Chapter Two

SINGLE-PARTICLE MOTIONS

INTRODUCTION 2.1

What makes plasmas particularly difficult to analyze is the fact that the densities fall in an intermediate range. Fluids like water are so dense that the motions of individual molecules do not have to be considered. Collisions dominate, and the simple equations of ordinary fluid dynamics suffice. At the other extreme in very low-density devices like the alternating-gradient synchrotron, only single-particle trajectories need be considered; collective effects are often unimportant. Plasmas behave sometimes like fluids, and sometimes like a collection of individual particles. The first step in learning how to deal with this schizophrenic personality is to understand how single particles behave in electric and magnetic fields. This chapter differs from succeeding ones in that the **E** *and* **B** *fields are assumed to be prescribed* and not affected by the charged particles.

UNIFORM E AND B FIELDS 2.2

$\mathbf{E} = 0$ 2.2.1

In this case, a charged particle has a simple cyclotron gyration. The equation of motion is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B} \qquad [2\text{-}1]$$

Taking $\hat{\mathbf{z}}$ to be the direction of $\mathbf{B}$ ($\mathbf{B} = B\hat{\mathbf{z}}$), we have

$$m\dot{v}_x = qBv_y \qquad m\dot{v}_y = -qBv_x \qquad m\dot{v}_z = 0$$

$$\ddot{v}_x = \frac{qB}{m}\dot{v}_y = -\left(\frac{qB}{m}\right)^2 v_x \tag{2-2}$$

$$\ddot{v}_y = -\frac{qB}{m}\dot{v}_x = -\left(\frac{qB}{m}\right)^2 v_y$$

This describes a simple harmonic oscillator at the *cyclotron frequency*, which we define to be

$$\boxed{\omega_c \equiv \frac{|q|B}{m}} \tag{2-3}$$

By the convention we have chosen, ω_c is always nonnegative. B is measured in tesla, or webers/m^2, a unit equal to 10^4 gauss. The solution of Eq. [2-2] is then

$$v_{x,y} = v_\perp \exp\left(\pm i\omega_c t + i\delta_{x,y}\right)$$

the $\pm$ denoting the sign of q. We may choose the phase δ so that

$$v_x = v_\perp e^{i\omega_c t} = \dot{x} \tag{2-4a}$$

where $v_\perp$ is a positive constant denoting the speed in the plane perpendicular to $\mathbf{B}$. Then

$$v_y = \frac{m}{qB}\dot{v}_x = \pm\frac{1}{\omega_c}\dot{v}_x = \pm iv_\perp e^{i\omega_c t} = \dot{y} \tag{2-4b}$$

Integrating once again, we have

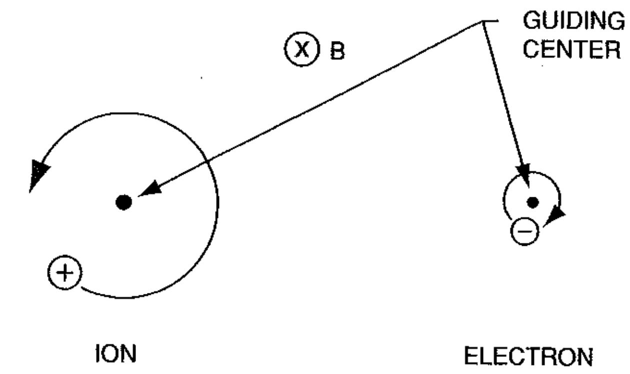
$$x - x_0 = -i\frac{v_\perp}{\omega_c}e^{i\omega_c t} \qquad y - y_0 = \pm\frac{v_\perp}{\omega_c}e^{i\omega_c t} \tag{2-5}$$

We define the *Larmor radius* to be

$$\boxed{r_L \equiv \frac{v_\perp}{\omega_c} = \frac{mv_\perp}{|q|B}} \tag{2-6}$$

Taking the real part of Eq. [2-5], we have

$$x - x_0 = r_L \sin\omega_c t \qquad y - y_0 = \pm r_L \cos\omega_c t \tag{2-7}$$



Larmor orbits in a magnetic field. **FIGURE 2-1**

This describes a circular orbit about a *guiding center* (x_0, y_0) which is fixed (Fig. 2-1). The direction of the gyration is always such that the magnetic field generated by the charged particle is opposite to the externally imposed field. Plasma particles, therefore, tend to *reduce* the magnetic field, and plasmas are *diamagnetic*. In addition to this motion, there is an arbitrary velocity v_z along $\mathbf{B}$ which is not affected by $\mathbf{B}$. The trajectory of a charged particle in space is, in general, a helix.

