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Plasma Simulation Algorithm for the Two-Fluid Plasma Model

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Abstract

Many current plasma simulation codes are based on the magnetohydrodynamic (MHD) model whose derivation involves several asymptotic and simplifying assumptions that severely limit its applicability, particularly for Hall effect physics. The two-fluid plasma model only assumes local thermodynamic equilibrium within each fluid (ion and electron fluid). Therefore, the two-fluid plasma model more accurately represents the appropriate physical processes. An algorithm is presented for the simulation of plasma dynamics using the two-fluid plasma model. The two-fluid equations are derived in conservative/divergence form and an approximate Riemann solver is developed. The algorithm uses the approximate Riemann solver to compute the fluxes of the electron and ion fluids at the computational cell interfaces and an upwind characteristic-based solver to compute the electromagnetic fields. The source terms that couple the fluids and fields are treated implicitly to relax the stiffness. The algorithm is validated with the coplanar Riemann problem, Langmuir plasma oscillations, and the electromagnetic shock problem. The electromagnetic shock problem has been simulated with the MHD plasma model. In nondimensional form the two-fluid plasma model has three physically relevant parameters that define the importance of the two-fluid effects, the electron to ion mass ratio m_e/m_i , the Debye length λ_D , and the ion Larmor radius r_L . The MHD model assumes $m_e/m_i = 0$, $\lambda_D = 0$, and $r_L = 0$. By adjusting the value of r_L , the two-fluid model can approach either the unmagnetized gas dynamic limit $r_L \rightarrow \infty$ or the MHD limit $r_L \rightarrow 0$. Comparisons are presented that show these limits for the electromagnetic shock problem. Furthermore, a physical analysis of the intermediate values of r_L are presented.

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Background and Motivation

Plasmas may be most accurately modeled using kinetic theory. The plasma is described by distribution functions in physical space, velocity space, and time, $f(\mathbf{x}, \mathbf{v}, t)$. The evolution of the plasma is then modeled by the Boltzmann equation.

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}_\alpha \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + \frac{q_\alpha}{m_\alpha} (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{v}_\alpha} = \left. \frac{\partial f_\alpha}{\partial t} \right|_{collisions}$$

for each plasma species $\alpha = \text{ions, electrons}$.

Coupled with Maxwell's equations this provides a complete description of the plasma dynamics.



Kinetic Model

- The Boltzmann equation is seven dimensional.
- As a consequence plasma simulations using the Boltzmann equation are only used in very limited applications with narrow distributions, small spatial extent, and short time durations. The seven dimensional space is further exacerbated by the high velocity space that is unused except for the tail of the distribution or energetic beams.
- Boundary conditions are difficult to implement in kinetic simulations.



Particle in Cell (PIC) Model

- Particle in cell (PIC) plasma model applies the Boltzmann equation to representative superparticles which are far fewer (10^7) than the number of particles in the actual plasma (10^{20}). The particles fill the six dimensional phase space and are tracked in time.
- PIC simulations have similar limitations as simulations using kinetic theory due to statistical errors caused by the relatively few superparticles.
- Boundary conditions can also be difficult to implement in PIC simulations.



Single-Fluid (MHD) Model

- Simpler plasma models are generated by taking moments of the Boltzmann equation and averaging over velocity space for each species.
 - Two-Fluid Plasma Model
- The two-fluid equations are combined using simplifying assumptions to form the single-fluid MHD model.
 - low frequency, zero electron mass, zero Larmor radii, zero Debye length
- Plasma simulation algorithms based on the MHD model have been successful in modeling plasma dynamics and other phenomena.



Limitations of the MHD Model

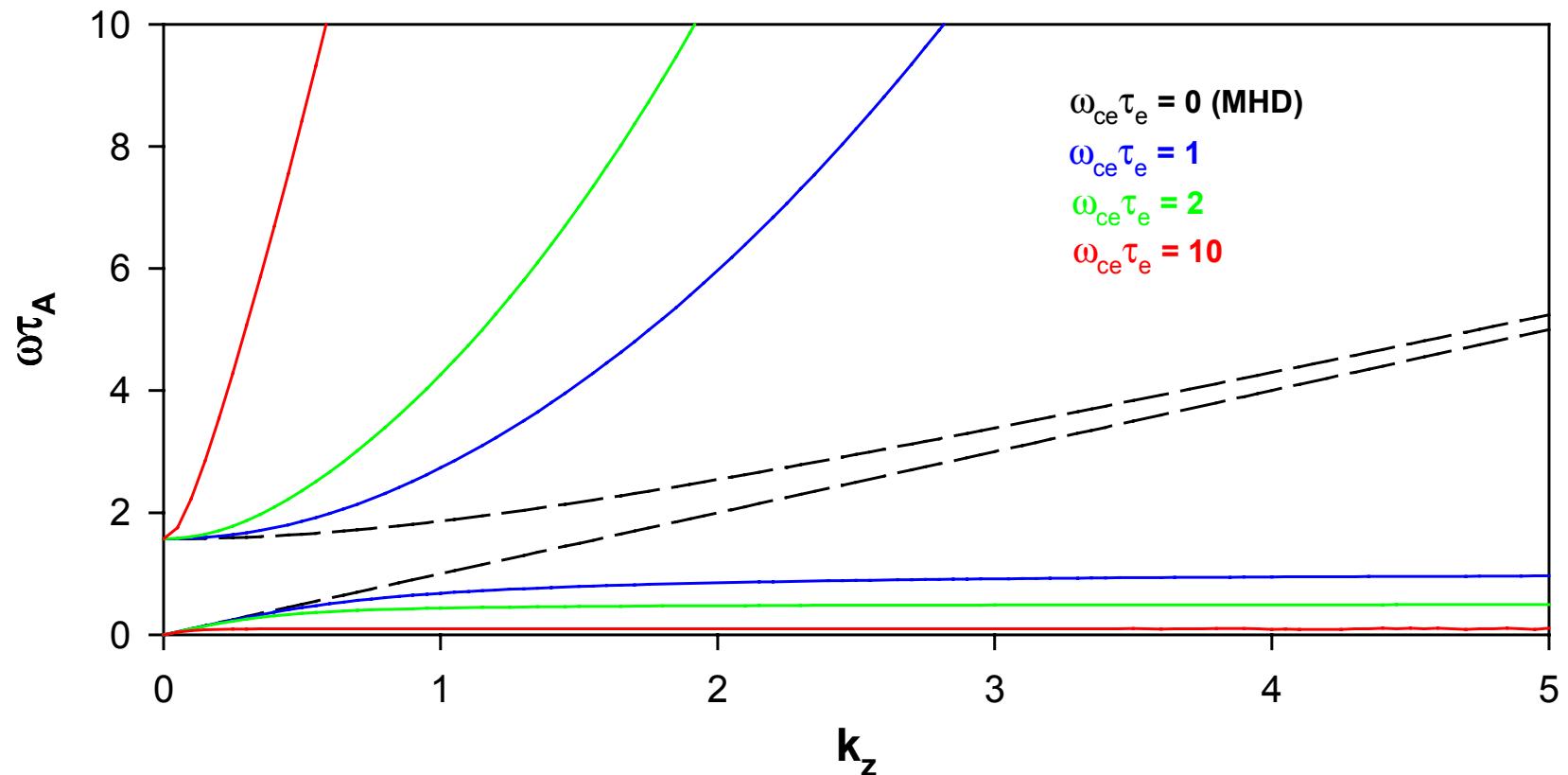
The MHD model has limitations that are introduced in its derivation.

- The generation of non-solenoidal magnetic fields.
- The Hall effect and diamagnetic terms are often neglected. These terms represent the separate motions of the ions and electrons and account for ion current and the finite Larmor radius of the plasma constituents.
 - Hall thrusters, MPD thrusters, Lorentz force thrusters
 - Anode and cathode fall
 - Plasma current drive
 - Magnetic relaxation
 - Space plasma effects



The Hall Effect

In general the Hall terms are difficult to stabilize because they lead to the whistler wave branch of the dispersion relation which can lead to large propagation speeds.





