Adding this to $\mathbf{v}_R$, we have the total drift in a curved vacuum field:

$$\boxed{\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)} \qquad [2\text{-}30]$$

It is unfortunate that these drifts add. This means that if one bends a magnetic field into a torus for the purpose of confining a thermonuclear plasma, the particles will drift out of the torus no matter how one juggles the temperatures and magnetic fields.

For a Maxwellian distribution, Eqs. [1-7] and [1-10] indicate that $\overline{v_\parallel^2}$ and $\frac{1}{2}\overline{v_\perp^2}$ are each equal to KT/m, since $v_\perp$ involves two degrees of freedom. Equations [2-3] and [1-6] then allow us to write the average curved-field drift as

$$\bar{\mathbf{v}}_{R+\nabla B} = \pm \frac{v_{\text{th}}^2}{R_c \omega_c}\hat{\mathbf{y}} = \pm \frac{\bar{r}_L}{R_c} v_{\text{th}} \hat{\mathbf{y}} \qquad [2\text{-}30a]$$

where $\hat{\mathbf{y}}$ here is the direction of $\mathbf{R}_c \times \mathbf{B}$. This shows that $\bar{\mathbf{v}}_{R+\nabla B}$ depends on the charge of the species but not on its mass.

2.3.3 $\nabla B \| \mathbf{B}$: Magnetic Mirrors

Now we consider a magnetic field which is pointed primarily in the z direction and whose magnitude varies in the z direction. Let the field be axisymmetric, with $B_\theta = 0$ and $\partial/\partial\theta = 0$. Since the lines of force converge and diverge, there is necessarily a component B_r (Fig. 2-7). We wish to show that this gives rise to a force which can trap a particle in a magnetic field.

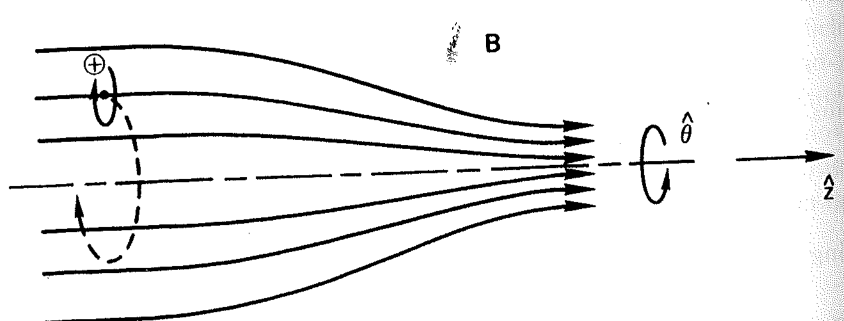


FIGURE 2-7 **Drift of a particle in a magnetic mirror field.**

We can obtain B_r from $\nabla \cdot \mathbf{B} = 0$:

$$\frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial B_z}{\partial z} = 0 \qquad [2\text{-}31]$$

If $\partial B_z/\partial z$ is given at $r = 0$ and does not vary much with r, we have approximately

$$rB_r = -\int_0^r r\frac{\partial B_z}{\partial z}\,dr \simeq -\frac{1}{2}r^2\left[\frac{\partial B_z}{\partial z}\right]_{r=0}$$
$$B_r = -\frac{1}{2}r\left[\frac{\partial B_z}{\partial z}\right]_{r=0} \qquad [2\text{-}32]$$

The variation of $|B|$ with r causes a grad-B drift of guiding centers about the axis of symmetry, but there is no radial grad-B drift, because $\partial B/\partial\theta = 0$. The components of the Lorentz force are

$$\begin{aligned} F_r &= q(v_\theta B_z - v_z \cancel{B_\theta}) \\ &\qquad\quad (1) \\ F_\theta &= q(-v_r B_z + v_z B_r) \\ &\qquad\quad (2) \qquad\ (3) \\ F_z &= q(v_r \cancel{B_\theta} - v_\theta B_r) \\ &\qquad\qquad\qquad (4) \end{aligned} \qquad [2\text{-}33]$$

Two terms vanish if $B_\theta = 0$, and terms 1 and 2 give rise to the usual Larmor gyration. Term 3 vanishes on the axis; when it does not vanish, this azimuthal force causes a drift in the radial direction. This drift merely makes the guiding centers follow the lines of force. Term 4 is the one we are interested in. Using Eq. [2-32], we obtain

$$F_z = \tfrac{1}{2}qv_\theta r(\partial B_z/\partial z) \qquad [2\text{-}34]$$

We must now average over one gyration. For simplicity, consider a particle whose guiding center lies on the axis. Then v_θ is a constant during a gyration; depending on the sign of q, v_θ is $\mp v_\perp$. Since $r = r_L$, the average force is

$$\bar{F}_z = \mp\frac{1}{2}qv_\perp r_L\frac{\partial B_z}{\partial z} = \mp\frac{1}{2}q\frac{v_\perp^2}{\omega_c}\frac{\partial B_z}{\partial z} = -\frac{1}{2}\frac{mv_\perp^2}{B}\frac{\partial B_z}{\partial z} \qquad [2\text{-}35]$$

We define the *magnetic moment* of the gyrating particle to be

$$\boxed{\mu \equiv \tfrac{1}{2}mv_\perp^2/B} \qquad [2\text{-}36]$$

so that

$$\bar{F}_z = -\mu(\partial B_z/\partial z) \qquad [2\text{-}37]$$

This is a specific example of the force on a diamagnetic particle, which in general can be written

$$\mathbf{F}_\parallel = -\mu\ \partial B/\partial \mathbf{s} = -\mu\ \nabla_\parallel B \qquad [2\text{-}38]$$

where ds is a line element along $\mathbf{B}$. Note that the definition [2-36] is the same as the usual definition for the magnetic moment of a current loop with area A and current I: $\mu = IA$. In the case of a singly charged ion, I is generated by a charge e coming around $\omega_c/2\pi$ times a second: $I = e\omega_c/2\pi$. The area A is $\pi r_L^2 = \pi v_\perp^2/\omega_c^2$. Thus

$$\mu = \frac{\pi v_\perp^2}{\omega_c^2}\frac{e\omega_c}{2\pi} = \frac{1}{2}\frac{v_\perp^2 e}{\omega_c} = \frac{1}{2}\frac{mv_\perp^2}{B}$$

