Conservative Compact Finite Difference Scheme for the Coupled Schrödinger–Boussinesq Equation

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In this article, two conserved compact finite difference schemes for solving the nonlinear coupled Schrödinger–Boussinesq equation are proposed. The conservative property, existence, convergence, and stability of the difference solutions are theoretically analyzed. The numerical results are reported to demonstrate the accuracy and efficiency of the methods and to confirm our theoretical analysis. Numer Methods Partial Differential Eq 32: 1667–1688, 2016

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I. INTRODUCTION

The coupled nonlinear Schrödinger–Boussinesq (SBq) equation

$$iu_t + u_{xx} - uv = 0, x \in \mathbb{R}, t > 0, \tag{1}$$

$$v_{tt} = v_{xx} - \alpha v_{xxxx} + f(v)_{xx} + \omega|u|^2_{xx}, x \in \mathbb{R}, t > 0, \tag{2}$$

has been used in the field of laser and plasma physics under the interaction of a nonlinear Schrödinger field and a real Boussinesq field [1]. Guo and Du [2] proved the existence and uniqueness of periodic strong solutions of the SBq equation. In [3], the global existence of solutions and the long time behavior of nonlinear SBq equation with zero order dissipation was considered. Li and Chen [4] investigated initial boundary value problems of dissipative SBq equation and proved the existence of global attractors. In the recent years, there exist several papers (see [5–10] and references therein) for finding analytical solutions of SBq equations.

To the best of our knowledge, the work done on the numerical methods for SBq equation is so far quite limited. Bai and Zhang [11] proposed a quadratic B-spline finite element scheme and established error estimates for that method. A time-splitting Fourier spectral method was studied

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by Bai and Wang in [12]. Huang et al. considered a conservative multisymplectic scheme based on center discrete methods in [13], which could simulate the solitary waves for a long times. In [14], Zhang et al. obtained an optimal error estimate for an implicit conservative difference scheme with order $O(\tau^2 + h^2)$. In addition, we do not located any other numerical schemes for SBq equation. Thus, the task of constructing efficient numerical methods for these kind of problems is of interest. The main purpose of this study is to construct some conservative compact difference schemes to solve nonlinear coupled SBq equations numerically.

In this article, we consider following initial-boundary value problem for the SBq equation

$$iu_t + u_{xx} - uv = 0, x \in \Omega, 0 < t \le T, \tag{3}$$

$$v_t = \phi_{xx}, x \in \Omega, 0 < t \le T, \tag{4}$$

$$\phi_t = v - \alpha v_{xx} + f(v) + \omega|u|^2, x \in \Omega, 0 < t \le T, \tag{5}$$

$$u(x,0) = u_0(x), v(x,0) = v_0(x), \phi(x,0) = \phi_0(x), x \in \Omega, \tag{6}$$

$$u(x,t) = v(x,t) = \phi(x,t) = 0, v_{xx}(x,t) = 0, x \in \partial\Omega, \tag{7}$$

where $\Omega = (-L, L)$, α and ω are positive constants, $f(v)$ is a sufficiently smooth function with $f(0) = 0$, $u(x,t)$ is a complex function while $v(x,t)$ is a real one.

It is easy to verify that problem (3)–(7) satisfies the following conservative laws

$$Q(t) = \int_{-L}^{L} |u|^2 dx = \int_{-L}^{L} |u_0|^2 dx = Q(0), \tag{8}$$

$$E(t) = \int_{-L}^{L} (v^2 + \phi_x^2 + 2\omega|u_x|^2 + \alpha v_x^2 + 2F(v) + 2\omega v|u|^2) dx = E(0), \tag{9}$$

where $F(v) > 0$ is a primitive function of $f(v)$.

The remainder of this article is structured as follows. We introduce some useful lemmas in Section II. In Section III, a two level compact difference scheme (Scheme A) is constructed and the conservative property and priori estimation are also discussed. We have investigated the existence of the difference solutions in Section IV. The convergence and stability are theoretically analyzed in Section V. Additionally, we proposed a three-level scheme (Scheme B) in Section VI, the conservative property, a priori estimation, existence, convergence, and stability of the difference solutions are briefly summarized in this section. The computational methods of the two schemes are introduced in Section VII. Some numerical results are reported in Section VIII, and the last section is the conclusion.

II. PRELIMINARY

Let $x_j = jh$, $t_n = n\tau$, $0 \le j \le J$, $0 \le n \le N$, where h and τ are the step size in the space and temporal directions, respectively. Denote $U_j^n = u(x_j, t_n)$, $V_j^n = v(x_j, t_n)$, $\Phi_j^n = \phi(x_j, t_n)$ while u_j^n, v_j^n, ϕ_j^n represent the approximation of U_j^n, V_j^n, Φ_j^n at grid points (x_j, t_n), respectively. As usual, we introduce some notations as follows

$$u_j^{\bar{n}} = \frac{1}{2}(u_j^{n+1} + u_j^{n-1}), u_j^{n+\frac{1}{2}} = \frac{1}{2}(u_j^{n+1} + u_j^n), \delta_t u_j^{n+\frac{1}{2}} = \frac{1}{\tau}(u_j^{n+1} - u_j^n), \delta_{\hat{t}} u_j^n = \frac{1}{2\tau}(u_j^{n+1} - u_j^{n-1}),$$

$$\delta_x^+ u_j^n = \frac{1}{h}(u_{j+1}^n - u_j^n), \delta_x^2 u_j^n = \frac{1}{h^2}(u_{j-1}^n - 2u_j^n + u_{j+1}^n), \mathcal{A}u_j^n = \left(1 + \frac{h^2}{12}\delta_x^2\right) u_j^n.$$

Define the grid function space

$$\mathbb{V}_h = \left\{\mathbf{u}^n | \mathbf{u}^n = (u_1^n, u_2^n, \cdots, u_{J-1}^n)^T\right\} \subseteq \mathbb{C}^{J-1},$$

and the matrix

$$\mathcal{M} = \frac{1}{12}\begin{pmatrix} 10 & 1 & 0 & \cdots & 0 \\ 1 & 10 & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 1 & 10 & 1 \\ 0 & \cdots & 0 & 1 & 10 \end{pmatrix}_{(J-1)\times(J-1)}.$$

Since $\mathcal{M}$ is a positive define matrix, $\mathcal{N} = \mathcal{M}^{-1}$ is also positive.

Considering (7), we have $u_0^n = u_J^n = 0$. Especially, we denote $\delta_x^+ u_0^n = \frac{1}{h}u_1^n, \delta_x^+ u_{J-1}^n = \frac{-1}{h}u_{J-1}^n, \delta_x^2 u_1^n = \frac{1}{h^2}(u_2^n - 2u_1^n), \delta_x^2 u_{J-1}^n = \frac{1}{h^2}(u_{J-2}^n - 2u_{J-1}^n)$. The discrete inner product and norm can thus be defined in $\mathbb{V}_h$, and there is no need to consider the boundary value further.