2.2.2 Finite E

If now we allow an electric field to be present, the motion will be found to be the sum of two motions: the usual circular Larmor gyration plus a drift of the guiding center. We may choose $\mathbf{E}$ to lie in the x–z plane so that $E_y = 0$. As before, the z component of velocity is unrelated to the transverse components and can be treated separately. The equation of motion is now

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v}\times\mathbf{B}) \tag{2-8}$$

whose z component is

$$\frac{dv_z}{dt} = \frac{q}{m}E_z$$

or

$$v_z = \frac{qE_z}{m}t + v_{z0} \tag{2-9}$$

This is a straightforward acceleration along **B**. The transverse components of Eq. [2-8] are

$$\frac{dv_x}{dt} = \frac{q}{m}E_x \pm \omega_c v_y \qquad \text{[2-10]}$$

$$\frac{dv_y}{dt} = 0 \mp \omega_c v_x$$

Differentiating, we have (for constant **E**)

$$\ddot{v}_x = -\omega_c^2 v_x \qquad \text{[2-11]}$$

$$\ddot{v}_y = \mp\omega_c\left(\frac{q}{m}E_x \pm \omega_c v_y\right) = -\omega_c^2\left(\frac{E_x}{B} + v_y\right)$$

We can write this as

$$\frac{d^2}{dt^2}\left(v_y + \frac{E_x}{B}\right) = -\omega_c^2\left(v_y + \frac{E_x}{B}\right)$$

so that Eq. [2-11] is reduced to the previous case if we replace v_y by $v_y + (E_x/B)$. Equation [2-4] is therefore replaced by

$$v_x = v_\perp e^{i\omega_c t} \qquad \text{[2-12]}$$

$$v_y = \pm i v_\perp e^{i\omega_c t} - \frac{E_x}{B}$$

The Larmor motion is the same as before, but there is superimposed a drift $\mathbf{v}_{gc}$ of the guiding center in the $-y$ direction (for $E_x > 0$) (Fig. 2-2).

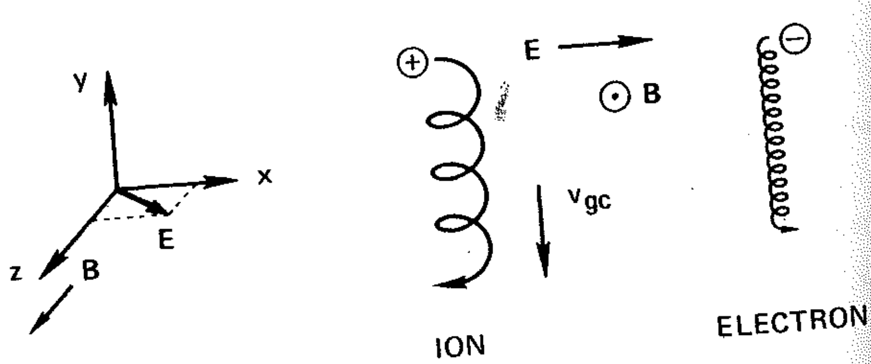


FIGURE 2-2 **Particle drifts in crossed electric and magnetic fields.**

To obtain a general formula for $\mathbf{v}_{gc}$, we can solve Eq. [2-8] in vector form. We may omit the $m\,d\mathbf{v}/dt$ term in Eq. [2-8], since this term gives only the circular motion at ω_c, which we already know about. Then Eq. [2-8] becomes

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0 \qquad \text{[2-13]}$$

Taking the cross product with **B**, we have

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{v} \cdot \mathbf{B}) \qquad \text{[2-14]}$$

The transverse components of this equation are

$$\mathbf{v}_{\perp gc} = \mathbf{E} \times \mathbf{B}/B^2 \equiv \mathbf{v}_E \qquad \text{[2-15]}$$

