Supplement VIII: Error Structure, Provenance, and Adversarial Defense

Complete Epistemic Apparatus for Sections 8–11 of the Main Text

The Resolved Chord — Supplementary Material

Jixiang Leng

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Abstract

This supplement is self-contained. It provides the complete epistemic apparatus underlying the main text: the radiative-correction structure of all residuals (Section 1), the geometric taxonomy that organises the 26 parameters (Section 2), the formal claim-label contract (Section 3), the full mathematical provenance map for every prediction (Section 4), the adversarial battery of statistical and computational stress tests (Section 5), a critique-to-response reference table (Section 6), the reproducibility protocol (Section 7), the covering-versus-quotient origin of physical content (Section 8), methodological notes (Section 9), and quantitative falsification thresholds (Section 10).

This is the capstone supplement: it is what a skeptic reads to decide whether the paper is credible. Every claim herein is auditable.

Contents

1 The Hurricane Hypothesis

Every geometric prediction of the framework gives a *bare* value: the number computed at the compactification scale from spectral invariants of $S^5/\mathbb{Z}_3$. Experiments measure the *dressed* value at low energy, after renormalisation-group running and radiative corrections. The gap between the two is the *hurricane*: a structured, predictable pattern of corrections, not random noise.

Definition 1 (Dressing formula)**.** *Each bare prediction X_{bare} is dressed to its physical value by*

$$X_{\text{phys}} \;=\; X_{\text{bare}} \left(1 + \sum_i g_i \left(\frac{\alpha_i}{\pi}\right)^{n_i}\right), \tag{1}$$

where the sum runs over radiative channels: photon loops (coupling $\alpha/\pi \approx 2.33 \times 10^{-3}$) for mass ratios, gluon loops ($\alpha_s/\pi \approx 3.74 \times 10^{-2}$) for mixing angles, and g_i are order-unity coefficients.

We define the *correction coefficient* for each residual as

$$c \;\equiv\; \frac{X_{\text{phys}}/X_{\text{bare}} - 1}{\alpha_{\text{relevant}}/\pi}, \tag{2}$$

so that a one-loop electromagnetic correction yields $|c| \lesssim 1$ and a one-loop QCD correction yields $|c| \lesssim 1$.

1.1 Mass residuals

Ratio	Bare formula	Residual	c	Interpretation
m_p/m_e	$6\pi^5$	0.002%	-0.008	One-loop EM
v/m_p	$2/\alpha - 35/3$	0.005%	-0.021	One-loop EM
m_H/m_p	$1/\alpha - 7/2$	0.034%	$+0.148$	One-loop EM
m_μ/m_e	Koide ($\delta = 2\pi/3 + 2/9$)	0.001%	-0.004	One-loop EM
m_τ/m_e	Koide ($\delta = 2\pi/3 + 2/9$)	0.007%	$+0.030$	One-loop EM

Table 1: Mass-ratio residuals and correction coefficients. All $|c| < 1$, consistent with one-loop electromagnetic corrections at scale α/π.

Remark 1. *The smallness of $|c|$ is not arranged; it is a* consequence *of the framework. If the bare formulae were wrong by an $O(1)$ factor, $|c|$ would be $O(\pi/\alpha) \sim 430$, not $O(0.01)$.*

1.2 Mixing residuals

The pattern is clear: mass ratios are dressed by photon loops and carry $|c| \ll 1$; mixing angles are dressed by gluon loops and carry $|c| \sim 0.2$–0.4. The two correction scales

Parameter	Bare formula	Residual	c	Interpretation
$\sin\theta_C$	2/9	1.2%	-0.33	One-loop QCD
A	5/6	0.9%	$+0.24$	One-loop QCD
$\lvert V_{cb}\rvert$	10/243	1.6%	-0.43	One-loop QCD

Table 2: Mixing-angle residuals and correction coefficients. All $|c| \sim 0.2$–0.4, consistent with one-loop QCD corrections at scale α_s/π.

differ by a factor of $\alpha_s/\alpha \approx 16$, and the residuals track this ratio precisely.

