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Trapping and Loss of Charged Particles in a Perturbed Magnetic Field

R. C. WINGERSON, T. H. DUPREE, AND D. J. ROSE
Department of Nuclear Engineering and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts
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The properties of a nonadiabatic magnetic system for trapping and confining charged particles are derived.The trapping mechanism is a helical perturbation that matches the gyrations of an incoming particle beam. These particles, in resonance with the structure, convert their axial energy into transverse energy even in the absence of a magnetic mirror. By the same scheme, a spiraling beam could be straightened out. The best injection system is a four-conductor set, which is analyzed in detail. Once confined, the particles experience small and almost random nonadiabatic perturbations in their motion; they diffuse in both real and velocity space. A linear operator that governs this diffusion is presented; it is applicable to a uniform magnetic field with small perturbations. The velocity-space diffusion in a magnetic mirror is calculated for a general perturbation and in particular for a helical perturbation. For this last case, the injected particles can be confined for several hundred axial transits. If the charged particles are modified after being initially trapped (e.g., dissociation of molecular ions), the loss in physically realizable systems can be made virtually negligible compared with charge transfer and other conventional losses.

I. INTRODUCTION

CONSIDER the spatial distribution of charged particles in a static magnetic-field configuration. The equilibrium distribution is everywhere uniform. Nevertheless, particles can be injected into some configurations and temporarily trapped there, because the orbits leading to escape are extremely circuitous. Our paper is concerned with the properties of a subclass of such configurations.

If particles are to be injected and confined for as long a time as possible without change of state, special requirements must be met. They are: (1) that the structure accepts particles with high probability via some narrow path in phase space, to which a beam is matched; (2) that the magnetic structure of the entrance path perturbs the temporarily trapped particles as little as possible. The degree of selectivity determines the duration of trapping, which can in principle be made arbitrarily long.

The problem naturally splits into two parts: the input scheme and its effect upon loss of already trapped particles. For the first, we analyze approximately the motion of charges in a solenoidal magnetic field with a resonant helical perturbation. For the second part, we present a theory of particle losses in solenoidal systems with small perturbations, then specialize to consider perturbed mirrors, and finally we consider a helically perturbed mirror.

A capsule summary of prior work in this area will be helpful in providing the context for our present work. The interesting class of magnetic field structures is that in which small field perturbations are located in spatial resonance with the cyclotron gyrations of selected charged particles moving in an otherwise uniform field. The synchronous perturbation of the affected particles can lead to large changes in their axial (and transverse) energy. Thus (for example) an axial beam can be trapped in a magnetic mirror. All such systems have ideally the property of being very nonadiabatic for particles in some small region of phase space and as adiabatic as possible for particles in all other regions.

The first configuration to perform these functions was an azimuthally symmetric perturbation, proposed by Sinel'nikov[1] and co-workers. Its properties are still being explored both in the USSR[2] and in the United Kingdom.[3] A disadvantage of this system is that already trapped particles moving in either direction can be resonantly "untrapped" by the perturbations.

A helically perturbed system was first proposed by Wingerson[4] to overcome this difficulty in part. It is this so-called corkscrew-mirror system that occupies our principal attention here. He showed that if the field pitch matched an axially incoming

[1] V. D. Fedorchenko, B. N. Rutkevich, and B. M. Chernyi, Zh. Tekhn. Fiz. **29**, 1212 (1959) [English transl.: Soviet Phys.—Tech. Phys. **4**, 1112 (1959)]. K. D. Sinel'nikov, B. N. Rutkevich, and V. D. Fedorchenko, Zh. Tekhn. Fiz. **30**, 249 (1960) [English transl.: Soviet Phys.—Tech. Phys. **5**, 229 (1960)]. K. D. Sinel'nikov, V. D. Fedorchenko, B. N. Rutkevich, B. M. Cherny, and B. G. Safronov, Zh. Tekhn. Fiz. **30**, 256 (1960) [English transl.: Soviet Phys.—Tech. Phys. **5**, 236 (1960)]. K. D. Sinel'nikov, B. N. Rutkevich and B. G. Safronov, *Nonadiabatic Traps for Charged Particles*, (Physico-Technical Institute, Academy of Sciences Ukraine SSR, Khar'kov, 1960); Atomic Energy Commission Report 4724.
[2] V. D. Fedorchenko, *et al*, private communication.
[3] A. E. Robson, Nucl. Fusion Suppl. Pt. 3, 1120 (1962).
[4] R. C. Wingerson, Phys. Rev. Letters **6**, 446 (1961).

particle orbit pitch (including the effects of "handedness" of the system), the particle transferred axial kinetic energy into transverse energy, and could in fact be reflected from a magnetic mirror at the far end of the system. Reflected particles have opposite orbit handedness and are minimally affected by the helical perturbation. Nevertheless, an already trapped particle traveling yet again in the forward direction will may experience an "untrapping" resonant interaction.

Since that time, this corkscrew system has been investigated by Dreicer *et al.*,[5] by Karr, Knapp, and Riensenfeld,[6] and is now being studied also by Baratov[7]. In Ref. 5, some trapping criteria were presented, and electron-beam experiments were reported in detail. In particular, the scattering was measured of already trapped electrons, as they passed again in the forward direction through the perturbing system; these results were compared with orbit calculations. A measure of the number of confined transits in a magnetic mirror containing a helical perturbation was obtained. In Ref. 6, the authors showed that a dense plasma blob injected into such a system from one end could be stopped, giving a transient trapped plasma with high ratio of (plasma pressure/plasma and magnetic pressure). It appears from those results that the helical perturbation looks like an ion cyclotron wave to the moving plasma blob, and the wave–plasma interaction slows the plasma blob.

In this paper, we are not concerned with high plasma-density phenomena, but rather with extensions of thought pertaining to single particle orbits and with the calculation of loss rates from a large class of nonadiabatic systems.

