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A conservative Fourier pseudo-spectral method for the nonlinear Schrödinger equation

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ABSTRACT

A Fourier pseudo-spectral method that conserves mass and energy is developed for a two-dimensional nonlinear Schrödinger equation. By establishing the equivalence between the semi-norm in the Fourier pseudo-spectral method and that in the finite difference method, we are able to extend the result in Ref. [56] to prove that the optimal rate of convergence of the new method is in the order of $O(N^{-r}+\tau^2)$ in the discrete L^2 norm without any restrictions on the grid ratio, where N is the number of modes used in the spectral method and τ is the time step size. A fast solver is then applied to the discrete nonlinear equation system to speed up the numerical computation for the high order method. Numerical examples are presented to show the efficiency and accuracy of the new method.
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1. Introduction

The nonlinear Schrödinger (NLS) equation is the most important equation in quantum mechanics [37]. It describes a wide range of physical phenomena [38,44,45,51], including plasma physics, nonlinear optics, self-focusing in laser pulses, propagation of heat pulses in crystals, and dynamics of Bose–Einstein condensates at the extremely low temperature. In this paper, we consider the following cubic NLS equation in two dimensional space:

$$iu_t + \Delta u + \beta|u|^2 u = 0,\ (x, y) \in \mathbb{R}\times\mathbb{R},\ 0 < t \le T, \tag{1.1}$$

subject to the (l_1, l_2)-periodic boundary condition

$$u(x, y, t) = u(x + l_1, y, t),\ u(x, y, t) = u(x, y + l_2, t),\ (x, y) \in \mathbb{R}\times\mathbb{R},\ 0 < t \le T, \tag{1.2}$$

and the initial condition

$$u(x, y, 0) = \varphi(x, y),\ (x, y) \in \mathbb{R}\times\mathbb{R}, \tag{1.3}$$

where $\Delta = \partial_{xx} + \partial_{yy}$ is the 2-dimensional Laplacian operator, $\beta \neq 0$ is a given real constant, $\varphi(x, y)$ is a given (l_1, l_2)-periodic complex-valued function. The NLS equation (1.1) is focusing for $\beta > 0$, and defocusing for $\beta < 0$, which is a generic model for the slowly varying envelop of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena. It is also

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obtained as the subsonic limit of the Zakharov model for Langmuir waves in plasma physics [59]. It has been shown that it is possible for solutions of the two-dimensional NLS equation to develop singularities in finite time [33].

The NLS equation has been studied extensively. For the derivation, well-posedness and dynamical properties of the NLS equation, readers are referred to [7,18,51] and the references therein. In fact, the solution of the initial-periodic boundary value problem (1.1)–(1.3) satisfies the following mass and energy conservation laws

$$Q(t) := \int_{\Omega} |u(x,y,t)|^2 dxdy \equiv Q(0), \tag{1.4}$$

and

$$E(t) := \int_{\Omega} \left[|\nabla u(x,y,t)|^2 - \frac{\beta}{2}|u(x,y,t)|^4 \right] dxdy \equiv E(0), \tag{1.5}$$

where $|\nabla u|^2 = |u_x|^2 + |u_y|^2$ and $\Omega = [0, l_1] \times [0, l_2]$. These conservation laws imply that, $\|u(\cdot,\cdot,t)\|_{H_p^1} < \infty$ for the defocusing case or for the focusing case with $\|\varphi\|^2 \le \frac{2-\widetilde{\varepsilon}}{4(1+\varepsilon)\beta}$ where ε and $\widetilde{\varepsilon}$ are two positive numbers which can be arbitrary small. The theory on existence of global solutions of the NLS equation [18,51] implies that, if $\|u(\cdot,\cdot,t)\|_{H_p^1} < \infty, u(x,y,t) \in L^\infty(\mathbb{R}^+, H_p^1(\Omega))$.

Numerically, various algorithms have been developed for solving the NLS equation, including the finite difference method [3,19,23,28,31,35,50,55–57,60], compact boundary value methods [27], finite element method [1,2,32,39], discontinuous Galerkin method [40,58], Runge–Kutta or Crank–Nicolson pseudo-spectral method [17,20,29,30,48], time-splitting pseudo-spectral method [5,6,8,53], meshless method [25,26] and radial basis function collocation method [24], etc. Some comparisons among the various numerical methods for the NLS equation can be found in [4,19,43,54] and the references therein.

For the NLS equation in one dimension, error estimates of different numerical methods have been established. For time-splitting methods, we refer to [9,22,42,46,53] for details. For the implicit Runge–Kutta finite element method, we refer to [2,47] for more details. Analyses on unconditional convergence of the conservative finite difference (CFD) method for the NLS equation in one dimension were conducted in [19,55]. In fact, their proofs for CFD scheme rely heavily on not only the discrete conservative property but also the discrete version of the Sobolev inequality in one dimension

$$\|f\|_{L^\infty} \le C\|f\|_{H^1}, \quad \forall f \in H_0^1(\Omega) \quad \text{or} \quad f \in H_p^1(\Omega) \quad \text{with} \quad \Omega \subset \mathbb{R}, \tag{1.6}$$

which immediately implies an a priori uniform bound for $\|f\|_{L^\infty}$. However, the extension of the discrete version of the above Sobolev inequality is no longer valid in two dimensions. Thus the techniques used in [19,55] cannot be extended to high dimensions without significant modifications. Due to the difficulty in obtaining an a priori, uniform estimate for the numerical solution, few error estimates are available in the literature for the NLS equation in two dimensions. Gao and Xie [31] proposed a fourth-order alternating direction implicit compact finite difference scheme for a two-dimensional Schrödinger equation, and also used induction argument to prove that their scheme is conditionally convergent in the discrete L^2 norm. In the analysis, they imposed a strict restriction on the grid ratio. In [56], Wang et al. developed a new technique to analyze a compact finite difference scheme for the NLS equation in two dimensions. They proved that their method is convergent in the order of $O(h^4 + \tau^2)$ in the discrete L^2 norm without any restrictions on the mesh ratio.

The spectral method is a classical, high order and widely used technique to solve differential equations, both theoretically and numerically [49]. Bridges and Reich [10] first introduced the idea of Fourier spectral discretization to construct multi-symplectic integrator for Hamiltonian systems. Based on their theory, Chen and Qin [20] proposed a multi-symplectic Fourier pseudo-spectral method for Hamiltonian PDEs and applied it to integrate the NLS equation with periodic boundary conditions. Later, different structure-preserving Fourier pseudo-spectral methods were developed [12,21,34,35,41]. However, few convergent results of the Fourier pseudo-spectral method are available. Recently, Cai et al. [13–15] developed and analyzed different structure-preserving Fourier pseudo-spectral methods for the 3D Maxwell's equation. But unconditionally convergent results on Fourier pseudo-spectral method for nonlinear PDEs have not been obtained. For nonlinear problems, once an a priori estimate of the numerical solution in discrete L^∞ norm is known, it seems that the error analysis can be carried out using the standard linear theory for evolutionary equations. However, it is hard to be substantiated. Therefore, there has not been a single proof on the unconditional error estimate of the Fourier pseudo-spectral method for nonlinear NLSs. In this paper, we develop a conservative Fourier pseudo-spectral method for solving the initial-periodic boundary value problem of the NLS equation in two dimensions. We show that the semi-norm in the Fourier pseudo-spectral method is equivalent to that in the finite difference method. Once this important connection is established, we prove the method is unconditionally convergent with the order of $O(N^{-r} + \tau^2)$ in the discrete L^2 norm, using the techniques developed in Ref. [56]. A fast solver is then applied to solve the discrete nonlinear equation system. We present some numerical examples in the end and demonstrate the effectiveness of the new high order method.

The rest of the paper is organized as follows. In Section 2, we define some notations to be used in the proof and present the conservative Fourier pseudo-spectral algorithm for the 2D NLS equation. In Section 3, we prove the existence of the numerical solution by the Browder fixed point theorem. Then, we show the numerical scheme satisfies discrete conservation laws and obtain an a priori estimate. In Section 4, the convergence property of the new scheme is analyzed.

Numerical examples are presented to show the efficiency and accuracy of the method in Section 5. Finally, we give a concluding remark in Section 6.

