginal stability in the pessimistic variation. The marginal analysis is based on the second adiabatic invariant, which is a dynamical constant for slow (marginal) motions. This invariant shares two distinct properties which are basic to the entropy. One is the dynamical property of being constant for a slowly varying system. The second property is convexity as a function of energy which characterizes equilibrium as the state of minimum energy at fixed entropy. The first property suggests the marginal stability hypothesis. The second leads to the pessimistic variation. The pessimistic variation and the second adiabatic argument lead to identical computations in some cases and to quite different conclusions in others. Where the transition to instability does occur through the origin, the adiabatic calculation will be correct. Where the adiabatic calculation gives a correct answer even

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* A. Kadish, private communication.

though the transitional state is not in slow motion [cf. (2.2)], it is only because it happens to agree with the pessimistic calculation. In cases where it is obvious that one cannot expect adiabatic behavior, such as with multiple turning points or in a uniform infinite plasma (the invariant cannot be defined), only the pessimistic variation is available. But there are less obvious cases where the adiabatic invariant can be defined but where the two procedures disagree violently, e.g., with non-monotone $f^{o}(\varepsilon)$.

The significance of the monotonicity criterion is shrouded in ambiguity. One reason is that there are two independent stability criteria involving partial derivatives of f with respect to energy, $\partial f/\partial \varepsilon$, holding different variables fixed. The two separate monotonicity requirements cure two distinct types of instability; in one case as a sufficient condition for interchange stability (Sec. 14), in the other as a necessary condition for local stability (Sec. 13). To complete the confusion, there is an important special case where the two partial derivatives coalesce. And the second adiabatic invariant argument which greatly simplifies the analysis of one instability, completely misses the other.

In addition to the cross-confusion, each of the partial derivatives $\partial f/\partial \varepsilon$ is subject to its own individual misunderstandings. With regard to interchange stability, the appropriate derivative $\partial f/\partial \varepsilon < 0$ is correctly taken as a sufficient condition. But, even though the literature abounds with counterexamples, it is incorrectly claimed to be sufficient for absolute stability (even where an author himself quotes a counterexample).

Turning to local stability, we find three distinct attitudes toward the criterion $\partial f/\partial \varepsilon < 0$. It is not mentioned at all as a requirement for stability by Rosenbluth and Rostoker in a marginal analysis [10] and by Taylor [21] and Andreoletti [26] in adiabatic arguments (but the latter two require the other derivative, $\partial f/\partial \varepsilon < 0$, for interchange stability). It is, in fact, impossible for any motion or variation which preserves the area of each energy contour (adiabaticity) to excite any instability which is generated by an inversion of f-values. The monotonicity condition is required by Kruskal and Oberman [13] but without reference to stability, simply to be able to carry out the calculations; they are neutral parties to the dispute. On the other hand, a variational analysis (using the pessimistic instead of the adiabatic variation) characterizes all non-monotone distributions as unstable [16].

An accurate study of boundedness of solutions of the equations of motion should give a result in between the completely optimistic marginal or adiabatic view and the completely pessimistic variational opinion. Partial results are known for non-monotone f only for the infinite uniform plasma [19]. The question for a contained plasma (using boundedness as the definition of stability) is not likely to yield lightly to analysis. However, some indications are available. Lack of existence of the neutralizing potential is as much a blow to the equations of motion as to the variational formulation; this gives certain necessary criteria on the degree of non-monotonicity of $f(\varepsilon)$ [Secs. 10 and 12]. The adiabatic equations (Sec. 5) are strictly interpreted as governing stability only for monotone $f(\varepsilon)$. But they describe a valid approximation to an actual motion whenever it is truly marginal (whatever is the sign of $\partial f/\partial\varepsilon$). Therefore, a prediction of adiabatic instability is a real instability if we independently verify that ω is zero at the transition. In a uniform plasma we have seen that an instability that transfers at large k can be found (without any marginal connotation) by setting $\omega = 0$. This indicates that a transfer of stability predicted by the adiabatic equations for small wavelength disturbances may also be legitimate. Even in the infinite plasma there are instabilities associated with non-monotone f which cannot be found in this way. In a contained plasma, a little reflection shows that there are probably many more.

A variational analysis has several advantages over one involving direct examination of the motion. It is sufficiently simpler so that some nonlinear results become accessible (cf. Secs. 11 and 14). It is also more complete in that a natural boundary condition is supplied if one is inadvertently or intentionally omitted. But the chief advantage is that we have a certain freedom in choosing the admissibility class according to mathematical expediency. This avoids the necessity of a detailed (and usually impossible) study of exactly which states are accessible via an actual motion. It may require dropping a constraint which is known to govern all motions in order to convert a lower bound to an attained minimum (cf. the pessimistic variation, Sec. 11). The admissibility class should be wide enough to include all states which can arise in a motion, and narrow enough not to unduly restrict the class of equilibria.

There are several analyses of GCP stability which lie somewhere between a variational and a full-blown study of actual motions, viz., by Trubnikov [15] and by Kruskal and Oberman [13]. Trubnikov considers the time derivatives $\dot{K}$ and $\ddot{K}$ of the kinetic energy of a motion as it passes through a state of equilibrium at $t = 0$. He verifies that $\dot{K} = 0$ and takes $\ddot{K} < 0$ as the definition of stability. At first glance this looks very similar to the condition $\delta^2\Phi > 0$. One basic objection in principle is that it says nothing about the stability of a state which is close to equilibrium but which is not an immediate antecedent or successor in time to the exact equilibrium. It is exactly this question of accessibility of states during a motion which is avoided by taking the more contrete admissibility class in a true variational analysis.

A second more elementary difficulty is that a growing oscillation (complex eigenvalue) would inevitably be classified as stable. Thus self-adjointness seems to be a prerequisite.

But perhaps the most interesting difficulty with the Trubnikov procedure is an acute sensitivity to how one separates the variables which describe the plasma state into <u>coordinates</u> and <u>rates</u>. In a classical Hamiltonian formulation there is an unambiguous one-to-one correspondence between coordinates and momenta. The coordinates would be fixed at an equilibrium and $\ddot{K}$ would be evaluated for all initial momenta. We describe three different choices of the coordinates that can be taken for the GCP, to each of which one can apply the Trubnikov procedure.

The most obvious course of action is to take the magnetic field configuration and the distribution function as coordinates, leaving only the magnetic line velocity as a rate. This is an unbalanced but natural choice. If we follow Trubnikov's recipe with this assignment, we do not get his answer, but we obtain a much more special formula, viz., the <u>transverse</u> second variation of Sec. 11. This is a special variation in which f is not varied at all; it arises in this context because not only K but even $\dot{f}$ (which is equivalent to the first variation of f) vanishes at $t = 0$.

Trubnikov actually evaluates $\ddot{K}$ using a special approximation involving the second adiabatic invariant, and in doing so obtains the pessimistic second variation of Sec. 11. But this is not an incidental approximation; it amounts to a different choice of the state variables and of the equations of motion. Taking the adiabatic invariant as constant implies that a unique distribution function is assigned to any given magnetic configuration. The distribution function is no longer an independent coordinate, and the resulting adiabatic equations of motion [pessimistic if $f^{O}(\varepsilon)$ is monotone] involve only the magnetic field. The displacement and velocity of field lines form a balanced set of coordinates and momenta.

A third choice of coordinates is that taken by Kulsrud in his symmetric version of the equations of motion (see Sec. 5). Symmetry is found by splitting $f = \overline{f} + \hat{f}$ into even and odd components in the parallel component of velocity, taking field displacement and $\overline{f}$ as coordinates and field velocity and $\hat{f}$ as momenta. The Trubnikov method applied here gives still a different result; the full second variation $\delta^2\Phi$ but with $\overline{f}$ instead of f as the argument.

The Trubnikov recipe is very flexible but it seems to require independent confirmation of the calculation by a more fundamental method.

Several interesting and ingenious definitions of stability have been introduced by Kruskal and Oberman. The first criterion is simply that the given equilibrium be the unique state which is compatible with "all regular time-independent constants of the motion." In a classical system with finite degrees of freedom, uniqueness is essentially equivalent to stability. The significance with respect to more complicated systems is not clear. Uniqueness is investigated by varying the total energy, kinetic plus potential, subject to the stated constraints. At a later stage this criterion is altered by removing the positive kinetic and electrostatic energy and examining the sign of the remainder, called δW. To investigate the value of δW subject to <u>all</u> constants of the motion means to take as the admissible class only actual motions. This would be impossible to pursue. But the admissible class is unwittingly widened in two directions. First the equation governing the motion of the field lines (momentum conservation) is ignored, making the field displacement and magnetic line velocity independent of the distribution function and losing undetermined constants of the motion. If we were to literally keep all remaining constants, the admissible class would now be a class of virtual motions in which the field variation is chosen arbitrarily as a function of time, and the distribution function evolves in accordance with the correct kinetic equation. This procedure is the same as Trubnikov's in the first version described above and would lead to the <u>transverse</u> variation for δW. But Kruskal and

Oberman actually constrain f by a very specific analytic expression which is far from all inclusive, and, in so doing, obtain an expression comparable to the full second variation, $\delta^2\Phi$. A final formal minimization with respect to the distribution function yields a formula similar to the pessimistic variation. Analysis of the minimization shows that the minimum can be achieved in general only by relaxing even the explicit constraint formula of Kruskal and Oberman.

An unusual formulation has been introduced by Taylor [21] and Andreoletti [23], [28]. They take particle orbits as governed by constant second adiabatic invariant as well as magnetic moment and take flux line coordinates as the dynamical variables. These dynamical equations of motion describe a higher order GC "drift" of particles which are not frozen to magnetic lines (as they are in the GC theory described in the present paper). But this is combined with a flux equation governing the magnetic field variation [see (3.9)] which is compatible only with particles which are frozen to magnetic lines. Energy variations are computed for stability purposes assuming that the particles are frozen; in fact, for the partially higher order GC approximation used, there is no relevant energy conservation law. Since the dynamical (drift) equations are used only to delimit the class of equilibria, not to govern motion or stability, all results derived from this procedure are equivalent to a marginal zero order GC theory or to the adiabatic equations of motion (Sec. 5).

It should be clear from this discussion that the unadorned terms stable, unstable, necessary, sufficient are subject to an infinite variety of interpretations. We consider this not as a fault but as evidence of the infinite richness of the subject. But this wealth must elude us if it is not subject to acceptable auditing standards.

3. Orbits and Electromagnetic Potentials

The classical equations of motion of a charged particle are

$$m \frac{dv}{dt} = e(E + v \times B) \tag{3.1}$$

where $E(x,t)$ and $B(x,t)$ are assumed to be given. The particle motion is closely tied to Maxwell's equations. For example, a Lagrangian exists only if E and B satisfy the two homogeneous Maxwell equations,

$$\begin{aligned} \operatorname{div} B &= 0 \\ \frac{\partial B}{\partial t} + \operatorname{curl} E &= 0 \end{aligned} \tag{3.2}$$

In this case, the Lagrangian and Hamiltonian are given by

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} mv^2 + ev \cdot A - e\phi \\ \mathcal{H} &= \frac{1}{2m} (p - eA)^2 + e\phi \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} B &= \operatorname{curl} A \\ E &= - \frac{\partial A}{\partial t} - \nabla\phi \quad . \end{aligned} \tag{3.4}$$

It will be convenient to discuss the electromagnetic field before turning to the orbits. First suppose there is given an incompressible vector field, div B = 0, which varies smoothly as a function of a parameter, $B(x,t)$. In a domain which is simply covered by the vector field, Fig. 1,

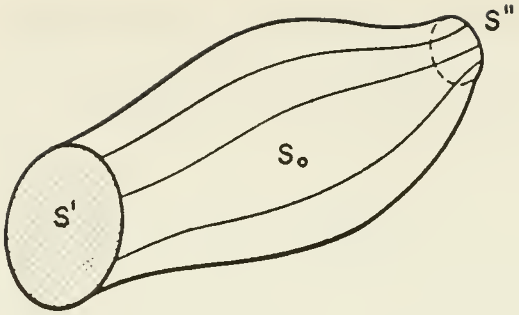

FIGURE 1

we can introduce flux coordinates $\alpha(x)$, $\beta(x)$ such that

$$B = \nabla\alpha \times \nabla\beta \ . \tag{3.5}$$

On a transverse surface, $B \cdot dS = d\alpha d\beta =$ flux. Since α and β are constant on each line, they can be considered as coordinates for the field lines. There is considerable flexibility in choosing α and β. For example, α can be chosen arbitrarily on one transverse surface (S' in Fig. 1) and β still has some freedom. This arbitrariness holds for each value of t, and we only require $\alpha(x,t)$ and $\beta(x,t)$ to be smooth in t. After making a choice, we adopt the expedient of identifying fixed values $\alpha = \alpha_o$, $\beta = \beta_o$ as the <u>same</u> line, moving with t. It now becomes possible to speak of the velocity, U, of a given line, and we may clearly set $U \cdot B = 0$ to fix U uniquely. To be specific, we introduce a Lagrangian parameter σ along each line (e.g., arclength at $t = 0$) such that the curve $\sigma =$ constant on a given moving line is orthogonal to the field lines. The complete set of Lagrangian coordinates with respect to the flow U is (α,β,σ) and we have

$$\frac{D\alpha}{Dt} \equiv \frac{\partial\alpha}{\partial t} + U\cdot\nabla\alpha = 0$$
$$\frac{D\beta}{Dt} \equiv \frac{\partial\beta}{\partial t} + U\cdot\nabla\beta = 0 \tag{3.6}$$

where $\partial/\partial t$ and ∇ refer to the Eulerian variables (x,t). A little vector manipulation yields the formulas

$$B \times U = \frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta \tag{3.7}$$

$$U = \frac{1}{B^2}(\nabla\alpha\nabla\beta - \nabla\beta\nabla\alpha)\cdot\left(\frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta\right) \tag{3.8}$$

$$\frac{\partial B}{\partial t} + \mathrm{curl}(B \times U) = 0 \tag{3.9}$$

We repeat that the velocity U which describes a given field variation $B(x,t)$ is far from unique and depends on a quite flexible assignment of the coordinates (α,β) at each t.

Now suppose we are given a pair of vector fields $E(x,t)$, $B(x,t)$ which satisfy the field equations (3.2) and also $E\cdot B = 0$ for all t. We define a vector U by $U = E \times B/B^2$ or

$$E + U \times B = 0\ ,\ U\cdot B = 0\ . \tag{3.10}$$

With this definition, U and B satisfy (3.9). It is a well known consequence of this differential equation that magnetic lines are mapped onto magnetic lines by the motion U, so that coordinates (α,β) which are carried by the flow can be introduced, $B = \nabla\alpha \times \nabla\beta$ and $D\alpha/Dt = D\beta/Dt = 0$. By the previous analysis we conclude that

$$E = \frac{\partial\beta}{\partial t}\nabla\alpha - \frac{\partial\alpha}{\partial t}\nabla\beta \ . \tag{3.11}$$

This relation holds not for arbitrary flux coordinates but only for a proper choice of (α,β) since U has been uniquely chosen in this case.

We present an independent derivation of (3.11) without making use of the theory of the equation (3.9). Using the previous analysis of B alone, we introduce arbitrary parameters $(\overline{\alpha},\overline{\beta})$ associated with a magnetic flow velocity $\overline{U}$. Since U, defined in (3.10), and $\overline{U}$ each satisfy (3.9), we have

$$E = \frac{\partial\overline{\beta}}{\partial t}\nabla\overline{\alpha} - \frac{\partial\overline{\alpha}}{\partial t}\nabla\overline{\beta} + \nabla\phi \ . \tag{3.12}$$

From $E\cdot B = 0$ we find $B\cdot\nabla\phi = 0$ and conclude that ϕ is a function of $(\overline{\alpha},\overline{\beta},t)$ only. Thus

$$E = \left(\frac{\partial\overline{\beta}}{\partial t} + \frac{\partial\phi}{\partial\overline{\alpha}}\right)\nabla\overline{\alpha} - \left(\frac{\partial\overline{\alpha}}{\partial t} - \frac{\partial\phi}{\partial\overline{\beta}}\right)\nabla\overline{\beta} \ . \tag{3.13}$$

If we now choose α and β to satisfy

$$\begin{aligned} \frac{D\alpha}{Dt} &= \frac{\partial(\phi,\ \alpha)}{\partial(\overline{\alpha},\ \overline{\beta})} \\ \frac{D\beta}{Dt} &= \frac{\partial(\phi,\ \beta)}{\partial(\overline{\alpha},\ \overline{\beta})} \end{aligned} \tag{3.14}$$

and the initial condition

$$\frac{\partial(\alpha,\beta)}{\partial(\bar{\alpha},\bar{\beta})} = 1 \ , \tag{3.15}$$

then a small computation shows that E has the form (3.11). Equation (3.14) for α and β is a Liouville equation in the $(\bar{\alpha},\bar{\beta})$-plane with $\phi(\bar{\alpha},\bar{\beta},t)$ as Hamiltonian. The derivative D/Dt is to be taken with $(\bar{\alpha},\bar{\beta})$ fixed. Equation (3.14) states that the values of α and β are carried with the flow velocity $(-\partial\phi/\partial\bar{\beta}, -\partial\phi/\partial\bar{\alpha})$, and the Jacobian relation (3.15) is easily seen to hold for all t.

Finally we turn to a pair of vector fields E(x,t) and B(x,t) which satisfy (3.2) and $E\cdot B = \mathcal{O}(\delta)$ where δ is an extraneous parameter taken to be small. If we define $U = E \times B/B^2$, then the flux conservation equation (3.9) is satisfied only to within $\mathcal{O}(\delta)$, and magnetic lines are carried by the velocity field U only to within $\mathcal{O}(\delta)$. It will be much more satisfactory to modify the definition of U by a small amount $\mathcal{O}(\delta)$ in order to preserve (3.9) exactly.

To do this we construct a function $\phi^*(x,t)$ with the property that $E\cdot B = \nabla\phi^*\cdot B$. To find ϕ^*, we merely integrate $\phi^* = \int E\cdot dx$ along each magnetic line from an arbitrary transverse surface on which $\phi^* = 0$. From the construction, ϕ^* itself is $\mathcal{O}(\delta)$. The vector field

$$E^* = E - \nabla\phi^* \tag{3.16}$$

differs from E by $\mathcal{O}(\delta)$ and satisfies Maxwell's equation (3.2) as well as $E^* \cdot B = 0$. We construct the velocity field U from E^* and observe that (3.6)-(3.9) are satisfied together with

$$
\begin{aligned}
E + U \times B &= \nabla \phi^* \\
E &= \frac{\partial \beta}{\partial t} \nabla \alpha - \frac{\partial \alpha}{\partial t} \nabla \beta + \nabla \phi^*
\end{aligned}
\tag{3.17}
$$

The parallel component of $\nabla \phi^*$ is unique but ϕ^* itself is not unique to within an arbitrary added function of (α,β,t) [which we restrict to be $\mathcal{O}(\delta)$]. Thus U is unique to within $\mathcal{O}(\delta)$.

After these preliminaries, we turn to the particle orbits. To be definite we consider m to be fixed and study the orbits in the limit of large e. We assume that E and B are bounded, B is bounded away from zero, and require that v remain bounded. The last requirement, bounded energy, implies that the component of E in the direction B approaches zero with 1/e. In other words, there is no satisfactory limiting behavior as $e \rightarrow \infty$ unless

$$
E \cdot B = \mathcal{O}\left(\frac{1}{e}\right) . \tag{3.18}
$$

In order to study the behavior of individual orbits, we merely postulate this relation. In order to apply the result to a plasma it will have to be established that the collective particle motions give rise to such an electric field.

This we leave to later.

In the limit of large e, a particle will execute high frequency tight oscillations about a guiding center (GC). The asymptotic description governs the motion of the GC and the energy of the oscillation. As is usual in an asymptotic expansion, the parameters which appear in the asymptotic description (e.g., the GC coordinates) are not simply related to the variables (x,v) taken from the exact equations. This is just the familiar question of assigning a definite asymptotic series to given initial values. We do have an asymptotic formula which relates the asymptotic coordinates to the instantaneous state, and in a few special cases (primarily a constant magnetic field region), the relation is exact.

The motion of a GC transverse to the magnetic field is given by the velocity $E \times B/B^2$ to lowest order. This is true for all particles, independent of mass, sign or magnitude of e, or energy. Consistent with this description, we can introduce the velocity U of (3.17) (not exactly $E \times B/B^2$) which carries magnetic lines into themselves and state that a GC is constrained to lie on a given (moving and deforming) magnetic line. The motion of the GC along the magnetic line can be described as that of a classical constrained particle subject to the additional potential $\mu B + e\phi^*$ where μ is a constant. The potential ϕ^* [as in (3.17)] is unique within an added constant on a given line, and from

(3.18) and the preceding analysis, $e\phi^*$ is of order unity. The constant μ (magnetic moment) is related to the oscillatory energy by the asymptotic expression,

$$\mu = \frac{1}{2} m v_\perp^2 / B \ . \qquad (3.19)$$

The fact that μ is constant is the equation that governs the variation of the oscillatory energy; the variation of B on a trajectory is determined by the GC motion.

The Lagrangian of a freely moving constrained particle is $\frac{1}{2} mv^2 + \frac{1}{2} mU^2$ where the second term is to be interpreted as the constraining potential. Therefore, for the GC motion along a line we have

$$L = \frac{1}{2} m \frac{\dot{\sigma}^2}{\zeta^2} + \frac{1}{2} mU^2 - \mu B \mp \phi \qquad (3.20)$$

$$H = \frac{1}{2m} \zeta^2 p^2 - \frac{1}{2} mU^2 + \mu B \pm \phi \ . \qquad (3.21)$$

We have introduced σ, the Lagrangian coordinate following U, and with it

$$\zeta = \frac{\partial \sigma}{\partial s}$$

$$v = \frac{\dot{\sigma}}{\zeta} = p \frac{\zeta}{m} \ . \qquad (3.22)$$

Also, we have incorporated the factor e into the potential ϕ (which is now of order unity); $e\phi^*$ is written $+\phi$ for ions and $-\phi$ for electrons.

The perpendicular and parallel components of the electric field enter quite differently into the GC formulas. In the decomposition (3.16) E^*, of order unity, determines the transverse GC motion through $E^* + U \times B = 0$. The potential ϕ^*, of order $1/e$, enters the lowest order Hamiltonian (3.21). It is therefore necessary to consider E^* and $e\phi^* = \pm\,\phi$ as independent lowest order entities, forgetting their common origin. We shall use E henceforth instead of E^*; thus E is a vector which is orthogonal to B, and ϕ is an unrelated scalar which is defined only to within an added constant on each line. The entire collective plasma analysis to follow will treat E and ϕ distinctly.

To summarize, the leading term in the asymptotic expantion for large e describes a GC which is constrained to follow a given magnetic line, moving along the line according to the Hamiltonian (3.21). The fact that μ, appearing in the Hamiltonian, is a constant governs the oscillatory energy. To complete the identification of the asymptotic coordinates we should also introduce the oscillatory phase

$$\theta = \frac{e}{m} \int B \, dt \ , \qquad (3.23)$$

oriented relative to fixed (α,β). The six coordinates (x,v) are related in an elementary way [using the gyro radius $\lambda = (2m\mu/B)^{1/2}/e$] with the GC coordinates $(\alpha,\beta,\sigma,v,\mu,\theta)$. The parameter θ will not enter into any of the subsequent analysis.

Most orbit analyses go much deeper into the theory than the leading term which is all we require for this theory; general references are [31] and [32] and for proofs [33].

4. The Vlasov Equations--Charge Neutrality and Galilean Invariance

We take as a kinetic equation the Liouville equation formed with the "exact" Hamiltonian (3.3),

$$\frac{\partial F}{\partial t} = \frac{\partial(\mathcal{H},F)}{\partial(\underline{x},\underline{p})} \tag{4.1}$$

Here $F(\underline{x},\underline{p},t)$ is the particle density in the phase space $(\underline{x},\underline{p})$ of a single particle. There are two equations (4.1) for F_+ and F_- in terms of $\mathcal{H}_+$ and $\mathcal{H}_-$. The two kinetic equations (4.1) together with Maxwell's equations (3.2) and

$$\begin{aligned} -\frac{1}{c^2}\frac{\partial E}{\partial t} + \text{curl } B &= \mu_o J \\ \kappa_o \text{div } E &= q \ , \end{aligned} \tag{4.2}$$

and the identification of the electromagnetic source terms

$$\begin{aligned} q &= e(n_+ - n_-) \\ J &= e(n_+u_+ - n_-u_-) \end{aligned} \tag{4.3}$$

(the density n and the flow velocity u are the usual moments of F) comprise the Vlasov equations. There are some special existence theorems for this system of integro-differential equations,* and there is a certain amount of evidence to indicate that the system is mathematically sound and none to the contrary at the present time. On the other hand, the Vlasov equations are inelegant since they are invariant under neither Galilean nor Lorentz transformations. The system can be made Lorentz invariant by introducing a relativistic Hamiltonian. Or it can be made Galilean invariant in a charge-neutral approximation as described below.

