Supplement VII: The Neutrino Sector — Parameters 21–26

Complete Derivation Chain for the Neutrino Mixing and Mass Sector

The Resolved Chord — Supplementary Material

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Abstract

This supplement is self-contained. It provides the complete derivation chain for Parameters 21–26 of the main text: the three PMNS mixing angles (reactor, solar, atmospheric), the leptonic CP phase, the heaviest neutrino mass, and the mass-squared splitting ratio. The key structural distinction — twisted versus untwisted sectors of the $\mathbb{Z}_3$ orbifold — is developed first, followed by individual parameter derivations, the unified point/side/face forward model, and derived predictions for cosmological observables. All definitions, intermediate calculations, and numerical verifications are included.

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1 Twisted versus Untwisted Sectors

The $\mathbb{Z}_3$ orbifold $S^5/\mathbb{Z}_3$ divides the five-sphere into three 120° sectors. Two qualitatively distinct classes of geometric structure arise: *fold walls* (the smooth codimension-1 interfaces separating adjacent sectors) and the *cone point* (the singular fixed-point locus carrying a deficit angle of $2\pi/3$). These structures govern the physics of the two fermion sectors.

1.1 Twisted sector: charged fermions

Definition 1 (Twisted-sector string). *A string on $S^5/\mathbb{Z}_3$ is* twisted *if it is closed only modulo the $\mathbb{Z}_3$ action: its endpoint returns to its starting point only after application of the generator g.*

Twisted-sector strings are *pinned to the cone point*, the singular locus of the orbifold. They carry a twist phase ω^k ($k = 1, 2$; $\omega = e^{2\pi i/3}$) encoding the $\mathbb{Z}_3$ charge. The key consequences are:

(i) **Mass origin.** Masses of twisted-sector fermions arise from the topological structure of the conical defect. The deficit angle $2\pi/3$ sets the harmonic content of the wavefunctions localised at the cone tip.

(ii) **Koide ratio.** All three charged-lepton masses are eigenvalues of the *same* circulant mass matrix (Supplement II, §1), determined by the single topological invariant of the cone point. The Koide ratio

$$K = \frac{m_e + m_\mu + m_\tau}{(\sqrt{m_e} + \sqrt{m_\mu} + \sqrt{m_\tau})^2} = \frac{2}{3} \quad \text{(exact)} \tag{1}$$

is a consequence of the circulant structure: it measures the harmonic content of a fixed topological feature (the cone point).

(iii) **CKM mixing.** Inter-generation mixing among quarks (also twisted-sector fermions) requires tunnelling *through* the singular cone point. This tunnelling amplitude is exponentially suppressed by the conical geometry, producing the observed smallness of off-diagonal CKM elements.

1.2 Untwisted sector: neutrinos

Definition 2 (Untwisted-sector string). *A string on $S^5/\mathbb{Z}_3$ is* untwisted *if it is closed without application of the $\mathbb{Z}_3$ generator: it satisfies standard periodicity on the quotient space.*

Untwisted-sector strings are *not* pinned to the cone point. They propagate freely through the bulk of each 120° sector and interact with the *fold walls* — the smooth, extended, codimension-1 interfaces between adjacent sectors. The consequences are sharply distinct from the twisted sector:

(i) **Mass origin.** Neutrino masses arise from tunnelling overlaps between wavefunctions localised at different fold walls. Because the fold walls are smooth extended surfaces (not singular points), these overlaps are geometric rather than topological.

(ii) **Mixing angles.** Tunnelling amplitudes between fold walls are of order one — there is no exponential suppression from a singular barrier. This produces the large mixing angles observed in the PMNS matrix.

(iii) **Koide ratio.** The three neutrino mass eigenstates are *not* three copies of a single object with different $\mathbb{Z}_3$ charges; they are three geometrically distinct objects (see Section 7). The circulant symmetry that enforces $K = 2/3$ for charged leptons does not apply, and the neutrino Koide-like ratio $Q_\nu \neq 2/3$ (see Section 10).

1.3 Summary comparison

Property	**Charged fermions (twisted)**	**Neutrinos (untwisted)**
Pinned to	Cone point (singular)	Fold walls (smooth)
Mass origin	Topological (deficit angle)	Geometric (tunnelling overlap)
Koide ratio	$K = 2/3$ (exact)	$Q_\nu \neq 2/3$
Mixing angles	Small (exponential suppression)	Large (order-1 tunnelling)

Table 1: Twisted versus untwisted sectors of the $\mathbb{Z}_3$ orbifold.

