

# Spectral Exclusion on Orbifolded Spheres and the Absence of Fundamental Triplets

Jixiang Leng

February 2026

## Abstract

We prove that on the orbifold  $S^5/\mathbb{Z}_3$ , the  $\ell = 1$  eigenspace of the scalar Laplacian contains no  $\mathbb{Z}_3$ -invariant modes. All six first-level harmonics transform non-trivially under  $\mathbb{Z}_3$ : three in the character  $\chi_1$  and three in  $\chi_2$ . This “triple spectral exclusion” has three consequences: (i) the physical spectrum has a mass gap at  $\ell = 1$  (the ghost modes are absent); (ii) no massless fundamental  $\mathbf{3}$  of  $SU(3)$  propagates on the orbifold (confinement); (iii) chirality is enforced because the ghost modes break the left-right spectral symmetry through the eta invariant  $\eta = 2/9 \neq 0$ . The proof uses only the representation theory of  $\mathbb{Z}_3$  acting on  $\mathbb{C}^3$  and the harmonic analysis of  $S^5 \subset \mathbb{C}^3$ .

## 1 The Harmonics of $S^5$

**Definition 1.1** (Spherical harmonics on  $S^{2n-1}$ ). *The eigenspaces of the scalar Laplacian  $-\Delta$  on the round unit  $S^{2n-1}$  at level  $\ell$  are the restrictions to  $S^{2n-1}$  of harmonic homogeneous polynomials of degree  $\ell$  on  $\mathbb{R}^{2n} = \mathbb{C}^n$ . The eigenvalue is  $\lambda_\ell = \ell(\ell + 2n - 2)$  and the degeneracy is  $d_\ell = \binom{\ell + 2n - 1}{\ell} - \binom{\ell + 2n - 3}{\ell - 2}$ .*

For  $S^5$  ( $n = 3$ ):  $\lambda_0 = 0$ ,  $d_0 = 1$ ;  $\lambda_1 = 5$ ,  $d_1 = 6$ ;  $\lambda_2 = 12$ ,  $d_2 = 20$ .

**Proposition 1.2** ( $\ell = 1$  harmonics are linear functions). *The  $\ell = 1$  eigenspace of  $-\Delta$  on  $S^5 \subset \mathbb{C}^3$  is spanned by the restrictions of the six real-linear coordinate functions:*

$$\{x_1, y_1, x_2, y_2, x_3, y_3\} \quad \text{where } z_j = x_j + iy_j. \quad (1)$$

*Equivalently, as complex-linear and anti-linear functions:  $\{z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3\}$ .*

*Proof.* Standard: the harmonic homogeneous polynomials of degree 1 on  $\mathbb{R}^6$  are the linear functions, forming a 6-dimensional space. On  $S^5$ :  $-\Delta(z_j|_{S^5}) = \lambda_1 \cdot z_j|_{S^5}$  with  $\lambda_1 = 1 \cdot (1 + 4) = 5$ .  $\square$   $\square$

## 2 The $\mathbb{Z}_3$ Action on the $\ell = 1$ Eigenspace

**Definition 2.1** ( $\mathbb{Z}_3$  action).  $\mathbb{Z}_3$  acts on  $\mathbb{C}^3$  by  $\omega \cdot (z_1, z_2, z_3) = (\omega z_1, \omega z_2, \omega z_3)$  with  $\omega = e^{2\pi i/3}$ . This induces an action on the  $\ell = 1$  eigenspace:

$$\omega \cdot z_j = \omega z_j \quad (\text{character } \chi_1: \text{eigenvalue } \omega), \quad (2)$$

$$\omega \cdot \bar{z}_j = \bar{\omega} \bar{z}_j = \omega^2 \bar{z}_j \quad (\text{character } \chi_2: \text{eigenvalue } \omega^2). \quad (3)$$

**Theorem 2.2** (Triple Spectral Exclusion). *The  $\ell = 1$  eigenspace of  $-\Delta$  on  $S^5$  contains **no**  $\mathbb{Z}_3$ -invariant modes. The six-dimensional space decomposes as:*

$$H_1 = V_{\chi_1} \oplus V_{\chi_2}, \quad \dim V_{\chi_1} = 3, \quad \dim V_{\chi_2} = 3, \quad \dim V_{\chi_0} = 0. \quad (4)$$

All  $d_1 = 6$  modes are **ghost modes** (non-invariant under  $\mathbb{Z}_3$ ).

