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Exponential integrators preserving local conservation laws of PDEs with time-dependent damping/driving forces

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ABSTRACT

Structure-preserving algorithms for solving conservative PDEs with added linear dissipation are generalized to systems with time dependent damping/driving terms. This study is motivated by several PDE models of physical phenomena, such as Korteweg–de Vries, Klein–Gordon, Schrödinger, and Camassa–Holm equations, all with damping/driving terms and time-dependent coefficients. Since key features of the PDEs under consideration are described by local conservation laws, which are independent of the boundary conditions, the proposed (second-order in time) discretizations are developed with the intent of preserving those local conservation laws. The methods are respectively applied to a damped-driven nonlinear Schrödinger equation and a damped Camassa–Holm equation. Numerical experiments illustrate the structure-preserving properties of the methods, as well as favorable results over other competitive schemes.

1. Introduction

When modeling physical phenomena, it is important to maintain as many properties of the physical system as possible. Numerical models that fail to maintain aspects of the physical behavior are actually modeling something other than the intended physics. This idea has been instrumental in the development of structure-preserving algorithms for solving differential equations that model conservative dynamics. Many successful time-stepping schemes for solving conservative differential equations are constructed to ensure that aspects of the qualitative solution behavior are preserved. However, many physical systems are commonly effected by external forces or by the dissipative effects of friction, and methods that preserved the structure in standard conservative systems fail to preserve the structure when damping/driving forces are included in the model.

The need for numerical methods that preserve dissipative properties of physical systems has resulted in the development of some useful discretization techniques. In some cases, guaranteeing monotonic dissipation of a desired quantity (numerically) may be sufficient (cf. [1]), but in this work we focus on a special form of linear damping that can be preserved more precisely. Numerical methods that preserve the conformal symplectic structure of conformal Hamiltonian systems [2] are naturally known as conformal symplectic methods, and they were first constructed for ODEs using splitting techniques [3,4], by solving the dissipative part exactly and the conservative part with a symplectic method, and then composing the flow maps. In subsequent work, various second order conformal symplectic methods were proposed [5,6], which included linear stability analysis, backward error analysis, additional results on structure-preservation, and favorable numerical results for both ODE and PDE examples.

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In fact, there are several works that use conformal symplectic integrators to construct structure preserving methods for PDEs. Conformal symplectic Euler methods were used to solve damped nonlinear wave equations [7]. Then, the ideas were extended to general linearly damped multi-symplectic PDEs, with application to damped semi-linear waves equations and damped NLS equations, where midpoint-type discretizations (similar to the Preissmann box scheme and midpoint discrete gradient methods) were shown to preserve various dissipative properties of the governing equations [8]. Other similar methods have been developed based on an even–odd Strang splitting for a modified (non-diffusive) Burger's equation [5], the average vector field method for a damped NLS equation [9], and an explicit fourth-order Nyström method with composition techniques for damped acoustic wave equations [10].

More recently, conformal symplectic integration has been generalized in a few different ways. All of the aforementioned works on methods for PDEs have hinged on conformal symplectic time discretizations, but the ideas have been fully generalized to PDEs, including conformal symplectic spatial discretizations, with several example model problems [11]. In the context of conformal Hamiltonian ODEs, general classes of exponential Runge–Kutta methods and partitioned Runge–Kutta methods provide higher order methods, where sufficient conditions on the coefficient functions guarantee preservation of conformal symplecticity or other quadratic conformal invariants [12]. The framework proposed there also provides straightforward approaches to constructing conformal symplectic methods, using a Lawson-type transformation. As shown in [12], many of the conformal symplectic exponential Runge–Kutta methods are applicable to differential equations with time-dependent linear damping, generalizing previous work on conformal symplectic methods, which only allows constant coefficients on the damping terms.

The aim of the present article is to develop structure-preserving methods (multi-conformal-symplectic and/or momentum-preserving) for PDEs, which have damping coefficients that are necessarily functions of time and may be subject to external or parametric forcing. Similar to [7], we consider a multi-symplectic PDE with additional (linear) forcing and damping terms

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z,t) - a(t)\mathbf{K}z + \mathbf{F}(x,t), \tag{1}$$

where $\mathbf{F}$ is an external driving force and subscripts denote usual derivatives, but in this case we allow the damping coefficient $a(t)$ and the smooth function S to depend on time. The corresponding variational equation takes the form

$$\mathbf{K}dz_t + \mathbf{L}dz_x = S_{zz}(z,t)dz - a(t)\mathbf{K}dz$$

where $S_{\zeta\zeta}$ is the Hessian. Taking the wedge product with dz and multiplying through by $e^{\theta(t)}$ with $\theta(t) := \int_0^t a(s)ds$ yields

$$\partial_t \left(e^{2\theta(t)}(dz \wedge \mathbf{K}dz)\right) + \partial_x \left(e^{2\theta(t)}(dz \wedge \mathbf{L}dz)\right) = 0, \tag{2}$$

which is a multi-conformal-symplectic conservation law [11]. We call a discretization of (1) *multi-conformal-symplectic* if it satisfies a discrete version of (2). Other similar conservation laws may exist in the special case $\mathbf{F}(x,t) = 0$. Taking the inner product of the equation

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z,t) - a(t)\mathbf{K}z, \tag{3}$$

with z_x [8], and then multiplying the result through by $e^{\theta(t)}$ yields a conformal momentum conservation law

$$\partial_t \left(e^{2\theta(t)}\left(\frac{1}{2}z^T\mathbf{K}z_x\right)\right) + \partial_x \left(e^{2\theta(t)}\left(\frac{1}{2}z_t^T\mathbf{K}z + S(z,t)\right)\right) = 0. \tag{4}$$

Finally, if $\mathbf{B}$ defines an action under which the system (3) is invariant in the special case $a(t) = 0$, so that $(\mathbf{B}z)^T\nabla S(z,t) = 0$, then solutions of the system with any $a(t)$ also satisfy

$$\partial_t \left(e^{2\theta(t)}z^T\mathbf{KB}z\right) + \partial_x \left(e^{2\theta(t)}z^T\mathbf{LB}z\right) = 0. \tag{5}$$

We say a numerical method preserves a conservation law if it satisfies a discrete version of it.

