Introduction and Basic Implementation for Finite Element Methods

Chapter 7: Finite elements for 2D steady Navier-Stokes equation

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Outline

Outline

Target problem

- Consider the 2D Navier-Stokes equation:

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} & in \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 & in \quad \Omega, \\ \mathbf{u} = \mathbf{g} & on \quad \partial\Omega. \end{cases}$$

where

$$\mathbf{u}(x,y) = (u_1,\ u_2)^t,\ \mathbf{g}(x,y) = (g_1,\ g_2)^t,\ \mathbf{f}(x,y) = (f_1,\ f_2)^t.$$

- The nonlinear advection is defined as

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = \begin{pmatrix} u_1\frac{\partial u_1}{\partial x} + u_2\frac{\partial u_1}{\partial y} \\ u_1\frac{\partial u_2}{\partial x} + u_2\frac{\partial u_2}{\partial y} \end{pmatrix}.$$

Target problem

- The stress tensor $\mathbb{T}(\mathbf{u}, p)$ is defined as

$$\mathbb{T}(\mathbf{u}, p) = 2\nu\mathbb{D}(\mathbf{u}) - p\mathbb{I}$$

where ν is the viscosity and the deformation tensor

$$\mathbb{D}(\mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^t).$$

- In more details, the deformation tensor can be written as

$$\mathbb{D}(\mathbf{u}) = \begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix}.$$

- Hence the stress tensor can be written as

$$\mathbb{T}(\mathbf{u}, p) = \begin{pmatrix} 2\nu\frac{\partial u_1}{\partial x} - p & \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \nu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & 2\nu\frac{\partial u_2}{\partial y} - p \end{pmatrix}.$$

Weak formulation

- Since p appears in the equation without any derivative, then, if $(\mathbf{u}, p)$ is a solution, then $(\mathbf{u}, p+c)$ is also a solution where c is a constant. Hence we need to impose additional condition for p. Here are three regular choices:
- (1) Fix p at one point in the domain Ω.
- (2) Apply a stress or Robin boundary condition (at least in the normal direction) on part of the boundary $\partial\Omega$.
- (3) Apply $\int_\Omega p dxdy = 0$.

Weak formulation

- First, take the inner product with a vector function $\mathbf{v}(x,y) = (v_1,\ v_2)^t$ on both sides of the Navier-Stokes equation:

$$
\begin{aligned}
& (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \quad \text{in } \Omega \\
\Rightarrow\quad & (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} - (\nabla\cdot\mathbb{T}(\mathbf{u},p))\cdot\mathbf{v} = \mathbf{f}\cdot\mathbf{v} \quad \text{in } \Omega \\
\Rightarrow\quad & \int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy - \int_\Omega (\nabla\cdot\mathbb{T}(\mathbf{u},p))\cdot\mathbf{v}\ dxdy = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy.
\end{aligned}
$$

- Second, multiply the divergence free equation by a function $q(x,y)$:

$$
\begin{aligned}
\nabla\cdot\mathbf{u} = 0 \quad \Rightarrow\quad & (\nabla\cdot\mathbf{u})q = 0 \\
\Rightarrow\quad & \int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.
\end{aligned}
$$

- $\mathbf{u}(x,y)$ and $p(x,y)$ are called trail functions and $\mathbf{v}(x,y)$ and $q(x,y)$ are called test functions.

Weak formulation

- Using integration by parts in multi-dimension:

$$\int_\Omega (\nabla \cdot \mathbb{T}) \cdot \mathbf{v} \; dxdy = \int_{\partial\Omega} (\mathbb{T}\mathbf{n}) \cdot \mathbf{v} \; ds - \int_\Omega \mathbb{T} : \nabla \mathbf{v} \; dxdy,$$

where $\mathbf{n} = (n_1, \; n_2)^t$ is the unit outer normal vector of $\partial\Omega$, we obtain

$$\begin{aligned} &\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} \; dxdy \\ &- \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy. \end{aligned}$$

Here,

$$\begin{aligned} A : B &= \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \\ &= a_{11}b_{11} + a_{12}b_{12} + a_{21}b_{21} + a_{22}b_{22}. \end{aligned}$$

Weak formulation

- Using the above definition for $A : B$, it is not difficult to verify (an independent study project topic) that

$$\begin{aligned} \mathbb{T}(\mathbf{u}, p) : \nabla \mathbf{v} &= (2\nu \mathbb{D}(\mathbf{u}) - p\mathbb{I}) : \nabla \mathbf{v} \\ &= 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) - p(\nabla \cdot \mathbf{v}). \end{aligned}$$

- Hence we obtain

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy$$
$$- \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0.$$

Here we multiply the second equation by -1 in order to keep the matrix formulation symmetric later.

Weak formulation

- Since the solution on the domain boundary $\partial\Omega$ are given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega$.
- Hence

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy$$

$$- \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0.$$

Weak formulation

- Weak formulation in the vector format: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy$$
$$- \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u})q \; dxdy = 0,$$

for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $q \in L^2(\Omega)$.

Weak formulation

- Define

$$
\begin{aligned}
c(\mathbf{w},\mathbf{u},\mathbf{v}) &= \int_\Omega (\mathbf{w}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy,\\
a(\mathbf{u},\mathbf{v}) &= \int_\Omega 2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy,\\
b(\mathbf{u},q) &= -\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy,\\
(\mathbf{f},\mathbf{v}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy.
\end{aligned}
$$

- Weak formulation: find $\mathbf{u}\in H^1(\Omega)\times H^1(\Omega)$ and $p\in L^2(\Omega)$ such that

$$
\begin{aligned}
c(\mathbf{u},\mathbf{u},\mathbf{v}) + a(\mathbf{u},\mathbf{v}) + b(\mathbf{v},p) &= (\mathbf{f},\mathbf{v}),\\
b(\mathbf{u},q) &= 0,
\end{aligned}
$$

for any $\mathbf{v}\in H_0^1(\Omega)\times H_0^1(\Omega)$ and $q\in L^2(\Omega)$.

Weak formulation

- In more details,

$$
\begin{aligned}
&\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= &\begin{pmatrix} \frac{\partial u_1}{\partial x} & \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) & \frac{\partial u_2}{\partial y} \end{pmatrix} \\
&: \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) & \frac{\partial v_2}{\partial y} \end{pmatrix} \\
= &\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) \\
&+\frac{1}{4}\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right)\left(\frac{\partial v_1}{\partial y} + \frac{\partial v_2}{\partial x}\right) + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y}.
\end{aligned}
$$

Weak formulation

- Hence

$$
\begin{aligned}
& \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \\
= \; & \frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
& + \frac{1}{2}\frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{1}{2}\frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x}.
\end{aligned}
$$

- Then

$$
\begin{aligned}
& \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy \\
= \; & \int_\Omega \nu \Big(2\frac{\partial u_1}{\partial x}\frac{\partial v_1}{\partial x} + 2\frac{\partial u_2}{\partial y}\frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y}\frac{\partial v_1}{\partial y} \\
& + \frac{\partial u_1}{\partial y}\frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x}\frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x}\frac{\partial v_2}{\partial x} \Big) \; dxdy.
\end{aligned}
$$

Weak formulation

- We also have

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy$$

$$= \int_\Omega \left(u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right) \; dxdy,$$

$$\int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy = \int_\Omega \left(p \frac{\partial v_1}{\partial x} + p \frac{\partial v_2}{\partial y} \right) \; dxdy,$$

$$\int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy = \int_\Omega (f_1 v_1 + f_2 v_2) \; dxdy,$$

$$\int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = \int_\Omega \left(\frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right) \; dxdy.$$

Weak formulation

- Weak formulation in the scalar format: find $u_1 \in H^1(\Omega)$, $u_2 \in H^1(\Omega)$, and $p \in L^2(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega \left(u_1 \frac{\partial u_1}{\partial x} v_1 + u_2 \frac{\partial u_1}{\partial y} v_1 + u_1 \frac{\partial u_2}{\partial x} v_2 + u_2 \frac{\partial u_2}{\partial y} v_2 \right) \ dxdy \\
&+ \int_\Omega \nu \Big(2 \frac{\partial u_1}{\partial x} \frac{\partial v_1}{\partial x} + 2 \frac{\partial u_2}{\partial y} \frac{\partial v_2}{\partial y} + \frac{\partial u_1}{\partial y} \frac{\partial v_1}{\partial y} \\
&+ \frac{\partial u_1}{\partial y} \frac{\partial v_2}{\partial x} + \frac{\partial u_2}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2}{\partial x} \frac{\partial v_2}{\partial x} \Big) \ dxdy \\
&- \int_\Omega \left(p \frac{\partial v_1}{\partial x} + p \frac{\partial v_2}{\partial y} \right) \ dxdy = \int_\Omega (f_1 v_1 + f_2 v_2) \ dxdy. \\
&- \int_\Omega \left(\frac{\partial u_1}{\partial x} q + \frac{\partial u_2}{\partial y} q \right) \ dxdy = 0.
\end{aligned}
$$

for any $v_1 \in H_0^1(\Omega)$, $v_2 \in H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

Galerkin formulation

- Consider a finite element space $U_h \subset H^1(\Omega)$ for the velocity and a finite element space $W_h \subset L^2(\Omega)$ for the pressure. Define U_{h0} to be the space which consists of the functions of U_h with value 0 on the Dirichlet boundary.

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\begin{aligned} c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Galerkin formulation

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\begin{aligned} c(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}_h) + a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h), \\ b(\mathbf{u}_h, q_h) &= 0, \end{aligned}$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Galerkin formulation

- In more details of the vector format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy$$
$$- \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Galerkin formulation

- In our numerical example, $U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$ are chosen to be the finite element spaces with the quadratic global basis functions $\{\phi_j\}_{j=1}^{N_b}$ and linear global basis functions $\{\psi_j\}_{j=1}^{N_{bp}}$, which are defined in Chapter 2. They are called Taylor-Hood finite elements.
- Why do we choose the pairs of finite elements in this way?
- Stability of mixed finite elements: inf-sup condition.

$$\inf_{0 \neq q_h \in W_h} \sup_{0 \neq \mathbf{u}_h \in U_h \times U_h} \frac{b(\mathbf{u}_h, q_h)}{\|\nabla \mathbf{u}_h\|_0 \, \|q_h\|_0} > \beta,$$

  where $\beta > 0$ is a constant independent of mesh size h.
- See other course materials and references for the theory and more examples of stable mixed finite elements for Navier-Stokes equation.

Galerkin formulation

- In the scalar format, the Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later) is to find $u_{1h} \in U_h$, $u_{2h} \in U_h$, and $p_h \in W_h$ such that

$$\int_\Omega \left(u_{1h}\frac{\partial u_{1h}}{\partial x} v_{1h} + u_{2h}\frac{\partial u_{1h}}{\partial y} v_{1h} + u_{1h}\frac{\partial u_{2h}}{\partial x} v_{2h} + u_{2h}\frac{\partial u_{2h}}{\partial y} v_{2h} \right) \; dxdy$$

$$+ \int_\Omega \nu \Big(2\frac{\partial u_{1h}}{\partial x}\frac{\partial v_{1h}}{\partial x} + 2\frac{\partial u_{2h}}{\partial y}\frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{1h}}{\partial y}$$

$$+ \frac{\partial u_{1h}}{\partial y}\frac{\partial v_{2h}}{\partial x} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}}{\partial x}\frac{\partial v_{2h}}{\partial x}\Big) \; dxdy$$

$$- \int_\Omega \left(p_h\frac{\partial v_{1h}}{\partial x} + p_h\frac{\partial v_{2h}}{\partial y} \right) \; dxdy = \int_\Omega (f_1 v_{1h} + f_2 v_{2h}) \; dxdy.$$

$$- \int_\Omega \left(\frac{\partial u_{1h}}{\partial x} q_h + \frac{\partial u_{2h}}{\partial y} q_h \right) \; dxdy = 0.$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

Outline

Newton's iteration

- How to handle the nonlinear terms in the weak formulation and Galerkin formulation?
- Newton's iteration!
- References:

[1]M. Gunzburger, Finite element methods for viscous incompressible flows. A guide to theory, practice, and algorithms. Academic Press, 1989.
[2] V. Girault and P. A. Raviart, Finite element methods for Navier-Stokes equations. Theory and algorithms. Springer-Verlag, 1986.