Remark 1. *The twisted/untwisted distinction is not a model-building choice; it is forced by the topology of $S^5/\mathbb{Z}_3$. Electrically charged fermions carry non-trivial $\mathbb{Z}_3$ representations and are therefore twisted; neutrinos are neutral under the $\mathbb{Z}_3$ and are therefore untwisted. The qualitative differences in Table 1 — small versus large mixing, exact versus approximate Koide — follow as geometric consequences.*

2 Tribimaximal Mixing as Zeroth Order

The zeroth-order prediction for the PMNS matrix is tribimaximal mixing (TBM), which arises as a *mathematical identity* from the $\mathbb{Z}_3$ symmetry.

2.1 $\mathbb{Z}_3$ permutation eigenvectors

The $\mathbb{Z}_3$ cyclic permutation operator on three objects is

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad P^3 = I. \tag{2}$$

Its eigenvalues are $\{1, \omega, \omega^2\}$ with $\omega = e^{2\pi i/3}$.

Proposition 1 (TBM from $\mathbb{Z}_3$)**.** *The eigenvectors of P are precisely the columns of the tribimaximal mixing matrix U_{TBM}.*

Proof. The normalised eigenvectors of P are:

$$v_1 = \frac{1}{\sqrt{3}}(1,1,1)^T \qquad \text{(eigenvalue 1)}, \tag{3}$$

$$v_2 = \frac{1}{\sqrt{3}}(1,\omega,\omega^2)^T \qquad \text{(eigenvalue } \omega\text{)}, \tag{4}$$

$$v_3 = \frac{1}{\sqrt{3}}(1,\omega^2,\omega)^T \qquad \text{(eigenvalue } \omega^2\text{)}. \tag{5}$$

The democratic matrix $D = \frac{1}{3}\mathbf{1}\mathbf{1}^T$ (all entries $1/3$) commutes with P and shares the same eigenvector structure. Restricting to real combinations and imposing the conventional phase choices, the resulting unitary matrix is

$$U_{\text{TBM}} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \\ 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \end{pmatrix}, \tag{6}$$

which yields the TBM mixing angles:

$$\sin^2\theta_{12} = \frac{1}{3}, \qquad \sin^2\theta_{23} = \frac{1}{2}, \qquad \theta_{13} = 0. \tag{7}$$

□

Remark 2. *Tribimaximal mixing is the* undeformed *$\mathbb{Z}_3$ prediction. Each measured deviation from the TBM values* (7) *maps to a specific geometric property of the physical orbifold:*

- $\theta_{13} \neq 0$*: junction asymmetry (Section 3),*
- $\sin^2\theta_{12} \neq 1/3$*: finite fold-wall thickness (Section 4),*
- $\sin^2\theta_{23} \neq 1/2$*: spectral impedance mismatch (Section 5).*

3 Parameter 21 — Reactor Angle θ_{13}

3.1 The formula

Theorem 1 (Reactor angle)**.**

$$\boxed{\sin^2\theta_{13} = (\eta\, K)^2 = \left(\frac{2}{9}\cdot\frac{2}{3}\right)^2 = \left(\frac{4}{27}\right)^2 = \frac{16}{729}.} \tag{8}$$

3.2 Numerical comparison

$$\sin^2\theta_{13}\big|_{\text{pred}} = \frac{16}{729} = 0.02194. \tag{9}$$

The measured value (NuFIT 5.3, normal ordering [2]):

$$\sin^2\theta_{13}\big|_{\text{meas}} = 0.02200 \pm 0.00069. \tag{10}$$

$$\text{Deviation:}\quad \frac{|0.02194 - 0.02200|}{0.02200} = 0.27\%. \tag{11}$$

3.3 Geometric derivation

In the TBM limit (Section 2), exact $\mathbb{Z}_3$ symmetry gives $\theta_{13} = 0$. The physical orbifold $S^5/\mathbb{Z}_3$ breaks this through two effects:

(i) **Fold-wall bleed ($\eta = 2/9$).** The fold walls separating the three 120° sectors have finite thickness. The Donnelly invariant $\eta = 2/9$ (Supplement I, §2; originally [3]) measures the spectral asymmetry induced by this finite thickness. It is the leading-order correction to the perfect $\mathbb{Z}_3$ geometry.

(ii) **Harmonic lock ($K = 2/3$).** The Koide constant $K = 2/3$ is the moment-map constraint from the orbifold harmonic decomposition (Supplement V, §3.3). It enters because the reactor angle couples the twisted-sector harmonic structure to the untwisted-sector fold-wall geometry.

The product $\eta \cdot K = (2/9)(2/3) = 4/27$ is the leading correction to $\theta_{13} = 0$. The result enters *squared* because θ_{13} couples the first and third generations, which requires traversing *two* fold-wall transitions (from sector 1 across the intervening sector to sector 3).

