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THEORY OF CUSPED GEOMETRIES
I. General Survey

by

Harold Grad

December 1, 1957

Reissued January 1, 1959

Institute of Mathematical Sciences

NEW YORK UNIVERSITY

NEW YORK, NEW YORK

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ADDENDUM - January 1, 1959

This report was originally issued as a classified document on December 1, 1957. No changes have been made in the manuscript, but some minor corrections and additional references are given below.

p. 19 l. 16: read "35" for "15"

p. 22 eq.(4): "$Q = 10^{18}$ RH"

p. 22 l. 17: read "Fig. 4c" for Fig. 5c"

p. 22 l. 18: read "Fig. 4d" for "Fig. 5d"

p. 22 eq.(5): "$\tau = 10^{-7}$ HR^2/T

p. 23 lines 1-3: read "For example, with H = 10,000 gauss, R = 10 cm, T = 10 ev, we have $\tau = 0.01$ sec. Or with H = 100,000 gauss, R = 100 cm, T = 10 kev, we have $\tau = 0.01$ sec. again"

p. 33 eq.(6): "$P_c = 0.08$ RTH"

p. 34 Table is somewhat modified according to the above changes.

Section 7: Since this report was issued, high density plasmas (for which cusped containment is appropriate) have been achieved experimentally; e.g. A. C. Kolb, Magnetic Compression of Shock Preheated Deuterium, Paper submitted to the second United Nations International Conference on the Peaceful Uses of Atomic Energy.

p. 42 l. 5 from bottom and following: The monocusp is the optimum shape.

ADDITIONAL REFERENCES

In the Proceedings of the second United Nations International Conference on the Peaceful Uses of Atomic Energy:

J. Berkowitz, K. O. Friedrichs, H. Goertzel, H. Grad, J. Killeen, and E. Rubin - Cusped Geometries

J. Berkowitz, H. Grad, H. Rubin - Problems in Magnetohydrodynamic Stability

C. S. Gardner, H. Goertzel, H. Grad, C. S. Morawetz, M. H. Rose, and H. Rubin - Hydromagnetic Shock Waves in High-Temperature Plasmas

PREFACE

This report is a survey of and introduction to some of the more important theoretical aspects of those cusped geometries which may be feasible as thermonuclear devices. The possibilities are still far from being completely explored, and the situation may still be expected to change. Quantitative analyses of many of the specific problems mentioned in this report will follow, but in view of the large number of inter-related phenomena which must be simultaneously analyzed, these results should be considered to be tentative until verified experimentally.

The text of this report is essentially identical to a draft which was written in October 1956. It was felt to be advisable to issue this material in a tentative form rather than to delay, possibly indefinitely, for the completion and improvement of the theory.

THEORY OF CUSPED GEOMETRIES

I. GENERAL SURVEY

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1. Introduction

The point of departure for these investigations is a certain idealized plasma-field configuration. This is a free boundary surface or interface separating a plasma domain in which there is no electromagnetic field from a vacuum domain in which there is no plasma. The original advantage of this model was its mathematical simplicity which allows deeper study and more quantitative analysis than is easily possible when plasma and field intermingle. Subsequently, however, it was found that such a sharp separation (provided that is can be realized experimentally) may have definite practical advantages over similar configurations without the sharp boundary (Ref. [12]).

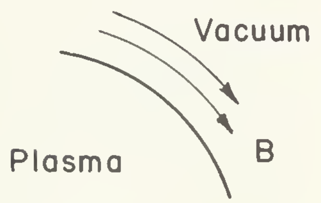

Fig. 1

Among the theoretical questions that should be asked about a given configuration presumed to be operating as a thermonuclear device are its stability, rate of loss of particles and of energy, and experimental feasibility. In order to answer all but the first question, the original idealization must be modified to take into account a continuous transition from plasma to vacuum. However, the picture we have in mind is still that of a thin transition zone of minimum theoretical thickness separating a reasonable approximation to a vacuum from an almost field-free plasma. It is this feature that distinguishes the configurations analyzed in this report. Treatment of the general case will be left to later reports but will be occasionally referred to for comparison.

2. Description of the Geometries

The basic two-dimensional stable geometries are indicated in Fig. 2. These shapes have been computed analytically by conformal mapping (Ref.[14]). The essential mathematical feature of

Fig. 2

the problem of the determination of such interfaces is that the position of the interface itself must be found as well as the accompanying magnetic field configuration, but the latter is determined by the surface currents on the unknown interface as well as by the given external coils. An equilibrium free-surface configuration is distinguished from any other magnetic field surrounding a perfect conductor by the fact that the magnitude of B must be a constant on the entire surface of the plasma. This restricts the possible shapes. The special case of Fig. 2a in which the external coils (infinite line currents) are placed at the corners of a square and move out to infinity yields a hypocycloid as the free boundary, $x^{2/3} + y^{2/3} = 1$.

In actual practice, the coils will have finite dimensions, and numerical computation will have to be resorted to. This type of computation is feasible.

For a given external coil system, there is a one parameter family of equilibrium free-surface configurations. One can imagine that plasma is 'poured in', increasing the size of the

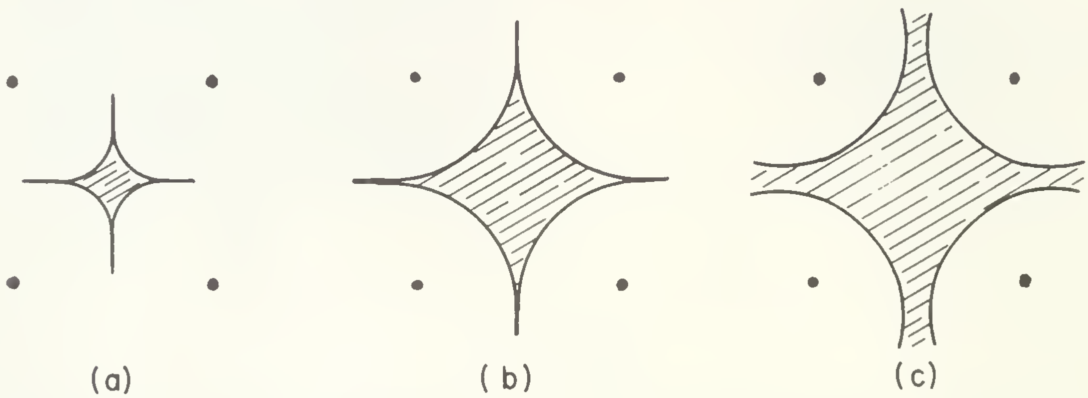

Fig. 3

diamond until a certain maximum size is reached (at which point the cusps extend to slightly beyond the coil position, Fig. 3b), after which the plasma 'spills out', and there is no bounded solution, in other words, no containment.

The shape of the interface is unchanged if we construct a <u>crossed-field</u> configuration by inserting in the plasma a constant longitudinal magnetic field which is perpendicular to the plane of the external field (Fig. 4a); this field is supposed to be confined to the plasma and does not extend to the vacuum domain.

In three dimensions, there are a number of cusped configurations which are suggested by Fig. 2. We say 'suggested' because the conformal mapping technique is not available in three dimensions and the existence of these configurations, although plausible, is not yet proved. In some cases, the existence has been empirically demonstrated by the apparent convergence of numerical computation (Ref. [15]).

