THE STABILITY OF ISOLATED CLUSTERS

Lyman Spitzer, Jr., Ph.D.

(Communicated by Dr. Harlow Shapley, Assoc. R.A.S.)

Introduction.—The internal structure of globular clusters presents problems of considerable theoretical interest. Heckmann and Siedentopf * have pointed out that encounters between stars of such a cluster should establish an isothermal condition in times comparable with the short-time scale; it is therefore relevant to investigate the structure of an isothermal gaseous sphere. Unfortunately, both the mass and radius of such a sphere are infinite. To meet this difficulty Heckmann and Siedentopf advanced the assumption, originally made by Martens, of a continuous distribution of galactic field stars, unobservable because of their small masses. These stars would presumably form a local condensation within the cluster, and would provide the extra gravitational attraction necessary to give the rest of the cluster a finite mass and radius.

There is, however, another possible escape from this dilemma. Consider an isolated cluster of known mass and radius. Encounters between stars will gradually set up a Maxwellian velocity distribution; as the cluster approaches isothermal conditions, what will happen to it?

The relation between the velocity of the cluster stars and the velocity of escape from the cluster determines the answer to this question. From the virial theorem it follows that the kinetic energy of the cluster stars is one-half the total gravitational energy, measuring this in the negative sense. This assumes, of course, that the cluster is in a statistically steady state. It is shown in section 3 that one may neglect in this connection the deviations from the steady state produced by the phenomenon of evaporation considered here. The mean kinetic energy of a cluster star is therefore one-half the potential energy of the cluster per star, and one-fourth † the mean potential energy of the cluster stars. It is evident that if the kinetic energy of any star exceeds its potential energy the star will escape. Hence if the kinetic energy of any star exceeds four times the mean kinetic energy of all the stars, that star will leave the cluster, provided that its potential energy is not greater than the mean potential energy of the cluster stars. In other

* O. Heckmann and H. Siedentopf, *Zs. f. Ap.*, **1**, 43, 1930.

† The additional factor of one-half may be derived as follows. Consider a cluster of mass M with some arbitrary density distribution. Let the energy required to remove a single gram of matter to infinity be equal, on the average, to ψGM, where ψ is some function of the size and density distribution of the cluster, and G is the usual gravitational constant. Then the potential energy of the cluster as a whole will be the integral over dM of $\psi GMdM$, or simply $\frac{1}{2}\psi GM^2$. Hence the potential energy of the cluster per unit mass is $\frac{1}{2}\psi GM$, or one-half of the mean potential energy of a unit mass.

words, the mean square velocity of the cluster stars is one-fourth the mean square velocity of escape.

If the kinetic energy of the cluster is divided among the stars by the usual Maxwell-Boltzmann formula, some stars will clearly have more than enough energy to leave the cluster entirely. The proportion of stars with such velocities will at any time be small, but as long as the members of the cluster behave as mass points with a random isothermal distribution of velocities, the cluster will lose stars by this process of "evaporation" and will continually contract. This is the dynamical explanation of the fact that an isothermal gaseous sphere in a steady state and in an infinite space must have an infinite mass and an infinite radius.

The problem of relevance to globular clusters is clearly the determination of the length of time involved in such a process. To calculate this we must know the time of relaxation for an enclosed system of particles—the time required to establish a Maxwellian velocity distribution. Let this quantity be denoted by τ, a function, in general, of the density and mean square velocity of the particles. Let K denote the fraction of stars whose velocities would be greater than the velocity of escape if the velocity distribution were Maxwellian; this will vary only with the ratio of the velocity of escape to the root mean square velocity. At any epoch t_0 consider all those stars within some region of the cluster which have velocities less than the velocity of escape from that region. If the region were enclosed, then at an epoch $t_0+\tau$ a fraction K of these stars would have attained velocities exceeding the velocity of escape. Since the system is not enclosed, however, and since the mean free path of a cluster star is greater than the radius of the cluster, we assume that this fraction K escapes during the interval τ; hence the probability per unit time that any particular star escapes from the cluster will be K/τ. Since K is small this procedure should give a valid estimate of the rate of escape.

To calculate Λ, the actual rate of evaporation, we should average K/τ over all regions of the cluster. The true density distribution of clusters is not well known, however, and to obtain an order-of-magnitude result we shall consider an idealized isothermal cluster of uniform density. The time of relaxation will be constant throughout such a cluster and only K need be averaged. The results of the calculations may then be applied to an approximate discussion of actual clusters.

In the first of the following six sections the value of K is calculated for this simplified model. Section 2 is devoted to a rough determination of the best value of τ. The resultant value of Λ is given in section 3 together with an evaluation of $F(r)$, the number of escaping stars within a radius r of the cluster centre; a determination is also given of the deviation from a steady state introduced by the escape of stars from the cluster. Section 4 extends the analysis to centrally concentrated clusters; section 5 examines the effect of a dispersion in stellar masses. In section 6 the relevance of these results to the age and probable fate of globular and galactic clusters is briefly examined.

1. The idealized isothermal cluster may be assumed to have a radius R

and a total mass M, while the individual stars may be assumed each to have the same mass m. All quantities will be given in terms of a macroscopic system of units with the parsec, the solar mass, and the sidereal year replacing the centimetre, the gram and the second, respectively. In these units the gravitational constant G has the value $4{\cdot}49 \times 10^{-15}$.