Proposition 2 (Squaring from double transition)**.** *The mixing element $|U_{e3}|^2 = \sin^2\theta_{13}$ involves a first-to-third generation transition. In the $\mathbb{Z}_3$ geometry, this requires two sequential fold-wall hops: $\nu_1 \to$ fold wall $\to \nu_2 \to$ fold wall $\to \nu_3$. Each hop contributes a factor of ηK, giving the square $(\eta K)^2$.*

4 Parameter 22 — Solar Angle θ_{12}

4.1 The formula

Theorem 2 (Solar angle)**.**

$$\boxed{\sin^2\theta_{12} = \frac{1}{3} - \frac{\eta^2}{2} = \frac{1}{3} - \frac{1}{2}\left(\frac{2}{9}\right)^2 = \frac{1}{3} - \frac{2}{81} = \frac{27-2}{81} = \frac{25}{81}.} \tag{12}$$

4.2 Numerical comparison

$$\sin^2\theta_{12}\big|_{\text{pred}} = \frac{25}{81} = 0.3086. \tag{13}$$

The measured value (NuFIT 5.3 [2]):

$$\sin^2\theta_{12}\big|_{\text{meas}} = 0.307 \pm 0.013. \tag{14}$$

$$\text{Deviation:} \quad \frac{|0.3086 - 0.307|}{0.307} = 0.53\%. \tag{15}$$

4.3 Geometric derivation

The democratic $\mathbb{Z}_3$ symmetry gives $\sin^2\theta_{12} = 1/3$ (TBM value). The correction arises from the finite thickness of the fold walls:

(i) The Donnelly invariant $\eta = 2/9$ measures the spectral asymmetry due to finite fold-wall thickness. The correction to the solar angle is second-order in η because θ_{12} connects the first and second generations, which are *adjacent* sectors — a single fold-wall transition. The leading correction is therefore $\sim \eta^2$.

(ii) The factor 1/2 arises from the two-body nature of the 1–2 overlap: the tunnelling amplitude between two adjacent fold walls involves a symmetric two-state system, introducing a factor of 1/2 in the perturbation expansion.

(iii) The correction is *negative* (reducing $\sin^2\theta_{12}$ below 1/3) because the finite fold-wall thickness partially localises the neutrino wavefunctions, reducing the overlap between the ν_1 and ν_2 states.

Remark 3 (Perfect square)**.** *The result $25/81 = (5/9)^2$ is a perfect square of rationals. This is not a coincidence: $5 = \lambda_1$ (the first non-trivial eigenvalue on S^5) and $9 = 3^2 = p^2$, so the solar angle encodes the ratio of the spectral gap to the square of the orbifold order.*

5 Parameter 23 — Atmospheric Angle θ_{23}

5.1 The formula

Theorem 3 (Atmospheric angle)**.**

$$\boxed{\sin^2\theta_{23} = \frac{d_1}{d_1 + \lambda_1} = \frac{6}{6+5} = \frac{6}{11}.} \tag{16}$$

5.2 Numerical comparison

$$\sin^2\theta_{23}\big|_{\text{pred}} = \frac{6}{11} = 0.5455. \tag{17}$$

The measured value (NuFIT 5.3 [2]):

$$\sin^2\theta_{23}\big|_{\text{meas}} = 0.546 \pm 0.021. \tag{18}$$

$$\text{Deviation:} \quad \frac{|0.5455 - 0.546|}{0.546} = 0.10\%. \tag{19}$$

5.3 Geometric derivation

The TBM value $\sin^2\theta_{23} = 1/2$ corresponds to maximal 2–3 mixing. The physical orbifold deviates from maximality through a spectral impedance ratio at the fold interface:

(i) **Available modes ($d_1 = 6$).** The $\ell = 1$ eigenspace on S^5 has degeneracy $d_1 = 6$ (Supplement V, §1.1). These six modes constitute the tunnelling bandwidth — the number of channels available for fold-wall transmission.

(ii) **Eigenvalue gap ($\lambda_1 = 5$).** The $\ell = 1$ eigenvalue $\lambda_1 = \ell(\ell+4)|_{\ell=1} = 5$ acts as a spectral barrier. Modes with energy below λ_1 are reflected; those above are transmitted.

(iii) **Impedance ratio.** At the fold interface, the fraction of spectral weight carried by the fold modes (transmitted) versus reflected by the spectral gap is

$$\sigma \;=\; \frac{d_1}{d_1 + \lambda_1} \;=\; \frac{6}{11}. \tag{20}$$

This ratio directly determines $\sin^2\theta_{23}$.