Hall Effect Physics

- Semi-implicit techniques have been implemented to treat the Hall effect terms.*
- The method corrects the ideal MHD evolution of the magnetic field and energy to account for the Hall effect.
- The operator uses a 45 point stencil in 3D.
- The method works for small Hall parameters but becomes slow to converge or unstable for large Hall parameters.

* Harned and Mikic, JCP **83**, 1 (1989).



Two-Fluid Plasma Model

- The two-fluid model is derived by taking moments of the Boltzmann equation for each species.
- The model has the same dimensionality as the MHD model except there are two fluids.
- The only approximation made is local thermodynamic equilibrium of each fluid but not with the other fluid.
- The model consists of governing equations for the continuity, momentum, and energy of the electrons and ions. Maxwell's equations complete the model.



Conservation Form

$$\frac{\partial}{\partial t} \begin{bmatrix} n_i \\ n_e \\ \mathbf{j}_i \\ \mathbf{j}_e \\ \mathcal{E}_i \\ \mathcal{E}_e \end{bmatrix} + \nabla \cdot \begin{bmatrix} \frac{\mathbf{j}_i}{e} \\ -\frac{\mathbf{j}_e}{e} \\ \frac{\mathbf{j}_i \mathbf{j}_i}{en_i} + \frac{e}{m_i} p_i \mathbf{I} \\ -\frac{\mathbf{j}_e \mathbf{j}_e}{en_e} - \frac{e}{m_e} p_e \mathbf{I} \\ (\mathcal{E}_i + p_i) \frac{\mathbf{j}_i}{en_i} \\ -(\mathcal{E}_e + p_e) \frac{\mathbf{j}_e}{en_e} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{e}{m_i} (en_i \mathbf{E} + \mathbf{j}_i \times \mathbf{B} + \mathbf{R}_{ei}) \\ \frac{e}{m_e} (en_e \mathbf{E} - \mathbf{j}_e \times \mathbf{B} + \mathbf{R}_{ei}) \\ \mathbf{j}_i \cdot \left(\mathbf{E} + \frac{\mathbf{R}_{ei}}{en_i} \right) \\ \mathbf{j}_e \cdot \left(\mathbf{E} + \frac{\mathbf{R}_{ei}}{en_e} \right) \end{bmatrix}$$



Characteristics in 1D

$$\frac{\partial \mathbf{Q}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = \frac{\partial \mathbf{Q}}{\partial t} + \mathbf{A} \frac{\partial \mathbf{Q}}{\partial x} = 0$$

- Hyperbolic equation set, so the flux Jacobian \mathbf{A} have complete sets of eigenvalues

$$\lambda = \left(\frac{j_{ix}}{en_i}, \frac{j_{ix}}{en_i} \pm \sqrt{\frac{\gamma T_i}{m_i}}, -\frac{j_{ex}}{en_e}, -\frac{j_{ex}}{en_e} \pm \sqrt{\frac{\gamma T_e}{m_e}} \right)$$

- First order fluxes are split and upwind differenced to form a Roe-type,* approximate Riemann solver

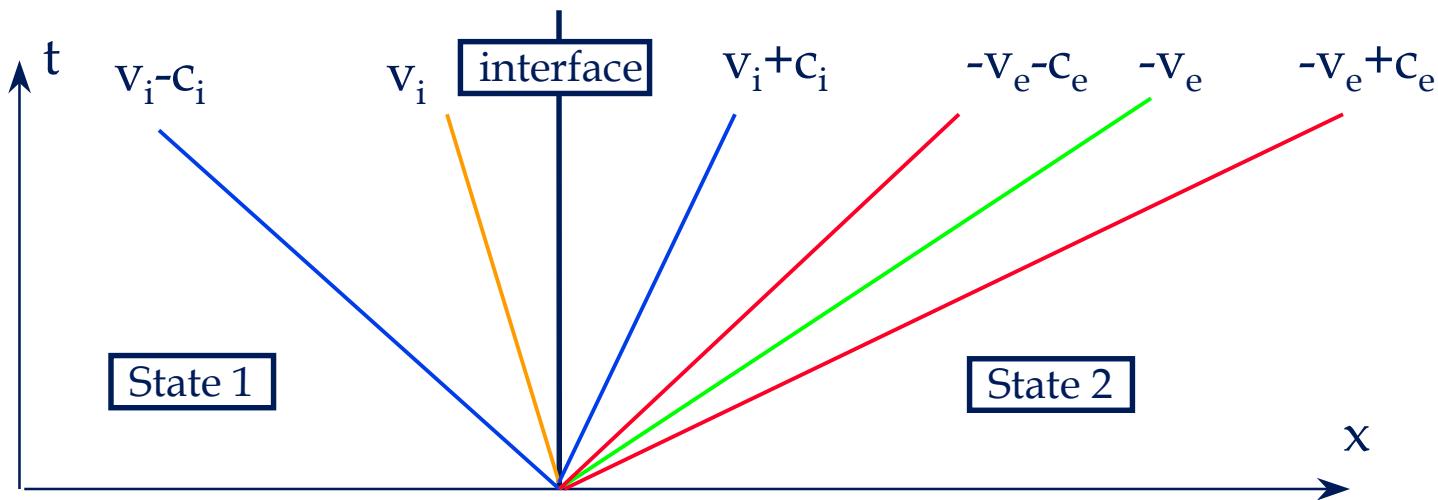
$$\mathbf{F}_{i+1/2} = \frac{1}{2} (\mathbf{F}_{i+1} + \mathbf{F}_i) - \frac{1}{2} \sum_k \mathbf{l}_k (\mathbf{Q}_{i+1} - \mathbf{Q}_i) |\lambda_k| \mathbf{r}_k$$

* Roe, JCP **135**, 250 (1997).



Approximate Riemann Solver for Two-Fluid Plasma Model

- Overall solution is composed of the solutions to the Riemann problem defined at the grid interfaces.



- Information propagates along characteristics.



Electromagnetic Field Model

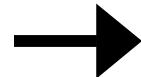
Maxwell's equations for the electromagnetic fields are required to compute the RHS and complete the plasma model.

$$\nabla \cdot \mathbf{E} = e(n_i - n_e) / \varepsilon_o$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} / \mu_o - \mathbf{j}_i - \mathbf{j}_e$$



$$\nabla^2 \phi - \mu_o \varepsilon_o \frac{\partial^2 \phi}{\partial t^2} = -e(n_i - n_e) / \varepsilon_o$$

$$\nabla^2 \mathbf{A} - \mu_o \varepsilon_o \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_o (\mathbf{j}_i + \mathbf{j}_e)$$

where the magnetic vector potential \mathbf{A} and the electric scalar potential ϕ are introduced.



Electromagnetic Field Model

For the electromagnetic field model, Maxwell's equations are solved

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E} = 0$$

$$\varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} - \nabla \times \mathbf{B} / \mu_0 = -\mathbf{j}_i - \mathbf{j}_e$$

subject to the initial conditions

$$\nabla \cdot \mathbf{E} = e(n_i - n_e) / \varepsilon_0 \quad \text{and} \quad \nabla \cdot \mathbf{B} = 0$$

The field equations are solved with a finite-volume, upwind scheme*
where the equations are written as

$$\frac{\partial \mathbf{Q}}{\partial t} + \nabla \cdot \mathbf{F} = \mathbf{S}$$

* Shang and Fithen, JCP **125**, 378 (1996).



Multidimensional Simulations

Satisfying the divergence constraint conditions on the electromagnetic fields becomes more difficult due to in-plane fields. Correction potentials* are added to the fields algorithm.

