

NEW STABILITY ESTIMATES FOR AN UNFITTED FINITE ELEMENT METHOD FOR TWO-PHASE STOKES PROBLEM

ERNESTO CÁCERES[†], JOHNNY GUZMÁN[†], AND MAXIM OLSHANSKII[‡]

ABSTRACT. The paper addresses stability and finite element analysis of the stationary two-phase Stokes problem with a piecewise constant viscosity coefficient experiencing a jump across the interface between two fluid phases. We first prove *a priori* estimates for the individual terms of the Cauchy stress tensor with stability constants independent of the viscosity coefficient. Next, this stability result is extended to the approximation of the two-phase Stokes problem by a finite element method. In the method considered, the interface between the phases does not respect the underlying triangulation, which put the finite element method into the class of unfitted discretizations. The finite element error estimates are proved with constants independent of viscosity. Numerical experiments supporting the theoretical results are provided.

1. INTRODUCTION

We are interested in the analysis and a finite element method for the two-phase Stokes problem (also known in the literature as *the Stokes interface problem*). The system of equations is posed in a bounded Lipschitz domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, decomposed in two subdomains (phases) Ω^\pm . The interface Γ between two phases is a closed hypersurface immersed in Ω , i.e., $\Gamma \subset \Omega$ and $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$. We assume Γ is Lipschitz smooth. The Stokes interface problem reads as follows: Given a force field $\mathbf{f} \in L^2(\Omega)^d$, a source term $g \in L^2(\Omega)$, an interface force $\boldsymbol{\lambda} \in L^2(\Gamma)^d$, and viscosity coefficient ν^\pm constant and positive in each subdomain, find the fluid velocity \mathbf{u} and the normalized kinematic pressure p such that

$$(1.1) \quad \begin{aligned} -\operatorname{div}(\nu^\pm D(\mathbf{u})) + \nabla p^\pm &= \mathbf{f}^\pm && \text{in } \Omega^\pm, \\ \operatorname{div} \mathbf{u} &= g && \text{in } \Omega \\ \llbracket \mathbf{u} \rrbracket &= \mathbf{0} && \text{on } \Gamma, \\ \llbracket \sigma(\mathbf{u}, p) \mathbf{n} \rrbracket &= \boldsymbol{\lambda} && \text{on } \Gamma, \\ \mathbf{u} &= \mathbf{0} && \text{on } \partial\Omega, \end{aligned}$$

where $D(\mathbf{u}) := \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^T)$ is the rate-of-strain tensor, $\sigma(\mathbf{u}, p) = \nu D(\mathbf{u}) - p \mathbb{I}$ is the Cauchy stress tensor, and \mathbf{n} is a unit vector on Γ pointing from Ω^+ to Ω^- . For any $f \in L^1(\Omega)$ we use notations f^\pm for the restriction of f on Ω^\pm , i.e., $f^\pm = f|_{\Omega^\pm}$; same convention is used for vector functions. The jumps on the interface are then defined as $\llbracket \sigma(\mathbf{u}, p) \mathbf{n} \rrbracket = \sigma(\mathbf{u}^+, p^+) \mathbf{n} - \sigma(\mathbf{u}^-, p^-) \mathbf{n}$ and $\llbracket \mathbf{u} \rrbracket = \mathbf{u}^+ - \mathbf{u}^-$.

The studies of the Stokes interface problem are motivated by continuum models of two-phase flows. If the fluid is treated as Newtonian incompressible with immiscible phases separated by the

Partially supported by NSF through the Division of Mathematical Sciences grant 1620100 .

Partially supported by NSF through the Division of Mathematical Sciences grant 1717516.

sharp interface, then the system (1.1) is a reasonable model problem for the limit case of highly viscous fluid; see, e.g., [19, 6, 17, 18]. It also appears as an auxiliary problem in numerical simulations of two-phase incompressible flows [8]. According to the continuum surface force model, cf. [3], the effect of interfacial forces, such as the surface tension, are taken into account by using a localized force term at the interface, i.e., $\boldsymbol{\lambda}$ in (1.1).

Problem (1.1) is linear and a standard weak formulation (2.1) renders it as a saddle-point problem, thus yielding the well-posedness result and leading to Galerkin numerical methods; see, e.g., [7, 4]. This textbook analysis, however, does not provide an explicit information on the dependence of the stability and numerical errors estimates on the viscosity coefficient, in particular, on the ratio ν^+/ν^- . This robustness question becomes important if one addresses numerical stability of Galerkin methods, such as the finite element method, for the case of high variation in viscosity coefficient between two phases. The ν -dependence of stability and finite element error estimates for (1.1) have been studied in the literature only recently; see [14, 13, 10, 11]. In those studies, stability and error analysis was done for the natural energy norm of the problem. In particular, under certain further assumptions on Ω^\pm , the *a priori* estimate from [14] (proved there for $g = 0$, $\boldsymbol{\lambda} = 0$) reads

$$(1.2) \quad \|\nu^{\frac{1}{2}} D\mathbf{u}\|_{L^2(\Omega)} + \|\nu^{-\frac{1}{2}} p\|_{L^2(\Omega)} \leq C \|\nu^{-\frac{1}{2}} \mathbf{f}\|_{L^2(\Omega)},$$

with C independent of ν . Note that for *single* phase Stokes problem, a simple scaling argument provides uniform estimates for the quantities $\nu \mathbf{u}$ and p . Similar result does not follow from (1.2) for the velocity and pressure in each of the phases. For example, for $\nu^+ \mathbf{u}^+$ and p^+ the estimate (1.2) yields

$$(1.3) \quad \|\nu^+ D\mathbf{u}^+\|_{L^2(\Omega^+)} + \|p^+\|_{L^2(\Omega^+)} \leq C(\|\mathbf{f}\|_{L^2(\Omega^+)} + \sqrt{\frac{\nu^+}{\nu^-}} \|\mathbf{f}\|_{L^2(\Omega^-)}).$$

We see that the right-hand side blows up for $\nu^- \rightarrow 0$. In the present paper, we prove the following stability result for the solution of (1.1):

$$(1.4) \quad \|\nu D\mathbf{u}\|_{L^2(\Omega)} + \|p\|_{L^2(\Omega)} \leq C(\|\mathbf{f}\|_{L^2(\Omega)} + \|\boldsymbol{\lambda}\|_{L^2(\Gamma)} + \|\nu g\|_{L^2(\Omega)}),$$

The improvement over (1.3) is clear: the re-scaled solution components, $\nu^\pm \mathbf{u}^\pm$ and p^\pm , enjoy uniform estimates in the corresponding subdomains, just as for the single phase problem. The estimate (1.4) can be also seen as *the uniform estimate for the components of Cauchy stress tensor*, an important quantity in practical fluid mechanics.

In the same spirit as (1.4) improves over the energy estimate (1.2), the finite element analysis developed in this paper extends the existing one by deriving robust in ν stability estimates and error estimates for the components of the finite element Cauchy stress tensor. Following [10], for the discretization of (1.1) we consider a geometrically unfitted finite element method known as Nitsche-XFEM or cutFEM. Geometrically unfitted methods use a fixed background mesh which does not respect the position of the interface. The main advantage of unfitted FEM is the relative ease of handling time-dependent domains, implicitly defined interfaces and problems with strong geometric deformations [2]. We prove uniform with respect to ν stability and error estimates for the unfitted FEM. These results hold for a family of bulk LBB-stable finite element Stokes pairs defined on the background mesh. These pairs include $P_{k+1} - P_k$, $k \geq 1$, and $P_{k+d} - P_k^{\text{disc}}$ for $k \geq 0$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and several other elements. We are able to accomplish this by combining ideas from the two papers [9, 5]. In [9] an unfitted FEM for the single phase Stokes

problem was analyzed. We borrow some crucial inf-sup stability estimates from that paper. In [5] similar stability results were proved the Poisson interface problem. The chief tool was to use extension operators in Sobolev spaces. Similarly, here extension operators are essential, however, the pressure terms and the div-free condition add several new difficulties.

We organized the paper in five sections. In section 2 a notion of the weak solution is introduced and estimate (1.3) is proved. Section 3 describes the finite element method and proves the analogue of (1.3) for the finite element solution. In section 4 a ν -independent optimal order error estimate is proved. Finally, a few illustrative results of numerical experiments are given in section 5.

2. *A priori* ANALYSIS FOR (1.1)

2.1. Preliminaries and problem setting. We introduce a variational formulation of (1.1) and some notations to be used throughout the paper. For an open set $\mathcal{O} \subset \mathbb{R}^d$ denote by $(\cdot, \cdot)_{\mathcal{O}}$ the L^2 inner product in \mathcal{O} , and by $\|\cdot\|_{\mathcal{O}}$ the corresponding norm. For the mixed variational formulation of (1.1), we set $\mathbf{V} := \mathbf{H}_0^1(\Omega)$, where $\mathbf{H}_0^1(\Omega) = [H_0^1(\Omega)]^d$ for the space the vector field \mathbf{u} belongs to, whereas for the pressure p we set $M = L_0^2(\Omega)$, with $L_0^2(\mathcal{O}) = \{p \in L^2(\mathcal{O}) : (p, 1)_{\mathcal{O}} = 0\}$. We let $\|\cdot\|_{1,\mathcal{O}}$ denote the $H^1(\mathcal{O})$ -norm. The norm of \mathbf{V}^* , the dual of \mathbf{V} , is denoted by $\|\cdot\|_{-1}$, and $\langle \cdot, \cdot \rangle_{-1}$ denotes the pairing, with respect to the L^2 -duality.

We consider the abstract mixed formulation: Find $(\mathbf{u}, p) \in \mathbf{V} \times M$ such that

$$(2.1) \quad \begin{aligned} a(\mathbf{u}, \mathbf{v}) + b(\mathbf{v}, p) &= \langle \hat{\mathbf{f}}, \mathbf{v} \rangle_{-1} & \forall \mathbf{v} \in \mathbf{V}, \\ b(\mathbf{u}, q) &= -(g, q)_{\Omega} & \forall q \in M, \end{aligned}$$

where

$$a(\mathbf{u}, \mathbf{v}) := (\nu D(\mathbf{u}), D(\mathbf{v}))_{\Omega}, \quad b(\mathbf{v}, q) := -(\operatorname{div} \mathbf{v}, q)_{\Omega}, \quad \text{and } \hat{\mathbf{f}} \in \mathbf{V}^*.$$

The problem (2.1) is the weak formulation of the Stokes interface equation (1.1) if we let

$$(2.2) \quad \langle \hat{\mathbf{f}}, \mathbf{v} \rangle_{-1} := \int_{\Omega} \mathbf{f} \cdot \mathbf{v} + \int_{\Gamma} \boldsymbol{\lambda} \cdot \mathbf{v}.$$

2.2. Stability estimates for the weak solution. In this section, we analyze the variational formulation (2.1) of the Stokes interface problem (1.1). We are interested in the following stability result for the solution $(\mathbf{u}, p) \in \mathbf{V} \times M$ of (2.1):

$$(2.3) \quad \|\nu D(\mathbf{u})\|_{\Omega} + \|p\|_{\Omega} \leq C(\|\hat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}),$$

for some constant $C > 0$ independent of ν and depending only on Ω and Γ . A standard energy norm argument gives estimates for $\|\sqrt{\nu} D(\mathbf{u})\|_{\Omega}$ with a the constant C on the right hand side dependent on ν . Without any loss of generality we shall always assume $\nu^- \leq \nu^+$. The key to get the improved result (2.3) is using an energy argument to estimate $\|\nu^- D(\mathbf{u})\|_{\Omega^-}$, and then using an extension operator to estimate $\|\nu^+ D(\mathbf{u})\|_{\Omega^+}$. This strategy is similar to the one taken in [5] for the Poisson interface problem. However, here the pressure term and the divergence condition require careful treatment.

In what follows, for an open set \mathcal{O} and a function $q \in L^2(\mathcal{O})$, we denote its average over \mathcal{O} by the expression $\operatorname{avg}_{\mathcal{O}}(q) = |\mathcal{O}|^{-1}(q, 1)_{\mathcal{O}}$. We start by proving the following results.

Lemma 2.1. *Let $\mathbf{u} \in \mathbf{V}$ and $p \in M$ solve (2.1). Then there exists $C > 0$, depending only on Ω and Γ , such that*

$$(2.4) \quad \|p^\pm - \text{avg}_{\Omega^\pm}(p^\pm)\|_{\Omega^\pm} \leq C(\|\nu^\pm D(\mathbf{u}^\pm)\|_{\Omega^\pm} + \|\widehat{\mathbf{f}}\|_{-1}),$$

$$(2.5) \quad \|p\|_\Omega \leq C(\|\nu D(\mathbf{u})\|_\Omega + \|\widehat{\mathbf{f}}\|_{-1}).$$

Proof. Since $p^\pm - \text{avg}_{\Omega^\pm}(p^\pm) \in L_0^2(\Omega^\pm)$, then there exists $\mathbf{v}^\pm \in \mathbf{H}_0^1(\Omega^\pm)$ such that $\text{div } \mathbf{v}^\pm = p^\pm - \text{avg}_{\Omega^\pm}(p^\pm)$ in Ω^\pm , and $\|\mathbf{v}^\pm\|_{1,\Omega^\pm} \leq C\|p^\pm - \text{avg}_{\Omega^\pm}(p^\pm)\|_\Omega$, for some $C > 0$ depending only on Ω [1]. Therefore, using the first equation of (2.1) with \mathbf{v}^\pm in Ω^\pm extended by zero on Ω^\mp and since $(\text{div } \mathbf{v}^\pm, 1)_{\Omega^\pm} = 0$, we get

$$\begin{aligned} \|p^\pm - \text{avg}_{\Omega^\pm}(p^\pm)\|_{\Omega^\pm}^2 &= (p^\pm - \text{avg}_{\Omega^\pm}(p^\pm), \text{div } \mathbf{v}^\pm)_{\Omega^\pm} = (\text{div } \mathbf{v}^\pm, p^\pm)_\Omega \\ &= (\nu^\pm D(\mathbf{u}^\pm), D(\mathbf{v}^\pm))_{\Omega^\pm} - \langle \widehat{\mathbf{f}}, \mathbf{v}^\pm \rangle_{-1} \leq (\|\nu^\pm D(\mathbf{u}^\pm)\|_{\Omega^\pm} + \|\widehat{\mathbf{f}}\|_{-1}) \|\mathbf{v}^\pm\|_{1,\Omega^\pm} \\ &\leq C(\|\nu^\pm D(\mathbf{u}^\pm)\|_{\Omega^\pm} + \|\widehat{\mathbf{f}}\|_{-1}) \|p^\pm - \text{avg}_{\Omega^\pm}(p^\pm)\|_{\Omega^\pm}, \end{aligned}$$

which gives (2.4). On the other hand, since $p \in M$, there exists $\mathbf{v} \in \mathbf{V}$ such that $\text{div } \mathbf{v} = p$ in Ω , and $\|\mathbf{v}\|_{1,\Omega} \leq C\|p\|_\Omega$, for some $C > 0$, depending only on Ω^\pm . Therefore, using the first equation of (2.1) with $\mathbf{v} \in \mathbf{V}$, we get

$$\|p\|_\Omega^2 = (\text{div } \mathbf{v}, p)_\Omega = (\nu D(\mathbf{u}), D(\mathbf{v}))_\Omega - \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_{-1} \leq C\|p\|_\Omega (\|\nu D(\mathbf{u})\|_\Omega + \|\widehat{\mathbf{f}}\|_{-1}),$$

which proves (2.5). \square

In order to prove (2.3), we start by estimating $\nu^- D(\mathbf{u}^-)$.