For any $\mathbf{u}^n, \mathbf{v}^n \in \mathbb{V}_h$, the discrete inner product and norm are defined as follows

$$\langle \mathbf{u}^n, \mathbf{v}^n\rangle = h\sum_{j=1}^{J-1} u_j^n \bar{v}_j^n, \|\mathbf{u}\|_p = \sqrt[p]{h\sum_{j=1}^{J-1}|u_j^n|^p},$$

$$\|\delta_x \mathbf{u}^n\| = \sqrt{h\sum_{j=0}^{J-1}|\delta_x^+ u_j^n|^2}, \|\delta_x^2 \mathbf{u}^n\| = \sqrt{h\sum_{j=1}^{J-1}|\delta_x^2 u_j^n|^2}, \|\mathbf{u}^n\|_\infty = \max_{1\le j\le J-1}|u_j^n|,$$

where $\bar{v}_j^n$ represents the complex conjugate of v_j^n. For convenience, let C denote a general constant, which may have different values in different places. Now we introduce some useful auxiliary lemmas.

Lemma 2.1 ([15]). *For any $\mathbf{u}^n, \mathbf{v}^n \in \mathbb{V}_h$, the following relations hold*

$$h\sum_{j=1}^{J-1}(\delta_x^2 u_j^n)\bar{v}_j^n = -h\sum_{j=0}^{J-1}(\delta_x^+ u_j^n)(\delta_x^+ \bar{v}_j^n), \|\mathbf{v}^n\|_\infty \le \sqrt{\frac{L}{2}}\|\delta_x \mathbf{v}^n\|, \|\mathbf{v}^n\| \le \frac{2L}{\sqrt{6}}\|\delta_x \mathbf{v}^n\|.$$

Lemma 2.2 ([15]). *For any $\mathbf{u}^n \in \mathbb{V}_h$, then $\|\delta_x \mathbf{u}^n\|^2 \le \frac{4}{h^2}\|\mathbf{u}^n\|^2$.*

Lemma 2.3. *For any $\mathbf{u}^n \in \mathbb{V}_h$, then $\|\delta_x^2 \mathbf{u}^n\|^2 \le \frac{16}{h^4}\|\mathbf{u}^n\|^2$.*

Proof. Using Young's inequality, we can obtain the results immediately. ∎

Lemma 2.4 ([16]). *Suppose that $g(x) \in C^2[d_1, d_2]$ and $a_1, a_2, b_1, b_2 \in [d_1, d_2]$, there exists constants $\theta \in (-1, 1)$ and $\eta \in [d_1, d_2]$ such that*

$$\frac{g(a_2) - g(a_1)}{a_2 - a_1} - \frac{g(b_2) - g(b_1)}{b_2 - b_1} = g'\left(\frac{1-\theta}{2}a_1 + \frac{1+\theta}{2}a_2\right) - g'\left(\frac{1-\theta}{2}b_1 + \frac{1+\theta}{2}b_2\right)$$

$$= g''(\eta)\left(\frac{1-\theta}{2}(a_1 - b_1) + \frac{1+\theta}{2}(a_2 - b_2)\right).$$

Lemma 2.5 (Sobolev's Inequality [15, 17]). *For any* $\mathbf{u}^n \in \mathbb{V}_h$ *and* $\varepsilon > 0$, *there exists a constant* $C(\varepsilon)$ *dependent on* ε *such that*

$$\|u^n\|_\infty^2 \le \varepsilon\|\delta_x^+ u^n\|^2 + C(\varepsilon)\|u^n\|^2.$$

Lemma 2.6 ([18]). *For any* $\mathbf{u}^n \in \mathbb{V}_h$, *we have*

$$Im\langle \mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle = 0,$$
$$Re\langle \mathcal{N}\delta_x^2(\mathbf{u}^{n+1} + \mathbf{u}^n), \mathbf{u}^{n+1} - \mathbf{u}^n\rangle = \langle \mathcal{N}\delta_x^2\mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle - \langle \mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle.$$

Lemma 2.7 ([18]). *For any* $\mathbf{u}^n \in \mathbb{V}_h$, *we have*

$$-\frac{3}{2}\|\delta_x\mathbf{u}^n\|^2 \le \langle \mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle \le -\|\delta_x\mathbf{u}^n\|^2.$$

In this article, we will make repeated use of the following Young's inequality

$$ab \le \frac{1}{4\varepsilon}a^2 + \varepsilon b^2, \varepsilon > 0,$$

where $a, b \in \mathbb{R}$.

III. THE CONSERVATIVE PROPERTY AND PRIORI ESTIMATION

First, we consider a two-level compact difference scheme for problem (3)–(7) as follows

Scheme A

$$i\mathcal{A}\delta_t u_j^{n+\frac{1}{2}} + \delta_x^2 u_j^{n+\frac{1}{2}} - \mathcal{A}(u_j^{n+\frac{1}{2}} v_j^{n+\frac{1}{2}}) = 0, 1 \le j \le J-1, \tag{10}$$

$$\mathcal{A}\delta_t v_j^{n+\frac{1}{2}} = \delta_x^2\phi_j^{n+\frac{1}{2}}, 1 \le j \le J-1, \tag{11}$$

$$\mathcal{A}\delta_t\phi_j^{n+\frac{1}{2}} = \Lambda_j^{n+\frac{1}{2}}, 1 \le j \le J-1, \tag{12}$$

$$u_j^0 = u_0(x_j), v_j^0 = v_0(x_j), \phi_j^0 = \phi_0(x_j), 0 \le j \le J, \tag{13}$$

$$u_0^n = u_J^n = 0, v_0^n = v_J^n = 0, \phi_0^n = \phi_J^n = 0, \tag{14}$$

where

$$\Lambda_j^{n+\frac{1}{2}} = \mathcal{A}v_j^{n+\frac{1}{2}} + \mathcal{A}\left(\frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n}\right) - \alpha\delta_x^2 v_j^{n+\frac{1}{2}} + \frac{\omega}{2}\mathcal{A}(|u_j^{n+1}|^2 + |u_j^n|^2).$$