We define this to be $\mathbf{v}_E$, the electric field drift of the guiding center. In magnitude, this drift is

$$v_E = \frac{E(\mathrm{V/m})}{B(\mathrm{tesla})}\ \frac{\mathrm{m}}{\mathrm{sec}} \qquad \text{[2-16]}$$

It is important to note that $\mathbf{v}_E$ is independent of q, m, and $v_\perp$. The reason is obvious from the following physical picture. In the first half-cycle of the ion's orbit in Fig. 2-2, it gains energy from the electric field and increases in $v_\perp$ and, hence, in r_L. In the second half-cycle, it loses energy and decreases in r_L. This difference in r_L on the left and right sides of the orbit causes the drift v_E. A negative electron gyrates in the opposite direction but also gains energy in the opposite direction; it ends up drifting in the same direction as an ion. For particles of the same velocity but different mass, the lighter one will have smaller r_L and hence drift less per cycle. However, its gyration frequency is also larger, and the two effects exactly cancel. Two particles of the same mass but different energy would have the same ω_c. The slower one will have smaller r_L and hence gain less energy from **E** in a half-cycle. However, for less energetic particles the fractional change in r_L for a given change in energy is larger, and these two effects cancel (Problem 2-4).

The three-dimensional orbit in space is therefore a slanted helix with changing pitch (Fig. 2-3).

Gravitational Field 2.2.3

The foregoing result can be applied to other forces by replacing $q\mathbf{E}$ in the equation of motion [2-8] by a general force **F**. The guiding center

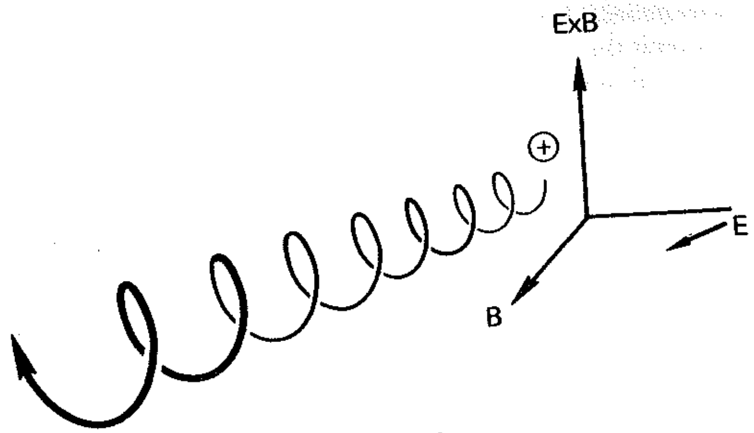


FIGURE 2-3 **The actual orbit of a gyrating particle in space.**

drift caused by **F** is then

$$\boxed{\mathbf{v}_f = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2}} \qquad [2\text{-}17]$$

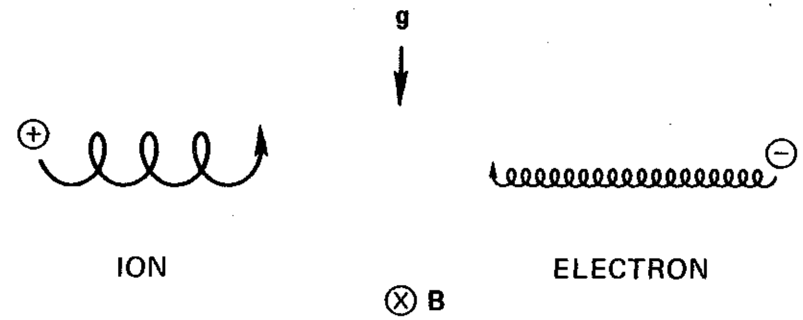
In particular, if **F** is the force of gravity $m\mathbf{g}$, there is a drift

$$\boxed{\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B}}{B^2}} \qquad [2\text{-}18]$$

This is similar to the drift $\mathbf{v}_E$ in that it is perpendicular to both the force and **B**, but it differs in one important respect. The drift $\mathbf{v}_g$ changes sign with the particle's charge. Under a gravitational force, ions and electrons drift in opposite directions, so there is a net current density in the plasma given by

$$\mathbf{j} = n(M + m)\frac{\mathbf{g}\times\mathbf{B}}{B^2} \qquad [2\text{-}19]$$