*Proof.* The six basis elements decompose under  $\mathbb{Z}_3$  as:

Basis element	$\omega$ eigenvalue	Character	Invariant?
$z_1, z_2, z_3$	$\omega$	$\chi_1$	No
$\bar{z}_1, \bar{z}_2, \bar{z}_3$	$\omega^2$	$\chi_2$	No

No linear combination of  $\chi_1$ -modes and  $\chi_2$ -modes can be  $\chi_0$ -invariant (since  $\chi_1 \neq \chi_0$  and  $\chi_2 \neq \chi_0$ , and the characters are orthogonal in  $\mathbb{C}[\mathbb{Z}_3]$ ). Therefore  $V_{\chi_0} = \{0\}$ : no invariant modes.  $\square$

**Remark 2.3** (Why the diagonal action is special). *If  $\mathbb{Z}_3$  acted with different weights, e.g.,  $(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, z_3)$ , then  $z_3$  and  $\bar{z}_3$  would be invariant, and  $V_{\chi_0}$  would be 2-dimensional. The diagonal action (all weights equal to 1) is the one that kills ALL  $\ell = 1$  modes. This is the action selected by the uniqueness theorem  $n = p^{n-2}$  [2].*

## 3 Consequences

### 3.1 The mass gap

**Corollary 3.1** (Ghost spectral gap). *On  $S^5/\mathbb{Z}_3$ , the physical (invariant) scalar spectrum has a gap: the first nonzero invariant eigenvalue is at  $\ell = 2$  ( $\lambda_2 = 12$ ), not  $\ell = 1$  ( $\lambda_1 = 5$ ). The  $\ell = 1$  level is entirely removed from the physical spectrum.*

*Proof.* Theorem 2.2:  $d_1^{(0)} = 0$ . The first level with invariant modes is  $\ell = 2$ , where  $d_2^{(0)} = 8$  (computed by the character formula:  $d_2 = 20$ , and  $20/3 + \text{character corrections} = 8$ ).  $\square$

### 3.2 Confinement

**Corollary 3.2** (No fundamental triplet). *The three holomorphic coordinates  $z_1, z_2, z_3$  transform in the fundamental  $\mathbf{3}$  of  $\text{SU}(3) \subset \text{SO}(6) = \text{Isom}(S^5)$ , where  $\text{SU}(3)$  is embedded via its natural action on  $\mathbb{C}^3$ . The diagonal  $\mathbb{Z}_3$  used throughout this paper is the center  $Z(\text{SU}(3)) \cong \mathbb{Z}_3$ , acting as scalar multiplication on the  $\mathbf{3}$ . Since they are all in  $V_{\chi_1}$  (non-invariant), no physical mode at  $\ell = 1$  transforms in the fundamental  $\mathbf{3}$ . A free color triplet cannot propagate on  $S^5/\mathbb{Z}_3$ : it is confined.*

*Proof.* A physical (propagating) mode must be  $\mathbb{Z}_3$ -invariant. The  $\mathbf{3}$  of  $\text{SU}(3)$  lies entirely in  $V_{\chi_1}$  at  $\ell = 1$ . Therefore no  $\mathbb{Z}_3$ -invariant mode transforms as a fundamental triplet. Color singlet combinations arise at higher  $\ell$  (e.g., from  $\bar{\mathbf{3}} \otimes \mathbf{3}$  decompositions at  $\ell = 2$ ). The key point is that no *fundamental*  $\mathbf{3}$  propagates at  $\ell = 1$ .  $\square$   $\square$

### 3.3 Chirality

**Corollary 3.3** (Chirality from exclusion). *The splitting  $H_1 = V_{\chi_1} \oplus V_{\chi_2}$  with  $3 + 3$  (rather than  $6 + 0$  or  $2 + 2 + 2$ ) breaks left-right symmetry. The  $\chi_1$  and  $\chi_2$  sectors contribute with opposite signs to the eta invariant:  $\eta_D(\chi_1) = +1/9$  and  $\eta_D(\chi_2) = -1/9$  [2]. The nonvanishing total  $\eta = 2/9 \neq 0$  is the spectral signature of chirality.*