Figure 1 illustrates the trapping problem to be analyzed. Current flowing in the helical ribbon conductor generates a field perturbation with a transverse component that rotates in space. If a charged particle trajectory matches this helical perturbation (resonance), the transverse particle energy can be altered monotonically by changing the helical pitch. The resonance condition that the pitch of the particle and the field perturbation must match is

$$2\pi/\omega_c = -2\pi m/qB_0 = p/v_z. \tag{1}$$

Here, B_0 is the main axial field (the perturbation is small), q and m are charge and mass, v_z is axial velocity, and ω_c is the cyclotron frequency. The quantity p is the pitch of the transverse field lines and has a sign (negative in Fig. 1). In the figure, p is shown as the pitch of the windings. This is exactly true if p does not change down the helix; but there will be slight differences if p is variable with position. We discuss this matter again later. Reversal of two signs of quantities in Eq. (1) is required to preserve the possibility of resonance. The structure of Fig. 1 could be resonant to ions traveling from left to right (q, $v_z > 0$) or to electrons traveling from right to left (q, $v_z < 0$). Reversal of B_0 or p reverses selectivity either to charge or to direction. In Secs. II and III to follow, we analyze the magnetic field and the motion of charges being trapped via the resonance implicit in Eq. (1).

Sections IV–VI display the basic theory of particle losses in such systems. The field perturbations produce small fluctuations in particle directions and in positions of guiding centers; the resulting statistical effect leads effectively to a particle diffusion both in velocity and configuration space; eventually they escape. Minimizing this diffusion for a given initial trapping is important. The interrelation of the two processes leads to a complex optimization that is not yet fully understood, but the salient features seem clear.

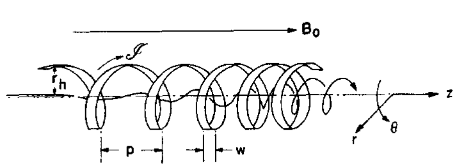

FIG. 1. Helical current $\mathscr{I}$ and ion trajectory in the elementary corkscrew trapping system.

II. EQUATIONS OF MOTION AND OF THE FIELD

A practical corkscrew field has little symmetry and its complete analysis can only be done in three dimensions. Exact solutions for either the field or the motion are very complicated. Thus a number of reasonable approximations are made to obtain analytic results; for any specific application, the approximate analysis guides the specific computation or experiment.

The equation of motion

$$\mathbf{F} = m\dot{\mathbf{v}} = q(\mathbf{v} \times \mathbf{B}) \tag{2}$$

has a z component

$$\begin{aligned} F_z &= (m/2)(d/dz)(v_z^2) \\ &= q(v_r B_\theta - v_\theta B_r) = q(\mathbf{v}_\perp \times \mathbf{B}). \end{aligned} \tag{3}$$

[5] H. Dreicer, H. J. Karr, E. A. Knapp, J. A. Phillips, E. J. Stovall, Jr., and J. L. Tuck, Nucl. Fusion Suppl. Pt. 1, 299 (1962).
[6] H. J. Karr, E. A. Knapp, and W. Riesenfeld, Phys. Fluids 6, 501 (1963).
[7] D. G. Baratov, private communication.

Here, $\mathbf{v}_\perp$ and $\mathbf{B}_\perp$ are components perpendicular to z. We note that B_θ and B_r are perturbations of about equal magnitude, and now make the approximation that the dominant perpendicular motion is gyration about the main axial field lines. Thus $v_\perp \approx v_\theta$ by suitable choice of an axis, and v_r is small. By energy conservation

$$v_z^2 + v_\perp^2 = v_0^2 = \text{const}; \tag{4}$$

then in our approximation, Eq. (3) reduces easily to the algebraic equation

$$dv_\perp/dz = (q/m)B_r; \tag{5}$$

$v_\perp$ has algebraic sign in Eq. (5), corresponding to the direction of v_θ. The equation is applicable whether the particle is in resonance with the field structure or not, and also applies to a particle initially entering the perturbing structure on axis, where $B_\perp \equiv B_r$. The quantity $\int B_r\, dz$ (with B_r as seen by the particle) determines its nonadiabatic history.

Although the field of a corkscrew with variable pitch is not simple, the field of an infinite uniform helix is well known.[8] For reasons to be given shortly, we consider not the field of the single conductor of Fig. 1, but that of a quadrifilar set. Figure 2 shows the arrangement of four ribbon conductors at a cross section normal to the z axis. A current $\mathscr{I}$ flows in each conductor in the direction indicated. The structure rotates by 2π in advancing the pitch distance p (out of the page). Define the radial variable

$$\rho = 2\pi r/p; \tag{6}$$

note that ρ is a weak function of z through the slowly changing $p(z)$. Define also the angle

$$\phi = 2\pi z/p \tag{7}$$

which rotates with the helix; and the angle

$$\chi = (2\pi z/p) - \theta = \phi - \theta \tag{8}$$

which follows the helical symmetry. The angle ϕ, a measure of the orientation of the transverse B lines, is an alternate definition of the pitch. Here again, p has algebraic sign, positive for a right-handed helix. For later use, define also the normalized helix radius ρ_h, and the width w of each ribbon (measured axially).

The field perturbation $\tilde{\mathbf{B}}$ is obtained from the scalar magnetic potential

$$\tilde{\mathbf{B}} = -\nabla\psi_m. \tag{9}$$

[8] H. Poritsky, J. Appl. Phys. **30**, 1828 (1959).

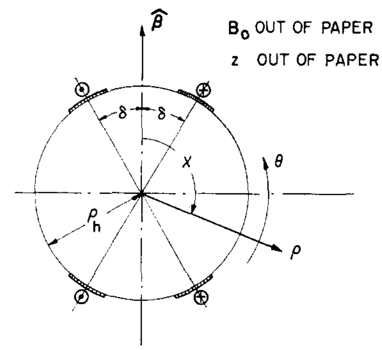

FIG. 2. Arrangement of four conductors at a cross section, and definition of quantities.

