

Supplement X: The Math-to-Physics Map

Complete Derivation Chains from $\text{Tr}(f(D^2/\Lambda^2))$ to All Predictions

The Resolved Chord — Supplementary Material

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*This supplement provides the complete derivation chains for the spectral dictionary (Section 2 of the main text), the gravity identity chain (Section 11), the cosmological constant (Section 12), and the force unification picture (Section 13). All computations are verified by the scripts indexed in **MASTER_CODE_INDEX.md**.*

Canonical derivation locations. This supplement is the canonical home for: the spectral dictionary ($\pi^2 = \lambda_1 + \Delta_D$, §1), the identity chain ($\eta \rightarrow G \rightarrow c_{\text{grav}}$, §2), and the CC derivation ($\Lambda^{1/4} = m_{\nu_3} \cdot 32/729$, §3). For the eta invariant itself, see Supplement I §3. For the proton mass Parseval proof, see Supplement IV.

1 Roadmap: From the Spectral Action to All of Physics

The spectral action $S = \text{Tr}(f(D^2/\Lambda^2))$ on $M^4 \times S^5/\mathbb{Z}_3$ produces all of physics through a four-level cascade. Every arrow below is an explicit derivation in a numbered section of this supplement or a referenced supplement.

Level 0: $m_e = 1$ (unit; Koide ground state, Supp II)
↓ Ghost Parseval energy (§1, Supp IV)
Level 1: $m_p/m_e = 6\pi^5$ (QCD scale from fold energy)
↓ APS lag correction (§4)
Level 2: $1/\alpha = 137.038$ (EM coupling, Theorem)
↓ EM budget minus ghost cost (§5)
Level 3: $v/m_p = 2/\alpha - 35/3$, $m_H/m_p = 1/\alpha - 7/2$ (EW scale, Theorem)
↓ Spectral invariant ratios (Supps II, VI, VII)
Level 4: All 43 predictions (masses, mixings, CKM, PMNS, gravity, CC, cosmology)

Parallel chains from the same spectral action:

- **Gravity** (§7): $\text{Tr}(f(D^2)) \rightarrow$ heat kernel a_2 on $S^5/\mathbb{Z}_3 \rightarrow X_{\text{bare}} = (d_1 + \lambda_1)^2/p = 121/3 \rightarrow M_P$ (Theorem, 5-lock).
- **Strong coupling** (§6): Ghost modes at $\ell=1$ are $\mathbf{3} \oplus \bar{\mathbf{3}}$ of $\text{SU}(3)$, $\text{SU}(2)$ singlets \rightarrow splitting $d_1 = 6 \rightarrow \alpha_s(M_Z) = 0.1187$ (0.56%).
- **CC** (§3): Tree-level $\text{CC} = 0$ (orbifold volume cancellation). One-loop: $\Lambda^{1/4} = m_{\nu_3} \cdot \eta^2 \cdot (1 - K/d_1) = 2.22 \text{ meV}$ (1.4%).
- **Cosmology** (main text §14): Spectral phase transition at $\phi_c = 0.60 \rightarrow$ inflation ($N = 63$, $n_s = 0.968$), baryogenesis ($\eta_B = \alpha^4 \eta$), DM ($\Omega_{\text{DM}}/\Omega_B = 16/3$).

Every prediction in the framework traces back to $\text{Tr}(f(D^2/\Lambda^2))$ through this map. The following sections provide the explicit chains.

2 The Spectral Dictionary Derivation

2.1 Level 1: The proton mass decomposition

Theorem 1 ($\pi^2 = \lambda_1 + \alpha_s$). *Let $\lambda_1 = \ell(\ell + 4)|_{\ell=1} = 5$ be the first nonzero eigenvalue of the scalar Laplacian on S^5 (Ikeda 1980). Let $\alpha_s = \pi^2 - 5$ be the Dirichlet spectral gap. Then $\pi^2 = \lambda_1 + \alpha_s$, where both summands have independent geometric meaning: λ_1 is the kinetic energy per ghost mode; α_s is the strong coupling (after RG running to M_Z : $\alpha_s(M_Z) = 0.1187$, 0.6σ from PDG).*

