High-electric-field limit for the Vlasov-Maxwell-Fokker-Planck system

Mihai Bostan*, Thierry Goudon†

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Abstract

In this paper we derive the high-electric-field limit of the three dimensional Vlasov-Maxwell-Fokker-Planck system. We use the modulated energy method which requires the smoothness of the solution of the limit problem. We obtain convergences of charge and current densities in the space of bounded measures.

Keywords: High-field limit, Vlasov-Maxwell-Fokker-Planck system, Modulated energy.

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1 Introduction

We consider a plasma in which the dilute charged particles interact both through collisions and through the action of their self-consistent electro-magnetic field. Actually, we are concerned with the evolution of the negative particles which are described in terms of a distribution function in phase space while the charge and current of the positive particles are given functions. Up to a dimensional analysis (postponed to the Appendix) the evolution of the plasma is governed by the following equations

$$\varepsilon(\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon) - (E_\varepsilon + \alpha\varepsilon(v\wedge B_\varepsilon))\cdot\nabla_v f_\varepsilon = \mathrm{div}_v(v f_\varepsilon + \nabla_v f_\varepsilon), \quad (t,x,v) \in]0,T[\times\mathbb{R}^3\times\mathbb{R}^3, \tag{1}$$

$$\partial_t E_\varepsilon - \mathrm{curl}_x B_\varepsilon = -(J - j_\varepsilon) = \int_{\mathbb{R}^3} v f_\varepsilon \, dv - J(t,x), \quad (t,x) \in]0,T[\times\mathbb{R}^3, \tag{2}$$

*Laboratoire de Mathématiques de Besançon, UMR CNRS 6623, Université de Franche-Comté, 16 route de Gray, 25030 Besançon Cedex France. E-mail : mbostan@math.univ-fcomte.fr

†Team SIMPAF–INRIA Futurs & Laboratoire Paul Painlevé, UMR CNRS 8524, Université des Sciences et Technologies Lille 1, Cité Scientifique, 59655 Villeneuve d'Ascq Cedex France. E-mail : Thierry.Goudon@math.univ-lille1.fr

$$\alpha\varepsilon\partial_t B_\varepsilon + \mathrm{curl}_x E_\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3, \tag{3}$$

$$\mathrm{div}_x E_\varepsilon = D(t,x) - \rho_\varepsilon(t,x) = D(t,x) - \int_{\mathbb{R}^3} f_\varepsilon \, dv, \;\; \mathrm{div}_x B_\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \tag{4}$$

The system (1), (2), (3), (4) is refered to as the Vlasov-Maxwell-Fokker-Planck (VMFP) system. Here $f_\varepsilon(t,x,v) \geq 0$ is the distribution function of the negative particles, $E_\varepsilon, B_\varepsilon$ stand for the electric and magnetic fields respectively while $D(t,x)$, $J(t,x)$ are the (given) charge and current densities of positive particles. They are supposed to satisfy the conservation law

$$\partial_t D + \mathrm{div}_x J = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \tag{5}$$

The dimensionless parameter $\varepsilon = \left(\frac{l}{\Lambda}\right)^2$ is the square of the ratio between the mean free path and the Debye length and $\alpha = \left(\frac{\Lambda}{\tau \cdot c_0}\right)^2$ is the square of the ratio between the Debye length and the distance travelled by the light during the relaxation time due to collisions. We are interested in the asymptotic regime

$$0 < \varepsilon \ll 1, \qquad \alpha = \mathcal{O}(1).$$

Notice that in (1) the nonlinear term $E_\varepsilon \cdot \nabla_v f_\varepsilon$ is of the same order of magnitude that the diffusion Fokker-Planck term (which is due to the hypothesis that the mean free path l is much smaller than the Debye length Λ). We call this asymptotic regime for $\varepsilon \searrow 0$ the high-electric-field limit. The system is completed by prescribing initial conditions for the distribution function f_ε and the electro-magnetic field $(E_\varepsilon, B_\varepsilon)$

$$f_\varepsilon(0,x,v) = f_\varepsilon^0(x,v), \;\; (x,v) \in \mathbb{R}^3 \times \mathbb{R}^3, \tag{6}$$

$$E_\varepsilon(0,x) = E_\varepsilon^0(x), \;\; B_\varepsilon(0,x) = B_\varepsilon^0(x), \;\; x \in \mathbb{R}^3. \tag{7}$$

We suppose that initially the plasma is globally neutral *i.e.*,

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx = \int_{\mathbb{R}^3} D(0,x)\, dx, \tag{8}$$

and also that the initial conditions satisfy

$$\mathrm{div}_x E_\varepsilon^0 = D(0,x) - \int_{\mathbb{R}^3} f_\varepsilon^0(x,v)\, dv, \;\; \mathrm{div}_x B_\varepsilon^0 = 0, \;\; x \in \mathbb{R}^3. \tag{9}$$

After integration of (1) with respect to $v \in \mathbb{R}^3$ we deduce that the charge and current densities of the negative particles

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^3} f_\varepsilon \, dv, \qquad j_\varepsilon(t,x) = \int_{\mathbb{R}^3} v f_\varepsilon \, dv$$

verify the conservation law

$$\partial_t \rho_\varepsilon + \mathrm{div}_x j_\varepsilon = 0, \;\; (t,x) \in]0,T[\times\mathbb{R}^3. \tag{10}$$

By using (10), (5) and by taking the divergence with respect to x of equations (2), (3) we deduce that (4) are consequences of (9).

Now, note that the Fokker-Planck operator can be written as follows

$$L_{FP}(f) := \operatorname{div}_v(vf + \nabla_v f) = \operatorname{div}_v\left(e^{-\frac{|v|^2}{2}}\nabla_v\left(fe^{\frac{|v|^2}{2}}\right)\right), \tag{11}$$

and therefore the kinetic equation (1) becomes

$$\partial_t f_\varepsilon + v\cdot\nabla_x f_\varepsilon - \alpha(v\wedge B_\varepsilon)\cdot\nabla_v f_\varepsilon = \frac{1}{\varepsilon}\operatorname{div}_v\left(e^{-\frac{|v+E_\varepsilon(t,x)|^2}{2}}\nabla_v\left(f_\varepsilon e^{\frac{|v+E_\varepsilon(t,x)|^2}{2}}\right)\right). \tag{12}$$

From (12) we can expect that when $\varepsilon \searrow 0$, the distribution function f_ε converges to

$$f_\varepsilon \approx \frac{\rho(t,x)}{(2\pi)^{\frac{3}{2}}}e^{-\frac{|v+E(t,x)|^2}{2}}, \tag{13}$$

and therefore we can guess that

$$j_\varepsilon(t,x) = \int_{\mathbb{R}^3} vf_\varepsilon\, dv \approx -\rho(t,x)E(t,x).$$

Using the charge conservation law (10) together with (2), (3), (4), we are thus formally led to the following limit system

$$\begin{cases} \partial_t\rho - \operatorname{div}_x(\rho E) = 0, \quad (t,x)\in]0,T[\times\mathbb{R}^3, \\ \operatorname{div}_x E = D(t,x) - \rho(t,x), \quad \operatorname{curl}_x E = 0, \quad (t,x)\in]0,T[\times\mathbb{R}^3, \\ \partial_t E - \operatorname{curl}_x B = -J(t,x) - \rho(t,x)E(t,x), \quad \operatorname{div}_x B = 0, \quad (t,x)\in]0,T[\times\mathbb{R}^3. \end{cases} \tag{14}$$

The problem is motivated from plasma physics, as for instance in the theory of semiconductors, the evolution of laser-produced plasmas or the description of tokamaks. High-field asymptotics has been first analyzed in the kinetic theory of semiconductors in [30], see also [14]. Results for different physical models have been obtained in [1], [4], [17], [27]. Of course, the main difficulty relies on the non linear acceleration term $E_\varepsilon\cdot\nabla_v f_\varepsilon$. The problem slightly simplifies in the electrostatic case where the electric field is simply defined through the Poisson equation (complete (1) by $E_\varepsilon = -\nabla_x\Phi_\varepsilon$, $\Delta_x\Phi_\varepsilon = \rho_\varepsilon - D$ and $B_\varepsilon = 0$). This actually means that the electric field E_ε is defined by a convolution with $\rho_\varepsilon - D$. The resulting Vlasov-Poisson-Fokker-Planck (VPFP) can be seen, at least formally, as an asymptotic limit of the VMFP model in a physical regime where the light speed is large compared to the thermal velocity. The high-field limit of the VPFP system can be addressed by appealing to usual compactness methods; however, constraints on the space dimension appear, due to the singularity of the convolution kernel. It turns out that the strategy works in dimension 1 [28] and dimension 2 [21]. Another approach uses modulated energy (or relative entropy) methods, as introduced in [34]. With such an approach, we try to evaluate how far the solution is from the expected limit. This method has been used to treat various asymptotic questions in plasma physics [10], [11], [20], gas dynamics [32], [5], fluid-particles interaction [23]... Concerning the VPFP system, it allows to justify the L^2 strong convergence for the electric field and we can pass to the limit for any space dimension [21]. However, this method requires some smoothness on the solutions of the limit system.

The aim of this paper is therefore to analyze the high-electric-field limit of the three dimensional VMFP system by using the modulated energy method. This extension is interesting both from the viewpoint of physics: we are dealing with a more realistic and complete model; and those of mathematics: replacing the Poisson equation by the Maxwell system we cannot expect regularizing effects from the coupling, and this also shows how robust the modulated energy method is. We consider only smooth solutions of the VMFP system. Unfortunately, to our knowledge, there are no mathematical results concerning the existence/uniqueness of strong solution for the VMFP system. Indeed, the theory for VPFP is well established: existence of weak solutions can be found in [12], [33] while for existence and uniqueness results of strong solution we refer to [6], [7], [16], [29]; but coupling with the Maxwell equations leads to much more difficult analysis. Weak solutions for VMFP are dealt with in [18]. The collisionless case has been further investigated and global existence of classical solutions relies on the behavior of the tip of the support of the solution, as shown by different approaches in [19], [8], [25]. It is also worth mentioning the recent results when considering data close to equilibrium [24], [35] or reduced model [13]. Eventually, we point out that a low-field regime, where diffusion dominates the transport terms, can also be considered: for the VPFP system, we refer to [31], [22]. Here, our main result states as follows (see Section 4 for the definition of tight convergence).

