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On the H^1 Conforming Virtual Element Method for Time Dependent Stokes Equation

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Abstract In this paper, we present a virtual element method for the time-dependent Stokes equation by employing a mixed formulation involving the velocity and the pressure as primitive variables. The velocity is approximated using the H^1 conforming virtual element and the pressure is approximated by the discontinuous piecewise polynomial. In order to approximate the non-stationary part with optimal order of convergence, we need to compute the L^2 projection operator of the full order k. In view of this requirement, we modify the velocity space keeping the same dimension. The virtual space is discrete inf-sup stable for $k \geq 2$ and non-divergence free. We estimate the optimal order of convergence for the velocity and the pressure.

Keywords Virtual element method · Conforming methods · Polynomial space · Error estimates · Discrete inf-sup stability

Mathematics Subject Classification 65A · 65M · 35

1 Introduction

Non-stationary Stokes equation describes many physical phenomena of incompressible flow problems including modelling of weather predictions, movement of ocean currents, water flow in a pipe and air flow around a wing. Therefore, it necessitates to develop an efficient numerical technique for the non-stationary Stokes equation. One of the commonly employed numerical scheme is the finite element method, which relies on the spatial discretization involving simplex elements. The governing equations for the Stokes equation involves coupled partial differential equations with velocity and pressure as the unknown field variables. Two approaches employed for the solution of the Stokes equation are: stream-function approach [31] and mixed formulation [13] involving velocity and pressure as primary variables. Amongst the two approaches, the mixed formulation is commonly employed. However, the stability of the scheme is dictated by the inf-sup condition [5]. It requires that the approximation space for the velocity field must be sufficiently richer than the pressure field. Due to this constraint, the velocity-pressure approximation cannot be done arbitrarily. This has received considerable attention amongst researchers. Only recent contributions are highlighted here, for a more detailed discussion, interested readers are referred to [13] and references therein.

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Shang [30] studied the non-stationary Stokes equation considering linear element for the velocity approximation and piecewise constant function for the pressure approximation (Q_1–P_0 element). By employing the following regularity assumption, justified in [24,25], Shang [30] presented the error analysis in the L^2 and H^1 norm:

$$\begin{cases} \sup\limits_{0\leq t\leq T}\Big(\|\mathbf{u}_t(t)\|_0^2+\|\mathbf{u}(t)\|_2^2+\|p(t)\|_1^2\Big)\leq C, \\ \sup\limits_{0\leq t\leq T}\sigma(t)\ \|\mathbf{u}_t(t)\|_1^2+\displaystyle\int_0^T\sigma(t)\Big(\|\mathbf{u}_{tt}(t)\|_0^2+|\mathbf{u}_t(t)|_2^2+\|p(t)\|_1^2\Big)\,\mathrm{d}t\leq C. \end{cases}$$

where $\mathbf{u}$ and p represents the velocity and the pressure field, respectively. The boundedness of $\mathbf{u}_t(t)$ and $\mathbf{u}(t)$ in the L^2 and H^1 norms were also studied in [30]. The (Q_1–P_0) element for the velocity and the pressure field is not discrete inf-sup stable and the numerical technique requires additional stabilizers. Some of the commonly used approaches include: stream upwind Petrov–Galerkin (SUPG) method [18], Brezzi–Pitkaranta method [17], the Douglas–Wang method [21], the well-known Galerkin least square (GLS) method [22] and the method of bubble function enrichment [15].

In the last decade, the focus was on the study of the lowest equal-order finite element pair $P_1 - P_1$ (linear function on triangle and tetrahedron), $Q_1 - Q_1$ (bilinear functions) or $Q_1 - P_0$ (linear functions on quadrilateral) using constant projection operator for pressure variable [12,27]. The above said stabilized finite element technique does not require stabilization parameters and calculation of high order derivatives. Therefore, this technique has drawn attention of several researchers. Li et al. [28] studied the non-stationary Navier–Stokes equation by employing the lowest 'equal' order finite element for the velocity-pressure pair. This pair is unstable and the existence of the divergence free Fortin operator cannot be guaranteed. To alleviate this, a discrete Stokes projection operator was introduced in [28] to estimate the theoretical results. Heywood and Rannacher [26] proposed a fully implicit Crank–Nicolson scheme for Navier–Stokes equation in [26] and proved that the scheme is unconditionally stable and converges optimally. An error analysis for the Crank–Nicolson extrapolation scheme of time discretization have been studied in [23], where they have utilized stabilized finite element approximation for the space variable.

Aforementioned studies were restricted to simplex elements. Inspired by the mimetic finite difference method [29], Brezzi and co-workers introduced the virtual element method [6]. The salient features of the VEM include: (a) simplicity in implementation; (b) easy to extend to higher dimensions without much change in the mathematical formulation and (c) the discrete bilinear form can be computed with the help of degrees of freedom, thus, circumventing the otherwise cumbersome numerical integration. Since its inception, the VEM has been applied to wide variety of problems, such as, elliptic equation [7,20], linear and nonlinear parabolic and hyperbolic problems [1,2,32,34], convection dominated diffusion reaction equation [11], plate bending problems [16], nonlinear elastic problems [9], to name a few. In particular, Antonietti et al. [4] introduced a stream virtual element formulation for the Stokes problems on polygonal mesh. The discrete scheme introduced is completely computable based on the information provided by the degrees of freedom. Beirão da Veiga, [10] presented a new VEM space that is divergence free and modified the stationary Stokes problem accordingly. The modified VEM space for the velocity contains polynomial of order k and the pressure space contains polynomial of order $k-1$. This velocity-pressure pair is inf-sup stable for $k \geq 2$. However, the primary drawback of this space is that the vector valued L^2 projection operator is not computable with optimal order. Moreover, in the same paper, author has designed reduced local virtual space which is computationally cheap. Vacca [33] presented a VEM space for the velocity variable, where the L^2 projection operator is completely computable and introduced the VEM discretization for Darcy and Brinkman equation. Nonconforming virtual element formulation for stationary Stokes equation has been studied by Cangiani et al. [19]. The virtual element space introduced in [19] is not divergent free but the projection operators are easy to compute from the edge and the cell moments.

In this paper, the primary focus is on the theoretical error estimation of the semi-discrete case for the time dependent Stokes equation. In order to analyze the semi-discrete case, we propose a new Stokes projection operator, which is compatible with the discrete virtual element space and derive the error estimates in the L^2 and H^1 norms.

We modify the VEM space in order to compute the L^2 projection operator $\Pi^0_{k,K}$ with optimal order k, this is an extension of the idea used in [3], where $\Pi^0_{k,K}$ is computed with the help of $\Pi^\nabla_{k,K}$ operator. For the pressure approximation, we consider the discontinuous polynomial space of order $k-1$ same as the finite element method. Since the virtual element space for the pressure is only a polynomial space, the discrete formulation associated with the pressure variable will be same as the finite element approximation and is directly computable from the degrees of freedom (DOF). Although, the discrete virtual element space presented in this work is not divergence free, the space is discrete inf-sup stable for $k \geq 2$ and provide optimal order of convergence in the L^2 and H^1 norms. Moreover, as the space is discrete inf-sup stable, it is easy to construct the Fortin operator that reduces the complexity of the theoretical estimation.

The rest of the paper is organised as follows: Sect. 2 presents the governing equations and the corresponding weak form for the time dependent Stokes equation. The basic construction of virtual element space and the associated degrees of freedom are discussed in Sect. 3. In Sect. 4, we construct the discrete virtual element formulation for the model problem. The discrete Stokes projection and the a priori error estimates for velocity and pressure fields are presented in Sect. 5, the optimal error estimates in the L^2 and H^1 norm are also derived, followed by concluding remarks in the last section.

2 Preliminaries and Governing Equations

Consider the time dependent Stokes equation

$$\begin{cases} \partial_t \mathbf{u} - \Delta \mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega \times (0, T) \\ \text{div}\, \mathbf{u} = 0 \quad \text{in} \quad \Omega \times (0, T) \\ \mathbf{u} = 0 \quad \text{on} \quad \partial\Omega \times (0, T) \\ \mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Omega \times \{0\}, \end{cases} \tag{2.1}$$

where the vector variable $\mathbf{u}$ and scalar variable p, represents the velocity and the pressure field, respectively. Moreover, we adopt the standard notation for the Laplacian, the divergence and the gradient operator as Δ, div, ∇. Let us denote the continuous velocity and the pressure space by $\mathcal{V}$ and $\mathcal{Q}$ respectively, where

$$\mathcal{V} := [H^1_0(\Omega)]^2; \quad \mathcal{Q} := L^2_0(\Omega) = \left\{ q \in L^2(\Omega) \; s.t. \int_\Omega q \, d\Omega = 0 \right\}$$

equipped with the natural norms

$$\|\mathbf{v}\|_1^2 := \|\mathbf{v}\|_{[H^1(\Omega)]^2}, \quad \|q\|_{\mathcal{Q}} := \|q\|_{L^2(\Omega)}.$$

We deduce that the force function $\mathbf{f} \in [L^2(\Omega)]^2$ and the bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$ are defined as

$$a(\mathbf{u}, \mathbf{v}) := \int_\Omega \nabla \mathbf{u} : \nabla \mathbf{v} \, d\mathbf{x}, \qquad b(\mathbf{v}, q) := \int_\Omega q \nabla \cdot \mathbf{v} \, d\mathbf{x},$$

where ':' represents the tensor product of two matrices. Exploiting the above two bilinear forms, we represent the continuous formulation: find $(\mathbf{u}, p) \in L^2(0, T; \mathcal{V}) \times L^2(0, T; \mathcal{Q})$ s.t.