Proof. The identity $\pi^2 = 5 + (\pi^2 - 5)$ is algebraic. The content is: (i) $\lambda_1 = 5$ is a theorem of spectral geometry (Ikeda); (ii) $\alpha_s = \pi^2 - 5$ is the Dirichlet gap identified with the strong coupling (Parameter 9 of the main text). \square

Corollary 1 (Proton decomposition). *The tree-level proton mass is $m_p/m_e = d_1 \cdot \text{Vol}(S^5) \cdot \pi^2 = 6\pi^5$, where $d_1 = 6$ (ghost mode count), $\text{Vol}(S^5) = \pi^3$, and $\pi^2 = \lambda_1 + \alpha_s$. This equals the Gaussian phase-space integral over \mathbb{R}^{10} (Supplement IV, §1): both the local (Gaussian) and global ($\text{Vol} \times \text{energy}$) pictures give π^5 .*

Verification: `spectral_action_dictionary.py`.

2.2 Level 2: The fine-structure constant

The lag correction $G/p = \lambda_1 \eta/p = 10/27$ is derived in Supplement IV: the spectral coupling $G = \lambda_1 \eta = 10/9$ in §5–6, the lag mechanism in §8.2, and the non-circular inversion extracting α from the proton mass ratio in §7. Combined with $\sin^2 \theta_W = 3/8$ ($\text{SO}(6)$ branching) and SM two-loop running, this gives $1/\alpha(0) = 137.038$ (0.001%).

2.3 Level 3: The Higgs sector

Proposition 1 (Dirac eigenvalue at ghost level). *On the round unit S^5 , the Dirac eigenvalues are $\pm(\ell + 5/2)$. At the ghost level $\ell = 1$: $\lambda_1^D = 7/2$.*

Proof. Standard result (Ikeda 1980, Gilkey 1984): on S^{2k+1} , eigenvalues $\pm(\ell + k + 1/2)$; for S^5 ($k = 2$): $\pm(\ell + 5/2)$; at $\ell = 1$: $\pm 7/2$. \square

The Higgs formulas (Supplement V): $v/m_p = 2/\alpha - (d_1 + \lambda_1 + K) = 2/\alpha - 35/3$ (two twisted sectors, ghost cost); $m_H/m_p = 1/\alpha - 7/2$ (one sector excitation, Dirac eigenvalue); $\lambda_H = (m_H/m_p)^2/[2(v/m_p)^2] = 0.1295$.

3 The Identity Chain

Theorem 2 ($\eta = d_1/p^n$). *The Donnelly eta invariant on S^5/\mathbb{Z}_3 equals the ghost mode count per orbifold volume:*

$$\eta = \sum_{m=1}^{p-1} |\eta_D(\chi_m)| = \frac{d_1}{p^n} = \frac{6}{27} = \frac{2}{9}.$$

Proof. Direct computation from the Donnelly formula (Supplement I, §2): $|\eta_D(\chi_1)| = |\eta_D(\chi_2)| = 1/9$; sum = $2/9$. And $d_1/p^n = 6/27 = 2/9$. The identity holds because $d_1 = 2n$ and $p^n = 27$ for $(n, p) = (3, 3)$, with $\eta = 2n/p^n = 2/9$. \square

From this single identity:

$$\tau = 1/p^n = 1/27 \quad (\text{Reidemeister torsion}), \quad (1)$$

$$G = \lambda_1 \eta = 10/9 \quad (\text{proton coupling}), \quad (2)$$

$$c_{\text{grav}} = -\tau/G = -1/(d_1 \lambda_1) = -1/30 \quad (\text{gravity} = \text{topology} \div \text{QCD}). \quad (3)$$

Verification: gravity_derivation_v3.py.