For a locally constant pitch helix,

$$\psi_m = \frac{p}{\pi} \sum_{n=1,3,5}^{\infty} \frac{\beta_n}{n} (\cos n\chi) I_n(n\rho); \tag{10}$$

$$\beta_n = -\frac{4\mathscr{I}\mu_0}{|p|}[\sin n\delta][n\rho_h K_n'(n\rho_h)]\left[\frac{\sin(n\pi w/p)}{n\pi w/p}\right]. \tag{11}$$

The radial field is then

$$B_r = \sum_{n=1,3,5\cdots}^{\infty} \beta_n[\cos n\chi][I_{n-1}(n\rho) + I_{n+1}(n\rho)]. \tag{12}$$

Here, μ_0 is the permeability of free space, and I_n and K_n are the modified Bessel functions defined by their asymptotic forms

$$\begin{aligned} I_n(x) &= e^x/(2\pi x)^{\frac{1}{2}}, \qquad x \to \infty, \\ K_n(x) &= e^{-x}(\pi/2x)^{\frac{1}{2}}, \qquad x \to \infty. \end{aligned} \tag{13}$$

A prime indicates differentiation with respect to the argument. These equations follow directly from the work of Ref. 8.

We now make two approximations to Eqs. (11) and (12). First, the pitch of the helix changes slowly ($d\,|p|/dz \ll p/r_h$), whence the field at any point is determined by the local characteristics of the helix. Thus we use the uniform helix equations, but allow p and β to change slowly with position. In this approximation, the quantities p, ϕ, and χ refer either to the winding or to the field, and we no longer distinguish the minor difference.

For the second approximation, we retain only the first term in the summation of Eq. (12). In justification, we note that windings could be added to generate a pure axial harmonic: but more important, the β_n's for higher n decrease almost exponentially with n (i.e., the exact shape of the source conductors at the helix radius has little effect upon the field near the axis). For narrow current ribbons, $(\sin \pi w/p)/(\pi w/p) \approx 1$, and we take

$$\beta = -\frac{4\mathcal{J}\mu_0}{|p|}\, \rho_h K_1'(\rho_h) \sin \delta, \tag{14}$$

$$B_r = \beta \cos \chi [I_0(\rho) + I_2(\rho)]. \tag{15}$$

The quantity β is the magnitude of B_r on the axis in the direction β on Fig. 2. The choice of a quadrifilar winding with angular adjustment δ is desirable because of the form of Eq. (14) and from Fig. 2. Besides having high symmetry consistent with finite B_r on the axis, the magnitude of B_r can be changed conveniently by adjusting δ. Smooth field transitions can then be made down the length of the helix.

III. PERFECT RESONANCE

For perfect resonance of a particle down the length of the corkscrew, $v_\perp$ of the particle will increase or decrease monotonically. From Eq. (5), B_r must have unchanging sign over the orbit, and $\cos \chi$ for the orbit must have constant sign. For example, in the structure of Fig. 1, B_r must be locally negative at the orbit to decelerate an ion axially. (The field lines point toward the axis at the orbit trajectory, the particle acts as if it were moving into a magnetic mirror, and $v_\perp \approx v_\theta$ becomes increasingly negative.) Then, since β is positive, we must have $\pi/2 < \chi < 3\pi/2$ from Eq. (15).

The stability of the quadrants can be demonstrated by a simple topological argument. Consider a line in Fig. 2 from the origin in the direction $\chi = \pi$, which is the direction of maximum negative B_r. This line $\chi = \pi$ developed down the helix generates a helical streamer, as shown solid in Fig. 3 (it has the same pitch as Fig. 1, for comparison). At $\chi = 3\pi/2$, another streamer with $B_r = 0$ exists and is shown dashed in Fig. 3. In this quadrant, an ion is axially decelerated in traveling from left to right. If it is insufficiently decelerated, it runs ahead, thereby encountering increased B_r and increased axial deceleration; similarly, if it falls behind, the deceleration is less. Thus the auger-like volume between the two streamers, marked quadrant 1 in the figure, is stable decelerating. Similarly, quadrant 2 is unstable decelerating for ions from the left. If the pitch had increased toward the right, the structure would be topologically suitable to accelerate spiralling ions entering from the left, stably in quadrant 4, and unstably in quadrant 3. The effect of slowly varying pitch, variable angle of the windings, etc., will slightly distort the regions and the magnitude of B_r. The problem is reminiscent of some stability problems in particle accelerators.

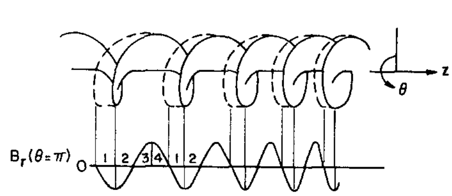

FIG. 3. Region of stable trapping orbits in a corkscrew.

In practice, the particle may hunt throughout this stable auger region, about some mean value χ_0. The mean value can be adjusted by adjusting the rate of change of pitch (for example, in a region of constant pitch $\chi = 3\pi/2$ or $\beta = 0$). In order to keep the perturbing field to a minimum, we desire to have χ_0 as close to π as possible, without having the particle hunt past the stable limits of χ. This possibility of hunting is analyzed in a companion paper.[9]

A convenient recipe exists for calculating consistent orbits and perturbation fields. Define the Larmor radius

$$r_c(z) = v_\perp(z)/\omega_c = -mv_\perp/qB_0 \tag{16}$$

for the particle, and the quantity

$$r_0 = |mv_0/qB_0| \tag{17}$$

which would be the Larmor radius if all the energy were transverse. Consider now a particle entering the system with $v_\perp = 0$. For resonance, Eqs. (1), (4), and (5) yield

$$dr_c/dz = -B_r/B_0 = \gamma(z) \tag{18}$$

$$r_c^2 = r_0^2\{1 - [p(z)/p_0]^2\}, \tag{19}$$

where p_0 is the entrance pitch. Because the numerator of Eq. (18) is B_r as seen by the particle at its position $r(z)$ or $\rho(z)$, these equations are not very tractable for arbitrary winding configurations. For example, if a particle is to have its axial speed reduced to $\frac{1}{2}v_0$, it is easy to show that the particle leaves the perturbing system at $\rho = \sqrt{3}$, and expansions about the axis converge poorly.

