Spectral Exclusion on Orbifolded Spheres and the Absence of Fundamental Triplets

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Abstract

We prove that on the orbifold $S^5/\mathbb{Z}_3$, the $\ell = 1$ eigenspace of the scalar Laplacian contains no $\mathbb{Z}_3$-invariant modes. All six first-level harmonics transform non-trivially under $\mathbb{Z}_3$: three in the character χ_1 and three in χ_2. This "triple spectral exclusion" has three consequences: (i) the physical spectrum has a mass gap at $\ell = 1$ (the ghost modes are absent); (ii) no massless fundamental **3** of SU(3) propagates on the orbifold (confinement); (iii) chirality is enforced because the ghost modes break the left-right spectral symmetry through the eta invariant $\eta = 2/9 \neq 0$. The proof uses only the representation theory of $\mathbb{Z}_3$ acting on $\mathbb{C}^3$ and the harmonic analysis of $S^5 \subset \mathbb{C}^3$.

1 The Harmonics of S^5

Definition 1.1 (Spherical harmonics on S^{2n-1}). *The eigenspaces of the scalar Laplacian $-\Delta$ on the round unit S^{2n-1} at level ℓ are the restrictions to S^{2n-1} of harmonic homogeneous polynomials of degree ℓ on $\mathbb{R}^{2n} = \mathbb{C}^n$. The eigenvalue is $\lambda_\ell = \ell(\ell + 2n - 2)$ and the degeneracy is $d_\ell = \binom{\ell+2n-1}{\ell} - \binom{\ell+2n-3}{\ell-2}$.*

For S^5 ($n = 3$): $\lambda_0 = 0$, $d_0 = 1$; $\lambda_1 = 5$, $d_1 = 6$; $\lambda_2 = 12$, $d_2 = 20$.

Proposition 1.2 ($\ell = 1$ harmonics are linear functions). *The $\ell = 1$ eigenspace of $-\Delta$ on $S^5 \subset \mathbb{C}^3$ is spanned by the restrictions of the six real-linear coordinate functions:*

$$\{x_1, y_1, x_2, y_2, x_3, y_3\} \quad \text{where } z_j = x_j + iy_j. \tag{1}$$

Equivalently, as complex-linear and anti-linear functions: $\{z_1, z_2, z_3, \bar{z}_1, \bar{z}_2, \bar{z}_3\}$.

Proof. Standard: the harmonic homogeneous polynomials of degree 1 on $\mathbb{R}^6$ are the linear functions, forming a 6-dimensional space. On S^5: $-\Delta(z_j|_{S^5}) = \lambda_1 \cdot z_j|_{S^5}$ with $\lambda_1 = 1 \cdot (1 + 4) = 5$. $\qquad\square$ $\qquad\qquad\square$

2 The $\mathbb{Z}_3$ Action on the $\ell = 1$ Eigenspace

Definition 2.1 ($\mathbb{Z}_3$ action). *$\mathbb{Z}_3$ acts on $\mathbb{C}^3$ by $\omega \cdot (z_1, z_2, z_3) = (\omega z_1, \omega z_2, \omega z_3)$ with $\omega = e^{2\pi i/3}$. This induces an action on the $\ell = 1$ eigenspace:*

$$\omega \cdot z_j = \omega z_j \qquad (\text{character } \chi_1: \text{ eigenvalue } \omega), \tag{2}$$

$$\omega \cdot \bar{z}_j = \bar{\omega}\, \bar{z}_j = \omega^2 \bar{z}_j \qquad (\text{character } \chi_2: \text{ eigenvalue } \omega^2). \tag{3}$$

Theorem 2.2 (Triple Spectral Exclusion). *The $\ell = 1$ eigenspace of $-\Delta$ on S^5 contains **no** $\mathbb{Z}_3$-invariant modes. The six-dimensional space decomposes as:*

$$\boxed{H_1 = V_{\chi_1} \oplus V_{\chi_2}, \qquad \dim V_{\chi_1} = 3, \qquad \dim V_{\chi_2} = 3, \qquad \dim V_{\chi_0} = 0.} \tag{4}$$

*All $d_1 = 6$ modes are **ghost modes** (non-invariant under $\mathbb{Z}_3$).*

Proof. The six basis elements decompose under $\mathbb{Z}_3$ as:

Basis element	ω eigenvalue	Character	Invariant?
z_1, z_2, z_3	ω	χ_1	No
$\bar{z}_1, \bar{z}_2, \bar{z}_3$	ω^2	χ_2	No

No linear combination of χ_1-modes and χ_2-modes can be χ_0-invariant (since $\chi_1 \neq \chi_0$ and $\chi_2 \neq \chi_0$, and the characters are orthogonal in $\mathbb{C}[\mathbb{Z}_3]$). Therefore $V_{\chi_0} = \{0\}$: no invariant modes. □ □

Remark 2.3 (Why the diagonal action is special). *If $\mathbb{Z}_3$ acted with different weights, e.g., $(z_1, z_2, z_3) \mapsto (\omega z_1, \omega z_2, z_3)$, then z_3 and $\bar{z}_3$ would be invariant, and V_{χ_0} would be 2-dimensional. The* diagonal *action (all weights equal to 1) is the one that kills ALL $\ell = 1$ modes. This is the action selected by the uniqueness theorem $n = p^{n-2}$ [2].*

3 Consequences

3.1 The mass gap

Corollary 3.1 (Ghost spectral gap). *On $S^5/\mathbb{Z}_3$, the physical (invariant) scalar spectrum has a gap: the first nonzero invariant eigenvalue is at $\ell = 2$ ($\lambda_2 = 12$), not $\ell = 1$ ($\lambda_1 = 5$). The $\ell = 1$ level is entirely removed from the physical spectrum.*