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} - \nabla \psi$$

$$\varepsilon_o \frac{\partial \mathbf{E}}{\partial t} = \nabla \times \mathbf{B} / \mu_o - \mathbf{j}_i - \mathbf{j}_e - \varepsilon_o \nabla \phi$$

$$\frac{\partial \phi}{\partial t} - \nabla \cdot \mathbf{E} = -e(n_i - n_e) / \varepsilon_o$$

$$\frac{\partial \psi}{\partial t} + \nabla \cdot \mathbf{B} = 0$$

* Munz, Schneider, and Vos, SIAM JSC **22**, 449 (2000).



Treatment of the Two-Fluid Source Terms

The two-fluid equation set contains source terms which result from the electromagnetic fields: the Lorentz force and Joule heating (fluids) and current densities (fields).

When the source terms are large (large Lorentz force), they are calculated implicitly with an iterative method.

$$\frac{\mathbf{Q}_i^{n+1} - \mathbf{Q}_i^n}{\Delta t} = - \left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n \right) + \mathbf{S}_i^{n+1}$$

The source terms are linearized, and a local source Jacobian $\partial \mathbf{S} / \partial \mathbf{Q}$ is defined.

$$\left(\frac{1}{\Delta t} - \frac{\partial \mathbf{S}^k}{\partial \mathbf{Q}^k} \right) (\mathbf{Q}_i^{n+1} - \mathbf{Q}_i^k) = - \frac{\mathbf{Q}_i^k - \mathbf{Q}_i^n}{\Delta t} - \left(\mathbf{F}_{i+1/2}^n - \mathbf{F}_{i-1/2}^n \right) + \mathbf{S}_i^k$$



Normalization of the Two-Fluid Model

In the normalization process several important physical parameters are identified.

- Ionization state, Z (set to 1).
- Ion / Electron mass ratio, m_i/m_e (set to 1836, hydrogen plasma).
- Ion Larmor radius, r_{Li} , where $r_L = v_{Th}/\omega_c = v_{Th}m/eB$.
- Debye length, λ_D , where $\lambda_D = v_{Th}/\omega_p = \sqrt{\varepsilon_0 kT / ne^2}$.



Benchmarks and Validation Tests

The algorithm has been applied to the classical 1-D MHD shock problem and reveals an effect of two-fluid physics and the transition between a gas dynamic shock and an MHD shock.

- 1-D Electromagnetic Plasma Shock*

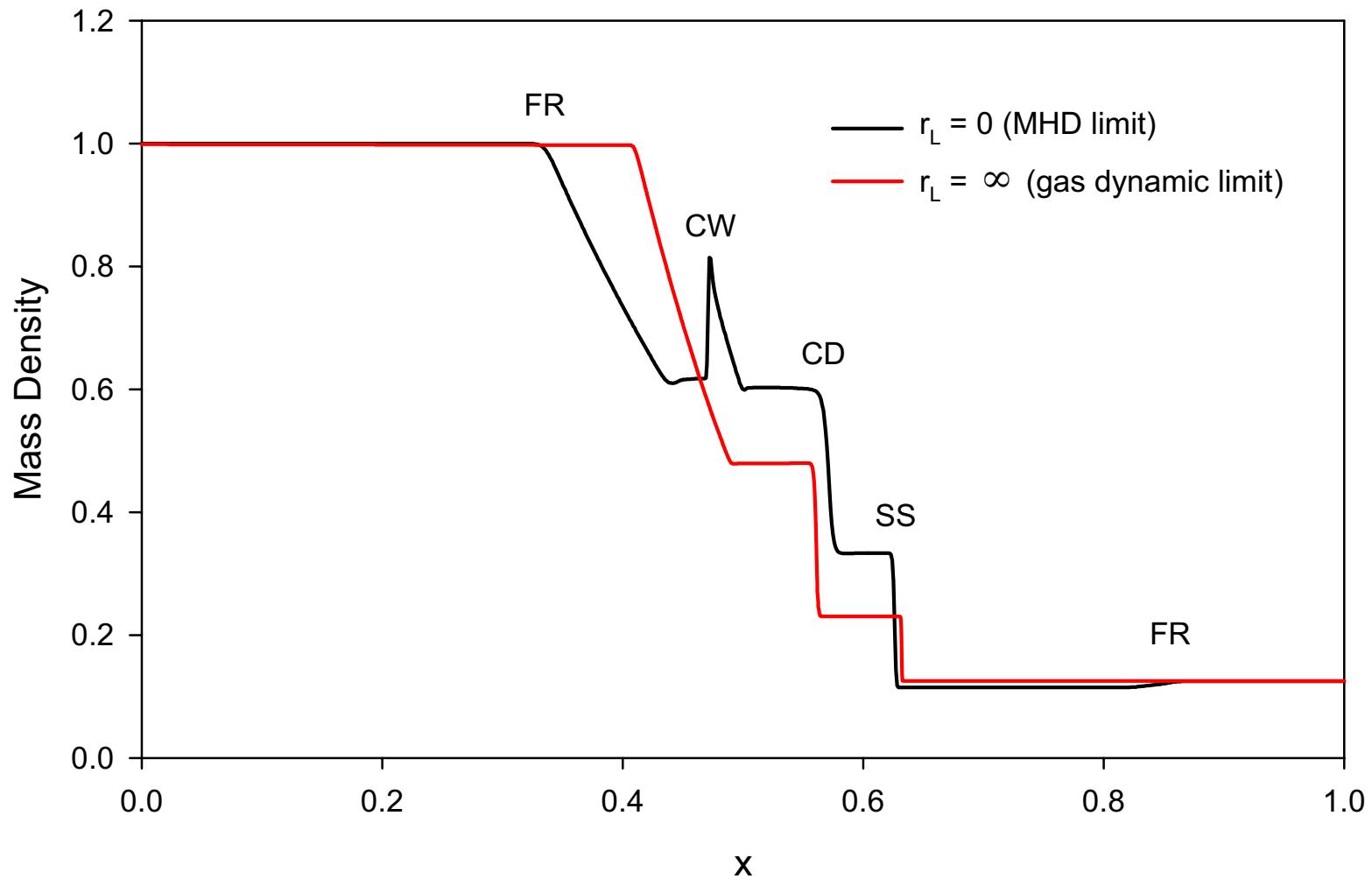
The algorithm has also been applied to several 2-D problems and compared to published results.

- 2-D Cylindrical Electromagnetic Plasma Shock
- Collisionless Magnetic Reconnection

* Brio and Wu, JCP **75**, 400 (1988).