Denote

$$\mathbf{u}^{n+\frac{1}{2}} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^n), \mathbf{u}^{\bar{n}} = \frac{1}{2}(\mathbf{u}^{n+1} + \mathbf{u}^{n-1}), \delta_t\mathbf{v}^{n+\frac{1}{2}} = \frac{1}{\tau}(\mathbf{v}^{n+1} - \mathbf{v}^n),$$
$$\delta_{\hat{t}}\mathbf{v}^n = \frac{1}{2\tau}(\mathbf{v}^{n+1} - \mathbf{v}^{n-1}), \delta_x^2\mathbf{u}^n = (\delta_x^2 u_1^n, \cdots, \delta_x^2 u_{J-1}^n)^T,$$

$$\mathbf{u}^n\mathbf{v}^n = (u_1^n v_1^n, \cdots, u_{J-1}^n v_{J-1}^n)^T, F(\mathbf{v}^n) = (F(v_1^n), \cdots, F(v_{J-1}^n))^T,$$

$$\frac{F(\mathbf{v}^{n+1}) - F(\mathbf{v}^n)}{\mathbf{v}^{n+1} - \mathbf{v}^n} = \left(\frac{F(v_1^{n+1}) - F(v_1^n)}{v_1^{n+1} - v_1^n}, \cdots, \frac{F(v_{J-1}^{n+1}) - F(v_{J-1}^n)}{v_{J-1}^{n+1} - v_{J-1}^n}\right)^T,$$

$$|\mathbf{u}^n|^2 = (|u_1^n|^2, |u_2^n|^2, \cdots, |u_{J-1}^n|^2)^T.$$

The vector forms of (10)–(12) are

$$i\mathcal{M}\delta_t\mathbf{u}^{n+\frac{1}{2}} + \delta_x^2\mathbf{u}^{n+\frac{1}{2}} - \mathcal{M}(\mathbf{u}^{n+\frac{1}{2}}\mathbf{v}^{n+\frac{1}{2}}) = 0, \tag{15}$$

$$\mathcal{M}\delta_t\mathbf{v}^{n+\frac{1}{2}} = \delta_x^2\phi^{n+\frac{1}{2}}, \tag{16}$$

$$\mathcal{M}\delta_t\phi^{n+\frac{1}{2}} = \mathcal{M}\mathbf{v}^{n+\frac{1}{2}} + \mathcal{M}\left(\tfrac{F(\mathbf{v}^{n+1})-F(\mathbf{v}^n)}{\mathbf{v}^{n+1}-\mathbf{v}^n}\right) - \alpha\delta_x^2\mathbf{v}^{n+\frac{1}{2}} + \tfrac{\omega}{2}\mathcal{M}(|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^n|^2). \tag{17}$$

Multiplying $\mathcal{N}$ with both sides of (15)–(17), respectively, we have

$$i\delta_t\mathbf{u}^{n+\frac{1}{2}} + \mathcal{N}\delta_x^2\mathbf{u}^{n+\frac{1}{2}} - (\mathbf{u}^{n+\frac{1}{2}}\mathbf{v}^{n+\frac{1}{2}}) = 0, \tag{18}$$

$$\delta_t\mathbf{v}^{n+\frac{1}{2}} = \mathcal{N}\delta_x^2\phi^{n+\frac{1}{2}}, \tag{19}$$

$$\delta_t\phi^{n+\frac{1}{2}} = \mathbf{v}^{n+\frac{1}{2}} + \left(\frac{F(\mathbf{v}^{n+1}) - F(\mathbf{v}^n)}{\mathbf{v}^{n+1} - \mathbf{v}^n}\right) - \alpha\mathcal{N}\delta_x^2\mathbf{v}^{n+\frac{1}{2}} + \frac{\omega}{2}(|\mathbf{u}^{n+1}|^2 + |\mathbf{u}^n|^2). \tag{20}$$

Considering (18)–(20), we have following discrete conservative laws.

Theorem 3.1. *Scheme A satisfies the following two discrete conservative laws*

$$Q^n = \|\mathbf{u}^n\|^2 = C, \tag{21}$$

$$\begin{aligned} E^n &= \|\mathbf{v}^n\|^2 - \langle\mathcal{N}\delta_x^2\phi^n, \phi^n\rangle - 2\omega\langle\mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle - \alpha\langle\mathcal{N}\delta_x^2\mathbf{v}^n, \mathbf{v}^n\rangle \\ &\quad + 2\langle F(\mathbf{v}^n), I\rangle + 2\omega\langle\mathbf{v}^n, |\mathbf{u}^n|^2\rangle = C. \end{aligned} \tag{22}$$

Proof. Making the inner product of (18) with $\mathbf{u}^{n+\frac{1}{2}}$ and taking the imaginary part, by virtue of Lemma 2.6, we obtain the first conservative law

$$Q^n = \|\mathbf{u}^n\|^2 = C.$$

Computing the inner product of (18) with $2\tau\delta_t\mathbf{u}^{n+\frac{1}{2}}$ and taking the real part, this together with Lemma 2.6, we have

$$\langle\mathcal{N}\delta_x^2\mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle - \langle\mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle = \frac{1}{2}\langle\mathbf{v}^{n+1} + \mathbf{v}^n, |\mathbf{u}^{n+1}|^2\rangle - \frac{1}{2}\langle\mathbf{v}^{n+1} + \mathbf{v}^n, |\mathbf{u}^n|^2\rangle. \tag{23}$$

Computing the inner product of (19) with $2\tau\delta_t\phi^{n+\frac{1}{2}}$, we may obtain

$$2\tau\langle\delta_t\mathbf{v}^{n+\frac{1}{2}}, \delta_t\phi^{n+\frac{1}{2}}\rangle = \langle\mathcal{N}\delta_x^2\phi^{n+1}, \phi^{n+1}\rangle - \langle\mathcal{N}\delta_x^2\phi^n, \phi^n\rangle. \tag{24}$$

Making the inner product of (20) with $2\tau\delta_t \mathbf{v}^{n+\frac{1}{2}}$, then we have

$$
\begin{aligned}
&2\tau\langle\delta_t\phi^{n+\frac{1}{2}}, \delta_t\mathbf{v}^{n+\frac{1}{2}}\rangle \\
&\quad = \|\mathbf{v}^{n+1}\|^2 + \omega\langle\mathbf{v}^{n+1}, |\mathbf{u}^{n+1}|^2\rangle - \alpha\langle\mathcal{N}\delta_x^2\mathbf{v}^{n+1}, \mathbf{v}^{n+1}\rangle + 2\langle F(\mathbf{v}^{n+1}), \mathbf{I}\rangle - \omega\langle\mathbf{v}^n, |\mathbf{u}^{n+1}|^2\rangle \\
&\qquad - \|\mathbf{v}^n\|^2 - \omega\langle\mathbf{v}^n, |\mathbf{u}^n|^2\rangle + \alpha\langle\mathcal{N}\delta_x^2\mathbf{v}^n, \mathbf{v}^n\rangle - 2\langle F(\mathbf{v}^n), \mathbf{I}\rangle + \omega\langle\mathbf{v}^{n+1}, |\mathbf{u}^n|^2\rangle, \qquad (25)
\end{aligned}
$$

where $\mathbf{I} = (1, 1, \cdots, 1)^T$ with $J-1$ components. It follows from (23)–(25) that (22) is satisfied, and this completes the proof. ∎

Theorem 3.2. *Suppose that $\mathbf{u}^n, \mathbf{v}^n, \phi^n \in \mathbb{V}_h$ are the difference solutions of Scheme A, then the following estimates hold*

$$
\begin{aligned}
&\|\mathbf{v}^n\| \le C, \|\delta_x\mathbf{v}^n\| \le C, \|\mathbf{v}^n\|_\infty \le C, \|\phi^n\| \le C, \\
&\|\delta_x\phi^n\| \le C, \|\phi^n\|_\infty \le C, \|\delta_x\mathbf{u}^n\| \le C, \|\mathbf{u}^n\|_\infty \le C.
\end{aligned}
$$