Newton's iteration for the weak formulation

- Initial guess: $\mathbf{u}^{(0)}$ and $p^{(0)}$.
- Newton's iteration for the weak formulation: for $l = 1, 2, \cdots, L$, find $\mathbf{u}^{(l)} \in H^1(\Omega) \times H^1(\Omega)$ and $p^{(l)} \in L^2(\Omega)$ such that

$$
\begin{aligned}
& c(\mathbf{u}^{(l)}, \mathbf{u}^{(l-1)}, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l)}, \mathbf{v}) + a(\mathbf{u}^{(l)}, \mathbf{v}) + b(\mathbf{v}, p^{(l)}) \\
& = (\mathbf{f}, \mathbf{v}) + c(\mathbf{u}^{(l-1)}, \mathbf{u}^{(l-1)}, \mathbf{v}), \\
& b(\mathbf{u}^{(l)}, q) = 0,
\end{aligned}
$$

for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $q \in L^2(\Omega)$.

Newton's iteration for the weak formulation

- Initial guess: $\mathbf{u}^{(0)}$ and $p^{(0)}$.
- Newton's iteration for the weak formulation in the vector format: for $l = 1, 2, \cdots, L$, find $\mathbf{u}^{(l)} \in H^1(\Omega) \times H^1(\Omega)$ and $p^{(l)} \in L^2(\Omega)$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}^{(l)} \cdot \nabla)\mathbf{u}^{(l-1)} \cdot \mathbf{v} \, dxdy + \int_\Omega (\mathbf{u}^{(l-1)} \cdot \nabla)\mathbf{u}^{(l)} \cdot \mathbf{v} \, dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}^{(l)}) : \mathbb{D}(\mathbf{v}) \, dxdy - \int_\Omega p^{(l)} (\nabla \cdot \mathbf{v}) \, dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dxdy + \int_\Omega (\mathbf{u}^{(l-1)} \cdot \nabla)\mathbf{u}^{(l-1)} \cdot \mathbf{v} \, dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}^{(l)}) q \, dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v} \in H_0^1(\Omega) \times H_0^1(\Omega)$ and $q \in L^2(\Omega)$.

Newton's iteration for the weak formulation

- Initial guess: $u_1^{(0)}$, $u_2^{(0)}$, and $p^{(0)}$.
- Newton's iteration for the weak formulation in the scalar format: for $l = 1, 2, \cdots, L$, find $u_1^{(l)} \in H^1(\Omega)$, $u_2^{(l)} \in H^1(\Omega)$, and $p^{(l)} \in L^2(\Omega)$ such that

$$\int_\Omega \left(u_1^{(l)} \frac{\partial u_1^{(l-1)}}{\partial x} v_1 + u_2^{(l)} \frac{\partial u_1^{(l-1)}}{\partial y} v_1 + u_1^{(l)} \frac{\partial u_2^{(l-1)}}{\partial x} v_2 + u_2^{(l)} \frac{\partial u_2^{(l-1)}}{\partial y} v_2 \right) \, dxdy$$
$$+ \int_\Omega \left(u_1^{(l-1)} \frac{\partial u_1^{(l)}}{\partial x} v_1 + u_2^{(l-1)} \frac{\partial u_1^{(l)}}{\partial y} v_1 + u_1^{(l-1)} \frac{\partial u_2^{(l)}}{\partial x} v_2 + u_2^{(l-1)} \frac{\partial u_2^{(l)}}{\partial y} v_2 \right) \, dxdy$$
$$+ \int_\Omega \nu \Big(2 \frac{\partial u_1^{(l)}}{\partial x} \frac{\partial v_1}{\partial x} + 2 \frac{\partial u_2^{(l)}}{\partial y} \frac{\partial v_2}{\partial y} + \frac{\partial u_1^{(l)}}{\partial y} \frac{\partial v_1}{\partial y} + \frac{\partial u_1^{(l)}}{\partial y} \frac{\partial v_2}{\partial x}$$
$$+ \frac{\partial u_2^{(l)}}{\partial x} \frac{\partial v_1}{\partial y} + \frac{\partial u_2^{(l)}}{\partial x} \frac{\partial v_2}{\partial x} \Big) \, dxdy - \int_\Omega \left(p^{(l)} \frac{\partial v_1}{\partial x} + p^{(l)} \frac{\partial v_2}{\partial y} \right) \, dxdy$$
$$= \int_\Omega (f_1 v_1 + f_2 v_2) \, dxdy + \int_\Omega \Big(u_1^{(l-1)} \frac{\partial u_1^{(l-1)}}{\partial x} v_1 + u_2^{(l-1)} \frac{\partial u_1^{(l-1)}}{\partial y} v_1$$
$$+ u_1^{(l-1)} \frac{\partial u_2^{(l-1)}}{\partial x} v_2 + u_2^{(l-1)} \frac{\partial u_2^{(l-1)}}{\partial y} v_2 \Big) \, dxdy,$$

Newton's iteration for the weak formulation

- Continued formulation:

$$-\int_\Omega \left(\frac{\partial u_1^{(l)}}{\partial x} q + \frac{\partial u_2^{(l)}}{\partial y} q \right) \, dxdy = 0.$$

for any $v_1 \in H_0^1(\Omega)$, $v_2 \in H_0^1(\Omega)$, and $q \in L^2(\Omega)$.

Newton's iteration for Galerkin formulation

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- Newton's iteration for Galerkin formulation: for $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ such that

$$
\begin{aligned}
& c(\mathbf{u}_h^{(l)}, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h) + c(\mathbf{u}_h^{(l-1)}, \mathbf{u}_h^{(l)}, \mathbf{v}_h) + a(\mathbf{u}_h^{(l)}, \mathbf{v}_h) + b(\mathbf{v}_h, p_h^{(l)}) \\
& = (\mathbf{f}, \mathbf{v}_h) + c(\mathbf{u}_h^{(l-1)}, \mathbf{u}_h^{(l-1)}, \mathbf{v}_h), \\
& b(\mathbf{u}_h^{(l)}, q_h) = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Newton's iteration for Galerkin formulation

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- Newton's iteration for Galerkin formulation in the vector format: for $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h^{(l)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \, dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{(l)}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{(l)} (\nabla \cdot \mathbf{v}_h) \, dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h^{(l)}) q_h \, dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Newton's iteration for Galerkin formulation

- Initial guess: $u_{1h}^{(0)}$, $u_{2h}^{(0)}$, and $p_h^{(0)}$.
- Newton's iteration for Galerkin formulation in the scalar format: for $l = 1, 2, \cdots, L$, find $u_{1h}^{(l)} \in U_h$, $u_{2h}^{(l)} \in U_h$, and $p_h^{(l)} \in W_h$ such that

$$\int_\Omega \left(u_{1h}^{(l)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} v_{1h} + u_{2h}^{(l)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} v_{1h} + u_{1h}^{(l)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} v_{2h} + u_{2h}^{(l)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} v_{2h} \right) dxdy$$
$$+ \int_\Omega \left(u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l)}}{\partial x} v_{1h} + u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l)}}{\partial y} v_{1h} + u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l)}}{\partial x} v_{2h} + u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l)}}{\partial y} v_{2h} \right) dxdy$$
$$+ \int_\Omega \nu \Big(2 \frac{\partial u_{1h}^{(l)}}{\partial x} \frac{\partial v_{1h}}{\partial x} + 2 \frac{\partial u_{2h}^{(l)}}{\partial y} \frac{\partial v_{2h}}{\partial y} + \frac{\partial u_{1h}^{(l)}}{\partial y} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{1h}^{(l)}}{\partial y} \frac{\partial v_{2h}}{\partial x}$$
$$+ \frac{\partial u_{2h}^{(l)}}{\partial x} \frac{\partial v_{1h}}{\partial y} + \frac{\partial u_{2h}^{(l)}}{\partial x} \frac{\partial v_{2h}}{\partial x} \Big) dxdy - \int_\Omega \left(p_h^{(l)} \frac{\partial v_{1h}}{\partial x} + p_h^{(l)} \frac{\partial v_{2h}}{\partial y} \right) dxdy$$
$$= \int_\Omega (f_1 v_{1h} + f_2 v_{2h})\, dxdy + \int_\Omega \left(u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} v_{1h} + u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} v_{1h} \right.$$
$$\left. + u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} v_{2h} + u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} v_{2h} \right) dxdy,$$

Newton's iteration for Galerkin formulation

- Continued formulation:

$$-\int_{\Omega}\left(\frac{\partial u_{1h}^{(l)}}{\partial x}q_h+\frac{\partial u_{2h}^{(l)}}{\partial y}q_h\right)\, dxdy = 0.$$

for any $v_{1h} \in U_h$, $v_{2h} \in U_h$, and $q_h \in W_h$.

Outline

Discretization formulation

Recall the following definitions from Chapter 2:

- N: number of mesh elements.
- N_m: number of mesh nodes.
- E_n $(n = 1, \cdots, N)$: mesh elements.
- Z_k $(k = 1, \cdots, N_m)$: mesh nodes.
- N_l: number of local mesh nodes in a mesh element.
- P:information matrix consisting of the coordinates of all mesh nodes.
- T: information matrix consisting of the global node indices of the mesh nodes of all the mesh elements.

Discretization formulation

- We only consider the nodal basis functions (Lagrange type) in this course.
- N_{lb}: number of local finite element nodes (=number of local finite element basis functions) in a mesh element.
- N_b: number of the finite element nodes (= the number of unknowns = the total number of the finite element basis functions).
- X_j $(j = 1, \cdots, N_b)$: finite element nodes.
- P_b: information matrix consisting of the coordinates of all finite element nodes.
- T_b: information matrix consisting of the global node indices of the finite element nodes of all the mesh elements.

Discretization formulation

- Since $u_{1h}^{(l)}, \ u_{2h}^{(l)} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h^{(l)} \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j, \quad u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j, \quad p_h^{(l)} = \sum_{j=1}^{N_{bp}} p_j^{(l)} \psi_j$$

  for some coefficients $u_{1j}^{(l)}, \ u_{2j}^{(l)} \ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)} \ \ (j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for $u_{1j}^{(l)}$, $u_{2j}^{(l)} \ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)} \ \ (j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{(l)} = (u_{1h}^{(l)}, u_{2h}^{(l)})^t$ and $p_h^{(l)}$ at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration.