If $-\Omega$ represents the gravitational energy of the cluster and T the total kinetic energy of all the cluster stars, then we have by the virial theorem *

$$T = \frac{1}{2}\Omega + \frac{1}{4}\frac{d^2I}{dt^2}, \qquad (1)$$

where I is the moment of inertia of the cluster about its origin. The last term will be neglected here since, as shown in section 3, its effect on T is small. For a uniform sphere we have

$$\Omega = \frac{3GM^2}{5R}, \qquad (2)$$

where G is the gravitational constant; if we define w^2 as the mean square velocity of the cluster stars, the kinetic energy is given by

$$T = \tfrac{1}{2}Mw^2. \qquad (3)$$

From (1), (2) and (3) we find

$$w^2 = 0{\cdot}6\frac{GM}{R}. \qquad (4)$$

Let $P(v)dv$ be the probability that the velocity of any particular cluster star lies between v and $v+dv$. Then, if the mean square velocity is w^2, the Maxwellian distribution is given by

$$P(v) = 3\left(\frac{6}{\pi}\right)^{\frac{1}{2}}\frac{v^2}{w^3}\exp\left(-\frac{3v^2}{2w^2}\right). \qquad (5)$$

For any value of the escape velocity v_∞, K is simply

$$K = \int_{v_\infty}^{\infty} P(v)dv; \qquad (6)$$

the introduction of (5) into (6) and the use of the substitution

$$z = \frac{3v_\infty{}^2}{2w^2} \qquad (7)$$

yields

$$K = 2\left(\frac{z}{\pi}\right)^{\frac{1}{2}}e^{-z} + 1 - \operatorname{erf} z^{\frac{1}{2}}. \qquad (8)$$

The function erf x is the usual error function of x.

Since at the surface $v_\infty{}^2 = 2GM/R$, it follows that at the boundary of a cluster of uniform density z equals five; from (8) we find that K in this case equals $1{\cdot}86 \times 10^{-2}$. When the mean square velocity of escape, which was shown in the introduction to equal $4w^2$, is used in (7), z equals 6 and K becomes $7{\cdot}4 \times 10^{-3}$. In averaging K over a uniform sphere we must take the usual polytropic function † for the polytropic index zero. In the usual notation we find for the velocity of escape as a function of distance from the centre of the cluster

$$v_\infty{}^2 = \frac{2GM}{R}(1 + \tfrac{1}{2}\theta_0); \qquad (9)$$

with the use of (4) this yields, in (7),

$$z = 5(1 + \tfrac{1}{2}\theta_0). \qquad (10)$$

The substitution of (10) into (8) and integration over the mass of the cluster is tedious but straightforward, involving integration by a series expansion. The final average value of K is found to equal $8{\cdot}8 \times 10^{-3}$.

2. The time of relaxation τ is a rough measure of the time required for a gaseous assembly to attain an approximately Maxwellian distribution of velocities. A precise definition would be of little use here since the calculation of τ is not accurate to much better than half an order of magnitude. A rigorous analysis has been carried through only for the case of inverse fifth-power forces between the particles of an assembly, and even then only in so far as the actual distribution of velocities can be neglected. In such a special case, τ, defined as the time in which deviations from a Maxwellian distribution fall to $1/e$ of their original value, is comparable with the mean time between collisions. This relationship should hold approximately for other laws of force as well. The point is essentially that by the time each particle has lost its original velocity, on the average, the Maxwellian distribution will have prevailed.

In the present instance it is difficult to know precisely what a collision, or an encounter, is, since in the case of inverse-square attraction between stars the deflection produced by a single encounter decreases so slowly with increasing distance of closest approach between the two stars that the cumulative effect of many distant encounters per unit time is actually greater than the effect of a few close encounters. One may take for τ the time required on the average for a single star to be deflected through an angle of $\pi/2$. The analysis for this case has been given by Jeans † and by Smart. ‡

We let τ_s denote the value of τ found from a consideration of single deflections alone. If ν is the number of stars per cubic parsec, and V is their relative velocity, we have Smart's formula

$$\tau_s = \frac{V^3}{4\pi G^2 m^2 \nu}. \qquad (11)$$

* H. Poincaré, *Leçons sur les hypothèses cosmogoniques*, p. 94; A. S. Eddington *M.N.*, **76**, 525, 1916.

* S. Chandrasekhar, *Stellar Structure* (University of Chicago Press, 1939), 91.

† J. H. Jeans, *Astronomy and Cosmogony* (Cambridge University Press, 1929), 318. The formulæ as given by Jeans are not in the most convenient form, since they refer to velocities and distances relative to the centre of mass of the two stars. In these units, furthermore, his use of the formula $\pi\nu Vp^2$ for the frequency of collisions is apparently incorrect and leads to a value of τ_s eight times too great.

‡ W. M. Smart, *Stellar Dynamics* (Cambridge University Press, 1938), 318.

The average value of V^3 may be found from the previous section. If we assume again that all stars have the same mass m, then the mean square of the relative velocity V equals twice w^2, the mean square of the individual velocity, and we have from (4)

$$\overline{V^2}=2w^2=1\cdot 2\frac{GM}{R}. \tag{12}$$

From the Maxwellian distribution law it may be shown that

$$\overline{V^3}=\frac{2^{7/2}(\overline{V^2})^{3/2}}{3^{3/2}\pi^{1/2}}=3\cdot 47w^3. \tag{13}$$

Since ν equals $3N/4\pi R^3$, where N, the total number of stars in the cluster, is equal to M/m, we have finally from (11), (12) and (13)

$$\tau_s=0\cdot 538N^{1/2}R^{3/2}/m^{1/2}G^{1/2}. \tag{14}$$

The consideration of single deflections alone gives rather a poor approximation, since the cumulative effect of the more distant encounters is actually more important. Jeans * has shown that such encounters will on the average produce a cumulative deflection of $\pi/2$ in a time † which may be expressed in the form

$$\tau=\frac{\pi^2}{32ln(\pi/2\delta)}\tau_s, \tag{15a}$$

where ln denotes the natural logarithm ; δ is the deflection produced by an encounter with a star at the interstellar distance $\nu^{-1/3}$. Expressing δ in terms of G, m, ν and V^2, and determining $\overline{V^2}$ from (12), we find

$$\tau=\frac{0\cdot 20}{\log N-0\cdot 18}\tau_s, \tag{15b}$$

where $\log N$ is the common logarithm of N.

The time τ has also been calculated by Schwarzschild ‡ from rather different considerations. This other approach rests on the assumption that τ is equal to the time in which the average interchange of energy between the stars is just equal to the average value of the kinetic energy per star. The method leads to the formula

$$\tau=\frac{0\cdot 044}{\log N-1\cdot 6}\tau_s, \tag{16}$$

provided that $\log N$ is considerably greater than unity, and that τ_s is taken as a parameter whose value is defined by (14).