Lemma 2.2. *Let $\mathbf{u} \in \mathbf{V}$ and $p \in M$ solve (2.1), then there exists $C > 0$, depending only on Ω , such that*

$$(2.6) \quad \|\nu^- D(\mathbf{u}^-)\|_{\Omega^-} \leq C(\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_\Omega).$$

Proof. We use the first equation of (2.1) with $\mathbf{v} = \mathbf{u}$ and the second equation with $q = p$ to get

$$(2.7) \quad \|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega^2 = \langle \widehat{\mathbf{f}}, \mathbf{u} \rangle_{-1} + (g, p)_\Omega.$$

We start by estimating the first term on the right-hand side

$$(2.8) \quad \langle \widehat{\mathbf{f}}, \mathbf{u} \rangle_{-1} \leq C\|\widehat{\mathbf{f}}\|_{-1}\|D(\mathbf{u})\|_\Omega \leq \frac{C\|\widehat{\mathbf{f}}\|_{-1}}{\sqrt{\nu^-}} \|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega$$

where we used Poincaré's inequality, Korn's inequality and the fact that $0 < \nu^- \leq \nu^+$. To estimate the second term, we use the following decomposition

$$\begin{aligned} (g, p)_\Omega &= (g^-, p^- - \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (g^-, \text{avg}_{\Omega^-}(p^-))_{\Omega^-} \\ &\quad + (g^+, p^+ - \text{avg}_{\Omega^+}(p^+))_{\Omega^+} + (g^+, \text{avg}_{\Omega^+}(p^+))_{\Omega^+}. \end{aligned}$$

By (2.4) we have

$$\begin{aligned} &(g^-, p^- - \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (g^+, p^+ - \text{avg}_{\Omega^+}(p^+))_{\Omega^+} \\ &\leq C(\sqrt{\nu^-}\|g^-\|_{\Omega^-} \|\sqrt{\nu^-} D(\mathbf{u}^-)\|_{\Omega^-} + \sqrt{\nu^+}\|g^+\|_{\Omega^+} \|\sqrt{\nu^+} D(\mathbf{u}^+)\|_{\Omega^+}) + C\|g\|_\Omega \|\widehat{\mathbf{f}}\|_{-1} \\ &\leq C(\|\nu^{\frac{1}{2}} g\|_\Omega \|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega + \|g\|_\Omega \|\widehat{\mathbf{f}}\|_{-1}). \end{aligned}$$

From $g \in M$ it follows that $(g, 1)_\Omega = 1$ and so $|\Omega^+| \text{avg}_{\Omega^+}(p^+) = -|\Omega^-| \text{avg}_{\Omega^-}(p^-)$. We employ this equality below to obtain

$$\begin{aligned} & (g^-, \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (g^+, \text{avg}_{\Omega^+}(p^+))_{\Omega^+} \\ &= (\text{avg}_{\Omega^-}(g^-), \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (\text{avg}_{\Omega^+}(g^+), \text{avg}_{\Omega^+}(p^+))_{\Omega^+} \\ &= -\frac{|\Omega^+|}{|\Omega^-|} (\text{avg}_{\Omega^+}(g^+), \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (\text{avg}_{\Omega^+}(g^+), \text{avg}_{\Omega^+}(p^+))_{\Omega^+}. \end{aligned}$$

Hence, we have

$$\begin{aligned} & (g^-, \text{avg}_{\Omega^-}(p^-))_{\Omega^-} + (g^+, \text{avg}_{\Omega^+}(p^+))_{\Omega^+} \leq C \|g^+\|_{\Omega^+} \|p\|_\Omega \leq C \|g^+\|_{\Omega^+} (\|\nu D(\mathbf{u})\|_\Omega + \|\widehat{\mathbf{f}}\|_{-1}) \\ & \leq C \sqrt{\nu^+} \|g^+\|_{\Omega^+} \|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega + C \|g\|_\Omega \|\widehat{\mathbf{f}}\|_{-1}, \end{aligned}$$

where we used (2.5) and the fact that $\nu^- \leq \nu^+$. Thus, we have shown that

$$(2.9) \quad (g, p)_\Omega \leq C (\|\nu^{\frac{1}{2}} g\|_\Omega \|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega + \|g\|_\Omega \|\widehat{\mathbf{f}}\|_{-1}).$$

Combining (2.7), (2.8), (2.9) and using that $\nu^- \leq \nu^+$ we arrive at

$$\|\nu^{\frac{1}{2}} D(\mathbf{u})\|_\Omega \leq C \left(\frac{1}{\sqrt{\nu^-}} \|\widehat{\mathbf{f}}\|_{-1} + \|\nu^{\frac{1}{2}} g\|_\Omega \right).$$

This implies the result due to $\nu^- \leq \nu^+$. \square

An immediate consequence of Lemma 2.2 and (2.4) is the following result.

Lemma 2.3. *Let $\mathbf{u} \in \mathbf{V}$ and $p \in M$ solve (2.1), then there exists $C > 0$, depending only on Ω , such that*

$$\|p^- - \text{avg}_{\Omega^-}(p^-)\|_{\Omega^-} \leq C (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_\Omega).$$

In order to estimate $\nu D(\mathbf{u})$ over Ω , we consider an extension operator. A detailed construction of this operator can be found on [16, Chapter VI, Section 3.3]. Let $E : \mathbf{H}^1(\Omega^+) \rightarrow \mathbf{H}^1(\Omega)$ be the bounded extension operator from Ω^+ to Ω . That is, if $\mathbf{w} \in \mathbf{H}^1(\Omega^+)$, then

$$(2.10) \quad E\mathbf{w} \in \mathbf{H}^1(\Omega), \quad E\mathbf{w} = \mathbf{w} \quad \text{in } \Omega^+, \quad \text{and} \quad \|E\mathbf{w}\|_{1,\Omega} \leq C \|\mathbf{w}\|_{1,\Omega^+},$$

for some constant $C > 0$, depending only on Ω^+ and Ω . Moreover, one can assume that $E\mathbf{w} \in \mathbf{V}$ for $\mathbf{w} \in \mathbf{V}$. Further, we note that if $\mathbf{w}(\mathbf{x}) := \mathbf{a} + \mathbf{B}\mathbf{x} \in \mathbb{RM}(\Omega)$, then

$$D(\mathbf{w}) = \mathbf{0}, \quad \text{and hence } \text{div } \mathbf{w} = \text{tr}(D(\mathbf{w})) = 0.$$

Here $\mathbb{RM}(\mathcal{O})$ is the space of rigid body motions defined on \mathcal{O} . We denote by $\mathcal{P}_{\mathcal{O}}^{RM}(\mathbf{v})$ the L^2 -projection of $\mathbf{v} \in L^2(\mathcal{O})^d$ onto the subspace of rigid body motions, i.e. $(\mathcal{P}_{\mathcal{O}}^{RM}(\mathbf{v}), \mathbf{w})_{\mathcal{O}} = (\mathbf{v}, \mathbf{w})_{\mathcal{O}}$ for all $\mathbf{w} \in \mathbb{RM}(\mathcal{O})$.

Lemma 2.4. *Let $\mathbf{u} \in \mathbf{V}$ and $p \in M$ solve (2.1), then there exists $C > 0$, depending only on Ω , such that*

$$(2.11) \quad \|\nu^+ D(\mathbf{u}^+)\|_{\Omega^+} \leq C (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_\Omega).$$

Proof. Define $\mathbf{v} \in \mathbf{V}$ by

$$\mathbf{v} = \begin{cases} E\mathbf{u}^+, & \text{if } |\partial\Omega^+ \cap \partial\Omega| > 0, \\ E(\mathbf{u}^+ - \mathcal{P}_{\Omega^+}^{RM}(\mathbf{u}^+)), & \text{if } |\partial\Omega^+ \cap \partial\Omega| = 0. \end{cases}$$

The boundeness of the extension operator in (2.10), Poincare's and Korn's inequalities imply

$$(2.12) \quad \|\mathbf{v}\|_{1,\Omega} \leq C \|D(\mathbf{u}^+)\|_{\Omega^+}.$$

The first equation of (2.1) with the above \mathbf{v} gives

$$(2.13) \quad \nu^+ \|D(\mathbf{u}^+)\|_{\Omega^+}^2 = \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_{-1} - (\nu^- D(\mathbf{u}^-), D(\mathbf{v}^-))_{\Omega^-} + (\operatorname{div} \mathbf{v}, p)_{\Omega},$$

where we used that $\operatorname{div} \mathbf{v}^+ = \operatorname{div} \mathbf{u}^+$. We now bound each term on the right-hand side of (2.13). Using (2.12) we have

$$(2.14) \quad \langle \widehat{\mathbf{f}}, \mathbf{v} \rangle_{-1} \leq C \|\widehat{\mathbf{f}}\|_{-1} \|D(\mathbf{u}^+)\|_{\Omega^+}.$$

Thanks to (2.12) and (2.6) we obtain

$$(2.15) \quad -(\nu^- D(\mathbf{u}^-), D(\mathbf{v}^-))_{\Omega^-} \leq C (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \|D(\mathbf{u}^+)\|_{\Omega^+}.$$

Finally,

$$(\operatorname{div} \mathbf{v}, p)_{\Omega} = (\operatorname{div} \mathbf{v}^-, p^- - \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} + (\operatorname{div} \mathbf{v}^-, \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} + (\operatorname{div} \mathbf{u}^+, p^+)_{\Omega^+}$$

Using (2.3) and (2.12) we estimate

$$(\operatorname{div} \mathbf{v}^-, p^- - \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} \leq C (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \|D(\mathbf{u}^+)\|_{\Omega^+}.$$

For the next term we use $(\operatorname{div} \mathbf{v}, 1)_{\Omega} = 0$ and the second equation in (1.1) to write

$$\begin{aligned} & (\operatorname{div} \mathbf{v}^-, \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} \\ &= (\operatorname{avg}_{\Omega^-}(\operatorname{div} \mathbf{v}^-), \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} = -\frac{|\Omega^+|}{|\Omega^-|} (\operatorname{avg}_{\Omega^+}(\operatorname{div} \mathbf{v}^+), \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} \\ &= -\frac{|\Omega^+|}{|\Omega^-|} (\operatorname{avg}_{\Omega^+}(\operatorname{div} \mathbf{u}^+), \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} = -\frac{|\Omega^+|}{|\Omega^-|} (\operatorname{avg}_{\Omega^+}(g^+), \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-}. \end{aligned}$$

Hence, with the help of (2.5) and (2.6) we get

$$\begin{aligned} (\operatorname{div} \mathbf{v}^-, \operatorname{avg}_{\Omega^-}(p^-))_{\Omega^-} &\leq C \|g^+\|_{\Omega^+} \|p\|_{\Omega} \leq C \|g^+\|_{\Omega^+} (\|\nu D(\mathbf{u})\|_{\Omega} + \|\widehat{\mathbf{f}}\|_{-1}) \\ &\leq C \nu^+ \|g^+\|_{\Omega^+} \|D(\mathbf{u}^+)\|_{\Omega^+} + C \|g^+\|_{\Omega^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}). \end{aligned}$$

Similarly we have,

$$(\operatorname{div} \mathbf{u}^+, p^+)_{\Omega^+} \leq C \nu^+ \|g^+\|_{\Omega^+} \|D(\mathbf{u}^+)\|_{\Omega^+} + C \|g^+\|_{\Omega^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}).$$

We combine the last two estimates to obtain the bound

$$(\operatorname{div} \mathbf{v}, p)_{\Omega} \leq (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \|D(\mathbf{u}^+)\|_{\Omega^+} + C \|g^+\|_{\Omega^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}).$$

This estimate together with (2.15), (2.14), (2.13) leads to another bound

$$\begin{aligned} \nu^+ \|D(\mathbf{u}^+)\|_{\Omega^+}^2 &\leq (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \|D(\mathbf{u}^+)\|_{\Omega^+} + C \|g^+\|_{\Omega^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \\ &\leq \frac{1}{2\nu^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega})^2 + \frac{\nu^+}{2} \|D(\mathbf{u}^+)\|_{\Omega^+}^2 + C \|g^+\|_{\Omega^+} (\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}). \end{aligned}$$

This implies (2.11) after multiplication all through by ν^+ and doing simple computations. \square

Collecting (2.5), (2.6) and (2.11) we obtain the main result of this section.

Theorem 2.1. *Let $\mathbf{u} \in \mathbf{V}$ and $p \in M$ solve (2.1), then there exists $C > 0$, depending only on Ω and Γ , such that*

$$(2.16) \quad \|\nu D(\mathbf{u})\|_{\Omega} + \|p\|_{\Omega} \leq C(\|\widehat{\mathbf{f}}\|_{-1} + \|\nu g\|_{\Omega}) \leq C(\|\mathbf{f}\|_{\Omega} + \|\boldsymbol{\lambda}\|_{\Gamma} + \|\nu g\|_{\Omega}).$$

The second inequality in (2.16) follows from the definition of the functional $\widehat{\mathbf{f}}$ in (2.2), the Poincare inequality and the trace inequality, $\|\mathbf{v}\|_{\Gamma} \leq C\|\mathbf{v}\|_{1,\Omega}$.

3. THE FINITE ELEMENT METHOD

3.1. Preliminaries and problem setting. For the discretization purpose, we assume that Ω is polygonal/polyhedral. Let $\{\mathcal{T}_h\}_{h>0}$ be an admissible family of triangulations of Ω . We adopt the convention that the elements T and edges e are open sets, and use over-line symbol to refer to their closure. For each simplex $T \in \mathcal{T}_h$, let h_T denote its diameter and define the global parameter of the triangulation by $h = \max_T h_T$. We assume that \mathcal{T}_h is shape regular, that is, there exists $\kappa > 0$ such that for every $T \in \mathcal{T}_h$, the radius ρ_T of the inscribed sphere satisfies

$$(3.1) \quad \rho_T > \frac{h_T}{\kappa}.$$

The sets of elements intersecting Ω^{\pm} and the set of elements cutting the interface Γ are of interest. These are defined by

$$\mathcal{T}_h^{\pm} := \{T \in \mathcal{T}_h : T \cap \Omega^{\pm} \neq \emptyset\}, \quad \text{and} \quad \mathcal{T}_h^{\Gamma} := \{T \in \mathcal{T}_h : \overline{T} \cap \Gamma \neq \emptyset\}.$$

We also define the sets of elements interior to each of subdomains Ω^{\pm} , $\mathcal{T}_{h,i}^{\pm} = \{T \in \mathcal{T}_h^{\pm} : T \subset \Omega^{\pm}\}$. Finally, we let $\mathcal{E}_{h,i}^{\pm}$ be the collection of $d-1$, sub-simplexes of $\mathcal{T}_{h,i}^{\pm}$ (faces for $d=3$ and edges for $d=2$).

For $T \in \mathcal{T}_h^{\Gamma}$, we denote $T_{\Gamma} := \overline{T} \cap \Gamma$. Under these definitions we define the h -dependent domains

$$\Omega_h^{\pm} := \text{int}\left(\bigcup_{T \in \mathcal{T}_h^{\pm}} \overline{T}\right), \quad \text{and} \quad \Omega_{h,i}^{\pm} := \text{int}\left(\bigcup_{T \in \mathcal{T}_{h,i}^{\pm}} \overline{T}\right).$$

We also consider the layer of elements cut by the interface:

$$\omega_h := \text{int}\left(\bigcup_{T \in \mathcal{T}_h^{\Gamma}} \overline{T}\right), \quad \text{and} \quad \Omega_h := \Omega \setminus \overline{\omega_h},$$

and define the set of faces (edges) of \mathcal{T}_h^{Γ} restricted to the interior of Ω_h^{\pm} :

$$\mathcal{E}_h^{\Gamma,\pm} := \{e = \text{int}(\partial T_1 \cap \partial T_2) : T_1, T_2 \in \mathcal{T}_h^{\pm} \text{ and } T_1 \cap \Gamma \neq \emptyset \text{ or } T_2 \cap \Gamma \neq \emptyset\}.$$

For a piecewise smooth function vector valued function \mathbf{v} , the jump across an interior face $e = \text{int}(\partial T_1 \cap \partial T_2)$ is defined by $[\![\mathbf{v}]\!] = \mathbf{v}|_{T_1} \cdot \mathbf{n}_1 + \mathbf{v}|_{T_2} \cdot \mathbf{n}_2$, where \mathbf{n}_1 and \mathbf{n}_2 are the unit normal vectors to e , pointing outwards to T_1 and T_2 , respectively. For a scalar function, we define $[\![p]\!] = p|_{T_1} \mathbf{n}_1 + p|_{T_2} \mathbf{n}_2$.