2. Fourier pseudo-spectral method

We develop a numerical method to solve the initial-periodic boundary value problem (1.1)–(1.3) in a finite domain $\Omega \times [0, T]$. First, we introduce some notations. For two positive even integers N_x and N_y, we define two step sizes in space: $h_1 = l_1/N_x$, $h_2 = l_2/N_y$. Then, the spatial grid points are defined as follows: $\Omega_h = \{(x_j, y_k)|j = 0, 1, \ldots, N_x - 1; k = 0, 1, \ldots, N_y - 1\}$, where $x_j = jh_1$, $0 \le j \le N_x - 1$, $y_k = kh_2$, $0 \le k \le N_y - 1$. For a positive integer N_t, we define the time-step: $\tau = T/N_t$. The grid points in space and time are given by $\Omega_h^\tau = \Omega_h \times \Omega_\tau$, where $\Omega_\tau = \{t_n|t_n = n\tau, n = 0, 1, \ldots, N_t\}$. Given a grid function $u = \{u_{j,k}^n|(x_j, y_k, t_n) \in \Omega_h^\tau\}$, we denote

$$\delta_x^+ u_{j,k}^n = \frac{u_{j+1,k}^n - u_{j,k}^n}{h_1},\ \delta_y^+ u_{j,k}^n = \frac{u_{j,k+1}^n - u_{j,k}^n}{h_2},\ \delta_t^+ u_{j,k}^n = \frac{u_{j,k}^{n+1} - u_{j,k}^n}{\tau},\ u_{j,k}^{n+1/2} = \frac{u_{j,k}^n + u_{j,k}^{n+1}}{2}.$$

Let $\mathbb{V}_h = \{u|u = (u_{j,k}), (x_j, y_k) \in \Omega_h\}$ be the space of grid functions defined on Ω_h. For any two grid functions $u, v \in \mathbb{V}_h$, we define the discrete inner product as follows:

$$(u, v)_h = h_1 h_2 \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} u_{j,k} \overline{v}_{j,k}.$$

The discrete L^2 norm of v and its difference quotients are defined, respectively, as

$$\|v\|_h = \sqrt{(v, v)_h},\ \|\delta_x^+ v\|_h = \sqrt{(\delta_x^+ v, \delta_x^+ v)_h},\ \|\delta_y^+ v\|_h = \sqrt{(\delta_y^+ v, \delta_y^+ v)_h},\ |v|_{h,1} = \sqrt{\|\delta_x^+ v\|_h^2 + \|\delta_y^+ v\|_h^2}.$$

We also define the discrete $L^p (1 \le p < \infty)$ norm as

$$\|v\|_{h,p} = \left(h_1 h_2 \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} |v_{j,k}|^p \right)^{1/p},$$

and the discrete L^∞ norm as

$$\|v\|_{h,\infty} = \max_{(x_j, y_k) \in \Omega_h} |v_{j,k}|.$$

We note that the discrete norm $\|\delta_x^+ v\|_h, \|\delta_y^+ v\|_h, |v|_{h,1}$ defined above are semi-norms.

For simplicity, we denote $u_{j,k}^n = u(x_j, y_k, t_n)$ and $U_{j,k}^n$ as the exact value of $u(x, y, t)$ and its numerical approximation at (x_j, y_k, t_n), respectively.

2.1. Fourier pseudo-spectral approximation of spatial derivatives

We define

$$S_N'' = \text{span}\{g_j(x) g_k(y), j = 0, 1, \ldots, N_x - 1; k = 0, 1, \ldots, N_y - 1\}$$

as the interpolation space, where $g_j(x)$ and $g_k(y)$ are trigonometric polynomials of degree $N_x/2$ and $N_y/2$, given respectively by

$$g_j(x) = \frac{1}{N_x} \sum_{p=-N_x/2}^{N_x/2} \frac{1}{a_p} e^{ip\mu_1(x - x_j)}, \quad g_k(y) = \frac{1}{N_y} \sum_{q=-N_y/2}^{N_y/2} \frac{1}{b_q} e^{iq\mu_2(y - y_k)}, \tag{2.1}$$

where $a_p = \begin{cases} 1, |p| < N_x/2, \\ 2, |p| = N_x/2, \end{cases} \quad \mu_1 = 2\pi/l_1,\ b_q = \begin{cases} 1, |q| < N_y/2, \\ 2, |q| = N_y/2, \end{cases} \quad \mu_2 = 2\pi/l_2.$

We define the interpolation operator $I_N : L^2(\Omega) \to S_N''$ as follows:

$$I_N u(x, y) = \sum_{j=0}^{N_x-1} \sum_{k=0}^{N_y-1} u_{j,k} g_j(x) g_k(y), \tag{2.2}$$

where $u_{j,k} = u(x_j, y_k)$, $g_j(x_l) = \delta_l^j$, $g_k(y_m) = \delta_m^k$.

To obtain derivative $\partial_x^{s_1}\partial_y^{s_2} I_N u(x,y)$ at the collocation points (x_j, y_k), we differentiate (2.2) and evaluate the resulting expressions at the points (x_j, y_k):

$$\partial_x^{s_1}\partial_y^{s_2} I_N u(x_j,y_k) = \sum_{l=0}^{N_x-1}\sum_{m=0}^{N_y-1} u_{l,m}\frac{d^{s_1}g_l(x_j)}{dx^{s_1}}\frac{d^{s_2}g_m(y_k)}{dy^{s_2}} = \left(D_{s_1}^x \mathbf{u}(D_{s_2}^y)^T\right)_{j,k},$$

where $D_{s_1}^x$ is an $N_x \times N_x$ matrix and $D_{s_2}^y$ an $N_y \times N_y$ matrix, respectively, with elements given by

$$(D_{s_1}^x)_{j,l} = \frac{d^{s_1}g_l(x_j)}{dx^{s_1}}, \quad (D_{s_2}^y)_{k,m} = \frac{d^{s_2}g_m(y_k)}{dy^{s_2}},$$

and $\mathbf{u} = (u_{j,k})$ is an $N_x \times N_y$ matrix. Note that D_s^x, D_s^y are real antisymmetric matrices when s is odd; D_s^x, D_s^y are real symmetric matrices when s is even. More details can be found in [20,34]. For second derivatives, we have

$$\partial_{xx} I_N u(x_j,y_k) = (D_2^x \mathbf{u})_{j,k}, \quad \partial_{yy} I_N u(x_j,y_k) = (\mathbf{u} D_2^y)_{j,k}, \tag{2.3}$$

where symmetry of D_2^y is used.

2.2. Conservative Fourier pseudo-spectral method

We discretize the NLS equation (1.1)–(1.3) using the Fourier pseudo-spectral method in space and the Crank–Nicolson method in time to arrive at a full-discrete system:

$$i\delta_t^+ U_{j,k}^n + (D_2^x U^{n+1/2})_{j,k} + (U^{n+1/2}D_2^y)_{j,k} + \frac{\beta}{2}\left(|U_{j,k}^n|^2 + |U_{j,k}^{n+1}|^2\right)U_{j,k}^{n+1/2} = 0, \quad U^n \in \mathbb{V}_h, \tag{2.4}$$

where $j = 0, 1, \ldots, N_x - 1$, $k = 0, 1, \ldots, N_y - 1$. For convenience, scheme (2.4) can be written in a matrix form

$$i\delta_t^+ U^n + D_2^x U^{n+1/2} + U^{n+1/2}D_2^y + F(U^n, U^{n+1}) = 0, \quad U^n \in \mathbb{V}_h, \tag{2.5}$$

where $U^n = (U_{jk}^n)$, $F(U^n, U^{n+1}) = (F(U_{j,k}^n, U_{j,k}^{n+1})) = (\frac{\beta}{4}\left(|U_{j,k}^n|^2 + |U_{j,k}^{n+1}|^2\right)\left(U_{j,k}^n + U_{j,k}^{n+1}\right))$. Here, we define a new semi-norm:

$$|v|_h = \sqrt{(-D_2^x v, v)_h + (-vD_2^y, v)_h}, \quad v \in \mathbb{V}_h. \tag{2.6}$$

Next, we show that $|\cdot|_h$ is equivalent to $|\cdot|_{h,1}$. Thus, definition (2.6) is meaningful.

A matrix in the form of

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

is called a circulant matrix [36]. Because the matrix A is determined by the entries in the first row, the matrix can be denoted as

$$A = C(a_0, a_1, \ldots, a_{n-1}).$$

According to summation-by-parts formula

$$(\delta_\alpha^+ f, g)_h + (f, \delta_\alpha^- g)_h = 0, \ \alpha = x \text{ or } y,$$

we have

$$\|\delta_x^+ u\|_h^2 = -(\delta_x^-\delta_x^+ u, u)_h = (-A_1 u, u)_h,$$
$$\|\delta_y^+ u\|_h^2 = -(\delta_y^-\delta_y^+ u, u)_h = (-uA_2, u)_h,$$

where

$$A_1 = \frac{1}{h_1^2}C(\underbrace{-2, 1, 0, \ldots, 0, 1}_{N_x}), \quad A_2 = \frac{1}{h_2^2}C(\underbrace{-2, 1, 0, \ldots, 0, 1}_{N_y}).$$

Lemma 2.1 *([36]). For a real circulant matrix $A = C(a_0, a_1, \ldots, a_{n-1})$, all eigenvalues of A are given by*

$$f(\varepsilon_k), \quad k = 0, 1, \ldots, n-1,$$

where $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_{n-1} x^{n-1}$, and $\varepsilon_k = e^{i\frac{2k\pi}{n}}$.