These two results (15b) and (16) differ by a factor of 4·5, which is largely accounted for by the different average values of V which are used in the two formulæ. In Schwarzschild's analysis V^{-2} and V^2 are averaged over a Maxwellian velocity distribution, whereas in the derivation of (15a) from

* J. H. Jeans, *loc. cit.*

† W. M. Smart, *Stellar Dynamics* (Cambridge University Press, 1938), 320.

‡ K. Schwarzschild, *Seeliger Festschrift*, 94, 1924.

Jeans's formula, the average value of V^3 has been used, increasing τ by a factor of 3·7. The encounters with low relative velocity are certainly not so effective as the Schwarzschild treatment assumes, since other stars will usually intervene before a very slow encounter has progressed very far. On the other hand, in computing (14) one should average the square of the total deflection per unit time over all velocities, rather than the time necessary to produce a given deflection. This would decrease the value of V used in (14) and would lead to a smaller τ. Unless some method were used, however, to eliminate the effect of the lowest-velocity encounters, this process would yield the average value of V^{-3}, which is infinite. This approach is hence not a convenient one. To obtain an order of magnitude result we may take a rough average of the two formulæ, and set

$$\tau=\frac{0\cdot 10}{\log N-0\cdot 5}\tau_s, \tag{17a}$$

with an uncertainty of somewhat less than half an order of magnitude. Substituting from (14) for τ_s we see that (17a) becomes

$$\tau=8\cdot 0\times 10^5\frac{N^{1/2}R^{3/2}}{m^{1/2}(\log N-0\cdot 5)}\text{ years}, \tag{17b}$$

where R and m again denote the radius of the cluster in parsecs and the mass of a cluster star in units of the solar mass.

The values of τ found from (17b), with m set equal to unity, are shown in the table below for various values of N and R. These are maximum values for any isothermal cluster with a finite radius R, and an average mass m equal to unity, since any change in the density distribution will increase ν in (11) more than it will increase V^3. If one considers a polytrope with n equal to 3, for instance, the average value of V^3 will be 4·0 times as great as in the case of a homogeneous cluster with the same mass and radius ; the value of ν averaged over the mass of the cluster, however, will increase by a factor of about 10.

Time of Relaxation in a Globular Cluster of Uniform Density

N	$R=1$	$R=3$	$R=10$	$R=30$	$R=100$
10^2	$5\cdot 3\cdot 10^6$	$2\cdot 8\cdot 10^7$	$1\cdot 7\cdot 10^8$	$8\cdot 7\cdot 10^8$	$5\cdot 3\cdot 10^9$
10^4	$2\cdot 3\cdot 10^7$	$1\cdot 2\cdot 10^8$	$7\cdot 2\cdot 10^8$	$3\cdot 8\cdot 10^9$	$2\cdot 3\cdot 10^{10}$
10^6	$1\cdot 5\cdot 10^8$	$7\cdot 6\cdot 10^8$	$4\cdot 6\cdot 10^9$	$2\cdot 4\cdot 10^{10}$	$1\cdot 5\cdot 10^{11}$
10^8	$1\cdot 1\cdot 10^9$	$5\cdot 6\cdot 10^9$	$3\cdot 4\cdot 10^{10}$	$1\cdot 7\cdot 10^{11}$	$1\cdot 1\cdot 10^{12}$
10^{10}	$8\cdot 4\cdot 10^9$	$4\cdot 4\cdot 10^{10}$	$2\cdot 7\cdot 10^{11}$	$1\cdot 4\cdot 10^{12}$	$8\cdot 4\cdot 10^{12}$

The values of the radius R are given in parsecs ; N is the number of stars in the cluster. The values of τ give the number of years required to establish a Maxwellian velocity distribution among stars of solar mass. For a centrally concentrated cluster R is the radius containing half the mass.

The values in the table are somewhat smaller than those usually given. Heckmann and Siedentopf, for instance, give $1\cdot 6\times 10^{10}$ years for the time of relaxation in a cluster with N equal to 10^6 and R equal to 10 parsecs.

The analysis by Rosseland *, from which their value is derived, is based on the same physical principles as that by Schwarzschild. Rosseland's analysis, however, assumes that one of the stars in each encounter is at rest, and his results give an upper limit for τ.

3. As we have seen in the introduction, we may divide K by τ to find Λ, the probability of escaping from the cluster per unit time per star. This procedure assumes that in the time τ a Maxwellian distribution is established for energies four or five times the average. If the initial distribution were one in which all velocities were equal, this assumption would certainly be incorrect, since several encounters would be necessary to give some stars the relevant energies. The situation contemplated, however, is one in which a cut-off Maxwellian velocity distribution already exists, the cut-off coming at the escape velocity. In this case K/τ should give a fairly close approximation to Λ.

Combining (17*b*), then, with the value of K found at the end of section 1, we have

$$\Lambda = 1{\cdot}1 \times 10^{-8} m^{1/2} (\log N - 0{\cdot}5)/N^{1/2} R^{3/2}. \tag{18}$$

The stipulation that the cluster does not lose an appreciable fraction of its mass during the short-time scale of 2×10^9 years leads to the condition that $2 \times 10^9 \Lambda$ is less than unity, or that

$$\frac{NR^3}{(\log N - 0{\cdot}5)^2} > 4{\cdot}8 \times 10^2 m. \tag{19}$$

The number of stars evaporating from the cluster per unit time is simply ΛN. The root mean square velocity of the escaping stars will be roughly equal to that of the stars in the cluster. Hence the star density $\nu(r)$ will be given by

$$4\pi r^2 \nu(r) w = \Lambda N. \tag{20}$$

This expression is valid when r is large compared to four times R, the cluster radius. A consideration of the distribution of velocities reduces the effective velocity in (20) by a factor of $\pi^{1/2}$. If we substitute from (18) for Λ, a simple integration yields for $F(r)$, the number of such stars within a sphere of radius r,

$$F(r) = 0{\cdot}38 (\log N - 0{\cdot}5) r/R, \tag{21}$$

since, for a uniform sphere, w^2 is $0{\cdot}6GM/R$. If we let $G(r)$ equal the number of stars within a cylinder of radius r (the axis of which passes through the centre of the cluster), then $G(r)$, as found from the usnal integral equation, is also given by (21), with 0·59 replacing 0·38.

