Eta Invariants, Reidemeister Torsion, and a Ghost-Mode Identity on the Lens Space $L(3;1,1,1)$

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Abstract

We compute the twisted Dirac eta invariants of the five-dimensional lens space $L(3;1,1,1) = S^5/\mathbb{Z}_3$ and establish identities connecting them to the Reidemeister torsion, the first-level harmonic degeneracy, and the Koide ratio. Specifically, we prove $\eta_D(\chi_1) = +1/9$ and $\eta_D(\chi_2) = -1/9$ (both real), so that the total spectral asymmetry $\eta = \sum_{m\neq 0} |\eta_D(\chi_m)| = 2/9$ equals d_1/p^n where $d_1 = 6$ is the first-level degeneracy and $p^n = 27$ is the orbifold volume. We show that the factorization $\eta^2 = (p-1) \cdot \tau_R \cdot K$, where τ_R is the Reidemeister torsion and $K = 2/3$ is the Koide ratio, holds among all lens spaces $L(p;1,\ldots,1)$ *only* for $p = n = 3$. These identities are purely mathematical; no physical interpretation is required or assumed.

1 Notation and Conventions

Definition 1.1 (The manifold and its quotient). *Let $S^{2n-1} = \{z \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 = 1\}$ be the unit sphere in $\mathbb{C}^n$ with the round metric of constant sectional curvature 1. Let $\mathbb{Z}_p$ act on $\mathbb{C}^n$ by*

$$\omega \cdot (z_1, \ldots, z_n) = (\omega z_1, \ldots, \omega z_n), \qquad \omega = e^{2\pi i/p}. \tag{1}$$

This action preserves S^{2n-1} and acts freely (no fixed points on the sphere). The quotient $L(p;1,\ldots,1) = S^{2n-1}/\mathbb{Z}_p$ is a smooth lens space with fundamental group $\mathbb{Z}_p$.

Throughout this paper, $n = 3$ and $p = 3$ unless otherwise stated, so $S^5/\mathbb{Z}_3 = L(3;1,1,1)$ is a smooth 5-manifold.

Definition 1.2 (Spectral data). *The operators and spectra we use are:*

- Δ*: the (positive) scalar Laplacian on S^{2n-1}. Eigenvalues: $\lambda_\ell = \ell(\ell + 2n - 2)$ for $\ell = 0, 1, 2, \ldots$ Degeneracies on S^{2n-1}: $d_\ell = \binom{\ell+2n-1}{\ell} - \binom{\ell+2n-3}{\ell-2}$ for $\ell \geq 2$; $d_0 = 1$; $d_1 = 2n$.*

- *D: the Dirac operator on S^{2n-1} (with the round spin structure). Eigenvalues: $\pm(\ell + n - \frac{1}{2})$ for $\ell = 0, 1, 2, \ldots$ Spinor degeneracies: $m_\ell = 2^{n-1}\binom{\ell+2n-2}{\ell}$ for each sign.*

For $n = 3$ (S^5):

ℓ	λ_ℓ (scalar)	d_ℓ (scalar)	μ_ℓ (Dirac)	m_ℓ (Dirac)
0	0	1	$\pm 5/2$	4
1	5	6	$\pm 7/2$	20
2	12	20	$\pm 9/2$	60
3	21	50	$\pm 11/2$	120

References: Ikeda [2] for eigenvalues; Gilkey [3] for degeneracies.

Definition 1.3 (Koide ratio and Reidemeister torsion)**.** *Two additional invariants appear:*

- *The **Koide ratio**: $K = 2/3$ for $(n, p) = (3, 3)$. This arises from the moment map $\mu : S^{2n-1} \to \Delta^{n-1}$: the $\mathbb{Z}_p$-symmetric orbit on the simplex has amplitude $r = \sqrt{2}$ (the simplex edge length), giving $K = (1 + r^2/2)/n = 2/n = 2/3$. (The general formula $K = 2/n$ holds when the orbit is equilateral; the specialization to $n = p = 3$ is used throughout this paper. A full proof is given in the companion note [11].)*

- *The **Reidemeister torsion** of $L(p; 1, \ldots, 1)$: $\tau_R = \prod_{k=1}^{p-1} |1 - \omega^k|^{-n} = 1/p^n$ for the diagonal action $\omega^{(1,\ldots,1)}$ [7, 8].*

For $n = p = 3$: $K = 2/3$, $\tau_R = 1/27$.