The physical reason for this drift (Fig. 2-4) is again the change in Larmor radius as the particle gains and loses energy in the gravitational field. Now the electrons gyrate in the opposite sense to the ions, but the force on them is in the same direction, so the drift is in the opposite direction. The magnitude of $\mathbf{v}_g$ is usually negligible (Problem 2-6), but when the lines of force are curved, there is an effective gravitational force due to



The drift of a gyrating particle in a gravitational field. **FIGURE 2-4**

centrifugal force. This force, which is *not* negligible, is independent of mass; this is why we did not stress the m dependence of Eq. [2-18]. Centrifugal force is the basis of a plasma instability called the "gravitational" instability, which has nothing to do with real gravity.

PROBLEMS

2-1. Compute r_L for the following cases if $v_\parallel$ is negligible:

(a) A 10-keV electron in the earth's magnetic field of 5×10^{-5} T.

(b) A solar wind proton with streaming velocity 300 km/sec, $B = 5 \times 10^{-9}$ T.

(c) A 1-keV He^+ ion in the solar atmosphere near a sunspot, where $B = 5 \times 10^{-2}$ T.

(d) A 3.5-MeV He^{++} ash particle in an 8-T DT fusion reactor.

2-2. In the TFTR (Tokamak Fusion Test Reactor) at Princeton, the plasma will be heated by injection of 200-keV neutral deuterium atoms, which, after entering the magnetic field, are converted to 200-keV D ions ($A = 2$) by charge exchange. These ions are confined only if $r_L \ll a$, where $a = 0.6$ m is the minor radius of the toroidal plasma. Compute the maximum Larmor radius in a 5-T field to see if this is satisfied.

2-3. An ion engine (see Fig. 1-6) has a 1-T magnetic field, and a hydrogen plasma is to be shot out at an $\mathbf{E}\times\mathbf{B}$ velocity of 1000 km/sec. How much internal electric field must be present in the plasma?

2-4. Show that v_E is the same for two ions of equal mass and charge but different energies, by using the following physical picture (see Fig. 2-2). Approximate the right half of the orbit by a semicircle corresponding to the ion energy after acceleration by the **E** field, and the left half by a semicircle corresponding to the energy after deceleration. You may assume that **E** is weak, so that the fractional change in $v_\perp$ is small.

2-5. Suppose electrons obey the Boltzmann relation of Problem 1-5 in a cylindrically symmetric plasma column in which $n(r)$ varies with a scale length λ; that is, $\partial n/\partial r \simeq -n/\lambda$.

(a) Using $\mathbf{E} = -\nabla\phi$, find the radial electric field for given λ.

(b) For electrons, show that finite Larmor radius effects are large if v_E is as large as v_{th}. Specifically, show that $r_L = 2\lambda$ if $v_E = v_{\mathrm{th}}$.

(c) Is (b) also true for ions?

Hint: Do *not* use Poisson's equation.

2-6. Suppose that a so-called Q-machine has a uniform field of 0.2 T and a cylindrical plasma with $KT_e = KT_i = 0.2$ eV. The density profile is found experimentally to be of the form

$$n = n_0 \exp\left[\exp\left(-r^2/a^2\right) - 1\right]$$

Assume the density obeys the electron Boltzmann relation $n = n_0 \exp(e\phi/KT_e)$.

(a) Calculate the maximum v_E if $a = 1$ cm.

(b) Compare this with v_g due to the earth's gravitational field.

(c) To what value can B be lowered before the ions of potassium ($A = 39$, $Z = 1$) have a Larmor radius equal to a?

2-7. An unneutralized electron beam has density $n_e = 10^{14}\ \mathrm{m}^{-3}$ and radius $a = 1$ cm and flows along a 2-T magnetic field. If $\mathbf{B}$ is in the $+z$ direction and $\mathbf{E}$ is the electrostatic field due to the beam's charge, calculate the magnitude and direction of the $\mathbf{E} \times \mathbf{B}$ drift at $r = a$. (See Fig. P2-7.)

