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MATHEMATICAL PROBLEMS IN MAGNETO-FLUID DYNAMICS AND PLASMA PHYSICS⁽¹⁾

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It is my particular pleasure to have the honor to present an address on the subject of magneto-fluid dynamics in the home of the founders of the subject, Alfvén, Lundquist, Herlofson, and their associates. This is a very new subject, not only in mathematics but even in physics. Generally a young subject in physics is found to follow somewhat pragmatic rules of evidence rather than strict mathematical deduction; historically, it is only at a later stage that we find a structure which is sufficiently formalized to be able to qualify as mathematics. But, because of the very rapid pace of modern science, this subject has moved very quickly into the realm of mathematics. This is not to say that the subject is now ready to be embalmed by Nicholas Bourbaki, but rather that there exists, at least in some aspects, a recognizable mathematical structure. Although I cannot hope to cover the entire field in this lecture, I shall attempt to briefly survey most of the mathematically significant high points, mainly work done in New York University at the Courant Institute. We shall examine, on the one hand, places where familiar analytical tools appear in a new setting and, on the other hand, situations where the mathematical structure is not so familiar but where, if history is a guide, we may expect the growth of new legitimate mathematical enterprises. In particular, we shall find that the classical trichotomy of partial differential equations into elliptic, hyperbolic, and parabolic types leaves uncovered a significant part of the subject. Although even the more unusual equations we encounter have solutions in the most general sense [1], the deeper question of the manifold of *all* solutions (described, as usual, in terms of well-posed initial and boundary value problems) is only beginning to be developed.

1. The equations

The physical problem concerns the flow of an electrically conducting medium in the presence of an electromagnetic field. On the one hand, the field exerts a force on the fluid; on the other hand, the motion generates charge and current sources which reflect back on the electromagnetic field. Perhaps the most striking feature of this interaction is the interplay between the scalar pressure of the fluid and the extremely anisotropic electromagnetic stress.

The equations we use are essentially those introduced by Lundquist [2]:

$$\frac{d\rho}{dt} + \rho \operatorname{div} \mathbf{u} = 0, \quad (1.1)$$

$$\rho \frac{d\mathbf{u}}{dt} + \nabla p = \mathbf{J} \times \mathbf{B}, \quad \mathbf{J} = \frac{1}{\mu_0} \operatorname{curl} \mathbf{B}, \quad (1.2)$$

$$\frac{d\eta}{dt} = 0, \quad (1.3)$$

$$\left. \begin{aligned} \frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} \mathbf{E} &= 0 \\ \mathbf{E} + \mathbf{u} \times \mathbf{B} &= 0 \end{aligned} \right\} \frac{\partial \mathbf{B}}{\partial t} + \operatorname{curl} (\mathbf{B} \times \mathbf{u}) = 0, \quad (1.4)$$

$$\operatorname{div} \mathbf{B} = 0. \quad (1.5)$$

We use $d/dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ for the Lagrangian derivative following the fluid. These equations are Galilean invariant as a consequence of the particular form taken for the electromagnetic equations and the electromagnetic forces. Specifically, the displacement current and electric component of the force are suppressed in (1.2); this allows \mathbf{J} to be eliminated in favor of \mathbf{B} .⁽¹⁾ The statement of perfect conductivity, $\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0$, combined with the other Maxwell equation, yields the equation (1.4) for the conservation of magnetic flux. This equation (together with (1.5)), familiar from the theory of vorticity, states that individual magnetic lines are carried with the fluid at the velocity \mathbf{u} , and the flux per unit area (strength of a magnetic tube) is constant.

The complete system consists of the conservation of mass (1.1), of momentum (1.2), of energy (1.3), and of flux (1.4) and (1.5), together with an appropriate equation of state relating ρ , p , and η . Qualitatively, this system describes a magnetic field which is carried with the fluid while reacting back on the motion through the force per volume

$$\mathbf{J} \times \mathbf{B} = \frac{1}{\mu_0} \mathbf{B} \cdot \nabla \mathbf{B} - \nabla (B^2/2\mu_0). \quad (1.6)$$

The mathematical significance of this system of equations lies in the fact that it is symmetric hyperbolic and of conservation form [6]. From the hyperbolicity we have access to familiar existence and uniqueness theorems for initial and boundary value problems; in addition, the concepts of domain of dependence and wave propagation are relevant. From the conservation form, we predict some properties of weak solutions, in particular shock waves. But, despite the general familiarity of the mathematical structure, we shall find the detailed properties to be very strange—principally because of the anisotropy introduced by the magnetic field.

The statement that the system of equations is hyperbolic requires some slight elaboration. The system (1.1)–(1.4) (ignoring (1.5)) can be seen to be symmetric hyperbolic by inspection; (in terms of the variable p rather than ρ , we replace (1.1) by $dp/dt + a^2 \rho \operatorname{div} \mathbf{u} = 0$, where a^2 is the ordinary adiabatic sound speed). If the initial data are taken to satisfy $\operatorname{div} \mathbf{B} = 0$, and if a suitable boundary condition is imposed (on a moving boundary receding from

⁽¹⁾ The material on Plasma Physics has been omitted from the written manuscript; for a survey of this subject we refer to H. Grad, *Modern Kinetic Theory of Plasmas*, in the Proc. of the Fifth International Conference on Ionization Phenomena in Gases, Munich 1961, North-Holland Publishing Co.

⁽¹⁾ A fully relativistic fluid description taken with the complete Maxwell equations yields a very similar general mathematical structure, cf. [3] and [4]. But the combination of conventional fluid equations with the Lorentz-invariant Maxwell equations yields a very undesirable differential system [5].

the fluid), a subclass of solutions is found satisfying $\text{div } \mathbf{B} = 0$ and subject to a certain smaller domain of dependence. Alternatively, we can parametrize $\text{div } \mathbf{B} = 0$ in terms of the two Euler stream functions,

$$\mathbf{B} = \nabla \alpha \times \nabla \beta, \quad (1.7)$$

and replace (1.4) by

$$\left. \begin{aligned} \frac{d\alpha}{dt} &= 0, \\ \frac{d\beta}{dt} &= 0. \end{aligned} \right\} \quad (1.8)$$

In the variables $(\varrho, \eta, \mathbf{u}, \alpha, \beta)$ the system (1.1)–(1.3) and (1.8) is hyperbolic, but it is not recognizably symmetric.

The theory of this hyperbolic system is outlined in Sect. 2. Of course, the statement that the time-dependent system is hyperbolic contains no implication with regard to problems of steady flow or static equilibrium; these will be investigated separately in Sects. 4 and 6 respectively. Each of these general formulations (time-dependent, steady flow, static equilibrium) is followed by a special case which can be analysed in more detail (transverse flow in Sect. 3, aligned flow in Sect. 5, the static free boundary in Sect. 7). These special cases all feature the property that they are exact mathematical analogues of problems in fluid dynamics (this is even true of the general static equilibrium). The transverse flow analogue allows the transfer of a large body of fluid dynamic theory to magnetofluid dynamics; in the aligned flow problem this transfer is limited by singularities in the transformation to the analogue variables; in the general static equilibrium, with the recent development of the magnetic version, the analogue is more appropriately used in the opposite direction! Sect. 7 also contains a complete description of the stability of magnetic free boundary configurations.

