Equivariant Spectral Decomposition with Coefficient One

A Universal Tool for Orbifold Spectral Theory

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Abstract

Let D be a self-adjoint operator with compact resolvent on a Hilbert space $\mathcal{H}$, and let G be a finite group acting on $\mathcal{H}$ by unitaries that commute with D. We prove that for any bounded Borel function f and any minimal central idempotent e_m of $\mathbb{C}[G]$, the identity $\mathrm{Tr}(f(D)\, e_m) = \mathrm{Tr}(f(D_m))$ holds with coefficient exactly 1, where $D_m = D|_{e_m\mathcal{H}}$. The proof is three lines of functional calculus, but the result is not trivial: we exhibit an explicit counterexample (a non-isometric $\mathbb{Z}_2$ action on S^1) where the coefficient becomes cutoff-dependent. We then demonstrate five applications where this "coefficient one" identity eliminates ambiguities that would otherwise undermine physical or mathematical conclusions: (i) spectral action sector decomposition on orbifolds, (ii) heat kernel asymptotics and one-loop effective actions, (iii) Casimir energy on orbifold compactifications, (iv) equivariant index theory and generation counting, and (v) band structure in crystallographic point groups. We also discuss obstructions (non-finite groups, continuous spectrum, anti-unitary actions) and a connection to equivariant K-theory via the Baum–Connes assembly map.

1 Introduction

A recurring situation in spectral geometry and mathematical physics: an operator D acts on a Hilbert space $\mathcal{H}$, a finite group G acts by symmetries, and the spectrum decomposes into sectors labeled by the irreducible representations of G. A spectral quantity — a trace, a zeta value, a heat coefficient, an eta invariant — is computed for each sector. The question arises:

Does the per-sector quantity depend on the regularization scheme?

If it does, the decomposition is an artifact; if it does not, it is a geometric invariant. The purpose of this paper is to provide a single, general theorem that settles this question for all finite isometric group actions, and to demonstrate its reach through a catalog of applications.

The theorem itself is elementary — three steps of functional calculus — and experts in operator algebra may regard it as "well known." However, we argue that its value lies not in the proof but in the *generality of application*. Like a wrench, the tool is simple; the art is knowing which bolts it fits. We exhibit five such bolts from different areas of mathematics and physics, and one explicit counterexample showing that the result has content: it fails when the isometry condition is violated.

2 Setup and Notation

Definition 2.1 (Spectral triple with finite group action)**.** *Let $(\mathcal{H}, D)$ be a spectral datum consisting of:*

- *A separable Hilbert space $\mathcal{H}$.*
- *A self-adjoint operator D with compact resolvent (hence discrete spectrum $\{\lambda_n\}_{n\in\mathbb{N}}$ with $|\lambda_n| \to \infty$).*

*Let G be a finite group acting on $\mathcal{H}$ by unitary operators $\rho : G \to \mathcal{U}(\mathcal{H})$ satisfying the **isometry condition**:*

$$\rho(g)\, D = D\, \rho(g) \qquad \text{for all } g \in G. \tag{1}$$

Definition 2.2 (Minimal central idempotents)**.** *The group algebra $\mathbb{C}[G]$ decomposes as a direct sum of matrix algebras, one for each irreducible representation π_m of G ($m = 0, 1, \ldots, r-1$ where r is the number of irreducible representations). The **minimal central idempotents** are:*

$$e_m = \frac{\dim \pi_m}{|G|} \sum_{g\in G} \overline{\chi_m(g)}\, \rho(g), \tag{2}$$

where χ_m is the character of π_m. These satisfy:

$$e_m^2 = e_m, \tag{3}$$

$$e_m e_{m'} = 0 \quad \text{for } m \neq m', \tag{4}$$

$$\sum_{m=0}^{r-1} e_m = \mathbf{1}. \tag{5}$$

Definition 2.3 (Sector Hilbert space and restricted operator)**.** *The m-th sector Hilbert space is $\mathcal{H}_m = e_m\mathcal{H}$. The restricted operator is $D_m = D|_{\mathcal{H}_m}$.*

3 The Main Theorem

Theorem 3.1 (Coefficient One)**.** *Let* $(\mathcal{H}, D, G, \rho)$ *be as in Definition 2.1, satisfying the isometry condition (1). Let* $f : \mathbb{R} \to \mathbb{C}$ *be a Borel function such that* $f(D)$ *is trace-class*[1]*. Then for each minimal central idempotent* e_m*:*

$$\boxed{\mathrm{Tr}(f(D)\, e_m) = \mathrm{Tr}(f(D_m)).} \tag{6}$$

In particular, the coefficient of every per-sector spectral quantity (eta invariant, heat coefficient, zeta value) is exactly 1*, independent of* f*.*

Proof. The proof proceeds in three steps.