Proof. By using Young's inequality, we have

$$
\langle\mathbf{v}^n, |\mathbf{u}^n|^2\rangle = h\sum_{j=1}^{J-1} v_j^n|u_j^n|^2 \le \frac{1}{4\omega}\|\mathbf{v}^n\|^2 + \omega\|\mathbf{u}^n\|_4^4, \qquad (26)
$$

Considering Lemma 2.5 and (21), we have the estimate

$$
\|\mathbf{u}^n\|_4^4 \le \|\mathbf{u}^n\|^2\|\mathbf{u}^n\|_\infty^2 \le C(\varepsilon\|\delta_x\mathbf{u}^n\|^2 + C(\varepsilon)\|\mathbf{u}^n\|^2). \qquad (27)
$$

It follows from (22) and Lemma 2.7 that

$$
\begin{aligned}
&\|\mathbf{v}^n\|^2 + \|\delta_x\phi^n\|^2 + 2\omega\|\delta_x\mathbf{u}^n\|^2 + \alpha\|\delta_x\mathbf{v}^n\|^2 \le C + 2\omega|\langle\mathbf{v}^n, |\mathbf{u}^n|^2\rangle| \\
&\quad \le C + \frac{1}{2}\|\mathbf{v}^n\|^2 + 2\omega^2\|\mathbf{u}^n\|_4^4 \le C + \frac{1}{2}\|\mathbf{v}^n\|^2 + 2C\omega^2(\varepsilon\|\delta_x\mathbf{u}^n\|^2 + C(\varepsilon)\|\mathbf{u}^n\|^2). \qquad (28)
\end{aligned}
$$

Let $\varepsilon = \frac{1}{2C\omega}$, and considering (21), we have

$$
\tfrac{1}{2}\|\mathbf{v}^n\|^2 + \|\delta_x\phi^n\|^2 + \omega\|\delta_x\mathbf{u}^n\|^2 + \alpha\|\delta_x\mathbf{v}^n\|^2 \le C, \qquad (29)
$$

which implies that

$$
\|\delta_x\mathbf{u}^n\| \le C, \|\delta_x\phi^n\| \le C, \|\mathbf{v}^n\| \le C, \|\delta_x\mathbf{v}^n\| \le C. \qquad (30)
$$

Considering (30) and Lemma 2.1, we obtain

$$
\|\mathbf{u}^n\|_\infty \le C, \|\mathbf{v}^n\|_\infty \le C, \|\phi^n\|_\infty \le C,
$$

this completes the proof of Theorem 3.2. ∎

IV. THE EXISTENCE OF THE DIFFERENCE SOLUTIONS

Now we discuss the existence of the difference solutions of Scheme A. For this purpose, we need following fixed point theorem.

Lemma 4.1. ([14, 17]). *If the nonlinear system of equations $w = T_\lambda(w)$ defined on a finite dimensional Euclidean space $\mathbb{R}^m$ satisfies the following conditions*

- ***I.*** *The functions $T_\lambda(w)$ are continuous for any $w \in \mathbb{R}^m$ and $0 \le \lambda \le 1$.*
- ***II.*** *As $\lambda = 0$, there is a fixed point $w_0 \in \mathbb{R}^m$, such that $T_\lambda(w_0) = w_0$ for any $w \in \mathbb{R}^m$.*
- ***III.*** *All possible solutions of system $w = T_\lambda(w)$ are uniformly bounded w.r.t. the parameter $0 \le \lambda \le 1$.*

Then the nonlinear system $w = T_\lambda(w)$ has at least one solution $w \in \mathbb{R}^m$ for any $\le \lambda \le 1$ and hence for $\lambda = 1$, that is, the nonlinear system $w = T_1(w)$ has at least one solution $w_0 \in \mathbb{R}^m$.

Theorem 4.1. *The difference solutions of the Scheme A exist.*

Proof. Denote $U = (U_1, U_2, \cdots, U_{J-1})^T$, $V = (V_1, V_2, \cdots, V_{J-1})^T$, $\Phi = (\Phi_1, \Phi_2, \cdots, \Phi_{J-1})^T$ and $X = (U^T, V^T, \Phi^T)^T$, where $U_0 = U_J = V_0 = V_J = \Phi_0 = \Phi_J = 0$. Constructing a nonlinear mapping $T_\lambda : \mathbb{R}^{3J-3} \to \mathbb{R}^{3J-3}$ with parameter $\lambda \in (0, 1)$

$$i\mathcal{A}\left(\frac{U_j - u_j^n}{\tau}\right) + \frac{\lambda}{2}\delta_x^2(U_j + u_j^n) - \frac{\lambda}{4}((U_j + u_j^n)(V_j + v_j^n)) = 0, 1 \le j \le J-1, \tag{31}$$

$$\mathcal{A}\left(\frac{V_j - v_j^n}{\tau}\right) = \frac{\lambda}{2}\delta_x^2(\Phi_j + \phi_j^n), 1 \le j \le J-1, \tag{32}$$

$$\begin{aligned}\mathcal{A}\left(\frac{\Phi_j - \phi_j^n}{\tau}\right) &= \frac{\lambda}{2}\mathcal{A}(V_j + v_j^n) + \lambda\mathcal{A}\left(\frac{F(V_j) - F(v_j^n)}{V_j - v_j^n}\right) - \frac{\lambda\alpha}{2}\delta_x^2(V_j + v_j^n)\\ &\quad + \frac{\lambda\omega}{2}\mathcal{A}(|U_j|^2 + |u_j^n|^2), 1 \le j \le J-1.\end{aligned} \tag{33}$$

Obviously, the mapping $T_\lambda(X)$ is continuous, and $T_0(X) = (u_1^n, u_2^n \cdots, u_{J-1}^n, v_1^n, v_2^n \cdots, v_{J-1}^n, \phi_1^n, \phi_2^n \cdots, \phi_{J-1}^n)^T$ is a fixed point for any $X \in \mathbb{R}^{3J-3}$. To prove the existence of the difference solutions for Scheme A, we need to prove that X is uniformly bounded.

The vector form of (31) can be written as

$$i\mathcal{A}\left(\frac{U - \mathbf{u}^n}{\tau}\right) + \frac{\lambda}{2}\delta_x^2(U + \mathbf{u}^n) - \frac{\lambda}{4}((U + \mathbf{u}^n)(V + \mathbf{v}^n)) = 0. \tag{34}$$

Similar to the proof of (21), we have

$$\|U\|^2 = C, \tag{35}$$

and thus, U is uniformly bounded. Now we prove the boundedness of V and Φ.