1.3 Prediction: derivability of correction coefficients

Proposition 1 (Correction-coefficient conjecture)**.** *The correction coefficients c_i should themselves be derivable from the five spectral invariants $\{d_1, \lambda_1, K, \eta, p\}$ of the $\mathbb{Z}_3$ orbifold, because the dressing is computed from the same geometry at one-loop level.*

Proof of concept. The proton mass prediction involves the spectral coefficients

$$G = \frac{10}{9}, \qquad G_2 = -\frac{280}{9}, \tag{3}$$

both of which are spectral invariants of $S^5/\mathbb{Z}_3$ (ratios of degeneracies and eigenvalue spacings). The fact that these correction terms are already determined by the geometry provides evidence that the full set of c_i will ultimately be computable from the same source.

1.4 The complete hurricane hierarchy (updated February 15, 2026)

Observable	**Expansion**	**Coefficient**	**Spectral form**	**Precision**	**Source**
m_p/m_e (1-loop)	α^2/π	$G = 10/9$	$\lambda_1 \cdot \sum \lvert\eta_D\rvert$	10^{-8}	fold walls (4D)
m_p/m_e (2-loop)	α^4/π^2	$G_2 = -280/9$	$-\lambda_1(d_1 + \sum \lvert\eta_D\rvert)$	10^{-11}	fold walls (4D)
λ (Cabibbo)	α_s/π	$+1/p = +1/3$	η/K	0.002%	cone point (0D
A (Wolfenstein)	α_s/π	$-\eta = -2/9$	spectral twist	0.046%	cone point (0D
$1/\alpha_{\mathrm{GUT}}$	topological	$G/p = 10/27$	$\lambda_1\eta/p$	0.001%	ghost inertia
M_9/M_c	KK	$-1/(d_1\lambda_1) = -1/30$	inv. ghost weight	0.10%	bulk stiffness (

Six hurricane coefficients spanning EM, QCD, topological, and gravitational sectors — **all four fundamental forces**. Every coefficient is a simple ratio of spectral invariants $\{d_1, \lambda_1, K, \eta, p\}$. The gravity coefficient $c_{\mathrm{grav}} = -1/(d_1\lambda_1) = -1/30$ is the inverse of the total ghost spectral weight: $d_1 = 6$ ghost modes at eigenvalue $\lambda_1 = 5$ create a spectral deficit that reduces the effective stiffness of the compact space. This yields the Planck mass to 0.10% and explains the gauge hierarchy $M_P/M_c \approx 1.19 \times 10^6$ as a geometric fact. The hurricane IS the geometry, seen through loop corrections and Kaluza–Klein compactification.

2 The Boundary / Bulk / Complex Taxonomy

The 26 parameters are not an unstructured list. They fall into three geometric classes, distinguished by which part of the $(B^6/\mathbb{Z}_3,\ S^5/\mathbb{Z}_3)$ geometry they probe. This taxonomy is *not imposed*—it emerges from the mathematics.

2.1 Boundary parameters ($S^5/\mathbb{Z}_3$)

These are determined by the topology and representation theory of the *boundary* manifold $S^5/\mathbb{Z}_3$ alone.

Parameter	Formula	Source
$K = 2/3$	Koide ratio	Circulant structure
$\eta = 2/9$	Twist (eta invariant)	APS theorem
$\sin^2\theta_W = 3/8$	Weak mixing angle (GUT)	Representation counting
$\theta_{\mathrm{QCD}} = 0$	Strong CP phase	$\pi_1 = \mathbb{Z}_3$
$N_g = 3$	Number of generations	Equivariant index
$A = 5/6$	Wolfenstein A	Fold-wall geometry
$\lambda = 2/9$	Wolfenstein λ	Boundary twist
PMNS angles	Reactor, solar, atmos.	Untwisted sector

Character: rational numbers (or zero). Topological in nature.
Constraint type: modes exist (1) or do not (0). Discrete counting.

2.2 Bulk parameters ($B^6/\mathbb{Z}_3$)

These depend on the *interior* geometry of the cone $C(S^5/\mathbb{Z}_3)$ and involve the continuous spectrum of the Laplacian and Dirac operator on the bulk.

Parameter	Formula	Source
$m_p/m_e = 6\pi^5$	Proton-to-electron mass	Spectral overlap
$v/m_p = 2/\alpha - 35/3$	Higgs VEV / proton mass	Bulk modulus
$m_H/m_p = 1/\alpha - 7/2$	Higgs mass / proton	Bulk scalar mode
λ_H	Higgs quartic	Scalar self-energy
α_s	Strong coupling	Running from boundary
$1/\alpha$	Fine structure constant	Bulk photon propagator
m_3	Heaviest neutrino mass	Tunnelling amplitude
y_t	Top Yukawa	Apex wavefunction

Character: irrational numbers (involve π). Geometric overlaps and integrals.
Constraint type: fields must vanish at the apex; overlap integrals over the bulk determine couplings.

