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TWO ENERGY-CONSERVED SPLITTING METHODS FOR THREE-DIMENSIONAL TIME-DOMAIN MAXWELL'S EQUATIONS AND THE CONVERGENCE ANALYSIS*

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Abstract. We devote the present paper to high-accuracy energy-preserving S-AVF(2) and S-AVF(4) schemes for the three-dimensional time-domain Maxwell's equations, based on the exponential operator splitting technique, the Fourier pseudospectral method, and the averaged vector field method. To obtain the present schemes, the key is to propose the splitting methods for Maxwell's equations, in which all subsystems should hold the same Hamiltonian. The proposed schemes are energy-preserving, high-order accurate, and unconditionally stable, while being implemented explicitly. Both schemes capture four energy invariants simultaneously. Rigorous error estimates of the schemes are established in the discrete L^2-norm. The theoretical results show that the S-AVF(2)/S-AVF(4) scheme converges with spectral accuracy in space and second-order/fourth-order accuracy in time, respectively. Numerical results support the theoretical analysis.

Key words. Maxwell's equations, time-split, composition, averaged vector field method, error estimate

AMS subject classifications. 65M12, 65M15

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1. Introduction.

Maxwell's equations in an isotropic, homogeneous, and lossless medium are

$$\frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon}\mathbf{curl}\ \vec{H}, \quad \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu}\mathbf{curl}\ \vec{E}, \tag{1.1}$$

$$\mathbf{div}(\varepsilon \vec{E}) = 0, \quad \mathbf{div}(\mu \vec{H}) = 0, \tag{1.2}$$

where ε is the electric permittivity, μ is the magnetic permeability, $\vec{E} = (E_x, E_y, E_z)^T$ and $\vec{H} = (H_x, H_y, H_z)^T$ represent electric and magnetic fields. In this paper, we consider initial values

$$\vec{E}_0(x, y, z) = \vec{E}(x, y, z, 0) \quad \text{and} \quad \vec{H}_0(x, y, z) = \vec{H}(x, y, z, 0), \tag{1.3}$$

on the cuboid domain $\Omega = [a_1, a_2] \times [b_1, b_2] \times [c_1, c_2]$ and periodic boundary conditions on $\partial\Omega \times [0, T]$. If the initial data are suitably smooth, Maxwell's equations (1.1) have

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unique smooth solutions for all time [1], and if the initial fields are divergence-free, the electric and magnetic fields are always divergence-free for all time.

THEOREM 1.1. *If $(\vec{E},\vec{H})^T$ is the solution of Maxwell's equations* (1.1) *in a lossless medium and satisfy periodic boundary conditions, then the energy conservation laws (ECLs) hold, $w = x, y, z$,*

$$\frac{d}{dt}\int_\Omega \left(\varepsilon\left|\vec{E}\right|^2 + \mu\left|\vec{H}\right|^2\right) dxdydz = 0, \tag{1.4}$$

$$\frac{d}{dt}\int_\Omega \left(\varepsilon\left|\partial_t\vec{E}\right|^2 + \mu\left|\partial_t\vec{H}\right|^2\right) dxdydz = 0, \tag{1.5}$$

$$\frac{d}{dt}\int_\Omega \left(\varepsilon\left|\partial_w\vec{E}\right|^2 + \mu\left|\partial_w\vec{H}\right|^2\right) dxdydz = 0, \tag{1.6}$$

$$\frac{d}{dt}\int_\Omega \left(\varepsilon\left|\partial_{tw}^2\vec{E}\right|^2 + \mu\left|\partial_{tw}^2\vec{H}\right|^2\right) dxdydz = 0. \tag{1.7}$$

These physical invariants are important for Maxwell's equations. Naturally it is desirable to propose a numerical scheme preserving them in the discrete sense.

Generally it is rather difficult to solve the three-dimensional Maxwell's equations directly with finite difference methods, finite element methods, or spectral methods because of a large scale of computational cost and memory requirement during computations. Hence it is necessary to develop efficient numerical methods. In order to overcome the computational difficulty, many efforts related to the alternating direction implicit (ADI) technique and local one-dimensional method have been made in computations of Maxwell's equations. Holland [2] studied the two-dimensional Maxwell's equations by combining the ADI method and Yee's finite difference time-domain (FDTD) scheme [3], and later Namiki [4] and Zheng, Chen, and Zhang [5] proposed unconditionally stable ADI-FDTD schemes for the three-dimensional Maxwell's equations. Other relevant works were reported in [6, 7, 8], and so on. These proposed methods are efficient, but not energy conserved. Recently, energy preserving and unconditionally stable splitting FDTD schemes [9, 10] have been introduced for two- and three-dimensional Maxwell's equations.

For the splitting schemes [9, 10], it requires complicated computations to obtain their energy-preserving properties. Naturally it is desirable to propose a method preserving the energy automatically. Recently, Celledoni et al. [11] have investigated the following partial differential equation (PDE) with $u \in \mathrm{R}^m$ and $(x,t) \in \mathrm{R}^d \times \mathrm{R}$:

$$\frac{du}{dt} = \mathcal{D}\frac{\delta\mathcal{H}}{\delta u}, \tag{1.8}$$

where $\mathcal{D}$ is a linear skew-adjoint differential operator and $\mathcal{H}[u] = \int_\Omega H(x; u^{(n)})dx$. The authors gave a systematic method for discretizing the Hamiltonian PDE (1.8) with constant symplectic structure, while preserving the Hamiltonian exactly.

LEMMA 1.2 (see [11]). *Let $\bar{\mathcal{H}}\Delta x$ be any consistent approximation to $\mathcal{H}$ (where $\Delta x := \Delta x_1\Delta x_2\cdots\Delta x_d$) with N degrees of freedom. Then in the finite-dimensional Hilbert space R^{N} with the Euclidean inner product, the variational derivative $\frac{\delta}{\delta u}(\bar{\mathcal{H}}\Delta x)$ is given by $\nabla\bar{\mathcal{H}}$.*

Let $\bar{\mathcal{D}}$ be any consistent constant skew matrix approximation to $\mathcal{D}$. From (1.8) we have a Hamiltonian system of ordinary differential equations (ODEs), $\frac{d\mathbf{u}}{dt} = f(\mathbf{u})$,

$f(\mathbf{u}) = \bar{\mathcal{D}}\nabla\bar{\mathcal{H}}(\mathbf{u})$. Next we use the averaged vector field (AVF) method [14] to integrate it and then obtain a fully discrete scheme

$$\frac{\mathbf{u}^{n+1}-\mathbf{u}^n}{\tau} = \left(\delta_j^i + \alpha\tau^2 f_k^i(\hat{\mathbf{u}}) f_j^k(\hat{\mathbf{u}})\right) \int_0^1 f^j((1-\xi)\mathbf{u}^n + \xi\mathbf{u}^{n+1})d\xi, \tag{1.9}$$

where α is an arbitrary constant, δ_j^i is the Kronecker delta, the subscripts denote partial derivatives, and repeated indices imply summation. For $\alpha = 0$ it is a second-order method. For $\alpha = -\frac{1}{12}$ and $\hat{\mathbf{u}} = \mathbf{u}^n$ the method is third order. For $\alpha = -\frac{1}{12}$ and $\hat{\mathbf{u}} = \frac{\mathbf{u}^n+\mathbf{u}^{n+1}}{2}$ the method is of fourth-order accuracy. The AVF method (1.9) preserves the Hamiltonian automatically, i.e., $\bar{\mathcal{H}}^n = \bar{\mathcal{H}}^{n-1} = \cdots = \bar{\mathcal{H}}^0$, since the matrix $\bar{\mathcal{D}}$ is nonsingular and antisymmetric. The second-order AVF method was first written down in [13] and identified as an energy-preserving method in [14]. Other works on the AVF method can be found in [12, 15, 16, 17].

The energy-preserving schemes [9, 10] are efficient only for geometries with moderate wavelengths because of low-order accuracy in both time and space. We need to refine the computational grid if we expect more accurate solution. For the large scale problems, for problems requiring long-time integration, or for problems of wave propagations over longer distances, it is desirable to develop higher-order schemes that produce smaller dispersion or phase error for a given mesh resolution. In this paper, in order to establish efficient high-order numerical schemes, we propose a second-order (fourth-order) splitting method for Maxwell's equations. Each subproblem of the splitting method amounts to three pairs of mutually uncoupled one-dimensional equations. All subproblems of the splitting method are Hamiltonian systems with the same Hamiltonian $\mathcal{H}$, which is crucial for establishing conservative schemes. For each subproblem, by using the Fourier pseudospectral method in space, and then applying the AVF method for the obtained ODEs, we propose efficient, high-order accurate and energy-preserving S-AVF(2) and S-AVF(4) schemes for Maxwell's equations. The schemes preserve the discrete Hamiltonian automatically, and also hold the four discrete energies simultaneously. For multidimensional PDEs, the error estimates of the spectral and pseudospectral methods [21, 22, 23, 24] have been proposed, but as we know the error estimate of the splitting spectral or pseudospectral method has not been established. In fact it is difficult to obtain the error estimates of the present splitting schemes since the schemes involve the intermediate variables. In the present paper, combining with the energy method, we strictly establish the error estimates for the splitting schemes. Although the S-AVF(2) and S-AVF(4) schemes are not divergence-free, they have second- and fourth-order approximations to the divergence-free fields, respectively. The proposed schemes seem implicit and inefficient, but they can be rewritten in fully explicit forms by using the discrete fast Fourier transform (DFFT). Numerical experiments show the good performance of the proposed schemes.

The present work is organized as follows. In section 2, we consider several splitting methods for Maxwell's equations, and then rewrite them in the Hamiltonian forms. We propose the S-AVF(2) and S-AVF(4) schemes for Maxwell's equations in section 3. The conservative properties are displayed in section 4. Section 5 includes error estimates of the two proposed schemes. Identities for the divergence terms are given in section 6 and we derive the explicit forms of the proposed schemes in section 7. We present some numerical experiments in section 8. In section 9, we draw conclusions.

2. Time-splitting methods and corresponding properties. In this section, we propose several splitting and composition methods for Maxwell's equations. All

subproblems of the splitting method share the same Hamiltonian, which is very important for constructing energy-preserving schemes.

2.1. Time-splitting and composition techniques. The three-dimensional Maxwell's equations (1.1) can be rewritten as

$$\frac{\partial}{\partial t}\begin{pmatrix} \sqrt{\varepsilon}\vec{E} \\ \sqrt{\mu}\vec{H} \end{pmatrix} = c\mathbf{A}\begin{pmatrix} \sqrt{\varepsilon}\vec{E} \\ \sqrt{\mu}\vec{H} \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{0} & \mathbf{curl} \\ -\mathbf{curl} & \mathbf{0} \end{pmatrix}, \tag{2.1}$$

with $c = 1/\sqrt{\varepsilon\mu}$. Obviously, the **curl** operator is self-adjoint ($\mathbf{curl}^* = \mathbf{curl}$) and the $\mathbf{A}$ operator is skew-adjoint ($\mathbf{A}^* = -\mathbf{A}$). Choosing $\sqrt{\varepsilon}\vec{E}$ and $\sqrt{\mu}\vec{H}$ as the basic variables yields

$$\begin{pmatrix} \sqrt{\varepsilon}\vec{E}(t_{n+1}) \\ \sqrt{\mu}\vec{H}(t_{n+1}) \end{pmatrix} = \exp(c\tau\mathbf{A})\begin{pmatrix} \sqrt{\varepsilon}\vec{E}(t_n) \\ \sqrt{\mu}\vec{H}(t_n) \end{pmatrix}, \tag{2.2}$$

where $\tau = t_{n+1} - t_n$. Note that the **curl** operator can be split into the sum of two operators $\mathbf{curl}_+$ and $\mathbf{curl}_-$, i.e., $\mathbf{curl} = \mathbf{curl}_+ + \mathbf{curl}_-$ with

$$\mathbf{curl}_+ = \begin{pmatrix} 0 & 0 & \partial_y \\ \partial_z & 0 & 0 \\ 0 & \partial_x & 0 \end{pmatrix}, \quad \mathbf{curl}_- = \begin{pmatrix} 0 & -\partial_z & 0 \\ 0 & 0 & -\partial_x \\ -\partial_y & 0 & 0 \end{pmatrix}.$$

Obviously we have $\mathbf{curl}_\pm^* = \mathbf{curl}_\mp$. The splitting **curl** will lead to a decomposition of the $\mathbf{A}$ operator, i.e., $\mathbf{A} = \mathbf{A}_+ + \mathbf{A}_-$,

$$\mathbf{A}_+ = \begin{pmatrix} \mathbf{0} & \mathbf{curl}_+ \\ -\mathbf{curl}_+^* & \mathbf{0} \end{pmatrix}, \quad \mathbf{A}_- = \begin{pmatrix} \mathbf{0} & \mathbf{curl}_- \\ -\mathbf{curl}_-^* & \mathbf{0} \end{pmatrix}.$$

It is clear that $\mathbf{A}_\pm^* = -\mathbf{A}_\pm$.