As the particle moves into regions of stronger or weaker $\mathbf{B}$, its Larmor radius changes, but μ *remains invariant*. To prove this, consider the component of the equation of motion along $\mathbf{B}$:

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s} \qquad [2\text{-}39]$$

Multiplying by $v_\parallel$ on the left and its equivalent ds/dt on the right, we have

$$mv_\parallel\frac{dv_\parallel}{dt} = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{\partial B}{\partial s}\frac{ds}{dt} = -\mu\frac{dB}{dt} \qquad [2\text{-}40]$$

Here dB/dt is the variation of B as seen by the particle; B itself is constant. The particle's energy must be conserved, so we have

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \frac{1}{2}mv_\perp^2\right) = \frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2 + \mu B\right) = 0 \qquad [2\text{-}41]$$

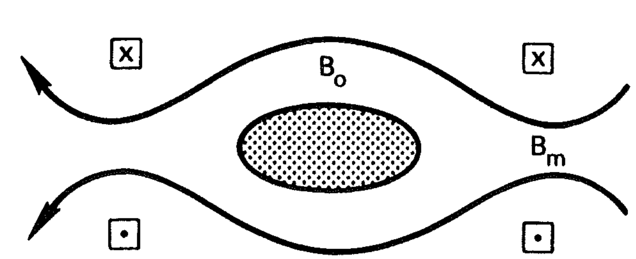
With Eq. [2-40] this becomes

$$-\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0$$

so that

$$d\mu/dt = 0 \qquad [2\text{-}42]$$

The invariance of μ is the basis for one of the primary schemes for plasma confinement: the *magnetic mirror*. As a particle moves from a weak-field region to a strong-field region in the course of its thermal



A plasma trapped between magnetic mirrors. **FIGURE 2-8**

motion, it sees an increasing B, and therefore its $v_\perp$ must increase in order to keep μ constant. Since its total energy must remain constant, $v_\parallel$ must necessarily decrease. If B is high enough in the "throat" of the mirror, $v_\parallel$ eventually becomes zero; and the particle is "reflected" back to the weak-field region. It is, of course, the force $\mathbf{F}_\parallel$ which causes the reflection. The nonuniform field of a simple pair of coils forms two magnetic mirrors between which a plasma can be trapped (Fig. 2-8). This effect works on both ions and electrons.

The trapping is not perfect, however. For instance, a particle with $v_\perp = 0$ will have no magnetic moment and will not feel any force along $\mathbf{B}$. A particle with small $v_\perp/v_\parallel$ at the midplane ($B = B_0$) will also escape if the maximum field B_m is not large enough. For given B_0 and B_m, which particles will escape? A particle with $v_\perp = v_{\perp 0}$ and $v_\parallel = v_{\parallel 0}$ at the midplane will have $v_\perp = v_\perp'$ and $v_\parallel = 0$ at its turning point. Let the field be B' there. Then the invariance of μ yields

$$\tfrac{1}{2}mv_{\perp 0}^2/B_0 = \tfrac{1}{2}mv_\perp'^2/B' \qquad [2\text{-}43]$$

Conservation of energy requires

$$v_\perp'^2 = v_{\perp 0}^2 + v_{\parallel 0}^2 \equiv v_0^2 \qquad [2\text{-}44]$$

Combining Eqs. [2-43] and [2-44], we find

$$\frac{B_0}{B'} = \frac{v_{\perp 0}^2}{v_\perp'^2} = \frac{v_{\perp 0}^2}{v_0^2} \equiv \sin^2\theta \qquad [2\text{-}45]$$

where θ is the pitch angle of the orbit in the weak-field region. Particles with smaller θ will mirror in regions of higher B. If θ is too small, B' exceeds B_m; and the particle does not mirror at all. Replacing B' by B_m in Eq. [2-45], we see that the smallest θ of a confined particle is given by

$$\sin^2\theta_m = B_0/B_m \equiv 1/R_m \qquad [2\text{-}46]$$

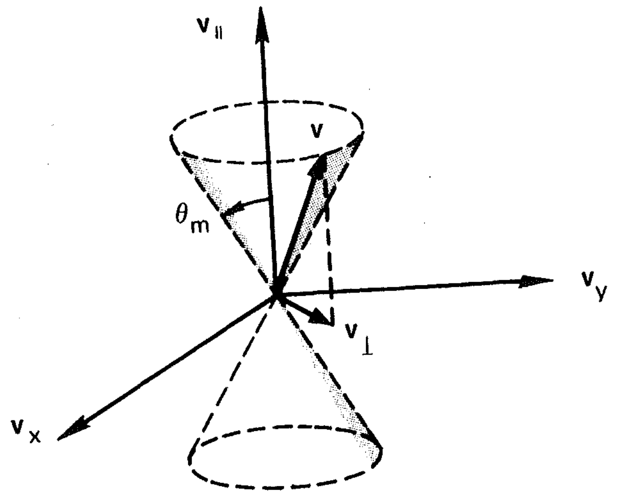


FIGURE 2-9 **The loss cone.**

where R_m is the *mirror ratio*. Equation [2-46] defines the boundary of a region in velocity space in the shape of a cone, called a *loss cone* (Fig. 2-9). Particles lying within the loss cone are not confined. Consequently, a mirror-confined plasma is never isotropic. Note that the loss cone is independent of q or m. Without collisions, both ions and electrons are equally well confined. When collisions occur, particles are lost when they change their pitch angle in a collision and are scattered into the loss cone. Generally, electrons are lost more easily because they have a higher collision frequency.

The magnetic mirror was first proposed by Enrico Fermi as a mechanism for the acceleration of cosmic rays. Protons bouncing between magnetic mirrors approaching each other at high velocity could gain energy at each bounce. How such mirrors could arise is another story. A further example of the mirror effect is the confinement of particles in the Van Allen belts. The magnetic field of the earth, being strong at the poles and weak at the equator, forms a natural mirror with rather large R_m.

PROBLEMS

2-8. Suppose the earth's magnetic field is 3×10^{-5} T at the equator and falls off as $1/r^3$, as for a perfect dipole. Let there be an isotropic population of 1-eV protons and 30-keV electrons, each with density $n = 10^7$ m^{-3} at $r = 5$ earth radii in the equatorial plane.

(a) Compute the ion and electron ∇B drift velocities.

(b) Does an electron drift eastward or westward?