The space of discontinuous and continuous finite element pressures are given by

$$M_{h,\text{disc}} := \{q \in L^2(\Omega) : q|_T \in P_{k_p}(T) \quad \forall T \in \mathcal{T}_h\} \quad \text{and} \quad M_{h,\text{cont}} := M_{h,\text{disc}} \cap C(\Omega),$$

where integer $k_p \geq 0$ is a fixed polynomial degree. Throughout this paper, $M_h^{\text{bulk}} = M_{h,\text{disc}}$ for $k_p \geq 0$ or $M_h^{\text{bulk}} = M_{h,\text{cont}}$ for $k_p \geq 1$. We define $M_h^\pm := M_h^{\text{bulk}} \cap L^2(\Omega_h^\pm)$. Finally, our pressure space is given by

$$M_h := \{q := (q^-, q^+) \in M_h^- \times M_h^+ : (q^-, 1)_{\Omega^-} + (q^+, 1)_{\Omega^+} = 0\}.$$

Note that every element from \mathcal{T}_h^Γ supports two finite element pressures corresponding to different phases. Only the restriction of these pressures to Ω^+ or Ω^- , respectively, makes sense as a numerical approximation of the true pressure solving the original problem (1.1). Same comment will be valid for the finite element velocity fields defined next. We consider the vector finite element space for $k \geq 1$,

$$\mathbf{W}_h^k = \{\mathbf{w} \in \mathbf{H}_0^1(\Omega) : \mathbf{w} \in \mathbf{P}_k(T), \text{ for all } T \in \mathcal{T}_h\}.$$

Next we consider a background velocity finite element space $\mathbf{V}_h^{\text{bulk}}$ such that

$$\mathbf{W}_h^{k_u} \subset \mathbf{V}_h^{\text{bulk}} \subset \mathbf{W}_h^s,$$

for some integers $s \geq k_u \geq 1$. Let $\mathbf{V}_h^\pm := \mathbf{V}_h^{\text{bulk}} \cap \mathbf{H}^1(\Omega_h^\pm)$. Finally our velocity space will be

$$\mathbf{V}_h := \{\mathbf{v} := (\mathbf{v}^-, \mathbf{v}^+) \in \mathbf{V}_h^- \times \mathbf{V}_h^+\}.$$

Also, we denote a generic element $\mathbf{v}_h \in \mathbf{V}_h$ by $\mathbf{v}_h := (\mathbf{v}_h^-, \mathbf{v}_h^+)$.

Functions from M_h , \mathbf{V}_h and their derivatives are multivalued in ω_h , the overlap of Ω_h^+ and Ω_h^- . Below we use the $L^2(\Omega)$ norm notion for such functions to denote the norm of single-valued functions obtained by restricting Ω_h^+ -components on $\Omega^+ \subset \Omega_h^+$ and Ω_h^- -components on $\Omega^- \subset \Omega_h^-$. For example,

$$\|p_h\|_\Omega^2 = \|p_h^+\|_{\Omega^+}^2 + \|p_h^-\|_{\Omega^-}^2 \quad \text{or} \quad \|\nu D(\mathbf{u}_h)\|_\Omega^2 = \|\nu^+ D(\mathbf{u}_h^+)\|_{\Omega^+}^2 + \|\nu^- D(\mathbf{u}_h^-)\|_{\Omega^-}^2,$$

for $p_h \in M_h$, $\mathbf{u}_h \in \mathbf{V}_h$, and so forth. The jump of a multivalued function over the interface is defined as the difference of components coming from Ω_h^+ and Ω_h^- , i.e. $[[\mathbf{v}_h]] = \mathbf{v}_h^+ - \mathbf{v}_h^-$ on Γ .

We consider a discrete norm $\|\cdot\|_{1,h}$, defined on \mathbf{V}_h , as follows: for all $\mathbf{v}_h \in \mathbf{V}_h$,

$$\|\mathbf{v}_h\|_{1,h}^2 := \|D(\mathbf{v}_h)\|_\Omega^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\|[[\mathbf{v}_h]]\|_{T_\Gamma}^2}{h_T} + \sum_{e \in \mathcal{E}_h^{\Gamma,-}} \sum_{\ell=1}^s |e|^{2\ell-1} \|[\partial_{\mathbf{n}}^\ell \mathbf{v}_h^-]\|_e^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} \sum_{\ell=1}^s |e|^{2\ell-1} \|[\partial_{\mathbf{n}}^\ell \mathbf{v}_h^+]\|_e^2,$$

where $\partial_{\mathbf{n}}^\ell$ is the ℓ -th order normal derivative. We define a scaled norm

$$\begin{aligned} \|\mathbf{v}_h\|_{1,h,\nu}^2 &:= \|\nu D(\mathbf{v}_h)\|_\Omega^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\|\nu^- [[\mathbf{v}_h]]\|_{T_\Gamma}^2}{h_T} \\ &+ \sum_{e \in \mathcal{E}_h^{\Gamma,-}} \sum_{\ell=1}^s |e|^{2\ell-1} \|\nu^- [\partial_{\mathbf{n}}^\ell \mathbf{v}_h^-]\|_e^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,+}} \sum_{\ell=1}^s |e|^{2\ell-1} \|\nu^+ [\partial_{\mathbf{n}}^\ell \mathbf{v}_h^+]\|_e^2, \end{aligned}$$

and the augmented scaled norm

$$(3.2) \quad \|\mathbf{v}_h\|_{1,h,\nu,\star}^2 := \|\mathbf{v}_h\|_{1,h,\nu}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} h_T \|\nu^- D(\mathbf{v}_h)\|_{T_\Gamma}^2.$$

We use the notation $\|\cdot\|_{\pm,h}$ to define a discrete norm on M_h , for all $q_h = (q_h^-, q_h^+) \in M_h$, by

$$\|q_h^\pm\|_{\pm,h}^2 := \|q_h^\pm\|_{\Omega^\pm}^2 + \sum_{e \in \mathcal{E}_h^{\Gamma,\pm}} \sum_{\ell=0}^{k_p} |e|^{2\ell+1} \|\llbracket \partial_{\mathbf{n}}^\ell q_h^\pm \rrbracket\|_e^2, \quad \|q_h\|_h^2 = \|q_h^+\|_{+,h}^2 + \|q_h^-\|_{-,h}^2.$$

Spaces \mathbf{V}_h^* and M_h^* are dual to \mathbf{V}_h and M_h^\pm with respect to $\|\cdot\|_{1,h}$ and $\|\cdot\|_{\pm,h}$. For $\mathbf{F}_h \in \mathbf{V}_h^*$ and $G_h \in M_h^*$, we have by definition that

$$\|\mathbf{F}_h\|_{-1,h} = \sup_{\mathbf{v}_h \in \mathbf{V}_h} \frac{\mathbf{F}_h(\mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,h}}, \quad \text{and} \quad \|G_h\|_{-1,h} = \sup_{q_h \in M_h} \frac{G_h(q_h)}{\|q_h\|_h},$$

where we agree that \sup_x is taken over non-zero elements, if x appears in the denominator. In this paper, we only consider bulk spaces $\mathbf{V}_h^{\text{bulk}}, M_h^{\text{bulk}}$ which form inf-sup stable pairs. This is listed as an assumption.

Assumption 1. There exists a constant $\xi > 0$ such that

$$\xi \|q\|_\Omega \leq \sup_{\mathbf{v} \in \mathbf{V}_h^{\text{bulk}}} \frac{(\text{div } \mathbf{v}, q)_\Omega}{\|\mathbf{v}\|_{1,\Omega}} \quad \text{for all } q \in M_h^{\text{bulk}} \cap L_0^2(\Omega).$$

We end this section by considering further assumptions on the mesh and on the pair of spaces $\{\mathbf{V}_h, M_h\}$. These assumptions are essentially the ones made in [9]. For a generic set of elements $\mathcal{T} \subset \mathcal{T}_h$, denote $\omega(\mathcal{T}) \subset \mathcal{T}_h$ the set of all tetrahedra having at least one vertex in \mathcal{T} .

Assumption 2. For any $T \in \mathcal{T}_h^\Gamma$ we assume that the sets $W^\pm(T) = \mathcal{T}_{h,i}^\pm \cap \omega(\omega(T))$ are not empty.

We note that this assumption can be weakened by allowing in $W(T)$ neighbors of T of degree L , with some finite and mesh independent $L \geq 2$.

Given $T \in \mathcal{T}_h^\Gamma$, we associate arbitrary but fixed $K_T^\pm \in W^\pm(T)$, which can be reached from T by crossing faces in $\mathcal{E}_h^{\Gamma,\pm}$. More precisely, there exists simplexes $T = K_1^\pm, K_2^\pm, \dots, K_M^\pm = K_T^\pm$ with $K_j^\pm \in \mathcal{T}_h^\Gamma$ for $j < M$. The number M is uniformly bounded and only depends on the shape regularity of the mesh. Moreover, note that by (3.1) there exists a constant c only depending on the shape regularity constant κ such that

$$\frac{h_T}{c} \leq h_{K_T^\pm} \leq ch_T.$$

For $T \in \mathcal{T}_{h,i}^\pm$, we set $K_T^\pm = T$.

Assumption 3. Let $F \in \mathcal{E}_h^{\Gamma,\pm}$, with $F = \partial T_1 \cap \partial T_2$. Assume K_{T_2} can be reached from K_{T_1} by crossing a finite, independent of h , number of faces of tetrahedra from $\mathcal{T}_{h,i}^\pm$.

The following assumption is also a type of inf-sup condition but restricted to interior elements in two phases, i.e. those lying inside $\Omega_{h,i}^\pm$.

Assumption 4. There exists a constant $\beta > 0$ such that

$$\beta |q^\pm|_{H_{h,i}^{1,\pm}} \leq \sup_{\mathbf{v} \in \mathbf{V}_h^{\text{bulk}} \cap \mathbf{H}_0^1(\Omega_{h,i}^\pm)} \frac{(\text{div } \mathbf{v}, q^\pm)_{\Omega_{h,i}^\pm}}{\|\mathbf{v}\|_{1,\Omega_{h,i}^\pm}} \quad \text{for all } q \in M_h^{\text{bulk}}.$$

where

$$|q^\pm|_{H_{h,i}^{1,\pm}}^2 = \sum_{T \in \mathcal{T}_{h,i}^\pm} h_T^2 \|\nabla q^\pm\|_{L^2(T)}^2 + \sum_{e \in \mathcal{E}_{h,i}^\pm} h_e \|\llbracket q^\pm \rrbracket\|_e^2.$$

Examples of pair of spaces $\mathbf{V}_h^{\text{bulk}} - M_h^{\text{bulk}}$ that satisfy the Assumption 4 can be found in [9, Section 6]; they include $P_{k+1} - P_k$, $k \geq 1$, and $P_{k+d} - P_k^{\text{disc}}$ for $k \geq 0$, $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, and several other elements. In particular, if a pair $\mathbf{V}_h^{\text{bulk}} - M_h^{\text{bulk}}$ satisfies these assumptions, we have the following result.

Theorem 3.1. *Suppose Assumptions 2–4 hold. There exists a constant $\theta > 0$, independent of h and q , and a constant $h_0 > 0$ such that for all $q \in L_0^2(\Omega_{h,i}^\pm) \cap M_h^{\text{bulk}}$ and $h \leq h_0$, we have*

$$\theta \|q\|_{\Omega_{h,i}^\pm} \leq \sup_{\mathbf{v} \in \mathbf{V}_h^\Gamma \cap \mathbf{H}_0^1(\Omega_{h,i}^\pm)} \frac{(\text{div } \mathbf{v}, q)_{\Omega_{h,i}^\pm}}{\|\mathbf{v}\|_{1, \Omega_{h,i}^\pm}}.$$

Proof. See [9, Theorem 1]. □

We will also need trace and inverse inequalities which can be found, for example, in [9].

Lemma 3.1. *Let $T \in \mathcal{T}_h$, then it holds*

$$(3.3) \quad \|v\|_{L^2(\partial T)} \leq C(h_T^{-1/2} \|v\|_T + h_T^{1/2} \|\nabla v\|_T^2), \quad \text{for all } v \in H^1(T)$$

$$(3.4) \quad \|v\|_{L^2(T \cap \Gamma)} \leq C(h_T^{-1/2} \|v\|_T + h_T^{1/2} \|\nabla v\|_T^2), \quad \text{for all } v \in H^1(T)$$

$$(3.5) \quad \|v\|_{L^2(\partial T)} \leq C h_T^{-1/2} \|v\|_T \quad \text{for all } v \in P_k(T),$$

$$(3.6) \quad \|v\|_{L^2(T \cap \Gamma)} \leq C h_T^{-1/2} \|v\|_T \quad \text{for all } v \in P_k(T),$$

with a constant C independent on how Γ intersects T .

3.2. Discrete Extension Operator. As in the continuous setting, a chief tool will be an extension operator. In this section we provide a discrete analogue of the extension operator in (2.10). Widlund [20] provided a discrete extension operator when the mesh fits the interface. For non-fitted meshes, as is the case here, a discrete extension operator can be found in [5] for piecewise-linear finite element functions and smooth interface.

In Lemma 3.2 below, we prove the result for unfitted meshes and finite elements of arbitrary degree that admit the existence of a *local nodal basis*. Let us make this assumption precise. The velocity bulk space is the space of vector functions, i.e. $\mathbf{V}_h^{\text{bulk}} = \otimes_{j=1}^d V_{j,h}$. For each component space $V_{j,h}$ we assume that (i) there is a set of points (nodes) $\mathcal{N}(\mathcal{T}_h) = \{\mathbf{y}_1, \dots, \mathbf{y}_\ell\}$ such that $v \in V_{j,h}$ is uniquely determined by $v(\mathbf{y}_i)$ for $1 \leq i \leq \ell$; (ii) for each T there exists a local subset $\mathcal{N}(T) = \{\mathbf{y} \in \mathcal{N}(\mathcal{T}_h) : \mathbf{y} \in \overline{T}\}$ such that $v|_T$ is uniquely determined by the values there; and (iii) if $\Phi : T \rightarrow \widehat{T}$ is the affine mapping to the reference simplex, then $\Phi(\mathcal{N}(T))$ is independent of $T \in \mathcal{T}_h$. The existence of the local nodal basis is, of course, standard if $\mathbf{V}_h^{\text{bulk}} = \mathbf{W}_h^k$ for some integer $k \geq 1$. Moreover, we assume only Lipschitz regularity of the interface.