It will be convenient to distinguish the charge and current that appear in Maxwell's equations (q_M, J_M) from the moments of F which are obtained from the kinetic equations (q_K, J_K). Each set independently [before making the identifications $q_K = q_M$, $J_K = J_M$ implied by (4.3)] satisfies charge conservation,

$$\frac{\partial q}{\partial t} + \operatorname{div} J = 0 . \tag{4.4}$$

From (4.4) we see that the identification $J_K = J_M$ implies $q_K = q_M$ provided that the latter is satisfied initially. It will be even more convenient to identify $\partial J_K/\partial t$ with $\partial J_M/\partial t$, in which case both the q and J identifications become initial restrictions only. From Maxwell's equations we compute

* D. Gorman and H. Weitzner, private communication.

$$\mu_o \frac{\partial J_M}{\partial t} = - \frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} - \text{curl curl } E \qquad (4.5)$$

and by taking moments of the kinetic equations,

$$\mu_o \frac{\partial J_K}{\partial t} = X + \frac{\Omega^2}{c^2} E \qquad (4.6)$$

where Ω is the plasma frequency and X does not contain E explicitly. Interpreting X, qualitatively, as an inhomogeneous term, the identification $\partial J_K/\partial t = \partial J_M/\partial t$,

$$\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} + \text{curl curl } E + \frac{\Omega^2}{c^2} E + X = 0 , \qquad (4.7)$$

describes electromagnetic waves coupled with plasma oscillations. We can take as equivalent to the original Vlasov system the two kinetic equations (4.1), Maxwell's equations, and equation (4.7) (plus initial data) to replace the source identification (4.3).

Before continuing, we remark on a qualitative interpretation of the <u>Debye length</u>

$$d = C_{-}/\Omega , \qquad (4.8)$$

(C_{-} is a mean electron thermal speed) and the <u>magnetic shielding length</u>

$$\delta = c/\Omega . \qquad (4.9)$$

The Debye length is the largest distance over which a finite fractional depletion of ions or electrons will give rise to a finite electrostatic field. Any reasonable electric field on a much larger length scale can be created by a small charge unbalance which leaves n_+ essentially equal to n_-. Intuitively (and in common finite difference schemes) one considers the charge separation q to give rise to E. But if d is a macroscopically small quantity (Ω is large), this is a very sensitive operation; numerically, the time step would have to be very small (comparable to $1/\Omega$). The charge neutral (or quasi-neutral) approximation consists of inverting this process. We set $n_+ = n_-$ as a constraint and compute what E must be in order to maintain neutrality. The constraint serves to determine the evolution of E rather than the time derivative of E (displacement current) which is dropped.

A similar situation holds with regard to the magnetic shielding distance, δ. It can be interpreted as the largest distance over which a finite relative motion J between ions and electrons gives rise to a finite magnetic field B. Any magnetic field which varies on a much larger length scale than δ can be produced by a current which represents a small relative motion (compared to mean thermal speed) between ions and electrons. Since magnetic forces are smaller than electric by a relativistic factor, the relation $\delta/d = c/C_-$ is revealing.

Formally, the charge neutral approximation in Maxwell's equations consists in dropping displacement current, curl $B = \mu_o J_M$, and completely ignoring the source relation

$\kappa_o \text{div } E = q_M$. We retain the <u>pre-Maxwell</u> system[34] ($J_M \rightarrow J_P$)

$$\frac{\partial B}{\partial t} + \text{curl } E = 0 \ , \ \text{div } B = 0 \tag{4.10}$$

$$\text{curl } B = \mu_o J_P \quad , \quad \text{div } J_P = 0 \tag{4.11}$$

To be compatible with the kinetic equations, E must be chosen to insure $q_K = 0$, div $J_K = 0$. This is accomplished by setting the pre-Maxwell value

$$\mu_o \frac{\partial J_P}{\partial t} = - \text{curl curl } E \tag{4.12}$$

equal to $\mu_o \partial J_K / \partial t$ as given in (4.6). The equation for E (heuristically uncoupled by looking at X as an "inhomogeneous" term) is now elliptic.

$$\text{curl curl } E + E/\delta^2 + X = 0 \ . \tag{4.13}$$

If X is given, E can be solved in an interior domain with a boundary condition on E_t, and in an exterior domain with suitable regularity at infinity. If by any means we obtain a solution, with q_K = div J_K = 0 initially, of the system of kinetic equations (4.1), pre-Maxwell equations (4.10), and constraint (4.13), then q_K = div J_K will vanish for all time since curl curl E has zero divergence.

There are no proofs of the mathematical legitimacy of this charge-neutral plasma formulation, but there is a certain amount of empirical evidence. For example, there is numerical evidence of a satisfactory solution in one dimension for a piston moved into a cold plasma, except that under certain circumstances one is forced to refer to a generalized weak solution with internal discontinuities [35].

A further reduction is obtained by considering "magnetic neutrality" with δ small (in a non-relativistic plasma this is always more restrictive than small d). We can interpret δ as a singular parameter in the elliptic equation (4.13). Except near a boundary or in a shock, we may set

$$E = -\delta^2 X \tag{4.14}$$

Making the charge neutral approximation in X, this takes the more explicit form

$$\frac{1}{\mu_0 \delta^2} (E + u \times B) = u \cdot \nabla J + J \text{ div } u + J \cdot \nabla u + \frac{m_+ - m_-}{e m_+ m_-} J \times B + \text{div } Q \tag{4.15}$$

$$Q_{ij} = \frac{e}{m_+} P^+_{ij} - \frac{e}{m_-} P^-_{ij} - \frac{m_+ - m_-}{ne(m_+ + m_-)} J_i J_j \tag{4.16}$$

[this expression is also correct for insertion into (4.13)].

Each of the forms (4.7), (4.13), (4.15) can be considered to be a collisionless version of Ohm's law. It is evident by inspection that (4.15) (and the entire coupled system of equations) is Galilean invariant provided that B and J are taken to be invariant under a change of frame and E transforms such that $E + u \times B$ is invariant. A slight computation shows that the expression (4.13) is similarly invariant. Transfer to a frame with a constant relative velocity u_o gives a term in curl curl E which is canceled by $u_o \cdot \nabla J$ in (4.15).

In the GC limit we let e become large (note that there are factors e in Ω, $1/d$, and $1/\delta$). The speed of light, c, is an independent parameter; taking it finite means using Maxwell's equations and letting it become large yields the pre-Maxwell equations (for definiteness we can take $\mu_o = 1$ and $\kappa_o = 1/c^2$). But examination of the complete source identification relation (4.7) yields a result which is independent of c to the two lowest orders in $1/e$. Whether or not c is kept in Maxwell's equations, it does not appear in the source compatibility equation. To lowest order we find $E + u \times B = 0$ (consistent with the GC orbit analysis), and to the next order we would need to include only a Hall term $J \times B$ and a term in div P from (4.15). The parallel component of E vanishes to lowest order; this confirms from the plasma equations the postulate that was needed in the GC orbit theory. The first order parallel component of E is given by the term in div P, about which more later.

We have based this discussion on a nonsteady plasma motion. The analysis must be reexamined in a steady flow since the casually dropped "initial restrictions" become dominant, and dominant terms such as curl curl E vanish.

5. Guiding Center Plasma Equations of Motion

The dominant feature of the GC motion is the constraint to a given magnetic line. Evidently the transverse velocity U is macroscopic since every particle, whether ion or electron, and without regard to mass or energy moves similarly. Any kinetic behavior occurs one-dimensionally along each magnetic line. To describe this motion we introduce the one-dimensional Liouville equation with a GC Hamiltonian (3.21),

$$\frac{\partial f}{\partial t} = [H,f] \equiv \frac{\partial H}{\partial \sigma}\frac{\partial f}{\partial p} - \frac{\partial H}{\partial p}\frac{\partial f}{\partial \sigma} \tag{5.1}$$

Here f is a function of $(\alpha,\beta,\sigma,p,\mu,t)$ where (σ,p) are the dynamical variables and (α,β,μ) are parameters. There are two such kinetic equations for ions and electrons. The GC distribution f is normalized as a number density in $\Omega = (\alpha,\beta,\sigma,p,\mu)$ and is related to the Vlasov distribution $F(\underline{x},\underline{p})$ by

$$f = 2\pi mF \tag{5.2}$$

through the Jacobian relations

$$\begin{aligned} d\underline{x} &= d\alpha d\beta d\sigma/\zeta B \\ d\underline{p} &= 2\pi m\, \zeta B\, dpd\mu \\ F\, d\underline{x}d\underline{p} &= f\, d\Omega \\ d\Omega &= d\alpha d\beta d\sigma dpd\mu \end{aligned} \tag{5.3}$$

An obvious way of interpreting this plasma is as a one-dimensional gas with a stress tensor

$$p_{ij} = p_1 b_i b_j \tag{5.4}$$

$$b_i = \frac{B_i}{B} \tag{5.5}$$

and with "molecules" which have internal magnetic moments μ and internal energy μB as well as translational kinetic energy $p^2/2m$. We have $\langle p^2/2m \rangle = \frac{1}{2}\, kT_1 = \frac{1}{2}\, p_1/n$. This substance is a very unusual magnetic material whose magnetic polarization is essentially unrelated (except in direction) to the applied field [2]. But, by reinterpreting the internal molecular energy as translational with a corresponding two-dimensionally isotropic stress component $p_2(\delta_{ij} - b_i b_j)$,

$$p_{ij} = p_1 b_i b_j + p_2(\delta_{ij} - b_i b_j) \tag{5.6}$$

we obtain an equivalent set of equations with a more familiar appearance.

The kinetic equations (5.1) govern the gas motion along a magnetic line, given its constraining motion. But, since U is a macroscopic velocity, the evident equation to govern its rate of change is the perpendicular component of the macroscopic equation of momentum conservation (which is universally valid for microscopic or macroscopic descriptions),

$$b \times [\rho(\frac{\partial u}{\partial t} + u \cdot \nabla u) + \operatorname{div} P - J \times B] = 0 \,. \tag{5.7}$$

Here u is the complete macroscopic velocity,

$$u = U + V \ , \ b \times V = 0 \ , \qquad (5.8)$$

and ρ and P are the sum of electron and ion mass densities and stress tensors respectively. The macroscopic moments for electrons and ions individually are defined as

$$\begin{aligned}
n &= \rho/m = \zeta B \int f \, dp d\mu \\
mV &= (\zeta^2 B/n) \int p \, f \, dp d\mu \\
\rho e_1 &= \tfrac{1}{2} nkT_1 = \tfrac{1}{2} p_1 = \zeta B \int \frac{(p-mV)^2}{2m} f \, dp d\mu \qquad (5.9) \\
\rho e_2 &= nkT_2 = p_2 = \zeta B \int \mu B \, f \, dp d\mu
\end{aligned}$$

Adding the electromagnetic equations

$$\begin{aligned}
\frac{\partial B}{\partial t} + \mathrm{curl}(B \times U) &= 0 \\
\mathrm{div}\ B &= 0 \qquad (5.10) \\
\mathrm{curl}\ B &= \mu_0 J
\end{aligned}$$

to the kinetic equations (5.1) and momentum constraint (5.7), we observe by counting variables that the system appears to be complete with the exception of the extra variable ϕ in the particle Hamiltonian. We have already committed ourselves to a charge neutral description [by dropping $\partial E/\partial t$ in (5.10) and by dropping the electric part of the Lorentz force terms in (5.7)]. This also determines ϕ. In the perpendicular direction, the charge neutrality constraint has already been satisfied by our choice of a common velocity, U, for ions and electrons. Taking $n_+ = n_-$ and $V_+ = V_-$ as initial conditions, we determine ϕ (or $b \cdot E$) by the constraint $\partial V_+/\partial t = \partial V_-/\partial t$. These derivatives are obtained as moment equations of the kinetic equations and we find the constraint,

$$(m_+ + m_-)n \frac{\partial \phi}{\partial s} = b \cdot [m_+ \text{div } P_- - m_- \text{div } P_+] \ . \qquad (5.11)$$

At any instant, ϕ is determined on each line within an irrelevant added constant.

To summarize, the GCP equations consist of kinetic equations (5.1) governing the parallel motion of the particles along a constraining magnetic line, a macroscopic momentum balance (5.7) which governs the transverse motion of the constraint, the flux equation (5.10), and the constraint (5.11) for the potential ϕ. If we wish, we can explicitly eliminate the variables J, E, and ϕ from the equations in favor of curl B, $U \times B$, and stresses. This leaves B as the only evident electromagnetic variable.

A more primitive version of the GCP theory is obtained as a one-fluid formulation ignoring the potential ϕ. This is strictly correct only if $m_+ = m_-$ and the ions and electrons are given identical initial conditions. Otherwise a nontrivial ϕ will generally develop during the motion. But the one-fluid theory, setting $\phi = 0$ in the GC Hamiltonian, does serve the purpose of a simpler theory which has many of the features of the more exact theory.

We remark that the charge neutrality condition is a necessity, not a choice. Within the formal GC expansion, charge and current, $q \sim ne$, $J \sim nev$, are potentially large compared to n and v. In order to avoid infinite charge and current as $e \to \infty$, we must have both charge and current neutrality, $n_+ = n_-$ and $u_+ = u_-$, but the resulting finite values of q and J are indeterminate from the kinetic equations. The choice $q = 0$, $\mu_o J = \text{curl } B$ results from completing the system by the pre-Maxwell equations. One also has the choice of retaining displacement current and the full Maxwell's equations. This depends on a decision with respect to the speed of light.which is independent of the GC parameter e. Both constraints, $E + U \times B$ and (5.11) for ϕ are independent of the disposition of c. But with finite c, the momentum equation takes the full Lorentz force term, $qE + J \times B$. In this expression we substitute $q = \kappa_o \text{div } E$ and $\mu_o J = \text{curl } B - \dot{E}/c^2$, remembering that $E = B \times U$. In the complete nonlinear system of equations the result is considerably more complex than the pre-Maxwell system. But in a linearization about a static equilibrium we lose qE as a second order term, and the only effect of displacement current in the entire system of equations is to increase the mass transverse to B. In the term $b_o \times \rho_o \partial u/\partial t$ in the linearized momentum equation, ρ_o is replaced by

$$\rho_* = \rho_o + B_o^2/\mu_o c^2 = \rho_o(1 + A_o^2/c^2) \quad . \tag{5.12}$$

We shall find that stability of a GC plasma is completely insensitive to the value of c (i.e., to the choice of Maxwell or pre-Maxwell formulation) in the nonlinear as well as the linear case even though the equations of motion are altered.

We recall that the CG orbit theory requires as a postulate that the parallel electric field be small. But the charge neutrality condition (5.11) yields as a conclusion that this is so for any GCP solution (the factor e has been removed from the potential). The more detailed physical mechanism involves high frequency plasma (and modified plasma) oscillations which are not explicit in the GC theory. Any parallel fields larger than $O(1/e)$ give rise to large velocities, oppositely directed for ions and electrons, which act to restore charge neutrality (except possibly in certain unstable situations, Sec. 10). More precisely, the neutralizing mechanism involves oscillations which are presumed to exist at only a low energy level since their presence would invalidate the orbit analysis.

If desired, the full momentum equation [bracket in (5.7)] can be taken as a governing equation, since the parallel momentum balance is a consequence of the kinetic equations. In this case, the identification of the parallel velocity V as a moment of f need only be made initially.

In addition to conservation of momentum one can derive, after considerable manipulation, a relation implying the conservation of energy. In almost all analyses of the GCP, conservation of energy is taken as an article of faith without verification. For the single fluid it was verified in [2], for two fluids in [16], and for the linearized equations (in which even the formula for the energy is quite nonintuitive) it has not been previously verified at all.

Energy conservation for the GCP is a much more subtle affair than the equivalent relation for the Vlasov equations. In the latter case, taking $\frac{1}{2}(\kappa_o E^2 + B^2/\mu_o)$ for the Maxwell and $B^2/2\mu_o$ for the pre-Maxwell electromagnetic energy density, we obtain an electromagnetic source term $J \cdot E$ and a balancing kinetic source term $J \times B \cdot U$. On the microscopic level, a particle gains energy as it moves (creating a current) in the direction of E. But a guiding center particle always moves transverse to E; it is a small drift velocity of order 1/e (not in the GC theory) which, when converted to a finite current by multiplication by the large parameter e, gives the conventional energy source $E \cdot J$. This energy source is missing in the kinetic part of the GC theory. But there is an entirely different energy component, μB, which varies when the particle is carried to a region of different B. It is a deeply hidden virtue of the singular GC limit that energy, which one would imagine is essentially concerned with higher order drifts, is conserved within the lowest order theory. As a matter of fact, if one were to compute the higher order drifts and include the corresponding currents and energy sources, the plasma energy source would be counted twice. Working backwards, one can use the conservation of energy alone to compute the first order drifts directly from the lowest order GC results without further recourse to the differential equations.

Specifically, we take

$$\mathcal{K} = \int \frac{1}{2} \rho U^2 dx \tag{5.13}$$

$$\mathcal{U} = \int (\varepsilon_+ f_+ + \varepsilon_- f_-) d\Omega \tag{5.14}$$

$$\mathcal{M} = \int \frac{1}{2\mu_o} B^2 dx \tag{5.15}$$

for the kinetic, internal, and magnetic energies respectively;

$$\varepsilon = \frac{p^2}{2m} + \mu B \tag{5.16}$$

is <u>not</u> the Hamiltonian; the term $\frac{1}{2} mU^2$ must not be kept, and the terms $\pm \phi$ do not contribute to the integral (5.14). For the rate of change of kinetic energy, one can show

$$\frac{d\mathcal{K}}{dt} = \int U \cdot (J \times B - \operatorname{div} \overline{P}) dx - \int \rho V \frac{\partial}{\partial s} \left(\frac{1}{2} U^2\right) dx \tag{5.17}$$

where

$$\begin{aligned} \overline{P}_{ij} &= \overline{p}_1 b_i b_j + p_2 (\delta_{ij} - b_i b_j) \\ \overline{p}_1 &= p_1 + \rho V^2 = \zeta B \int (p^2/m) f \, dp d\mu \ . \end{aligned} \tag{5.18}$$

For the total plasma energy we obtain

$$\frac{d}{dt} (\mathcal{K} + \mathcal{U}) = \int U \cdot J \times B \, dx \tag{5.19}$$

and including the magnetic energy,

$$\frac{d}{dt}(\mathcal{K} + \mathcal{U} + \mathcal{M}) = -\frac{1}{\mu_o} \oint E \times B \cdot dS \ . \tag{5.20}$$

The boundary term takes this form only if the boundary is fixed. If the plasma itself terminates at a fixed wall, $E = B \times U$ in this integral. If the plasma is surrounded by an electromagnetic field, the interface conditions are such as to cancel in the energy balance, and E is evaluated at the fixed outer boundary of the entire plasma plus field domain (e.g., $E_t = 0$ at a perfectly conducting wall).

The subdivision of plasma energy into $\mathcal{K}$ and $\mathcal{U}$ is peculiar to the GCP formulation. In a macroscopic or Vlasov version, one would usually include a component $\int \frac{1}{2} \rho V^2$ in kinetic rather than internal energy.

The single fluid GCP equations linearized about an arbitrary static equilibrium are

$$\begin{aligned} \frac{\partial f}{\partial t} &= [H_o, f] + [H, f^o] \\ \rho_o \frac{\partial U}{\partial t} + (\mathrm{div}\ P)_\perp &= \frac{1}{\mu_o} \mathrm{curl}\ B \times B_o \\ \frac{\partial B}{\partial t} + \mathrm{curl}(B_o \times U) &= 0 \end{aligned} \tag{5.21}$$

The perturbed Hamiltonian and stress tensor are most easily written as rates,

$$\begin{aligned} \frac{\partial H}{\partial t} + U \cdot \nabla H_o &= -mv^2 (b_o \cdot \frac{\partial U}{\partial s}) + \mu B_o (b_o \cdot \frac{\partial U}{\partial s} - \mathrm{div}\ U) \\ \frac{\partial p_1}{\partial t} + U \cdot \nabla p_1^o &= -p_1^o\ \mathrm{div}\ U + B_o \int m^2 v^2 \frac{\partial f}{\partial t}\, dv d\mu \\ \frac{\partial p_2}{\partial t} + U \cdot \nabla p_2^o &= p_2^o (b_o \cdot \frac{\partial U}{\partial s} - 2\ \mathrm{div}\ U) + B_o^2 \int m\mu \frac{\partial f}{\partial t}\, dv d\mu \end{aligned} \tag{5.22}$$

These can be integrated with respect to t by introducing the Lagrangian displacement $x = \xi(\alpha,\beta,\sigma,t)$ with $U = D\xi/Dt \sim \partial\xi/\partial t$,

$$
\begin{aligned}
H &= -\mu\xi\cdot\nabla B_o - mv^2(b_o\cdot\frac{\partial\xi}{\partial s}) + \mu B_o(b_o\cdot\frac{\partial\xi}{\partial s} - \operatorname{div}\xi) \\
p_1 &= -\operatorname{div}(\xi p_1^o) + B_o\int m^2v^2\, f\, dv d\mu \\
p_2 &= -\operatorname{div}(\xi p_2^o) + p_2^o(b_o\cdot\frac{\partial\xi}{\partial s} - \operatorname{div}\xi) + B_o^2\int m\mu\, f\, dv d\mu \\
B/B_o &= \frac{\partial\xi}{\partial s} - b_o \operatorname{div}\xi
\end{aligned}
\qquad (5.23)
$$

which leaves as the only remaining differential equations

$$
\begin{aligned}
\frac{\partial f}{\partial t} &= [H_o, f] + [H, f^o] \\
\rho_o\frac{\partial^2\xi}{\partial t^2} + (\operatorname{div} P)_\perp &= \frac{1}{\mu_o}\operatorname{curl} B\times B_o
\end{aligned}
\qquad (5.24)
$$

These linear equations have an energy constant, but it is much more formidable in appearance than for the nonlinear system. The simplest computation is to take the second variation $\delta^2\Phi$ of the nonlinear potential energy from Sec. 11. This, expressed as a function of the perturbed distribution $\delta f \sim f$ and with ξ replacing U gives the conservation law [cf. (11.17)]

$$
\begin{aligned}
&\int \frac{1}{2}\rho_o U^2 dx + T(\xi) + 2\int g\, f\, d\Omega \\
&\quad - \int f^2(\partial f^o/\partial\varepsilon)^{-1} d\Omega = \text{const.}
\end{aligned}
\qquad (5.25)
$$

where T is defined in (11.18) and $g = H + \mu\xi\cdot\nabla B_o$ is given by (11.11) or (5.23).

The energy constant for this system does not exist for arbitrary f when f° is not monotone. But if initially $f = 0$ where $\partial f^{\circ}/\partial \varepsilon = 0$, this property will maintain itself in time. Thus the energy constant exists for a well-defined special class. But there is no evident singular behavior of the equations of motion where the energy constant is nonexistent.

Kulsrud shows how to write these equations in a symmetric form when f° is a single-valued monotone function of ε. We first illustrate this with the system of ordinary differential equations with constant coefficients

$$\ddot{w} = Lw \tag{5.26}$$

where the matrix L is arbitrary. It is assumed that every solution satisfies the quadratic "energy" relation

$$Q = \frac{1}{2}\, wAw + \dot{w}Bw + \frac{1}{2}\, \dot{w}C\dot{w} = \text{const.} \tag{5.27}$$

We may take A and C to be symmetric. Since

$$\dot{Q} = wA\dot{w} + wL'Bw + \dot{w}B\dot{w} + \dot{w}CLw = 0$$

for all initial values w and $\dot{w}$, we have $\dot{w}B\dot{w} = 0$ setting $w = 0$ and $wL'Bw = 0$ from $\dot{w} = 0$. This leaves $\dot{w}(CL + A)w = 0$ whence $CL = -A$ and (5.26) can be rewritten as

$$C\ddot{w} = -Aw\ . \tag{5.28}$$

In other words, multiplication of (5.26) by the kinetic energy matrix C makes both left and right sides of the equation symmetric. This reduction has significant consequences (principally that the spectrum is real) only when at least one of the matrices A and C is definite.