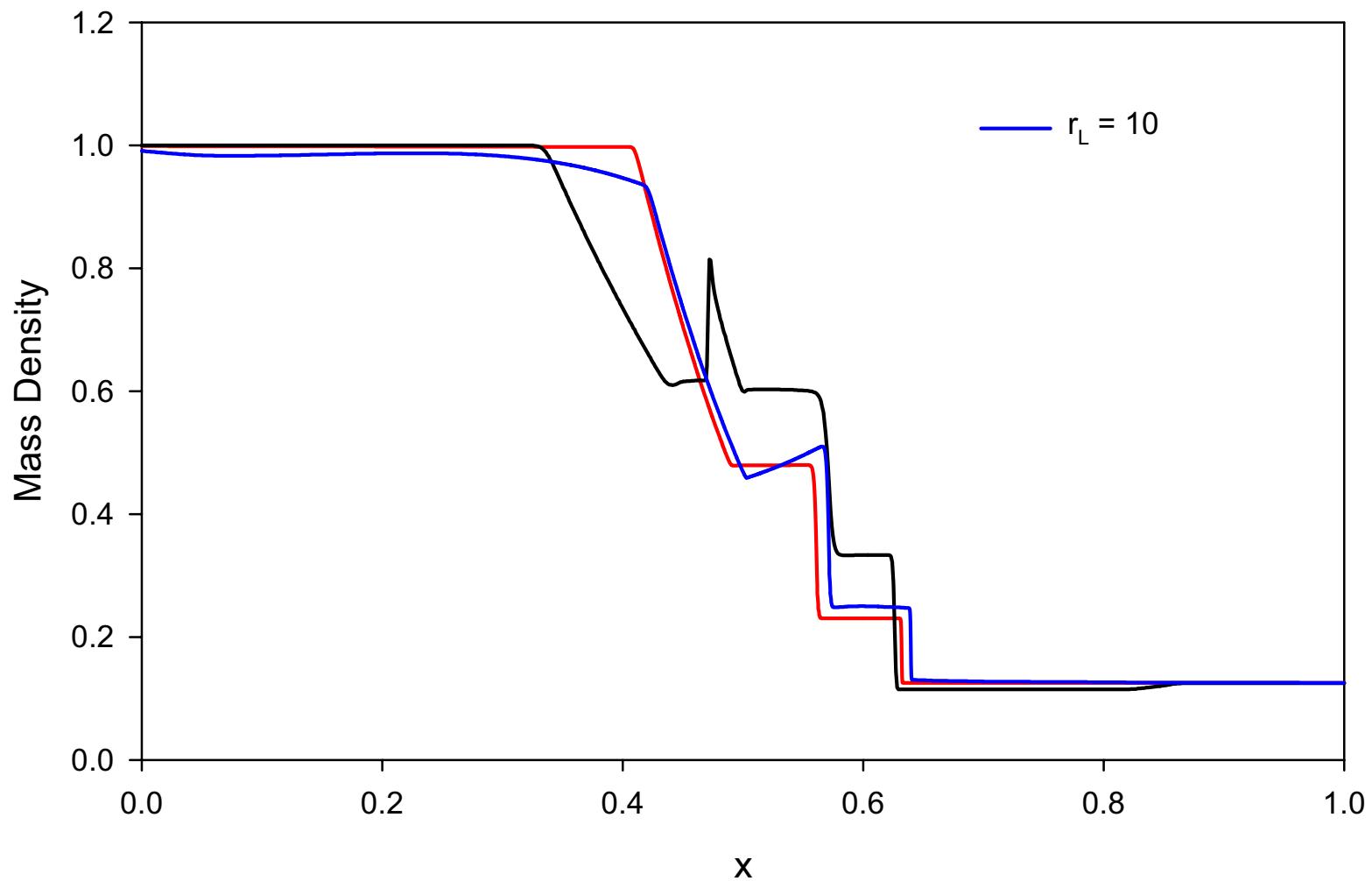


Electromagnetic Plasma Shock



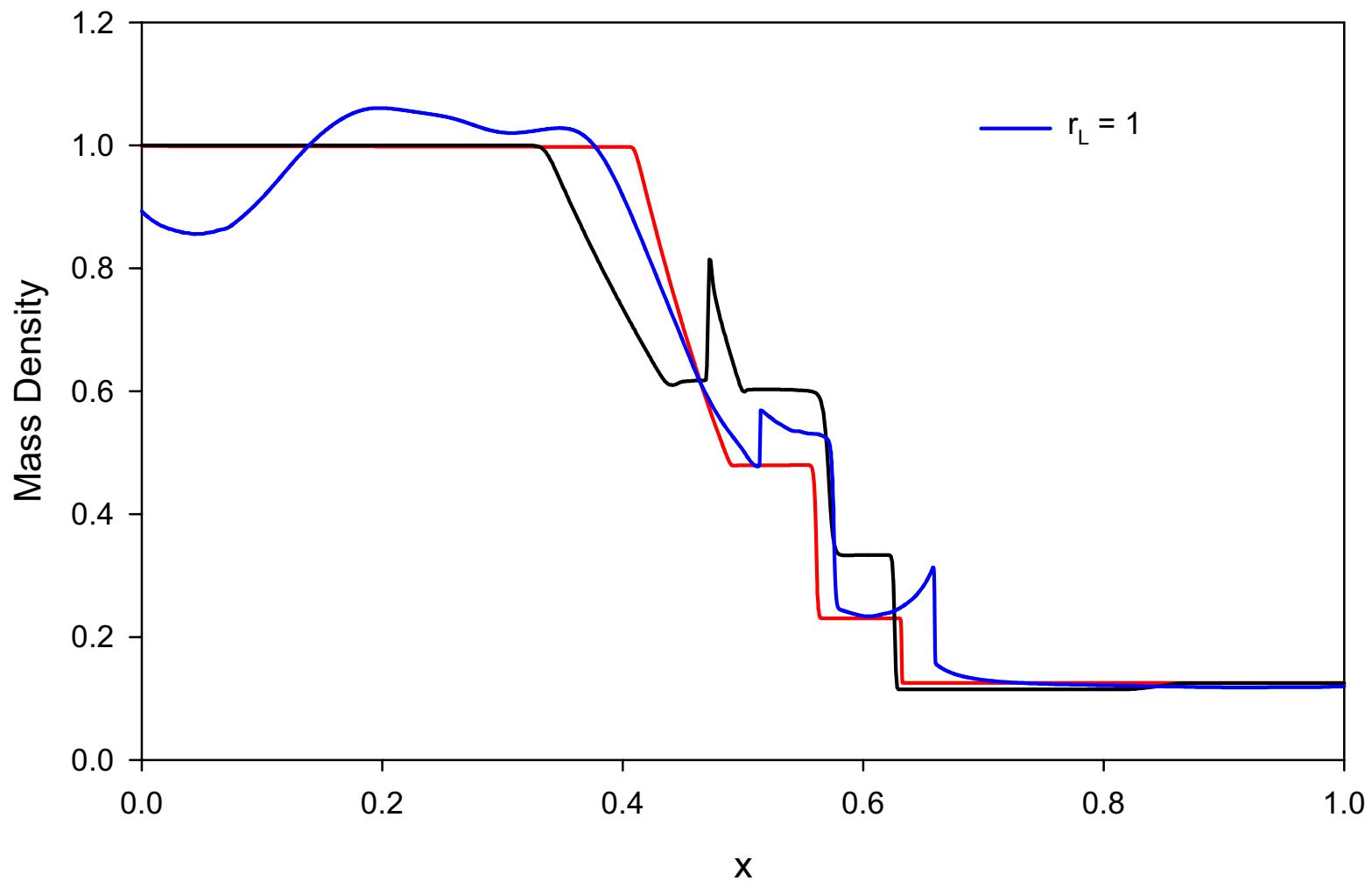


Electromagnetic Plasma Shock



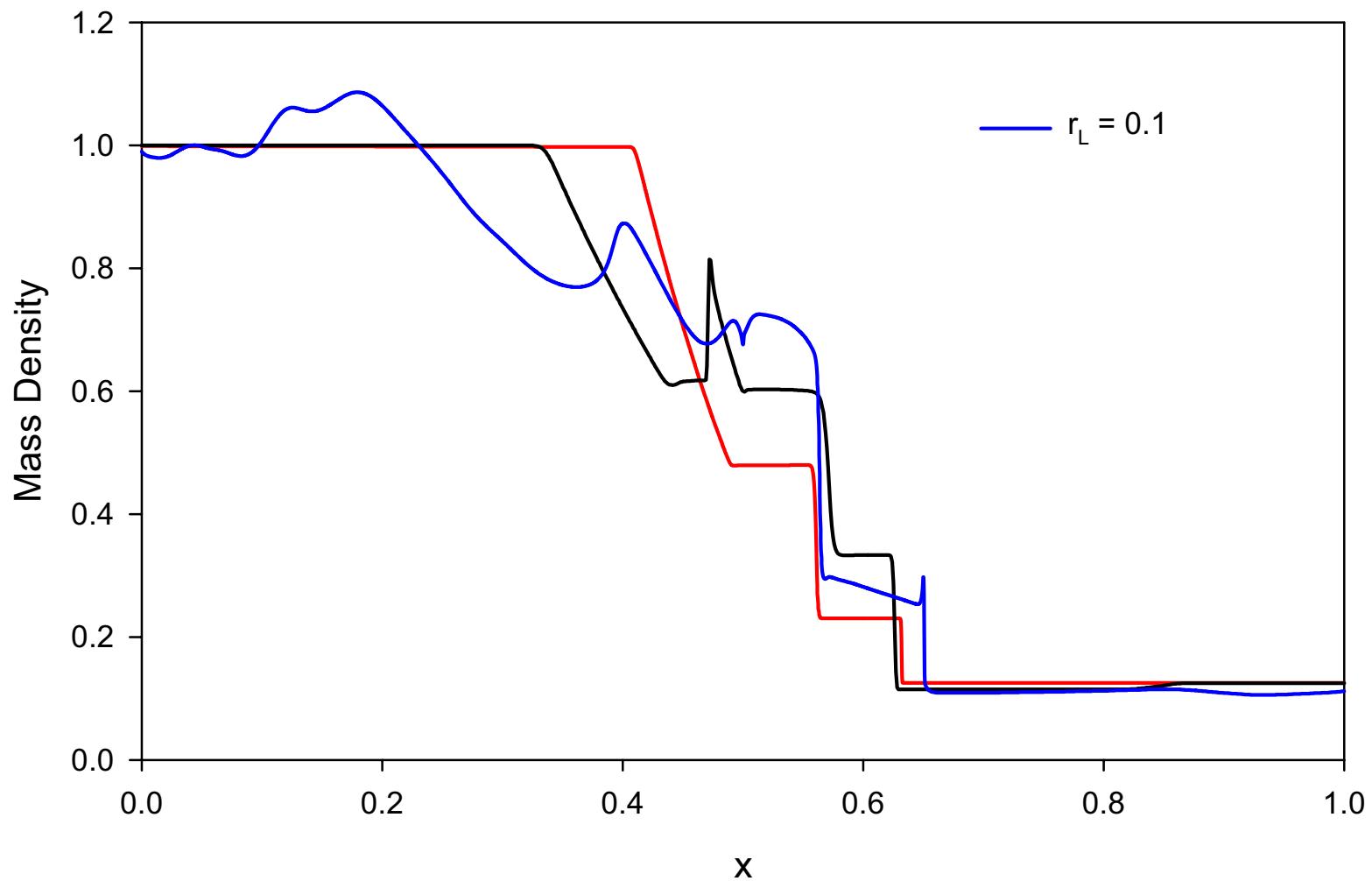


Electromagnetic Plasma Shock