(c) How long does it take an electron to encircle the earth?

(d) Compute the ring current density in A/m^2.

Note: The curvature drift is not negligible and will affect the numerical answer, but neglect it anyway.

2-9. An electron lies at rest in the magnetic field of an infinite straight wire carrying a current **I**. At $t = 0$, the wire is suddenly charged to a positive potential ϕ without affecting **I**. The electron gains energy from the electric field and begins to drift.

(a) Draw a diagram showing the orbit of the electron and the relative directions of **I**, **B**, $\mathbf{v}_E$, $\mathbf{v}_{\nabla B}$, and $\mathbf{v}_R$.

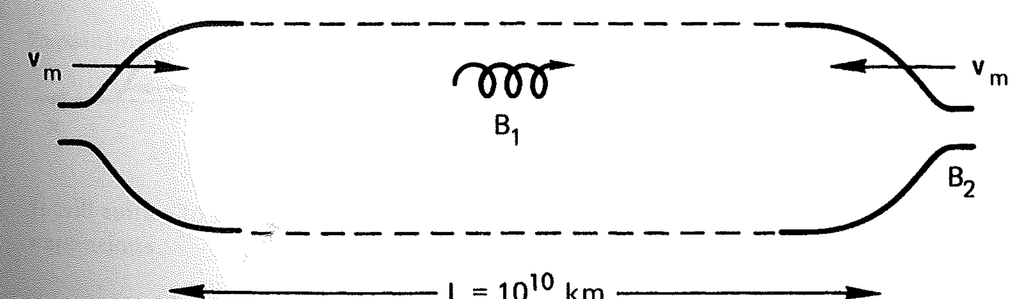
(b) Calculate the magnitudes of these drifts at a radius of 1 cm if $I = 500$ A, $\phi = 460$ V, and the radius of the wire is 1 mm. Assume that ϕ is held at 0 V on the vacuum chamber walls 10 cm away.

Hint: A good intuitive picture of the motion is needed in addition to the formulas given in the text.

2-10. A 20-keV deuteron in a large mirror fusion device has a pitch angle θ of 45° at the midplane, where $B = 0.7$ T. Compute its Larmor radius.

2-11. A plasma with an isotropic velocity distribution is placed in a magnetic mirror trap with mirror ratio $R_m = 4$. There are no collisions, so the particles in the loss cone simply escape, and the rest remain trapped. What fraction is trapped?

2-12. A cosmic ray proton is trapped between two moving magnetic mirrors with $R_m = 5$ and initially has $W = 1$ keV and $v_\perp = v_\parallel$ at the midplane. Each mirror moves toward the midplane with a velocity $v_m = 10$ km/sec (Fig. 2-10).



Acceleration of cosmic rays. **FIGURE 2-10**

(a) Using the loss cone formula and the invariance of μ, find the energy to which the proton will be accelerated before it escapes.

(b) How long will it take to reach that energy?

1. Treat the mirrors as flat pistons and show that the velocity gained at each bounce is $2v_m$.
2. Compute the number of bounces necessary.
3. Compute the time T it takes to traverse L that many times. Factor-of-two accuracy will suffice.

2.4 NONUNIFORM E FIELD

Now we let the magnetic field be uniform and the electric field be nonuniform. For simplicity, we assume **E** to be in the x direction and to vary sinusoidally in the x direction (Fig. 2-11):

$$\mathbf{E} = E_0(\cos kx)\hat{\mathbf{x}} \qquad [2\text{-}47]$$

This field distribution has a wavelength $\lambda = 2\pi/k$ and is the result of a sinusoidal distribution of charges, which we need not specify. In practice, such a charge distribution can arise in a plasma during a wave motion. The equation of motion is

$$m(d\mathbf{v}/dt) = q[\mathbf{E}(x) + \mathbf{v} \times \mathbf{B}] \qquad [2\text{-}48]$$

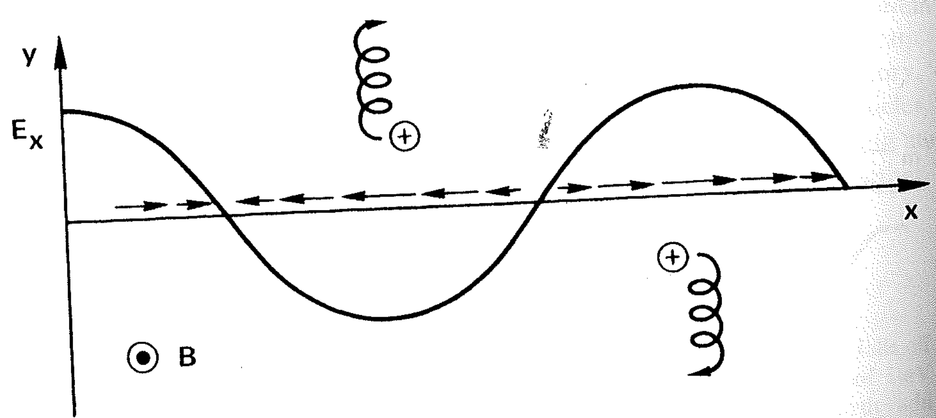


FIGURE 2-11 **Drift of a gyrating particle in a nonuniform electric field.**

whose transverse components are

$$\dot{v}_x = \frac{qB}{m}v_y + \frac{q}{m}E_x(x) \qquad \dot{v}_y = -\frac{qB}{m}v_x \qquad [2\text{-}49]$$

$$\ddot{v}_x = -\omega_c^2 v_x \pm \omega_c \frac{\dot{E}_x}{B} \qquad [2\text{-}50]$$

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_x(x)}{B} \qquad [2\text{-}51]$$

Here $E_x(x)$ is the electric field at the position of the particle. To evaluate this, we need to know the particle's orbit, which we are trying to solve for in the first place. If the electric field is weak, we may, as an approximation, use the *undisturbed orbit* to evaluate $E_x(x)$. The orbit in the absence of the E field was given in Eq. [2-7]:

$$x = x_0 + r_L \sin \omega_c t \qquad [2\text{-}52]$$

From Eqs. [2-51] and [2-47], we now have

$$\ddot{v}_y = -\omega_c^2 v_y - \omega_c^2 \frac{E_0}{B} \cos k(x_0 + r_L \sin w_c t) \qquad [2\text{-}53]$$