A similar analysis (but only formal and implicit since no properties have been established for the operators analogous to the matrices L, A, B, C) reduces the GCP equations to symmetric form. To obtain a second order equation for f, Kulsrud introduces even and odd components of f with respect to v,

$$f = \overline{f} + \hat{f} \ . \tag{5.29}$$

Since H_o, H, and f^o are even, and the operator $[H_o, *]$ reverses parity, we have

$$\begin{aligned} \frac{\partial \overline{f}}{\partial t} &= [H_o, \hat{f}] \\ \frac{\partial \hat{f}}{\partial t} &= [H_o, \overline{f}] + [H, f^o] \end{aligned} \tag{5.30}$$

Only $\overline{f}$ is coupled to the remaining equations through the even moments n, p_1, p_2; in the linearization the odd moment V drops out. We therefore eliminate $\hat{f}$,

$$\frac{\partial^2 \overline{f}}{\partial t^2} - [H_o,[H_o,\overline{f}]] = [H_o,[H,f^o]] \ . \tag{5.31}$$

We now have two second order coupled equations in the variables ξ and $\overline{f}$. But they cannot be made symmetric by only algebraic manipulations. We illustrate this with the simple example of a perturbation about a uniform plasma with constant B_o, p_1^o, p_2^o. We have

$$\rho_o \frac{\partial^2 \xi}{\partial t^2} - (B_o^2/\mu_o + p_2^o - p_1^o) \frac{\partial^2 \xi}{\partial x^2} - (B_o^2/\mu_o + 2p_2^o)\nabla_\perp \mathrm{div}\, \xi$$

$$= - B_o^2 \nabla_\perp \int \mu\, f\, dp d\mu \qquad (5.32)$$

$$\frac{\partial^2 \overline{f}}{\partial t^2} - v^2 \frac{\partial^2 \overline{f}}{\partial x^2} = \mu B_o v^2 \frac{\partial f^o}{\partial \varepsilon} \frac{\partial^2}{\partial x^2} (\mathrm{div}\ \xi) .$$

The "diagonal" terms (on the left) coupling ξ with itself and $\overline{f}$ with itself are evidently symmetric; but it is clear that no algebraic manipulation can symmetrize the cross terms which involve a third derivative of ξ and a first derivative of $\overline{f}$. But symmetry can be achieved by introducing either $\partial\xi/\partial x = \eta$ or $\int \overline{f} dx = \overline{g}$ as a new dependent variable, then dividing through (5.32) by $v^2 \partial f^o/\partial\varepsilon$. We see that the operator equivalent to C in (5.28) can be quite complicated. Also the "kinetic" energy in this formulation is more complicated than the natural kinetic energy. The second variation $\delta^2 \Phi$ has $f = \overline{f} + \hat{f}$ as its argument. The present interpretation amounts to splitting $\delta^2 \Phi$ into two parts, with the $\hat{f}$ component taken as kinetic and the $\overline{f}$ component as potential. One can expect the spectrum to be real if $\partial f^o/\partial\varepsilon < 0$, but there will always be a continuous spectrum extending to the origin (this is easily seen from the related "Rayleigh's principle").

The symmetric form cannot be derived if f^o is not a function of v^2, even if it is monotone on both sides.

The reason is that the Hamiltonian couples to both $\overline{f}$ and $\hat{f}$ in (5.30). In a mirror configuration f^o must be even. In a torus it need not. Both cases are represented by an infinite uniform plasma.

We may ask what the kinetic equation yields if we take an adiabatic approximation, $\partial f/\partial t \sim i\omega f \rightarrow 0$. From (5.24) we derive

$$0 = [H_o,f] + [H,f^o]$$

$$= [H_o,f] - \left(\frac{\partial f^o}{\partial H_o}\right)[H_o,H]$$

from which

$$f = H\,\frac{\partial f^o}{\partial H_o} + \psi(H_o)\ . \tag{5.33}$$

This formula associates a specific function f with any given field perturbation ξ through the perturbed Hamiltonian $H(\xi)$. We shall see in Sec. 11 that this formula [with the function $\psi(H_o)$ specified] is related to the pessimistic variation.

With the adiabatic interpretation of this formula, we cannot insert this expression into the magnetic field equation without consistently setting $\omega = 0$ throughout, e.g., to seek marginal modes. But with the pessimistic interpretation [which gives rise to the same formula (5.33) by a different argument] we are not so limited. Inserting this form of f into the remainder of the system (5.24) gives a symmetric second order system in ξ alone. In terms of the matrix analogue (5.28),

$$
\begin{aligned}
C_1\ddot{w}_1 &= -(A_{11}w_1 + A_{12}w_2) \\
C_2\ddot{w}_2 &= -(A_{21}w_1 + A_{22}w_2)
\end{aligned}
\tag{5.34}
$$

we drop $\ddot{w}_2$ from the second equation and eliminate w_2 to obtain the symmetric "adiabatic" system

$$C_1\ddot{w}_1 = (A_{12}A_{22}^{-1}A_{21} - A_{11})w_1 \tag{5.35}$$

The inversion A_{22}^{-1} is explicit through the formula (5.33) in the actual plasma case. Monotonicity of f^o is irrelevant in deriving the adiabatic equations since this enters only in the analogue of the matrix C_2 which has been dropped.

We can carry out this procedure explicitly for the uniform plasma. In this case the adiabatic approximation is

$$f = -\mu B_o \operatorname{div} \xi \frac{\partial f^o}{\partial \varepsilon} \tag{5.36}$$

and the adiabatic equation is

$$\frac{\partial^2 \xi}{\partial t^2} = c_1^2 \frac{\partial^2 \xi}{\partial x^2} + c_2^2 \nabla_\perp \operatorname{div} \xi \tag{5.37}$$

where

$$
\begin{aligned}
c_1^2 &= \frac{1}{\rho_o} [B_o^2/\mu_o + p_2^o - p_1^o] \\
c_2^2 &= \frac{1}{\rho_o} [B_o^2/\mu_o + 2p_2^o + B_o^3 \int \mu^2 \frac{\partial f^o}{\partial \varepsilon} \, dp d\mu]
\end{aligned}
\tag{5.38}
$$

We shall come across the stability criteria $c_1^2 > 0$ and $c_2^2 > 0$ again. The anisotropic wave equation (5.37) does not properly describe any actual motion of the plasma since ω^2 is bounded away from zero, cf. (2.2). But its connection with the pessimistic variation establishes that when $\partial f^0/\partial\varepsilon < 0$ the solutions of this wave equation are stable or unstable together with the actual motions. On the other hand, the connection with the adiabatic variation implies that even when $\partial f^0/\partial\varepsilon \nless 0$, the adiabatic equation (5.37) [or its equivalent in the general case] would give information about stability in any case where one verified that $\omega \sim 0$ or $k \sim \infty$ for the modes in question. This adiabatic equation is interpreted as an actual equation of motion by Andreoletti [23].

The equation (5.37) is a macroscopic one. We shall find that the macroscopic GCF equations predict a more complicated anisotropic behavior in the uniform plasma. The reason is that the GCF equations do describe actual motions, albeit in a crude macroscopic approximation.

It is instructive to mention the reason why the pessimistic variation gives a correct stability criterion even though it does not approximate the actual marginal motion (which is not adiabatic). The pessimistic variation minimizes the numerator alone in a Rayleigh's principle based on the symmetric equations; the correct marginal eigenvalue and eigenmode arise from minimizing the entire fraction which contains a complicated "kinetic energy" contribution from f in the denominator. But the sign of the minimum value is the same in both cases.

6. The Macroscopic Guiding Center Fluid

There are standard expedients for converting a microscopic model to any number of approximating macroscopic models [36]. The simplest is to write the complete equations of conservation of mass, momentum, and energy (these are exact but not a closed system), and then systematically remove quantities such as the stress deviator and heat flow which are not represented in the fluid state; another is to add new moment equations. In the GCP the fluid state has two pressures (p_1, p_2) instead of one. After writing the stress tensor as in (5.6) and dropping heat flow from the equation of conservation of energy, there is still one equation missing. Clearly this should relate to the conservation of magnetic moment. We easily establish that the mean magnetic moment per particle is given by

$$\langle \mu \rangle = kT_2/B = p_2/nB = mp_2/\rho B \ . \tag{6.1}$$

Thus we are led to adjoin the equation $(d/dt = \partial/\partial t + u\cdot\nabla)$

$$\frac{d}{dt}\left(\frac{p_2}{\rho B}\right) = 0 \tag{6.2}$$

$p_2/\rho B$ is constant on a particle path. This equation would also result from the exact moment equation for $\langle \mu \rangle$ after dropping a term analogous to heat flow (net magnetic moment flow with respect to the mean flow).

As in ordinary fluid dynamics or magneto-fluid dynamics, the total energy equation is equivalent to an entropy equation. But the entropy in this case turns out to be

$$\log \eta \sim \frac{1}{2} \log p_1 + \log p_2 - \frac{5}{2} \log \rho \tag{6.3}$$

instead of the usual

$$\log \eta \sim c_V \log p - c_p \log \rho \tag{6.4}$$

This can be obtained by manipulation of the energy equation, and it also arises as $\int f \log f$ for an equilibrium f which is a product of two Maxwellians, one degree of freedom at T_1 and two at T_2.

Our final system is therefore

$$\begin{gathered} \frac{d\rho}{dt} + \rho \operatorname{div} u = 0 \\ \rho \frac{du}{dt} + \operatorname{div} P = \frac{1}{\mu_o} \operatorname{curl} B \times B \\ \frac{dB}{dt} - B \cdot \nabla u + B \operatorname{div} u = 0 \ , \end{gathered} \tag{6.5}$$

plus two energy or entropy equations. For the total energy per mass

$$e = \frac{1}{2} RT_1 + RT_2 = p_1/2\rho + p_2/\rho \tag{6.6}$$

we may take the conservation equation

$$\rho \frac{de}{dt} + P_{ij} \frac{\partial u_i}{\partial x_j} = 0 \tag{6.7}$$

or the equivalent entropy equation [cf. (6.3)]

$$\frac{d}{dt} (p_1 p_2^2/\rho^5) = 0 \; . \tag{6.8}$$

For magnetic moment, we may take the conservation equation

$$\frac{d}{dt} (\frac{p_2}{B}) + \frac{p_2}{B} \operatorname{div} u = 0 \tag{6.9}$$

or the particle path invariant

$$\frac{d}{dt} (\frac{p_2}{\rho B}) = 0 \; . \tag{6.10}$$

The two particle path invariants can be chosen as any two of the following:

$$p_2/\rho B \; , \; p_1 B^2/\rho^3 \; , \; p_1 p_2^2/\rho^5 \; , \; p_2^3/p_1 B^5 \; . \tag{6.11}$$

The stability of these GCF equations for small perturbations about a constant state was first studied by analyzing the growth of plane waves (normal modes) [37], [38]. Instability is found whenever the unperturbed constant state is too anisotropic. But a more careful analysis of these results showed that they are technically not instabilities; they describe breakdown of the equations of motion [11]. In the unstable regime the equations become elliptic and the initial value problem is not well posed. We choose to interpret this breakdown as an instability, but this is a physical interpretation and not a mathematical deduction.

Since the GCF system (as distinguished from the GCP) consists of partial differential equations, we can investigate well-posedness in complete generality by computing the characteristics. The most convenient representation is in terms of the normal speed locus. Taking c as the characteristic speed relative to the mean motion u, the characteristic equations for (6.5), (6.8), (6.10) become

$$
\begin{aligned}
&-c\delta\rho + \rho\delta u_n = 0 \\
&-\rho c\delta u + n\cdot\delta P = \frac{1}{\mu_o}(n \times \delta B) \times B \\
&-c\delta B - B_n\delta u + B\delta u_n = 0 \qquad\qquad (6.12) \\
&\frac{\delta p_1}{p_1} + 2\frac{\delta p_2}{p_2} - 5\frac{\delta\rho}{\rho} = 0 \\
&\frac{\delta p_2}{p_2} - \frac{\delta\rho}{\rho} - \frac{\delta B}{B} = 0
\end{aligned}
$$

where n is the normal to the wavefront. The normal speed c is the eigenvalue of this homogeneous linear system. Ignoring particle path invariants, c = 0, we find the modified Alfvén or transverse wave

$$c^2 = \frac{1}{\rho}\left(\frac{B^2}{\mu_o} + p_2 - p_1\right)\cos^2\theta = A^2\cos^2\theta\left[1 + \frac{p_2 - p_1}{B^2/\mu_o}\right] \qquad (6.13)$$

and the modified compressive (or slow plus fast) wave as the solution of the biquadratic

$$\rho^2 c^4 - X\rho c^2 + Y = 0$$

$$X = B^2/\mu_o + 2p_2 + (2p_1 - p_2)\cos^2\theta$$

$$Y = \cos^2\theta(Y_1 \sin^2\theta + Y_2 \cos^2\theta) \tag{6.14}$$

$$Y_1 = 3p_1(B^2/\mu_o + 2p_2 - p_2^2/3p_1)$$

$$Y_2 = 3p_1(B^2/\mu_o + p_2 - p_1)$$

Here $\cos\theta = b \cdot n$ is the angle between B and the wave normal.

Although the normal speed c as a function of θ gives the characteristic cone only after some manipulation, loss of hyperbolicity follows immediately if c is not real for all θ. From the modified Alfvén wave (which, as in MH, represents strictly one-dimensional propagation along field lines [39]) we obtain the necessary condition for "stability"

$$B^2/\mu_o + p_2 > p_1 \ . \tag{6.15}$$

Since X and the discriminant $X^2 - 4Y$ are always positive, the single condition $Y > 0$ is necessary and sufficient for $c^2 > 0$ in (6.14). Since $Y_2 > 0$ is implied by (6.15), we need adjoin the single additional condition $Y_1 > 0$,

$$B^2/\mu_o + 2p_2 > p_2^2/3p_1 \ . \tag{6.16}$$

The two conditions (6.15) and (6.16) are necessary for the system to be hyperbolic; to show that they are also sufficient requires verification that the fast cone (giving the domain of dependence) is convex. The inequality (6.16) is a consequence of (6.15) and can be dropped if desired when

$$\frac{p_2}{p_1} < \frac{1}{2}\,(3 + \sqrt{21}) \sim 3.79 \; . \tag{6.17}$$

These conditions are necessary for stability in an arbitrary nonlinear, nonuniform, nonsteady flow. The equations are improper if hyperbolicity is violated anywhere in the flow. The enormous generality of this as a stability criterion is a result of the disastrous nature of the instability. With any more conventional interpretation of stability it is not clear how to even define stability in nonsteady flows. There is a further deduction that follows from our interpretation of non-posedness as instability, viz., that such instabilities are micro-instabilities; we shall return to this point in the discussion of the GCP.

A similar situation occurs in the common Rayleigh-Taylor instability, e.g., of an inverted glass of water. Here too, the linear equations of motion are not well posed. The difficulty is resolved either by examining the nonlinear equations or by inserting surface tension in the linear equations. For the GCF, we have seen that the difficulty is not removed in the nonlinear version. We shall see that it also appears (although quantitatively modified) in the microscopic GCP. It is apparently a consequence of the singular GC limit, but this remains only a plausible conjecture until better existence theorems are established in the finite gyro radius and Debye radius Vlasov formulations.

We conclude this very brief review of GCF theory with the remark that the classical scalar pressure MH results do not follow from setting $p_1 = p_2$ in these equations. The Alfvén wave does formally reduce to the MH result, but this is not true of the compressive waves. The reason is that an isotropic initial state will not remain so, even for the linear equations. The constraint (6.2) is quite distinct from anything which appears in the MH theory. Two simple cases which are easily compared with the MH theory are propagation perpendicular to the field ($\cos\theta = 0$) and parallel ($\cos\theta = 1$). In the first case we have

$$c^2 = 0 \ , \ c^2 = B^2/\mu_o\rho + 2p_2/\rho \quad . \tag{6.18}$$

The zero root agrees with the slow wave, and the finite root corresponds to the fast wave in a two-dimensional gas ($\gamma = 2$). For parallel propagation we have

$$c^2 = B^2/\mu_o\rho + (p_2 - p_1)/\rho \ , \ c^2 = 3p_1/\rho \quad . \tag{6.19}$$

The modified Alfvén root is repeated, as in the MH case, and there is an acoustic speed corresponding to a one-dimensional gas, $\gamma = 3$. Thus the anisotropy introduced by the magnetic field in MH is compounded by the anisotropy of the gas in the GCF; even with $p_1 = p_2$, the various degrees of freedom are uncoupled and have individual gas constants $\gamma = 2, 3$ instead of a combined $\gamma = 5/3$. It is also an interesting fact that the transverse (Alfvén) speed does not always separate the slow and fast speeds as it does in MH; other conclusions await a detailed calculation of the GCF characteristic cones.

7. The Infinite Uniform Plasma

The linearized motion about a constant state is well suited for analysis by Fourier and Laplace techniques [17], [40], [3], [18], but has been evaluated only recently by Kadish using the strict criterion of boundedness of solutions [19]. First of all, it is found that "instability" is equivalent to non-well posedness for every discrete normal mode. This is easily seen qualitatively. Since there is no length or time parameter in the equations of motion, the dispersion relation is a function of $(\omega/k_1, \omega/k_2)$. If this ratio takes any complex or imaginary value, the growth rate, Im ω, becomes arbitrarily large for small wavelength disturbances. On the other hand, for any stable set of parameters (described below) the existence and uniqueness of a solution to the initial value problems is established.

The constant unperturbed state is defined by an equilibrium distribution function $f^o(\varepsilon,\mu)$ [$\varepsilon = \frac{1}{2}mv^2 + \mu B$], and a constant vector B_o, or by two distributions $f^o_\pm(\varepsilon.\mu)$; in the uniform plasma there is no alternative to $\phi = 0$ in equilibrium. There is a continuous spectrum on the entire real ω-axis. This continuum gives a non-exponential decaying contribution and does not give rise to any instability if $\partial f^o/\partial\varepsilon$ is Hölder continuous. For such smooth functions f^o, Kadish establishes the following <u>necessary</u> conditions for stability for the single fluid

$$B^2/\mu_o + p_2 > p_1 \tag{7.1}$$

$$B^2/\mu_o + 2p_2 > -mB^3 \int \mu^2(\partial f_o/\partial\varepsilon)dvd\mu \tag{7.2}$$

The first is the same as in the GCF (6.15), and the second becomes comparable to the second GCF condition (6.16) when we remark on the inequality (for $\partial f^o/\partial\varepsilon < 0$ only)

$$-mB^3 \int \mu^2(\partial f_o/\partial\varepsilon)dvd\mu > p_2^2/3p_1 \quad . \tag{7.3}$$

The two conditions (7.1) and (7.2) become sufficient as well as necessary for stability if f^o is monotone in ε (bell shaped in v). But there are definite counterexamples that show lack of sufficiency when f^o is not monotone.

In the two-fluid case the second condition (7.2) is replaced by

$$B^2/\mu_o + 2p_2 > B^3 C_* \tag{7.4}$$

where C_* is a certain complicated combination of moments of $\partial f^o_\pm/\partial\varepsilon$ [see (10.16) and (10.24)]. C_* also satisfies the inequality (7.3) for the total pressures $p_2 = p_2^+ + p_2^-$, $p_1 = p_1^+ + p_1^-$ if both distributions are monotone.

The continuous spectrum can give rise to unstable solutions if $\partial f^o/\partial\varepsilon$ is not sufficiently smooth; the growth is not exponential for such solutions. With f^o not monotone in ε, the situation is not completely known. But explicit examples can be given of complex (not pure imaginary) poles for ω; and examples can be given of such poles making their appearance at some point removed from $\omega = 0$.* It has not been established whether there is any continuum in addition to the one on the real ω-axis; but there is no continuum in the upper(exponentially unstable) half plane.

Writing $\partial f^o/\partial\varepsilon = (mv)^{-1}\, \partial f^o/\partial v$ we see that the criterion (7.2) depends sensitively on the behavior of $f^o(v)$ near $v = 0$. This observation will take on much more significance in a nonuniform contained equilibrium (Sec. 13).

With the variational definition of stability to be described, the results turn out to be the same as for boundedness of solutions except for the monotonicity criterion and the contribution from the continuum. No instability from the continuum can be found variationally because the continuum is on the real ω-axis. Also, the variational analysis categorizes all non-monotone distributions as unstable.

---

* A. Kadish, private communication.

Thus the necessary and sufficient condition for variational stability of the infinite uniform plasma is [29]

$$\partial f^{o}/\partial \varepsilon < 0$$

$$B^2/\mu_o + p_2 - p_1 > 0 \tag{7.5}$$

$$B^2/\mu_o + 2p_2 - B^3 C_* > 0$$

This is also the exact necessary and sufficient condition for variational stability of a nonuniform plasma with unidirectional B and transverse variation subject to $p_2 + B^2/2\mu_o = \text{const}$.

8. Equilibrium--MH

We are concerned primarily with GC equilibria but present a brief summary of the scalar pressure theory for comparison.* The equations are

$$\nabla p = J \times B$$

$$\text{div } B = 0 \tag{8.1}$$

$$\mu_o J = \text{curl } B$$

---

*For a more complete account of the material in this section, including toroidal geometries which we do not discuss here, see [41]. The natural boundary conditions and potentials ζ and q are discussed in [42].

By inspection we see that the magnetic lines and the current lines lie in constant pressure surfaces. From div $B = 0$ and $J \cdot \nabla p = 0$ we can conclude that the magnetic field is a two-dimensional weighted harmonic vector field in each pressure surface.

$$\begin{aligned} B &= |\nabla p| n \times \nabla \omega \\ B &= \nabla' \phi = \nabla \phi - n(n \cdot \nabla \phi) \; . \end{aligned} \tag{8.2}$$

(without the weight $|\nabla p|$, the scalars ϕ and ω would be exactly conjugate surface harmonics). Given the set of pressure surfaces, the vector field B within each surface is subject to an elliptic equation; thus B is determined either by boundary conditions or, on a topologically complex surface (e.g., a cylinder) by periods.

The real characteristics of the system (8.1) are the magnetic lines counted twice, and there is, in addition, a second order elliptic characteristic cone. Identification of the characteristics allows us to guess at plausible well-posed problems. These guesses are refined by the variational analysis to follow and by the recognition of B as a surface harmonic within a pressure surface. In special cases where the real characteristics are explicitly integrated out, we can refer to actual existence theorems for purely elliptic equations.

Consider the tubular domain of Fig. 1. The boundary conditions B_n = given (e.g., $B_n > 0$ on S', $B_n < 0$ on S", $B_n = 0$ on S_o) are appropriate to the elliptic cone. The real characteristics require two initial values at one end of the tube or one at each end. We may specify p and J_n (the normal current flow) on S' or specify p at both ends. In the latter case, the two p specifications must be compatible with the B_n specification

$$\int_{p<p_o} B_n dS' = \int_{p<p_o} B_n dS'' \tag{8.3}$$

and, in addition, a period of B must be given on each p-surface if it is closed (this is not immediately evident by counting characteristics).

A variational formulation allows more precise evaluation of permissible boundary conditions than is given by the mere counting of characteristics. If we take

$$\begin{aligned} \operatorname{div} B &= 0 \\ B \cdot \nabla p &= 0 \end{aligned} \tag{8.4}$$

as admissibility conditions, variation of

$$\Phi(B,p) = \int (B^2/2\mu_o - p)dx \tag{8.5}$$

gives the first line of (8.1) as an Euler-Lagrange equation.

The admissibility conditions (8.4) can be more conveniently parametrized by taking

$$\begin{aligned} B &= \nabla\alpha \times \nabla\beta \\ p &= p(\alpha,\beta) \end{aligned} \tag{8.6}$$

where $p(\alpha,\beta)$ is a given function. This is essentially a new formulation with the variational function

$$\Phi(\alpha,\beta) = \int [\frac{1}{2\mu_o} |\nabla\alpha \times \nabla\beta|^2 - p(\alpha,\beta)]\, dx \tag{8.7}$$

An interior variation of (8.7) again gives $\nabla p = J \times B$. The boundary variation vanishes if α and β are fixed on S' and S" and $B_n = 0$ on S_o. Note that specification of α and β is stronger than just giving B_n. For example, if the p-surfaces are tubular, fixing the ends of the magnetic lines determines the "twist", i.e., the period of the harmonic vector B in the pressure surface.

Relaxation of a boundary condition will give rise to a corresponding natural boundary condition. For example, suppose the pressure contours assigned on S' are concentric curves and take α = const. on these contours, $p = p(\alpha)$. We fix α on S' but drop the boundary condition on β. The natural boundary condition turns out to be zero twist,

$$\oint_p B \cdot dx = 0 \tag{8.8}$$

on each pressure contour.

The more drastic relaxation of both α and β on S', keeping B_n fixed, leads to the natural boundary condition $\oint B \cdot dx = 0$ on any contour within S'; in other words,

$$J_n = 0 \text{ on } S' . \tag{8.9}$$

This result gives the connection between (8.5) and the classical Dirichlet principle. If we choose the function $p(\alpha,\beta) = 0$, the variational equation

$$\text{curl } B \times B = 0 \tag{8.10}$$

describes a <u>force-free field</u> or <u>pressureless plasma</u>. The variational function is Dirichlet's integral, $\Phi = \int \frac{1}{2} B^2$, but the boundary conditions, α and β fixed, are more restrictive than the proper boundary condition, B_n = given, for a Dirichlet problem. The equations (8.10) can be written

$$\text{curl } B = \sigma B \tag{8.11}$$

where $\sigma(\alpha,\beta)$ is constant on each line. The natural boundary condition (8.9), obtained by relaxing the boundary conditions from (α,β) = given to B_n = given, yields $\sigma = 0$; thus curl $B = 0$, and B is harmonic as in the Dirichlet problem.

