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NONCONFORMING FINITE ELEMENT PENALTY METHOD FOR STOKES EQUATION

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Abstract. a special penalty method is presented to improve the accuracy of the standard penalty method for solving Stokes equation with nonconforming finite element. It is shown that this method with a larger penalty parameter can achieve the same accuracy as the standard method with a smaller penalty parameter. The convergence rate of the standard method is just half order of this penalty method when using the same penalty parameter, while the extrapolation method proposed by Faik et al can not yield so high accuracy of convergence. At last, we also get the super-convergence estimates for total flux.

§ 1 Introduction

The penalty method with conforming finite element is often applied to numerical discretization of Stokes equation and Navier-Stokes equation since it gives a reduction of the size of system of equations, see [1], [2] and [7]. It has been shown that the difference between the solution of Stokes equation and the penalty approximation is $O(\lambda)$, ($\lambda>0$ is a penalty parameter). Thus, λ must be sufficiently small to obtain an accurate approximation. In particular for unsteady flow computations, one needs very high-order accuracy in the continuity equation at each timestep to satisfy the mass conservation law, but if λ is too small, the penalty approximation will become unstable due to round-off error. So San Yinlin [6] proposed the modified penalty method and proved that the approximation is more accurate in conforming finite element cases. However, up to the present, there are seldom papers that can be found in the literature which consider the nonconforming finite element approximations. Richar Falk et al in [5] proposed a penalty extrapolation procedure avoiding the problem of the construction of trial functions satisfying div $u=0$ and $u=0$ on $\partial\Omega$ in the approximation of stationary Stokes equation, but this method can not achieve the optimal order, and can not be applied to Navier-Stokes equation.

In this paper, we will apply the techniques in [6] to a class of nonconforming finite elements which satisfy some boundary approximation properties. It is shown that the

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high-order accuracy can be achieved the same as in [1] and [2]. The convergence rate is optimal and the results obtained in this paper are still valid for Navier-Stokes equation.

§ 2 Stokes Equation

We consider the Stokes equation with homogeneous houndary-value problem, i. e.

$$
\begin{aligned}
-\Delta u+\nabla p &= f \quad \text{in}\Omega, \\
\nabla\cdot u &= 0 \quad \text{in}\Omega, \\
u &= 0 \quad \text{on } \partial\Omega,
\end{aligned} \tag{1}
$$

where u is the velocity, p the pressure, f a given body force, and Ω a bounded domain in R^2 with sufficiently smooth boundary $\partial\Omega$. We introduce

$$
\begin{aligned}
\gamma_0 &= \{\langle v,g\rangle \mid v\in(H_0^1(\Omega)^2, g\in L^2(\Omega)\}, \\
H_0^1(\Omega) &= \{v\in H^1(\Omega)\mid v=0 \text{ on } \partial\Omega\}, \\
(u,v) &= \int_\Omega uv dx, \\
a(u,v) &= (\nabla u,\nabla v), \quad \forall u,v\in(H^1(\Omega))^2, \\
b(r,v) &= (r,\nabla\cdot v), r\in L^2(\Omega), v\in(H^2(\Omega))^2,
\end{aligned}
$$

$$B(\langle w,r\rangle,\langle v,q\rangle)=a(w,v)-b(r,v)-b(q,w) \quad \forall\ \langle w,r\rangle,\langle v,q\rangle\in\gamma_0.$$

Let $C_0>0$ be such that

$$B(\langle w,r\rangle,\langle v,q\rangle)\leqslant C_0\|\langle w,r\rangle\|_{1,0}\|\langle v,q\rangle\|_{1,0}, \quad \langle w,r\rangle,\langle v,q\rangle\in\gamma_0,$$

where $\|\langle v,q\rangle\|_{1,0}^2=\|v\|_1^2+\|g\|_0^2, \forall\ v\in(H^1(\Omega))^2, g\in L^2(\Omega)$. The analysis shows that there exists a positive constant C_1, such that

$$\sup\frac{B(\langle w,r\rangle,\langle v,q\rangle)}{\|\langle v,q\rangle\|_{1,0}}\geqslant C_1\|\langle w,r\rangle\|_{1,0}, \ \langle w,r\rangle\in\gamma_0. \tag{2}$$

Now suppose that $f\in(L^2(\Omega))^2$, it follows from (2) that there exists an unique solution $\langle u,p\rangle\in\gamma_0$, such that

$$B(\langle u,p\rangle,\langle v,q\rangle)=(f,v)=\int_\Omega fv dx, \quad \forall\ \langle v,q\rangle\in\gamma_0. \tag{3}$$

(3) is equivalent to the following equations

$$
\begin{aligned}
a(u,v)-b(p,v) &= (f,v), &\quad \forall\ v\in(H_0^1(\Omega))^2, \\
b(q,u) &= 0, &\quad \forall\ q\in L^2(\Omega).
\end{aligned} \tag{4}
$$