The difficulty can be avoided. We show in a later section that minimum scatter of an already trapped particles occurs for perturbation fields whose magnitude drops gradually to zero at the ends. Thus choose $\gamma(z)$ to be a suitable function, integrate Eq. (18), and determine $p(z)$ from Eq. (19). The quantity $\rho(z)$ at the particle orbit is then determined; and from Eq. (19) plus the field

[9] L. M. Lidsky, Phys. Fluids 7, 1484 (1964).

Eqs. (14) and (15) (or other appropriate field equations), we discover the proper parameters.

We illustrate the calculation by example. Let

$$\gamma(z) = (\pi\zeta r_0/2L)\sin(\pi z/L), \tag{20}$$

where

$$\zeta = r_c(z = L)/r_0 < 1, \tag{21}$$

represents the amount of synchronous windup of the particle. We have

$$r_c(z) = \tfrac{1}{2}\zeta r_0[1 - \cos(\pi z/L)] = \zeta r_0 \sin^2(\pi z/2L). \tag{22}$$

The pitch, from Eq. (19), is

$$p(z) = p_0[1 - \zeta^2 \sin^4(\pi z/2L)]^{\frac{1}{2}}, \tag{23}$$

and

$$\rho(z) \equiv \frac{2\pi r(z)}{p(z)} = \frac{\zeta \sin^2(\pi z/2L)}{[1 - \zeta^2 \sin^4(\pi z/2L)]^{\frac{1}{2}}}. \tag{24}$$

Finally, using Eqs. (14) and (15), we have

$$4\mu_0 \mathcal{I}\left[\frac{2\pi r_h}{p(z)}\right]K_1'\left[\frac{2\pi r_h}{p(z)}\right]\sin\delta\cos\chi_0 = \frac{[\pi\zeta B_0 r_0 p(z)/2L]\sin(\pi z/L)}{I_0[\rho(z)] + I_2[\rho(z)]}. \tag{25}$$

Any of the parameters $\mathcal{I}$, r_h, δ, or χ_0 are available to satisfy Eq. (25); we have found it convenient to fix r_h and χ_0, and to adjust $\mathcal{I}$ and $\delta(z)$.

If $\gamma(z)$ in Eq. (18) is symmetric about $z = \frac{1}{2}L$, one-quarter of the eventual energy transfer takes place in the first half of the corkscrew. Since it is possible to inject into a magnetic mirror at an angle through the ends, an appreciable saving in system length might be achieved by such a stratagem.

The dimensionless ratio controlling the amount of windup is the quantity $(\beta_0 \cos\chi_0)L/r_0B_0$ where $\beta_0 \cos\chi_0$ is a mean value of the radial field as seen by the particle.

IV. SCATTERING OPERATOR FOR SLIGHTLY NONADIABATIC MOTION

The scattering of an already trapped particle could be determined from the deceptively simple Eq. (5). Here, B_r as seen by the nonresonant particle is a rapidly fluctuating quantity and causes a succession of small changes in the adiabatic invariants of motion. The cumulative effect is a slow diffusion both in direction of velocity and radially in configuration space.

This diffusion has been approximately calculated by single-particle orbit theory. However, we present here an analysis for a distribution function, based on the Liouville or collisionless Boltzmann equation. Ultimately, the quantity of interest in a confinement problem is the distribution in velocity and space; furthermore, the theory to be given yields much more general and precise results than orbit theory does conveniently. Where calculations have been made by both methods, the results have agreed.

We state the plan of the calculation. The distribution function is $\mathfrak{F}(\mathbf{r}, \mathbf{v})$, by which is meant the density of guiding centers at $\mathbf{r}$ and of particle velocities at $\mathbf{v}$. The function $\mathfrak{F}$ is now disassembled into its two parts flowing in the positive and negative axial directions. The perturbing region is bounded by two planes z_1 and z_2, and the part of $\mathfrak{F}$ flowing from z_1 toward z_2 is $f(\mathbf{r}, \mathbf{v})$. We need to know the relation between f at z_1 and z_2 in steady state. This relation can be expressed by the linear operator T:

$$f(x, y, z_2, \mathbf{v}) = (1 + T)f(x, y, z_1, \mathbf{v}). \tag{26}$$

The most convenient velocity coordinate system is

$$v = |\mathbf{v}| \tag{27}$$

$$s = (v_x^2 + v_y^2)/v_0^2 = v_\perp^2/v^2 \tag{28}$$

$$\alpha = \tan^{-1} v_y/v_x; \tag{29}$$

the element of volume is

$$d\mathbf{v} = v^2\,dv\,d\alpha\,ds/2(1 - s)^{\frac{1}{2}}. \tag{30}$$

We assume that each mirror reflection randomizes the phase α of a particle. This assumption appears reasonable from an intuitive point of view. A particle makes many gyrations during the time it takes it to travel from the perturbing region throughout the system and to return again. The number of gyrations and hence the phase angle with which it re-enters is a sensitive function of the axial velocity v_z. Furthermore, the axial velocity is itself changed on each transit of the perturbing region, and this change is a sensitive function of both v_z and α. With this state of affairs, it is reasonable to assume that the set of entrance phase angles shows little significant correlations with each other. It is, of course, conceivable that long-time correlations might develop if a particle stayed trapped indefinitely.[10] However, correlation on such time scales does not effect particle loss which takes place on a shorter time scale. Under these conditions, f is independent of α, and it is per-

[10] A. Garren, *et al.*, in *Proceedings of the Second United Nations Conference on the Peaceful Uses of Atomic Energy* (United Nations, Geneva, 1958), Vol. 31, p. 65.

missible to use the operator $\bar{T}$ averaged over α. Because speed is preserved, $\bar{T}$ operates in velocity space only upon s.