As the evaporation of stars from the cluster proceeds, each escaping star will carry away on the average a positive energy $\frac{1}{2}mw^2$ †, and the energy of the cluster will accordingly diminish by the same amount. But from the virial theorem it follows that the total energy $U = T - \Omega$ is equal to $-T$, or $-\frac{1}{2}Mw^2$.

* S. Rosseland, *M.N.*, **88**, 208, 1928.

† This is an overestimate and gives an upper limit for γ in (30).

Hence we have for the rate of loss of energy by the cluster the two equations

$$\frac{dU}{dt} = \tfrac{1}{2} w^2 \frac{dM}{dt}, \tag{22}$$

$$\frac{dU}{dt} = \frac{d}{dt}\left(-\tfrac{1}{2} M w^2\right). \tag{23}$$

Combining these two equations we have

$$w^2 \frac{dM}{dt} = -w^2 \frac{dM}{dt} - M \frac{dw^2}{dt}, \tag{24}$$

$$\frac{2}{M}\frac{dM}{dt} = -\frac{1}{w^2}\frac{dw^2}{dt}, \tag{25}$$

$$w \propto \frac{1}{M}. \tag{26}$$

Since w is proportional to $1/M$, the kinetic energy T will also vary as $1/M$. Since by the virial theorem T and Ω are proportional, we have

$$\Omega \propto \frac{GM^2}{R} \propto \frac{1}{M}, \tag{27}$$

from which it follows that

$$R \propto M^3, \tag{28}$$

$$I \propto MR^2 \propto M^7. \tag{29}$$

From this proportionality we may readily calculate d^2I/dt^2 in (1). We find from (18), (28) and (29) that (1) then becomes approximately

$$T = \tfrac{1}{2}\Omega\left\{1 + \gamma\left(\frac{\log N - 0{\cdot}5}{N}\right)^2\right\}, \tag{30}$$

where γ is a numerical constant depending on the distribution of stars within the cluster; if the star density is assumed to be uniform, γ equals 0·22. It is evident that if N is greater than 20, this correction factor is less than one-tenth of 1 per cent. and quite negligible. Hence the use of the virial theorem leads to consistent results, and the relationship between T and Ω is essentially unchanged by the escape of stars.

4. An actual cluster may differ from the ideal configuration analysed here in at least three important respects. In the first place the motions of the stars may not be at random. This will be the case, in part at least, for any rotating cluster. In the second place the stars may be much more concentrated towards the centre of the cluster than they would be in the homogeneous model discussed here. Finally the different stars may have different masses. Each of these possibilities will be discussed in turn in this section and the next.

In connection with the first point, it may be noted that if the kinetic energy of the cluster stars appeared largely in the form of a rapid rotation of the cluster about some fixed axis, the effective velocity temperature would be very much lower than would otherwise be the case, since this temperature refers to the relative motion of neighbouring stars. Hence the random kinetic energy per star would be very much less than the energy of escape, and practically no stars would ever leave the cluster through this evaporation process. In such a case viscosity would act to accelerate the outer layers and eventually the cluster would eject stars, but at a very slow rate. Since such configurations would be very oblate, however, it is evident that they do not correspond to the observed clusters, as these are nearly spherical.

The case of a cluster with a considerable concentration of stars towards its centre is a much more important one. To extend the analysis to such a case we may apply (19) to the inner core of a cluster, taking this core to include half the total mass M of the cluster: the radius of the core will be denoted by $R_{1/2}$. One may expect that this core will not exhibit a marked density concentration towards its centre. Even for the limiting case of the polytrope with n equal to five, the ratio of the central density to the mean density in a sphere with half the total mass is only 4·4. Such a polytrope, it may be noted, corresponds to Schuster's law of density distribution for globular clusters, and gives apparently better agreement with the available observations than any other simple law.*

Because of this relatively slight density concentration, such an inner core will have properties not unlike those of a uniform sphere. At the boundary of this core the value of v_∞^2 should be roughly equal to its average value for the cluster as a whole, which, as we have already seen, is four times the mean square velocity w^2. This expectation is verified by simple calculations based on the Emden function for the polytrope n equals 5.† It is readily shown that when $M(r)$ equals $\frac{1}{2}M$ in this polytrope, v_∞^2 equals 1·03 times its average value for the entire polytrope. Hence the value of K for this inner core should not exceed 10^{-2}.

The average value of V^2 given by (12) should be approximately correct even for this more general case, provided that $\frac{1}{2}M$ and the corresponding radius $R_{1/2}$ are used instead of M and R. We may again make a comparison with the polytrope for n equal to 5, for which we find by use of the virial theorem that the kinetic energy of the entire polytrope per unit mass is $3\pi(2^{2/3}-1)^{-1/2}/32 \times G(\frac{1}{2}M)/R_{1/2}$ or $0{\cdot}384G(\frac{1}{2}M)/R_{1/2}$, which is only 28 per cent. greater than the corresponding quantity for an isothermal sphere of mass $\frac{1}{2}M$ and radius $R_{1/2}$.

If, then, we apply to the core of a cluster the formulæ developed for a uniform sphere, K will be substantially correct, the value of ν used will be too small by a factor of not more than 2 or 3, and the average value of V^3 will be too small by some 50 per cent.; the resultant value of Λ calculated from (18) will accordingly be too small by a factor of not more than 2, which is less than the inaccuracy inherent in the analysis in any case. Unless the inner half of a cluster is more centrally concentrated than the corresponding inner half of the polytrope for which n equals 5, we may

* P. ten Bruggencate, *Sternhaufen* (J. Springer, 1927), 38 *et seq.*

† S. Chandrasekhar, *loc. cit.*, 93.

conclude that the general order of magnitude given by (18) is substantially correct.

5. The assumption that all stars have the same mass is obviously not very realistic. It is clear that stars whose mass is less than one-fourth of m, the average mass, will tend to leave the cluster, since the root mean square equilibrium velocity for such stars will be greater than the average velocity of escape. To find the rate of escape, however, one must investigate more closely the exchange of energy between stars.