Theorem 1.1 *Let $\rho^0 \geq 0$ and $D \geq 0$ such that $\rho^0 \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$, and $D \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap W^{1,\infty}(]0,T[\times\mathbb{R}^3)$, $\partial_t D \in L^\infty(]0,T[; L^1(\mathbb{R}^3))$. Let $J \in L^2(]0,T[; L^2(\mathbb{R}^3))^3$, $\partial_t J \in L^2(]0,T[; H^{-1}(\mathbb{R}^3))^3$ such that $\partial_t D + \mathrm{div}_x J = 0$. Consider (ρ, E, B) the unique solution of (14) with the initial condition ρ^0. Let $f_\varepsilon^0 \geq 0$ be a sequence of smooth distribution functions verifying*

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx = \int_{\mathbb{R}^3} D(0,x)\, dx, \quad \sup_{\varepsilon>0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon^0 \, dv\, dx < +\infty, \tag{15}$$

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \varepsilon \left(\frac{|v|^2}{2} + |\ln f_\varepsilon^0| \right) f_\varepsilon^0 \, dv\, dx = 0, \tag{16}$$

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} \{ |E_\varepsilon^0 - E^0|^2 + \alpha\varepsilon |B_\varepsilon^0|^2 \}\, dx = 0, \tag{17}$$

where E^0 is the solution of $\mathrm{div}_x E^0 = D(0,x) - \rho^0(x)$, $\mathrm{curl}_x E^0 = 0$, $x \in \mathbb{R}^3$. We assume that $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ are strong solutions of the VMFP system (1), (2), (3), (4), (6), (7). Then $(E_\varepsilon)_{\varepsilon>0}$ converges to E in $L^\infty(]0,T[; L^2(\mathbb{R}^3))^3$, $(\sqrt{\varepsilon} B_\varepsilon)_{\varepsilon>0}$ converges to 0 in $L^\infty(]0,T[; L^2(\mathbb{R}^3))^3$, $(B_\varepsilon)_{\varepsilon>0}$ converges to B in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)^3$, $(\rho_\varepsilon)_{\varepsilon>0}$

converges to ρ *in* $C^0([0,T],\mathcal{M}^1_+(\mathbb{R}^3) - $ *tight*) *and* $(j_\varepsilon)_{\varepsilon>0}$ *converges to* $-\rho E$ *in* $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$*- tight.*

The paper is organized as follows. In Section 2 we establish some priori estimates satisfied by smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ of the three dimensional VMFP system. In the next section, we introduce the modulated energy and calculate its time evolution. There, we also analyze the well-posedness of the limit equation. In Section 4 we detail the passage to the limit. The dimension analysis of the equations and the physical meaning of the different parameters are detailed in the Appendix.

2 A priori estimates

In this section we establish a priori estimates for the smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ of VMFP, uniformly with respect to $\varepsilon > 0$.

Proposition 2.1 *Let* $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ *be a smooth solution of the problem* (1), (2), (3), (4), (6), (7) *where the initial conditions satisfy* $f^0_\varepsilon \geq 0$, *and*

$$M^0_\varepsilon := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^0_\varepsilon \, dv\, dx < +\infty,$$

$$W^0_\varepsilon := \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f^0_\varepsilon \, dv\, dx + \frac{1}{2}\int_{\mathbb{R}^3} |E^0_\varepsilon|^2 \, dx + \frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3} |B^0_\varepsilon|^2 \, dx < +\infty,$$

$$L^0_\varepsilon := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f^0_\varepsilon \, dv\, dx < +\infty,$$

$$H^0_\varepsilon := \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^0_\varepsilon |\ln f^0_\varepsilon| \, dv\, dx < +\infty.$$

We assume also that $J \in L^1(]0,T[;L^2(\mathbb{R}^3))^3$. *Then, we have for any* $0 < t < T < \infty$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\, dv\, dx = \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^0_\varepsilon \, dv\, dx < +\infty,$$

$$\begin{aligned}\sup_{0\leq t\leq T}\Big\{\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon(t,x,v)\, dv\, dx \; &+ \; \frac{1}{2}\int_{\mathbb{R}^3} |E_\varepsilon(t,x)|^2\, dx + \frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3}|B_\varepsilon(t,x)|^2\, dx\Big\} \\ &+ \; \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \, dv\, dx\, dt \\ &\leq \; \left((2W^0_\varepsilon + 6TM^0_\varepsilon)^{\frac{1}{2}} + \sqrt{2}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right)^2,\end{aligned}$$

$$\sup_{0\leq t\leq T}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon(t,x,v)\, dv\, dx \leq C_T(M^0_\varepsilon + W^0_\varepsilon + L^0_\varepsilon + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}),$$

$$\sup_{0\le t\le T} \varepsilon \int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} f_\varepsilon |\ln f_\varepsilon|(t,x,v)\, dv\, dx \;+\; \int_0^T\!\!\int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} |\nabla_v \sqrt{f_\varepsilon}|^2\, dv\, dx\, dt \le C_T(\varepsilon + M_\varepsilon^0 + W_\varepsilon^0 + \varepsilon L_\varepsilon^0 + H_\varepsilon^0 + \|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}).$$

Before starting our computations let us state the following lemma, based on classical arguments due to Carleman.

Lemma 2.1 *Assume that $f = f(x,v)$ satisfies $f \ge 0$, $(|x| + |v|^2 + |\ln f|)f \in L^1(\mathbb{R}^N \times \mathbb{R}^N)$. Then for all $k > 0$ we have*

$$f|\ln f| \le f \ln f + 2k(|x| + |v|^2)f + 2Ce^{-\frac{k}{2}(|x|+|v|^2)}, \quad \textit{with } C = \sup_{0<y<1}\{-\sqrt{y}\ln y\},$$

and

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f|\ln f|\, dv\, dx \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f\ln f\, dv\, dx + 2k\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(|x| + |v|^2)f\, dv\, dx + C_k,$$

with $C_k = 2C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{k}{2}(|x|+|v|^2)}\, dv\, dx$.

Proof. Since $f|\ln f| = f\ln f + 2f(\ln f)_-$, it is sufficient to estimate $f(\ln f)_-$. Take $k > 0$ and let $C = \sup_{0<y<1}\{-\sqrt{y}\ln y\} < +\infty$. We have

$$\begin{aligned} f(\ln f)_- &= -f\ln f \cdot \mathbf{1}_{\{0<f<e^{-k(|x|+|v|^2)}\}} - f\ln f\cdot \mathbf{1}_{\{e^{-k(|x|+|v|^2)}\le f<1\}} \\ &\le Ce^{-\frac{k}{2}(|x|+|v|^2)} + k(|x|+|v|^2)f, \;\; \forall\, (x,v) \in \mathbb{R}^N\times\mathbb{R}^N. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} f(\ln f)_-\, dv\, dx \le k\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}(|x|+|v|^2)f\, dv\, dx + C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} e^{-\frac{k}{2}(|x|+|v|^2)}\, dv\, dx,$$

and the conclusion follows easily. $\square$

Proof of Proposition 2.1. Integrating (1) with respect to $(x,v) \in \mathbb{R}^3\times\mathbb{R}^3$ yields

$$\frac{d}{dt}\int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\, dv\, dx = 0, \;\; t \in]0,T[,$$

which implies that

$$\int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\, dv\, dx = \int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} f_\varepsilon^0\, dv\, dx = M_\varepsilon^0, \;\; t\in]0,T[. \tag{18}$$

Note that integrating (5) with respect to x implies $\frac{d}{dt}\int_{\mathbb{R}^3} D(t,x)\, dx = 0$ and therefore we deduce that if initially the plasma is globally neutral, *i.e.*, $\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0\, dv\, dx = \int_{\mathbb{R}^3} D(0,x)\, dx$, then it remains globally neutral for all $t \in]0,T[$

$$\int_{\mathbb{R}^3}\!\!\int_{\mathbb{R}^3} f_\varepsilon(t,x,v)\, dv\, dx = \int_{\mathbb{R}^3} D(t,x)\, dx.$$

Multiplying (1) by $\frac{|v|^2}{2}$ and integrating with respect to (x,v) implies

$$\varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{|v|^2}{2}f_\varepsilon\;dv\,dx+\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}E_\varepsilon\cdot v f_\varepsilon\;dv\,dx=-\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|v|^2 f_\varepsilon\;dv\,dx+3\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon\;dv\,dx. \tag{19}$$

Multiplying (2), (3) by E_ε, resp. B_ε and integrating with respect to x yields

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|E_\varepsilon|^2+\alpha\varepsilon|B_\varepsilon|^2)\;dx=-\int_{\mathbb{R}^3}E_\varepsilon\cdot(J-j_\varepsilon)\;dx. \tag{20}$$

By combining (19), (20) we obtain

$$\begin{aligned}\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|v|^2 f_\varepsilon\;dv\,dx \;&+\; \frac{d}{dt}\left(\varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{|v|^2}{2}f_\varepsilon\;dv\,dx+\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon|^2+\alpha\varepsilon|B_\varepsilon|^2)\;dx\right)\\ &=\; -\int_{\mathbb{R}^3}E_\varepsilon\cdot J\;dx+3\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon^0\;dv\,dx,\end{aligned}$$

and therefore we have

$$\begin{aligned}\int_0^t\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|v|^2 f_\varepsilon\;dv\,dx\,ds \;&+\; \varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{|v|^2}{2}f_\varepsilon\;dv\,dx+\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon|^2+\alpha\varepsilon|B_\varepsilon|^2)\;dx\\ &\le\; \varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{|v|^2}{2}f_\varepsilon^0\;dv\,dx+\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon^0|^2+\alpha\varepsilon|B_\varepsilon^0|^2)\;dx\\ &+\; \int_0^t\left(\int_{\mathbb{R}^3}|E_\varepsilon(s,x)|^2\;dx\right)^{\frac{1}{2}}\cdot\left(\int_{\mathbb{R}^3}|J(s,x)|^2\;dx\right)^{\frac{1}{2}}\;ds\\ &+\; 3t\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon^0\;dv\,dx.\end{aligned}$$