To approximate the problem (3), we let h_1 and h_2 denote two discretization parameters tending to zero, and let $V_{h_1}^K\subset H^1(\Omega)$ and $M_{h_2}^K\subset L^2(\Omega)$ be finite element spaces defined in [8], which satisfy the following conditions (5)-(8). $\forall\ u\in(H^s(\Omega)\cap H_0^1(\Omega))^2, k\geqslant 3$,

$$\inf_{\chi\in V_{h_1}^K}\{\|u-\chi\|_0+h_1\|u-\chi\|_1\}\leqslant ch_1^s\|u\|_s, \quad 2\leqslant s\leqslant k, \tag{5}$$

and the boundary approximation

$$|\langle \gamma_i u, \chi\rangle_\Gamma| \leqslant c h_1^{t-i}\|u\|_t\|\chi\|_1, \quad \forall\, u \in H^t(\Omega), \chi \in V_{h_1}^K, 1 \leqslant t \leqslant k, \tag{6}$$

where γ_i are trace operators and c is a constant independent of h_1 and h_2.

$$\inf_{\chi\in M_{h_2}^K}\{\|p-\chi\|_0 + h_2\|p-\chi\|_1\} \leqslant c h_2^s\|p\|_s \quad \forall\, p \in H^s(\Omega), 1 \leqslant s \leqslant k-1, \tag{7}$$

and

$$\|\chi\|_1 \leqslant c h_2^{-1}\|\chi\|_0, \quad \forall\, \chi \in M_{h_2}^K. \tag{8}$$

It is shown that there are many spaces satisfy the above properties, see [8]. Let $h = \max(h_1, h_2)$, and $H_h = V_{h_1}^K \times (M_{h_2}^K \backslash R)$. Then from [4], there exist two constants $C_A > 0$ and $C_B > 0$, such that

$$\inf_{\substack{\langle u_h,p_h\rangle\in H_h\\ \|\langle u_h,p_h\rangle\|_{1,0}=1}}\ \sup_{\substack{\langle v_h,q_h\rangle\in H_h\\ \|\langle v_h,q_h\rangle\|_{1,0}\leqslant 1}} B(\langle u_h,p_h\rangle,\langle v_h,q_h\rangle) \geqslant C_A > 0, \tag{9}$$

$$\sup_{\substack{\langle u_h,p_h\rangle\in H_h\\ \|\langle u_h,p_h\rangle\|_{1,0}=1}} B(\langle u_h,p_h\rangle,\langle v_h,q_h\rangle) > 0, 0 \neq \langle v_h,q_h\rangle \in H_h, \tag{10}$$

$$B(\langle u_h,p_h\rangle,\langle v_h,q_h\rangle) \leqslant C_B\|\langle u_h,p_h\rangle\|_{1,0}\|\langle v_h,q_h\rangle\|_{1,0}. \tag{11}$$

Theorem 1. *If* $0<\lambda<\frac{C_1}{2}\delta_0$ ($\delta_0<1$ *is a positive number*), *then the penalty problem*

$$B(\langle u_h^\lambda,p_h^\lambda\rangle,\langle v_h,q_h\rangle) - \lambda(p_h^\lambda,q_h) = (f,v_h), \quad \forall\, \langle v_h,q_h\rangle \in H_h \tag{12}$$

has a unique solution $\langle u_h^\lambda,p_h^\lambda\rangle \in H_h$, *such that*

$$\|\langle u-u_p^\lambda, p-p_h^\lambda\rangle\|_{1,0} \leqslant c\{\inf_{\langle w_h,r_h\rangle\in H_h}\|\langle u-w_p, p-r_h\rangle\|_{1,0}$$
$$+ h^{s-1}(\|u\|_s + \|p\|_{s-1}) + \lambda\|p\|_0\},\ s\geqslant 1, \tag{13}$$

Moreover, if we assume that $\langle u,p\rangle \in (H^k(\Omega)\cap H_0^1(\Omega))^2 \times H^{k-1}(\Omega)\cap L^2(\Omega)$ $(k\geqslant 3)$, *then*

$$\|\langle u-u_h^\lambda, p-p_h^\lambda\rangle\|_{1,0} = O(h^{k-1}+\lambda). \tag{14}$$