$$\begin{cases} (\mathbf{u}_t, \mathbf{v}) + a(\mathbf{u}, \mathbf{v}) - b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) \quad \forall \quad \mathbf{v} \in V \\ \qquad\qquad\qquad\qquad b(\mathbf{u}, q) = 0 \qquad \forall \quad q \in Q, \end{cases} \tag{2.2}$$

where $(\cdot,\cdot)$ is the L^2-inner product on Ω. It can be easily verified that $b(\cdot,\cdot)$ satisfies the inf-sup condition, i.e. there exists a positive constant $C_\alpha > 0$ such that the following estimation holds

$$C_\alpha \, \|l\|_{\mathcal{Q}} \leq \sup_{\mathbf{v} \in \mathcal{V} \; \mathbf{v} \neq 0} \frac{b(\mathbf{u}, l)}{\|\mathbf{v}\|_1}, \forall \quad l \in \mathcal{Q}. \tag{2.3}$$

The bilinear forms $a(\cdot,\cdot)$ and $b(\cdot,\cdot)$ are continuous

$$\begin{aligned} |a(\mathbf{u},\mathbf{v})| &\leq C\ \|\mathbf{u}\|_{\mathcal{V}}\ \|\mathbf{v}\|_{\mathcal{V}} \quad \forall \quad \mathbf{u},\mathbf{v}\in\mathcal{V};\\ |b(\mathbf{u},l)| &\leq C\ \|\mathbf{u}\|_{\mathcal{V}}\ \|l\|_{\mathcal{Q}} \quad \forall \quad \mathbf{v}\in\mathcal{V} \text{ and } l\in\mathcal{Q}; \end{aligned} \tag{2.4}$$

where C is a generic constant. Moreover, $a(\cdot,\cdot)$ satisfies coercivity on $\mathcal{V}$, i.e., there exists a positive constant C such that

$$a(\mathbf{v},\mathbf{v}) \geq C\ \|\mathbf{v}\|_{\mathcal{V}}^2; \quad \forall\, \mathbf{v}\in\mathcal{V}. \tag{2.5}$$

Equations (2.3)–(2.5) ensure that the model problem (2.2) has a unique solution $(\mathbf{u}, p)$ [13] satisfying

$$\sup_{0\leq t\leq T} (\|\mathbf{u}(t)\|_2 + \|p(t)\|_1) \leq C. \tag{2.6}$$

For $\mathbf{u}\in[H^k]^2$, we define the norm as

$$\|\mathbf{u}\|_k := \left(\sum_{1\leq i\leq 2}\ \sum_{0\leq|\alpha|\leq k} \|D^\alpha \mathbf{u}_i\|^2\right)^{1/2}.$$

Furthermore, $L^p(0,T;[H^s]^2)$, $1\leq p\leq\infty$, $s\geq 0$ represent the Hilbert space of all L^p integrable functions $\psi(t):[0,T]\to H^s(\Omega)$ with the standard norm $\|\psi\|_{L^p(0,T;[H^s(\Omega)]^2)} := \left(\int_0^T \|\psi(t)\|^p_{[H^s(\Omega)]^2}\right)^{1/p}$ for $p\in[1,0)$ with standard modification at $p=\infty$.

3 Virtual Element Spaces

In this section, we present the basic construction of the local and the global virtual element space. The virtual element space is constructed such that the space will be unisolvent w.r.t. a set of functionals referred to as the degrees of freedom (DoFs) [16]. Moreover, the space satisfies all the assumptions which we will be inferred in order to analyze the theoretical estimations. Further, the decompositions of Ω satisfies the assumption on mesh regularity, given by:

Assumption 1 (A_1) Every element $K\in\mathcal{T}_h$ is star shaped with respect to a ball of radius greater than γh_K;
(A_2) For every element K, we assume every edge $e\subset\partial K$ satisfies $h_e\geq\gamma_* h_K$,

where $\mathcal{T}_h$ is the decomposition of the domain into non-overlapping elements, h_K is the measure of the element, K and $h=\max(h_K)$ and γ and γ_* are two positive real numbers.

In order to perform the convergence analysis, we introduce two basic tools: (a) the L^2 orthogonal projection operator $\Pi^0_{k,K}$ and (b) the energy projection operator $\Pi^\nabla_{k,K}$:

$$\Pi^\nabla_{k,K} : [H^1(K)]^2 \to [\mathbb{P}_k(K)]^2.$$

defined by

$$\begin{cases} \displaystyle\int_K \nabla\mathbf{w}_k : \nabla(\mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h)\, dK = 0 \quad \forall\, \mathbf{w}_k\in[\mathbb{P}_k(K)]^2,\\ \mathbf{P}_0(\mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h) = \mathbf{0}, \end{cases}$$

where $\mathbf{P}_0$ is the orthogonal L^2 projection operator onto constant functions which is defined as

$$\mathbf{P}_0\mathbf{v} := \frac{1}{|\partial K|}\int_{\partial K} \mathbf{v}.$$

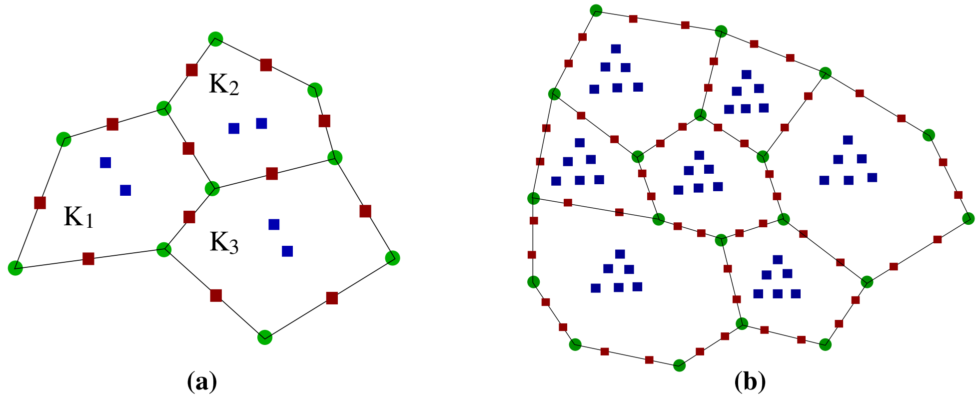

Fig. 1 Typical degrees of freedom of polygonal elements; (D_1) degrees of freedom are indicated by green circles; (D_2) are indicated by red squares;cell moments are indicated by blue squares (Color figure online)

The projection operator $\Pi^{\nabla}_{k,K}$ can be directly evaluated from the DOF. Moreover, for subsequent discussion, we require the L^2 orthogonal projection operator $\Pi^0_{k,K} : [L^2(K)]^2 \to [\mathbb{P}_k(K)]^2$ which is defined as

$$\int_K \mathbf{w}_k \cdot (\mathbf{v}_h - \Pi^0_{k,K}\mathbf{v}_h)\, dK = 0 \quad \text{for all } \mathbf{w}_k \in [\mathbf{P}_k(K)]^2.$$

Local virtual element space We consider the following local virtual element space $Z^k(K)$ which is already defined in [8] for the elasticity problem.

$$Z^k(K) := \Big\{ \mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2,\ \Delta\mathbf{u} \in [\mathbb{P}_{k-2}(K)]^2 \Big\},$$

where $[\mathbb{B}_k(\partial K))] := \{v \in C^0(\partial K)\ s.t.\ v|_e \in \mathbb{P}_k(e)\ \forall e \in \partial K\}$. Moreover, for a function $\mathbf{v} \in [H^1(K)]^2$, we defined the set of functional $\mathcal{F}_Z$ as

- ($\mathbf{D_1}$) Values of $\mathbf{v}$ at $V(K)$ vertexes of K, where $V(K)$ is a set of vertices of element K.
- ($\mathbf{D_2}$) For $k > 1$, the values of $\mathbf{v}$ at $k-1$ uniformly spaced points on each edge $e \subset \partial K$.
- ($\mathbf{D_3}$) For $k > 1$, the moments $\frac{1}{|K|}\int_K \mathbf{w}(\mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} \quad \forall\, \mathbf{w}(\mathbf{x}) \in [\mathbb{P}_{k-2}(K)]^2$.

In order to depict degrees of freedom, we exhibit mesh decomposition in Fig. 1.

Lemma 3.1 *$Z^k(K)$ is unisolvent with respect to $\mathcal{F}_Z$.*

Proof See [8] for details. □

For a positive integer k, the local virtual element space $Z^k(K)$ consists of functions v which are polynomial of degree k on ∂K and Δv is a polynomial of degree $k-2$ inside each polygon K. It is well known that $\dim([\mathbb{B}_k(\partial K)]^2) = 2N_K k$ and $\dim([\mathbb{P}_{k-2}(K)]^2) = k(k-1)$. Therefore, the dimension of the virtual element space $Z^k(K) = 2N_K k + k(k-1)$, where N_K denotes the number of vertices of an element K.

Computation of $\Pi^{\nabla}_{k,K}$ operator on $Z^k(K)$ Let $\boldsymbol{\phi}_i \in Z^k(K)$ be a local basis function. Then, $\Pi^{\nabla}_{k,K}\boldsymbol{\phi}_i$ is an element of $[\mathbb{P}_k(K)]^2$. Assume that $\mathbf{q} \in [\mathbb{P}_k(K)]^2$ is an arbitrary element. Then, from the definition of $\Pi^{\nabla}_{k,K}$, we can write

$$\begin{aligned}\int_K \nabla\Pi^{\nabla}_{k,K}\boldsymbol{\phi}_i : \nabla\mathbf{q} &= \int_K \nabla\boldsymbol{\phi}_i : \nabla\mathbf{q}\\ &= \underbrace{-\int_K \Delta\mathbf{q}\cdot\boldsymbol{\phi}_i}_{T_1} + \underbrace{\int_{\partial K} (\nabla\mathbf{q}\mathbf{n})\cdot\phi_i}_{T_2}.\end{aligned}$$

Since $\Delta \mathbf{q} \in [\mathbb{P}_{k-2}(K)]^2$, the term T_1 can directly be evaluated from the internal momentum and the other term (T_2) can be computed with ($\mathbf{D_1}$) and ($\mathbf{D_2}$) degrees of freedom.