Discretization formulation

- For the first equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Discretization formulation

- Set $\mathbf{v}_h = (\phi_i, 0)^t$, i.e., $v_{1h} = \phi_i$ and $v_{2h} = 0$ ($i = 1, \cdots, N_b$), in the first equation of the Galerkin formulation at the step l ($l = 1, 2, \cdots, L$) of Newton's iteration. Then

$$
\begin{aligned}
&\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\phi_j\right)\phi_i\ dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\phi_j\right)\phi_i\ dxdy \\
&+\int_\Omega u_{1h}^{(l-1)}\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\frac{\partial\phi_j}{\partial x}\right)\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\frac{\partial\phi_j}{\partial y}\right)\phi_i\ dxdy \\
&+2\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\frac{\partial\phi_j}{\partial x}\right)\frac{\partial\phi_i}{\partial x}\ dxdy + \int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\frac{\partial\phi_j}{\partial y}\right)\frac{\partial\phi_i}{\partial y}\ dxdy \\
&+\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\frac{\partial\phi_j}{\partial x}\right)\frac{\partial\phi_i}{\partial y}\ dxdy - \int_\Omega\left(\sum_{j=1}^{N_{bp}} p_j^{(l)}\psi_j\right)\frac{\partial\phi_i}{\partial x}\ dxdy \\
&= \int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\ dxdy.
\end{aligned}
$$

Discretization formulation

- Set $\mathbf{v}_h = (0, \phi_i)^t$, i.e., $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$, in the first equation of the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration. Then

$$
\begin{aligned}
&\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x}\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\phi_j\right)\phi_i\ dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y}\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\phi_j\right)\phi_i\ dxdy\\
&+\int_\Omega u_{1h}^{(l-1)}\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\frac{\partial \phi_j}{\partial x}\right)\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\frac{\partial \phi_j}{\partial y}\right)\phi_i\ dxdy\\
&+2\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{1j}^{(l)}\frac{\partial \phi_j}{\partial y}\right)\frac{\partial \phi_i}{\partial x}\ dxdy\\
&+\int_\Omega \nu\left(\sum_{j=1}^{N_b} u_{2j}^{(l)}\frac{\partial \phi_j}{\partial x}\right)\frac{\partial \phi_i}{\partial x}\ dxdy - \int_\Omega \left(\sum_{j=1}^{N_{bp}} p_j^{(l)}\psi_j\right)\frac{\partial \phi_i}{\partial y}\ dxdy\\
&=\int_\Omega f_2\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{2h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{2h}^{(l-1)}}{\partial y}\phi_i\ dxdy.
\end{aligned}
$$

Discretization formulation

- Set $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$ in the second equation of the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration. Then

$$-\int_{\Omega} \left(\sum_{j=1}^{N_b} u_{1j}^{(l)} \frac{\partial \phi_j}{\partial x} \right) \psi_i \; dxdy - \int_{\Omega} \left(\sum_{j=1}^{N_b} u_{2j}^{(l)} \frac{\partial \phi_j}{\partial y} \right) \psi_i \; dxdy = 0.$$

Discretization formulation

- Simplify the above three sets of equations, we obtain

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)}\Big(2\int_\Omega \nu \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial\phi_j}{\partial y}\frac{\partial\phi_i}{\partial y}\ dxdy \\
&+ \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial\phi_j}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial\phi_j}{\partial y}\phi_i\ dxdy\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)}\left(\int_\Omega \nu \frac{\partial\phi_j}{\partial x}\frac{\partial\phi_i}{\partial y}\ dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)}\left(-\int_\Omega \psi_j\frac{\partial\phi_i}{\partial x}\ dxdy\right) \\
&= \int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\ dxdy,
\end{aligned}
$$

Discretization formulation

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \ dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_j \phi_i \ dxdy \right)$$

$$+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \Big(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \ dxdy$$

$$+ \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_j \phi_i \ dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \ dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \ dxdy \Big)$$

$$+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \ dxdy \right)$$

$$= \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \ dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \ dxdy,$$

$$\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(- \int_\Omega \frac{\partial \phi_j}{\partial x} \psi_i \ dxdy \right) + \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(- \int_\Omega \frac{\partial \phi_j}{\partial y} \psi_i \ dxdy \right) + \sum_{j=1}^{N_{bp}} p_j^{(l)} * 0 =$$

Matrix formulation

- Define

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_4 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}},$$

$$A_7 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial x}\psi_i\, dxdy\right]_{i=1,j=1}^{N_{bp},N_b}, \quad A_8 = \left[\int_\Omega -\frac{\partial \phi_j}{\partial y}\psi_i\, dxdy\right]_{i=1,j=1}^{N_{bp},N_b}.$$

- Define a zero matrix $\mathbb{O}_1 = [0]_{i=1,j=1}^{N_{bp},N_{bp}}$ whose size is $N_{bp} \times N_{bp}$. Then

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_4 & 2A_2 + A_1 & A_6 \\ A_7 & A_8 & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Each matrix above can be obtained by Algorithm I-3 in Chapter 3.
- It is not difficult to verify (an independent study project topic) that

$$A_4 = A_3^t, \quad A_7 = A_5^t, \quad A_8 = A_6^t.$$

- Hence the matrix A is actually symmetric:

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Define

$$AN_1 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_2 = \left[\int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\ dxdy\right]_{i,j=1}^{N_b},$$

$$AN_3 = \left[\int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_4 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},$$

$$AN_5 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_6 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b}.$$

- Define a zero matrix $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$. Then

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_2 \\ \mathbb{O}_2^t & \mathbb{O}_2^t & \mathbb{O}_1 \end{pmatrix}$$

Matrix formulation

- Define the load vector

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy \right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy \right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Matrix formulation

- Define the vector

$$\overrightarrow{bN} = \begin{pmatrix} \overrightarrow{bN}_1 + \overrightarrow{bN}_2 \\ \overrightarrow{bN}_3 + \overrightarrow{bN}_4 \\ \vec{0} \end{pmatrix}$$

where

$$\overrightarrow{bN}_1 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} \phi_i \ dxdy \right]_{i=1}^{N_b}, \ \overrightarrow{bN}_2 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} \phi_i \ dxdy \right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_3 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \ dxdy \right]_{i=1}^{N_b}, \ \overrightarrow{bN}_4 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \ dxdy \right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

Matrix formulation

- Define the unknown vector

$$\vec{X}^{(l)} = \begin{pmatrix} \vec{X}_1^{(l)} \\ \vec{X}_2^{(l)} \\ \vec{X}_3^{(l)} \end{pmatrix}$$

where

$$\vec{X}_1^{(l)} = \left[u_{1j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_2^{(l)} = \left[u_{2j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_3^{(l)} = \left[p_j^{(l)}\right]_{j=1}^{N_{bp}}.$$

- Define

$$A^{(l)} = A + AN, \ \vec{b}^{(l)} = \vec{b} + \overrightarrow{bN}.$$

- For step l ($l = 1, 2, \cdots, L$) of the Newton's iteration, we obtain the linear algebraic system

$$A^{(l)}\vec{X}^{(l)} = \vec{b}^{(l)}.$$

Assembly of the matrix for an integral with a finite element coefficient function

Recall Algorithm I-3, which is to assemble the matrix for an integral with a given coefficient function c:

- Initialize the matrix: $A = sparse(N_b, N_b)$;
- Compute the integrals and assemble them into A:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \alpha = 1, \cdots, N_{lb}$:

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad\quad$ Compute $r = \int_{E_n} c \frac{\partial^{r+s}\psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad\quad$ Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

$\quad\quad END$

$\quad END$

END

Assembly of the matrix for an integral with a finite element coefficient function

- How to slightly modify Algorithm I-3 to assemble the matrix for an integral with a finite element coefficient function?

- Replace c by

$$\frac{\partial^{d+e} c_h}{\partial x^d \partial y^e}!$$

- How to implement this idea?

- For the coefficient function part of the Gauss quadrature subroutine, call the subroutine for the finite element function evaluation, which was already coded for the error computation in Chapter 3, instead of calling the subroutine for function c.

Assembly of the matrix for an integral with a finite element coefficient function

Algorithm VIII:

- Initialize the matrix: $A = sparse(N_b, N_b)$;
- Compute the integrals and assemble them into A:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \alpha = 1, \cdots, N_{lb}$:

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad\quad$ Compute $r = \int_{E_n} \frac{\partial^{d+e} c_h}{\partial x^d \partial y^e} \frac{\partial^{r+s} \psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad\quad$ Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

$\quad\quad END$

$\quad END$

END

Assembly of the matrix for an integral with a finite element coefficient function

- In Chapter 3, the subroutine for the finite element function evaluation is coded for $\sum\limits_{k=1}^{N_{lb}} u_{T_b(k,n)} \frac{\partial^{\alpha_1+\alpha_2}\psi_{nk}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$, which is the restriction of a finite element function $\frac{\partial^{\alpha_1+\alpha_2}u_h}{\partial x^{\alpha_1}\partial y^{\alpha_2}} = \sum\limits_{j=1}^{N_b} u_j \frac{\partial^{\alpha_1+\alpha_2}\phi_j}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$ on the n^{th} element.

- Here $u_{T_b(k,n)}$ is the coefficient in the linear combination of the finite element function for the k^{th} basis function on the n^{th} element.

Assembly of the matrix for an integral with a finite element coefficient function

- Compared with the subroutine for function c, the subroutine for $\sum_{k=1}^{N_{lb}} u_{T_b(k,n)} \frac{\partial^{\alpha_1+\alpha_2}\psi_{nk}}{\partial x^{\alpha_1}\partial y^{\alpha_2}}$ requires more input parameters which need to be provided by the Gauss quadrature subroutine. And Gauss quadrature subroutine will obtain these parameters from its mother subroutine, which is the matrix/vector assembly subroutines.

- Parameters needed by the subroutine for the finite element function evaluation: coordinates, the coefficients in the linear combination of a finite element function, the n^{th} element's vertices, basis type, derivative orders for basis functions.

Assembly of the vector for an integral with two finite element coefficient functions

Recall Algorithm II-3, which is to assemble the vector for an integral with a given coefficient function f:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Assembly of the vector for an integral with two finite element coefficient functions

- How to slightly modify Algorithm II-3 to assemble the vector for an integral with two finite element coefficient functions?

- Replace f by

$$\frac{\partial^{d+e} f_{1h}}{\partial x^d \partial y^e} \frac{\partial^{r+s} f_{2h}}{\partial x^r \partial y^s}!$$

- How to implement this idea?

- For the coefficient function part of the Gauss quadrature subroutine, call the subroutine for the finite element function evaluation, which was already coded for the error computation in Chapter 3, twice, instead of calling the subroutine for function f.

Assembly of the vector for an integral with two finite element coefficient functions

Algorithm IX:

- Initialize the matrix: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} \frac{\partial^{d+e} f_{1h}}{\partial x^d \partial y^e} \frac{\partial^{r+s} f_{2h}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Outline

Dirichlet boundary condition

- Basically, the Dirichlet boundary condition $\mathbf{u} = \mathbf{g}$ (i.e., $u_1 = g_1$ and $u_2 = g_2$) provides the solutions at all boundary finite element nodes.
- Since the coefficient $u_{1j}^{(l)}$ and $u_{2j}^{(l)}$ in the finite element solutions $u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j$ and $u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j$ are actually the numerical solutions at the finite element node X_j $(j = 1, \cdots, N_b)$ when nodal basis functions are used, we actually know those $u_{1j}^{(l)}$ and $u_{2j}^{(l)}$ which are corresponding to the boundary finite element nodes.
- Recall that boundarynodes(2,:) store the global node indices of all boundary finite element nodes.
- If $m \in$ $boundarynodes(2,:)$, then the m^{th} equation is called a boundary node equation for u_1 and the $(N_b + m)^{th}$ equation is called a boundary node equation for u_2.
- Set nbn to be the number of boundary nodes;

Dirichlet boundary condition

- One way to impose the Dirichlet boundary condition is to replace the boundary node equations in the linear system by the following equations

$$u_{1m}^{(l)} = g_1(X_m)$$
$$u_{2m}^{(l)} = g_2(X_m).$$

for all $m \in$ *boundarynodes*$(2,:)$. This is similar to $u_m = g(X_m)$ in Chapter 3. We have discussed about this in Chapter 6 and Chapter 7.

- Since the Dirichlet boundary condition only involves u_1 and u_2, not p, only the first two rows of the 3×3 block matrix $A^{(l)}$ need to be modified for the Dirichlet boundary condition. This is similar to how we handle Dirichlet boundary condition in Chapter 6. We have discussed about this in Chapter 7.