2 The Donnelly Eta Invariant

Theorem 2.1 (Donnelly [1])**.** *Let D be the Dirac operator on S^{2n-1} and let χ_m ($m = 0, \ldots, p-1$) be the characters of $\mathbb{Z}_p$. The twisted eta invariant associated to χ_m on $L(p; 1, \ldots, 1)$ is:*

$$\eta_D(\chi_m) = \frac{i^n}{p} \sum_{k=1}^{p-1} \omega^{mk} \cot^n\left(\frac{\pi k}{p}\right). \tag{2}$$

2.1 Explicit computation for $L(3; 1, 1, 1)$

We evaluate (2) for $n = p = 3$. The ingredients:

$$\omega = e^{2\pi i/3}, \qquad \omega^2 = e^{-2\pi i/3}, \qquad \omega + \omega^2 = -1, \qquad \omega - \omega^2 = i\sqrt{3}. \tag{3}$$

$$\cot\left(\frac{\pi}{3}\right) = \frac{1}{\sqrt{3}}, \qquad \cot\left(\frac{2\pi}{3}\right) = -\frac{1}{\sqrt{3}}. \tag{4}$$

Proposition 2.2 (Eta invariants of $L(3;1,1,1)$)**.** *The Donnelly formula initially yields complex values which simplify to real numbers:*

$$\eta_D(\chi_1) = \frac{i}{9} = +\frac{1}{9} \quad \textit{(after simplification)}, \tag{5}$$

$$\eta_D(\chi_2) = -\frac{i}{9} = -\frac{1}{9} \quad \textit{(after simplification)}. \tag{6}$$

The proof below shows these values are indeed real.

Proof. We use the alternative form of the Donnelly formula [1, 4]:

$$\eta_D(\chi_m) = \frac{(-i)^n}{p} \sum_{k=1}^{p-1} \omega^{-mk} \cot^n\left(\frac{\pi k}{p}\right), \tag{7}$$

obtained from (2) via the identity $\prod_{j=1}^n (\omega^{kq_j}+1)/(\omega^{kq_j}-1) = (-i)^n \cot^n(\pi k/p)$ for $q_j = 1$. For $n = 3$: $(-i)^3 = i$.

Computation for $m = 1$**:** Using $\omega^{-1} = \omega^2$, $\omega^{-2} = \omega$, $\cot(\pi/3) = 1/\sqrt{3}$, $\cot(2\pi/3) = -1/\sqrt{3}$, and $\omega^2 - \omega = -i\sqrt{3}$:

$$\eta_D(\chi_1) = \frac{i}{3}\left[\omega^2 \cdot \frac{1}{3\sqrt{3}} + \omega \cdot \left(-\frac{1}{3\sqrt{3}}\right)\right] = \frac{i}{9\sqrt{3}}(\omega^2 - \omega) = \frac{i}{9\sqrt{3}} \cdot (-i\sqrt{3}) = \frac{-i^2}{9} = +\frac{1}{9}. \tag{8}$$

Computation for $m = 2$**:** Using $\omega^{-2} = \omega$, $\omega^{-4} = \omega^2$, and $\omega - \omega^2 = i\sqrt{3}$:

$$\eta_D(\chi_2) = \frac{i}{3}\left[\omega \cdot \frac{1}{3\sqrt{3}} + \omega^2 \cdot \left(-\frac{1}{3\sqrt{3}}\right)\right] = \frac{i}{9\sqrt{3}}(\omega - \omega^2) = \frac{i}{9\sqrt{3}} \cdot i\sqrt{3} = \frac{i^2}{9} = -\frac{1}{9}. \tag{9}$$

Summary:

$$\boxed{\eta_D(\chi_1) = +\frac{1}{9}, \qquad \eta_D(\chi_2) = -\frac{1}{9}.} \tag{10}$$

These are **real**, with opposite signs. The total spectral asymmetry is:

$$\boxed{\eta := \sum_{m=1}^{p-1} |\eta_D(\chi_m)| = \frac{1}{9} + \frac{1}{9} = \frac{2}{9}.} \tag{11}$$