The perturbation magnetic field $\tilde{\mathbf{B}}_\perp$ normal to the z-axis is expressed in cylindrical coordinates. Its magnitude is β, Eqs. (11) or (14), but more convenient here is the quantity

$$\tilde{\omega}_\perp = -q\tilde{B}_\perp/m. \tag{31}$$

The angle ϕ, corresponding to $\tan^{-1} B_y/B_x$, is defined by Eq. (7); here, ϕ refers specifically to the local transverse direction of the field, no matter how the windings are arranged. The axial perturbation is not required. To avoid unnecessary analytic complications, it is assumed that $\tilde{\omega}_\perp$ and ϕ are independent of x and y. Thus, in this approximation, lines of $\tilde{\mathbf{B}}_\perp$ are straight and parallel at any cross section z = constant.

Particle trajectories in the perturbed-field region are complicated. Although the perturbation is small, it acts synchronously on particles that are being initially trapped and quickly produces large changes in their motion. However, in a finely tuned system, the volume of phase space representing such entering particles is negligibly small, and the neighboring phase space is large inaccessible to the trapped distribution. Thus by far the greatest amount of phase space corresponds to particles whose trajectories are only slightly perturbed. The operator $\bar{T}$ can then be expressed in powers of the small perturbing parameter $\tilde{\omega}_\perp/\omega_c$, where ω_c represents the main axial field B_0 (see Eq. 1); only the lowest order nonvanishing term is retained. A rather involved analysis[11] shows that $\bar{T}$ has a very simple form when expressed in (v, s, α) space. It is

$$\bar{T} = (\partial/\partial s)sD(\partial/\partial s) + \nabla_\perp\cdot(v^2D/4\omega_c^2)\nabla_\perp, \tag{32}$$

where

$$D = \left|\frac{1}{v}\int_{z_1}^{z_2} dz\, \tilde{\omega}_\perp(z) \exp i\left[\phi(z) + \frac{\omega_c z}{v_z}\right]\right|^2, \tag{33}$$

and

$$\nabla_\perp = \hat{\mathbf{x}}(\partial/\partial x) + \hat{\mathbf{y}}(\partial/\partial y). \tag{34}$$

According to Eq. (32), $\bar{T}$ is a diffusion operator in s space and $\mathbf{r}_\perp$ space, with coefficients sD and $v^2D/4\omega_c^2$, respectively. Time does not appear dimensionally in these coefficients, because diffusion is calculated per axial transit. As $v_z \to 0$ $(s \to 1)$, the exponential in Eq. (33) oscillates to give $D \to 0$, which means that no particles diffuse past the $v_z = 0$ point and reverse their motion along the z axis.

Rather than work with the distribution function f itself, it is very convenient to change now to the function

$$j = vf. \tag{35}$$

Since $\bar{T}$ does not operate on v, $\bar{T}$ transforms j properly. The convenience is evident when we note that $v_z = v(1 - s)^{\frac{1}{2}}$, and write the total axial particle current integrated over a cross section

$$\begin{aligned} J &= \int dx\, dy\, d\mathbf{v}\, v_z f \\ &= \int dx\, dy\, vf(ds\, v^2\, dv\, d\alpha/2) \\ &= \int dx\, dy\, j\, d\tau. \end{aligned} \tag{36}$$

Here, $d\tau$ represents the element of velocity space shown in parentheses. The total current J must be constant over every x–y plane even in the presence of radial diffusion, provided only that f is negligibly small at the radial boundary. From the final form of Eq. (36), we interpret j as an axial-current-density distribution function, if it is measured in s space itself (whose volume element is $d\tau$). Therefore, by using j and s space, we utilize the simple conservation properties and interpretation of the axial current; also we avoid the factor $(1 - s)^{\frac{1}{2}}$ that appears in Eq. (30) and elsewhere when f and $d\mathbf{v}$ are used.

A conservation equation relating the divergence of the flow of j to the rate of change of j is useful. For j in s and $\mathbf{r}_\perp$ space (the other coordinates do not matter), the rate of change per transit is $\bar{T}_j$. The sense of the flow is positive in the positive s direction and outward in $\mathbf{r}_\perp$ space. The flow density K in s space derives from the first term on the right side of Eq. (32). The divergence theorem then gives

$$(\partial/\partial s)K(s) = -(\partial/\partial s)sD(\partial/\partial s)j(s), \tag{37}$$

or

$$K(s) = -sD(\partial/\partial s)j(s). \tag{38}$$

Similarly, the spatial part of the flow is

$$\mathbf{P}(s) = -(v^2D/4\omega_c^2)\nabla_\perp j(s); \tag{39}$$

here, $\mathbf{n}\cdot\mathbf{P}\, dl\, d\tau$ is the number of guiding centers in $d\tau$ that pass per unit time through a line ele-

[11] H. T. Dupree, to be published.

ment dl in the x–y plane, and $\mathbf{n}$ is the outward normal to the line element.

The number of particles that leave the perturbing region per unit time must equal the number that enter. However, the average value of s in $j(s)$ changes during one transit by an amount

$$\Delta\bar{s} = \int_0^1 ds\, s\bar{T}j(s)$$

$$= \int_0^1 ds\, s\frac{\partial}{\partial s} sD \frac{\partial}{\partial s} j(s) \tag{40}$$

$$= \int_0^1 ds \left(\frac{\partial}{\partial s} sD\right) j(s). \tag{41}$$

Equation (41) follows from Eq. (40) via two integrations by parts, and the vanishing of s or D at each limit. We conclude that on the average a particle experiences a change in s of

$$\overline{\Delta s} = (\partial/\partial s)sD \tag{42}$$

in one transit. By similar means $j(s)$ may be used to compute average changes in other particle parameters.