Anticipating the result, we look for a solution which is the sum of a gyration at ω_c and a steady drift v_E. Since we are interested in finding an expression for v_E, we take out the gyratory motion by averaging over a cycle. Equation [2-50] then gives $\bar{v}_x = 0$. In Eq. [2-53], the oscillating term $\ddot{v}_y$ clearly averages to zero, and we have

$$\bar{\ddot{v}}_y = 0 = -\omega_c^2 \bar{v}_y - \omega_c^2 \frac{E_0}{B} \overline{\cos k(x_0 + r_L \sin \omega_c t)} \qquad [2\text{-}54]$$

Expanding the cosine, we have

$$\cos k(x_0 + r_L \sin \omega_c t) = \cos(kx_0)\cos(kr_L \sin \omega_c t) - \sin(kx_0)\sin(kr_L \sin \omega_c t) \qquad [2\text{-}55]$$

It will suffice to treat the small Larmor radius case, $kr_L \ll 1$. The Taylor expansions

$$\cos \epsilon = 1 - \tfrac{1}{2}\epsilon^2 + \cdots$$
$$\sin \epsilon = \epsilon + \cdots \qquad [2\text{-}56]$$

allow us to write

$$\cos k(x_0 + r_L \sin \omega_c t) \approx (\cos kx_0)(1 - \tfrac{1}{2}k^2 r_L^2 \sin^2 \omega_c t) - (\sin kx_0) k r_L \sin \omega_c t$$

The last term vanishes upon averaging over time, and Eq. [2-54] gives

$$\bar{v}_y = -\frac{E_0}{B}(\cos kx_0)\left(1 - \frac{1}{4}k^2 r_L^2\right) = -\frac{E_x(x_0)}{B}\left(1 - \frac{1}{4}k^2 r_L^2\right) \quad [2\text{-}57]$$

Thus the usual $\mathbf{E} \times \mathbf{B}$ drift is modified by the inhomogeneity to read

$$\mathbf{v}_E = \frac{\mathbf{E} \times \mathbf{B}}{B^2}\left(1 - \frac{1}{4}k^2 r_L^2\right) \quad [2\text{-}58]$$

The physical reason for this is easy to see. An ion with its guiding center at a maximum of $\mathbf{E}$ actually spends a good deal of its time in regions of weaker $\mathbf{E}$. Its average drift, therefore, is less than E/B evaluated at the guiding center. In a linearly varying $\mathbf{E}$ field, the ion would be in a stronger field on one side of the orbit and in a field weaker by the same amount on the other side; the correction to $\mathbf{v}_E$ then cancels out. From this it is clear that the correction term depends on the *second derivative* of $\mathbf{E}$. For the sinusoidal distribution we assumed, the second derivative is always negative with respect to $\mathbf{E}$. For an arbitrary variation of $\mathbf{E}$, we need only replace ik by ∇ and write Eq. [2-58] as

$$\boxed{\mathbf{v}_E = \left(1 + \frac{1}{4}r_L^2 \nabla^2\right)\frac{\mathbf{E} \times \mathbf{B}}{B^2}} \quad [2\text{-}59]$$

The second term is called the *finite-Larmor-radius effect*. What is the significance of this correction? Since r_L is much larger for ions than for electrons, $\mathbf{v}_E$ is no longer independent of species. If a density clump occurs in a plasma, an electric field can cause the ions and electrons to separate, generating another electric field. If there is a feedback mechanism that causes the second electric field to enhance the first one, $\mathbf{E}$ grows indefinitely, and the plasma is unstable. Such an instability, called a *drift instability*, will be discussed in a later chapter. The grad-B drift, of course, is also a finite-Larmor-radius effect and also causes charges to separate. According to Eq. [2-24], however, $\mathbf{v}_{\nabla B}$ is proportional to kr_L, whereas the correction term in Eq. [2-58] is proportional to $k^2 r_L^2$. The nonuniform-E-field effect, therefore, is important at relatively large k, or small scale lengths of the inhomogeneity. For this reason, drift instabilities belong to a more general class called *microinstabilities*.

2.5 TIME-VARYING E FIELD

Let us now take $\mathbf{E}$ and $\mathbf{B}$ to be uniform in space but varying in time. First, consider the case in which $\mathbf{E}$ alone varies sinusoidally in time, and let it lie along the x axis:

$$\mathbf{E} = E_0 e^{i\omega t}\,\hat{\mathbf{x}} \quad [2\text{-}60]$$

Since $\dot{E}_x = i\omega E_x$, we can write Eq. [2-50] as

$$\ddot{v}_x = -\omega_c^2\left(v_x \mp \frac{i\omega}{\omega_c}\frac{\tilde{E}_x}{B}\right) \quad [2\text{-}61]$$

Let us define

$$\tilde{v}_p \equiv \pm\frac{i\omega}{\omega_c}\frac{\tilde{E}_x}{B} \qquad \tilde{v}_E \equiv -\frac{\tilde{E}_x}{B} \quad [2\text{-}62]$$

where the tilde has been added merely to emphasize that the drift is oscillating. The upper (lower) sign, as usual, denotes positive (negative) q. Now Eqs. [2-50] and [2-51] become

$$\ddot{v}_x = -\omega_c^2(v_x - \tilde{v}_p) \qquad \ddot{v}_y = -\omega_c^2(v_y - \tilde{v}_E) \quad [2\text{-}63]$$

By analogy with Eq. [2-12], we try a solution which is the sum of a drift and a gyratory motion:

$$v_x = v_\perp e^{i\omega_c t} + \tilde{v}_p \qquad v_y = \pm i v_\perp e^{i\omega_c t} + \tilde{v}_E \quad [2\text{-}64]$$

If we now differentiate twice with respect to time, we find

$$\ddot{v}_x = -\omega_c^2 v_x + (\omega_c^2 - \omega^2)\tilde{v}_p \qquad \ddot{v}_y = -\omega_c^2 v_y + (\omega_c^2 - \omega^2)\tilde{v}_E \quad [2\text{-}65]$$

This is not the same as Eq. [2-63] unless $\omega^2 \ll \omega_c^2$. If we now make the assumption that $\mathbf{E}$ varies slowly, so that $\omega^2 \ll \omega_c^2$, then Eq. [2-64] is the approximate solution to Eq. [2-63].

