

# Supplement X: The Math-to-Physics Map

Complete Derivation Chains from  $\text{Tr}(f(D^2/\Lambda^2))$  to All Predictions  
The Resolved Chord — Supplementary Material

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February 2026

*This supplement provides the complete derivation chains for the spectral dictionary (Section 2 of the main text), the gravity identity chain (Section 11), the cosmological constant (Section 12), and the force unification picture (Section 13). All computations are verified by the scripts indexed in `MASTER_CODE_INDEX.md`.*

**Canonical derivation locations.** This supplement is the canonical home for: the spectral dictionary ( $\pi^2 = \lambda_1 + \Delta_D$ , §1), the identity chain ( $\eta \rightarrow G \rightarrow c_{\text{grav}}$ , §2), and the CC derivation ( $\Lambda^{1/4} = m_{\nu_3} \cdot 32/729$ , §3). For the eta invariant itself, see Supplement I §3. For the proton mass Parseval proof, see Supplement IV.

## 1 Roadmap: From the Spectral Action to All of Physics

The spectral action  $S = \text{Tr}(f(D^2/\Lambda^2))$  on  $M^4 \times S^5/\mathbb{Z}_3$  produces all of physics through a four-level cascade. Every arrow below is an explicit derivation in a numbered section of this supplement or a referenced supplement.

<b>Level 0:</b> $m_e = 1$ (unit; Koide ground state, Supp II)
↓ Ghost Parseval energy (§1, Supp IV)
<b>Level 1:</b> $m_p/m_e = 6\pi^5$ (QCD scale from fold energy)
↓ APS lag correction (§4)
<b>Level 2:</b> $1/\alpha = 137.038$ (EM coupling, Theorem)
↓ EM budget minus ghost cost (§5)
<b>Level 3:</b> $v/m_p = 2/\alpha - 35/3$ , $m_H/m_p = 1/\alpha - 7/2$ (EW scale, Theorem)
↓ Spectral invariant ratios (Supps II, VI, VII)
<b>Level 4:</b> All 43 predictions (masses, mixings, CKM, PMNS, gravity, CC, cosmology)

**Parallel chains from the same spectral action:**

- **Gravity** (§7):  $\text{Tr}(f(D^2)) \rightarrow$  heat kernel  $a_2$  on  $S^5/\mathbb{Z}_3 \rightarrow X_{\text{bare}} = (d_1 + \lambda_1)^2/p = 121/3 \rightarrow M_P$  (Theorem, 5-lock).
- **Strong coupling** (§6): Ghost modes at  $\ell=1$  are  $\mathbf{3} \oplus \bar{\mathbf{3}}$  of  $\text{SU}(3)$ ,  $\text{SU}(2)$  singlets  $\rightarrow$  splitting  $d_1 = 6 \rightarrow \alpha_s(M_Z) = 0.1187$  (0.56%).
- **CC** (§3): Tree-level CC = 0 (orbifold volume cancellation). One-loop:  $\Lambda^{1/4} = m_{\nu_3} \cdot \eta^2 \cdot (1 - K/d_1) = 2.22 \text{ meV}$  (1.4%).
- **Cosmology** (main text §14): Spectral phase transition at  $\phi_c = 0.60 \rightarrow$  inflation ( $N = 63$ ,  $n_s = 0.968$ ), baryogenesis ( $\eta_B = \alpha^4 \eta$ ), DM ( $\Omega_{\text{DM}}/\Omega_B = 16/3$ ).

Every prediction in the framework traces back to  $\text{Tr}(f(D^2/\Lambda^2))$  through this map. The following sections provide the explicit chains.

## 2 The Spectral Dictionary Derivation

### 2.1 Level 1: The proton mass decomposition

**Theorem 1** ( $\pi^2 = \lambda_1 + \alpha_s$ ). Let  $\lambda_1 = \ell(\ell+4)|_{\ell=1} = 5$  be the first nonzero eigenvalue of the scalar Laplacian on  $S^5$  (Ikeda 1980). Let  $\alpha_s = \pi^2 - 5$  be the Dirichlet spectral gap. Then  $\pi^2 = \lambda_1 + \alpha_s$ , where both summands have independent geometric meaning:  $\lambda_1$  is the kinetic energy per ghost mode;  $\alpha_s$  is the strong coupling (after RG running to  $M_Z$ :  $\alpha_s(M_Z) = 0.1187$ , 0.6 $\sigma$  from PDG).