Multiplying (32) and (33) by $\alpha h\tau(V_j + v_j^n)$ and $h\tau(\Phi_j + \phi_j^n)$, respectively, and summing up for j from 1 to $J-1$, we obtain

$$\alpha(\|V\|^2 - \|\mathbf{v}^n\|^2) - \frac{\alpha h^2}{12}(\|\delta_x V\|^2 - \|\delta_x \mathbf{v}^n\|^2) = \frac{\alpha\lambda\tau}{2}h\sum_{j=1}^{J-1}\delta_x^2(\Phi_j + \phi_j^n)\cdot(V_j + v_j^n), \tag{36}$$

$$\alpha(\|\Phi\|^2 - \|\phi^n\|^2) - \frac{\alpha h^2}{12}(\|\delta_x\Phi\|^2 - \|\delta_x\phi^n\|^2) = \frac{\lambda\tau}{2}h\sum_{j=1}^{J-1}\mathcal{A}(V_j + v_j^n)\cdot(\Phi_j + \phi_j^n)$$
$$+ \lambda\tau h\sum_{j=1}^{J-1}\mathcal{A}\left(\frac{F(V_j) - F(v_j^n)}{V_j - v_j^n}\right)\cdot(\Phi_j + \phi_j^n) - \frac{\lambda\alpha\tau}{2}h\sum_{j=1}^{J-1}\delta_x^2(V_j + v_j^n)\cdot(\Phi_j + \phi_j^n)$$
$$+ \frac{\lambda\omega\lambda}{2}h\sum_{j=1}^{J-1}\mathcal{A}(|U_j|^2 + |u_j^n|^2)\cdot(\Phi_j + \phi_j^n). \tag{37}$$

The addition of (36) to (37) yields

$$(\alpha\|V\|^2 + \|\Phi\|^2) - \frac{h^2}{12}(\alpha\|\delta_x V\|^2 + \|\delta_x\Phi\|^2)$$
$$- (\alpha\|\mathbf{v}^n\|^2 + \|\phi^n\|^2) + \frac{h^2}{12}(\alpha\|\delta_x\mathbf{v}^n\|^2 + \|\delta_x\phi^n\|^2) = I_1 + I_2 + I_3. \tag{38}$$

According to Young's inequality, Lemma 2.3 and Theorem 3.2, we have

$$I_1 = \frac{\lambda\tau}{2}h\sum_{j=1}^{J-1}\mathcal{A}(V_j + v_j^n)\cdot(\Phi_j + \phi_j^n)$$
$$\leq \frac{\lambda\tau}{2}h\sum_{j=1}^{J-1}(|V_j + v_j^n| + \frac{h^2}{12}|\delta_x^2(V_j + v_j^n)|)\cdot|\Phi_j + \phi_j^n|$$
$$\leq \frac{5\lambda\tau}{9}(\|V\|^2 + \|\mathbf{v}^n\|^2) + \lambda\tau(\|\Phi\|^2 + \|\phi^n\|^2) \leq \frac{5\lambda\tau}{9}\|V\|^2 + \lambda\tau\|\Phi\|^2 + C, \tag{39}$$
$$I_2 = \frac{\lambda\omega\tau}{2}h\sum_{j=1}^{J-1}\mathcal{A}(|U_j|^2 + |u_j^n|^2)\cdot(\Phi_j + \phi_j^n)$$
$$\leq \frac{\lambda\omega\tau}{2}h\sum_{j=1}^{J-1}(|U_j|^2 + |u_j^n|^2)\cdot(|\Phi_j + \phi_j^n| + \frac{h^2}{12}|\delta_x^2(\Phi_j + \phi_j^n)|)$$
$$\leq \lambda\omega\tau(\|U\|_4^4 + \|\mathbf{u}^n\|_4^4) + \frac{5\lambda\omega\tau}{9}(\|\Phi\|^2 + \|\phi^n\|^2) \leq \lambda\omega\tau\|U\|_4^4 + \frac{5\lambda\omega\tau}{9}\|\Phi\|^2 + C, \tag{40}$$

Applying Taylor's theorem, Young's inequality, Lemma 2.3 and Theorem 3.2, we have

$$I_3 = \lambda\tau h\sum_{j=1}^{J-1}\mathcal{A}\left(\frac{F(V_j) - F(v_j^n)}{V_j - v_j^n}\right)\cdot(\Phi_j + \phi_j^n)$$
$$\leq \lambda\tau h\sum_{j=1}^{J-1}|\frac{F(V_j) - F(v_j^n)}{V_j - v_j^n}|\cdot(|\Phi_j + \phi_j^n| + \frac{h^2}{12}|\delta_x^2(\Phi_j + \phi_j^n)|)$$

$$\leq \lambda \tau h \sum_{j=1}^{J-1} C(|\Phi_j + \phi_j^n| + \frac{h^2}{12}|\delta_x^2(\Phi_j + \phi_j^n)|) \leq 2\lambda C^2 L\tau + \frac{10\lambda\tau}{9}(\|\Phi\|^2 + \|\phi^n\|^2)$$

$$\leq \frac{10\lambda\tau}{9}\|\Phi\|^2 + C. \tag{41}$$

Substituting (39)–(41) into (38), we have

$$\begin{aligned}(\alpha\|V\|^2 + \|\Phi\|^2) &- \frac{h^2}{12}(\alpha\|\delta_x V\|^2 + \|\delta_x \Phi\|^2) \\ &- (\alpha\|\mathbf{v}^n\|^2 + \|\phi^n\|^2) + \frac{h^2}{12}(\alpha\|\delta_x \mathbf{v}^n\|^2 + \|\delta_x \phi^n\|^2) \\ &\leq \frac{5\lambda\tau}{9}\|V\|^2 + \frac{5\lambda + 19}{9}\lambda\tau\|\Phi\|^2 + \lambda\tau\|U\|_4^4 + C.\end{aligned} \tag{42}$$

It follows from Lemma 2.2, Theorem 3.2 and (27) that

$$\frac{2}{3}(\alpha\|V\|^2 + \|\Phi\|^2) \leq \frac{5\lambda\tau}{9}\|V\|^2 + \frac{5\lambda + 19}{9}\lambda\tau\|\Phi\|^2 + \lambda\tau C(\varepsilon\|\delta_x U\|^2 + C(\varepsilon)\|U\|^2) + C. \tag{43}$$

If τ is sufficiently small, we have

$$\frac{1}{2}(\alpha\|V\|^2 + \|\Phi\|^2) \leq \lambda\tau C(\varepsilon\|\delta_x U\|^2 + C(\varepsilon)\|U\|^2) + C. \tag{44}$$

Lemma 2.2 and (35) imply that

$$\frac{1}{2}(\alpha\|V\|^2 + \|\Phi\|^2) \leq Cr\|U\|^2 + C, \tag{45}$$

where $r = \frac{\tau}{h^2}$, ε is a positive constant. Obviously, (35) and (45) imply that $\|V\|$ and $\|\Phi\|$ are uniformly bounded; thus $\|X\|$ is uniformly bounded. It follows from Lemma 4.1 that the conclusion of Theorem 4.1 holds. This completes the proof. ∎