Proof. From (9)-(11), and $\lambda<\frac{C_1}{2}\delta$, it is easy to check that

$$\widetilde{B}(\langle u_h,p_h\rangle,\langle v_h,q_h\rangle) = B(\langle u_h,p_h\rangle,\langle v_h,q_h\rangle) - \lambda(p_h,q_h)$$

satisfies the properties of (9)-(11) with the constants $\widetilde{C}_A, \widetilde{C}_B$, (i.e. C_A, C_B are replaced by $\widetilde{C}_A$ and $\widetilde{C}_B$), thus, from the Lax-Milgram theorem, the problem (12) has a unique solution $\langle u_h^\lambda,p_h^\lambda\rangle$. Note that $\forall\, v_h \in V_{h_1}^K$

$$a(u,v_h) - b(p,v_h) = (f,v_h) + \int_\Gamma \frac{\partial u}{\partial n} v_h ds - \int_\Gamma p v_h \cdot n ds,$$

so

$$a(u_h^\lambda - u, v_h) - b(p_h^\lambda - p, v_h) = \int_\Gamma p v_h \cdot n ds - \int_\Gamma \frac{\partial u}{\partial n} v_h ds,$$
$$B(\langle u_h^\lambda - w_h, p_h^\lambda - r_h\rangle,\langle v_h,q_h\rangle) = B(\langle u-u_h^\lambda, p-r_h\rangle,\langle v_h,q_h\rangle)$$
$$-\lambda(p_h^\lambda,q_h) + \int_\Gamma p v_h \cdot n ds - \int_\Gamma \frac{\partial u}{\partial n} v_h ds. \tag{15}$$

By using the property of (6) for $s\geqslant 1$, we have

$$\|\langle w_h - u_h^\lambda, r_h - p_h^\lambda\rangle\|_{1,0} \leqslant c\{\|\langle u-u_h^\lambda, p-r_h\rangle\|_{1,0}$$
$$+ h^{s-1}\|u\|_s + h^{s-1}\|p\|_{s-1} + \lambda\|p_h^\lambda\|_0\}.$$

Thus

$$\|\langle u-u_h^\lambda, p-p_h^\lambda\rangle\|_{1,0}\leqslant\|\langle u-w_h, p-r_h\rangle\|_{1,0}+\|\langle w_h-u_h^\lambda, r_h-p_h^\lambda\rangle\|_{1,0}$$
$$\leqslant c\Big(\inf_{\langle w_h, r_h\rangle}\|\langle u-w_h, p-r_h\rangle\|_{1,0}$$
$$+\lambda\|p\|_0+h^{s-1}(\|u\|_s+\|p\|_{s-1})\Big),$$

this means that (13) holds.

By putting $s=k\geqslant 3$, and using (5), we have

$$\|\langle u-u_h^\lambda, p-p_h^\lambda\rangle\|_{1,0}=O(h^{k-1}+\lambda).$$

§ 3 Modified Penalty Techniques

Based on the estimates of (13) and (14), we now introduce other pair $\langle u_{mn}^h, p_{mn}^h\rangle$ to approximate the solution $\langle u, p\rangle$

$$u_{mn}^h=u_h^{\lambda_n}-\lambda_n\frac{u_h^{\lambda_m}-u_h^{\lambda_n}}{\lambda_m-\lambda_n},$$
$$p_{mn}^h=p_h^{\lambda_n}-\lambda_n\frac{p_h^{\lambda_m}-p_h^{\lambda_n}}{\lambda_m-\lambda_n}, \tag{16}$$

where $\langle u_h^{\lambda_m}, p_h^{\lambda_m}\rangle$ and $\langle u_h^{\lambda_n}, p_h^{\lambda_n}\rangle$ are the solutions of (12), w. r. t. $\lambda=\lambda_m$ and λ_n.

Remark 1. (16) is something like the standard extrapolation formula, when $\lambda_m=4h$, $\lambda_n=h$.

Theorem 2. *Let* $\lambda_m, \lambda_n<\frac{C_1}{2}\delta$, *and* $\lambda_m=k\lambda_n$, *suppose*

$$\langle u, p\rangle\in(H^k(\Omega)\cap H_0^1(\Omega))^2\times(H^{k-1}(\Omega)\cap L^2(\omega)), k\geqslant 3,$$

then there holds

$$\|\langle u-u_{mn}^h, p-p_{mn}^h\rangle\|_{1,0}=O(h^{k-1}+\lambda_n\lambda_m). \tag{17}$$

Proof. It is easy to see that $\langle u_{mn}^h, p_{mn}^h\rangle$ satisfies the following equations:

$$a(u_{mn}^h, v)-b(p_{mn}^h, v)=(f, v)\quad v\in V_{h_1}^K,$$
$$b(q, u_{mn}^h)=\frac{\lambda_m\lambda_n}{\lambda_m-\lambda_n}(p_h^{\lambda_m}-p_h^{\lambda_n}, q)\quad\forall\, q\in M_h. \tag{18}$$