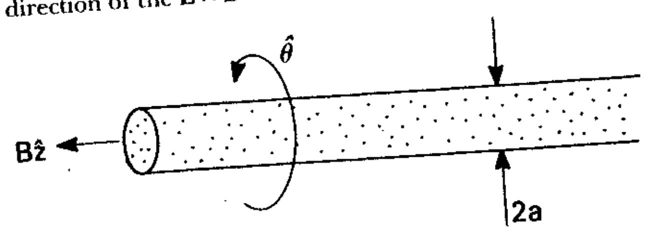
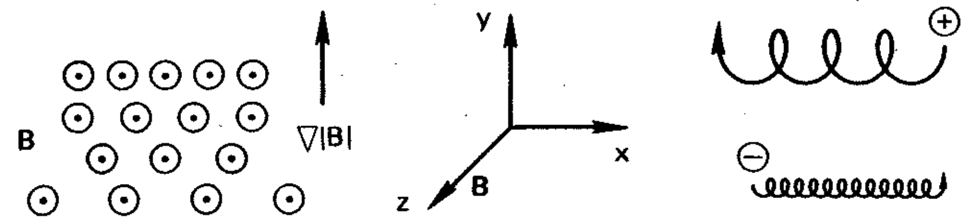


FIGURE P2-7

2.3 NONUNIFORM B FIELD

Now that the concept of a guiding center drift is firmly established, we can discuss the motion of particles in inhomogeneous fields—$\mathbf{E}$ and $\mathbf{B}$ fields which vary in space or time. For uniform fields we were able to obtain exact expressions for the guiding center drifts. As soon as we introduce inhomogeneity, the problem becomes too complicated to solve



The drift of a gyrating particle in a nonuniform magnetic field. FIGURE 2-5

exactly. To get an approximate answer, it is customary to expand in the small ratio r_L/L, where L is the scale length of the inhomogeneity. This type of theory, called *orbit theory*, can become extremely involved. We shall examine only the simplest cases, where only one inhomogeneity occurs at a time.

$\nabla B \perp \mathbf{B}$: Grad-$B$ Drift 2.3.1

Here the lines of force* are straight, but their density increases, say, in the y direction (Fig. 2-5). We can anticipate the result by using our simple physical picture. The gradient in $|B|$ causes the Larmor radius to be larger at the bottom of the orbit than at the top, and this should lead to a drift, in opposite directions for ions and electrons, perpendicular to both $\mathbf{B}$ and ∇B. The drift velocity should obviously be proportional to r_L/L and to $v_\perp$.

Consider the Lorentz force $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$, averaged over a gyration. Clearly, $\bar{F}_x = 0$, since the particle spends as much time moving up as down. We wish to calculate $\bar{F}_y$, in an approximate fashion, by using the *undisturbed orbit* of the particle to find the average. The undisturbed orbit is given by Eqs. [2-4] and [2-7] for a uniform $\mathbf{B}$ field. Taking the real part of Eq. [2-4], we have

$$F_y = -qv_x B_z(y) = -qv_\perp(\cos\omega_c t)\left[B_0 \pm r_L(\cos\omega_c t)\frac{\partial B}{\partial y}\right] \qquad [2\text{-}20]$$

where we have made a Taylor expansion of $\mathbf{B}$ field about the point $x_0 = 0$, $y_0 = 0$ and have used Eq. [2-7]:

$$\mathbf{B} = \mathbf{B}_0 + (\mathbf{r}\cdot\nabla)\mathbf{B} + \cdots$$
$$B_z = B_0 + y(\partial B_z/\partial y) + \cdots \qquad [2\text{-}21]$$

*The magnetic field lines are often called "lines of force." They are not lines of force. The misnomer is perpetuated here to prepare the student for the treacheries of his profession.