The main idea in the split-step method is to approximate the exact solution of (2.2) by solving the obtained splitting equations in a given sequential order, in which the solution of one subproblem is employed as an initial condition for the next subproblem. This could be achieved by using a solution operator $\varphi_n(\tau)$ that includes an appropriate combination of products of the exponential operators $\exp(c\tau\mathbf{A}_-)$ and $\exp(c\tau\mathbf{A}_+)$. This produces a splitting error to the noncommutativity of the operators $\mathbf{A}_-$ and $\mathbf{A}_+$. Generally, the first- and second-order splitting methods are used. The first-order solution operator is given by $\varphi_1(\tau) = \exp(c\tau\mathbf{A}_+)\exp(c\tau\mathbf{A}_-)$ and the second-order solution operator is approximated by

$$\varphi_2(\tau) = \exp\left(\frac{c\tau}{2}\mathbf{A}_-\right)\exp(c\tau\mathbf{A}_+)\exp\left(\frac{c\tau}{2}\mathbf{A}_-\right), \tag{2.3}$$

which is symmetric, that is $\varphi_2(\tau)\varphi_2(-\tau) = \text{identity}$. In this paper, we consider the second-order operator for the three-dimensional Maxwell's equations. Sometimes, we may need higher-order splitting methods. Approximations of higher order, that preserve the symmetry, can be constructed by a proper composition of the second-order symmetric splitting [18]. A fourth-order splitting operator is given in the form

$$\varphi_4(\tau) = \varphi_2(\omega\tau)\varphi_2((1-2\omega)\tau)\varphi_2(\omega\tau) \tag{2.4}$$

with $\omega = (2 + 2^{1/3} + 2^{-1/3})$. Further, it can be rewritten as

$$\begin{aligned}\varphi_4(\tau) = &\exp(l_1\tau c\mathbf{A}_-)\exp(l_2\tau c\mathbf{A}_+)\exp(l_3\tau c\mathbf{A}_-)\exp(l_4\tau c\mathbf{A}_+)\\ &\times\exp(l_3\tau c\mathbf{A}_-)\exp(l_2\tau c\mathbf{A}_+)\exp(l_1\tau c\mathbf{A}_-),\end{aligned}\tag{2.5}$$

where $(l_1, l_2, l_3, l_4) = \left(\frac{\omega}{2}, \omega, \frac{1-\omega}{2}, (1-2\omega)\right)$. Note that the number of products of exponential operators increases with the order of decay of splitting error. In general, $\mathbf{A}_+$ and $\mathbf{A}_-$ in the above exponential operators may be interchanged without affecting the order of the method.

2.2. ECLs for the splitting method. Consider the following three-stage process corresponding to the second-order splitting method:

- Stage 1. Solve $\exp(\frac{\tau}{2}c\mathbf{A}_-)$, i.e., $\begin{pmatrix}\sqrt{\varepsilon}\vec{E}(t_n)\\ \sqrt{\mu}\vec{H}(t_n)\end{pmatrix} \Longrightarrow \begin{pmatrix}\sqrt{\varepsilon}\vec{E}^{(1)}\\ \sqrt{\mu}\vec{H}^{(1)}\end{pmatrix}$.
- Stage 2. Solve $\exp(\tau c\mathbf{A}_+)$, i.e., $\begin{pmatrix}\sqrt{\varepsilon}\vec{E}^{(1)}\\ \sqrt{\mu}\vec{H}^{(1)}\end{pmatrix} \Longrightarrow \begin{pmatrix}\sqrt{\varepsilon}\vec{E}^{(2)}\\ \sqrt{\mu}\vec{H}^{(2)}\end{pmatrix}$.
- Stage 3. Solve $\exp(\frac{\tau}{2}c\mathbf{A}_-)$, i.e., $\begin{pmatrix}\sqrt{\varepsilon}\vec{E}^{(2)}\\ \sqrt{\mu}\vec{H}^{(2)}\end{pmatrix} \Longrightarrow \begin{pmatrix}\sqrt{\varepsilon}\vec{E}(t_{n+1})\\ \sqrt{\mu}\vec{H}(t_{n+1})\end{pmatrix}$.

Stage 1 shows that we obtain $(\vec{E}^{(1)}, \vec{H}^{(1)})^T$ by advancing $\frac{\partial\vec{E}}{\partial t} = \frac{1}{\varepsilon}\mathbf{curl}_-\vec{H}$ and $\frac{\partial\vec{H}}{\partial t} = -\frac{1}{\mu}\mathbf{curl}_-^*\vec{E}$ with initial value $(\vec{E}(t_n), \vec{H}(t_n))^T$ and $\frac{\tau}{2}$ time increment. It is clear that

$$\left\langle \varepsilon\frac{\partial\vec{E}}{\partial t}, \vec{E}\right\rangle + \left\langle \mu\frac{\partial\vec{H}}{\partial t}, \vec{H}\right\rangle = \langle\mathbf{curl}_-\vec{H}, \vec{E}\rangle - \langle\mathbf{curl}_-^*\vec{E}, \vec{H}\rangle = \int_{\partial\Omega}(\vec{E}\times\vec{n})\cdot\vec{H}dS,\tag{2.6}$$

where $\langle\cdot,\cdot\rangle$ denotes the standard $L^2(\Omega)$ inner product and $\vec{n}$ is the outward normal vector on $\partial\Omega$. Thanks to the periodic boundaries, the term $\int_{\partial\Omega}(\vec{E}\times\vec{n})\cdot\vec{H}dS$ vanishes and then we have

$$\int_\Omega \left(\varepsilon|\vec{E}^{(1)}|^2 + \mu|\vec{H}^{(1)}|^2\right)dxdydz = \int_\Omega \left(\varepsilon|\vec{E}(t_n)|^2 + \mu|\vec{H}(t_n)|^2\right)dxdydz.$$

The analogous argument can be applied to stages 2 and 3. Hence, the second-order splitting method possesses the ECL (1.4). Similarly, we can derive the ECLs (1.5), (1.6), and (1.7).

THEOREM 2.1. *Let $\vec{E}$ and $\vec{H}$ be the solutions of Maxwell's equations* (1.1) *in a lossless medium and satisfy periodic boundaries, then the splitting methods* (2.3) *and* (2.4) *hold for the four ECLs* (1.4)–(1.7).

2.3. Hamiltonian formulations for the splitting methods. Maxwell's equations have an infinite-dimensional Hamiltonian formulation [19],

$$\frac{\partial}{\partial t}\begin{pmatrix}\vec{E}\\ \vec{H}\end{pmatrix} = \begin{pmatrix}0 & (\varepsilon\mu)^{-1}\mathbf{curl}\\ -(\varepsilon\mu)^{-1}\mathbf{curl} & 0\end{pmatrix}\begin{pmatrix}\varepsilon\vec{E}\\ \mu\vec{H}\end{pmatrix} = \mathcal{D}\begin{pmatrix}\frac{\delta\mathcal{H}}{\delta\vec{E}}\\ \frac{\delta\mathcal{H}}{\delta\vec{H}}\end{pmatrix}$$

with the quadratic Hamiltonian functional

$$\mathcal{H} = \int_\Omega\left(\frac{1}{2}\mu\vec{H}\cdot\vec{H} + \frac{1}{2}\varepsilon\vec{E}\cdot\vec{E}\right)dxdydz.\tag{2.7}$$

Next we show that each subsystem of the splitting method also holds the Hamiltonian $\mathcal{H}$. Note that the operators $\exp(c\tau\mathbf{A}_-)$ and $\exp(c\tau\mathbf{A}_+)$ are repeatedly advanced

by different time increments in the splitting methods. To obtain the Hamiltonian formulations of all subsystems, we only need to investigate the Hamiltonian forms of

$$\text{(2.8)}\qquad \begin{cases} \frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon}\mathbf{curl}_- \vec{H}, \\ \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu}\mathbf{curl}_-^* \vec{E}, \end{cases} \quad \text{and} \quad \begin{cases} \frac{\partial \vec{E}}{\partial t} = \frac{1}{\varepsilon}\mathbf{curl}_+ \vec{H}, \\ \frac{\partial \vec{H}}{\partial t} = -\frac{1}{\mu}\mathbf{curl}_+^* \vec{E}. \end{cases}$$

It is clear that the Hamiltonian formulas are given by, respectively,

$$\text{(2.9)}\qquad \frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1}\mathbf{curl}_- \\ -(\varepsilon\mu)^{-1}\mathbf{curl}_-^* & 0 \end{pmatrix}\begin{pmatrix} \varepsilon\vec{E} \\ \mu\vec{H} \end{pmatrix} = \mathcal{D}_1 \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \vec{E}} \\ \frac{\delta \mathcal{H}}{\delta \vec{H}} \end{pmatrix},$$

and

$$\text{(2.10)}\qquad \frac{\partial}{\partial t}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1}\mathbf{curl}_+ \\ -(\varepsilon\mu)^{-1}\mathbf{curl}_+^* & 0 \end{pmatrix}\begin{pmatrix} \varepsilon\vec{E} \\ \mu\vec{H} \end{pmatrix} = \mathcal{D}_2 \begin{pmatrix} \frac{\delta \mathcal{H}}{\delta \vec{E}} \\ \frac{\delta \mathcal{H}}{\delta \vec{H}} \end{pmatrix},$$

where both $\mathcal{D}_1$ and $\mathcal{D}_2$ operators are skew-adjoint and satisfy the Jacobi identity.

Remark. All subsystems possess the same Hamiltonian $\mathcal{H}$, which ensures that the splitting method also holds $\mathcal{H}$.

3. Construction of conservative schemes. In this section, we propose two schemes for the three-dimensional Maxwell's equations which preserve the Hamiltonian automatically, based on the Fourier pseudospectral method and the AVF method. Define a series of collocation points $x_i = a_1 + ih_x$, $y_j = b_1 + jh_y$, $z_k = c_1 + kh_z$, $i = 0, 1, \cdots, N_x - 1$, $j = 0, 1, \cdots, N_y - 1$, $k = 0, 1, \cdots, N_z - 1$, where $h_x = (a_2 - a_1)/N_x$, $h_y = (b_2 - b_1)/N_y$, $h_z = (c_2 - c_1)/N_z$ with even integers N_x, N_y, and N_z. Let $\tau = T/N$ be the time step. The approximation of the value of the function $U(x, y, z, t)$ at the node (x_i, y_j, z_k, t_n) is denoted by $U^n_{i,j,k}$.

Let $u(x, t)$ be a one-dimensional smooth function defined on $[a_1, a_2]$. Interpolating it at collocation points x_i gives $P_c u(x, t) = \sum_{i=0}^{N_x - 1} u_i g_i(x)$, where $u_i = u(x_i, t)$ and $g_i(x)$ is a trigonometric polynomial of degree $N_x/2$ satisfying $g_i(x_j) = \delta_i^j$. In fact, $g_i(x)$ is given explicitly by $g_i(x) = \frac{1}{N_x}\sum_{p=-N_x/2}^{N_x/2} \frac{1}{c_p} e^{lp\mu_x(x - x_i)}$, $l = \sqrt{-1}$, $\mu_x = 2\pi/(a_2 - a_1)$, $c_p = 1$ $(p \neq \pm N_x/2)$, $c_{\pm N_x/2} = 2$. It is noticed that the first-order partial differential operator ∂_x yields a skew-symmetric Fourier spectral differential matrix B_1^x whose entries are

$$(B_1^x)_{r,s} = \begin{cases} \frac{1}{2}\mu_x (-1)^{r+s} \cot(\mu_x \frac{x_r - x_s}{2}), & r \neq s, \\ 0, & r = s, \end{cases}$$

with $r, s = 0, 1, \cdots, N_x - 1$. The extension of this to multidimensional problems utilizes tensor products, i.e., a three-dimensional function is represented as

$$\text{(3.1)}\qquad P_c U(x, y, z, t) = \sum_{i=0}^{N_x - 1}\sum_{j=0}^{N_y - 1}\sum_{z=0}^{N_z - 1} U_{i,j,k} g_i(x) g_j(y) g_k(z).$$

Differentiating it with respect to x, y, or z, and evaluating the resulting expression at the collocation points, we have $\partial_w P_c U(x_i, y_j, z_k, t) = (A_w \vec{U})_{i,j,k}$, $w = x, y, z$, where

$$\vec{U} = (U_{1,1,1}, U_{2,1,1}, \cdots, U_{N_x,1,1}, U_{1,2,1}, U_{2,2,1}, \cdots, U_{N_x,2,1}, \cdots, U_{1,N_y,1}, U_{2,N_y,1}, \cdots, U_{N_x,N_y,1}, \cdots, U_{1,1,2}, U_{2,1,2}, \cdots, U_{N_x,1,2}, \cdots, U_{1,N_y,N_z}, U_{2,N_y,N_z}, \cdots, U_{N_x,N_y,N_z})^T,$$

and $A_x = I_{N_z} \otimes I_{N_y} \otimes B_1^x$, $A_y = I_{N_z} \otimes B_1^y \otimes I_{N_x}$, $A_z = B_1^z \otimes I_{N_y} \otimes I_{N_x}$ with identity matrix $I_{N_w} \in \mathcal{R}^{N_w \times N_w}$.

To propose schemes for the present splitting methods, we consider

$$A_- = \begin{pmatrix} 0 & -A_z & 0 \\ 0 & 0 & -A_x \\ -A_y & 0 & 0 \end{pmatrix} \quad \text{and} \quad A_+ = \begin{pmatrix} 0 & 0 & A_y \\ A_z & 0 & 0 \\ 0 & A_x & 0 \end{pmatrix} \tag{3.2}$$

as discretizations for the $\mathbf{curl}_-$ and $\mathbf{curl}_+$ operators, respectively. It is clear that $A = A_- + A_+$ is symmetric and $A_+^T = A_-$ since B_1^w is skew-symmetric.

3.1. Spatial semi-discretization. In order to utilize the AVF method for the second- or fourth-order splitting methods, we should first make a spatial semi-discretization for each subproblem. In the splitting methods, (2.9) and (2.10) are repeatedly advanced by different time increments, so that we just need to investigate the semidiscretizations for them.