A few numerical schemes that preserve the conformal structures of PDEs with parametric and/or external forcing and damping coefficients that depend on time are presented in Section 2. Specific methods are applied to damped-driven NLS and damped Camassa–Holm equations in Section 3, where multi-conformal-symplectic methods are used to demonstrate advantageous numerical results.

2. Structure-preserving integrators

Here, we present our main contributions. First, a general framework for constructing structure-preserving algorithms for (1) and (3), is given. With this framework, the construction of a method begins with a known structure-preserving algorithm for a conservative PDE, which we call the *underlying scheme*, and follows the approach of Lawson [13] for constructing exponential integrators. Then, we present three discretization methods, which result from this framework, and show that each method preserves one or more of the conservation laws (2), (4), and/or (5). The three methods presented only differ with regard to the three spatial discretizations that are used, which are all standard for structure-preserving schemes. Each of the three methods uses the same time-stepping scheme, which is an exponential integrator with exponentials that depend explicitly on time.

2.1. Constructing a scheme

There are two main approaches for constructing conformal symplectic methods for solving conformal Hamiltonian ODEs. The first, proposed by McLachlan and Quispel [3,4], is a splitting approach in which the vector field is split into a Hamiltonian part and a linearly damped part. Then the exact flow map of the damped equation is composed with a symplectic flow map that is used to solve the conservative equation, resulting in a flow map that is conformal symplectic. Another approach relies on choosing the coefficient functions of an exponential Runge–Kutta method (or a partitioned exponential Runge–Kutta method) in a way that guarantees preservation of conformal symplecticity [12]. From another point of view, a Birkhoffian formulation of the governing equations leads to schemes that preserve many of the same properties [14]. Each approach has advantages and lead to many of the same methods that have been studied in previous works.

Approaches for constructing conformal symplectic integrators are based on an underlying symplectic method, which results in the absence of damping/driving forces, i.e. $a = 0$ and $\mathbf{F} = \mathbf{0}$. Unfortunately, the resulting conformal symplectic method is not guaranteed to maintain the order of the underlying method (cf. [8,15]). Though splitting methods can be constructed in such a way that the order is known, a simpler approach is to apply a change of variables, similar to a Lawson transformation [13], and use multi-symplectic method on the transformed equation. This approach was used to construct structure-preserving integrating-factor methods for systems of ODEs [12]. For the PDE (1) the approach can be delineated as follows. Introduce the change of variables $\zeta = e^{\theta(t)}z$ where $\theta(t) := \int_0^t a(s)ds$, and write Eq. (1) as

$$\mathbf{K}\zeta_t + \mathbf{L}\zeta_x = \nabla\tilde{S}(\zeta, t) + \mathbf{G}(x, t), \tag{6}$$

where $\mathbf{G}(x, t) = e^{\theta(t)}\mathbf{F}(x, t)$ and $\tilde{S}(\zeta, t) = e^{\theta(t)}S(e^{-\theta(t)}\zeta, t)$. Then, a structure-preserving algorithm, which is the so-called underlying method, may be applied to this transformed equation. It is expected that transforming back to the original variables will result in methods that preserve conformal versions of the properties, provided they exist, and the order of the underlying method is maintained, in general.

Here, we present discretization methods that preserve one or more of the conservation laws (2), (4), and (5), and each of the methods can be constructed using the approach delineated in the previous paragraph. If structure preservation is important to the practitioner who is solving equations (1) or (3), then it is essential to use an exponential method for the time discretization, due to the damping/driving terms. Indeed, there are many advantageous possibilities provided by a special class of exponential Runge–Kutta methods [12]. For our purposes, we consider only one time discretization. Let $t_{i+1} = t_i + \Delta t$ and define

$$A^a_{t_i} z^i = \frac{1}{2}\left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} z^{i+1} + e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} z^i\right)$$

and

$$D^a_{t_i} z^i = \frac{1}{\Delta t}\left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} z^{i+1} - e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} z^i\right).$$

where z^i denotes an approximation of $z(x, t_i)$. An exponential time-discretization, used here to solve (1), is given by

$$\mathbf{K}D^a_{t_i} z^i + \mathbf{L}A^a_{t_i} z^i = \nabla_z S(A^a_{t_i} z^i, t_{i+1/2}) + \mathbf{F}(x, t_{i+1/2}).$$

Since the scheme reduces to implicit midpoint in the case $a = 0$, the underlying method here is the implicit midpoint rule. In the case $a > 0$ is constant, the time discretization reduces to the conformal implicit midpoint rule, which has been shown to be second order, unconditionally stable, and structure-preserving for a damped semi-linear wave equation and a damped nonlinear transport equation [5].

Many standard methods can be used to discretize (1) or (3) in space. In fact, we expect that the spatial discretization will guarantee preservation of the desired conservation law, as long as it preserves the same conservation law in the absence of damping/driving forces, meaning finite-element, pseudospectral, and finite-difference, methods are all possibilities. But, as a introduction to structure-preserving discretizations for PDEs with time-dependent damping/driving forces, we present only three finite-difference spatial discretizations in the following subsections. Thus, we define $x_{n+1} = x_n + \Delta x$ with

$$z^i_{n+1/2} = \frac{z^i_{n+1} + z^i_n}{2}, \qquad \delta_x^- z^i_n = \frac{z^i_n - z^i_{n-1}}{\Delta x}, \qquad \text{and} \qquad \delta_x^+ z^i_n = \frac{z^i_{n+1} - z^i_n}{\Delta x}.$$

In space, we use (i) symplectic Euler, which is the spatial discretization used in the Euler Box scheme [7,16,17]; (ii) implicit midpoint, which is the spatial discretization used in the Preissmann box scheme [8,16,18]; and (iii) discrete gradient spatial discretizations [8].

Indeed, the present work signifies an improvement over each of the methods in [7,8], as the time discretizations of these previous works are first order for general nonlinear equations, but the time discretization proposed here are second-order. More importantly, the methods presented here signify generalizations to problems with time-dependent a and S. A primary reason for this generalization hinges on the usefulness of structure-preserving algorithms in long-time simulation. When $a > 0$ is constant, the solutions are damped out, and structure-preserving algorithms are only expected to be advantageous when a is small, meaning the PDE is close to conservative. However, when a depends on time, the solutions may be driven in some time intervals and damped in others, so that long-time simulations become more challenging and the need for structure-preserving methods becomes more substantial.