Lemma 2.2. *For matrices A_1, A_2, D_2^x and D_2^y, there exist relations*

$$A_1 = F_{N_x}^H \Lambda_1 F_{N_x}, \quad A_2 = F_{N_y}^H \Lambda_2 F_{N_y}, \tag{2.7}$$

$$D_2^x = F_{N_x}^H \Lambda_3 F_{N_x}, \quad D_2^y = F_{N_y}^H \Lambda_4 F_{N_y}, \tag{2.8}$$

where F_N is the discrete Fourier transform with elements $(F_N)_{j,k} = \frac{1}{\sqrt{N}} e^{-i\frac{2\pi}{N}jk}$, F_N^H is the conjugate transpose matrix of F_N, and

$$\Lambda_1 = diag(\lambda_{A_1,0}, \lambda_{A_1,1}, \ldots, \lambda_{A_1,N_x-1}),\ \lambda_{A_1,j} = -\frac{4}{h_1^2}\sin^2\frac{j\pi}{N_x},$$

$$\Lambda_2 = diag(\lambda_{A_2,0}, \lambda_{A_2,1}, \ldots, \lambda_{A_2,N_y-1}),\ \lambda_{A_2,k} = -\frac{4}{h_2^2}\sin^2\frac{k\pi}{N_y},$$

$$\Lambda_3 = diag(\lambda_{D_2^x,0}, \lambda_{D_2^x,1}, \ldots, \lambda_{D_2^x,N_x-1}),\ \lambda_{D_2^x,j} = \begin{cases} -(j\mu_1)^2, & 0 \le j \le N_x/2, \\ -\big((j-N_x)\mu_1\big)^2, & N_x/2 < j < N_x, \end{cases}$$

$$\Lambda_4 = diag(\lambda_{D_2^y,0}, \lambda_{D_2^y,1}, \ldots, \lambda_{D_2^y,N_y-1}),\ \lambda_{D_2^y,k} = \begin{cases} -(k\mu_2)^2, & 0 \le k \le N_y/2, \\ -\big((k-N_y)\mu_2\big)^2, & N_y/2 < k < N_y. \end{cases}$$

Furthermore, we have

$$0 \le -\frac{4}{\pi^2}\lambda_{D_2^x,j} \le -\lambda_{A_1,j} \le -\lambda_{D_2^x,j},\ \forall 0 \le j < N_x, \tag{2.9}$$

$$0 \le -\frac{4}{\pi^2}\lambda_{D_2^y,k} \le -\lambda_{A_2,k} \le -\lambda_{D_2^y,k},\ \forall 0 \le k < N_y. \tag{2.10}$$

Proof. According to Lemma 2.1, we have

$$\lambda_{A_1,j} = \frac{1}{h_1^2}\big(-2 + e^{i\frac{2j\pi}{N_x}} + (e^{i\frac{2j\pi}{N_x}})^{N_x-1}\big) = \frac{1}{h_1^2}(-2 + 2\cos\frac{2j\pi}{N_x}) = -\frac{4}{h_1^2}\sin^2\frac{j\pi}{N_x}.$$

Similarly, we can compute $\lambda_{A_2,k}$. Since A_1 and A_2 are circulant matrices [36], we have (2.7). For the proof of (2.8), please refer to [34].

Using the inequality

$$0 \le \frac{2}{\pi}x \le \sin x \le x,\ \forall x \in [0, \frac{\pi}{2}],$$

we readily deduce (2.9)–(2.10). □

Lemma 2.3. *For $A \in \mathbb{C}_{N_x \times N_x}$, $B \in \mathbb{C}_{N_y \times N_y}$, and $u, v \in \mathbb{V}_h$, there exist identities*

$$(Au, v)_h = (u, A^H v)_h, \tag{2.11}$$

$$(uB, v)_h = (u, vB^H)_h. \tag{2.12}$$

Lemma 2.4. *For any grid function $u \in \mathbb{V}_h$, the following inequalities hold:*

$$|u|_{h,1} \le |u|_h \le \frac{\pi}{2}|u|_{h,1}. \tag{2.13}$$

Proof. It follows from Lemmas 2.2 and 2.3 that

$$\begin{aligned} \|\delta_x^+ u\|_h^2 &= (-A_1 u, u)_h = (-F_{N_x}^H \Lambda_1 F_{N_x} u, u)_h = (-\Lambda_1 F_{N_x} u, F_{N_x} u)_h \\ &= -h_1 h_2 \sum_{j=0}^{N_x-1}\sum_{k=0}^{N_y-1} \lambda_{A_1,j}|(F_{N_x}u)_{j,k}|^2 \\ &\le -h_1 h_2 \sum_{j=0}^{N_x-1}\sum_{k=0}^{N_y-1} \lambda_{D_2^x,j}|(F_{N_x}u)_{j,k}|^2 = (-D_2^x u, u)_h. \end{aligned} \tag{2.14}$$

Similarly, we have

$$\|\delta_x^+ u\|_h^2 \geq \frac{4}{\pi^2}(-D_2^x u, u)_h, \tag{2.15}$$

and

$$\frac{4}{\pi^2}(-uD_2^y, u)_h \leq \|\delta_y^+ u\|_h^2 \leq (-uD_2^y, u)_h. \tag{2.16}$$

Eqs. (2.14)–(2.16) imply (2.13). □

3. Solution existence and conservation of the scheme

In this section, we show scheme (2.4) is solvable and it conserves the total mass and energy at the discrete level. Then, we give an a priori estimate for the scheme. First, we introduce some useful lemmas.

Lemma 3.1. *For approximation $U^n \in \mathbb{V}_h$, there exist identities:*

$$\mathrm{Im}\left(D_2^x U^{n+1/2} + U^{n+1/2} D_2^y, U^{n+1/2}\right)_h = 0, \tag{3.1}$$

$$\mathrm{Re}\left(D_2^x U^{n+1/2} + U^{n+1/2} D_2^y, \delta_t^+ U^n\right)_h = -\frac{1}{2\tau}\left(|U^{n+1}|_h^2 - |U^n|_h^2\right), \tag{3.2}$$

where "$\mathrm{Im}(s)$" and "$\mathrm{Re}(s)$" are the imaginary and the real part of a complex number s, respectively.

Proof. From definition (2.6), we have

$$\mathrm{Im}(D_2^x U^{n+1/2} + U^{n+1/2} D_2^y, U^{n+1/2})_h = \mathrm{Im}\left(-|U^{n+1/2}|_h^2\right) = 0.$$

Using Lemma 2.3 and symmetry of real matrix D_2^x, we obtain

$$\begin{aligned}\mathrm{Re}\left(D_2^x U^{n+1/2}, \delta_t^+ U^n\right)_h &= \frac{1}{2\tau}\mathrm{Re}\left(D_2^x U^{n+1} + D_2^x U^n, U^{n+1} - U^n\right)_h \\ &= \frac{1}{2\tau}\left((D_2^x U^{n+1}, U^{n+1})_h - (D_2^x U^n, U^n)_h\right).\end{aligned} \tag{3.3}$$

Similarly, we have

$$\mathrm{Re}\left(U^{n+1/2} D_2^y, \delta_t^+ U^n\right)_h = \frac{1}{2\tau}\left((U^{n+1} D_2^y, U^{n+1})_h - (U^n D_2^y, U^n)_h\right). \tag{3.4}$$

Adding (3.3) and (3.4) gives (3.2). □

Lemma 3.2 *([56]). For any grid function $u \in \mathbb{V}_h$, the following inequality holds*

$$\|u\|_{h,4}^4 \leq \|u\|_h^2 \left(2|u|_{h,1} + \frac{1}{l}\|u\|_h\right)^2, \tag{3.5}$$

where $l = \min\{l_1, l_2\}$.

3.1. Existence

Lemma 3.3 *(Browder Fixed Point Theorem [2,11]). Let $\left(H, \langle\cdot,\cdot\rangle\right)$ be a finite dimensional inner product space, $\|\cdot\|$ the associated norm, and $g : H \to H$ a continuous function. If*

$$\exists \alpha > 0, \forall z \in H, \|z\| = \alpha, \ s.t. \ \mathrm{Re}\langle g(z), z\rangle \geq 0,$$

there exists a $z^* \in H$, $\|z^*\| \leq \alpha$ such that $g(z^*) = 0$.