If we consider a single encounter between star A and star B, of mass m_A and m_B, respectively, then the gain of energy of star A is given by

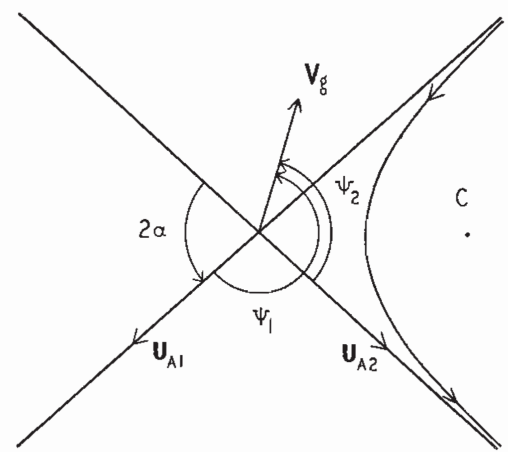
$$\Delta E_A = \tfrac{1}{2}m_A(\mathbf{V}_g + \mathbf{U}_{A2})^2 - \tfrac{1}{2}m_A(\mathbf{V}_g + \mathbf{U}_{A1})^2, \qquad (31a)$$

where $\mathbf{V}_g$ is the velocity of the centre of mass of the two stars; $\mathbf{U}_{A1}$ and $\mathbf{U}_{A2}$ are the initial and final velocities of star A relative to the centre of mass. The orbit of star A relative to the centre of mass is a simple hyperbola; U_{A1}, the scalar value of the vector $\mathbf{U}_{A1}$, is clearly equal to the corresponding U_{A2}.

If we introduce ψ_1, the angle between $\mathbf{U}_{A1}$ and $\mathbf{V}_g$, and ψ_2, that between $\mathbf{U}_{A2}$ and $\mathbf{V}_g$, we have from (31a),

$$\Delta E_A = m_A U_{A1} V_g(\cos\psi_2 - \cos\psi_1). \qquad (31b)$$

From the figure it is evident that $\psi_1 - \psi_2$ equals $\pi - 2\alpha$, where 2α is the angle between the asymptotes of the relative orbit.



The centre of mass of the two stars is actually at C, but to clarify the meaning of the angles involved, its vector velocity $\mathbf{V}_g$ *is shown by an arrow drawn at the intersection of the asymptotes of the orbit of one of the stars. The other star is not shown, but its orbit relative to C is a hyperbola of the same shape.*

To determine α we have the relationship derived* from simple

* Smart, *loc. cit.*, 317.

geometry and the conservation of energy and angular momentum

$$\cot \alpha = \frac{G(m_A + m_B)}{pV^2}, \tag{32}$$

where p is the distance at which the stars would have passed had there been no gravitational attraction. Hence we have from (31*b*), eliminating ψ_2, and using (32) to evaluate $\sin 2\alpha$ and $\cos 2\alpha$,

$$\Delta E_A = 2m_A U_{A1} V_g \left\{ \frac{Y \sin \psi_1 - \cos \psi_1}{1 + Y^2} \right\}, \tag{33}$$

where

$$Y = \tan \alpha. \tag{34}$$

To find τ by Schwarzschild's method *, one computes the average value of $(\Delta E_A)^2$. In the present case, however, the average value of ΔE_A is not zero. Although the average value of $\sin \psi_1$ obviously vanishes, since positive and negative values of ψ_1 are equally probable, $\cos \psi_1$ may have a non-zero average. From the definition of a scalar product we see that

$$\begin{aligned} U_{A1} V_g \cos \psi_1 &= \mathbf{U}_{A1} \cdot \mathbf{V}_g = (\mathbf{v}_{A1} - \mathbf{V}_g) \cdot \mathbf{V}_g \\ &= \frac{m_B}{(m_A + m_B)^2} \{ m_A v_{A1}{}^2 - m_B v_{B1}{}^2 + (m_B - m_A) \mathbf{v}_{A1} \cdot \mathbf{v}_{B1} \}, \end{aligned} \tag{35}$$

where $\mathbf{v}_{A1}$ and $\mathbf{v}_{B1}$ are the original velocities of the two stars in the inertial frame of the cluster. In the derivation of (35) use has been made of the usual formula for $\mathbf{V}_g$,

$$\mathbf{V}_g = \frac{m_A}{(m_A + m_B)} \mathbf{v}_{A1} + \frac{m_B}{(m_A + m_B)} \mathbf{v}_{B1}. \tag{36}$$

To find the total energy gained per second by star A we must first multiply (33) by $2\pi\nu V p dp$, the number of times per year that star A encounters a star of relative velocity V in such a way that p lies between p and $p + dp$; ν is the number of stars of mass m_B per unit volume. Then we must integrate over all relevant values of p. Since the infinite integral diverges, we may take $\nu^{-1/3}$ as the upper limit of integration over p, since in general encounters such that p is greater than this limit will be screened by other intervening stars. This somewhat crude procedure is apparently the only way by which an analysis of single encounters can be made to apply to the present problem. Finally the resultant expression must be averaged over all values of θ, the angle between $\mathbf{v}_{A1}$ and $\mathbf{v}_{B1}$, and averaged also over v_{B1}.