It can be seen that the projection operator $\Pi^0_{k-2,K}$ is computable on $Z^k(K)$. Moreover the right-hand side term (c.f. Eq. 2.1) and the time dependent parts are approximated with the help of $\Pi^0_{k,K}$ operator. However, the projection operator $\Pi^0_{k,K}$ is not computable over $Z^k(K)$. This is because, we have the internal momentum of order upto $k-2$. Upon employing an analogous idea from [3], we will recast the local virtual element space $Z^k(K)$, where the projection operator $\Pi^0_{k,K}$ is fully computable. In view of this requirement, we start by introducing the local classical space $\mathcal{V}^k(K)$.

On each polygon $K \in \mathcal{T}_h$, we define the classical local space as

$$\mathcal{V}^k(K) := \left\{ \mathbf{v} \in [H^1(K)]^2 \text{ s.t. } \mathbf{v}|_{\partial K} \in [\mathbb{B}_k(\partial K)]^2,\ \Delta \mathbf{u} \in [\mathbb{P}_k(K)]^2 \right\}.$$

Exploiting the elliptic operator $\Pi^\nabla_{k,K}$, we recast the local modified virtual element space, which is basically the restriction of $\mathcal{V}^k(K)$

$$\mathcal{W}^k(K) := \left\{ \mathbf{v} \in \mathcal{V}^k(K)\ s.t.\ \int_K \mathbf{q}_k \cdot (\mathbf{v} - \Pi^\nabla_{k,K} \mathbf{v}) = 0 \quad \forall\, \mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2 \right\},$$

where the symbol $[\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2$ represents the linear space spanned by the polynomials of degree k and $k-1$. A global virtual element space is then defined as

$$\mathcal{W}^k_h := \left\{ \mathbf{v} \in [H^1_0(\Omega)]^2\ s.t.\ \mathbf{v}|_K \in \mathcal{W}^k(K) \right\}.$$

In the classical virtual element space $\mathcal{V}^k(K)$, we consider the Laplacian of $\mathbf{w} \in \mathcal{V}^k(K)$ is an element of $[\mathbb{P}_k(K)]^2$. The modified VEM spaces are designed as a restriction of the functions of the space $\mathcal{V}^k(K)$. It seems that the dimension of $\mathcal{W}^k(K)$ is more than dimension of $Z^k(K)$. However, this prediction is no longer true and we will show that $\mathcal{F}_Z$ forms the degrees of freedom for $\mathcal{W}^k(K)$. In view of this, we continue our discussion with the following lemma.

Lemma 3.2 *The dimension of $\mathcal{V}^k(K)$ is*

$$dim(\mathcal{V}^k(K)) = 2N_K k + (k+1)(k+2).$$

Furthermore, the set of functional $\mathcal{F}_Z$ along with the moments $\mathcal{F}_V$ which is defined as

$$\mathcal{I}(\mathbf{v}) := \int_K \mathbf{v} \cdot \mathbf{q}_k \quad \textit{for all} \quad \mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2,$$

form the degrees of freedom for $\mathcal{V}^k(K)$.

Proof The framework of the proof is analogous to the case of the scalar valued functions mentioned in [3]. Hence, briefly we present the abstract framework of the proof. Let $\mathbf{w}_h$ be an element of $\mathcal{V}^k(K)$. We show that

$$\mathbf{w}_h|_{\partial K} = 0 \quad \text{and} \quad \Pi^0_{k,K} \mathbf{w}_h = 0$$

imply $\mathbf{w}_h = 0$. Basically, these two conditions indicate $\mathbf{A}\mathbf{w}_h = 0$, where the operator $\mathbf{A}$ is defined as

$$\mathbf{A}\mathbf{w} := \begin{bmatrix} \Delta w_1 \\ \Delta w_2 \end{bmatrix}, \qquad \mathbf{w} := \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}.$$

Proceeding same as in [6,8], we can construct a one-to-one mapping $\mathbf{R} : [\mathbb{P}_k(K)]^2 \to [\mathbb{P}_k(K)]^2$. This implies isomorphism between the internal moments of function $\mathbf{w} \in [H^1_0(K)]^2$ and their Laplacian which is a component-wise polynomial of order k. The mapping $\mathbf{R}$ is defined as

$$\mathbf{R}\mathbf{q} := \Pi^0_{k,K} \mathbf{A}^{-1} \mathbf{q}. \quad \text{for all} \quad \mathbf{q} \in [\mathbb{P}_k(K)]^2.$$

Hence the dimension of $\mathcal{V}^k(K)$ is

$$\begin{aligned}\dim(\mathcal{V}^k(K)) &= \dim([\mathbb{B}_k(\partial K)]^2) + \dim([\mathbb{P}_k(K)]^2).\\ &= 2N_K k + (k+1)(k+2).\end{aligned}$$

$\square$

Remark 3.1 On $\mathcal{W}^k(K)$, the L^2 projection operator $\Pi^0_{k,K}$ is fully computable. We evaluate $\Pi^0_{k,K}$ operator employing $\Pi^\nabla_{k,K}$ operator. Moreover, for each $\mathbf{v} \in [H^1(K)]^2$, $\Pi^0_{k,K}\mathbf{v}$ and $\Pi^\nabla_{k,K}\mathbf{v}$ are polynomial approximations in $[\mathbb{P}_k(K)]^2$. Hence, we can assume that $\Pi^0_{k,K}\mathbf{v} \approx \Pi^\nabla_{k,K}\mathbf{v}$ for $k = 1, 2$. Furthermore, from the construction of the VEM space $\mathcal{W}^k(K)$, it can be deduced that $(\mathbf{q}_k, \mathbf{v}_h - \Pi^\nabla_{k,K}\mathbf{v}_h) = 0$ for $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2$. This condition reduces the dimension of the space $\mathcal{W}^k(K)$.

Lemma 3.3 *$\mathcal{W}^k(K)$ is unisolvent with respect to set of functions $\mathcal{F}_Z$.*

Proof In the modified VEM space $\mathcal{W}^k(K)$, we have additional conditions that are $\dim([\mathbb{P}_k(K)]^2) - \dim([\mathbb{P}_{k-2}(K)]^2) = 4k+2$. Upon removing these additional conditions, we get

$$\dim(\mathcal{W}^k(K)) \geq \dim[\mathcal{V}^k(K)] - (4k+2) = \dim(\mathcal{Z}^k(K)).$$

In order to show that $\mathcal{W}^k(K)$ is unisolvent with respect to $\mathcal{F}_Z$, we show that an element $\mathbf{w}_h \in \mathcal{W}^k(K)$ with $\mathcal{F}_Z(\mathbf{w}_h) = 0$, is identically zero. Since $\mathcal{F}_Z(\mathbf{w}_h) = 0$ implies $\Pi^\nabla_{k,K}\mathbf{w}_h = 0$. Furthermore, from the definition of the VEM space $\mathcal{W}^k(K)$, we have

$$\int_K \mathbf{w}_h \mathbf{q}_k = \int_K \Pi^\nabla_{k,K}\mathbf{w}_h \mathbf{q}_k \quad \forall \quad \mathbf{q}_k \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2.$$

Hence, we deduce that $\mathcal{F}_V(\mathbf{w}_h) = 0$. And from Lemma 3.2, we can conclude that $\mathbf{w}_h = 0$. $\square$

Computation of L^2 projection operator $\Pi^0_{k,K}$ on the modified VEM space $\mathcal{W}^k(K)$

Lemma 3.4 *The orthogonal L^2 projection operator $\Pi^0_{k,K}$ is computable on $\mathcal{W}^k(K)$.*

Proof Let $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2$ be an arbitrary function that can be decomposed as

$$\mathbf{q}_k = \mathbf{q_1} + \mathbf{q_2},$$

where $\mathbf{q_1} \in [\mathbb{P}_{k-2}(K)]^2$ and $\mathbf{q_2} \in [\mathbb{P}_k(K)]^2/[\mathbb{P}_{k-2}(K)]^2$. Let $\boldsymbol{\phi}_i \in \mathcal{W}^k(K)$ be a local basis function. Hence, we write the following estimation

$$\begin{aligned}\int_K \Pi^0_{k,K}\boldsymbol{\phi}_i \cdot \mathbf{q}_k &= \underbrace{\int_K \boldsymbol{\phi}_i \cdot \mathbf{q}_k}_{\text{Def. of } \Pi^0_{k,K}}\\ &= \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_1} + \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_2}\\ &= \int_K \boldsymbol{\phi}_i \cdot \mathbf{q_1} + \underbrace{\int_K \Pi^\nabla_{k,K}\, \boldsymbol{\phi}_i \cdot \mathbf{q_2}}_{\text{Def. of } \mathcal{W}^k(K)}.\end{aligned}$$

It is well know that on the space $\mathcal{W}^k(K)$, the energy projection operator $\Pi^\nabla_{k,K}$ is computable. Hence, we can compute $\Pi^\nabla_{k,K}\,\boldsymbol{\phi}_i$. The first term involving polynomial of degree $k-2$ is computable and the second term involving the integration $\int_K \Pi^\nabla_{k,K}\,\boldsymbol{\phi}_i \cdot \mathbf{q_2}$ is also computable. Therefore, we can evaluate $\Pi^0_{k,K}\phi$ for $\phi \in \mathcal{W}^k(K)$. $\square$

For the pressure approximation, we consider the standard finite dimensional space

$\mathcal{Q}_k(K) := \mathbb{P}_{k-1}(K).$

Globally, the pressure space is defined as

$\mathcal{Q}_h^k := \{l \in L_0^2(\Omega) \text{ s.t. } l|_K \in \mathcal{Q}_k(K) \quad \text{for all } K \in \mathcal{T}_h\}.$

Moreover, for the local pressure $q \in \mathcal{Q}_k(K)$, we consider the linear operator D_Q, which is the moment upto order $k-1$ of q, i.e.

$$\int_K q\ p_{k-1} dK \quad \text{for all} \quad p_{k-1} \in \mathbb{P}_{k-1}(K).$$

It can be easily observed that $\mathcal{Q}_k(K)$ is unisolvent w.r.t. D_Q.