By using Bellman's lemma we obtain for all $0\le t\le T$

$$\begin{aligned}\left(\int_{\mathbb{R}^3}|E_\varepsilon(t,x)|^2\;dx\right)^{\frac{1}{2}} \;&\le\; \left(\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}(\varepsilon|v|^2+6T)f_\varepsilon^0\;dv\,dx+\int_{\mathbb{R}^3}(|E_\varepsilon^0|^2+\alpha\varepsilon|B_\varepsilon^0|^2)\;dx\right)^{\frac{1}{2}}\\ &+\; \int_0^T\left(\int_{\mathbb{R}^3}|J(s,x)|^2\;dx\right)^{\frac{1}{2}}\;ds\\ &=\; (2W_\varepsilon^0+6TM_\varepsilon^0)^{\frac{1}{2}}+\|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))}.\end{aligned}$$

Finally we get

$$\begin{aligned}\int_0^T\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|v|^2 f_\varepsilon\;dv\,dx\,dt \;&+\; \sup_{0\le t\le T}\left\{\varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{|v|^2}{2}f_\varepsilon\;dv\,dx+\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon|^2+\alpha\varepsilon|B_\varepsilon|^2)\;dx\right\}\\ &\le\; \left((2W_\varepsilon^0+6TM_\varepsilon^0)^{\frac{1}{2}}+\sqrt{2}\|J\|_{L^1(]0,T[;L^2(\mathbb{R}^3))}\right)^2.\end{aligned} \tag{21}$$

We multiply now (1) by $|x|$ and we obtain after integration with respect to (x,v)

$$\frac{d}{dt}\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|x|f_\varepsilon\;dv\,dx-\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}\frac{(v\cdot x)}{|x|}f_\varepsilon\;dv\,dx=0.$$

We deduce that

$$\begin{aligned}\sup_{0\le t\le T}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|x|f_\varepsilon\,dv\,dx &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|x|f_\varepsilon^0\,dv\,dx+\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|v|f_\varepsilon\,dv\,dx\,dt\\ &\le \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|x|f_\varepsilon^0\,dv\,dx+\int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{1}{2}(|v|^2+1)f_\varepsilon\,dv\,dx\,dt\\ &\le C_T(M_\varepsilon^0+W_\varepsilon^0+L_\varepsilon^0+\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}). \qquad (22)\end{aligned}$$

We multiply now (1) by $(1+\ln f_\varepsilon)$ and after integration with respect to (x,v) we get

$$\begin{aligned}\varepsilon\frac{d}{dt}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon\ln f_\varepsilon\,dv\,dx &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}(vf_\varepsilon+\nabla_v f_\varepsilon)\frac{\nabla_v f_\varepsilon}{f_\varepsilon}\,dv\,dx\\ &= 3\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon\,dv\,dx-4\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\nabla_v\sqrt{f_\varepsilon}|^2\,dv\,dx.\end{aligned}$$

Finally we deduce that

$$\begin{aligned}\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon\ln f_\varepsilon\,dv\,dx+4\int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\nabla_v\sqrt{f_\varepsilon}|^2\,dv\,dx\,ds &= \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon^0\ln f_\varepsilon^0\,dv\,dx\\ &+ 3t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon^0\,dv\,dx. \qquad (23)\end{aligned}$$

Combining (21), (22), (23) and Lemma 2.1 with $k=1$ yields

$$\begin{aligned}\sup_{0\le t\le T}\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon|\ln f_\varepsilon|\,dv\,dx &+ \int_0^T\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\nabla_v\sqrt{f_\varepsilon}|^2\,dv\,dx\,dt\le C_T(\varepsilon+M_\varepsilon^0+W_\varepsilon^0\\ &+ \varepsilon L_\varepsilon^0+H_\varepsilon^0+\|J\|^2_{L^1(]0,T[;L^2(\mathbb{R}^3))}).\end{aligned}$$

□

3 The modulated energy method

In this section we introduce the modulated energy (see [10], [34]) which allows us to show that the electro-magnetic field $(E_\varepsilon,\sqrt{\alpha\varepsilon}B_\varepsilon)$ converges strongly in $L^2(\mathbb{R}^3)^6$, uniformly in time. The proof requires some regularity properties of the limit solutions (ρ,E,B) of (14) as well as the convergence of the initial electro-magnetic field

$$\lim_{\varepsilon\searrow 0}\left(\int_{\mathbb{R}^3}|E_\varepsilon^0(x)-E(0,x)|^2\,dx+\alpha\varepsilon\int_{\mathbb{R}^3}|B_\varepsilon^0(x)|^2\,dx\right)=0.$$

Let us define the modulated energy $\mathcal{H}_\varepsilon$ and the positive modulated energy $\mathcal{H}_{\varepsilon,p}$ as follows

$$\mathcal{H}_\varepsilon(t)=\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}f_\varepsilon(\ln f_\varepsilon+\frac{1}{2}|v+E|^2)\,dv\,dx+\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon-E|^2+\alpha\varepsilon|B_\varepsilon-B|^2)\,dx,$$

and

$$\mathcal{H}_{\varepsilon,p}(t) = \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(|\ln f_\varepsilon| + \frac{1}{2}|v+E|^2)\, dv\, dx + \frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2)\, dx,$$

where $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ are smooth solutions of (1), (2), (3), (4), (6), (7) and (ρ, E, B) is a smooth solution of (14). According to the derivation of the high-field limit of the VPFP system in [21], the idea is to study the time variation of $\mathcal{H}_\varepsilon(t)$ (which is also called relative entropy), which in turn will give information on $\mathcal{H}_{\varepsilon,p}$.

3.1 Analysis of the limit system

We start with the analysis of the system (14). Note that this system can be split into two problems. First solve for (ρ, E)

$$\begin{cases} \partial_t\rho - \mathrm{div}_x(\rho E) = 0, & (t,x) \in]0,T[\times\mathbb{R}^3, \\ \mathrm{div}_x E = D(t,x) - \rho(t,x), & (t,x) \in]0,T[\times\mathbb{R}^3, \\ \mathrm{curl}_x E = 0, & (t,x) \in]0,T[\times\mathbb{R}^3, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}^3, \end{cases} \tag{24}$$

and secondly find B solution of

$$\begin{cases} \partial_t E - \mathrm{curl}_x B = -J(t,x) - \rho(t,x)E(t,x), & (t,x) \in]0,T[\times\mathbb{R}^3, \\ \mathrm{div}_x B = 0, & (t,x) \in]0,T[\times\mathbb{R}^3, \end{cases} \tag{25}$$

where the charge and current densities D, J are given functions satisfying $\partial_t D + \mathrm{div}_x J = 0$. We give here an existence result for (24) which is a direct consequence of the existence result obtained in [28], see also [21].

Proposition 3.1 *Let $\rho^0 \in W^{1,1}(\mathbb{R}^3) \cap W^{1,\infty}(\mathbb{R}^3)$, $D \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap W^{1,\infty}(]0,T[\times\mathbb{R}^3)$, $\partial_t D \in L^\infty(]0,T[; L^1(\mathbb{R}^3))$. Then there is a unique solution for* (24) *satisfying*

$$\rho \in W^{1,\infty}(]0,T[\times\mathbb{R}^3), \quad E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3.$$

Proof. We introduce the exterior electric field E_0 given by $\mathrm{div}_x E_0 = D$, $\mathrm{curl}_x E_0 = 0$, so that $\mathrm{div}_x(E - E_0) = -\rho$, $\mathrm{curl}_x(E - E_0) = 0$. The hypotheses imply that $E_0 \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$. Following the arguments of Theorem 3 and Lemma 8 of [28] we deduce that there is a unique strong solution (ρ, E) for (24) verifying $\rho \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$, $D - \rho \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$, $E \in L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))^3$. By differentiating the first equation of (24) with respect to x we check that $\nabla_x\rho \in L^\infty(]0,T[; L^1(\mathbb{R}^3))^3$ and since $\partial_t\rho = E\cdot\nabla_x\rho + \rho(D-\rho) \in L^\infty(]0,T[; L^1(\mathbb{R}^3)) \cap L^\infty(]0,T[; L^\infty(\mathbb{R}^3))$ we deduce that $\partial_t E \in L^\infty(]0,T[; W^{1,p}(\mathbb{R}^3))^3 \subset L^\infty(]0,T[; L^\infty(\mathbb{R}^3))^3$ for all $p > 3$. Finally we obtain that $\rho \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)$ and $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$. In fact, since $D - \rho \in L^\infty(]0,T[; W^{1,1}(\mathbb{R}^3)) \cap L^\infty(]0,T[; W^{1,\infty}(\mathbb{R}^3))$ we have $E \in L^\infty(]0,T[; W^{2,p}(\mathbb{R}^3))^3$ for all $1 < p < +\infty$. On the other hand since $\partial_t D - \partial_t\rho \in L^\infty(]0,T[; L^1(\mathbb{R}^3)) \cap L^\infty(]0,T[; L^\infty(\mathbb{R}^3))$ we have $\partial_t E \in L^\infty(]0,T[; W^{1,p}(\mathbb{R}^3))^3$ for all $1 < p < +\infty$. In particular we obtain that $E, \partial_t E \in L^\infty(]0,T[; L^2(\mathbb{R}^3))^3$. $\square$

Once we find (ρ, E) it is easy to solve (25).

Proposition 3.2 *Under the hypotheses of Proposition* 3.1 *assume also that* $J, \partial_t J \in L^2(]0,T[; H^{-1}(\mathbb{R}3))^3$ *and* $\partial_t D + \mathrm{div}_x J = 0$ *in* $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$. *Then there is a unique solution* B *for* (25) *verifying* $B, \partial_t B \in L^2(]0,T[; L^2(\mathbb{R}^3))^3$.