Lemma 3.2 (Finite element extension). *Assume Γ is Lipschitz, the meshes $\{\mathcal{T}_h\}$ are shape regular and satisfy Assumptions 2, and assume $\mathbf{V}_h^{\text{bulk}}$ has a local nodal basis. There exists an extension operator $E_h : \mathbf{V}_h^+ \rightarrow \mathbf{V}_h^{\text{bulk}}$ with the following properties:*

- a) $E_h \mathbf{v}_h = \mathbf{v}_h$ on Ω_h^+ ,
 b) There exists $C > 0$, independent of h and position of Γ against the underlying mesh, such that for all $\mathbf{v}_h \in \mathbf{V}_h^+$,

$$\|E_h \mathbf{v}_h\|_{1,\Omega} \leq C \|\mathbf{v}_h\|_{1,\Omega_h^+}.$$

Proof. The proof is given in the Appendix. \square

3.3. The discrete variational formulation. In this section, we define the discrete counterpart of (2.1). The jumps over the interface are enforced weakly, and a term is added to enforce the symmetry of the bilinear form a . A discrete variational analogue of (2.1) is given by the problem of finding $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ such that

$$(3.7) \quad \begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h) &= \mathbf{F}_h(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}_h, q_h) - J_h(p_h, q_h) &= G_h(q_h) \quad \forall q_h \in M_h, \end{aligned}$$

with bilinear forms given below. For all $\mathbf{u}_h = (\mathbf{u}_h^-, \mathbf{u}_h^+)$, $\mathbf{v}_h = (\mathbf{v}_h^-, \mathbf{v}_h^+) \in \mathbf{V}_h$ and $p_h = (p_h^-, p_h^+)$, $q_h = (q_h^-, q_h^+) \in M_h$, we define

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{v}_h) &:= (\nu^- D(\mathbf{u}_h^-), D(\mathbf{v}_h^-))_{\Omega^-} + (\nu^+ D(\mathbf{u}_h^+), D(\mathbf{v}_h^+))_{\Omega^+} + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\gamma}{h_T} \nu^- ([\mathbf{u}_h], [\mathbf{v}_h])_{T_\Gamma} \\ &\quad - (\nu^- D(\mathbf{u}_h^-) \mathbf{n}, [\mathbf{v}_h])_\Gamma - (\nu^- D(\mathbf{v}_h^-) \mathbf{n}, [\mathbf{u}_h])_\Gamma + \mathbf{J}_h(\mathbf{u}_h, \mathbf{v}_h), \end{aligned}$$

with

$$\mathbf{J}_h(\mathbf{u}_h, \mathbf{v}_h) = \mathbf{J}_h^-(\mathbf{u}_h, \mathbf{v}_h) + \mathbf{J}_h^+(\mathbf{u}_h, \mathbf{v}_h), \quad \mathbf{J}_h^\pm(\mathbf{u}_h, \mathbf{v}_h) = \sum_{\ell=1}^s |e|^{2\ell-1} \sum_{e \in \mathcal{E}_h^{\Gamma, \pm}} \gamma_{\mathbf{u}}^\pm \nu^\pm ([\partial_n^\ell \mathbf{u}_h^\pm], [\partial_n^\ell \mathbf{v}_h^\pm])_e,$$

$$b_h(\mathbf{v}_h, q_h) := -(q_h^-, \operatorname{div} \mathbf{v}_h^-)_{\Omega^-} - (q_h^+, \operatorname{div} \mathbf{v}_h^+)_{\Omega^+} + (q_h^-, [\mathbf{v}_h] \cdot \mathbf{n})_\Gamma,$$

and

$$J_h(p_h, q_h) = J_h^-(p_h, q_h) + J_h^+(p_h, q_h), \quad J_h^\pm(p_h, q_h) := \frac{\gamma_p^\pm}{\nu^\pm} \sum_{e \in \mathcal{E}_h^{\Gamma, \pm}} \sum_{\ell=0}^{k_p} |e|^{2\ell+1} ([\partial_n^\ell p_h^\pm], [\partial_n^\ell q_h^\pm])_e.$$

Stabilization parameters γ_p^\pm , $\gamma_{\mathbf{u}}^\pm$ and γ are all assumed to be independent of ν , h and position of Γ against the underlying mesh. Parameter γ needs to be large enough to provide the bilinear form $a(\cdot, \cdot)$ with coercivity. For the purpose of analysis, we set $\gamma_p^\pm = \gamma_{\mathbf{u}}^\pm = 1$. In practice, these parameters can be tuned for better numerical performance (see section 5 for numerical examples) and the analysis below remains valid if all γ_p^\pm and $\gamma_{\mathbf{u}}^\pm$ are $O(1)$ parameters.

The right-hand side $\mathbf{F}_h \in \mathbf{V}_h^*$ will be defined later on, and we assume $G_h \in M_h^*$ is given by

$$G_h(q_h) := G_h^-(q_h) + G_h^+(q_h), \quad \text{such that} \quad G_h^\pm(q_h) = G_h^\pm(q_h^\pm).$$

It is straightforward to check that the norm of linear bounded functions G_h^\pm can be expressed in terms of M_h^\pm spaces. More precisely, it holds

$$\|G_h^\pm\|_{-1,h} = \sup_{q_h^\pm \in M_h^\pm} |G_h^\pm(q_h^\pm)| / \|q_h^\pm\|_{\pm,h}.$$

Also, we assume G_h satisfies $G_h(1) = 0$. In particular, this implies that if the second equation of (3.7) is satisfied by $q_h \in M_h$, then it is also satisfied by $q_h + c$ for any constant function c .

3.4. Well-posedness of the discrete scheme. In this section, we deduce the discrete version of (2.3). That is, for (\mathbf{u}_h, p_h) solving (3.7), we are interested in the following stability result:

$$(3.8) \quad \|\nu D(\mathbf{u}_h)\|_\Omega + \|p_h\|_\Omega \leq C(\|\mathbf{F}_h\|_{-1,h} + \nu^- \|G_h^-\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h})$$

where, similarly, the constant $C > 0$ is independent of ν^\pm , but also it is independent of h and the position of Γ against the bulk mesh.

We start by establishing the continuity of the bilinear forms a_h and b_h , and the \mathbf{V}_h -coercivity of a_h . To this end, define the natural energy norm for a_h , given for all $\mathbf{v}_h = (\mathbf{v}_h^-, \mathbf{v}_h^+) \in \mathbf{V}_h$ by

$$(3.9) \quad \|\mathbf{v}_h\|_{\mathbf{V}_h}^2 := \|\nu^{\frac{1}{2}} D(\mathbf{v}_h)\|_\Omega^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\|\sqrt{\nu^-} \llbracket \mathbf{v}_h \rrbracket\|_{T_\Gamma}^2}{h_T} + \mathbf{J}_h(\mathbf{v}_h, \mathbf{v}_h).$$

We will need the following technical lemma which is essentially found in [12, Lemma 5.1]. The main difference is the use of Korn's inequality and projections onto the rigid body motions instead of the Poincaré inequality and projection onto the constants.

Lemma 3.3. *There exists $C > 0$, independent of h and ν^\pm , such that for every $q_h \in M_h$, it holds*

$$(3.10) \quad \|q_h^\pm\|_{\Omega_h^\pm}^2 \leq C \left(\|q_h^\pm\|_{\Omega_{h,i}^\pm}^2 + \nu^- J_h^-(q_h^-, q_h^-) + \nu^+ J_h^+(q_h^+, q_h^+) \right),$$

and for all $\mathbf{v}_h \in \mathbf{V}_h$,

$$(3.11) \quad \|D(\mathbf{v}_h^\pm)\|_{\Omega_h^\pm}^2 \leq C \left(\|D(\mathbf{v}_h^\pm)\|_{\Omega_{h,i}^\pm}^2 + \frac{1}{\nu^-} \mathbf{J}_h^-(\mathbf{v}_h^-, \mathbf{v}_h^-) + \frac{1}{\nu^+} \mathbf{J}_h^+(\mathbf{v}_h^+, \mathbf{v}_h^+) \right) \leq C \|\mathbf{v}_h\|_{1,h}^2.$$

The following result discusses the continuity and coercivity of the bilinear form a_h . We omit the proof since they are by now standard and can easily be proved using Lemma 3.1. See, for example, similar results in [9].

Lemma 3.4. *Let $\mathbf{v}_h, \mathbf{w}_h \in \mathbf{V}_h$ and $q_h \in M_h$. Then, there exists $C > 0$ and h_0 , independent of h and ν^\pm , \mathbf{v}_h and \mathbf{w}_h , such that for all $h \leq h_0$ it holds*

$$(3.12) \quad |a_h(\mathbf{v}_h, \mathbf{w}_h)| \leq C \|\mathbf{v}_h\|_{\mathbf{V}_h} \|\mathbf{w}_h\|_{\mathbf{V}_h},$$

$$|a_h(\mathbf{v}_h, \mathbf{w}_h)| \leq C \|\mathbf{v}_h\|_{1,h,\nu} \|\mathbf{w}_h\|_{1,h}.$$

Additionally, for all $\mathbf{v} \in [\mathbf{H}^{s+1}(\Omega_h^-) \times \mathbf{H}^{s+1}(\Omega_h^+)] \times \mathbf{V}_h$ we have

$$(3.13) \quad |a_h(\mathbf{v}, \mathbf{w}_h)| \leq C \|\mathbf{v}\|_{1,h,\nu,\star} \|\mathbf{w}_h\|_{1,h}.$$

Finally, there exists $\alpha > 0$, independent of h and ν^\pm , such that

$$\alpha \|\mathbf{v}_h\|_{\mathbf{V}_h}^2 \leq a_h(\mathbf{v}_h, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h.$$

The following result is the discrete analogue of Lemma 2.1.

Lemma 3.5. *Let $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in M_h$ solve (3.7). Then, there exists $C, h_0 > 0$, depending only on Ω and Γ , such that for $h \leq h_0$ it holds*

$$(3.14) \quad \|p_h^\pm - \text{avg}_{\Omega_{h,i}^\pm}(p_h^\pm)\|_{\Omega^\pm} \leq C(\|\nu^\pm D(\mathbf{u}_h^\pm)\|_{\Omega^\pm} + (\nu^\pm J_h(p_h^\pm, p_h^\pm))^{1/2} + \|\mathbf{F}_h\|_{-1,h}),$$

and

$$(3.15) \quad \|p_h\|_\Omega \leq C(\|\mathbf{u}_h\|_{1,h,\nu} + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2} + (\nu^+ J_h^+(p_h^+, p_h^+))^{1/2} + \|\mathbf{F}_h\|_{-1,h}).$$

Proof. Let $q_h^\pm := p_h^\pm - \text{avg}_{\Omega_{h,i}^\pm}(p_h^\pm)$. Noting that $J_h^\pm(p_h^\pm, p_h^\pm) = J_h(q_h^\pm, q_h^\pm)$ and employing (3.10) from Lemma 3.3, we get

$$(3.16) \quad C\|q_h^\pm\|_{\Omega^\pm}^2 \leq \|q_h\|_{\Omega_{h,i}^\pm}^2 + \nu^\pm J_h^\pm(p_h^\pm, p_h^\pm).$$

On the other hand, since $q_h^\pm \in L_0^2(\Omega_{h,i}^\pm)$, Theorem 3.1 implies that there exist $\theta, h_0 > 0$, depending only on Ω , such that for all $h \leq h_0$, it holds

$$(3.17) \quad \theta\|q_h^\pm\|_{\Omega_{h,i}^\pm} \leq \sup_{\mathbf{w}_h^\pm \in \mathbf{V}_h^{\text{bulk}} \cap \mathbf{H}_0^1(\Omega_{h,i}^\pm)} \frac{(\text{div } \mathbf{w}_h^\pm, q_h^\pm)_{\Omega_{h,i}^\pm}}{\|\mathbf{w}_h^\pm\|_{1,\Omega_{h,i}^\pm}} = \sup_{\mathbf{w}_h^\pm \in \mathbf{V}_h^{\text{bulk}} \cap \mathbf{H}_0^1(\Omega_{h,i}^\pm)} \frac{(\text{div } \mathbf{w}_h^\pm, p_h^\pm)_{\Omega_{h,i}^\pm}}{\|\mathbf{w}_h^\pm\|_{1,\Omega_{h,i}^\pm}}.$$

However, for $\mathbf{w}_h^\pm \in \mathbf{V}_h^{\text{bulk}} \cap \mathbf{H}_0^1(\Omega_{h,i}^\pm)$ we notice equalities

$$(3.18) \quad \frac{(\text{div } \mathbf{w}_h^\pm, p_h^\pm)_{\Omega_{h,i}^\pm}}{\|\mathbf{w}_h^\pm\|_{1,\Omega_{h,i}^\pm}} = \frac{b_h(\mathbf{v}_h, p_h)}{\|\mathbf{v}_h\|_{1,h}} = \frac{\mathbf{F}_h(\mathbf{v}_h) - a_h(\mathbf{v}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{1,h}},$$

where $\mathbf{v}_h = (\mathbf{w}_h^-, 0)$ or $\mathbf{v}_h = (0, \mathbf{w}_h^+)$. Because \mathbf{v}_h is supported on $\Omega_{h,i}^\pm$ we have

$$(3.19) \quad a_h(\mathbf{u}_h, \mathbf{v}_h) := (\nu^- D(\mathbf{u}_h^-), D(\mathbf{v}_h^-))_{\Omega^-} + (\nu^+ D(\mathbf{u}_h^+), D(\mathbf{v}_h^+))_{\Omega^+} + \mathbf{J}_h(\mathbf{u}_h, \mathbf{v}_h),$$

With the help of the Cauchy-Schwarz inequality, inverse estimates and Korn's inequality we obtain

$$(3.20) \quad |a_h(\mathbf{u}_h, \mathbf{v}_h)| \leq C\|\nu^\pm D(\mathbf{u}_h^\pm)\|_{\Omega^\pm} \|\mathbf{v}_h\|_{1,h}.$$

Using (3.18)–(3.19) in (3.17), we arrive at

$$(3.21) \quad \|q_h^\pm\|_{\Omega^\pm} \leq C(\|\nu^\pm D(\mathbf{u}_h^\pm)\|_{\Omega^\pm} + (\nu^\pm J_h^\pm(p_h^\pm, p_h^\pm))^{1/2} + \|\mathbf{F}_h\|_{-1,h}),$$

which leads to (3.14).

In order to prove (3.15), we consider $\alpha_h = p_h - q_h$ and observe that $\|\alpha_h\|_{L^\infty(\Omega)} \leq C\|p_h\|_\Omega$. Moreover, a simple calculation shows that

$$(\alpha_h, 1)_\Omega = \frac{|\Omega^+|}{|\Omega_{h,i}^+|} (p_h^+, 1)_{\Omega^+} + \frac{|\Omega^+|}{|\Omega_{h,i}^+|} (p_h^-, 1)_{\Omega^-} - \frac{|\Omega^+|}{|\Omega_{h,i}^+|} (p_h^+, 1)_{\Omega^+ \setminus \Omega_{h,i}^+} - \frac{|\Omega^-|}{|\Omega_{h,i}^-|} (p_h^-, 1)_{\Omega^- \setminus \Omega_{h,i}^-}.$$

Hence, using that $(p_h^-, 1)_{\Omega^-} + (p_h^+, 1)_{\Omega^+} = 0$ and that $|\Omega_h^\pm \setminus \Omega_{h,i}^\pm| = |\Omega_h^\pm| - |\Omega_{h,i}^\pm| \leq Ch$ we have

$$(3.22) \quad \begin{aligned} |(\alpha_h, 1)_\Omega| &\leq C \left(|(p_h^+, 1)_{\Omega^+ \setminus \Omega_{h,i}^+}| + |(p_h^-, 1)_{\Omega^- \setminus \Omega_{h,i}^-}| \right) + \left| \frac{|\Omega^+|}{|\Omega_{h,i}^+|} - \frac{|\Omega^-|}{|\Omega_{h,i}^-|} \right| |(p_h^+, 1)_{\Omega^+}| \\ &\leq C(\|p_h^+\|_{\Omega^+} \|1\|_{\Omega^+ \setminus \Omega_{h,i}^+} + \|p_h^-\|_{\Omega^-} \|1\|_{\Omega^- \setminus \Omega_{h,i}^-} + h\|p_h^+\|_{L^1(\Omega^+)}) \leq Ch^{\frac{1}{2}}\|p_h\|_\Omega. \end{aligned}$$

In the case that $M_h^{\text{bulk}} = M_{h,\text{disc}}$ we let $r_h \in M_h^{\text{bulk}}$ to be the L^2 projection of α_h onto piecewise constants with respect to the mesh \mathcal{T}_h . In the case, $M_h^{\text{bulk}} = M_{h,\text{cont}}$ we let $r_h \in M_h^{\text{bulk}}$ be the continuous piecewise linear function such that $r_h(x) = \alpha_h^+(x)$ if x is a vertex and $x \in \Omega^+$, and $r_h(x) = \alpha_h^-(x)$ if x is a vertex and $x \in \Omega^-$. In either case, $r_h|_{\Omega_{h,i}^\pm} = \alpha_h^\pm$.