The integration over p yields

$$\begin{aligned} 2\pi\nu V \int_0^{\nu^{-1/3}} \Delta E_A p dp &= -4\pi\nu V m_A U_{A1} V_g \cos \psi_1 \\ &\times \frac{1}{2} \frac{G^2(m_A + m_B)^2}{V^4} ln\left(1 + \frac{V^4}{G^2(m_A + m_B)^2 \nu^{2/3}} \right); \end{aligned} \tag{37}$$

* K. Schwarzschild, *Seeliger Festschrift*, 94, 1924.

if we substitute for $\cos \psi_1$ from (35) and drop the subscript 1 from v_{A1} and v_{B1}, we find that (37) becomes

$$\begin{aligned} 2\pi\nu V \int_0^{\nu^{-1/3}} \Delta E_A p dp &= 2\pi\nu V m_A m_B G^2 ln\left(1 + \frac{V^4}{G^2(m_A + m_B)^2 \nu^{2/3}} \right) \\ &\times \left\{ \frac{m_B v_B{}^2 - m_A v_A{}^2 + (m_A - m_B) \mathbf{v}_A \cdot \mathbf{v}_B}{V^3} \right\}. \end{aligned} \tag{38}$$

Let θ equal the angle between $\mathbf{v}_A$ and $\mathbf{v}_B$. Then we have

$$V^2 = v_A{}^2 + v_B{}^2 - 2\mathbf{v}_A \cdot \mathbf{v}_B, \tag{39a}$$

$$\mathbf{v}_A \cdot \mathbf{v}_B = v_A v_B \cos \theta. \tag{39b}$$

To average over θ we multiply (38) by $\frac{1}{2} \sin \theta d\theta$ and integrate from zero to π. Unless v_A is nearly equal to v_B the logarithmic term will not change appreciably in the course of the integration, and we may take it outside the integral with its value for $\cos \theta$ equal to unity. Since the relative change in the logarithm with θ becomes less as ν decreases, this procedure becomes asymptotically correct with decreasing ν. If we let dE_A/dt denote the average of (38) over θ, and let x equal $\cos \theta$, then we find

$$\frac{dE_A}{dt} = \tfrac{1}{2} Q \int_{-1}^{+1} \frac{m_B v_B{}^2 - m_A v_A{}^2 + (m_A - m_B) v_A v_B x}{(v_A{}^2 + v_B{}^2 - 2v_A v_B x)^{3/2}} dx, \tag{40}$$

where

$$Q = 2\pi\nu m_A m_B G^2 ln\left(1 + \frac{(v_A - v_B)^4}{G^2(m_A + m_B)^2 \nu^{2/3}} \right). \tag{41}$$

The integration is elementary and yields

$$\begin{aligned} \frac{dE_A}{dt} = \tfrac{1}{2} Q \Bigg\{ &\frac{m_B v_B{}^2 - m_A v_A{}^2}{v_A v_B} \left(\frac{1}{|v_A - v_B|} - \frac{1}{v_A + v_B} \right) \\ &+ \frac{m_A - m_B}{v_A v_B} \left(\frac{v_A{}^2 + v_B{}^2 - v_A v_B}{|v_A - v_B|} - \frac{v_A{}^2 + v_B{}^2 + v_A v_B}{v_A + v_B} \right) \Bigg\}, \end{aligned} \tag{42}$$

where $|v|$ denotes the absolute value of v. If v_A is greater or less than v_B respectively, (42) becomes

$$\frac{dE_A}{dt} = \begin{cases} -Q m_A / v_A & v_A > v_B, \\ Q m_B / v_B & v_A < v_B. \end{cases} \tag{43}$$

If (43) is integrated over a Maxwellian distribution for v_B, one obtains the average rate of increase in energy for stars of mass m_A and velocity v_A. This rate is a quantity of not much significance, however, since it bears little relation to the rate of increase in the number of stars with velocities greater than v_A. It is rather the average of the rate of increase of E_A over all values of v_A as well that gives the true rate of approach towards equipartition. The velocity distribution, over which this average should be taken, is unknown. To carry through the integration we shall assume a Maxwellian distribution with, of course, a different mean square velocity from that which equipartition of energy with the more massive stars would establish. It is readily shown that other reasonable distributions do not change the results appreciably.

Let $\bar{E}_A$ and $\bar{E}_B$ denote the average values of E_A and E_B respectively; from (43) we find by straightforward integration,

$$\frac{d\bar{E}_A}{dt} = \left(\frac{3}{\pi}\right)^{1/2} Q \frac{\bar{E}_B - \bar{E}_A}{(\bar{E}_A/m_A + \bar{E}_B/m_B)^{3/2}}. \tag{44}$$

We may assume that ν, $\overline{v_B^2}$ and m_B are equal to the number of all stars per cubic parsec, the mean square velocity w^2 and the average mass m (or M/N) respectively. Then if we substitute from (41) for Q, and recall formulæ (11), (13) and (17a) for τ, (44) becomes, on integration,

$$t = 0{\cdot}68\tau \int \frac{(1+y)^{3/2}}{1 - m_A y/m_B} dy, \tag{45}$$

where we have introduced the new variable

$$y = \frac{m_B \bar{E}_A}{m_A \bar{E}_B} = \frac{\overline{v_A^2}}{\overline{v_B^2}}. \tag{46}$$

In deriving (45) from (44) one should replace the constant $-0{\cdot}5$ in the denominator of (17b) by $-0{\cdot}7$; this change, however, is negligible. Since (44) is much more accurate than any of the formulæ derived for the time of relaxation τ, (45) may be regarded simply as a parameter whose value is defined by (11) and (17a).

When m_A is greater than m_B, and y is less than unity, the factor $(1+y)^{3/2}$ in the integrand of (45) may be neglected, and we have

$$\frac{m_A}{m_B} y = \frac{\bar{E}_A}{\bar{E}_B} = 1 - C \exp\left(-1{\cdot}47 \frac{m_A t}{m_B \tau}\right), \tag{47}$$

where C is a constant of integration. Hence equipartition of energy for stars more massive than the average will be attained in the time τ or less. It follows that stars of relatively high mass will rarely be ejected from the cluster. A solution similar to (47) may be found when y is large, provided that $m_A y/m_B$ is nearly unity; in this case $(m_A/m_B)^{5/2}$ replaces m_A/m_B in the exponent.

When $m_A y/m_B$ is small we may expand the denominator in (45). In this way we find

$$t = 0{\cdot}27\tau(1+y)^{5/2}\left\{1 + \frac{1}{7}\frac{m}{m_B}(5y-2) + \frac{1}{63}\left(\frac{m_A}{m_B}\right)^2(35y^2 - 20y + 8)\ \ldots\right\}. \tag{48}$$

It is evident from (48) that within a time τ stars of small mass will increase their average energy by about 50 per cent., provided that all stars had initially the same velocity. This would account for a considerable stratification of stars of different mass in the extended atmospheres of globular clusters.