Equation [2-64] tells us that the guiding center motion has two components. The y component, perpendicular to $\mathbf{B}$ and $\mathbf{E}$, is the usual $\mathbf{E} \times \mathbf{B}$ drift, except that v_E now oscillates slowly at the frequency ω. The x component, a new drift *along the direction of* $\mathbf{E}$, is called the *polarization drift*. By replacing $i\omega$ by $\partial/\partial t$, we can generalize Eq. [2-62] and define the polarization drift as

$$\boxed{\mathbf{v}_p = \pm \frac{1}{\omega_c B} \frac{d\mathbf{E}}{dt}} \qquad [2\text{-}66]$$

Since $\mathbf{v}_p$ is in opposite directions for ions and electrons, there is a *polarization current*; for $Z = 1$, this is

$$\mathbf{j}_p = ne(v_{ip} - v_{ep}) = \frac{ne}{eB^2}(M + m)\frac{d\mathbf{E}}{dt} = \frac{\rho}{B^2}\frac{d\mathbf{E}}{dt} \qquad [2\text{-}67]$$

where ρ is the mass density.

The physical reason for the polarization current is simple (Fig. 2-12). Consider an ion at rest in a magnetic field. If a field $\mathbf{E}$ is suddenly applied, the first thing the ion does is to move in the direction of $\mathbf{E}$. Only after picking up a velocity $\mathbf{v}$ does the ion feel a Lorentz force $e\mathbf{v} \times \mathbf{B}$ and begin to move downward in Fig. (2-12). If $\mathbf{E}$ is now kept constant, there is no further $\mathbf{v}_p$ drift but only a $\mathbf{v}_E$ drift. However, if $\mathbf{E}$ is reversed, there is again a momentary drift, this time to the left. Thus $\mathbf{v}_p$ is a startup drift due to inertia and occurs only in the first half-cycle of each gyration during which $\mathbf{E}$ changes. Consequently, $\mathbf{v}_p$ goes to zero with ω/ω_c.

The polarization effect in a plasma is similar to that in a solid dielectric, where $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P}$. The dipoles in a plasma are ions and electrons separated by a distance r_L. But since ions and electrons can move around to preserve quasineutrality, the application of a steady $\mathbf{E}$ field does not result in a polarization field $\mathbf{P}$. However, if $\mathbf{E}$ oscillates, an oscillating current $\mathbf{j}_p$ results from the lag due to the ion inertia.

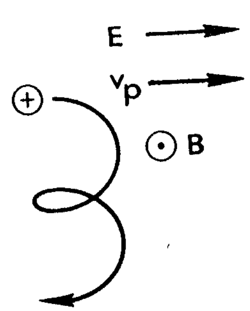


FIGURE 2-12 **The polarization drift.**

TIME-VARYING B FIELD 2.6

Finally, we allow the magnetic field to vary in time. Since the Lorentz force is always perpendicular to $\mathbf{v}$, a magnetic field itself cannot impart energy to a charged particle. However, associated with $\mathbf{B}$ is an electric field given by

$$\nabla \times \mathbf{E} = -\dot{\mathbf{B}} \qquad [2\text{-}68]$$

and this can accelerate the particles. We can no longer assume the fields to be completely uniform. Let $\mathbf{v}_\perp = d\mathbf{l}/dt$ be the transverse velocity $\mathbf{l}$ being the element of path along a particle trajectory (with $v_\parallel$ neglected). Taking the scalar product of the equation of motion [2-8] with $\mathbf{v}_\perp$, we have

$$\frac{d}{dt}\left(\frac{1}{2}mv_\perp^2\right) = q\mathbf{E} \cdot \mathbf{v}_\perp = q\mathbf{E} \cdot \frac{d\mathbf{l}}{dt} \qquad [2\text{-}69]$$

The change in one gyration is obtained by integrating over one period:

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \int_0^{2\pi/\omega_c} q\mathbf{E} \cdot \frac{d\mathbf{l}}{dt}\,dt$$

If the field changes slowly, we can replace the time integral by a line integral over the unperturbed orbit:

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \oint q\mathbf{E} \cdot d\mathbf{l} = q\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S}$$

$$= -q\int_S \dot{\mathbf{B}} \cdot d\mathbf{S} \qquad [2\text{-}70]$$

Here $\mathbf{S}$ is the surface enclosed by the Larmor orbit and has a direction given by the right-hand rule when the fingers point in the direction of $\mathbf{v}$. Since the plasma is diamagnetic, we have $\mathbf{B} \cdot d\mathbf{S} < 0$ for ions and >0 for electrons. Then Eq. [2-70] becomes

$$\delta\left(\frac{1}{2}mv_\perp^2\right) = \pm q\dot{B}\pi r_L^2 = \pm q\pi\dot{B}\frac{v_\perp^2}{\omega_c}\frac{m}{\pm qB} = \frac{\frac{1}{2}mv_\perp^2}{B} \cdot \frac{2\pi\dot{B}}{\omega_c} \qquad [2\text{-}71]$$

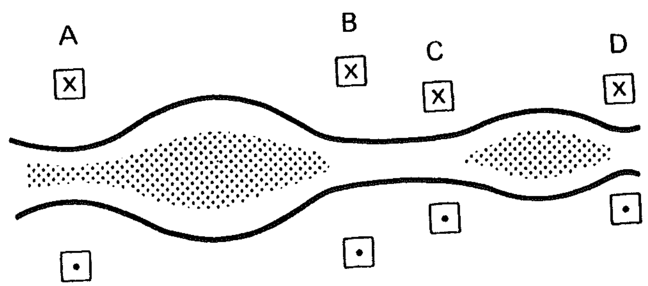


FIGURE 2-13 **Two-stage adiabatic compression of a plasma.**

The quantity $2\pi\dot{B}/\omega_c = \dot{B}/f_c$ is just the change δB during one period of gyration. Thus

$$\delta(\tfrac{1}{2}mv_\perp^2) = \mu\,\delta B \tag{2-72}$$

Since the left-hand side is $\delta(\mu B)$, we have the desired result

$$\delta\mu = 0 \tag{2-73}$$

The magnetic moment is invariant in slowly varying magnetic fields.

As the B field varies in strength, the Larmor orbits expand and contract, and the particles lose and gain transverse energy. This exchange of energy between the particles and the field is described very simply by Eq. [2-73]. The invariance of μ allows us to prove easily the following well-known theorem:

The magnetic flux through a Larmor orbit is constant.