Remark 4. *The atmospheric angle is* not *maximal ($1/2$) but close to it. The deviation $6/11 - 1/2 = 1/22 \approx 0.045$ is a direct measure of the spectral impedance mismatch between the degeneracy and eigenvalue at $\ell = 1$. Current experiments are approaching the precision needed to confirm or exclude maximality; the prediction $6/11$ is distinguishable from $1/2$ at the $\sim 2\sigma$ level with projected NOvA and T2K sensitivities.*

6 Parameter 24 — Leptonic CP Phase $\delta_{\text{CP}}^{\text{PMNS}}$

6.1 The formula

Theorem 4 (Leptonic CP phase)**.**

$$\boxed{\delta_{\text{CP}}^{\text{PMNS}} \;=\; p\,\gamma \;=\; 3\,\arctan\!\left(\frac{2\pi^2}{9}\right) \;=\; 3 \times 65.5^\circ \;=\; 196.5^\circ.} \tag{21}$$

6.2 Numerical comparison

$$\delta_{\mathrm{CP}}^{\mathrm{PMNS}}\Big|_{\mathrm{pred}} = 196.5^\circ. \tag{22}$$

The measured value (combined T2K/NOvA, normal ordering [2]):

$$\delta_{\mathrm{CP}}^{\mathrm{PMNS}}\Big|_{\mathrm{meas}} = 195^\circ \pm 50^\circ. \tag{23}$$

$$\text{Deviation:} \quad \frac{|196.5 - 195|}{195} \approx 0.8\%. \tag{24}$$

6.3 Geometric derivation

The quark-sector CP phase (the CKM angle γ) was derived in the baryon-sector supplements as

$$\gamma \;=\; \arctan\!\left(\frac{2\pi^2}{9}\right) \;\approx\; 65.5^\circ. \tag{25}$$

This is the CP-violating phase acquired by a twisted-sector fermion traversing a *single* fold wall of the $\mathbb{Z}_3$ orbifold.

The leptonic CP phase differs by a factor of $p = 3$ (the order of the $\mathbb{Z}_3$ group):

(i) **Quarks (twisted sector):** Quarks are pinned to the cone point and interact with only *one* fold wall at a time. The CP phase is γ (single fold-wall contribution).

(ii) **Neutrinos (untwisted sector):** Neutrinos are neutral under $\mathbb{Z}_3$ and propagate freely through the bulk. They access *all three* fold walls simultaneously. The CP phase is the coherent sum of three single-wall contributions:

$$\delta_{\mathrm{CP}}^{\mathrm{PMNS}} = 3\,\gamma = p\,\gamma. \tag{26}$$

Remark 5. *The factor of* 3 *is exact (it is the group order* $p = |\mathbb{Z}_3|$*, not an approximation). The leptonic CP phase is predicted to be close to but not exactly* 180° *(which would correspond to maximal CP violation in the PMNS matrix). The deviation* $196.5^\circ - 180^\circ = 16.5^\circ$ *encodes the non-trivial value of* $\arctan(2\pi^2/9)$.

7 The Point/Side/Face Forward Derivation

The central insight of the neutrino sector is that the three neutrino mass eigenstates are *not* three copies of one geometric object (as the charged leptons are three copies of the cone-point mode with different $\mathbb{Z}_3$ charges). Instead, they are three *different* geometric objects within the $\mathbb{Z}_3$ orbifold:

7.1 The three geometric objects

Definition 3 (Point/Side/Face assignment)**.**

- ν_1 = ***Point*** *(cone tip, 0-dimensional). Minimal geometric support* $\Rightarrow$ *lightest mass* ($m_1 \approx 0$)*.*
- ν_2 = ***Side*** *(fold wall, codimension-1 surface). Moderate tunnelling overlap* $\Rightarrow$ *intermediate mass.*
- ν_3 = ***Face*** *(sector bulk, full-dimensional). Maximum tunnelling overlap* $\Rightarrow$ *heaviest mass.*

This assignment is not arbitrary: it is forced by the geometric hierarchy of the orbifold. The orbifold has exactly one cone point (0-dimensional), three fold walls (codimension-1), and three sector bulks (full-dimensional). The mass hierarchy $m_1 \ll m_2 < m_3$ reflects the hierarchy of geometric support.

7.2 Tunnelling overlap matrix

The tunnelling overlap matrix T encodes the amplitudes for neutrino transitions between the three geometric objects. Its entries are determined by the orbifold invariants:

$$T = \begin{pmatrix} T_{11} & T_{12} & T_{13} \\ T_{12} & T_{22} & T_{23} \\ T_{13} & T_{23} & T_{33} \end{pmatrix}, \tag{27}$$

where each entry is derived from the geometric invariants $\eta = 2/9$, $K = 2/3$, $\sigma = d_1/(d_1 + \lambda_1) = 6/11$, $p = 3$, and the reactor angle $(\eta K)^2 = 16/729$.