For the Hamiltonian formulation (2.9), we consider

$$\bar{\mathcal{D}}_1 = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1} A_- \\ -(\varepsilon\mu)^{-1} A_-^T & 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathcal{H}} = \frac{\mu}{2}\vec{H}\cdot\vec{H} + \frac{\varepsilon}{2}\vec{E}\cdot\vec{E} \tag{3.3}$$

as the discretizations of $\mathcal{D}_1$ and $\mathcal{H}$, respectively, where $\vec{H} = [\vec{H}_x^T, \vec{H}_y^T, \vec{H}_z^T]^T$, $\vec{E} = [\vec{E}_x^T, \vec{E}_y^T, \vec{E}_z^T]^T$, and "$\cdot$" means dot product of two vectors. The resulting system of ODEs is

$$\frac{d}{dt}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \bar{\mathcal{D}}_1 \nabla \bar{\mathcal{H}} = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1} A_- \\ -(\varepsilon\mu)^{-1} A_-^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon\vec{E} \\ \mu\vec{H} \end{pmatrix}, \tag{3.4}$$

which is a Hamiltonian system and possesses the first integral $\bar{\mathcal{H}}$, i.e., $\frac{d}{dt}\bar{\mathcal{H}} = 0$.

For the Hamiltonian formulation (2.10), choosing

$$\bar{\mathcal{D}}_2 = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1} A_+ \\ -(\varepsilon\mu)^{-1} A_+^T & 0 \end{pmatrix} \quad \text{and} \quad \bar{\mathcal{H}} = \frac{\mu}{2}\vec{H}\cdot\vec{H} + \frac{\varepsilon}{2}\vec{E}\cdot\vec{E} \tag{3.5}$$

results in a Hamiltonian system of ODEs

$$\frac{d}{dt}\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \bar{\mathcal{D}}_2 \nabla \bar{\mathcal{H}} = \begin{pmatrix} 0 & (\varepsilon\mu)^{-1} A_+ \\ -(\varepsilon\mu)^{-1} A_+^T & 0 \end{pmatrix} \begin{pmatrix} \varepsilon\vec{E} \\ \mu\vec{H} \end{pmatrix}, \tag{3.6}$$

with Hamiltonian $\bar{\mathcal{H}}$, i.e., $\frac{d}{dt}\bar{\mathcal{H}} = 0$.

3.2. Temporal second-order AVF scheme. In this section, we derive a numerical scheme for the second-order splitting method (2.3). We use the AVF method (1.9) ($\alpha = 0$) to repeatedly advance the ODEs (3.4) and (3.6) by certain time increments, and then obtain a temporal second-order three-stage splitting scheme for the three-dimensional Maxwell's equations as follows.

S-AVF(2) scheme:

• Stage 1.

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)} - \varepsilon\vec{E}^n \\ \mu\vec{H}^{(1)} - \mu\vec{H}^n \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)} + \vec{E}^n \\ \vec{H}^{(1)} + \vec{H}^n \end{pmatrix}. \tag{3.7}$$

• Stage 2.

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(2)}-\varepsilon\vec{E}^{(1)} \\ \mu\vec{H}^{(2)}-\mu\vec{H}^{(1)} \end{pmatrix}=\frac{1}{2}\begin{pmatrix} 0 & A_+ \\ -A_+^T & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(2)}+\vec{E}^{(1)} \\ \vec{H}^{(2)}+\vec{H}^{(1)} \end{pmatrix}. \tag{3.8}$$

• Stage 3.

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{n+1}-\varepsilon\vec{E}^{(2)} \\ \mu\vec{H}^{n+1}-\mu\vec{H}^{(2)} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{n+1}+\vec{E}^{(2)} \\ \vec{H}^{n+1}+\vec{H}^{(2)} \end{pmatrix}. \tag{3.9}$$

The application of the AVF method implies that the schemes (3.7)–(3.9) hold the discrete Hamiltonians, respectively, $\bar{\mathcal{H}}^{(1)}=\bar{\mathcal{H}}^n$, $\bar{\mathcal{H}}^{(2)}=\bar{\mathcal{H}}^{(1)}$, and $\bar{\mathcal{H}}^{n+1}=\bar{\mathcal{H}}^{(2)}$, where $\bar{\mathcal{H}}^n=\frac{\mu}{2}\vec{H}^n\cdot\vec{H}^n+\frac{\varepsilon}{2}\vec{E}^n\cdot\vec{E}^n$ and $\bar{\mathcal{H}}^{(i)}=\frac{\mu}{2}\vec{H}^{(i)}\cdot\vec{H}^{(i)}+\frac{\varepsilon}{2}\vec{E}^{(i)}\cdot\vec{E}^{(i)}$ with $i=1,2$. Obviously, the S-AVF(2) scheme holds the discrete Hamiltonian

$$\bar{\mathcal{H}}^n=\bar{\mathcal{H}}^{n-1}=\cdots=\bar{\mathcal{H}}^0. \tag{3.10}$$

Remark. The S-AVF(2) is just the midpoint/trapezoidal scheme since the second-order AVF method coincides with the midpoint/trapezoidal rule when applied to linear autonomous systems of ODEs.

The S-AVF(2) scheme seems to be inconvenient to be applied, but it only seems like this. Denote $\vec{U}_{,j,k}=(U_{0,j,k},\cdots,U_{N_x-1,j,k})^T$, $\vec{U}_{i,,k}=(U_{i,0,k},\cdots,U_{i,N_y-1,k})^T$, and $\vec{U}_{i,j,}=(U_{i,j,0},\cdots,U_{i,j,N_z-1})^T$. The scheme (3.7) can be decomposed into three independent equations, i.e.,

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{y_{,j,k}}-\varepsilon\vec{E}^{n}_{y_{,j,k}} \\ \mu\vec{H}^{(1)}_{z_{,j,k}}-\mu\vec{H}^{n}_{z_{,j,k}} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 0 & -B_1^x \\ -B_1^x & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{y_{,j,k}}+\vec{E}^{n}_{y_{,j,k}} \\ \vec{H}^{(1)}_{z_{,j,k}}+\vec{H}^{n}_{z_{,j,k}} \end{pmatrix}, \tag{3.11}$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{z_{i,,k}}-\varepsilon\vec{E}^{n}_{z_{i,,k}} \\ \mu\vec{H}^{(1)}_{x_{i,,k}}-\mu\vec{H}^{n}_{x_{i,,k}} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 0 & -B_1^y \\ -B_1^y & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{z_{i,,k}}+\vec{E}^{n}_{z_{i,,k}} \\ \vec{H}^{(1)}_{x_{i,,k}}+\vec{H}^{n}_{x_{i,,k}} \end{pmatrix}, \tag{3.12}$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{x_{i,j,}}-\varepsilon\vec{E}^{n}_{x_{i,j,}} \\ \mu\vec{H}^{(1)}_{y_{i,j,}}-\mu\vec{H}^{n}_{y_{i,j,}} \end{pmatrix}=\frac{1}{4}\begin{pmatrix} 0 & -B_1^z \\ -B_1^z & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{x_{i,j,}}+\vec{E}^{n}_{x_{i,j,}} \\ \vec{H}^{(1)}_{y_{i,j,}}+\vec{H}^{n}_{y_{i,j,}} \end{pmatrix}, \tag{3.13}$$

which are more suitable for coding. The method for the treatment of (3.7) is also applicable to (3.8) and (3.9). In section 7, the S-AVF(2) scheme will be further rewritten in fully explicit forms by using the DFFT technique.

3.3. Temporal fourth-order AVF scheme. Now we consider the fourth-order AVF method for the splitting method (2.5). The integral in the fourth-order AVF method for the ODEs (3.4) and (3.6) with various time increments can be calculated exactly to give a fourth-order seven-stage scheme as follows.

S-AVF(4) scheme:

• Stage q. $(q=1,2,\cdots,7)$.

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(q)}-\varepsilon\vec{E}^{(q-1)} \\ \mu\vec{H}^{(q)}-\mu\vec{H}^{(q-1)} \end{pmatrix}=\frac{p_q}{2}\begin{pmatrix} 0 & \tilde{A}_s \\ -\tilde{A}_s^T & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(q)}+\vec{E}^{(q-1)} \\ \vec{H}^{(q)}+\vec{H}^{(q-1)} \end{pmatrix}, \tag{3.14}$$

where $(p_1,p_2,p_3,p_4,p_5,p_6,p_7)=(l_1,l_2,l_3,l_4,l_3,l_2,l_1)$, $s=\mathrm{mod}(q,2)$, $\tilde{A}_1=A_-+\frac{(cp_q\tau)^2}{12}\tilde{A}_-$, $\tilde{A}_0=A_++\frac{(cp_q\tau)^2}{12}\tilde{A}_+$ with $\tilde{A}_\pm=A_\pm A_\pm^T A_\pm$, and $\vec{U}^{(0)}=\vec{U}^n$, $\vec{U}^{(7)}=\vec{U}^{n+1}$

with $U = E, H$. S-AVF(4) also preserves the discrete Hamiltonian (3.10) automatically. The scheme in stage q can be rewritten in equivalent forms, just like (3.11)–(3.13) for (3.7). For example, the scheme in stage 1 can be represented as

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{y,j,k} - \varepsilon\vec{E}^{n}_{y,j,k} \\ \mu\vec{H}^{(1)}_{z,j,k} - \mu\vec{H}^{n}_{z,j,k} \end{pmatrix} = \frac{l_1}{2}\begin{pmatrix} 0 & B_1^x \\ \underline{B_1^x} & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{y,j,k} + \vec{E}^{n}_{y,j,k} \\ \vec{H}^{(1)}_{z,j,k} + \vec{H}^{n}_{z,j,k} \end{pmatrix}, \tag{3.15}$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{z_{i,},k} - \varepsilon\vec{E}^{n}_{z_{i,},k} \\ \mu\vec{H}^{(1)}_{x_{i,},k} - \mu\vec{H}^{n}_{x_{i,},k} \end{pmatrix} = \frac{l_1}{2}\begin{pmatrix} 0 & B_1^y \\ \underline{B_1^y} & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{z_{i,},k} + \vec{E}^{n}_{z_{i,},k} \\ \vec{H}^{(1)}_{x_{i,},k} + \vec{H}^{n}_{x_{i,},k} \end{pmatrix}, \tag{3.16}$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\vec{E}^{(1)}_{x_{i,j,}} - \varepsilon\vec{E}^{n}_{x_{i,j,}} \\ \mu\vec{H}^{(1)}_{y_{i,j,}} - \mu\vec{H}^{n}_{y_{i,j,}} \end{pmatrix} = \frac{l_1}{2}\begin{pmatrix} 0 & B_1^z \\ \underline{B_1^z} & 0 \end{pmatrix}\begin{pmatrix} \vec{E}^{(1)}_{x_{i,j,}} + \vec{E}^{n}_{x_{i,j,}} \\ \vec{H}^{(1)}_{y_{i,j,}} + \vec{H}^{n}_{y_{i,j,}} \end{pmatrix}, \tag{3.17}$$

where $\underline{B_1^w} = -B_1^w + \frac{(cl_1\tau)^2}{12}(B_1^w)^3$ with $w = x$, y, or z. The S-AVF(4) scheme can also be rewritten by fully explicit forms (see section 7).

4. Discrete energy conservations and stability. In this section, we prove the S-AVF(2) and S-AVF(4) schemes preserve four ECLs in the discrete sense simultaneously. For this we need some notation. For any grid scalar functions $U^n_{i,j,k}$, define a discrete L_2 norm as $\|\vec{U}^n\|_2^2 = \sum_{i,j,k} |U^n_{i,j,k}|^2 h_x h_y h_z$, and denote $\|\vec{U}^n\|_{B_1^x} = \|A_x\vec{U}^n\|_2$, $\|\vec{U}^n\|_{B_1^y} = \|A_y\vec{U}^n\|_2$, and $\|\vec{U}^n\|_{B_1^z} = \|A_z\vec{U}^n\|_2$. For vector $\vec{U}^n = ((\vec{U}^n_x)^T, (\vec{U}^n_y)^T, (\vec{U}^n_z)^T)^T$, we define the norm $\|\vec{U}^n\|^2 = \|\vec{U}^n_x\|_2^2 + \|\vec{U}^n_y\|_2^2 + \|\vec{U}^n_z\|_2^2$ and denote $\|\vec{U}^n\|^2_{B_1^w} = \|\vec{U}^n_x\|^2_{B_1^w} + \|\vec{U}^n_y\|^2_{B_1^w} + \|\vec{U}^n_z\|^2_{B_1^w}$, $w = x, y, z$.

THEOREM 4.1 (energy conservation I). *For the integers $n \geq 0$, let $\vec{E}^n = (\vec{E}^n_x, \vec{E}^n_y, \vec{E}^n_z)^T$ and $\vec{H}^n = (\vec{H}^n_x, \vec{H}^n_y, \vec{H}^n_z)^T$ be the solutions of the S-AVF(2) or S-AVF(4) scheme. Then there exist discrete ECLs*

$$\mathcal{E}_1^{n+1} = \mathcal{E}_1^n = \cdots = \mathcal{E}_1^1 = \mathcal{E}_1^0, \qquad \mathcal{E}_1^n = \frac{\varepsilon}{2}\|\vec{E}^n\|^2 + \frac{\mu}{2}\|\vec{H}^n\|^2, \tag{4.1}$$

and

$$\mathcal{E}_2^{n+1} = \mathcal{E}_2^n = \cdots = \mathcal{E}_2^1 = \mathcal{E}_2^0, \qquad \mathcal{E}_2^n = \frac{\varepsilon}{2}\|\delta_t\vec{E}^{n-\frac{1}{2}}\|^2 + \frac{\mu}{2}\|\delta_t\vec{H}^{n-\frac{1}{2}}\|^2, \tag{4.2}$$

where $\delta_t\vec{U}^{n+\frac{1}{2}} = (\vec{U}^{n+1} - \vec{U}^n)/\tau$.