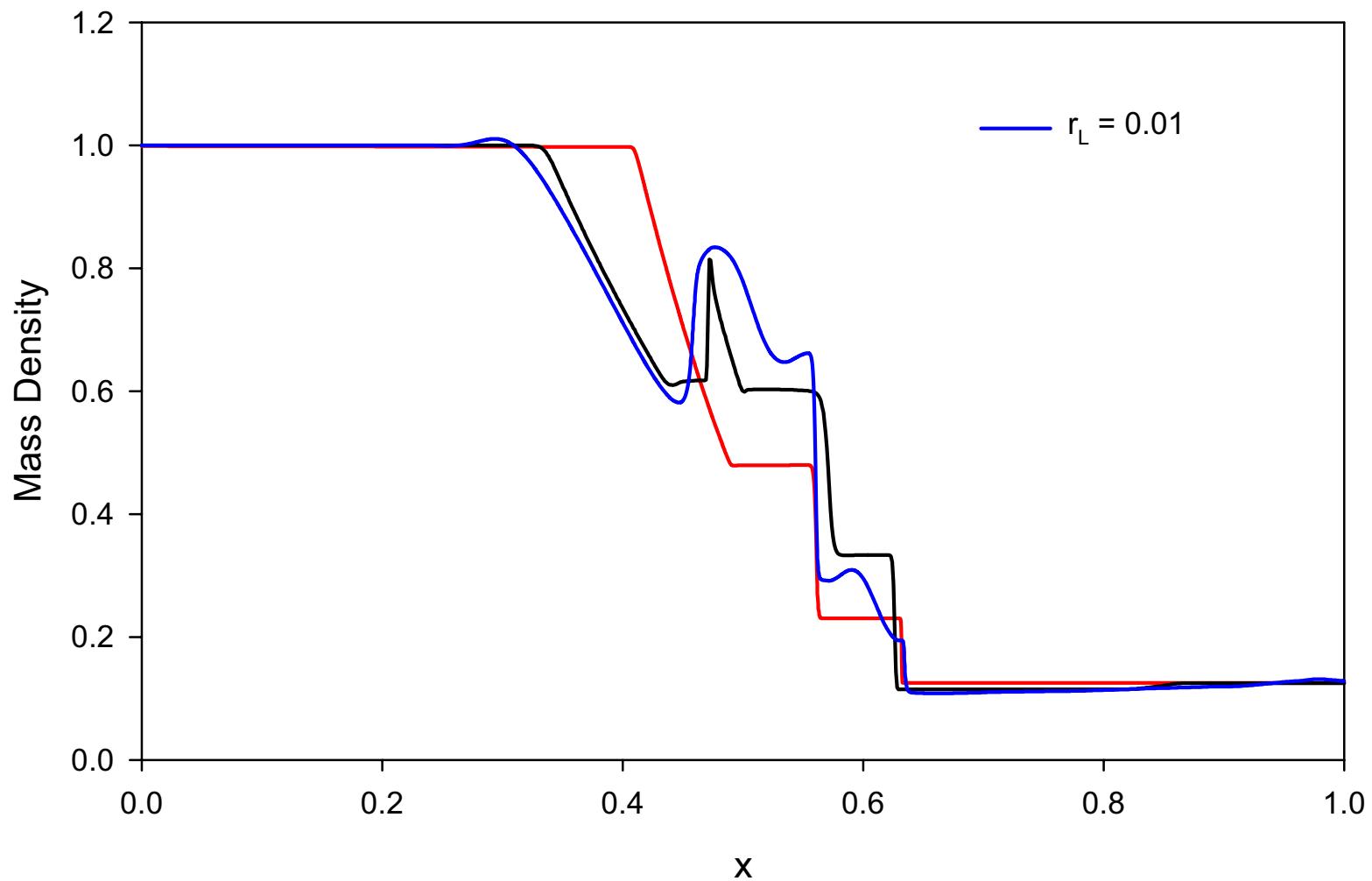


Electromagnetic Plasma Shock



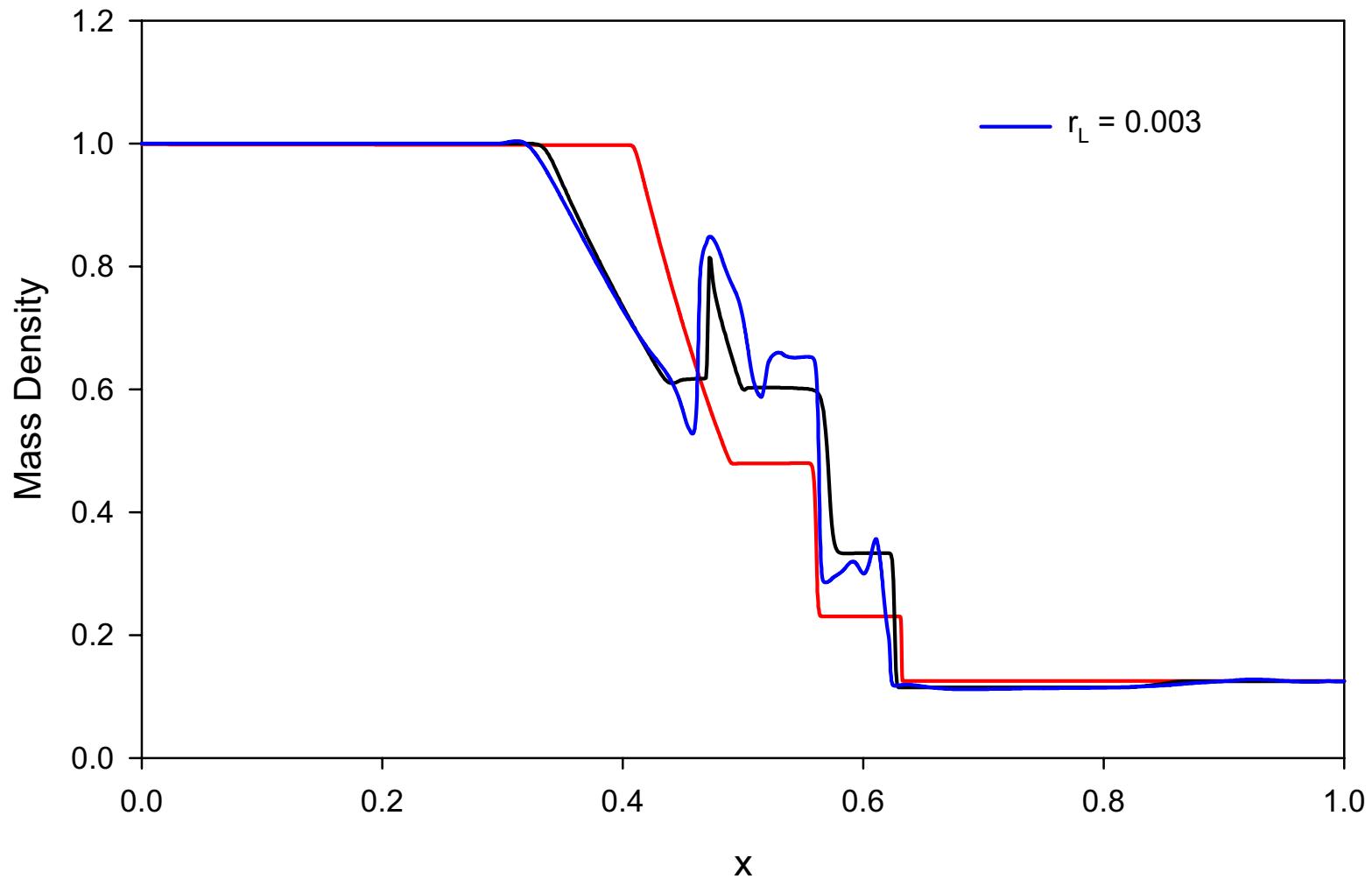


Electromagnetic Plasma Shock



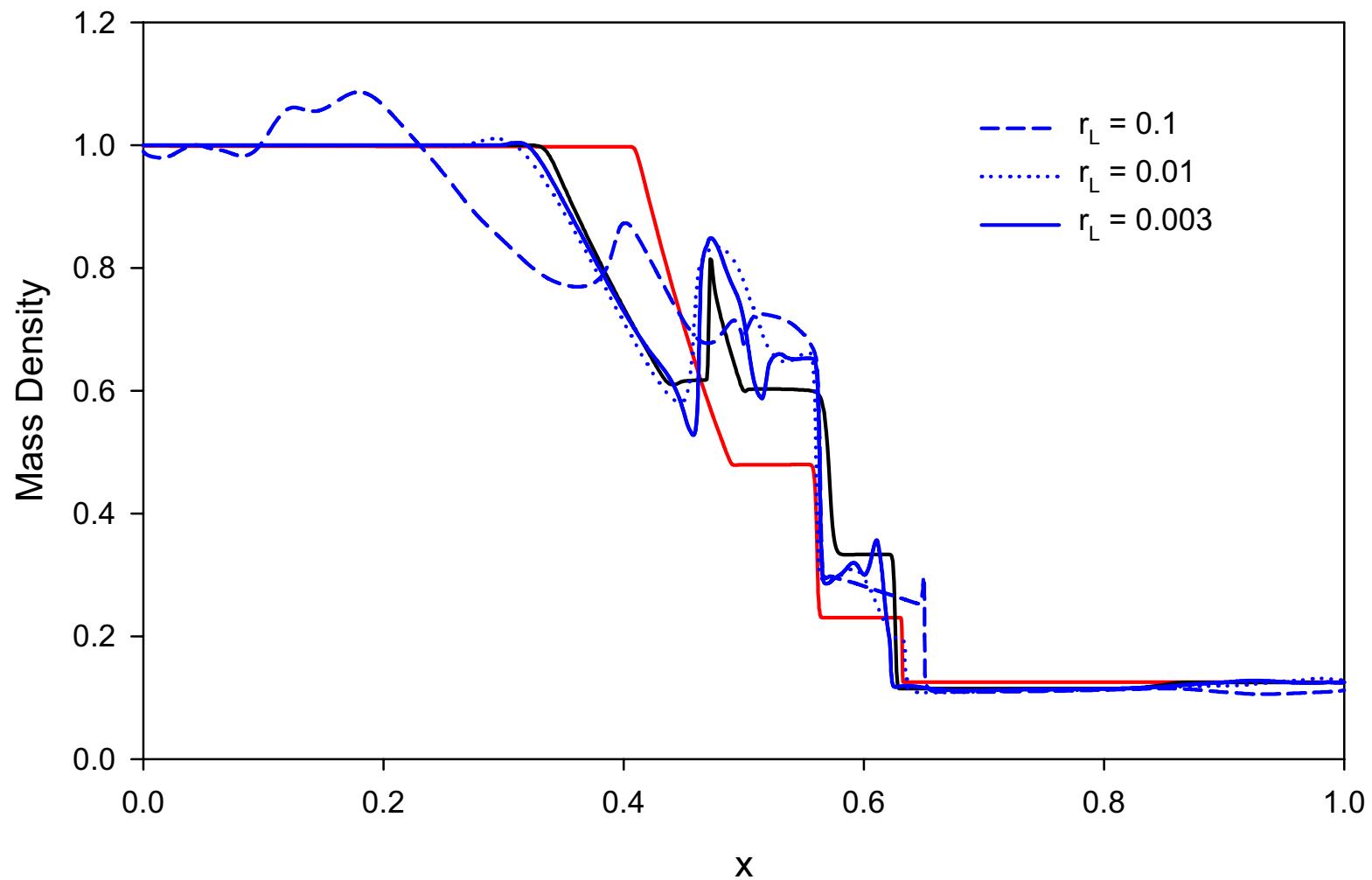


Electromagnetic Plasma Shock





Electromagnetic Plasma Shock





Fast Wave Analysis in the Plasma Shock

The wave vector is perpendicular to the discontinuity and parallel to the longitudinal magnetic field, $\mathbf{k} \parallel \mathbf{B}$.