Theorem 3.4. *The nonlinear equation system in scheme (2.4) is solvable.*

Proof. For a fixed n, (2.5) can be written as

$$U^{n+1/2} - U^n - i\frac{\tau}{2}\left[D_2^x U^{n+1/2} + U^{n+1/2} D_2^y + G(U^n, U^{n+1/2})\right] = 0, \tag{3.6}$$

$$U^n \in \mathbb{V}_h, \quad U^{n+1/2} \in \mathbb{V}_h, \tag{3.7}$$

where $G(U^n, U^{n+1/2}) = (G(U_{j,k}^n, U_{j,k}^{n+1/2})) = (\frac{\beta}{2}\left(|U_{j,k}^n|^2 + |2U_{j,k}^{n+1/2} - U_{j,k}^n|^2\right) U_{j,k}^{n+1/2})$. We define a mapping $\mathcal{F}: \mathbb{V}_h \to \mathbb{V}_h$ as follows

$$\mathcal{F}w = w - U^n - i\frac{\tau}{2}\left[D_2^x w + w D_2^y + G(U^n, w)\right], \tag{3.8}$$

which is obviously continuous. By a straightforward calculation, we obtain

$$(D_2^x w + w D_2^y + G(U^n, w), w)_h = -|w|_h^2 + \frac{\beta}{2}(|U^n|^2 + |2w - U^n|^2, |w|^2)_h, \tag{3.9}$$

which is a real number. Computing the inner product of (3.8) with w and taking the real part, then using (3.9) and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
\mathrm{Re}(\mathcal{F}w, w)_h &= \|w\|_h^2 - \mathrm{Re}(U^n, w)_h \\
&\geq \|w\|_h^2 - |(U^n, w)_h| \\
&\geq \|w\|_h^2 - \|U^n\|_h \|w\|_h \\
&\geq \frac{1}{2}(\|w\|_h^2 - \|U^n\|_h^2).
\end{aligned} \tag{3.10}$$

Hence, taking $\alpha = \sqrt{\|U^n\|_h^2 + 1}$, for $\|w\|_h = \alpha$, we have $\mathrm{Re}(\mathcal{F}w, w)_h \geq \frac{1}{2}$. Thus, the existence of $U^{n+1/2}$ follows from Lemma 3.3 and consequently the existence of U^{n+1} is established. □

3.2. Conservation

Theorem 3.5. *Scheme (2.4) is conservative in the sense that*

$$Q^n = Q^0, \ t_n \in \Omega_\tau, \tag{3.11}$$

$$E^n = E^0, \ t_n \in \Omega_\tau, \tag{3.12}$$

where

$$Q^n = \|U^n\|_h^2, \quad E^n = |U^n|_h^2 - \frac{\beta}{2}\|U^n\|_{h,4}^4,$$

are the discrete total mass and discrete total energy, respectively.

Proof. Computing the discrete inner product of (2.5) with $U^{n+1/2}$, then taking the imaginary part, we obtain

$$\frac{1}{2\tau}\left(\|U^{n+1}\|_h^2 - \|U^n\|_h^2\right) = 0, \ t_n \in \Omega_\tau, \tag{3.13}$$

where Lemma 3.1 is used. This gives (3.11).

Computing the discrete inner product of (2.5) with $\delta_t^+ U^n$, then taking the real part, we obtain

$$-\frac{1}{2\tau}\left[\left(|U^{n+1}|_h^2 - \frac{\beta}{2}\|U^{n+1}\|_{h,4}^4\right) - \left(|U^n|_h^2 - \frac{\beta}{2}\|U^n\|_{h,4}^4\right)\right] = 0, \ t_n \in \Omega_\tau, \tag{3.14}$$

where Lemma 3.1 is used. This yields (3.12). □

3.3. A priori estimate

Lemma 3.6. *If one of the following two conditions are satisfied:*

$$(a)\ \varphi \in H_p^1(\Omega), \quad \beta < 0, \tag{3.15}$$

$$(b)\ \varphi \in H_p^1(\Omega), \quad \|\varphi\|_h^2 \leq \frac{2 - \widetilde{\varepsilon}}{4(1+\varepsilon)\beta}, \quad \beta > 0, \tag{3.16}$$

where φ is the initial condition in (1.3), ε and $\widetilde{\varepsilon}$ are two positive numbers which can be arbitrarily small, there exists a constant C so that for the solution calculated using scheme (2.4):

$$\|U^n\|_h \le C, \quad |U^n|_{h,1} \le C, \quad t_n \in \Omega_\tau. \tag{3.17}$$

Proof. It follows from (3.11) that

$$\|U^n\|_h = \|\varphi\|_h \le C, \quad t_n \in \Omega_\tau. \tag{3.18}$$

First, under condition (a), we obtain directly from (3.12) that

$$|U^n|_h \le C, \quad t_n \in \Omega_\tau.$$

This, together with Lemma 2.4, gives

$$|U^n|_{h,1} \le C, \quad t_n \in \Omega_\tau. \tag{3.19}$$

Second, under condition (b), using Lemma 3.2 and Theorem 3.5, we obtain

$$\begin{aligned}\|U^n\|_{h,4}^4 &\le \|U^n\|_h^2 \left(2|U^n|_{h,1} + \frac{1}{l}\|U^n\|_h\right)^2 \le 4(1+\varepsilon)\|U^n\|_h^2|U^n|_{h,1}^2 + (1+\varepsilon^{-1})\frac{1}{l^2}\|U^n\|_h^4 \\ &= 4(1+\varepsilon)\|\varphi\|_h^2|U^n|_{h,1}^2 + (1+\varepsilon^{-1})\frac{1}{l^2}\|\varphi\|_h^4 \le \frac{2-\widetilde{\varepsilon}}{\beta}|U^n|_{h,1}^2 + (1+\varepsilon^{-1})\frac{1}{l^2}\|\varphi\|_h^4,\end{aligned} \tag{3.20}$$

where the inequality $(a+b)^2 \le (1+\varepsilon)a^2 + (1+\varepsilon^{-1})b^2$ is used. This, together with Lemma 2.4 and Theorem 3.5, gives

$$\begin{aligned}|U^n|_{h,1}^2 &\le |U^n|_h^2 = E^0 + \frac{\beta}{2}\|U^n\|_{h,4}^4 \\ &\le \frac{2-\widetilde{\varepsilon}}{2}|U^n|_{h,1}^2 + C,\end{aligned} \tag{3.21}$$

which implies $|U^n|_{h,1} \le C$, $t_n \in \Omega_\tau$. □

4. Convergence of the scheme

Different from the finite difference method, error estimates of spectral or pseudo-spectral method often rely on the projection error and interpolation error in the spectral space. In this section, we present some basic approximation results for the trigonometric polynomials and prove the unconditional convergence of scheme (2.4) in the discrete L^2 norm.

For simplicity, we let $\Omega = [0, 2\pi] \times [0, 2\pi]$, $L^2(\Omega)$ with the inner product $(\cdot,\cdot)$ and the norm $\|\cdot\|$ defined previously. For any positive integer r, the semi-norm and the norm of $H^r(\Omega)$ are denoted by $|\cdot|_r$ and $\|\cdot\|_r$, respectively. Let $C_p^\infty(\Omega)$ be the set of infinitely differentiable functions with $(2\pi, 2\pi)$-period. $H_p^r(\Omega)$ is the closure of $C_p^\infty(\Omega)$ in $H^r(\Omega)$. Let $N_x = N_y = N$, the interpolation space S_N'' can be written as

$$S_N'' = \left\{ u \middle| u(x,y) = \sum_{|l|\le N/2} \sum_{|m|\le N/2} \frac{\widehat{u}_{l,m}}{c_l c_m} e_{l,m}, \widehat{u}_{\frac{N}{2},m} = \widehat{u}_{-\frac{N}{2},m}, \widehat{u}_{l,\frac{N}{2}} = \widehat{u}_{l,-\frac{N}{2}}, |l| \le \frac{N}{2}, |m| \le \frac{N}{2} \right\},$$

where $e_{l,m} = e^{i(lx+my)}$, $c_l = \begin{cases} 1, |l| < N/2, \\ 2, |l| = N/2. \end{cases}$

We denote

$$S_N = \left\{ u \middle| u(x,y) = \sum_{|l|\le N/2} \sum_{|m|\le N/2} \widehat{u}_{l,m} e_{l,m} \right\}.$$

It is obviously that $S_N'' \subseteq S_N$. We denote the orthogonal projection operator as $P_N : L^2(\Omega) \to S_N$ and recall the interpolation operator $I_N : L^2(\Omega) \to S_N''$. Note that P_N and I_N satisfy the following properties:

- $P_N \partial_z u = \partial_z P_N u$, $I_N \partial_z u \ne \partial_z I_N u$, $z = x$ or y;
- $P_N u = u$, $\forall u \in S_N$; $I_N u = u$, $\forall u \in S_N''$.