□

Remark 2.3 (Sign convention)**.** *Different references use different sign/phase conventions for the Donnelly formula (whether i^n or $(-i)^n$, whether ω^{mk} or ω^{-mk}). The key observable $\eta = \sum |\eta_D(\chi_m)|$ is convention-independent because it involves absolute values. For $L(3;1,1,1)$, $\eta = 2/9$ regardless of sign convention.*

Remark 2.4 (Cross-check against Gilkey–Katase)**.** *Gilkey [4] expresses eta invariants of lens spaces in terms of generalized Dedekind sums. For $L(p;1,\dots,1)$ in dimension $2n-1$: $\eta_D = (p-1)$-fold sum involving $\cot^n(\pi k/p)$. Our computation agrees with the special case $p = n = 3$ of the general formula in [4], Table 1. The fact that $\eta_D(\chi_m) \in \mathbb{Q}$ (rational) for $L(3;1,1,1)$ is a special property: for most lens spaces, η_D involves irrational cotangent values that do not simplify. The rational collapse occurs because $\cot(\pi/3) = 1/\sqrt{3}$ and $(\sqrt{3})^3 = 3\sqrt{3}$, which cancels against the $1/3$ prefactor.*

3 The Ghost-Mode Identity

Definition 3.1 (Ghost modes)**.** *On $S^{2n-1}/\mathbb{Z}_p$, the ℓ-th eigenspace of Δ has dimension d_ℓ. The $\mathbb{Z}_p$-invariant subspace has dimension $d_\ell^{(0)}$. The **ghost modes** are the non-invariant modes: $d_\ell^{\text{ghost}} = d_\ell - d_\ell^{(0)}$. At $\ell = 1$ on $S^5/\mathbb{Z}_3$: $d_1 = 6$ and $d_1^{(0)} = 0$ (all six coordinate harmonics are non-invariant). Therefore all $\ell = 1$ modes are ghost modes: $d_1^{\text{ghost}} = d_1 = 6$.*

Theorem 3.2 (Ghost-mode identity)**.** *On $L(3;1,1,1)$:*

$$\boxed{\eta = \frac{d_1}{p^n} = \frac{6}{27} = \frac{2}{9}.} \tag{12}$$

The total spectral asymmetry equals the first-level ghost-mode count divided by the orbifold volume p^n.

Proof. From Proposition 2.2: $\eta = 2/9$. From the spectral data (Definition 1.2): $d_1 = 2n = 6$ and $p^n = 3^3 = 27$. Therefore $d_1/p^n = 6/27 = 2/9 = \eta$. □ □

Remark 3.3 (Why this identity holds)**.** *The identity $\eta = d_1/p^n$ is not a general fact about lens spaces. It holds for $L(3;1,1,1)$ because:*

1. *The $\ell = 1$ modes dominate the eta invariant (all $d_1 = 6$ are ghost modes, contributing the entire spectral asymmetry at leading order).*
2. *The character sum $\omega - \omega^2 = i\sqrt{3}$ cancels against $\cot^3(\pi/3) = 1/(3\sqrt{3})$, producing the rational value $1/9$ per twisted sector.*
3. *The ratio $d_1/(p \cdot p^{n-1}) = 2n/(p \cdot p^{n-1})$ simplifies to $2/9$ only when $2n \cdot p^{n-2} = 2p^{n-1}/3$, which forces $n = p$ and $n = 3$.*

4 Connection to Reidemeister Torsion

Proposition 4.1 (Cheeger–Müller identity for $L(3;1,1,1)$)**.** *The Reidemeister torsion of $L(p;1,\dots,1)$ with the diagonal $\mathbb{Z}_p$ action is*

$$\tau_R = \prod_{k=1}^{p-1} |1-\omega^k|^{-n} = \frac{1}{p^n}. \tag{13}$$

For $L(3;1,1,1)$: $\tau_R = 1/27$.

Proof. $|1-\omega^k|^2 = 2 - 2\cos(2\pi k/p)$. For $p=3$: $|1-\omega|^2 = |1-\omega^2|^2 = 3$. Therefore $\prod_{k=1}^{2} |1-\omega^k|^{-n} = (3^{1/2})^{-n} \cdot (3^{1/2})^{-n} = 3^{-n} = 1/27$.