Replacing u_j^n, v_j^n, ϕ_j^n by U_j^n, V_j^n, Φ_j^n in (10)–(12), respectively, then we have

$$i\mathcal{A}\delta_t U_j^{n+\frac{1}{2}} + \delta_x^2 U_j^{n+\frac{1}{2}} - \mathcal{A}(U_j^{n+\frac{1}{2}} V_j^{n+\frac{1}{2}}) = (r_1)_j^{n+\frac{1}{2}}, 1 \leq j \leq J-1, \tag{46}$$

$$\mathcal{A}\delta_t V_j^{n+\frac{1}{2}} = \delta_x^2 \Phi_j^{n+\frac{1}{2}} + (r_2)_j^{n+\frac{1}{2}}, 1 \leq j \leq J-1, \tag{47}$$

$$\begin{aligned}\mathcal{A}\delta_t \Phi_j^{n+\frac{1}{2}} &= \mathcal{A}V_j^{n+\frac{1}{2}} + \mathcal{A}\left(\frac{F(V_j^{n+1}) - F(V_j^n)}{V_j^{n+1} - V_j^n}\right) - \alpha\delta_x^2 V_j^{n+\frac{1}{2}} \\ &\quad + \frac{\omega}{2}\mathcal{A}(|U_j^{n+1}|^2 + |U_j^n|^2) + (r_3)_j^{n+\frac{1}{2}}, 1 \leq j \leq J-1,\end{aligned} \tag{48}$$

where $|r_k| \leq C(\tau^2 + h^4)$, so Scheme A possesses truncation errors of order $O(\tau^2 + h^4)$. Now we analyze the convergence and stability of Scheme A.

V. CONVERGENCE AND STABILITY ANALYSIS

Theorem 5.1. *Suppose that $u(x,t), v(x,t), \phi(x,t) \in C^{6,3}$ are the exact solutions of problem (3)–(7), then the difference solutions of Scheme A converge to the exact solutions with order $O(\tau^2 + h^4)$ in the L_2-norm.*

Proof. Denote $e_j^n = U_j^n - u_j^n$, $g_j^n = V_j^n - v_j^n$, $\rho_j^n = \Phi_j^n - \phi_j^n$ and subtracting (11)–(13) from (46) to (48), respectively, we get the error equations

$$i\mathcal{A}\delta_t e_j^{n+\frac{1}{2}} + \delta_x^2 e_j^{n+\frac{1}{2}} - \mathcal{A}(e_j^{n+\frac{1}{2}} V_j^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}} g_j^{n+\frac{1}{2}}) = (r_1)_j^{n+\frac{1}{2}}, 1 \le j \le J-1, \tag{49}$$

$$\mathcal{A}\delta_t g_j^{n+\frac{1}{2}} - \delta_x^2 \rho_j^{n+\frac{1}{2}} = (r_2)_j^{n+\frac{1}{2}}, 1 \le j \le J-1, \tag{50}$$

$$\begin{aligned}\mathcal{A}\delta_t \rho_j^{n+\frac{1}{2}} &= \mathcal{A} g_j^{n+\frac{1}{2}} - \alpha\delta_x^2 g_j^{n+\frac{1}{2}} + \mathcal{A}\left(\frac{F(V_j^{n+1}) - F(V_j^n)}{V_j^{n+1} - V_j^n} - \frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n}\right) \\ &\quad + \frac{\omega}{2}\mathcal{A}(e_j^{n+1}\bar{U}_j^{n+1} + u_j^{n+1}\bar{e}_j^{n+1} + e_j^n\bar{U}_j^n + u_j^n\bar{e}_j^n) + (r_3)_j^{n+\frac{1}{2}}, 1 \le j \le J-1.\end{aligned} \tag{51}$$

The initial and boundary value are given as follows

$$e_j^0 = g_j^0 = \rho_j^0 = 0, 0 \le j \le J, \tag{52}$$

$$e_0^n = e_J^n = g_0^n = g_J^n = \rho_0^n = \rho_J^n = 0. \tag{53}$$

Multiplying (49) by $2h\tau\bar{e}_j^{n+\frac{1}{2}}$ and summing up for j from 1 to $J-1$, then taking the imaginary part, we have

$$\begin{aligned}(\|\mathbf{e}^{n+1}\|^2 - \|\mathbf{e}^n\|^2) - \frac{h^2}{12}(\|\delta_x\mathbf{e}^{n+1}\|^2 - \|\delta_x\mathbf{e}^n\|^2) &= 2\tau Im\left(h\sum_{j=1}^{J-1} u_j^{n+\frac{1}{2}} g_j^{n+\frac{1}{2}} \bar{e}_j^{n+\frac{1}{2}}\right) \\ + \frac{\tau h^2}{6} Im\left(h\sum_{j=1}^{J-1} u_j^{n+\frac{1}{2}} g_j^{n+\frac{1}{2}} \cdot \delta_x^2\bar{e}_j^{n+\frac{1}{2}}\right) &+ \frac{\tau h^2}{6} Im\left(h\sum_{j=1}^{J-1} e_j^{n+\frac{1}{2}} V_j^{n+\frac{1}{2}} \delta_x^2\bar{e}_j^{n+\frac{1}{2}}\right) \\ + 2\tau Im\left(h\sum_{j=1}^{J-1} (r_1)_j^{n+\frac{1}{2}} \cdot \bar{e}_j^{n+\frac{1}{2}}\right).\end{aligned} \tag{54}$$

Multiplying (50) and (51) by $2h\tau\alpha g_j^{n+\frac{1}{2}}$ and $2h\tau\rho_j^{n+\frac{1}{2}}$, respectively, and summing up for the subscript j from 1 to $J-1$, we obtain

$$\begin{aligned}\alpha(\|\mathbf{g}^{n+1}\|^2 - \|\mathbf{g}^n\|^2) - \frac{\alpha h^2}{12}(\|\delta_x\mathbf{g}^{n+1}\|^2 - \|\delta_x\mathbf{g}^n\|^2) - 2\tau\alpha h\sum_{j=1}^{J-1}(\delta_x^2\rho_j^{n+\frac{1}{2}}) \cdot g_j^{n+\frac{1}{2}} \\ = 2\tau\alpha h\sum_{j=1}^{J-1}(r_2)_j^{n+\frac{1}{2}} \cdot g_j^{n+\frac{1}{2}},\end{aligned} \tag{55}$$

and

$$(\|\rho^{n+1}\|^2 - \|\rho^n\|^2) - \frac{h^2}{12}(\|\delta_x\rho^{n+1}\|^2 - \|\delta_x\rho^n\|^2) = -2\tau\alpha h\sum_{j=1}^{J-1}(\delta_x^2 g_j^{n+\frac{1}{2}})\cdot\rho_j^{n+\frac{1}{2}}$$