Lemma 4.1. *For any $u \in S_N''$, $\|u\| \le \|u\|_h \le 2\|u\|$.*

Proof. For any $u \in S_N''$, noticing that $(e_{l,m}, e_{j,k}) = 4\pi^2 \delta_l^j \delta_m^k$, we obtain

$$\begin{aligned}
\|u\|^2 &= \Big(\sum_{|l|\le N/2}\sum_{|m|\le N/2} \frac{\widehat{u}_{l,m}}{c_l c_m} e_{l,m}, \sum_{|l|\le N/2}\sum_{|m|\le N/2} \frac{\widehat{u}_{l,m}}{c_l c_m} e_{l,m}\Big) \\
&= 4\pi^2 \sum_{|l|\le N/2}\sum_{|m|\le N/2} \frac{|\widehat{u}_{l,m}|^2}{(c_l c_m)^2} \\
&= 4\pi^2 \left(\frac{1}{4}|\widehat{u}_{N/2,N/2}|^2 + \frac{1}{2}\sum_{|l|<N/2} |\widehat{u}_{l,N/2}|^2 + \frac{1}{2}\sum_{|m|<N/2} |\widehat{u}_{N/2,m}|^2 + \sum_{|l|<N/2}\sum_{|m|<N/2} |\widehat{u}_{l,m}|^2 \right).
\end{aligned} \tag{4.1}$$

Similarly, noticing that

$$(e_{l,m}, e_{j,k})_N = \begin{cases} 4\pi^2, & l \equiv j (\mathrm{mod}\ N) \text{ and } m \equiv k (\mathrm{mod}\ N), \\ 0, & \text{other}, \end{cases}$$

we obtain

$$\|u\|_h^2 = 4\pi^2 \left(|\widehat{u}_{N/2,N/2}|^2 + \sum_{|l|<N/2} |\widehat{u}_{l,N/2}|^2 + \sum_{|m|<N/2} |\widehat{u}_{N/2,m}|^2 + \sum_{|l|<N/2}\sum_{|m|<N/2} |\widehat{u}_{l,m}|^2 \right). \tag{4.2}$$

Combining (4.1) and (4.2), we have

$$\|u\| \le \|u\|_h \le 2\|u\|. \qquad \square$$

Lemma 4.2 *([16]). If $0 \le l \le r$ and $u \in H_p^r(\Omega)$,*

$$\|P_N u - u\|_l \le C N^{l-r} |u|_r,$$
$$\|P_N u\|_l \le C\|u\|_l;$$

if, in addition, $r > 1$,

$$\|I_N u - u\|_l \le C N^{l-r} |u|_r,$$
$$\|I_N u\|_l \le C\|u\|_l.$$

Lemma 4.3. *For $u \in H_p^r(\Omega)$, $r > 1$, let $u^* = P_{N-2}u$, then $\|u^* - u\|_h \le C N^{-r}|u|_r$.*

Proof. From Lemma 4.1 and Lemma 4.2, we obtain

$$\begin{aligned}
\|u^* - u\|_h &= \|I_N(u^* - u)\|_h \\
&\le 2\|I_N(u^* - u)\| \\
&\le C\|u^* - u\| \\
&\le C(N-2)^{-r}|u|_r = C(1 - \frac{2}{N})^{-r} N^{-r}|u|_r \\
&\le C N^{-r}|u|_r,
\end{aligned}$$

where the last inequality holds if $N \ge 4$. $\square$

Lemma 4.4 *([52]). For any complex numbers U, V, u, v, the following inequality holds*

$$\Big| |U|^2 V - |u|^2 v \Big| \le \Big(\max\{|U|, |V|, |u|, |v|\} \Big)^2 \cdot \big(2|U - u| + |V - v|\big). \tag{4.3}$$

Lemma 4.5 *(Gronwall Inequality [61]). Suppose that the nonnegative discrete function $\{\omega^n | n = 0, 1, 2, \ldots, N_t; N_t\tau = T\}$ satisfies the inequality*

$$\omega^n - \omega^{n-1} \le A\tau\omega^n + B\tau\omega^{n-1} + C_n\tau,$$

where A, B and $C_n (n = 1, 2, \ldots, N_t)$ are nonnegative constants. Then

$$\max_{1\le n\le N_t} \omega^n \le \left(\omega^0 + \tau \sum_{l=1}^{N_t} C_l\right) e^{2(A+B)T},$$

where τ is sufficiently small, such that $(A+B)\tau \le \frac{N_t-1}{2N_t}$ $(N_t > 1)$.

Theorem 4.6. *Under assumptions (a) and (b) of Lemma 3.6, if*

$$u(x, y, t) \in C^3\Big(0, T; H_p^r(\Omega)\Big),\ r > 1,$$

the Fourier pseudo-spectral solution of scheme (2.4) converges, without any restrictions on the grid ratio, to the solution of the initial-periodic boundary value problem (1.1)–(1.3) in the order of $O(N^{-r} + \tau^2)$ in the discrete L^2 norm.

Proof. We denote

$$u^* = P_{N-2}u, \quad f = f(u) = \beta|u|^2 u, \quad f^* = P_{N-2}f.$$

The projection of Eq. (1.1) is written as

$$i\partial_t u^* + \partial_{xx}u^* + \partial_{yy}u^* + f^* = 0. \tag{4.4}$$

We define

$$\eta_{j,k}^n = i\delta_t^+ u_{j,k}^{*n} + \left(D_2^x u^{*n+1/2}\right)_{j,k} + \left(u^{*n+1/2} D_2^y\right)_{j,k} + f_{j,k}^{*n+1/2}. \tag{4.5}$$

Noticing $u^* \in S_J''$, $\partial_{xx}u^*(x_j, y_k, t_n) = \left(D_2^x u^{*n}\right)_{j,k}$, $\partial_{yy}u^*(x_j, y_k, t_n) = \left(u^{*n} D_2^y\right)_{j,k}$, we obtain

$$\eta_{j,k}^n = i\left(\delta_t^+ u_{j,k}^{*n} - \partial_t u_{j,k}^{*n+1/2}\right). \tag{4.6}$$

Using the Taylor expansion, we obtain

$$|\eta_{j,k}^n| \le C\tau^2 \tag{4.7}$$

for some constant C.

Denote $e_{j,k}^n = u_{j,k}^{*n} - U_{j,k}^n$. Subtracting (2.4) from (4.5) yields the following error equation

$$i\delta_t^+ e^n + D_2^x e^{n+1/2} + e^{n+1/2} D_2^y + G^n = \eta^n, \tag{4.8}$$

$$e^0 = u^{*0} - u^0, \tag{4.9}$$

where

$$G_{j,k}^n = f_{j,k}^{*n+1/2} - F(U_{j,k}^n, U_{j,k}^{n+1}).$$

Denoting

$$(G_1)_{j,k}^n = f_{j,k}^{*n+1/2} - f_{j,k}^{n+1/2}, \quad (G_2)_{j,k}^n = f_{j,k}^{n+1/2} - F(u_{j,k}^n, u_{j,k}^{n+1}),$$

$$(G_3)_{j,k}^n = F(u_{j,k}^n, u_{j,k}^{n+1}) - F(u_{j,k}^{*n}, u_{j,k}^{*n+1}), \quad (G_4)_{j,k}^n = F(u_{j,k}^{*n}, u_{j,k}^{*n+1}) - F(U_{j,k}^n, U_{j,k}^{n+1}),$$

we have

$$G_{j,k}^n = (G_1)_{j,k}^n + (G_2)_{j,k}^n + (G_3)_{j,k}^n + (G_4)_{j,k}^n.$$

According to Lemma 4.3, we have

$$\|G_1^n\|_h \le CN^{-r}. \tag{4.10}$$

Using the Taylor expansion with an integral remainder, there exist complex numbers c_k, $|c_k| \le C$, $k = 1, 2, 3$ such that

$$\frac{1}{2}(|u^n|^2 u^n + |u^{n+1}|^2 u^{n+1}) = |u(x, y, t_{n+1/2})|^2 u(x, y, t_{n+1/2}) + c_1\tau^2, \tag{4.11}$$

$$\frac{1}{2}(|u^n|^2 + |u^{n+1}|^2) = |u(x, y, t_{n+1/2})|^2 + c_2\tau^2, \tag{4.12}$$

$$u^{n+1/2} = u(x, y, t_{n+1/2}) + c_3\tau^2. \tag{4.13}$$

Therefore we have

$$|(G_2)_{j,k}^n| = \left|\frac{\beta}{2}\left(|u_{j,k}^n|^2 u_{j,k}^n + |u_{j,k}^{n+1}|^2 u_{j,k}^{n+1}\right) - \frac{\beta}{2}\left(|u_{j,k}^n|^2 + |u_{j,k}^{n+1}|^2\right) u_{j,k}^{n+1/2}\right| \le C\tau^2. \tag{4.14}$$

From Lemma 4.4, we deduce

$$|(G_3)^n_{j,k}| \le C\big(|u^n_{j,k} - u^{*n}_{j,k}| + |u^{n+1}_{j,k} - u^{*n+1}_{j,k}|\big). \tag{4.15}$$