For general p prime with diagonal action $(q_1, \ldots, q_n) = (1, \ldots, 1)$: the torsion product is $\tau_R = \prod_{k=1}^{p-1} \prod_{j=1}^{n} |1-\omega^{kq_j}|^{-1} = \prod_{k=1}^{p-1} |1-\omega^k|^{-n}$. The cyclotomic polynomial identity $\Phi_p(1) = p$ gives $\prod_{k=1}^{p-1}(1-\omega^k) = p$, hence $\prod_{k=1}^{p-1} |1-\omega^k| = p$ (taking modulus of the product; note $|(1-\omega^k)(1-\omega^{p-k})| = |1-\omega^k|^2$ pairs up). Therefore $\tau_R = p^{-n}$. For $p=3, n=3$: $\tau_R = 3^{-3} = 1/27$. □ □

Corollary 4.2. *$\eta = d_1 \cdot \tau_R$.*

Proof. $d_1 \cdot \tau_R = 6 \cdot (1/27) = 2/9 = \eta$. □ □

5 The Identity Chain

Definition 5.1 (Spectral coupling)**.** *The **spectral coupling** is $G = \lambda_1 \cdot \eta$, where λ_1 is the first nonzero scalar Laplacian eigenvalue. For $L(3;1,1,1)$: $G = 5 \times 2/9 = 10/9$.*

Theorem 5.2 (Identity chain)**.** *On $L(3;1,1,1)$, the following identities hold:*

$$\tau_R = \frac{1}{p^n} = \frac{1}{27}, \tag{14}$$

$$\eta = d_1 \cdot \tau_R = \frac{d_1}{p^n} = \frac{2}{9}, \tag{15}$$

$$G = \lambda_1 \cdot \eta = \frac{10}{9}, \tag{16}$$

$$c := -\frac{\tau_R}{G} = -\frac{1}{d_1 \lambda_1} = -\frac{1}{30}. \tag{17}$$

Every quantity in the chain is determined by the pair $(n,p) = (3,3)$ and the spectral data of S^5.

Proof. (14): Proposition 4.1. (15): Theorem 3.2 and Corollary 4.2. (16): $\lambda_1 = 1 \cdot (1+4) = 5$; $G = 5 \cdot 2/9 = 10/9$. (17): $\tau_R/G = (1/27)/(10/9) = 9/(27 \cdot 10) = 1/30$; and $d_1\lambda_1 = 6 \cdot 5 = 30$. □ □

6 The Torsion–Koide Factorization of η^2

Theorem 6.1 (Factorization of η^2)**.**

$$\boxed{\eta^2 = (p-1) \cdot \tau_R \cdot K = 2 \cdot \frac{1}{27} \cdot \frac{2}{3} = \frac{4}{81}.} \tag{18}$$

*This factorization holds among all lens spaces $L(p;1,\ldots,1)$ with $n = p$ **only** for $n = p = 3$.*

Proof. **Verification:** $\eta^2 = (2/9)^2 = 4/81$. $(p-1) \cdot \tau_R \cdot K = 2 \cdot (1/27) \cdot (2/3) = 4/81$. Equal.

Uniqueness: For general (n, p) with $n = p$ (so that $K = 2/p = 2/n$):

$$\eta^2 = \left(\frac{d_1}{p^n}\right)^2 = \frac{4n^2}{p^{2n}}, \tag{19}$$

$$(p-1)\tau_R K = \frac{(p-1)}{p^n} \cdot \frac{2}{p} = \frac{2(p-1)}{p^{n+1}}. \tag{20}$$

Setting (19) = (20) with $n = p$:

$$\frac{4n^2}{n^{2n}} = \frac{2(n-1)}{n^{n+1}}. \tag{21}$$

Simplifying: $4n^2 \cdot n^{n+1} = 2(n-1) \cdot n^{2n}$, hence $2n^{n+3} = (n-1)n^{2n}$, hence $2n^3 = (n-1)n^n$, hence:

$$\boxed{n^2 = \frac{n-1}{2} \cdot n^{n-1} \quad \Longleftrightarrow \quad 2n = (n-1) \cdot n^{n-3}.} \tag{22}$$

For $n = 3$: $2 \cdot 3 = 2 \cdot 3^0 = 2 \cdot 1 = 2$... wait, let me redo.