2. Characteristics and wave propagation

The qualitative and many quantitative features of a hyperbolic system are elucidated by an examination of the characteristic cones. For the system (1.1)–(1.5), these were computed by Friedrichs [6]. There are two degenerate first order characteristic lines immediately recognizable, viz., the particle paths for η and the lines $x = \text{constant}$ for $\text{div } \mathbf{B}$. In addition there are three distinct nontrivial cones corresponding to a system of sixth order. In Fig. 1 are shown the characteristic loci, i.e., the traces of the characteristic cones at unit time. The figures are surfaces of revolution about the direction of \mathbf{B} . These loci are determined by the ratio of two parameters (local properties of the medium), the adiabatic sound speed

$$a = (\partial p / \partial \varrho)^{1/2} \quad (2.1)$$

and the Alfvén speed

$$A = B / (\mu_0 \varrho)^{1/2}. \quad (2.2)$$

The characteristic loci are subject to several interpretations. First, the outer convex locus identifies the domain of dependence. Each of the figures describes the propagation of discontinuities, either of derivatives of any order or of small discontinuities in the variables themselves (weak shocks).

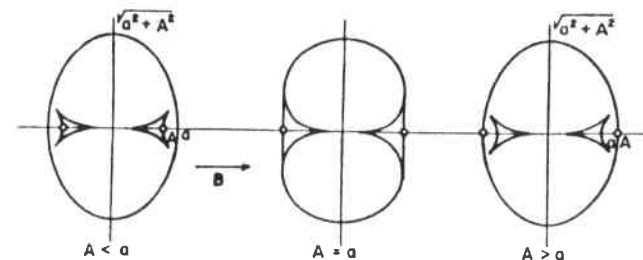


Fig. 1. Characteristic loci.

The various loci in Fig. 1 also separate different analytic pieces of the fundamental solution; the figure is a snapshot of the result of a point explosion. Finally, the various loci in Fig. 1 are to be used instead of spheres (as in an isotropic medium) in the application of Huyghen's construction of "wavelets". We shall also see in Sect. 4 how the characteristics in a steady flow (Mach cones) can be constructed from this figure.

It will be convenient to consider three types of question in order of complexity. First we shall examine the propagation of plane waves in a linear perturbation of a constant state. This is an essentially algebraic problem. Then we observe the behavior of discontinuities in general solutions or, the equivalent, the propagation of small discontinuous wave fronts superposed on an arbitrary solution. This involves the study of a single first order equation (Hamilton–Jacobi) whose theory reduces to ordinary differential equations. Only then do we examine properties of the full (sixth order) system of partial differential equations.

The propagation of plane waves is completely described by the *normal speed* diagram, Fig. 2. In a given direction are plotted radii proportional to the normal speeds of plane waves with the given orientation. There are three speeds (each with a double multiplicity $\pm c$) in each direction, the *intermediate speed*

$$c^2 = A^2 \cos^2 \theta \quad (2.3)$$

and the *slow* and *fast* speeds (together referred to as *compressive*) given by

$$c^4 - c^2(a^2 + A^2) + a^2 A^2 \cos^2 \theta = 0, \quad (2.4)$$

where θ is the angle between the magnetic field and the plane wave normal. In every direction the slow and fast speeds bracket the intermediate speed. The fast wave is only moderately anisotropic (the maximum ratio of major to minor axis is 2^4). Both the intermediate and slow waves are extremely anisotropic and apparently do not propagate at all in the direction perpendicular to \mathbf{B} . We remark that the characteristic loci, Fig. 1, can be obtained from the normal speed loci, Fig. 2, by taking envelopes of the indicated plane waves; (the compressive loci are given by a certain tenth degree algebraic surface [7]).

Next we turn to the propagation of arbitrary wave fronts (discontinuity surfaces) [8], [9] and examine the characteristic loci. Again we observe that the outer fast (or *magnetosonic*) locus is somewhat anisotropic but is otherwise unexciting. The locus for the intermediate (or *transverse* or *Alfvén*)

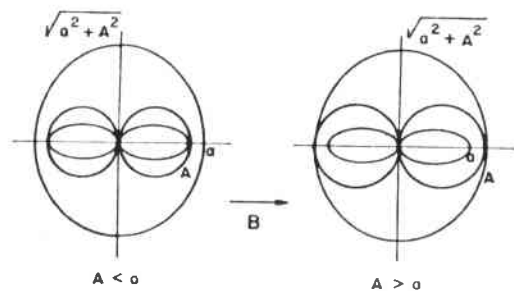


Fig. 2. Normal speed diagram.

wave is extremely degenerate and consists of the two encircled points at $\pm A$ together with the included line segment. This degenerate locus represents strictly one-dimensional propagation along magnetic lines. A transverse wave front is forever contained within a fixed flux tube (but its shape can be distorted in an inhomogeneous medium or a nonuniform field). The slow locus is even more singular. It consists of two disconnected cusped regions together with the intercepted line segment (Fig. 1). The qualitative significance is most easily appreciated by the example, in Fig. 3, of the evolution of an originally spherical slow wavefront in a uniform medium. The slow front lags behind the transverse front (which remains spherical) so long as it is smooth. It overtakes the transverse wave (in apparent contradiction to the implications of the normal speed loci, Fig. 2) after singularities develop in the wave front. Summarizing, the intermediate wave propagates one-dimensionally along flux tubes, the slow wave front is confined to a certain cone surrounding the magnetic field direction, and only the fast wave front propagates in all directions (but not isotropically).

The classical theory of ray optics can be employed to make more quantitative this description of the propagation of wave fronts [8]. A discontinuity or characteristic surface $\phi(x, t) = 0$ satisfies the partial differential equation (cf. (2.3) and (2.4))

$$(\phi_t - \mathbf{A} \cdot \nabla \phi)^2 [\phi_t^4 - (a^2 + A^2) \phi_t^2 (\nabla \phi)^2 + a^2 (\nabla \phi)^2 (\mathbf{A} \cdot \nabla \phi)^2] = 0, \quad (2.5)$$

where \mathbf{A} is the vector Alfvén speed pointing in the direction of \mathbf{B} . Each root of (2.5) is a first order partial differential equation which, in turn, has characteristics which can be described as curves rather than cones; these

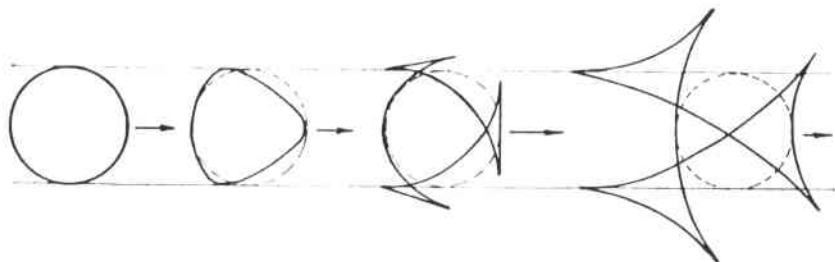


Fig. 3. Slow wave front.

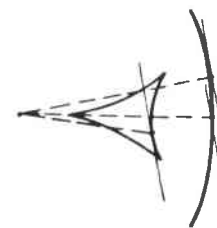


Fig. 4. Ray construction.

are the *rays* or *bicharacteristics*. A given initial discontinuity front splits into six components traveling in both directions along each of the three sets of rays. For each of the categories—slow, intermediate, fast—in a given problem we have a direction field of rays (possibly multiply covering the plane), along which the wave fronts propagate. There is also for each ray an ordinary differential equation governing the variation of the magnitude of the discontinuity front [8]. The singular behavior which results from multi-valued direction fields requires special handling [10].

The ray direction for a given orientation of the wave front is determined as indicated in Fig. 4. For propagation in a homogeneous medium (equation (2.5) has constant coefficients), the ray construction, Fig. 4, is valid in the large. An element of a wave front remains parallel to itself and moves the distance shown along the appropriate straight-line ray; (this remark yields a simple quantitative construction of Fig. 3).