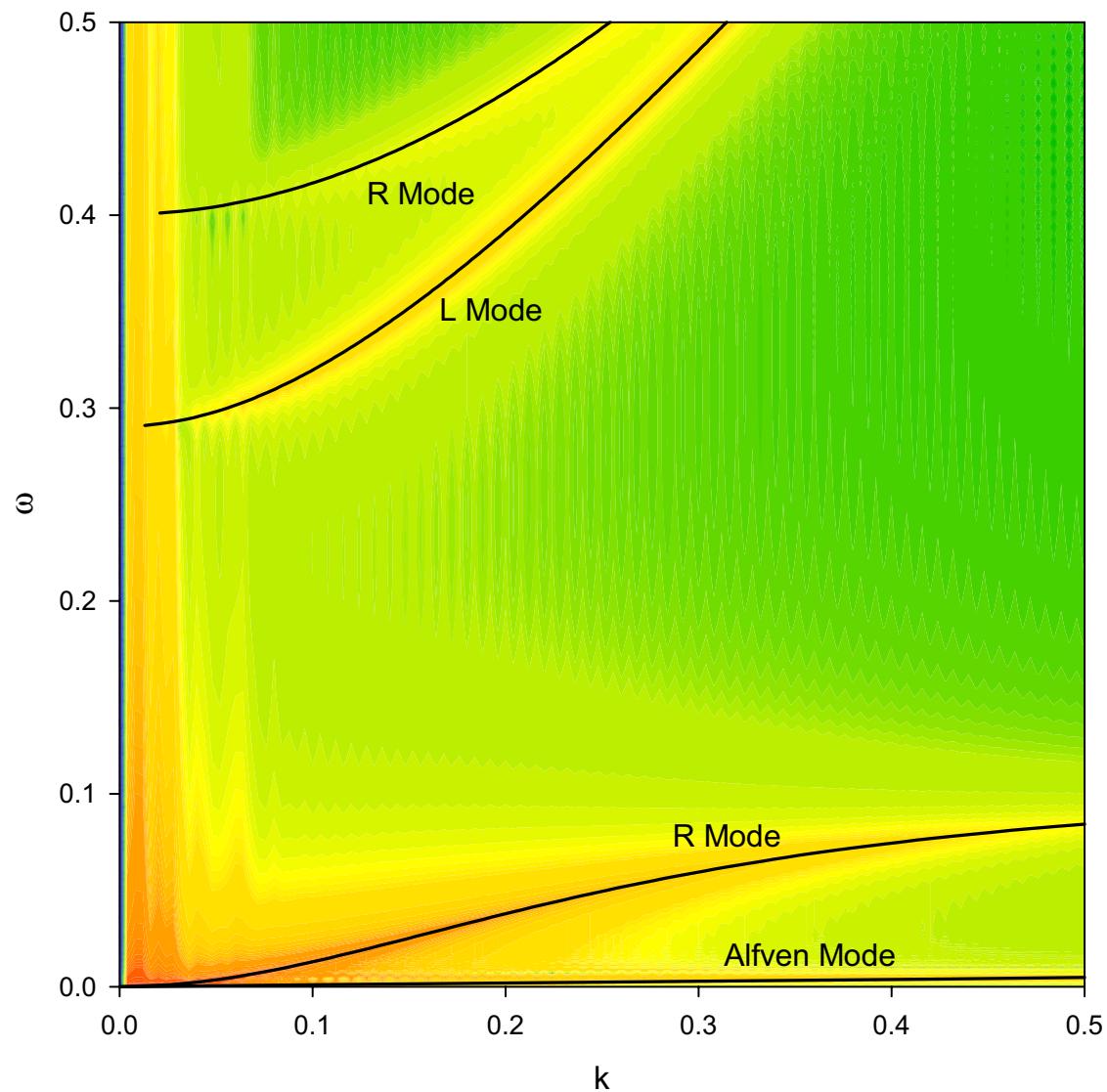
The plasma dispersion relation for this case yields the left and right circularly polarized waves (L mode and R mode), in addition to the slower Alfvén waves.

$$\frac{c^2 k^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \pm \omega_{ce})}$$

The low frequency portion of the lower branch of the R mode is the whistler wave.



Electromagnetic Plasma Shock



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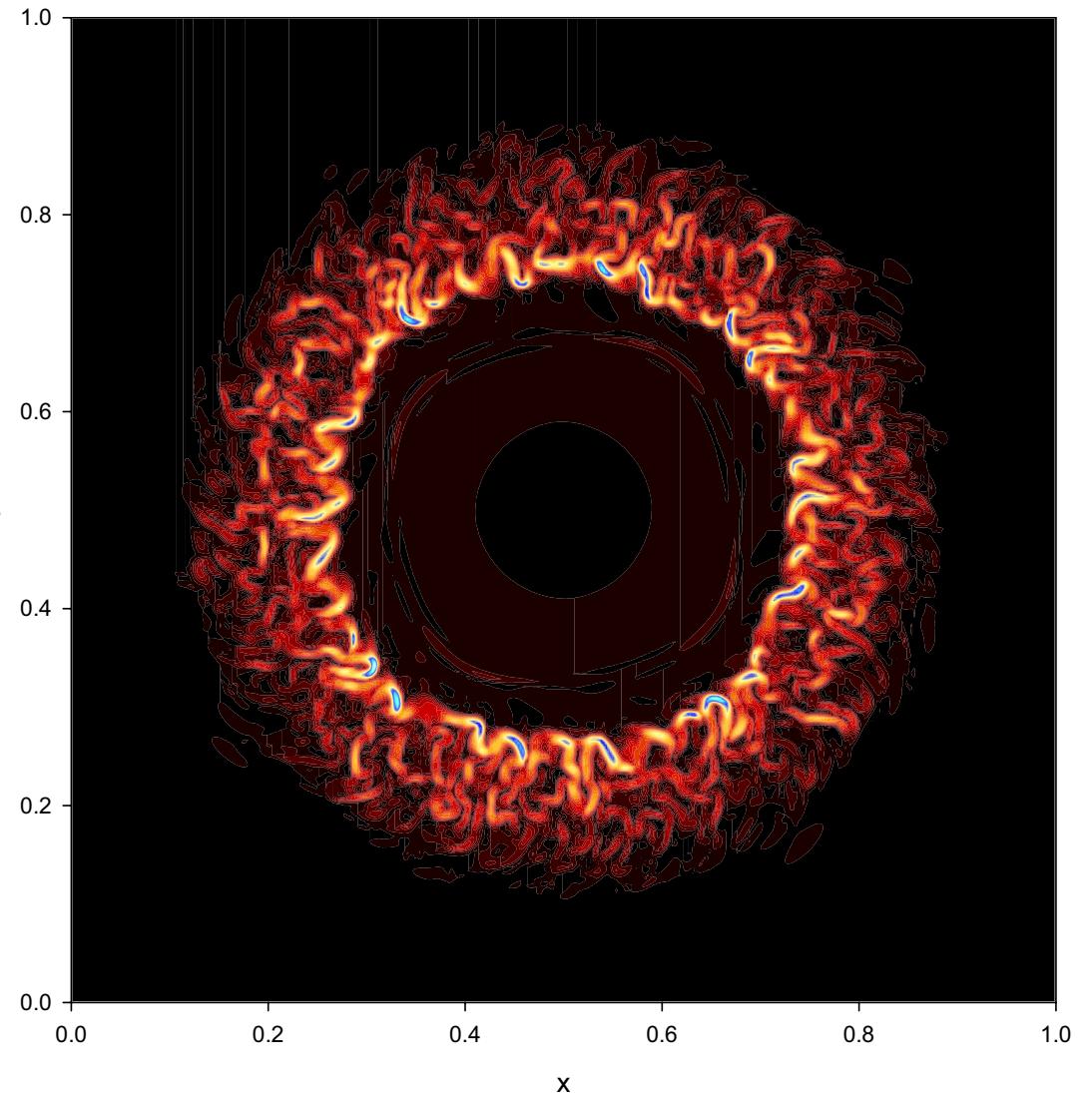


Cylindrical Plasma Shock

The electromagnetic plasma shock simulation is repeated in a cylindrical geometry. No longitudinal magnetic field is applied.

An azimuthal current sheet supports the discontinuous jump in the magnetic field.

The current sheet breaks due to a Kelvin-Helmholtz instability when the sheet thickness decreases below the electron skin depth (c/ω_{pe}).



