On the Essential Spectrum of Ideal Magnetohydrodynamics

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Dedicated to Harold Grad on the occasion of his sixtieth birthday

Abstract

The essential spectrum of magnetohydrodynamics (MHD) is shown to arise from waves propagating one-dimensionally along magnetic field lines. Different polarizations of these waves give rise to the "Alfven" and "ballooning" spectra. The essential spectrum of an axisymmetric equilibrium when a single azimuthal mode number is considered consists of the Alfven spectrum only, while the ballooning modes appear as intervals of accumulation of discrete eigenvalues with different mode numbers. We derive some necessary and one sufficient conditions for stability and show some examples where the criteria coincide to yield a simple condition for stability.

1. Introduction

The ideal magnetohydrodynamics model (MHD) is the mathematically most accessible description of a magnetized plasma, the ionized gas which is expected to be the medium for controlled thermonuclear fusion reactions. An important theoretical question is the stability of the plasma, and under the assumption of only a small deviation from an equilibrium state it can be determined by linearizing the MHD equations about the equilibrium state and then applying the Laplace transform in time to the equations. Equivalently, one assumes an exponential dependence $\exp\{i\omega t\}$ on the time t. This gives rise to an eigenvalue equation, with ω as the eigenparameter. It is known (see [1]) that when the equilibrium state contains no mass flow, it is possible to cast the spectral problem in a formally selfadjoint form with eigenvalue ω^2. Thus ω^2 is real.

Investigation of the spectrum for a special case (see [2]) already exhibited some apparent singularities, which Grad [3] identified as a manifestation of the existence of a continuous spectrum. Heuristic arguments were then used in [4], [5] to derive part of the continuous spectrum, known as the Alfven spectrum, for axisymmetric configuations. The non-symmetric case was also similarly treated (see [6]). An understanding as to the origin of the continuous spectrum and its rigorous derivation in some cases were described by this author in [7], [8], but without giving full details. A different mathematical derivation of the Alfven spectrum in axisymmetry was given by Descloux and Geymonat [9].

Another peculiar property of the MHD spectrum is the existence of "ballooning" modes, named in analogy with the instability developing at a weak spot in a pressurized elastic container. As will be shown in this paper these modes form part of the essential spectrum of MHD. They were first derived as a stability

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criterion for axisymmetric plasmas (see [10], [11]) by considering the limit as $m \to \infty$, where m is the aximuthal Fourier mode number. This was later shown (see [12]) to extend to the non-symmetric case. The relation between ballooning modes and the essential spectrum was first noticed by the author in [8]. Similar equations for the case considered in [8] were later derived by Spies [13].

In these works, only configurations where all magnetic field lines close on themselves were treated but, in contrast with other works, not only was a stability criterion derived but the ballooning modes were obtained from some reduced eigenvalue problem, indicating their relation to the spectrum. More recently, Dewar and Glasser [14] derived formally, in an independent way, similar-looking equations for the more interesting case of an ergodic magnetic field. Their work stimulated us to extend our mathematical approach to the general magnetic field configuration.

In this paper we intend to present a rigorous and unified treatment of the MHD essential spectrum, including both the "Alfven" and "ballooning" spectra, and accessible to mathematicians and physicists alike. The two spectra will be treated similarly, and will be shown to arise from waves propagating one-dimensionally along magnetic field lines, but with different polarizations. Let us use for now the definition of the essential spectrum (there are many (see [15])) as the whole spectrum except for isolated eigenvalues of finite algebraic multiplicity. An essential spectrum can arise if one deals with a non-elliptic operator. As an illuminating example, compare the equations $\Delta u = \lambda u$ and $\mathbf{B} \cdot \nabla u = \lambda u$, both considered in a $2\pi \times 2\pi$ square in x, y-plane, with $\mathbf{B}$ a constant vector (B_x, B_y), and u doubly periodic. The eigenfunctions are the Fourier functions $u_{kl} = \exp\{i(kx + ly)\}$, where k, l range over all integers, $\lambda_{kl} = -(k^2 + l^2)$ in the first (elliptic) case, and $\lambda_{kl} = i(kB_x + lB_y)$ in the second (hyperbolic) case. In the elliptic case all eigenvalues are of finite multiplicity and have no finite accumulation points, while in the hyperbolic case if B_x/B_y is irrational the eigenvalues λ_{kl} are dense on the imaginary axis. The spectrum, which is always a closed set, is the entire axis. If B_x/B_y is a rational number, there is a pair (k_0, l_0) such that $k_0 B_x + l_0 B_y = 0$. Thus integers (k, l) and $(k + Nk_0, l + Nl_0)$, N being any integer, yield the same eigenvalue λ which is then of infinite multiplicity. In any case we get an essential spectrum for the hyperbolic equation under consideration.

The MHD spectral equations also exhibit non-ellipticity. To see it, observe that after linearization and replacement of the time derivatives by $i\omega$, the system of equations is of the same type as the system which determines an equilibrium state. The equilibrium question was considered by Grad [16] who found that, for an equilibrium without flow, the system is of mixed type, neither fully elliptic nor fully hyperbolic. Indeed, Alfven and slow magnetosonic waves propagate along magnetic field lines and give rise to the hyperbolic part of the equations. The structure of the magnetic field is then of great importance for this work, and we shall describe it in some detail in the next section. But first we recall some basic facts from spectral theory (see [17]) for the benefit of our mixed readership.

Let L be a closed linear operator in a Banach space X defined on the set of functions $D(L)$ and with a range set $R(L)$. The *nullity* $\nu(L)$ is the dimensionality of the null space of L. The *deficiency* $\delta(L)$ is the codimension of $R(L)$, i.e., the number of independent functions necessary to linearly complete $R(L)$ to X. We use

DEFINITION. λ is not in the spectrum of L if a solution to $(L-\lambda)u = f$ exists for all f in X, and u depends continuously on f. That is, if the operator $(L-\lambda)^{-1}$ can be defined on all of X and is bounded.

DEFINITION. λ is in the essential spectrum of L if at least one of the following requirements is not satisfied:

(i) The range $R(L-\lambda)$ is a closed set.
(ii) The nullity $\nu(L-\lambda)$ is finite.
(iii) The deficiency $\delta(L-\lambda)$ is finite.

When conditions (i)-(iii) are satisfied, $L-\lambda$ is said to be a *Fredholm* operator. If conditions (i) and either (ii) or (iii) are satisfied, $L-\lambda$ is called *semi-Fredholm.* The definition we use for the essential spectrum, i.e., that $L-\lambda$ not be a Fredholm operator, reduces for selfadjoint L to the previously mentioned definition of the set of limit points of the spectrum.

One way of showing that λ is in the essential spectrum is to construct a *singular sequence* of approximate eigenfunctions $\{u_n\}$ with norm $\|u_n\| = 1$, and $(L-\lambda)u_n \to 0$ as $n \to \infty$, and such that the sequence contains no convergent subsequence. In the selfadjoint case the existence of a singular sequence is necessary and sufficient for λ to be in the essential spectrum, while in the general case it is a sufficient condition only. That λ is in the spectrum if such a sequence exists is evident. For otherwise, define $f_n = (L-\lambda)u_n$, where $f_n \to 0$, so that $\|u_n\| \leqq \|(L-\lambda)^{-1}\| \times \|f_n\| \to 0$ since $(L-\lambda)^{-1}$ is bounded, in contradiction to the normalization of u_n. Let the *approximate nullity* $\nu'(L)$ be the greatest integer $m \leqq \infty$ such that for any $\varepsilon > 0$ there exists an m-dimensional subspace in $D(L)$ with the property that $\|Lu\| \leqq \varepsilon \|u\|$ for every u in the subspace. (Clearly $\nu'(L) \geqq \nu(L)$). It is known (see [17]) that

(i) if $R(L-\lambda)$ is closed, then $\nu'(L-\lambda) = \nu(L-\lambda)$;
(ii) if $R(L-\lambda)$ is not closed, then $\nu'(L-\lambda) = \infty$;
(iii) $\nu'(L-\lambda) = \infty$ if and only if a singular sequence exists.

From this we find that if a singular sequence exists, then either $R(L-\lambda)$ is not closed, or $\nu(L-\lambda) = \infty$ and λ is of infinite multiplicity. In both cases, λ must be in the essential spectrum. This concludes the description of background mathematical material.

The plan of the paper is as follows: In Section 2 we present the MHD model, give a detailed description of the structure of experimentally common toroidal magnetic fields, and derive the spectral problem. Section 3 deals with the "Alfven" spectrum in axisymmetric configurations. We are in fact able to show that it forms the whole essential spectrum if only a single azimuthal mode number is considered. In Section 4 we extend the previous treatment to the ballooning modes and then, in Section 5, derive a simple criterion for ballooning stability. Finally, in Section 6 we present some cases, all of them with closed magnetic field lines, where ballooning stability is tantamount to global stability.