A number of the three-dimensional possibilities are sketched in Fig. 4. At (a) we have merely reproduced Fig. 2a in three-dimensional perspective. The external coil system consists of four parallel wires. This can be compared with a conventional linear pinch or with a B_z-pinch if the indicated crossed field is inserted in the plasma. We obtain (b) by bending (a) to close the figure into a torus. The shape of the cross-section will be somewhat modified by the bending and more so if there is an interior crossed field. The external coil system is now four circular wires. It is interesting to note that if we invert the degrees of importance which we attach to the external cusp field and to the internal crossed field, i.e. consider the internal field to be basic and the cusped field auxiliary, then this configuration becomes similar to some of the helical Stellarator geometries. Figure 4c is 2a rotated about an axis through the cusps; this is best compared to the Mirror Machine. The external current system could be two opposed Helmholtz coils. Figure 4d is Figure 2b rotated about a horizontal axis or, equivalently, a number of 4c's placed end to end. The coil

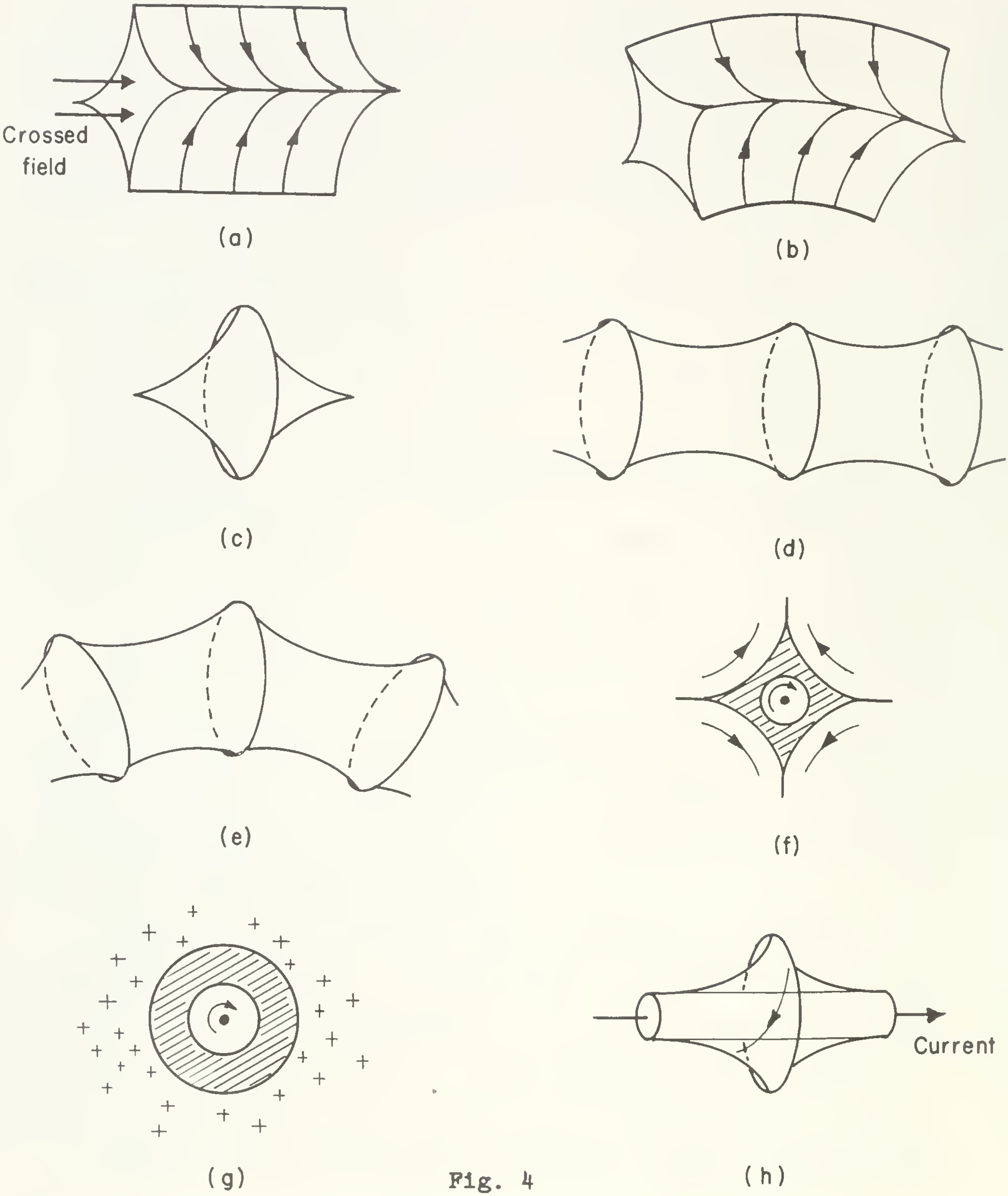

Fig. 4

system is a succession of closed loops encircling the plasma. In 4e we bend 4d around to get a closed configuration. These have been called hexacusp, octocusp, etc. depending on the number of segments, and, by analogy, 4c is a monocusp.*

Figures 4a and 4d can be further modified by cutting out a hole in the plasma using an internal conductor as indicated in 4f and 4g respectively. The purpose of this is economy of magnetic energy and will be discussed later. This modification also provides a means of introducing a crossed field into 4d or 4g by allowing the field of this central conductor to penetrate somewhat into the plasma. In Fig. 4h, we have sketched such a modification of 4c with a crossed field. The possibility of cutting a hole in a torus such as 4b or 4e depends on passing conductor leads inside; this also seems to be theoretically possible, but the sketch is rather formidable. The more complex of the above suggestions are, of course, only tentative, subject to either theoretical or experimental confirmation.

M. Levine has proposed certain configurations which bear a resemblance to that of Fig. 2a, but with six coils instead of four.

All of these geometries are idealized in that a sharp separation surface is postulated. There is a corresponding family of more general configurations exhibiting a continuous

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*This was originally referred to as the cuspidor, reference [12].

transition. For example, with plasma and field intermingled, Fig. 2b represents the original Picket Fence configuration (Ref. [11]).

3. Stability

There is one important feature shared by virtually all the geometries just mentioned, and that is absolute stability against all perturbations.* In the absence of a crossed field, this property has been proved using the energy principle described in Ref. [1] and with crossed fields using the more general energy principle of Frieman (Ref. [3]). The crux of the argument which demonstrates this stability is, roughly speaking, that the magnetic potential (or some equivalent potential) is a minimum for the configurations of Section 2. With only slight modifications, the argument can be applied to any conventional conducting fluid and to any conducting gas, e.g. one governed by the conventional equations of gas dynamics or by the Boltzmann equation. By suitable modifications of this argument, stability can be demonstrated to hold for <u>finite</u> perturbations as well as for infinitesimal and for a variety of external constraints including certain external circuitry (Ref. [7]). The sole property that the fluid (or collection of particles) must possess is that the field and fluid do not mix; i.e. the transition zone must remain thin for an appreciable length of time. In particular, it should be emphasized that the cusp itself is stable against displacements or deformations.

All the geometries of Section 2 are stable whether or not there are additional external fixed conducting shells or internal

*References [2], [4], [7].

crossed fields but not necessarily when accelerated. Inward (compressive) acceleration tends to destabilize and outward (expansive) stabilizes. In a periodically oscillating plasma, the acceleration will be stabilizing when the plasma is most compressed and destabilizing when it is expanded. In the absence of a crossed field, the maximum inward acceleration that can be sustained is given by

$$2aR < A_o^2 = B_o^2/\mu\rho \tag{1}$$

where a is the acceleration, R is the maximum radius of curvature of a magnetic line on the interface, and A_o is the Alfven speed constructed from the plasma density ρ and vacuum field value B_o at the interface (see [4]). The maximum value of R is found midway between cusps and is, generally, about equal to the radius of the plasma.