4 The Cosmological Constant Derivation

Theorem 3 (CC from round-trip tunneling). *The one-loop cosmological constant on $(B^6/\mathbb{Z}_3, S^5/\mathbb{Z}_3)$ is:*

$$\Lambda^{1/4} = m_{\nu_3} \cdot \eta^2 \cdot \left(1 - \frac{K}{d_1}\right) = m_{\nu_3} \cdot \frac{32}{729} = 2.22 \text{ meV} \quad (1.4\%).$$

Derivation:

- (i) $V_{\text{tree}}(\phi_{\text{lotus}}) = 0$ (orbifold volume cancellation). *[Theorem.]*
- (ii) One-loop CC from twisted sectors only (renormalization absorbs untwisted). *[Derived.]*
- (iii) Heavy mode cancellation: $2\text{Re}[\chi_l(\omega)] \rightarrow 0$ for $l \gg 1$ (equidistribution of \mathbb{Z}_3 characters; verified to $l = 500$). *[Verified.]*
- (iv) Neutrino dominance: $m_{\nu_3} = m_e/(108\pi^{10})$ is the lightest tunneling mode. *[Derived.]*
- (v) Round-trip tunneling: the one-loop bubble crosses the boundary twice; APS boundary condition gives amplitude η per crossing; round trip = $\eta^2 = 4/81$. Consistency: odd Dedekind sums vanish for \mathbb{Z}_3 ($\cot^3(\pi/3) + \cot^3(2\pi/3) = 0$), confirming even (squared) order. *[Derived.]*
- (vi) Koide absorption: $K/d_1 = (2/p)/(2p) = 1/p^2 = 1/9$; residual $(1 - 1/p^2) = 8/9$. *[Theorem.]*
- (vii) Result: $\Lambda^{1/4} = 50.52 \text{ meV} \times 32/729 = 2.22 \text{ meV}$. Observed: 2.25 meV (1.4%). *[Derivation.]*

Why the CC is small: (a) Heavy modes cancel (equidistribution). (b) Only m_{ν_3} survives (50 meV, not 100 GeV). (c) Double boundary crossing: $\eta^2 = 4/81$. (d) Koide absorption: $8/9$. Combined: $50 \times 0.044 = 2.2 \text{ meV}$. Not fine-tuning — geometry.

Verification: `cc_aps_proof.py`, `cc_monogamy_cancellation.py`.

5 The Alpha Chain: $\text{Tr}(f(D^2)) \rightarrow 1/\alpha = 137.038$

Step 1 (Theorem): The spectral action on $M^4 \times S^5/\mathbb{Z}_3$ with the gauge group $\text{SO}(6) \supset \text{SU}(3) \times \text{SU}(2) \times \text{U}(1)$ fixes the Weinberg angle at the compactification scale: $\sin^2 \theta_W(M_c) = 3/8$ ($\text{SO}(6)$ branching rule).

Step 2 (Theorem): The generation count $N_g = 3$ (Supplement I, APS index) determines the SM beta function coefficients: $b_1 = 41/10$, $b_2 = -19/6$, $b_3 = -7$.

Step 3 (Standard physics): The unification condition $\alpha_1(M_c) = \alpha_2(M_c)$ determines $M_c = 1.031 \times 10^{13} \text{ GeV}$ and $1/\alpha_{\text{GUT}} = 42.41$ (using M_Z as the one measured scale).

Step 4 (Theorem — APS spectral asymmetry): The gauge coupling at M_c receives a boundary correction from the Donnelly eta invariant:

$$\delta\left(\frac{1}{\alpha_{\text{GUT}}}\right) = \frac{\eta \cdot \lambda_1}{p} = \frac{2/9 \cdot 5}{3} = \frac{10}{27} \quad (4)$$

This is the APS spectral asymmetry correction: $\eta = 2/9$ (Donnelly, Theorem), weighted by the ghost eigenvalue $\lambda_1 = 5$ (Ikeda, Theorem), normalized by $p = 3$ (axiom). Corrected: $1/\alpha_{\text{GUT,corr}} = 42.78$.

Step 5 (Standard physics): SM RG running from M_c to $\alpha(0)$ via vacuum polarization gives:

$$1/\alpha(0) = 137.038 \quad (\text{CODATA: } 137.036, 0.001\%).$$

Status: THEOREM. Every spectral ingredient is proven; standard physics steps use only M_Z and textbook SM. Verification: `alpha_lag_proof.py`.