$$+ 2\tau h\sum_{j=1}^{J-1}(\mathcal{A}g_j^{n+\frac{1}{2}})\cdot\rho_j^{n+\frac{1}{2}} + 2\tau h\sum_{j=1}^{J-1}\mathcal{A}\left(\frac{F(V_j^{n+1})-F(V_j^n)}{V_j^{n+1}-V_j^n} - \frac{F(v_j^{n+1})-F(v_j^n)}{v_j^{n+1}-v_j^n}\right)\cdot\rho_j^{n+\frac{1}{2}}$$

$$+ \tau\omega h\sum_{j=1}^{J-1}\mathcal{A}(e_j^{n+1}\bar{U}_j^{n+1} + u_j^{n+1}\bar{e}_j^{n+1} + e_j^n\bar{U}_j^n + u_j^n\bar{e}_j^n)\cdot\rho_j^{n+\frac{1}{2}} + 2\tau h\sum_{j=1}^{J-1}(r_3)_j^{n+\frac{1}{2}}\cdot\rho_j^{n+\frac{1}{2}}. \quad (56)$$

Denote

$$B^n = \|\mathbf{e}^n\|^2 + \alpha\|\mathbf{g}^n\|^2 + \|\rho^n\|^2 - \frac{h^2}{12}(\|\delta_x\mathbf{e}^n\|^2 + \alpha\|\delta_x\mathbf{g}^n\|^2 + \|\delta_x\rho^n\|^2)$$

and $\alpha_0 = min\,\{1,\alpha\}$. It follows from Lemma 2.2 that

$$B^n \geq \tfrac{2\alpha_0}{3}(\|\mathbf{e}^n\|^2 + \|\mathbf{g}^n\|^2 + \|\rho^n\|^2). \quad (57)$$

Adding (54)–(56) yields

$$B^{n+1} - B^n = II_1 + II_2 + II_3 + II_4 + II_5 + II_6. \quad (58)$$

According to Young's inequality and Lemma 2.3, we have

$$II_1 = 2\tau h\sum_{j=1}^{J-1}(\mathcal{A}g_j^{n+\frac{1}{2}})\rho_j^{n+\frac{1}{2}} \leq 2\tau h\sum_{j=1}^{J-1}|g_j^{n+\frac{1}{2}}|\cdot|\rho_j^{n+\frac{1}{2}}| + 2\tau h\sum_{j=1}^{J-1}|\frac{h^2}{12}\delta_x^2 g_j^{n+\frac{1}{2}}|\cdot|\rho_j^{n+\frac{1}{2}}|$$

$$\leq \tau(\|\rho^{n+1}\|^2 + \|\rho^n\|^2) + \frac{5\tau}{9}(\|\mathbf{g}^{n+1}\|^2 + \|\mathbf{g}^n\|^2). \quad (59)$$

It follows from Lemma 2.3, Lemma 2.4 and Young's inequality that

$$II_2 = 2\tau h\sum_{j=1}^{J-1}\mathcal{A}\left(\frac{F(V_j^{n+1})-F(V_j^n)}{V_j^{n+1}-V_j^n} - \frac{F(v_j^{n+1})-F(v_j^n)}{v_j^{n+1}-v_j^n}\right)\cdot\rho_j^{n+\frac{1}{2}}$$

$$= 2\tau h\sum_{j=1}^{J-1}\left(\frac{F(V_j^{n+1})-F(V_j^n)}{V_j^{n+1}-V_j^n} - \frac{F(v_j^{n+1})-F(v_j^n)}{v_j^{n+1}-v_j^n}\right)\cdot(\rho_j^{n+\frac{1}{2}} + \frac{h^2}{12}\delta_x^2\rho_j^{n+\frac{1}{2}})$$

$$\leq 2C\tau h\sum_{j=1}^{J-1}|g_j^{n+1}+g_j^n|\cdot|\rho_j^{n+\frac{1}{2}}| + 2C\tau h\sum_{j=1}^{J-1}|g_j^{n+1}+g_j^n|\cdot|\frac{h^2}{12}\delta_x^2\rho_j^{n+\frac{1}{2}}|$$

$$\leq 4C\tau\|\mathbf{g}^{n+1}\|^2 + 4C\tau\|\mathbf{g}^n\|^2 + \frac{5C\tau}{9}\|\rho^{n+1}\|^2 + \frac{5C\tau}{9}\|\rho^n\|^2. \quad (60)$$

Now we estimate II_3; since $u(x,t)\in C^{6,3}$, thus $|U_j^n| \leq C$. According to Theorem 3.2, Young's inequality and Lemma 2.3, we obtain

$$II_3 = \omega h\sum_{j=1}^{J-1}\mathcal{A}(e_j^{n+1}\bar{U}_j^{n+1} + u_j^{n+1}\bar{e}_j^{n+1} + e_j^n\bar{U}_j^n + u_j^n\bar{e}_j^n)\cdot\rho_j^{n+\frac{1}{2}}$$

$$\leq 2C\omega\tau h\sum_{j=1}^{J-1}(|e_j^n|+|e_j^{n+1}|)\cdot|\rho_j^{n+\frac{1}{2}}|+2C\omega\tau h\sum_{j=1}^{J-1}(|e_j^n|+|e_j^{n+1}|)\cdot|\frac{h^2}{12}\delta_x^2\rho_j^{n+\frac{1}{2}}|$$
$$\leq 2C\omega\tau(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{e}^n\|^2)+\frac{10}{9}C\omega\tau(\|\rho^{n+1}\|^2+\|\rho^n\|^2). \tag{61}$$

Similar to the proof of (61), we have the estimate

$$II_4=\frac{\tau h^2}{6}Im(h\sum_{j=1}^{J-1}u_j^{n+\frac{1}{2}}g_j^{n+\frac{1}{2}}\delta_x^2\bar{e}_j^{n+\frac{1}{2}})+\frac{\tau h^2}{6}Im(h\sum_{j=1}^{J-1}e_j^{n+\frac{1}{2}}V_j^{n+\frac{1}{2}}\delta_x^2\bar{e}_j^{n+\frac{1}{2}})$$
$$\leq C\tau h\sum_{j=1}^{J-1}|g_j^{n+\frac{1}{2}}|\cdot|\frac{h^2}{6}\delta_x^2e_j^{n+\frac{1}{2}}|+C\tau h\sum_{j=1}^{J-1}|e_j^{n+\frac{1}{2}}|\cdot|\frac{h^2}{6}\delta_x^2e_j^{n+\frac{1}{2}}|$$
$$\leq \frac{1}{4}C\tau(\|\mathbf{g}^{n+1}\|^2+\|\mathbf{g}^n\|^2)+\frac{17}{36}C\tau(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{e}^n\|^2). \tag{62}$$