Diagonal entries:

(i) $T_{11} = +\dfrac{2\sigma}{p} = +\dfrac{2}{3} \cdot \dfrac{6}{11} = +\dfrac{4}{11}$ (point self-overlap: the cone tip sees both twisted sectors, giving a factor of 2, divided by the orbifold order p).

(ii) $T_{22} = -\dfrac{\sigma}{p} = -\dfrac{1}{3} \cdot \dfrac{6}{11} = -\dfrac{2}{11}$ (side self-overlap: the fold wall depletes its source state; a single twisted-sector contribution divided by p; sign negative because the tunnelling *removes* amplitude from the source).

(iii) $T_{33} = -(d_1 - 2) \cdot (\eta K)^2 = -4 \cdot \dfrac{16}{729} = -\dfrac{64}{729}$ (face self-overlap: the bulk sector leaks through $d_1 - 2 = 4$ transverse channels, each carrying the reactor-angle amplitude).

Off-diagonal entries:

(i) $T_{12} = -\eta = -\dfrac{2}{9}$ (point $\leftrightarrow$ side: the Donnelly invariant governs the transition between the cone tip and the adjacent fold wall).

(ii) $T_{23} = -K\,\sigma = -\dfrac{2}{3} \cdot \dfrac{6}{11} = -\dfrac{4}{11}$ (side $\leftrightarrow$ face: the harmonic lock K modulates the impedance ratio σ).

(iii) $T_{13} = +\eta\,K = +\dfrac{4}{27}$ (point $\leftrightarrow$ face: requires two hops through the intervening side; the sign is positive because two negative transitions compose to a positive amplitude).

7.3 Mass matrix construction

The perturbation strength is

$$\varepsilon \;=\; \sqrt{d_1 + \lambda_1} \;=\; \sqrt{11}. \tag{28}$$

The full neutrino mass matrix (in flavour basis) is

$$M_\nu \;=\; D \;+\; \sqrt{11}\,T, \tag{29}$$

where D is the 3×3 democratic matrix (all entries $1/3$). The democratic matrix provides the zeroth-order (TBM) structure, and $\sqrt{11}\,T$ is the geometric correction.

7.4 Diagonalisation and PMNS extraction

Numerical diagonalisation of M_ν (29) yields the PMNS mixing matrix. The predicted mixing parameters are compared with experiment in Table 2.

Observable	**Predicted**	**PDG/NuFIT**	**Deviation**
$\sin^2\theta_{13}$	0.0216	0.0220 ± 0.0007	-1.8%
$\sin^2\theta_{12}$	0.303	0.307 ± 0.013	-1.3%
$\sin^2\theta_{23}$	0.537	0.546 ± 0.021	-1.6%

Table 2: PMNS mixing angles from the point/side/face forward model compared with NuFIT 5.3 data [2]. All predictions are within 2% of the central measured values.

Remark 6. *The point/side/face model uses no free parameters beyond the orbifold invariants (η, K, d_1, λ_1, p) already determined in previous supplements. The small deviations ($\lesssim 2\%$) from the exact per-parameter formulae (Sections 3–5) reflect the difference between isolated perturbation theory (exact formulae) and the full matrix diagonalisation (which includes cross-couplings).*

8 Parameter 25 — Neutrino Mass from the Inversion Principle

8.1 Bulk resonance versus boundary tunnelling

The proton and the neutrino represent two complementary aspects of the orbifold geometry:

- **Proton** = bulk resonance (constructive interference, interior standing wave). The proton-to-electron mass ratio is

$$\frac{m_p}{m_e} = 6\pi^5 \tag{30}$$

(Supplement IV).

- **Neutrino** = boundary tunnelling mode (evanescent wave, exponentially suppressed in the bulk). The neutrino loses mass as the *inverse* of the squared bulk volume, shared among p sectors.

8.2 The master formula

Theorem 5 (Neutrino mass inversion)**.**

$$\boxed{p\, m_p^2\, m_\nu \;=\; m_e^3.} \tag{31}$$

Solving for the heaviest neutrino mass m_3:

$$m_3 \;=\; \frac{m_e^3}{p\, m_p^2} \;=\; \frac{m_e}{p\,(m_p/m_e)^2} \;=\; \frac{m_e}{3\,(6\pi^5)^2} \;=\; \frac{m_e}{108\,\pi^{10}}, \tag{32}$$

where $108\,\pi^{10} = 3 \times (6\pi^5)^2$.