The separation of wave fronts into slow, intermediate, and fast waves is, of course, not possible for general solutions of the original differential system.⁽¹⁾ Even in the case of a linear perturbation about a constant state (constant coefficients), a fast wave front will leave behind it a trail of slow and transverse debris, and a slow wave front will send ahead of it fast and transverse signals (initiated by a wave front of higher order). Nevertheless, in a certain sense the intermediate wave can be partly separated from the compressive wave in some cases [11], [2], [9]. To observe this, consider a linear perturbation about a constant state characterized by parameters ϱ_0 and B_0 . The dimensionless perturbation variables

$$\left. \begin{aligned} \sigma &= \varrho / \varrho_0 \\ \beta &= \mathbf{B} \cdot \mathbf{B}_0 / B_0^2 \end{aligned} \right\} \quad (2.6)$$

satisfy the fourth order system

$$\left. \begin{aligned} \frac{\partial^2 \sigma}{\partial t^2} &= a_0^2 \left(\frac{\partial^2 \sigma}{\partial x^2} + \frac{\partial^2 \sigma}{\partial y^2} + \frac{\partial^2 \sigma}{\partial z^2} \right) + A_0^2 \left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} \right) \\ \frac{\partial^2 \beta}{\partial t^2} &= a_0^2 \left(\frac{\partial^2 \sigma}{\partial y^2} + \frac{\partial^2 \sigma}{\partial z^2} \right) + A_0^2 \left(\frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} \right) \end{aligned} \right\} \quad (2.7)$$

which has as characteristics the slow and fast cones. On the other hand, the components perpendicular to B_0 ,

⁽¹⁾ The decomposition is possible at the expense of a loss of domain of dependence attributes, e.g., by a Fourier expansion.

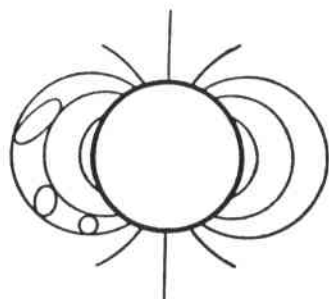


Fig. 5. Whistlers.

$$\left. \begin{aligned} \mathbf{u}' &= \mathbf{u} - \mathbf{B}_0(\mathbf{u} \cdot \mathbf{B}_0)/B_0^2 \\ B_0 \beta' &= B - B_0 \beta \end{aligned} \right\} \quad (2.8)$$

satisfy the inhomogeneous one-dimensional wave equation

$$\left. \begin{aligned} \frac{\partial \mathbf{u}'}{\partial t} - A_0^2 \frac{\partial \beta'}{\partial x} &= -A_0^2 \nabla' \beta - a_0^2 \nabla' \sigma \\ \frac{\partial \beta'}{\partial t} - \frac{\partial \mathbf{u}'}{\partial x} &= 0. \end{aligned} \right\} \quad (2.9)$$

Although (2.9) has the one-dimensional intermediate characteristics, the inhomogeneous term, found from (2.7), is different from zero within the larger compressive domain of dependence.

Despite the fact that an arbitrary initial state cannot be resolved into intermediate and compressive components, each with the appropriate domain of dependence, one can find large classes of initial data that propagate purely as a one-dimensional transverse wave or as a compressive wave. For example, taking $\sigma = \beta = \partial \sigma / \partial t = \partial \beta / \partial t = 0$ initially yields a purely transverse solution of (2.9) for \mathbf{u}' and \mathbf{B}' ($\partial \sigma / \partial t = \partial \beta / \partial t = 0$ can be satisfied by taking $\text{div } \mathbf{u} = 0$ and $\mathbf{u} \cdot \mathbf{B}_0 = 0$ initially). This simple separation does not occur with variable coefficients. As an illustration we consider the phenomenon of "whistlers" in the earth's magnetic field (taking for granted the relevance of these equations). A transverse wave front will propagate along a magnetic tube without radial attenuation and will, under appropriate conditions, be reflected at each end for an indefinite lifetime (Fig. 5). But there is no purely transverse solution of the full system of partial differential equations in a case of variable coefficients. Thus there must be some coupling of the disturbance with compressive waves which radiate away to infinity. The precise relation between the wave front which persists and the complete disturbance which apparently radiates has not yet been established.

An explicit formula has been obtained for the two-dimensional fundamental solution for the case of constant coefficients [12]. In three dimensions, some asymptotic properties of the fundamental solution have been obtained by Fourier analysis [13], [14], but most of these results are more easily obtained using the simpler ray optics techniques.

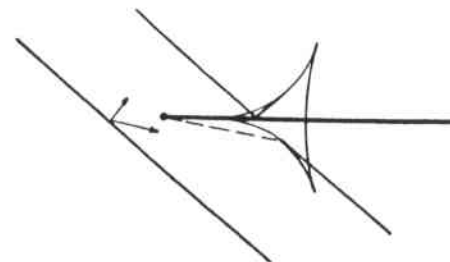


Fig. 6. Wave incident on a slit.

As a final illustration of the unusual wave motions embraced by this theory, we quote an explicit solution for the motion of a plane discontinuity front incident on a semi-infinite slit [15]. The qualitative behavior for a slow wave front is indicated in Fig. 6 (the unperturbed magnetic field is parallel to the slit). Because the ray direction (which is the direction of propagation of the wave) is not normal to the front, we observe that both the shadow and the reflected wave lie on the "wrong" sides of the slit.

3. Transverse flow [16]

There is a simple special configuration which, by appropriate identification of variables, can be made into an exact mathematical analogue with a very large class of ordinary fluid flows. The intriguing feature of this analogue is that it maps a large portion of the entire body of fluid dynamic theory (viz., all two-dimensional nonsteady flows which make no specialization with regard to the equation of state) onto a miniscule corner of magneto-fluid dynamics.

Specifically, we assume that all variables depend on (x, y, t) , the flow field is two-dimensional (u_x, u_y) , and the magnetic field is one-dimensional, having the single component B_z in the ignorable direction. The essential simplification is due to the fact that the Maxwell stress tensor is equivalent to an isotropic scalar pressure in the plane of the flow; cf. (1.6) where $(\mathbf{B} \cdot \nabla) \mathbf{B} = 0$. With this remark, the momentum equation

$$\rho \frac{d\mathbf{u}}{dt} + \nabla p_* = 0 \quad (3.1)$$

is formally the same as in fluid dynamics in terms of the "total pressure"

$$p_* = p + B^2/2\mu_0. \quad (3.2)$$

In the flux equation (1.4), we remark that \mathbf{B} can be reinterpreted as a scalar; in this notation (1.4) takes the form

$$\frac{dB}{dt} + B \text{ div } \mathbf{u} = 0. \quad (3.3)$$

In this geometry, conservation of mass (1.1) is identical to conservation of

flux, since the two-dimensional "volume" element is the same as the element of area. An alternative form to (3.3) is therefore

$$\frac{d\eta_*}{dt} = 0, \quad (3.4)$$

where

$$\eta_* = B/\varrho. \quad (3.5)$$

Finally, just as the conventional entropy equation (1.3) is equivalent to an energy equation, (3.4) can be replaced by

$$\varrho \frac{de_*}{dt} + p_* \operatorname{div} \mathbf{u} = 0 \quad (3.6)$$

in terms of the "total energy" per mass

$$e_* = e + B^2/2\mu_0\varrho. \quad (3.7)$$

The complete system of equations is the conservation of mass (1.1) and of momentum (3.1) together with the two particle path invariants η and η_* , (1.3) and (3.4).