2. The Magnetohydrodynamics Model

We begin with a description of the ideal MHD equations. Let $\mathbf{B}, \mathbf{J}, \mathbf{u}, p, \rho, s$ be the magnetic field, current density, and the plasma velocity, pressure, density, and specific entropy, respectively. They evolve according to

$$
\begin{aligned}
&\rho \mathbf{u}_t + \rho \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{J} \times \mathbf{B}, \\
&\mathbf{B}_t + \operatorname{curl} (\mathbf{B} \times \mathbf{u}) = 0, \\
&p_t + \mathbf{u} \cdot \nabla p + \rho \frac{\partial p}{\partial \rho} \operatorname{div} \mathbf{u} = 0, \\
&s_t + \mathbf{u} \cdot \nabla s = 0, \\
&\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{J} = \operatorname{curl} \mathbf{B}, \qquad p = p(\rho, s).
\end{aligned} \tag{2.1}
$$

The third equation replaces the usual continuity equation for ρ, and the last relation is an equation of state. For simplicity, and without affecting general properties, we shall use the particular equation of state appropriate for polytropic gases,

$$p = S(s)\rho^{\gamma} \tag{2.2}$$

with constant γ, usually taken as $\frac{5}{3}$ for plasmas. Now $\rho\, \partial p/\partial \rho = \gamma p$. We consider (2.1) in a toroidal domain bounded by a rigid perfect conductor, for which the boundary conditions are vanishing normal components of $\mathbf{B}$ and $\mathbf{u}$.

An equilibrium state is determined from

$$
\begin{aligned}
&\nabla p = \mathbf{J} \times \mathbf{B}, \\
&\operatorname{div} \mathbf{B} = 0, \qquad \mathbf{J} \equiv \operatorname{curl} \mathbf{B}.
\end{aligned} \tag{2.3}
$$

Since charged particles tend to follow magnetic field lines, improved confinement in toroidal devices can be achieved by taking magnetic fields which lie on surfaces, usually nested, as in Figure 1. If ψ is a label for the surfaces, $\mathbf{B} \cdot \nabla \psi = 0$. From (2.3) one has $\mathbf{B} \cdot \nabla p = 0$, thus the equilibrium pressure is constant along the field line. If $\mathbf{B}$ covers the surface ergodically, $p = p(\psi)$. One can define (see [18]) the "safety factor" $q(\psi)$ which determines the topology of field lines on the ψ-surfaces. A surface ψ_0 with rational q, $q(\psi_0) = n/m$, means that every field line on this surface closes on itself after n circuits the long way around and m circuits the

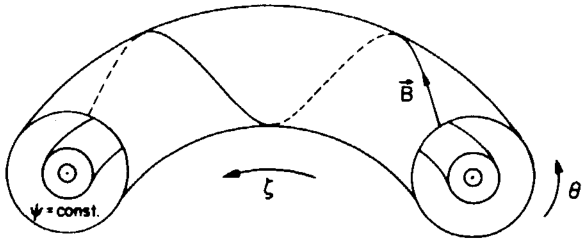

Figure 1. Structure of a toroidal magnetic field with nested flux surfaces.

short way around. If $q(\psi_0)$ is irrational, almost every field line covers the surface ergodically. In this paper we assume the existence of nested flux surfaces and sufficiently smooth equilibrium profiles. There is some doubt about their existence if no symmetry is assumed (see [19]). In the case of axisymmetry, however, one can represent $\mathbf{B}$ in the form

$$\mathbf{B}=\nabla\psi\times\nabla\zeta+I(\psi)\nabla\zeta, \tag{2.4}$$

where ζ is the ignorable toroidal angle, and get the flux surfaces by specifying $p(\psi)$ and $I(\psi)$ and solving the Grad-Shafranov equation

$$\frac{\partial^2\psi}{\partial r^2}-\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{\partial^2\psi}{\partial z^2}=-\left(r^2\frac{dp}{d\psi}+I\frac{dI}{d\psi}\right)\cdot$$

with the boundary condition $\psi=\text{const.}$ on the plasma boundary. In axisymmetry,

$$q(\psi)=\frac{1}{2\pi}\oint\frac{B_T}{rB_P}\,dl, \tag{2.5}$$

where B_T and B_P denote the toroidal and poloidal (long way and short way) components of $\mathbf{B}$, and dl is a poloidal length element on the ψ-surface.

Returning to the evolution equation (2.1) and writing $\mathbf{J}\times\mathbf{B}\equiv\mathbf{B}\cdot\nabla\mathbf{B}-\nabla(\frac{1}{2}\mathbf{B}^2)$ and $\operatorname{curl}(\mathbf{B}\times\mathbf{u})=\mathbf{u}\cdot\nabla\mathbf{B}-\mathbf{B}\cdot\nabla\mathbf{u}+\mathbf{B}\operatorname{div}\mathbf{u}$, we now linearize the system about an equilibrium state and take the Laplace transform. After some manipulations we get the eigenvalue problem:

$$\mathbf{B}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{B}-\nabla p_*-i\omega\rho\mathbf{u}=0, \tag{2.6a}$$

$$\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B}+\frac{1}{\gamma p}(\mathbf{u}\cdot\nabla p)\mathbf{B}-i\omega\left[\mathbf{b}+\frac{1}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{B}-\frac{1}{\gamma p}p_*\mathbf{B}\right]=0, \tag{2.6b}$$

$$\mathbf{B}\cdot(\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B})-\mathbf{u}\cdot\nabla p-(\gamma p+\mathbf{B}^2)\operatorname{div}\mathbf{u}-i\omega p_*=0, \tag{2.6c}$$

$$-\mathbf{u}\cdot\nabla s-i\omega s_1=0. \tag{2.6d}$$

Here $\mathbf{b}$ and $\mathbf{u}$ denote the perturbed magnetic field and velocity, respectively, s_1 is the perturbed entropy, and p_* is the perturbed total pressure $p+\frac{1}{2}\mathbf{B}^2$. All coefficients are equilibrium quantities. In writing (2.6b) we have taken a combination of the magnetic field and pressure evolution equations to eliminate the div $\mathbf{u}$ term there. This term now appears in (2.6c) only. Notice that (2.6d) decouples from the other equations and yields the point $\omega = 0$ in the spectrum. We shall ignore this fact for now.

It is common to give the problem a selfadjoint formulation (see [1]). One can substitute $\mathbf{b}$ and p_* from (2.6b, c) into (2.6a) to get a second-order system of the form

$$F\mathbf{u} = \omega^2 \rho \mathbf{u} \tag{2.7}$$

with F a Hermitian operator with respect to the usual L^2 norm. We shall not pursue this approach here for the simple reason that first-order systems are easier to handle and that selfadjointness has nothing to do with the existence of an essential spectrum. The important notion here is the part-hyperbolicity of the system. Indeed, one can use our approach to deal with equilibria with flow (see [21]) for which there are propagating waves but no selfadjoint formulation. The existence of a hyperbolic part can be seen (see [7]) from the property of (2.6) that all spatial derivatives are *within* ψ-surfaces, except in div $\mathbf{u}$ and ∇p_*. Thus, every ψ-surface is a characteristic surface of multiplicity six. If Cauchy data are prescribed on such a surface, only the normal derivatives of p_* and $\mathbf{u} \cdot \nabla \psi$ are known while all other quantities cannot be continued off the surface. The discussion in the following sections is indeed based on singling out these quantities.

3. The Axisymmetric Case

In this section we consider mostly axisymmetric configurations and restrict our attention to a single azimutal Fourier mode. Thus we take a dependence of $\exp\{im\zeta\}$ on the toroidal angle ζ. System (2.6) can be written symbolically as (see [7])

$$Av + Bw = 0, \tag{3.1a}$$

$$\frac{\partial}{\partial \psi} w + Cv + Dw = 0. \tag{3.1b}$$

Here w is a 2-vector which contains the two quantities differentiated in (2.6) across ψ-surfaces, namely p_* and $\mathbf{u} \cdot \nabla \psi$; v contains the other six dependent variables, and A, B, C, D are differential operators involving only first-order derivatives in the poloidal angle θ and depending linearly on ω. The toroidal derivative was eliminated by $\partial/\partial\zeta \to im$. Relation (3.1b) contains (2.6c) and the ψ component of (2.6a). We shall use the symbol ξ for the pair (v, w), and (3.1) will be denoted by the operation $L\xi$. We do not bother to specify a space and a norm for ξ for, as will become evident, almost any choice will yield the same result.