Dirichlet boundary condition

Based on Algorithm III-3 in Chapter 6, we obtain Algorithm III-4:

- Deal with the Dirichlet boundary conditions:

  $FOR\ k = 1, \cdots, nbn$:

  If $boundarynodes(1, k)$ shows Dirichlet condition, then

  $i = boundarynodes(2, k)$;

  $A^{(l)}(i, :) = 0$;

  $A^{(l)}(i, i) = 1$;

  $b^{(l)}(i) = g_1(P_b(:, i))$;

  $A^{(l)}(N_b + i, :) = 0$;

  $A^{(l)}(N_b + i, N_b + i) = 1$;

  $b^{(l)}(N_b + i) = g_2(P_b(:, i))$;

  $ENDIF$

  END

Additional treatment for the solution uniqueness

Recall:

- Since p appears in the equation without any derivative, then, if $(\mathbf{u}, p)$ is a solution, then $(\mathbf{u}, p + c)$ is also a solution where c is a constant. Hence we need to impose additional condition for p. Here are three regular choices:
- (1) Fix p at one point in the domain Ω.
- (2) Apply a stress or Robin boundary condition (at least in the normal direction) on part of the boundary $\partial\Omega$.
- (3) Apply $\int_\Omega p dxdy = 0$.

Outline

Universal framework of the finite element method

Recall from Chapter 3:

- Generate the mesh information: matrices P and T;
- Assemble the matrices and vectors: local assembly based on P and T only;
- Deal with the boundary conditions: boundary information matrix and local assembly;
- Solve linear systems: numerical linear algebra.

Algorithm

- Generate the mesh information matrices P and T.
- Assemble the stiffness matrix A by using Algorithm I-3.
- Assemble the load vector $\vec{b}$ by using Algorithm II-3.
- Newton iteration: $FOR\ l = 1, 2, \cdots, L$
- Assemble the matrix AN by using Algorithm VIII.
- Assemble the vector $\overrightarrow{bN}$ by using Algorithm IX.
- $A^{(l)} = A + AN$ and $\vec{b}^{(l)} = \vec{b} + \overrightarrow{bN}$
- Deal with the Dirichlet boundary condition for $A^{(l)}\vec{X}^{(l)} = \vec{b}^{(l)}$ by using Algorithm III-4.
- Fix the pressure at one point in the domain Ω.
- Solve $A^{(l)}\vec{X}^{(l)} = \vec{b}^{(l)}$ for $\vec{X}$ by using a direct or iterative method.

END

Algorithm

Recall Algorithm I-3 from Chapter 3:

- Initialize the matrix: $A = sparse(N_b, N_b)$;
- Compute the integrals and assemble them into A:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \alpha = 1, \cdots, N_{lb}$:

$\quad\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad\quad$ Compute $r = \int_{E_n} c \frac{\partial^{r+s}\psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad\quad$ Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

$\quad\quad END$

$\quad END$

END

Algorithm

- Call Algorithm I-3 with $r = 1$, $s = 0$, $p = 1$, $q = 0$, $c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_1.
- Call Algorithm I-3 with $r = 0$, $s = 1$, $p = 0$, $q = 1$, $c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_2.
- Call Algorithm I-3 with $r = 1$, $s = 0$, $p = 0$, $q = 1$, $c = \nu$, basis type of $\mathbf{u}$ for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_3.
- Call Algorithm I-3 with $r = 0$, $s = 0$, $p = 1$, $q = 0$, $c = -1$, basis type of p for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_5.
- Call Algorithm I-3 with $r = 0$, $s = 0$, $p = 0$, $q = 1$, $c = -1$, basis type of p for trial function, and basis type of $\mathbf{u}$ for test function, to obtain A_6.
- Generate a zero matrix $\mathbb{O}$ whose size is $N_{bp} \times N_{bp}$.
- Then the stiffness matrix
  $A = [A_1 + 2A_2 \quad A_3 \quad A_5; A_3^t \quad 2A_2 + A_1 \quad A6; A_5^t \quad A_6^t \quad \mathbb{O}]$.

Algorithm

Recall Algorithm II-3 from Chapter 3:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} f \frac{\partial^{p+q}\psi_{n\beta}}{\partial x^p \partial y^q}\, dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Algorithm

- Call Algorithm II-3 with $p = q = 0$ and $f = f_1$ to obtain b_1.
- Call Algorithm II-3 with $p = q = 0$ and $f = f_2$ to obtain b_2.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp} \times 1$.
- Then the load vector $\vec{b} = [b_1; b_2; \vec{0}]$.

Algorithm

Recall Algorithm VIII from this chapter:

- Initialize the matrix: $A = sparse(N_b, N_b)$;
- Compute the integrals and assemble them into A:

FOR $n = 1, \cdots, N$:

  FOR $\alpha = 1, \cdots, N_{lb}$:

    FOR $\beta = 1, \cdots, N_{lb}$:

      Compute $r = \int_{E_n} \frac{\partial^{d+e} c_h}{\partial x^d \partial y^e} \frac{\partial^{r+s} \psi_{n\alpha}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q} \, dxdy$;

      Add r to $A(T_b(\beta, n), T_b(\alpha, n))$.

    END

  END

END

Algorithm

- Call Algorithm VIII with $d = 1$, $e = 0$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_1.
- Call Algorithm VIII with $d = 0$, $e = 0$, $r = 1$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_2.
- Call Algorithm VIII with $d = 0$, $e = 0$, $r = 0$, $s = 1$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_3.
- Call Algorithm VIII with $d = 0$, $e = 1$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{1h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_4.

Algorithm

- Call Algorithm VIII with $d = 1$, $e = 0$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_5.
- Call Algorithm VIII with $d = 0$, $e = 1$, $r = 0$, $s = 0$, $p = 0$, $q = 0$, $c_h = u_{2h}^{(l-1)}$, basis type of **u** for both trial and test functions, to obtain AN_6.
- Generate a zero matrix $\mathbb{O}_1 = [0]_{i,j=1}^{N_{bp}}$, $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_{bp}}$.
- Then the stiffness matrix

$$A = [AN_1 + AN_2 + AN_3 \;\; AN_4 \;\; \mathbb{O}_2; AN_5 \;\; AN_6 + AN_2 + AN_3 \;\; \mathbb{O}_3; \mathbb{O}_2^t \;\; \mathbb{O}_3^t \;\; \mathbb{O}_1].$$

Algorithm

Recall Algorithm IX from this chapter:

- Initialize the vector: $b = sparse(N_b, 1)$;
- Compute the integrals and assemble them into b:

$FOR\ n = 1, \cdots, N$:

$\quad FOR\ \beta = 1, \cdots, N_{lb}$:

$\quad\quad$ Compute $r = \int_{E_n} \frac{\partial^{d+e} f_{1h}}{\partial x^d \partial y^e} \frac{\partial^{r+s} f_{2h}}{\partial x^r \partial y^s} \frac{\partial^{p+q} \psi_{n\beta}}{\partial x^p \partial y^q}\ dxdy$;

$\quad\quad b(T_b(\beta, n), 1) = b(T_b(\beta, n), 1) + r$;

$\quad END$

END

Algorithm

- Call Algorithm IX with $d=0$, $e=0$, $r=1$, $s=0$, $p=0$, $q=0$ and $f_{h1}=u_{1h}^{(l-1)}$, $f_{h2}=u_{1h}^{(l-1)}$ to obtain bN_1.
- Call Algorithm IX with $d=0$, $e=0$, $r=0$, $s=1$, $p=0$, $q=0$ and $f_{h1}=u_{2h}^{(l-1)}$, $f_{h2}=u_{1h}^{(l-1)}$ to obtain bN_2.
- Call Algorithm IX with $d=0$, $e=0$, $r=1$, $s=0$, $p=0$, $q=0$ and $f_{h1}=u_{1h}^{(l-1)}$, $f_{h2}=u_{2h}^{(l-1)}$ to obtain bN_3.
- Call Algorithm IX with $d=0$, $e=0$, $r=0$, $s=1$, $p=0$, $q=0$ and $f_{h1}=u_{2h}^{(l-1)}$, $f_{h2}=u_{2h}^{(l-1)}$ to obtain bN_4.
- Define a zero column vector $\vec{0}$ whose size is $N_{bp}\times 1$
- Then the load vector $\overrightarrow{bN}=[bN_1+bN2; bN_3+bN_4; \vec{0}]$.

Algorithm

Recall Algorithm III-4 from this chapter:

- Deal with the Dirichlet boundary conditions:

  FOR $k = 1, \cdots, nbn$:

  If $boundarynodes(1, k)$ shows Dirichlet condition, then

  $i = boundarynodes(2, k)$;

  $A^{(l)}(i, :) = 0$;

  $A^{(l)}(i, i) = 1$;

  $b^{(l)}(i) = g_1(P_b(:, i))$;

  $A^{(l)}(N_b + i, :) = 0$;

  $A^{(l)}(N_b + i, N_b + i) = 1$;

  $b^{(l)}(N_b + i) = g_2(P_b(:, i))$;

  ENDIF

  END

Measurements for errors

- L^∞ norm error:

$$\begin{aligned}
\|\mathbf{u}-\mathbf{u}_h\|_\infty &= \max\left(\|u_1-u_{1h}\|_\infty, \|u_2-u_{2h}\|_\infty\right), \\
\|u_1-u_{1h}\|_\infty &= \sup_\Omega |u_1-u_{1h}|, \\
\|u_2-u_{2h}\|_\infty &= \sup_\Omega |u_2-u_{2h}|, \\
\|p-p_h\|_\infty &= \sup_\Omega |p-p_h|.
\end{aligned}$$

Measurements for errors

- L^2 norm error:

$$\|\mathbf{u}-\mathbf{u}_h\|_0=\sqrt{\|u_1-u_{1h}\|_0^2+\|u_2-u_{2h}\|_0^2},$$

$$\|u_1-u_{1h}\|_0=\sqrt{\int_\Omega (u_1-u_{1h})^2 dxdy},$$

$$\|u_2-u_{2h}\|_0=\sqrt{\int_\Omega (u_2-u_{2h})^2 dxdy},$$

$$\|p-p_h\|_0=\sqrt{\int_\Omega (p-p_h)^2 dxdy}.$$

Measurements for errors

- H^1 semi-norm error:

$$|\mathbf{u}-\mathbf{u}_h|_1 = \sqrt{|u_1-u_{1h}|_1^2+|u_2-u_{2h}|_1^2},$$

$$|u_1-u_{1h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_1-u_{1h})}{\partial x}\right)^2+\left(\frac{\partial(u_1-u_{1h})}{\partial y}\right)^2 dxdy},$$

$$|u_2-u_{2h}|_1 = \sqrt{\int_\Omega \left(\frac{\partial(u_2-u_{2h})}{\partial x}\right)^2+\left(\frac{\partial(u_2-u_{2h})}{\partial y}\right)^2 dxdy},$$

$$|p-p_h|_1 = \sqrt{\int_\Omega \left(\frac{\partial(p-p_h)}{\partial x}\right)^2+\left(\frac{\partial(p-p_h)}{\partial y}\right)^2 dxdy}.$$

- Basic idea: call Algorithm IV and Algorithm V in Chapter 3 for each of u_1, u_2, and p; then plug the results into the above formulas for the errors of $\mathbf{u}$ and p.