4 Discrete Virtual Element Formulation

In this section, we construct the discrete virtual element formulation of the weak formulation (2.2). Upon employing the L^2 projection operator $\Pi_{k,K}^0$, we approximate the non-stationary part (u_t, v). Th local discrete formulations consist of two parts: a polynomial part and a non-polynomial part. The global formulation is an additive sum of the local formulation on each polygon, K. For $\mathbf{u}, \mathbf{v} \in [L_0^2(K)]^2$, we define the inner-product

$$m^K(\mathbf{u}, \mathbf{v}) := \int_K \mathbf{u} \cdot \mathbf{v}\, \mathrm{d}K,$$

Employing the inner-product $m^K(\cdot,\cdot)$, we define the local discrete formulation for the time derivative part as

$$m_h^K(\mathbf{u}_h, \mathbf{v}_h) := m^K(\Pi_{k,K}^0 \mathbf{u}_h,\ \Pi_{k,K}^0 \mathbf{v}_h) + \mathcal{S}_m^K\Big((I - \Pi_{k,K}^0)\mathbf{u}_h,\ (I - \Pi_{k,K}^0)\mathbf{v}_h\Big),$$

where $\mathcal{S}_m^K\Big(\cdot,\cdot\Big) : \mathcal{W}^k(K) \times \mathcal{W}^k(K) \to \mathbb{R}$ is a symmetric bilinear form that stabilizes the bilinear form $m_h^K(\cdot,\cdot)$ satisfying

$$\beta_*\, m^K(\mathbf{v}_h, \mathbf{v}_h) \le \mathcal{S}_m^K(\mathbf{v}_h, \mathbf{v}_h) \le \beta^* m^K(\mathbf{v}_h, \mathbf{v}_h) \quad \text{for all} \quad \mathbf{v}_h \in \ker(\Pi_{k,K}^0), \tag{4.1}$$

where $\ker(\Pi_{k,K}^0) \subset \mathcal{W}^k(K)$ denotes kernel of $\Pi_{k,K}^0$. Furthermore, employing the energy projection operator $\Pi_{k,K}^\nabla$, we discretize the bilinear form $a(\mathbf{u}, \mathbf{v})$ as

$$a_h^K(\mathbf{u}_h, \mathbf{v}_h) := a^K(\Pi_{k,K}^\nabla \mathbf{u}_h, \Pi_{k,K}^\nabla \mathbf{v}_h) + \mathcal{S}_a^K\Big((I - \Pi_{k,K}^\nabla)\mathbf{u}_h, (I - \Pi_{k,K}^\nabla)\mathbf{v}_h\Big),$$

where $\mathbf{u}_h, \mathbf{v}_h \in \mathcal{W}^k(K)$ and the stabilizer $\mathcal{S}_a^K\Big(\cdot,\cdot\Big) : \mathcal{W}^k(K) \times \mathcal{W}^k(K) \to \mathbb{R}$ satisfies

$$\alpha_*\, a^K(\mathbf{v}_h, \mathbf{v}_h) \le \mathcal{S}_a^K(\mathbf{v}_h, \mathbf{v}_h) \le \alpha^* a^K(\mathbf{v}_h, \mathbf{v}_h) \quad \text{for all} \quad \mathbf{v}_h \in \ker(\Pi_{k,K}^\nabla). \tag{4.2}$$

$\alpha_*, \alpha^*, \beta_*$ and β^* are positive constants independent of K and h. The global formulation is defined as adding the contribution from the local bilinear form over each polygon K, as:

$$a_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_K a_h^K(\mathbf{u}_h, \mathbf{v}_h),$$
$$m_h(\mathbf{u}_h, \mathbf{v}_h) := \sum_K m_h^K(\mathbf{u}_h, \mathbf{v}_h).$$

Stability An application of Eqs. (4.1) and (4.2) yields four positive constants independent of h and K, such that for $\mathbf{v}_h \in \mathcal{W}^k(K)$, it holds

$$\begin{aligned} \min\{\alpha_*, 1\}\, a^K(\mathbf{v}, \mathbf{v}) \ &\le\ a_h^K(\mathbf{v}, \mathbf{v}) \le\ \max\{\alpha^*, 1\}\, a^K(\mathbf{v}, \mathbf{v}); \\ \min\{\beta_*, 1\}\, m^K(\mathbf{v}, \mathbf{v})_K \ &\le\ m_h^K(\mathbf{v}, \mathbf{v}) \le\ \max\{\beta^*, 1\}\, m^K(\mathbf{v}, \mathbf{v}), \end{aligned} \tag{4.3}$$

In order to derive the error estimation for the semi-discrete case in the L^2 and H^1 norm, we require the polynomial consistency property of the discrete bilinear forms $a_h^K(\cdot,\cdot)$ and $m_h^K(\cdot,\cdot)$ which can be stated as

Lemma 4.1 *For all polygonal element $K \in \mathcal{T}_h$, the bilinear form $m_h^K(\cdot,\cdot)$ and $a_h^K(\cdot,\cdot)$ satisfy the following consistency property*

$$a_h^K(\mathbf{q}_k, \mathbf{v}_h) = a^K(\mathbf{q}_k, \mathbf{v}_h),$$
$$m_h^K(\mathbf{q}_k, \mathbf{v}_h) = m^K(\mathbf{q}_k, \mathbf{v}_h),$$

for all $\mathbf{q}_k \in [\mathbb{P}_k(K)]^2$ and $\mathbf{v}_h \in \mathcal{W}_h^k(K)$.

Proof Since $\int_K (\nabla \Pi_{k,K}^{\nabla} \mathbf{v}_h - \nabla \mathbf{v}_h) : \nabla \mathbf{q}_k = 0$ for all $\mathbf{q}_k \in \mathbb{P}_k(K)$, we have $\mathcal{S}_a^K(\mathbf{q}_k, \mathbf{v}_h) = 0$. Similarly, applying definition of L^2 projection operator $\Pi_{k,K}^0$, we deduce that $\mathcal{S}_m^K(\mathbf{q}_k, \mathbf{v}_h) = 0$. Moreover, both the operators are identity on polynomial space $[\mathbb{P}_k(K)]^2$, i.e.,
$\Pi_{k,K}^{\nabla}[\mathbb{P}_k(K)]^2 = [\mathbb{P}_k(K)]^2$ and $\Pi_{k,K}^0[\mathbb{P}_k(K)]^2 = [\mathbb{P}_k(K)]^2$ which gives the final thesis. □

Approximation of the right-hand side term $(\mathbf{f}_h, \mathbf{v}_h)$ Now, we discuss the discretization of the right-hand side load term $(\mathbf{f}, \mathbf{v})$, where $\mathbf{f}$ denotes the force function. For all $K \in \mathcal{T}_h$, exploiting the $\Pi_{k,K}^0$ operator, we approximate the load term $\mathbf{f}_h$ as

$$\mathbf{f}_h|_K := \Pi_{k,K}^0 \mathbf{f}, \quad \text{for all } K \in \mathcal{T}_h.$$

Hence, utilizing the orthogonality property of $\Pi_{k,K}^0$ operator, we can recast the right hand side as:

$$(\mathbf{f}_h, \mathbf{v}_h) := \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f}_h \cdot \mathbf{v}_h \, \mathrm{d}K = \sum_{K \in \mathcal{T}_h} \int_K \Pi_{k,K}^0 \mathbf{f} \cdot \mathbf{v}_h \, \mathrm{d}K = \sum_{K \in \mathcal{T}_h} \int_K \mathbf{f} \cdot \Pi_{k,K}^0 \mathbf{v}_h \, \mathrm{d}K.$$

In contrast with the FEM, the convergence analysis depends on the regularity of the force function $\mathbf{f}$. An approximation property of the projection operator $\Pi_{k,K}^0$ ensures optimal order of convergence. On $\mathcal{W}_h^k$, the projection operator $\Pi_{k,K}^0$ is fully computable. Moreover, $\Pi_{k,K}^0 \mathbf{v}_h$ can be written in terms of a polynomial. Hence, the right-hand side reduces to integration of a known function, which can be evaluated by applying appropriate quadrature rule.

The semidiscrete variational formulation of (2.2) is defined as follows: find $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ s.t.

$$\begin{cases} m_h(\partial_t \mathbf{u}_h, \mathbf{v}_h) + a_h(\mathbf{u}_h, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h), & \forall\, v_h \in \mathcal{W}_h^k \\ \qquad b(\mathbf{u}_h, q_h) = 0 \quad \forall\, q_h \in \mathcal{Q}_h^k. \end{cases} \tag{4.4}$$

Moreover, the pair $(\mathcal{W}_h^k, \mathcal{Q}_h^k)$ satisfies discrete inf-sup condition. Since p_h is a polynomial over the element K, the term $b(\mathbf{v}_h, p_h)$ is computable from the degrees of freedom.

Lemma 4.2 *The family of virtual element spaces $\{(\mathcal{W}_h^k, \mathcal{Q}_h^k)\}_{h>0}$, $k \geq 2$, satisfies the discrete inf-sup condition, i.e., there exists a positive constant $\mathcal{B} > 0$ such that the following holds*

$$\mathcal{B}\, \|\mathbf{q}_h\|_{\mathcal{Q}} \leq C \sup_{\substack{\mathbf{w}_h \in \mathcal{W}_h^k \\ \mathbf{w}_h \neq 0}} \frac{b(\mathbf{w}_h, \mathbf{q}_h)}{\|\mathbf{w}_h\|_{\mathcal{V}}} \quad \forall \quad \mathbf{q}_h \in \mathcal{Q}_h^k. \tag{4.5}$$

Proof See [8] for details. □

Remark 4.1 The velocity-pressure space considered here is a canonical extension of P_k/P_{k-1} to the virtual element method. This space is discrete inf-sup stable for $k \geq 2$. We derive the a priori error estimates for discrete inf-sup stable case.