Proof. Both the S-AVF(2) and S-AVF(4) schemes hold the discrete Hamiltonian (3.10), which implies the discrete ECL (4.1). Let $\vec{E}^{(1)+1}$, $\vec{H}^{(1)+1}$, $\vec{E}^{(2)+1}$, and $\vec{H}^{(2)+1}$ be the intermediate values of $\vec{E}^{(1)}$, $\vec{H}^{(2)}$, $\vec{E}^{(2)}$, and $\vec{H}^{(2)}$ at time level $t = t_{n+1}$, respectively. Denote $\delta_t\vec{U}^{(1)+\frac{1}{2}} = (\vec{U}^{(1)+1} - \vec{U}^{(1)})/\tau$ and $\delta_t\vec{U}^{(2)+\frac{1}{2}} = (\vec{U}^{(2)+1} - \vec{U}^{(2)})/\tau$. Applying the difference operator δ_t to the schemes (3.7)–(3.9) gives

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\delta_t\vec{E}^{(1)+\frac{1}{2}} - \varepsilon\delta_t\vec{E}^{n+\frac{1}{2}} \\ \mu\delta_t\vec{H}^{(1)+\frac{1}{2}} - \mu\delta_t\vec{H}^{n+\frac{1}{2}} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}\begin{pmatrix} \delta_t\vec{E}^{(1)+\frac{1}{2}} + \delta_t\vec{E}^{n+\frac{1}{2}} \\ \delta_t\vec{H}^{(1)+\frac{1}{2}} + \delta_t\vec{H}^{n+\frac{1}{2}} \end{pmatrix},$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\delta_t\vec{E}^{(2)+\frac{1}{2}} - \varepsilon\delta_t\vec{E}^{(1)+\frac{1}{2}} \\ \mu\delta_t\vec{H}^{(2)+\frac{1}{2}} - \mu\delta_t\vec{H}^{(1)+\frac{1}{2}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & A_+ \\ -A_+^T & 0 \end{pmatrix}\begin{pmatrix} \delta_t\vec{E}^{(2)+\frac{1}{2}} + \delta_t\vec{E}^{(1)+\frac{1}{2}} \\ \delta_t\vec{H}^{(2)+\frac{1}{2}} + \delta_t\vec{H}^{(1)+\frac{1}{2}} \end{pmatrix},$$

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon\delta_t\vec{E}^{n+\frac{3}{2}} - \varepsilon\delta_t\vec{E}^{(2)+\frac{1}{2}} \\ \mu\delta_t\vec{H}^{n+\frac{3}{2}} - \mu\delta_t\vec{H}^{(2)+\frac{1}{2}} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}\begin{pmatrix} \delta_t\vec{E}^{n+\frac{3}{2}} + \delta_t\vec{E}^{(2)+\frac{1}{2}} \\ \delta_t\vec{H}^{n+\frac{3}{2}} + \delta_t\vec{H}^{(2)+\frac{1}{2}} \end{pmatrix}.$$

Thanks to the skew symmetry of the matrix $\begin{pmatrix} 0 & A_{\pm} \\ -A_{\pm}^T & 0 \end{pmatrix}$, we obtain (4.2) immediately. Similarly shows the ECL (4.2) for S-AVF(4). This ends the proof. ∎

COROLLARY 4.2. *Discrete ECL* (4.1) *implies that the numerical solutions are bounded in the* L_2 *norm and do not blow up. Therefore, both S-AVF(2) and S-AVF(4) are unconditionally stable.*

THEOREM 4.3 (energy conservation II). *Both the AVF(2) and AVF(4) schemes possess discrete ECLs*

$$\mathcal{E}_{3w}^{n+1} = \mathcal{E}_{3w}^{n} = \cdots = \mathcal{E}_{3w}^{1} = \mathcal{E}_{3w}^{0}, \quad \mathcal{E}_{3w}^{n} = \varepsilon\|\vec{E}^n\|_{B_1^w}^2 + \mu\|\vec{H}^n\|_{B_1^w}^2, \tag{4.3}$$

and

$$\mathcal{E}_{4w}^{n+1} = \mathcal{E}_{4w}^{n} = \cdots = \mathcal{E}_{4w}^{1} = \mathcal{E}_{4w}^{0}, \quad \mathcal{E}_{4w}^{n} = \varepsilon\|\delta_t\vec{E}^{n-\frac{1}{2}}\|_{B_1^w}^2 + \mu\|\delta_t\vec{H}^{n-\frac{1}{2}}\|_{B_1^w}^2, \tag{4.4}$$

where $w = x$, y, *or* z.

Proof. In the following, we only display the proof for the case of $w = x$. Let $M = \mathrm{diag}(A_x, A_x, A_x)$ be a block diagonal matrix. By using the property of the Kronecker product, we have $MA_- = A_-M$. Left multiplying both sides of (3.7) with $\begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$ reads

$$\frac{1}{\tau}\begin{pmatrix} \varepsilon M\vec{E}^{(1)} - \varepsilon M\vec{E}^n \\ \mu M\vec{H}^{(1)} - \mu M\vec{H}^n \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}\begin{pmatrix} M\vec{E}^{(1)} + M\vec{E}^n \\ M\vec{H}^{(1)} + M\vec{H}^n \end{pmatrix}. \tag{4.5}$$

It follows from (4.5) that $\varepsilon\|M\vec{E}^{(1)}\|^2 + \mu\|M\vec{H}^{(1)}\|^2 = \varepsilon\|M\vec{E}^n\|^2 + \mu\|M\vec{H}^n\|^2$. Thanks to $\|M\vec{E}^n\|^2 = \|\vec{E}^n\|_{B_1^x}^2$ and $\|M\vec{H}^n\|^2 = \|\vec{H}^n\|_{B_1^x}^2$, we have

$$\varepsilon\|\vec{E}^{(1)}\|_{B_1^x}^2 + \mu\|\vec{H}^{(1)}\|_{B_1^x}^2 = \varepsilon\|\vec{E}^n\|_{B_1^x}^2 + \mu\|\vec{H}^n\|_{B_1^x}^2. \tag{4.6}$$

Similarly, from (3.8) and (3.9), we have

$$\begin{aligned} \varepsilon\|\vec{E}^{(2)}\|_{B_1^x}^2 + \mu\|\vec{H}^{(2)}\|_{B_1^x}^2 &= \varepsilon\|\vec{E}^{(1)}\|_{B_1^x}^2 + \mu\|\vec{H}^{(1)}\|_{B_1^x}^2, \\ \varepsilon\|\vec{E}^{n+1}\|_{B_1^x}^2 + \mu\|\vec{H}^{n+1}\|_{B_1^x}^2 &= \varepsilon\|\vec{E}^{(2)}\|_{B_1^x}^2 + \mu\|\vec{H}^{(2)}\|_{B_1^x}^2. \end{aligned} \tag{4.7}$$

Combining (4.6) and (4.7) gives the ECL (4.3). Thanks to the commutativity of the matrices M and $A_{\mp} + \frac{(cl_1\tau)^2}{12}\tilde{A}_{\mp}$, we also derive the ECL (4.3) for S-AVF(4). Next, by the same argument as detailed in Theorem 4.1, we obtain (4.4). The proof is completed. ∎

5. Convergence analysis. In this section, we focus on the error estimates of the proposed S-AVF(2) and S-AVF(4) schemes. For simplicity we consider the cubic domain $\Omega = [0, 2\pi]^3$ with the spatial grid points $N_x = N_y = N_z = \widetilde{N}$. A general cuboid domain can be linearly mapped into $\Omega = [0, 2\pi]^3$. Let $C_p^\infty(\Omega)$ be the set of infinitely differentiable periodic functions with period 2π, and $H_p^r(\Omega)$ be the closure of $C_p^\infty(\Omega)$ in $H^r(\Omega)$. Define the inner product by

$$\langle u, v\rangle = \frac{1}{8\pi^3}\int_\Omega u(x,y,z)\overline{v(x,y,z)}dxdydz,$$

and the discrete inner product and norm by, respectively,

$$\langle u, v\rangle_{\widetilde{N}} = \frac{1}{\widetilde{N}^3}\sum_{i=0}^{\widetilde{N}-1}\sum_{j=0}^{\widetilde{N}-1}\sum_{k=0}^{\widetilde{N}-1} u(x_i, y_j, z_k)\overline{v(x_i, y_j, z_k)}, \qquad \|u\|_{\widetilde{N}}^2 = \langle u, u\rangle_{\widetilde{N}}.$$

The norm and seminorm of $H^r(\Omega)$ are denoted by $\|\cdot\|_r$ and $|\cdot|_r$, respectively. In particular, $\|\cdot\|_0 = \|\cdot\|$. Let $l = \sqrt{-1}$,

$$\mathcal{V}_{\widetilde{N}} = \left\{ u = \sum_{|p|,|q|,|r|\le \frac{\widetilde{N}}{2}} \hat{u}_{p,q,r} e^{l(px+qy+rz)} : \ \overline{\hat{u}}_{p,q,r} = \hat{u}_{-p,-q,-r} \right\},$$

and

$$\mathcal{W}_{\widetilde{N}} = \left\{ u = \sum_{|p|,|q|,|r|\le \frac{\widetilde{N}}{2}} \hat{u}_{p,q,r} e^{l(px+qy+rz)} : \ \overline{\hat{u}}_{p,q,r} = \hat{u}_{-p,-q,-r}; \right.$$
$$\left. \hat{u}_{\frac{\widetilde{N}}{2},\frac{\widetilde{N}}{2},\frac{\widetilde{N}}{2}} = \hat{u}_{-\frac{\widetilde{N}}{2},-\frac{\widetilde{N}}{2},-\frac{\widetilde{N}}{2}} \right\} \subset \mathcal{V}_{\widetilde{N}}.$$

It is clear that both $\mathcal{V}_{\widetilde{N}}$ and $\mathcal{W}_{\widetilde{N}}$ are real, and $\mathcal{V}_{\widetilde{N}-2} \subset \mathcal{W}_{\widetilde{N}}$. By careful calculations, we find $\mathcal{W}_{\widetilde{N}} = \mathrm{span}\{g_i(x)g_j(y)g_k(z) : \ i, j, k = 0, 1\cdots, \widetilde{N}-1\}$.

PROPOSITION 5.1. *For all* $\vec{u} \in [\mathcal{W}_{\widetilde{N}}]^3$, $\|\vec{u}\| \le \|\vec{u}\|_{\widetilde{N}} \le 2\sqrt{2}\|\vec{u}\|$.

PROPOSITION 5.2. *If* $u, v \in \mathcal{W}_{\widetilde{N}}$, *then* $\langle \partial_w u, v\rangle_{\widetilde{N}} = -\langle u, \partial_w v\rangle_{\widetilde{N}}$, $w = x,\ y,$ *or* z.

Let $P_{\widetilde{N}}$: $[L^2(\Omega)]^3 \to [\mathcal{V}_{\widetilde{N}}]^3$ be the orthogonal projection operator, i.e.,

$$\langle P_{\widetilde{N}}\vec{u}, \varphi\rangle = \langle \vec{u}, \varphi\rangle \quad \forall \varphi \in [\mathcal{V}_{\widetilde{N}}]^3,$$

and P_c: $[C(\overline{\Omega})]^3 \to [\mathcal{W}_{\widetilde{N}}]^3$ be the interpolation operator. An application of Proposition 5.2 yields the commutativity of the $\mathbf{curl}_{\pm}$ and $P_{\widetilde{N}}$ operators.

LEMMA 5.3 (see [21, 22]). *If* $0 \le \gamma \le \sigma$ *and* $\vec{u} \in [H^\sigma(\Omega)]^3$, *then* $\|P_{\widetilde{N}}\vec{u} - \vec{u}\|_\gamma \le C\widetilde{N}^{\gamma-\sigma}|\vec{u}|_\sigma$, *and further* $\sigma > \frac{3}{2}$, *then* $\|P_c\vec{u} - \vec{u}\|_\gamma \le C\widetilde{N}^{\gamma-\sigma}|\vec{u}|_\sigma$.

THEOREM 5.4 (convergence of S-AVF(2)). *Suppose that the exact solution components* $\vec{E}$ *and* $\vec{H}$ *are smooth enough:* $\vec{E}$, $\vec{H} \in C^3\left(0, T; [H_p^r(\overline{\Omega})]^3\right)$, $r > \frac{5}{2}$. *For* $n \ge 0$, *let* $\vec{E}_c^n$ *and* $\vec{H}_c^n$ *be the numerical solutions of the S-AVF(2) scheme. Then for any fixed* $T = N\tau$, *there exists a positive constant* C *independent of* τ, h_x, h_y, *and* h_z *such that*

$$\max_{0\le n\le N}\left(\varepsilon\left\|\vec{E}(t_n) - \vec{E}_c^n\right\|_{\widetilde{N}}^2 + \mu\left\|\vec{H}(t_n) - \vec{H}_c^n\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}} \le C(\tau^2 + \widetilde{N}^{1-r}). \tag{5.1}$$

Proof. Let C be a positive constant independent of τ, h_x, h_y, and h_z which may be different in different cases. As we know, the second-order splitting method (2.3), which is implemented with initial value $(\underline{\vec{E}}^{(0)}, \underline{\vec{H}}^{(0)})^T = (\vec{E}(t_n), \vec{H}(t_n))^T$, is a three-stage method. Since the solution of one subproblem is employed as an initial condition for the next subproblem, we let $(\underline{\vec{E}}^{(1)}, \underline{\vec{H}}^{(1)})^T$, $(\underline{\vec{E}}^{(2)}, \underline{\vec{H}}^{(2)})^T$, and $(\underline{\vec{E}}(t_{n+1}), \underline{\vec{H}}(t_{n+1}))^T$ be the analytical solutions of the three subproblems, respectively. We also let $(\vec{E}_c^{(1)}, \vec{H}_c^{(1)})^T$, $(\vec{E}_c^{(2)}, \vec{H}_c^{(2)})^T$, and $(\vec{E}_c(t_{n+1}), \vec{H}_c(t_{n+1}))^T$ be the numerical

solutions of (3.7), (3.8), and (3.9), respectively. It is clear that $(\vec{E}_c(t_{n+1}), \vec{H}_c(t_{n+1}))^T$ is the numerical solution of the S-AVF(2) scheme.