Numerical example

- Example 1: Use the finite element method to solve the following equation on the domain $\Omega = [0,1] \times [-0.25, 0]$:

$$
\begin{aligned}
(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \mathbb{T}(\mathbf{u}, p) &= \mathbf{f} \quad \text{on } \Omega, \\
\nabla \cdot \mathbf{u} &= 0 \qquad in \quad \Omega, \\
u_1 &= e^{-y} \quad \text{on } x = 0, \\
u_1 &= y^2 + e^{-y} \quad \text{on } x = 1, \\
u_1 &= \frac{1}{16}x^2 + e^{0.25} \quad \text{on } y = -0.25, \\
u_1 &= 1 \quad \text{on } y = 0, \\
u_2 &= 2 \quad \text{on } x = 0, \\
u_2 &= -\frac{2}{3}y^3 + 2 \quad \text{on } x = 1, \\
u_2 &= \frac{1}{96}x + 2 - \pi \sin(\pi x) \quad \text{on } y = -0.25, \\
u_2 &= 2 - \pi \sin(\pi x) \quad \text{on } y = 0.
\end{aligned}
$$

Numerical example

- Here

$$
\begin{aligned}
f_1 &= -2\nu x^2 - 2\nu y^2 - \nu e^{-y} + \pi^2 \cos(\pi x)\cos(2\pi y) \\
&\quad +2xy^2(x^2y^2 + e^{-y}) + (-2xy^3/3 + 2 - \pi\sin(\pi x))(2x^2y - e^{-y}), \\
f_2 &= 4\nu xy - \nu\pi^3 \sin(\pi x) + 2\pi(2 - \pi\sin(\pi x))\sin(2\pi y) \\
&\quad +(x^2y^2 + e^{-y})(-2y^3/3 - \pi^2\cos(\pi x)) \\
&\quad +(-2xy^3/3 + 2 - \pi\sin(\pi x))(-2xy^2).
\end{aligned}
$$

  We can also verify f_1 and f_2 above by plugging the analytic solutions below into the Navier-Stokes equation.
- The analytic solution of this problem is

$$
\begin{aligned}
&u_1 = x^2y^2 + e^{-y}, \quad u_2 = -\frac{2}{3}xy^3 + 2 - \pi\sin(\pi x), \\
&p = -(2 - \pi\sin(\pi x))\cos(2\pi y),
\end{aligned}
$$

  which can be used to compute the errors between the numerical solution and the analytic solution.

Numerical example

- Let's code for the Taylor-Hood finite elements for the 2D Navier-Stokes equation together!
- Taylor-Hood finite elements: linear finite elements for the pressure and quadratic finite elements for the velocity.
- Open your Matlab!

Numerical example

h	$\|\mathbf{u}-\mathbf{u}_h\|_\infty$	$\|\mathbf{u}-\mathbf{u}_h\|_0$	$\vert\mathbf{u}-\mathbf{u}_h\vert_1$
$1/8$	1.6853×10^{-3}	3.5640×10^{-4}	2.0429×10^{-2}
$1/16$	2.0224×10^{-4}	4.4016×10^{-5}	5.0681×10^{-3}
$1/32$	2.5167×10^{-5}	5.4798×10^{-6}	1.2623×10^{-3}
$1/64$	3.1048×10^{-6}	6.8421×10^{-7}	3.1523×10^{-4}

Table: The numerical errors for quadratic finite elements of the velocity.

- Any Observation?
- Third order convergence $O(h^3)$ in L^2/L^∞ norm and second order convergence $O(h^2)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise quadratic functions.

Numerical example

h	$\|p-p_h\|_\infty$	$\|p-p_h\|_0$	$\|p-p_h\|_1$
$1/8$	1.3616×10^{-1}	2.2577×10^{-2}	1.2648×10^{0}
$1/16$	4.5862×10^{-2}	8.6669×10^{-3}	6.3069×10^{-1}
$1/32$	1.2533×10^{-2}	2.4764×10^{-3}	3.1369×10^{-1}
$1/64$	3.2510×10^{-3}	6.5584×10^{-4}	1.5658×10^{-1}

Table: The numerical errors for linear finite elements of the pressure.

- Any Observation?
- Second order convergence $O(h^2)$ in L^2/L^∞ norm and first order convergence $O(h)$ in H^1 semi-norm, which match the optimal approximation capability expected from piecewise linear functions.

Outline

Stress boundary condition

- Consider

$$\begin{cases} (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \quad in \quad \Omega, \\ \nabla\cdot\mathbf{u} = 0 \quad in \quad \Omega, \\ \mathbb{T}(\mathbf{u},p)\mathbf{n} = \mathbf{p} \quad on \quad \partial\Omega. \end{cases}$$

where $\mathbf{n} = (n_1,\ n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and

$$\mathbf{p}(x,y) = (p_1,\ p_2)^t,\ \mathbf{f}(x,y) = (f_1,\ f_2)^t.$$

- Recall

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy$$

$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$

$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

Stress boundary condition

- Hence

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$
$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}\mathbf{p}\cdot\mathbf{v}\ ds = \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$
$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

- The solution is unique for the Navier-Stokes equation with pure stress boundary condition!

- If $\mathbf{u} = (u_1, u_2)^t$ is a solution, then $\mathbf{u}+\mathbf{c}$ is not a solution because of the nonlinear term $\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy$.

Stress boundary condition

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbb{T}(\mathbf{u},p)\mathbf{n} = \mathbf{p} \quad \text{on } \Gamma_S \subset \partial\Omega,\\
&\mathbf{u} = \mathbf{g} \quad \text{on } \Gamma_D = \partial\Omega/\Gamma_S.
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\; dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\; dxdy\\
&-\int_\Omega p(\nabla\cdot\mathbf{v})\; dxdy - \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\; ds \;=\; \int_\Omega \mathbf{f}\cdot\mathbf{v}\; dxdy,\\
&-\int_\Omega (\nabla\cdot\mathbf{u})q\; dxdy = 0.
\end{aligned}
$$

Stress boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \quad & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\partial\Omega/\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \quad & \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \; ds.
\end{aligned}
$$

Stress boundary condition

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \, dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \, dxdy$$
$$- \int_\Omega p(\nabla \cdot \mathbf{v}) \, dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \, dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \, ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \, dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here

$$\int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \, ds = \int_{\Gamma_S} p_1 v_1 \, ds + \int_{\Gamma_S} p_2 v_2 \, ds,$$
$$H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Stress boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Stress boundary condition

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h (\nabla \cdot \mathbf{v}_h) \; dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition

Newton's iteration for Galerkin formulation in the vector format:

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- For $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h^{(l)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \, dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{(l)}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{(l)} (\nabla \cdot \mathbf{v}_h) \, dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v}_h \, ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h^{(l)}) q_h \, dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition

- Since $u_{1h}^{(l)},\ u_{2h}^{(l)} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j, \quad u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j, \quad p_h^{(l)} = \sum_{j=1}^{N_{bp}} p_j^{(l)} \psi_j$$

  for some coefficients $u_{1j}^{(l)},\ u_{2j}^{(l)}$ $(j = 1, \cdots, N_b)$, and $p_j^{(l)}$ $(j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for $u_{1j}^{(l)}$, $u_{2j}^{(l)}$ $(j = 1, \cdots, N_b)$, and $p_j^{(l)}$ $(j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{(l)} = (u_{1h}^{(l)}, u_{2h}^{(l)})^t$ and $p_h^{(l)}$ at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration.

Stress boundary condition

- For the first equation in the Galerkin formulation at the step l ($l = 1, 2, \cdots, L$) of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ ($i = 1, \cdots, N_b$) and $\mathbf{v}_h = (0, \phi_i)^t$ ($i = 1, \cdots, N_b$). That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ ($i = 1, \cdots, N_b$) and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ ($i = 1, \cdots, N_b$).

- For the second equation in the Galerkin formulation at the step l ($l = 1, 2, \cdots, L$) of Newton's iteration, we choose $q_h = \psi_i$ ($i = 1, \cdots, N_{bp}$).

Stress boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \Big(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy \\
&+ \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\ dxdy\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right) \\
= \ &\int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\ dxdy \\
&+ \int_{\Gamma_S} p_1\phi_i\ ds,
\end{aligned}
$$

Stress boundary condition

- Continued formulation:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_j \phi_i \, dxdy \right) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \Big(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dxdy \\
&+ \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_j \phi_i \, dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dxdy \Big) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \, dxdy \right) \\
= \; & \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \, dxdy \\
&+ \int_{\Gamma_S} p_2 \phi_i \, ds,
\end{aligned}
$$

Stress boundary condition

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial x} \psi_i \; dxdy \right) + \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial y} \psi_i \; dxdy \right)$$

$$+ \sum_{j=1}^{N_{bp}} p_j^{(l)} * 0 = 0.$$

Stress boundary condition

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}},$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

- Each matrix A_i can be obtained by Algorithm I-3 in Chapter 3.

Stress boundary condition

- Recall

$$AN_1 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_2 = \left[\int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\ dxdy\right]_{i,j=1}^{N_b},$$

$$AN_3 = \left[\int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_4 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},$$

$$AN_5 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},\quad AN_6 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right]_{i,j=1}^{N_b},$$

and

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_3 \\ \mathbb{O}_2^t & \mathbb{O}_3^t & \mathbb{O}_1 \end{pmatrix}$$

with zero matrices $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_{bp}}$.

Stress boundary condition

- Each matrix AN_i can be obtained by Algorithm VIII in this chapter.
- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1\phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2\phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Stress boundary condition

- Recall

$$\overrightarrow{bN} = \begin{pmatrix} \overrightarrow{bN}_1 + \overrightarrow{bN}_2 \\ \overrightarrow{bN}_3 + \overrightarrow{bN}_4 \\ \vec{0} \end{pmatrix}$$

where

$$\overrightarrow{bN}_1 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} \phi_i \; dxdy\right]_{i=1}^{N_b}, \; \overrightarrow{bN}_2 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} \phi_i \; dxdy\right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_3 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \; dxdy\right]_{i=1}^{N_b}, \; \overrightarrow{bN}_4 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \; dxdy\right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each matrix bN_i can be obtained by Algorithm IX in this chapter.

Stress boundary condition

- Recall the unknown vector

$$\vec{X}^{(l)} = \begin{pmatrix} \vec{X}_1^{(l)} \\ \vec{X}_2^{(l)} \\ \vec{X}_3^{(l)} \end{pmatrix}$$

where

$$\vec{X}_1 = \left[u_{1j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_2 = \left[u_{2j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_3 = \left[p_j^{(l)}\right]_{j=1}^{N_{bp}}.$$

- Recall

$$A^{(l)} = A + AN, \quad \vec{b}^{(l)} = \vec{b} + \overrightarrow{bN}.$$

Stress boundary condition

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 \\ \vec{v}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_1 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{v}_2 = \left[\int_{\Gamma_S} p_2 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}}^{(l)} = \vec{b} + \vec{v} + \overrightarrow{bN}$.

Stress boundary condition

- For step l $(l = 1, 2, \cdots, L)$ of the Newton's iteration, we obtain the linear algebraic system

$$A^{(l)}\vec{X}^{(l)} = \widetilde{\vec{b}}^{(l)}.$$

- Similar to Chapter 6, we essentially only need repeat the code of Neumman condition in Chapter 3 for $\vec{v}_1$ and $\vec{v}_2$. We have discussed about this in Chapter 7 and obtained Algorithm VI-4 in Chapter 7 based on VI-2 in Chapter 6.