Recalling the notation $\omega_h = \Omega \setminus (\Omega_{h,i}^- \cup \Omega_{h,i}^+)$, we then note that

$$(3.23) \quad \|r_h - \alpha_h\|_\Omega = \|r_h - \alpha_h\|_{\omega_h} \leq Ch^{\frac{1}{2}} \|r_h - \alpha_h\|_{L^\infty(\omega_h)} \leq Ch^{\frac{1}{2}} \|\alpha_h\|_{L^\infty(\omega_h)} \leq Ch^{\frac{1}{2}} \|p_h\|_\Omega.$$

We let $\tilde{r}_h = r_h - \text{avg}_\Omega(r_h)$. From (3.22) and (3.23) we have that

$$(3.24) \quad \|\tilde{r}_h - r_h\|_\Omega \leq Ch^{\frac{1}{2}} \|p_h\|_\Omega \quad \text{which implies} \quad \|\tilde{r}_h - \alpha_h\|_\Omega \leq Ch^{\frac{1}{2}} \|p_h\|_\Omega.$$

Assumption 1 provides us with $\mathbf{v}_h \in \mathbf{V}_h^{\text{bulk}}$ such that

$$(3.25) \quad \xi \|\tilde{r}_h\|_\Omega \leq (\text{div } \mathbf{v}_h, \tilde{r}_h)_\Omega \quad \text{and} \quad \|\mathbf{v}_h\|_{1,\Omega} = 1.$$

Let $\mathbf{w}_h \in \mathbf{V}_h$ be given by $\mathbf{w}_h = (\mathbf{v}_h|_{\Omega_h^-}, \mathbf{v}_h|_{\Omega_h^+})$. It holds $\|\mathbf{w}_h\|_{1,h} \leq C \|\mathbf{v}_h\|_{1,\Omega}$. To verify the last inequality, we note that the first term in the definition of $\|\mathbf{w}_h\|_{1,h}$ vanish, while the second jump term can be estimated with the help of the finite element trace and inverse inequalities. Hence, we get

$$\begin{aligned} (\text{div } \mathbf{v}_h, \tilde{r}_h)_\Omega &= (\text{div } \mathbf{v}_h, p_h^-)_{\Omega^-} + (\text{div } \mathbf{v}_h, p_h^+)_{\Omega^+} \\ &\quad + (\text{div } \mathbf{v}_h, \tilde{r}_h^- - p_h^-)_{\Omega^-} + (\text{div } \mathbf{v}_h, \tilde{r}_h^+ - p_h^+)_{\Omega^+} \\ &= a_h(\mathbf{u}_h, \mathbf{w}_h) - \mathbf{F}_h(\mathbf{w}_h) + (\text{div } \mathbf{v}_h, \tilde{r}_h^- - p_h^-)_{\Omega^-} + (\text{div } \mathbf{v}_h, \tilde{r}_h^+ - p_h^+)_{\Omega^+}, \end{aligned}$$

where we used the first equation of (3.7). Thanks to (3.12) and (3.25) we have

$$(3.26) \quad \begin{aligned} \xi \|\tilde{r}_h\|_\Omega &\leq C(\|\mathbf{u}_h\|_{1,h,\nu} + \|\mathbf{F}_h\|_{-1,h} + \|\tilde{r}_h^- - p_h^-\|_{\Omega^-} + \|\tilde{r}_h^+ - p_h^+\|_{\Omega^+}) \\ &\leq C(\|\mathbf{u}_h\|_{1,h,\nu} + \|\mathbf{F}_h\|_{-1,h} + \|\tilde{r}_h - \alpha_h\|_\Omega + \|q_h^-\|_{\Omega^-} + \|q_h^+\|_{\Omega^+}). \end{aligned}$$

We then use the triangle inequality, (3.21), (3.24), and (3.26) to get

$$\begin{aligned} \|p_h\|_\Omega &\leq \|q_h^-\|_{\Omega^-} + \|q_h^+\|_{\Omega^+} + \|\alpha_h - \tilde{r}_h\|_\Omega + \|\tilde{r}_h\|_\Omega \\ &\leq C(\|\mathbf{u}_h\|_{1,h,\nu} + \|\mathbf{F}_h\|_{-1,h} + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2} + (\nu^+ J_h^+(p_h^+, p_h^+))^{1/2} + h^{\frac{1}{2}} \|p_h\|_\Omega). \end{aligned}$$

The result now follows after taking h_0 small enough. \square

In order to prove (3.8), we start by obtaining an estimate for $\|\nu^- D(\mathbf{u}_h^-)\|_{\Omega^-}$ using the energy norm (3.9).

Lemma 3.6. *Let $(\mathbf{u}_h, p_h) \in V_h \times M_h$ be a solution of (3.7). Then, there exists $C > 0$, independent of h and ν^\pm , such that*

$$(3.27) \quad \begin{aligned} \|\nu^- D(\mathbf{u}_h^-)\|_{\Omega^-}^2 + \nu^- \mathbf{J}_h^-(\mathbf{u}_h^-, \mathbf{u}_h^-) + \nu^- J_h^-(p_h^-, p_h^-) \\ \leq C(\|\mathbf{F}_h\|_{-1,h} + \nu^- \|G_h^-\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h})^2. \end{aligned}$$

Proof. We use the first and second equation of (3.7) with $\mathbf{v}_h = \mathbf{u}_h$ and $q_h = p_h$, respectively, and the coercivity of a_h for γ large enough to get

$$(3.28) \quad \alpha \|\mathbf{u}_h\|_{\mathbf{V}_h}^2 + J_h(p_h, p_h) \leq a_h(\mathbf{u}_h, \mathbf{u}_h) + J_h(p_h, p_h) \leq |\mathbf{F}_h(\mathbf{u}_h)| + |G_h(p_h)|,$$

with some $\alpha > 0$ independent of ν and h . By definition of the norms $\|\cdot\|_{1,h}$ and $\|\cdot\|_{\mathbf{V}_h}$, and since $\nu^- \leq \nu^+$, we get

$$(3.29) \quad |\mathbf{F}_h(\mathbf{u}_h)| \leq \|\mathbf{F}_h\|_{-1,h} \|\mathbf{u}_h\|_{1,h} \leq \frac{\|\mathbf{F}_h\|_{-1,h} \|\mathbf{u}_h\|_{\mathbf{V}_h}}{\sqrt{\nu^-}},$$

so that it only remains to estimate $|G_h(p_h)|$. In order to do this, we will decompose the expression $G_h(p_h) = G_h^-(p_h^-) + G_h^+(p_h^+)$ into three terms:

$$G_h(p_h) = G_h^-(p_h^- - \alpha_h^-) + G_h^+(p_h^+ - \alpha_h^+) + G_h(\alpha_h),$$

where $\alpha_h^\pm := \text{avg}_{\Omega_{h,i}^\pm}(p_h^\pm)$. Then, using (3.14) and $\nu^- \leq \nu^+$ we get

$$|G_h^\pm(p_h^\pm - \alpha_h^\pm)| \leq C\sqrt{\nu^\pm} \|G_h^\pm\|_{\pm,h} (\|\sqrt{\nu^\pm} D(\mathbf{u}_h^\pm)\|_{\Omega^\pm} + J_h^\pm(p_h^\pm, p_h^\pm)^{1/2}) + C\|G_h^\pm\|_{\pm,h} \|\mathbf{F}_h\|_{-1,h}.$$

On the other hand,

$$|G_h(\alpha_h)| = |G_h^+(\alpha_h^+) + G_h^-(\alpha_h^-)| = |G_h^+(\alpha_h^+ - \alpha_h^-)| \leq \|G_h^+\|_{+,h} \|\alpha_h^+ - \alpha_h^-\|_{\Omega^+} \leq C\|G_h^+\|_{+,h} \|p_h\|_{\Omega},$$

where we used our assumption that $G_h^+(1) = -G_h^-(1)$. Therefore, we have

(3.30)

$$|G_h(p_h)| \leq C(\sqrt{\nu^-} \|G_h^-\|_{-1,h} + \sqrt{\nu^+} \|G_h^+\|_{-1,h}) (\|\mathbf{u}_h\|_{\mathbf{V}_h} + J_h(p_h, p_h)^{1/2} + \frac{1}{\sqrt{\nu^-}} \|\mathbf{F}_h\|_{-1,h} + \frac{1}{\sqrt{\nu^+}} \|p_h\|_{\Omega}),$$

where we used that $\nu^- \leq \nu^+$. After re-scaling by $\frac{1}{\sqrt{\nu^+}}$ the estimate (3.15) yields for the last term on the right-hand side of (3.30) the bound,

$$\frac{1}{\sqrt{\nu^+}} \|p_h\|_{\Omega} \leq C(\|\mathbf{u}_h\|_{\mathbf{V}_h} + J_h(p_h, p_h)^{1/2} + \frac{1}{\sqrt{\nu^-}} \|\mathbf{F}_h\|_{-1,h}),$$

again due to $\nu^- \leq \nu^+$. Using now (3.29) and (3.30) to bound from above the right-hand side of (3.28) leads after some calculations to

$$\alpha \|\mathbf{u}_h\|_{\mathbf{V}_h}^2 + J_h(p_h, p_h) \leq C(\sqrt{\nu^-} \|G_h^-\|_{-1,h} + \sqrt{\nu^+} \|G_h^+\|_{-1,h}) (\|\mathbf{u}_h\|_{\mathbf{V}_h} + J_h(p_h, p_h)^{1/2} + \frac{1}{\sqrt{\nu^-}} \|\mathbf{F}_h\|_{-1,h}).$$

Therefore, we have

$$\|\mathbf{u}_h\|_{\mathbf{V}_h}^2 + J_h(p_h, p_h) \leq C(\sqrt{\nu^-} \|G_h^-\|_{-1,h} + \sqrt{\nu^+} \|G_h^+\|_{-1,h} + \frac{1}{\sqrt{\nu^-}} \|\mathbf{F}_h\|_{-1,h})^2.$$

The result now follows by multiplying both sides by ν^- and using that $\nu^- \leq \nu^+$. \square

As an immediate consequence of Lemma 3.6 and (3.14), we have the following result.

Lemma 3.7. *Let $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in M_h$ solve (3.7). Then, there exist $C, h_0 > 0$, depending only on Ω and Γ , such that for $h \leq h_0$,*

$$(3.31) \quad \|p_h^- - \text{avg}_{\Omega_{h,i}^-}(p_h^-)\|_{\Omega^-} \leq C(\nu^- \|G_h^-\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h} + \|\mathbf{F}_h\|_{-1,h}).$$

The next result is the discrete analogue of Lemma 2.4.

Lemma 3.8. *Let $\mathbf{u}_h \in \mathbf{V}_h$ and $p_h \in M_h$ solve (3.7). Then, there exist $C, h_0 > 0$, depending only on Ω and Γ , such that for $h \leq h_0$ it holds*

$$(3.32) \quad \|\nu^+ D(\mathbf{u}_h^+)\|_{\Omega_h^+} + (\nu^+ J_h^+(p_h^+, p_h^+))^{1/2} \leq C(\nu^- \|G_h^-\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h} + \|\mathbf{F}_h\|_{-1,h}).$$

Proof. We first note that the rigid body motions belong to the velocity finite element space, and using Lemma 3.2, we consider the discrete extension $E_h : \mathbf{V}_h^+ \rightarrow \mathbf{V}_h^{\text{bulk}}$ and define $\mathbf{w}_h \in \mathbf{V}_h^{\text{bulk}}$ by

$$\mathbf{w}_h := \begin{cases} E_h \mathbf{u}_h^+ & \text{if } |\partial\Omega \cap \partial\Omega_h^+| > 0, \\ E_h(\mathbf{u}_h^+ - \mathcal{P}_{\Omega_h^+}^{\text{RM}}(\mathbf{u}_h^+)) & \text{if } |\partial\Omega \cap \partial\Omega_h^+| = 0. \end{cases}$$

Let $h_0 > 0$ be sufficiently small such that if Ω_h is the inclusion, then $\partial\Omega \cap \partial\Omega_h^+ = \emptyset$. Then, if Ω_h^+ is not the inclusion, we have by the boundedness of the extension E_h and by Korn's inequality that $\|\mathbf{w}_h\|_{1,\Omega} \leq \|\mathbf{u}_h^+\|_{1,\Omega_h^+} \leq C\|D(\mathbf{u}_h^+)\|_{\Omega_h^+}$. On the other hand, if Ω_h^+ is the inclusion, we have by the boundedness of the discrete extension E_h , Korn's inequality and since

$$(3.33) \quad D(\mathbf{w}_h^+) = D(\mathbf{u}_h^+) \quad \text{in } \Omega_h^+,$$

that $\|\mathbf{w}_h\|_{1,\Omega} \leq C\|\mathbf{u}_h^+ - \mathcal{P}_{\Omega_h^+}^{\text{RM}}(\mathbf{u}_h^+)\|_{1,\Omega_h^+} \leq C\|D(\mathbf{u}_h^+)\|_{\Omega_h^+}$. Either case, it holds

$$(3.34) \quad \|\mathbf{w}_h\|_{1,\Omega} \leq C\|D(\mathbf{u}_h^+)\|_{\Omega_h^+}.$$

We let $\mathbf{v}_h \in \mathbf{V}_h$ be given by $\mathbf{v}_h = (\mathbf{w}_h|_{\Omega_h^-}, \mathbf{w}_h|_{\Omega_h^+})$. By the same arguments as in the proof of Lemma 3.5, one shows $\|\mathbf{v}_h\|_{1,h} \leq C\|\mathbf{w}_h\|_{1,\Omega}$. First equation from (3.7) yields

$$(3.35) \quad \nu^+ \|D(\mathbf{u}_h^+)\|_{\Omega^+}^2 + \mathbf{J}_h^+(\mathbf{u}_h, \mathbf{u}_h) = a_h(\mathbf{u}_h, \mathbf{v}_h) + R = F_h(\mathbf{v}_h) - b_h(\mathbf{v}_h, p_h) + R,$$

where

$$R = -(\nu^- D(\mathbf{u}_h^-), D(\mathbf{v}_h))_{\Omega^-} - (\nu^- D(\mathbf{v}_h^-) \mathbf{n}^-, \llbracket \mathbf{u}_h \rrbracket)_{\Gamma} - \mathbf{J}_h^-(\mathbf{u}_h, \mathbf{v}_h).$$

We bound every term on the right hand side of (3.35). With the help of (3.6) and (3.11), we get for the last term

$$(3.36) \quad R \leq C\Xi \|\mathbf{v}_h\|_{1,h},$$

where

$$\Xi := \|\nu^- D(\mathbf{u}_h^-)\|_{\Omega^-}^2 + \sum_{T \in \mathcal{T}_h^\Gamma} \frac{\|\nu^- \llbracket \mathbf{u}_h \rrbracket\|_{T_\Gamma}^2}{h_T} + \nu^- \mathbf{J}_h^-(\mathbf{u}_h, \mathbf{u}_h)$$

consists of terms already estimated in Lemma 3.6. For the first term on the right-hand side of (3.35) we have

$$(3.37) \quad \mathbf{F}_h(\mathbf{v}_h) \leq \|\mathbf{F}_h\|_{-1,h} \|\mathbf{v}_h\|_{1,h} \leq \|\mathbf{F}_h\|_{-1,h} \|\mathbf{w}_h\|_{1,\Omega}.$$

It remains to estimate the second term on the right-hand side of (3.35). Using that (3.33) also implies $\text{div } \mathbf{w}_h^+ = \text{div } \mathbf{u}_h^+$ and $\Omega^+ \subset \Omega_h^+$, we get

$$-b_h(\mathbf{v}_h, p_h) = (\text{div } \mathbf{v}_h^-, p_h^-)_{\Omega^-} + (\text{div } \mathbf{v}_h^+, p_h^+)_{\Omega^+} = (\text{div } \mathbf{w}_h^-, p_h^-)_{\Omega^-} + (\text{div } \mathbf{u}_h^+, p_h^+)_{\Omega^+}.$$