The scheme (3.7) is equivalent to finding the numerical solution $(\vec{E}_c^{(1)}, \vec{H}_c^{(1)})^T \in [\mathcal{W}_{\widetilde{N}}]^6$ such that

$$\left\langle \delta_{t/2} \begin{pmatrix} \varepsilon \vec{E}_c^{(0)} \\ \mu \vec{H}_c^{(0)} \end{pmatrix}, \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} \right\rangle_{\widetilde{N}} = \left\langle \mathbf{A}_- \begin{pmatrix} \vec{E}_c^{(0)+\frac{1}{2}} \\ \vec{H}_c^{(0)+\frac{1}{2}} \end{pmatrix}, \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} \right\rangle_{\widetilde{N}} \tag{5.2}$$

for all $(\vec{\psi}, \vec{\phi})^T \in [\mathcal{W}_{\widetilde{N}}]^6$, where $\vec{E}_c^{(0)+\frac{1}{2}} = \frac{1}{2}(\vec{E}_c^{(1)} + \vec{E}_c^{(0)})$, $\vec{H}_c^{(0)+\frac{1}{2}} = \frac{1}{2}(\vec{H}_c^{(1)} + \vec{H}_c^{(0)})$, and

$$\delta_{t/2} \begin{pmatrix} \vec{E}_c^{(0)} \\ \vec{H}_c^{(0)} \end{pmatrix} = \frac{1}{\tau/2} \left[\begin{pmatrix} \vec{E}_c^{(1)} \\ \vec{H}_c^{(1)} \end{pmatrix} - \begin{pmatrix} \vec{E}_c^{(0)} \\ \vec{H}_c^{(0)} \end{pmatrix} \right].$$

The first stage of the splitting method (2.3) means that we should obtain $(\underline{\vec{E}}^{(1)}, \underline{\vec{H}}^{(1)})^T$ by solving

$$\frac{\partial}{\partial t} \begin{pmatrix} \varepsilon \vec{E} \\ \mu \vec{H} \end{pmatrix} = \mathbf{A}_- \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \tag{5.3}$$

with the initial value $(\underline{\vec{E}}^{(0)}, \underline{\vec{H}}^{(0)})^T$ and $\frac{\tau}{2}$ time step. Obviously, the orthogonal projection equation associated with (5.3) is

$$\frac{\partial}{\partial t} \begin{pmatrix} \varepsilon (P_{\widetilde{N}-2} \underline{\vec{E}})^{(0)+\frac{1}{2}} \\ \mu (P_{\widetilde{N}-2} \underline{\vec{H}})^{(0)+\frac{1}{2}} \end{pmatrix} = \mathbf{A}_- \begin{pmatrix} (P_{\widetilde{N}-2} \underline{\vec{E}})^{(0)+\frac{1}{2}} \\ (P_{\widetilde{N}-2} \underline{\vec{H}})^{(0)+\frac{1}{2}} \end{pmatrix}. \tag{5.4}$$

Denote $\vec{\mathcal{H}} = \vec{H}_c - P_{\widetilde{N}-2} \underline{\vec{H}}$ and $\vec{\mathcal{E}} = \vec{E}_c - P_{\widetilde{N}-2} \underline{\vec{E}}$. It follows from (5.2) and (5.4) that

$$\left\langle \delta_{t/2} \begin{pmatrix} \varepsilon \vec{\mathcal{E}}^{(0)} \\ \mu \vec{\mathcal{H}}^{(0)} \end{pmatrix}, \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} \right\rangle_{\widetilde{N}} = \left\langle \mathbf{A}_- \begin{pmatrix} \vec{\mathcal{E}}^{(0)+\frac{1}{2}} \\ \vec{\mathcal{H}}^{(0)+\frac{1}{2}} \end{pmatrix} + \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix}, \begin{pmatrix} \vec{\phi} \\ \vec{\psi} \end{pmatrix} \right\rangle_{\widetilde{N}} \tag{5.5}$$

for all $(\vec{\psi}, \vec{\phi})^T \in [\mathcal{W}_{\widetilde{N}}]^6$, where $\vec{\xi} = (\xi_x, \xi_y, \xi_z)^T$, $\vec{\eta} = (\eta_x, \eta_y, \eta_z)^T$, and

$$\begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} = \frac{\partial}{\partial t} \begin{pmatrix} \varepsilon (P_{\widetilde{N}-2} \underline{\vec{E}})^{(0)+\frac{1}{2}} \\ \mu (P_{\widetilde{N}-2} \underline{\vec{H}})^{(0)+\frac{1}{2}} \end{pmatrix} - \delta_{t/2} \begin{pmatrix} \varepsilon (P_{\widetilde{N}-2} \underline{\vec{E}})^{(0)} \\ \mu (P_{\widetilde{N}-2} \underline{\vec{H}})^{(0)} \end{pmatrix}.$$

Further combining (5.3) and (5.4), we have

$$\begin{aligned} \begin{pmatrix} \vec{\xi} \\ \vec{\eta} \end{pmatrix} = & \left[\frac{\partial}{\partial t} \begin{pmatrix} \varepsilon \underline{\vec{E}}^{(0)+\frac{1}{2}} \\ \mu \underline{\vec{H}}^{(0)+\frac{1}{2}} \end{pmatrix} - \delta_{t/2} \begin{pmatrix} \varepsilon \underline{\vec{E}}^{(0)} \\ \mu \underline{\vec{H}}^{(0)} \end{pmatrix} \right] + \delta_{t/2} \begin{pmatrix} \varepsilon (\underline{\vec{E}} - P_{\widetilde{N}-2} \underline{\vec{E}})^{(0)} \\ \mu (\underline{\vec{H}} - P_{\widetilde{N}-2} \underline{\vec{H}})^{(0)} \end{pmatrix} \\ & - \begin{pmatrix} \left(\mathbf{curl}_- \underline{\vec{H}} - P_{\widetilde{N}-2} (\mathbf{curl}_- \underline{\vec{H}}) \right)^{(0)+\frac{1}{2}} \\ \left(-\mathbf{curl}_-^* \underline{\vec{E}} + P_{\widetilde{N}-2} (\mathbf{curl}_-^* \underline{\vec{E}}) \right)^{(0)+\frac{1}{2}} \end{pmatrix}. \end{aligned}$$

Making use of Lemma 5.3, we have $\|\vec{\xi}\|, \|\vec{\eta}\| \le C(\tau^2 + \widetilde{N}^{1-r})$. Note that both $\vec{\xi}$ and $\vec{\eta}$ belong to $[\mathcal{V}_{\widetilde{N}}]^3$ ($\subset [\mathcal{W}_{\widetilde{N}}]^3$). An application of Proposition 5.1 yields

$$\|\vec{\xi}\|_{\widetilde{N}}, \|\vec{\eta}\|_{\widetilde{N}} \le C(\tau^2 + \widetilde{N}^{1-r}).$$

Note that the $\mathbf{A}_-$ operator is skew-adjoint. In (5.5), taking $(\vec{\phi},\vec{\psi})^T=(\vec{\mathcal{E}}^{(0)+\frac{1}{2}},\vec{\mathcal{H}}^{(0)+\frac{1}{2}})^T$ leads to

$$\varepsilon\|\vec{\mathcal{E}}^{(1)}\|_{\widetilde{N}}^2+\mu\|\vec{\mathcal{H}}^{(1)}\|_{\widetilde{N}}^2-\varepsilon\|\vec{\mathcal{E}}^{n}\|_{\widetilde{N}}^2-\mu\|\vec{\mathcal{H}}^{n}\|_{\widetilde{N}}^2=\frac{\tau}{2}\left\langle\vec{\xi},\vec{\mathcal{E}}^{(1)}+\vec{\mathcal{E}}^{n}\right\rangle_{\widetilde{N}}+\frac{\tau}{2}\left\langle\vec{\eta},\vec{\mathcal{H}}^{(1)}+\vec{\mathcal{H}}^{n}\right\rangle_{\widetilde{N}}.$$

An application of the formulation of completing the square yields

(5.6)
$$\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(1)}-\frac{\tau}{4\sqrt{\varepsilon}}\vec{\xi}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{(1)}-\frac{\tau}{4\sqrt{\mu}}\vec{\eta}\right\|_{\widetilde{N}}^2=\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n}+\frac{\tau}{4\sqrt{\varepsilon}}\vec{\xi}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{n}+\frac{\tau}{4\sqrt{\mu}}\vec{\eta}\right\|_{\widetilde{N}}^2.$$

As in the proof of (5.6), it follows from the stages 2 and 3 (see (3.8) and (3.9)) of the S-AVF(2) scheme that

$$\begin{aligned}(5.7)\qquad&\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(2)}-\frac{\tau}{2\sqrt{\varepsilon}}\vec{\tilde{\xi}}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{(2)}-\frac{\tau}{2\sqrt{\mu}}\vec{\tilde{\eta}}\right\|_{\widetilde{N}}^2\\&=\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(1)}+\frac{\tau}{2\sqrt{\varepsilon}}\vec{\tilde{\xi}}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{(1)}+\frac{\tau}{2\sqrt{\mu}}\vec{\tilde{\eta}}\right\|_{\widetilde{N}}^2,\end{aligned}$$

and

$$\begin{aligned}(5.8)\qquad&\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n+1}-\frac{\tau}{4\sqrt{\varepsilon}}\vec{\tilde{\xi}}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{n+1}-\frac{\tau}{4\sqrt{\mu}}\vec{\tilde{\eta}}\right\|_{\widetilde{N}}^2\\&=\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(2)}+\frac{\tau}{4\sqrt{\varepsilon}}\vec{\tilde{\xi}}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{(2)}+\frac{\tau}{4\sqrt{\mu}}\vec{\tilde{\eta}}\right\|_{\widetilde{N}}^2,\end{aligned}$$

where $\|\vec{\tilde{\xi}}\|_{\widetilde{N}}$, $\|\vec{\tilde{\eta}}\|_{\widetilde{N}}$, $\|\vec{\tilde{\xi}}\|_{\widetilde{N}}$, $\|\vec{\tilde{\eta}}\|_{\widetilde{N}}\le C(\tau^2+\widetilde{N}^{1-r})$. Next, by using the inequality $(a^2+b^2)^{\frac{1}{2}}\le\big((a-c)^2+(b-d)^2\big)^{\frac{1}{2}}+(c^2+d^2)^{\frac{1}{2}}$ and the triangle inequality of the discrete norm, we obtain

(5.9)
$$\begin{aligned}&\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n+1}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{n+1}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\\&\le\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n+1}-\frac{\tau}{4\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{n+1}-\frac{\tau}{4\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+\frac{\tau}{4}\left(\|\frac{1}{\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\frac{1}{\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\\&=\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(2)}+\frac{\tau}{4\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{(2)}+\frac{\tau}{4\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+\frac{\tau}{4}\left(\|\frac{1}{\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\frac{1}{\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\\&\le\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(2)}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{(2)}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+\frac{\tau}{2}\left(\|\frac{1}{\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\frac{1}{\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}.\end{aligned}$$

As in the procedure of (5.9), we also have the relationships

$$\begin{aligned}(5.10)\qquad\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(2)}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{(2)}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}&\le\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(1)}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{(1)}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\\&\quad+\tau\left(\|\frac{1}{\sqrt{\varepsilon}}\vec{\tilde{\xi}}\|_{\widetilde{N}}^2+\|\frac{1}{\sqrt{\mu}}\vec{\tilde{\eta}}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}},\end{aligned}$$

and

$$\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(1)}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{(1)}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\le\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{n}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+\frac{\tau}{2}\left(\|\frac{1}{\sqrt{\varepsilon}}\vec{\xi}\|_{\widetilde{N}}^2+\|\frac{1}{\sqrt{\mu}}\vec{\eta}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}. \tag{5.11}$$

Inequalities (5.9)–(5.11) imply

$$\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n+1}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{n+1}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\le\left(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n}\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^{n}\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+C\tau(\tau^2+\widetilde{N}^{1-r}).$$

Further it follows from the above inequality that

$$\left(\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{n}\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\le\left(\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{0}\right\|_{\widetilde{N}}^2+\left\|\sqrt{\mu}\vec{\mathcal{H}}^{0}\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}+Cn\tau(\tau^2+\widetilde{N}^{1-r}). \tag{5.12}$$