Crossed fields have an additional stabilizing effect*, and this effect is enhanced by surrounding the plasma with a conducting shell; but without the crossed field, the external conductor has no effect at all on stability. The relative magnitudes of these effects on stability can be described by comparing them as equivalent accelerations. In the presence of crossed fields, the results vary with the geometry, and we shall not attempt to cover

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* This is rigorously true if the crossed field is uniform, e.g. in Fig. 4a, but is an oversimplification if the crossed field is curved as in Fig. 4b.

all cases. For example, in the geometry of Fig. 4a, the crossed field effect has order of magnitude

(2) $$a_x \sim A_1^2/L$$

where A_1 is the Alfven speed associated with the internal field, B_1, and L has the order of the length of the plasma. This should be compared with the curvature effect which is

(3) $$a_c = A_o^2/2R \quad .$$

The maximum acceleration that a given configuration can withstand and remain stable is given by the sum $a_c + a_x$. Of course, in an actual experiment, the duration of a destabilizing acceleration would also be a significant parameter.

4. Particle Losses

What we wish to discuss in this section is the containment time of a typical stable cusped geometry. The perfectly conducting fluid model with a mathematically sharp interface has an infinite time of containment since it is stable and there is no diffusion across the field. This fluid model could be modified macroscopically by introducing finite conductivity, but it is more appropriate to do a direct particle analysis.

First, consider those cases in which there is no crossed field. The diffusion of particles <u>across the field</u> turns out to be a negligible loss compared to the loss of particles through the cusps <u>in the direction of B</u>. The latter loss is a direct consequence of adopting a finite transition layer rather than an abrupt discontinuity between plasma and field. It is not immediately related to a mirror-type loss since the magnitude of B does not increase for a particle approaching a cusp; however, in a certain generalized sense, the mechanism is the same as in a mirror (Ref. [13]). The magnitude of this loss is found to be directly related to the thickness of the transition layer, and the best result is obtained with the thinnest possible layer. The gyro radius represents, roughly speaking, the shortest distance within which a particle can be turned back by a magnetic field; this represents, therefore, a lower bound on the thickness of the layer. There is still a considerable range, however, between the electron and the ion gyro radii (a factor of 60 for deuterium). It can be made plausible by appropriate theoretical arguments that, by judicious experimental control, the minimum theoretical thickness, i.e. the electron gyro radius, can be approximated quite closely. It is this thin transition zone which seems to give the high density (large β^*) cusped configurations an advantage over the low density picket fence configurations.

The estimation of particle losses requires (among other things) the solution of an extremely complex self-consistent field problem involving the determination of the electromagnetic field in the boundary layer (including the neighborhood of the cusps) so as to be consistent with the particle trajectories which give rise to this electromagnetic field. What we shall do is present a sequence of models of increasing complexity until we obtain one which we believe gives a reasonable approximation to the actual loss rate. The details will be presented in later reports.

For the time being, to simplify the arguments, we assume that ions and electrons have the same temperature and a mono-energetic (i.e. fixed speed) isotropic velocity distribution. Any plasma which is dense enough to be interesting thermonuclear-wise must be neutral on the whole; this does not eliminate the possibility of non-neutrality in very thin regions on the order

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$^*\beta$ is the ratio of plasma energy to magnetic energy.

of the Debye length, but does require neutrality in any domain as large as the gyro radius, e.g. in the boundary layer.

The first step is to take the magnetic field configuration of the fluid model with a discontinuity in B and compute particle trajectories, cf. Fig. 5a. The trajectories are straight lines joined by cycloidal pieces. The limiting case

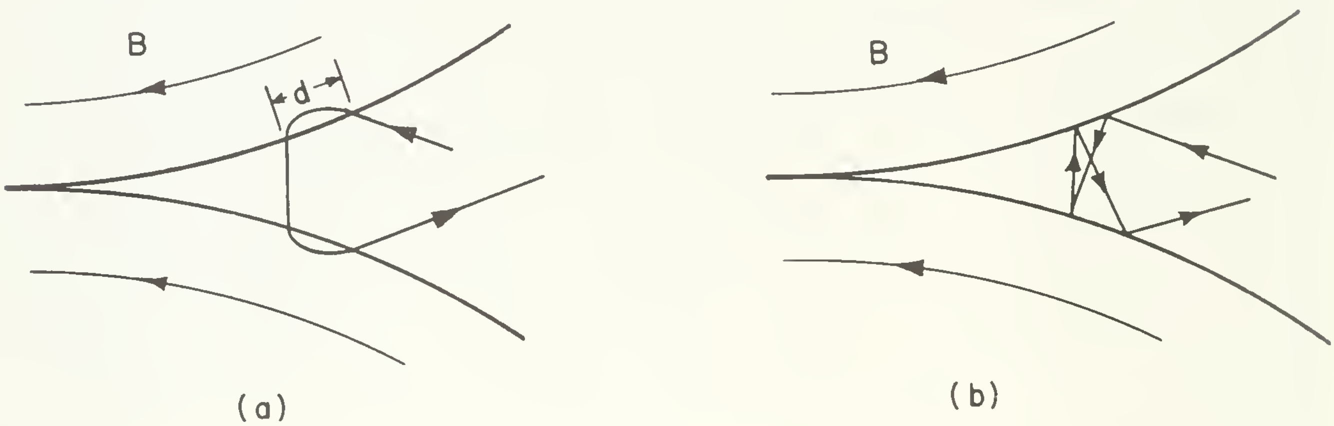

Fig. 5

of zero gyro radius is a billiard ball model, cf. Fig. 5b. In this idealized case, every particle will be turned back before reaching the cusp unless it is on the axis and aimed directly at the cusp. There is no loss of particles; the containment time is infinite which is consistent with the original fluid model. On the other hand, the 'soft' billiard table, Fig. 5a, does allow certain particles to reach the cusp and escape; (this will be adopted, temporarily, as the definition of 'escape'). More precisely, those particles which are aimed near enough to the cusp are lost. Analysis of these trajectories leads to the conclusion that the rate of loss of particles of a given speed is proportional to the mean distance of travel toward the cusp in a single encounter with the boundary layer ('d' in Fig. 5a); this however, is proportional to the gyro radius. Since electrons are faster by a factor 60 and have a smaller gyro radius by the same factor, the loss rates of ions and of electrons are equal on the basis of this computation.

This computation, using the original discontinuous magnetic field, is not self-consistent in two ways; the finite penetration distance produces a current layer rather than a current sheet, ultimately yielding a continuous transition of the magnetic field, and the greater penetration of ions produces an electric field which further modifies the trajectories. Of these two effects, the first produces only minor quantitative changes, but the second is crucial. The reason for this is that charge separation of an amount indicated by the difference between electron and ion gyro radii would produce gigantic electric fields. To a very good approximation, the electric field must be such as to produce equal mean penetration of ions and of electrons, or more precisely, equal densities of both particles everywhere in the layer. Solution of this self-consistent problem yields the result that the electrons are pulled out only slightly but the ions are pulled in to almost the original electron penetration distance. Although the mean penetrations are now the same for the two particles, the mean distances travelled in the direction of the cusp are not quite the same since the actual trajectories of ions and electrons are no longer similar in the presence of the electric field. However, a very rough estimate of the loss rates can be computed by assuming that electron and ion orbits are geometrically similar. Since the electron penetration is roughly the same as in the original non-self-consistent model, while the ions have been pulled in by a factor 60, we conclude that the electron loss rate is roughly the same as was originally computed, but the ion loss rate has been cut down 60 times by the charge separation electric field. More precise computation yields the factor 15 rather than 60. (These computations will be made in succeeding reports).