Make the analysis as in Theorem 1, we have $\forall\ \langle w_h, r_h\rangle, \langle v_h, q_h\rangle\in H_h$,

$$B(\langle u_{mn}^h-w_h, p_{mn}^h-r_h\rangle, \langle v_h, q_h\rangle)=B(\langle u-w_h, p-r_h\rangle, \langle v_h, q_h\rangle)$$
$$-\frac{\lambda_m\lambda_n}{\lambda_m-\lambda_n}(p_h^{\lambda_m}-p_h^{\lambda_n}, q_h)+\int_\Gamma pv_h\cdot n ds-\int_\Gamma\frac{\partial u}{\partial n}v_h ds,$$

since $u\in(H^k(\Omega)\cap H_0^1(\Omega))^2, p\in H^{k-1}(\Omega)\cap L^2(\Omega)$. Applying the boundary approximation condition (6) and putting $s=k$ yield

$$\left|\int_\Gamma pv_h\cdot n ds\right|\leqslant\left(\int_\Gamma v_h^2 ds\right)^{\frac12}\left(\int_\Gamma p^2 ds\right)^2\leqslant h^{k-1}\|v_h\|_1\|p\|_1,$$
$$\left|\int_\Gamma\frac{\partial u}{\partial n}v_h ds\right|\leqslant\left(\int_\Gamma v_h^2 ds\right)^{\frac12}\left(\int_\Gamma\left(\frac{\partial u}{\partial n}\right)^2 ds\right)^{\frac12}\leqslant h^{k-1}\|v_h\|_1\|u\|_2.$$

Thus

$$\|\langle u_{mn}^h - w_h, p_{mn}^h - r_h\rangle\|_{1,0} \leqslant c\|\langle u - w_h, p - r_h\rangle\|_{1,0} + \frac{\lambda_m\lambda_n}{\lambda_m - \lambda_n}\|p_h^{\lambda_m} - p_h^{\lambda_n}\|_0 + h^{k-1}(\|p\|_1 + \|u\|_2)).$$

Therefore

$$\|\langle u - u_{mn}^h, p - p_{mn}^h\rangle\|_{1,0} \leqslant c\Big(\inf_{\langle w_h, r_h\rangle\in H_h}\|\langle u - w_h, p - r_h\rangle\|_{1,0} + (h^{k-1} + \lambda_m\lambda_n)(\|u\|_2\|p\|_1)\Big).$$

Using (5) and (7) for $s=k\geqslant 3$, we have

$$\|\langle u - u_{mn}^h, p - p_{mn}^h\rangle\|_{1,0} \leqslant c(h^{k-1} + \lambda_m\lambda_n)(\|u\|_k + \|p\|_{k-1}) = O(h^{k-1} + \lambda_m\lambda_n).$$

The proof of Theorem 2 is completed.

From estimate (17), we can see that if $\lambda_m = O\left(h^{\frac{k-1}{2}}\right)$, the convergence order is of $O(h^{k-1})$. It is $\left(\frac{k-1}{2}\right)$ order higher than that of the standard penalty method (see Theorem 1).

Remark 2. In order to achieve high order $O(h^2)$. λ must be chosen as $O(h^2)$, when the standard penalty method is used, this sometime may make the stiffness matrix in a bad condition. But now, this problem is overcome, since we need only to choose $\lambda_m = O(h), \lambda_n = O(h)$. On the other hand, by comparing with the results of extrapolation method in [5], it may be reduced to that the convergence order is of $O(h^{\frac{4}{3}})$ for $k=3$. It is $\frac{1}{2}$ order lower than that of our paper, and is not an optimal order.

It needs to mention that we also have the following super-convergence error estimate for the approximation of the total flux.

Theorem 3. *Let $\lambda_m, \lambda_n < \frac{c_1}{2}\delta$ and $\lambda_m > \lambda_n, k \geqslant 3$, then we have*

$$\|\nabla u_{mn}^h\|_0 = O(\lambda_n(\lambda_m + h^{k-1}))$$

and

$$\left|\int_{\partial\Omega} u\cdot n ds - \int_{\partial\Omega} u_{mn}^h \cdot n ds\right| \leqslant O(\lambda_n(\lambda_m + h^{k-1})).$$

Proof. The proof is the same as Theorem 3 in [6].

It can be seen that if $k = 3, \lambda_m = O(h^{\frac{3}{2}}), \lambda_n = O(h^{\frac{3}{2}})$ then

$$\|\nabla u_{mn}^h\|_0 = O(h^3), \left|\int_{\partial\Omega} u\cdot n ds - \int_{\partial\Omega} u_{mn}^h\cdot n ds\right| = O(h^3),$$

just one order higher than that of the standard method.

Remark 3. Obviously, our results are valid for conforming finite elements. On the other hand, we can apply the above method to nonlinear Navier-Stokes equation with slight modification of the above proof procedure.

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