Stress boundary condition

Recall Algorithm VI-4 from Chapter 7:

- Initialize the vector: $v = sparse(2N_b + N_{bp}, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ \ k = 1, \cdots, nbe$:

$\quad IF\ \ boundaryedges(1, k)$ shows stress boundary, $THEN$

$\quad\quad n_k = boundaryedges(2, k)$;

$\quad\quad FOR\ \ \beta = 1, \cdots, N_{lb}$:

$\quad\quad\quad$ Compute $r = \int_{e_k} p_1 \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b}\, ds$;

$\quad\quad\quad v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r$;

$\quad\quad\quad$ Compute $r = \int_{e_k} p_2 \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b}\, ds$;

$\quad\quad\quad v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r$;

$\quad\quad END$

$\quad ENDIF$

END

Robin boundary conditions

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\mathbf{u} = \mathbf{q} \ \text{ on } \Gamma_R \subseteq \partial\Omega,\\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/\Gamma_R.
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy\\
&-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,\\
&-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.
\end{aligned}
$$

Robin boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_R$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds + \int_{\partial\Omega/\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= \; & \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds.
\end{aligned}
$$

Robin boundary condition

- The weak formulation is find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy$$
$$+ \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds = \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u})q \; dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here

$$\int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} q_1 v_1 \; ds + \int_{\Gamma_R} q_2 v_2 \; ds,$$
$$\int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds = \int_{\Gamma_R} r u_1 v_1 \; ds + \int_{\Gamma_R} r u_2 v_2 \; ds,$$
$$H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}.$$

Robin boundary condition

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy$$

$$- \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy + \int_{\Gamma_R} r\mathbf{u}_h \cdot \mathbf{v}_h \; ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h \; ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Robin boundary condition

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h (\nabla \cdot \mathbf{v}_h) \; dxdy + \int_{\Gamma_R} r\mathbf{u}_h \cdot \mathbf{v}_h \; ds \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v}_h \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Robin boundary condition

Newton's iteration for Galerkin formulation in the vector format:

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- For $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ such that

$$\int_\Omega (\mathbf{u}_h^{(l)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \; dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \; dxdy$$

$$+ \int_\Omega 2\nu\mathbb{D}(\mathbf{u}_h^{(l)}) : \mathbb{D}(\mathbf{v}_h) \; dxdy - \int_\Omega p_h^{(l)}(\nabla \cdot \mathbf{v}_h) \; dxdy + \int_{\Gamma_R} r\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \; ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_S} \mathbf{q} \cdot \mathbf{v}_h \; ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}_h^{(l)}) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Robin boundary condition

- Since $u_{1h}^{(l)},\ u_{2h}^{(l)} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j, \quad u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j, \quad p_h^{(l)} = \sum_{j=1}^{N_{bp}} p_j^{(l)} \psi_j$$

  for some coefficients $u_{1j}^{(l)},\ u_{2j}^{(l)}\ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)}\ \ (j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for $u_{1j}^{(l)}$, $u_{2j}^{(l)}\ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)}\ \ (j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{(l)} = (u_{1h}^{(l)}, u_{2h}^{(l)})^t$ and $p_h^{(l)}$ at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration.

Robin boundary condition

- For the first equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Robin boundary condition

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \Big(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\, dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\, dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\, dxdy \\
&+ \int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\, dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\, dxdy + \int_{\Gamma_R} r\phi_j\phi_i\, ds\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\, dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\, dxdy\right) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\, dxdy\right) \\
= \quad & \int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\, dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\, dxdy \\
&+ \int_{\Gamma_S} q_1\phi_i\, ds,
\end{aligned}
$$

Robin boundary condition

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_j \phi_i \, dxdy \right)$$

$$+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \Big(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_j \phi_i \, dxdy$$

$$+ \int_\Omega u_{1h}^{(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dxdy + \int_{\Gamma_R} r \phi_j \phi_i \, ds \Big)$$

$$+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \, dxdy \right)$$

$$= \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \, dxdy$$

$$+ \int_{\Gamma_S} q_2 \phi_i \, ds,$$

Robin boundary condition

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial x} \psi_i \; dxdy \right) + \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial y} \psi_i \; dxdy \right)$$

$$+ \sum_{j=1}^{N_{bp}} p_j^{(l)} * 0 = 0.$$

Robin boundary condition

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega -\psi_j \frac{\partial \phi_i}{\partial y}\ dxdy\right]_{i=1,j=1}^{N_b,N_{bp}},$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

- Each matrix A_i can be obtained by Algorithm I-3 in Chapter 3.

Stress boundary condition

- Recall

$$AN_1 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_2 = \left[\int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

$$AN_3 = \left[\int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_4 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

$$AN_5 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_6 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

and

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_3 \\ \mathbb{O}_2^t & \mathbb{O}_3^t & \mathbb{O}_1 \end{pmatrix}$$

with zero matrices $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_{bp}}$.

Robin boundary condition

- Each matrix AN_i can be obtained by Algorithm VIII in this chapter.
- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Robin boundary condition

- Recall

$$\overrightarrow{bN} = \begin{pmatrix} \overrightarrow{bN}_1 + \overrightarrow{bN}_2 \\ \overrightarrow{bN}_3 + \overrightarrow{bN}_4 \\ \vec{0} \end{pmatrix}$$

where

$$\overrightarrow{bN}_1 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} \phi_i \; dxdy\right]_{i=1}^{N_b}, \quad \overrightarrow{bN}_2 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} \phi_i \; dxdy\right]_{i=1}^{N_b},$$

$$\overrightarrow{bN}_3 = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \; dxdy\right]_{i=1}^{N_b}, \quad \overrightarrow{bN}_4 = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \; dxdy\right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each matrix bN_i can be obtained by Algorithm IX in this chapter.

Robin boundary condition

- Recall the unknown vector

$$\vec{X}^{(l)} = \begin{pmatrix} \vec{X}_1^{(l)} \\ \vec{X}_2^{(l)} \\ \vec{X}_3^{(l)} \end{pmatrix}$$

where

$$\vec{X}_1 = \left[u_{1j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_2 = \left[u_{2j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_3 = \left[p_j^{(l)}\right]_{j=1}^{N_{bp}}.$$

- Recall

$$A^{(l)} = A + AN, \ \vec{b}^{(l)} = \vec{b} + \overrightarrow{bN}.$$

Robin boundary condition

- Define the additional vector from the Robin boundary condition:

$$\vec{w} = \begin{pmatrix} \vec{w}_1 \\ \vec{w}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{w}_1 = \left[\int_{\Gamma_S} q_1 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{w}_2 = \left[\int_{\Gamma_S} q_2 \phi_i \ ds \right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}}^{(l)} = \vec{b} + \vec{w} + \overrightarrow{bN}$.
- Since each of $\vec{w}_1$ and $\vec{w}_2$ is similar to the $\vec{w}$ for the Robin condition in Chapter 3, we essentially only need repeat the code of $\vec{w}$ in Chapter 3 for $\vec{w}_1$ and $\vec{w}_2$.

Robin boundary condition

- Define the additional matrix from the Robin boundary condition

$$R = [r_{ij}]_{i,j=1}^{N_b} = \left[\int_{\Gamma_R} r \phi_j \phi_i \; ds \right]_{i,j=1}^{N_b}.$$

- Since R is the same as the R in Chapter 3, the code for R is the same. But R needs to be added to the matrix A twice as showed below to obtain $\widetilde{A}$.

Robin boundary condition

- Define

$$\widetilde{A} = \begin{pmatrix} 2A_1 + A_2 + R & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 + R & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

and

$$A^{(l)} = \widetilde{A} + AN.$$

- Then we obtain the linear algebraic system

$$A^{(l)} \vec{X}^{(l)} = \widetilde{\vec{b}}^{(l)}.$$

- Pseudo code? (Part of a project for you)

Dirichlet/stress/Robin mixed boundary condition

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbb{T}(\mathbf{u},p)\mathbf{n} = \mathbf{p} \ \text{ on } \Gamma_S \subset \partial\Omega,\\
&\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\mathbf{u} = \mathbf{q} \ \text{ on } \Gamma_R \subseteq \partial\Omega,\\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/(\Gamma_S\cup\Gamma_R).
\end{aligned}
$$

- Recall

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy\\
&-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,\\
&-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition

- Since the solution on $\Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R)$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/(\Gamma_S \cup \Gamma_R)$.
- Then

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \ ds + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \ ds - \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \ ds.
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition

- The weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy$$

$$- \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy + \int_{\Gamma_R} r\mathbf{u} \cdot \mathbf{v} \; ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy + \int_{\Gamma_R} \mathbf{q} \cdot \mathbf{v} \; ds + \int_{\Gamma_S} \mathbf{p} \cdot \mathbf{v} \; ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0,$$

  for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$. Here $H^1_{0D}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_D\}$.
- Code? Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.

Stress boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n} = p_n,\ \tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n} = p_\tau \ \text{ on } \Gamma_S\subset\partial\Omega,\\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/\Gamma_S.
\end{aligned}
$$

where $\mathbf{n} = (n_1,\ n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Stress boundary condition in normal/tangential directions

- Recall

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$
$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds \ = \ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$
$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_S$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x, y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_S$.

Stress boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
&\int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= &\int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\partial\Omega/\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= &\int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= &\int_{\Gamma_S} \left[(\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n})\mathbf{n} + (\tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n})\tau\right] \cdot \left[(\mathbf{n}^t \mathbf{v})\mathbf{n} + (\tau^t \mathbf{v})\tau\right] \ ds \\
= &\int_{\Gamma_S} (\mathbf{n}^t \mathbb{T}(\mathbf{u}, p)\mathbf{n})(\mathbf{n}^t \mathbf{v}) \ ds + \int_{\Gamma_S} (\tau^t \mathbb{T}(\mathbf{u}, p)\mathbf{n})(\tau^t \mathbf{v}) \ ds \\
= &\int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \ ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}) \ ds.
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\begin{aligned}
&\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy \\
&- \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}) \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}) q \; dxdy = 0,
\end{aligned}$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.

Stress boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h (\nabla \cdot \mathbf{v}_h) \; dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_S} p_n (\mathbf{n}^t \mathbf{v}_h) \; ds + \int_{\Gamma_S} p_\tau (\tau^t \mathbf{v}_h) \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Stress boundary condition in normal/tangential directions

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_S} p_n(\mathbf{n}^t \mathbf{v}_h) \; ds + \int_{\Gamma_S} p_\tau(\tau^t \mathbf{v}_h) \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition in normal/tangential directions

Newton's iteration for Galerkin formulation in the vector format:

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- For $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ such that

$$\int_\Omega (\mathbf{u}_h^{(l)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \, dxdy$$
$$+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{(l)}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{(l)} (\nabla \cdot \mathbf{v}_h) \, dxdy$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy$$
$$+ \int_{\Gamma_S} p_n (\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_S} p_\tau (\tau^t \mathbf{v}_h) \, ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h^{(l)}) q_h \, dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Stress boundary condition in normal/tangential directions

- Since $u_{1h}^{(l)}$, $u_{2h}^{(l)} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j, \quad u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j, \quad p_h^{(l)} = \sum_{j=1}^{N_{bp}} p_j^{(l)} \psi_j$$

for some coefficients $u_{1j}^{(l)}$, $u_{2j}^{(l)}$ $(j = 1, \cdots, N_b)$, and $p_j^{(l)}$ $(j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for $u_{1j}^{(l)}$, $u_{2j}^{(l)}$ $(j = 1, \cdots, N_b)$, and $p_j^{(l)}$ $(j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{(l)} = (u_{1h}^{(l)}, u_{2h}^{(l)})^t$ and $p_h^{(l)}$ at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration.