This second model is internally self-consistent, but the difference in electron and ion loss rates must be compensated externally by a change in the potential of the whole plasma. An equilibrium will be reached in which the ion and electron loss rates are equal, but the crucial question is whether this is accomplished mostly by increasing the ion loss rate or by decreasing the electron loss rate. The answer to this question involves solving another self-consistent problem involving the above-mentioned boundary layer, the cusp neighborhood where two boundary layers merge and an external (possibly penetrating the plasma) field necessitated by the unequal loss rates computed in its absence. The basic issue is as follows. The external electrostatic potential accelerates the emerging ions to the walls but it turns back a certain number of electrons. Some of the returning electrons will reenter the plasma, cutting down the net electron loss rate, but some will be slightly deflected (a distance of one gyro radius) and will miss the cusp, entering the boundary layer (cf. Fig. 6). Now, any electron space charge in the boundary layer decreases the charge separation field and allows the ion trajectories to move out, thereby increasing the ion loss rate (cf. Fig. 5a; the larger the penetration, the larger is the distance d toward the cusp). In addition, if this electrostatic potential penetrates inside the plasma through the cusp, it can directly increase the loss rate of ions by accelerating them toward the cusp.

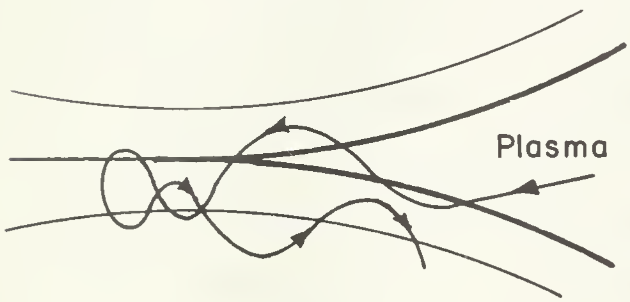

Fig. 6

Qualitative arguments can be given to show that the latter effect, i.e. penetration of the field, is small; this remains to be investigated quantitatively. With regard to the former effect, it is clear that the equilibrium electron space charge in the boundary layer (i.e. the thickness of this layer) is determined by a balance between entering electrons which have missed returning to the cusp plus entering electrons which diffuse from the plasma to the boundary layer and leaving electrons which diffuse back into the plasma or out to the walls of the apparatus. It is possible to conclude that a balance will be reached in which the boundary layer does not thicken indefinitely but reaches a limiting thickness which is not much larger than the electron gyro radius. To see this qualitatively, we note first that in virtue of the external electrostatic field, electrons are trapped on a magnetic line in the boundary layer whereas ions are immediately lost to the external surroundings; in other words the space charge is purely electronic. Next, although the electron density in the plasma may be considerably larger than that of the electrons trapped in the boundary layer, the diffusion rate inward can be considerably larger than the diffusion rate outward; (if not for this, the boundary layer would thicken indefinitely). This happens because particle deflections are predominantly the result of a cumulation of small deflections rather than isolated large deflections. The diffusion into the plasma of a boundary layer electron is such a gradual affair. However, the outward diffusion of a plasma electron to the boundary layer must be accomplished during a single gyro period, while it traverses the layer; at its next encounter with the layer, it has 'forgotten' any small deflections it had previously received. A rough quantitative estimate based on these ideas yields an equilibrium boundary layer thickness which is not considerably larger than the electron gyro radius.

According to these arguments, the electric field configuration will be somewhat as shown in Fig. 7; the lower figure is a plot of the electric potential as a function of radius along the dashed line section indicated.

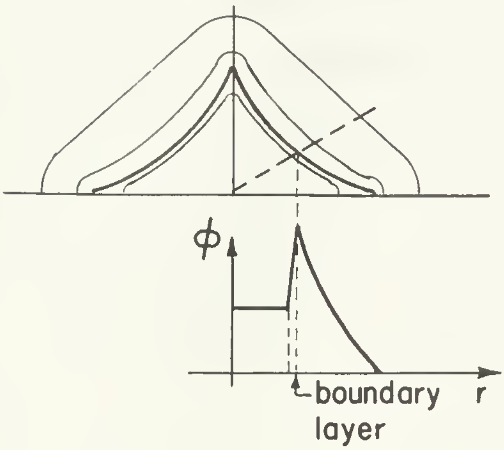

Fig. 7

Making these estimates, we obtain a formula for the net loss rate,

(4) $$Q = 10^{17} R H ;$$

here Q is in ions per second, R is a representative radius of the plasma in cm., and H is the magnetic field in gauss. This computation applies to a geometry similar to Fig. 5c or to a single section of Fig. 5d. The density of the plasma is implicitly contained in H since the magnetic field must be such as to balance the plasma pressure. The lifetime of the plasma is given by

(5) $$\tau = 10^{-6} H R^2/T \qquad ;$$

τ is in seconds and T is the temperature in electron volts.

For example with $H = 10{,}000$ guass, $R = 10$ cm., $T = 10$ ev., we have $\tau = 0.1$ second. Or, with $H = 100{,}000$ guass, $R = 100$ cm., $T = 10$ kev., we have $\tau = 0.1$ second again.

These numerical formulas are only representative; the results depend within a factor of 2 or 3 on which of the various cusped geometries is used (the above formulas are representative of the monocusp, Fig. 4c) and by possibly even larger factors depending on the ultimate effect of the electrostatic field surrounding the plasma on the structure of the boundary layer.

The loss picture is entirely different in the presence of a crossed field (Refs. [13] and [16]). To illustrate this difference, consider the following three different particle loss mechanisms:

(1) diffusion across the field

(2) collision dominated mirror type losses

(3) non-adiabatically dominated losses.

The first is self-explanatory and is caused by interparticle collisions. The second involves the concept of adiabatic or slow space and time variation of the magnetic field compared to the gyro radius and frequency. This assumption has the consequence that a particle remembers its past and, if once reflected by a mirror, will be in the same situation at every future encounter with the mirror, and will be perpetually reflected. This ideal result is modified by collisions which cause a particle to forget its past. The third loss mechanism applies to the same mirror geometry when the field changes appreciably compared to the gyro radius. In this case, even without collisions, a particle may be lost after several traverses because of a loss of memory. This is exactly the situation in the cusped geometry just analyzed; the fact that a particle is reflected at a cusp yields no information about the result of a later encounter with the cusp.* We could subdivide case (3) further depending on whether there is a sharp interface (smaller loss rate) or not (larger loss rate)--the difference is at least one order of magnitude. A non-adiabatically dominated geometry will usually exhibit losses which are one or two orders of magnitude larger than a mirror-dominated geometry, and this in turn will usually be one or two orders of magnitude higher than those of a diffusion-dominated geometry. The total spread is at least three to four orders of magnitude!

In general, all three mechanisms will be operative, but frequently a single one is dominant. However, it is even possible for the different mechanisms to be dominant in different parts of the same machine. For example, consider a monocusp geometry (Fig. 4c) with field and plasma intermingled. A mirror ratio can be assigned to ecah magnetic line and the numerical values will range from say 2 or 3 on an outside magnetic line to a very large value on a line which approaches the center of the apparatus (where $B = 0$). The losses from the outer layers

*More precisely, there is only a slight persistence of memory, Ref. [13].

will then be collision dominated as in the conventional Mirror Machine, whereas the losses from the inner layers can be considerably larger since they are non-adiabatically dominated (see [13] for a more complete discussion).