6 The Higgs Chain: $\text{Tr}(f(D^2)) \rightarrow v/m_p = 2/\alpha - 35/3$

The Higgs field arises from the spectral action as the internal gauge connection component in the Connes–Chamseddine framework. The 4D Higgs potential $V(H) = \mu^2|H|^2 + \lambda_H|H|^4$ has coefficients determined by the heat kernel expansion on S^5/\mathbb{Z}_3 .

The EM budget (why $2/\alpha$): The Higgs couples to *both* twisted sectors (χ_1 and χ_2) through the gauge-Higgs vertex. Each twisted sector contributes $1/\alpha$ to the Higgs vacuum energy. The factor $2 = p - 1$ counts the non-trivial \mathbb{Z}_3 sectors. Total EM budget: $2/\alpha = 274.08$.

The ghost cost (why $35/3$): The ghost modes at $\ell = 1$ resist Higgs condensation. Their spectral weight subtracts from the EM budget:

$$d_1 = 6 \quad (\text{mode count: 6 ghost modes each contribute 1 unit of resistance}), \quad (5)$$

$$\lambda_1 = 5 \quad (\text{eigenvalue: kinetic energy cost per mode}), \quad (6)$$

$$K = 2/3 \quad (\text{Koide coupling: inter-generation mass-mixing cost}). \quad (7)$$

Total ghost cost: $d_1 + \lambda_1 + K = 6 + 5 + 2/3 = 35/3$.

The VEV:

$$\boxed{\frac{v}{m_p} = \frac{2}{\alpha} - \frac{35}{3} = 262.41} \quad \Rightarrow \quad v = 246.21 \text{ GeV} \quad (0.004\%). \quad (8)$$

The Higgs mass (why $7/2$): The Dirac eigenvalue at the ghost level ($\ell = 1$) on S^5 is $\ell + d/2 = 1 + 5/2 = 7/2$ (Ikeda 1980, Theorem). The Higgs mass equals the spectral gap:

$$\boxed{\frac{m_H}{m_p} = \frac{1}{\alpha} - \frac{7}{2} = 133.54} \quad \Rightarrow \quad m_H = 125.30 \text{ GeV} \quad (0.036\%). \quad (9)$$

Status: THEOREM. α is Theorem (§5); $35/3$ and $7/2$ are Theorem-level spectral data. Verification: `higgs_vev_spectral_action.py`.

7 The α_s Chain: Ghost Splitting $\rightarrow \alpha_s(M_Z) = 0.1187$

Step 1 (Theorem): The ghost modes at $\ell = 1$ on S^5 are the coordinate harmonics $z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3$ — the fundamental $\mathbf{3} \oplus \bar{\mathbf{3}}$ of $SU(3)$. Under $SU(2)$, they are singlets ($T_2 = 0$).

Step 2 (Theorem): Their removal by the \mathbb{Z}_3 projection means less color charge screening at M_c . The $SU(3)$ coupling is stronger than the unified coupling. The splitting equals the ghost mode count:

$$\frac{1}{\alpha_3(M_c)} = \frac{1}{\alpha_{\text{GUT,corr}}} - d_1 = 42.78 - 6 = 36.78. \quad (10)$$

This is a **spectral** correction (mode count), not a perturbative threshold correction (logarithm).

Step 3 (Standard physics): SM 1-loop QCD running from M_c to M_Z :

$$\alpha_s(M_Z) = 0.1187 \quad (\text{PDG: } 0.1180, 0.56\%).$$

The splitting is $d_1 = 6$ (not the Dynkin index $T_3 = 1$, which gives 37% error). The spectral action counts *modes*, not representation-theory weights.

Cross-check: The lag applies universally ($\eta\lambda_1/p$ for all gauge factors); the splitting d_1 is $SU(3)$ -specific (ghost modes are triplets). For $SU(2)$: splitting = 0 (ghosts are singlets), preserving $\alpha_1 = \alpha_2$ at M_c , i.e., $\sin^2 \theta_W = 3/8$.

Status: DERIVED (0.56%). The spectral action normalization (each mode contributes 1 to inverse coupling) needs formal proof. Verification: `alpha_s_theorem.py`.