Applications of Proposition 5.1 and Lemma 5.3 give $\|\vec{\mathcal{E}}^0\|_{\widetilde{N}}\le 2\sqrt{2}\big(\|P_c\vec{E}(0)-\vec{E}(0)\|+\|\vec{E}(0)-P_{\widetilde{N}-2}\vec{E}(0)\|\big)\le C\widetilde{N}^{-r}$ and $\|\vec{\mathcal{H}}^0\|_{\widetilde{N}}\le C\widetilde{N}^{-r}$. Because of $n\tau<T$, it follows from (5.12) that $(\|\sqrt{\varepsilon}\vec{\mathcal{E}}^n\|_{\widetilde{N}}^2+\|\sqrt{\mu}\vec{\mathcal{H}}^n\|_{\widetilde{N}}^2)^{\frac{1}{2}}\le C(\tau^2+\widetilde{N}^{1-r})$. By using the inequality $a+b\le\sqrt{2(a^2+b^2)}$, we have

$$\sqrt{\varepsilon}\left\|\vec{\mathcal{E}}^n\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\vec{\mathcal{H}}^n\right\|_{\widetilde{N}}\le C(\tau^2+\widetilde{N}^{1-r}). \tag{5.13}$$

Now we can establish an error estimate for the S-AVF(2) scheme as follows:

$$\begin{aligned}
&\left(\varepsilon\left\|\vec{E}(t_n)-\vec{E}_c^n\right\|_{\widetilde{N}}^2+\mu\left\|\vec{H}(t_n)-\vec{H}_c^n\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\\
&\le\sqrt{\varepsilon}\left\|\vec{E}(t_n)-\vec{E}_c^n\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\vec{H}(t_n)-\vec{H}_c^n\right\|_{\widetilde{N}}\\
&\le\sqrt{\varepsilon}\left\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\varepsilon}\left\|\underline{\vec{E}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\varepsilon}\left\|\vec{\mathcal{E}}^n\right\|_{\widetilde{N}}\\
&\quad+\sqrt{\mu}\left\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\underline{\vec{H}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\vec{\mathcal{H}}^n\right\|_{\widetilde{N}}\\
&=\sqrt{\varepsilon}\left\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\varepsilon}\left\|P_c\underline{\vec{E}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\varepsilon}\left\|\vec{\mathcal{E}}^n\right\|_{\widetilde{N}}\\
&\quad+\sqrt{\mu}\left\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|P_c\underline{\vec{H}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\vec{\mathcal{H}}^n\right\|_{\widetilde{N}}\\
&\le\sqrt{\varepsilon}\left\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+2\sqrt{2\varepsilon}\left\|P_c\underline{\vec{E}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{E}}(t_n)\right\|+\sqrt{\varepsilon}\left\|\vec{\mathcal{E}}^n\right\|_{\widetilde{N}}\\
&\quad+\sqrt{\mu}\left\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}+2\sqrt{2\mu}\left\|P_c\underline{\vec{H}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{H}}(t_n)\right\|+\sqrt{\mu}\left\|\vec{\mathcal{H}}^n\right\|_{\widetilde{N}}\\
&\le\sqrt{\varepsilon}\left\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}}+2\sqrt{2\varepsilon}\left\|P_c\underline{\vec{E}}(t_n)-\underline{\vec{E}}(t_n)\right\|\\
&\quad+2\sqrt{2\varepsilon}\left\|\underline{\vec{E}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{E}}(t_n)\right\|+\sqrt{\varepsilon}\left\|\vec{\mathcal{E}}^n\right\|_{\widetilde{N}}+\sqrt{\mu}\left\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}\\
&\quad+2\sqrt{2\mu}\left\|P_c\underline{\vec{H}}(t_n)-\underline{\vec{H}}(t_n)\right\|+2\sqrt{2\mu}\left\|\underline{\vec{H}}(t_n)-P_{\widetilde{N}-2}\underline{\vec{H}}(t_n)\right\|+\sqrt{\mu}\left\|\vec{\mathcal{H}}^n\right\|_{\widetilde{N}}.
\end{aligned} \tag{5.14}$$

Note that the temporal second-order splitting method (2.3) implies $\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\|_\infty$, $\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\|_\infty\le C\tau^2$. We immediately have

$$\left\|\vec{E}(t_n)-\underline{\vec{E}}(t_n)\right\|_{\widetilde{N}},\quad\left\|\vec{H}(t_n)-\underline{\vec{H}}(t_n)\right\|_{\widetilde{N}}\le C\tau^2. \tag{5.15}$$

Thanks to Lemma 5.3, (5.13), (5.15), and (5.14), we complete the proof. ∎

THEOREM 5.5 (convergence S-AVF(4)). *Suppose that the exact solution components $\vec{E}$ and $\vec{H}$ are smooth enough: $\vec{E}$, $\vec{H} \in C^5\left(0,T;[H_p^r(\overline{\Omega})]^3\right)$, $r > \frac{9}{2}$. For $n \geq 0$, let $\vec{E}_c^n$ and $\vec{H}_c^n$ be the numerical solutions of the S-AVF(4) scheme. Then for any fixed $T = N\tau$, there exists a positive constant C independent of τ, h_x, h_y, and h_z such that*

$$\max_{0\leq n\leq N}\left(\varepsilon\left\|\vec{E}(t_n)-\vec{E}_c^n\right\|_{\widetilde{N}}^2+\mu\left\|\vec{H}(t_n)-\vec{H}_c^n\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}}\leq C(\tau^4+\widetilde{N}^{3-r}). \tag{5.16}$$

Proof. The idea of the proof is similar to that in the preceding theorem, but the details are a bit more complicated. Let

$$\widetilde{\mathbf{A}}_-=\begin{pmatrix}0 & \widetilde{\mathbf{curl}}_-\\ -\widetilde{\mathbf{curl}}_-^* & 0\end{pmatrix}\quad\text{and}\quad\widetilde{\mathbf{A}}_+=\begin{pmatrix}0 & \widetilde{\mathbf{curl}}_+\\ -\widetilde{\mathbf{curl}}_+^* & 0\end{pmatrix},$$

where $\widetilde{\mathbf{curl}}_- = \mathbf{curl}_-\mathbf{curl}_-^*\mathbf{curl}_-$ and $\widetilde{\mathbf{curl}}_+ = \mathbf{curl}_+\mathbf{curl}_+^*\mathbf{curl}_+$. For the scheme in stage 1 of S-AVF(4), the numerical solution $(\vec{E}_c^{(1)},\vec{H}_c^{(1)})^T \in [\mathcal{W}_{\widetilde{N}}]^6$ satisfies

$$\left\langle\delta_{l_1t}\begin{pmatrix}\varepsilon\vec{E}_c^{(0)}\\ \mu\vec{H}_c^{(0)}\end{pmatrix},\begin{pmatrix}\vec{\phi}\\ \vec{\psi}\end{pmatrix}\right\rangle_{\widetilde{N}}=\left\langle\left(\mathbf{A}_-+\frac{(cl_1\tau)^2}{12}\widetilde{\mathbf{A}}_-\right)\begin{pmatrix}\vec{E}_c^{(0)+\frac{1}{2}}\\ \vec{H}_c^{(0)+\frac{1}{2}}\end{pmatrix},\begin{pmatrix}\vec{\phi}\\ \vec{\psi}\end{pmatrix}\right\rangle_{\widetilde{N}}, \tag{5.17}$$

for all $(\vec{\phi},\vec{\psi})^T \in [\mathcal{W}_{\widetilde{N}}]^6$, where $\delta_{l_1t}\begin{pmatrix}\vec{E}_c^{(0)}\\ \vec{H}_c^{(0)}\end{pmatrix} = \frac{1}{l_1\tau}\left[\begin{pmatrix}\vec{E}_c^{(1)}\\ \vec{H}_c^{(1)}\end{pmatrix}-\begin{pmatrix}\vec{E}_c^{(0)}\\ \vec{H}_c^{(0)}\end{pmatrix}\right]$. Combining (5.17) and (5.4) gives

$$\begin{aligned}&\left\langle\delta_{l_1t}\begin{pmatrix}\varepsilon\vec{\mathcal{E}}^{(0)}\\ \mu\vec{\mathcal{H}}^{(0)}\end{pmatrix},\begin{pmatrix}\vec{\phi}\\ \vec{\psi}\end{pmatrix}\right\rangle_{\widetilde{N}}\\ &\quad=\left\langle\mathbf{A}_-\begin{pmatrix}\vec{\mathcal{E}}^{(0)+\frac{1}{2}}\\ \vec{\mathcal{H}}^{(0)+\frac{1}{2}}\end{pmatrix}+\frac{(cl_1\tau)^2}{12}\widetilde{\mathbf{A}}_-\begin{pmatrix}\vec{\mathcal{E}}_c^{(0)+\frac{1}{2}}\\ \vec{\mathcal{H}}_c^{(0)+\frac{1}{2}}\end{pmatrix}+\begin{pmatrix}\vec{\xi}\\ \vec{\eta}\end{pmatrix},\begin{pmatrix}\vec{\phi}\\ \vec{\psi}\end{pmatrix}\right\rangle_{\widetilde{N}},\end{aligned} \tag{5.18}$$

where

$$\begin{aligned}\begin{pmatrix}\vec{\xi}\\ \vec{\eta}\end{pmatrix}&=\frac{\partial}{\partial t}\begin{pmatrix}\varepsilon(P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)+\frac{1}{2}}\\ \mu(P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)+\frac{1}{2}}\end{pmatrix}-\delta_{l_1t}\begin{pmatrix}\varepsilon(P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)+\frac{1}{2}}\\ \mu(P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)+\frac{1}{2}}\end{pmatrix}\\ &\quad+\frac{(cl_1\tau)^2}{12}\widetilde{\mathbf{A}}_-\begin{pmatrix}(P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)+\frac{1}{2}}\\ (P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)+\frac{1}{2}}\end{pmatrix}=f_1+f_2\end{aligned}$$

with $f_1=-\delta_{l_1t}\begin{pmatrix}\varepsilon\underline{\vec{E}}^{(0)}\\ \mu\underline{\vec{H}}^{(0)}\end{pmatrix}+\mathbf{A}_-\begin{pmatrix}\underline{\vec{E}}^{(0)+\frac{1}{2}}\\ \underline{\vec{H}}^{(0)+\frac{1}{2}}\end{pmatrix}+\frac{(cl_1\tau)^2}{12}\widetilde{\mathbf{A}}_-\begin{pmatrix}\underline{\vec{E}}^{(0)+\frac{1}{2}}\\ \underline{\vec{H}}^{(0)+\frac{1}{2}}\end{pmatrix}$ and

$$\begin{aligned}f_2&=\delta_{l_1t}\begin{pmatrix}\varepsilon(\underline{\vec{E}}-P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)}\\ \mu(\underline{\vec{H}}-P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)}\end{pmatrix}-\mathbf{A}_-\begin{pmatrix}(\underline{\vec{E}}-P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)+\frac{1}{2}}\\ (\underline{\vec{H}}-P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)+\frac{1}{2}}\end{pmatrix}\\ &\quad-\frac{(cl_1\tau)^2}{12}\widetilde{\mathbf{A}}_-\begin{pmatrix}(\underline{\vec{E}}-P_{\widetilde{N}-2}\underline{\vec{E}})^{(0)+\frac{1}{2}}\\ (\underline{\vec{H}}-P_{\widetilde{N}-2}\underline{\vec{H}})^{(0)+\frac{1}{2}}\end{pmatrix}.\end{aligned}$$

It is clear that f_1 is the truncation error of a scheme obtained by applying the fourth-order AVF method to (5.3) with the initial value $(\underline{\vec{E}}^{(0)}, \underline{\vec{H}}^{(0)})^T$ and $l_1\tau$ time increment. Immediately one has $\|f_1\|_\infty \le C\tau^4$. From the error equation f_2, we obtain $\|f_2\| \le C(\tau^4 + \widetilde{N}^{3-r})$. We thus have $\|\vec{\xi}\|, \|\vec{\eta}\| \le C(\tau^4 + \widetilde{N}^{3-r})$, and further obtain $\|\vec{\xi}\|_{\widetilde{N}}, \|\vec{\eta}\|_{\widetilde{N}} \le C(\tau^4 + \widetilde{N}^{3-r})$. Since both the $\mathbf{A}_-$ and $\widetilde{\mathbf{A}}_-$ operators are skew-adjoint, it follows from (5.18) that

(5.19)
$$\left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{(1)} - \frac{l_1\tau}{2\sqrt{\varepsilon}}\vec{\xi}\right\|_{\widetilde{N}}^2 + \left\|\sqrt{\mu}\vec{\mathcal{H}}^{(1)} - \frac{l_1\tau}{2\sqrt{\mu}}\vec{\eta}\right\|_{\widetilde{N}}^2 = \left\|\sqrt{\varepsilon}\vec{\mathcal{E}}^{n} + \frac{l_1\tau}{2\sqrt{\varepsilon}}\vec{\xi}\right\|_{\widetilde{N}}^2 + \left\|\sqrt{\mu}\vec{\mathcal{H}}^{n} + \frac{l_1\tau}{2\sqrt{\mu}}\vec{\eta}\right\|_{\widetilde{N}}^2.$$

From the qth stage ($q = 2, 3, 4, 5, 6, 7$) of the S-AVF(4) scheme, we can also obtain a similar energy equality to (5.19). Next as in the proof of Theorem 5.4, we have $\sqrt{\varepsilon}\|\vec{\mathcal{E}}^n\|_{\widetilde{N}} + \sqrt{\mu}\|\vec{\mathcal{H}}^n\|_{\widetilde{N}} \le C(\tau^4 + \widetilde{N}^{3-r})$. Note that the fourth-order splitting method (2.5) implies $\|\vec{E}(t_n) - \underline{\vec{E}}(t_n)\|_{\widetilde{N}}, \|\vec{H}(t_n) - \underline{\vec{H}}(t_n)\|_{\widetilde{N}} \le C\tau^4$. Substituting these results into (5.14), we complete the proof. ∎

Remark. (1) From the Theorems 5.4 and 5.5, one can see that the convergence order in space increases with the smoothness of the solutions $\vec{E}$ and $\vec{H}$. (2) The S-AVF(4) scheme needs stronger regularities of $\vec{E}$ and $\vec{H}$ than the S-AVF(2) scheme.