As it stands, this system is very similar to the ordinary fluid equations (there are the same characteristic cones, with the exception of an extra particle path), and the similarity can be made into an identity by a slight further specialization. We consider three cases:

- (1) $\eta_* = \text{constant}$,
- (2) $\eta = \text{constant}$,
- (3) $\eta_* = f(\eta)$.

In the first case, we take $\eta_* = B/\varrho$ to be a constant over the entire flow instead of along each particle path. The mass, momentum, and entropy equations are exactly as in fluid dynamics provided that we complete the formulation with an equation of state relating (p_*, ϱ, η) , viz.,

$$p_*(\varrho, \eta) = p(\varrho, \eta) + \eta_*^2 \varrho^2 / 2\mu_0. \quad (3.8)$$

It is a simple matter to verify that this equation of state is compatible with conventional thermodynamic convexity requirements, and, in particular, the adiabatic sound speed is given by

$$a_*^2 = \partial p_* / \partial \varrho = a^2 + A^2. \quad (3.9)$$

In the second case listed above, isentropic flow with η constant, we obtain a conventional system of fluid equations in (p_*, ϱ, η_*) , taking η_* as the fluid entropy. Again, we need only an equation of state, in this case

$$p_*(\varrho, \eta_*) = p(\varrho) + \eta_*^2 \varrho^2 / 2\mu_0. \quad (3.10)$$

It is again possible to verify that the thermodynamic structure is standard, and the sound speed takes the same form (3.9).

In the third case, we interpret $\eta_* = f(\eta)$ quite generally to mean that the surfaces $\eta = \text{constant}$ coincide with $\eta_* = \text{constant}$. For example, in one-

dimensional nonsteady flow, variables (x, t) , this is no specialization; the relation $\eta_* = f(\eta)$ is obtained by eliminating x between the given initial values of $\eta_*(x)$ and $\eta(x)$. Eliminating η_* in favor of η , we obtain a conventional fluid system with the equation of state

$$p_*(\varrho, \eta) = p(\varrho, \eta) + f^2(\eta) \varrho^2 / 2\mu_0 \quad (3.11)$$

and the same sound speed (3.9).

The thermodynamic structure in each of the above three special cases can be derived from the elementary identity

$$T d\eta + (B/\mu_0) d\eta_* = de_* + p_* d(1/\varrho). \quad (3.12)$$

In the first example the analogue temperature is $T_* = T$, in the second example it is $T_* = B/\mu_0$, and in the third example it is $T_* = T + Bf'(\eta)/\mu_0$.

The conventional laws of Bernoulli and the conservation of circulation are easily derived. The restricted Bernoulli's law (for a steady flow) is valid in complete generality,

$$h_* + \frac{1}{2} u^2 = a(\psi), \quad (3.13)$$

where a is constant on each streamline (as are η and η_*) and the "total enthalpy" is

$$h_* = e_* + p_*/\varrho = h + B^2/\mu_0\varrho. \quad (3.14)$$

Conservation of circulation results under the condition that p_* depend on ϱ alone; this is guaranteed, for example, if both η and η_* are constant throughout the flow. Under the same restrictions potential flow

$$\mathbf{u} = \nabla \phi \quad (3.15)$$

is validated, and we obtain the strong form of Bernoulli's law

$$\frac{\partial \phi}{\partial t} + h_* + \frac{1}{2} u^2 = a(t). \quad (3.16)$$

From the analogy with ordinary fluid dynamics we can transfer the relevant theory of shock waves, simple waves and interactions, flows around objects and bends, Riemann invariants, cavitation, etc. But we may not transfer essentially three-dimensional properties related to stability and turbulence, nor dissipative properties connected with boundary layers, etc.

4. Steady flow

The system of equations is as before, (1.1)–(1.5), but this time with the time derivatives suppressed. As a result of this specialization we find the scene dominated by more unusual mathematical structures. A typical problem involves a higher order system which is partly elliptic and partly hyperbolic; e.g. a sixth order system with two real characteristic cones and one imaginary. A more conventional structure which is encountered is a fully hyperbolic system, but purely elliptic problems are not found (except

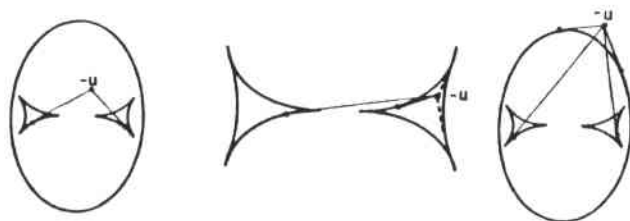


Fig. 7. Steady flow characteristics.

under very special circumstances, see Sect. 5). The classical fluid dichotomy into subsonic (elliptic) and supersonic (hyperbolic) does not generalize. One reason is that the fast and slow characteristics are two real traces of a single complex algebraic surface, and the concepts subsonic and supersonic cannot be unambiguously applied to the slow and fast speeds individually. As to the transverse wave, since propagation is one-dimensional, it is only possible to distinguish subsonic from supersonic in very special (essentially one-dimensional) flows.

The steady-flow characteristic (Mach) cones are obtained from the characteristic loci (Fig. 1) by a simple geometrical construction [16], [17], [18]. With reference to a specific location in a general steady flow, we take the non-steady locus, Fig. 1, with the appropriate local values of a and A , and mark the terminus of the reversed local flow velocity vector $-u$, on this diagram (Fig. 7). The real characteristic cones are obtained by drawing tangents from $-u$ to the characteristic loci;⁽¹⁾ in two dimensions we obtain lines and in three dimensions cones.

The intermediate characteristic is a degenerate triangular disc (collapsed cone) in three dimensions and a pair of lines in two dimensions. For the fast locus we obtain a real cone (or pair of lines) when u is outside the locus. The slow locus yields two distinct real cones if u lies inside either of the two cusped regions and a single real cone everywhere else; (a slow cone which has a trace on both cusped regions also contains a triangular "collapsed" disc which joins the two pieces). Summarizing, the sixth order system (neglecting the entropy-induced real streamline characteristic) has three real cones if the velocity lies outside the fast locus or inside the cusped regions (only the former is clearly supersonic). There are only two real cones (and one elliptic imaginary cone) if the velocity falls between the cusped regions and the fast locus. One can distinguish a "fast" from a "slow" Mach cone when P lies within a cusped region by analytic continuation, but it is not clear that this has any significance.

Since there is no substantial general theory, we shall be content to mention a few illustrative examples. First we recall the elementary problem of the linearized supersonic gas flow past a thin airfoil. This is translated as the problem of solving the wave equation in a slit domain (Fig. 8a) with a given boundary condition on the slit (describing the shape of the airfoil). A unique result is obtained only after one imposes an additional regularity condition at infinity upstream. But the mathematical boundary value problem would be equally well-posed if the regularity condition were im-

⁽¹⁾ In doubtful or singular cases, one should take the envelope of tangent planes rather than tangent lines.

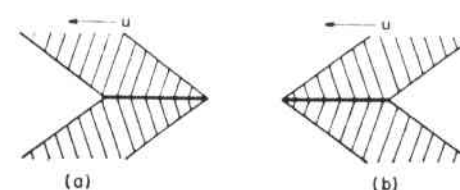


Fig. 8. Airfoil flow.

posed at infinity downstream (Fig. 8b), and one can even satisfy the boundary condition on the slit with an arbitrary linear combination of the two solutions indicated in Figs. 8a and 8b. Of course, one must turn to the time-dependent problem to obtain the correct regularity condition at infinity. The interesting point is that with a certain choice of parameters, there is a linearized magnetic flow problem past an airfoil which reduces to the same mathematical problem just described, but the correct choice of the domain of dependence lies *upstream* of the body [17].