5 Convergence Analysis

In this section, we perform the error estimations in the L^2 and H^1 norms employing the discrete Stokes projection operator (Π_h^s, Π_h^p). On the virtual element space $\mathcal{W}_h^k$, we have the following approximation property.

Lemma 5.1 *For all h, let $K \in \mathcal{T}_h$ and k be a natural number. Then, for all $\mathbf{w} \in [H^{m+1}(K)]^2$ where $0 \leq m \leq k$, there exists a polynomial function $\mathbf{u}_\pi \in [\mathbb{P}_k(K)]^2$, such that*

$$\|\mathbf{u} - \mathbf{u}_\pi\|_{0,K} + h_K\ |\mathbf{u} - \mathbf{u}_\pi|_{1,K} \leq\ C\ h_K^{m+1}\ |\mathbf{u}|_{m+1,K}.$$

Proof The result follows from classical result by Scott–Dupont [14]. □

Moreover, since the pair $(\mathcal{W}_h^k, \mathcal{Q}_h^k)$ satisfies the discrete inf-sup condition, proceeding analogously as [8], we can prove the the following result.

Lemma 5.2 *For each enough regular $\mathbf{u} \in \mathcal{V}$, there exists $\mathbf{u}_I \in \mathcal{W}_h^k$, s.t. the following condition holds*

$$\Pi_{k-1,K}^0(\nabla \cdot \mathbf{u}_I) = \Pi_{k-1,K}^0(\nabla \cdot \mathbf{u})\ \forall\ K \in \mathcal{T}_h,$$
$$\|\mathbf{u} - \mathbf{u}_I\|_{\mathcal{V}} \leq\ C \inf_{\mathbf{v}_h \in \mathcal{W}_h^k} \|\mathbf{u} - \mathbf{v}_h\|_{\mathcal{V}}.$$

Proof See [8] for details. □

Next, we define the discrete Stokes projection operator which will be utilized to derive the error estimate for the velocity and the pressure here and the text below.

Discrete Stokes projection Let $(\mathbf{u}, p)$ be the solution of model problem (2.1), then discrete Stokes projection is defined as $(\Pi_h^s\mathbf{u}, \Pi_h^p p) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$

$$\begin{aligned} a_h(\Pi_h^s\mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, \Pi_h^p p) &= a(\mathbf{u}, \mathbf{v}_h) - b(\mathbf{v}_h, p)\ \forall\ \mathbf{v}_h \in \mathcal{W}_h^k \\ b(\Pi_h^s\mathbf{u}, q_h) &= b(\mathbf{u}, q_h)\ \forall\ q_h \in \mathcal{Q}_h^k. \end{aligned} \tag{5.1}$$

For each $\mathbf{u}$, the approximation $\Pi_h^s\mathbf{u} \in \mathcal{W}_h^k$ converges with an optimal order in the L^2 and the H^1 norm. An application of Lemma 5.2, and the discrete inf-sup condition yields the following result.

Lemma 5.3 *Let $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies Eq. (2.1) and $(\Pi_h^s\mathbf{u},\ \Pi_h^p p)$ be the discrete Stokes projection. Then, the following estimations hold*

$$\|\mathbf{u} - \Pi_h^s\mathbf{u}\|_0 + h\ |\mathbf{u} - \Pi_h^s\mathbf{u}|_1 \leq\ C\ h^{k+1}\ \Big(|\mathbf{u}|_{k+1} + |p|_k\Big),$$
$$\|p - \Pi_h^p p\|_0 \leq\ C\ h^k\ \Big(|\mathbf{u}|_{k+1} + |p|_k\Big).$$

Proof We split the error $\mathbf{u} - \Pi_h^s\mathbf{u}$ as follows

$$\mathbf{u} - \Pi_h^s\mathbf{u} = \mathbf{u} - \mathbf{u}_I + \mathbf{u}_I - \Pi_h^s\mathbf{u},$$

where $\mathbf{u}_I$ be the Fortin operator defined in Lemma 5.2. Let $\eta = \Pi_h^s\mathbf{u} - \mathbf{u}_I$ be an element of $\mathcal{W}_h^k$. Hence, employing discrete coercivity of $a_h(\cdot, \cdot)$, we have

$$\begin{aligned} |\eta|_1^2 &\leq C(\alpha_*) a_h(\Pi_h^s\mathbf{u} - \mathbf{u}_I, \eta) \\ &= a_h(\Pi_h^s\mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) \\ &= a(\mathbf{u}, \eta) + b(\eta, \Pi_h^p p - p) - a_h(\mathbf{u}_I, \eta) \\ &= a(\mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) + b(\eta, \Pi_h^p p - p_\pi + p_\pi - p) \\ &= a(\mathbf{u}, \eta) - a_h(\mathbf{u}_I, \eta) + b(\eta, p_\pi - p) \\ &\leq\ C\ h^k\ |\mathbf{u}|_{k+1}\ |\eta|_1 + C\ h^k\ |p|_k\ |\eta|_1. \end{aligned}$$

where p_π, interpolant of p on discrete space, we have

$|\eta|_1 \leq C\, h^k\, (|\mathbf{u}|_{k+1} + |p|_k)$.

Together with the estimation $|\mathbf{u} - \mathbf{u}_I|_1 \leq\ C\, h^k\, |\mathbf{u}|_{k+1}$, we deduce

$$|\mathbf{u} - \Pi_h^s \mathbf{u}|_1 \leq C\, h^k \Big(|\mathbf{u}|_{k+1} + |p|_k\Big). \tag{5.2}$$

Now, we proceed to estimate $\| p - \Pi_h^p p\|_0$. Let q_h be an arbitrary element of $\mathcal{Q}_h^k$, then from discrete inf-sup condition (Lemma 4.2), we have

$$\begin{aligned}
\beta\, \|\Pi_h^p p - q_h\|_0 &\leq \sup_{\mathbf{v}_h \in V_h \setminus \{0\}} \frac{b(\mathbf{v}_h, \Pi_h^p p - q_h)}{\|\mathbf{v}_h\|_1} \\
&= \sup_{\mathbf{v}_h \in V_h \setminus \{0\}} \frac{b(\mathbf{v}_h, \Pi_h^p p - p) + b(\mathbf{v}_h, p - q_h)}{\|\mathbf{v}_h\|_1}.
\end{aligned}$$

An application of definition of Stokes projection [Eq. (5.1)] helps to derive

$$\begin{aligned}
b(\mathbf{v}_h, \Pi_h^p p - p) &= a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h) \\
&= \sum_{K \in \mathcal{T}_h} \Big(a_h^K(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h) + a^K(\mathbf{u}_\pi - \mathbf{u}, \mathbf{v}_h) \Big).
\end{aligned}$$

With the help of estimation (5.2) and Lemma 5.1, we can write

$$b(\mathbf{v}_h, \Pi_h^p p - p) \leq C\, h^k\, (|\mathbf{u}|_{k+1} + |p|_k)\, |\mathbf{v}_h|_1. \tag{5.3}$$

Therefore, the error $\|p - \Pi_h^p p\|$ can be rewritten as

$$\|p - \Pi_h^p p\|_0 \leq C\, h^k\, (|\mathbf{u}|_{k+1} + |p|_k) + \inf_{q_h \in \mathcal{Q}_h^k} \|p - q_h\|. \tag{5.4}$$

In the above expression, if we choose $q_h = p_\pi$, where p_π is the standard interpolation operator, we obtain the required results. In order to estimate $\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0$, we apply the duality argument. In this direction, we consider the following auxiliary problem

$$\begin{aligned}
-\Delta \boldsymbol{\Psi} + \nabla r &= \mathbf{u} - \Pi_h^s \mathbf{u} \\
\nabla \cdot \boldsymbol{\Psi} &= 0.
\end{aligned}$$

where $\boldsymbol{\Psi}$ and r belongs to $[H_0^1(\Omega)]^2 \cap [H^2(\Omega)]^2$ and $L_0^2(\Omega) \cap H^1(\Omega)$, respectively. Moreover, we have the following regularity result

$$\|\boldsymbol{\Psi}\|_2 + |r|_1 \leq C\, \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0.$$

We have

$$\begin{aligned}
\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0^2 &= a(\boldsymbol{\Psi}, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) \\
&= a(\boldsymbol{\Psi} - \boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) + a(\boldsymbol{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r).
\end{aligned} \tag{5.5}$$

The first term of estimation (5.5), can be estimated as

$$a(\mathbf{\Psi} - \mathbf{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) \leq \ C\ h^{k+1}\ \|\mathbf{\Psi}\|_2 \left(|\mathbf{u}|_{k+1} + |p|_k\right).$$

The other two terms of estimation (5.5) can be bounded as

$$\begin{aligned} a(\mathbf{\Psi}_I, \mathbf{u} - \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) &= a(\mathbf{\Psi}_I, \mathbf{u}) - a(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}) + a_h(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}) \\ &\quad - a_h(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r). \end{aligned}$$

We denote

$$H_1 = a_h(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}) - a(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}),$$

and

$$H_2 = a(\mathbf{\Psi}_I, \mathbf{u}) - a_h(\mathbf{\Psi}_I, \Pi_h^s \mathbf{u}) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r).$$

An application of definition of projection operators implies

$$\begin{aligned} H_2 &= b(\mathbf{\Psi}_I, p) - b(\mathbf{\Psi}_I, \Pi_h^p p) - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r) \\ &= b(\mathbf{\Psi}_I - \mathbf{\Psi}, p - \Pi_h^p p) + \underbrace{b(\mathbf{\Psi}, p - \Pi_h^p p)}_{0} - b(\mathbf{u} - \Pi_h^s \mathbf{u}, r - r_\pi). \end{aligned}$$