Setting $\alpha_h^- = \text{avg}_{\Omega_{h,i}^-}(p_h^-)$ and using that $(\text{div } \mathbf{w}_h, 1)_{\Omega} = 0$, we have

$$\begin{aligned} (\text{div } \mathbf{w}_h^-, p_h^-)_{\Omega^-} &= (\text{div } \mathbf{w}_h^-, p_h^- - \alpha_h^-)_{\Omega^-} + (\text{div } \mathbf{w}_h^-, \alpha_h^-)_{\Omega^-} \\ &= (\text{div } \mathbf{w}_h^-, p_h^- - \alpha_h^-)_{\Omega^-} - \alpha_h^- (\text{div } \mathbf{w}_h^+, 1)_{\Omega^+} \\ &= (\text{div } \mathbf{w}_h^-, p_h^- - \alpha_h^-)_{\Omega^-} - (\text{div } \mathbf{u}_h^+, \alpha_h^-)_{\Omega^+} \\ &= (\text{div } \mathbf{w}_h^-, p_h^- - \alpha_h^-)_{\Omega^-} - G_h^+(\alpha_h^-) \\ &\leq \|\mathbf{w}_h\|_{1,\Omega^-} \|p_h^- - \alpha_h^-\|_{\Omega^-} + \|G_h^+\|_{-1,h} \|\alpha_h^-\|_{\Omega^+}. \end{aligned}$$

We also have from the second equation of (3.7) with $q_h = (0, p_h^+)$

$$(\text{div } \mathbf{u}_h^+, p_h^+)_{\Omega^+} = -G_h^+(p_h^+) - J_h^+(p_h^+, p_h^+) \leq \|G_h^+\|_{-1,h} \|p_h^+\|_{+,h} - J_h^+(p_h^+, p_h^+).$$

Now we see that

$$(3.38) \quad -b_h(\mathbf{v}_h, p_h) \leq \|\mathbf{w}_h\|_{1,\Omega^-} \|p_h^- - \alpha_h^-\|_{\Omega^-} + \|G_h^+\|_{-1,h} (\|p_h^+\|_{+,h} + \|\alpha_h^-\|_{\Omega^+}) - J_h^+(p_h^+, p_h^+).$$

We combine (3.38), (3.37), (3.36), and (3.35) to get

$$(3.39) \quad L \leq (C\Xi + \|\mathbf{F}_h\|_{-1,h} + \|p_h^- - \alpha_h^-\|_{\Omega^-})\nu^+ \|\mathbf{w}_h\|_{1,\Omega} + \nu^+ \|G_h^+\|_{-1,h} (\|p_h^+\|_{+,h} + \|\alpha_h^-\|_{\Omega^+}),$$

where

$$L = \|\nu^+ D(\mathbf{u}_h^+)\|_{\Omega^+}^2 + \nu^+ \mathbf{J}_h^+(\mathbf{u}_h, \mathbf{u}_h) + \nu^+ J_h^+(p_h^+, p_h^+).$$

It remains to estimate the solution-dependent terms on the right-hand side of (3.39). Thanks to (3.15) we get

$$\begin{aligned} \|p_h^+\|_{+,h} + \|\alpha_h^-\|_{\Omega^+} &\leq (\|p_h\|_{\Omega} + (\nu^+ J_h^+(p_h^+, p_h^+))^{1/2}) \\ &\leq C(\|\mathbf{u}_h\|_{1,h,\nu} + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2} + (\nu^+ J_h^+(p_h^+, p_h^+))^{1/2} + \|\mathbf{F}_h\|_{-1,h}) \\ &\leq C(\Xi + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2} + \|\mathbf{F}_h\|_{-1,h}) + C\sqrt{L}. \end{aligned}$$

From (3.34) we conclude that $\nu^+ \|\mathbf{w}_h\|_{1,\Omega} \leq C\nu^+ \|D(\mathbf{u}_h^+)\|_{\Omega^+} \leq C\sqrt{L}$ and hence we have

$$\begin{aligned} L &\leq C(\Xi + \|p_h^- - \alpha_h^-\|_{\Omega^-} + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2} + \|\mathbf{F}_h\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h})\sqrt{L} \\ &\quad + C\nu^+ \|G_h^+\|_{-1,h} (\|\mathbf{F}_h\|_{-1,h} + \Xi + (\nu^- J_h^-(p_h^-, p_h^-))^{1/2}). \end{aligned}$$

The result now follows after using (3.31) and (3.27). \square

As an immediate consequence of Lemma 3.8 and (3.15), we have the following result

Theorem 3.2. *Let $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times M_h$ be a solution of the finite element method (3.7). Then, there exists $C, h_0 > 0$, independent of h , position of Γ against the mesh and ν^\pm , such that for $h \leq h_0$, it holds*

$$\|\nu D(\mathbf{u}_h)\|_{\Omega} + \|p_h\|_{\Omega} \leq C(\nu^- \|G_h^-\|_{-1,h} + \nu^+ \|G_h^+\|_{-1,h} + \|\mathbf{F}_h\|_{-1,h}).$$

4. FINITE ELEMENT ERROR ESTIMATES

We use the stability results from the previous sections to obtain error estimates for both $\|\nu D(\mathbf{u} - \mathbf{u}_h)\|_{\Omega}$ and $\|p - p_h\|_{\Omega}$. We will assume our solutions are sufficiently smooth to make the method consistent. In particular, we assume that $\mathbf{u}^\pm \in \mathbf{H}^{s+1}(\Omega^\pm)$ and $p^\pm \in H^{k_p+1}(\Omega^\pm)$. In such a case there exist extensions of \mathbf{u} and p , from Ω^\pm to Ω^\mp , denoted by \mathbf{u}_E^\pm and p_E^\pm and with the property $\mathbf{u}_E^\pm|_{\Omega^\pm} = \mathbf{u}^\pm$ and $p_E^\pm|_{\Omega^\pm} = p^\pm$, such that $\mathbf{u}_E^\pm \in \mathbf{H}^{s+1}(\Omega)$, $p_E^\pm \in H^{k_p+1}(\Omega)$,

$$\|\mathbf{u}_E^\pm\|_{s+1,\Omega} \leq C\|\mathbf{u}^\pm\|_{s+1,\Omega^\pm}, \quad \text{and} \quad \|p_E^\pm\|_{k_p+1,\Omega} \leq C\|p^\pm\|_{k_p+1,\Omega^\pm},$$

where C depends only on Ω and Γ . From now on, we will identify \mathbf{u}^\pm and p^\pm with there extensions.

For the error analysis in addition to the augmented norm for the velocity (3.2) we also define

$$\|q_h\|_{h,\star}^2 := \|q_h\|_h^2 + \sum_{T \in \mathcal{T}_h^\Gamma} h_T (\|q_h^-\|_{T_\Gamma}^2 + \|q_h^+\|_{T_\Gamma}^2).$$

Multiplying the first equation of (1.1) with \mathbf{v}^\pm , using that $\llbracket \sigma(\mathbf{u}, p) \mathbf{n} \rrbracket = \boldsymbol{\lambda}$, and taking into account the choice of the weights for the average $\{\mathbf{v}\}_\Gamma$ in the definition of a_h , we define \mathbf{F}_h by

$$(4.1) \quad \mathbf{F}_h(\mathbf{v}_h) := (\mathbf{f}^-, \mathbf{v}_h^-)_{\Omega^-} + (\mathbf{f}^+, \mathbf{v}_h^+)_{\Omega^+} + (\boldsymbol{\lambda}, \mathbf{v}_h^+)_{\Gamma}.$$

Then, we have the main result of this section.

Theorem 4.1. *Let $(\mathbf{u}, p) \in \mathbf{V} \times M$ be a solution of (2.1) with $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $g \in L_0^2(\Omega)$. Furthermore, let (\mathbf{u}_h, p_h) be the approximation that solves (3.7) with \mathbf{F}_h given by (4.1) and $G_h^\pm(q_h^\pm) = -(g, q_h^\pm)_{\Omega^\pm}$. Assume that $\mathbf{u}^\pm \in \mathbf{H}^{s+1}(\Omega^\pm)$ and $p^\pm \in H^{k_p+1}(\Omega^\pm)$. Then there exists a constant C independent of ν, h, \mathbf{u}, p such that*

$$(4.2) \quad \|\nu D(\mathbf{u} - \mathbf{u}_h)\|_\Omega + \|p - p_h\|_\Omega \leq C \inf_{\mathbf{w}_h \in \mathbf{V}_h, r_h \in M_h} (\|\mathbf{u} - \mathbf{w}_h\|_{1,h,\nu,\star} + \|p - r_h\|_{h,\star}).$$

Proof. As discussed above we let \mathbf{u}^\pm and p^\pm be the extensions of the same functions to the entire Ω . Then, by using (4.1), we one easily checks the consistency result:

$$\begin{aligned} a_h(\mathbf{u}, \mathbf{v}_h) + b_h(\mathbf{v}_h, p) &= \mathbf{F}_h(\mathbf{v}_h), \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h, \\ b_h(\mathbf{u}, q_h) - J_h(p, q_h) &= G_h(q_h), \quad \text{for all } q_h \in M_h, \end{aligned}$$

Hence, we have for an arbitrary $\mathbf{w}_h \in \mathbf{V}_h$ and $r_h \in Q_h$ the following consistency result:

$$(4.3) \quad a_h(\mathbf{u}_h - \mathbf{w}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p_h - r_h) = \mathbf{L}_h(\mathbf{v}_h) \quad \text{for all } \mathbf{v}_h \in \mathbf{V}_h,$$

$$(4.4) \quad b_h(\mathbf{u}_h - \mathbf{w}_h, q_h) - J_h(p_h - r_h, q_h) = Q_h(q_h) \quad \text{for all } q_h \in M_h.$$

where

$$\mathbf{L}_h(\mathbf{v}_h) := a_h(\mathbf{u} - \mathbf{w}_h, \mathbf{v}_h) + b_h(\mathbf{v}_h, p - r_h), \quad Q_h(q_h) := Q_h^-(q_h) + Q_h^+(q_h),$$

and

$$\begin{aligned} Q_h^-(q_h) &:= -(q_h^-, \operatorname{div}(\mathbf{u}^- - \mathbf{w}_h^-))_{\Omega^-} - (q_h^-, \llbracket \mathbf{u} - \mathbf{w}_h \rrbracket \cdot \mathbf{n}^-)_{\Gamma} - J_h^-(p^- - r_h^-, q_h^-), \\ Q_h^+(q_h) &:= -(q_h^+, \operatorname{div}(\mathbf{u}^+ - \mathbf{w}_h^+))_{\Omega^+} - J_h^+(p^+ - r_h^+, q_h^+). \end{aligned}$$

We can easily show, using for example (3.13), the following bound

$$|\mathbf{L}_h(\mathbf{v}_h)| \leq (\|\mathbf{u} - \mathbf{w}_h\|_{1,h,\nu,\star} + \|p - r_h\|_{h,\star}) \|\mathbf{v}_h\|_{1,h},$$

which implies

$$\|\mathbf{L}_h\|_{-1,h} \leq (\|\mathbf{u} - \mathbf{w}_h\|_{1,h,\nu,\star} + \|p - r_h\|_{h,\star}).$$

Similarly,

$$\nu^\pm |Q_h^\pm(q_h)| \leq C(\|\mathbf{u} - \mathbf{w}_h\|_{1,h,\nu,\star} + \|p - r_h\|_{h,\star}) \|q_h^\pm\|_{\pm,h}.$$

Hence,

$$\nu^- \|Q_h^-\|_{-1,h} + \nu^+ \|Q_h^+\|_{-1,h} \leq C(\|\mathbf{u} - \mathbf{w}_h\|_{1,h,\nu,\star} + \|p - r_h\|_{h,\star}).$$

The result follows after applying Theorem 3.2 and triangle inequality. \square

Using (3.3) and (3.4) with standard interpolation properties of finite element functions and the definition of the norms of the right-hand side of (4.2) the next results follows from the theorem.

Corollary 4.1. Under the same assumptions as in Theorem 4.1, it holds:

$$\begin{aligned} (4.5) \quad & \|\nu D(\mathbf{u} - \mathbf{u}_h)\|_\Omega + \|p - p_h\|_\Omega \\ & \leq Ch^{\min\{k_u, k_p+1\}} (\|\nu \mathbf{u}\|_{k_u+1,\Omega} + \|p\|_{k_p+1,\Omega}) + Ch^{k_u} \sum_{\ell=2}^{s+1-k_u} h^{\ell-1} \|\nu \mathbf{u}\|_{k_u+\ell,\Omega}, \end{aligned}$$

with a constant C independent of ν , h and the position of the interface against the background mesh. The solution norms on the right-hand side of (4.5) are the norms in the broken Sobolev spaces $H^\ell(\Omega^-) \times H^\ell(\Omega^+)$,

$$\|q\|_{\ell,\Omega}^2 = \|q\|_{\ell,\Omega^-}^2 + \|q\|_{\ell,\Omega^+}^2, \quad \text{for } q \in H^\ell(\Omega^-) \times H^\ell(\Omega^+),$$

and similar for vector functions from $H^\ell(\Omega^-)^d \times H^\ell(\Omega^+)^d$.

5. NUMERICAL EXPERIMENTS

Example 1. Consider the squared domain $\Omega := (-1, 1) \times (-1, 1)$ and the embedded interface $\Gamma : x_1^2 + x_2^2 = R^2$ for $R = 1/\sqrt{\pi}$. We define $\Omega^- = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < R\}$, and $\Omega^+ = \Omega \setminus \overline{\Omega}^-$. We choose the data $\mathbf{f} \in \mathbf{L}^2(\Omega)$ so that the exact solution (\mathbf{u}, p) is given for all $\mathbf{x} = (x_1, x_2)$ by

$$(5.1) \quad \mathbf{u}(\mathbf{x}) = \begin{cases} \frac{R^2 - r^2}{\nu^-} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & |\mathbf{x}| < R \\ \frac{R^2 - r^2}{\nu^+} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & |\mathbf{x}| \geq R \end{cases}, \quad \text{and} \quad p(\mathbf{x}) = x_2^2 - x_1^2.$$

We observe that $[\![\mathbf{u}]\!] = \mathbf{0}$ and $[\![\nu D(\mathbf{u})]\!] = 0$, and since p is continuous, we get $[\![\sigma(\mathbf{u}, p)\mathbf{n}]\!] = \mathbf{0}$. Also, \mathbf{u} is divergence free. We will show that the errors $\|\nu D(\mathbf{u} - \mathbf{u}_h)\|_\Omega$ and $\|p - p_h\|_\Omega$ are independent of ν . In order to do this, we consider a uniform diagonal triangular decomposition of Ω , and test the code with two pairs of spaces $\mathbf{V}_h^{\text{bulk}} \times M_h^{\text{bulk}}$ that satisfy Assumption 4. These are the Mini-element and the pair $P_2 - P_0$. We choose a mesh with $N = 160092$ degrees of freedom for the Mini-element and $N = 42199$ degrees of freedom for the pair $P_2 - P_0$.

We denote the errors by $\mathbf{e}(\mathbf{u}) := \|\nu D(\mathbf{u} - \mathbf{u}_h)\|_\Omega$ and $\mathbf{e}(p) := \|p - p_h\|_\Omega$ and compute them experimentally, with decreasing values of ν^- and increasing values of ν^+ . As a good balance between stability and conditioning, we set the stabilization parameters to $\gamma = 25$, $\gamma_{\mathbf{u}}^\pm = 15$ and $\gamma_p^\pm = 20$ for the Mini-element and $\gamma = 20$, $\gamma_{\mathbf{u}}^\pm = 15$ and $\gamma_p^\pm = 0.1$ for the pair $P_2 - P_0$. The numerical results are summarized in Table 5.1.