The crucial feature of the crossed field is that it reduces the losses from group (3) to either group (1) or group (2); in the case of a sharp interface to group (1) and, more generally, to group (2). The explanation is roughly that in the presence of the crossed field, the resultant field is never zero, and there is a finite mirror ratio on each magnetic line.

5. Additional Considerations

In this section we shall discuss a number of complicating features which must be considered in the analysis of an operating cusped thermonuclear device.

First is the possibility of an ion temperature which is different from the electron temperature. It is clear from the preceding analysis that an electron temperature which is lower than the ion temperature reduces the net ion loss rate. As an extreme example, if the electron and ion velocities are equal ($T_+/T_- = 3600$), the loss rate is smaller than that given in Equation (4) by a factor of about five. A small effect in this direction is provided by Bremstrahlung. A more significant mechanism is provided by the selective heating of ions which results from some methods of heating (see below). However, it is unlikely that for relevant densities and temperatures the equipartition rate between ions and electrons will allow an appreciable temperature difference to subsist long enough to be useful as such. Nevertheless, there is another possible advantage in having the electron temperature low, even for a short while, because with a very small electron gyro radius, it is more likely that a plasma of the type postulated (with a thin boundary layer) can be set up experimentally.

There are a few aspects of the question of heating which are somewhat different from those in other geometries. Magnetic pumping as applied, for example, to a Stellerator geometry depends on an irreversible influx of energy arising from an alternating magnetic field variation near the gyro frequency of the ions or the transit time through a bulge. A similar situation is found in any of the cusped geometries, but, in this case, the frequency should be on the order of the transit frequency of ions across the apparatus. The simplest model for computing this type of heating is a moving magnetic wall. Computations based on this model show it to be quite feasible. However, this computation is too naive on at least two counts. First of all, rapid heating requires large acceleration of the plasma boundary which may lead to instability. The restriction to stable values of acceleration (Equation (1)) allows the moving wall to reach on the order of the ion speed in accelerating over a distance comparable to the radius of the plasma; this allows the plasma energy to increase by a factor significantly greater than one in each cycle, which is rapid enough heating to be practical; (any configuration which is interesting thermonuclearwise must persist for many ion transit times). One particularly intriguing possibility for heating is to deposit enough energy into the magnetic field to be able to heat the plasma to thermonuclear energies in one shot (cf. the pinch). Using the same model, we find that the magnetic wall executes damped oscillations. The initial acceleration is, of course, too high for stability. However, within one or two cycles, it drops to a value on the order of the maximum stable value. This situation is obviously not one in which the linear (i.e. small perturbation) stability analysis can be relied on but, roughly speaking, the possibility of practical stability, i.e. within a cycle or two, seems to be reasonable.

The second major defect of the moving wall model is the likelihood of creating shock waves in the plasma. In such an event, the moving wall merely accelerates the adjacent plasma, and energy is converted into thermal energy of the plasma via the shock. It is not definitely verified as yet that shocks can exist under relevant conditions. Certainly a classical shock whose width is comparable to the mean-free-path cannot exist since the mean-free-path is large compared to the apparatus dimensions. However, there are indications that shocks with thickness on the order of the gyro radius may be observed.*
Assuming that shocks governed by the gyro radius exist, Hanan Rubin has shown that, for shocks of medium strength, ions are heated more than electrons and under appropriate conditions even much more. Arguments have been given by Stirling Colgate (Ref. [20]) and by Marshall Rosenbluth and Conrad Longmire (Ref. [19]) to indicate the opposite conclusion for strong shocks. Insofar as all these results are reliable, they might indicate a preference for heating by a sequence of several medium strength shocks rather than by a single large one.

If the situation is dominated by shock waves, the stability analysis has to be reconsidered. Instead of an acceleration of

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*At the present time (December 1957) the evidence is very good that damped shocks exist without the intervention of particle collisions.

the plasma, we are presented with an applied impulse or sequence of impulses. This analysis has not yet been done.

The single-shot heating process is basically the same as in the conventional pinch, especially for the geometry of Fig. 4a. Roughly speaking, in adopting a cusped configuration for the pinch, we achieve stability at the expense of greater losses. The insertion of a crossed field (B_z) increases stability in both cases, and it also cuts down losses in the cusped case. We still maintain greater stability in the cusped case than in the conventional pinch, but we also, in all probability, have greater losses here as before. It is difficult to judge at the present time whether stability or particle losses is more important in obtaining large containment times. It is even possible that the relative importance could reverse when comparing scale models and full scale operating devices.

The next question concerns the extent to which the velocity distribution is Maxwellian. The cusp loss formulas show that fast particles are lost more quickly than slow ones. This effect will tend to cut off the Maxwell tail and so reduce the number of thermonuclear reactions. At a given temperature in the range of 10 - 100 kev., the effect is much more serious with DD than with DT. Especially in preliminary (low temperature) experiments when almost all reactions originate in the tail of the distribution will this effect be important. The situation becomes more complicated under operating conditions in the presence of hot (charged) reaction products. The primary effect on the structure of the boundary layer (due to the large gyro radius of the hot ions) is to cut down the loss rate of the cold particles. This effect is probably more than enough to compensate for the diminished Maxwell tail. On the debit side, the hot particles themselves are lost very quickly. First of all, this makes unlikely the possibility of directly recovering the released thermonuclear energy by expansion of the plasma against the containing field since the excess pressure due to the hot particles is small. Secondly, this may cut down on the total thermonuclear yield by reducing the number of secondary reactions (in the case of D-D). Since the computation of thermonuclear yield is fairly sensitive to the exact shape of the Maxwell tail, only rough estimates of these factors can be made at present. In the presence of a crossed field, the result is even more complex because we are faced with a combination of non-adiabatic, cusp-type losses which are larger for hot particles and mirror-type losses which are smaller for hot particles. This subject remains to be investigated.

The fact that most of the particle loss is in a directed beam might eventually turn out to be important in recovering this streaming energy directly as electrical energy.

One of the most difficult questions to answer theoretically is the effect of magnetic field penetration into the plasma. This factor is important since we do not expect to easily construct a plasma which is practically field-free. The quantitative loss computations which have already been done can be definitely relied upon only if the internal gyro radius is large compared to the apparatus dimensions. This is a difficult situation to achieve. At the opposite extreme, namely complete field penetration, we have the picket fence, for which the losses may be much larger but for which no quantitative results have yet been obtained.* The question is how close to the former (cusp) configuration can experimental techniques bring us. With the crossed field, there is the same distinction between sharp and gradual transitions, but the distinction may not be as important in this case, namely in comparing 'good' with 'better' rather than 'bad' with 'worse'.

An intriguing feature is the possibility of controlling the boundary layer thickness by external means. The particle loss situation is improved if we can draw off electrons from the boundary layer. Alternatively, what we wish to do is lower the potential of the plasma with respect to the potential of the boundary layer. This goal would seem to be possible, based on the suggestive concept of high conductivity of a plasma parallel to the field and low conductivity perpendicular to the field (Hall effect). Specifically, instead of the potential distribution of Fig. 7, we might be able to obtain that of Fig. 8. The equipotentials would not close around the plasma, but would

*The formulas of Hartland Snyder give upper estimates only.