Employing Lemma 5.2 and estimation (5.4), we have

$$\begin{aligned} |H_2| &\leq |\mathbf{\Psi}_I - \mathbf{\Psi}|_1\ \|p - \Pi_h^p p\|_0 + |\mathbf{u} - \Pi_h^s \mathbf{u}|_1\ \|r - r_\pi\|_0 \\ &\leq\ C\ h^{k+1} \left(|\mathbf{u}|_{k+1} + |p|_k\right) \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0. \end{aligned} \tag{5.6}$$

An application of polynomial consistency property of $a_h(\cdot,\cdot)$ and Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} a_h(\Pi_h^s \mathbf{u}, \mathbf{\Psi}_I) - a(\Pi_h^s \mathbf{u}, \mathbf{\Psi}_I) &= \sum_{K \in \mathcal{T}_h} \left(a_h^K(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{\Psi}_I - \mathbf{\Psi}_\pi) - a(\Pi_h^s \mathbf{u} - \mathbf{u}_\pi, \mathbf{\Psi}_I - \mathbf{\Psi}_\pi)\right) \\ &\leq\ C\ |\Pi_h^s \mathbf{u} - \mathbf{u}_\pi|_1\ |\mathbf{\Psi}_I - \mathbf{\Psi}_\pi| \\ &\leq\ C\ h^{k+1} \left(|\mathbf{u}|_{k+1} + |p|_k\right) \|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0. \end{aligned} \tag{5.7}$$

Substituting the estimations (5.6) and (5.7) in (5.5), we obtain

$$\|\mathbf{u} - \Pi_h^s \mathbf{u}\|_0 \leq\ C\ h^{k+1}\left(|\mathbf{u}|_{k+1} + |p|_k\right). \tag{5.8}$$

An application of (5.8) and (5.2) yield the desired thesis. □

5.1 Error Estimation for the Velocity

Optimal L^2-error estimation

Theorem 5.4 *Let* $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ *satisfies* (2.1) *and* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ *satisfies* (4.4). *Then the following estimation holds*

$$\|(\mathbf{u}-\mathbf{u}_h)(t)\|_0 \leq \|(\mathbf{u}-\mathbf{u}_h)(0)\|_0 + C\, h^{k+1} \Big(|\mathbf{u}_0|_{k+1} + |\mathbf{u}(t)|_{k+1} + |p(t)|_k + \|\mathbf{f}\|_{L^2(0,T;[H^{k+1}(\Omega)]^2)} + \|\mathbf{u}_t\|_{L^2(0,T;[H^{k+1}(\Omega)]^2)}\Big).$$

Proof In order to estimate $\|\mathbf{u}-\mathbf{u}_h\|_0$, we split the term with the help of Stokes projection $\Pi_h^s \mathbf{u}$.

$$\mathbf{u}-\mathbf{u}_h = \mathbf{u} - \Pi_h^s\mathbf{u} + \Pi_h^s\mathbf{u} - \mathbf{u}_h.$$

The estimation of $\|\mathbf{u}-\Pi_h^s\mathbf{u}\|_0$ is known from Lemma 5.3. Now, we proceed to estimate $\boldsymbol{\zeta} = \mathbf{u}_h - \Pi_h^s\mathbf{u}$. Putting $\boldsymbol{\zeta}$ in the semi-discrete approximation (4.4), we get

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s\mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s\mathbf{u}, \mathbf{v}_h). \tag{5.9}$$

We choose $\mathbf{v}_h = \boldsymbol{\zeta}$ in estimation (5.9) which reduces to

$$m_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}) + a_h(\boldsymbol{\zeta}, \boldsymbol{\zeta}) - b(\boldsymbol{\zeta}, p_h) = (\mathbf{f}_h, \boldsymbol{\zeta}) - m_h(\Pi_h^s\mathbf{u}_t, \boldsymbol{\zeta}) - a_h(\Pi_h^s\mathbf{u}, \boldsymbol{\zeta}). \tag{5.10}$$

Since $b(\boldsymbol{\zeta}, q_h) = 0$ for all $q_h \in Q_h$ and the discrete bilinear form $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ satisfies stability property (4.3), estimation (5.10) reduces to

$$\frac{d}{dt}\|\boldsymbol{\zeta}\|_0^2 + |\boldsymbol{\zeta}|_1^2 \leq C_1 \underbrace{\Big((\mathbf{f}_h, \boldsymbol{\zeta}) - (\mathbf{f}, \boldsymbol{\zeta})\Big)}_{T_1} + C_2 \underbrace{\Big(-m_h(\Pi_h^s\mathbf{u}_t, \boldsymbol{\zeta}) + (\mathbf{u}_t, \boldsymbol{\zeta})\Big)}_{T_2}. \tag{5.11}$$

Exploiting an approximation property of the L^2 projection operator $\Pi_{k,K}^0$ and Cauchy–Schwarz inequality, we have

$$\begin{aligned} |T_1| &= |(\mathbf{f}_h, \boldsymbol{\zeta}) - (\mathbf{f}, \boldsymbol{\zeta})| \\ &\leq \sum_K |(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta})_K| \\ &\leq \sum_K \|\Pi_{k,K}^0\mathbf{f} - \mathbf{f}\|_K \ \|\boldsymbol{\zeta}\|_K \\ &\leq C \sum_K h_K^{k+1}|\mathbf{f}|_{k+1,K} \ \|\boldsymbol{\zeta}\|_{0,K} \\ &\leq C\, h^{k+1}\ |\mathbf{f}|_{k+1}\ \|\boldsymbol{\zeta}\|. \end{aligned} \tag{5.12}$$

Non-stationary part can be estimated with the help of polynomial consistency property of bilinear form $m_h(\cdot,\cdot)$

$$\begin{aligned} |T_2| &= |-m_h(\Pi_h^s\mathbf{u}_t, \boldsymbol{\zeta}) + (\mathbf{u}_t, \boldsymbol{\zeta})| \\ &\leq \sum_K |m_h^K(\Pi_h^s\mathbf{u}_t - \Pi_{k,K}^0\mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0\mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K| \\ &\leq C \sum_K \Big(|m_h^K(\Pi_h^s\mathbf{u}_t - \mathbf{u}_t + \mathbf{u}_t - \Pi_{k,K}^0\mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0\mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K|\Big) \\ &\leq C \sum_K \Big(|m_h^K(\Pi_h^s\mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})| + |m_h^K(\mathbf{u}_t - \Pi_{k,K}^0\mathbf{u}_t, \boldsymbol{\zeta})| + |(\Pi_{k,K}^0\mathbf{u}_t - \mathbf{u}_t, \boldsymbol{\zeta})_K|\Big) \\ &\leq C \sum_K \|\Pi_{k,K}^0\mathbf{u}_t - \mathbf{u}_t\|_{0,K}\ \|\boldsymbol{\zeta}\|_{0,K} \\ &\leq C\, h^{k+1}\ |\mathbf{u}_t|_{k+1}\ \|\boldsymbol{\zeta}\|_{0,K}. \end{aligned} \tag{5.13}$$

Inserting (5.12) and (5.13) into (5.11), we have

$$\|\boldsymbol{\zeta}\|_0 \, \frac{d}{dt}\|\boldsymbol{\zeta}\|_0 + |\boldsymbol{\zeta}|_1^2 \leq \; C\, h^{k+1}\Big(|\mathbf{f}|_{k+1} + |\mathbf{u}_t|_{k+1}\Big)\, \|\boldsymbol{\zeta}\|_0.$$

Since the term $|\boldsymbol{\zeta}|_1^2$ is positive, we can estimate as

$$\frac{d}{dt}\|\boldsymbol{\zeta}\|_0 \leq \; C\, h^{k+1}\Big(|\mathbf{f}|_{k+1} + |\mathbf{u}_t|_{k+1}\Big). \tag{5.14}$$

Taking integration from 0 to t, we have

$$\|\boldsymbol{\zeta}(t)\|_0 \leq \; \|\boldsymbol{\zeta}(0)\|_0 + C\, h^{k+1}\, \Big(|\mathbf{f}|_{L^2(0,T;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_t|_{L^2(0,T;[H^{k+1}(\Omega)]^2)}\Big). \tag{5.15}$$

Moreover, we have

$$\|\boldsymbol{\zeta}(0)\| \leq \|\mathbf{u}(0) - \mathbf{u}_h(0)\|_0 + \; C\, h^{k+1}\, |\mathbf{u}(0)|_{H^{k+1}(\Omega)}. \tag{5.16}$$

Inserting (5.16) into (5.15) and with the help of Lemma 5.3, we obtain final estimation

$$\begin{aligned}\|(\mathbf{u}-\mathbf{u}_h)(t)\| \leq \|(\mathbf{u}-\mathbf{u}_h)(0)\| + \; C\, h^{k+1}\, \Big(&|\mathbf{u}(0)|_{H^{k+1}(\Omega)} + |\mathbf{u}(t)|_{k+1} + |p(t)|_k \\ &+ |\mathbf{f}|_{L^2(0,T,[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_t|_{L^2(0,T,[H^{k+1}(\Omega)]^2)}\Big).\end{aligned}$$

□

Next, we move to the convergence analysis in the H^1-norm.