Step 1: $f(D)$ commutes with e_m.

Since G acts by isometries, $\rho(g)D = D\rho(g)$ for all $g \in G$. By the functional calculus for self-adjoint operators, for any bounded Borel function f:

$$\rho(g)\, f(D) = f(D)\, \rho(g) \qquad \text{for all } g \in G. \tag{7}$$

Proof of (7): Let $D = \int \lambda \, dE(\lambda)$ be the spectral decomposition, so $f(D) = \int f(\lambda)\, dE(\lambda)$. Since $\rho(g)$ commutes with D, it commutes with every spectral projection $E(B)$ for Borel sets $B \subset \mathbb{R}$ (standard result: see [4], Theorem VIII.5). Therefore $\rho(g)$ commutes with $\int f(\lambda)\, dE(\lambda) = f(D)$.

Since e_m is a $\mathbb{C}$-linear combination of the $\rho(g)$ (equation (2)), it follows that:

$$[f(D),\, e_m] = 0. \tag{8}$$

Step 2: The trace decomposes.

Since e_m is a projection ($e_m^2 = e_m$) and commutes with $f(D)$, the operator $f(D)\, e_m = e_m\, f(D)\, e_m$ acts on $\mathcal{H}_m = e_m \mathcal{H}$. Let $\{|\psi_n^{(m)}\rangle\}$ be an orthonormal eigenbasis of D_m in $\mathcal{H}_m$ with eigenvalues $\{\lambda_n^{(m)}\}$. Then:

$$\begin{aligned}
\mathrm{Tr}(f(D)\, e_m) &= \sum_n \langle \psi_n^{(m)} | f(D)\, e_m | \psi_n^{(m)} \rangle && (9)\\
&= \sum_n \langle \psi_n^{(m)} | f(D) | \psi_n^{(m)} \rangle \qquad (\text{since } e_m|\psi_n^{(m)}\rangle = |\psi_n^{(m)}\rangle) && (10)\\
&= \sum_n f(\lambda_n^{(m)}) \qquad (\text{since } |\psi_n^{(m)}\rangle \text{ is an eigenstate of } D) && (11)\\
&= \mathrm{Tr}(f(D_m)). && (12)
\end{aligned}$$

Step 3: The coefficient is 1.

Equation (12) shows $\mathrm{Tr}(f(D)\, e_m) = \mathrm{Tr}(f(D_m))$ with coefficient exactly 1, for any trace-class $f(D)$. No normalization, no f-dependent prefactor. □ □

[1] This holds for heat kernels $f(x) = e^{-tx^2}$, resolvents $f(x) = (x^2 + m^2)^{-s}$ with $\mathrm{Re}(s)$ sufficiently large, smooth cutoffs with sufficient decay, and more generally any f for which $\sum_n |f(\lambda_n)| < \infty$. The eta function case $f(x) = \mathrm{sign}(x)|x|^{-s}$ is handled by analytic continuation.

4 Why the Result Is Not Trivial: A Counterexample

One might suspect that Theorem 3.1 is vacuous — that "the coefficient is always 1" regardless of assumptions. We show this is false: when the isometry condition (1) is violated, the coefficient becomes f-dependent.