Stress boundary condition in normal/tangential directions

- For the first equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Stress boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \Big(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy \\
&+ \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\ dxdy\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy\right) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(-\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\ dxdy\right) \\
= &\int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\ dxdy \\
&+ \int_{\Gamma_S} p_n\phi_i n_1\ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_1\ ds,
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Continued formulation:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \ dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_j \phi_i \ dxdy \right) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \Bigg(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \ dxdy \\
&+ \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_j \phi_i \ dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \ dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \ dxdy \Bigg) \\
&+ \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \ dxdy \right) \\
= \ & \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \ dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \ dxdy \\
&+ \int_{\Gamma_S} p_n \phi_i n_2 \ ds + \int_{\Gamma_S} p_\tau \phi_i \tau_2 \ ds,
\end{aligned}
$$

Stress boundary condition in normal/tangential directions

- Continued formulation:

$$
\sum_{j=1}^{N_b} u_{1j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial x} \psi_i \ dxdy \right) + \sum_{j=1}^{N_b} u_{2j}^{(l)} \left(-\int_\Omega \frac{\partial \phi_j}{\partial y} \psi_i \ dxdy \right)
$$

$$
+ \sum_{j=1}^{N_{bp}} p_j^{(l)} * 0 = 0.
$$

Stress boundary condition in normal/tangential directions

- Recall

$$A_1 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i,j=1}^{N_b}, \quad A_2 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_3 = \left[\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i,j=1}^{N_b},$$

$$A_5 = \left[\int_\Omega \psi_j \frac{\partial \phi_i}{\partial x}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}}, \quad A_6 = \left[\int_\Omega \psi_j \frac{\partial \phi_i}{\partial y}\, dxdy\right]_{i=1,j=1}^{N_b,N_{bp}},$$

and

$$A = \begin{pmatrix} 2A_1 + A_2 & A_3 & A_5 \\ A_3^t & 2A_2 + A_1 & A_6 \\ A_5^t & A_6^t & \mathbb{O}_1 \end{pmatrix}$$

where $\mathbb{O}_1$ is a zero matrix whose size is $N_{bp} \times N_{bp}$.

- Each matrix A_i can be obtained by Algorithm I-3 in Chapter 3.

Stress boundary condition

- Recall

$$AN_1 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_2 = \left[\int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

$$AN_3 = \left[\int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_4 = \left[\int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

$$AN_5 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b}, \quad AN_6 = \left[\int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y}\phi_j\phi_i \; dxdy\right]_{i,j=1}^{N_b},$$

and

$$AN = \begin{pmatrix} AN_1 + AN_2 + AN_3 & AN_4 & \mathbb{O}_2 \\ AN_5 & AN_6 + AN_2 + AN_3 & \mathbb{O}_3 \\ \mathbb{O}_2^t & \mathbb{O}_3^t & \mathbb{O}_1 \end{pmatrix}$$

with zero matrices $\mathbb{O}_2 = [0]_{i=1,j=1}^{N_b,N_{bp}}$ and $\mathbb{O}_3 = [0]_{i=1,j=1}^{N_b,N_{bp}}$.

Stress boundary condition in normal/tangential directions

- Each matrix AN_i can be obtained by Algorithm VIII in this chapter.
- Recall

$$\vec{b} = \begin{pmatrix} \vec{b}_1 \\ \vec{b}_2 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{b}_1 = \left[\int_\Omega f_1 \phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{b}_2 = \left[\int_\Omega f_2 \phi_i dxdy\right]_{i=1}^{N_b}, \quad \vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Each of $\vec{b}_1$ and $\vec{b}_2$ can be obtained by Algorithm II-3 in Chapter 3.

Stress boundary condition in normal/tangential directions

- Recall

$$\overrightarrow{bN} = \begin{pmatrix} \overrightarrow{bN_1} + \overrightarrow{bN_2} \\ \overrightarrow{bN_3} + \overrightarrow{bN_4} \\ \vec{0} \end{pmatrix}$$

where

$$\overrightarrow{bN_1} = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial x} \phi_i \ dxdy \right]_{i=1}^{N_b}, \ \overrightarrow{bN_2} = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{1h}^{(l-1)}}{\partial y} \phi_i \ dxdy \right]_{i=1}^{N_b},$$

$$\overrightarrow{bN_3} = \left[\int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \ dxdy \right]_{i=1}^{N_b}, \ \overrightarrow{bN_4} = \left[\int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \ dxdy \right]_{i=1}^{N_b}.$$

Here the size of the zero vector is $N_{bp} \times 1$. That is, $\vec{0} = [0]_{i=1}^{N_{bp}}$.

- Each matrix bN_i can be obtained by Algorithm IX in this chapter.

Stress boundary condition in normal/tangential directions

- Recall the unknown vector

$$\vec{X}^{(l)} = \begin{pmatrix} \vec{X}_1^{(l)} \\ \vec{X}_2^{(l)} \\ \vec{X}_3^{(l)} \end{pmatrix}$$

where

$$\vec{X}_1 = \left[u_{1j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_2 = \left[u_{2j}^{(l)}\right]_{j=1}^{N_b}, \quad \vec{X}_3 = \left[p_j^{(l)}\right]_{j=1}^{N_{bp}}.$$

- Recall

$$A^{(l)} = A + AN, \ \vec{b}^{(l)} = \vec{b} + \overrightarrow{bN}.$$

Stress boundary condition in normal/tangential directions

- Define the additional vector from the stress boundary condition:

$$\vec{v} = \begin{pmatrix} \vec{v}_1 + \vec{v}_2 \\ \vec{v}_3 + \vec{v}_4 \\ \vec{0} \end{pmatrix}$$

where

$$\vec{v}_1 = \left[\int_{\Gamma_S} p_n \phi_i n_1 \ ds \right]_{i=1}^{N_b}, \ \vec{v}_2 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_1 \ ds \right]_{i=1}^{N_b},$$

$$\vec{v}_3 = \left[\int_{\Gamma_S} p_n \phi_i n_2 \ ds \right]_{i=1}^{N_b}, \ \vec{v}_4 = \left[\int_{\Gamma_S} p_\tau \phi_i \tau_2 \ ds \right]_{i=1}^{N_b}$$

$$\vec{0} = [0]_{i=1}^{N_{bp}}.$$

- Define the new vector $\widetilde{\vec{b}}^{(l)} = \vec{b} + \vec{v} + \overrightarrow{bN}$.

Stress boundary condition in normal/tangential directions

- For step l ($l = 1, 2, \cdots, L$) of the Newton's iteration, we obtain the linear algebraic system

$$A^{(l)}\vec{X}^{(l)} = \tilde{\vec{b}}^{(l)}.$$

- Similar to Chapter 6, we essentially only need repeat the code of Neumman condition in Chapter 3 for $\vec{v}_1$ and $\vec{v}_2$. We have discussed about this in Chapter 7 and obtained Algorithm VI-5 in Chapter 7 based on VI-3 in Chapter 6.

- The major difference between $\vec{v}_i$ ($i = 1, 2, 3, 4$) here and the $\vec{v}$ for the Neumann condition in Chapter 3 is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix *boundaryedges*.

Stress boundary condition in normal/tangential directions

Recall Algorithm VI-5 from Chapter 7:

- Initialize the vector: $v = sparse(2N_b + N_{bp}, 1)$;
- Compute the integrals and assemble them into v:

$FOR\ k = 1, \cdots, nbe$:

$IF\ boundaryedges(1, k)$ shows stress boundary in normal/tangential directions, $THEN$

$n_k = boundaryedges(2, k)$;

$FOR\ \beta = 1, \cdots, N_{lb}$:

Compute $r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} n_1\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} \tau_1\ ds$;

$v(T_b(\beta, n_k), 1) = v(T_b(\beta, n_k), 1) + r$;

Compute $r = \int_{e_k} p_n \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} n_2\ ds + \int_{e_k} p_\tau \frac{\partial^{a+b}\psi_{n_k\beta}}{\partial x^a \partial y^b} \tau_2\ ds$;

$v(N_b + T_b(\beta, n_k), 1) = v(N_b + T_b(\beta, n_k), 1) + r$;

END

$ENDIF$

END

Robin boundary conditions in normal/tangential directions

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n,\ \ \tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \ \text{ on } \Gamma_R \subseteq \partial\Omega,\\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/\Gamma_R.
\end{aligned}
$$

where $\mathbf{n} = (n_1,\ n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Robin boundary conditions in normal/tangential directions

- Recall

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$
$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy - \int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds\ =\ \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy,$$
$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0.$$

- Since the solution on $\Gamma_D = \partial\Omega/\Gamma_R$ is given by $\mathbf{u} = \mathbf{g}$, then we can choose the test function $\mathbf{v}(x,y)$ such that $\mathbf{v} = 0$ on $\partial\Omega/\Gamma_R$.

Robin boundary condition in normal/tangential directions

- Using the above conditions, orthogonal decomposition of a vector, and the definition of unit normal/tangential vector, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_R} (\mathbb{T}(\mathbf{u},p)\mathbf{n}) \cdot \mathbf{v} \; ds \\
= & \int_{\Gamma_R} \left[(\mathbf{n}^t \mathbb{T}(\mathbf{u},p)\mathbf{n})\mathbf{n} + (\tau^t \mathbb{T}(\mathbf{u},p)\mathbf{n})\tau \right] \cdot \left[(\mathbf{n}^t \mathbf{v})\mathbf{n} + (\tau^t \mathbf{v})\tau \right] \; ds \\
= & \int_{\Gamma_S} (\mathbf{n}^t \mathbb{T}(\mathbf{u},p)\mathbf{n})(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_S} (\tau^t \mathbb{T}(\mathbf{u},p)\mathbf{n})(\tau^t \mathbf{v}) \; ds \\
= & \left[\int_{\Gamma_R} q_n (\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_R} q_\tau (\tau^t \mathbf{v}) \; ds \right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u})(\mathbf{n}^t \mathbf{v}) \; ds + \int_{\Gamma_R} (r\tau^t \mathbf{u})(\tau^t \mathbf{v}) \; ds \right],
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Then the weak formulation is to find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ such that

$$\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy + \int_\Omega 2\nu\mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v})\ dxdy$$
$$-\int_\Omega p(\nabla\cdot\mathbf{v})\ dxdy + \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v})\ ds$$
$$= \int_\Omega \mathbf{f}\cdot\mathbf{v}\ dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v})\ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v})\ ds,$$
$$-\int_\Omega (\nabla\cdot\mathbf{u})q\ dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.

Robin boundary condition in normal/tangential directions

- Then the Galerkin formulation is to find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy \\
&- \int_\Omega p_h (\nabla \cdot \mathbf{v}_h) \; dxdy \\
&+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u}_h)(\mathbf{n}^t\mathbf{v}_h) \; ds + \int_{\Gamma_R} (r\tau^t\mathbf{u}_h)(\tau^t\mathbf{v}_h) \; ds \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}_h) \; ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}_h) \; ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,
\end{aligned}
$$

for any $\mathbf{v}_h \in U_{h0} \times U_{h0}$ and $q_h \in W_h$.

Robin boundary condition in normal/tangential directions

- For an easier implementation, we use the following Galerkin formulation (without considering the Dirichlet boundary condition, which will be handled later): find $\mathbf{u}_h \in U_h \times U_h$ and $p_h \in W_h$ such that

$$\int_\Omega (\mathbf{u}_h \cdot \nabla)\mathbf{u}_h \cdot \mathbf{v}_h \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h) : \mathbb{D}(\mathbf{v}_h) \; dxdy$$
$$- \int_\Omega p_h(\nabla \cdot \mathbf{v}_h) \; dxdy$$
$$+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u}_h)(\mathbf{n}^t\mathbf{v}_h) \; ds + \int_{\Gamma_R} (r\tau^t\mathbf{u}_h)(\tau^t\mathbf{v}_h) \; ds$$
$$= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \; dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}_h) \; ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}_h) \; ds,$$
$$- \int_\Omega (\nabla \cdot \mathbf{u}_h) q_h \; dxdy = 0,$$

for any $\mathbf{v}_h \in U_h \times U_h$ and $q_h \in W_h$.