If α and β are relaxed at both ends, S' and S", we of course obtain $J_n = 0$ at both ends. A somewhat weaker relaxation of the original boundary condition is the <u>interchange</u>. This allows (α,β) to vary arbitrarily (subject to fixed B_n) at one end, but forces an identical variation of (α,β) at the other end. To be precise, we note that any admissible (α',β') carrying the same B_n as (α,β) is obtained by an incompressible mapping,

$$\frac{\partial(\alpha',\beta')}{\partial(\alpha,\beta)} = 1 \ . \tag{8.12}$$

We require the transformation $(\alpha,\beta) \rightarrow (\alpha',\beta')$ to be the same on S' and S", but it is otherwise arbitrary. For a given admissible field B, any transformation (8.12) is trivial and leaves B(x) unchanged. What it does is alter the relative assignment of p-values to magnetic lines. Thus an interchange does not alter B but maps $p(\alpha,\beta)$ incompressibly in the (α,β) plane. The resulting natural boundary condition is that the current input and output are equal at the two ends of each magnetic tube, specifically

$$j_n' + j_n'' = 0 \tag{8.13}$$

where

$$j_n = J_n/B_n \tag{8.14}$$

is the current density with respect to flux,

$$j_n d\alpha d\beta = J_n dS \tag{8.15}$$

Just as B can be parametrized by $B = \nabla p \times \nabla \omega$, J can also be written

$$J = \nabla \zeta \times \nabla p \tag{8.16}$$

where $J \cdot dS = d\zeta dp$. The stream function ζ (which is constant on a J-line) is closely related to the indefinite integral

$$q(\alpha,\beta,s) = \int_{s'}^{s} \frac{ds}{B(\alpha,\beta,s)} \tag{8.17}$$

along a magnetic line; the lower limit is taken on S'. It is easily shown that ζ differs from q by a function of α and β alone, i.e., they differ by a constant on each magnetic line [42]. With closed p-contours, ζ can be multivalued; the "no-twist" condition (8.8) makes ζ single-valued. If $J_n = 0$ on S' we can take $\zeta = q$ and write

$$J = \nabla q \times \nabla p \tag{8.18}$$

In general, ζ differs from q by the initial value of ζ at one end; this is closely related to the initial J_n on S'.

If we have $J_n = 0$ on S' and S", we conclude that

$$\overline{q}(\alpha,\beta) = q(\alpha,\beta,s'') = \int_{s'}^{s''} ds/B \qquad (8.19)$$

is a function of $p(\alpha,\beta)$; $(\nabla\overline{q} \times \nabla p)\cdot dS = J\cdot dS = 0$. Thus a special equilibrium with $J_n = 0$ at both ends is characterized by p contours which coincide with $\overline{q}$ contours. This is also true of the more general interchange equilibria in which the (α,β) specification is only partially relaxed as described above. Allowing α and β to be free at both ends requires p to readjust so that it corresponds to $\overline{q}$. The natural boundary condition $j_n' + j_n'' = 0$ is equivalent to the global condition $p \sim \overline{q}$.

The boundary condition that (α,β) be fixed at an end wall S' has the physical interpretation of a perfectly conducting wall; the perfectly conducting boundary condition $E_t = 0$ implies no tangential motion $u_t = 0$ in the presence of $B_n \neq 0$. An insulating end wall will allow the ends of the magnetic lines to move. The two physical facts associated with an insulator, motion of magnetic lines and inability to draw current, are seen to be related mathematically as conjugate boundary conditions. The constraint B_n = fixed is a mathematical convenience to isolate the given domain from the external magnetic field which can then be considered to be unaffected by any internal motions. It could be realized by a perfectly conducting wall faced by a thin insulator in contact with the plasma.

The interchange condition (as distinguished from both ends moving independently) has no apparent physical realization. It is related to the mathematical realization of a toroidal system as an open one with identified ends. The condition that $\overline{q}$ be a function of p is a restriction on the class of open-ended equilibria, but it is a property of <u>all</u> toroidal equilibria [43] as an alternative statement of the fact that J_n is single valued.

In a two-dimensional or axially symmetric problem, the real characteristics can be integrated out explicitly, leaving a purely elliptic problem with known theory. We introduce the stream function ψ for B in an axially symmetric geometry and consider $p = p(\psi)$ as a given function. The equation for ψ is

$$\frac{1}{\mu_o} \Delta^* \psi = -r^2 p'(\psi)$$

$$\Delta^* = \partial^2/\partial z^2 + \partial^2/\partial r^2 - \frac{1}{r} \partial/\partial r \ . \tag{8.20}$$

For sufficiently well-behaved functions $p(\psi)$ and a domain which is not too large, solutions of this nonlinear elliptic equation exist and are unique. The more general axially symmetric problem with $B_\theta \neq 0$ exhibits a similar reduction. Writing

$$B = \frac{1}{r} n \times \nabla \psi + n B_\theta \tag{8.21}$$

where n is the azimuthal unit vector, we discover that $r^2 B_\theta^2$ is constant on flux tubes. Setting

$$g(\psi) = \frac{1}{2\mu_o} r^2 B_\theta \tag{8.22}$$

we find

$$\frac{1}{\mu_o} \Delta^* \psi = -r^2 p'(\psi) - g'(\psi) \tag{8.23}$$

with the same theory as the special case (8.20). The function $g(\psi)$ is closely related to the boundary value of J_n; thus the two functions $p(\psi)$ and $g(\psi)$ amount to a direct integration of the two boundary conditions p = given, J_n = given along the real characteristics. With this degree of symmetry, p given at one end determines p at the other end, so this alternative formulation is lost.

The differential equation (8.23) can frequently be solved by iteration. This suggests a procedure for solving problems in a general geometry [41]. Consider the tubular domain, Fig. 1, with B_n given and p and J_n given at one end. Taking an initial approximation to B, which satisfies div B = 0 and the boundary condition B_n, we find p in the entire domain by extending the values along B-lines. Next we compute J from the system

$$\begin{aligned} \nabla p &= J \times B \\ \operatorname{div} J &= 0 \end{aligned} \tag{8.24}$$

taking p and B as given. To do this most directly we use

(8.16), $J = \nabla\zeta \times \nabla p$, where

$$\zeta = \zeta' + q = \zeta(\alpha,\beta,s') + \int_{s'}^{s} ds/B \; ; \tag{8.25}$$

ζ' is determined by the boundary value of J_n. Having obtained J, we compute the next approximation to B from

$$\begin{aligned} \text{curl } B &= J/\mu_o \\ \text{div } B &= 0 \end{aligned} \tag{8.26}$$

and continue.

An iteration process should preferably have the property of smoothing with regard to derivatives. The iteration described does not seem to be smoothing, although this is so in axial symmetry. The construction of $p(\alpha,\beta)$ is such as to plausibly imply that ∇p is comparable to B ($= \nabla\alpha \times \nabla\beta$). But $\nabla\zeta$ also involves a derivative of B (in the direction normal to B). Thus inversion of (8.26), which gains one derivative, seems to preserve but not improve the level of differentiability in a complete iteration.

A possibly superior form for the iteration would be to solve for B from

$$\begin{aligned} \frac{1}{\mu_o} \text{curl } B - \sigma B &= \frac{B \times \nabla p}{B^2} \\ \text{div } B &= 0 \end{aligned} \tag{8.27}$$

assuming that the right side is given and both B and σ are to be found on the left. This is a nonstandard differential system which seems to gain a derivative for B over ∇p and B on the right. This differs from the previous iteration in that the qualifications parallel and perpendicular refer to the new B rather than to the old.

There are a number of plausible iteration processes based on the property that B satisfies an elliptic two-dimensional equation in any flux surface; but in the absence of a convergence proof there is little point in describing these. In summary, we can refer to actual theorems on nonlinear elliptic partial differential equations to prove existence in the axially symmetric (and the two-dimensional) problem; but we can only rely on plausibility arguments for the general case.

9. Equilibrium--GCF

For the GCF we have the equilibrium equations

$$\begin{aligned} \operatorname{div} P &= J \times B \\ \operatorname{div} B &= 0 \\ \mu_o J &= \operatorname{curl} B \end{aligned} \tag{9.1}$$

The parallel and perpendicular components of the pressure balance take the form

$$\frac{\partial p_1}{\partial s} = (p_1 - p_2)\,\frac{1}{B}\,\frac{\partial B}{\partial s} \tag{9.2}$$

$$(p_1 - p_2)\kappa + \nabla_\perp p_2 = J \times B \tag{9.3}$$

where $\kappa = \partial b/\partial s$ is the magnetic line curvature and $\nabla_\perp p_2 = (B \times \nabla p_2 \times B)/B^2$. Taking B as the independent variable along the line instead of arclength, (9.2) takes the form

$$\frac{\partial}{\partial B}\left(\frac{p_1}{B}\right) = -\frac{p_2}{B^2} \quad . \tag{9.4}$$

Expanding

$$\begin{aligned} \text{curl } B \times B &= B\cdot\nabla B - \nabla \tfrac{1}{2} B^2 \\ &= \kappa B^2 - \nabla_\perp \tfrac{1}{2} B^2 \end{aligned} \tag{9.5}$$

and introducing

$$p_* = p_2 + B^2/2\mu_o \ , \tag{9.6}$$

the perpendicular equation (9.3) becomes

$$\nabla_\perp p_* = \kappa(B^2/\mu_o + p_2 - p_1) \ . \tag{9.7}$$

Since this system involves two pressures instead of only one as in the MH version, we will expect to supply an additional relation. This system reduces exactly to the MH system if we take the additional relation as the statement that p_1 is constant on a magnetic line. From (9.2) we immediately conclude that $p_1 = p_2$, and everything else follows. If instead we assume that p_2 is constant on a line, we conclude from (9.4) only that

$$p_1 = p_2 + c(\alpha,\beta)B \tag{9.8}$$

which is slightly more general. In the next section we shall see that specification of a distribution function, as is appropriate to a GCP equilibrium, will assign both p_1 and p_2 as functions of B along each line, but in such a way that they are automatically compatible with (9.2). Thus we can consider that either $p_1(B,\alpha,\beta)$ or $p_2(B,\alpha,\beta)$ is given arbitrarily and we can then find the other from (9.4). [In an equilibrium contained by mirrors, the integration constant in p_1 is determined by the fact that p_1 and p_2 vanish simultaneously at the mirror.] We shall show that a solution can be expected to exist given $p_2(B,\alpha,\beta)$ and boundary conditions on (α,β). One special case (MH) is if the given function $p_2(\alpha,\beta)$ is independent of B. Two other interesting special cases which we shall consider are $p_2(B)$ independentof (α,β) and $p_*(\alpha,\beta)$ independent of B.

Another way of completing the system is to adjoin one of the particle path invariants appropriate to a motion. Since ρ does not appear in (9.1) we adjoin the relation (6.11)

$$p_2^3/p_1B^5 = a(x) \tag{9.9}$$

where a(x) is an arbitrary given function. We can incorporate both types of supplementary relation in the very general form,

$$g(p_1,p_2,B,x) = 0 \quad . \tag{9.10}$$

An elementary computation of characteristics shows that, in addition to real characteristics along the field lines, we obtain an elliptic characteristic cone provided that

$$B^2/\mu_o + p_2 - p_1 > 0 \tag{9.11}$$

and

$$\frac{\partial g}{\partial p_2}\left[\frac{B^2}{\mu_o}\frac{\partial g}{\partial p_2} + (p_2 - p_1)\frac{\partial g}{\partial p_1} - B\frac{\partial g}{\partial B}\right] > 0 \ . \tag{9.12}$$

The first relation has already been found as a condition for the initial value problem to make sense. Although not compelling, it is natural to require that a static equilibrium formulation be elliptic (e.g., to be able to impose B_n as a boundary condition). We therefore choose to interpret the equilibrium problem as being well-posed only when it is elliptic (that is in addition to the twice covered real magnetic lines), and we further characterize non-ellipticity as instability.

First, taking the particle path invariant (9.9), $3 \log p_2 - \log p_1 - 5 \log B - \log a \equiv g(p_1, p_2, B, x)$, we have

$$\frac{\partial g}{\partial p_1} = -\frac{1}{p_1} \ , \ \frac{\partial g}{\partial p_2} = \frac{3}{p_2} \ , \ \frac{\partial g}{\partial B} = -\frac{5}{B} \tag{9.13}$$

and (9.12) takes the form

$$B^2/\mu_o + 2p_2 > p_2^2/3p_1 \tag{9.14}$$

which is exactly (6.16). In other words, with the supplementary choice (9.9), <u>the same conditions are found for an elliptic equilibrium as for a hyperbolic motion</u>. This is not surprising.

Next take the supplementary condition

$$p_2 = P(B, \alpha, \beta) \tag{9.15}$$

we have

$$\frac{\partial g}{\partial p_1} = 0 \ , \ \frac{\partial g}{\partial p_2} = 1 \ , \ \frac{\partial g}{\partial B} = - \frac{\partial P}{\partial B} \tag{9.16}$$

and the stability condition is

$$B^2/\mu_o + B \frac{\partial P}{\partial B} > 0 \tag{9.17}$$

which can be rewritten as

$$\frac{\partial p_*}{\partial B} > 0 \ . \tag{9.18}$$

This "stability" condition is completely different from (9.14) and is apparently unrelated to stability of a motion; but the two concepts will be tied together in the next section.

The condition (9.18) can be obtained for a larger class of supplementary relations, viz., if in (9.10) we assume that g is not a general function of x but only a function of (α,β),

$$g(p_1, p_2, B, \alpha, \beta) = 0 \ . \tag{9.19}$$

Differentiating (9.19) "totally" with respect to B (i.e., keeping only α and β fixed)

$$\frac{\partial g}{\partial p_1} \frac{\partial p_1}{\partial B} + \frac{\partial g}{\partial p_2} \frac{\partial p_2}{\partial B} + \frac{\partial g}{\partial B} = 0 \, , \tag{9.20}$$

and inserting $\partial p_1/\partial B = (p_1 - p_2)/B$ yields (9.18) again.

The constraint (9.9) can be brought into the form (9.19) if a(x) is taken to be a function of (α,β) alone;

the particle path invariant is constant on each magnetic line. In this case a simple computation shows the equivalence of (9.14) and (9.18). But this special case is very uninteresting since it is never stable. From

$$\frac{p_2^2}{p_1} = a(\alpha,\beta)\,\frac{B^5}{p_2} \tag{9.21}$$

we see that p_2^2/p_1 is unbounded at the mirror ($p_2 \longrightarrow 0$) which violates (9.14). In other words, the two constraints (9.9) and (9.19) are independent, as are the corresponding stability criteria (9.14) and (9.18), except in an uninteresting special case.

Just as in MH, we can obtain an explicit elliptic system by specializing to axial symmetry. We suppose that $p_2(\psi,B)$ is a given function. We have

$$\begin{aligned} J \times B &= -\frac{1}{4\pi^2 r^2}\,\nabla\psi(\Delta^*\psi) \\ \nabla_\perp p_2 &= \frac{\partial p_2}{\partial\psi}\,\nabla\psi + \frac{\partial p_2}{\partial B}\,\nabla_\perp B \end{aligned} \tag{9.22}$$

The term $\Delta^*\psi$ does not determine the nature of the differential system because there are second derivatives implicit in $\nabla_\perp B$ as well as in κ. A more appropriate form for the equation is (9.7). The function $p_*(\psi,B) = p_2 + B^2/2\mu_o$ is known, and we write (9.7) as

$$\kappa(B^2/\mu_o + p_2 - p_1) - \frac{\partial p_*}{\partial B}\,\nabla_\perp B = \frac{\partial p_*}{\partial \psi}\,\nabla\psi\ . \tag{9.23}$$

The right side of the equation and the coefficients of

$\nabla_\perp B$ and of κ involve only first derivatives of ψ.

Expanding $\kappa = \partial b/\partial s$ we find

$$\kappa = -(\psi_r^2 + \psi_z^2)^{-3/2}[\psi_r^2\psi_{zz} - 2\psi_r\psi_z\psi_{rz} + \psi_z^2\psi_{rr}] \tag{9.24}$$

The differential operator

$$\psi_r^2 \frac{\partial^2}{\partial z^2} - 2\psi_r\psi_z \frac{\partial^2}{\partial r\partial z} + \psi_z^2 \frac{\partial^2}{\partial r^2} \tag{9.25}$$

is, within first derivative terms, the square of

$$\psi_r \frac{\partial}{\partial z} - \psi_z \frac{\partial}{\partial r} = 2\pi rB \frac{\partial}{\partial s} . \tag{9.26}$$

Insofar as characteristics are concerned, the dominant second order contribution in κ is the degenerate hyperbolic operator $-\partial^2/\partial s^2$. A similar computation shows that the dominant term in $\nabla_\perp B$ is essentially $\partial^2/\partial n^2$ where n is in the direction $\nabla\psi$. The equation (9.23) is therefore elliptic as a linear combination with positive coefficients of $\partial^2/\partial s^2$ and $\partial^2/\partial n^2$.

Another special case which turns out to be purely elliptic is when $p_2(B)$ is independent of α and β (Northrop and Whiteman [44]); the same will be true for $p_*(B)$. An elementary computation [using (9.4)] shows that

$$B \times \operatorname{curl}(\sigma B) = p_*' \nabla_\perp B - B^2\sigma\kappa \tag{9.27}$$

where

$$\sigma(B) = (B^2/\mu_o + p_2 - p_1)/B^2 . \tag{9.28}$$

We can therefore write (9.7) in the form

$$B \times \mathrm{curl}(\sigma B) = 0 \ . \tag{9.29}$$

Outside the plasma, where $p_1 = p_2 = 0$, we have $\sigma = 1/\mu_o$. A boundary condition $J_n = 0$ therefore implies that

$$\mathrm{curl}(\sigma B) = 0 \tag{9.30}$$

which, together with div $B = 0$ is a nonlinear elliptic system for the vector B.

A final special example which simplifies in an entirely different way is the marginal case with $p_*(\alpha,\beta)$ constant on each magnetic line. One of the criteria for stability is just met (the significance will be discussed in Sec. 14). It is convenient to introduce the flux coordinate ψ on which $p_*(\alpha,\beta) = \text{const}$. Thus we can take $p_*(\psi)$ as given, but with ψ itself unidentified. The equation is

$$\nabla p_* = p_*' \nabla \psi = \kappa(B^2/\mu_o + p_2 - p_1) \ . \tag{9.31}$$

We see that κ is orthogonal to a ψ-surface; the magnetic lines are geodesics on these surfaces. The differential equation is degenerate hyperbolic and one can formulate an initial value problem, given the vector B as well as p_* on an initial transverse surface, say midway between S' and S" in Fig. 1 (the domain D of Fig. 1 is not known beforehand in this problem). First we note that $p_1(B,\psi)$ and $p_2(B,\psi)$ are both explicit once $p_*(\psi)$ is known [using (9.4)]

$$p_2(B,\psi) = p_* - B^2/2\mu_o$$
$$p_1(B,\psi) = p_* + B^2/2\mu_o - B(2p_*/\mu_o)^{1/2} \tag{9.32}$$

On the given transverse surface S, the initial values of the vector B and of $p_* > B^2/2\mu_o$ determine p_1 and p_2 as well as ψ. We compute ∇p_* as being perpendicular to B from its projection on S. Thus we have the vector κ on S. We can move to a neighboring surface S_1 with a corrected direction for B and employ div B = 0 to compute the increment in the magnitude of B. Although the hyperbolic operator is degenerate and involves only derivatives along a line, the term ∇p_* provides coupling between lines.

A very degenerate special case occurs as a combination of the two previous examples, viz., p_* = constant in the plasma domain. The equilibrium equation says only that $\kappa = 0$; all magnetic lines are straight! In both this case and the previous case, $p_*(\alpha,\beta)$, where one cannot give boundary values, the plasma domain can only be joined to an external field at a boundary with a surface current. A more appropriate formulation is as a very unusual free boundary problem in which continuity of the vector B eliminates the surface current. For example, with axial symmetry one would look for a solution as in Fig. 2.

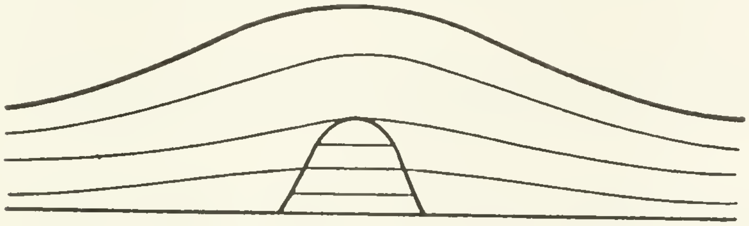

Figure 2

A solution will presumably be defined uniquely by specifying $p_*(\psi)$ in the plasma region and magnetic field boundary conditions (vacuum or force free field) at the exterior boundary.

10. Equilibrium--GCP

The equations to be satisfied are the macroscopic momentum balance (9.3), the two Liouville equations for the distribution functions, and the constraint equation for the potential ϕ. Solution of this problem can be separated into an analysis of the distribution functions and potential ϕ followed by a macroscopic analysis; the latter has already been done in the previous section.

Consider first the one-fluid theory. A time-independent solution of Liouville's equation is $f = f^{o}(\varepsilon,\mu,\alpha,\beta)$ where

$$\varepsilon = \frac{1}{2} mv^2 + \mu B \quad . \tag{10.1}$$

If the energy contours are not simple in the (s,v)-plane, then f^{o} need not be a single-valued function of ε. The macroscopic connection is given by

$$p_1(B,\alpha,\beta) = m^2 B \int v^2 f^{o}(\tfrac{1}{2} mv^2 + \mu B,\mu,\alpha,\beta)dvd\mu \tag{10.2}$$

$$p_2(B,\alpha,\beta) = mB^2 \int \mu f^{o}(\tfrac{1}{2} mv^2 + \mu B,\mu,\alpha,\beta)dvd\mu \tag{10.3}$$

Insert, page 82 (after Sec. 9)

In the general case of an arbitrary GCF equilibrium, introduction of the quantity $\sigma(B,\alpha,\beta) = 1/\mu_o + (p_2 - p_1)/B^2$ as in (9.28) yields the equation

$$B \times \mathrm{curl}(\sigma B) = -\frac{\partial p_1}{\partial \alpha} \nabla\alpha - \frac{\partial p_1}{\partial \beta} \nabla\beta \tag{9.33}$$

where σ, p_1, p_2 are given functions of (B,α,β). Since the highest order derivatives enter only on the left side of this equation, plausible iterations can be devised as in the scalar pressure case.

For a generalized axially symmetric problem we introduce a stream function ψ and B_θ as in (8.21). Instead of rB_θ = constant on ψ-surfaces, we now find $\sigma r B_\theta$ = constant. Setting

$$\frac{1}{2} \sigma^2 r^2 B_\theta^2 = g(\psi) \tag{9.34}$$

we find an equation for ψ,

$$\Delta^* \psi + \nabla\psi \cdot \nabla \log \sigma = -\frac{1}{\sigma^2} g'(\psi) - \frac{r^2}{\sigma} \frac{\partial p_1}{\partial \psi} \tag{9.35}$$

The left side is second order and elliptic while the right side is first order through the dependence of σ and p_1 on B.

The limits of integration are $0 < \mu < \infty$ and $0 < \frac{1}{2} mv^2 < \mu(B_1 - B)$ where B_1 is the largest value of B taken on the given line. In a plasma contained within mirrors, $f^o(\varepsilon,\mu)$ takes values in a sector of the (ε,μ) plane, $\varepsilon/\mu < B_1$, and f^o must vanish before reaching $\varepsilon/\mu = B_1$. At a given point on the line, the range of integration is $B < \varepsilon/\mu < B_1$.

To establish that p_1 and p_2 are compatible with equilibrium along a magnetic line, we compute

$$\frac{\partial}{\partial B}\left(\frac{p_1}{B}\right) = m^2 \int v^2\mu \frac{\partial f^o}{\partial \varepsilon} dv d\mu. \tag{10.4}$$

There is no contribution from the variable limits of integration since f^o vanishes there. Noting that

$$\frac{\partial f_o}{\partial \varepsilon} = \frac{1}{mv}\frac{\partial f^o}{\partial v} \tag{10.5}$$

and integrating by parts (the limits again vanish),

$$\frac{\partial}{\partial B}\left(\frac{p_1}{B}\right) = -m \int \mu f^o dv d\mu = - p_2/B^2 \tag{10.6}$$

which is the required relation, (9.4). The remaining analysis of the equilibrium equations follows as in Sec. 9 with $p_1(B,\alpha,\beta)$ and $p_2(B,\alpha,\beta)$ given compatibly.