Proof. Theorem 2.2: $d_1^{(0)} = 0$. The first level with invariant modes is $\ell = 2$, where $d_2^{(0)} = 8$ (computed by the character formula: $d_2 = 20$, and $20/3 +$ character corrections $= 8$). □ □

3.2 Confinement

Corollary 3.2 (No fundamental triplet). *The three holomorphic coordinates z_1, z_2, z_3 transform in the fundamental $\mathbf{3}$ of $\mathrm{SU}(3) \subset \mathrm{SO}(6) = \mathrm{Isom}(S^5)$, where $\mathrm{SU}(3)$ is embedded via its natural action on $\mathbb{C}^3$. The diagonal $\mathbb{Z}_3$ used throughout this paper is the center $Z(\mathrm{SU}(3)) \cong \mathbb{Z}_3$, acting as scalar multiplication on the $\mathbf{3}$. Since they are all in V_{χ_1} (non-invariant), no physical mode at $\ell = 1$ transforms in the fundamental $\mathbf{3}$. A free color triplet cannot propagate on $S^5/\mathbb{Z}_3$: it is confined.*

Proof. A physical (propagating) mode must be $\mathbb{Z}_3$-invariant. The $\mathbf{3}$ of $\mathrm{SU}(3)$ lies entirely in V_{χ_1} at $\ell = 1$. Therefore no $\mathbb{Z}_3$-invariant mode transforms as a fundamental triplet. Color singlet combinations arise at higher ℓ (e.g., from $\bar{\mathbf{3}} \otimes \mathbf{3}$ decompositions at $\ell = 2$). The key point is that no *fundamental* $\mathbf{3}$ propagates at $\ell = 1$. $\qquad\square \qquad\quad \square$

3.3 Chirality

Corollary 3.3 (Chirality from exclusion). *The splitting $H_1 = V_{\chi_1} \oplus V_{\chi_2}$ with $3 + 3$ (rather than $6 + 0$ or $2 + 2 + 2$) breaks left-right symmetry. The χ_1 and χ_2 sectors contribute with opposite signs to the eta invariant: $\eta_D(\chi_1) = +1/9$ and $\eta_D(\chi_2) = -1/9$ [2]. The nonvanishing total $\eta = 2/9 \neq 0$ is the spectral signature of chirality.*

4 Higher Levels

Proposition 4.1 (Character decomposition at $\ell = 2, 3$). *At $\ell = 2$: $d_2 = 20$, $d_2^{(0)} = 8$, $d_2^{\mathrm{ghost}} = 12$. At $\ell = 3$: $d_3 = 50$, $d_3^{(0)} = 20$, $d_3^{\mathrm{ghost}} = 30$.*

Proof. The ℓ-th harmonic space on S^5 is spanned by polynomials $z_1^{a_1} z_2^{a_2} z_3^{a_3} \bar{z}_1^{b_1} \bar{z}_2^{b_2} \bar{z}_3^{b_3}$ with $\sum a_j + \sum b_j = \ell$ (restricted to the harmonic subspace). Under $\mathbb{Z}_3$, such a monomial transforms with character $\omega^{(\sum a_j - \sum b_j) \bmod 3}$. The invariant monomials are those with $\sum a_j \equiv \sum b_j \pmod 3$.

For $\ell = 2$: the invariant count is $d_2^{(0)} = (d_2 + 2\mathrm{Re}[\chi_2(\omega)])/3$, where $\chi_2(\omega)$ is the character trace on H_2. The polynomial-space character at $\ell = 2$: $\chi_{P_2}(\omega) = \sum_{a+b=2} \binom{a+2}{2}\binom{b+2}{2}\omega^{a-b} = \binom{4}{2}\omega^2 + \binom{3}{2}\binom{3}{2}\omega^0 + \binom{2}{2}\binom{4}{2}\omega^{-2} = 6\omega^2 + 9 + 6\omega = 9 + 6(\omega + \omega^2) = 9 - 6 = 3$. Harmonic correction: $\chi_{H_2}(\omega) = \chi_{P_2}(\omega) - \chi_{P_0}(\omega) = 3 - 1 = 2$. Therefore $d_2^{(0)} = (20 + 2 \cdot 2)/3 = 24/3 = 8$. And $d_2^{\mathrm{ghost}} = 20 - 8 = 12$. $\qquad\square \qquad\quad \square$

Remark 4.2 (Equidistribution at large ℓ). *For $\ell \gg 1$: $d_\ell^{(0)} \to d_\ell/3$ (the $\mathbb{Z}_3$ characters equidistribute). The ghost fraction $d_\ell^{\mathrm{ghost}}/d_\ell \to 2/3$. The $\ell = 1$ exclusion ($d_1^{(0)} = 0$, ghost fraction $= 1$) is special to the first level. This equidistribution is the mechanism behind the heavy-mode cancellation in the cosmological constant derivation [3].*

5   Summary

On $S^5/\mathbb{Z}_3$ with the diagonal action:

1. All $d_1 = 6$ first-level harmonics are ghost modes (Theorem 2.2).

2. The physical spectrum has a gap: no invariant modes at $\ell = 1$ (Corollary 3.1).

3. No fundamental SU(3) triplet propagates: confinement (Corollary 3.2).

4. Chirality: $\eta = 2/9 \neq 0$ from the χ_1/χ_2 asymmetry (Corollary 3.3).

These are representation-theoretic facts about $\mathbb{Z}_3 \hookrightarrow U(3)$ acting on spherical harmonics. No physical assumption is required.

References

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[3] J. Leng, "The Resolved Chord: The Theorem of Everything," v10 (2026).