This, together with Lemma 4.3, gives

$$\|G^n_3\|_h \le CN^{-r}. \tag{4.16}$$

Notice that

$$\begin{aligned}(G_4)^n_{j,k} &= \frac{\beta}{2}\left(|u^{*n}_{j,k}|^2 + |u^{*n+1}_{j,k}|^2\right)u^{*n+1/2}_{j,k} - \frac{\beta}{2}\left(|U^n_{j,k}|^2 + |U^{n+1}_{j,k}|^2\right)U^{n+1/2}_{j,k}\\ &= \frac{\beta}{2}\left(|u^{*n}_{j,k}|^2 + |u^{*n+1}_{j,k}|^2 - |U^n_{j,k}|^2 - |U^{n+1}_{j,k}|^2\right)u^{*n+1/2}_{j,k} + \frac{\beta}{2}\left(|U^n_{j,k}|^2 + |U^{n+1}_{j,k}|^2\right)e^{n+1/2}_{j,k}\\ &:= \phi^n_{j,k} + \psi^n_{j,k},\end{aligned} \tag{4.17}$$

where

$$\begin{aligned}\phi^n_{j,k} &= \frac{\beta}{2}\left(|u^{*n}_{j,k}|^2 + |u^{*n+1}_{j,k}|^2 - |u^{*n}_{j,k} - e^n_{j,k}|^2 - |u^{*n+1}_{j,k} - e^{n+1}_{j,k}|^2\right)u^{*n+1/2}_{j,k}\\ &= \frac{\beta}{2}\left(u^{*n}_{j,k}\overline{e}^n_{j,k} + \overline{u^*}^n_{j,k}e^n_{j,k} + u^{*n+1}_{j,k}\overline{e}^{n+1}_{j,k} + \overline{u^*}^{n+1}_{j,k}e^{n+1}_{j,k} - |e^n_{j,k}|^2 - |e^{n+1}_{j,k}|^2\right)u^{*n+1/2}_{j,k},\end{aligned} \tag{4.18}$$

$$\psi^n_{j,k} = \frac{\beta}{2}\left(|U^n_{j,k}|^2 + |U^{n+1}_{j,k}|^2\right)e^{n+1/2}_{j,k}. \tag{4.19}$$

Computing the discrete inner product of (4.8) with $e^{n+1/2}$, then taking the imaginary part, we obtain

$$\frac{1}{2\tau}(\|e^{n+1}\|^2_h - \|e^n\|^2_h) + \mathrm{Im}(G^n_1 + G^n_2 + G^n_3 + G^n_4, e^{n+1/2})_h = \mathrm{Im}(\eta^n, e^{n+1/2})_h, \tag{4.20}$$

where Lemma 3.1 was used. Using Cauchy–Schwartz inequality, we obtain

$$\left|(G^n_s, e^{n+1/2})_h\right| \le \frac{1}{2}\|G^n_s\|^2_h + \frac{1}{4}(\|e^n\|^2_h + \|e^{n+1}\|^2_h),\ \ s = 1, 2, 3, \tag{4.21}$$

$$\left|\mathrm{Im}(G^n_4, e^{n+1/2})_h\right| = \left|\mathrm{Im}(\phi^n, e^{n+1/2})_h\right| \le C(\|e^n\|^2_h + \|e^{n+1}\|^2_h + \|e^n\|^4_{h,4} + \|e^{n+1}\|^4_{h,4}), \tag{4.22}$$

$$\left|(\eta^n, e^{n+1/2})_h\right| \le \frac{1}{2}\|\eta^n\|^2_h + \frac{1}{4}(\|e^n\|^2_h + \|e^{n+1}\|^2_h). \tag{4.23}$$

It follows from Lemma 3.6, Lemma 4.1 and Lemma 4.2 that

$$\|e^n\|_h = \|u^{*n} - U^n\|_h \le \|u^{*n}\|_h + \|U^n\|_h \le 2\|u^{*n}\| + \|U^n\|_h \le C. \tag{4.24}$$

Similarly, we obtain

$$|e^n|_{h,1} \le C. \tag{4.25}$$

Eqs. (4.24)–(4.25), together with Lemma 3.2, give

$$\|e^n\|^4_{h,4} \le C\|e^n\|^2_h. \tag{4.26}$$

This, together with (4.22), gives

$$\left|\mathrm{Im}(G^n_4, e^{n+1/2})_h\right| \le C(\|e^n\|^2_h + \|e^{n+1}\|^2_h). \tag{4.27}$$

Substituting (4.21), (4.23) and (4.27) into (4.20) yields

$$\frac{1}{2\tau}(\|e^{n+1}\|^2_h - \|e^n\|^2_h) \le C(\|e^n\|^2_h + \|e^{n+1}\|^2_h) + \frac{1}{2}\left(\|G^n_1\|^2_h + \|G^n_2\|^2_h + \|G^n_3\|^2_h + \|\eta^n\|^2_h\right). \tag{4.28}$$

This, together with Lemma 4.5 and (4.10), (4.14), (4.16), gives that, for a sufficiently small τ,

$$\|e^n\|^2_h \le \left(\|e^0\|^2_h + CT(N^{-2r} + \tau^4)\right)e^{4CT}. \tag{4.29}$$

This, together with

$$\|e^0\|_h = \|u^{*0} - u^0\|_h \le CN^{-r},$$

Table 1
Convergence test in space with $\tau = 1.0\text{e-}6$ at $T = 1$.

$N_x \times N_y$	Error		Maximum iterations	CPU(s)
	L^∞	L^2		
8×8	1.2341e-10	6.2680e-10	3	59.4
16×16	4.6966e-11	2.2571e-10	3	138.6
32×32	7.9636e-11	4.5296e-10	3	359.8
64×64	1.4757e-11	5.0357e-11	3	1386.0
128×128	3.7748e-11	1.8009e-10	3	3765.4

gives

$$\|e^n\|_h \le C(N^{-r} + \tau^2). \tag{4.30}$$

According to Lemma 4.3 and (4.30), we obtain

$$\|u^n - U^n\|_h \le \|u^n - u^{*n}\|_h + \|u^{*n} - U^n\|_h \le C(N^{-r} + \tau^2).$$

This completes the proof. □

5. Numerical examples

In this section, we present some numerical examples to substantiate our theoretical analyses on convergence and conservation of mass and energy.

First, we present a fast approach to solving nonlinear system (2.5) [49]. At time step n, the nonlinear system defined in scheme (2.5) is computed by the following fixed-point iteration method

$$i\frac{U_{s+1}^{n+1} - U^n}{\tau} + D_2^x \frac{U_{s+1}^{n+1} + U^n}{2} + \frac{U_{s+1}^{n+1} + U^n}{2} D_2^y + F(U^n, U_s^{n+1}) = 0, \tag{5.1}$$

where the initial iteration $U_0^{n+1} = U^n$. We iterate the solution until the following criteria is satisfied

$$\|U_{s+1}^{n+1} - U_s^{n+1}\|_{h,\infty} < 10^{-14}.$$

For iteration step s, system (5.1) can be simplified into

$$iU_{s+1}^{n+1} + \frac{\tau}{2} D_2^x U_{s+1}^{n+1} + \frac{\tau}{2} U_{s+1}^{n+1} D_2^y = B, \tag{5.2}$$

where $B = iU^n - \frac{\tau}{2} D_2^x U^n - \frac{\tau}{2} U^n D_2^y - \tau F(U^n, U_s^{n+1})$. Let $\widetilde{U}_{s+1}^{n+1} = F_{N_x} U_{s+1}^{n+1} F_{N_y}^H$, $\widetilde{B} = F_{N_x} B F_{N_y}^H$. According to Lemma 2.2 and $F_N^H F_N = I$, multiplying F_{N_x} and $F_{N_y}^H$ on both sides of (5.2), respectively, we have

$$i\widetilde{U}_{s+1}^{n+1} + \frac{\tau}{2} \Lambda_3 \widetilde{U}_{s+1}^{n+1} + \frac{\tau}{2} \widetilde{U}_{s+1}^{n+1} \Lambda_4 = \widetilde{B}, \tag{5.3}$$

which gives

$$(\widetilde{U}_{s+1}^{n+1})_{j,k} = \frac{\widetilde{B}_{j,k}}{i + \frac{\tau}{2}\lambda_{D_2^x,j} + \frac{\tau}{2}\lambda_{D_2^y,k}}. \tag{5.4}$$

Solving the above equations gives the matrix $\widetilde{U}_{s+1}^{n+1}$. Using the relation $U_{s+1}^{n+1} = F_{N_x}^H \widetilde{U}_{s+1}^{n+1} F_{N_y}$ yields the solution matrix U_{s+1}^{n+1}. Note that we can apply the Fast Fourier Transform (FFT) algorithm to the above process.

Example 5.1. We consider the 2D NLS equation (1.1) with a progressive plane wave solution [58]

$$u(x, y, t) = A \exp\big(i(k_1 x + k_2 y - \omega t)\big), \tag{5.5}$$

where $\omega = k_1^2 + k_2^2 - \beta|A|^2$.