To show that the discretizations in the following subsections are structure-preserving when a depends on time, it is useful to have a discrete product rule

$$D_{t_i}^{2a}\left[(z^i)^T y^i\right] = \left(D_{t_i}^a z^i\right)^T A_{t_i}^a y^i + \left(A_{t_i}^a z^i\right)^T D_{t_i}^a y^i \tag{7}$$

which follows from

$$\begin{aligned} D_{t_i}^{2a}\left[(z^i)^T y^i\right] =& \frac{1}{\Delta t}\left[e^{\int_{t_{i+1/2}}^{t_{i+1}} 2a(s)ds}(z^{i+1})^T y^{i+1} - e^{-\int_{t_i}^{t_{i+1/2}} 2a(s)ds}(z^i)^T y^i\right] \\ =& \frac{1}{2\Delta t}\left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} z^{i+1} - e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} z^i\right)^T \left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} y^{i+1} + e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} y^i\right) \\ &+ \frac{1}{2\Delta t}\left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} z^{i+1} + e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} z^i\right)^T \left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} y^{i+1} - e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} y^i\right). \end{aligned}$$

In addition, notice that

$$\begin{aligned} \delta_t^+\left(e^{\int_0^{t_i} a(s)ds} y^i\right) =& \frac{1}{\Delta t}\left(e^{\int_0^{t_{i+1}} a(s)ds} y^{i+1} - e^{\int_0^{t_i} a(s)ds} y^i\right) \\ =& \frac{1}{\Delta t} e^{\int_0^{t_{i+1/2}} a(s)ds}\left(e^{\int_{t_{i+1/2}}^{t_{i+1}} a(s)ds} y^{i+1} - e^{-\int_{t_i}^{t_{i+1/2}} a(s)ds} y^i\right). \end{aligned}$$

which leads to a useful relationship between $D_{t_i}^a$ and the standard forward difference operator given by

$$\delta_t^+\left(e^{\theta_i} y^i\right) = e^{\theta_{i+1/2}} D_{t_i}^a y^i \tag{8}$$

where we have introduced $\theta_i := \int_0^{t_i} a(s)ds$ for the sake of convenience.

2.2. Symplectic Euler spatial discretization

Let us rewrite (1) as follows

$$\mathbf{K}z_t + \mathbf{L}_+ z_x + \mathbf{L}_- z_x = \nabla_z S(z,t) - a(t)\mathbf{K}z + \mathbf{F}(x,t), \tag{9}$$

where $\mathbf{L}_+ + \mathbf{L}_- = \mathbf{L}$ and $\mathbf{L}_+ = -\mathbf{L}_-^T$. Now, consider the discretization method

$$\mathbf{K}D_{t_i}^a z_n^i + \mathbf{L}_+ A_{t_i}^a \delta_x^+ z_n^i + \mathbf{L}_- A_{t_i}^a \delta_x^- z_n^i = \nabla_z S(A_{t_i}^a z_n^i, t_{i+1/2}) + \mathbf{F}(x_n, t_{i+1/2}). \tag{10}$$

Here, the underlying method (i.e. the method that results when $a = 0$ and $F = 0$) is a mixed box scheme with implicit midpoint in time and symplectic Euler in space (cf. [19, Chapter 4]).

We now show that (10) is a multi-conformal-symplectic scheme. The variational equation associated with this scheme is

$$\mathbf{K}D_{t_i}^a dz_n^i + \mathbf{L}_+ A_{t_i}^a \delta_x^+ dz_n^i + \mathbf{L}_- A_{t_i}^a \delta_x^+ dz_{n-1} = S_{zz}(A_{t_i}^a z_n^i, t_{i+1/2}) A_{t_i}^a dz_n^i.$$

where S_{zz} is the Hessian of S. Taking the wedge product of $A_{t_i}^a dz_n^i$ with this equation we get

$$A_{t_i}^a dz_n^i \wedge \mathbf{K}D_{t_i}^a dz_n^i + A_{t_i}^a dz_n^i \wedge \mathbf{L}_+ A_{t_i}^a \delta_x^+ dz_n^i + A_{t_i}^a dz_n^i \wedge \mathbf{L}_- A_{t_i}^a \delta_x^+ dz_{n-1} = 0, \tag{11}$$

because $A_{t_i}^a dz_n^i \wedge S_{zz}(A_{t_i}^a z_n^i, t_{i+1/2}) A_{t_i}^a dz_n^i = 0$, due to the symmetry of S_{zz}. Using the discrete product rule, we get

$$A_{t_i}^a dz_n^i \wedge \mathbf{K}D_{t_i}^a dz_n^i = D_{t_i}^{2a}\left(\tfrac{1}{2}(dz_n^i \wedge \mathbf{K}dz_n^i)\right),$$

and

$$A_{t_i}^a dz_n^i \wedge \mathbf{L}_+ A_{t_i}^a \delta_x^+ dz_n^i + A_{t_i}^a dz_n^i \wedge \mathbf{L}_- A_{t_i}^a \delta_x^+ dz_{n-1} = \delta_x^+(A_{t_i}^a dz_{n-1}^i \wedge \mathbf{L}_+ A_{t_i}^a dz_n^i).$$

Then substituting back into (11) gives

$$D_{t_i}^{2a}\left(\tfrac{1}{2}(dz_n^i \wedge \mathbf{K}dz_n^i)\right) + \delta_x^+(A_{t_i}^a dz_{n-1}^i \wedge \mathbf{L}_+ A_{t_i}^a dz_n^i) = 0,$$

which is a discrete version of the multi-conformal-symplectic conservation law, proving that (10) is a multi-conformal-symplectic scheme.

The scheme does not, in general, preserve the other conservation laws, (4) and (5). However, the scheme does preserve other conformal conservation laws in special cases, as is demonstrated in the following section. It is also important to notice that the external forcing plays no role in the variational equation, so any multi-conformal-symplectic scheme will also preserve the conservation law (2) in the presence of external forcing.