Fully hyperbolic problems can be solved using standard means in two dimensions (if linear, using Riemann invariants [16]), and should offer no qualitative surprises in three dimensions [19], except for the unusual orientation and strange shapes of the Mach cones [18]. Although the equations are not unfamiliar, experience from fluid dynamics may be irrelevant. For example, the flow around a bend cannot be solved in terms of shocks or simple waves. The reason is that the boundary condition (at an insulating wall) that the magnetic field is continuous implies that we must solve a static magnetic field problem (elliptic) in the complementary domain and match it to the fluid problem, rather than apply only a local boundary condition [20]. Another feature of the magnetic problem is that, with decreasing magnetic field, the flow does not approach the ordinary fluid flow uniformly [16].

"Mixed" elliptic and hyperbolic problems are another story. The linearized flow past a thin airfoil in such a mixed regime has been successfully carried out [21]. This solution is based on the observation that the variables can be separated into two sets which satisfy elliptic and hyperbolic equations respectively. Since the boundary conditions are applied to certain linear combinations of the elliptic and hyperbolic variables, it becomes necessary to solve an integral equation on the boundary to obtain the solution. We recall that the classical elliptic fluid problem in this slit domain (subsonic flow) is not uniquely determined without specifying a value for the circulation around the body. The magnetic problem is similarly nonunique, but consideration of the singularities leads to a plausible requirement that applies a Kutta condition sometimes at the leading edge and sometimes at the trailing edge, depending on the parameters.

To complete the picture we remark on the possibility that we may encounter transitions of type in different domains ("mixed" or transonic flows in aerodynamic terminology). The combination of mixed equations (partly elliptic, partly hyperbolic in a given domain) together with changes of type from one part of the domain to another seems to offer limitless possibilities for complications. A special case, which is simple enough to yield some information will be discussed in the next section. From the characteris-

tic diagram Fig. 7, we observe that both the transverse and slow Mach cones collapse upon the streamline when u is parallel to B . If this alignment is satisfied in the entire flow, then certain variables can be integrated out of the problem (just as we can take $\eta = \text{constant}$ in the entire flow), and we are left with a second order system. But from Fig. 7 we easily see that this second order system yields three transitions—elliptic to hyperbolic to elliptic to hyperbolic—as u increases from zero to a large value.

5. Aligned flow

Consider the special case of a flow in which the vectors u and B are aligned throughout the flow. From (1.4) we see that this special restriction is significant only in a steady flow. For simplicity, we look first at an incompressible flow and set

$$B = \sigma u \quad (5.1)$$

Since $\text{div } B = \text{div } u = 0$, we conclude that σ is constant on a streamline. By inspection of

$$\rho(u \cdot \nabla)u + \nabla(p + B^2/2\mu_0) = \frac{1}{\mu_0}(B \cdot \nabla)B \quad (5.2)$$

we recognize the analogue variables

$$\left. \begin{aligned} p_* &= p + B^2/2\mu_0 \\ \rho_* &= \rho - \sigma^2/\mu_0 \end{aligned} \right\} \quad (5.3)$$

which satisfy the ordinary fluid system

$$\left. \begin{aligned} \rho_*(u \cdot \nabla)u + \nabla p_* &= 0 \\ \text{div } u &= 0. \end{aligned} \right\} \quad (5.4)$$

It is not necessary to refer to an equation of state in an incompressible problem. The pressure can be eliminated from the problem unless it appears explicitly, e.g. in a boundary condition as in the problem of water waves under gravity. The analogy extends to such problems with a free surface in air since the "total" pressure p_* acts exactly as does p at interface which is a flux surface. Of course, a specifically electromagnetic boundary condition (e.g., that caused by a vacuum magnetic field in the exterior domain) would destroy the analogy.

A slightly more complicated identification of variables yields a similar analogue in a compressible parallel flow [22], [16]. This time $\text{div}(\rho u) = 0$ and we set

$$B = \alpha \rho u, \quad (5.5)$$

where α is constant on a stream line. If we insist on

$$\rho_* u_* = \rho u, \quad (5.6)$$

we immediately detect, from (5.2), the identifications

$$\left. \begin{aligned} u_* &= u(1 - \alpha^2 \rho/\mu_0) = u(1 - 1/M^2), \\ \rho_* &= \rho/(1 - \alpha^2 \rho/\mu_0) = \rho/(1 - 1/M^2), \end{aligned} \right\} \quad (5.7)$$

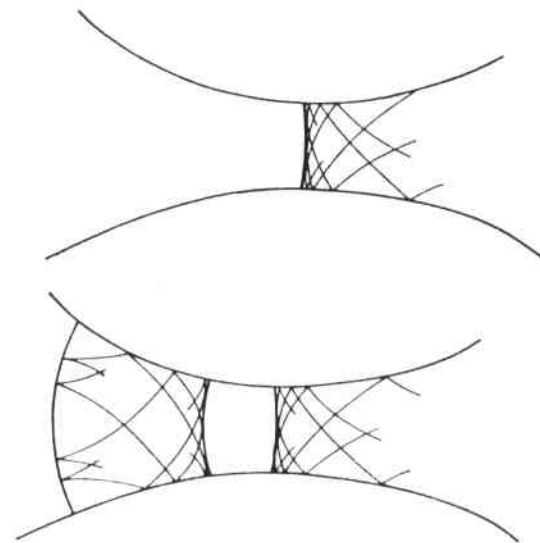


Fig. 9. Nozzle flow.

where M is the Alfvén "Mach" number (a function of ρ and α)

$$M^2 = u^2/A^2 = \mu_0/\alpha^2 \rho. \quad (5.8)$$

For a compressible flow we do require an equation of state. We are forced to accept an equation giving p_* as a function of ρ_* on each individual streamline, rather than universally. It is obtained by appealing to Bernoulli's law which is valid in the original variables (and also in the new variables with a suitable definition of the enthalpy),

$$h + \frac{1}{2} u^2 = h_0, \quad (5.9)$$

where h_0 is constant on a streamline. The "equation of state" is

$$p_*(\rho_*; \eta, \alpha, h_0) = p(\rho, \eta) + \frac{\alpha^2}{\mu_0} [h_0 - h(\rho, \eta)] \rho^2 \quad (5.10)$$

after ρ is eliminated in favor of ρ_* and α from (5.7). The analogue sound speed is

$$a_*^2 = \partial p_*/\partial \rho_* = (1 - 1/M^2)^2 [a^2(1 - 1/M^2) + A^2], \quad (5.11)$$

and the entire analogue thermodynamic structure can be computed [16], but it is evidently unconventional for $M < 1$ (ρ_* is negative and the analogue flow is in the opposite direction to the actual flow) and even more so for $M^2 < a^2/(a^2 + A^2)$ (the analogue sound speed is imaginary).

Thus we see that the analogue is useful only to a limited extent in studying parallel flow. In particular, it sheds no light on the unusual transitional flows mentioned at the close of Sect. 4. Let us, as an example, consider the flow in a converging-diverging channel. In Fig. 9 we show the qualitative features of a normal transitional flow in a gas and a conjectured magnetic

parallel flow; the characteristics are sketched in the hyperbolic regions. We recall that there are three transitions—elliptic to hyperbolic to elliptic to hyperbolic; these occur at $u^2 = a^2$, $u^2 = A^2$, $u^2 = a^2 A^2 (a^2 + A^2)$. The second transition, $M^2 = 1$, is a singularity of the transformation (5.7), and the third transition indicates the onset of the imaginary analogue sound speed. Therefore the normal sonic transition, $u^2 = a^2$, is the only one that is not complicated by the introduction of the analogue variables.