In order to determine the essential spectrum for (3.1) we use a singular sequence of functions localized about a particular surface $\psi=\psi_0$. Take $\varepsilon>0$ and let

$$\hat{\psi}=\frac{\psi-\psi_0}{\varepsilon},\qquad \frac{\partial}{\partial\psi}=\frac{1}{\varepsilon}\frac{\partial}{\partial\hat{\psi}}. \tag{3.2}$$

Now, expand each operator in (3.1) in Taylor series in ψ. For example, $A=A_0+(\psi-\psi_0)A_1+(\psi-\psi_0)^2A_2+\cdots$, where every A_j, $j=0,1,\cdots$, can be written as $A_j=A_j^0(\theta)+A_j^1(\theta)\,\partial/\partial\theta$. The A_j^k are matrices. We pick the yet unnormalized sequence

$$v_\varepsilon=f(\hat{\psi})v_0(\theta),\qquad w_\varepsilon=\varepsilon g(\hat{\psi})w_0(\theta); \tag{3.3}$$

f and g are smooth and vanish outside $-1\leqq\hat{\psi}\leqq 1$, or $|\psi-\psi_0|\leqq\varepsilon$. System (3.1) now has the property

$$L\xi_\varepsilon=\begin{pmatrix} fA_0v_0\\ g'w_0+fC_0v_0\end{pmatrix}+O(\varepsilon), \tag{3.4}$$

where $g'=dg/d\hat{\psi}$ and $O(\varepsilon)$ denotes terms formally smaller than v_ε by order ε. (3.4) follows by noting that every operator, say A, can be written as $A=A_0+O(\varepsilon)$ when applied to our trial functions; ξ_ε will give an approximate eigenfunction if we eliminate the leading terms in (3.4). Thus pick v_0 such that

$$A_0\left(\theta,\frac{\partial}{\partial\theta};\psi_0,m,\omega\right)v_0=0, \tag{3.5}$$

and have $f=g'$, g arbitrary, and $w_0=-C_0v_0$. Equation (3.5) is an eigenvalue equation in ω with periodic boundary condition in θ, yielding a sequence $\omega_n(\psi_0)$. To show that for each ω_n and the corresponding v_{0n}, ξ_ε forms a singular sequence, one has to show that $\|L\xi_\varepsilon\|/\|\xi_\varepsilon\|\to 0$ as $\varepsilon\to 0$, which is obvious. Also the normalized ξ_ε does not contain a convergent subsequence since it converges to zero pointwise everywhere except on $\psi=\psi_0$. The limit function could only have been zero in the usual spaces of measurable functions, but the normalization of ξ_ε precludes this possibility.

We have found many points $\omega_n(\psi_0)$ in the essential spectrum of MHD. As ψ_0 varies one gets continuous pieces of the spectrum. To get the explicit form of (3.5), we drop equation (3.1b) and set the variables w in (3.1a) equal to zero. Thus we get, from (2.6),

$$\begin{gathered}\mathbf{P}\{\mathbf{B}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{B}\}=i\omega\rho\mathbf{u},\\ \mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B}=i\omega\left[\mathbf{b}+\frac{1}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{b}\right],\\ -\mathbf{u}\cdot\nabla s=i\omega s_1.\end{gathered} \tag{3.6}$$

$\mathbf{P}$ is a projection operator which annihilates the component normal to a ψ-surface, and $\mathbf{u}$ is within the surface, $\mathbf{u} \cdot \nabla\psi = 0$. The last equation in (3.6) contributes a point $\omega = 0$ to the spectrum and may be ignored. Similarly, the second equation when multiplied by $\nabla\psi$ yields

$$\mathbf{B} \cdot \nabla(\mathbf{u} \cdot \nabla\psi) = i\omega(\mathbf{b} \cdot \nabla\psi). \tag{3.7}$$

Again, either $\omega = 0$ or $\mathbf{b} \cdot \nabla\psi = 0$. Eliminating the two roots $\omega = 0$ one gets

$$\begin{aligned} &\mathbf{P}\{\mathbf{B} \cdot \nabla\mathbf{b} + \mathbf{b} \cdot \nabla\mathbf{B}\} = i\omega\rho\mathbf{u}, \\ &\mathbf{P}\{\mathbf{B} \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{B}\} = i\omega\left[\mathbf{b} + \frac{1}{\gamma p}(\mathbf{B} \cdot \mathbf{b})\mathbf{B}\right]. \end{aligned} \tag{3.8}$$

Both $\mathbf{u}$ and $\mathbf{b}$ are now vectors within the fiux surface. We summarize our result as a theorem.

THEOREM 3.1. *In an axisymmetric configuration with a smooth magnetic field forming nested flux surfaces, every eigenvalue of system* (3.8), *considered on any flux surface and with assumed azimuthal behavior of* $\exp\{im\zeta\}$, *determines a stable point in the essential spectrum of the linearized MHD.*

System (3.8) is equivalent to equations derived formally (see [4], [5]) for the "Alfven" spectrum, where the fact that ω must be real (stable) was also shown. A simple proof for the reality of ω based on viewing (3.8) as an anti-Hermitian problem and also valid without symmetry, can be found in [7]. Derivation of the Alfven spectrum for the non-symmetric case will be discussed in the next section, but it will again be determined by equation (3.8). We point out the identity of the modes we found. Ideal MHD is a hyperbolic system of equations (see [22], [23]) which describes six waves plus two disturbances carried by the fluid velocity field. In the linearized system with zero equilibrium velocity the latter disturbances yield the two roots $\omega = 0$. Also, four of the waves propagate along magnetic field lines in a one-dimensional manner, namely the Alfven and slow magnetosonic waves, each type containing two waves moving in either positive or negative directions. These give rise to (3.8), where the waves are polarized so that $\mathbf{u}$ and $\mathbf{b}$ lie within a ψ-surface. Ballooning modes, to be discussed in the next section, involve the same waves polarized differently.

The natural question to ask is whether (3.6) yields the whole essential spectrum for a given mode number m. The answer appears to be positive and is discussed in the following. As described in the introduction, one expects only the hyperbolic part of (2.6) to contribute to the essential spectrum. Let us prove it for a similar looking system and then return to the MHD case. Consider the eigenvalue equation

$$(L - \lambda)\xi \equiv \begin{pmatrix} \mathbf{H} & \mathbf{B} \\ \mathbf{C} & \mathbf{E} \end{pmatrix} \begin{pmatrix} v \\ w \end{pmatrix} = 0. \tag{3.9}$$

Here there are two spatial dimensions, a periodic coordinate x, and a bounded y, and all entries in the matrix are first-order operators. Moreover, $\mathbf{H}$ involves only x-derivatives, $\mathbf{H}=\mathbf{H}_0+\mathbf{H}_1\,\partial/\partial x$, and the matrix $\mathbf{H}_1$ is non-singular. Also, $\mathbf{B}$ and $\mathbf{C}$ contain no y-derivatives. It is possible to arrange things so that $\mathbf{B}$ and $\mathbf{C}$ involve no x-derivatives as well and are merely matrix multiplication operators. This can be done by first adding combinations of the first row to the second one, thus eliminating $\partial/\partial x$ in $\mathbf{C}$, and then redefining variables, say $v'=v-Sw$, $w'=w$, with an appropriate matrix S. With $\mathbf{B}$ and $\mathbf{C}$ multiplication operators, assume $\mathbf{E}$ to be a uniformly elliptic operator. Thus $\mathbf{H}$ and $\mathbf{E}$ contain the hyperbolic and elliptic parts, respectively, of (3.9). Let us rewrite this equation as

$$\left\{\begin{pmatrix}\mathbf{H} & 0\\ \mathbf{C} & \mathbf{E}\end{pmatrix}+\begin{pmatrix}0 & \mathbf{B}\\ 0 & 0\end{pmatrix}\right\}\begin{pmatrix}v\\ w\end{pmatrix}=0, \tag{3.10}$$

where we view the second matrix as a perturbation of the first. It is known that a "relatively compact" perturbation leaves the essential spectrum unchanged (see [16]). Here we use the following:

DEFINITION. M is relatively compact with respect to L if for every sequence $\{\xi_n\}$ in $D(L)$ such that $\|\xi_n\|$ and $\|L\xi_n\|$ are bounded, there is a convergent subsequence of $\{M\xi_n\}$.

It is simple to verify the well-known result that M is relatively compact if $(L+K)^{-1}$ exists for some bounded operator K, and $M(L+K)^{-1}$ is compact. (Simply consider the bounded sequence $(L+K)\xi_n$). In our case, $L+K$ is the first matrix operator in (3.10) evaluated at some λ_0 for which this operator is invertible (as will soon become evident such λ_0 exists), and M is the perturbing matrix. We compute

$$M(L+K)^{-1}=\begin{pmatrix}0 & \mathbf{B}\\ 0 & 0\end{pmatrix}\begin{pmatrix}\mathbf{H}^{-1} & 0\\ -\mathbf{E}^{-1}\mathbf{C}\mathbf{H}^{-1} & \mathbf{E}^{-1}\end{pmatrix}=\begin{pmatrix}-\mathbf{B}\mathbf{E}^{-1}\mathbf{C}\mathbf{H}^{-1} & \mathbf{B}\mathbf{E}^{-1}\\ 0 & 0\end{pmatrix}. \tag{3.11}$$

The inverse of an elliptic operator is compact with respect to the usual norms. $\mathbf{E}^{-1}$ is multiplied by various bounded operators, where we notice that $\mathbf{H}^{-1}$ and $\mathbf{E}^{-1}$ exist as part of the invertibility of $L+K$. Thus, (3.11) is a compact operator. We conclude that the essential spectrum of (3.9) is given by the essential spectrum of the same operator with $\mathbf{B}$ set to zero. The spectrum of this operator is seen from (3.11) to be the union of the spectra of $\mathbf{H}$ and $\mathbf{E}$. However, as a uniformly elliptic operator, $\mathbf{E}$ does not have an essential spectrum, while the spectrum of $\mathbf{H}$ is in the essential spectrum of the unperturbed operator, as was shown for the Alfven spectrum in MHD. We conclude that the spectrum of $\mathbf{H}$ forms the essential spectrum of (3.9).