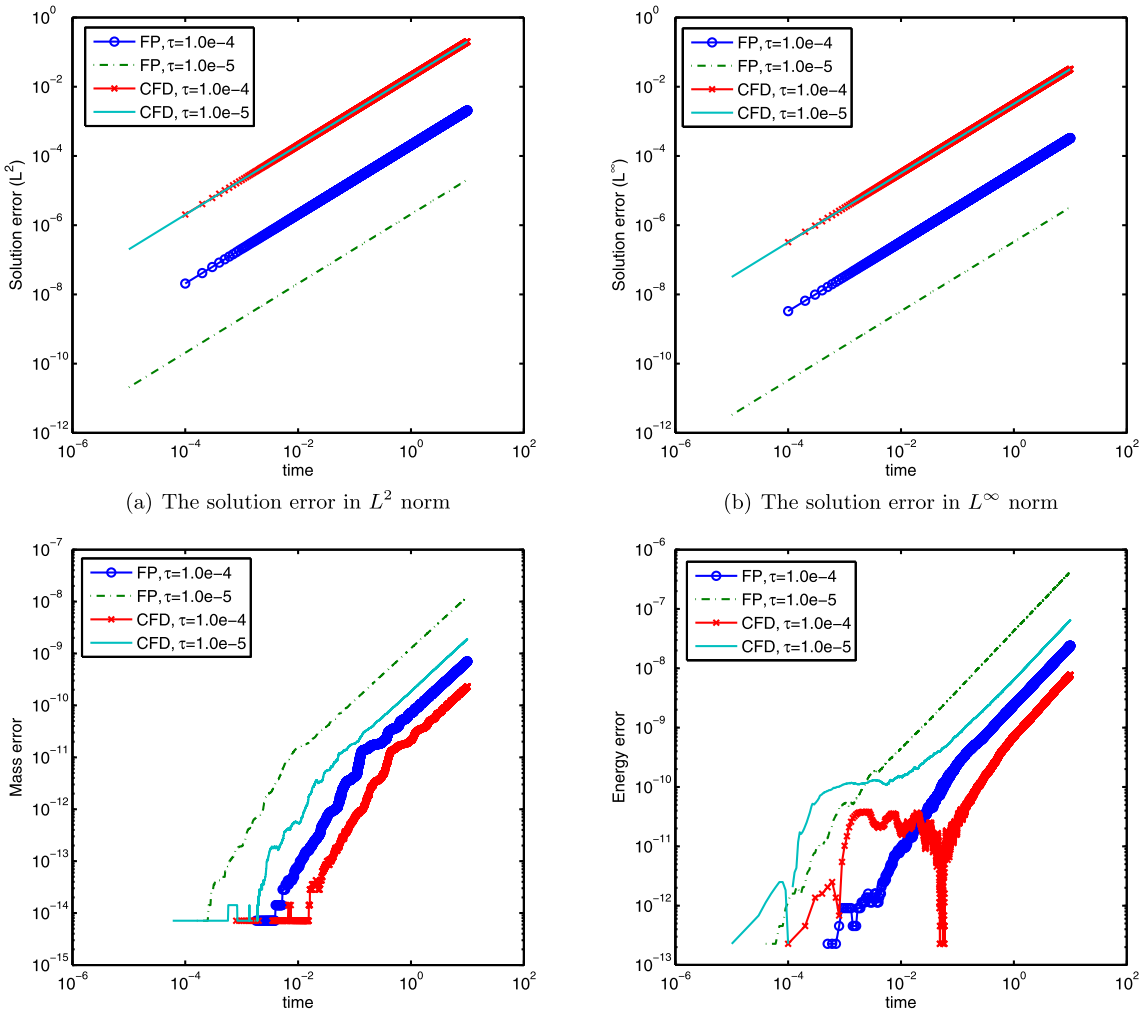
Firstly, for the convergence test, the problem is solved on domain $[0, 2\pi] \times [0, 2\pi]$ until $T = 1$ with $A = 1$, $k_1 = k_2 = 1$, $\beta = -1$. The convergence order is calculated using the formula

$$Order = \frac{\ln\big(error_1/error_2\big)}{\ln(\delta_1/\delta_2)}, \tag{5.6}$$

where δ_l, $error_l$ $(l = 1, 2)$ are step size and the error with step size δ_l, respectively. Table 1 shows that the spatial error is very small and almost negligible, and the error is dominated by the time discretization error. It confirms that, for the sufficiently smooth problem, the Fourier pseudo-spectral method is of arbitrary order of accuracy. Table 2 clearly indicates

Table 2
Convergence test in time with $N_x = N_y = 64$ at $T = 1$.

τ	Error		Order		Maximum iterations	CPU(s)
	L^∞	L^2	L^∞	L^2		
0.02	8.9951e-04	5.6518e-03	–	–	9	0.3
0.01	2.2497e-04	1.4135e-03	1.9994	1.9994	8	0.6
0.005	5.6248e-05	3.5342e-04	1.9999	1.9999	7	1.0
0.0025	1.4062e-05	8.8357e-05	2.0000	2.0000	6	1.7
0.00125	3.5156e-06	2.2089e-05	2.0000	2.0000	5	2.7



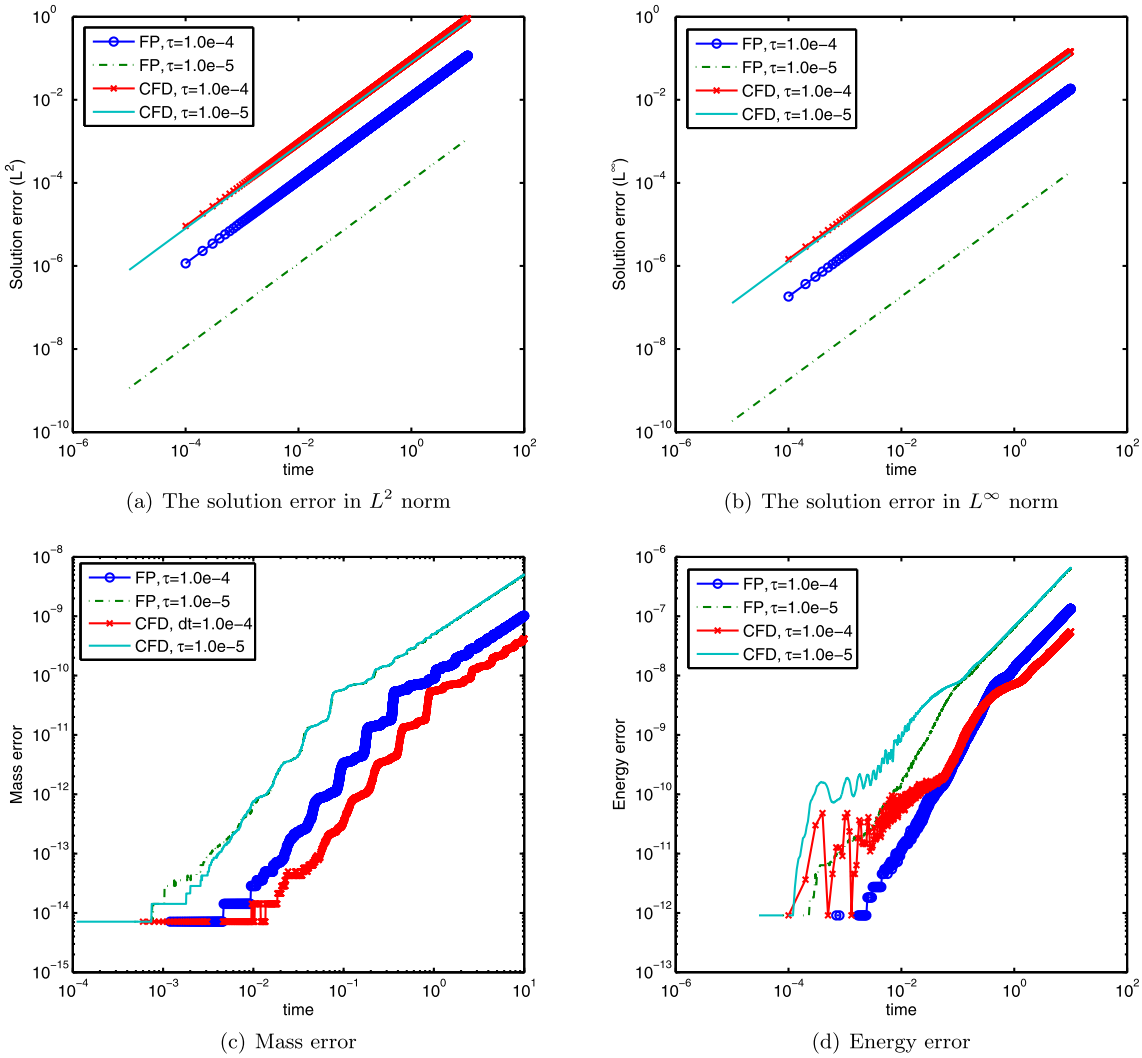
(a) The solution error in L^2 norm (b) The solution error in L^∞ norm

(c) Mass error (d) Energy error

Fig. 1. Error comparison between the FP and the CFD method ($\beta = -2$, $A = 1$, $k_1 = k_2 = 4$, $N_x = N_y = 64$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

that the method is of the second-order in time. These numerical results confirm the result in Theorem 4.6. The maximum iterations and the total CPU time are also given in Tables 1–2, which show that our algorithm is very efficient in this practical computation. It is worth noting that the maximum number of iterations reduces as time step τ decreases.