Some formal analysis has been done indicating that the flow sketched in Fig. 9 may indeed be legitimate [23], but a mathematically satisfactory answer to this multiple transition question is not yet at hand.

It is interesting to note that the elementary hydraulic approximation to the channel flow in the original variables gives the identical answer as in ordinary gas dynamics; the only facts that are employed in this approximation are the conservation of mass, constant entropy, and Bernoulli's law. In other words, provided that this unusual and difficult flow problem has a solution, the answer is closely approximated by the well-known ordinary gas flow.

A "parallel-transverse" flow in which a two-dimensional aligned flow has a third component of B superposed (in the ignorable direction), can also be transformed into an ordinary fluid problem [16]. But this time the flow blows up near $M = 1$ even in the original variables. This indicates that a transition across $M = 1$ in the simple aligned flow is quite special.

6. Static equilibrium and force free fields [24]

The problem of static equilibrium is in principle a special case of the previous theories, but they are, in fact, completely unrelated. Fortunately, in the present case there is an exact analogue with a problem in fluid dynamics. But progress in the magnetic problem has carried it beyond the earlier status of the fluid version, so the analogy is, in essence, reversed. The equations are

$$\left. \begin{aligned} \nabla p &= \mathbf{J} \times \mathbf{B}, \\ \mathbf{J} &= \frac{1}{\mu_0} \text{curl } \mathbf{B}, \\ \text{div } \mathbf{B} &= 0. \end{aligned} \right\} \quad (6.1)$$

By inspection we see that the pressure is constant along a magnetic line and also along a current line; or we can say that a pressure surface is covered by magnetic lines and by current lines. In the form

$$\left. \begin{aligned} \nabla p_* &= \frac{1}{\mu_0} (\mathbf{B} \cdot \nabla) \mathbf{B} \\ p_* &= p + B^2/2\mu_0 \\ \text{div } \mathbf{B} &= 0 \end{aligned} \right\} \quad (6.2)$$

we recognize an exact equivalence with the steady rotational flow of an incompressible fluid⁽¹⁾

(1) This analogue was first noted by Lundquist [2].

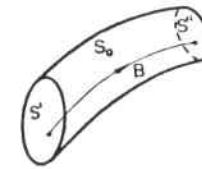


Fig. 10. Tubular equilibrium.

$$\left. \begin{aligned} \rho(\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \hat{p} &= 0, \\ \text{div } \mathbf{u} &= 0. \end{aligned} \right\} \quad (6.3)$$

The velocity and magnetic field are identified (within a factor); the "total" pressure p_* is identified with $-\hat{p}$ or, what is equivalent, $-\hat{p}$ is identified with the Bernoulli constant $\hat{p} + \frac{1}{2}\rho u^2$.

By combining plausible arguments with proved special cases, we are able to discuss the theory of this system. Consider the tubular domain D of Fig. 10. Roughly speaking, a well-posed boundary value problem will result from the specification of B_n on the boundary of D (say $B_n > 0$ on S' , $B_n = 0$ on S_0 , and $B_n < 0$ on S''), and also the specification of p on the two ends S' and S'' . To be more precise, we suppose that values of B_n and p are given at the end S' . We then take an arbitrary smooth one-one mapping of S' onto S'' . The mapping is used to carry the given values of p over to S'' , and B_n is multiplied by the Jacobian of the mapping so that $B_n dS' = B_n' dS''$. The statement is then that a solution of (6.1) exists which takes the given boundary values of B_n and p and also has the property that the two ends of each magnetic line are related by the given mapping between S' and S'' .

The significance of the identification of the ends of the magnetic lines can be made clear by two examples. If the lines $p = \text{constant}$ cover S' simply (Fig. 11a), then the specification of B_n independently on S' and S'' (subject to $\int B_n dS' + \int B_n' dS'' = 0$) and the specification of compatible p values at both ends (i.e.,

$$\int_{p < p_*} B_n dS' + \int_{p < p_*} B_n' dS'' = 0$$

for all p_0) uniquely determines a mapping of S' onto S'' . On the other hand, if the lines $p = \text{constant}$ are closed curves (Fig. 11b), then specification of compatible boundary values of B_n and p at both ends does not fix the mapping. In addition to these boundary values, we also specify a given "twist" from S' to S'' of each constant p curve.

The necessity for this twist becomes clear when we note that a vector field \mathbf{B} taken from a solution of (6.1) is, in a certain generalized sense, a

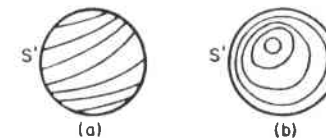


Fig. 11. Pressure boundary condition.

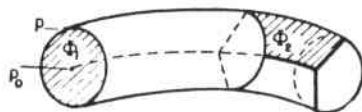


Fig. 12. Toroidal equilibrium.

harmonic vector field within a pressure surface [24]. In particular, such a surface harmonic is uniquely determined by boundary values when the pressure surface is simple, Fig. 11a, but it requires the specification of certain *periods* (e.g., circulation) when the pressure surface is not simple, Fig. 11b.

Making use of this concept, we are led to a formulation in a toroidal domain. In this case there are no boundaries (except the trivial one on which $B_n = 0$) and we can specify *only* periods on each pressure surface. Specifically, a solution of the system (6.1) with nested toroidal pressure surfaces can be expected to be determined by the specification of two functions $\Phi_1(p), \Phi_2(p), p_0 < p < p_1$, which give the total flux intercepted in the two independent directions between the axis p_0 (a closed curve) and a given surface p (Fig. 12). An alternative formulation is to fix the two fluxes $\Phi_1(V), \Phi_2(V)$ as functions of the volume V which parametrizes the nested toruses (p will be determined only within an added constant). It should be remembered that the family of nested toruses must be found as well as the associated \mathbf{B} and p ; only the outer boundary is known beforehand.

The special case of (6.1) with $p = 0$,

$$\left. \begin{aligned} \text{curl } \mathbf{B} \times \mathbf{B} &= 0, \\ \text{div } \mathbf{B} &= 0, \end{aligned} \right\} \quad (6.4)$$

is called a *Force-Free Field* (in the fluid version, the Bernoulli constant is constant over the entire flow and we have a Beltrami field). This is a significant concept in the theory of ionized gases because it is possible for a gas to be very rarefied ($p \sim 0$) and still carry an appreciable current. In the tubular domain, Fig. 10, the statement of a well-posed problem is the same as before, omitting the specification of p . A mapping of S' onto S'' is given and also the value of B_n ; a solution will take the given value of B_n and will satisfy the end coordination. In a torus, p serves only as a parameter to identify the nested toruses. There is no difference between the pressure and volume parametrization, and the parametric representation $\Phi_1(p), \Phi_2(p), p_0 < p < p_1$, is equivalent to specifying Φ_1 as a function of Φ_2 over a specified range $0 < \Phi_2 < \Phi^*$.

It is illuminating to compare a general equilibrium (6.1), a Force-Free Field (6.4), and a harmonic vector ($\text{curl } \mathbf{B} = 0, \text{div } \mathbf{B} = 0$) in the domain between two concentric toruses. For the system (6.1) we specify $\Phi_1(p)$ and $\Phi_2(p)$; for the system (6.4) we specify only $\Phi_1(\Phi_2)$; for the harmonic field we specify two constants, Φ_1 and Φ_2 .