V. STEADY STATE PARTICLE DENSITY IN A MAGNETIC MIRROR SYSTEM

As a first example of its application, the operator $\bar{T}$ is used for a calculation of the steady state distribution function $\mathfrak{F}(s)$ in a magnetic-mirror system which contains an arbitrary perturbation field. Let the velocity space loss cone be defined by $s = s_e$. Then all particles with $s > s_e$ would remain trapped indefinitely were it not for the perturbation field $\tilde{\mathbf{B}}$. During each traversal of the mirror system, a particle experiences a small change in s, the cumulative effect of which may eventually enable the particle to escape from the mirror loss cones. According to Eq. (32), there is also a diffusion of particles in real space accompanying the diffusion in velocity space, so that conceivably a particle might strike the walls of a finite radius mirror system before it escapes through the loss cones. For simplicity, it will be assumed that the radius R of the mirror system is large enough so that this loss is negligible. The assumption is not essential; the analysis of diffusion losses in real space is completely analogous to that in velocity space. The ratio of the two loss rates for particles of speed v is approximately, from Eqs. (38) and (39),

$$\frac{\text{rate of particle loss to walls}}{\text{rate of escape through loss cones}} \approx \frac{2\pi(R - r_c)P(s)}{\pi(R - r_c)^2 K(s)}$$

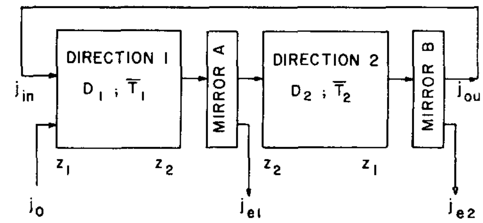

FIG. 4. Schematic representation of a magnetic mirror trap with perturbation field. Perturbations represented by D and $\bar{T}$ cause losses j_e, balanced by a trapped input current j_0.

$$\approx \frac{2\pi(R - r_c)\left[\dfrac{v^2 D}{4\omega_c^2} \dfrac{j}{(R - r_c)}\right]}{\pi(R - r_c)^2[sDj/s]} \approx \left(\frac{r_c}{R - r_c}\right)^2. \tag{43}$$

A schematic representation of a magnetic mirror system is shown in Fig. 4. Here, j_0 represents the source current of particles which are entering the system by one means or another at a value of $s = s_0$, where $s_0 > s_e$. These particles approach the perturbing region at z_1, together with the incoming current j_{in} of previously trapped particles that have just been reflected from a mirror. They all pass through the perturbing region to z_2 in direction 1. The subsequent encounter with mirror A reflects all particles with $s > s_e$. Particles with $s < s_e$ constitute the escape current j_{e1}. The reflected particles pass back through the perturbing region in direction 2 after which they encounter mirror B, where further loss j_{e2} ensues. The reflected particles comprise the current j_{out} which re-enters region 1, etc. Note that for a collisionless toroidal system, regions 1 and 2 close on themselves separately, and the right- and left-handed parts of the distribution $\mathfrak{F}$ are uncoupled.

If we put

$$j_0(s) = a_0\delta(s - s_0), \tag{44}$$

$$j_e(s) = j_{e1} + j_{e2} = a_e\delta(s - s_e), \tag{45}$$

then j_{in} is related to j_{out} by

$$(1 + \bar{T}_2)(1 + \bar{T}_1)j_{in}(s) + a_0\delta(s - s_0) - a_e\delta(s - s_e) = j_{out}(s). \tag{46}$$

But in the steady state we must also have

$$j_{in} = j_{out} = j, \tag{47}$$

$$a_0 = a_e. \tag{48}$$

If the quadratic term $\bar{T}_2\bar{T}_1$ is neglected, Eq. (46) becomes

$$(\bar{T}_1 + \bar{T}_2)j + a_0\delta(s - s_0) - a_0\delta(s - s_e) = 0. \tag{49}$$

Equation (49) will be satisfied if

$$\begin{aligned} sD(\partial/\partial s)j(s) = -K(s) &= 0, \qquad s < s_e, \\ &= a_0, \quad s_0 < s < s_e, \\ &= 0, \qquad s > s_0, \end{aligned} \tag{50}$$

where

$$D = D_1 + D_2; \tag{51}$$

it then follows that

$$\begin{aligned} j(s) &= 0, \qquad 0 < s < s_e, \\ &= a_0 \int_{s_e}^{s} \frac{ds}{sD}, \qquad s_e < s < s_0, \\ &= a_0 \int_{s_e}^{s_0} \frac{ds}{sD}, \qquad s_0 < s < 1. \end{aligned} \tag{52}$$

Using Eq. (35) and adding the distribution function for the two directions, we obtain

$$\mathfrak{F}(s) = 2f(s) = 2j(s)/v. \tag{53}$$

Finally, the density is given by

$$n = 2\pi \int_0^\infty dv \int_0^1 ds\, vj(s)/(1-s)^{\frac{1}{2}}. \tag{54}$$

According to Eq. (52), the result of increasing D is always to reduce particle confinement. (Incidentally, we conclude that mirror losses due to particle–particle interactions cannot be reduced by the introduction of a small nonadiabatic field.)

VI. DIFFUSION COEFFICIENT IN A CORKSCREW-MIRROR SYSTEM

A. Stationary Phase Case

In this case the dominant contribution to the integral comes from the region in which the phase is stationary or nearly so. Therefore, we would expect that the argument of the exponential may be represented by the first few terms of a power series expansion about the stationary point z_0:

$$\phi(z) + \frac{\omega_c z}{v_z} = \phi(z_0) + \frac{\omega_c z_0}{v_z} + \left[\phi'(z_0) + \frac{\omega_c}{v_z}\right][z - z_0] + \tfrac{1}{2}\phi''(z_0)(z - z_0)^2 + \cdots. \tag{55}$$

The linear term in $z - z_0$ vanishes because the relation

$$\phi'(z_0) + (\omega_c/v_z) = 0 \tag{56}$$

determines z_0. Substituting Eq. (55) into Eq. (33), we obtain

$$D = \left|\frac{\tilde{\omega}_\perp(z_0)}{v} \int_0^L \exp\left[i\tfrac{1}{2}\phi''(z_0)(z - z_0)^2\right] dx\right|^2. \tag{57}$$

If z_0 is not too close to 0 or L, then Eq. (57) is approximately

$$D \approx \left|\frac{\tilde{\omega}_\perp(z_0)}{v} \int_{-\infty}^{\infty} dz \exp\left[i\phi''(z_0)z^2\right]\right|^2 = \frac{2\pi\tilde{\omega}_\perp^2(z_0)}{v^2\,|\phi''(z_0)|}. \tag{58}$$

If $x_0 = 0$ or L, D is one-quarter as large. The method fails if ϕ'' also happens to become small at z_0; then Eq. (55) must be expanded to higher powers. This difficulty seldom arises in practice, because the vanishing of ϕ'' implies a local region of the corkscrew designed for no nonadiabatic effect.