Proof. Observe that we have $\mathrm{div}_x(\partial_t E + \rho E + J) = 0$ and that $\partial_t E + \rho E + J \in L^2(]0,T[; H^{-1}(\mathbb{R}^3))^3$. Therefore there is a unique $B \in L^2(]0,T[; L^2(\mathbb{R}^3))^3$ such that $\partial_t E + \rho E + J = \mathrm{curl}_x B$, $\mathrm{div}_x B = 0$. In order to estimate $\partial_t B$ in $L^2(]0,T[; L^2(\mathbb{R}^3))^3$ it is sufficient to estimate $\partial_t(\partial_t E + \rho E + J)$ in $L^2(]0,T[; H^{-1}(\mathbb{R}^3))^3$. We have

$$\mathrm{div}_x(\partial_t^2 E) = \partial_t^2(D - \rho) = \partial_t(-\mathrm{div}_x J - \mathrm{div}_x(\rho E)),$$

and thus

$$\begin{aligned}
\|\partial_t^2 E\|_{L^2(]0,T[;H^{-1}(\mathbb{R}^3))} &\leq C \cdot \|\partial_t(J + \rho E)\|_{L^2(]0,T[;H^{-1}(\mathbb{R}^3))} \\
&\leq C \cdot \{\|\partial_t J\|_{L^2(]0,T[;H^{-1}(\mathbb{R}^3))} + \|\rho\|_{L^\infty} \cdot \|\partial_t E\|_{L^2(]0,T[;L^2(\mathbb{R}^3))} \\
&+ \|\partial_t \rho\|_{L^\infty} \cdot \|E\|_{L^2(]0,T[;L^2(\mathbb{R}^3))}\},
\end{aligned}$$

and the conclusion follows. □

3.2 Evolution of the modulated energy

This section is devoted to the study of the evolution of the modulated energy. Let us multiply (1) by $(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2})$ where (ρ, E, B) is solution of (14). It is convenient to rewrite (1) as follows

$$\varepsilon(\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon) - \mathrm{div}_v(f_\varepsilon(v + E) + \nabla_v f_\varepsilon) = \mathrm{div}_v\{(E_\varepsilon - E + \alpha\varepsilon(v \wedge B_\varepsilon))f_\varepsilon\}. \quad (26)$$

We perform our computations in three steps.

Lemma 3.1 *Assume that* $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$. *Let us set*

$$Q_1(t) := \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon) \cdot \left(1 + \ln f_\varepsilon + \frac{|v + E|^2}{2}\right) \, dv \, dx.$$

Then we have

$$\begin{aligned}
Q_1(t) &= \varepsilon \frac{d}{dt} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \left(\ln f_\varepsilon + \frac{|v + E|^2}{2}\right) \, dv \, dx \\
&\quad -\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(v + E) \cdot (\partial_t E + (\mathrm{D}_x E)v) \, dv \, dx,
\end{aligned}$$

where $\mathrm{D}_x E$ *stands for the jacobian matrix of* E. *Moreover, there is a constant* $C = C(\|E\|_{W^{1,\infty}(]0,T[\times\mathbb{R}^3)})$ *such that*

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(v + E) \cdot (\partial_t E + (\mathrm{D}_x E)v) \, dv \, dx\right| \leq C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \left(1 + \frac{|v + E|^2}{2}\right) \, dv \, dx.$$

Proof. We can write

$$
\begin{aligned}
(\partial_t f_\varepsilon &+ v \cdot \nabla_x f_\varepsilon) \cdot \left(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2}\right) = \partial_t (f_\varepsilon \ln f_\varepsilon) + v \cdot \nabla_x (f_\varepsilon \ln f_\varepsilon) \\
&+ \partial_t \left(f_\varepsilon \frac{|v+E|^2}{2}\right) + v \cdot \nabla_x \left(f_\varepsilon \frac{|v+E|^2}{2}\right) - f_\varepsilon (v+E) \cdot (\partial_t E + (\mathrm{D}_x E) v),
\end{aligned}
$$

where $\mathrm{D}_x E = \left(\frac{\partial E_i}{\partial x_j}\right)_{1 \le i,j \le 3}$. After integration with respect to (x, v) we get

$$
\begin{aligned}
Q_1(t) &= \varepsilon \frac{d}{dt} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon \left(\ln f_\varepsilon + \frac{|v+E|^2}{2}\right) \, dv \, dx \\
&\quad - \varepsilon \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon (v+E) \cdot (\partial_t E + (\mathrm{D}_x E) v) \, dv \, dx.
\end{aligned}
$$

The last term in the above equation can be estimated by

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon (v+E) \cdot \partial_t E \, dv \, dx \right| &\le \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon |v+E|^2 \, dv \, dx + \frac{1}{2} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon |\partial_t E|^2 \, dv \, dx \\
&\le C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon \left(1 + \frac{|v+E|^2}{2}\right) \, dv \, dx,
\end{aligned}
$$

and

$$
\begin{aligned}
\left| \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon (v+E) \cdot (\mathrm{D}_x E) v \, dv \, dx \right| &\le \|\mathrm{D}_x E\|_{L^\infty} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \left(\frac{3}{2}|v+E|^2 + \frac{1}{2}|E|^2\right) f_\varepsilon \, dv \, dx \\
&\le C \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} f_\varepsilon \left(1 + \frac{|v+E|^2}{2}\right) \, dv \, dx.
\end{aligned}
$$

$\square$

Lemma 3.2 *Let us set*

$$
Q_2(t) := - \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \mathrm{div}_v (f_\varepsilon (v+E) + \nabla_v f_\varepsilon) \left(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2}\right) \, dv \, dx.
$$

Then, we have

$$
Q_2(t) = \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2 \nabla_v \sqrt{f_\varepsilon}|^2 \, dv \, dx.
$$

Proof. By using the formula

$$
\mathrm{div}_v (f_\varepsilon (v+E) + \nabla_v f_\varepsilon) = \mathrm{div}_v \left\{ e^{-\frac{|v+E|^2}{2}} \nabla_v \left(f_\varepsilon e^{\frac{|v+E|^2}{2}} \right) \right\},
$$

we deduce that

$$\begin{aligned}
Q_2(t) &= -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{div}_v \left\{ e^{-\frac{|v+E|^2}{2}} \nabla_v \left(f_\varepsilon e^{\frac{|v+E|^2}{2}} \right) \right\} \ln \left(f_\varepsilon e^{\frac{|v+E|^2}{2}} \right) \, dv\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{e^{-|v+E|^2}}{f_\varepsilon} \left| \nabla_v \left(f_\varepsilon e^{\frac{|v+E|^2}{2}} \right) \right|^2 \, dv\, dx \\
&= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}|^2 \, dv\, dx.
\end{aligned}$$

□

Lemma 3.3 *Let* (ρ, E, B) *be a solution of* (14) *satisfying* $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, $B, \partial_t B, J \in L^2(]0,T[; L^2(\mathbb{R}^3))^3$. *Let us set*

$$Q_3(t) := -\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathrm{div}_v((E_\varepsilon - E + \alpha\varepsilon(v \wedge B_\varepsilon))f_\varepsilon) \left(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2} \right) \, dv\, dx.$$

Then, we have

$$\begin{aligned}
Q_3(t) &= \frac{d}{dt}\left\{ \frac{1}{2}\int_{\mathbb{R}^3} |E_\varepsilon - E|^2 \, dx + \frac{\alpha\varepsilon}{2}\int_{\mathbb{R}^3} |B_\varepsilon - B|^2 \, dx \right\} \\
&- \int_{\mathbb{R}^3} \{(E_\varepsilon - E)\mathrm{div}_x(E_\varepsilon - E) - (E_\varepsilon - E) \wedge \mathrm{curl}_x(E_\varepsilon - E)\} \cdot E \, dx \\
&- \alpha\varepsilon \int_{\mathbb{R}^3} \{B_\varepsilon \mathrm{div}_x B_\varepsilon - B_\varepsilon \wedge \mathrm{curl}_x B_\varepsilon\} \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t B \cdot (B_\varepsilon - B) \, dx \\
&+ \alpha\varepsilon \int_{\mathbb{R}^3} (J \wedge B_\varepsilon) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t (E_\varepsilon \wedge B_\varepsilon) \cdot E \, dx.
\end{aligned}$$

Moreover, there is a constant $C = C(\|E\|_{W^{1,\infty}})$ *such that*

$$\left| \int_{\mathbb{R}^3} \{(E_\varepsilon - E)\mathrm{div}_x(E_\varepsilon - E) - (E_\varepsilon - E) \wedge \mathrm{curl}_x(E_\varepsilon - E)\} \cdot E \, dx \right| \le C \int_{\mathbb{R}^3} |E_\varepsilon - E|^2 \, dx,$$

$$\left| \int_{\mathbb{R}^3} \{B_\varepsilon \mathrm{div}_x B_\varepsilon - B_\varepsilon \wedge \mathrm{curl}_x B_\varepsilon\} \cdot E \, dx \right| \le C \int_{\mathbb{R}^3} |B_\varepsilon|^2 \, dx \le C \int_{\mathbb{R}^3} \{|B_\varepsilon - B|^2 + |B|^2\} \, dx.$$

Proof. We can write

$$\begin{aligned}
\mathrm{div}_v\{f_\varepsilon(E_\varepsilon - E &+ \alpha\varepsilon(v \wedge B_\varepsilon))\} \left(1 + \ln f_\varepsilon + \frac{|v+E|^2}{2} \right) \\
&= \mathrm{div}_v\{f_\varepsilon \ln f_\varepsilon (E_\varepsilon - E + \alpha\varepsilon(v \wedge B_\varepsilon))\} \\
&\quad + \mathrm{div}_v\{f_\varepsilon(E_\varepsilon - E + \alpha\varepsilon(v \wedge B_\varepsilon))\} \frac{|v+E|^2}{2}.
\end{aligned}$$

After integration with respect to (x,v) we get

$$\begin{aligned}
Q_3(t) &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(E_\varepsilon - E + \alpha\varepsilon(v\wedge B_\varepsilon))\cdot(v+E)\,dv\,dx \\
&= \int_{\mathbb{R}^3}(E_\varepsilon - E)\cdot j_\varepsilon\;dx + \int_{\mathbb{R}^3}\rho_\varepsilon(E_\varepsilon - E)\cdot E\;dx + \alpha\varepsilon\int_{\mathbb{R}^3}(j_\varepsilon\wedge B_\varepsilon)\cdot E\;dx \\
&= \int_{\mathbb{R}^3}(E_\varepsilon - E)\cdot(j_\varepsilon + \rho E)\,dx + \int_{\mathbb{R}^3}(\rho_\varepsilon-\rho)(E_\varepsilon - E)\cdot E\,dx \\
&\quad +\alpha\varepsilon\int_{\mathbb{R}^3}(j_\varepsilon\wedge B_\varepsilon)\cdot E\,dx \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