In the MHD case the elliptic part consists of the equations for p_*, $\mathbf{u}\cdot\nabla\psi$, and $\mathbf{b}\cdot\nabla\psi$, that is, the ψ-component of (2.6a), and equations (2.6c) and (3.7).

These are three equations rather than the expected two. To write the operator **E** explicitly, define $\mathbf{N} = \nabla\psi/|\nabla\psi|^2$, $u_\psi = \mathbf{u}\cdot\nabla\psi$, $b_\psi = \mathbf{b}\cdot\nabla\psi$, and replace the component of $\mathbf{b}$ parallel to $\mathbf{B}$ by $\mathbf{B}\cdot\mathbf{b} - p_*$ (which is minus the perturbed pressure), so as to get only derivatives of this quantity in the parallel component of (2.6a). One then gets

$$\mathbf{E}\begin{pmatrix} p_* \\ u_\psi \\ b_\psi \end{pmatrix} = \begin{pmatrix} \operatorname{div}(B^2\mathbf{N}u_\psi) + u_\psi\, dp/d\psi + i\omega p_* \\ \nabla\psi\cdot\nabla p_* - \mathbf{B}\cdot\nabla b_\psi - b_\psi\nabla\psi\cdot(\mathbf{B}\cdot\nabla\mathbf{N} + \mathbf{N}\cdot\nabla\mathbf{B}) + i\omega\rho u_\psi \\ -\mathbf{B}\cdot\nabla u_\psi + i\omega b_\psi \end{pmatrix}. \tag{3.12}$$

Notice that the first equation was derived after expressing $\mathbf{B}\cdot\nabla$ derivatives of surface components of $\mathbf{u}$ according to (2.6a). The ellipticity of (3.12) is seen when the first and last expressions are substituted into the middle one (for $\omega \neq 0$), to form a second-order expression in u_ψ. The second-order derivatives are $-|\nabla\psi|^2 B^2 \mathbf{N}\cdot\nabla(\mathbf{N}\cdot\nabla u_\psi) - \mathbf{B}\cdot\nabla(\mathbf{B}\cdot\nabla u_\psi)$. This notion of ellipticity is in agreement with the one introduced in [23]. Notice that the norms of p_* and b_ψ should require lower differentiability than u_ψ. We note that at the magnetic axis $\nabla\psi = 0$, and (3.12) is singular. As a result there is no uniform ellipticity and we cannot exclude the possibility that **E** has a non-empty essential spectrum. Only a plasma with an internal wall will not present this uncertainty. The singularity due to the magnetic axis was pointed out in [9], while we overlooked this point in our earlier work [8]. It was suggested in [9] that the singularity does indeed introduce an additional part to the essential spectrum. To conclude, we state

THEOREM 3.2. *The essential spectrum of* (2.6) *for an axisymmetric equilibrium with nested flux surfaces and bounded between two perfectly conducting tori which exclude the magnetic center, and for a particular mode number* m, *consists of the point* $\omega = 0$ *and all eigenvalues of* (3.8) *on all flux surfaces.*

THEOREM 3.3. *The point* $\omega = 0$ *lies in the Alfven spectrum* (3.8) *for some* m.

Proof: Set in (3.8) $\mathbf{b} = 0$, $\mathbf{u} = \mathbf{B}$, $\omega = 0$. The equation is clearly satisfied. Another solution is obtained by $\mathbf{b} = 0$, $u = \mathbf{J}$, $\omega = 0$, when we recall that $\mathbf{B}\cdot\nabla\mathbf{J} - \mathbf{J}\cdot\nabla\mathbf{B} = 0$, which follows from taking the curl of (2.3). The solutions we found are suitable for $m = 0$. For $m \neq 0$, and if at least one rational surface exists, we can still find two solutions $\mathbf{u}$ of $\mathbf{P}\{\mathbf{B}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{B}\} = 0$, coupled with $\mathbf{b} = 0$ and $\omega = 0$. Try $\mathbf{u} = f e^{im\zeta}\boldsymbol{\nu}$, $\boldsymbol{\nu}$ is either $\mathbf{B}$ or $\mathbf{J}$. The scalar function f is independent of ζ and satisfies $\mathbf{B}_p\cdot\nabla f + imfB_T/r = 0$. The solution is

$$f = f_0 \exp\left\{-im\int_0^l \frac{B_T}{rB_P}\,dl\right\}, \tag{3.13}$$

where all quantities are defined as in (2.5). By (3.13), f is defined as a single-valued function on the rational flux surface with $q(\psi) = n/m$, n some integer.

Theorem 3.2 determines the essential spectrum, namely the points ω for which (2.6) is not a Fredholm operator. It is interesting to know whether or not the spectrum contains additional continuous sets, curves or even open sets, for which however (2.6) is Fredholm. This we now indicate does not happen in MHD. Returning to the perturbation argument used before, we know that the spectrum of the unperturbed operator in (3.10) consists of the essential spectrum plus discrete eigenvalues. The question is then what happens when the perturbation is added. It is known (see [17]) that the points ω for which (2.6) is semi-Fredholm are the union of disconnected open sets whose boundaries are in the essential spectrum, and in each of which the nullity and deficiency of the operator are constant, except perhaps at isolated points with *higher* nullity and deficiency. Moreover, the boundaries of these open sets are unchanged under relatively compact perturbations. Since the essential spectrum in MHD for a specific mode number m is real (even for many equilibria with flow; see [25]), we have at most two such open sets, containing the upper and lower half ω-plane. Since MHD is a hyperbolic system, one may find by energy methods a bound on growth rates of the time dependent solutions. Thus, points ω with sufficiently large $|\mathcal{I}m\ \omega|$ must be in the resolvent set, with zero nullity and deficiency, and therefore the whole complex plane except for the essential spectrum and a discrete set of points forms the resolvent set. This answers the question posed before. We do not state this result as a theorem because of some remaining questions on its validity for various norms one may want to use.

The last question we deal with in this section is the structure of the spectrum of (2.6) if one considers all possible Fourier modes rather than a single mode m. Even though the Fourier functions $\{e^{im\zeta}\}$ span the space under consideration, it is not clear that the spectrum σ can be obtained from the union of the spectra σ_m corresponding to all Fourier modes. Clearly, $\sigma \supset \overline{\bigcup_m \sigma_m} \equiv \sigma_\cup$ (the closure appears because σ is always a closed set), but equality is not certain. Consider, for example, the sequence of $n \times n$ matrices $A_m = \lambda_0 I + mJ$, where λ_0 is a constant, I is the identity matrix, and J has entries 1 in the first upper diagonal and zeros everywhere else (as in a canonical Jordan block). All A_m have a spectrum consisting of the point λ_0 only. Yet, $(A_m - \lambda I)^{-1}$ is of order m^{n-1} for all $\lambda \neq \lambda_0$), and $\|(A_m - \lambda I)^{-1}\|$ is not uniformly bounded as $m \to \infty$. If we define A to be the direct sum of all the A_m, its spectrum is the whole complex λ-plane and not merely $\{\lambda_0\}$.

In view of the last example we have to consider the possibility that σ strictly contains $\sigma_\cup$. We claim, however, that *σ does not contain an additional bounded set disconnected* from $\sigma_\cup$. This is because ω is in the resolvent set of (2.6) if and only if the inverse of (2.6) for a particular m is bounded in norm, uniformly in m. As such inverses, say $R_m(\omega)$, are analytic operator-valued functions in ω, and because of the maximum principle for analytic functions, if $\|R_m(\omega)\|$ are uniformly bounded on a closed curve, enclosing the supposed set in $\sigma - \sigma_\cup$, they are uniformly bounded in the interior of the curve, which cannot then be in σ.

In our particular case of ideal MHD without flow, and if one uses the selfadjoint formulation (2.7), one has the same bound on $\|(F-\rho\omega^2)^{-1}\|$ in terms of the distance of ω^2 from the spectrum, independently of m. Thus no new points can be added to the spectrum when all modes are considered. The spectrum σ is the closure of the union of σ_m, each of which consisting of a continuous set on the real ω-axis, and some discrete points. Any continuous part of σ not obtained from the Alfven spectrum, as are the ballooning modes considered in the next section, must be formed from the discrete modes becoming *dense* in the plane as $m \to \infty$.

4. Ballooning Modes

Our work in the previous section suggests that every family of flux surfaces has points in the essential spectrum associated with it. The $\psi = \text{const.}$ surfaces, the single-valued family, is not the only one in existence. In fact, if we draw a curve in a poloidal cross section of the torus and continue it toroidally by following field lines, we generate a magnetic surface. Of couse, the surface will not be regular in the large, and will eventually selfintersect, but will nevertheless give rise to more essential spectrum. For this part of the work we consider general, non-symmetric, configurations with a smooth magnetic field. It is known that a divergence-free vector field can be represented (see [26]) as

$$\mathbf{B} = \nabla\psi \times \nabla\alpha, \qquad \alpha = \zeta - q(\psi)\theta, \tag{4.1}$$

where ζ and θ are toroidal and poloidal angle coordinates, respectively, with period 2π, and the Jacobian $\nabla\psi \times \nabla\zeta \cdot \nabla\theta$ is bounded and bounded away from zero.