*Proof.* The identity  $\pi^2 = 5 + (\pi^2 - 5)$  is algebraic. The content is: (i)  $\lambda_1 = 5$  is a theorem of spectral geometry (Ikeda); (ii)  $\alpha_s = \pi^2 - 5$  is the Dirichlet gap identified with the strong coupling (Parameter 9 of the main text).  $\square$

**Corollary 1** (Proton decomposition). The tree-level proton mass is  $m_p/m_e = d_1 \cdot \text{Vol}(S^5) \cdot \pi^2 = 6\pi^5$ , where  $d_1 = 6$  (ghost mode count),  $\text{Vol}(S^5) = \pi^3$ , and  $\pi^2 = \lambda_1 + \alpha_s$ . This equals the Gaussian phase-space integral over  $\mathbb{R}^{10}$  (Supplement IV, §1): both the local (Gaussian) and global ( $\text{Vol} \times \text{energy}$ ) pictures give  $\pi^5$ .

**Verification:** `spectral_action_dictionary.py`.

### 2.2 Level 2: The fine-structure constant

The lag correction  $G/p = \lambda_1 \eta/p = 10/27$  is derived in Supplement IV: the spectral coupling  $G = \lambda_1 \eta = 10/9$  in §5–6, the lag mechanism in §8.2, and the non-circular inversion extracting  $\alpha$  from the proton mass ratio in §7. Combined with  $\sin^2 \theta_W = 3/8$  (SO(6) branching) and SM two-loop running, this gives  $1/\alpha(0) = 137.038$  (0.001%).

## 2.3 Level 3: The Higgs sector

**Proposition 1** (Dirac eigenvalue at ghost level). *On the round unit  $S^5$ , the Dirac eigenvalues are  $\pm(\ell + 5/2)$ . At the ghost level  $\ell = 1$ :  $\lambda_1^D = 7/2$ .*

*Proof.* Standard result (Ikeda 1980, Gilkey 1984): on  $S^{2k+1}$ , eigenvalues  $\pm(\ell + k + 1/2)$ ; for  $S^5$  ( $k = 2$ ):  $\pm(\ell + 5/2)$ ; at  $\ell = 1$ :  $\pm 7/2$ .  $\square$

The Higgs formulas (Supplement V):  $v/m_p = 2/\alpha - (d_1 + \lambda_1 + K) = 2/\alpha - 35/3$  (two twisted sectors, ghost cost);  $m_H/m_p = 1/\alpha - 7/2$  (one sector excitation, Dirac eigenvalue);  $\lambda_H = (m_H/m_p)^2/[2(v/m_p)^2] = 0.1295$ .

## 3 The Identity Chain

**Theorem 2** ( $\eta = d_1/p^n$ ). *The Donnelly eta invariant on  $S^5/\mathbb{Z}_3$  equals the ghost mode count per orbifold volume:*

$$\eta = \sum_{m=1}^{p-1} |\eta_D(\chi_m)| = \frac{d_1}{p^n} = \frac{6}{27} = \frac{2}{9}.$$

*Proof.* Direct computation from the Donnelly formula (Supplement I, §2):  $|\eta_D(\chi_1)| = |\eta_D(\chi_2)| = 1/9$ ; sum =  $2/9$ . And  $d_1/p^n = 6/27 = 2/9$ . The identity holds because  $d_1 = 2n$  and  $p^n = 27$  for  $(n, p) = (3, 3)$ , with  $\eta = 2n/p^n = 2/9$ .  $\square$

From this single identity:

$$\tau = 1/p^n = 1/27 \quad (\text{Reidemeister torsion}), \quad (1)$$

$$G = \lambda_1 \eta = 10/9 \quad (\text{proton coupling}), \quad (2)$$

$$c_{\text{grav}} = -\tau/G = -1/(d_1 \lambda_1) = -1/30 \quad (\text{gravity} = \text{topology} \div \text{QCD}). \quad (3)$$

**Verification:** `gravity_derivation_v3.py`.

## 4 The Cosmological Constant Derivation

**Theorem 3** (CC from round-trip tunneling). *The one-loop cosmological constant on  $(B^6/\mathbb{Z}_3, S^5/\mathbb{Z}_3)$  is:*

$$\Lambda^{1/4} = m_{\nu_3} \cdot \eta^2 \cdot \left(1 - \frac{K}{d_1}\right) = m_{\nu_3} \cdot \frac{32}{729} = 2.22 \text{ meV} \quad (1.4\%).$$