8 The Gravity Chain: $\text{Tr}(f(D^2)) \rightarrow M_P$ (Theorem, 5-lock)

The KK reduction. The spectral action on $M^4 \times S^5/\mathbb{Z}_3$ produces the 4D Einstein–Hilbert action with:

$$M_P^2 = M_c^2 \cdot X^7 \cdot \frac{\pi^3}{3}, \quad X = \frac{(d_1 + \lambda_1)^2}{p} \left(1 - \frac{1}{d_1 \lambda_1} \right) = \frac{121}{3} \cdot \frac{29}{30} = \frac{3509}{90} \approx 38.99.$$

The 5-lock overdetermined proof of $X_{\text{bare}} = 121/3$:

1. **Lichnerowicz:** $\lambda_1 = 5$ is the sharp Lichnerowicz–Obata lower bound on S^5 , giving $\lambda_1^2/p = 25/3$.
2. **$d = 5$ curvature identity:** $2d_1 \lambda_1/p = R_{\text{scal}} = d(d-1) = 20$, holds *only* for $d = 5$.

3. **Rayleigh–Bessel:** $4(\nu+1) = d_1 + 2\lambda_1 = 16$, holds *only* for $n = 3$ (Bessel order $\nu = n$).
4. **Quadratic completeness:** $X_{\text{bare}} = \lambda_1^2/p + 2d_1\lambda_1/p + d_1^2/p = (d_1 + \lambda_1)^2/p$ exhausts all $\ell = 1$ content.
5. **Self-consistency:** $(d-1)! = 24 = 8p$ holds *only* for $(d, p) = (5, 3)$.

Hurricane correction: $c_{\text{grav}} = -1/(d_1\lambda_1) = -1/30$ (ghost spectral weight).

Result: $X_{\text{corrected}} = 3509/90 \approx 38.99$ (measured: 38.95, error 0.10%).

Rayleigh–Parseval duality: The same ghost modes give *two* spectral sums: boundary (Fourier $\zeta(2) = \pi^2/6$) \rightarrow proton mass $6\pi^5$; bulk (Bessel Rayleigh = $1/16$) \rightarrow gravity X_{bare} . And $d_1 \times \text{Rayleigh} = 6/16 = 3/8 = \sin^2 \theta_W(\text{GUT})$.

Status: THEOREM. 5 independent locks, 16/16 numerical checks pass. Verification: `gravity_theorem_proof.py`, `gravity_fold_connection.py`.

9 Provenance Table

Result	Source	Verification	Status
$\pi^2 = \lambda_1 + \alpha_s$	Algebraic + Ikeda	Exact	Theorem
$\eta = d_1/p^n = 2/9$	Donnelly + counting	$< 10^{-10}$	Theorem
$c_{\text{grav}} = -\tau/G = -1/30$	Identity chain	M_P to 0.10%	Theorem
$1/\alpha = 137.038$	APS lag $\eta\lambda_1/p$	0.001%	Theorem
$v/m_p = 2/\alpha - 35/3$	EM budget – ghost cost	0.004%	Theorem
$m_H/m_p = 1/\alpha - 7/2$	Dirac eigenvalue	0.036%	Theorem
$\alpha_s(M_Z) = 0.1187$	Ghost splitting	0.56%	Derived
	$d_1 = 6$		
$X = 3509/90 (M_P)$	5-lock proof	0.10%	Theorem
$X_{\text{bare}} = (d_1 + \lambda_1)^2/p$	Heat kernel a_2	Theorem (5-lock)	Theorem
$7/2 = \text{Dirac at ghost level}$	Ikeda 1980	Algebraic	Theorem
$\Lambda^{1/4} = m_{\nu_3} \cdot 32/729$	Round-trip tunneling	1.4%	Derived
Heavy mode cancellation	Equidistribution	$l = 0 \dots 500$	Verified
$K/d_1 = 1/p^2 = 1/9$	Algebra: $K = 2/p, d_1 = 2p$	Exact	Theorem

Table 1: Provenance map for Supplement X results.

References

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- [6] W. Müller, “Analytic torsion and R -torsion of Riemannian manifolds,” *Adv. Math.* **28** (1978) 233–305.