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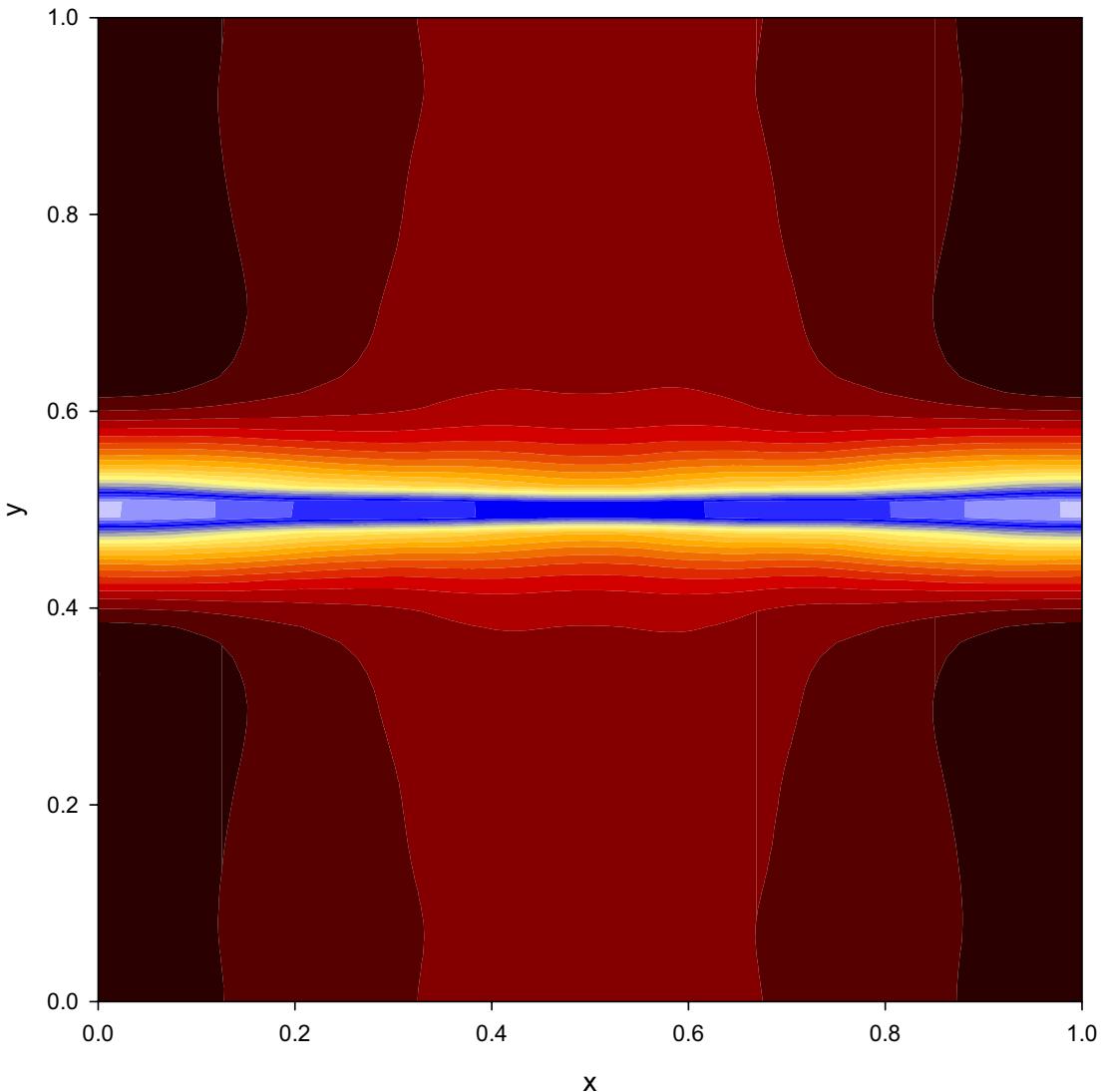


Collisionless Magnetic Reconnection

Magnetic reconnection in a current sheet tears and reconnects magnetic flux through the original current layer.

Reconnection usually requires resistivity, but it can also occur without collisions if two-fluid effects are included.*

Shown is the magnitude of B_x revealing reconnected flux through the current layer.



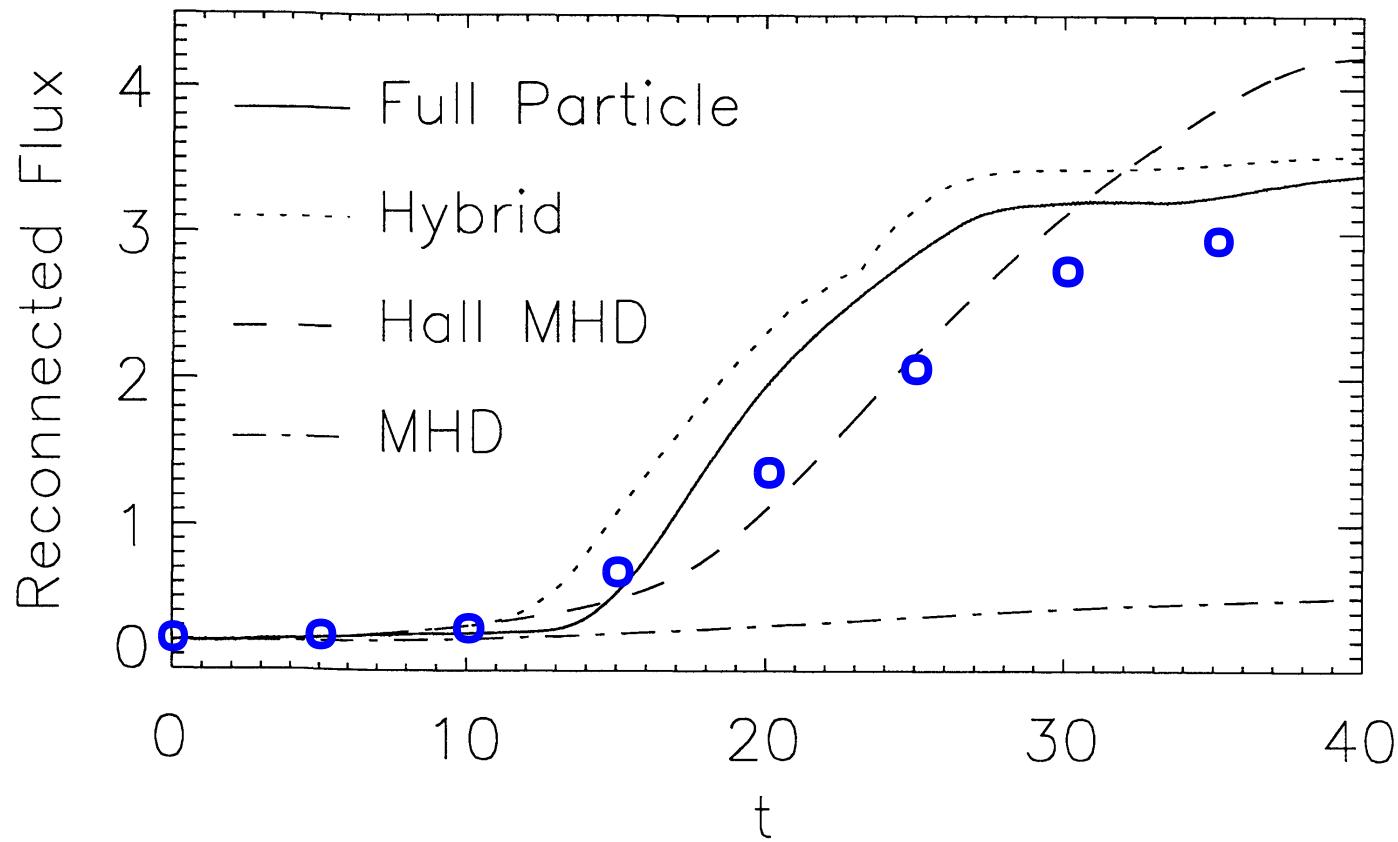
* Birn *et al.*, JGR **106**, 3715 (2001).



Collisionless Magnetic Reconnection

Birn *et al.* tracks the time evolution of the reconnected magnetic flux for different plasma models. Two-fluid model results are also shown.

BIRN ET AL.: GEM RECONNECTION CHALLENGE



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Summary

- The two-fluid plasma model relaxes some of the constraining assumptions used in MHD and models more complete physics.
- We have developed a new algorithm based on a Roe-type approximate Riemann solver for the two-fluid plasma model.
- We have implemented an electromagnetic field solver and an implicit treatment of stiff source terms.
- The algorithm has been benchmarked to published problems and produces accurate results.
- Simulations have revealed important physical insight and demonstrate the importance of two-fluid effects.
 - MHD Plasma Shock (1-D and 2-D)
 - Collisionless Magnetic Reconnection