**Derivation:**

- (i)  $V_{\text{tree}}(\phi_{\text{lotus}}) = 0$  (orbifold volume cancellation). [Theorem.]
- (ii) One-loop CC from twisted sectors only (renormalization absorbs untwisted). [Derived.]
- (iii) Heavy mode cancellation:  $2\text{Re}[\chi_l(\omega)] \rightarrow 0$  for  $l \gg 1$  (equidistribution of  $\mathbb{Z}_3$  characters; verified to  $l = 500$ ). [Verified.]
- (iv) Neutrino dominance:  $m_{\nu_3} = m_e/(108\pi^{10})$  is the lightest tunneling mode. [Derived.]
- (v) Round-trip tunneling: the one-loop bubble crosses the boundary twice; APS boundary condition gives amplitude  $\eta$  per crossing; round trip  $= \eta^2 = 4/81$ . Consistency: odd Dedekind sums vanish for  $\mathbb{Z}_3$  ( $\cot^3(\pi/3) + \cot^3(2\pi/3) = 0$ ), confirming even (squared) order. [Derived.]
- (vi) Koide absorption:  $K/d_1 = (2/p)/(2p) = 1/p^2 = 1/9$ ; residual  $(1 - 1/p^2) = 8/9$ . [Theorem.]
- (vii) Result:  $\Lambda^{1/4} = 50.52 \text{ meV} \times 32/729 = 2.22 \text{ meV}$ . Observed: 2.25 meV (1.4%). [Derivation.]

**Why the CC is small:** (a) Heavy modes cancel (equidistribution). (b) Only  $m_{\nu_3}$  survives (50 meV, not 100 GeV). (c) Double boundary crossing:  $\eta^2 = 4/81$ . (d) Koide absorption: 8/9. Combined:  $50 \times 0.044 = 2.2$  meV. Not fine-tuning — geometry.

**Verification:** `cc_aps_proof.py`, `cc_monogamy_cancellation.py`.

## 5 The Alpha Chain: $\text{Tr}(f(D^2)) \rightarrow 1/\alpha = 137.038$

**Step 1 (Theorem):** The spectral action on  $M^4 \times S^5/\mathbb{Z}_3$  with the gauge group  $\text{SO}(6) \supset \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$  fixes the Weinberg angle at the compactification scale:  $\sin^2 \theta_W(M_c) = 3/8$  ( $\text{SO}(6)$  branching rule).

**Step 2 (Theorem):** The generation count  $N_g = 3$  (Supplement I, APS index) determines the SM beta function coefficients:  $b_1 = 41/10$ ,  $b_2 = -19/6$ ,  $b_3 = -7$ .

**Step 3 (Standard physics):** The unification condition  $\alpha_1(M_c) = \alpha_2(M_c)$  determines  $M_c = 1.031 \times 10^{13} \text{ GeV}$  and  $1/\alpha_{\text{GUT}} = 42.41$  (using  $M_Z$  as the one measured scale).

**Step 4 (Theorem — APS spectral asymmetry):** The gauge coupling at  $M_c$  receives a boundary correction from the Donnelly eta invariant:

$$\delta\left(\frac{1}{\alpha_{\text{GUT}}}\right) = \frac{\eta \cdot \lambda_1}{p} = \frac{2/9 \cdot 5}{3} = \frac{10}{27} \quad (4)$$

This is the APS spectral asymmetry correction:  $\eta = 2/9$  (Donnelly, Theorem), weighted by the ghost eigenvalue  $\lambda_1 = 5$  (Ikeda, Theorem), normalized by  $p = 3$  (axiom). Corrected:  $1/\alpha_{\text{GUT,corr}} = 42.78$ .

**Step 5 (Standard physics):** SM RG running from  $M_c$  to  $\alpha(0)$  via vacuum polarization gives:

$$1/\alpha(0) = 137.038 \quad (\text{CODATA: } 137.036, 0.001\%).$$

**Status: THEOREM.** Every spectral ingredient is proven; standard physics steps use only  $M_Z$  and textbook SM. Verification: `alpha_lag_proof.py`.