Secondly, we compare the Fourier pseudo-spectral (FP) method (2.5) with the compact finite difference (CFD) method proposed in [56]. In this comparison, we focus on the long-time behavior of solutions in capturing high-frequency waves. The computations are conducted on $[0, 2\pi] \times [0, 2\pi]$ until $T = 10$. Given the spatial grid size $N_x = N_y = 64$ and the parameter values $A = 1$, $k_1 = k_2 = 4$, $\beta = -2$, we compute the discrete L^2 and L^∞ errors between the exact solution and the numerical solution obtained by the two methods, respectively, as shown in Fig. 1 a–b. For $t \in [0, 10]$, the solution error increases linearly over time and the numerical approximate results obtained using the FP method are much better than that



(a) The solution error in L^2 norm

(b) The solution error in L^∞ norm

(c) Mass error

(d) Energy error

Fig. 2. Error Comparison between the FP and the CFD method ($\beta = -2$, $A = 1$, $k_1 = k_2 = 8$, $N_x = N_y = 128$). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

of the CFD method. When the time step reduces by one order of magnitude ($\tau = 1.0e^{-5}$), the error using the FP method reduces further while that using the CFD does not show much improvement. The errors in the global mass and energy are also plotted in Fig. 1 c–d. Notice that the errors in mass and energy are very small, despite of a linear growth in the solution, which might be due to the temporal finite differencing. Similar phenomena are shown in Fig. 2, where a high-frequency wave with $A = 1$, $k_1 = k_2 = 8$, $\beta = -2$ is approximated.

Example 5.2. In this example, we show a singular solution for the 2D NLS equation

$$iu_t + \Delta u + |u|^2 u = 0, \tag{5.7}$$

with initial condition

$$u(x, y, 0) = (1 + \sin x)(2 + \sin y), \ (x, y) \in [0, 2\pi] \times [0, 2\pi],$$

and the $(2\pi, 2\pi)$-periodic boundary condition. Fig. 3 shows the singular solution, which matches the result in [58] very well. The errors in mass and energy are plotted in Fig. 4, which are conserved very well.

6. Conclusions

In this paper, we have developed a high order numerical method for solving the 2D NLS equation, where the Crank–Nicolson method in time and Fourier pseudo-spectral method in space discretization is implemented. We show that it

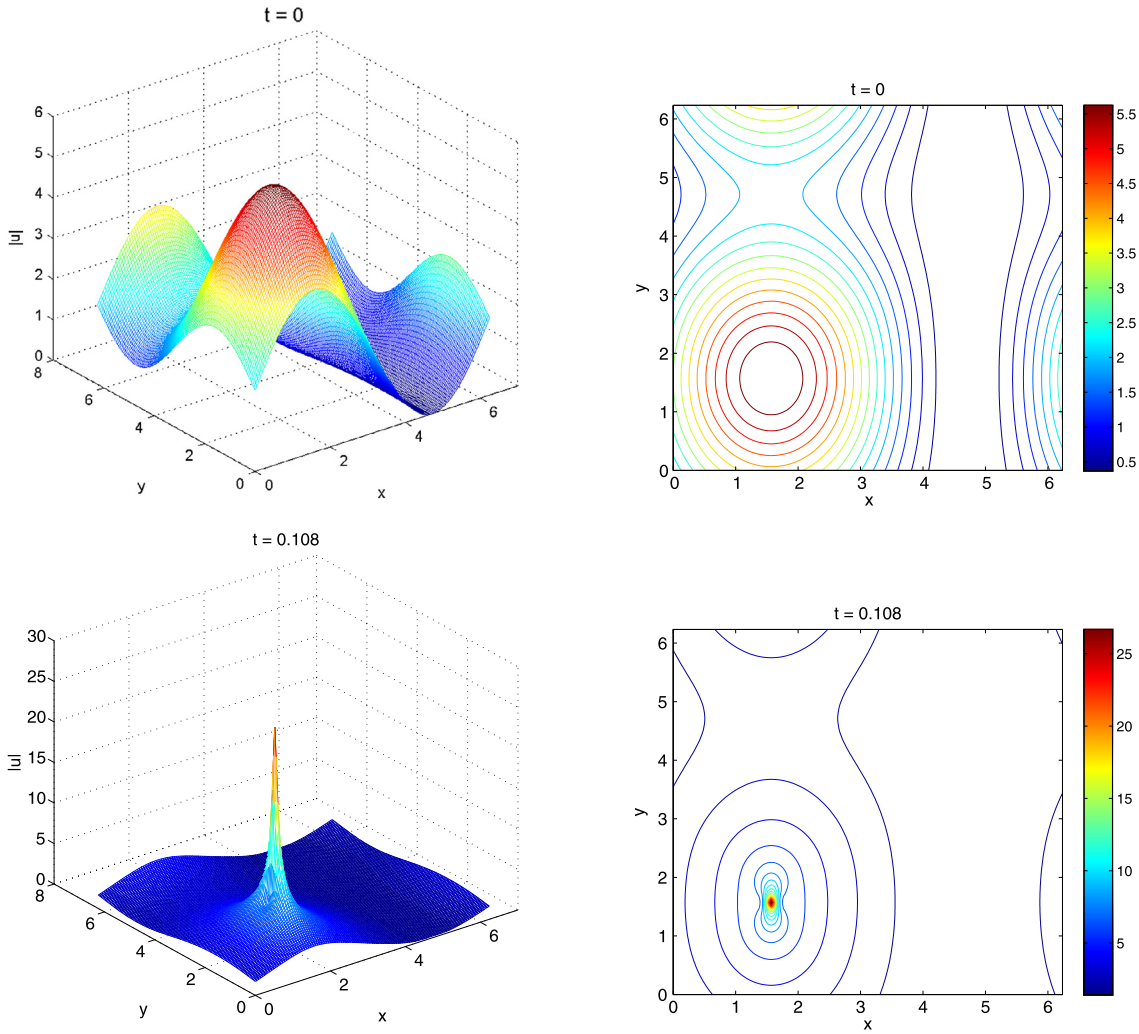


Fig. 3. Profiles and contours of modulus of initial data (top) and the singular solution, where the singularity develops at $t = 0.108$ (bottom) with $N_x = N_y = 128$, $\tau = 0.0001$. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

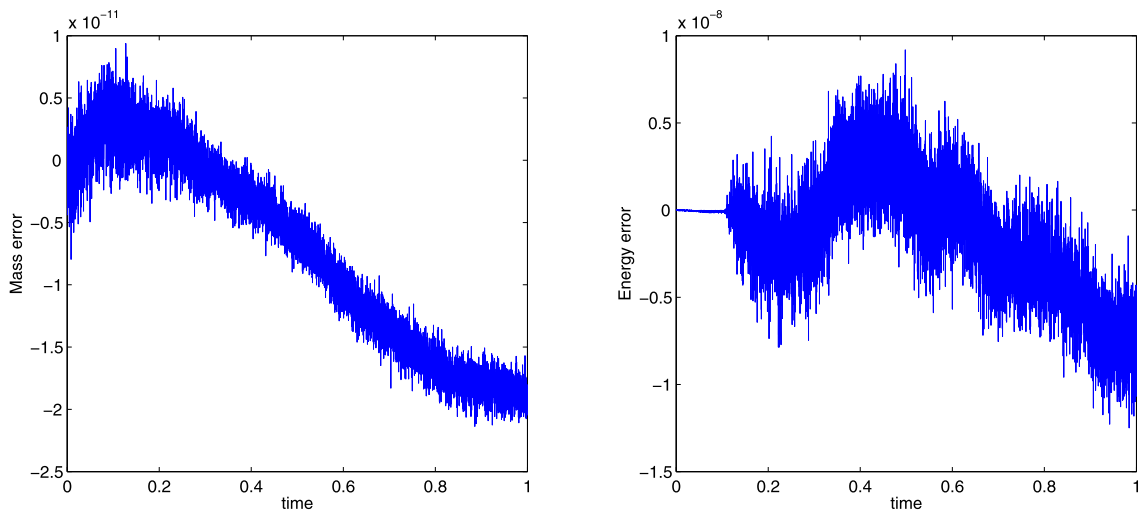


Fig. 4. The error of the numerical singular solution in mass (left) and in energy (right).

conserves mass and energy at the discrete level, consistent with the continuum theory. We first establish the fact that the semi-norm in the Fourier pseudo-spectral method is equivalent to that in the finite difference method. This enables us to apply the techniques developed in [56] to analyze the Fourier pseudo-spectral method. The pseudo-spectral method is shown to be convergent in the order of $O(N^{-r} + \tau^2)$ in the discrete L^2 norm. However, the proposed method is implicit and nonlinear. To obtain the solution U^{n+1} at time level $n+1$, a nonlinear iteration for U^{n+1} need to be done. A simple fixed-point iteration method is employed to solve the nonlinear equation systems defined in scheme (2.5), where the FFT is used to speed up the computation. Two numerical examples are presented to illustrate the efficiency and accuracy of the new scheme in the end, where it shows that the pseudo-spectral method is much better than the CFD method in accuracy.

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