We can obtain another interesting piece of information by differentiating p_2,

$$\frac{\partial}{\partial B}\left(\frac{p_2}{B^2}\right) = m \int \mu^2 \frac{\partial f^o}{\partial \varepsilon} dv d\mu \equiv - C_2 \tag{10.7}$$

But this moment of $\partial f^o/\partial \varepsilon$ is exactly what appears in the stability criterion (7.2) for an infinite homogeneous plasma (we shall also confirm it as a necessary condition in a general static equilibrium in Sec. 13). Thus we can rewrite the stability criterion (7.2) as

$$B^2/\mu_o + 2p_2 + B^3 \frac{\partial}{\partial B} \left(\frac{p_2}{B^2}\right) > 0 \tag{10.8}$$

or

$$\frac{\partial p_*}{\partial B} > 0 \ . \tag{10.9}$$

We now summarize the complex situation with regard to well-posed problems. The ubiquitous condition $B^2/\mu_o + p_2 - p_1 > 0$ is necessary for well-posed equilibrium or well-posed motion, for both the GCP and the GCF. The microscopic GCP theory has a second stability condition which is again the same in equilibrium and in a motion, but it can be written in two different but equivalent forms, (10.9) and

$$B^2/\mu_o + 2p_2 - B^3 C_2 > 0 \tag{10.10}$$

respectively. In the macroscopic GCF theory there are two similar criteria for equilibrium and motion, viz., (10.9) again for equilibrium and

$$B^2/\mu_o + 2p_2 - p_2^2/3p_1 > 0 \tag{10.11}$$

instead of (10.10) for motion; but these two criteria are independent in this theory. There is no way to relate the derivative $\partial p_*/\partial B$ to the quantity $p_2^2/3p_1$ (even as an inequality) without recourse to a distribution function. On the other hand, a different constraint in the macroscopic theory, viz., (9.9), brings the static and dynamic conditions together in the common form (10.11).

The GCF equilibrium equations are essentially incomplete. When they are completed in one way they yield well-posedness criteria which agree with the GCF equations of motion. When completed in another way the conditions agree with the GCP equations of motion (cf. Sec. 13).

We now turn to the two-fluid theory. Again, solution of Liouville's equation gives $f_{\pm} = f^{o}_{\pm}(\varepsilon_{\pm})$ where

$$\varepsilon_{\pm} = \frac{1}{2} m_{\pm} v^2 + \mu B \pm \phi \ . \tag{10.12}$$

We must choose ϕ to insure charge neutrality. By inspection of

$$n_{+} = m_{+} B \int f^{o}_{+}(\tfrac{1}{2} m_{+} v^2 + \mu B + \phi) dv d\mu \tag{10.13}$$

we conclude that n_{+} is a monotone decreasing function of ϕ (with B fixed) if f^{o} is monotone in ε, $\partial f^{o}/\partial \varepsilon < 0$. Both the integrand and the domain of integration decrease with an increase in ϕ. The opposite is true of n_{-}. Since $n_{\pm} \longrightarrow 0$ as $\phi \longrightarrow \pm\infty$, there is a unique value of ϕ at which $n_{+} = n_{-}$. This value depends on the parameter B, $\phi = \phi(B)$. Inserting this neutralizing value of the potential makes $n_{+} = n_{-}$ a function of B, and the same is true for $p_1^{\pm}$ and $p_2^{\pm}$. We may now continue with the macroscopic determination of the equilibrium configuration exactly as before. This argument depends on the fact that the determination of $\phi(B)$ is entirely local (on each magnetic line and at each location B) and can be done first before analysing the global pressure balance as in Sec. 9.

The situation is more complicated if f^o is not monotone in ε. For example, as shown in Fig. 3, there could be

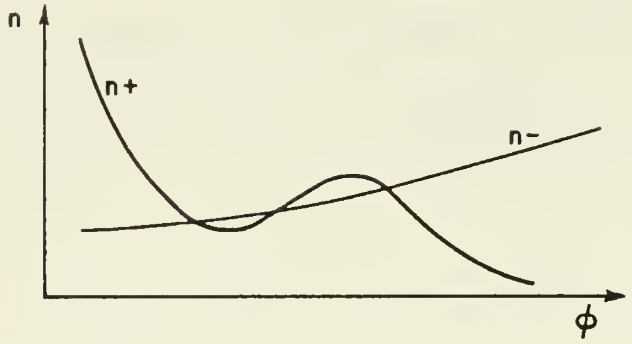

Figure 3

several solutions for ϕ which yield $n_+ = n_-$. If we look at the original physical picture from which the quasi-neutral formulation was extracted, we see that the middle value in Fig. 3 is unstable (the restoring electric field enhances any unbalanced charge), and both extremes are stable. In an infinite homogeneous plasma, there would be no difficulty in selecting either of the stable choices. But in a more general equilibrium with varying B along a line, we can find a situation in which there is no continuous solution $\phi(B)$. There are two choices for intermediate B, and a unique choice for large and small B, but there is no choice over the whole range which is physically satisfactory. Physically one would expect plasma oscillations of finite amplitude, at least in the intermediate section. Again we conclude mathematically that the problem is not well posed and we interpret this physically as an instability. We note that not every non-monotone profile $f^o(\varepsilon)$ produces a mon-monotone $n(\phi)$, not every non-monotone $n(\phi)$ produces non-unique values ϕ, and even if ϕ is multi-valued, limited ranges of B can give reasonable (even though non-unique) solutions.

There is a great deal of interesting information that can be obtained by differentiating the expressions for n, p_1, and p_2, just as in the one-fluid case. We have

$$
\begin{aligned}
n_{\pm} &= m_{\pm} B \int f^o_{\pm} \, dv d\mu \\
p_1^{\pm} &= m_{\pm}^2 B \int v^2 f^o_{\pm} \, dv d\mu \\
p_2^{\pm} &= m_{\pm} B^2 \int \mu f^o_{\pm} \, dv d\mu
\end{aligned}
\tag{10.14}
$$

and

$$
\begin{aligned}
\frac{\partial}{\partial B}\left(\frac{n_{\pm}}{B}\right) &= - C_1^{\pm} \mp C_o^{\pm} \frac{\partial\phi}{\partial B} \\
\frac{\partial}{\partial B}\left(\frac{p_1^{\pm}}{B}\right) &= - \frac{p_2^{\pm}}{B^2} \mp \frac{n_{\pm}}{B} \frac{\partial\phi}{\partial B} \\
\frac{\partial}{\partial B}\left(\frac{p_2^{\pm}}{B^2}\right) &= - C_2^{\pm} \mp C_1^{\pm} \frac{\partial\phi}{\partial B}
\end{aligned}
\tag{10.15}
$$

where

$$C_r^{\pm} = - m_{\pm} \int (\partial f^o/\partial \varepsilon)_{\pm} \, \mu^r \, dv d\mu \tag{10.16}$$

We recall that $\partial/\partial B$ is a derivative along the line keeping (α,β) fixed. We set

$$\begin{aligned}
& n = n^+ + n^- \\
& p_1 = p_1^+ + p_1^- \qquad \hat{p}_1 = p_1^+ - p_1^- \\
& p_2 = p_2^+ + p_2^- \qquad \hat{p}_2 = p_2^+ - p_2^- \\
& C_r = C_r^+ + C_r^- \qquad \hat{C}_r = C_r^+ - C_r^- .
\end{aligned} \tag{10.17}$$

Adding and subtracting the equations (10.15), and setting $n_+ = n_- = \frac{1}{2} n$, we obtain

$$\frac{\partial}{\partial B}\left(\frac{n}{B}\right) = - C_1 - \hat{C}_o \frac{\partial \phi}{\partial B} \tag{10.18}$$

$$0 = \hat{C}_1 + C_o \frac{\partial \phi}{\partial B} \tag{10.19}$$

$$\frac{\partial}{\partial B}\left(\frac{p_1}{B}\right) = - \frac{p_2}{B^2} \tag{10.20}$$

$$\frac{\partial}{\partial B}\left(\frac{\hat{p}_1}{B}\right) = - \frac{\hat{p}_2}{B^2} - \frac{n}{B}\frac{\partial \phi}{\partial B} \tag{10.21}$$

$$\frac{\partial}{\partial B}\left(\frac{p_2}{B^2}\right) = - C_2 - \hat{C}_1 \frac{\partial \phi}{\partial B} \tag{10.22}$$

$$\frac{\partial}{\partial B}\left(\frac{\hat{p}_2}{B^2}\right) = - \hat{C}_2 - C_1 \frac{\partial \phi}{\partial B} \tag{10.23}$$

First we note that (10.20) is the compatibility relation between p_1 and p_2 for the macroscopic equations. Any of the other equations can be considered as nonlinear integral equations for the determination of $\phi(B)$ (the parameter ϕ is implicit in the moments n, p_1, p_2, C_r). The relation (10.19) is equivalent to the one which was used to prove existence of $\phi(B)$. Formula (10.21) is the equivalent (in equilibrium) of the general constraint equation (5.11) for ϕ during a motion. The simpler appearance here (absence of factors $m_\pm$) results from the equilibrium relation $b \cdot \operatorname{div} P^+ + b \cdot \operatorname{div} P^- = 0$. Eliminating $\partial\phi/\partial B$ between (10.19) and (10.22) gives the two fluid analogue of (10.7),

$$\begin{aligned} &\frac{\partial}{\partial B}\left(\frac{p_2}{B^2}\right) = -C_* \\ &C_* = C_2 - \frac{\hat{C}_1^2}{C_o} \end{aligned} \qquad (10.24)$$

The inequality

$$B^3 C_* > \frac{p_2^2}{3p_1} \qquad (10.25)$$

for monotone f^o is stronger than the similar single fluid result with C_2. It can be proved by integrating the explicitly negative expression

$$I = \Sigma_\pm m_\pm \int \left(\mu B - \frac{1}{3}\frac{p_2}{p_1} m_\pm v^2 \mp \frac{B\hat{C}_1}{C_o}\right)^2 \left(\frac{\partial f_\pm^o}{\partial \varepsilon_\pm}\right) dv d\mu < 0 \ . \qquad (10.26)$$

The constant C_* plays the same role in two-fluid stability that C_2 does in one fluid.

11. Variational Formulation--Single Fluid*

Our starting point is the observation that the total energy $\mathcal{K} + \Phi$ is a constant of the motion subject to appropriate boundary conditions (5.20). If $\Phi = \Phi_o$ is a minimum of the potential within a certain specified class of neighboring states, we conclude that there is stability in the form

$$\begin{aligned} 0 < \mathcal{K} < \delta \\ \Phi_o < \Phi < \Phi_o + \delta \end{aligned} \tag{11.1}$$

for a perturbation with energy constant $\mathcal{K} + \Phi = \Phi_o + \delta$. The relation between existence of a minimum Φ_o and a score of other definitions of stability is, as we have seen in Sec. 2, a subtle and complex affair. We therefore accept the minimum property as the definition of variational stability.

The utility of this concept depends entirely on the precise class of admissible functions which are allowed into competition. The purpose of a variational approach is to replace the complex study of those states which can arise during a motion with a simpler class that is mathematically accessible.

We consider a tubular domain (Fig. 1) within which a plasma is contained by mirrors, i.e., there is a

*For details of this theory, see [16].

relatively large value of B at each end of the tube. There is not necessarily a single minimum of B on each line; this means that the energy contours in (s,v) are not necessarily a family of simple closed curves. In any event, even with a single minimum of B, the electric potential ϕ can be such as to complicate the topology. And in a torus, the energy contours are never simple.

We introduce the admissible magnetic fields and admissible distribution functions independently. For B we require that it cover the domain simply and satisfy

$$\operatorname{div} B = 0 \ , \ B = \nabla \alpha \times \nabla \beta \tag{11.2}$$

and a boundary condition; either

BCI: $B_n = 0$ on S_o, α and β given compatibly on S' and S''

BCII: $B_n = 0$ on S_o, compatible $B_n > 0$ on S' and $B_n < 0$ on S''.

The compatibility is taken care of (as is conventional in all variational formulations) by requiring that there exist at least one admissible vector field B. In BCII this implies only that $\int_{S'} B_n + \int_{S''} B_n = 0$, while in BCI there is a restriction on the topology of the curves α = const., β = const. as well as on the total flux, $\int_{S'} d\alpha d\beta = \int_{S''} d\alpha d\beta$.

BCI corresponds to conducting end walls with tied lines and BCII is appropriate for insulating walls or a vacuum surrounding the plasma. For a contained plasma, the pressure is required to approach zero before reaching any physical walls. The distinction between BCI and BCII is between a pressureless plasma (force free field) or a vacuum acting as a buffer zone around the primary plasma. Whether a given physical plasma is a vacuum or a force free field depends on the ratio of two small quantities, the plasma energy compared to magnetic energy, $\beta_o = (\frac{1}{2} p_1 + p_2)/(B^2/2\mu_o)$, and the gyro radius compared to a plasma dimension, $\eta = \lambda/R$. If $\beta_o/\eta >> 1$, then sufficient current can be carried to appreciably alter B from a vacuum field (curl B = 0) to a force free field (curl $B \times B = 0$). Since η is the GCP expansion parameter, it is implicit that $\beta_o/\eta >> 1$ wherever the GCP equations are used. A vacuum domain must be treated separately, using a form of Maxwell's equations. On the other hand, a vacuum can arise as a special case, curl B = 0, even when the fluid itself is considered to be a pressureless plasma, e.g., as a result of the boundary condition, $J_n = 0$.

The distribution function f is fixed in its dependence on (α,β,μ) and is varied only in its dependence on the dynamical coordinates (σ,p). This is consistent with the equations of motion which preserve (α,β,μ). Any motion subject to Liouville's equation is incompressible, and preserves area in (σ,p) or (s,v).* On a given line (α,β) and for a given value μ, we shall consider as admissible all functions $f(\sigma,p)$ which are compatible, through an incompressible mapping, with a given reference function. To be precise, take a fixed monotone function $f^*(A)$ and consider as admissible any function $f(\sigma,p)$ which is equimeasurable with f^*; if A_o is the area of the domain on which $f(\sigma,p) > f_o$, then $f_o = f^*(A_o)$. The domain $f > f_o$ need not be connected, and two admissible functions need not be topologically similar. More completely, the reference function f^* is a function of (A,μ,α,β).

The restrictions on B and f are not entirely independent; the field must be such that f does not "spill" over the mirrors.

For some purposes, e.g., to obtain first and second variations, it is more suitable to consider deformations of B and f rather than mappings. This gives a more restricted admissibility class, e.g., the topology of f-contours is fixed. Although any increased flexibility introduced by mapping as compared to deformation is irrelevant physically and incompatible with solutions of

---

*For purposes of defining the admissible class the distinction between s and σ (3.22) can be ignored, $\zeta = 1$. The parameter ζ is essential, however, in performing a continuous variation of an admissible function depending on a parameter.

Liouville's equation, we shall on occasion find this flexibility to be useful, and we shall even generalize the admissible deformations to correspond. The reason is that certain auxiliary minima which are not attained by any member of the lesser admissible class are attained (without alteration of the minimum value) in the greater admissible class.

The variation of B and f is most conveniently described in terms of mappings of the underlying space, $x \longrightarrow x'$ carrying the magnetic field lines and $(\sigma,p) \longrightarrow (\sigma',p')$ carrying f. We have seen in Sec. 3 that any deformation of a solenoidal field, div B = 0, can be represented by a "velocity" U such that

$$\frac{\partial B}{\partial t} + \mathrm{curl}(B \times U) = 0 \ , \ U \cdot B = 0 \ . \tag{11.3}$$

This is a variational equation and t is merely the parameter which describes the family $B(x,t)$; $\partial B/\partial t$ at $t = 0$ would be more conventionally represented as δB and U is a displacement δx. We similarly represent a deformation of f in terms of the incompressible velocity field w in $(\sigma,p) \equiv z$

$$\begin{gathered} \frac{\partial f}{\partial t} + w \cdot \frac{\partial f}{\partial z} = 0 \\ \frac{\partial}{\partial z} \cdot w = 0 \end{gathered} \tag{11.4}$$

A variation of (B,f) is described by a pair (U,w). U is arbitrary subject to $U \cdot B = 0$ and a boundary condition

BCI: $U_n = 0$ on S_o

$U = 0$ on S', S''

BCII: $U_n = 0$ on S_o

$\oint_C U \times B \cdot dx = 0$ on S', S''

for every closed curve C on S' and S''. Note that U_n may be different from zero on S' using BCII; U does not represent motion of plasma, only of field lines. The end of the plasma is determined by $f = 0$; this point is not carried perpendicular to B with U. The velocity w is arbitrary subject to $(\partial/\partial z) \cdot w = 0$.

For the one fluid theory we take

$$\Phi[B,f] = \int \frac{1}{2\mu_o} B^2 dx + \int \varepsilon f d\Omega \tag{11.5}$$

as the variational function. Performing a first variation with respect to f alone, holding B fixed, yields the variational condition that f be a function of ε,

$$f = f^o(\varepsilon,\mu,\alpha,\beta) \ . \tag{11.6}$$

This is a local condition, and f^o need not take the same value on disjoint ε-contours. But a very simple analysis shows that f is an absolute minimum if and only if f^o is single-valued and a monotone decreasing function of ε,

$$\frac{\partial f^o}{\partial \varepsilon} \leq 0 \tag{11.7}$$

Examination of the second variation of Φ (also varied with respect to f alone) reveals a minimum, $\delta^2\Phi > 0$, even if $f^o(\varepsilon)$ is not single-valued, provided that each branch is locally monotone. Such a multivalued equilibrium f^o provides a minimum in the linear theory but is not a minimum with respect to finite perturbations, no matter how small.

We can now state that a necessary condition that any state [B,f] be stable is that f^o be monotone in ε; single-valued and monotone in a finite theory, and piecewise monotone in an infinitesimal theory.

Performing a first variation with respect to B yields the result

$$B \times \left\{ \operatorname{div} P - \frac{1}{\mu_o} \operatorname{curl} B \times B \right\} = 0 \qquad (11.8)$$

for an interior variation U. The boundary variation vanishes identically under BCI, tied ends, and yields the natural boundary condition

$$J_n = 0 \qquad (11.9)$$

under BCII, free ends. Thus we obtain variationally exactly what was termed an equilibrium state in Secs. 9 and 10, together with a natural boundary condition when it is appropriate. The natural boundary condition $J_n = 0$ implies that the buffer zone surrounding the plasma satisfies curl B = 0 even if it is a force-free field and is capable of carrying current.

The previous analysis of the variation of Φ with respect to f alone holds not only for an equilibrium magnetic field but for any given admissible B. We can therefore minimize Φ in two steps. For any fixed admissible B we have a unique function $f = f_\pi$, depending on B, which provides an absolute minimum. If we associate this f_π with B, then Φ becomes a functional of B alone. We call this special class of admissible pairs $[B,f_\pi]$ a pessimistic variation. Even though this class of variations is special, it is evident that all minima of Φ can be found within this class.

The existence of a pessimistic variation depends vitally on allowing mappings rather than deformations in the (s,v)-plane. In a complex topology with islands (Fig. 4), a variation of B may require the transfer of f-values from the center of one island to the center of the other without

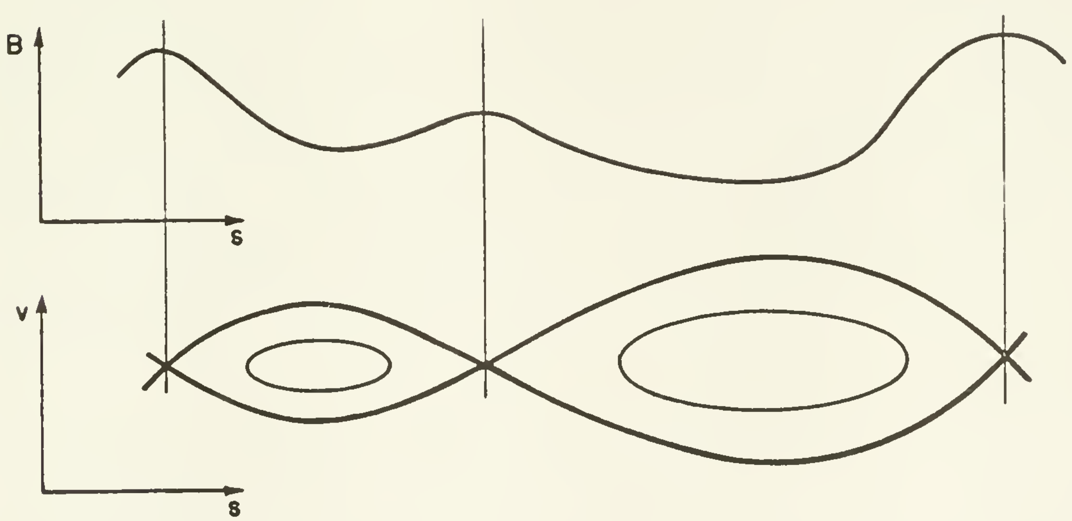

Figure 4

disturbing the intermediate values. Of course, such a mapping could be approximated by a flow, Fig. 5. Restriction

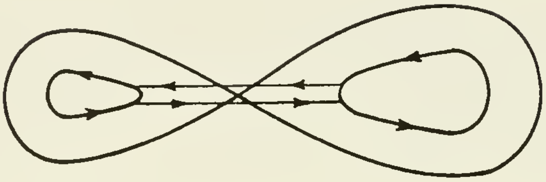

Figure 5

of the admissible class to the more physically relevant class of deformations would yield a functional Φ which has only a lower bound but no minimum. The wider class of mappings is what allows the simple characterization of the minimum with respect to f and the introduction of the pessimistic variation in a complex geometry. It is evident that these mappings cannot be represented by an incompressible flow w. A suitably generalized class of flows can be introduced. but we shall not require it.

It is not immediately evident that one can introduce the concept of a pessimistic variation in an infinitesimal theory (first and second variations) where one is necessarily restricted to deformations, not mappings. In any finite pessimistic variation it is only the sum of the areas of distinct f-contours which is preserved rather than each individual area as would follow from an incompressible w. But, if we examine the second variation $\delta^2 \Phi$ we are able to construct an explicit and unique minimizing w in terms of the given variation U simply by completing squares. This pessimistic $w = w_\pi$ turns out to be incompressible, except for a dipole type singularity at the critical point where two islands join. The reason for this is geometrically clear. For an infinitesimal variation, the interior of each island is isolated from the others and can be assigned an incompressible w. This preserves the area of each individual interior f-contour. Since a variation of B can transfer area between islands, the singularity in w is required. Although the area of every interior f-contour is preserved, the areas of the critical contours are not. We remark that, although the transferred area is of first order, the change in energy induced by the transfer of f values between islands is of order higher than second (i.e., negligible in the second variation); otherwise no minimizing w_π could be found for a piecewise monotone $f^o(\varepsilon)$. Even if $\partial f^o/\partial\varepsilon$ is discontinuous between two branches, f^o is stationary as a function of (s,v) at a critical point.

There is another problem where the Hamiltonian requirement of constant area of each f-contour must be dropped to obtain a pessimistic variation. In the infinite homogeneous plasma the energy contours are horizontal straight lines in equilibrium. For finite energy perturbations we consider variations of B which are bounded in extent (compact support). The rule for a pessimistic variation, placing a value of f on the energy contour of appropriate area, cannot be applied because the areas are infinite. Formal minimization of the second variation with respect to w gives an explicit result, but one which violates the area criterion. To interpret this result, we choose a large region surrounding the perturbed domain, minimize within this region, and then let the region grow. Within the large region the minimum is given by placing the largest f-values on the smallest ε-contours, preserving area. The minimizing f will be discontinuous on the boundary of the large region (Fig. 6).

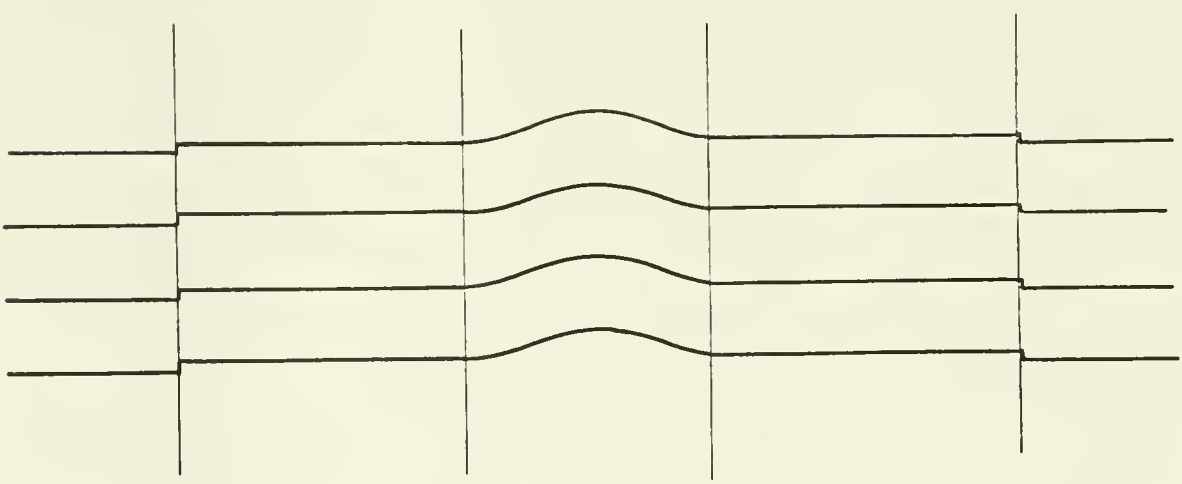

Figure 6

As this region grows, the value of f which is placed on a given ε-contour will approach the original unperturbed value, $f^0(\varepsilon)$. But the limiting result $f = f^0(\varepsilon)$ (i.e., the original values of f applied to the perturbed ε-contours) violates the area constraint. The area discrepancy which is caused by the perturbation of B on a given contour is finite, but it is distributed over a longer and longer length. Area is not preserved in a nonuniform limit, but the energy, $\int \varepsilon f$, can be seen to converge uniformly; (since $\int \varepsilon f$ is stationary for $f = f^0$, the energy discrepancy is of second order in the length of the region).