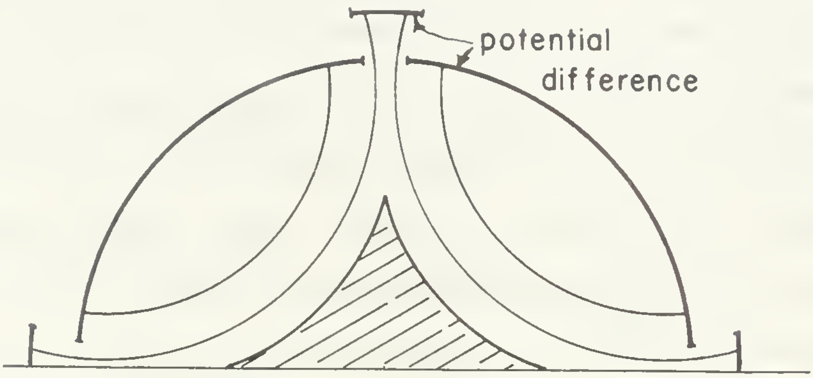


Fig. 8

follow the magnetic lines and be fixed by the external potentials maintained where the magnetic lines hit an external conductor.

6. Feasibility Estimates

An estimate of the feasibility of a given geometry as a thermonuclear device can be arrived at by balancing thermonuclear power input against various losses to get a positive net output. As a rough estimate we consider cusp losses and coil heating losses only, and envision a steady state. The principal omissions are radiation losses and the effect of a duty cycle. We shall use the particle loss estimates given in Section 4 for the monocusp, Fig. 4c. These are subject to revision, up or down, depending on the accuracy of the self-consistent field computation described in Section 4. More optimistic results can be expected in crossed-field configurations.

We can specify the state of the plasma, once the geometry is decided, by three parameters, e.g. volume, pressure and temperature. Actually we choose R (a representative plasma radius in cm.), H (the magnetic field in gauss), and T (the temperature in electron volts). For the energy loss through the cusp, in watts, we find (Ref. [12])

(6) $$P_C = .03\ RTH \quad ,$$

for the ohmic losses,

(7) $$P_O = 10^{-4}\ RH^2$$

and for the thermonuclear yield (charged particles only),

$$
(8)\quad
\left.
\begin{aligned}
P_N &= 2 \times 10^{-15}\, R^3 H^4 , \quad T = 10^4 \\
\text{"} &= 2 \times 10^{-16}\, R^3 H^4 , \quad T = 10^5
\end{aligned}
\right\} \text{DT}
$$

$$
\left.
\begin{aligned}
\text{"} &= 3 \times 10^{-17}\, R^3 H^4 , \quad T = 10^4 \\
\text{"} &= 6 \times 10^{-18}\, R^3 H^4 , \quad T = 10^5
\end{aligned}
\right\} \text{DD}
$$

Clearly, P_N can always by made to dominate $P_C + P_R$ simply by making R and H large. Let us define 'breakeven' by the condition $P_N = 3(P_C + P_R)$ (implying energy withdrawal by means of a thermal cycle).* A few representative breakeven figures are as follows:

Reaction	T	H	R	n(ions/cc)	τ(μ sec.)	ion transits	P_C(KW)
DT	10^4	10^5	22	3×10^{16}	5,000	10^4	6×10^5
DD	10^5	10^6	40	3×10^{18}	16,000	3×10^4	10^8

P_C (the cusp loss power) represents what must be inserted to generate the plasma and is also comparable to the expected net output of the device. The DT figure for power output is comparable to that of a conventional generating station, but the DD figure would seem to be impractical. The power output scales as $1/\sqrt{H}$, but H has already been chosen as high as is feasible (or possibly higher)!

*This is optimistic in virtue of the presence of a duty cycle but pesimistic in that the energy supplied to create the plasma is assumed to be completely lost.

In the case of a crossed-field configuration, no quantitative particle loss results are yet at hand. As a rough estimate, particle losses could be computed by interpolation between formulas representative of the Mirror Machine and the Stellerator geometries. The breakeven figure for the cusped geometry would then be quite encouraging since it could be computed for $\beta = 1$.

7. Possible Experiments

In this section we shall give a brief description of some experiments, not thermonuclear in scope, which might shed light on the feasibility of constructing cusped plasma configurations.

One significant feature of these geometries is the sharp interface. This can be achieved only if the gyro radius is small compared to the size of the apparatus. As a matter of fact, it is easily verified that the average number of transits made by a particle before being lost is directly proportional to the ratio of plasma dimension to gyro radius. As an example of a crude experiment, take the nominal ion gyro radius equal to the plasma dimension; from the analysis of Section 4, we might expect an optimum boundary layer thickness of about .03 times the plasma size. At a plasma temperature of 10 volts and ion density of 10^{14}, we compute a containment time of about 300 microseconds which is about 4500 electron transits or 75 ion transits. On the basis of free particle motion in the vacuum field produced by the external field coils alone (i.e. for a picket fence), one would expect a containment time of only several ion transit times or several hundred electron transit times (this estimate is tentative and must be checked by numerical computation of representative orbits). This experiment should therefore allow the two theories to be differentiated. With the figures given above, the field strength at the plasma surface is about 150 gauss. The radius of the plasma is about 5 cm. The ion loss rate through the cusps is 7.5×10^{19} per second or about 10 amperes. This indicates the size of ion source or plasma gun which would be required to create such a plasma.

It is important to realize that, if such a plasma is to be inserted into an already existing field of appropriate geometry, this must be done very quickly. Otherwise, the loss rate while the plasma is being created is the larger value corresponding to the picket fence mechanism. A plasma gun rather than an ion source is therefore indicated. A possible configuration is indicated in Fig. 9.

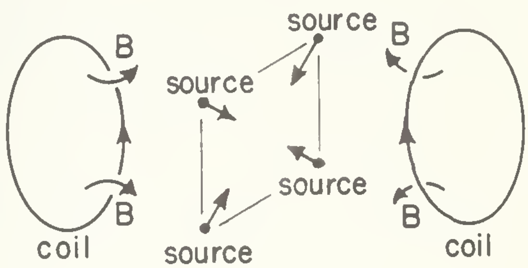


Fig. 9

An alternative procedure is to first create the plasma and then apply the confining field. For example, a shock can be passed through a tube, ionizing the gas, after which the magnetic field is applied. This would seem to be an efficient way of constructing the desired sharp interface. To obtain a stationary plasma, one might utilize two shocks, colliding head-on in the center of the tube. Any of the linear (i.e. non-toroidal) magnetic fields of Fig. 4 could be applied to such a plasma including the ones with internal (crossed) fields and internal conductors (holes). In addition to confining, the magnetic field could be used to compress the plasma and also to remove it from the walls. It should not be difficult to experimentally determine the effect of a crossed field on containment and the efficiency of an internal conductor for compression (see Section 8).

Another method of creating a plasma is by a discharge as in the pinch. This has the feature that the same mechanism both creates and then confines the plasma. The linear geometry, Fig.4a, is probably most appropriate for initial experiments of this type. The discharge required to produce this configuration is quite unusual. It consists of four parallel line discharges, alternating in direction, Fig. 10. It is obviously impossible to

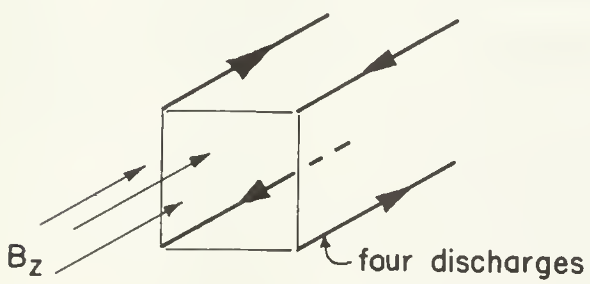


Fig. 10

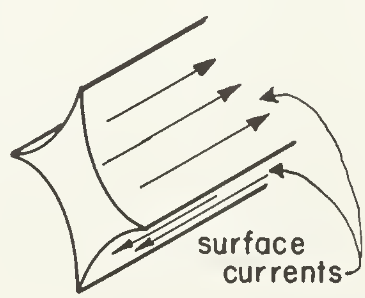


Fig. 11

maintain such an array of discharges in the absence of a strong longitudinal field, B_z. With such a field, it is not clear that the discharges would flow as shown rather than the shorter way, across the magnetic field, but such a possibility does not seem to be ruled out. This experiment could probably be tried rather easily. A preliminary experiment might try to run two discharges separated by a distance of several ion gyro radii.