This conjectured existence theory quoted above is based on a study of the characteristics of the system (6.1), on a number of proved special cases, and on deductions from a variational formulation of the problem. The system has both real and imaginary characteristics. But in contrast with the mixed elliptic and hyperbolic problems which appear in steady flow problems, the

real cones are all degenerate and fall on the streamlines. This suggests the specification of B_n as a boundary condition and p as a separate "initial condition" at each end but the conditions involving periods on nonsimple pressure surfaces can only be discovered after the interpretation of \mathbf{B} as a surface harmonic. The solvable special cases are two-dimensional and axially symmetric problems (and include the toroidal case by imposing periodicity on an axially symmetric problem). The variational formulation for the tubular domain, Fig. 10, is to make the variational function

$$\int (B^2/2\mu_0 - p) dV \quad (6.5)$$

stationary subject to an admissible class of functions (B, p) satisfying

$$\left. \begin{aligned} \text{div } \mathbf{B} &= 0 \\ \mathbf{B} \cdot \nabla p &= 0 \end{aligned} \right\} \quad (6.6)$$

together with the boundary conditions on B_n and p and the mapping restriction at the ends of the magnetic lines. Under these conditions an interior variation yields the Euler-Lagrange equation $\nabla p = \mathbf{J} \times \mathbf{B}$, and there is no contribution from the boundary variation.

In the toroidal problem, the same variational function (6.5) is used and an admissible (B, p) satisfies (6.6) as well as the given flux requirements $\Phi_1(p), \Phi_2(p)$. In other words, an admissible pair (B, p) is obtained from a set of nested toruses by assigning values of p and a tangential magnetic field compatible with the given flux. In the case of given $\Phi_1(V), \Phi_2(V)$, we use

$$F = \int B^2/2\mu_0 dV \quad (6.7)$$

as the variational function. Given a set of concentric toruses, the parameter V is known and we choose a tangential field B compatible with the given fluxes. The pressure does not enter this variational formulation and takes the form of a Lagrangian multiplier in (6.5) if we wish to specify $\Phi_1(p), \Phi_2(p)$ instead of $\Phi_1(U), \Phi_2(U)$.

As a final illustration, let us consider the classical Dirichlet principle for the tubular domain, Fig. 10. A harmonic vector is uniquely determined by the given boundary condition on B_n . Variationally, we minimize F in (6.7) subject to the admissibility condition

$$\left. \begin{aligned} \text{div } \mathbf{B} &= 0 \\ B_n &= \text{given} \end{aligned} \right\} \quad (6.8)$$

to obtain the variational condition

$$\text{curl } \mathbf{B} = 0. \quad (6.9)$$

A simple way of incorporating the condition $\text{div } \mathbf{B} = 0$ is to introduce the vector potential

$$\mathbf{B} = \text{curl } \mathbf{A}, \quad (6.10)$$

after which (6.9) is obtained by the usual integration by parts. But, if we parametrize $\text{div } \mathbf{B} = 0$ by

$$\mathbf{B} = \nabla \alpha \times \nabla \beta \quad (6.11)$$

(the existence of such α and β is both necessary and sufficient for $\text{div } \mathbf{B} = 0$), an integration by parts in the variation of F yields the interior variation

$$\text{curl } \mathbf{B} \times \mathbf{B} = 0 \quad (6.12)$$

instead of $\text{curl } \mathbf{B} = 0$. Since the variational function is the same as in Dirichlet's principle and the variational condition (6.12) is wider, the admissibility class must be (unwittingly) more restricted. Indeed, performing the boundary variation we discover that the boundary condition $B_n = \text{given}$ is not sufficient to make F stationary. Fixing α and β at both ends of the tube does make F stationary, and this is exactly the mapping condition on the ends of the magnetic lines (both α and β are constant along a magnetic line).

The resolution of the surprising distinction between the two parametrizations of $\text{div } \mathbf{B} = 0$ is that, although they are equivalent in the large, a local variation of α and β keeps intact the identification of magnetic lines across the varied domain. In the general equilibrium problem, (6.5), the parametrization

$$\mathbf{B} = \nabla p \times \nabla \omega \quad (6.13)$$

(where ω may be multivalued) accomplishes the same purpose.

7. Free boundary equilibrium and stability [25]

We now turn to a problem which was one of the first in magneto-fluid dynamics to be studied in depth. The reason this could be done is that the electromagnetic field is almost completely separated from the fluid dynamics. Specifically, we consider an interface which separates a pure electromagnetic field in vacuo from a conducting fluid within which there is no electromagnetic field. By the perfect conductivity, this separation persists in time. The fluid provides boundary conditions for the electromagnetic field, viz.

$$\left. \begin{aligned} B_n &= 0 \\ (\mathbf{E} + \mathbf{u} \times \mathbf{B})_t &= 0 \end{aligned} \right\} \quad (7.1)$$

and the field supplies a boundary condition for the fluid motion

$$p = B^2/2\mu_0. \quad (7.2)$$

To provide a suitable formulation it is necessary to use a Galilean invariant form of the electromagnetic equations [26]. The result is a theory which is basically elliptic (in particular, \mathbf{B} is determined at any instant by the instantaneous geometrical configuration plus the specification of certain, possibly time-varying, periods such as fluxes or currents). Heuristically, we conclude that the driving pressure (7.2) on the fluid is determined by the instantaneous position of the interface; thus the fluid motion is determined.

The simplest of these problems is the static equilibrium of a fluid at rest

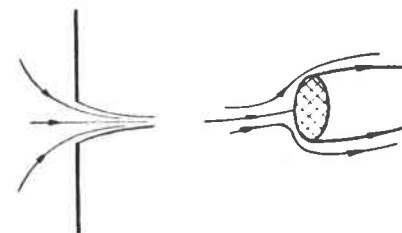


Fig. 13. Free boundary flows.

($p = \text{constant}$) balanced by a magnetic field. This is, of course, a special case of the static equilibria studied in the previous section. It is therefore an analogue of a fluid flow problem, viz., the classical free boundary flow. In this problem we have a potential flow, $\text{div } \mathbf{u} = 0$, $\text{curl } \mathbf{u} = 0$, separated at a stream surface from either stagnant water or a cavity. The additional boundary condition, $|\mathbf{u}| = \text{constant}$ at the interface (obtained from Bernoulli's law) serves to determine the shape of the interface. In the magnetic version, the vacuum field $\text{curl } \mathbf{B} = 0$, $\text{div } \mathbf{B} = 0$ is also provided with an extra boundary condition, $|\mathbf{B}| = \text{constant}$ at the interface.

Although the two problems are mathematically identical, the solutions which are of interest in the two cases are not the same. For example, the fluid flow from an orifice and the cavitation flow around an obstacle (Fig. 13) are of very little interest magnetically. On the other hand, important magnetic configurations such as the *pinch* and the *cusped geometry* (Fig. 14) are unimportant fluid dynamically. One important reason for this distinction is the question of stability. The stability problem involves time-dependent equations which are *not* analogous in the magnetic and fluid versions. A very complete analysis of the magnetic free boundary stability problem has been done. The end result is a simple criterion which is necessary and sufficient for stability against finite amplitude displacements in terms of easily verified properties of the equilibrium configuration. The great generality of this theory results from the existence of a variational formulation. The appropriate variational function is

$$F = \int_{D_v} (B^2/2\mu_0) dV + \int_{D_f} \rho dV - \sum_1^m I_r \Phi_r, \quad (7.3)$$

where the first integral (over the vacuum domain) is the magnetic energy, the second integral (over the fluid domain) is the fluid energy, and the summation is carried over those magnetic circuits $1 \dots m$ for which the currents I_r are fixed constraints; in the remaining magnetic circuits, $m+1, \dots, n$, the

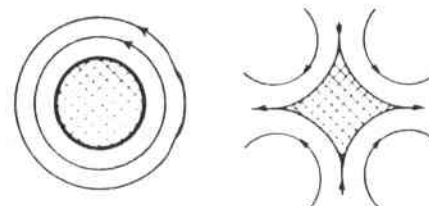


Fig. 14. Magnetic free boundary solutions.