Every function $\chi = \chi(\psi, \alpha)$ yields a family of flux surfaces since $\mathbf{B} \cdot \nabla\chi = 0$. Because of the singular behavior of the magnetic surface in the large, it is necessary to localize our test functions not only about the χ-surface but also within it. The localization is done within a flux tube about a particular *ergodic* field line characterized by $\psi = \psi_0$, $\chi = \chi_0$ (with χ not coinciding with ψ, $\partial\chi/\partial\alpha \neq 0$ on the field line). The tube will eventually selfintersect, however the thinner it starts the longer it will be before selfintersection. We thus localize in the region $|\chi - \chi_0| \leqq \varepsilon$, $|\psi - \psi_0| \leqq \varepsilon^n$, $|\zeta| \leqq L(\varepsilon)$, where $L(\varepsilon)$ is essentially the half-length of the tube before selfintersection, $L(\varepsilon) \to \infty$ as $\varepsilon \to 0$, and we expect $n < 1$ so that localization in χ is the stronger one. Using coordinates (ψ, χ, ζ), we find $\mathbf{B} \cdot \nabla = (\mathbf{B} \cdot \nabla\zeta)\partial/\partial\zeta$. As in equation (3.1) we split the variables and equations such that only two w-variables, p_* and $\mathbf{u} \cdot \nabla\chi$, are differentiated with respect to χ. The corresponding equations are (2.6c) and the χ-component of (2.6a). The system now looks exactly like (3.1) except that $\partial w/\partial\psi$ is replaced by $\partial w/\partial\chi$.

Defining

$$\hat{\chi} = \frac{\chi - \chi_0}{\varepsilon}, \qquad \hat{\psi} = \frac{\psi - \psi_0}{\varepsilon^n}, \tag{4.2}$$

our test functions are

$$
\begin{aligned}
v &= F\left(\frac{\zeta}{L(\varepsilon)}\right) f_1(\hat{\psi}) f_2(\hat{\chi}) v_0(\zeta), \\
w &= F\left(\frac{\zeta}{L(\varepsilon)}\right) \{\varepsilon^k g_1(\hat{\psi}) g_2(\hat{\chi}) w_0(\zeta) + \varepsilon h_1(\hat{\psi}) h_2(\hat{\chi}) y_0(\zeta)\},
\end{aligned}
\tag{4.3}
$$

where all the cutoff functions F, f_i, g_i, h_i vanish outside the range $|\tau| \leqq 1$ of their argument τ. Every coefficient $a(\psi, \chi, \zeta)$ is expanded as

$$a(\psi, \chi, \zeta) = a(\psi_0, \chi_0, \zeta) + \varepsilon \hat{\chi} \frac{\partial a_0}{\partial \chi} + \varepsilon^n \hat{\psi} \frac{\partial a_0}{\partial \psi} + \text{higher order.}$$

Notice that $(\partial/\partial\zeta)\{Fv_0\} = F \partial v_0/\partial\zeta + O(1/L)$ so that to leading order in ε, F is not differentiated. Equation (3.1a) yields formally

$$Av + Bw = Ff_1 f_2 A_0(\zeta, \partial/\partial\zeta) v_0 + O(1/L + \varepsilon + \varepsilon^n) + O(\varepsilon^{k-n}),$$

where A_0 is the operator A evaluated on the particular field line, and the last O-term comes from $\partial p_*/\partial\psi$. We set

$$k > n, \tag{4.4}$$

$$A_0(\zeta, \partial/\partial\zeta) v_0 = 0. \tag{4.5}$$

Equation (3.1b) yields

$$
\begin{aligned}
\frac{\partial w}{\partial \chi} + Cv + Dw = F\{&\varepsilon^{k-1} g_1 g_2' w_0 + h_1 h_2' y_0 + f_1 f_2 C_0^0 v_0 \\
&+ \varepsilon^{-n} f_1' f_2 [C_0^1 + \varepsilon^n \hat{\psi} C_1^1] v_0\} + \text{higher order.}
\end{aligned}
$$

Here we have written $C = C^0 \partial/\partial\zeta + C^1 \partial/\partial\psi$ and the lower indices of the C^j correspond to expansion about ψ_0. Primes of the cut-off functions denote their derivatives with respect to their arguments. To eliminate the lowest-order terms we need

$$k - 1 = -n, \tag{4.6}$$

$$w_0 = -C_0^1 v_0, \tag{4.7}$$

and we choose $g_1 = f_1'$, $f_2 = g_2'$. There are three $O(1)$ terms left. We replace $h_1 h_2 y_0$ by two similar terms, say $h_1 h_2 y_0 + l_1 l_2 z_0$, and take $h_1 = f_1'$, $h_2' = f_2$, $y_0 = -C_0^0 v_0$, and $l_1 = \hat{\psi} f_1'$, $l_2' = f_2$, $z_0 = -C_1^1 v_0$. Considering (4.4) and (4.6), we need

$$0 < n < \tfrac{1}{2}, \qquad \tfrac{1}{2} < k = 1 - n < 1. \tag{4.8}$$

It is worthwhile to mention that when using coordinates (ψ, χ, ζ), some coefficients are of order ζ which itself is $O(L)$. This is because field lines diverge in the flux tube but not on the same ψ-surface. To see it note that $\nabla\chi =$

$(\partial\chi/\partial\psi)\nabla\psi+(\partial\chi/\partial\alpha)(\nabla\zeta-q\nabla\theta-q'\theta\nabla\psi)$, and $\theta=O(\zeta)$. However, this fact does not invalidate the formal ordering in our analysis. We can always take $L(\varepsilon)$ small enough such that $\varepsilon^a L\to 0$ for any $a>0$ used, and still have $L(\varepsilon)\to\infty$. The $O(\zeta)$ terms also do not interfere with the introduction of the cut-off function F, because we always find the terms $\partial F/\partial\zeta$, which do not arise from $\mathbf{B}\cdot\nabla$ operations, to be multiplied by "small" terms of order higher than 1.

Equation (4.5) determines the ballooning modes. Its structure is similar to that of (3.6), and after the same reduction as in (3.7) with ψ replaced by χ, it reads

$$\begin{gathered}\mathbf{P}\{\mathbf{B}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{B}\}=i\omega\rho\mathbf{u},\\ \mathbf{P}\left\{\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B}+\frac{1}{\gamma p}\mathbf{B}\mathbf{u}\cdot\nabla p\right\}=i\omega\left[\mathbf{b}+\frac{1}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{B}\right].\end{gathered}\tag{4.9}$$

$\mathbf{P}$ is now a projection operator which eliminates the component perpendicular to the χ-surface. The ballooning mode equations (4.9) are of fourth order and can be brought to the form used in [14] in contrast with the original form of a single second-order equation (see [10]-[12]) which is only correct as a stability criterion, i.e., for marginal modes. It remains to specify the boundary conditions appropriate for (4.9). We shall not pursue this question in full here, except to note that if a solution of the equation blows up as $|\zeta|\to\infty$, then the formal ordering used before is invalid. In particular, the term $L^{-1}F'(\zeta/L)v_0$ may become larger than Fv_0, even in norm. It is sufficient to require that the solution of (4.9) should remain *bounded* (pointwise) as $|\zeta|\to\infty$ and then our previous discussion holds. To summarize, we state

THEOREM 4.1. *A point ω for which there exists a solution of* (4.9) *that is bounded as $|\zeta|\to\infty$, for a particular ergodic field line and magnetic projection* $\mathbf{P}$, *is in the essential* (*ballooning*) *spectrum of MHD.*

A special direction of localization we have not considered in this section is the case $\chi=\psi$; it is the same as in the Alfven spectrum discussed before. Our treatment based on (4.3) can go through with minor technical changes, like switching the role of ψ and some independent χ so that the main localization is across the ψ-surface. One then gets again (4.9) which, when $\mathbf{P}$ projects on the ψ-surface, reduces to (3.8). We particularly point out that even though the ergodic field line fills the ψ-surface which is well defined globally, the condition on the solution is (at most) the boundedness we mentioned, and *not* double periodicity on the toroidal surface, unlike the view taken in [27].

A striking feature of the ballooning spectrum is its high multiplicity. Every field line generates a one-parameter family of frequencies ω, the parameter being the polarization of the wave expressed by the orientation of the χ-surface, or χ_ψ/χ_α, which is constant on the field line. When one moves from one field line to its neighbor, it is expected that by a slight change in the direction of χ, the same frequency ω will be obtained. Interestingly, in the axisymmetric case one gets infinite multiplicity even for the spectrum corresponding to the same field line. To see it, note first that the direction of localization always tends to the normal to the ψ-surface as $|\zeta|\to\infty$. This follows from the relation $\nabla\alpha = \nabla\zeta - q\nabla\theta - \theta q'\nabla\psi$ when noting that $|\theta|\to\infty$ with $|\zeta|$. If, in the axisymmetric case, we follow a field line from some initial point $\mathbf{x}_0$ to a point $\mathbf{x}_1$ which is on the same poloidal location as $\mathbf{x}_0$, $\theta_1 = \theta_0 + 2\pi$, the direction of $\nabla\chi$ will change in a monotone way. Yet, we could have considered $\mathbf{x}_1$ to be our initial point and $\nabla\chi(\mathbf{x}_1)$ the initial polarization, and would have obtained the same eigenvalue ω. Thus $\nabla\chi(\mathbf{x}_0) + 2\pi nq'\nabla\psi$, with arbitrary integer n, all define directions of polarization at $\mathbf{x}_0$ which yield the same ballooning frequency in the axisymmetric case. This fact was observed algebraically in [14]. Note that the degeneracy of the ballooning spectrum may disappear in non-symmetric configurations.