Robin boundary condition in normal/tangential directions

Newton's iteration for Galerkin formulation in the vector format:

- Initial guess: $\mathbf{u}_h^{(0)}$ and $p_h^{(0)}$.
- For $l = 1, 2, \cdots, L$, find $\mathbf{u}_h^{(l)} \in U_h \times U_h$ and $p_h^{(l)} \in W_h$ s.t.

$$
\begin{aligned}
&\int_\Omega (\mathbf{u}_h^{(l)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l)} \cdot \mathbf{v}_h \, dxdy \\
&+ \int_\Omega 2\nu \mathbb{D}(\mathbf{u}_h^{(l)}) : \mathbb{D}(\mathbf{v}_h) \, dxdy - \int_\Omega p_h^{(l)} (\nabla \cdot \mathbf{v}_h) \, dxdy \\
&+ \int_{\Gamma_R} (r\mathbf{n}^t \mathbf{u}_h^{(l)})(\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} (r\tau^t \mathbf{u}_h^{(l)})(\tau^t \mathbf{v}_h) \, ds \\
&= \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dxdy + \int_\Omega (\mathbf{u}_h^{(l-1)} \cdot \nabla)\mathbf{u}_h^{(l-1)} \cdot \mathbf{v}_h \, dxdy \\
&+ \int_{\Gamma_R} q_n (\mathbf{n}^t \mathbf{v}_h) \, ds + \int_{\Gamma_R} q_\tau (\tau^t \mathbf{v}_h) \, ds, \\
&- \int_\Omega (\nabla \cdot \mathbf{u}_h^{(l)}) q_h \, dxdy = 0, \quad \text{for any } \mathbf{v}_h \in U_h \times U_h \text{ and } q_h \in W_h.
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Since $u_{1h}^{(l)},\ u_{2h}^{(l)} \in U_h = span\{\phi_j\}_{j=1}^{N_b}$ and $p_h \in W_h = span\{\psi_j\}_{j=1}^{N_{bp}}$, then

$$u_{1h}^{(l)} = \sum_{j=1}^{N_b} u_{1j}^{(l)} \phi_j, \quad u_{2h}^{(l)} = \sum_{j=1}^{N_b} u_{2j}^{(l)} \phi_j, \quad p_h^{(l)} = \sum_{j=1}^{N_{bp}} p_j^{(l)} \psi_j$$

  for some coefficients $u_{1j}^{(l)},\ u_{2j}^{(l)}\ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)}\ \ (j = 1, \cdots, N_{bp})$.

- If we can set up a linear algebraic system for $u_{1j}^{(l)}$, $u_{2j}^{(l)}\ \ (j = 1, \cdots, N_b)$, and $p_j^{(l)}\ \ (j = 1, \cdots, N_{bp})$, then we can solve it to obtain the finite element solution $\mathbf{u}_h^{(l)} = (u_{1h}^{(l)}, u_{2h}^{(l)})^t$ and $p_h^{(l)}$ at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration.

Robin boundary condition in normal/tangential directions

- For the first equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $\mathbf{v}_h = (\phi_i, 0)^t$ $(i = 1, \cdots, N_b)$ and $\mathbf{v}_h = (0, \phi_i)^t$ $(i = 1, \cdots, N_b)$. That is, in the first set of test functions, we choose $v_{1h} = \phi_i$ $(i = 1, \cdots, N_b)$ and $v_{2h} = 0$; in the second set of test functions, we choose $v_{1h} = 0$ and $v_{2h} = \phi_i$ $(i = 1, \cdots, N_b)$.

- For the second equation in the Galerkin formulation at the step l $(l = 1, 2, \cdots, L)$ of Newton's iteration, we choose $q_h = \psi_i$ $(i = 1, \cdots, N_{bp})$.

Robin boundary condition in normal/tangential directions

- Then by the same procedure to derive the matrix formulation before, we obtain the following linear system:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)}\Big(2\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial x}\ dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial y}\frac{\partial \phi_i}{\partial y}\ dxdy \\
&+ \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_j\phi_i\ dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial \phi_j}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial \phi_j}{\partial y}\phi_i\ dxdy \\
&+ \int_{\Gamma_R}(rn_1\phi_j)(\phi_i n_1)\ ds + \int_{\Gamma_R}(r\tau_1\phi_j)(\phi_i\tau_1)\ ds\Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)}\Big(\int_\Omega \nu \frac{\partial \phi_j}{\partial x}\frac{\partial \phi_i}{\partial y}\ dxdy + \int_\Omega \frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_j\phi_i\ dxdy \\
&+ \int_{\Gamma_R}(rn_2\phi_j)(\phi_i n_1)\ ds + \int_{\Gamma_R}(r\tau_2\phi_j)(\phi_i\tau_1)\ ds\Big) + \sum_{j=1}^{N_{bp}} p_j^{(l)}\left(-\int_\Omega \psi_j\frac{\partial \phi_i}{\partial x}\ dxdy\right) \\
= &\int_\Omega f_1\phi_i dxdy + \int_\Omega u_{1h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial x}\phi_i\ dxdy + \int_\Omega u_{2h}^{(l-1)}\frac{\partial u_{1h}^{(l-1)}}{\partial y}\phi_i\ dxdy \\
&+ \int_{\Gamma_R} q_n\phi_i n_1\ ds + \int_{\Gamma_R} q_\tau\phi_i\tau_1\ ds,
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Continued formulation:

$$
\begin{aligned}
&\sum_{j=1}^{N_b} u_{1j}^{(l)} \Big(\int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_j \phi_i \, dxdy \\
&+ \int_{\Gamma_R} (r n_1 \phi_j)(\phi_i n_2) \, ds + \int_{\Gamma_R} (r \tau_1 \phi_j)(\phi_i \tau_2) \, ds \Big) \\
&+ \sum_{j=1}^{N_b} u_{2j}^{(l)} \Big(2 \int_\Omega \nu \frac{\partial \phi_j}{\partial y} \frac{\partial \phi_i}{\partial y} \, dxdy + \int_\Omega \nu \frac{\partial \phi_j}{\partial x} \frac{\partial \phi_i}{\partial x} \, dxdy + \int_\Omega \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_j \phi_i \, dxdy \\
&+ \int_\Omega u_{1h}^{(l-1)} \frac{\partial \phi_j}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial \phi_j}{\partial y} \phi_i \, dxdy \\
&+ \int_{\Gamma_R} (r n_2 \phi_j)(\phi_i n_2) \, ds + \int_{\Gamma_R} (r \tau_2 \phi_j)(\phi_i \tau_2) \, ds \Big) + \sum_{j=1}^{N_{bp}} p_j^{(l)} \left(- \int_\Omega \psi_j \frac{\partial \phi_i}{\partial y} \, dxdy \right) \\
= \; & \int_\Omega f_2 \phi_i dxdy + \int_\Omega u_{1h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial x} \phi_i \, dxdy + \int_\Omega u_{2h}^{(l-1)} \frac{\partial u_{2h}^{(l-1)}}{\partial y} \phi_i \, dxdy \\
&+ \int_{\Gamma_R} q_n \phi_i n_2 \, ds + \int_{\Gamma_R} q_\tau \phi_i \tau_2 \, ds,
\end{aligned}
$$

Robin boundary condition in normal/tangential directions

- Continued formulation:

$$\sum_{j=1}^{N_b} u_{1j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial x}\psi_i \; dxdy\right) + \sum_{j=1}^{N_b} u_{2j}\left(-\int_\Omega \frac{\partial \phi_j}{\partial y}\psi_i \; dxdy\right)$$

$$+\sum_{j=1}^{N_{bp}} p_j * 0 = 0.$$

Robin boundary condition in normal/tangential directions

- Matrix formulation? Pesudo code? (Part of a project for you)
- Similar to the previous ones for Robin condition, we need to add eight sub-matrices and four sub-vectors into the block linear system.
- The major difference is that here we need to provide the unit normal/tangential vectors. That is, we need to provide $\mathbf{n} = (n_1, n_2)^t$ and $\tau = (\tau_1, \tau_2)^t$, in the information matrix *boundaryedges*.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Consider

$$
\begin{aligned}
&(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\mathbb{T}(\mathbf{u},p) = \mathbf{f} \qquad in \quad \Omega,\\
&\nabla\cdot\mathbf{u} = 0 \qquad in \quad \Omega,\\
&\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n} = p_n,\ \ \tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n} = p_\tau \ \text{ on } \Gamma_S \subset \partial\Omega,\\
&\mathbf{n}^t\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\mathbf{n}^t\mathbf{u} = q_n,\ \ \tau^t\mathbb{T}(\mathbf{u},p)\mathbf{n} + r\tau^t\mathbf{u} = q_\tau \ \text{ on } \Gamma_R \subseteq \partial\Omega,\\
&\mathbf{u} = \mathbf{g} \ \text{ on } \Gamma_D = \partial\Omega/(\Gamma_S \cup \Gamma_R).
\end{aligned}
$$

where $\mathbf{n} = (n_1,\ n_2)^t$ is the unit outer normal vector of $\partial\Omega$ and $\tau = (\tau_1, \tau_2)^t$ is the corresponding unit tangential vector of $\partial\Omega$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Recall

$$\int_{\Omega}(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\ dxdy+\int_{\Omega}2\nu\mathbb{D}(\mathbf{u}):\mathbb{D}(\mathbf{v})\ dxdy$$
$$-\int_{\Omega}p(\nabla\cdot\mathbf{v})\ dxdy-\int_{\partial\Omega}(\mathbb{T}(\mathbf{u},p)\mathbf{n})\cdot\mathbf{v}\ ds\ =\ \int_{\Omega}\mathbf{f}\cdot\mathbf{v}\ dxdy,$$
$$-\int_{\Omega}(\nabla\cdot\mathbf{u})q\ dxdy=0.$$

- Since the solution on $\Gamma_D=\partial\Omega/(\Gamma_S\cup\Gamma_R)$ is given by $\mathbf{u}=\mathbf{g}$, then we can choose the test function $\mathbf{v}(x,y)$ such that $\mathbf{v}=0$ on $\partial\Omega/(\Gamma_S\cup\Gamma_R)$.

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Combining the above derivation for stress and Robin boundary conditions in normal/tangential directions, we obtain

$$
\begin{aligned}
& \int_{\partial\Omega} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \int_{\Gamma_S} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds + \int_{\Gamma_R} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
& + \int_{\partial\Omega/(\Gamma_S \cup \Gamma_R)} (\mathbb{T}(\mathbf{u}, p)\mathbf{n}) \cdot \mathbf{v} \ ds \\
= \ & \left[\int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& + \left[\int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \ ds \right] \\
& - \left[\int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \ ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \ ds \right],
\end{aligned}
$$

Dirichlet/stress/Robin mixed boundary condition in normal/tangential directions

- Weak formulation: find $\mathbf{u} \in H^1(\Omega) \times H^1(\Omega)$ and $p \in L^2(\Omega)$ s.t.

$$\int_\Omega (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} \; dxdy + \int_\Omega 2\nu \mathbb{D}(\mathbf{u}) : \mathbb{D}(\mathbf{v}) \; dxdy - \int_\Omega p(\nabla \cdot \mathbf{v}) \; dxdy$$

$$+ \int_{\Gamma_R} (r\mathbf{n}^t\mathbf{u})(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} (r\tau^t\mathbf{u})(\tau^t\mathbf{v}) \; ds$$

$$= \int_\Omega \mathbf{f} \cdot \mathbf{v} \; dxdy + \int_{\Gamma_R} q_n(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_R} q_\tau(\tau^t\mathbf{v}) \; ds$$

$$+ \int_{\Gamma_S} p_n(\mathbf{n}^t\mathbf{v}) \; ds + \int_{\Gamma_S} p_\tau(\tau^t\mathbf{v}) \; ds,$$

$$- \int_\Omega (\nabla \cdot \mathbf{u})q \; dxdy = 0,$$

for any $\mathbf{v} \in H^1_{0D}(\Omega) \times H^1_{0D}(\Omega)$ and $q \in L^2(\Omega)$.

- Code? Combine all of the subroutines for Dirichlet/stress/Robin boundary conditions.