Parameters		$P_2 - P_0$		Mini-element	
ν^-	ν^+	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(p)$
$1E - 01$	$1E + 01$	0.023580	0.004914	0.062001	0.004099
$1E - 02$	$1E + 02$	0.023580	0.004914	0.062001	0.004099
$1E - 03$	$1E + 03$	0.023580	0.004914	0.062001	0.004099
$1E - 04$	$1E + 04$	0.023580	0.004914	0.062001	0.004099

Table 5.1. Example 1 with Ω^- completely interior. Errors are shown for a fixed mesh, increasing values of ν^+ and decreasing values of ν^- .

From Table 5.1, we observe that the errors $\mathbf{e}(\mathbf{u})$ and $\mathbf{e}(p)$ remain unchanged for a fixed mesh when ν^- decreases and ν^+ increases.

Now we switch the role of Ω^+ and Ω^- , and define $\Omega^+ = \{\mathbf{x} \in \mathbb{R}^2 : |\mathbf{x}| < R\}$, $\Omega^- = \Omega \setminus \overline{\Omega}^+$. Observe that Ω^+ is now the inclusion. We choose the data $\mathbf{f} \in \mathbf{L}^2(\Omega)$ so that the exact solution

(\mathbf{u}, p) is given by

$$\mathbf{u}(\mathbf{x}) = \begin{cases} \frac{R^2 - r^2}{\nu^+} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & |\mathbf{x}| < R \\ \frac{R^2 - r^2}{\nu^-} \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, & |\mathbf{x}| \geq R \end{cases}, \quad \text{and} \quad p(\mathbf{x}) = x_2^2 - x_1^2.$$

We choose a mesh with $N = 318782$ degrees of freedom for the Mini-element and $N = 42199$ degrees of freedom for the pair $P_2 - P_0$, and as a good balance between stability and conditioning we set the stabilization parameters to $\gamma = 60$, $\gamma_{\mathbf{u}}^{\pm} = 30$ and $\gamma_p^{\pm} = 35$ for the Mini-element, and $\gamma = 25$, $\gamma_{\mathbf{u}}^{\pm} = 15$ and $\gamma_p^{\pm} = 0.01$ for the pair $P_2 - P_0$, consider decreasing values of ν^- and increasing values of ν^+ , and summarize the results in Table 5.2.

Parameters		$P_2 - P_0$		Mini-element	
ν^-	ν^+	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{e}(\mathbf{u})$	$\mathbf{e}(p)$
$1E - 01$	$1E + 01$	0.023559	0.005172	0.046267	0.001663
$1E - 02$	$1E + 02$	0.023559	0.005172	0.046267	0.001663
$1E - 03$	$1E + 03$	0.023559	0.005172	0.046267	0.001663
$1E - 04$	$1E + 04$	0.023559	0.005172	0.046267	0.001663

Table 5.2. Example 1 with Ω^+ completely interior. Errors are shown for a fixed mesh, increasing values of ν^+ and decreasing values of ν^- .

Similarly, we observe that the errors $\mathbf{e}(\mathbf{u})$ and $\mathbf{e}(p)$ remain unchanged for a fixed mesh when ν^- decreases and ν^+ increases.

Example 2. We consider the same exact solution (\mathbf{u}, p) given by (5.1), and the finite element errors

$$\mathbf{e}(\mathbf{u}) := \|\nu D(\mathbf{u} - \mathbf{u}_h)\|_{\Omega}, \quad \mathbf{e}(\mathbf{p}) := \|p - p_h\|_{\Omega}, \quad \text{and} \quad \mathbf{e}(\mathbf{u}, p) = \|\nu D(\mathbf{u} - \mathbf{u}_h)\|_{\Omega} + \|p - p_h\|_{\Omega}.$$

We test the method for $P_2 - P_0$ bulk spaces and fixed viscosity and stabilization parameters $\nu^- = 0.5$, $\nu^+ = 20$, $\gamma = 20$, $\gamma_{\mathbf{u}}^{\pm} = 10$ and $\gamma_p^{\pm} = 15$. We consider a sequence uniform triangular meshes with decreasing mesh size. The experimental rates of convergence are defined by

$$\mathbf{r}(\mathbf{u}) = \frac{\log(\mathbf{e}(\mathbf{u}_j)/\mathbf{e}(\mathbf{u}_{j-1}))}{\log(h_j/h_{j-1})}, \quad \mathbf{r}(\mathbf{p}) = \frac{\log(\mathbf{e}(p_j)/\mathbf{e}(p_{j-1}))}{\log(h_j/h_{j-1})}, \quad \mathbf{r}(\mathbf{u}, p) = \frac{\log(\mathbf{e}(\mathbf{u}_j, p_j)/\mathbf{e}(\mathbf{u}_{j-1}, p_{j-1}))}{\log(h_j/h_{j-1})}.$$

The error norms and experimental rates are shown in Table 5.3. Also, we show plots of the approximate solution (\mathbf{u}_h, p_h) in Figure 5.1.

We repeat the experiment using the Mini-element and the same set parameters and meshes. The computed error norms and experimental rates are shown in Table 5.4.

Example 3. As a final example, we consider $\nu^- = 0.5$, $\nu^+ = 2$, and the exact solution (\mathbf{u}, p) given by

$$\mathbf{u}(\mathbf{x}) = \frac{1}{\pi} \begin{pmatrix} \sin \pi x_1 \sin \pi x_2 \\ \cos \pi x_1 \cos \pi x_2 \end{pmatrix}, \quad \text{and} \quad p(\mathbf{x}) = \begin{cases} x_1^2 + x_2^2, & |\mathbf{x}| < R, \\ -\frac{1}{6\pi}, & |\mathbf{x}| > R \end{cases}.$$

dofs	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$	$\mathbf{e}(\mathbf{u}, p)$	$\mathbf{r}(\mathbf{u}, p)$
815	$5.9E - 01$	—	$9.8E - 01$	—	$1.1E + 00$	—
2867	$2.6E - 01$	1.201	$2.2E - 01$	2.157	$3.4E - 01$	1.761
10867	$1.1E - 01$	1.163	$1.0E - 01$	1.106	$1.5E - 01$	1.138
42199	$5.2E - 02$	1.139	$4.6E - 02$	1.154	$6.9E - 02$	1.145
166303	$2.5E - 02$	1.039	$2.2E - 02$	1.035	$3.4E - 02$	1.037
660243	$1.3E - 02$	1.010	$1.1E - 02$	0.998	$1.7E - 02$	1.005
1481863	$8.4E - 03$	1.004	$7.5E - 03$	0.996	$1.1E - 02$	1.000

Table 5.3. Example 2, errors for a sequence of uniform meshes, and fixed values of ν^\pm . The solution (\mathbf{u}_h, p_h) is approximated with $P_2 - P_0$ elements.

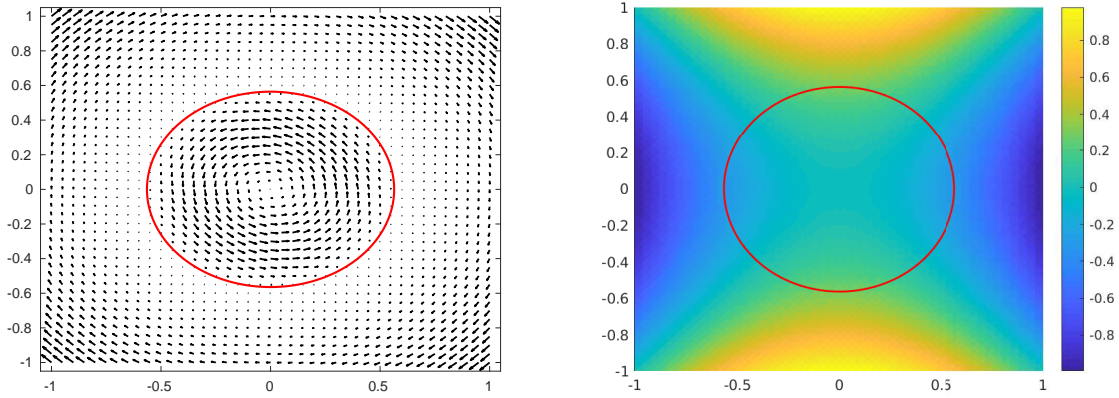


Figure 5.1. \mathbf{u}_h (left) and p_h (right) using the pair $P_2 - P_0$.

dofs	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$	$\mathbf{e}(\mathbf{u}, p)$	$\mathbf{r}(\mathbf{u}, p)$
604	$1.8E + 00$	—	$9.5E - 01$	—	$2.0E + 00$	—
2072	$7.5E - 01$	1.226	$2.0E - 01$	2.269	$7.8E - 01$	1.362
7736	$3.3E - 01$	1.189	$6.0E - 02$	1.716	$3.4E - 01$	1.213
29796	$1.5E - 01$	1.121	$2.0E - 02$	1.587	$1.5E - 01$	1.132
116924	$7.4E - 02$	1.043	$7.0E - 03$	1.507	$7.4E - 02$	1.049
463192	$3.7E - 02$	1.012	$2.5E - 03$	1.470	$3.7E - 02$	1.015
1038836	$2.4E - 02$	1.005	$1.4E - 03$	1.460	$2.4E - 02$	1.006

Table 5.4. Example 2, errors for a sequence of uniform meshes, and fixed values of ν^\pm . The solution (\mathbf{u}_h, p_h) is approximated with the Mini-element.

We observe that $[\![\mathbf{u}]\!] = \mathbf{0}$, and the jump $[\![\sigma(\mathbf{u}, p)\mathbf{n}]\!]$ is non-zero and is given by

$$[\![\sigma(\mathbf{u}, p)\mathbf{n}]\!] = \frac{-3 \cos \pi x_1 \sin \pi x_2}{2R} \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} - \frac{6\pi R^2 + 1}{6\pi R} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} =: \boldsymbol{\lambda}(\mathbf{x}).$$

Additionally, \mathbf{u} is divergence-free and $(p, 1)_\Omega = 0$. We test the code with the stabilization parameters $\gamma = 30$, $\gamma_{\mathbf{u}}^\pm = 25$ and $\gamma_p^\pm = 25$ for the Mini-element, $\gamma = 30$, $\gamma_{\mathbf{u}}^\pm = 25$ and $\gamma_p^\pm = 20$ for the pair $P_2 - P_0$, decompose the domain by a sequence of uniform meshes with decreasing size, and summarize the results in the Tables 5.5 and 5.6.

dofs	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$	$\mathbf{e}(\mathbf{u}, p)$	$\mathbf{r}(\mathbf{u}, p)$
815	$1.4E + 00$	—	$5.8E + 00$	—	$5.9E + 00$	—
2867	$4.6E - 01$	1.568	$2.9E - 01$	4.321	$5.4E - 01$	3.445
10867	$1.6E - 01$	1.484	$8.1E - 02$	1.824	$1.8E - 01$	1.564
42199	$7.2E - 02$	1.192	$3.3E - 02$	1.309	$7.9E - 02$	1.213
166303	$3.3E - 02$	1.130	$1.3E - 02$	1.359	$3.5E - 02$	1.164
660243	$1.6E - 02$	1.033	$6.0E - 03$	1.104	$1.7E - 02$	1.042
1481863	$1.1E - 02$	0.999	$4.0E - 03$	0.962	$1.1E - 02$	0.995

Table 5.5. Example 2, errors for a sequence of uniform meshes, and fixed values of ν^\pm . The solution (\mathbf{u}_h, p_h) is approximated with $P_2 - P_0$ elements.

dofs	$\mathbf{e}(\mathbf{u})$	$\mathbf{r}(\mathbf{u})$	$\mathbf{e}(p)$	$\mathbf{r}(p)$	$\mathbf{e}(\mathbf{u}, p)$	$\mathbf{r}(\mathbf{u}, p)$
604	$3.0E + 00$	—	$1.1E + 01$	—	$1.1E + 01$	—
2072	$1.5E + 00$	1.003	$1.7E + 00$	2.639	$2.3E + 00$	2.288
7736	$5.9E - 01$	1.329	$4.5E - 01$	1.927	$7.4E - 01$	1.609
29796	$2.4E - 01$	1.285	$1.4E - 01$	1.717	$2.8E - 01$	1.415
116924	$1.1E - 01$	1.159	$4.1E - 02$	1.732	$1.2E - 01$	1.261
463192	$5.2E - 02$	1.064	$1.0E - 02$	2.014	$5.3E - 02$	1.134
1038836	$3.4E - 02$	1.021	$4.8E - 03$	1.850	$3.4E - 02$	1.043

Table 5.6. Example 2, errors for a sequence of uniform meshes, and fixed values of ν^\pm . The solution (\mathbf{u}_h, p_h) is approximated with the Mini-element.

We finish by showing plots of the computed finite element solution (\mathbf{u}_h, p_h) in Figure 5.2.

REFERENCES

- [1] M. BOGOVSKII, *Solution of the first boundary value problem for the equation of continuity of an incompressible medium*, in Dokl. Akad. Nauk SSSR, vol. 248, 1979, pp. 1037–1040.
- [2] S. P. BORDAS, E. BURMAN, M. G. LARSON, AND M. A. OLSHANSKII, eds., *Geometrically Unfitted Finite Element Methods and Applications*, vol. 121 of Lecture Notes in Computational Science and Engineering, Springer, 2018.
- [3] J. U. BRACKBILL, D. B. KOTHE, AND C. ZEMACH, *A continuum method for modeling surface tension*, Journal of computational physics, 100 (1992), pp. 335–354.
- [4] F. BREZZI AND M. FORTIN, *Mixed and hybrid finite element methods*, vol. 15, Springer Science & Business Media, 1991.
- [5] E. BURMAN, J. GUZMÁN, M. SÁNCHEZ, AND M. SARKIS, *Robust flux error estimation of an unfitted Nitsche method for high-contrast interface problems*, IMA J. Numer. Anal., 38 (2017), pp. 646–668.
- [6] D. A. DREW, *Mathematical modeling of two-phase flow*, Annual review of fluid mechanics, 15 (1983), pp. 261–291.

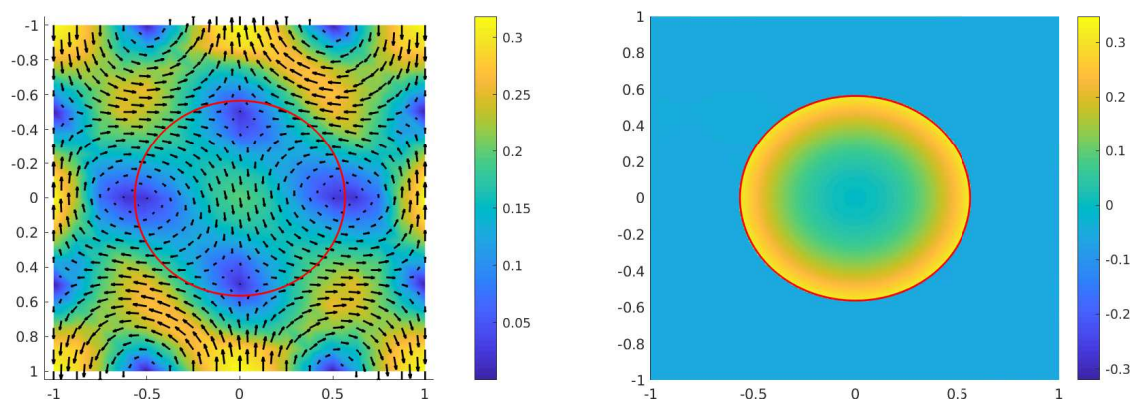


Figure 5.2. \mathbf{u}_h (left) and p_h (right) using the Mini-element. The colormap for the velocity vector field \mathbf{u}_h is $\|\mathbf{u}_h\|_2$ and for the pressure field p_h is the height $p_h(\mathbf{x})$.