An infinitesimal analysis gives the same result; the pessimistic f is the unperturbed function, f^0, of the perturbed argument ε. There is no proper minimum within the original admissibility class, but the value of the minimum is unchanged when we extend this class to include the pessimistic variation.

Although minimization with respect to f is essentially explicit, the subsequent minimization with respect to B is not and requires intensive study. We list several forms for the second variation, $\delta^2\Phi$, both for a general variation and a pessimistic variation. The pessimistic variation w_π is given in terms of U through the formula

$$w_\pi \cdot \frac{\partial \varepsilon}{\partial z} = \langle g \rangle_\varepsilon - g \tag{11.10}$$

where

$$g = -mv^2\left(b \cdot \frac{\partial U}{\partial s}\right) + \mu B\left(b \cdot \frac{\partial U}{\partial s} - \operatorname{div} U\right) \tag{11.11}$$

and

$$\langle \psi \rangle_{\varepsilon} = \frac{\int \psi[\varepsilon - \mu B(s)]^{-1/2}\, ds}{\int [\varepsilon - \mu B(s)]^{1/2}\, ds} \tag{11.12}$$

is a microcanonical average. Although the formula (11.10) gives only the component of w_{π} normal to the energy contour, incompressibility serves to complete the definition; moreover, the normal component is all that appears in any of the later formulas. In the infinite uniform plasma, the pessimistic variation merely drops the term $\langle g \rangle_{\varepsilon}$ in (11.10). But in a uniform plasma with finite (e.g., periodic) boundary conditions, the full expression (11.10) would be kept.

The formula (11.10) is easily interpreted once we identify g as the pessimistic variation of ε (at fixed area) induced by the displacement U. A pessimistic variation requires that an ε-contour map into another ε-contour,

$$\frac{\partial \varepsilon}{\partial t} + w \cdot \frac{\partial \varepsilon}{\partial z} = \psi(\varepsilon) \quad . \tag{11.13}$$

But a simple computation shows that $\langle w \cdot \partial\varepsilon/\partial z \rangle = 0$; therefore $\psi(\varepsilon) = \langle \partial\varepsilon/\partial t \rangle$ and (11.10) follows.

For the second variation we find

$$\delta^2 \Phi = T(U) - Q(w_{\pi}) + Q(w - w_{\pi}) \tag{11.14}$$

where

$$Q(w) = -\int \left(w \cdot \frac{\partial \varepsilon}{\partial z}\right)^2 \frac{\partial f^{\circ}}{\partial \varepsilon}\, d\Omega \tag{11.15}$$

and we shall define T(U) below. Q is positive if $\partial f^{\circ}/\partial \varepsilon < 0$, and $w = w_\pi$ clearly minimizes $\delta^2\Phi$ at fixed U. If f^o is not monotone, we can take U = 0 and easily find a variation w which makes $\delta^2\Phi$ negative. The pessimistic second variation [defined only for piecewise monotone $f^o(\varepsilon)$] is

$$\begin{aligned}\delta^2\Phi_\pi &= T(U) - Q(w_\pi) \\ &= T(U) + \int (g - \langle g\rangle)^2 \frac{\partial f^o}{\partial\varepsilon}\, d\Omega \qquad (11.16) \\ &= T(U) + \int (g^2 - \langle g\rangle^2) \frac{\partial f^o}{\partial\varepsilon}\, d\Omega \, .\end{aligned}$$

An expression for $\delta^2\Phi$ directly in terms of δf rather than w is

$$\delta^2\Phi = T(U) + 2\int g(\delta f)d\Omega - \int (\delta f)^2(\partial f^o/\partial\varepsilon)^{-1}\, d\Omega \qquad (11.17)$$

The transverse variation T(U) is a special variation taking w = 0; in other words, f as a function of the canonical coordinates (σ, p) is frozen on a given line. It happens to be identical to the macroscopic GCF transverse variation (but in the GCF a general allowed displacement U would also have a component parallel to B). We have

$$T(U) = \int [p_1G_1 + p_2G_2 + (U\cdot\nabla U)\cdot \mathrm{div}\, P + G_m]dx \qquad (11.18)$$

$$\begin{aligned}&= \int [p_1G_1 + p_2G_2 + \tilde{G}_m]dx \qquad (11.19) \\ &\quad - \frac{1}{2\mu_o}\oint (B\cdot\nabla U^2)B\cdot dS - \frac{1}{2\mu_o}\oint B^2(U\cdot\nabla U)\cdot dS\end{aligned}$$

where

$$
\begin{aligned}
G_1 &= 4\left(b \cdot \frac{\partial U}{\partial s}\right)^2 - \left(\frac{\partial U}{\partial s}\right)^2 \\
G_2 &= \left(\frac{\partial U}{\partial s}\right)^2 - 2\left(b \cdot \frac{\partial U}{\partial s}\right)^2 + \left(b \cdot \frac{\partial U}{\partial s} - \operatorname{div} U\right)^2 + \frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} \\
G_m &= \frac{1}{\mu_o}\left\{[\operatorname{curl}(U \times B)]^2 + U \cdot \operatorname{curl}(U \times B) \times \operatorname{curl} B\right\} \\
\tilde{G}_m &= \frac{1}{2\mu_o} B^2\left[\frac{\partial U_i}{\partial x_j}\frac{\partial U_j}{\partial x_i} - (\operatorname{div} U)^2\right] + [B \cdot \nabla U - B \operatorname{div} U]^2
\end{aligned}
\tag{11.20}
$$

Boundary terms have been dropped whenever they vanish for both BCI and BCII. The first boundary term in (11.19) vanishes on the lateral boundary, S_o; both terms vanish at the ends S' and S" for BCI. But at least one boundary term is always present, except for a local disturbance in which U becomes zero before reaching the boundary.

The second form (11.19) is useful for making certain estimates since it contains no derivatives of the equilibrium parameters p_1, p_2, and B. Interchanges are more easily studied with the first form (11.18). It is also fortunate that (11.18) is insensitive to whether the ends are tied or free; this is what allows the Kruskal and Oberman formula (originally derived for tied ends) to be used by Taylor and others for interchange stability.

Another useful form is obtained for the pessimistic variation by expanding the term in g^2,

$$\delta^2\Phi_\pi = \int (B^2/2\mu_o + p_2 - p_1)(b \times \frac{\partial U}{\partial s})^2 dx$$
$$+ \int (B^2/2\mu_o + 2p_2 - B^3C_2)(b \cdot \frac{\partial U}{\partial s} - \text{div } U)^2 dx \qquad (11.21)$$
$$+ \int \nabla p_* \cdot (U \text{ div } U - U\cdot\nabla U) dx + \int \langle g\rangle^2 \frac{\partial f^o}{\partial \varepsilon} d\Omega$$
$$- \frac{1}{2\mu_o} \oint (B\cdot\nabla U^2) B\cdot dS - \frac{1}{2\mu_o} \oint B^2 (\text{div } U) U\cdot dS$$

In this form the boundary terms appear only at the ends S' and S" and only for BCII.

We have introduced a variational formulation as originally suggested by the equations of motion. But this procedure can be reversed. The pessimistic variation $\delta^2\Phi_\pi$ is a quadratic functional of the magnetic displacement U. From this we can obtain a symmetric system of pessimistic equations of motion, taking $\delta^2\Phi_\pi$ as the potential and $\int \frac{1}{2}\rho_o \dot{U}^2 dx$ as the kinetic energy [the notation is more conventional if we denote the displacement by ξ and take $U = \dot{\xi}$ as the velocity].

The pessimistic variation of f,

$$\delta f_\pi = -w_\pi \cdot \frac{\partial f^o}{\partial z} = -w_\pi \cdot \frac{\partial \varepsilon}{\partial z}\frac{\partial f^o}{\partial \varepsilon} = [g - \langle g\rangle] \frac{\partial f^o}{\partial \varepsilon} \qquad (11.22)$$

is clearly the same as the adiabatic variation (5.33) (g is the perturbed Hamiltonian). Thus the pessimistic equations of motion agree in form with the adiabatic equations of motion which were introduced in Sec. 5, even though the motivation and the range of application differ widely.

From their derivation, the pessimistic equations of motion are limited to monotone $f^{o}(\varepsilon)$. Variational boundedness as we have defined it is necessary and sufficient for exponential boundedness of these equations; any nonexponentially growing solutions are extraneous. There is no claim that pessimistic motions approximate actual motions, only that the stability criteria are related.

With the adiabatic interpretation of these equations (e.g., as used by Andreoletti [23]), we are not limited to monotone $f^{o}(\varepsilon)$. On the other hand, they give information about stability only if an unstable adiabatic motion is verified to be slow; we have seen that this can be easily violated even when $f^{o}(\varepsilon)$ is monotone.

Choosing the most useful interpretation in each case, exponential stability of the adiabatic equations is <u>equivalent</u> to variational stability when $f^{o}(\varepsilon)$ is monotone; and exponential stability of the adiabatic equations is <u>necessary</u> for exponential stability of the actual equations of motion for general $f^{o}(\varepsilon)$, but only if it is further verified that the marginal motion is slow.

12. Two-Fluid Variation--Existence of the Potential

For the two-fluid version we specify two reference functions $f^*_{\pm}(A,\mu,\alpha,\beta)$ subject to

$$\int f^*_+ \, dAd\mu = \int f^*_- \, dAd\mu \ . \tag{12.1}$$

In other words, there are equal numbers of electrons and ions on each line. Admissible functions $f_{\pm}$ are chosen to be compatible with $f^*_{\pm}$ and also subject to charge neutrality at each point on the line

$$\int f_+ d\mu dp = \int f_- d\mu dp \ . \tag{12.2}$$

Note that the electrostatic potential is not mentioned in this formulation of a charge neutral admissibility class.

The charge neutrality requirement (12.2) means that the two variations (w_+, w_-) are not independent; they are subject to the constraint

$$\int \left(w_+ \cdot \frac{\partial f_+}{\partial z} - w_- \cdot \frac{\partial f_-}{\partial z}\right) dp d\mu = 0 \tag{12.3}$$

or, setting $w = (w_\sigma, w_p)$,

$$\int w^+_\sigma f^+ dp d\mu = \int w^-_\sigma f^- dp d\mu \ . \tag{12.4}$$

Writing

$$\begin{aligned} \bar{\varepsilon}_{\pm} &= \frac{1}{2} m_{\pm} v^2 + \mu B \\ \varepsilon_{\pm} &= \frac{1}{2} m_{\pm} v^2 + \mu B \pm \emptyset \end{aligned} \tag{12.5}$$

we have

$$\Phi = \int \frac{1}{2\mu_0} B^2 dx + \int (\bar{\varepsilon}_+ f_+ + \bar{\varepsilon}_- f_-) d\Omega \tag{12.6}$$

$$= \int \frac{1}{2\mu_0} B^2 dx + \int (\varepsilon_+ f_+ + \varepsilon_- f_-) d\Omega \tag{12.7}$$

The terms involving ϕ implicit in $\varepsilon_\pm$ cancel out of (12.7) because of charge neutrality, (12.2), for any choice of the function $\phi(\sigma)$. It is an easy matter to verify that Φ is stationary if f_+ and f_- are functions of ε_+ and ε_- respectively. The converse, that all charge neutral stationary states take this form, is more difficult; this amounts to justifying the Lagrange multiplier rule (which yields this result instantly). To verify this, it is sufficient to show that given an arbitrary w_+, we can find a related w_- satisfying (12.3) which makes the second term vanish in

$$\delta(\Phi - \eta) = -\int (\varepsilon_+ w_+ \cdot \frac{\partial f_+}{\partial z} + \varepsilon_- w_- \cdot \frac{\partial f_-}{\partial z}) d\Omega \ . \tag{12.8}$$

Alternatively, it suffices to find a class of functions $w^-(\sigma,p)$ which is general enough to allow us to assign an arbitrarily given value to the second term in (12.8) while satisfying $\int w_\sigma^- f^- dp d\mu = 0$. The trial function $w = [a'(p)b(\sigma), -a(p)b'(\sigma)]$ is easily seen to satisfy these requirements.

We note that $\phi(\sigma)$ is arbitrary in this entire analysis of the first variation. The function $\phi(\sigma)$ is determined by the condition that the equilibrium be charge neutral: to any given pair $f_\pm^*(A)$, a potential ϕ can be found which will make $f_\pm(\varepsilon_\pm)$ charge neutral. We recall that we were able to prove a similar theorem in Sec. 10. There the functions $f_+(\varepsilon_+)$ and $f_-(\varepsilon_-)$ were assumed to be given, and the neutralizing $\phi(s)$ was found locally, at each point s. But if $f_\pm$ are fixed as functions of A instead of ε, variation of $\phi(s)$ changes $n_\pm(s)$ along the entire line, and $\phi(s)$ must be found from a nonlinear integral equation which we shall present later.

Just as it is necessary to prove the existence of a neutralizing potential in equilibrium, it is necessary to find the varied potential in a pessimistic variation. Note that the potential ϕ has no significance in a generel variation, but it must be defined in a pessimistic variation where f is constrained by the auxiliary minimization to be a function of ε. The relation (5.11) which determines ϕ in a motion has no relevance to the variational analysis which does not employ the dynamical equations of conservation of momentum. The varied potential, $\psi = \delta\phi$, is found to satisfy the integral equation [16]

$$L[\psi] = \chi \tag{12.9}$$

where

$$L = L_+ + L_-$$

$$L_\pm[\psi] = m_\pm \int (\psi - \langle\psi\rangle_\pm)(\partial f^o/\partial\varepsilon)_\pm \, dv d\mu \tag{12.10}$$

and

$$\chi = L_-[g_-] - L_+[g_+] \tag{12.11}$$

and the $g_\pm$ are given functions (11.11) in terms of U. The mean values $\langle\ \rangle_\pm$ are taken on distinct contours $\varepsilon_\pm$ = const. The term ψ can be taken outside the integral,

$$L_\pm[\psi] = -C_o^{\pm}\ \psi + K_\pm[\psi] \tag{12.12}$$

where $C_o^{\pm}$ is the previously defined moment (10.16) and $K_\pm$ is a symmetric integral operator with the kernel

$$k_\pm(s,s') =$$

$$\int [\varepsilon-\mu B(s) \mp \phi(s)]^{-1/2}[\varepsilon-\mu B(s') \mp \phi(s')]^{-1/2}\ \tau_\pm(\varepsilon,\mu)d\varepsilon d\mu$$

$$\tau_\pm(\varepsilon,\mu) = -\frac{\partial f_\pm^*}{\partial A} = -\frac{\partial f_\pm^o}{\partial \varepsilon_\pm}\Big/\frac{\partial A}{\partial \varepsilon_\pm} \tag{12.13}$$

The quantity $\partial A/\partial\varepsilon$ is essentially the orbital period of a particle with given ε,

$$\frac{\partial A}{\partial \varepsilon} = \left(\frac{2}{m}\right)^{1/2} \int [\varepsilon - \mu B \mp \phi]^{-1/2}\ ds = \frac{2}{m}\int \frac{ds}{v} \tag{12.14}$$

The integral equation is to be solved for $\phi(s)$ on a given line; α and β are parameters.

We consider only monotone $f^o(\varepsilon)$, since this is a necessary condition for stability. This implies that the kernel $k(s,s')$ is a positive function. It also follows easily that both -L and K are positive operators,

$$(\psi,L\psi) = \int \psi L[\psi]ds = m\int (\psi - \langle\psi\rangle)^2 \frac{\partial f_o}{\partial \varepsilon} dvd\mu ds < 0$$
$$(\psi,K\psi) = \int \psi K[\psi]ds = \frac{m}{2}\int [\int \frac{\psi ds}{v}]^2 \tau d\varepsilon d\mu > 0 \qquad (12.15)$$

L is not definite; $L\psi = 0$ if and only if $\psi = \langle\psi\rangle$, i.e., ψ is constant. We verify that the inhomogeneous term χ in (12.11) is always orthogonal to a constant. The solution ψ will therefore be unique only up to an added constant.

The spectrum (and in particular any discrete eigenvalue) of K relative to C_o does not exceed unity,

$$K[\psi] = \lambda C_o \psi$$
$$0 < \lambda \leq 1 \qquad (12.16)$$

This is an immediate consequence of the positivity of K and -L,

$$(\psi,K\psi) \leq (\psi,C_o\psi) \quad . \qquad (12.17)$$

It is easily verified (this amounts to $\lambda = 1$ being an eigenvalue) that

$$\int k(s,s')ds' = C_o(s) \ . \qquad (12.18)$$

Therefore the operator

$$M[\psi] = \frac{1}{C_o} K[\psi] \qquad (12.19)$$

is a mean value with positive weight, and the norm of M is not greater than unity taking a maximum value norm on ψ. From this fact alone it follows that iterations of

$$\psi = \frac{1}{C_o} K[\psi] - \frac{\chi}{C_o} \tag{12.20}$$

will converge in absolute value if χ/C_o is bounded. To estimate the speed of convergence we must show that the eigenvalue $\lambda = 1$ in (12.16) is isolated. In terms of the new variables

$$\begin{aligned} C_o^{1/2}\,\psi &= \bar{\psi} \\ C_o^{-1/2}\,\chi &= \bar{\chi} \end{aligned} \tag{12.21}$$

we have the integral equation

$$\bar{\psi} = \bar{K}[\bar{\psi}] - \bar{\chi} \tag{12.22}$$

[with the same spectrum as (12.16)] where

$$\bar{k}(s,s') = [C_o(s)C_o(s')]^{-1/2}\, k(s,s') \tag{12.23}$$

Examination of the kernel (for example, taking τ to be bounded) shows that it is bounded except near $s = s'$ and near $s = s_1$ or $s' = s_1$ where s_1 is the limit of the plasma, $f^o(s_1) = 0$. Near $s = s'$ there is a logarithmic singularity $\bar{k} \sim \log|s - s'|$; $\bar{k}$ is bounded at the boundary of the domain except at the corners, $s = s' = s_1$ where $\bar{k}$ grows at most as $(ss')^{-1/4}$. Thus $\bar{k}$ is square integrable, and the Neumann series of (12.22) converges uniformly in L_2. Given the existence of an L_2 solution $\bar{\psi}$ of (12.22), it follows that

$\bar{K}[\bar{\Psi}]$ is continuous and bounded, and $\bar{\Psi}$ is also continuous provided that this is true of the inhomogeneous term.

The existence of a ϕ to neutralize $f^*_{\pm}(A,\mu)$ in equilibrium depends on the solution of a nonlinear integral equation which is closely related to the linear equation just studied. We have

$$\begin{aligned} n_{\pm}/B &= m_{\pm} \int f^{\circ}_{\pm}\, dv d\mu \\ &= (\tfrac{1}{2}\, m_{\pm})^{1/2} \int f^{\circ}_{\pm}[\varepsilon - \mu B \mp \phi]^{-1/2} d\varepsilon d\mu \\ &= -(2m_{\pm})^{1/2} \int [\varepsilon - \mu B \mp \phi]^{1/2} (\frac{\partial f^{\circ}_{\pm}}{\partial \varepsilon}) d\varepsilon d\mu \\ &= -(2m_{\pm})^{1/2} \int [\varepsilon_{\pm} - \mu B \mp \phi]^{1/2} (\frac{\partial f^{*}_{\pm}}{\partial A}) dA d\mu \end{aligned} \tag{12.24}$$

Here $n_{\pm}$ depends on ϕ not only explicitly but in ε when expressed as a function of A and μ. The charge neutrality condition is $G[\phi] = 0$ where

$$\begin{aligned} G[\phi] = {} & (m_+)^{1/2} \int [\varepsilon_+ - \mu B - \phi]^{1/2} \frac{\partial f^*_+}{\partial A}\, dA d\mu \\ & - (m_-)^{1/2} \int [\varepsilon_- - \mu B + \phi]^{1/2} \frac{\partial f^*_-}{\partial A}\, dA d\mu \end{aligned} \tag{12.25}$$

If we set $\phi = 0$ in this expression we obtain a definite function $G[0] = N[s]$, essentially the residual difference $n_+ - n_-$ when the potential is ignored. We consider $\phi(s,t)$ to be the solution of the integral equation

$$G[\phi] = (1 - t)N(s) \ . \tag{12.26}$$

At $t = 0$, $\phi = 0$ is a known solution; we wish to extend the solution to $t = 1$. Differentiating with respect to t (i.e., performing a variation) we obtain

$$L[\frac{\partial\phi}{\partial t}] = 2^{1/2}N(s) \ . \tag{12.27}$$

This is a linear integral equation (the one we have already studied) for $\partial\phi/\partial t$ with a kernel which depends on ϕ. In other words, it is an ordinary differential equation for $\phi(t)$ in a function space. We shall not carry the analysis further, but it seems quite likely that existence can be proved in this way.

The analysis of the second variation is similar to the one-fluid case, except that a pessimistic variation involves the variation $\delta\phi$ which must be obtained from the integral equation (12.9). The pessimistic variation is given by

$$\begin{aligned} w^{\pm}_{\pi} \cdot \frac{\partial \varepsilon^{\pm}}{\partial z} &= \langle \delta\varepsilon_{\pm} \rangle_{\pm} - \delta\varepsilon_{\pm} \\ &= \langle g_{\pm} \rangle_{\pm} - g_{\pm} \pm (\langle \delta\phi \rangle_{\pm} - \delta\phi) \end{aligned} \tag{12.28}$$

where g is given by (11.11) as before. For a general variation ($\delta\phi$ does not appear), formulas (11.14), (11.15), (11.17) are merely summed over the two fluids, and (11.18), (11.19), (11.20) are identical. For a pessimistic variation

we have

$$\delta^2 \Phi_\pi = T(U) - Q_\pi(U) \tag{12.29}$$

where T(U) is identical to the one-fluid expression and

$$\begin{aligned} Q_\pi &= -\Sigma_\pm \int \left\{ g_\pm - \langle g_\pm \rangle_\pm \right\}^2 (\partial f^o/\partial\varepsilon)_\pm d\Omega \\ &+ \Sigma_\pm \int \left\{ \delta\phi - \langle \delta\phi \rangle_\pm \right\}^2 (\partial f^o/\partial\varepsilon)_\pm d\Omega \\ &= \Sigma_\pm \int \left\{ (\delta\phi)^2 - g_\pm^2 + \langle g_\pm \rangle_\pm^2 - \langle \delta\phi \rangle_\pm^2 \right\} (\partial f^o/\partial\varepsilon)_\pm d\Omega \ . \end{aligned} \tag{12.30}$$

To $\delta^2 \Phi_\pi$ in (11.21) summed over + and - we must only add the term

$$\begin{aligned} &\Sigma_\pm \int [\delta\phi - \langle \delta\phi \rangle_\pm]^2 (\partial f^o/\partial\varepsilon)_\pm d\Omega \\ &\qquad = \int (\delta\phi)^2 C_o B \, dx + \Sigma_\pm \int \langle \delta\phi \rangle_\pm^2 (\partial f_o/\partial\varepsilon)_\pm d\Omega \end{aligned} \tag{12.31}$$

For a sufficient condition we can drop this entire positive term, taking (11.21) as given; for a necessary condition we can drop the $\langle \delta\phi \rangle^2$ term and keep only the term in $(\delta\phi)^2$.

13. Local Stability

It would seem plausible to expect that the stability of a general contained plasma with respect to localized (or small wavelength) disturbances should be somehow related to stability of the infinite homogeneous plasma with the local value of the distribution function and magnetic field. To be more precise, the nonuniform plasma can be expected to be unstable if the infinite plasma is unstable for small wavelengths; and when the infinite plasma is stable, the general plasma should be stable with respect to all sufficiently localized disturbances.

For the scalar pressure plasma, this is a known result. The infinite homogeneous plasma is always stable; it is in absolute thermodynamic equilibrium. And there is a theorem, proved by H. Rubin [45],that any smooth equilibrium is stable with respect to sufficiently localized disturbances.