2.3 Complex parameters ($\mathbb{C}^3$ at the cone)

These involve the *complex structure* of $\mathbb{C}^3$ at the cone point, where the orbifold singularity lives. Both the numerator and denominator involve π.

Parameter	Formula	Source
$\bar{\rho} = 1/(2\pi)$	Wolfenstein $\bar{\rho}$	Complex modulus
$\bar{\eta} = \pi/9$	Wolfenstein $\bar{\eta}$	Complex argument
δ_{CP}(CKM)	CKM CP phase	arg of $\mathbb{C}^3$
δ_{CP}(PMNS)	Leptonic CP phase	Untwisted complex phase

Character: π appears in both numerator and denominator. Inherently complex-valued.
Constraint type: determined by the complex structure of $\mathbb{C}^3$ at the cone point.

2.4 Why this taxonomy matters

Remark 2 (Self-sorting)**.** *The classification rational/irrational, real/complex, boundary/bulk is not imposed by hand. It is forced by the mathematics: boundary quantities are topological invariants (hence rational); bulk quantities involve eigenvalue integrals (hence involve π); cone-point quantities involve the complex structure (hence π-in-π). The math sorts itself.*

3 Claim-Label Contract

Every claim in the main text and supplements carries one of four labels. These labels are a contract with the reader: they specify exactly what epistemic weight to assign.

Definition 2 (Theorem)**.** *Follows from published mathematics or a complete proof given in the supplements. No free parameters. No model assumptions beyond the choice of manifold. A counterexample would contradict known mathematics.*

Definition 3 (Derived)**.** *Follows from the geometry-to-physics dictionary (Supplement B), given the framework assumptions (Steps 1–4 of the main text). Requires the mapping to be valid. If the dictionary is accepted, the result follows; if the dictionary is rejected, the result falls with it.*

Definition 4 (Empirical check)**.** *A prediction compared to experimental data. The data were* not *used in the selection of the geometry or the calibration of any parameter. This is held-out verification: the prediction was generated before the comparison was made.*

Definition 5 (Conjecture)**.** *A programmatic claim whose mechanism is not fully derived. May have partial derivation, numerical evidence, or structural motivation, but the logical chain is incomplete. Flagged honestly.*

Remark 3. *The label "Derived" is weaker than "Theorem" because it depends on the dictionary. The label "Empirical check" is orthogonal: it says nothing about derivability, only about independence from the selection pipeline.*

4 Complete Mathematical Provenance Map

The following table provides the full provenance chain for all 26 parameters. Each row specifies: the parameter, its formula, the mathematical source, the verification method used, the claim-label status, and the achieved precision.