From (2), (3), (14) we have

$$\partial_t(E_\varepsilon - E) - \mathrm{curl}_x(B_\varepsilon - B) = j_\varepsilon + \rho E, \tag{27}$$

$$\alpha\varepsilon\partial_t(B_\varepsilon - B) + \mathrm{curl}_x(E_\varepsilon - E) = -\alpha\varepsilon\partial_t B. \tag{28}$$

By multiplying (27), (28) by $E_\varepsilon - E$, and $B_\varepsilon - B$ respectively, we find after integration with respect to x

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3}(|E_\varepsilon - E|^2 + \alpha\varepsilon|B_\varepsilon - B|^2)\;dx &= \int_{\mathbb{R}^3}(j_\varepsilon + \rho E)\cdot(E_\varepsilon - E)\;dx \\
&- \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon - B)\;dx \\
&= I_1 - \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon - B)\;dx. \qquad (29)
\end{aligned}$$

From (4), (14) we have $\mathrm{div}_x(E_\varepsilon - E) = -(\rho_\varepsilon - \rho)$ and thus

$$I_2 = \int_{\mathbb{R}^3}(\rho_\varepsilon - \rho)(E_\varepsilon - E)\cdot E\;dx = -\int_{\mathbb{R}^3}\mathrm{div}_x(E_\varepsilon - E)(E_\varepsilon - E)\cdot E\;dx. \tag{30}$$

Now by using (2), (3) we deduce

$$\begin{aligned}
I_3 &= \alpha\varepsilon\int_{\mathbb{R}^3}(j_\varepsilon\wedge B_\varepsilon)\cdot E\;dx = \alpha\varepsilon\int_{\mathbb{R}^3}((\partial_t E_\varepsilon - \mathrm{curl}_x B_\varepsilon + J)\wedge B_\varepsilon)\cdot E\;dx \\
&= \alpha\varepsilon\int_{\mathbb{R}^3}(B_\varepsilon\wedge\mathrm{curl}_x B_\varepsilon)\cdot E\;dx + \alpha\varepsilon\int_{\mathbb{R}^3}(J\wedge B_\varepsilon)\cdot E\;dx \\
&+ \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t(E_\varepsilon\wedge B_\varepsilon)\cdot E\;dx - \alpha\varepsilon\int_{\mathbb{R}^3}(E_\varepsilon\wedge\partial_t B_\varepsilon)\cdot E\;dx \\
&= \alpha\varepsilon\int_{\mathbb{R}^3}(B_\varepsilon\wedge\mathrm{curl}_x B_\varepsilon)\cdot E\;dx + \alpha\varepsilon\int_{\mathbb{R}^3}(J\wedge B_\varepsilon)\cdot E\;dx \\
&+ \alpha\varepsilon\int_{\mathbb{R}^3}\partial_t(E_\varepsilon\wedge B_\varepsilon)\cdot E\;dx + \int_{\mathbb{R}^3}(E_\varepsilon\wedge\mathrm{curl}_x E_\varepsilon)\cdot E\;dx. \qquad (31)
\end{aligned}$$

Note also that since $\mathrm{curl}_x E = 0$ we have

$$\int_{\mathbb{R}^3} (E_\varepsilon \wedge \mathrm{curl}_x E_\varepsilon) \cdot E \, dx = \int_{\mathbb{R}^3} ((E_\varepsilon - E) \wedge \mathrm{curl}_x (E_\varepsilon - E)) \cdot E \, dx.$$

The equalities (30), (31) imply

$$\begin{aligned} I_2 + I_3 &= -\int_{\mathbb{R}^3} \{(E_\varepsilon - E)\mathrm{div}_x(E_\varepsilon - E) - (E_\varepsilon - E) \wedge \mathrm{curl}_x(E_\varepsilon - E)\} \cdot E \, dx \\ &- \alpha\varepsilon \int_{\mathbb{R}^3} \{B_\varepsilon \mathrm{div}_x B_\varepsilon - B_\varepsilon \wedge \mathrm{curl}_x B_\varepsilon\} \cdot E \, dx \\ &+ \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t (E_\varepsilon \wedge B_\varepsilon) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} (J \wedge B_\varepsilon) \cdot E \, dx. \end{aligned}$$

Finally we get

$$\begin{aligned} Q_3(t) &= \frac{1}{2}\frac{d}{dt} \int_{\mathbb{R}^3} \{|E_\varepsilon - E|^2 + \alpha\varepsilon |B_\varepsilon - B|^2\} \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t B \cdot (B_\varepsilon - B) \, dx \\ &- \int_{\mathbb{R}^3} \{(E_\varepsilon - E)\mathrm{div}_x(E_\varepsilon - E) - (E_\varepsilon - E) \wedge \mathrm{curl}_x(E_\varepsilon - E)\} \cdot E \, dx \\ &- \alpha\varepsilon \int_{\mathbb{R}^3} \{B_\varepsilon \mathrm{div}_x B_\varepsilon - B_\varepsilon \wedge \mathrm{curl}_x B_\varepsilon\} \cdot E \, dx \\ &+ \alpha\varepsilon \int_{\mathbb{R}^3} \partial_t (E_\varepsilon \wedge B_\varepsilon) \cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} (J \wedge B_\varepsilon) \cdot E \, dx. \end{aligned}$$

Now, using the formula

$$A_i(u,u) \quad := \quad (u \mathrm{div}_x u - u \wedge \mathrm{curl}_x u)_i = \sum_{j=1}^{3} \frac{\partial}{\partial x_j}(u_i u_j) - \frac{1}{2}\frac{\partial}{\partial x_i}|u|^2, \quad 1 \le i \le 3,$$

one gets easily after integration by parts that

$$\left| \int_{\mathbb{R}^3} A(E_\varepsilon - E, E_\varepsilon - E) \cdot E \, dx \right| \le C(\|E\|_{W^{1,\infty}}) \int_{\mathbb{R}^3} |E_\varepsilon - E|^2 \, dx,$$

and similarly

$$\begin{aligned} \left| \int_{\mathbb{R}^3} A(B_\varepsilon, B_\varepsilon) \cdot E \, dx \right| &\le C(\|E\|_{W^{1,\infty}}) \int_{\mathbb{R}^3} |B_\varepsilon|^2 \, dx \\ &\le C(\|E\|_{W^{1,\infty}}) \left(\int_{\mathbb{R}^3} |B_\varepsilon - B|^2 \, dx + \int_{\mathbb{R}^3} |B|^2 \, dx \right). \end{aligned}$$

□

From (26) we know that $Q_1(t) + Q_2(t) + Q_3(t) = 0$ for all $0 \le t \le T$. Therefore, combining Lemmas 3.1, 3.2, 3.3 characterizes the evolution of the modulated energy as follows.

Proposition 3.3 *Let $f_\varepsilon^0 \geq 0$ verify the assumptions of Proposition 2.1. Let $D \geq 0$, $D \in L^\infty(]0,T[;L^1(\mathbb{R}^3))$ and $J \in L^2(]0,T[;L^2(\mathbb{R}^3))^3$ verify $\partial_t D + \mathrm{div}_x J = 0$. Let $(f_\varepsilon, E_\varepsilon, B_\varepsilon)$ be a smooth solution of the VMFP system (1), (2), (3), (4) with the initial conditions $f_\varepsilon^0, E_\varepsilon^0, B_\varepsilon^0$ satisfying (9). Assume that the solution (ρ, E, B) of (14) verifies $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, $E, B, \partial_t E, \partial_t B \in L^2(]0,T[;L^2(\mathbb{R}^3))^3$. Then the balance of the modulated energy is given by*

$$\begin{aligned}\frac{d}{dt}\mathcal{H}_\varepsilon(t) \quad &+ \quad \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v\sqrt{f_\varepsilon}|^2 \, dv\, dx \\ &= \quad \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(v+E)\cdot(\partial_t E + (\mathrm{D}_x E)v)\, dv\, dx \\ &\quad + \int_{\mathbb{R}^3} A(E_\varepsilon - E, E_\varepsilon - E)\cdot E \, dx + \alpha\varepsilon \int_{\mathbb{R}^3} A(B_\varepsilon, B_\varepsilon)\cdot E\, dx \qquad (32) \\ &\quad -\alpha\varepsilon \int_{\mathbb{R}^3} \partial_t B\cdot(B_\varepsilon - B)\, dx - \alpha\varepsilon\int_{\mathbb{R}^3} (J\wedge B_\varepsilon)\cdot E\, dx \\ &\quad -\alpha\varepsilon\int_{\mathbb{R}^3} \partial_t(E_\varepsilon \wedge B_\varepsilon)\cdot E\, dx,\end{aligned}$$

where, for a given $u : \mathbb{R}^3 \to \mathbb{R}^3$, $A(u,u)$ denotes the vector $u\ \mathrm{div}_x u - u \wedge \mathrm{curl}_x u$.

In turn, we are able to establish a useful estimate for the positive modulated energy.