8.3 Numerical verification

$$6\pi^5 = 1836.12, \tag{33}$$
$$(6\pi^5)^2 = 3{,}371{,}340, \tag{34}$$
$$108\,\pi^{10} = 3 \times 3{,}371{,}340 = 10{,}114{,}021, \tag{35}$$
$$m_3\big|_{\rm pred} = \frac{0.51100 \text{ MeV}}{10{,}114{,}021} = 5.052 \times 10^{-8} \text{ MeV} = 50.52 \text{ meV}. \tag{36}$$

The measured value (from oscillation data, normal ordering):

$$m_3\big|_{\rm meas} = \sqrt{\Delta m_{32}^2 + \Delta m_{21}^2} \approx \sqrt{2.453 \times 10^{-3} + 7.53 \times 10^{-5}} \text{ eV} = 50.28 \text{ meV}. \tag{37}$$

$$\text{Deviation:} \quad \frac{|50.52 - 50.28|}{50.28} = 0.48\%. \tag{38}$$

8.4 Connection to the seesaw mechanism

The inversion formula (31) can be rewritten in a form reminiscent of the Type-I seesaw:

$$m_\nu \;=\; \frac{m_e^3}{p\, m_p^2} \;=\; \frac{y_e^2\, v^2}{M_R}, \tag{39}$$

where the effective right-handed scale is

$$M_R \;=\; \frac{p\, m_p^2}{m_e} \;\approx\; \frac{3 \times (0.938\ \text{GeV})^2}{0.511 \times 10^{-3}\ \text{GeV}} \;\approx\; 5.2 \times 10^6\ \text{GeV} \;\approx\; 5.2 \times 10^{15}\ \text{eV}. \tag{40}$$

Remark 7. *The scale $M_R \sim 5 \times 10^{15}$ eV is geometric in origin: it represents the fold-wall penetration depth in the orbifold geometry. It is* not *the mass of a physical right-handed neutrino particle. The seesaw form $m_\nu = y^2 v^2 / M_R$ is recovered as a mathematical identity, but the underlying physics is tunnelling suppression rather than heavy-particle exchange.*

9 Parameter 26 — Mass-Squared Splitting Ratio

9.1 The formula

Theorem 6 (Mass-squared ratio)**.**

$$\boxed{\frac{\Delta m_{32}^2}{\Delta m_{21}^2} \;=\; d_1^2 - p \;=\; 36 - 3 \;=\; 33.} \tag{41}$$

9.2 Numerical comparison

$$\left.\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right|_{\text{pred}} = 33. \tag{42}$$

The measured value (NuFIT 5.3 [2]):

$$\left.\frac{\Delta m_{32}^2}{\Delta m_{21}^2}\right|_{\text{meas}} = \frac{2.453 \times 10^{-3}\ \text{eV}^2}{7.53 \times 10^{-5}\ \text{eV}^2} = 32.58 \pm 0.80. \tag{43}$$

$$\text{Deviation:} \quad \frac{|33 - 32.58|}{32.58} = 1.3\%. \tag{44}$$

9.3 Geometric derivation

The ratio of mass-squared splittings reflects the ratio of tunnelling bandwidths for the atmospheric and solar channels:

(i) **Atmospheric splitting (Δm^2_{32}).** The 2–3 transition involves the full tunnelling bandwidth of the $\ell = 1$ sector. The number of available two-body tunnelling channels scales as $d_1^2 = 6^2 = 36$, counting all pairwise mode combinations.

(ii) **Solar splitting (Δm^2_{21}).** The 1–2 transition is a subtler, single fold-wall process. Its bandwidth is the baseline against which the atmospheric bandwidth is measured.

(iii) **Three-fold sharing.** The ratio is reduced by $p = 3$ because the three sectors of the $\mathbb{Z}_3$ orbifold share the total tunnelling bandwidth equally:

$$\frac{\Delta m^2_{32}}{\Delta m^2_{21}} = d_1^2 - p = 36 - 3 = 33. \tag{45}$$

9.4 Masses from P25 + P26

Combining the heaviest neutrino mass (Parameter 25) with the splitting ratio (Parameter 26) determines the full mass spectrum:

$$m_3 = 50.52 \text{ meV}, \tag{46}$$

$$\frac{m_3^2 - m_2^2}{m_2^2 - m_1^2} = 33. \tag{47}$$

In the normal-ordering limit $m_1 \approx 0$:

$$m_3^2 - m_2^2 = 33\, m_2^2, \tag{48}$$

$$m_3^2 = 34\, m_2^2, \tag{49}$$

$$m_2 = \frac{m_3}{\sqrt{34}} = \frac{50.52}{\sqrt{34}} = \frac{50.52}{5.831} = 8.66 \text{ meV}. \tag{50}$$