Example 4.1 (Non-isometric $\mathbb{Z}_2$ action on S^1)**.** *Let $\mathcal{H} = L^2(S^1)$ with the standard Dirac operator $D_0 = -i\, d/d\theta$, spectrum $\{n : n \in \mathbb{Z}\}$. Let $\mathbb{Z}_2$ act by the reflection $\theta \mapsto -\theta$, which is an isometry of the round S^1. In this case, $[\rho(\sigma), D_0] = 0$ and Theorem 3.1 applies: all spectral quantities decompose with coefficient* 1*.*

Now deform the metric: let $D_\epsilon = -i\, h(\theta)^{-1}\, d/d\theta$ where $h(\theta) = 1 + \epsilon\cos\theta$ ($|\epsilon| < 1$). The reflection $\theta \mapsto -\theta$ preserves h (since $h(-\theta) = h(\theta)$), so it is still an isometry and $[\rho(\sigma), D_\epsilon] = 0$. Theorem 3.1 still applies.

But consider instead the non-isometric *action $\theta \mapsto \theta + \pi$ (translation by half-period) with the* same *deformed metric $h(\theta) = 1 + \epsilon\cos\theta$. Since $h(\theta + \pi) = 1 - \epsilon\cos\theta \neq h(\theta)$ for $\epsilon \neq 0$, this action does* not *preserve the metric. Therefore $[\rho(\sigma), D_\epsilon] \neq 0$, and:*

$$\mathrm{Tr}(f(D_\epsilon)\, e_{\text{even}}) \neq \mathrm{Tr}(f((D_\epsilon)_{\text{even}})) \quad \textit{for generic } f. \tag{13}$$

Explicitly: the eigenvalues of D_ϵ are $\mu_n = n + \epsilon\, c_n + O(\epsilon^2)$ where the perturbation coefficients c_n depend on the parity of n asymmetrically. The even-sector trace $\sum_{n\,\text{even}} f(\mu_n)$ differs from $\mathrm{Tr}(f(D_\epsilon)\, e_{\text{even}})$ by $O(\epsilon)$ corrections whose sign and magnitude depend on f. The "coefficient" is no longer 1 *but a function of the cutoff.*

Remark 4.2 (The isometry condition has teeth)**.** *The counterexample shows that Theorem 3.1 is genuinely a theorem about isometric group actions. The word "isometry" cannot be weakened to "diffeomorphism," "conformal map," or "homeomorphism." The wrench fits only isometric bolts.*

5 Application 1: Spectral Action on Orbifolds

Corollary 5.1 (Cutoff independence of sector corrections)**.** *In the Connes–Chamseddine spectral action [1, 2], the bosonic action on a Riemannian orbifold M/G is $S = \mathrm{Tr}(f(D/\Lambda))$ where f is a cutoff function and Λ a scale. The sector decomposition*

$$S = \sum_{m=0}^{r-1} S_m, \qquad S_m = \mathrm{Tr}(f(D_m/\Lambda)), \tag{14}$$

holds with coefficient 1 *per sector, independent of the choice of f.*

This is the original motivation for the theorem. Applied to $S^5/\mathbb{Z}_3$: the three $\mathbb{Z}_3$ sectors (χ_0, χ_1, χ_2) contribute their respective eta invariants $\eta_D(\chi_m)$ to the Yukawa coupling phase with coefficient exactly 1, regardless of whether f is a sharp cutoff, a smooth Schwartz function, or a heat kernel. This eliminates a potential source of scheme dependence in the derivation of the Koide phase $\delta = 2\pi/3 + 2/9$ from the spectral geometry of $S^5/\mathbb{Z}_3$ [9].

Remark 5.2. *The Seeley–DeWitt expansion* $\mathrm{Tr}(f(D/\Lambda)) \sim \sum_k f_k \Lambda^{d-k} a_k(D)$ *involves "moments"* $f_k = \int_0^\infty f(u)\, u^{k/2-1}\, du$ *that* do *depend on* f*. The theorem does not say these moments are universal — it says the* sector decomposition $a_k = \sum_m a_k^{(m)}$ *is universal. The moments multiply the total; the decomposition multiplies with coefficient* 1*.*

6 Application 2: Heat Kernel Asymptotics

Corollary 6.1 (Heat trace decomposition)**.** *Let* D *be the Dirac operator on a compact Riemannian orbifold* M/G*. The heat trace*

$$K(t) = \mathrm{Tr}(e^{-tD^2}) = \sum_n e^{-t\lambda_n^2} \tag{15}$$

decomposes by G*-representation sector as:*

$$K(t) = \sum_{m=0}^{r-1} K_m(t), \qquad K_m(t) = \mathrm{Tr}(e^{-tD_m^2}), \tag{16}$$

with coefficient 1 *per sector, for all* $t > 0$*.*

Proof. Apply Theorem 3.1 with $f(x) = e^{-tx^2}$, which is a bounded Borel function making $f(D)$ trace-class (since D has compact resolvent and e^{-tx^2} decays rapidly). □ □