Corollary 3.1 *Let us denote $E^0 := E(0,\cdot)$, $B^0 := B(0,\cdot)$. Then, for all $0 \leq t \leq T$ the following inequality holds*

$$\begin{aligned}\mathcal{H}_{\varepsilon,p}(t) \quad &+ \quad \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\sqrt{f_\varepsilon}(v+E) + 2\nabla_v\sqrt{f_\varepsilon}|^2\, dv\, dx\, ds \leq C\int_{\mathbb{R}^3}\{|E_\varepsilon^0 - E^0|^2 + \alpha\varepsilon|B_\varepsilon^0|^2\} \\ &+ \quad C\varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0\left(|\ln f_\varepsilon^0| + 1 + |x| + \frac{|v|^2}{2}\right)\, dv\, dx \\ &+ \quad C\left(\varepsilon + (\varepsilon + \alpha\varepsilon + \sqrt{\alpha\varepsilon})\|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \sqrt{\alpha\varepsilon}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0\, dv\, dx\right) \\ &+ \quad (\varepsilon + \sqrt{\alpha\varepsilon})\left(\frac{1}{2}\int_{\mathbb{R}^3}(|E_\varepsilon^0|^2 + \alpha\varepsilon|B_\varepsilon^0|^2)\, dx + \varepsilon\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|v|^2}{2} f_\varepsilon^0\, dv\, dx\right) \\ &+ \quad C\alpha\varepsilon\left(\|B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \|\partial_t B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}\right) + C\int_0^t \mathcal{H}_{\varepsilon,p}(s)\, ds. \qquad (33)\end{aligned}$$

Proof. We use the estimates proved in Lemmas 3.1 and 3.3

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon(v+E)\cdot(\partial_t E + (\mathrm{D}_x E)v)\, dv\, dx\right| \leq C\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon\left(1 + \frac{|v+E|^2}{2}\right)\, dv\, dx,$$

$$\begin{aligned}\left|\int_{\mathbb{R}^3} A(E_\varepsilon - E, E_\varepsilon - E)\cdot E\, dx\right| \quad &+ \quad \alpha\varepsilon\left|\int_{\mathbb{R}^3} A(B_\varepsilon, B_\varepsilon)\cdot E\, dx\right| \leq C\int_{\mathbb{R}^3}|E_\varepsilon - E|^2\, dx \\ &+ \quad C\alpha\varepsilon\left(\int_{\mathbb{R}^3}|B_\varepsilon - B|^2\, dx + \int_{\mathbb{R}^3}|B|^2\, dx\right),\end{aligned}$$

and the trivial inequalities

$$\alpha\varepsilon\left|\int_{\mathbb{R}^3}\partial_t B\cdot(B_\varepsilon-B)\,dx\right|\le\sqrt{\alpha\varepsilon}\left(\int_{\mathbb{R}^3}|\partial_t B|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}\alpha\varepsilon|B_\varepsilon-B|^2\,dx\right)^{\frac12},$$

$$\begin{aligned}\alpha\varepsilon\left|\int_{\mathbb{R}^3}(J\wedge B_\varepsilon)\cdot E\,dx\right| &\le C\sqrt{\alpha\varepsilon}\left(\int_{\mathbb{R}^3}|J|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}\alpha\varepsilon|B_\varepsilon-B|^2\,dx\right)^{\frac12}\\ &\quad +C\alpha\varepsilon\left(\int_{\mathbb{R}^3}|J|^2\,dx\right)^{\frac12}\left(\int_{\mathbb{R}^3}|B|^2\,dx\right)^{\frac12},\end{aligned}$$

where $C=C(\|E\|_{W^{1,\infty}})$. After integration of (32) with respect to $t\in]0,T[$ we obtain

$$\begin{aligned}\mathcal{H}_\varepsilon(t) &+ \int_0^t\!\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|\sqrt{f_\varepsilon}(v+E)+2\nabla_v\sqrt{f_\varepsilon}|^2\,dv\,dx\,ds\le\mathcal{H}_\varepsilon(0)\\ &+ C\varepsilon\int_0^t\!\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon\left(1+\frac{|v+E|^2}{2}\right)\,dv\,dx\,ds\\ &+ C\int_0^t\int_{\mathbb{R}^3}\left(\frac12|E_\varepsilon-E|^2+\frac{\alpha\varepsilon}{2}|B_\varepsilon-B|^2\right)\,dxds\\ &+ C\alpha\varepsilon(\|B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}+\|\partial_t B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}+\|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))})\\ &- \alpha\varepsilon\int_0^t\int_{\mathbb{R}^3}\partial_t(E_\varepsilon\wedge B_\varepsilon)\cdot E\,dx\,ds. \end{aligned}\tag{34}$$

Note that we have

$$\begin{aligned}\mathcal{H}_\varepsilon(0) &\le C\varepsilon\int_0^t\!\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon^0\left(|\ln f_\varepsilon^0|+1+\frac{|v|^2}{2}\right)\,dv\,dx\,ds\\ &+ \frac12\int_{\mathbb{R}^3}\{|E_\varepsilon^0(x)-E(0,x)|^2+\alpha\varepsilon|B_\varepsilon^0(x)-B(0,x)|^2\}\,dx.\end{aligned}\tag{35}$$

By using Proposition 2.1 we deduce also that

$$\begin{aligned}\left|\alpha\varepsilon\int_0^t\!\!\int_{\mathbb{R}^3}\partial_t(E_\varepsilon\wedge B_\varepsilon)\cdot E\,dx\,ds\right| &\le \alpha\varepsilon\left|\int_{\mathbb{R}^3}(E_\varepsilon\wedge B_\varepsilon)\cdot E\,dx\Big|_0^t\right|\\ &+ \alpha\varepsilon\left|\int_0^t\!\!\int_{\mathbb{R}^3}(E_\varepsilon\wedge B_\varepsilon)\cdot\partial_t E\,dx\,ds\right|\\ &\le C\sqrt{\alpha\varepsilon}(M_\varepsilon^0+W_\varepsilon^0+\|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}).\end{aligned}\tag{36}$$

Finally, combining (34), (35), (36) yields for all $0\le t\le T$

$$\begin{aligned}\mathcal{H}_\varepsilon(t) &+ \int_0^t\!\!\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}|\sqrt{f_\varepsilon}(v+E)+2\nabla_v\sqrt{f_\varepsilon}|^2\,dv\,dx\,ds\le C\varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon^0\,dv\,dx\\ &\quad +C\varepsilon\int_{\mathbb{R}^3}\!\int_{\mathbb{R}^3}f_\varepsilon^0\left(|\ln f_\varepsilon^0|+\frac{|v|^2}{2}\right)\,dv\,dx+\frac12\int_{\mathbb{R}^3}\{|E_\varepsilon^0-E^0|^2+\alpha\varepsilon|B_\varepsilon^0-B^0|^2\}\\ &+ C\alpha\varepsilon(\|B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}+\|\partial_t B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}+\|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))})\\ &+ C\sqrt{\alpha\varepsilon}(M_\varepsilon^0+W_\varepsilon^0+\|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))})+C\int_0^t\mathcal{H}_{\varepsilon,p}(s)\,ds.\end{aligned}\tag{37}$$

Now we can show exactly as in the proof of Lemma 2.1 that

$$\begin{aligned} f_\varepsilon |\ln f_\varepsilon| &\leq f_\varepsilon \ln f_\varepsilon + \frac{1}{2}(|x| + |v+E|^2) f_\varepsilon \\ &+ 2C e^{-\frac{1}{8}(|x|+|v+E|^2)}, \;\; \forall\, (t,x,v) \in]0,T[\times \mathbb{R}^3 \times \mathbb{R}^3. \end{aligned} \tag{38}$$

After integration with respect to (x,v) we deduce

$$\begin{aligned} \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon |\ln f_\varepsilon|\, dv\, dx &\leq \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \left(\ln f_\varepsilon + \frac{|v+E|^2}{2} \right) dv\, dx \\ &+ \frac{\varepsilon}{2} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |x| f_\varepsilon \, dv\, dx + 2\varepsilon C \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} e^{-\frac{1}{8}(|x|+|v|^2)}\, dv\, dx. \end{aligned} \tag{39}$$

Using Proposition 2.1 and (39) we obtain the inequality

$$\mathcal{H}_{\varepsilon,p}(t) \leq \mathcal{H}_\varepsilon(t) + C\varepsilon(1 + M_\varepsilon^0 + W_\varepsilon^0 + L_\varepsilon^0 + \|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))}). \tag{40}$$

Therefore, from (37), (40) we deduce

$$\begin{aligned} \mathcal{H}_{\varepsilon,p}(t) &+ \int_0^t \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}|^2 \, dv\, dx\, ds \leq C \int_{\mathbb{R}^3} \{|E_\varepsilon^0 - E^0|^2 + \alpha\varepsilon |B_\varepsilon^0|^2\} \\ &+ C\varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \left(|\ln f_\varepsilon^0| + 1 + |x| + \frac{|v|^2}{2} \right) dv\, dx \\ &+ C\left(\varepsilon + (\varepsilon + \alpha\varepsilon + \sqrt{\alpha\varepsilon}) \|J\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \sqrt{\alpha\varepsilon} \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon^0 \, dv\, dx \right) \\ &+ (\varepsilon + \sqrt{\alpha\varepsilon}) \left(\frac{1}{2} \int_{\mathbb{R}^3} (|E_\varepsilon^0|^2 + \alpha\varepsilon |B_\varepsilon^0|^2)\, dx + \varepsilon \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon^0 \, dv\, dx \right) \\ &+ C\alpha\varepsilon \left(\|B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \|\partial_t B\|^2_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \|B^0\|_{L^2(\mathbb{R}^3)} \right) \\ &+ C \int_0^t \mathcal{H}_{\varepsilon,p}(s)\, ds, \;\; \forall\, 0 \leq t \leq T. \end{aligned} \tag{41}$$

The conclusion follows easily by observing that

$$\|B\|_{L^\infty(]0,T[;L^2(\mathbb{R}^3))} \leq C\{\|B\|_{L^2(]0,T[;L^2(\mathbb{R}^3))} + \|\partial_t B\|_{L^2(]0,T[;L^2(\mathbb{R}^3))}\}.$$

□

4 Asymptotics

In this section we analyze the asymptotic behavior of smooth solutions $(f_\varepsilon, E_\varepsilon, B_\varepsilon)_{\varepsilon>0}$ of the VMFP system (1), (2), (3), (4), (6), (7) when the parameter $\varepsilon \searrow 0$. We will prove that $(E_\varepsilon)_{\varepsilon>0}$ converges to E strongly in $L^\infty(]0,T[;L^2(\mathbb{R}^3))^3$, $(B_\varepsilon)_{\varepsilon>0}$ converges to B in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)^3$ and $\rho_\varepsilon \to \rho$, $j_\varepsilon \to -\rho E$ in a sense that will be precised, where

(ρ, E, B) is the unique strong solution of (14). Throughout the paper we denote by $\mathcal{M}^1(\mathbb{R}^3)$ the set of bounded Radon measures on $\mathbb{R}^3$, while $\mathcal{M}^1_+(\mathbb{R}^3)$ stands for its positive cone. We recall some definitions and compactness properties in measure spaces (see [9] for more details).