We would like to comment on ballooning modes arising from non-ergodic field lines. Since these modes express some wave propagation phenomenon, they are expected to be associated with closed lines as well, and to be described by the same equation (4.9). Note, however, that the direction of localization is not periodic for sheared systems ($dq/d\psi \neq 0$), even when the field line closes on itself. Thus (4.9) will have to be considered on the infinite domain $-\infty < \zeta < \infty$. We do not pursue here the full analysis for this case and also leave open the interesting question of the correct boundary conditions at infinity.

Finally, we return to the last remark of the previous section. Since ballooning modes are part of the essential spectrum not visible when only a particular Fourier mode is considered, their presence implies accumulation of the spectral points as $m\to\infty$. This statement is rigorous when the selfadjoint formulation is used. Such accumulation was known to occur in a special case (see [28]), and the prevalence of this phenomenon was the basis of the surprising result discussed in [21], where it was shown that ballooning modes could not be observed from a rotating frame of coordinates.

5. Ballooning Stability Criteria

In this section we follow the selfadjoint formulation of MHD (2.7), and look for energy inequalities which would guarantee ballooning stability. It turns out that the stability criterion can be expressed in terms of the first two eigenvalues of a single second-order ordinary differential operator. The trick is, as usual, to express $\mathbf{b}$ in terms of $\mathbf{u}$ in (4.9). This procedure was carried out previously in [20] for the same system, but without the projection $\mathbf{P}$ which was replaced by the identity operator. We shall first use the same procedure for the system (2.6).

Defining the operators

$$\alpha(\mathbf{u}) = \mathbf{B} \cdot \nabla\mathbf{u} - \mathbf{u} \cdot \nabla\mathbf{B} + \frac{1}{\gamma p}(\mathbf{u} \cdot \nabla p)\mathbf{B},$$

$$\beta(\mathbf{u}) = \mathbf{B} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{B}, \tag{5.1}$$

$$\mathbf{Q}\mathbf{u} = \mathbf{u} + \frac{1}{\gamma p}(\mathbf{B} \cdot \mathbf{u})\mathbf{B}, \quad Q^{-1}\mathbf{u} = \mathbf{u} - \frac{1}{\gamma p + \mathbf{B}^2}(\mathbf{B} \cdot \mathbf{u})\mathbf{B},$$

one obtains from (2.6b)

$$i\omega\mathbf{b} = Q^{-1}\alpha\mathbf{u} + \frac{i\omega}{\gamma p + \mathbf{B}^2} p_*. \tag{5.2}$$

When substituted into (2.6a) this yields

$$\omega^2 \rho\mathbf{u} = -\beta Q^{-1}\alpha\mathbf{u} + G(p_*). \tag{5.3}$$

The operator G operates on $\mathbf{u}$ only through $i\omega p_*$ expressed by (2.6c). It was found in [20] that G is positive definite in the L^2 inner product,

$$(G(p_*), \mathbf{u}) = \int \frac{1}{\gamma p + \mathbf{B}^2} |i\omega p_*|^2 \, d^3\mathbf{x}.$$

The operator F in (2.7) is of course equal to $-\beta Q^{-1}\alpha + G$ and its positivity is equivalent to linear stability. A similar procedure carried out for (4.9) yields expression (5.3) without the $G(p_*)$ term, which is consistent with our notion of setting $p_* \to 0$ for ballooning modes. The ballooning equation is then

$$\mathbf{P}\hat{F}\mathbf{P}\mathbf{u} = \omega^2 \rho\mathbf{u}, \qquad \hat{F} = -\beta Q^{-1}\alpha. \tag{5.4}$$

Using the antisymmetry of the operator $\mathbf{B} \cdot \nabla$ and the identity

$$(\mathbf{u} \cdot \nabla\mathbf{B}) \cdot \mathbf{v} = \mathbf{u} \cdot (\mathbf{v} \cdot \nabla\mathbf{B}) + \mathbf{u} \times \mathbf{v} \cdot \operatorname{curl} \mathbf{B}$$

for any vector fields $\mathbf{u}, \mathbf{v}, \mathbf{B}$, we get the adjoint β^* of β,

$$\beta^*(\mathbf{u}) = -\mathbf{B} \cdot \nabla\mathbf{u} + \mathbf{u} \cdot \nabla\mathbf{B} - \mathbf{J} \times \mathbf{u}. \tag{5.5}$$

Thus, using (5.1),

$$(\hat{F}\mathbf{u}, \mathbf{u}) = \int \{Q^{-1}\alpha(\mathbf{u}) \cdot \alpha(\mathbf{u}) + \mathbf{u} \times \alpha(\mathbf{u}) \cdot \mathbf{J}\} \, d^3\mathbf{x}, \tag{5.6}$$

and we can use real vectors $\mathbf{u}$ since $\hat{F}$ is real. Because of the positivity of G the positivity of (5.6) for all admissible $\mathbf{u}$ (in the domain of the MHD operator) is sufficient for global plasma stability (see [20], [29]). Note that $\hat{F}$ is a differential operator on a field line, with directional derivatives of the form $\mathbf{B} \cdot \nabla$ only. Using (4.1) one has for a volume element $d^3\mathbf{x} = d\psi \, d\alpha \, dl/B$ ($B = |\mathbf{B}|$) so that the measure of integration in (5.6) may be replaced by dl/B, where dl is a length element along the field line. As a reminder, we change the notation of our volume inner product to the reduced inner product on a field line,

$$\langle \mathbf{u}, \mathbf{v}\rangle = \int \mathbf{u} \cdot \mathbf{v}\, dl/B, \tag{5.7}$$

and $\langle \hat{F}\mathbf{u}, \mathbf{u}\rangle \geqq 0$ for all $\mathbf{u}$ in the domain of F and all field lines is sufficient for plasma stability. Returning to the ballooning equation (5.4), one gets stability if $\langle \hat{F}\mathbf{Pu}, \mathbf{Pu}\rangle \geqq 0$ for all admissible ballooning eigenfunctions $\mathbf{u}$ (a class we have not completely determined in Section 4). To summarize, we state loosely our

CONCLUSION. *Stability of the sixth-order system* (4.9), *with* $\mathbf{P}$ *replaced by the identity operator, on all field lines, is sufficient for global stability. Stability of the fourth-order system* (4.9) *for all magnetic projections* $\mathbf{P}$ *and all field lines is necessary for ballooning stability.*

In the next section we shall discuss some cases where the necessity and sufficiency conditions coincide. Here we would like to further simplify the ballooning stability criterion. We express $\mathbf{Pu}$ in the form

$$\mathbf{Pu} = X\mathbf{N} + Y\mathbf{B}, \tag{5.8}$$

where $\mathbf{N}$ is normal to $\mathbf{B}$ *within* the flux surface $\chi = \text{const.}$, $\mathbf{N} \cdot \mathbf{B} = 0$, $\mathbf{N} \cdot \nabla\chi = 0$. For normalization we use $\mathbf{N} \cdot \nabla\psi = 1$ and we exclude from this treatment the Alfven spectrum which corresponds to $\chi = \psi$, and which is known to be stable (see [7]). From previous remarks we can conclude that $\mathbf{N} \cdot \nabla\psi \neq 0$ on an entire field line if the inequality is satisfied at one point on the line. For the following manipulations it is useful to notice that

$$\begin{aligned} &\boldsymbol{\alpha}(\mathbf{B}) = 0, \\ &\boldsymbol{\alpha}(\mathbf{N}) = \operatorname{curl}(\mathbf{N} \times \mathbf{B}) + \left[\operatorname{div}\mathbf{N} + \frac{1}{\gamma p}\mathbf{N} \cdot \nabla p\right]\mathbf{B}, \\ &\boldsymbol{\alpha}(\mathbf{Pu}) = X'\mathbf{N} + X\boldsymbol{\alpha}(\mathbf{N}) + Y'\mathbf{B}, \end{aligned} \tag{5.9}$$

where from now on the prime denotes $\mathbf{B} \cdot \nabla$, e.g. $X' \equiv \mathbf{B} \cdot \nabla X$.