2.3. Implicit midpoint spatial discretization

Another method for solving (1) is given by

$$\mathbf{K}D^a_{t_i} z^i_{n+1/2} + \mathbf{L}A^a_{t_i}\delta^+_x z^i_n = \nabla S\left(A^a_{t_i} z^i_{n+1/2}, t_{i+1/2}\right) + \mathbf{F}(x_{n+1/2}, t_{i+1/2}), \tag{12}$$

and the underlying method in this case is the well-known Preissmann box scheme. To show that (12) is a multi-conformal-symplectic scheme, we take the wedge product of the corresponding variational equation with $A^a_{t_n} dz^i_{n+1/2}$ to get

$$A^a_{t_i} dz^i_{n+1/2} \wedge \mathbf{K}D^a_{t_i} z^i_{n+1/2} + A^a_{t_i} dz^i_{n+1/2} \wedge \mathbf{L}A^a_{t_i}\delta^+_x z^i_n = 0.$$

Since the wedge product is skew-symmetric, we may use the discrete product rule to obtain

$$D^{2a}_{t_i}\left(dz^i_{n+1/2} \wedge \mathbf{K}z^i_{n+1/2}\right) + \delta^+_x\left(A^a_{t_i} dz^i_n \wedge \mathbf{L}A^a_{t_i} z^i_n\right) = 0.$$

Then, formula (8) implies

$$\delta^+_t\left(e^{2\theta_i} dz^i_{n+1/2} \wedge \mathbf{K}z^i_{n+1/2}\right) + \delta^+_x\left(e^{2\theta_{i+1/2}} A^a_{t_i} dz^i_n \wedge \mathbf{L}A^a_{t_i} z^i_n\right) = 0,$$

which is a discrete version of the multi-conformal-symplectic conservation law (2).

To show that method (12) also satisfies a discrete version of the conservation law (5) when $F = 0$, take the inner product of (12) with $\mathbf{B}A^a_{t_i} z^i_{n+1/2}$, and rearrange the terms to get

$$\left(A^a_{t_i} z^i_{n+1/2}\right)^T \mathbf{KB}D^a_{t_i} z^i_{n+1/2} + \left(A^a_{t_i} z^i_{n+1/2}\right)^T \mathbf{LB}A^a_{t_i}\delta^+_x z^i_n = 0.$$

Then, the discrete product rule implies

$$D^{2a}_{t_i}\left(\left(z^i_{n+1/2}\right)^T \mathbf{KB}z^i_{n+1/2}\right) + \delta^+_x\left(\left(A^a_{t_i} z^i_n\right)^T \mathbf{LB}A^a_{t_i} z^i_n\right) = 0$$

and property (8) yields the following discrete version of (5)

$$\delta^+_t\left(e^{2\theta_i}\left(z^i_{n+1/2}\right)^T \mathbf{KB}z^i_{n+1/2}\right) + \delta^+_x\left(e^{2\theta_{i+1/2}}\left(A^a_{t_i} z^i_n\right)^T \mathbf{LB}A^a_{t_i} z^i_n\right) = 0.$$

2.4. Discrete gradient spatial discretizations

In some cases it may be more desirable to preserve a conformal momentum conservation law, rather than the multi-conformal-symplectic conservation law. Thus, we also propose discrete gradient discretizations, and define $\overline{\nabla} S$ according to the properties

$$\lim_{\hat{z}\to z} \overline{\nabla} S(\hat{z}, z, t) = \overline{\nabla} S(z, z, t) = \nabla S(z, t) \qquad \text{and} \qquad \overline{\nabla} S(\hat{z}, z, t)^T(\hat{z} - z) = S(\hat{z}, t) - S(z, t).$$

A discrete gradient method for solving (3) is given by

$$\mathbf{K}D^a_{t_i} z^i_{n+1/2} + \mathbf{L}A^a_{t_i}\delta^+_x z^i_n = \overline{\nabla} S\left(A^a_{t_i} z^i_{n+1}, A^a_{t_i} z^i_n, t_{i+1/2}\right). \tag{13}$$

To show that this method satisfies a discrete version of the momentum conservation law (4), take the inner product of (13) with $A^a_{t_i}\delta^+_x z^i_n$ to get

$$\left(A^a_{t_i}\delta^+_x z^i_n\right)^T \mathbf{K}D^a_{t_i} z^i_{n+1/2} = \frac{1}{\Delta x}\left(S\left(A^a_{t_i} z^i_{n+1}, t_{i+1/2}\right) - S\left(A^a_{t_i} z^i_n, t_{i+1/2}\right)\right) = \delta^+_x S\left(A^a_{t_i} z^i_n, t_{i+1/2}\right).$$

Then, using the discrete product rule gives

$$D^{2a}_{t_i}\left(\frac{1}{2}\left(\delta^+_x z^i_n\right)^T \mathbf{K}z^i_{n+1/2}\right) + \delta^+_x\left(\frac{1}{2}\left(A^a_{t_i} z^i_n\right)^T \mathbf{K}D^a_{t_i} z^i_n\right) = \delta^+_x S\left(A^a_{t_i} z^i_n, t_{i+1/2}\right).$$

Rearranging terms and using property (8) yields

$$\delta^+_t\left(e^{2\theta_i}\frac{1}{2}\left(z^i_{n+1/2}\right)^T \mathbf{K}\delta^+_x z^i_n\right) + \delta^+_x\left(e^{2\theta_{i+1/2}}\left(\frac{1}{2}\left(D^a_{t_i} z^i_n\right)^T \mathbf{K}A^a_{t_i} z^i_n + S\left(A^a_{t_i} z^i_n, t_{i+1/2}\right)\right)\right) = 0.$$

3. Applications

Overall, there are many PDE models that could be characterized under the premise of this article, along with many competitive discretizations for each model, and it is not possible to give a thorough exposition here. But, to illustrate the usefulness of the proposed discretizations, we perform numerical experiments using the method (10) for a damped-driven NLS equation, and a damped-driven Camassa–Holm equation.