Definition 4.1 *Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1(\mathbb{R}^3)$. We say that*
1) $(\rho_n)_{n\in\mathbb{N}}$ converges vaguely to ρ iff

$$\lim_{n\to+\infty} \int_{\mathbb{R}^3} \varphi \, d\rho_n = \int_{\mathbb{R}^3} \varphi \, d\rho, \tag{42}$$

for any continuous function with compact support $\varphi \in C^0_c(\mathbb{R}^3)$ (actually the convergence holds for any continuous function φ vanishing at infinity i.e., $\lim_{|x|\to+\infty} \varphi(x) = 0$) ;
2) $(\rho_n)_{n\in\mathbb{N}}$ converges tightly to ρ iff (42) *holds for any continuous and bounded function $\varphi \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.*

We have the following classical results.

Proposition 4.1 *1) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1_+(\mathbb{R}^3)$ which converges vaguely to ρ. Assume also that $\lim_{n\to+\infty}\rho_n(\mathbb{R}^3) = \rho(\mathbb{R}^3)$. Then $(\rho_n)_{n\in\mathbb{N}}$ converges to ρ tightly.*
2) Let $(\rho_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{M}^1(\mathbb{R}^3)$ verifying $\sup_n |\rho_n|(\mathbb{R}^3) < +\infty$ and such that for any $\eta > 0$ there exists a compact set $K_\eta \subset \mathbb{R}^3$ satisfying $\sup_n |\rho_n|(\mathbb{R}^3 - K_\eta) \le \eta$. Then $(\rho_n)_{n\in\mathbb{N}}$ is relatively compact for the tight topology.

We recall also the following compactness result, cf. [21].

Proposition 4.2 *Assume that $(\rho_\varepsilon)_{\varepsilon>0}$, $(j_\varepsilon)_{\varepsilon>0}$ satisfy $\rho_\varepsilon \ge 0$, $\partial_t \rho_\varepsilon + \mathrm{div}_x j_\varepsilon = 0$ in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)$, $\forall \varepsilon > 0$ and*

$$\sup_{\varepsilon>0} \sup_{t\in[0,T]} \int_{\mathbb{R}^3} \rho_\varepsilon(t,x)(1+|x|) \, dx < +\infty,$$

$$\sup_{\varepsilon>0} \int_0^T \left(\int_{\mathbb{R}^3} |j_\varepsilon(t,x)| \, dx \right)^2 dt < +\infty,$$

$$\sup_{\varepsilon>0} \int_0^T \int_{\mathbb{R}^3} (1+\sqrt{|x|})|j_\varepsilon(t,x)| \, dx dt < +\infty.$$

Then $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^3) -$ tight $)$ and $(j_\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$- tight.

Now we are ready to end the proof of Theorem 1.1.

Proof of Theorem 1.1. From Proposition 3.3 and the hypotheses (15), (16), (17) we deduce that for $0 \le t \le T$ we have

$$\mathcal{H}_{\varepsilon,p}(t) \;+\; \int_0^t\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v\sqrt{f_\varepsilon}|^2 \, dv\, dx\, ds \le r(\varepsilon) + C\int_0^t \mathcal{H}_{\varepsilon,p}(s)\, ds,$$

with $\lim_{\varepsilon \searrow 0} r(\varepsilon) = 0$. By Gronwall's Lemma we obtain

$$\mathcal{H}_{\varepsilon,p}(t) \leq r(\varepsilon) e^{Ct}, \qquad 0 \leq t \leq T, \;\; \varepsilon > 0,$$

and

$$\int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}|^2 \, dv\, dx\, dt \leq r(\varepsilon) e^{CT}, \qquad \forall\, \varepsilon > 0.$$

In particular we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} \{|E_\varepsilon(t,x) - E(t,x)|^2 + \alpha\varepsilon |B_\varepsilon(t,x)|^2\} \, dx = 0, \;\; \text{uniformly for } t \in [0,T], \tag{43}$$

and

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}|^2 \, dv\, dx\, dt = 0. \tag{44}$$

By Proposition 2.1 and the hypotheses (15), (16), (17) we have

$$\sup_{\varepsilon>0} \int_0^T \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|v|^2}{2} f_\varepsilon \, dv\, dx\, dt < +\infty, \qquad \sup_{\varepsilon>0} \sup_{t\in[0,T]} \int_{\mathbb{R}^3} (1+|x|)\rho_\varepsilon(t,x)\, dx < +\infty.$$

Moreover we have the inequalities

$$\begin{aligned}
\int_0^T \|j_\varepsilon(t)\|^2_{L^1(\mathbb{R}^3)}\, dt &\leq \int_0^T \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v| f_\varepsilon \, dv\, dx \right)^2 dt \\
&\leq \int_0^T \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \, dv\, dx \right) \cdot \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \, dv\, dx \right) dt \\
&\leq \left(\int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |v|^2 f_\varepsilon \, dv\, dx\, dt \right) \cdot \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f^0_\varepsilon \, dv\, dx \right) \\
&\leq C, \qquad \forall\, \varepsilon > 0,
\end{aligned}$$

and

$$\begin{aligned}
\int_0^T \int_{\mathbb{R}^3} (1+\sqrt{|x|})|j_\varepsilon(t,x)|\, dxdt &\leq \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} (1+\sqrt{|x|})|v| f_\varepsilon \, dv\, dx\, dt \\
&\leq \frac{1}{2} \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \{(1+|v|^2) + (|x| + |v|^2)\} f_\varepsilon \, dv\, dx\, dt \\
&\leq C, \qquad \forall\, \varepsilon > 0.
\end{aligned}$$

Therefore, by using Proposition 4.2 we deduce that $(\rho_\varepsilon)_{\varepsilon>0}$ is relatively compact in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^3)-\text{tight})$ and $(j_\varepsilon)_{\varepsilon>0}$ is relatively compact in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight. Since $\mathrm{div}_x(E_\varepsilon - E) = (D - \rho_\varepsilon) - (D - \rho)$ we obtain that $\rho_\varepsilon \to \rho$ in $C^0([0,T], \mathcal{M}^1_+(\mathbb{R}^3)-$ tight). Using now (44) yields

$$\int_0^T \int_{\mathbb{R}^3} |j_\varepsilon + \rho_\varepsilon E|\, dx\, dt \leq \int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \sqrt{f_\varepsilon} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}| \, dv\, dx\, dt \tag{45}$$

$$\leq \left(\int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} f_\varepsilon \, dv\, dx\, dt \right)^{\frac{1}{2}} \left(\int_0^T \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} |\sqrt{f_\varepsilon}(v+E) + 2\nabla_v \sqrt{f_\varepsilon}|^2 \, dv\, dx\, dt \right)^{\frac{1}{2}},$$

and thus we have for all continuous bounded function θ

$$\begin{aligned}\left|\int_0^T\int_{\mathbb{R}^3}(j_\varepsilon+\rho E)\theta\,dx\,dt\right| &\le \left|\int_0^T\int_{\mathbb{R}^3}(j_\varepsilon+\rho_\varepsilon E)\theta\,dx\,dt\right|+\left|\int_0^T\int_{\mathbb{R}^3}(\rho-\rho_\varepsilon)E\theta\,dx\,dt\right|\\ &\le \|\theta\|_{L^\infty}\int_0^T\int_{\mathbb{R}^3}|j_\varepsilon+\rho_\varepsilon E|\,dx\,dt\\ &+ \left|\int_0^T\int_{\mathbb{R}^3}(\rho_\varepsilon-\rho)E\theta\,dx\,dt\right|. \qquad (46)\end{aligned}$$

Since $E\theta$ is bounded and continuous we have $\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(\rho_\varepsilon-\rho)E\theta\,dxdt=0$ and thus (44), (45), (46) imply that $j_\varepsilon\to-\rho E$ in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight.

In order to prove that $(B_\varepsilon)_{\varepsilon>0}$ converges to B in $\mathcal{D}'(]0,T[\times\mathbb{R}^3)^3$ we will show that

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(B_\varepsilon-B)\cdot\varphi\,dx\,dt=0,$$

for all function $\varphi\in C^2_c(]0,T[\times\mathbb{R}^3)^3$. Take φ such a function and observe that in particular we have $\varphi,\partial_t\varphi\in L^2(]0,T[;H^1(\mathbb{R}^3))^3$. By using the decomposition

$$\varphi=\nabla_x\varphi_1+\mathrm{curl}_x\varphi_2,$$

with $\varphi_1,\partial_t\varphi_1\in L^2(]0,T[;H^2(\mathbb{R}^3))$ and $\varphi_2,\partial_t\varphi_2\in L^2(]0,T[;H^2(\mathbb{R}^3))^3$, it is sufficient to prove that

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(B_\varepsilon-B)\cdot\nabla_x\varphi_1\,dx\,dt=0, \qquad (47)$$

and

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(B_\varepsilon-B)\cdot\mathrm{curl}_x\varphi_2\,dx\,dt=0. \qquad (48)$$

The convergence (47) follows easily since $\mathrm{div}_x B_\varepsilon=\mathrm{div}_x B=0$. To justify (48) we use the equations

$$\partial_t E_\varepsilon-\mathrm{curl}_x B_\varepsilon=j_\varepsilon-J,\qquad \partial_t E-\mathrm{curl}_x B=-\rho E-J.$$

After multiplication by the test function φ_2 we find

$$-\int_0^T\int_{\mathbb{R}^3}(E_\varepsilon-E)\cdot\partial_t\varphi_2\,dx\,dt-\int_0^T\int_{\mathbb{R}^3}(B_\varepsilon-B)\cdot\mathrm{curl}_x\varphi_2\,dx\,dt=\int_0^T\int_{\mathbb{R}^3}(j_\varepsilon+\rho E)\cdot\varphi_2\,dx\,dt.$$

Since $\partial_t\varphi_2\in L^2(]0,T[;L^2(\mathbb{R}^3))^3$ and $\lim_{\varepsilon\searrow 0}E_\varepsilon=E$ in $L^2(]0,T[;L^2(\mathbb{R}^3))^3$ the first integral in the left hand side vanishes as $\varepsilon\searrow 0$. To deal with the right hand side, observe that φ_2 is a continuous bounded function since $\varphi_2,\partial_t\varphi_2\in L^2(]0,T[;H^2(\mathbb{R}^3))^3$ imply $\varphi_2\in C^0([0,T];H^2(\mathbb{R}^3))^3\subset C^0([0,T]\times\mathbb{R}^3)^3\cap L^\infty(]0,T[\times\mathbb{R}^3)^3$. By using the convergence $\lim_{\varepsilon\searrow 0}j_\varepsilon=-\rho E$ in $\mathcal{M}^1([0,T]\times\mathbb{R}^3)^3$-tight we deduce

$$\lim_{\varepsilon\searrow 0}\int_0^T\int_{\mathbb{R}^3}(j_\varepsilon+\rho E)\cdot\varphi_2\,dx\,dt=0,$$

and therefore (48) holds. $\square$

Let us end with the following remark, which makes clear the formal result (13).