We claim that $\boldsymbol{\alpha}(\mathbf{N}) = a\mathbf{B}$, with some scalar function a. To see this notice that, for any flux function f, $\mathbf{B} \cdot \nabla f = 0$, we have

$$\nabla f \cdot \operatorname{curl}(\mathbf{N} \times \mathbf{B}) \equiv \operatorname{div}[(\mathbf{N} \times \mathbf{B}) \times \nabla f] = \operatorname{div}[(\mathbf{N} \cdot \nabla f)\mathbf{B}].$$

Taking $f = \psi$ with $\mathbf{N} \cdot \nabla\psi = 1$ or $f = \chi$ with $\mathbf{N} \cdot \nabla\chi = 0$, this expression vanishes. Thus $\boldsymbol{\alpha}(\mathbf{N}) \cdot \nabla f = 0$ for two independent flux functions and $\boldsymbol{\alpha}(\mathbf{B})$ is parallel to $\mathbf{B}$. Computing $a = B^{-2}\boldsymbol{\alpha}(\mathbf{B}) \cdot \mathbf{B}$, we get

$$\boldsymbol{\alpha}(\mathbf{N}) = a\mathbf{B}, \qquad a = -2\boldsymbol{\kappa} \cdot \mathbf{N} + \frac{1}{\beta B^2}\dot{p}(\psi); \tag{5.10}$$

$\boldsymbol{\kappa}$ is the curvature vector of the magnetic line, $\dot{p}=dp/d\psi$, and

$$\beta\equiv\frac{\gamma p}{\gamma p+B^2}\cdot$$

Substituting (5.8) into (5.6) yields

$$\langle\hat{F}\mathbf{u},\mathbf{u}\rangle=\int\{X'^2|\mathbf{N}|^2-2\dot{p}(\boldsymbol{\kappa}\cdot\mathbf{N})X^2+\beta B^2[Y'-2\boldsymbol{\kappa}\cdot\mathbf{N}X]^2\}\frac{dl}{B}\cdot\tag{5.11}$$

The form of the last term suggests that the worst case for stability occurs when $Y'=2\boldsymbol{\kappa}\cdot\mathbf{N}X$. For given X, however, a solution Y to this equation may not be admissible. Nevertheless, we can minimize (5.11) with respect to functions Y satisfying $Y(\zeta_0)=Y(\zeta_1)=0$ and such that Y vanishes identically outside the interval $\zeta_0\leqq\zeta\leqq\zeta_1$ on the field line. If we then let $\zeta_0\to-\infty$, $\zeta_1\to+\infty$, the last term of (5.11) will drop out if $X\to 0$ fast enough as $|\zeta|\to\infty$. To see this, note that minimizing with respect to Y one obtains the Euler equation $\mathbf{B}\cdot\nabla[\beta B^2(Y'-2\boldsymbol{\kappa}\cdot\mathbf{N}X)]=0$, or

$$Y'=2\boldsymbol{\kappa}\cdot\mathbf{N}X+\frac{c}{\beta B^2},\qquad c=\text{const.}\tag{5.12}$$

The boundary conditions at $\zeta=\zeta_0$ and $\zeta=\zeta_1$ imply

$$c=-\left\{\int_{\zeta_0}^{\zeta_1}(\beta B^2)^{-1}\frac{dl}{B}\right\}^{-1}\int_{\zeta_0}^{\zeta_1}2\boldsymbol{\kappa}\cdot\mathbf{N}X\frac{dl}{B},$$

and the contribution to (5.11) of the last term is

$$\int_{\zeta\notin[\zeta_0,\zeta_1]}\beta B^2[2\boldsymbol{\kappa}\cdot\mathbf{N}X]^2\frac{dl}{B}+\left\{\int_{\zeta_0}^{\zeta_1}(\beta B^2)^{-1}\frac{dl}{B}\right\}^{-1}\left\{\int_{\zeta_0}^{\zeta_1}2\boldsymbol{\kappa}\cdot\mathbf{N}X\frac{dl}{B}\right\}^2\tag{5.13}$$

which becomes negligible with respect to the other terms in the limit if $|\mathbf{N}|X=O(|\zeta|^{-p})$, $p>\frac{1}{2}$, as $|\zeta|\to\infty$. (We tacitly assumed that βB^2 does not vanish on the line, the common experimental situation.) The quadratic form to treat for the ergodic line is then

$$\langle\hat{F}\mathbf{u},\mathbf{u}\rangle=\int\{X'^2|\mathbf{N}|^2-2\dot{p}(\boldsymbol{\kappa}\cdot\mathbf{N})X^2\}\frac{dl}{B}\cdot\tag{5.14}$$

Positivity of this form guarantees ballooning stability. More work is needed in order to determine the class of admissible functions X in (5.14) which, presumably, will give rise to a selfadjoint operator $\hat{F}$ such that the positivity of (5.14) will be both necessary and sufficient for ballooning stability.

As a last remark, we point out that all possible orientations of the χ-surfaces can be taken account of by using $\nabla\chi=\nabla(\zeta-q\theta)+\lambda\nabla\psi$, with $-\infty<\lambda<\infty$ the constant direction parameter. Clearly $\mathbf{N}$ is linear in λ, $\mathbf{N}=\nabla\chi\times\mathbf{B}/B^2$, and (5.14) has the form $\langle\hat{F}\mathbf{u},\mathbf{u}\rangle=\lambda^2a+2\lambda b+c$, with a, b, c quadratic forms in X. Positivity of (5.14) for all orientations requires $ac-b^2 \geqq 0$ for all X. This condition is more difficult to check than the positivity of (5.14) for a specific λ.

6. Closed Line Systems

In this section we deal with configurations where all field lines in the plasma close on themselves. Experimental devices with this property are the field reversed mirrors, the Elmo Bumpy Torus, and some shearless stellarators. Our derivation of the ballooning mode equation (4.9) is valid in this case as well, and is in fact made easier by not having to localize the test functions to a finite portion of the field line. The boundary conditions on eigenfunctions are simply singlevaluedness (periodicity) on the closed field line, and with them the operator $\hat{F}$ of the previous section is selfadjoint. The last term of (5.11) does contribute to the stability criterion as in (5.13), and one gets

$$\begin{aligned}\langle \hat{F}\mathbf{u}, \mathbf{u}\rangle = &\oint [X'^2|\mathbf{N}|^2 - 2(\mathbf{N}\cdot\nabla p)(\mathbf{N}\cdot\boldsymbol{\kappa})X^2] \\ &+ \left\{\oint (\beta B^2)^{-1}\right\}^{-1}\left\{\oint 2(\mathbf{N}\cdot\boldsymbol{\kappa})X\right\}^2,\end{aligned} \tag{6.1}$$

where all integrals are taken along the closed field line with respect to the measure dl/B. We notice that $\mathbf{N}$ itself is periodic and $\mathbf{N}\cdot\nabla p$ is constant on the field line. Positivity of (6.1) for all periodic and differentiable X is a necessary and sufficient condition for ballooning stability.

The positivity of (6.1) can be determined by considering the related eigenvalue problem

$$\begin{aligned}TX \equiv &-(|\mathbf{N}|^2X')' - 2(\mathbf{N}\cdot\nabla p)(\mathbf{N}\cdot\boldsymbol{\kappa})X \\ &+4\left\{\oint (\beta B^2)^{-1}\right\}^{-1}\left\{\oint \mathbf{N}\cdot\boldsymbol{\kappa}X\right\}\mathbf{N}\cdot\boldsymbol{\kappa} = \Lambda X.\end{aligned} \tag{6.2}$$

Form (6.1) is positive semi-definite if and only if the lowest eigenvalue $\Lambda_1 \geqq 0$. Using the fact that the last term of (6.2) is of rank one, projecting all functions X in the direction of $\mathbf{N}\cdot\boldsymbol{\kappa}$, it is possible (see [1], [20]) to determine the positivity of T from the first two eigenvalues $\lambda_1 < \lambda_2$ of the neighboring operator T_0,

$$T_0X \equiv (|\mathbf{N}|^2X')' - 2(\mathbf{N}\cdot\nabla p)(\mathbf{N}\cdot\boldsymbol{\kappa})X = \lambda X. \tag{6.3}$$

We state the result as a theorem.

THEOREM 6.1. *For closed line systems the plasma is ballooning stable if and only if for each field line and each magnetic projection (which determines* $\mathbf{N}$*) one of the following two conditions is satisfied:*

(i) $\lambda_1 \geqq 0$,

(ii) $\lambda_1 < 0 \leqq \lambda_2$ *and* $2(\mathbf{N}\cdot\nabla p)^{-1}\oint \mathbf{N}\cdot\boldsymbol{\kappa} \geqq \oint (\beta B^2)^{-1}$.

The rest of the section will be devoted to the investigation of the sufficient condition for global stability introduced in Section 5 and, in particular, we shall describe some cases where it coincides with the ballooning stability criterion, rendering it a necessary and sufficient condition for stability. We recall that the criterion may be stated as the stability of the sixth-order system with periodic boundary conditions:

$$\mathbf{B}\cdot\nabla\mathbf{b}+\mathbf{b}\cdot\nabla\mathbf{B}=i\omega\rho\mathbf{u}, \tag{6.4a}$$

$$\mathbf{B}\cdot\nabla\mathbf{u}-\mathbf{u}\cdot\nabla\mathbf{B}+\frac{1}{\gamma p}(\mathbf{u}\cdot\nabla p)\mathbf{B}=i\omega\left[\mathbf{b}+\frac{1}{\gamma p}(\mathbf{B}\cdot\mathbf{b})\mathbf{B}\right]. \tag{6.4b}$$

Clearly, this condition becomes necessary for stability if every unstable eigenfunction $\{\mathbf{u},\mathbf{b}\}$ lies in some magnetic surface χ, such that $\mathbf{u}=\mathbf{Pu}$ for the corresponding projection operator; the instability then is of the ballooning type. Notice that, as in (3.7), equation (6.4b) yields

$$\mathbf{B}\cdot\nabla(\mathbf{u}\cdot\nabla\chi)=i\omega(\mathbf{b}\cdot\nabla\chi), \tag{6.5}$$

and $\mathbf{b}=\mathbf{Pb}$ as well. We demonstrate the coincidence of the two criteria in the following cases.