Consequence for Seeley–DeWitt coefficients. The small-t expansion $K_m(t) \sim \sum_{k\geq 0} a_k^{(m)}\, t^{(k-d)/2}$ gives Seeley–DeWitt coefficients $a_k^{(m)}$ for each sector. These are the building blocks of one-loop effective actions on orbifold backgrounds. Theorem 3.1 guarantees:

$$a_k = \sum_{m=0}^{r-1} a_k^{(m)}, \tag{17}$$

with no renormalization of the per-sector contributions. In particular, the equivariant Euler characteristic, the equivariant signature, and the equivariant $\hat{A}$-genus all decompose with coefficient 1.

Remark 6.2 (One-loop determinants in string theory)**.** *On orbifold string backgrounds* $\mathcal{M}/G$*, the one-loop partition function* $Z = (\det D^2)^{-1/2}$ *factorizes by twisted sector. The coefficient* 1 *theorem ensures that no sector receives an anomalous weight — the twisted-sector contributions to the vacuum energy are exactly* $\log\det(D_m^2)$ *with no cutoff-dependent normalization. This is implicit in standard orbifold CFT calculations [5] but is not usually stated as a general theorem.*

7 Application 3: Casimir Energy on Orbifolds

Corollary 7.1 (Casimir energy decomposition)**.** *The regularized vacuum energy of a quantum field on M/G is*

$$E_{\text{Cas}} = \frac{1}{2}\zeta_D'(0), \qquad \zeta_D(s) = \sum_{\lambda_n > 0} \lambda_n^{-2s}, \tag{18}$$

where the sum is over positive eigenvalues of D. This decomposes as:

$$E_{\text{Cas}} = \sum_{m=0}^{r-1} E_{\text{Cas}}^{(m)}, \qquad E_{\text{Cas}}^{(m)} = \frac{1}{2}(\zeta_{D_m})'(0), \tag{19}$$

with coefficient 1 per sector, independent of the regularization prescription.

Proof. The zeta function $\zeta_D(s) = \operatorname{Tr}(D^{-2s}\,\Pi_+)$ (where Π_+ projects onto positive eigenvalues) decomposes by sector via Theorem 3.1 applied to $f(x) = |x|^{-2s}$ in the region of absolute convergence $\operatorname{Re}(s) > d/2$, then extended to $s = 0$ by analytic continuation. The continuation preserves the coefficient because it acts independently in each sector (the sectors are spectrally disjoint). □ □

Remark 7.2 (Randall–Sundrum and orbifold GUTs)**.** *In Randall–Sundrum models [6] and orbifold GUT compactifications [7], the Casimir energy on $S^1/\mathbb{Z}_2$ stabilizes the extra dimension. The $\mathbb{Z}_2$-even and $\mathbb{Z}_2$-odd sectors contribute independently to the Casimir force, and the coefficient 1 theorem guarantees that no regularization artifact contaminates the even/odd decomposition. This is physically important: the hierarchy between the Planck and TeV scales depends on the* ratio *of even to odd Casimir contributions, and a scheme-dependent coefficient would destroy the prediction.*

8 Application 4: Equivariant Index Theory

Corollary 8.1 (Equivariant APS index)**.** *Let $(M, \partial M)$ be a compact Riemannian manifold with boundary, G a finite group of isometries, and D the Dirac operator with APS boundary conditions. The equivariant index in the m-th sector is:*

$$\operatorname{ind}_m(D) = \operatorname{Tr}(\gamma_5\, e_m) = \operatorname{Tr}(\gamma_5|_{\mathcal{H}_m}), \tag{20}$$

with coefficient 1, independent of the regularization used to define the index.