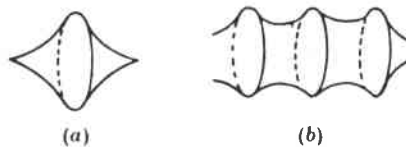


Fig. 15. Stable equilibria.

fluxes Φ , are fixed constraints. An equilibrium configuration is characterized by stationary F ; ⁽¹⁾ a stable configuration is characterized by a minimum value of F .

Specifically, the theorem [25]⁽²⁾ is that a given configuration is stable if there exists a domain D_1 which encloses the fluid domain D_0 and within which the magnitude B of the magnetic field is larger than the constant value B_0 taken on the interface, $B > B_0$ in $D_1 - D_0$. It is unstable if there is a region adjacent to D_0 where $B < B_0$. This criterion is valid for any geometrical and topological fluid and confining coil configuration; for any position of external fixed conductors, touching the fluid or not; for constant flux or constant current constraints (or any combination) on the confining fields; for any fluid properties, incompressible or compressible with any equation of state, or even subject to the Boltzmann equation. The essential mathematical content of this problem is the domain variation of harmonic vectors in regions of higher topological structure.

There is a simple relation between the above criterion for stability in the large and the more elementary criterion for positivity of the second variation of the appropriate variational function. The latter condition is simply that $\partial B^2/\partial n$ be positive on the entire interface for stability, and negative anywhere for instability. This criterion has a simple geometrical interpretation. Since

$$\text{curl } \mathbf{B} \times \mathbf{B} = (\mathbf{B} \cdot \nabla) \mathbf{B} - \nabla(\frac{1}{2} B^2) = 0 \quad (7.4)$$

for a harmonic vector, the sign of $\partial B^2/\partial n$ is associated with the sign of the curvature of a magnetic line on the interface (the magnetic lines are easily seen to be geodesics, so this is the curvature of the surface in the given direction). From this remark one can easily prove the theorem that no finite fluid domain with a smooth boundary can be stable (at a point of tangency with a support plane, the curvature must be in the unstable direction). This stability analysis led to the discovery of the stable cusped configuration of Fig. 14.

There are many stable three-dimensional configurations [29] of which we indicate only two in Fig. 15. These configurations are stable once they have been shown to exist. Two-dimensional equilibria are easily found by conformal mapping. The periodic structure, Fig. 15b, can be shown to exist by perturbation from the known two-dimensional solution, but the singularity on the axis of Fig. 15a has so far eluded an existence proof (although the figure has been empirically computed, numerically).

It is instructive to compare several common definitions of stability:

- (1) boundedness of solutions of the nonlinear partial differential equations,
- (2) verification of a minimum in the large for F , eq. (7.3),

⁽¹⁾ This is a modification of a variational principle used by Friedrichs to show existence of an axially symmetric fluid free boundary [27].

⁽²⁾ For a more complete account see [28].

- (3) determination of the sign of the second variation of F ,
- (4) determination of the characteristic frequencies of a linearized solution of the partial differential equations, assumed to be exponential in time.

The approach (1) has not been attempted. The relation between (2) and (3) is the simple one stated above. There is also a simple connection between (3) and (4); there is a variational formulation for the characteristic frequencies which uses the second variation of F , just as in classical vibration problems.

In this last formulation, the free boundary stability problem has been generalized (for the linearized problem) to many cases of general equilibria, $\nabla p = \mathbf{J} \times \mathbf{B}$ [30]. This problem is more complicated than the free boundary because it can be shown that there is no local property of the equilibrium (as there is in the free boundary case) that can characterize stability [25]. In particular, there are no results concerning finite amplitude stability.

Historical note

Many of the early results in this subject (prior to 1958) were first presented in restricted conferences concerned with Project Sherwood controlled thermonuclear research⁽¹⁾. Most of the fluid analogues (transverse flow, Sect. 3, incompressible parallel flow, Sect. 5, general static⁽²⁾ and free boundary equilibria, Sects. 6 and 7), also the variational formulation of the many static equilibria, Sect. 6 (including a tentative formulation for the toroidal geometry) as well as the variational formulation of the linear and nonlinear free boundary stability problem, Sect. 7, were presented by the author in lectures at New York University,⁽³⁾ Livermore,⁽⁴⁾ and Princeton⁽⁵⁾ in 1954. The Galilean invariant electromagnetic theory [26] was developed along two different lines by K. O. Friedrichs and the author in 1954.⁽⁶⁾ The characteristic theory, Sect. 2, was presented by K. O. Friedrichs in lectures at New York University, Los Alamos,⁽⁷⁾ and Princeton⁽⁸⁾ in 1954. The theorem relating the linear stability of a free boundary configuration to the curvature of the boundary as well as the theorem of nonexistence of smooth stable equilibria, Sect. 7, were announced and proved by the author in 1955.⁽⁹⁾ An independent heuristic stability argument involving particle orbits leading to a roughly similar criterion was given by Rosenbluth and Longmire [31]. The variational free boundary stability analysis was subsequently extended by E. Frieman and others to general equilibria (without an

⁽¹⁾ An excellent general account is given in A. S. Bishop, *Project Sherwood*, Addison-Wesley Publishing Co., 1958.

⁽²⁾ This analogue was noted earlier by Lundquist [2].

⁽³⁾ Mimeographed lecture notes on magnetohydrodynamics (I. *General Equations*, II, *Transients*, III, *Ohm's Law*, IV, *Potential Flows and Similarities with Fluid Dynamics*), New York University, August 10, 1954.

⁽⁴⁾ *Note on Magnetohydrostatics* (mimeographed), Livermore, August 30, 1954.

⁽⁵⁾ U.S. Atomic Energy Commission Report WASH-184, p. 144 (issued January 1955).

⁽⁶⁾ WASH-184, pp. 144 and 148.

⁽⁷⁾ *Nonlinear Wave Motion in Magnetohydrodynamics*, Los Alamos Report LAMS-2015 (written Sept. 1954, issued 1957).

⁽⁸⁾ WASH-184, p. 148.

⁽⁹⁾ U.S. Atomic Energy Commission Report WASH-289, p. 115 (issued June 1955).

interface) for the vibrational form of the linear problem [30]⁽¹⁾ (also independently in Germany [32] and in the U.S.S.R. [33]). The existence of the two-dimensional cusped equilibrium was shown by Friedrichs.⁽²⁾ The non-existence of a simple local stability criterion in any but a free boundary equilibrium was shown by H. Rubin in 1955.⁽³⁾ The free boundary stability theorem in the large was established by the author and A. A. Blank in 1955.⁽⁴⁾ The toroidal equilibrium theory was further developed by the author and H. Rubin in 1955 and 1956 [24], and a somewhat similar simultaneous development was carried out by Kruskal and Kulsrud [34].

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⁽¹⁾ WASH-289, p. 343. This formulation, for vibrational stability, was originally introduced by Lundquist [2].

⁽²⁾ WASH-289, p. 115. A somewhat similar configuration had been described earlier in the fluid version, as a collision of jets but was dismissed (properly) as nonphysical.

⁽³⁾ U.S. Atomic Energy Commission Report TID-7503, p. 247 (issued February 1956).

⁽⁴⁾ TID-7503, p. 238.

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