Most stars of small mass will escape from the cluster if y is about 4,

since, as we have seen in the introduction, the root mean square velocity of the average cluster star is one-half the average velocity of escape. The increased rate of evaporation with increasing velocity, however, will eliminate most of these less massive stars before y has reached so high a value. More specifically, a star of zero mass and originally of average velocity will require a time equal to 14τ to double its velocity; on the other hand, a time of $2{\cdot}7\tau$ will suffice to increase y from 1 to 2, increasing K—the probability of evaporation of a single star during a time equal to the time of relaxation τ —by approximately 10. Since the average value of K is about 0·01, as we have seen in section 1, it is evident that at this increased rate a time of 10τ will suffice to eliminate most of the less massive stars. While this effect will be diminished by the tendency of the less massive stars to rise further from the centre of the cluster into regions of low star density, one may nevertheless conclude that by the time 20 per cent. of the average stars have been lost, most of the stars less massive than one-fourth the average will have left the cluster. If the cluster is sufficiently old, this effect should produce a sharp cut-off at the lower end of the mass distribution function.

6. These results may be applied to actual clusters, both globular and galactic. The values of N and $R_{1/2}$ for globular clusters are not well known. A lower limit of 10^5 may safely be adopted for N *; since $R_{1/2}$, the radius containing half the mass of the cluster, is scarcely less than 5 parsecs, and m is not much greater than unity, (19) is obviously satisfied by a wide margin for all globular clusters. Similarly, it is evident from (18) that for such clusters Λ is less than $1{\cdot}5 \times 10^{-11}$ years^{-1}; hence a globular cluster will lose not more than roughly half its mass in 10^{11} years.

Even for the long-time scale of 2×10^{12} years the loss of stars by evaporation is not necessarily serious. In this case, the right-hand side of (19) is increased by a factor of 10^6. Even with the extreme value of R above, a cluster with m equal to 0·2 need contain only some 10^8 stars to be stable during so long an interval. The loss of the less massive stars, however, which is just on the verge of becoming important in 10^9 years, would presumably be complete in this latter case.

It is evident from (21) that the number of escaping stars in the neighbourhood of a cluster depends only logarithmically on the total population of the cluster. If $\log N$ is 6·5 and $R_{1/2}$ is 6 parsecs, $G(r)$, the number of stars on the photographic plate within a radius r parsecs from the centre of the cluster, should be equal to $0{\cdot}6r$; thus there should be 120 stars moving outwards from the cluster within a radius of 200 parsecs. This quantity depends on the determination of τ, and is therefore uncertain to within half an order of magnitude. Any dispersion in the masses of cluster stars will increase $G(r)$; the results here give a lower limit for this quantity.

It has been known for some time that a few cluster-type Cepheids may be observed far from the centre of globular clusters. Recent investigations by Kopal † indicate that the above order of magnitude is at least a lower limit for the distribution of these short-period variables. Star counts in

* H. Shapley, *Mt. Wilson Contr.*, **152**, 27, 1918.

† Z. Kopal, unpublished work, to appear shortly as a *Harvard Circular*.

the vicinity of globular clusters would be valuable in determining the total number and distribution of the stars associated with these clusters and in providing a test for the theory.

It is of interest to note that a mass of 10^9 suns is an upper limit for the observed clusters, since otherwise we should frequently observe direct collisions between cluster stars. If two stars, each of radius r_0, pass within a periastron distance q of each other, where q equals $0{\cdot}29r_0$, corresponding to an $M(r)$ equal to $\frac{1}{2}M(r_0)$ on the polytropic model n equal to 3, then an enormous liberation of energy will certainly take place. As Whipple has suggested *, this is perhaps the mechanism responsible for supernovæ. At any rate, such a cataclysm would lead to a much greater increase in the rate of energy radiation than is observed for ordinary novæ; one may assume that if more than one such collision occurred every twenty years in the globular clusters of our own galaxy, at least one such would have been observed.

Following Whipple, we have for Γ the total number of collisions per year in a homogeneous cluster of uniform density

$$\Gamma = \frac{2(3\pi)^{1/2}GmqS\nu^2}{w}, \tag{49}$$

where m is the mass and w is the root mean square velocity of a cluster star; S is the volume of the cluster and ν is again the number of stars per cubic parsec. As in section 2, we may determine w from formula (4), and if R is again the radius of the cluster, we have

$$\Gamma = \frac{3}{2}\left(\frac{5}{\pi}\right)^{1/2}\frac{G^{1/2}M^{3/2}q}{R^{5/2}m}. \tag{50}$$

Let t_c be defined as the average interval in years between such collisions in each cluster; then we have simply

$$t_c = 1/\Gamma = 3{\cdot}5 \times 10^{14}R^{5/2}m/M^{3/2}q', \tag{51}$$

where q' denotes the value of $q/R_\odot$.

If we set q equal to $0{\cdot}29r_0$, and assume that the stars under consideration have the same mean density as the Sun, then we find for M,

$$M = 1{\cdot}1 \times 10^{10}R^{5/3}m^{4/9}t_c^{-2/3}. \tag{52}$$

If seventy globular clusters produce one such encounter every twenty years, t_c equals 1400 years. The integrated magnitudes of globular clusters indicate that if M is as great as 10^8, m must be considerably less than unity; we may accordingly set m equal to $0{\cdot}1$. As in section 3, these results may be applied to the inner core containing half the mass of the cluster. Since the radius of such a core will not exceed 10 parsecs in general, (37) gives for M the upper limit

$$M < 1{\cdot}5 \times 10^9. \tag{53}$$

Any rotation of the cluster as a whole will decrease the relative velocity and

* F. L. Whipple, *Proc. Nat. Acad. Sciences*, **25**, 118, 1939.

will decrease M in (53). Since in addition the observations probably indicate a considerably greater value of t_c than 1400 years, one may infer that (53) is a generous upper limit for M.

The galactic clusters present an essentially different picture. As Bok * has recently shown, a galactic cluster will gain energy by encounters with field stars; this will lead to an expansion of the cluster until it can no longer hold together under the disruptive influence of galactic rotation. The effect is greatest for extended, diffuse configurations. The process of evaporation analysed here, however, leads to a contraction of the cluster and is greatest for the densest clusters. Hence galactic clusters may be divided into two classes: the loose clusters, which are expanding and will gradually break up, and the dense clusters, which are contracting and ejecting stars.