Proof. The index can be expressed as $\operatorname{ind}_m = \operatorname{Tr}(\gamma_5\, e^{-tD^2}\, e_m)$ for any $t > 0$ (McKean–Singer formula). Since G acts by isometries, γ_5 commutes with the G-action (the chirality grading is preserved by orientation-preserving isometries). By Theorem 3.1, the trace decomposes with coefficient 1. Taking $t \to 0^+$ gives the index. □ □

Connection to generation counting. In the companion paper [10], the equivariant APS index on $B^6/\mathbb{Z}_3$ is computed to give $N_g = 1+1+1 = 3$ — one chiral zero mode per $\mathbb{Z}_3$ sector. The coefficient 1 theorem is the reason this count does not depend on the choice of heat-kernel regulator t or any other regularization parameter. The generation count $N_g = 3$ is a topological invariant precisely *because* the equivariant decomposition has coefficient 1.

Without the isometry condition, one could imagine an anomalous weighting where sector 1 receives weight $1+\epsilon$ and sector 2 receives weight $1-\epsilon$ for some ϵ depending on the regularization. The theorem rules this out.

9 Application 5: Crystallographic Point Groups

Corollary 9.1 (Band structure decomposition)**.** *Let $\mathcal{H} = L^2(\mathbb{R}^3/\Lambda)$ be the Hilbert space of a crystalline solid with lattice Λ, and let G be the point group of the crystal (a finite subgroup of $\mathrm{O}(3)$). The Hamiltonian $H = -\nabla^2 + V(x)$, where V is G-invariant, satisfies $[\rho(g), H] = 0$. The density of states per symmetry channel decomposes as:*

$$\rho(\epsilon) = \sum_m \rho_m(\epsilon), \qquad \rho_m(\epsilon) = \mathrm{Tr}(\delta(\epsilon - H_m)), \tag{21}$$

with coefficient 1, independent of any smearing or broadening prescription.

Proof. Apply Theorem 3.1 with $f(x) = \delta(\epsilon - x^2)$ (distributional limit of smooth approximations), or equivalently with $f(x) = \chi_{[\epsilon,\epsilon+d\epsilon]}(x^2)$. In practice, one uses a Lorentzian or Gaussian broadening $f_\sigma(x) = (\pi\sigma)^{-1}(\sigma^2 + (x^2-\epsilon)^2)^{-1}$; the theorem says the per-irrep decomposition is independent of σ. □ □

Remark 9.2 (Tight-binding models)**.** *In tight-binding models on molecules or clusters with point-group symmetry G, the molecular orbitals classify by irreps of G. The total energy $E_{\mathrm{tot}} = \sum_m E_m$ decomposes by irrep with coefficient 1. This is exploited routinely in computational chemistry (e.g., symmetry-adapted basis sets), but the underlying mathematical guarantee is precisely Theorem 3.1. When symmetry-breaking perturbations are introduced (e.g., Jahn–Teller distortions), the isometry condition* (1) *is violated and the clean decomposition is lost — consistent with the counterexample in §4.*

10 Non-Examples and Obstructions

The hypotheses of Theorem 3.1 — finite group, compact resolvent, isometry condition — are sharp. We catalog the failure modes.

Non-Example 10.1 (Non-finite groups)**.** *Let $G = \mathrm{U}(1)$ act on $L^2(S^1)$ by rotation. The group algebra $\mathbb{C}[\mathrm{U}(1)]$ is infinite-dimensional and has no minimal central idempotents in the algebraic sense. The decomposition into Fourier modes (irreps of* $\mathrm{U}(1)$*) still works* *spectrally, but the idempotent construction* (2) *involves an integral over G rather than a finite sum. Theorem 3.1 extends to compact groups via the Peter–Weyl theorem, but the proof requires the additional hypothesis that the multiplicity spaces are finite-dimensional (automatic for compact G on compact M, but subtle for non-compact M).*

Non-Example 10.2 (Continuous spectrum)**.** *Let $D = -i\,d/dx$ on $L^2(\mathbb{R})$ (continuous spectrum, no compact resolvent). Let $\mathbb{Z}_2$ act by $x \mapsto -x$. The even/odd decomposition of $L^2(\mathbb{R})$ is well-defined, but $\mathrm{Tr}(f(D)\,e_{\mathrm{even}})$ is not defined for most f because $f(D)$ is not trace-class. The theorem requires compact resolvent precisely to ensure trace-class regularity.*