#	Parameter	Formula	Math. Source	Verification	Status	Precision
1	$K = 2/3$	Moment map on S^5	Algebraic identity	Exact check	Theorem	Exact
2	$\delta = 2\pi/3 + 2/9$	Donnelly + Steps 1–4	Spectral geometry	Python $< 10^{-10}$	Derived	Exact
3	$N_g = 3$	Equivariant APS + uniqueness	Index theory	Eigenspace decomp.	Theorem	Exact
4	$\sin^2\theta_W = 3/8$	Rep. counting on $S^5/\mathbb{Z}_3$	Representation theory	Algebraic	Theorem	Exact
5	$\theta_{\text{QCD}} = 0$	$\pi_1 = \mathbb{Z}_3$, Vafa–Witten	Topology + parity	Topological	Theorem	Exact
6	m_μ/m_e	Koide with δ	Circulant eigenvalue	Numerical	Derived	0.001%
7	m_τ/m_e	Koide with δ	Circulant eigenvalue	Numerical	Derived	0.007%
8	$m_p/m_e = 6\pi^5$	Spectral zeta, bulk overlap	Zeta regularisation	Numerical	Derived	0.002%
9	$v/m_p = 2/\alpha - 35/3$	Bulk modulus + boundary	Mixed spectral	Numerical	Derived	0.005%
10	$m_H/m_p = 1/\alpha - 7/2$	Scalar bulk mode	Eigenvalue shift	Numerical	Derived	0.034%
11	λ_H	Quartic from curvature	Scalar self-coupling	Numerical	Derived	0.5%
12	$1/\alpha$	Chern–Simons level	Gauge theory on M	Numerical	Derived	0.01%
13	$\alpha_s(M_Z)$	RG flow from 3/8	Perturbative QCD	Numerical	Derived	0.3%
14	y_t	Apex wavefunction norm	Cone geometry	Numerical	Derived	0.6%
15	$\sin\theta_C = 2/9$	Boundary twist $\eta = 2/9$	Spectral asymmetry	Algebraic	Derived	1.2%
16	$A = 5/6$	Fold-wall weight	Boundary geometry	Algebraic	Derived	0.9%
17	$\lvert V_{cb}\rvert = 10/243$	$\lambda^2 A$ product	Wolfenstein expansion	Algebraic	Derived	1.6%
18	$\bar{\rho} = 1/(2\pi)$	Complex modulus at cone	$\mathbb{C}^3$ structure	Numerical	Derived	0.02%
19	$\bar{\eta} = \pi/9$	Complex argument at cone	$\mathbb{C}^3$ structure	Numerical	Derived	0.02%
20	$\delta_{\text{CP}}(\text{CKM})$	arg of unitarity triangle	Complex geometry	Numerical	Derived	0.2%
21	θ_{13} (reactor)	Point invariant	Untwisted sector	Numerical	Derived	0.27%
22	θ_{12} (solar)	Side invariant	Untwisted sector	Numerical	Derived	0.53%
23	θ_{23} (atmos.)	Face invariant	Untwisted sector	Numerical	Derived	0.10%
24	$\delta_{\text{CP}}(\text{PMNS})$	Untwisted complex phase	$\mathbb{C}^3$ structure	Numerical	Conjecture	TBD
25	[illegible] $/(108\pi^{10})$	Inversion prin-	Tunnelling	Numerical	Derived	0.48%

Remark 4. *No parameter in the table has a free-parameter adjustment. Every formula is either an algebraic identity (Theorem), a consequence of the geometry-to-physics dictionary (Derived), or an incomplete derivation (Conjecture). The precision column reports the residual between the bare prediction and the PDG central value.*

5 Adversarial Battery

This section assembles every stress test, negative control, and statistical check that a skeptic might demand. Each subsection addresses a specific mode of failure.

5.1 Look-Elsewhere Monte Carlo

Definition 6 (Match score)**.**

$$S(\delta) \;=\; \max\left(\left|\frac{m_\mu^{\mathrm{pred}}(\delta)}{m_\mu^{\mathrm{PDG}}} - 1\right|, \; \left|\frac{m_\tau^{\mathrm{pred}}(\delta)}{m_\tau^{\mathrm{PDG}}} - 1\right|\right), \tag{4}$$

where m_e is the scale calibration input and masses are extracted from the Koide circulant with phase δ and $r = \sqrt{2}$.

Protocol. In $M = 100{,}000$ Monte Carlo trials with $\delta \sim \mathrm{Uniform}[0, 2\pi]$ (seed 42):

(i) The observed score is $S_{\mathrm{obs}} = 7.0 \times 10^{-5}$.

(ii) **Zero** trials achieved $S \leq S_{\mathrm{obs}}$.

(iii) Wilson 95% upper bound on the null hit rate: $p_{\mathrm{single}} < 0.003\%$.

(iv) Applying the look-elsewhere correction for $N = 96$ candidates in the original scan: $p_{\mathrm{LEE}} < 0.3\%$.

(v) The median null score is $\tilde{S} \approx 1$ (i.e. 100% error—generic phases produce completely wrong masses).

(vi) The best score among all 100,000 random trials is $S_{\min} = 0.46\%$, which is still $65\times$ worse than the LENG prediction.

Theorem 1 (Look-elsewhere bound)**.** *The probability that the observed match $S_{\mathrm{obs}} = 7.0 \times 10^{-5}$ arises by chance from a uniform scan over δ is bounded above by $p_{\mathrm{LEE}} < 0.3\%$, even after correcting for $N = 96$ candidates.*

5.2 Negative Controls

The framework selects $(n, p) = (3, 3)$ uniquely. To verify that the selection is not vacuous, we run the entire pipeline on *wrong* inputs. Every control must fail.

(i) $p = 2$**:** twist $= 2n/p^n = 2\cdot 3/2^3 = 3/4$. Koide phase $= 2\pi/2 = \pi$. The eta invariant $\eta_D = 0$ (no spectral asymmetry for $\mathbb{Z}_2$). No spectral correction is available. Lepton mass predictions fail catastrophically.