A. Axisymmetric mirror machine. Assume the magnetic field to have no toroidal component, $B_T\equiv r(\mathbf{B}\cdot\nabla\zeta)=0$. The toroidal components of (6.4a, b) decouple and read

$$\begin{aligned}&\mathbf{B}\cdot\nabla b_T+\frac{1}{r}(\mathbf{B}\cdot\nabla r)b_T=i\omega\rho u_T,\\&\mathbf{B}\cdot\nabla u_T-\frac{1}{r}(\mathbf{B}\cdot\nabla r)u_T=i\omega b_T.\end{aligned} \tag{6.6}$$

Casting (6.6) in matrix form, we have

$$\begin{pmatrix}0 & \partial+r^{-1}(\mathbf{B}\cdot\nabla r)\\ \partial-r^{-1}(\mathbf{B}\cdot\nabla r) & 0\end{pmatrix}\begin{pmatrix}u_T\\ b_T\end{pmatrix}=i\omega\begin{pmatrix}\rho & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_T\\ b_T\end{pmatrix}, \tag{6.7}$$

where $\partial\equiv\mathbf{B}\cdot\nabla$ is antisymmetric in the L^2 inner product with integration measure dl/B. The matrix operator in (6.7) is then antisymmetric in the $L^2\oplus L^2$-space, with purely imaginary eigenvalues. Thus ω in (6.7) must be real. Any instability of (6.4) must have eigenfunctions $\{\mathbf{u},\mathbf{b}\}$ with $u_T=b_T=0$, that is, they must be projections with $\chi=\zeta$, where as we note, $\zeta=$ const. is a flux surface since $\mathbf{B}\cdot\nabla\zeta=0$. We conclude that the stability of the ballooning modes with this particular polarization is necessary and sufficient for the plasma stability. Our result can be reduced to the energy criterion developed for this case in [1].

B. Low pressure. Consider β to be a small parameter, $\mathbf{B}=O(1)$, $p=O(\beta)$, $\mathbf{J}=O(\beta)$, and normalize the eigenfunction so that $\mathbf{u}=O(1)$. From (5.6), when read as $\langle \hat{F}\mathbf{u}, \mathbf{u}\rangle = \omega^2\langle \rho\mathbf{u}, \mathbf{u}\rangle$, we see that the only negative contribution to $\hat{F}$ is $O(\beta)$ so that an unstable ω is at most $O(\sqrt{\beta})$. The positive term in (5.6), $\oint Q^{-1}\boldsymbol{\alpha}(\mathbf{u})\cdot\boldsymbol{\alpha}(\mathbf{u})$, can be at most $O(\beta)$, to allow an instability. Since Q^{-1} leaves the components perpendicular to $\mathbf{B}$ unchanged while multiplying the parallel component by $\gamma p/(\gamma p + B^2)$, we see that the perpendicular components of $\boldsymbol{\alpha}(\mathbf{u})$ are $O(\sqrt{\beta})$. Using the identity $\nabla\chi\cdot\boldsymbol{\alpha}(\mathbf{u}) \equiv \mathbf{B}\cdot\nabla(\mathbf{u}\cdot\nabla\chi)$ for every flux function χ, we get to leading order, $\mathbf{u}\cdot\nabla\chi_j = \text{const.}$ for two independent flux functions $\chi_j, j = 1, 2$. A linear combination χ of χ_1, χ_2 yields $\mathbf{u}\cdot\nabla\chi = 0$ and the unstable eigenfunction is a projection. Hence stability of (6.4) is necessary and sufficient for plasma stability, to leading order in the growth rate.

C. The closed-line screw-pinch. Lest the reader gets the impression that the ballooning criterion is always sufficient for stability, the following case will serve as a simple counter-example. Consider the screw-pinch configuration of a straight cylindrical plasma of length L, with its axis in the z direction. The two ends of the cylinder are identified to produce a topological torus. It is assumed that the magnetic surfaces are the concentric cylinders $r = \text{const.}$, $\mathbf{B}\cdot\nabla r = 0$, and all equilibrium quantities depend on r only, where we use cylindrical coordinates r, θ, z. In such a configuration the equilibrium equation (2.3) reduces to

$$\frac{d}{dr}(p + \tfrac{1}{2}B^2) + \frac{B_\theta^2}{r} = 0, \tag{6.8}$$

and the safety factor q satisfies

$$q(r) = \frac{2\pi}{L}\frac{rB_z}{B_\theta}, \tag{6.9}$$

where subscripts denote components. The closed line case has for q a constant rational number.

We first deal with the ballooning condition (6.1). It is possible to consider as test functions the Fourier functions $X(\theta, z) = \hat{X}\exp\{i(m\theta + kz)\}$, where $\hat{X}$ is constant, $k = 2\pi n/L$ and m, n are any integers. The last (rank one) term of (6.1) contributes only when $\mathbf{k}\cdot\mathbf{B} = 0$, where $\mathbf{k}\cdot\mathbf{B} \equiv mB_\theta/r + kB_z$. For modes with $\mathbf{k}\cdot\mathbf{B} \neq 0$ the ballooning criterion has the form

$$\oint [X'^2|\mathbf{N}|^2 - 2(\mathbf{N}\cdot\nabla p)\mathbf{N}\cdot\boldsymbol{\kappa})X^2] \geqq 0. \tag{6.10}$$

Recall that $\mathbf{N} = \nabla\chi\times\mathbf{B}/B^2$, where $\nabla\chi = \nabla(2\pi z/L - q\theta) + \lambda\nabla r$, with λ the constant polarization parameter. Clearly $\mathbf{N}$ has constant cylindrical coordinates. Also,

$$\boldsymbol{\kappa} = -\frac{B_\theta^2}{rB^2}\nabla r, \qquad \nabla p = \frac{dp}{dr}\nabla r,$$

and typically $dp/dr<0$. Condition (6.10) can be written as

$$\oint\left[X'^2+2\frac{dp}{dr}\frac{B_\theta^2}{rB^2}\frac{(\mathbf{N}\cdot\nabla r)^2}{|\mathbf{N}|^2}X^2\right]\geqq 0. \tag{6.11}$$

The most unstable case is when $\lambda=0$, or $(\mathbf{N}\cdot\nabla r)^2=|\mathbf{N}|^2$. Using (6.8) and the constancy of q, one gets the ballooning stability condition

$$\oint\left\{X'^2+\left[\frac{4B_\theta^2B_z^2}{r^2B^2}-\frac{2}{r^2}B_\theta\frac{d}{dr}(rB_\theta)\right]X^2\right\}\geqq 0. \tag{6.12}$$

Our sufficient condition for stability (5.6) was reduced in [20] to the form

$$\oint\left[\frac{1}{|\nabla p|^2}X'^2+2\boldsymbol{\nu}\times\mathbf{J}\cdot\boldsymbol{\nu}'X^2\right]+\text{low rank integral}\geqq 0, \tag{6.13}$$

where $\boldsymbol{\nu}=\nabla p/|\nabla p|^2$, $\boldsymbol{\nu}'=\mathbf{B}\cdot\nabla\boldsymbol{\nu}$. The low rank integral vanishes for modes with $\mathbf{k}\cdot\mathbf{B}\neq 0$, and then (6.13) can be written as

$$\oint\left[X'^2-\frac{2}{r^2}B_\theta\frac{d}{dr}(rB_\theta)X^2\right]\geqq 0. \tag{6.14}$$

Obviously, this sufficient condition is stricter than the ballooning condition (6.12).

7. Conclusion

The article presents a spectral-theoretic treatment of the essential spectrum of MHD. Our investigation of this part of the spectrum was based upon the understanding that it arises as a result of wave motion along rays which never intersect the boundary and which therefore represent a localized phenomenon. We have indeed described the Alfven and ballooning spectra, generally thought of as "unrelated" parts of the spectrum, as the effects of the same plasma waves but with different polarizations. One should not *a priori* exclude the possibility that an additional essential spectrum, of a different origin, exists. However, the proof we gave for the hard core axisymmetric plasma where the Alfven spectrum is all of the essential spectrum, indicates that this possibility is unlikely.

Some questions concerning ballooning modes were left open. Although we formulated a boundary condition on solutions of system (4.9) which will guarantee the existence of such a mode, it is probably possible (and even not difficult) to relax the condition so that (4.9) can be made equivalent to a selfadjoint eigenvalue problem, the spectrum of which yields the ballooning spectrum of MHD. We also did not discuss ballooning modes corresponding to singular surfaces, e.g., separatrices. Finally, we would like to call attention to the expected fundamental effects of nonlinearities on the ballooning modes. The slow magnetosonic wave is no longer one-dimensional and our reasoning for the existence of localized disturbances fails. Nevertheless, in view of the fact that the ballooning modes are approximated by eigenmodes with sufficiently high mode numbers (at least in the axisymmetric case), we believe that the linear theory still gives a meaningful approximation to the more general case.

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