3.1. Damped-driven NLS equation

Several variations of the damped-driven nonlinear Schrödinger (DDNLS) equation with variable coefficients have applications in mathematical physics (see for example [20–22]), and in many cases DDNLS equations can also be stated in the form (3) (see for example [11]). As an example, consider the damped NLS equation with parametric forcing [23–25], given by

$$i\psi_t + \psi_{xx} + i\gamma\psi + ce^{i\omega t}\psi^* + V'(|\psi|^2)\psi = 0, \tag{14}$$

where ψ^* denotes the complex conjugate of ψ. Setting $\psi = p + iq$, $\alpha = c\cos(\omega t)$ and $\beta = c\sin(\omega t)$ implies

$$\begin{aligned} p_t + q_{xx} + \gamma p + \beta p - \alpha q + V'(p^2+q^2)q &= 0 \\ -q_t + p_{xx} - \gamma q + \beta q + \alpha p + V'(p^2+q^2)p &= 0. \end{aligned}$$

Eq. (14) may also be cast in the form (3) by taking $z = [p, q, v, w]^T$,

$$\mathbf{K} = \begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{L} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \qquad \mathbf{J} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{15}$$

with

$$S(z,t) = \frac{1}{2}(v^2 + w^2 + V(p^2+q^2) + \alpha(t)(p^2+q^2) + 2\beta(t)qp).$$

In this case, we have set $a(t) = \gamma$. This system of equations does not satisfy a so called conformal norm conservation law, but it does satisfy the conformal momentum conservation law

$$\partial_t\left(e^{2\gamma t}(pq_x - qp_x)\right) + \partial_x\left(e^{2\gamma t}\left(p_x^2 + q_x^2 - pq_t + qp_t - \alpha(q^2-p^2) + 2\beta qp + V(p^2+q^2)\right)\right) = 0. \tag{16}$$

Since this conservation law is a special case of (4), it is preserved by the discretization (13).

Another important example is the damped-driven NLS equation [26,27], given by

$$i\psi_t + \psi_{xx} + i\gamma\psi + ce^{i\omega t}\psi + V'(|\psi|^2)\psi = 0, \tag{17}$$

which we explore in more detail, since the damping/driving term is time-dependent. Let $i\gamma\psi + ce^{i\omega t}\psi = (\alpha + i\beta)\psi$, where $\alpha = \alpha(t) = c\cos(\omega t)$ and $\beta = \beta(t) = \gamma + c\sin(\omega t)$. Setting $\psi = p + iq$, Eq. (17) can be rewritten as

$$q_t - p_{xx} = V'(p^2+q^2)p + \alpha p - \beta q \tag{18}$$

$$-p_t - q_{xx} = V'(p^2+q^2)q + \alpha q + \beta p, \tag{19}$$

or, to state it in the form of Eq. (3), set $z = [p, q, v, w]^T$, $a(t) = \beta(t)$, and

$$S(z,t) = \frac{1}{2}(v^2 + w^2 + V(p^2+q^2) + \alpha(t)(p^2+q^2)),$$

and use the matrices defined in (15). Therefore, it is clear that the equation satisfies a multi-conformal-symplectic conservation law (2). Indeed, it also satisfies a conformal norm conservation law

$$\partial_t N + \partial_x M = -2\beta N \quad \Longleftrightarrow \quad \partial_t(e^{2\theta(t)}N) + \partial_x(e^{2\theta(t)}M) = 0, \tag{20}$$

which is a special form of (5), for $N = p^2 + q^2$, $M = 2(pq_x - qp_x)$, and $\theta(t) = \int_0^t \beta(s)ds = \gamma t + \frac{c}{\omega}(1 - \cos(\omega t))$.

In addition to a multi-conformal-symplectic conservation law, one can also show that discretization (10) (with $F = 0$) preserves the norm conservation law (20) for Eq. (17). Writing the discretization for the system (18)–(19) gives

$$D_{t_i}^{\beta} q_n^i - A_{t_i}^{\beta}\delta_x^2 p_n^i - V'\left((A_{t_i}^{\beta}p_n^i)^2 + (A_{t_i}^{\beta}q_n^i)^2\right)A_{t_i}^{\beta}p_n^i - \alpha(t_{i+1/2})A_{t_i}^{\beta}p_n^i = 0 \tag{21}$$

$$D_{t_i}^{\beta} p_n^i + A_{t_i}^{\beta}\delta_x^2 q_n^i + V'\left((A_{t_i}^{\beta}p_n^i)^2 + (A_{t_i}^{\beta}q_n^i)^2\right)A_{t_i}^{\beta}q_n^i + \alpha(t_{i+1/2})A_{t_i}^{\beta}q_n^i = 0, \tag{22}$$

where $\delta_x^2 := \delta_x^+\delta_x^- = \delta_x^-\delta_x^+$. (In this case, the symplectic Euler spatial discretization reduces to standard central differences.) Multiplying the first equation through by $A_{t_i}^{\beta}q_n^i$ and the second equation by $A_{t_i}^{\beta}p_n^i$ and adding them together implies

$$A_{t_i}^{\beta}q_n^i D_{t_i}^{\beta}q_n^i - A_{t_i}^{\beta}q_n^i A_{t_i}^{\beta}\delta_x^2 p_n^i + A_{t_i}^{\beta}p_n^i D_{t_i}^{\beta}p_n^i + A_{t_i}^{\beta}p_n^i A_{t_i}^{\beta}\delta_x^2 q_n^i = 0.$$

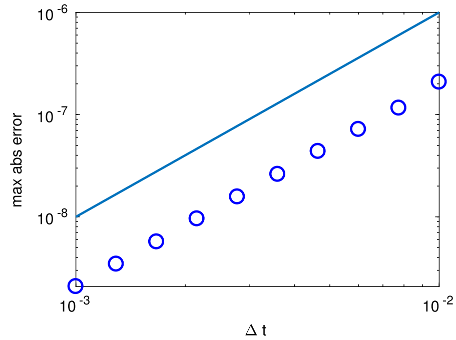


Fig. 1. Maximum absolute errors (scattered points) in the plane wave solution due to method (21)–(22) showing its second order accuracy and continuous line of slope 2 for reference.