The four currents shown in Fig. 10 are eventually to be surface currents on the plasma, (Fig.11), and the circuits must be completed by introducing external conductors as returns. The complete circuit might appear as in Fig. 12; the outer shells represent currents in conductors and the inner $\pm$ signs represent discharges. This configuration bears a strong resemblance to the stabilized (B_z) pinch. Instead of a single discharge and concentric return, there are four discharges and returns. Now, the striking feature of the stabilized pinch configuration is that ionization is produced by the initial stages of the same field which later produces the pinch and containment. Following this line, it may be possible to apply the external (shell) currents in Fig. 12 so quickly that the concomitant large electric field ionizes and initiates a discharge as shown.

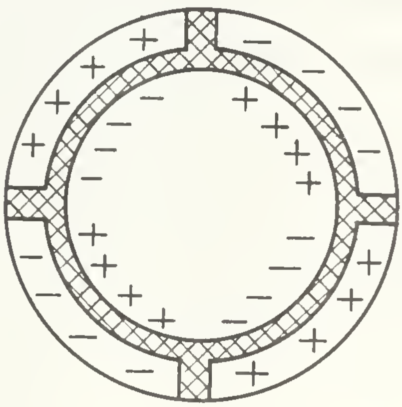
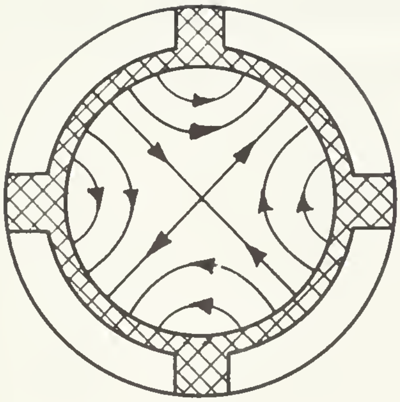


Fig. 12

The initial stages of this cusped B_z pinch might take place as follows. Upon application of a potential (e.g. by condenser discharge), the initial E in the unionized gas will be somewhat



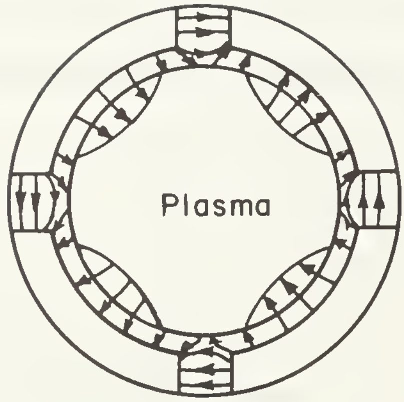


Fig. 13

as shown in Fig. 13a. This initial E-field is orthogonal to the B-field that would eventually arise in a vacuum and the magnitude of E varies linearly along the length of the tube from zero at one end to the applied E at the other; (it is this linear variation which provides a curl E, consequently a $\partial B/\partial t$). Upon ionization, since the current paths that would be produced by this E intersect the insulating walls, E is immediately reduced to zero inside the plasma and is concentrated in the insulator, Fig. 13b. Henceforth , B grows only in this insulator and eventually builds up to a value sufficient to push the plasma away from the wall as a <u>cusped pinch</u>. Any B_z that was present in the unionized gas is trapped in the plasma. A current in a central conductor (Fig. 4f) should be initiated only after ionization.

The losses at the ends of a cusped pinch can be cut down considerably by using <u>insulated ends</u> for the cylinder. The four current paths on the surface of the plasma will than close in a thin layer of plasma near the insulated end and will pull away to provide a secondary <u>end pinch</u> somewhat as in Fig. 14. This

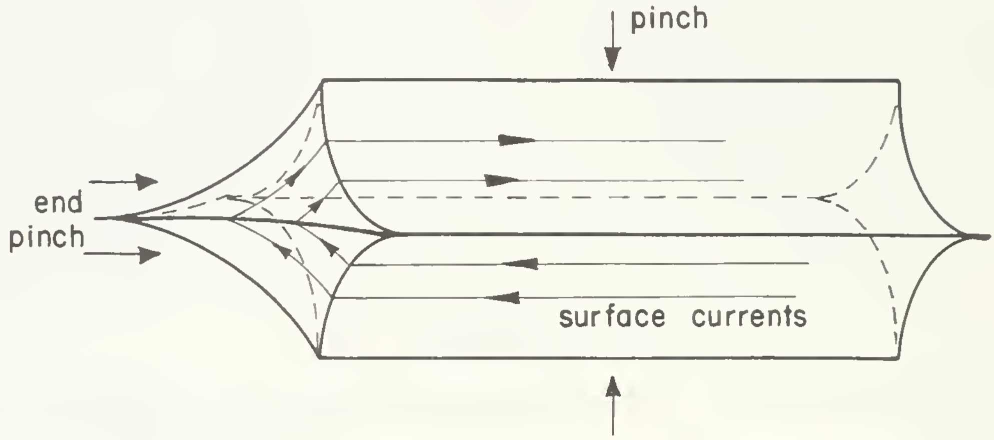


Fig. 14

might be useful in increasing the containment time of a preliminary linear pinch experiment, but, ultimately, a toroidal geometry would probably have to be resorted to.

The various modifications with internal conductors and holes (cf. Figs. 4f, 4g, 4h) should cause no difficulty in small scale experiments.

8. Comparison of Geometries

In this section we present a semi-quantitative (frequently qualitative) comparison between the various cusped geometries and some of the other, at present further advanced, thermonuclear devices, viz. the pinch, the Stellerator, and the Mirror Machine. In addition to the major factors of stability and loss rates, there are questions involving methods of heating, extraction of energy, initiation, ease of construction of experiments, and many others which can at least be analyzed qualitatively in the absence of any experimental experience with the cusped geometries.

First consider geometrical scaling of the various cusped geometries. For the monocusp (Fig. 4c), it is clear that one should try to minimize the cusp perimeter compared to the volume. This implies a long thin shape. There is a limit to how far one can go in this direction because the stability will be thereby reduced. Of course this configuration is always stable against arbitrarily small perturbations, but the restriction on the amplitude of a stable disturbance becomes more severe as the configuration elongates. Also, the amount of acceleration that can be withstood is reduced. A reasonable proportion might be a length four times the diameter. Given such an optimum shape, it is clear that the multicusp of Fig. 4d is somewhat better because of greater volume per line cusp. On the other hand, for a given plasma volume, a long multicusp will have a relatively small ratio of volume to cusp length, so the number of sections should be taken quite small, probably only two or three, ending with a point cusp (Fig. 15).