For the GCP there is no natural length scale. Therefore stability of the infinite plasma is independent of wavelength. It turns out that local stability of a general contained plasma is essentially identical to variational stability of the infinite plasma. We recall that for the infinite plasma variational stability disagrees with boundedness of solutions with respect to the continuous spectrum and with respect to monotonicity of $f^{o}(\varepsilon)$. We cannot expect anything different for the contained plasma, so we compare only variational stability for the two cases. Specifically, the following conditions (at each point of the plasma) are <u>necessary</u> for stability in an arbitrary contained plasma

$$\partial f^{o}/\partial \varepsilon < 0$$

$$B^2/\mu_o + p_2 - p_1 > 0 \qquad (13.1)$$

$$B^2/\mu_o + 2p_2 - B^3 C_* > 0$$

where C_* is defined in (10.24) and (10.16). The same three conditions are also <u>sufficient</u> for stability of every <u>localized</u> disturbance in an arbitrary contained plasma if we add a smoothness condition; e.g., bounded ∇p_2 is adequate [29]. The degree of localization required for stability can be estimated from the equilibrium parameters.

As immediate corollaries of this theorem of local stability, any short plasma with tied ends is stable if (13.1) is satisfied; and any long and only slightly curved plasma with tied ends will also be stable.

The proof of the local stability theorem depends on the particularly apposite form (11.21) of the second variation and on the possibility of solving the integral equation for $\delta\phi$ for a localized disturbance. It is shown that the first two terms of (11.21) and the first term in (12.27) dominate in any local disturbance.

Just as in the MH theory, one may expect a discontinuity surface to provide additional sources of local instability. But in the GC case the question is open. It is not surprising that in the two-fluid theory the constant C_* replaces C_2. The GCF equilibrium theory, Sec. 9, is identical for one and for two fluids. And the condition for the equilibrium to be well posed has been given in terms of $\partial p_2/\partial B$ which, by direct calculation, gives C_2 for one fluid (10.7) and C_* for two (10.24).

Although this has not been proved, it seems likely that violation of (13.1) will result in non-posedness of the equations of motion as it does for the equilibrium equations and as it does for motion in the infinite uniform plasma. Assuming this, we can interpret violation of (13.1) as a micro-instability. The equations refuse to predict a growth rate; we conclude this must be related to omitted parameters such as the plasma frequency and gyro frequency. The identification of exactly which micro-instabilities are incorporated in the GCP theory is not complete. The advantage of this theory is that it gives a very broad criterion (13.1) applicable to arbitrary geometries, whereas all other micro-instability analyses are made in restricted geometries allowing at most small linear gradients. The disadvantage is that the GCP theory yields no information about the instability other than its presence or absence.

There is another advantage to this theory. When (13.1) is satisfied any instability requires interaction between distant parts of the plasma. We are justified in calling these macro-instabilities. Thus the GCP theory provides a clear-cut separation between micro (local) and macro (global) instabilities. We repeat that although it is not proved, it seems very likely that recourse to the equations of motion would yield catastrophic growth for the local and finite growth for the global instabilities. It is interesting to note that micro-instabilities appear even in the GCF which is a totally macroscopic theory.

The significance of the second condition (13.1) needs no elaboration; it is present in exactly the same form in both GCF and GCP theories and is required for a well-posed equilibrium formulation as well as initial value problem. The significance of the third condition, the <u>velocity gradient</u> instability, is deeper and lies hidden in the combination C_* of moments $f^o_{\pm}$. We set

$$B^3 C_* = \eta p_2^2 / 3p_1 \tag{13.2}$$

where η is a dimensionless constant which depends on $f^o_{\pm}$. We recall that for $\eta = 1$ this stability criterion becomes the same as in the GCF theory (6.16) and for monotone $f^o_{\pm}$ we always have $\eta > 1$. Thus local stability of the GCF is a necessary condition for local stability of the GCP. This is a special case of a general theorem which states simply that GCF stability is necessary for GCP stability (linear theory [13], nonlinear [16]).

The minimum value $\eta = 1$ can be approximated arbitrarily closely, but it is not attained by any stable contained distribution. In this sense, the GCF theory gives a best possible approximation to the GCP theory. We have $\eta = 1$ only if f is singular in μ and a step-function in ε; $f = \delta(\mu - \mu_o)H(\varepsilon_o - \varepsilon)$, where $\mu_o > 0$ and $H = 1$ for $\varepsilon < \varepsilon_o$, $H = 0$ for $\varepsilon > \varepsilon_o$. But for this distribution, C_* becomes unbounded near the edge of the plasma; thus it is always unstable, whatever the configuration of B. It is exactly this special limiting case which made the two constraints (9.9) and (9.19) coalesce in the equilibrium GCF theory. (In the infinite homogeneous case, the limiting distribution, $\eta = 1$, is not unstable.)

For an isotropic distribution, $f^o(\varepsilon)$ independent of μ (this cannot be contained in an open ended mirror system), we have $\eta = 6$ independent of what function f^o is of ε. Also, for a two-temperature Maxwellian [$f = a \exp(-b\varepsilon - c\mu)$], which can be anisotropic and can be contained by mirrors, we also have $\eta = 6$. The moment C_2 (and therefore C_*) is unbounded if $f^o(\varepsilon)$ has an infinite slope $f^o \sim (\varepsilon - \varepsilon_o)^{1/2 + \delta}$ on the left ($\varepsilon > \varepsilon_o$) or $f^o \sim (\varepsilon_o - \varepsilon)^{1/2}$ on the right ($\varepsilon < \varepsilon_o$), Fig. 7. This is to be compared with the infinite plasma, where the stability criterion is sensitive to bad behavior of f^o only at $v = 0$. But in a contained plasma, any bad behavior in f^o is reduced to $v = 0$ at some value of B. This

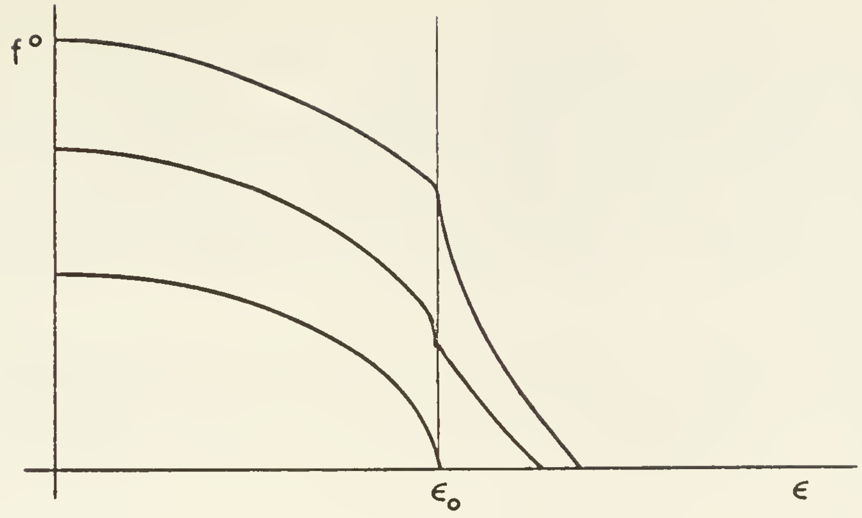

Figure 7

is the reason why the "optimal" distribution with $\eta = 1$ is allowable in the infinite case but not in any contained geometry.

These restrictions on f^o are quite relevant practically. Stated more precisely, instability results from a large gradient of f at a given value of ε/μ, i.e., near a certain turning point. Almost any laboratory injection of plasma will yield an initial distribution with discontinuities of f^o or very large phase space gradients across some critical turning point $B = \varepsilon/\mu$. Whether the plasma is instantly lost or quickly shifts to a more congenial contained situation depends on the circumstances; both cases can be observed experimentally.

Poor values of η (i.e., values in excess of six) can only arise together with anisotropy. And a very special case of this velocity gradient instability is the mirror instability which arises from large values of p_2^2/p_1. But the coefficient C_* can become unbounded even at a location where $p_2 = p_1$, so the measure of anisotropy which is implicit in C_* is a much more subtle one than just the ratio p_2/p_1.

We recall from (10.24) that C_* can be eliminated from the stability criterion in favor of a "total" pressure derivative,

$$\frac{\partial p_*}{\partial B} > 0 \ . \tag{13.3}$$

The indicated derivative is with B as the parameter along the magnetic line. This form of the stability criterion has the simple interpretation that not too much plasma can be placed on the line depending on the strength of the mirrors (Fig. 8). The maximum amount of plasma that can be placed on a line is obtained by filling in p_2 up to the maximum value $B = B_1$ at the mirror,

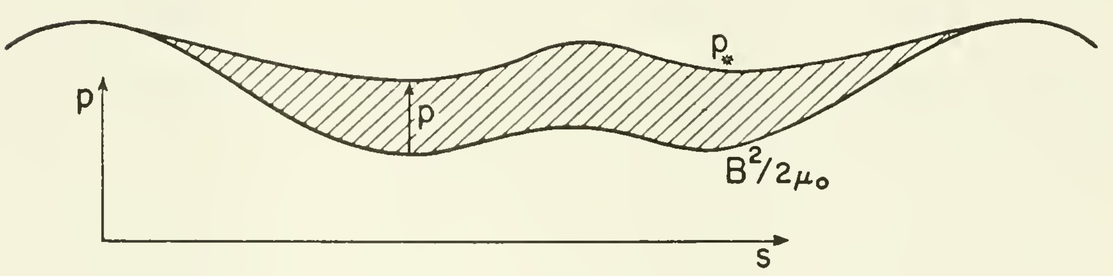

Figure 8

$$p_* = p_2 + B^2/2\mu_o = B_1^2/2\mu_o \ . \tag{13.4}$$

There are several disadvantages to examination of $\partial p_*/\partial B$ rather than C_*. First $\partial p_*/\partial B$ is completely unobservable experimentally whereas the distribution function is both measurable and is a primary input. In an approximately uniform field $\partial p_*/\partial B = (\partial p_*/\partial s)/(\partial B/\partial s)$ becomes indeterminate while the limitation on the distribution function remains viable. Second there is much more information revealed by examining the dependence of C_* on f^o. And third, the quantity $\partial p_2/\partial B$ is not local. The derivative $\partial p_2/\partial B$ at one point on the line is related to its values all along the line implicitly, through the common distribution function.

For example, (13.4) gives a legitimate upper estimate to the amount of plasma that can be contained (taking into account that the field will be altered by addition of plasma) [22]; but it is not a very good estimate. The reason is that assigning a specific function $p_2(B)$ on a magnetic line is very restrictive of the distribution function. We can write the moment p_2 as

$$p_2(B) = B^2 \int_B^{B_1} (x - B)^{-1/2} \, \zeta(x)dx \tag{13.5}$$

where

$$\zeta(x) = (2m)^{1/2} \int_0^{\infty} \mu^{3/2} \, f^o(\mu x, \mu)d\mu \tag{13.6}$$

But the integral equation (13.5) can be inverted uniquely for $\zeta(x)$ in terms of $p_2(B)$,

$$\zeta(x) = -\frac{1}{\pi}\int_x^{B_1} (B - x)^{-1/2} \frac{\partial}{\partial B}\left(\frac{p_2}{B^2}\right) dB \tag{13.7}$$

In other words, not only does the distribution function determine $p_2(B,\alpha,\beta)$, but the pressure function $p_2(B,\alpha,\beta)$ determines the distribution function $\zeta(x,\alpha,\beta)$ [and greatly restricts $f^o(\mu x,\mu)$]. In particular, the special function $p_2 = p_*(\alpha,\beta) - B^2/2\mu_o$ yields the explicit distribution

$$\zeta(x) = \frac{4p_*}{\pi}\left\{(B_1 - x)^{1/2}\left(\frac{1}{4xB_1^2} + \frac{3}{8x^2 B_1}\right) + \frac{3}{8x^3}\tan^{-1}\left(\frac{B_1}{x} - 1\right)^{1/2}\right\} . \tag{13.8}$$

Unless one is fortunate enough to inject this precise distribution of particles on each line, the optimum amount of plasma indicated by the formula (13.4) cannot be contained. In plasma stability criteria in which a maximum stable amount of plasma proportional to the well depth is obtained, $\delta p \sim \text{const.}\delta B^2$ [20], [25], the constant is more sensitive to the distribution function than to the geometry.

In the GCF theory, local instability implies non-posedness in an arbitrary nonsteady flow. We conjecture that the same is true of the GCP local instability criterion. In this case $\partial p_*/\partial B < 0$ and $B^2/\mu_o + 2p_2 < B^3 C_*$ are no longer equivalent. It is evident that the C_* formulation, in terms of the distribution function, will be the correct one.

Further details of this theory of local stability will be found in Ref. [29].

14. Mostly Interchange Stability

An interchange is a special variation (allowable only under BCII with insulated ends) in which the displacement U is equivalent to an incompressible flow in the $\lambda = (\alpha,\beta)$ plane. Writing u for the velocity in (α,β), we have

$$\begin{aligned} U\cdot\nabla &= u\cdot\frac{\partial}{\partial\lambda} \\ \frac{\partial}{\partial\lambda}\cdot u &= 0 \end{aligned} \tag{14.1}$$

The constraint curl $(U \times B) = 0$ is equivalent to $(\partial/\partial\lambda)\cdot u = 0$. In an interchange the magnetic field is not varied at all in physical space. Only the distribution function $f^{o}(\varepsilon,\mu,\lambda)$ is redistributed among the lines, subject to a given $f^{*}(A,\mu,\lambda)$. As in the general case, there is no loss of generality in restricting ourselves to the pessimistic variation. This means that

$$f^{o}(\varepsilon,\mu,\lambda) = f^{*}(J(\varepsilon,\mu,\lambda),\mu,\lambda) \tag{14.2}$$

where

$$J(\varepsilon,\mu,\lambda) = \oint p\,d\sigma = 2(2m)^{1/2}\int \sqrt{\varepsilon - \mu B(s)}\,ds \tag{14.3}$$

and $f^{*}(J,\mu,\lambda)$ is carried unchanged by the motion u.

As a special variation, stability with respect to interchanges can only give a necessary condition for stability. But this is a sufficiently important concept to warrant singling it out as a definition of <u>interchange stability</u>. Note that an interchange is a variation and not a motion; it is not clear that in any but very exceptional circumstances can an interchange be an actual motion.

In an interchange, the invariant function $f^*(J)$ changes as a function of ε through the variation of the dependence of J on ε when λ is varied. In a one-fluid formulation $J(\varepsilon,\mu,\lambda)$ depends only on the fixed magnetic field geometry and not on f; more precisely, $J/\mu^{1/2}$ is a function of the single variable ε/μ. But in the two-fluid theory, the potential ϕ (contained in $\varepsilon_{\pm}$) depends on $f^{o}_{\pm}$ as well as on the magnetic field. Thus $J(\varepsilon,\mu,\lambda)$ depends on f and not only on the field geometry. Almost all the simplification of an interchange analysis over an analysis of the full second variation is lost in the two-fluid theory.
Almost all interchange results are restricted to the one-fluid theory, although this is not usually recognized in the literature. We do obtain a sufficient condition for interchange stability by ignoring ϕ in the two-fluid theory. This only has significance if we take interchange stability as an independently defined concept; otherwise a sufficient condition for another condition which is only necessary for stability yields no definite conclusion.

The analytic manipulations (for finite interchanges as well as infinitesimal) are most easily carried out in the form [30]

$$\Phi = \int \varepsilon(J,\mu,\lambda)f^*(J,\mu,\lambda_0)dJd\mu d\alpha d\beta \qquad (14.4)$$

where λ is an Eulerian variable and λ_0 is a Lagrangian variable; both ε and f^* are fixed functions of the indicated variables and $d\alpha d\beta = d\alpha_0 d\beta_0$.

One important role played by interchange stability is in low pressure plasmas. It is frequently stated, even "proved", that at low pressure interchange stability implies absolute stability. The argument in support of the interchange hypothesis is as follows. The second variation is the sum of a magnetic contribution (non-negative since the vacuum field is an absolute minimum) and a small plasma contribution. For any given variation U which is not an interchange, the plasma contribution will be dominated by the positive magnetic contribution below some β_0 (we can consider β to be a parameter multiplying f). But an interchange gives a positive second variation by hypothesis. Therefore the second variation is positive for sufficiently small β. The fallacy, of course, is that the critical β_0 depends on the variation U, and β_0 may have the lower bound zero as U is varied over the admissible class.

The important question is, of course, not whether the argument is false but whether the conclusion is false. But it is so demonstrably false, with so many very familiar theoretical and experimental counterexamples, that it is an interesting psychological question how this belief has come to gain such wide acceptance. Experimentally, any micro-instability (well known to exist at low β) is an immediate counterexample. Perhaps the reason these experimental "counterexamples" have not been faced in this context is the lack of appreciation that the GCP is a microscopic theory. But the belief is strong enough so that when one author is faced with an immediate theoretical counterexample to his "proof" of sufficiency, he resolves the conflict by pragmatically strengthening the "sufficient" condition!

In the GCF and the GCP it is very easy to present counterexamples, and in the MH theory, more subtle ones. For example, one must always require local stability, since this is a universal necessary condition. Furthermore, local stability is independent of interchange stability and is not consequent upon low pressure. In the GCF we can easily exhibit equilibria in which p_2^2/p_1 is unbounded near the mirror. No matter how small is the value of β, local stability will be violated near the mirror. In the GCP version the same mirror instability is excited if f^o goes to zero slower than $(\varepsilon_1 - \varepsilon)^{1/2}$ at the edge of the plasma ($\varepsilon_1 = \mu B_1$). But we have much more flexibility than this in the GCP theory; the velocity gradient instability can be excited in the plasma interior whenever $f^o(\varepsilon)$ has a large gradient, and this can produce instability at arbitrarily low β and with bounded p_2/p_1. It is even possible to give unstable counterexamples where ∇p_2 and curl B go to zero with β. And there still remains the local instability associated with f^o not monotone in ε which (especially in the form of electrostatic oscillations as suggested in Sec. 10) remains at arbitrarily low β. These gaps are not only mathematical; there are very many experimental instabilities observed at very low β and under circumstances which are probably compatible with interchange stability (or the equivalent, tied ends). Although the experimental identification of instabilities is not yet a fully developed discipline, it is very likely that the GCP local instabilities lie among those observed.

A further gap in the interchange hypothesis lies in the possibility of non-smooth equilibria. Even in the MH theory, a shear surface across which the tangential component of B is discontinuous can be unstable at arbitrarily low β. We recall that sufficiency for local stability in the GCP is more demanding than necessity with regard to smoothness.

Although the simple form of the interchange hypothesis is false, it can be replaced by a slightly more sophisticated version which is much more plausible. Local stability involves small scale disturbances. Interchange stability involves, in a sertain sense, the largest scale disturbances; the scale length is not even restricted by the apparatus dimension, since the ends are free to move. Tying down stability at both extremes in wavelength would seem to give a good rough estimate with regard to absolute stability, We conjecture that the combination of local and interchange stability implies absolute stability under fairly wide circumstances. For example, there is plausible evidence that this is so under the further assumption of low β, or a long thin device, or a device with limited curvature to the magnetic lines, etc. The hypothesis of interchange stability can be dropped in a system with tied ends. We have already given several examples of absolute stability contingent on local stability and tied ends (Sec. 13). By far the largest and most diversified part of the current knowledge of GCP stability lies in this combination of interchange and local stability [30].

We interpolate a brief introduction describing interchange stability for the scalar pressure MH plasma. We have already mentioned the restriction that p be a function of $q = \int ds/B$ in order for the equilibrium to be compatible with free ends. These are the only equilibria for which we can study interchanges.* A simple analysis gives as a necessary and sufficient condition for stability with respect to finite or infinitesimal interchanges that pq^{γ}

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* This restriction on interchange equilibria is given in [42]. The same formula for a torus (at low β) was found earlier [43]. In a torus this is no restriction; it is a property of <u>all</u> equilibria.

be a monotone function of q.* Writing $p = p(q)$, the criterion is

$$p' + \gamma p/q > 0 . \qquad (14.5)$$

If $p = P(\psi)$ and $q = Q(\psi)$ are expressed as functions of a flux parameter, this takes the form

$$\left(\frac{P'}{P} + \gamma \frac{Q'}{Q}\right)Q' > 0 \qquad (14.6)$$

(there is no restriction to axial symmetry implicit in this description in terms of ψ). An obvious sufficient condition is

$$p' > 0 , \quad P'Q' > 0 \qquad (14.7)$$

or that p be a monotone function of q. For a contained plasma in which p is monotone in physical space, $P' < 0$, this requirement is that q be a maximum at the center of the plasma. We repeat that this universally cherished condition, that $\int ds/B$ be a maximum within the plasma, is mathematically neither necessary nor sufficient for absolute stability. It is sufficient for interchange stability and therefore may be sufficient for stability at low β or in other suitably restricted circumstances. At the edge of

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* The stability criterion (14.5) is also given in [43] for low β, toroidal equilibria and in [42] for general equilibria and finite perturbations.

the plasma, if both p and p' approach zero, the more stringent sufficient condition $p'(q) > 0$ approaches the necessary and sufficient condition (14.5). But it is an unnecessarily strong restriction within the plasma, and it is not required for a plasma which terminates at a finite gradient (we return to this point in the GCP analysis).

The scalar pressure theory is applicable only to a toroidal system or one where the plasma ends at material walls. In the latter case, the shape of the walls exerts a strong influence on the variation of q from line to line, and therefore on stability. We note that GCP stability for an isotropic equilibrium is not the same as MH stability even though the equilibrium theory is identical. And we reiterate that even in the MH case where it is quite plausible, it has not been proved that interchange stability implies stability at low β.

We now turn to GCP interchange stability. This theory is more complex than the MH theory in that an entire distribution function rather than a number, p, is assigned to each line. No simple universal necessary and sufficient condition for interchange stability such as (14.5) is available. But we can completely analyze several special classes of distribution functions which taken together seem to be wide enough to give a reliable evaluation of the general situation, and we shall be able to supplement this with some sufficient conditions, not restrictive of distribution function, which support the same general conclusions. A crucial point will be the assessment in each case of how representative the results may be.

First we remark that the special case

$$f = f^*(J,\mu) \ , \tag{14.8}$$

i.e., the same function of J and μ on each magnetic line, is completely neutral to all interchanges. This is evident, since it is the function $f^*(J,\mu)$ which is carried unchanged from line to line by the pessimistic variation. This is a mathematical rather than a practical example since $f^*(J,\mu)$ cannot be contained in any magnetic field. But since it is neutral, it is a useful basis for perturbations and for the determination of a transition between stability and instability.

The next special class of distributions is

$$f = \sigma(\lambda)\hat{f}(J,\mu) \ ; \tag{14.9}$$

the distribution is fixed in J and μ except for the amount placed on each line. This case can be examined in complete detail, for finite as well as infinitesimal stability, just as in MH. The condition that the equilibrium be compatible with free ends ($J_n = 0$) is that $\sigma(\lambda)$ be a function of

$$\begin{aligned} \hat{\varepsilon}(\lambda) &= \int \varepsilon(J,\mu,\lambda)\hat{f}(J,\mu)dJd\mu \\ &= \int (\tfrac{1}{2}\,\hat{p}_1 + \hat{p}_2)\,\frac{ds}{B} \ , \end{aligned} \tag{14.10}$$

i.e., σ = const. must be placed on $\hat{\varepsilon}(\lambda)$ = const.

A necessary and sufficient condition for finite interchange stability is that σ be a monotone decreasing function of $\hat{\varepsilon}$. This result is very similar to the scalar pressure sufficient condition, $p \sim q$, but the analogue is only mathematical since the parameters σ and $\hat{\varepsilon}$ are quite unrelated to p and q.

For this special class of equilibria there always exist surfaces, $\hat{\varepsilon}$ = const., which are neutral with respect to interchanges. Taking ψ to be the flux parameter on the neutral surfaces, we find

$$\hat{\varepsilon}'(\psi) = \int [\frac{p_2}{B}\frac{\partial B}{\partial \psi} + (\frac{B^2}{\mu_o} + p_2 - p_1)^{-1} p_1 \frac{\partial p_*}{\partial \psi}]\frac{ds}{B} > 0 \qquad (14.11)$$

as the criterion for stability [based on a monotone profile $\sigma'(\psi) < 0$]. The derivatives $\partial/\partial\psi$ are to be taken in the direction normal to the ψ surface. For a low β plasma (vacuum or force-free field) (14.11) simplifies to

$$\int (p_1 + p_2)\frac{\partial B}{\partial \psi}\frac{ds}{B^2} > 0 \qquad (14.12)$$

which is a simple anisotropic modification of the maximum $\int ds/B$ criterion since, for low β, one can verify that

$$\frac{\partial q}{\partial \psi} = -2\int \frac{\partial B}{\partial \psi}\frac{ds}{B^2} \; . \qquad (14.13)$$

We shall see in a moment that the validity of (14.12) depends on the precise definition of low β.