This expansion of course requires $r_L/L \ll 1$, where L is the scale length of $\partial B_z/\partial y$. The first term of Eq. [2-20] averages to zero in a gyration, and the average of $\cos^2 \omega_c t$ is $\frac{1}{2}$, so that

$$\bar{F}_y = \mp q v_\perp r_L \tfrac{1}{2}(\partial B/\partial y) \tag{2-22}$$

The guiding center drift velocity is then

$$\mathbf{v}_{gc} = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} = \frac{1}{q}\frac{\bar{F}_y}{|B|}\hat{\mathbf{x}} = \mp\frac{v_\perp r_L}{B}\frac{1}{2}\frac{\partial B}{\partial y}\hat{\mathbf{x}} \tag{2-23}$$

where we have used Eq. [2-17]. Since the choice of the y axis was arbitrary, this can be generalized to

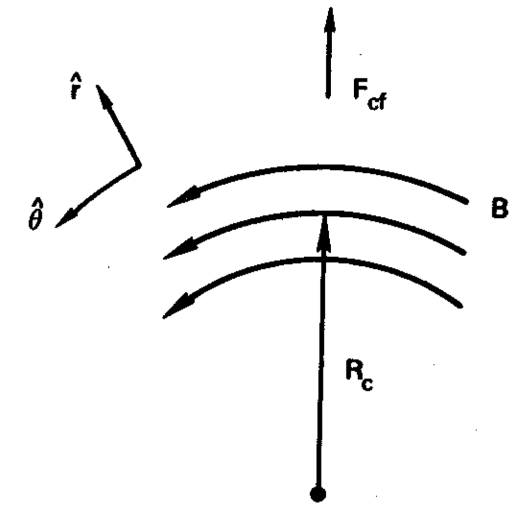
$$\boxed{\mathbf{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_L \frac{\mathbf{B}\times\nabla B}{B^2}} \tag{2-24}$$

This has all the dependences we expected from the physical picture; only the factor $\frac{1}{2}$ (arising from the averaging) was not predicted. Note that the $\pm$ stands for the sign of the charge, and lightface B stands for $|B|$. The quantity $\mathbf{v}_{\nabla B}$ is called the *grad-B drift*; it is in opposite directions for ions and electrons and causes a current transverse to $\mathbf{B}$. An exact calculation of $\mathbf{v}_{\nabla B}$ would require using the exact orbit, including the drift, in the averaging process.

2.3.2 Curved B: Curvature Drift

Here we assume the lines of force to be curved with a constant radius of curvature R_c, and we take $|B|$ to be constant (Fig. 2-6). Such a field does not obey Maxwell's equations in a vacuum, so in practice the grad-B drift will always be added to the effect derived here. A guiding center drift arises from the centrifugal force felt by the particles as they move along the field lines in their thermal motion. If $v_\parallel^2$ denotes the average square of the component of random velocity along $\mathbf{B}$, the average centrifugal force is

$$\mathbf{F}_{cf} = \frac{mv_\parallel^2}{R_c}\hat{\mathbf{r}} = mv_\parallel^2\frac{\mathbf{R}_c}{R_c^2} \tag{2-25}$$



A curved magnetic field. **FIGURE 2-6**

According to Eq. [2-17], this gives rise to a drift

$$\boxed{\mathbf{v}_R = \frac{1}{q}\frac{\mathbf{F}_{cf}\times\mathbf{B}}{B^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2}} \tag{2-26}$$

The drift $\mathbf{v}_R$ is called the *curvature drift.*

We must now compute the grad-B drift which accompanies this when the decrease of $|B|$ with radius is taken into account. In a vacuum, we have $\nabla\times\mathbf{B} = 0$. In the cylindrical coordinates of Fig. 2-6, $\nabla\times\mathbf{B}$ has only a z component, since $\mathbf{B}$ has only a θ component and ∇B only an r component. We then have

$$(\nabla\times\mathbf{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = 0 \qquad B_\theta \propto \frac{1}{r} \tag{2-27}$$

Thus

$$|B| \propto \frac{1}{R_c} \qquad \frac{\nabla|B|}{|B|} = -\frac{\mathbf{R}_c}{R_c^2} \tag{2-28}$$

Using Eq. [2-24], we have

$$\mathbf{v}_{\nabla B} = \mp\frac{1}{2}\frac{v_\perp r_L}{B^2}\mathbf{B}\times|B|\frac{\mathbf{R}_c}{R_c^2} = \pm\frac{1}{2}\frac{v_\perp^2}{\omega_c}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B} = \frac{1}{2}\frac{m}{q}v_\perp^2\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2 B^2} \tag{2-29}$$