Then, using the discrete product rules provides

$$D_{t_i}^{2\beta}\left((p_n^i)^2+(q_n^i)^2\right)+\frac{1}{\Delta x}\delta_x^+\left(A_{t_i}^\beta p_{n-1}^i A_{t_i}^\beta q_n^i - A_{t_i}^\beta p_n^i A_{t_i}^\beta q_{n-1}^i\right)=0,$$

or equivalently

$$\delta_t^+\left(e^{2\theta_i}\left((p_n^i)^2+(q_n^i)^2\right)\right)+\delta_x^+\left(\frac{e^{2\theta_{i+1/2}}}{\Delta x}\left(A_{t_i}^\beta p_{n-1}^i A_{t_i}^\beta q_n^i - A_{t_i}^\beta p_n^i A_{t_i}^\beta q_{n-1}^i\right)\right)=0,$$

which is a discrete version of the norm conservation law (20) for the damped-driven nonlinear Schrödinger equation.

The plane wave solution of damped NLS is given by

$$\psi(x,t)=A\exp\left(-2\gamma t+\mathrm{i}\lambda|A|^2\frac{1-\exp(-4\gamma t)}{4\gamma}\right)$$

where A is the wave amplitude, γ is the constant damping, and λ is the coefficient of nonlinearity V, i.e., $V(\eta)=\frac{1}{2}\lambda\eta^2$. To test the accuracy of the integrator (21)–(22), we plot the maximum absolute errors in the plane wave solution at various final times against Δt in Fig. 1 with

$$\Delta t/\Delta x=0.01,\ x\in[-1,1],\ t\in[0,T],\ T=0.1,\ \alpha(t)=0,\ \beta(t)=0.001,\ \lambda=2,\ \text{and}\ A=0.5.$$

The figure validates the (global) second order accuracy of the method (in time) with coupled space and time step-size reduction.

In all of the following experiments we set

$$\Delta x=0.1,\ \Delta t=0.001,\ x\in[-30,30],\ t\in[0,10],\ \alpha(t)=-0.2\cos(\pi t),\ \beta(t)=0.1-0.2\sin(\pi t),\ \text{and}\ V(\eta)=\eta^2$$

so that $\beta(t)$ fluctuates trigonometrically between -0.1 and 0.3. Fig. 2 shows propagation of two initial wave profiles with suitable boundary conditions. One can see that the amplitude of the propagating wave increases with negative values of the parameter β and decreases with its positive values. This causality becomes less pronounced over time as the initial profile evolves and gets dispersed over the entire spatial domain. Propagating a dark soliton with the method of (10) shows the solution disintegrating into a wave train [28] (left plot of Fig. 2). The solutions illustrate preservation of norm, by plotting the norm error, which is defined to be

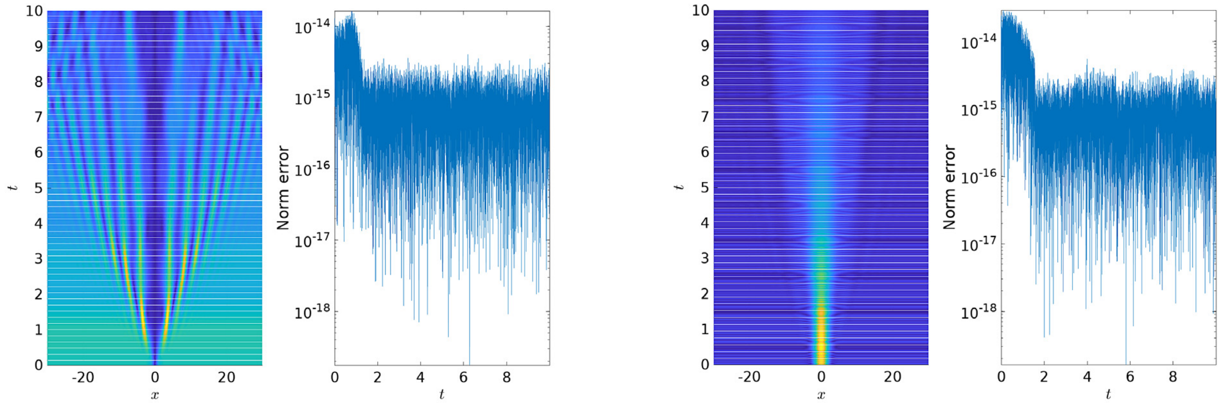
$$\text{Norm error }=\log\left(\frac{\sum_n|\psi_n^{i+1}|^2}{\sum_n|\psi_n^i|^2}\right)-4(\theta_{i+1}-\theta_i). \tag{23}$$

Fig. 3 shows an example where the equation is initialized with propagating solitons to produce a collision. As a result, dispersive waves of much smaller amplitude emit out of the collision. In this experiment, the error in norm for the exponential time-discretization is compared to that of the underlying method (implicit midpoint in time, symplectic Euler in space). Recent work proposed the implicit midpoint time-discretization for a similar equation [26], showing that the discretization is almost structure-preserving, though it does preserve the norm in the absence of weak damping. Only the exponential integrator shows the correct behavior when α and β are nonzero.

3.2. Damped-driven Camassa–Holm equation

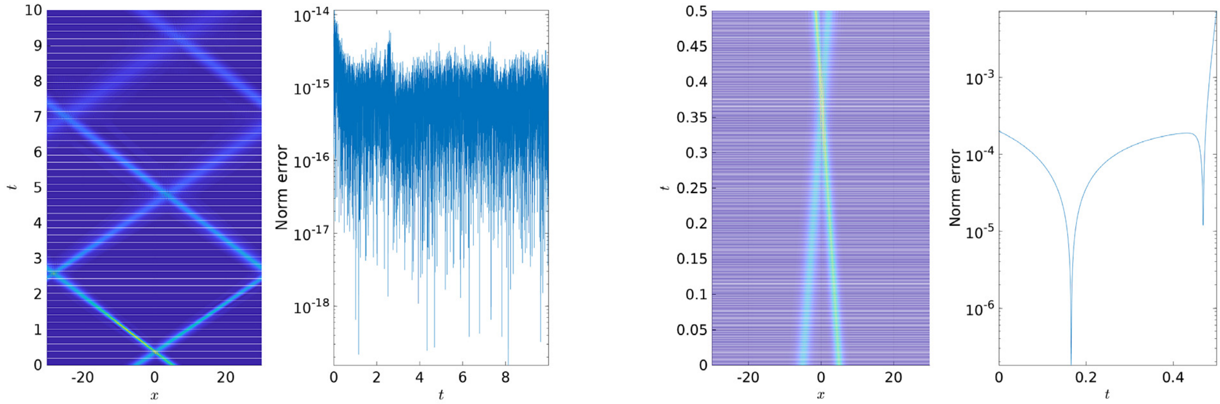
A dissipative Camassa–Holm equation with external forcing

$$u_t-u_{xxt}+3uu_x+\gamma(t)(u-u_{xx})-2u_xu_{xx}-uu_{xxx}=f(x,t),\quad u(x,0)=u_0, \tag{24}$$