The stability criterion (14.12) is very frequently quoted, and was originally found by Rosenbluth and Longmire [9] using heuristic arguments which hid the fact that the result depends on a very special assumption with regard to the distribution function. We shall also find that this criterion is unnecessarily pessimistic. For example, it is easily verified that the criterion (14.12) can never be satisfied near the axis in an axially symmetric mirror field. But the finite pressure formula (14.11) is much more easily satisfied. For example, in any plasma with fixed β, increasing the length to diameter ratio will eventually make it interchange stable. The criterion is roughly that stability is achieved if β exceeds some measure of the line curvature compared to plasma radius.

If both σ and $\sigma'(\psi)$ (i.e., p_2 and ∇p_2) approach zero at the edge of the plasma, then the low β formula (14.12) governs interchange stability at the edge of the plasma. This condition is very difficult to satisfy without special stabilizing windings. But the situation is quite different if σ approaches zero linearly in ψ. In this case the second variation must be carefully evaluated together with boundary terms at the interface between plasma and vacuum which represent interchanges between vacuum lines and lines bearing plasma. The same formula (14.11) is obtained but with the interpretation

$$\hat{\varepsilon}'(\psi) = \int \left[\frac{\hat{p}_2}{B}\frac{\partial B}{\partial \psi} + \frac{\hat{p}_1}{B^2/\mu_o}\frac{\partial p_*}{\partial \psi}\right]\frac{ds}{B} \,. \tag{14.14}$$

We have dropped p_1 and p_2 compared to B^2/μ_o where they are not differentiated. The derivatives $\partial B/\partial\psi$ and $\partial p_*/\partial\psi$ are evaluated for the actual equilibrium, but $\hat{p}_1$ and $\hat{p}_2$ are evaluated for the representative function $\hat{f}$; in particular, $\hat{p}_1$ and $\hat{p}_2$ are not zero even in the vacuum. The function $\hat{\varepsilon}(\psi)$ has a discontinuous slope just as does $\sigma(\psi)$. The conclusion is that the plasma is stable if $\hat{\varepsilon}'(\psi) > 0$ <u>on the plasma side</u>; a change in sign of $\hat{\varepsilon}'(\psi)$ across the interface is irrelevant to the sign of the second variation. This means that with finite β (i.e., finite gradients of pressure) we can easily stabilize the entire plasma out to the edge.

But the problem of interchange stability with the special distribution (14.9) is amenable to nonlinear analysis. If there is a change in sign of $\hat{\varepsilon}'(\psi)$ across the interface, the plasma is not stable to finite interchanges. Of course, no plasma is stable with respect to arbitrary finite perturbations; the most stable location for the plasma is at infinity. But in the present problem, the amplitude of the displacement which is required for a release of potential energy is proportional to the distance of the magnetic line from the plasma edge. It requires an arbitrarily small interchange close to the edge. With a finite amount of noise, a certain thin layer will be unstable.

But the energy released is also arbitrarily small. It is, for example, of too high an order to be visible in the stable second variation. A simple yes or no answer to the question of stability may be misleading. Whether the linear or nonlinear conclusion is more reliable could depend on the noise level.

One can also point out that any smoothing of the profile $\sigma(\psi)$ to make $\sigma'(\psi)$ continuous will create a technically unstable situation even in the linear analysis. But <u>every</u> stable plasma is arbitrarily close to another unstable one [e.g., by introducing an arbitrarily small change in $f^{\circ}(\varepsilon)$]. Also the instability created by smoothing $\sigma(\psi)$ will be very weak as well as localized.

To summarize the situation, using the same (linear) theory as in almost all other stability analyses, this configuration is stable. In a practical sense, one has reason to expect a weakly excited disturbance in a thin layer which depends on the noise level within the plasma. As in any analysis which is sensitive to a small region, the conclusion (and the instability) may be wiped out by finite gyro radius effects. It is certainly clear that interchange stability is enhanced by increasing β and by a sharp rather than a gradual transition at the edge.

Next we turn to a somewhat more flexible distribution function

$$\hat{f} = \sigma f(J/\tau, \mu) = \sigma \hat{f}(j,\mu) \tag{14.15}$$

with two scaling parameters $\sigma(\lambda)$ and $\tau(\lambda)$. This time we cannot do the analysis in the large and we examine the first and second variations. The pessimistic interchange variations are given by

$$\delta\Phi = -\int \varepsilon(u \cdot \frac{\partial f}{\partial \lambda}) \, dJ d\mu d\alpha d\beta \tag{14.16}$$

$$\delta^2\Phi = -\int (u \cdot \frac{\partial \varepsilon}{\partial \lambda})(u \cdot \frac{\partial f}{\partial \lambda}) \, dJ d\mu d\alpha d\beta \tag{14.17}$$

where $\partial/\partial\lambda$ is understood to imply that J and μ are fixed. For simplicity we assume that the contours $\sigma(\lambda) = \text{const.}$ agree with $\tau(\lambda) = \text{const.}$ There is a common flux parameter ψ (unspecified) such that $\sigma = \sigma(\psi)$, $\tau = \tau(\psi)$. This specialization implies the existence of a family of surfaces ($\psi = \text{const.}$) which are neutral to interchanges but is not restricted restricted to axial symmetry. The condition that the first variation vanish (equivalent to BCII) is that

$$\rho \equiv \varepsilon_1 \frac{\sigma'}{\sigma} + \varepsilon_2 \frac{\tau'}{\tau} \tag{14.18}$$

be a function of ψ, where

$$\begin{aligned} \varepsilon_1(\lambda,\tau) &= \int \varepsilon(\tau j,\mu,\lambda)\hat{f}(j,\mu) dj d\mu \\ \varepsilon_2(\lambda,\tau) &= -\int \varepsilon(\tau j,\mu,\lambda)[j \, \partial\hat{f}/\partial j] dj d\mu \, . \end{aligned} \tag{14.19}$$

We may, for example, suppose that $\sigma(\psi)$ and $\tau(\psi)$ are given functions. For a given value $\psi = \psi_o$ we examine the contours

$\rho(\lambda,\psi_o)$ = constant in the λ-plane and, assuming that they are concentric, choose the one with the flux value ψ_o and assign to it the value ψ_o. Varying ψ_o in $\rho(\lambda,\psi_o)$ gives a family of ρ-contours each with a value ψ_o, thereby specifying a function $\psi(\alpha,\beta)$. This construction can be carried out provided that the Jacobian

$$\frac{\partial(\rho,\psi)}{\partial(\alpha,\beta)} = \frac{\sigma'}{\sigma}\frac{\partial(\varepsilon_1,\psi)}{\partial(\alpha,\beta)} + \frac{\tau'}{\tau}\frac{\partial(\varepsilon_2,\psi)}{\partial(\alpha,\beta)} \tag{14.20}$$

does not vanish (note that the dependence on τ of ε_1 and ε_2 can be ignored in these Jacobians).

The second variation stability condition can be shown to reduce to

$$\frac{\sigma'}{\sigma}\varepsilon_3 + \frac{\tau'}{\tau}\varepsilon_4 < 0 \tag{14.21}$$

where

$$\begin{aligned} \varepsilon_3(\psi) &= \int \frac{\partial\varepsilon}{\partial\psi}\hat{f}\, dj d\mu \\ \varepsilon_4(\psi) &= -\int \frac{\partial\varepsilon}{\partial\psi}\, j\, \frac{\partial\hat{f}}{\partial j}\, dj d\mu \end{aligned} \tag{14.22}$$

by using the property that ρ in (14.18) is a function of ψ alone. (Note that ε_3 and ε_4 are not ψ-derivatives of ε_1 and ε_2.)

We are primarily interested in equilibria in which $\sigma'(\psi) < 0$; $\tau(\psi)$ is at our disposal. The stability condition $\varepsilon_3 < 0$ which results for $\tau = 1$ is exactly the same as the previous result (14.11). In an axially symmetric low β

field, $\partial\varepsilon/\partial\psi$ will be negative at the center of a magnetic line and positive towards the mirrors. Since f concentrates its largest values at the center, the central negative values of $\partial\varepsilon/\partial\psi$ tend to dominate, and (at low β) the ε_3 term will frequently be destabilizing. But in ε_4, f is replaced by $-j\ \partial f/\partial j = -J\ \partial f/\partial J$ which concentrates more at the large values of J; f must be monotone in J for local stability, but this is not required of $J\ \partial f/\partial J$. Thus it turns out to be possible to stabilize even an axially symmetric vacuum field by making the distribution $f(J/\tau)$ narrower in J as one moves out radially. On the other hand, this process cannot easily be carried to the edge of the plasma where σ'/σ is unbounded, since τ cannot be decreased below a certain value without changing the sign of ε_4 (one can arrange to shape the field so that τ can approach zero with σ, but this requires quite special fields).

In summary, we have described two stabilization mechanisms, finite β (which can even be carried to the edge of the plasma) and scaling f to be narrower outside (which works even at zero β, but not close to the edge).

We now turn to a different series of special distributions not based on the interchange neutrality of the function $f = f^*(J,\mu)$. Consider

$$f(J,\mu,\lambda) = f^{\circ}(\varepsilon(J,\mu,\lambda),\mu)\ . \tag{14.23}$$

In other words, place the same distribution of ε and μ on each magnetic line. The distribution $f^*(J,\mu)$ cannot be contained; $f^o(\varepsilon,\mu)$ is also severely limited, but there do exist special containing fields.

The local stability criterion, $\partial f^o/\partial\varepsilon < 0$, is seen by inspection to yield a positive second variation in (14.17), $\partial f/\partial\lambda = (\partial f^o/\partial\varepsilon)(\partial\varepsilon/\partial\lambda)$. All special distributions $f^o(\varepsilon,\mu)$ which are locally stable are automatically interchange stable. The condition for interchange stability alone is weaker, viz., that the 2×2 matrix

$$A_{ij}(\lambda) = \int \frac{\partial\varepsilon}{\partial\lambda_i} \frac{\partial\varepsilon}{\partial\lambda_j} \frac{\partial f^o}{\partial\varepsilon}\, dJd\mu \tag{14.24}$$

be negative definite on each line, or the equivalent, that

$$\int \left(\frac{\partial\varepsilon}{\partial\psi}\right)^2 \frac{\partial f^o}{\partial\varepsilon}\, dJd\mu \tag{14.25}$$

be negative for every directional derivative $\partial/\partial\psi$ in the λ-plane.

Directly from their definitions, $p_1(B)$ and $p_2(B)$ are seen to be independent of λ. For confinement, with concentric pressure contours reaching zero at the outer confines of the plasma, we must have concentric contours of constant B within a <u>minimum-B</u> or <u>magnetic well</u> field configuration. Note that this field configuration is required for containment, not for stability which is automatic.

The next example in this sequence is the more general function

$$f(J,\mu,\lambda) = f_1(\varepsilon(J,\mu,\lambda),J,\mu) \tag{14.26}$$

This class of functions is too general to yield a simple theory of interchange stability. But, as in the previous example, an evident sufficient condition for interchange stability is

$$\frac{\partial f_1}{\partial \varepsilon} < 0 \; . \tag{14.27}$$

And, as before, the weaker (but complicated) condition (14.24) or (14.25) is the more precise condition for interchange stability. But this time the local stability condition is entirely independent of the interchange condition. Setting

$$f^o(\varepsilon,\mu,\lambda) = f_1(\varepsilon,J(\varepsilon,\mu,\lambda),\mu) \tag{14.28}$$

we have

$$\frac{\partial f^o}{\partial \varepsilon} = \frac{\partial f_1}{\partial \varepsilon} + \frac{\partial f_1}{\partial J}\,\frac{\partial J}{\partial \varepsilon} < 0 \tag{14.29}$$

for local stability. The factor $\partial J/\partial \varepsilon$ (the orbital period) is positive, but $\partial f_1/\partial J$ is indeterminate. A <u>sufficient</u> condition for local and interchange stability would be

$$\frac{\partial f^o}{\partial \varepsilon} < 0 \quad , \quad \frac{\partial f_1}{\partial \varepsilon} < 0 \tag{14.30}$$

together with the inequalities (13.1).

The class of distributions (14.26) subject to (14.30) is subject to the same restriction on containment as $f^{o}(\varepsilon,\mu)$, viz., to minimum B. If we denote by $B_1(J,\mu,\lambda) = \varepsilon/\mu$ the turning point specified by the given parameters, we see that $f_1(\mu B_1,J,\mu)$ is constant on magnetic contours B_1 = const. for fixed J and μ. If f_1 is monotone in $\varepsilon = \mu B_1$, it is monotone in B_1 for fixed J and μ. For containment the contours of B_1 must be closed and monotone.

The significance of the interchange sufficient condition $\partial f_1/\partial\varepsilon < 0$ can be seen by introducing a <u>particle</u> interchange (up to now we have considered <u>line</u> interchanges). Suppose we consider fixed (J,μ) in $f(J,\mu,\lambda)$ and interchange the values of $f(\lambda)$ incompressibly in the λ-plane to minimize $\int \varepsilon\, f\, d\alpha d\beta$. The minimum is obtained by transferring the largest f-values to smallest ε; in other words, f is a monotone decreasing function of $\varepsilon(\lambda)$ for the given (J,μ), $\partial f_1/\partial\varepsilon < 0$. This particle interchange violates the constraint that all particles on a line interchange together. The minimum is too low and $\partial f_1/\partial\varepsilon < 0$ is sufficient but not necessary. The stringent condition $\partial f_1/\partial\varepsilon < 0$ is violated by most of our previous examples of interchange stable systems. It appears to be unnecessarily strong for stability and unnecessarily restrictive of containment. But this criterion would seem to be the one that should arise in a consistent higher order "drift" GC theory in which particles are allowed to move from line to line [23], [21].

The main reason for introducing the special class $f_1(\varepsilon,J,\mu)$ is not its simplicity but its significance for long time containment. Some elements of a theory of equilibrium can be carried out for a long time scale $\mathcal{O}(e)$. In a static field the guiding centers will drift away from magnetic lines on this time scale, but will remain on drift surfaces J = const. Thus if there exists an unchanging equilibrium configuration over this time scale, the distribution function must take the form $f_1(\varepsilon,\mu,J)$. This automatically satisfies the interchange boundary condition, $J_n = 0$, but not conversely.

There are several reasons for carrying out the stability theory for the more general fast equilibria $f^o(\varepsilon,\mu,\lambda)$. First there is no apparent simplification in taking f_1, and all general results proved for f^o apply also to f_1. Secondly, no set of equations of motion has yet been formulated which is accurate on both the short and long time scale. There are higher order drift equations which uncover certain slow micro-instabilities not adequately described by the lowest order GC theory. But all long time formulations assume that J is constant and are thus incapable of describing fast motions. In particular, it is pointless to use slow "drift" equations to investigate interchanges which are strictly fast instabilities. As we have already mentioned in Sec. 2, the treatment of interchanges using higher order drifts uses an energy minimization for interchanges which properly belongs to the zero order GC theory. Finally there is the important practical question of the behavior of an actual system which has been created close to but not exactly in equilibrium. Assuming that there is an approach to a long term equilibrium $f_1(\varepsilon,J,\mu)$ of an incorrectly created plasma, the approach will be on the long time scale. The stability of the fast equilibria met along the way is just as important as the stability of the ultimate equilibrium state.

The relatively complete (necessary and sufficient) information which we have described for interchange stability of the special classes of functions $\sigma f(J/\tau,\mu)$ and $f^{o}(\varepsilon,\mu)$, can be supplemented by a sufficient condition which is valid quite generally [30],

$$\int \left\{ (B^2/\mu_o + p_2 - p_1)^{-1} \frac{\partial p_*}{\partial \Psi} \frac{dp_1}{d\Psi} + \frac{1}{B} \frac{\partial B}{\partial \Psi} \frac{dp_2}{d\Psi} \right\} \frac{ds}{B} < 0 \ . \qquad (14.31)$$

Here $\partial/\partial\Psi$ is a derivative normal to the line [as in (14.11)] and $d/d\Psi$ is a derivative holding B fixed [as when $p_2(B,\Psi)$ is given in equilibrium]

$$\frac{d}{d\Psi} = \frac{\partial}{\partial\Psi} - \frac{\partial B}{\partial \Psi}\frac{\partial}{\partial B} \ . \qquad (14.32)$$

If there are neutral surfaces, Ψ = const., the formula is unambiguous. If not, $\partial/\partial\Psi$ is a directional derivative and (14.31) must be satisfied for all directions (or we can write the condition as a 2×2 matrix involving derivatives $\partial/\partial\alpha$ and $\partial/\partial\beta$).

We confirm qualitatively from this sufficient condition the stabilizing effect of finite β, of slight field curvature, and of scaling the distribution to be narrower on the outside. For example, if we let the geometry become long and straight at any finite β (the mirror ratio can be finite, so long as the curvature is small), then $\partial p_*/\partial\psi$ becomes small and $\partial B/\partial\psi$ becomes positive; the sufficiency condition will always be satisfied.

There are a few results which have been proved using the full second variation as distinguished from interchange stability. First we recall the stability results with tied ends that follow from local stability (Sec. 13). With ends free, Taylor and Hastie [22] have shown that for the special distribution $f^o(\varepsilon,\mu)$ local stability is not only necessary but is also sufficient for stability in the one-fluid case. We recall that in this special case local stability trivially implies interchange stability (the partial derivatives $\partial f^o/\partial\varepsilon$ and $\partial f_1/\partial\varepsilon$ coalesce). In another direction, Taylor and Hastie carry out an ingenious singular expansion about a uniform magnetic field and low β in which p_2^2/p_1 and $\partial f^o/\partial\varepsilon$ remain finite as p_1, p_2, and f^o become small.* Limiting forms of the local necessary conditions (for low β and one fluid) are obtained, also a sufficient

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*The result is claimed to hold for finite β; this is somewhat exaggerated since it is the leading term in an expansion in β.

condition which can be written

$$\int (B_o^2/\mu_o - C_2/B_o)^{-1} \frac{\partial p_*}{\partial x_\perp} \cdot \left(\frac{C_2}{B_o^2} \frac{\partial B}{\partial x_\perp} + \frac{\partial p_2}{\partial x_\perp}\right) ds < 0 \qquad (14.33)$$

or

$$\int (B_o^2/\mu_o - C_2/B_o)^{-1} \frac{\partial p_*}{\partial x_\perp} \cdot \left[\frac{\partial p_*}{\partial x_\perp} - (B_o/\mu_o - C_2/B_o^2) \frac{\partial B}{\partial x_\perp}\right] ds < 0 \qquad (14.34)$$

Implicit in this form is the existence of neutral surfaces; the normal derivatives are normal to the unperturbed uniform field. This formula is quite similar to (14.31) but is disturbingly different. To compare the two we insert $p_1 << p_2 << B_o^2$ and the perturbation about uniform B_o into (14.31) [$\partial p_*/\partial x_\perp \sim \partial B/\partial x_\perp \sim \partial p_2/\partial x_\perp$ are equivalent in order], to obtain

$$B_o^{-4} \int \frac{\partial B}{\partial x_\perp} \cdot \left[\frac{\partial p_*}{\partial x_\perp} - (B_o/\mu_o - C_2/B_o^2) \frac{\partial B}{\partial x_\perp}\right] ds < 0 . \qquad (14.35)$$

Examination shows that (14.34) is a much more stringent condition. It does not at all predict the finite β improvement in stability. It is very difficult to satisfy (14.34) without a magnetic well; (14.35) can be satisfied in every long thin geometry, almost independent of the distribution function and field configuration. Strictly speaking the two criteria are mathematically independent; (14.34) is a sufficient condition for absolute stability, and (14.35) guarantees interchange and local stability, which are only necessary for absolute stability. Nevertheless, barring errors in computation, we must conclude that either local plus interchange stability at low β is not sufficient for stability, or the sufficient estimate given by Taylor and Hastie is very crude.

We conclude with a qualitative discussion of the practical implications with regard to stability of the presently available GCP theory. The possibilities are much more elaborate than in MH. In the latter, only the magnetic field configuration is subject to significant variation. In the GCP there is also the distribution function. Stability requires the proper correlation of a distribution function with a magnetic field, and there appear to be many such stable combinations.

Any theoretical analysis is severely constrained by the criterion of mathematical simplicity. A primary problem is to assess whether models which are chosen for their accessibility to analysis are representative of the general case. The simplest choice of distribution is the absolute Maxwellian. This is completely stable under all circumstances, but it cannot be contained by any magnetic field; pressure is constant everywhere. Next in simplicity is probably the isotropic distribution $f^{o}(\varepsilon)$ or $f^{o}(\varepsilon,\lambda)$. The former is relatively easily stabilized, but like the Maxwellian it cannot be contained. The latter can be contained only in a toroidal geometry, and we shall therefore disregard it in the present context. The simplest distribution that can be contained in any open-ended system is $f = f^{o}(\varepsilon,\mu)$. This turns out to be relatively easily stabilized but again is quite restricted in the containing fields which cannot be other than magnetic wells.

But the special assumption $f^{o}(\varepsilon,\mu)$ [and the more general $f_1(\varepsilon,\mu,J)$ with $\partial f_1/\partial\varepsilon < 0$] offers no clue in distinguishing a well as being more stable than any other field; all others are eliminated by lack of containment because of the specialization in the distribution function. To invert the problem, establishing containment first and then examining stability is more to the point (and more difficult). The Taylor and Hastie sufficient condition (14.34) for the low β shallow well expansion is much more relevant with regard to the value of magnetic wells, provided that its undue pessimism is not selective. But the analysis of scaled distributions and the general interchange sufficient condition (14.31) seem to offer a much more balanced view, especially since the scaled distributions investigate the neighborhood of the borderline distribution $f^{*}(J,\mu)$. We do find that there are, roughly speaking, fewer restrictions on the distribution function in a magnetic well, but it is neither true that a magnetic well is always stable nor that an axially symmetric geometry cannot be stabilized without cusped fields (both are opinions which seem to be very commonly accepted). In other words, modest advantages related to cost, ease of construction, diagnostic benefits, etc. may well dominate.

The belief in magnetic wells as a panacea is based partly on the available evidence and partly on a deep psychological need for simplicity. Certainly a result which can be simply stated and easily understood is greatly to be preferred to one whish is more complicated, provided only that it is not dangerously oversimplified. One strong psychological support comes from the comparison of a plasma with a diamagnetic body. There is one exact piece of mathematical evidence to bolster this analogue. If the magnetic field is entirely excluded from the plasma and there is a sharp separation surface between the plasma and the containing vacuum magnetic field, then a necessary and sufficient condition for absolute stability (even in the large) is that the plasma be placed at an absolute minimum of the field [45]. The significance of a relative minimum at a finite non-zero field value to preserve adiabatic orbits was early recognized as a qualitative desideratum [46]. The important fact that stability can be achieved by the GC theory as well as by the MH theory is a crucial recent discovery by Taylor [20]. But the GCP is much too complex a creature, with a distribution as well as a field, to be interpreted as a classical diamagnetic solid. Although the final answer is far from known, we can already see that the diamagnetic analogue is a qualitative tendency, not a good answer.

The entire question of interchange stability evaporates with tied ends. Even the question of injecting a proper equilibrium is enormously simpler. Any injection which is in local equilibrium on a line [i.e., $f^{o}(\varepsilon,\mu,\lambda)$] is (subject to existence proofs) an acceptable equilibrium. With ends free, a general $f^{o}(\varepsilon,\mu,\lambda)$ will set up large scale motions (in particular Alfvén waves) to try to achieve the natural boundary conditions. With ends tied, local stability, together with mild limitations on the distribution and distortion of field lines, should yield stability. For example, the expansion introduced by Taylor and Hastie yields, for tied ends, the conclusion that every plasma which is locally stable is absolutely stable!* From a theoretical viewpoint tied ends are so much superior that, even with admitted

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* We do not claim this as a proved theorem; nor is Taylor and Hastie's result more than formal. One explicit omission in the simple low β expansion (which leads to the "proof" that interchange stability suffices at low β) is remedied in this singular expansion. But it is not known what other singular behavior may appear at low β.

technical difficulties, this would appear to be as promising a pursuit as complex field configurations.

Interchange stability has been studied intensively: by Taylor in [20] for the special distribution $f^{o}(\varepsilon,\mu)$ and in [21] with the more general $f_1(\varepsilon,\mu,J)$, and in a very large number of interesting forms and approximations by Andreoletti [23]-[28]. In all of these interchange stability is interpreted as absolute stability at low β, ignoring local stability. The local criterion $\partial f^{o}/\partial\varepsilon < 0$ is implicit when the Kruskal and Oberman δW is used; but it is omitted when the interchange analysis is done directly. The additional single-fluid local criteria (7.1) and (7.2) are found by Taylor and Hastie [22] in the special forms appropriate to the assumption $f^{o}(\varepsilon,\mu)$ on the one hand and to the low β almost uniform field expansion on the other. The general necessary condition in two fluids and for an arbitrary contained equilibrium as well as its recognition as a micro-instability condition with strong dependence on the shape of f^{o} are given in [29]. The combination of local and interchange stability, in particular for scaled distributions, is analysed in [30], where one finds somewhat less advantage to magnetic wells and many possibilities of stability in other geometries, based on what seems to be approximately the same available evidence as in the other literature.

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