6. Identities for the divergence terms. For the Maxwell's equations (1.1), there exist two divergence-free fields (1.2). The following analysis shows that the S-AVF(2) and S-AVF(4) schemes cannot preserve divergence, but have second- and fourth-order approximations to the divergence-free fields.

Denote $\overline{\mathbf{div}} = (A_x, A_y, A_z)$ by the discrete divergence operator. It is clear that $\overline{\mathbf{div}}\left((A_- + A_+)\vec{U}\right) = 0$, $U = E, H$. Let $u = (\sqrt{\varepsilon}\vec{E}, \sqrt{\mu}\vec{H})$ and

$$\underline{A}_- = c\tau \begin{pmatrix} 0 & A_- \\ -A_-^T & 0 \end{pmatrix}, \quad \underline{A}_+ = c\tau \begin{pmatrix} 0 & A_+ \\ -A_+^T & 0 \end{pmatrix}.$$

Thus the schemes (3.7)–(3.9) can be rewritten as $u^{(1)} - u^n = \frac{\underline{A}_-}{4}(u^{(1)} + u^n)$, $u^{(2)} - u^{(1)} = \frac{\underline{A}_+}{2}(u^{(2)} + u^{(1)})$, and $u^{n+1} - u^{(2)} = \frac{\underline{A}_-}{4}(u^{n+1} + u^{(2)})$, respectively. By introducing the temporary variables $\overline{\vec{E}}^n = \frac{\vec{E}^{(2)} + \vec{E}^{(1)}}{2}$, $\overline{\vec{H}}^n = \frac{\vec{H}^{(2)} + \vec{H}^{(1)}}{2}$, and $\bar{u}^n = \left((\overline{\vec{E}}^n)^T, (\overline{\vec{H}}^n)^T\right)^T$, we have

$$\left(I - \frac{\underline{A}_-}{4}\right)\left(I - \frac{\underline{A}_+}{2}\right)\bar{u}^n = \left(I + \frac{\underline{A}_-}{4}\right)u^n, \tag{6.1}$$

$$\left(I + \frac{\underline{A}_-}{4}\right)\left(I + \frac{\underline{A}_+}{2}\right)\bar{u}^n = \left(I - \frac{\underline{A}_-}{4}\right)u^{n+1}. \tag{6.2}$$

Multiplying $\left(I - \frac{\underline{A}_-}{4}\right)$ on both sides of (6.1) and $\left(I + \frac{\underline{A}_-}{4}\right)$ on both sides of (6.2), respectively, we obtain

$$u^{n+1} - u^n = \frac{\underline{A}_-^2}{16}(u^{n+1} - u^n) + (\underline{A}_- + \underline{A}_+)\bar{u}^n + \frac{\underline{A}_-^2\underline{A}_+}{16}\bar{u}^n. \tag{6.3}$$

Left multiplying both side of (6.3) with $\overline{\mathbf{div}}$ yields the divergence relationships

$$\overline{\mathbf{div}}(\varepsilon\vec{E}^n) - \frac{\tau^2}{16\mu}\overline{\mathbf{div}}(A_-A_-^T\vec{E}^n) = \overline{\mathbf{div}}(\varepsilon\vec{E}^0) - \frac{\tau^2}{16\mu}\overline{\mathbf{div}}(A_-A_-^T\vec{E}^0) + S_1, \tag{6.4}$$

$$\overline{\mathbf{div}}(\mu\vec{H}^n) - \frac{\tau^2}{16\varepsilon}\overline{\mathbf{div}}(A_-^TA_-\vec{H}^n) = \overline{\mathbf{div}}(\mu\vec{H}^0) - \frac{\tau^2}{16\varepsilon}\overline{\mathbf{div}}(A_-^TA_-\vec{H}^0) + S_2, \tag{6.5}$$

where $S_1 = \frac{c^2\tau^3}{16}\sum_{l=1}^n \overline{\mathbf{div}}(A_-A_-^TA_+\overline{\vec{H}}^l)$ and $S_2 = \frac{c^2\tau^3}{16}\sum_{l=1}^n \overline{\mathbf{div}}(A_-^TA_-A_+^T\overline{\vec{E}}^l)$. Here the identities in (6.4) and (6.5) depend on the intermediate variables. From (6.4) and (6.5), we know the S-AVF(2) scheme has second-order approximation to the divergence-free fields. For the S-AVF(4) scheme, it is difficult to derive the identities for the divergence terms. In section 8, numerical results show that the S-AVF(4) scheme has fourth-order approximation to the divergence-free fields.

7. Fast computation. Now we further improve the computational efficiency of the S-AVF(2) and S-AVF(4) schemes. In fact, both schemes can be rewritten in fully explicit forms by using the relationship[1]

$$B_1^w = \mathcal{F}_{N_w}^{-1}B_{N_w}\mathcal{F}_{N_w}, \quad w = x, y, \text{ or } z, \tag{7.1}$$

where $\mathcal{F}_{N_w}$ is the matrix of discrete Fourier transform coefficients with entries given by $[\mathcal{F}_{N_w}]_{r,s} = \omega_{N_w}^{rs}$, $\omega_{N_w} = e^{-l2\pi/N_w}$, $[\mathcal{F}_{N_w}^{-1}]_{r,s} = \frac{1}{N_w}\omega_{N_w}^{-rs}$, $l = \sqrt{-1}$, and B_{N_w} is a diagonal matrix whose (nonzero) entries are the scaled wave numbers

$$\mathrm{diag}(B_{N_w}) = l\mu_w\left[0, 1, \cdots, \frac{N_w}{2} - 1, 0, -\frac{N_w}{2} + 1, \cdots, -2, -1\right].$$

Next we take the S-AVF(4) scheme as an example to show its explicit form. In the scheme (3.15), an application of the relationship (7.1) yields $-\underline{B_1}^x = \mathcal{F}_{N_x}^{-1}\Lambda_x\mathcal{F}_{N_x}$, $\Lambda_x = B_{N_x} - \frac{(cl_1\tau)^2}{12}B_{N_x}^3$. Further, we have

$$\begin{pmatrix} \frac{2\varepsilon}{l_1\tau}I & \pm\overline{B_1}^x \\ \pm\underline{B_1}^x & \frac{2\mu}{l_1\tau}I \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{N_x}^{-1} & 0 \\ 0 & \mathcal{F}_{N_x}^{-1} \end{pmatrix}\begin{pmatrix} \frac{2\varepsilon}{l_1\tau}I & \pm\Lambda_x \\ \pm\Lambda_x & \frac{2\mu}{l_1\tau}I \end{pmatrix}\begin{pmatrix} \mathcal{F}_{N_x} & 0 \\ 0 & \mathcal{F}_{N_x} \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{2\varepsilon}{l_1\tau}I & -B_1^x \\ -\underline{B_1^x} & \frac{2\mu}{l_1\tau}I \end{pmatrix}^{-1} = \begin{pmatrix} \mathcal{F}_{N_x}^{-1} & 0 \\ 0 & \mathcal{F}_{N_x}^{-1} \end{pmatrix}\begin{pmatrix} \frac{2\mu}{l_1\tau}I & -\Lambda_x \\ -\Lambda_x & \frac{2\varepsilon}{l_1\tau}I \end{pmatrix}\begin{pmatrix} \widetilde{\Lambda}_x & 0 \\ 0 & \widetilde{\Lambda}_x \end{pmatrix}\begin{pmatrix} \mathcal{F}_{N_x} & 0 \\ 0 & \mathcal{F}_{N_x} \end{pmatrix},$$

where I is an identity matrix and $\widetilde{\Lambda}_x = (\frac{4\varepsilon\mu}{(l_1\tau)^2}I - \Lambda_x^2)^{-1}$ is a diagonal matrix. Now it follows from (3.15) that

$$\begin{pmatrix} \vec{E}_{y,j,k}^{(1)} \\ \vec{H}_{z,j,k}^{(1)} \end{pmatrix} = \begin{pmatrix} \mathcal{F}_{N_x}^{-1}C_x\mathcal{F}_{N_x}\vec{E}_{y,j,k}^n + \mu\mathcal{F}_{N_x}^{-1}\widetilde{C}_x\mathcal{F}_{N_x}\vec{H}_{z,j,k}^n \\ \varepsilon\mathcal{F}_{N_x}^{-1}\widetilde{C}_x\mathcal{F}_{N_x}\vec{E}_{y,j,k}^n + \mathcal{F}_{N_x}^{-1}C_x\mathcal{F}_{N_x}\vec{H}_{z,j,k}^n \end{pmatrix}, \tag{7.2}$$

where both $C_x = \frac{4\varepsilon\mu}{(l_1\tau)^2}\widetilde{\Lambda}_x + \Lambda_x^2\widetilde{\Lambda}_x$ and $\widetilde{C}_x = -\frac{4}{l_1\tau}\Lambda_x\widetilde{\Lambda}_x$ are diagonal matrices. Similarly, (3.16) and (3.17) can be also rewritten in explicit forms. The above method is also applicable to the schemes in stages 2 to 7.

[1] In practical problem, we may use $\mathcal{F}_{N_w}^{-1}B_{N_w}\mathcal{F}_{N_w}\vec{u}$ instead of $B_1^w\vec{u}$. The computational cost will decrease to $\mathcal{O}(N\log N)$. Besides, $B_{N_w}^{-1}$ can be expressed explicitly, i.e., $B_{N_w}^{-1} = \mathcal{F}_{N_w}^{-1}B_{N_w}^{-1}\mathcal{F}_{N_w}$.

8. Numerical experiments. In this section, we present numerical experiments to show the performance of the S-AVF(2) and S-AVF(4) schemes. Maxwell's equations (1.1) have traveling wave solutions ($\varepsilon = \mu = 1$)

$$\begin{aligned} &E_x = \cos(2\pi(x+y+z) - 2\sqrt{3}\pi t), \quad E_y = -2E_x, \\ &E_z = E_x, \quad H_x = \sqrt{3}E_x, \quad H_y = 0, \quad H_z = -\sqrt{3}E_x, \end{aligned} \tag{8.1}$$

and also possess standing wave solutions

$$\begin{aligned} E_x &= \frac{k_y - k_z}{\varepsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\cos(k_x\pi x)\sin(k_y\pi y)\sin(k_z\pi z), \\ E_y &= \frac{k_z - k_x}{\varepsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\sin(k_x\pi x)\cos(k_y\pi y)\sin(k_z\pi z), \\ E_z &= \frac{k_x - k_y}{\varepsilon\sqrt{\mu}\omega}\cos(\omega\pi t)\sin(k_x\pi x)\sin(k_y\pi y)\cos(k_z\pi z), \\ H_x &= \sin(\omega\pi t)\sin(k_x\pi x)\cos(k_y\pi y)\cos(k_z\pi z), \\ H_y &= \sin(\omega\pi t)\cos(k_x\pi x)\sin(k_y\pi y)\cos(k_z\pi z), \\ H_z &= \sin(\omega\pi t)\cos(k_x\pi x)\cos(k_y\pi y)\sin(k_z\pi z), \end{aligned} \tag{8.2}$$

where $\omega^2 = (k_x^2 + k_y^2 + k_z^2)/(\varepsilon\mu)$.

Example I. We take the analytical solutions (8.1) at $T = 0$ on the unit cube $[0,1]^3$ as the initial condition.

Example II. We consider the initial values obtained from (8.2) with $T = 0$, $\Omega = [0,2]^3$, $k_x = 1$, $k_y = 2$, $k_z = -3$, and $\varepsilon = \mu = 1$. All simulations are done with $\tau = 0.005$ and spatial grid points $N_x = N_y = N_z = 16$ up to $T = 10$.

8.1. The error in the solution. In our simulations, we use norms defined by

$$L_\infty = \max\left\{\varepsilon \max_{i,j,k}\left|\vec{E}(t_n) - \vec{E}^n\right|, \mu \max_{i,j,k}\left|\vec{H}(t_n) - \vec{H}^n\right|\right\},$$

and

$$L_2 = \left(\varepsilon\left\|\vec{E}(t_n) - \vec{E}^n\right\|_{\widetilde{N}}^2 + \mu\left\|\vec{H}(t_n) - \vec{H}^n\right\|_{\widetilde{N}}^2\right)^{\frac{1}{2}},$$

to scale the maximal and average errors in solution, respectively. The numerical errors are listed in Tables 1 and 2. From the tables, we see that both the S-AVF(2) and S-AVF(4) schemes simulate the wave propagation well. We also see that the errors in the solution obtained by both S-AVF(2) and S-AVF(4) grow as time evolves, while S-AVF(4) provides more accurate solutions than S-AVF(2).

8.2. Energy conservation. In section 4, we have strictly proved that the S-AVF(2) and S-AVF(4) schemes preserve the four discrete energies simultaneously. In this section, the relative changes in energy invariants defined by $\mathrm{Re}(\mathcal{E}_i) = |\mathcal{E}_i^n - \mathcal{E}_i^0|/|\mathcal{E}_i^0|$, $i = 1,2$, and $\mathrm{Re}(\mathcal{E}_{iw}) = |\mathcal{E}_{iw}^n - \mathcal{E}_{iw}^0|/|\mathcal{E}_{iw}^0|$, $i = 3,4$, $w = x,y,z$, are monitored. For the two numerical experiments, the relative errors are displayed in Tables 3 and 4. Both the S-AVF(2) and S-AVF(4) schemes preserve the energy invariants exactly since the relative errors are within the roundoff error of the machine. The results support the theoretical analysis.