Theorem 5.5 *Let $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies (2.1) and $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ satisfies (4.4). Moreover, we assume that $\mathbf{f} \in L^2(0,T;[H^{k+1}(\Omega)]^2)$, $\mathbf{u} \in L^2(0,T;[H^{k+1}(\Omega)]^2)$ and $\mathbf{u}_t \in L^2(0,T;[H^{k+1}(\Omega)]^2)$. Then there exists a generic constant C such that the following estimation holds*

$$\begin{aligned}|\mathbf{u}(t) - \mathbf{u}_h(t)|_1 \leq \; &|\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + C\, h^k\Big(|\mathbf{u}(0)|_{k+1} + |\mathbf{u}(t)|_{k+1} + |p(t)|_k\Big) \\ &+ \; C\, h^{k+1}\, \Big(|\mathbf{f}|_{L^2(0,T;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_t|_{L^2(0,T;[H^{k+1}(\Omega)]^2)}\Big).\end{aligned}$$

Proof We review the following equation

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h). \tag{5.17}$$

The discrete Stokes projection operator Π_h^s defined in Eq. (5.1) commutes with time-derivative. Hence, we have the result

$$\boldsymbol{\zeta}_t = \frac{d}{dt}(\mathbf{u}_h - \Pi_h^s \mathbf{u}) = \mathbf{u}_{ht} - \Pi_h^s\, \mathbf{u}_t.$$

Choosing $\mathbf{v}_h = \boldsymbol{\zeta}_t$ in (5.17) and since $b(\boldsymbol{\zeta}_t, q_h) = 0$ for all $q_h \in Q_h$, we have

$$m_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}_t) + a_h(\boldsymbol{\zeta}, \boldsymbol{\zeta}_t) = (\mathbf{f}_h, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) - a_h(\Pi_h^s \mathbf{u}, \boldsymbol{\zeta}_t).$$

Exploiting stability property of discrete bilinear form $m_h(\cdot,\cdot)$ and $a_h(\cdot,\cdot)$ revealed in (4.3) and making use that time derivative which commutes with discrete bilinear form $a_h(\cdot,\cdot)$, we have

$$\|\boldsymbol{\zeta}_t\|_0^2 + \frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\,(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) + (\mathbf{u}_t, \boldsymbol{\zeta}_t). \tag{5.18}$$

An analogous estimation as (5.12) yields the following result

$$\|(\mathbf{f}_h - \mathbf{f}, \boldsymbol{\zeta}_t)\| \leq C\,h^{k+1}\,|\mathbf{f}|_{k+1}\,\|\boldsymbol{\zeta}_t\|_0. \tag{5.19}$$

Exploiting the polynomial consistency , continuity of discrete bilinear form $m_h(\cdot,\cdot)$ and a standard approximation property of projection operator Π_h^s and Cauchy–Schwartz inequality, we bound as

$$|-m_h(\Pi_h^s \mathbf{u}_t, \boldsymbol{\zeta}_t) + (\mathbf{u}_t, \boldsymbol{\zeta}_t)| \leq C\,h^{k+1}\,|\mathbf{u}_t|_{k+1}\,\|\boldsymbol{\zeta}_t\|_0. \tag{5.20}$$

Inserting (5.19) and (5.20) into (5.18), we obtain

$$\|\boldsymbol{\zeta}_t\|_0^2 + \frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\,h^{2k+2}\Big(|\mathbf{f}|_{k+1}^2 + |\mathbf{u}_t|_{k+1}^2\Big).$$

Since $\|\boldsymbol{\zeta}_t\|_0^2$ is positive quantity, we have

$$\frac{1}{2}\frac{d}{dt}|\boldsymbol{\zeta}|_1^2 \leq C\,h^{2k+2}\,\Big(|\mathbf{f}|_{k+1}^2 + |\mathbf{u}_t|_{k+1}^2\Big). \tag{5.21}$$

Integrating the above Eq. (5.21) form 0 to t, we have

$$|\boldsymbol{\zeta}(t)|_1 \leq |\boldsymbol{\zeta}(0)|_1 + C\,h^{k+1}\Big(|\mathbf{f}|_{L^2(0,T;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_t|_{L^2(0,T;[H^{k+1}(\Omega)]^2)}\Big). \tag{5.22}$$

An application of approximation property of Π_h^s operator yields

$$|\boldsymbol{\zeta}(0)|_1 \leq |\mathbf{u}_h(0) - \mathbf{u}(0)|_1 + C\,h^k\,|\mathbf{u}(0)|_{k+1}. \tag{5.23}$$

Inserting (5.23) into (5.22) and exploiting Lemma 5.3, we have

$$\begin{aligned} |\mathbf{u}(t) - \mathbf{u}_h(t)|_1 \leq\ & |\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + C_1\,h^k\Big(|\mathbf{u}(0)|_{k+1} + |\mathbf{u}(t)|_{k+1} + |p(t)|_k\Big) \\ & + C_2\,h^{k+1}\,\Big(|\mathbf{f}|_{L^2(0,T;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_t|_{L^2(0,T;[H^{k+1}(\Omega)]^2)}\Big). \end{aligned}$$

□

Now, we proceed to estimate $\|(\mathbf{u}_t - \mathbf{u}_{ht})(t)\|_0$.

Theorem 5.6 *Let $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies (2.1) and $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ be the corresponding discrete solution satisfying (4.4). Moreover, we assume that $\mathbf{f}_t(t) \in [H^{k+1}(\Omega)]^2$, $\mathbf{u}_{tt}(t) \in [H^{k+1}(\Omega)]^2$ and $\mathbf{u}_t(t) \in [H^{k+1}(\Omega)]^2$ for all $t \in [0, T]$. Then there exists a generic constant C such that the following estimation holds*

$$\begin{aligned} \|(\mathbf{u}_t - \mathbf{u}_{ht})(t)\|_0 \leq\ & \|\mathbf{u}_t(0) - \mathbf{u}_{ht}(0)\|_0 + C\,h^{k+1}\,\Big(|\mathbf{u}_t(0)|_{k+1} + |\mathbf{u}_t(t)|_{k+1} + |p_t(t)|_k\Big) \\ & + C\,h^{k+1}\,\Big(|\mathbf{f}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_{tt}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)}\Big). \end{aligned}$$

Proof We first consider the equation

$$m_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}, \mathbf{v}_h) - b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_t, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u}, \mathbf{v}_h). \tag{5.24}$$

Differentiating Eq. (5.24) with respect to t and since $\mathbf{v}_h \in \mathcal{W}_h^k$ is independent of temporal variable t and Stokes projection Π_h^s commutes with time variable t, we obtain

$$m_h(\boldsymbol{\zeta}_{tt}, \mathbf{v}_h) + a_h(\boldsymbol{\zeta}_t, \mathbf{v}_h) - b(\mathbf{v}_h, p_{ht}) = (\mathbf{f}_{ht}, \mathbf{v}_h) - m_h(\Pi_h^s \mathbf{u}_{tt}, \mathbf{v}_h) - a_h(\Pi_h^s \mathbf{u_t}, \mathbf{v}_h). \tag{5.25}$$

Replacing $\mathbf{v}_h$ by $\boldsymbol{\zeta}_t$ in Estimation (5.25) and since $b(\boldsymbol{\zeta}_t, p_{ht}) = 0$, we acquire

$$\begin{aligned} m_h(\boldsymbol{\zeta}_{tt}, \boldsymbol{\zeta}_t) + a_h(\boldsymbol{\zeta}_t, \boldsymbol{\zeta}_t) &= (\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_{tt}, \boldsymbol{\zeta}_t) - a_h(\Pi_h^s \mathbf{u_t}, \boldsymbol{\zeta}_t). \\ &= (\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - m_h(\Pi_h^s \mathbf{u}_{tt}, \boldsymbol{\zeta}_t) - (\mathbf{f}, \boldsymbol{\zeta}_t) + (\mathbf{u}_{tt}, \boldsymbol{\zeta}_t). \end{aligned} \tag{5.26}$$

Employing approximation property of Stokes projection Π_h^s and Cauchy–Schwarz inequality, we have

$$|(\mathbf{f}_{ht}, \boldsymbol{\zeta}_t) - (\mathbf{f}, \boldsymbol{\zeta}_t)| \leq C\, h^{k+1}\, |\mathbf{f}_t|_{k+1}\, \|\boldsymbol{\zeta}_t\|. \tag{5.27}$$

In view of polynomial consistency property of $m_h(\cdot, \cdot)$, standard approximation property of Stokes projection Π_h^s and L^2 projection $\Pi_{k,K}^0$, Cauchy–Schwarz inequality, we have

$$|m_h(\Pi_h^s u_{tt}, \boldsymbol{\zeta}_t) - (\mathbf{u}_{tt}, \boldsymbol{\zeta}_t)| \leq C\, h^{k+1}\, |\mathbf{u}_{tt}|_{k+1}\, \|\zeta_t\|. \tag{5.28}$$

Inserting (5.27) and (5.28) into (5.26), we get

$$\frac{d}{dt}\, \|\boldsymbol{\zeta}_t\|_0^2 + |\boldsymbol{\zeta}_t|_1^2 \leq C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1} \Big)\, \|\zeta_t\|.$$

Furthermore, we utilize stability property of bilinear forms $m_h(\cdot, \cdot)$ and $a_h(\cdot, \cdot)$ in order to derive the above estimation. Since $|\boldsymbol{\zeta}_t|_1^2$ is positive quantity, hence we can neglect this term and we obtain

$$\|\boldsymbol{\zeta}_t\|\, \frac{d}{dt}\, \|\boldsymbol{\zeta}_t\| \leq C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1} \Big) \|\boldsymbol{\zeta}_t\|.$$

Without loss of generality, we assume that $\|\boldsymbol{\zeta}_t\| \neq 0$. Therefore, the above equation reduces to

$$\frac{d}{dt}\, \|\boldsymbol{\zeta}_t\| \leq C\, h^{k+1} \Big(|\mathbf{f}_t|_{k+1} + |\mathbf{u}_{tt}|_{k+1} \Big).$$

A straightforward integration of the above equation from 0 to t implies

$$\|\boldsymbol{\zeta}_t(t)\| \leq \|\boldsymbol{\zeta}_t(0)\| + C\, h^{k+1} \Big(|\mathbf{f}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_{tt}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} \Big). \tag{5.29}$$

Furthermore, an application of approximation property of Stokes projection Π_h^s mentioned in Lemma 5.3 reduces the estimation as

$$\|\mathbf{u}_{ht}(t) - \Pi_h^s \mathbf{u}_t\| \leq C\, \|\mathbf{u}_{ht}(0) - \mathbf{u}_t(0)\| + C\, h^{k+1}\, |\mathbf{u}_t(0)|_{k+1}. \tag{5.30}$$

Utilizing (5.30), Lemma 5.3 and the Estimation (5.29), we obtain the desired result. □

5.2 Error Estimate for the Pressure Variable

Exploiting the operator Π_h^p and discrete inf-sup condition, we exhibit that discrete solution p_h converges optimally in L^2 norm.