The flux Φ is given by BS, with $S = \pi r_L^2$. Thus

$$\Phi = B\pi\frac{v_\perp^2}{\omega_c^2} = B\pi\frac{v_\perp^2 m^2}{q^2B^2} = \frac{2\pi m}{q^2}\frac{\frac{1}{2}mv_\perp^2}{B} = \frac{2\pi m}{q^2}\mu \tag{2-74}$$

Therefore, Φ is constant if μ is constant.

This property is used in a method of plasma heating known as *adiabatic compression*. Figure 2-13 shows a schematic of how this is done. A plasma is injected into the region between the mirrors A and B. Coils A and B are then pulsed to increase $\mathbf{B}$ and hence $v_\perp^2$. The heated plasma can then be transferred to the region C–D by a further pulse in A, increasing the mirror ratio there. The coils C and D are then pulsed to further compress and heat the plasma. Early magnetic mirror fusion devices employed this type of heating. Adiabatic compression has also been used successfully on toroidal plasmas and is an essential element of laser-driven fusion schemes using either magnetic or inertial confinement.

SUMMARY OF GUIDING CENTER DRIFTS 2.7

General force $\mathbf{F}$:

$$\mathbf{v}_f = \frac{1}{q}\frac{\mathbf{F}\times\mathbf{B}}{B^2} \tag{2-17}$$

Electric field:

$$\mathbf{v}_E = \frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{2-15}$$

Gravitational field:

$$\mathbf{v}_g = \frac{m}{q}\frac{\mathbf{g}\times\mathbf{B}}{B^2} \tag{2-18}$$

Nonuniform $\mathbf{E}$:

$$\mathbf{v}_E = \left(1 + \frac{1}{4}r_L^2\nabla^2\right)\frac{\mathbf{E}\times\mathbf{B}}{B^2} \tag{2-59}$$

Nonuniform $\mathbf{B}$ *field*

Grad-B drift:

$$\mathbf{v}_{\nabla B} = \pm\frac{1}{2}v_\perp r_L\frac{\mathbf{B}\times\nabla B}{B^2} \tag{2-24}$$

Curvature drift:

$$\mathbf{v}_R = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2B^2} \tag{2-26}$$

Curved vacuum field:

$$\mathbf{v}_R + \mathbf{v}_{\nabla B} = \frac{m}{q}\left(v_\parallel^2 + \frac{1}{2}v_\perp^2\right)\frac{\mathbf{R}_c\times\mathbf{B}}{R_c^2B^2} \tag{2-30}$$

Polarization drift:

$$\mathbf{v}_p = \pm\frac{1}{\omega_c B}\frac{d\mathbf{E}}{dt} \tag{2-66}$$

ADIABATIC INVARIANTS 2.8

It is well known in classical mechanics that whenever a system has a periodic motion, the action integral $\oint p\,dq$ taken over a period is a constant of the motion. Here p and q are the generalized momentum and coordinate which repeat themselves in the motion. If a slow change is made in the system, so that the motion is not quite periodic, the constant of the motion does not change and is then called an *adiabatic invariant*. By slow here we mean slow compared with the period of motion, so that the integral $\oint p\,dq$ is well defined even though it is strictly no longer an integral over a closed path. Adiabatic invariants play an important role in plasma physics; they allow us to obtain simple answers in many instances involving complicated motions. There are three adiabatic invariants, each corresponding to a different type of periodic motion.

2.8.1 The First Adiabatic Invariant, μ

We have already met the quantity

$$\mu = mv_\perp^2/2B$$

and have proved its invariance in spatially and temporally varying **B** fields. The periodic motion involved, of course, is the Larmor gyration. If we take p to be angular momentum $mv_\perp r$ and dq to be the coordinate $d\theta$, the action integral becomes

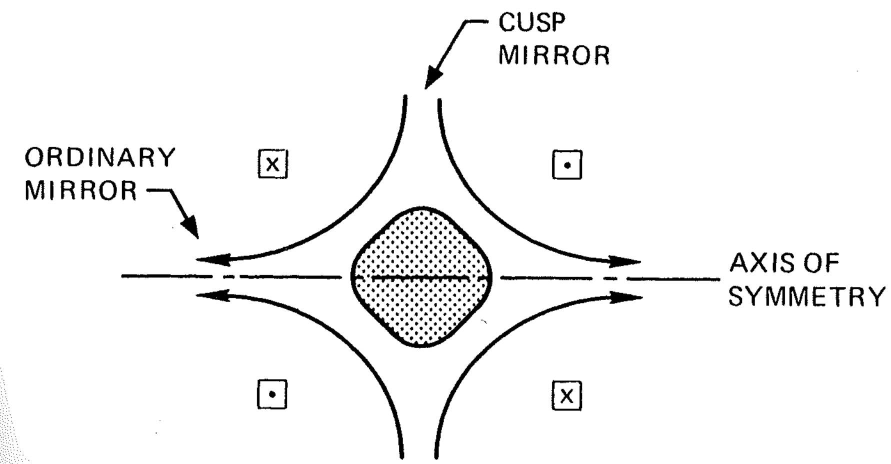
$$\oint p\,dq = \oint mv_\perp r_L\,d\theta = 2\pi r_L m v_\perp = 2\pi\frac{mv_\perp^2}{\omega_c} = 4\pi\frac{m}{|q|}\mu \qquad \text{[2-75]}$$

Thus μ is a constant of the motion as long as q/m is not changed. We have proved the invariance of μ only with the implicit assumption $\omega/\omega_c \ll 1$, where ω is a frequency characterizing the rate of change of **B** as seen by the particle. A proof exists, however, that μ is invariant even when $\omega \lesssim \omega_c$. In theorists' language, μ is invariant "to *all* orders in an expansion in ω/ω_c." What this means in practice is that μ remains much more nearly constant than **B** does during one period of gyration.

It is just as important to know when an adiabatic invariant does *not* exist as to know when it does. Adiabatic invariance of μ is violated when ω is not small compared with ω_c. We give three examples of this.

(A) *Magnetic Pumping.* If the strength of **B** in a mirror confinement system is varied sinusoidally, the particles' $v_\perp$ would oscillate; but there would be no gain of energy in the long run. However, if the particles make collisions, the invariance of μ is violated, and the plasma can be heated. In particular, a particle making a collision during the compression phase can transfer part of its gyration energy into $v_\parallel$ energy, and this is not taken out again in the expansion phase.