TABLE 1
The error in the solution at different times for Example I: $\tau = 0.005$, $N_x = N_y = N_z = 16$.

	S-AVF(2)		S-AVF(4)	
Time	L_∞	L_2	L_∞	L_2
T=1	2.0010e-3	2.5122e-3	4.3833e-6	5.4082e-6
T=2	3.9723e-3	4.9342e-3	8.6997e-6	1.0806e-5
T=5	9.9179e-3	1.2339e-2	2.1734e-5	2.7014e-5
T=8	1.5993e-2	1.9736e-2	3.5050e-5	4.3222e-5
T=10	2.0118e-2	2.4669e-2	4.4057e-5	5.4028e-5

TABLE 2
The error in the solution at different times for Example II: $\tau = 0.005$, $N_x = N_y = N_z = 16$.

	S-AVF(2)		S-AVF(4)	
Time	L_∞	L_2	L_∞	L_2
T=1	1.5350e-3	2.7116e-3	2.1955e-6	3.8584e-6
T=2	4.0608e-3	5.3475e-3	6.0233e-6	7.7738e-6
T=5	7.9894e-3	1.3396e-2	1.2376e-5	1.9392e-5
T=8	1.6146e-2	2.1394e-2	2.5189e-5	3.1053e-5
T=10	1.9478e-2	2.6709e-2	2.9554e-5	3.8845e-5

TABLE 3
The relative errors in energy invariants for Example I: $\tau = 0.005$, $N_x = N_y = N_z = 16$.

Method	Time	$\mathrm{Re}(\mathcal{E}_1)$	$\mathrm{Re}(\mathcal{E}_2)$	$\mathrm{Re}(\mathcal{E}_{3x})$	$\mathrm{Re}(\mathcal{E}_{4x})$
S-AVF(2)	T=1	3.4639e-14	3.3131e-14	1.1604e-14	1.3470e-12
	T=2	6.8390e-14	6.6902e-14	2.2968e-14	2.6838e-12
	T=5	1.7260e-13	1.7270e-13	5.7619e-14	6.8733e-12
	T=8	2.7859e-13	2.7833e-13	9.2991e-14	1.1027e-11
	T=10	3.5024e-13	3.4892e-13	1.1688e-13	1.3813e-11
S-AVF(4)	T=1	3.2715e-13	3.2773e-13	1.0910e-13	1.2915e-11
	T=2	6.5562e-13	6.5626e-13	2.1863e-13	2.5891e-11
	T=5	1.6436e-12	1.6439e-12	5.4804e-13	6.4875e-11
	T=8	2.6309e-12	2.6314e-12	8.7729e-13	1.0385e-10
	T=10	3.2897e-12	3.2895e-12	1.0968e-12	1.2983e-10

TABLE 4
The relative errors in energy invariants for Example II: $\tau = 0.005$, $N_x = N_y = N_z = 16$.

Method	Time	$\mathrm{Re}(\mathcal{E}_1)$	$\mathrm{Re}(\mathcal{E}_2)$	$\mathrm{Re}(\mathcal{E}_{3x})$	$\mathrm{Re}(\mathcal{E}_{4x})$
S-AVF(2)	T=1	2.2501e-14	2.1953e-14	1.5950e-15	2.1843e-13
	T=2	4.0856e-14	4.1573e-14	2.9156e-15	4.0832e-13
	T=5	1.0258e-13	9.8512e-14	7.3318e-15	9.7250e-13
	T=8	1.6357e-13	1.6245e-13	1.1688e-14	1.6014e-12
	T=10	2.0206e-13	2.0443e-13	1.4449e-14	2.0174e-12
S-AVF(4)	T=1	2.2826e-13	2.2345e-13	1.6272e-14	2.2068e-12
	T=2	4.5075e-13	4.5060e-13	3.2158e-14	4.4498e-12
	T=5	1.1283e-12	1.1227e-12	8.0570e-14	1.1082e-11
	T=8	1.8015e-12	1.7983e-12	1.2868e-13	1.7751e-11
	T=10	2.2487e-12	2.2518e-12	1.6064e-13	2.2224e-11

8.3. Convergence order. In section 5, we establish the error estimates for the S-AVF(2) and S-AVF(4) schemes. The regularities of (8.1) and (8.2) are infinite so that the solutions of both the S-AVF(2) and S-AVF(4) schemes converge with infinite-order accuracy in space. In the following, we only test the convergence orders in time. We implement the two schemes with various time steps up to $T = 1$. Figure 1 is a logarithm-logarithm plot. The slopes of the lines confirm the temporal convergence orders of S-AVF(2) and S-AVF(4) are 2 and 4, respectively.

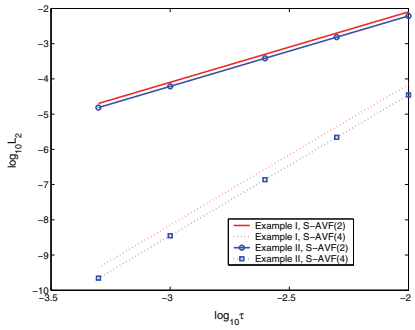

FIG. 1. *Convergence order of numerical solution in time for the S-AVF(2) and S-AVF(4) schemes.*

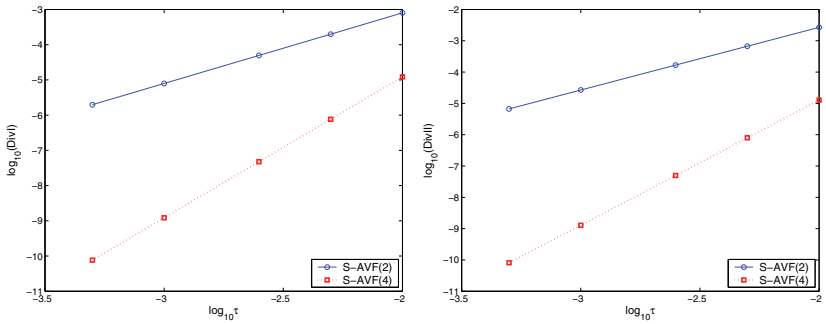

FIG. 2. *Convergence order of divergence of the S-AVF(2) and S-AVF(4) schemes. Left:* DivI*; right:* DivII.

8.4. The convergence rate of numerical divergence. We will check whether S-AVF(2) and S-AVF(4) preserve the divergence-free properties well for both the electric and magnetic fields, which means we need to compute the values of DivI $= \|\varepsilon(A_x\vec{E}_x^n + A_y\vec{E}_y^n + A_z\vec{E}_z^n)\|_\infty$ and DivII $= \|\mu(A_x\vec{H}_x^n + A_y\vec{H}_y^n + A_z\vec{H}_z^n)\|_\infty$ at $T = 1$. In section 6, we prove the S-AVF(2) scheme has second-order approximation to the divergence-free fields. This theoretical analysis is verified by the slopes of the blue solid lines shown in Figure 2. For the S-AVF(4) scheme, we cannot obtain the convergence order of the divergence-free fields theoretically. The slopes of the red dotted lines displayed in Figure 2 are about 4, which shows that the S-AVF(4) scheme has fourth-order approximation to the divergence-free fields. Although the proposed schemes are not divergence-free, both of them preserve divergence well.

8.5. Comparison with some existing methods. First, we make numerical comparisons of the splitting schemes of order 2 including S-AVF(2), the ADI-FDTD method [5], and the energy conserved splitting FDTD (EC-S-FDTD) methods EC-S-FDTD-II-1 and EC-S-FDTD-II-2 [9]. Consider the simulation of Example II with $\tau = h_x = h_y = h_z$. The results are displayed in Table 5. It is clear that the S-AVF(2) scheme performs better than the others in both accuracy of solution and convergence order.

Second, Zhu, Song, and Chen [25] proposed a second-order implicit energy-preserving multisymplectic wavelet collocation (MSWC) scheme (a nonsplitting scheme) for the three-dimensional Maxwell's equations, and simulated Example I

TABLE 5
Numerical comparison of the second-order splitting methods: $\tau = h_x = h_y = h_z$, $T = 1$.

	$\tau = 0.1$		$\tau = 0.05$		$\tau = 0.025$	
Method	L_2	Order	L_2	order	L_2	Order
S-AVF(2)	1.56e-1	-	3.74e-2	2.06	9.53e-3	1.97
ADI-FDTD	6.10e-1	-	1.76e-1	1.79	4.53e-2	1.95
EC-S-FDTD-II-1	5.05e-1	-	1.41e-1	1.84	3.62e-2	1.96
EC-S-FDTD-II-2	5.59e-1	-	1.61e-1	1.79	4.15e-2	1.95

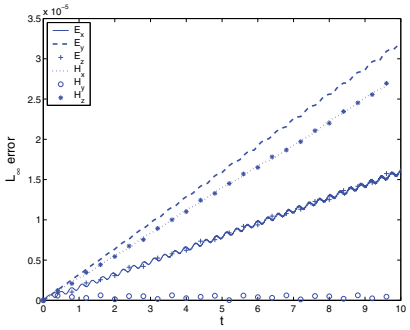

FIG. 3. *The maximal errors in all components of the solution, obtained by the S-AVF(2) scheme.*

with $\tau = 0.0002$ and $N_x = N_y = N_z = 32$. Figure 3 shows the maximal errors in all components of the solution of S-AVF(2). By comparing the figure with the ones in [25, p. 684], one can see that the S-AVF(2) scheme provides more accurate solutions than the MSWC scheme. In addition, the S-AVF(2) scheme is more efficient than the MSWC scheme. Sun and Tse also proposed some symplectic and multisymplectic finite difference schemes in [26]. The schemes are second-order, local energy preserving, and divergence preserving, but require lots of computing resources.

9. Conclusions. Our main objective is to develop high-order accurate and efficient schemes for the three-dimensional time-domain Maxwell's equations. To achieve this, we decompose the Maxwell's equations into a series of one-dimensional subproblems by using the exponential operator splitting technique. The application of the energy-preserving numerical method to each subsystem leads to the energy-conserving S-AVF(2) and S-AVF(4) schemes. Next we establish the error estimates. To improve the computational efficiency of the present methods, we rewrite them in the fully explicit forms by using the DFFT technique. The numerical experiments show the good performance of the S-AVF(2) and S-AVF(4) schemes.

In the present paper, the S-AVF(4) scheme is derived based on the fourth-order splitting method (2.5), which is a Yoshida's composition [18] of the second-order symmetric splitting method (2.3). An application of Suzuki's idea [20] to (2.3) will give another fourth-order splitting method, in which the two equations in (2.8) are repeatedly advanced by the time increments $\{\frac{r_1}{2}, r_1, r_1, r_1, \frac{r_1+r_2}{2}, r_2, \frac{r_1+r_2}{2}, r_1, r_1, r_1, \frac{r_1}{2}\}$, where $r_1 = \frac{\tau}{4-4^{1/3}}$ and $r_2 = \frac{4^{1/3}\tau}{4-4^{1/3}}$. It is an eleven-stage splitting method, in which all subproblems hold the same Hamiltonian $\mathcal{H}$. Applications of the Fourier pseudospectral method in space and the fourth-order AVF method for every subproblem will yield a new energy-preserving scheme (Suzuki-S-AVF(4)). Tables 6 and 7 list the numerical results obtained by Suzuki-S-AVF(4). One can see that the performance of Suzuki-S-AVF(4) is much better than that of S-AVF(4). Making use of the DFFT, the Suzuki-S-AVF(4) scheme can be also rewritten in an efficiently explicit form.

TABLE 6

Numerical results obtained by the Suzuki-S-AVF(4) scheme for Example II: $N_x = N_y = N_z = 16$, $T = 1$.

τ	L_2	Order	Div I	Order	Div II	Order
1.0e-2	9.2371e-7	-	5.3543e-8	-	8.2385e-8	-
5.0e-3	5.7777e-8	3.9989	3.3482e-9	3.9992	5.1539e-9	3.9986
2.5e-3	3.6118e-9	3.9997	2.0947e-10	3.9986	3.2257e-10	3.9980
1.0e-3	9.2493e-11	3.9996	5.7789e-12	3.9184	8.8303e-12	3.9268

TABLE 7

The relative errors in energy invariants for Example II, obtained by the Suzuki-S-AVF(4): $\tau = 0.005$, $N_x = N_y = N_z = 16$.

Time	$\lvert\mathcal{E}_1^n - \mathcal{E}_1^0\rvert/\lvert\mathcal{E}_1^0\rvert$	$\lvert\mathcal{E}_2^n - \mathcal{E}_2^0\rvert/\lvert\mathcal{E}_2^0\rvert$	$\lvert\mathcal{E}_{3_x}^n - \mathcal{E}_{3_x}^0\rvert/\lvert\mathcal{E}_{3_x}^0\rvert$	$\lvert\mathcal{E}_{4_x}^n - \mathcal{E}_{4_x}^0\rvert/\lvert\mathcal{E}_{4_x}^0\rvert$
T=1	8.5265e-14	8.8200e-14	6.0869e-15	8.6801e-13
T=2	1.7245e-13	1.7270e-13	1.2319e-14	1.7042e-12
T=5	4.2721e-13	4.2701e-13	3.0512e-14	4.2149e-12
T=8	6.8390e-13	6.8406e-13	4.8841e-14	6.7509e-12
T=10	8.5369e-13	8.5374e-13	6.0980e-14	8.4244e-12

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