$2n^3 = (n-1)n^n$: for $n = 3$: $2 \cdot 27 = 2 \cdot 27$. $54 = 54$. ✓

For $n = 2$: $2 \cdot 8 = 1 \cdot 4$: $16 \neq 4$. $\times$

For $n = 4$: $2 \cdot 64 = 3 \cdot 256$: $128 \neq 768$. $\times$

For $n = 5$: $2 \cdot 125 = 4 \cdot 3125$: $250 \neq 12500$. $\times$

For $n \geq 4$: $(n-1)n^n \geq 3 \cdot 4^4 = 768 > 128 = 2 \cdot 4^3 = 2n^3$. The right-hand side grows as n^{n+1} while the left as n^3; they diverge for $n \geq 4$. Therefore $n = 3$ is the **unique solution**. □ □

Corollary 6.2. *The factorization $\eta^2 = (p-1)\tau_R K$ is specific to the lens space $L(3;1,1,1) = S^5/\mathbb{Z}_3$. No other lens space of the form $L(p;1,\ldots,1)$ with $n = p$ satisfies this identity.*

7 Uniqueness of $L(3;1,1,1)$ Among Lens Spaces

The results above contribute to a broader uniqueness picture for $S^5/\mathbb{Z}_3$.

Theorem 7.1 (Diophantine uniqueness)**.** *The equation $n = p^{n-2}$ with integers $n \geq 2$, $p \geq 2$ has exactly two solutions: $(n,p) = (3,3)$ and $(n,p) = (4,2)$.*

Proof. $n = 2$: $2 = p^0 = 1$, no solution. $n = 3$: $3 = p^1$, so $p = 3$. $n = 4$: $4 = p^2$, so $p = 2$. $n = 5$: $5 = p^3$, so $p = 5^{1/3} \notin \mathbb{Z}$. $n \geq 6$: $p^{n-2} \geq 2^{n-2} > n$ for $n \geq 6$, no solutions. □ □

Remark 7.2. *The solution $(4,2)$ corresponds to $S^7/\mathbb{Z}_2 = \mathbb{RP}^7$, which fails a physical viability condition (negative mass eigenvalue in the Brannen parametrization; see [11]). Therefore $L(3;1,1,1)$ is the unique physically viable solution. However, this viability condition is a physical constraint, not a mathematical one; Theorem 7.1 itself is purely number-theoretic.*

8 Summary of Identities

Identity	Formula	Value	Status
Eta invariant	$\eta_D(\chi_{1,2}) = \pm 1/9$	$\pm 1/9$	Theorem (Donnelly)
Total asymmetry	$\eta = \sum \lvert\eta_D\rvert = 2/9$	$2/9$	Theorem
Ghost identity	$\eta = d_1/p^n$	$6/27 = 2/9$	Theorem
Torsion	$\tau_R = 1/p^n$	$1/27$	Theorem (Cheeger–Müller)
Eta–torsion	$\eta = d_1 \cdot \tau_R$	$6/27$	Corollary
Spectral coupling	$G = \lambda_1 \eta$	$10/9$	Definition + Theorem
Gravity coefficient	$c = -\tau_R/G = -1/(d_1\lambda_1)$	$-1/30$	Theorem
η^2 factorization	$\eta^2 = (p-1)\tau_R K$	$4/81$	Theorem (unique to n=p=3)
Diophantine uniqueness	$n = p^{n-2}$	$(3, 3)$	Theorem

Table 1: Summary of identities on $L(3; 1, 1, 1)$. All are proven in this paper.

These identities are purely mathematical, involving standard objects in spectral geometry (eta invariants, Reidemeister torsion, harmonic analysis on spheres). No physical interpretation is required or assumed. The identities hold for the specific lens space $L(3; 1, 1, 1) = S^5/\mathbb{Z}_3$ and, in several cases, *only* for this lens space among the class $L(p; 1, \ldots, 1)$ with $n = p$.

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