Theorem 5.7 *Let* $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ *satisfy* (2.1) *and* $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ *satisfy* (4.4)*. Moreover, we deduce that all assumptions of Theorem* 5.4*, Theorems* 5.5 *and* 5.6 *hold. Then there exists a positive constant C depending on regularity of* $\mathbf{u}$, $\mathbf{u}_t$, $\mathbf{u}_{tt}$, p, p_t, $\mathbf{f}$ *and* $\mathbf{f}_t$ *such that the following estimation holds*

$$\|p - p_h\| \leq C\, h^k.$$

Proof Let $q_h \in \mathcal{Q}_h^k$ be an arbitrary element. Then $(p_h(t) - q_h) \in \mathcal{Q}_h^k$. An application of discrete inf-sup condition (Lemma 4.2) implies that

$$\begin{aligned} \mathcal{B}\,\|p_h(t) - q_h\| &\leq \sup_{\substack{\mathbf{v}_h \in \mathcal{W}_h^k \\ \mathbf{v}_h \neq 0}} \frac{b(\mathbf{v}_h,\ p_h(t) - q_h)}{\|\mathbf{v}_h\|_1} \\ &= \sup_{\substack{\mathbf{v}_h \in \mathcal{W}_h^k \\ \mathbf{v}_h \neq 0}} \frac{b(\mathbf{v}_h,\ p_h(t) - p(t)) + b(\mathbf{v}_h, p(t) - q_h)}{\|\mathbf{v}_h\|_1}. \end{aligned} \tag{5.31}$$

Since $(\mathbf{u}, p) \in \mathcal{V} \times \mathcal{Q}$ satisfies non-stationary Stokes equation (2.1), we have

$$a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) = (\mathbf{f}, \mathbf{v}) - (\mathbf{u}_t, \mathbf{v}). \tag{5.32}$$

Again, since $(\mathbf{u}_h, p_h) \in \mathcal{W}_h^k \times \mathcal{Q}_h^k$ satisfies the discrete equation (4.4), we obtain

$$a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) = (\mathbf{f}_h, \mathbf{v}_h) - m_h(\mathbf{u}_{ht}, \mathbf{v}_h). \tag{5.33}$$

Replacing $\mathbf{v}$ by $\mathbf{v}_h$ in (5.32) and then subtracting (5.33) form (5.32), we have

$$b(\mathbf{v}_h, p - p_h) = \underbrace{(\mathbf{f} - \mathbf{f}_h, \mathbf{v}_h)}_{=:H_1} + \underbrace{m_h(\mathbf{u}_{ht}, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)}_{=:H_2} + \underbrace{a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h)}_{=:H_3}. \tag{5.34}$$

In order to reduce the cumbersome notation, we split the load term into three parts H_1, H_2 and H_3. Exploiting the approximation property of $\Pi_{k,K}^0$ operator and Cauchy–Schwartz inequality, we estimate that

$$\begin{aligned} |H_1| &= |\sum_K (f - \Pi_{k,K}^0 f, \mathbf{v}_h)_K| \\ &\leq \sum_K \|f - \Pi_{k,K}^0 f\|_{0,K}\ \|\mathbf{v}_h\|_{0,K} \\ &\leq C\, h^k\ |\mathbf{f}|_k\ \|\mathbf{v}_h\|. \end{aligned} \tag{5.35}$$

In order to estimate second term H_2, we proceed as follows

$$
\begin{aligned}
|H_2| &\le \sum_K \left| m_h^K(\mathbf{u}_{ht}, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)_K \right| \\
&\le \sum_K \left| m_h^K(\mathbf{u}_{ht}, \mathbf{v}_h) - m_h^K(\Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) + m_h^K(\Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) - (\mathbf{u}_t, \mathbf{v}_h)_K \right| \\
&\le \sum_K \Big(\left| m_h^K(\mathbf{u}_{ht} - \Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) \right| + \left| (\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}_h)_K \right| \Big) \\
&\le \sum_K \Big(\left| m_h^K(\mathbf{u}_{ht} - \mathbf{u}_t + \mathbf{u}_t - \Pi^0_{k,K}\mathbf{u}_t, \mathbf{v}_h) \right| + \left| (\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t, \mathbf{v}_h)_K \right| \Big) \\
&\le C \sum_K \Big(\|\mathbf{u}_{ht} - \mathbf{u}_t\|_{0,K} \ \|\mathbf{v}_h\|_{0,K} + \|\Pi^0_{k,K}\mathbf{u}_t - \mathbf{u}_t\|_{0,K} \ \|\mathbf{v}_h\|_{0,K} \Big) \\
&\le C \ \|\mathbf{u}_{ht} - \mathbf{u}_t\| \ \|\mathbf{v}_h\| + C \ h^k \ |\mathbf{u}_t|_k \ \|\mathbf{v}_h\|.
\end{aligned}
\tag{5.36}
$$

H_3 can be bounded as

$$
\begin{aligned}
|H_3| &= |a_h(\mathbf{u}_h, \mathbf{v}_h) - a(\mathbf{u}, \mathbf{v}_h)| \\
&\le \sum_K \Big(|a_h^K(\mathbf{u}_h - \mathbf{u} + \mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h)| + |a^K(\mathbf{u} - \mathbf{u}_\pi, \mathbf{v}_h)| \Big) \\
&\le \sum_K \Big(|\mathbf{u}_h - \mathbf{u}|_{1,K} \ |\mathbf{v}_h|_{1,K} + |\mathbf{u}_\pi - \mathbf{u}|_{1,K} \ |\mathbf{v}_h|_{1,K} \Big) \\
&\le C \Big(|\mathbf{u}_h - \mathbf{u}|_1 + h^k \ |\mathbf{u}|_{k+1} \Big) \ |\mathbf{v}_h|_1.
\end{aligned}
\tag{5.37}
$$

The estimation of $|\mathbf{u}_h - \mathbf{u}|_1$ can be evaluated from Theorem 5.5. Substituting the results (5.35), (5.36), and (5.37) into (5.34), we deduce that

$$
\begin{aligned}
\frac{b(\mathbf{v}_h, p - p_h)}{\|\mathbf{v}_h\|_1} &\le C \Big(|\mathbf{u}(0) - \mathbf{u}_h(0)|_1 + \|\mathbf{u}_{ht}(0) - \mathbf{u}_t(0)\| \Big) + C \ h^k \Big(|\mathbf{u}(0)|_{k+1} \\
&+ |\mathbf{u}_t(0)|_{k+1} + |\mathbf{u}(t)|_{k+1} + |p(t)|_k + |p_t(t)|_k + |\mathbf{f}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} \\
&+ |\mathbf{u}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} + |\mathbf{f}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} + |\mathbf{u}_{tt}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} \Big).
\end{aligned}
$$

Considering $\mathbf{u}_h(0) := I_h\mathbf{u}(0)$ and $\mathbf{u}_{ht}(0) := I_h\mathbf{u}_t(0)$, we get

$$
\frac{b(\mathbf{v}_h, p - p_h)}{\|\mathbf{v}_h\|_1} \le C \ h^k, \tag{5.38}
$$

where C is positive generic constant that depends on the regularity of $\mathbf{u}, \ \mathbf{f}, \ \mathbf{u}_t, \ \mathbf{u}_{tt}, \ p, \ p_t$ and $\mathbf{f}_t$. In particular

$$
\begin{aligned}
C = Const\Big(&|\mathbf{u}_t(0)|_{k+1}, \ |\mathbf{u}(t)|_{k+1}, \ |p(t)|_k, \ |p_t(t)|_k, |\mathbf{f}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)}, \\
&|\mathbf{u}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)}, \ |\mathbf{f}_t|_{L^2(0,t;[H^{k+1}(\Omega)]^2)}, \ |\mathbf{u}_{tt}|_{L^2(0,t;[H^{k+1}(\Omega)]^2)} \Big).
\end{aligned}
$$

Upon Substituting (5.38) into (5.31), we obtain

$$
\mathcal{B} \ \|p_h(t) - q_h\| \le C \ h^k + \|p - q_h\|.
$$

Again, we have

$$\begin{aligned} \|p - p_h\| &\le C \, \|p - q_h\| + \|p_h - q_h\| \\ &\le C \left(h^k + \|p - q_h\|\right), \end{aligned} \tag{5.39}$$

where q_h is an arbitrary element of $\mathcal{Q}_h^k$. As a consequence, the estimation (5.39) can be recast as

$$\|p - p_h\| \le C \, h^k + \inf_{q_h \in \mathcal{Q}_h^k} \|p - q_h\|,$$

and choosing $q_h = \Pi_h^p p$, and exploiting Lemma 5.3, we have the desired result. □

6 Conclusion

In this work, we have presented of the virtual element method for the time dependent Stokes equation. The VEM space presented here is discrete inf-sup stable for $k \ge 2$ and H^1 conforming space. Since the present VEM space $(\mathcal{W}_h^k, \mathcal{Q}_h^k)$ is discrete inf-sup stable, we can construct the Fortin operator that represents the analysis in a simpler form. As required by the theory, we modify the virtual space where we can compute the L^2 projection operator optimally. The primary contribution of this paper is the construction of discrete Stokes projection operator that helps to estimate the error $u - u_h$ in a canonical way. The approximation of the pressure space is same as the finite element method, where the piecewise discontinuous polynomials are employed. It is noted that the primary drawback is that our discrete space is not divergence free. It is further opined that a similar analysis could be carried out for the virtual element space presented in [10], which has an added advantage of being divergence free, which will be a topic for future communications.

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