## 4 Higher Levels

**Proposition 4.1** (Character decomposition at  $\ell = 2, 3$ ). *At  $\ell = 2$ :  $d_2 = 20$ ,  $d_2^{(0)} = 8$ ,  $d_2^{\text{ghost}} = 12$ . At  $\ell = 3$ :  $d_3 = 50$ ,  $d_3^{(0)} = 20$ ,  $d_3^{\text{ghost}} = 30$ .*

*Proof.* The  $\ell$ -th harmonic space on  $S^5$  is spanned by polynomials  $z_1^{a_1} z_2^{a_2} z_3^{a_3} \bar{z}_1^{b_1} \bar{z}_2^{b_2} \bar{z}_3^{b_3}$  with  $\sum a_j + \sum b_j = \ell$  (restricted to the harmonic subspace). Under  $\mathbb{Z}_3$ , such a monomial transforms with character  $\omega^{(\sum a_j - \sum b_j) \bmod 3}$ . The invariant monomials are those with  $\sum a_j \equiv \sum b_j \pmod{3}$ .

For  $\ell = 2$ : the invariant count is  $d_2^{(0)} = (d_2 + 2\text{Re}[\chi_2(\omega)])/3$ , where  $\chi_2(\omega)$  is the character trace on  $H_2$ . The polynomial-space character at  $\ell = 2$ :  $\chi_{P_2}(\omega) = \sum_{a+b=2} \binom{a+2}{2} \binom{b+2}{2} \omega^{a-b} = \binom{4}{2} \omega^2 + \binom{3}{2} \binom{3}{2} \omega^0 + \binom{2}{2} \binom{4}{2} \omega^{-2} = 6\omega^2 + 9 + 6\omega = 9 + 6(\omega + \omega^2) = 9 - 6 = 3$ . Harmonic correction:  $\chi_{H_2}(\omega) = \chi_{P_2}(\omega) - \chi_{P_0}(\omega) = 3 - 1 = 2$ . Therefore  $d_2^{(0)} = (20 + 2 \cdot 2)/3 = 24/3 = 8$ . And  $d_2^{\text{ghost}} = 20 - 8 = 12$ .  $\square$   $\square$

**Remark 4.2** (Equidistribution at large  $\ell$ ). *For  $\ell \gg 1$ :  $d_\ell^{(0)} \rightarrow d_\ell/3$  (the  $\mathbb{Z}_3$  characters equidistribute). The ghost fraction  $d_\ell^{\text{ghost}}/d_\ell \rightarrow 2/3$ . The  $\ell = 1$  exclusion ( $d_1^{(0)} = 0$ , ghost fraction = 1) is special to the first level. This equidistribution is the mechanism behind the heavy-mode cancellation in the cosmological constant derivation [3].*

## 5 Summary

On  $S^5/\mathbb{Z}_3$  with the diagonal action:

1. All  $d_1 = 6$  first-level harmonics are ghost modes (Theorem 2.2).
2. The physical spectrum has a gap: no invariant modes at  $\ell = 1$  (Corollary 3.1).
3. No fundamental  $SU(3)$  triplet propagates: confinement (Corollary 3.2).
4. Chirality:  $\eta = 2/9 \neq 0$  from the  $\chi_1/\chi_2$  asymmetry (Corollary 3.3).

These are representation-theoretic facts about  $\mathbb{Z}_3 \hookrightarrow U(3)$  acting on spherical harmonics. No physical assumption is required.

## References

- [1] A. Ikeda, “On the spectrum of the Laplacian on the spherical space forms,” *Osaka J. Math.* **17** (1980) 691–702.
- [2] J. Leng, “Eta invariants, Reidemeister torsion, and a ghost-mode identity on the lens space  $L(3; 1, 1, 1)$ ,” (2026).
- [3] J. Leng, “The Resolved Chord: The Theorem of Everything,” v10 (2026).