Such clusters as the Pleiades, the Hyades, Præsepe and Messier 37 are clearly loose clusters in this sense. Their radius is great enough to satisfy (19) by a considerable margin, while the galactic effects will, as shown by Bok, disrupt these clusters in about $2{\cdot}10^9$ years.

For some of the smaller aggregations, however, the galactic effects are negligible and such clusters are presumably contracting. Cuffey † has recently determined colour indices and hence parallaxes and true diameters for several of these objects. The smallest cluster on his programme, N.G.C. 2129, contains 40 stars, and is found to have a diameter of 0·8 parsec. The star density in this cluster is about 150 stars per cubic parsec, more than a thousand times the density in the neighbourhood of the Sun. So dense a cluster will not be disrupted by any of the galactic effects in a time comparable with 10^9 years. While the analysis leading to (15) and (16) breaks down when $\log N$ is of order unity, (14) still gives an upper limit for τ; setting $R_{1/2}$ equal to 0·3 and m equal to unity, we see that the time of relaxation is less than $8{\cdot}10^6$ years. Hence the cluster should lose 1 per cent. of its mass every 10^7 years, or one star every $2{\cdot}10^7$ years. If these views are correct, either the cluster must have been formed more recently than $2{\cdot}10^9$ years ago, or else we now observe a cluster which has lost at least half its original members. From the previous section it follows that this cluster should contain very few stars of mass much less than the average.

It is of some theoretical interest to investigate in this connection the eventual fate of an isolated cluster.‡ It is evident from section 3 that the loss

* B. J. Bok, *Harvard Circular* No. 384, 1934.

† J. Cuffey, *Harvard Annals*, **106**, No. 2, 1937.

‡ Evaporation of stars from a cluster and the resultant contraction and evolution of the cluster have been qualitatively discussed by R. H. Dicke (*Astr. J.*, **48**, 108, 1939) in a paper which appeared after this work had been completed.

The recent investigation by H. Mineur (*Annales d'Astrophysique*, **2**, 167, 1939) refers not to the loss of stars by evaporation but to the shearing effect of galactic rotation. His result that the rate of loss from this source alone becomes infinite as the cluster becomes more and more compact seems to contradict the present analysis, but actually results from the use of the steady-state approximation. As in the case of a nearly isothermal sphere, this approximation is not valid far out from the effective cluster radius. The radius of the critical equipotential surface S_1 discussed by Mineur (all stars passing through S_1 are assumed to leave the cluster) will clearly be independent of the cluster radius for a spherical cluster, and will in fact vary of stars from a cluster proceeds at a continually accelerating rate as the cluster contracts. On the other hand, it is obvious that the cluster has not sufficient energy to eject all its stars to infinity. It may be readily shown that the mean free path of a cluster star is always greater than the radius of the cluster, provided that the cluster has more than ten stars. Hence the evaporation of stars would presumably proceed until one of two alternatives occurred. Collisions between stars could become important. Or, as the cluster continned to lose stars, the remaining ones might conceivably find themselves in periodic orbits, with no encounters to upset things. If, for instance, the cluster possessed angular momentum, it would presumably become a highly oblate ellipsoid rotating with an angular velocity varying along the radius. If the stars in such a disc had very little random velocity, this would be very nearly a permanently stable configuration.

Summary

The fact that an isothermal gaseous sphere has an infinite mass and an infinite radius is shown to correspond to the dynamical fact that a finite, nearly isothermal sphere will eject matter to infinity. The fraction of stars which have velocities greater than the escape velocity is calculated for a simplified ideal clnster with uniform density by the use of the virial theorem and the Maxwell-Boltzmann distribution of velocities. A determination of the time of relaxation gives the rate at which an equilibrium velocity distribntion is established, and hence determines the rate at which stars leave the cluster through this process of "evaporation."

The validity of the formulæ is extended to include the case in which the density increases towards the centre of the cluster; computations are made for the polytrope n equal to 5, corresponding to the Schuster density law. A determination is made of the rate at which energy becomes equalized between stars of unequal masses; it is shown that a cluster which has lost 20 per cent. of its total mass through evaporation will contain very few stars of mass less than one-fourth the average.

The calculations indicate that even in 10^{11} years, however, a globular cluster will not lose more than half of its stars through this process of evaporation. For the short-time scale the effect on the evolution of such clusters is therefore quite negligible. The number of escaping stars within a radius r of the cluster should vary directly with r and should depend only logarithmically on the population of the cluster; if half the mass of the cluster is concentrated within a sphere of radius 6 parsecs, 120 such stars should be observed within 200 parsecs of the centre. The fact that no

only with thc cluster mass. Hence if the density of the cluster is high, the radius of the cluster will be very much less than the radius of S_1, and the steady-state approximation cannot be used to find ρ_1, the density at S_1, or ϵ, thc rate at which stars pass through S_1. In the case of a compact cluster most of the stars at S_1 will be those which have already left the cluster through the process of evaporation discussed here, and which are moving outwards with velocities greater than thc velocity of escape. The rate of loss in such a casc is given by (18), not by Mineur's formulæ.

stellar collisions have been observed in globular clusters leads to an upper limit of 10^9 suns for the mass of an average cluster.

In the galactic clusters the effect may be more important because of the lower average stellar velocities. Such clusters may be divided into two classes: the loose extended configurations expanding and dissipating under galactic influences, the denser ones ejecting stars and contracting. The cluster N.G.C. 2129, for instance, with 40 stars within a radius of 0·4 parsec, should eject one star every 2×10^7 years; the rate of loss, moreover, should increase with the time. The possible fate of such systems is briefly discussed.

The author is very grateful to Dr. Shapley for the use of the (Harvard) observatory facilities, to Dr. Schwarzschild and Dr. Kopal for several suggestions, to Dr. Bok for many valuable discussions and to the National Research Council for a fellowship during the past year.

Sloane Physics Laboratory,
Yale University:
1940 *January.*