(*B*) *Cyclotron Heating.* Now imagine that the B field is oscillated at the frequency ω_c. The induced electric field will then rotate in phase with some of the particles and accelerate their Larmor motion continuously. The condition $\omega \ll \omega_c$ is violated, μ is not conserved, and the plasma can be heated.



Plasma confinement in a cusped magnetic field. **FIGURE 2-14**

(*C*) *Magnetic Cusps.* If the current in one of the coils in a simple magnetic mirror system is reversed, a magnetic cusp is formed (Fig. 2-14). This configuration has, in addition to the usual mirrors, a spindle-cusp mirror extending over 360° in azimuth. A plasma confined in a cusp device is supposed to have better stability properties than that in an ordinary mirror. Unfortunately, the loss-cone losses are larger because of the additional loss region; and the particle motion is nonadiabatic. Since the B field vanishes at the center of symmetry, ω_c is zero there; and μ is not preserved. The local Larmor radius near the center is larger than the device. Because of this, the adiabatic invariant μ does not guarantee that particles outside a loss cone will stay outside after passing through the nonadiabatic region. Fortunately, there is in this case another invariant: the canonical angular momentum $p_\theta = mrv_\theta - erA_\theta$. This ensures that there will be a population of particles trapped indefinitely until they make a collision.

2.8.2 The Second Adiabatic Invariant, J

Consider a particle trapped between two magnetic mirrors: It bounces between them and therefore has a periodic motion at the "bounce frequency." A constant of this motion is given by $\oint mv_\parallel\,ds$, where ds is an element of path length (of the guiding center) along a field line. However, since the guiding center drifts across field lines, the motion is not exactly periodic, and the constant of the motion becomes an adiabatic invariant. This is called the *longitudinal invariant J* and is defined for a half-cycle

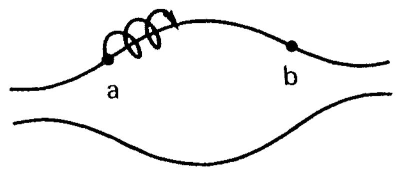


FIGURE 2-15 **A particle bouncing between turning points *a* and *b* in a magnetic field.**

between the two turning points (Fig. 2–15):

$$\boxed{J = \int_a^b v_\parallel \, ds} \qquad \text{[2-76]}$$

We shall prove that J is invariant in a static, nonuniform B field; the result is also true for a slowly time-varying B field.

Before embarking on this somewhat lengthy proof, let us consider an example of the type of problem in which a theorem on the invariance of J would be useful. As we have already seen, the earth's magnetic field mirror-traps charged particles, which slowly drift in longitude around the earth (Problem 2-8; see Fig. 2-16). If the magnetic field were perfectly symmetric, the particle would eventually drift back to the same line of force. However, the actual field is distorted by such effects as the solar wind. In that case, will a particle ever come back to the same line of force? Since the particle's energy is conserved and is equal to $\frac{1}{2}mv_\perp^2$ at the turning point, the invariance of μ indicates that $|B|$ remains the same at the turning point. However, upon drifting back to the same

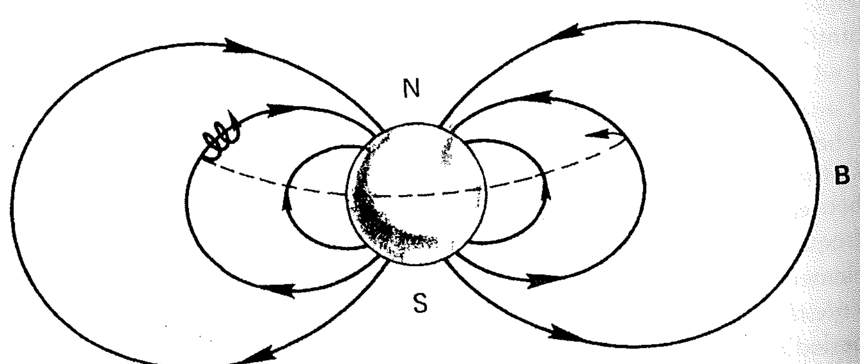


FIGURE 2-16 **Motion of a charged particle in the earth's magnetic field.**

longitude, a particle may find itself on another line of force at a different altitude. This cannot happen if J is conserved. J determines the length of the line of force between turning points, and no two lines have the same length between points with the same $|B|$. Consequently, the particle returns to the same line of force even in a slightly asymmetric field.

To prove the invariance of J, we first consider the invariance of $v_\parallel \, \delta s$, where δs is a segment of the path along $\mathbf{B}$ (Fig. 2-17). Because of guiding center drifts, a particle on s will find itself on another line of force $\delta s'$ after a time Δt. The length of $\delta s'$ is defined by passing planes perpendicular to $\mathbf{B}$ through the end points of δs. The length of δs is obviously proportional to the radius of curvature:

$$\frac{\delta s}{R_c} = \frac{\delta s'}{R_c'}$$

so that

$$\frac{\delta s' - \delta s}{\Delta t \, \delta s} = \frac{R_c' - R_c}{\Delta t R_c} \qquad \text{[2-77]}$$

The "radial" component of $\mathbf{v}_{gc}$ is just

$$\mathbf{v}_{gc} \cdot \frac{\mathbf{R}_c}{R_c} = \frac{R_c' - R_c}{\Delta t} \qquad \text{[2-78]}$$

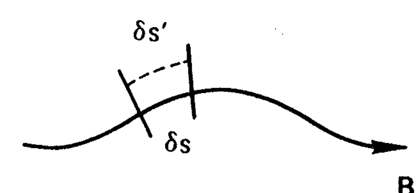
From Eqs. [2-24] and [2-26], we have

$$\mathbf{v}_{gc} = \mathbf{v}_{\nabla B} + \mathbf{v}_R = \pm \frac{1}{2} v_\perp r_L \frac{\mathbf{B} \times \nabla B}{B^2} + \frac{m v_\parallel^2}{q} \frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \qquad \text{[2-79]}$$