The approximate magnitude of D and the consequent confinement time can readily be obtained. From Eq. (38) for the escaping flux, and the fact that s is of order unity, we see that the number N of transits to escape is approximately

$$N \approx 1/D. \tag{59}$$

Thus we are bid to find

$$N \approx \frac{1}{2\pi}\left(\frac{r_0^2 B_0^2}{L^2 \tilde{B}_\perp^2}\right) L^2 \frac{d^2}{dz^2}\left(\frac{2\pi z}{p}\right) \tag{60}$$

suitably averaged over the interval $(0, L)$. The precise value can only be obtained by integration of Eq. (57) for specific cases; but the quantity $(r_0^2 B_0^2/L^2\tilde{B}_\perp^2)$ is of order unity, from the last paragraph of Sec. III. The second derivative ϕ'' will be of order π/Lr_0, whence $N \approx L/r_0$. The example starting with Eq. (20) can be used for illustration. For $\zeta = \frac{3}{2}$, and at $z = L/2$, we find $\phi'' = 1.06\,\pi/Lr_0$, and from Eq. (25) $B_0 r_0/\tilde{B}_\perp L = 0.8 \cos \chi_0$.

The quantity L/r_0 will be large for long systems; for a helical system of 50 turns, about 200 transits are possible.

The general features of a distribution of initially trapped particles can be seen from this calculation. A properly designed corkscrew always has the property that $\phi''[z_0(s)]$ is a positive and monotonically increasing function of s, such as that shown in Fig. 5(a). This property, together with the facts that D is finite at $s = 0$ and zero at $s = 1$, serves to fix the qualitative form of D as shown in Fig. 5(b). Figure 5(c) shows the diffusion coefficient $sD(s)$, and Fig. 5(d) shows the quantity $\overline{\Delta s}$ from Eq. (42). The average changes in transverse energy for a single pass obtained both by experiment and computer[5] agree qualitatively with Fig. 5(d). Finally, Fig. 5(e) shows $j(s)$ from Eq. (52).

B. Nonstationary Phase Case

A confinement time in the order of 200 transits is insufficient for long-time confinement. However, it is a long time for a process such as molecular ion dissociation. The resulting charged particle has a gyration frequency that never matches that of the perturbation field (made for molecular ions). That is, there is no region where the phase of the integrand in Eq. (33) is stationary, and we now show that the diffusion is much reduced. The integrand can be approximated by replacing the variable frequency phase of the exponential by some appropriate average frequency phase. That is

$$\phi(z) + (\omega_c z/v_z) \approx kz; \tag{61}$$

the coefficient k will be of order $1/r_0$.

For the case of sharp field boundaries,

$$\begin{aligned} \tilde{\omega}_\perp(z) &= \tilde{\omega}_\perp, \qquad 0 \leq z \leq L, \\ &= 0, \qquad \text{otherwise}, \end{aligned} \tag{62}$$

we find

$$D \approx \left| \frac{\tilde{\omega}_\perp}{v} \int_0^L e^{ikz}\, dz \right|^2 = \left| \frac{\tilde{\omega}_\perp}{v} \frac{e^{ikL} - 1}{ik} \right|^2. \tag{63}$$

Averaging over k gives

$$D \approx 4\tilde{\omega}_\perp^2/v^2\bar{k}^2; \tag{64}$$

again using $r_0 B_0/L\tilde{B}_\perp \approx 1$, we find a confinement of $N \approx 1/D \approx (L/2r_0)^2$ transits. Thus Eq. (64) represents substantially less diffusion than does Eq. (58). The diffusion can be thought of as arising in the following way. A particle suddenly enters the perturbing region and encounters a succession of equal magnitude perturbations of opposite sign. On the average, one such perturbation remains uncancelled as the particle leaves, and the net effect is expressed by the coefficient of Eq. (64).

This reasoning suggests that the diffusion can be substantially eliminated by designing the magnitude of the perturbation field to decrease gradually to zero at the ends of the perturbing region. Then the net effect will arise only from a lack of cancellation of perturbations at one end, where they are small. Choose, for example,

$$\begin{aligned} \tilde{\omega}_\perp(z) &= \tilde{\omega}_\perp(4z/L), \qquad 0 < z < \tfrac{1}{2}L, \\ &= \tilde{\omega}_\perp 4(L - z)L, \qquad \tfrac{1}{2}L < z < L, \\ &= 0, \qquad \text{otherwise}. \end{aligned} \tag{65}$$

The average value of $\tilde{\omega}_\perp(z)$ in the interval $(0, L)$ is still $\tilde{\omega}_\perp$. But a little algebra shows that Eq. (64) is replaced by

$$D = (6\tilde{\omega}_\perp^2/v^2\bar{k}^2)(16/\bar{k}^2L^2). \tag{66}$$

The diffusion is reduced by a factor of order $(4r_0/L)^2$, and the number of confined transits is $N \approx (L/2r_0)^4$. This is a very large number; for a helix of 50 turns, $N \approx 10^8$ transits; a perturbation field so designed has negligible effect on nonresonant trapped particles.