- [7] V. GIRAULT AND P.-A. RAVIART, *Finite element methods for Navier-Stokes equations: theory and algorithms*, vol. 5, Springer Science & Business Media, 1986.
- [8] S. GROSS AND A. REUSKEN, *Numerical methods for two-phase incompressible flows*, vol. 40, Springer Science & Business Media, 2011.
- [9] J. GUZMÁN AND M. OLSHANSKII, *Inf-sup stability of geometrically unfitted Stokes finite elements*, Math. Comp., 87 (2017), pp. 2091–2112.
- [10] P. HANSBO, M. G. LARSON, AND S. ZAHEDI, *A cut finite element method for a Stokes interface problem*, Applied Numerical Mathematics, 85 (2014), pp. 90–114.
- [11] M. KIRCHHART, S. GROSS, AND A. REUSKEN, *Analysis of an XFEM discretization for Stokes interface problems*, SIAM Journal on Scientific Computing, 38 (2016), pp. A1019–A1043.
- [12] A. MASSING, M. LARSON, A. LOGG, AND A. ROGNES, *Stabilized nitsche fictitious domain method for the Stokes problem*, J. Sci. Comput., 61 (2014), pp. 604–628.
- [13] K. OHMORI AND N. SAITO, *Some remarks on the flux-free finite element method for immiscible two-fluid flows*, Journal of computational and applied mathematics, 232 (2009), pp. 127–138.
- [14] M. A. OLSHANSKII AND A. REUSKEN, *Analysis of a Stokes interface problem*, Numer. Math., 103 (2006), pp. 129–149.
- [15] L. R. SCOTT AND S. ZHANG, *Finite element interpolation of nonsmooth functions satisfying boundary conditions*, Mathematics of Computation, 54 (1990), pp. 483–493.
- [16] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.
- [17] M. SUSSMAN, P. SMERKA, AND S. OSHER, *A level set approach for computing solutions to incompressible two-phase flow*, Journal of Computational physics, 114 (1994), pp. 146–159.
- [18] M. SUSSMAN, K. M. SMITH, M. Y. HUSSAINI, M. OHTA, AND R. ZHI-WEI, *A sharp interface method for incompressible two-phase flows*, Journal of computational physics, 221 (2007), pp. 469–505.
- [19] C. TRUESDELL AND R. TOUPIN, *The Classical Field Theories*, Springer Berlin Heidelberg, Berlin, Heidelberg, 1960, pp. 226–858.
- [20] O. B. WIDLUND, *An extension theorem for finite element spaces with three applications*, in Numerical Techniques in Continuum Mechanics, Springer, 1987, pp. 110–122.

APPENDIX A. FINITE ELEMENT EXTENSION.

The goal of this section is to provide the H^1 -bounded extension operator for finite elements of arbitrary degree, i.e. to prove Lemma 3.2. We build the desired extension for each velocity component independently. Let V_h^k be the $H_0^1(\Omega)$ -conformal FE space of degree k . Then $\mathbf{V}_h^{\text{bulk}} = \otimes_{i=1}^j V_{j,h}$, and each $V_{j,h}$ satisfies $V_h^1 \subset V_{j,h} \subset V_h^r$ for some integer $r > 0$. Fix arbitrary $j \in \{1, \dots, d\}$. Using the definition of the local nodal basis, mapping $T \in \mathcal{T}_h$ to the reference simplex and the equivalence of norms in a space of finite dimension, one shows

$$(A.1) \quad \|v\|_{L^\infty(T)} \leq C \max_{\mathbf{y} \in \mathcal{N}(T)} |v(\mathbf{y})| \quad \forall v \in V_{j,h}, T \in \mathcal{T}_h,$$

with some C independent of v and T .

The strategy will be to build an extension operator for piecewise linears first. Then to use that extension operator to build a general extension operator.

A.1. Extension operator for piecewise linears. Let $V_h^{1,+} = \{v|_{\Omega_h^+} : v \in V_h^1\}$. Consider $v \in V_h^{1,+}$ and let $Ev \in H_0^1(\Omega)$ be the Stein extension [16] such that

$$(A.2) \quad Ev = v \quad \text{on } \Omega^+ \quad \text{and} \quad \|Ev\|_{1,\Omega} \leq C\|v\|_{1,\Omega^+}.$$

We need $I^{\text{SZ}}(Ev)$, the Scott-Zhang interpolant of Ev onto V_h^1 . The construction of $I^{\text{SZ}}(Ev) \in V_h^1$ follows the standard procedure from [15]. However, some care is required to ensure that we are recovering the same P^1 function in all interior tetrahedra of Ω^+ ,

$$(A.3) \quad I^{\text{SZ}}Ev|_T = Ev|_T = v|_T \quad \text{for all } T \in \mathcal{T}_{h,i}^+.$$

To provide (A.3), we exploit a freedom in choosing the Scott-Zhang interpolant pointed out in [15]: For every vertex \mathbf{y} of \mathcal{T}_h we need to associate either a d -dimensional simplex to \mathbf{y} or a $d-1$ -dimensional simplex. If \mathbf{y} is a vertex for some $T \in \mathcal{T}_{h,i}^+$ and $\mathbf{y} \notin \partial\Omega$, then we associate one of these simplices from $\mathcal{T}_{h,i}^+$ with \mathbf{y} . By the stability property of the Scott-Zhang interpolant we have

$$(A.4) \quad \|I^{\text{SZ}}Ev\|_{1,\Omega} \leq C\|Ev\|_{1,\Omega}.$$

Now we define a discrete extension operator for piecewise linear $v \in V_h^{1,+}$:

$$E_h^1 v(\mathbf{y}) = \begin{cases} v(\mathbf{y}) & \text{if } \mathbf{y} \text{ is a vertex and } \mathbf{y} \in \overline{\Omega_h^+}, \\ I^{\text{SZ}}Ev(\mathbf{y}) & \text{if } \mathbf{y} \text{ is a vertex and } \mathbf{y} \notin \overline{\Omega_h^+}. \end{cases}$$

Note that

$$E_h^1 v = v \quad \text{on } \Omega_h^+.$$

We decompose $\mathcal{T}_{h,i}^- = \mathcal{T}_{h,i}^{-,1} \cup \mathcal{T}_{h,i}^{-,\text{int}}$ where $\mathcal{T}_{h,i}^{-,1} = \{T \in \mathcal{T}_{h,i}^- : T \text{ has a vertex } \mathbf{y} \text{ such that } \mathbf{y} \in K, \text{ for some } K \in \mathcal{T}_h^\Gamma\}$ and $\mathcal{T}_{h,i}^{-,\text{int}} = \mathcal{T}_{h,i}^- \setminus \mathcal{T}_{h,i}^{-,1}$.

We then see that

$$\begin{aligned}
\|\nabla E_h^1 v\|_{\Omega_{h,i}^-}^2 &= \sum_{T \in \mathcal{T}_{h,i}^-} \|\nabla E_h^1 v\|_T^2 \\
&= \sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla E_h^1 v\|_T^2 + \sum_{T \in \mathcal{T}_{h,i}^{-,\text{int}}} \|\nabla E_h^1 v\|_T^2 \\
&= \sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla E_h^1 v\|_T^2 + \sum_{T \in \mathcal{T}_{h,i}^{-,\text{int}}} \|\nabla I^{\text{SZ}} E v\|_T^2 \\
&\leq \sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla E_h^1 v\|_T^2 + \|I^{\text{SZ}} E v\|_{1,\Omega^-}^2.
\end{aligned}$$

Therefore, we are left to bound $\sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla E_h^1 v\|_T^2$. We use the triangle inequality to get

$$\begin{aligned}
\sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla E_h^1 v\|_T^2 &\leq \sum_{T \in \mathcal{T}_{h,i}^{-,1}} 2(\|\nabla I^{\text{SZ}} E v\|_T^2 + \|\nabla(E_h^1 v - I^{\text{SZ}} E v)\|_T^2) \\
&\leq 2\|I^{\text{SZ}} E v\|_{1,\Omega^-}^2 + 2 \sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla(E_h^1 v - I^{\text{SZ}} E v)\|_T^2.
\end{aligned}$$

For ease of notation we set $w = E_h^1 v - I^{\text{SZ}} E v$. Now note that if $T \in \mathcal{T}_{h,i}^{-,1}$ then w vanishes on all vertices that do not belong to $\overline{\Omega_h^+}$. Hence, using inverse estimates we get

$$\begin{aligned}
\|\nabla w\|_T^2 &\leq C h_T^{-2} \|w\|_T^2 \leq C h_T^{d-2} \|w\|_{L^\infty(T)}^2 \leq C h_T^{d-2} \sum_{K \in \mathcal{T}_h^\Gamma, \overline{K} \cap \overline{T} \neq \emptyset} \|w\|_{L^\infty(K)}^2 \\
&\leq C h_T^{-2} \sum_{K \in \mathcal{T}_h^\Gamma, \overline{K} \cap \overline{T} \neq \emptyset} \|w\|_K^2.
\end{aligned}$$

Hence, we will have

$$\sum_{T \in \mathcal{T}_{h,i}^{-,1}} \|\nabla w\|_T^2 \leq C \sum_{T \in \mathcal{T}_h^\Gamma} h_T^{-2} \|w\|_T^2.$$

Recalling Assumption 2, if $T \in \mathcal{T}_h^\Gamma$ there exists $K_T \in \mathcal{T}_{h,i}^+$ such that $T = K_1, K_2, \dots, K_\ell = K_T$ and K_j, K_{j+1} have a common $d-1$ face for $j < \ell$ and $K_j \subset \Omega_h^+$. The number $\ell \leq M$ where M is uniformly bounded and only depends on the shape regularity of the mesh. Then we see from a simple scaling argument that

$$\|w\|_T^2 \leq C(\|w\|_{K_T}^2 + \sum_{i=1}^{\ell} h_{K_i}^2 \|\nabla w\|_{K_i}^2).$$

Using (A.3) we have $w|_{K_T} \equiv 0$ since $K_T \in \mathcal{T}_{h,i}^+$ and so

$$\|w\|_T^2 \leq C \sum_{i=1}^{\ell} h_{K_i}^2 \|\nabla w\|_{K_i}^2 \leq C h_T^2 \sum_{i=1}^{\ell} \|\nabla w\|_{K_i}^2.$$

In the last inequality we used that by shape regularity $h_{K_i} \leq Ch_T$ where C depends on M and shape regularity constant. We then get

$$\sum_{T \in \mathcal{T}_h^\Gamma} h_T^{-2} \|w\|_T^2 \leq C \sum_{T \in \mathcal{T}_h^+} \|\nabla w\|_T^2 \leq C(\|\nabla v\|_{\Omega_h^+}^2 + \|\nabla(I^{\text{SZ}}Ev)\|_{\Omega_h^+}^2).$$

In the last step we used (A.1). Therefore, combining all the inequalities above we obtain

$$\|\nabla E_h^1 v\|_{\Omega_{h,i}^-}^2 \leq C(\|I^{\text{SZ}}Ev\|_{1,\Omega}^2 + \|\nabla v\|_{\Omega_h^+}^2).$$

We hence, get that after using (A.2) and (A.4) that

$$(A.5) \quad \|\nabla E_h^1 v\|_{\Omega} \leq C\|v\|_{1,\Omega_h^+}.$$

A.2. General Discrete Extension Operator. Building on the availability of E_h^1 we define the general extension operator. Let $V_h^+ = \{v|_{\Omega_h^+} : v \in V_{j,h}\}$ and consider the subspace $W_h^+ = \{v \in V_h^+ : v(\mathbf{y}) = 0 \text{ for all vertices } \mathbf{y} \text{ of } \mathcal{T}_h^+\}$. For $v \in W_h^+$ we consider the extension $Q_h v \in V_{j,h}$ by defining its nodal values as follows

$$Q_h v(\mathbf{y}) = \begin{cases} v(\mathbf{y}) & \text{if } \mathbf{y} \in \mathcal{N}(\mathcal{T}_h) \text{ and } \mathbf{y} \in \overline{\Omega_h^+} \\ 0 & \text{if } \mathbf{y} \in \mathcal{N}(\mathcal{T}_h) \text{ and } \mathbf{y} \notin \overline{\Omega_h^+} \end{cases}$$

Then we can easily prove the following lemma.

Lemma A.1. *For $v \in W_h^+$ it holds*

$$(A.6) \quad \|\nabla(Q_h v)\|_{\Omega_{h,i}^-} \leq C_1 \|\nabla v\|_{\Omega_h^+},$$

where the constant C_1 only depends on the shape regularity of the mesh.

Proof. Note that $Q_h v|_{\Omega_{h,i}^-}$ is supported on all simplices $T \in \mathcal{T}_h^-$ that have at least one edge belonging to $\partial\Omega_h^+$. Lets call this set of simplices ω_h^- :

$$\omega_h^- = \{T \in \mathcal{T}_h^- : \text{an edge of } T \text{ belongs to } \partial\Omega_h^+\},$$

and we also consider the set

$$\omega_h^+ = \{T \in \mathcal{T}_h^+ : \text{an edge of } T \text{ belongs to } \partial\Omega_h^+\}.$$

For each $T \in \omega_h^-$, let $\Xi(T) := \{\tau \in \mathcal{T}_h^+ : \tau \text{ and } T \text{ share a common edge}\}$. Then we have due to the finite element inverse estimates:

$$\|\nabla Q_h v\|_{\Omega_{h,i}^-}^2 = \sum_{T \in \omega_h^-} \|\nabla Q_h v\|_T^2 \leq C \sum_{T \in \omega_h^-} h_T^{d-2} \|Q_h v\|_{L^\infty(T)}^2.$$

At the same time, with the help of (A.1) we have for each $T \in \omega_h^+$,

$$\|Q_h v\|_{L^\infty(T)} \leq C \max_{\mathbf{y} \in \mathcal{N}(T)} \|Q_h v(\mathbf{y})\| \leq C \max_{\tau \in \Xi(T)} \|v\|_{L^\infty(\tau)}.$$

For the second inequality we used that $Q_h v$ vanishes on all nodes except the nodal points of T that belong ∂T and belong to the boundary of $\partial\Omega_h^+$. Since $v \in W_h^+$ and so v vanishes on all the vertices of such $\tau's$, we obtain

$$\|Q_h v\|_{L^\infty(T)} \leq Ch_T \max_{\tau \in \Xi(T)} \|\nabla v\|_{L^\infty(\tau)}.$$

Finally, applying inverse estimates gives

$$\|Q_h v\|_{L^\infty(T)} \leq C h_T h_T^{-d/2} \|\nabla v\|_\tau \quad \text{for some } \tau \in \Xi(T).$$

Hence,

$$\|\nabla Q_h v\|_{\Omega_{h,i}^-}^2 \leq C \sum_{\tau \in \omega_h^+} \|\nabla v\|_\tau^2 \leq C \|\nabla v\|_{\Omega_h^+}^2.$$

□

Let $I_h : C(\Omega_h^+) \rightarrow V_h^{1,+}$ be the Lagrange interpolant. In the above section we defined a stable discrete extension operator $E_h^1 : V_h^{1,+}(\Omega_h^+) \rightarrow V_h^1 \subset V_{j,h}$.

We finally can define our general *discrete extension operator*. For any $v \in V_{j,h}^+$, we define $E_h v \in V_{j,h}$ as follows

$$(A.7) \quad E_h v := E_h^1(I_h v) + Q_h(v - I_h v).$$

Note that $v - I_h v \in W_h^+$ so indeed this definition makes sense.

We use the boundedness of E_h^1 (A.5), (A.6) and the stability of I_h to obtain:

$$\begin{aligned} \|\nabla E_h v\|_{\Omega_{h,i}^-} &\leq \|\nabla(E_h^1(I_h v))\|_{\Omega_{h,i}^-} + \|\nabla(Q_h(v - I_h v))\|_{\Omega_{h,i}^-} \\ &\leq C \|I_h v\|_{1,\Omega_h^+} + C \|\nabla(v - I_h v)\|_{\Omega_h^+} \leq C \|v\|_{1,\Omega_h^+}, \end{aligned}$$

with some C independent of h and the position of Γ against the background mesh. Hence, we have proven Lemma 3.2.

[†] DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
E-mail address: ernesto_caceres_valenzuela@brown.edu

[†] DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RI 02912, USA
E-mail address: johnny_guzman@brown.edu

[‡] DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TX 77204, USA
E-mail address: molshan@math.uh.edu