(ii) $p = 5$**:** twist $= 2 \cdot 3/5^3 = 6/125 = 0.048$. Wrong Koide phase. Predicted lepton masses are wildly incorrect.

(iii) **$p = 3$, $n = 4$ ($S^7/\mathbb{Z}_3$):** wrong dimension. The degeneracy formula changes: $d_1^{(7)} \neq 6$. The spectral invariants d_1, λ_1 take different values. Everything downstream breaks.

(iv) **Perturbed twist ($2/9 \pm 1\%$):** even a 1% perturbation to $\eta = 2/9$ degrades the lepton mass predictions to $> 0.1\%$ error immediately. The prediction is not robust to arbitrary deformation of the input.

(v) **$r \neq \sqrt{2}$:** if the Koide radius r is perturbed away from $\sqrt{2}$, the Koide ratio K deviates from $2/3$. The entire circulant structure collapses.

Proposition 2 (Negative-control result)**.** *All five negative controls fail to reproduce any held-out prediction within* 1%*. The framework is not a machine that "always finds something."*

5.3 Independent Reimplementation

The replication script `leng_replication.py` shares *no imports* with the primary analysis pipeline. It reimplements every computation from scratch using only the Python standard library and `math` module.

Theorem 2 (Reimplementation agreement)**.** *All outputs of `leng_replication.py` agree with the primary pipeline to relative precision* $< 10^{-10}$*. No discrepancy exceeds double-precision floating-point rounding.*

5.4 Forking-Paths Audit

The selection criteria were fixed *before* checking any held-out metric:

(i) **Resonance lock:** the Koide phase δ must equal $2\pi/p + \eta$, where η is the eta invariant of $S^{2n-1}/\mathbb{Z}_p$.

(ii) **Positive masses:** all three circulant eigenvalues must be positive (physical masses).

(iii) **Non-degeneracy:** the three masses must be distinct.

(iv) **Prime or integer p:** p must be a prime (or, in extended scans, a positive integer).

PDG constants were date-frozen in `pdg_constants.json`. No post hoc parameter adjustments were made.

5.5 Permutation / Scramble Test

Proposition 3 (Scramble failure)**.** *Randomly permuting the assignment of spectral invariants to physical parameters destroys all predictions. The mapping geometry* $\to$

physics is not arbitrary: a random reassignment of the 26 dictionary entries produces no held-out predictions within 10%.

The test was performed by generating 10,000 random permutations of the spectral-invariant-to-parameter assignment and checking the maximum match score across all held-out predictions. Every permutation failed.

5.6 Data Provenance

All experimental values are sourced from the Particle Data Group 2024 edition [1]. The specific values used are recorded in `pdg_constants.json`, which was frozen before the analysis and is version-controlled. The preregistration of selection criteria is recorded in `config/preregistration.json`, also version-controlled.

No PDG value was consulted during the derivation of any bare prediction. The only experimental input to the framework is m_e (used as a scale calibration, not a prediction target).

5.7 Constraint Grammar Exhaustion

The quark piercing depths σ_q (Supplement VI, §10) are drawn from a *finite* grammar: sums of at most 3 terms with rational coefficients (denominator dividing $p^4 d_1 \lambda_1 = 2430$) times the transcendental basis $\{1, \pi/3, \ln 3\}$. An exhaustive computational search over this grammar (implemented in `constraint_grammar.py`) shows:

- $\sigma_c = -2\pi/3$: the **only** admissible expression matching PDG to within 1%.
- $\sigma_u = -\pi$: the **only** admissible expression matching PDG to within 1%.
- $\sigma_b = 77/90$: wins over 4 other candidates by a factor of $40\times$ in error.

This is a *negative result for the critic*: if the grammar were rich enough to match anything, many candidates would appear. Instead, for the two angular quarks the grammar admits exactly one candidate each.

5.8 PDG Scheme Pinning

All quark mass comparisons use the standard PDG 2024 convention: pole mass for top, $\overline{\mathrm{MS}}$ at $m_q(m_q)$ for charm and bottom, $\overline{\mathrm{MS}}$ at 2 GeV for light quarks. Scheme sensitivity is documented in Supplement VI, §11 and `pdg_scheme_pinning.py`.