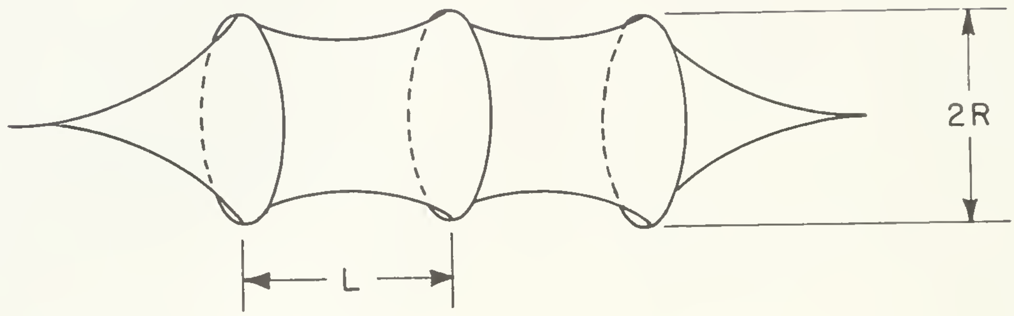


Fig. 15

For the linear geometry, Fig. 4a, neglecting end losses, we are led to a short squat configuration if we wish to minimize the loss rate for a given volume of plasma. Exactly the opposite will be true if we compute end losses only. The optimum shape is a length to diameter ratio about the same as the diameter to gyro radius ratio. Or we may cap the ends as indicated in Fig. 14, in which case the optimum result will be approximately cubical. In either case, the long, thin, open-ended configuration or the capped one, the optimization is not quite as good as in the previous case, Fig. 15.

Except in the case of large scale experiments approaching the thermonuclear in scope, these differences in geometry would probably be secondary to considerations of experimental simplicity.

In devices which do not include an internal crossed field, there is no point in considering toroidal configurations since they offer no advantage over Fig. 15. With an internal field, however, since this field can only be supplied in Fig. 15 by inserting an internal conductor (Fig. 4h), and it is not clear whether it can be applied at all with the capped linear geometry, the most serious contender might be the toroidal shape of Fig. 4b. If insertion of an internal conductor should prove to be feasible, the configuration of Fig. 4h (or its equivalent as in Fig. 15) would come to the fore again. The feasibility of an internal conductor may well be answered differently for a small scale and for a large scale experiment, e.g. depending on the importance of contaminants or of bombardment of the conductor.

Finally, we remark that for initial, exploratory experiments, it may well be possible to tolerate the open-ended linear configuration, Fig. 4a, operated as a pinch.

We turn now to a consideration of the scaling of four basic particle loss mechanisms. The basic domain is taken to be a cylinder of length L and radius R.

(1) Unimpeded end losses at a rate $\sim n\sqrt{T}$ through an area $\sim R^2$. The rate of loss of energy scales as

$$P_E \sim nR^2T^{3/2}$$

(2) Diffusion losses perpendicular to the field,

$$P_D \sim nL/T^{1/2}$$

(3) Mirror type losses,

$$P_M \sim n^2R^2L/T^{1/2}$$

(4) Cusp losses. For easy comparison, take the linear geometry of Fig. 4a,

$$P_C \sim n^{1/2} L T^{3/2}$$

It is illuminating to compare these energy loss rates with the thermonuclear input,

$$P_T \sim n^2 R^2 L \, f(T) \quad .$$

We have

$$P_E : P_D : P_M : P_C : P_T =$$

$$T^2/nL : 1/nR^2 : 1 : T^2/n^{3/2}R^2 : T^{1/2} f(T) \quad .$$

A number of elementary conclusions follow. Except in the case of mirror losses, the thermonuclear input can always be made to dominate either by taking n large or the dimensions large. The factor n is of most importance for cusp losses. The size factor is more important for diffusion and cusp losses than for end losses. Also, devices in which cusp losses or end losses are significant would tend to be operated at lower temperatures than diffusion or mirror-loss dominated machines.

In a qualitative analysis of this type it is unnecessary to consider radiation or ohmic losses in the field coils since these scale the same way in all cases; quantitatively there does exist a difference in that the actual numerical coefficients may be different.

The fundamental advantages of the cusped geometries lie in stability and high β. The chief disadvantages are large particle loss rate and inefficient use of magnetic field. To be definite, let us first compare the cusped linear pinch with the conventional stabilized pinch. Problems of initiation are not radically different. With regard to heating, the cusped pinch is superior in that there is no limit to the amount of stable compression, and if recourse is had to heating by intermixing of crossed fields, there is probably no fear of losing stability. On the other hand, the compressive efficiency of the magnetic field is less because the field is weaker at the plasma than at the coil, and this is only partially compensated by the insertion of an internal conductor. Particle losses are greater for the cusped pinch, except that for a small scale experiment, the capped pinch would probably have much lower losses than an open-ended conventional linear pinch. It seems likely (but this is still subject to verification) that the cusped pinch could operate at somewhat higher β than the stabilized pinch; this is based on an estimate that the optimum internal field required to cut particle losses is likely to be lower than the internal field required for stabilization.

For a preliminary experiment in which economy is unimportant, the cusped pinch offers the definite advantage of unlimited compression without endangering stability together with a relatively low particle loss rate at low temperatures (small compared to 10 kev) but the disadvantage of a depleted Maxwell tail which might be especially serious at low temperatures.

With regard to the Stellerator and the Mirror Machine, the chief advantage of a cusped geometry would seem to lie in a considerably higher stable value of β. The particle loss rate would probably be higher for a cusped geometry, but this is not certain if crossed fields are resorted to. It is conceivable that the cusped configuration could operate in a largely diffusion-dominated regime which would bring the losses below that of the Mirror Machine and conceivably even below that of the Stellerator by utilizing a larger volume to surface area factor in a linear geometry. However, this is at the present time sheer conjecture. The magnetic field utilization is likely to be somewhat less efficient for a cusped configuration but it might be made more efficient by the utilization of ferromagnetic cores.*

So far as initiation by pre-ionization, injection, pinching, shock waves, or any other method, there does not seem to be any significant difference between cusped and other geometries.

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*This possibility was point out by M. Levine.

SELECTED REFERENCES

Stability

[1]	H. Grad,	WASH 184, p. 144	
[2]	H. Grad	WASH 289, p. 115	Motion and stability of free surfaces.
[3]	E. Frieman	WASH 289, p. 343	An energy principle.
[4]	H. Grad	TID 7503, p. 238	Stability of free surfaces.
[5]	H. Rubin	TID 7503, p. 247	Local stability of an incompressible plasma mixed with a magnetic field.
[6]	J. Berkowitz	TID 7503, p. 241	Dynamical stabilization of the pinch.
[7]	H. Grad	TID 7520, p. 373	Some new developments in stability of free surfaces.
[8]	M. D. Kruskal and J. Tuck	LA 1716	Instability of a pinched fluid with a longitudinal magnetic field.
[9]	M. Rosenbluth	LA 2030	Stabilization of the pinch.
[10]	H. Grad	(to appear)	Motion and stability of free surfaces.

Theory of Cusped Surfaces

[11]	J. L. Tuck	WASH 184 p. 77	Picket fence.
[12]	H. Grad	TID 7503, p. 319	Theory of the cuspidor.
[13]	H. Grad	TID 7520, p. 148	Developments in the theory of cusped geometries.
[14]	J. Berkowitz	TID 7520, p. 394	Computation of two-dimensional cusped surfaces.
[15]	H. Goertzel	TID 7520, p. 400	Computation of three-dimensional cusped surfaces.

[16]	J. Killeen	TID 7536, p. 200	Particle losses in crossed fields
[17]	J. Berkowitz, H. Grad and E. L. Rubin	(to appear)	Cusped geometries II: Computation of cusp losses.
[18]	H. Grad and J. Killeen	(to appear)	Cusped geometries III: Cusp losses in crossed fields.

Shock Heating

[19]	M. Rosenbluth and C. L. Longmire	TID 7520, p. 131	Structure of hydromagnetic shocks
[20]	S. A. Colgate	UCRL-4829	Description of a shock wave
[21]	H. Grad	TID 7536, part I	Shock heating