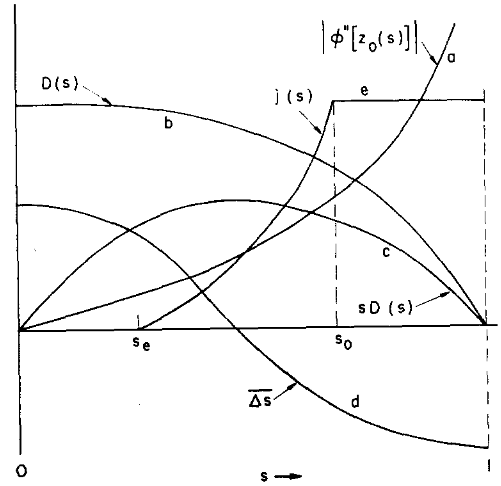

FIG. 5. Sketch of functions related to a particle distribution in a perturbed magnetic field, as functions of perpendicular energy $s = v_\perp^2/v^2$. (a) Second spatial derivative of phase angle, ϕ''. (b) Source function $D(s)$ for diffusion. (c) s space diffusion coefficient sD. (d) Average change $\overline{\Delta s}$ in perpendicular energy per particle per transit. (e) Axial current distribution function $j(s)$ for particles trapped in a magnetic mirror; injection is at $s = s_0$, and the escape cone is at $s = s_e$.

VII. CONCLUSION

The work of Secs. I–III discusses the initial trapping of charged particles in a magnetic mirror by a resonant helical-field perturbation. The calculation is relatively straightforward, and the following comments can be made about the scheme:

(1) The scheme is similar to the resonant injection in a periodic mirror, as proposed by Sinel'nikov, but the symmetry allows more efficient coupling;

(2) The system radius can be several times the Larmor radius; the form of Eq. (12) limits the system radius to perhaps four Larmor radii for appreciable perturbation field on the axis. This compares favorably with conventional ion injection schemes, where the system radius is limited approximately to the ion radius;

(3) Because B_r increases with radius [Eq. (15)], more trapping per unit perturbation might be achieved by injecting off-axis. The orbits would be of a hybrid shape with features of both a helix and a Sinel'nikov trap. On the other hand, the radial diffusion will be larger at large radius, and injection there will bring about more particle loss to the radial walls. These features have not yet been explored;

(4) Because substantial axial velocity remains for the trapped particles, the trapped distribution should not be susceptible to instabilities that appear to arise when $v_\parallel/v_\perp$ is very small;

(5) The injection system is outside the mirror;

(6) As with all spatially resonant systems, the injected beam must be well collimated;

(7) The perturbations should not be appreciably disturbed by the plasmas, so long as $2\mu_0 nkT/B_0^2 \ll 1$; but space charge will affect the energy of injected ions, and some allowance will have to be made in the injection energy. Reference 6 shows that even with large plasma diamagnetism, the trapping effect still exists;

(8) Resonant systems for other applications (e.g., injection into a torus) seem at least conceivable.

Regarding the loss of already trapped particles, an analysis has been given in Sec. IV that is applicable to a wide variety of perturbed systems. Equation (33) shows the importance of keeping the perturbation field $\tilde{B}$ (or $\tilde{\omega}_\perp$) no larger than required to achieve its desired purpose. For a given configuration, the confinement time $\propto 1/\tilde{B}^2$. Section V applies equally well to a resonant periodic mirror as to our case of a helically perturbed system.

Section VI shows that a helically perturbed system of the sort described here is not suitable for longtime trapping of the injected particles, because of the "resonance" diffusion described by Eq. (58). This difficulty is probably inherent in other resonant perturbation schemes, unless the trapped particles seldom encounter the perturbation region. The interesting possibility of effecting such a partial separation, for example by injection off axis into an azimuthally asymmetric structure, has not been investigated. Other nonresonant particles, formed for example by dissociation of the injected beam, are virtually unaffected by the perturbation, and their confinement time is set by other mechanisms.

ACKNOWLEDGMENTS

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Orbit Stability in a Helically Perturbed Magnetic Field

L. M. Lidsky

Department of Nuclear Engineering and Research Laboratory of Electronics, Massachusetts Institute of Technology, Cambridge, Massachusetts

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The stability of particle orbits in resonant helices ("corkscrews") is investigated for axial acceleration and deceleration of the injected particles. It is shown that deviations from the unperturbed orbit lead to growing oscillations about that orbit for a decelerating corkscrew. The growth rate is small in devices suitable for injection into mirror fields. The numerical results are in agreement with experimental observations of injection parameters and output properties. Deviations from the equilibrium orbit in an accelerating corkscrew lead to damped oscillations.

I. INTRODUCTION

The resonant transfer of longitudinal to transverse kinetic energy in a helically perturbed magnetic field ("corkscrew") has been demonstrated experimentally by Wingerson[1] and Dreicer, *et al.*,[2] and discussed in detail by Wingerson, Dupree, and Rose.[3] It has been shown that this resonance is stable to first order; that is, a particle displaced from the stable orbit will oscillate about the position of stability. We address ourselves here to the question of second-order stability: does an oscillation about the stable orbit grow or decay?

II. LINEARIZED EQUATIONS OF MOTION

For the coordinate system of Fig. 1, the orbital equations for $v_\perp$, the aximuthal velocity, and χ, the azimuthal angle between the particle position and the maximum of the helical field, are[4]

$$dv_\perp/dz = \omega_r(r, z) \cos \chi, \tag{1}$$

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[2] H. Dreicer, H. J. Karr, E. A. Knapp, J. A. Phillips, E. J. Stovall, Jr., and J. L. Tuck, Nucl. Fusion Suppl. Pt. 1, 299 (1962).

[3] R. C. Wingerson, T. H. Dupree, and D. J. Rose, Phys. Fluids **7**, 1475 (1964).

[4] The notation differs slightly from that of Wingerson, Dupree, and Rose. In particular, the direction of positive magnetic field in this paper is opposite in sign to that chosen by Wingerson, Dupree, and Rose. This simplifies the equations discussed here.