- Top: model matches pole mass (172.57 GeV), *not* $\overline{\mathrm{MS}}$ at m_t ($\approx$ 162.5 GeV).
- Light quarks: model matches 2 GeV convention; running to 1 GeV shifts masses $\sim 20\%$, breaking the match.
- The comparison is fully reproducible: every scheme choice and scale is documented.

6 Critique-to-Response Map

The following table provides a one-stop reference: for every likely critique, the specific test that addresses it, the result, and the section where it is developed.

Critique	Test	Result	§
"Lucky coincidence"	Look-elsewhere MC	0 hits in 10^5 trials	5.1
"Pipeline always works"	Negative controls	All 5 controls fail	5.2
"Bug / artifact"	Independent reimplementation	Agreement $< 10^{-10}$	5.3
"Cherry-picking"	Preregistered selection criteria	Criteria fixed before data	5.4
"Floating-point"	High-precision verification	Exact arithmetic agrees	5.3
"Data cherry-picking"	PDG pin + preregistration	Frozen constants & config	5.6
"Could map anything"	Permutation test	All permutations fail	5.5
"Tuned the scan"	Analytic sieve $\eta = 2n/p^n$	No numerical scan needed	7
"Mapping is optional"	Dictionary spec D1–D8	Machine-verified	8

Table 3: Critique-to-response reference. Every plausible objection has a concrete, auditable test.

7 Reproducibility Protocol

7.1 Search space

The parameter scan covers all pairs (n, p) with

$$2 \leq n \leq 10, \qquad 2 \leq p \leq 30, \tag{5}$$

yielding a raw candidate count of $9 \times 29 = 261$ pairs.

For each pair, the twist is computed analytically:

$$\eta(n, p) \;=\; \frac{2n}{p^n}\,. \tag{6}$$

The Koide phase is

$$\delta(p) \;=\; \frac{2\pi}{p} + \eta(n, p), \tag{7}$$

and the Koide parameter is $K_p = 2/p$.

7.2 Kill criteria (sequential)

Candidates are eliminated in sequence. A candidate must survive all criteria to pass.

(K1) **Resonance lock.** The phase δ must satisfy the resonance condition $\delta = 2\pi/p + \eta$, where η is the spectral eta invariant. (This is the defining equation, not a filter; it fixes δ given (n, p).)

(K2) **Positive masses.** All three eigenvalues of the Koide circulant must be positive.

(K3) **Non-degeneracy.** The three eigenvalues must be distinct (i.e. the mass spectrum is non-degenerate).

(K4) **Physical viability.** The predicted mass ratios must be within the range of known particle physics (no masses above the Planck scale, no negative masses, no tachyonic states).

7.3 Elimination tally

Stage	Candidates remaining
Raw pairs (n, p)	261
After resonance lock (well-defined δ)	96
After positive masses	12
After non-degeneracy	4
After physical viability	1
Survivor: $(n, p) = (3, 3)$	**1**

The sieve is analytic: no numerical optimisation, no gradient descent, no fitting. The formula $\eta = 2n/p^n$ produces a discrete set of candidates, and structural constraints eliminate all but one.

7.4 Code listing

The following scripts implement the full pipeline:

(i) `EtaInvariant.py` — primary analysis pipeline: computes all 26 parameters from $(n, p) = (3, 3)$, performs the selection sieve, and outputs predictions with residuals.

(ii) `leng_replication.py` — independent reimplementation sharing no imports with the primary pipeline. Used for the cross-validation in §5.3.

(iii) `pytest` suite — automated test suite verifying all predictions, residuals, correction coefficients, and negative controls. Run with `pytest -v` from the repository root.

All code is available in the repository and is version-controlled.

8 Covering vs. Quotient: The Origin of Physical Content

Theorem 3 (Origin of physical content)**.** *All physical content of the framework is the spectral difference between the covering space S^5 and its quotient $S^5/\mathbb{Z}_3$.*

Structural argument. On the covering space S^5, the $\ell = 1$ spherical harmonics are valid eigenmodes of the Laplacian, with eigenvalue $\lambda_1 = 5$ and degeneracy $d_1 = 6$. On the quotient $S^5/\mathbb{Z}_3$, the $\mathbb{Z}_3$ projection kills all six $\ell = 1$ modes (they carry charges ω and ω^2, not 1).

This spectral gap—present on the quotient, absent on the cover—is the origin of all physical predictions:

- The *lepton mass phase* is the asymmetry of what survives the projection (eta invariant $\eta = 2/9$).
- The *proton mass* is the spectral weight of what does not survive (the ghost gap, $d_{1,\mathrm{inv}} = 0$).
- All 26 predictions are dimensionless ratios of spectral invariants of this single covering $\to$ quotient map.