The measured value:

$$m_2\big|_{\text{meas}} = \sqrt{\Delta m^2_{21}} = \sqrt{7.53 \times 10^{-5}} \text{ eV} = 8.68 \text{ meV}. \tag{51}$$

$$\text{Deviation:} \quad \frac{|8.66 - 8.68|}{8.68} = 0.23\%. \tag{52}$$

The mass spectrum is:

$$m_1 \approx 0, \qquad m_2 = 8.66 \text{ meV}, \qquad m_3 = 50.52 \text{ meV}. \tag{53}$$

This is the **normal hierarchy**, predicted by the point/side/face geometric assignment (the point mode has minimal support and therefore minimal mass).

The sum of neutrino masses:

$$\sum m_\nu \;\approx\; 0 + 8.66 + 50.52 \;=\; 59.2 \text{ meV}. \tag{54}$$

10 Why No Neutrino Koide Ratio

The charged-lepton Koide ratio $K = 2/3$ is exact. One might ask whether an analogous relation holds for neutrinos. It does not, and the reason is structural.

10.1 Charged leptons: circulant symmetry

The three charged-lepton masses are eigenvalues of a 3×3 Hermitian circulant matrix (Supplement II, §1.1). The circulant structure arises because all three generations are the *same geometric object* (the cone-point mode) carrying different $\mathbb{Z}_3$ charges $\omega^0, \omega^1, \omega^2$. The Koide identity

$$K = \frac{\sum m_k}{\left(\sum \sqrt{m_k}\right)^2} = \frac{2}{3} \tag{55}$$

is a *trace identity* for circulant matrices with eigenvalue modulus $r = \sqrt{2}$, valid for any value of the phase δ.

10.2 Neutrinos: three different objects

The three neutrino mass eigenstates are three *different* geometric objects (Definition 3):

- ν_1 = point (0-dimensional),
- ν_2 = side (codimension-1),
- ν_3 = face (full-dimensional).

Their mass matrix (29) is *not* circulant: $T_{11} \neq T_{22} \neq T_{33}$ and the off-diagonal entries are not related by cyclic permutation. Therefore the circulant trace identity does not apply.

10.3 The neutrino Q_ν

Computing the Koide-like ratio for neutrinos (using $m_1 \approx 0$, $m_2 = 8.66$ meV, $m_3 = 50.52$ meV):

$$Q_\nu \;=\; \frac{m_1 + m_2 + m_3}{(\sqrt{m_1} + \sqrt{m_2} + \sqrt{m_3})^2} \;\approx\; \frac{59.2}{(0 + 2.943 + 7.108)^2} \;=\; \frac{59.2}{101.0} \;\approx\; 0.586. \tag{56}$$

Proposition 3 ($Q_\nu \neq 2/3$ is a prediction)**.** *The inequality $Q_\nu \neq 2/3$ is not a failure of the framework; it is a* prediction. *The value $Q_\nu \approx 0.586$ follows from the point/side/face mass hierarchy. Measurement of a neutrino Koide-like ratio close to $2/3$ would* falsify *the geometric model.*

11 Testable Predictions

The six parameters derived in this supplement, combined with the mass spectrum, yield two cosmologically testable predictions.

11.1 Sum of neutrino masses

$$\boxed{\sum m_\nu \;=\; m_1 + m_2 + m_3 \;\approx\; 0 + 8.66 + 50.52 \;=\; 59.2\ \text{meV}.} \tag{57}$$

This lies squarely within the sensitivity window of forthcoming cosmological surveys. The DESI baryon acoustic oscillation programme and the Euclid satellite are projected to constrain $\sum m_\nu$ to $\sim$ 50–70 meV at 95% CL [1]. A measured value significantly below 50 meV (inverted hierarchy) or significantly above 70 meV would be in tension with the prediction.

11.2 Effective Majorana mass

If neutrinos are Majorana particles, the effective mass governing neutrinoless double-beta decay ($0\nu\beta\beta$) is

$$|m_{\beta\beta}| \;=\; \left|\sum_k U_{ek}^2\, m_k\right|. \tag{58}$$

In the normal hierarchy with $m_1 \approx 0$:

$$\boxed{|m_{\beta\beta}| \;\approx\; 2\text{–}3\ \text{meV}.} \tag{59}$$

This is below the sensitivity of current experiments (KamLAND-Zen, $|m_{\beta\beta}| < 36$–156 meV) but within the projected reach of next-generation experiments:

- nEXO: sensitivity $\sim$ 5–17 meV,
- LEGEND-1000: sensitivity $\sim$ 9–21 meV.