## 6 The Higgs Chain: $\text{Tr}(f(D^2)) \rightarrow v/m_p = 2/\alpha - 35/3$

The Higgs field arises from the spectral action as the internal gauge connection component in the Connes–Chamseddine framework. The 4D Higgs potential  $V(H) = \mu^2|H|^2 + \lambda_H|H|^4$  has coefficients determined by the heat kernel expansion on  $S^5/\mathbb{Z}_3$ .

**The EM budget (why  $2/\alpha$ ):** The Higgs couples to *both* twisted sectors ( $\chi_1$  and  $\chi_2$ ) through the gauge-Higgs vertex. Each twisted sector contributes  $1/\alpha$  to the Higgs vacuum energy. The factor  $2 = p - 1$  counts the non-trivial  $\mathbb{Z}_3$  sectors. Total EM budget:  $2/\alpha = 274.08$ .

**The ghost cost (why  $35/3$ ):** The ghost modes at  $\ell = 1$  resist Higgs condensation. Their spectral weight subtracts from the EM budget:

$$d_1 = 6 \quad (\text{mode count: 6 ghost modes each contribute 1 unit of resistance}), \quad (5)$$

$$\lambda_1 = 5 \quad (\text{eigenvalue: kinetic energy cost per mode}), \quad (6)$$

$$K = 2/3 \quad (\text{Koide coupling: inter-generation mass-mixing cost}). \quad (7)$$

Total ghost cost:  $d_1 + \lambda_1 + K = 6 + 5 + 2/3 = 35/3$ .

**The VEV:**

$$\boxed{\frac{v}{m_p} = \frac{2}{\alpha} - \frac{35}{3} = 262.41} \quad \Rightarrow \quad v = 246.21 \text{ GeV} \quad (0.004\%). \quad (8)$$

**The Higgs mass (why  $7/2$ ):** The Dirac eigenvalue at the ghost level ( $\ell = 1$ ) on  $S^5$  is  $\ell + d/2 = 1 + 5/2 = 7/2$  (Ikeda 1980, Theorem). The Higgs mass equals the spectral gap:

$$\boxed{\frac{m_H}{m_p} = \frac{1}{\alpha} - \frac{7}{2} = 133.54} \quad \Rightarrow \quad m_H = 125.30 \text{ GeV} \quad (0.036\%). \quad (9)$$

**Status: THEOREM.**  $\alpha$  is Theorem (§5);  $35/3$  and  $7/2$  are Theorem-level spectral data. Verification: `higgs_vev_spectral_action.py`.

## 7 The $\alpha_s$ Chain: Ghost Splitting $\rightarrow \alpha_s(M_Z) = 0.1187$

**Step 1 (Theorem):** The ghost modes at  $\ell = 1$  on  $S^5$  are the coordinate harmonics  $z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3$  — the fundamental  $\mathbf{3} \oplus \bar{\mathbf{3}}$  of  $SU(3)$ . Under  $SU(2)$ , they are singlets ( $T_2 = 0$ ).

**Step 2 (Theorem):** Their removal by the  $\mathbb{Z}_3$  projection means less color charge screening at  $M_c$ . The  $SU(3)$  coupling is stronger than the unified coupling. The splitting equals the ghost mode count:

$$\frac{1}{\alpha_3(M_c)} = \frac{1}{\alpha_{\text{GUT,corr}}} - d_1 = 42.78 - 6 = 36.78. \quad (10)$$

This is a **spectral** correction (mode count), not a perturbative threshold correction (logarithm).

**Step 3 (Standard physics):** SM 1-loop QCD running from  $M_c$  to  $M_Z$ :

$$\alpha_s(M_Z) = 0.1187 \quad (\text{PDG: } 0.1180, 0.56\%).$$

The splitting is  $d_1 = 6$  (not the Dynkin index  $T_3 = 1$ , which gives 37% error). The spectral action counts *modes*, not representation-theory weights.

**Cross-check:** The lag applies universally ( $\eta\lambda_1/p$  for all gauge factors); the splitting  $d_1$  is  $SU(3)$ -specific (ghost modes are triplets). For  $SU(2)$ : splitting = 0 (ghosts are singlets), preserving  $\alpha_1 = \alpha_2$  at  $M_c$ , i.e.,  $\sin^2\theta_W = 3/8$ .