Corollary 4.1 *Let us set $M_E(t,x,v) = (2\pi)^{-3/2} e^{-|v+E(t,x)|^2/2}$. Under the assumptions of Theorem 1.1, then f_ε converges to ρM_E in the following sense*

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} \big(f_\varepsilon(t,x,v) - \rho(t,x) M_E(t,x,v)\big)\ \varphi(x)\ dx \right| dv\ dt = 0,$$

for any test function $\varphi \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$.

Proof. At first, we expand

$$f_\varepsilon - \rho M_E = (f_\varepsilon - \rho_\varepsilon M_E) + (\rho_\varepsilon - \rho) M_E.$$

Consider $\varphi \in C^0(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3)$. Since $E \in W^{1,\infty}(]0,T[\times\mathbb{R}^3)^3$, for all $(t,v) \in [0,T] \times \mathbb{R}^3$ the function $x \to M_E(t,x,v)\varphi(x)$ is continuous and bounded. We have already shown that $\rho_\varepsilon - \rho$ tends to 0 in $C^0([0,T]; \mathcal{M}^1(\mathbb{R}^3) - tight)$ and thus we have

$$\lim_{\varepsilon \searrow 0} \int_{\mathbb{R}^3} (\rho_\varepsilon(t,x) - \rho(t,x)) M_E(t,x,v) \varphi(x)\ dx = 0,\ \ \forall (t,v) \in [0,T] \times \mathbb{R}^3.$$

Moreover we have the inequality $|v + E(t,x)|^2 \geq \frac{1}{2}|v|^2 - |E(t,x)|^2 \geq \frac{1}{2}|v|^2 - \|E\|^2_{L^\infty}$ and thus $M_E(t,x,v) \leq C(\|E\|_{L^\infty}) e^{-|v|^2/4}$, $\forall\ (t,x,v) \in [0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$. We deduce that

$$\left| \int_{\mathbb{R}^3} (\rho_\varepsilon(t,x) - \rho(t,x)) M_E(t,x,v) \varphi(x)\ dx \right| \leq 2C(\|E\|_{L^\infty}) \|\varphi\|_{L^\infty} \int_{\mathbb{R}^3} D(0,x)\ dx\ e^{-|v|^2/4},$$

and by using the dominated convergence theorem we obtain

$$\lim_{\varepsilon \searrow 0} \int_0^T \int_{\mathbb{R}^3} \left| \int_{\mathbb{R}^3} (\rho_\varepsilon(t,x) - \rho(t,x)) M_E(t,x,v) \varphi(x)\ dx \right|\ dv\ dt = 0.$$

It remains to discuss $f_\varepsilon - \rho_\varepsilon M_E$. We appeal to the logarithmic Sobolev inequality, see e.g. [3], [2], which yields

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^N} \left\{ \frac{f_\epsilon}{\rho_\epsilon M_E} \ln\left(\frac{f_\epsilon}{\rho_\epsilon M_E}\right) - \frac{f_\epsilon}{\rho_\epsilon M_E} + 1 \right\} \rho_\epsilon\ M_E\ dv &= \int_{\mathbb{R}^N} f_\epsilon \ln\left(\frac{f_\epsilon}{\rho_\epsilon M_E}\right) dv \\ \leq \lambda \int_{\mathbb{R}^N} \left| \nabla_v \sqrt{f_\epsilon / M_E} \right|^2 M_E\ dv, \end{aligned}$$

for some $\lambda > 0$. By using (44), the integral with respect to time and space of this quantity tends to 0 as $\varepsilon \searrow 0$. Eventually, we conclude by using the Csiszar-Kullback-Pinsker inequality, see [15], [26], which implies that

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} |f_\epsilon - \rho_\epsilon M_E|\ dv\ dx \right)^2 \leq \mu \int_{\mathbb{R}^N} f_\epsilon \ln\left(\frac{f_\epsilon}{\rho_\epsilon M_E}\right) dv\ dx,$$

with $\mu > 0$. $\square$

5 Appendix

We detail here the dimensional analysis of the equations and the physical meaning of the different parameters introduced previously. Let us write the equations in physical variables. We distinguish the following physical constants
- ε_0 the vacuum permittivity ;
- μ_0 the vacuum permeability ;
- c_0 the vacuum light speed given by $\varepsilon_0 \mu_0 c_0^2 = 1$;
- q the charge of (negative) particles ;
- m the mass of particles ;
- τ the relaxation time which characterizes the interactions of the particles with the thermal bath ;
- K_B the Planck constant ;
- T_{th} the temperature of the thermal bath.

Let $f(t, x, v)$ denote the particle distribution function, which depends on the time $t > 0$, space coordinates $x \in \mathbb{R}^3$ and velocity coordinates $v \in \mathbb{R}^3$. The evolution of f is described by the Fokker-Planck equation

$$\partial_t f + v \cdot \nabla_x f + \frac{F}{m} \cdot \nabla_v f = L_{FP}(f), \;\; (t, x, v) \in]0, +\infty[\times \mathbb{R}^3 \times \mathbb{R}^3,$$

where the Fokker-Planck collision operator is given by

$$L_{FP}(f) = \frac{1}{\tau} \mathrm{div}_v \left(vf + \frac{K_B T_{th}}{m} \nabla_v f \right),$$

and $F(t, x, v) = q(E(t, x) + v \wedge B(t, x))$ represents the Lorentz force. The evolution of the electro-magnetic field (E, B) is given by the Maxwell equations

$$\partial_t E - c_0^2 \cdot \mathrm{curl}_x B = -\frac{j(t, x)}{\varepsilon_0}, \;\; \partial_t B + \mathrm{curl}_x E = 0, \;\; (t, x) \in]0, +\infty[\times \mathbb{R}^3,$$

$$\mathrm{div}_x E = \frac{\rho(t, x)}{\varepsilon_0}, \;\; \mathrm{div}_x B = 0, \;\; (t, x) \in]0, +\infty[\times \mathbb{R}^3,$$

where $\rho = q \int_{\mathbb{R}^3} f \; dv$ and $j = q \int_{\mathbb{R}^3} vf \; dv$ are respectively the charge and current densities. The plasma is characterized by the mean free path $l = \sqrt{\frac{K_B T_{th}}{m}} \cdot \tau$, which is the average distance travelled by a particle between two successive collisions, and the Debye length $\Lambda = \sqrt{\frac{\varepsilon_0 K_B T_{th} L^3}{q^2 \mathcal{N}}}$, which is the typical length of perturbations of a quasi-neutral plasma. Here $\mathcal{N}$ stands for a typical value for the number of particles in the plasma. In this paper we focus our attention on asymptotic regimes where the mean free path is much smaller than the Debye length *i.e.*, $l \ll \Lambda$. We set $\varepsilon = \left(\frac{l}{\Lambda}\right)^2$ which is a small parameter. We introduce time, length and velocity units

$$T = \frac{\tau}{\varepsilon}, \;\; L = \frac{l}{\varepsilon}, \;\; V = \sqrt{\frac{K_B T_{th}}{m}}.$$

Observe also that we have $L = TV$. We define dimensionless variables and unknowns by the relations

$$t = Tt', \quad x = Lx', \quad v = Vv',$$

$$f(t,x,v) = \frac{\mathcal{N}}{L^3V^3} f'(\frac{t}{T}, \frac{x}{L}, \frac{v}{V}), \quad E(t,x) = \frac{U_{th}}{L\varepsilon} E\,'(\frac{t}{T}, \frac{x}{L}), \quad B(t,x) = \frac{VU_{th}}{c_0^2 L\varepsilon} B\,'(\frac{t}{T}, \frac{x}{L}),$$

$$\rho(t,x) = \frac{q\mathcal{N}}{L^3} \rho\,'(\frac{t}{T}, \frac{x}{L}), \quad j(t,x) = \frac{qV\mathcal{N}}{L^3} j\,'(\frac{t}{T}, \frac{x}{L}),$$

where $U_{th} = \frac{K_B T_{th}}{q}$ is the thermal potential. After changing variables and unknowns, we obtain dropping the primes

$$\varepsilon(\partial_t f + v \cdot \nabla_x f) - (E(t,x) + \frac{V^2}{c_0^2} v \wedge B(t,x)) \cdot \nabla_v f = \mathrm{div}_v(vf + \nabla_v f),$$

$$\partial_t E - \mathrm{curl}_x B = -j(t,x), \quad \frac{V^2}{c_0^2} \partial_t B + \mathrm{curl}_x E = 0,$$

$$\mathrm{div}_x E = \rho(t,x), \quad \mathrm{div}_x B = 0,$$

where $\rho(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\, dv$, $j(t,x) = \int_{\mathbb{R}^3} v f(t,x,v)\, dv$. Notice that we have

$$\frac{V^2}{c_0^2} = \left(\frac{l}{\tau}\right)^2 \frac{1}{c_0^2} = \frac{\Lambda^2 \varepsilon}{\tau^2 c_0^2} = \alpha\varepsilon,$$

where $\alpha = \left(\frac{\Lambda}{\tau c_0}\right)^2$, so that we are interested in a regime where the light speed remains large compared to the velocity unit of observation.

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