Non-Example 10.3 (Anti-unitary actions)**.** *Time reversal T is anti-unitary: $T(c|\psi\rangle) = \bar{c}\,T|\psi\rangle$. The idempotent construction* (2) *uses $\mathbb{C}$-linear combinations of group elements, which fails for anti-unitary operators. For systems with time-reversal symmetry, the relevant decomposition is by* real *or* quaternionic *representations (Dyson's threefold way), and the coefficient* 1 *result must be replaced by the appropriate Kramers multiplicity.*

Non-Example 10.4 (Non-isometric diffeomorphisms)**.** *A diffeomorphism $\phi : M \to M$ that is not an isometry does not commute with the Laplacian or Dirac operator. The pullback ϕ^* acts on functions but $[\phi^*, \Delta] \neq 0$ in general (the Laplacian depends on the metric, which ϕ does not preserve). The counterexample in §4 is a concrete instance. In such cases, the "coefficient" in the sector decomposition becomes a function of the regularization, and per-sector spectral quantities are not intrinsic.*

11 Connection to Equivariant K-Theory

The decomposition $\mathcal{H} = \bigoplus_m e_m\mathcal{H}$ is a manifestation of the equivariant K-theory of the algebra of observables.

Proposition 11.1 (K-theoretic interpretation)**.** *Let $\mathcal{A} = C(M)$ (continuous functions on M) with the G-action by pullback. The equivariant K-group $K^0_G(M)$ decomposes as:*

$$K^0_G(M) \cong \bigoplus_{m=0}^{r-1} K^0(M/G) \otimes R(G)_m, \tag{22}$$

where $R(G)_m$ is the m-th component of the representation ring. The Chern character $\mathrm{ch} : K^0_G(M) \to H^*_G(M;\mathbb{Q})$ intertwines the idempotent decomposition: $\mathrm{ch}(e_m \cdot [D]) = e_m \cdot \mathrm{ch}([D])$. Theorem 3.1 is the operator-trace shadow *of this K-theoretic decomposition.*

Remark 11.2 (Baum–Connes for finite groups)**.** *For finite groups, the Baum–Connes assembly map $\mu : K^G_*(\underline{E}G) \to K_*(C^*_r G)$ is an isomorphism [8]. The coefficient* 1 *in Theorem 3.1 reflects the fact that the assembly map for finite groups is an* exact *isomorphism — no correction terms, no anomalous dimensions, no renormalization. For infinite discrete groups, the assembly map may fail to be surjective (the Baum–Connes* *conjecture), and the clean coefficient* 1 *decomposition would not hold in general. The finiteness of G is therefore not merely a technical convenience but a reflection of the exactness of the assembly map.*

12 Generality of the Result

Remark 12.1 (Summary of scope)**.** *Theorem 3.1 holds for:*

1. *Any finite group G (not just cyclic groups).*
2. *Any self-adjoint operator with compact resolvent (not just the Dirac operator).*
3. *Any bounded Borel function f (not just smooth cutoffs).*
4. *Any Riemannian manifold M on which G acts by isometries.*

The only requirement is the isometry condition (1)*: the group action must commute with the operator. For the Dirac operator on a Riemannian manifold, this is equivalent to requiring that G act by isometries — a geometric condition, not an analytic one.*

13 Summary

The coefficient one theorem (Theorem 3.1) is a three-line proof with five applications and one counterexample:

Application	**Eliminates**	**Section**
Spectral action (orbifolds)	Cutoff dependence in sector corrections	§5
Heat kernel asymptotics	Ambiguity in per-sector Seeley–DeWitt coeff.	§6
Casimir energy	Regularization artifact in sector forces	§7
Equivariant APS index	Regulator dependence in generation counting	§8
Band structure (crystals)	Broadening dependence in per-irrep DOS	§9
Counterexample (§4)	Shows isometry condition is necessary	§4
Non-examples (§10)	Infinite G, continuous spectrum, anti-unitary	§10
K-theory (§11)	Connects to Baum–Connes assembly map	§11

The theorem is the wrench. The applications are the bolts. The tool is simple; the art is knowing where it fits.

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