The last term has no component along $\mathbf{R}_c$. Using Eqs. [2-78] and [2-79], we can write Eq. [2-77] as

$$\frac{1}{\delta s} \frac{d}{dt} \delta s = \mathbf{v}_{gc} \cdot \frac{\mathbf{R}_c}{R_c^2} = \frac{1}{2} \frac{m}{q} \frac{v_\perp^2}{B^3} (\mathbf{B} \times \nabla B) \cdot \frac{\mathbf{R}_c}{R_c^2} \qquad \text{[2-80]}$$

This is the rate of change of δs as seen by the particle. We must now get the rate of change of $v_\parallel$ as seen by the particle. The parallel and



Proof of the invariance of *J*. **FIGURE 2-17**

perpendicular energies are defined by

$$W \equiv \tfrac{1}{2}mv_\parallel^2 + \tfrac{1}{2}mv_\perp^2 = \tfrac{1}{2}mv_\parallel^2 + \mu B \equiv W_\parallel + W_\perp \qquad [2\text{-}81]$$

Thus $v_\parallel$ can be written

$$v_\parallel = [(2/m)(W - \mu B)]^{1/2} \qquad [2\text{-}82]$$

Here W and μ are constant, and only B varies. Therefore,

$$\frac{\dot{v}_\parallel}{v_\parallel} = -\frac{1}{2}\frac{\mu \dot{B}}{W - \mu B} = -\frac{1}{2}\frac{\mu \dot{B}}{W_\parallel} = -\frac{\mu \dot{B}}{mv_\parallel^2} \qquad [2\text{-}83]$$

Since $\mathbf{B}$ was assumed static, $\dot{B}$ is not zero only because of the guiding center motion:

$$\dot{B} = \frac{dB}{d\mathbf{r}} \cdot \frac{d\mathbf{r}}{dt} = \mathbf{v}_{gc} \cdot \nabla B = \frac{mv_\parallel^2}{q}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2 B^2} \cdot \nabla B \qquad [2\text{-}84]$$

Now we have

$$\frac{\dot{v}_\parallel}{v_\parallel} = -\frac{\mu}{q}\frac{(\mathbf{R}_c \times \mathbf{B}) \cdot \nabla B}{R_c^2 B^2} = -\frac{1}{2}\frac{m}{q}\frac{v_\perp^2}{B}\frac{(\mathbf{B} \times \nabla B) \cdot \mathbf{R}_c}{R_c^2 B^2} \qquad [2\text{-}85]$$

The fractional change in $v_\parallel\,\delta s$ is

$$\frac{1}{v_\parallel\,\delta s}\frac{d}{dt}(v_\parallel\,\delta s) = \frac{1}{\delta s}\frac{d\delta s}{dt} + \frac{1}{v_\parallel}\frac{dv_\parallel}{dt} \qquad [2\text{-}86]$$

From Eqs. [2-80] and [2-85], we see that these two terms cancel, so that

$$v_\parallel\,\delta s = \text{constant} \qquad [2\text{-}87]$$

This is not exactly the same as saying that J is constant, however. In taking the integral of $v_\parallel\,\delta s$ between the turning points, it may be that the turning points on $\delta s'$ do not coincide with the intersections of the perpendicular planes (Fig. 2-17). However, any error in J arising from such a discrepancy is negligible because near the turning points, $v_\parallel$ is nearly zero. Consequently, we have proved

$$J = \int_a^b v_\parallel\,ds = \text{constant} \qquad [2\text{-}88]$$

An example of the violation of J invariance is given by a plasma heating scheme called *transit-time magnetic pumping*. Suppose an oscillating current is applied to the coils of a mirror system so that the mirrors alternately approach and withdraw from each other near the bounce frequency. Those particles that have the right bounce frequency will always see an approaching mirror and will therefore gain $v_\parallel$. J is not conserved in this case because the change of $\mathbf{B}$ occurs on a time scale not long compared with the bounce time.

Referring again to Fig. 2-16, we see that the slow drift of a guiding center around the earth constitutes a third type of periodic motion. The adiabatic invariant connected with this turns out to be the total magnetic flux Φ enclosed by the drift surface. It is almost obvious that, as $\mathbf{B}$ varies, the particle will stay on a surface such that the total number of lines of force enclosed remains constant. This invariant, Φ, has few applications because most fluctuations of $\mathbf{B}$ occur on a time scale short compared with the drift period. As an example of the violation of Φ invariance, we can cite some recent work on the excitation of hydromagnetic waves in the ionosphere. These waves have a long period comparable to the drift time of a particle around the earth. The particles can therefore encounter the wave in the same phase each time around. If the phase is right, the wave can be excited by the conversion of particle drift energy to wave energy.

PROBLEMS

2-13. Derive the result of Problem 2-12(b) directly by using the invariance of J.

(a) Let $\int v_\parallel\,ds \simeq v_\parallel L$ and differentiate with respect to time.

(b) From this, get an expression for T in terms of dL/dt. Set $dL/dt = -2v_m$ to obtain the answer.

2-14. In plasma heating by adiabatic compression, the invariance of μ requires that $KT_\perp$ increase as B increases. The magnetic field, however, cannot accelerate particles because the Lorentz force $q\mathbf{v} \times \mathbf{B}$ is always perpendicular to the velocity. How do the particles gain energy?

2-15. The polarization drift v_p can also be derived from energy conservation. If $\mathbf{E}$ is oscillating, the $\mathbf{E} \times \mathbf{B}$ drift also oscillates; and there is an energy $\frac{1}{2}mv_E^2$ associated with the guiding center motion. Since energy can be gained from an $\mathbf{E}$ field only by motion along $\mathbf{E}$, there must be a drift v_p in the $\mathbf{E}$ direction. By equating the rate of change of $\frac{1}{2}mv_E^2$ with the rate of energy gain from $\mathbf{v}_p \cdot \mathbf{E}$, find the required value of v_p.

2-16. A hydrogen plasma is heated by applying a radiofrequency wave with $\mathbf{E}$ perpendicular to $\mathbf{B}$ and with an angular frequency $\omega = 10^9$ rad/sec. The confining magnetic field is 1 T. Is the motion of (a) the electrons and (b) the ions in response to this wave adiabatic?

2-17. A 1-keV proton with $v_\parallel = 0$ in a uniform magnetic field $\mathbf{B} = 0.1$ T is accelerated as B is slowly increased to 1 T. It then makes an elastic collision with a heavy particle and changes direction so that $v_\perp = v_\parallel$. The $\mathbf{B}$-field is then slowly decreased back to 0.1 T. What is the proton's energy now?