(a) (b)

Fig. 2. Solutions (in the form $|\psi|$) of (17) by the method of (21)–(22) next to the error in the norm. (a) $\psi(x,0) = \tanh(x)$ with anti-periodic boundary conditions, (b) $\psi(x,0) = \sqrt{\frac{2}{3\sqrt{\pi}}}\exp(-(2x/3)^2/2)$ with periodic boundary conditions.



(a) (b)

Fig. 3. Left: Solution of (17) with $\psi(x,0) = \exp(8\mathrm{i}x)\,\mathrm{sech}(x+5) + 1.5\exp(-7\mathrm{i}x)\,\mathrm{sech}(1.5(x-5))$ and periodic boundary conditions. Right: Error in norm. (a) Results for the exponential method (21)–(22); (b) Results for the implicit midpoint time-discretization.

is used to model dispersive shallow water waves on a flat bottom [29]. Here, u is the fluid velocity in the x direction or the height of water's free surface above a flat bottom. The conservative Camassa–Holm equation ($\gamma = 0, f = 0$) is completely integrable, bi-Hamiltonian and possesses infinitely many conservation laws. As a result, several authors have constructed and analyzed structure-preserving algorithms for solving the equation, including an energy preserving numerical scheme [30], a multi-symplectic numerical scheme constructed through a variational approach [31], and multi-symplectic methods based on Euler box scheme [32], one of which is designed to handle conservative continuation of peakon–antipeakon collision.

Since the physical system generally exhibits some weak dissipation [29], we are interested in the dissipative ($\gamma \neq 0$) Camassa–Holm equation. It was shown in [33–35] that the global solutions of (24) with $f = 0$ exist and decay to zero as time goes to infinity for certain initial profiles and under certain restrictions on the associated potential $(1 - \partial_x^2)u_0$. Hence, (24) does not possess any traveling wave solutions although the equation does manifest the wave breaking phenomenon like its conservative counterpart. It was also shown there that the onset of blow-up is affected by the dissipative parameter γ but the blow-up rate is not affected by γ. We point out the following result from these articles

$$\|u(x,t)\|^2_{\mathbb{H}^1} = e^{-2\gamma t}\|u_0(x)\|^2_{\mathbb{H}^1} \text{ for all } t \in [0,T]$$

and $u_0 \in \mathbb{H}^s$, $s > \frac{3}{2}$ where $T > 0$ is the maximal existence time of the solution u. In [36], authors establish a global well-posedness result and existence of global attractors in the periodic case for a closely related equation called viscous Camassa–Holm equation.

Eq. (24) can be written in conformal multi-symplectic form (1) with $a(t) = \gamma(t)$,

$$\mathbf{K} = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{L} = \begin{bmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad z = \begin{bmatrix} u \\ v \\ w \\ q \\ p \end{bmatrix}, \quad \mathbf{F}(x,t) = \begin{bmatrix} f(x,t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$S(z,t) = -\frac{u^3}{2} - \frac{up^2}{2} - uw + qp + \frac{\gamma(t)}{2}up.$$

Thus, the solutions of (24) satisfy the multi-conformal-symplectic conservation law (2), as well as two other conservation laws, in the absence of external forcing. It is a straightforward calculation to show that the equation satisfies the *Casimir* conservation law given by

$$\partial_t \left(e^{\theta(t)}u\right) + \partial_x \left(e^{\theta(t)} \left(\frac{3}{2}u^2 - \frac{1}{2}u_x^2 - u_{xt} - uu_{xx} - \gamma(t)u_x\right)\right) = 0$$

and an energy conservation law given by

$$\partial_t \left(\frac{e^{2\theta(t)}}{2}\left(u^2 + u_x^2\right)\right) + \partial_x \left(e^{2\theta(t)}\left(u^3 - u^2u_{xx} - uu_{xt} - \gamma(t)uu_x\right)\right) = 0.$$

Integrating each of these conservation laws in space, with appropriate boundary conditions, yields

$$\partial_t \int \left(e^{\theta(t)}u\right) dx = 0 \qquad \text{and} \qquad \partial_t \int \left(e^{2\theta(t)}\left(u^2 + u_x^2\right)\right) dx = 0, \tag{25}$$

which signify global conservation of the Casimir and the energy, respectively.

Consider the discretization (10) applied to Eq. (24). Take $A_{x^n} = A_xA_xA_x \ldots (n \text{ times})$, $\delta_{x^n}^+ = \delta_x^+\delta_x^+\delta_x^+ \ldots (n \text{ times})$ and apply the method directly to (24) to get

$$\begin{aligned} &D_{t_i}^{\gamma} A_{x^2} u_{n+1/2}^i - D_{t_i}^{\gamma} \delta_{x^2}^+ u_{n+1/2}^i + 3A_x(\delta_x^+ A_{t_i}^{\gamma} u_{n+1/2}^i \ A_x A_{t_i}^{\gamma} u_{n+1/2}) \\ &= 3(\delta_x^+ A_x A_{t_i}^{\gamma} u_{n+1/2}^i \ \delta_{x^2}^+ A_{t_i}^{\gamma} u_{n+1/2}^i) - A_x(\delta_x^+ A_{t_i}^{\gamma} u_{n+1/2}^i \ \delta_{x^2}^+ A_{t_i}^{\gamma} u_n^i) + (A_{x^2} A_{t_i}^{\gamma} u_{n+1/2}^i \ \delta_{x^3}^+ A_{t_i}^{\gamma} u_n^i). \end{aligned} \tag{26}$$

It can be shown that this method preserves the conformal Casimir (i.e. $\sum_n e^{\theta(t^{i+1})}u_n^{i+1} = \sum_n e^{\theta(t^i)}u_n^i$).