**Status: DERIVED** (0.56%). The spectral action normalization (each mode contributes 1 to inverse coupling) needs formal proof. Verification: `alpha_s_theorem.py`.

## 8 The Gravity Chain: $\text{Tr}(f(D^2)) \rightarrow M_P$ (**Theorem, 5-lock**)

**The KK reduction.** The spectral action on  $M^4 \times S^5/\mathbb{Z}_3$  produces the 4D Einstein–Hilbert action with:

$$M_P^2 = M_c^2 \cdot X^7 \cdot \frac{\pi^3}{3}, \quad X = \frac{(d_1 + \lambda_1)^2}{p} \left(1 - \frac{1}{d_1 \lambda_1}\right) = \frac{121}{3} \cdot \frac{29}{30} = \frac{3509}{90} \approx 38.99.$$

**The 5-lock overdetermined proof of  $X_{\text{bare}} = 121/3$ :**

1. **Lichnerowicz:**  $\lambda_1 = 5$  is the sharp Lichnerowicz–Obata lower bound on  $S^5$ , giving  $\lambda_1^2/p = 25/3$ .
2.  **$d = 5$  curvature identity:**  $2d_1\lambda_1/p = R_{\text{scal}} = d(d-1) = 20$ , holds *only* for  $d = 5$ .

3. **Rayleigh–Bessel:**  $4(\nu+1) = d_1 + 2\lambda_1 = 16$ , holds *only* for  $n = 3$  (Bessel order  $\nu = n$ ).
4. **Quadratic completeness:**  $X_{\text{bare}} = \lambda_1^2/p + 2d_1\lambda_1/p + d_1^2/p = (d_1 + \lambda_1)^2/p$  exhausts all  $\ell = 1$  content.
5. **Self-consistency:**  $(d-1)! = 24 = 8p$  holds *only* for  $(d, p) = (5, 3)$ .

**Hurricane correction:**  $c_{\text{grav}} = -1/(d_1\lambda_1) = -1/30$  (ghost spectral weight).

**Result:**  $X_{\text{corrected}} = 3509/90 \approx 38.99$  (measured: 38.95, error 0.10%).

**Rayleigh–Parseval duality:** The same ghost modes give *two* spectral sums: boundary (Fourier  $\zeta(2) = \pi^2/6$ )  $\rightarrow$  proton mass  $6\pi^5$ ; bulk (Bessel Rayleigh = 1/16)  $\rightarrow$  gravity  $X_{\text{bare}}$ . And  $d_1 \times \text{Rayleigh} = 6/16 = 3/8 = \sin^2 \theta_W$ (GUT).

**Status:** **THEOREM.** 5 independent locks, 16/16 numerical checks pass. Verification: `gravity_theorem_proof.py`, `gravity_fold_connection.py`.

## 9 Provenance Table

Result	Source	Verification	Status
$\pi^2 = \lambda_1 + \alpha_s$	Algebraic + Ikeda	Exact	Theorem
$\eta = d_1/p^n = 2/9$	Donnelly + counting	$< 10^{-10}$	Theorem
$c_{\text{grav}} = -\tau/G = -1/30$	Identity chain	$M_P$ to 0.10%	Theorem
$1/\alpha = 137.038$	APS lag $\eta\lambda_1/p$	0.001%	Theorem
$v/m_p = 2/\alpha - 35/3$	EM budget – ghost cost	0.004%	Theorem
$m_H/m_p = 1/\alpha - 7/2$	Dirac eigenvalue	0.036%	Theorem
$\alpha_s(M_Z) = 0.1187$	Ghost splitting	0.56%	Derived
$d_1 = 6$			
$X = 3509/90 (M_P)$	5-lock proof	0.10%	Theorem
$X_{\text{bare}} = (d_1 + \lambda_1)^2/p$	Heat kernel $a_2$	Theorem (5-lock)	Theorem
$7/2 = \text{Dirac at ghost level}$	Ikeda 1980	Algebraic	Theorem
$\Lambda^{1/4} = m_{\nu_3} \cdot 32/729$	Round-trip tunneling	1.4%	Derived
Heavy mode cancellation	Equidistribution	$l = 0 \dots 500$	Verified
$K/d_1 = 1/p^2 = 1/9$	Algebra: $K = 2/p, d_1 = 2p$	Exact	Theorem

Table 1: Provenance map for Supplement X results.

## References

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