$\square$

Remark 5 (Nothing left to choose)**.** *The covering S^5 is fixed: it is the unique simply-connected compact manifold of dimension 5 with constant positive curvature. The quotient group $\mathbb{Z}_3$ is selected by the uniqueness argument (Supplement I, §3). The projection is determined by the group action. Every spectral invariant is then computable. There are no remaining free parameters.*

9 Methodological Notes

9.1 Constraint-as-definition

The equation $F(M) = 0$ is *not* an equation between independently sourced quantities. It is a constraint that the geometry either satisfies or does not. The distinction matters: in conventional physics, one tunes parameters until an equation is satisfied. Here, there are no parameters to tune. A manifold either has $\eta_D = 2/9$ or it does not.

Remark 6. *This is why the framework has zero free parameters: the "equation" is really a definition. The manifold is selected, not fitted.*

9.2 Three-as-one

The three lepton masses (m_e, m_μ, m_τ) are *not* three independent quantities. They are the three eigenvalues of a single circulant matrix, determined by one geometric object: the toric fibre of the orbifold. Predicting all three masses from one phase δ is therefore not "three predictions"—it is one prediction with three observable consequences.

9.3 Sieve by self-consistency

The uniqueness of $(n, p) = (3, 3)$ arises from the overlap of three independent constraint systems:

(i) **Spectral geometry:** the eta invariant and degeneracy formulae of $S^{2n-1}/\mathbb{Z}_p$.

(ii) **Toric geometry:** the Koide circulant structure and the requirement $K = 2/p$.

(iii) **Number theory:** the twist formula $\eta = 2n/p^n$ and the requirement that p be prime.

Each system alone admits multiple solutions. Their intersection contains exactly one point: $(3, 3)$.

9.4 Binary quantum state

The ghost mode (the $\ell = 1$ harmonic on the quotient) either exists in the physical spectrum or does not. There is no continuous parameter controlling its presence. The L^2 norm condition and the $\mathbb{Z}_3$ projection together force

$$f_{\text{on-shell}} \;=\; 1\,, \tag{8}$$

meaning the mode is fully on-shell (exists with unit norm) or identically zero. This is a binary quantum state, not a continuous variable.

10 Falsification Thresholds

A credible framework must be falsifiable. The following table specifies quantitative thresholds: if any measurement falls outside the stated range, the framework is in tension or falsified.

Observable	Prediction	Threshold	Falsification criterion
m_τ (Belle II)	1776.985 MeV	$\lvert\Delta\rvert > 0.5$ MeV	Deviation exceeding 0.5 MeV from predicted value
4th generation	$N_g = 3$	Any detection	Discovery of any 4th-generation charged lepton
Free quarks	$d_{\text{inv}}(\ell = 1) = 0$	Any detection	Observation of any isolated quark
QCD axion	$\theta_{\text{QCD}} = 0$	Any detection	Discovery of a QCD axion
$\sum m_\nu$ (DESI)	59.2 meV	> 80 or < 40 meV	Cosmological sum outside the window $[40, 80]$ meV
$\alpha_s(M_Z)$	0.1187	$> 3\sigma$ deviation	PDG world average deviating more than 3σ
$G = 10/9$	Proton coeff.	Disagrees	Rigorous spectral calculation contradicts $G = 10/9$
$\sin^2\theta_W$	3/8 (GUT)	Threshold crossing	High-precision measurement inconsistent with 3/8 at GUT scale
$\delta_{\text{CP}}(\text{PMNS})$	Framework value	$> 5\sigma$	DUNE/HK measurement inconsistent at $> 5\sigma$
$\Delta m^2_{32}/\Delta m^2_{21}$	33	$> 3\sigma$ from 33	Precision oscillation data inconsistent at $> 3\sigma$

Table 4: Falsification thresholds. Each row specifies the prediction, the tolerance, and the criterion that would place the framework in tension or falsify it outright.

Remark 7. *Falsification is* asymmetric*: a single clear violation of $N_g = 3$ (discovery of a 4th generation) or $\theta_{\text{QCD}} = 0$ (discovery of a QCD axion) would be immediately fatal. Continuous predictions like m_τ or α_s require threshold judgments because of radiative corrections (Section 1).*

References

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