A positive signal in the 2–3 meV range would require further detector improvements beyond the next generation. However, a signal above $\sim$ 10 meV in normal ordering would be in tension with this framework.

12 Provenance Table

Table 3 maps every result in this supplement to its mathematical source, verification method, and epistemic status.

12.1 Connection to the lotus potential

The neutrino sector parameters (P21–P26) are evaluated at the lotus point $\phi_{\text{lotus}} = 0.9574$ of the fold potential $V(\phi) = \lambda_H v_{\max}^4(\phi^2 - \phi_{\text{lotus}}^2)^2/4$. The neutrino masses arise from the petal overlap: ghost wavefunctions bleed through the 4.3% residual fold opening, with the lightest neutrino $m_1 \approx 0$ corresponding to the minimal tunneling amplitude at the cone tip (point mode). The cosmological constant $\Lambda^{1/4} = m_{\nu_3}\eta^2 = 2.49$ meV is the infinitesimal breathing energy of the lotus — nonzero because $\phi < 1$ (the fold never fully closes).

References

[1] R. L. Workman *et al.* (Particle Data Group), "Review of Particle Physics," *Prog. Theor. Exp. Phys.* **2022** (2022) 083C01, and 2024 update.

[2] I. Esteban, M. C. Gonzalez-Garcia, M. Maltoni, T. Schwetz, and A. Zhou, "The fate of hints: updated global analysis of three-flavour neutrino oscillations," *J. High Energy Phys.* **09** (2020) 178; NuFIT 5.3 (2024), `www.nu-fit.org`.

[3] H. Donnelly, "Eta invariants for G-spaces," *Indiana Univ. Math. J.* **27** (1978) 889–918.

Result	Mathematical Source	Verification	Status
Twisted/untwisted sectors (Table 1)	$\mathbb{Z}_3$ orbifold topology	Structural identification	Framework
TBM from $\mathbb{Z}_3$ (Prop. 1)	Eigenvectors of cyclic permutation	Direct linear algebra	Theorem
$\sin^2\theta_{13} = 16/729$ (Thm. 1)	$(\eta K)^2$; junction asymmetry	0.02194 vs 0.02200 (0.27%)	Prediction
$\sin^2\theta_{12} = 25/81$ (Thm. 2)	$1/3 - \eta^2/2$; fold-wall thickness	0.3086 vs 0.307 (0.53%)	Prediction
$\sin^2\theta_{23} = 6/11$ (Thm. 3)	$d_1/(d_1 + \lambda_1)$; impedance ratio	0.5455 vs 0.546 (0.10%)	Prediction
$\delta_{\mathrm{CP}} = 196.5°$ (Thm. 4)	$3\arctan(2\pi^2/9)$; $p\times$ quark phase	196.5° vs 195° (0.8%)	Prediction
Point/side/face model (Def. 3)	Orbifold geometric hierarchy	Table 2 (all < 2%)	Theorem
Tunnelling matrix T (Eq. 27)	Orbifold invariants η, K, σ, p	PMNS extraction	Theorem
$m_3 = m_e/(108\pi^{10})$ (Thm. 5)	Inversion: $p\, m_p^2\, m_\nu = m_e^3$	50.52 vs 50.28 meV (0.48%)	Prediction
$\Delta m^2_{32}/\Delta m^2_{21} = 33$ (Thm. 6)	$d_1^2 - p$; tunnelling bandwidth	33 vs 32.58 (1.3%)	Prediction
$m_2 = m_3/\sqrt{34}$ (derived)	P25 + P26 combination	8.66 vs 8.68 meV (0.23%)	Theorem
$Q_\nu \neq 2/3$ (Prop. 3)	Non-circulant mass matrix	$Q_\nu \approx 0.586$	Prediction
$\sum m_\nu \approx 59.2$ meV	Mass spectrum sum	DESI/Euclid window	Prediction
$\lvert m_{\beta\beta}\rvert \sim 2$–$3$ meV	PMNS elements + masses	nEXO/LEGEND-1000 reach	Prediction
Seesaw form recovered	$M_R = p\, m_p^2/m_e$	$M_R \sim 5.2 \times 10^{15}$ eV	Consistency

Table 3: Provenance map for Supplement VII results (Parameters 21–26 and derived predictions). "Theorem" entries follow from established mathematics or the spectral action on $S^5/\mathbb{Z}_3$. As of v12, all formerly "Derived" entries have been promoted to Theorem. "Prediction" entries are compared against PDG/NuFIT measurements. "Framework" entries depend on the orbifold identification. "Consistency" entries recover known structures.