Fig. 4, shows residuals (defined analogous to (23)) in spatially discrete versions of the Casimir and the energy (25) when method (26) and its underlying method are used to propagate smooth and non-smooth initial data

$$u_0 = 0.2 + 0.1\cos(3x), \text{ and } u_0 = e^{-|x|} \tag{27}$$

with

$$\Delta x = 0.07,\ \Delta t = 0.001,\ x \in [-\pi, \pi],\ t \in [0, 10],\ \gamma(t) = -0.2\sin(\pi t),\ \text{and } f(x,t) = 0.$$

The figures confirm Casimir preservation by the structure-preserving method (26). As expected, the non-smooth kink initial profile $e^{-|x|}$ evolves and produces persisting unstable modes early in the simulation because of the non-preservation of the decay in the energy or the $\mathbb{H}^1$ norm of the solution. The non-smooth solution maintains its shape and direction throughout the numerical simulations nonetheless. Although neither method preserves the energy of the system, the exponential integrator (26) has energy residual one order of magnitude lower than the underlying method. The result of the periodic input to the system is evident in all the plots. The solutions' amplitude and the energy residual due to the underlying method vary periodically in sync with the amplitude of γ.

4. Conclusion

We have presented general systems of PDEs with time-dependent coefficients, along with damping/driving forces, that have solutions which satisfy certain conservation laws. We have shown that several PDEs arising from physical applications have the general form we consider, including damped/driven NLS, and Camassa–Holm equations. Since these equations have conservation laws that are desirable to preserve numerically, we present three, second-order in time, discretizations that preserve certain conservation laws. Even in cases where the damping coefficients are constant, the preservation of these local conservation laws signifies a stronger result than the global (boundary-condition-dependent) conservation that was previously achieved with conformal symplectic methods. Ultimately, the number of PDEs that might be considered under this framework, along with the number of structure-preserving discretizations we might propose, is vast. To demonstrate the effectiveness of our approach, we numerically solve a damped-driven NLS equation and a damped-driven Camassa–Holm equation using discretizations that preserve particular properties of the equations. In addition to illustrating preservation of the desired properties, the results exhibit qualitatively correct behavior in other respects, such as energy, as well as favorable comparison to other methods that have been proposed for similar equations.

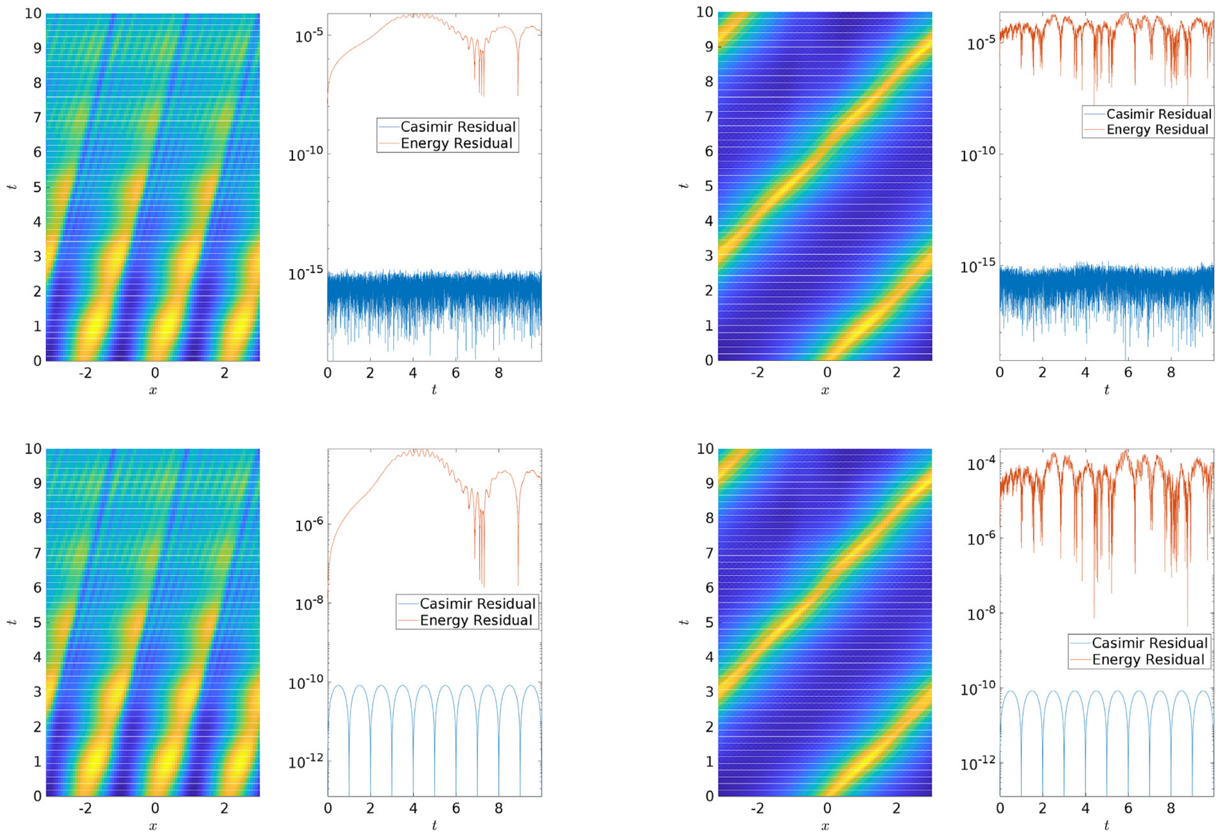


Fig. 4. Solutions of (24) next to the Casimir and energy residuals. The top row shows results from the conformal multi-symplectic method (26), while the bottom row shows results from the underlying method (implicit midpoint in time, centered difference in space). Left columns show results with $u_0 = 0.2 + 0.1\cos(3x)$, and the right columns show results for $u_0 = e^{-|x|}$.

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