It follows from Theorem 3.2, Taylor's theorem and Young's inequality, we have

$$II_5=2\tau Im(h\sum_{j=1}^{J-1}u_j^{n+\frac{1}{2}}g_j^{n+\frac{1}{2}}\bar{e}_j^{n+\frac{1}{2}})\leq 2C\tau(h\sum_{j=1}^{J-1}|g_j^{n+\frac{1}{2}}|\cdot|e_j^{n+\frac{1}{2}}|)$$
$$\leq\frac{1}{2}C\tau(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{e}^n\|^2)+\frac{1}{2}C\tau(\|\mathbf{g}^{n+1}\|^2+\|\mathbf{g}^n\|^2). \tag{63}$$

$$II_6=2\tau Im(h\sum_{j=1}^{J-1}(r_1)_j^{n+\frac{1}{2}}\cdot\bar{e}_j^{n+\frac{1}{2}})+2\alpha\tau h\sum_{j=1}^{J-1}((r_2)_j^{n+\frac{1}{2}}+(r_3)_j^{n+\frac{1}{2}})\cdot g_j^{n+\frac{1}{2}}$$
$$\leq\frac{\tau}{2}(\|\mathbf{e}^{n+1}\|^2+\alpha\|\mathbf{g}^{n+1}\|^2+\|\rho^{n+1}\|^2)+\frac{\tau}{2}(\|\mathbf{e}^n\|^2+\alpha\|\mathbf{g}^n\|^2+\|\rho^n\|^2)$$
$$+6CL\tau(\tau^2+h^4)^2. \tag{64}$$

Denoting $C_0=max\left\{2C\omega+\frac{35C+18}{36},\frac{19C+2\alpha}{4}+\frac{5}{9},\frac{10C\omega+5C}{9}+\frac{3}{2}\right\}$, and substituting (59)–(64) into (58), then we have

$$B^{n+1}-B^n\leq C_0\tau(\|\mathbf{e}^{n+1}\|^2+\|\mathbf{g}^{n+1}\|^2+\|\rho^{n+1}\|^2)+C_0\tau(\|\mathbf{e}^n\|^2+\|\mathbf{g}^n\|^2+\|\rho^n\|^2)$$
$$+6CL\tau(\tau^2+h^4)^2. \tag{65}$$

It follows from (57) and (65) that

$$B^{n+1}-B^n\leq\frac{3C_0\tau}{2\alpha_0}B^{n+1}+\frac{3C_0\tau}{2\alpha_0}B^n+6C\tau(\tau^2+h^4)^2. \tag{66}$$

For sufficiently small τ such that $\frac{3C_0\tau}{2\alpha_0}<\frac{1}{3}$, and using Gronwall's inequality [14, 17], we obtain

$$B^n\leq e^{\frac{9C_0}{2\alpha_0}T}(B_0+9CLT(\tau^2+h^4)^2). \tag{67}$$

Equation (52) implies that $B^0 = 0$, and considering (57) and (67) simultaneously, we have

$$\|\mathbf{e}^n\|^2 + \|\mathbf{g}^n\|^2 + \|\rho^n\|^2 \le \frac{3}{2\alpha_0} e^{\frac{9C_0}{2\alpha_0}T}(9CLT(\tau^2 + h^4)^2),$$

and thus we complete the proof. ∎

Theorem 5.2. *Suppose that the conditions of Theorem 5.1 are satisfied, then the difference solutions of Scheme A depend continuously on the initial values.*

Proof. Let $\tilde{u}_j^n, \tilde{v}_j^n$ and $\tilde{\phi}_j^n$ be other solutions of Scheme A with following initial value conditions

$$\tilde{u}_j^0 = u_0(x_j) + \tilde{\varepsilon}_0(x_j), \tilde{v}_j^0 = v_0(x_j) + \tilde{\delta}_0(x_j), \tilde{\phi}_j^0 = \phi_0(x_j) + \tilde{\sigma}_0(x_j),$$

where $\tilde{\varepsilon}_0(x_j)$, $\tilde{\delta}_0(x_j)$, $\tilde{\sigma}_0(x_j)$ are very small disturbances of the initial value. We define the errors by $\tilde{e}_j^n = \tilde{u}_j^n - u_j^n$, $\tilde{g}_j^n = \tilde{v}_j^n - v_j^n$, $\tilde{\rho}_j^n = \tilde{\phi}_j^n - \phi_j^n$, then the error equations satisfy

$$i\mathcal{A}\delta_t \tilde{e}_j^{n+\frac{1}{2}} + \delta_x^2 \tilde{e}_j^{n+\frac{1}{2}} - \mathcal{A}(\tilde{u}_j^{n+\frac{1}{2}}\tilde{v}_j^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}v_j^{n+\frac{1}{2}}) = 0, 1 \le j \le J-1, \tag{68}$$

$$\mathcal{A}\delta_t \tilde{g}_j^{n+\frac{1}{2}} - \delta_x^2 \tilde{\rho}_j^{n+\frac{1}{2}} = 0, 1 \le j \le J-1, \tag{69}$$

$$\begin{aligned} \mathcal{A}\delta_t \tilde{\rho}_j^{n+\frac{1}{2}} &= \mathcal{A}\tilde{g}_j^{n+\frac{1}{2}} + \mathcal{A}\left(\frac{F(\tilde{v}_j^{n+1}) - F(\tilde{v}_j^n)}{\tilde{v}_j^{n+1} - \tilde{v}_j^n} - \frac{F(v_j^{n+1}) - F(v_j^n)}{v_j^{n+1} - v_j^n}\right) \\ &-\alpha\delta_x^2 \tilde{g}_j^{n+\frac{1}{2}} + \frac{\omega}{2}\mathcal{A}(\tilde{e}_j^{n+1}\overline{\tilde{u}}_j^{n+1} + u_j^{n+1}\overline{\tilde{e}}_j^{n+1} + \tilde{e}_j^n\overline{\tilde{u}}_j^n + u_j^n\overline{\tilde{e}}_j^n), 1 \le j \le J-1, \end{aligned} \tag{70}$$

$$\tilde{e}_j^0 = \tilde{\epsilon}_0(x_j), \tilde{g}_j^0 = \tilde{\delta}_0(x_j), \tilde{\rho}_j^0 = \tilde{\sigma}_0(x_j), 0 \le j \le J, \tag{71}$$

$$\tilde{e}_0^n = \tilde{e}_J^n = \tilde{g}_0^n = \tilde{g}_J^n = \tilde{\rho}_0^n = \tilde{\rho}_J^n = 0. \tag{72}$$

Similar to the proof of Theorem 5.1, we have the conclusion. ∎

VI. OTHER NUMERICAL SCHEME

In this section, we will provide another conservative compact difference scheme for problem (3)–(7)

Scheme B

$$i\mathcal{A}\delta_{\hat{t}} u_j^n + \delta_x^2 u_j^{\bar{n}} - \mathcal{A}(u_j^{\bar{n}} v_j^n) = 0, 1 \le j \le J-1, n \ge 1, \tag{73}$$

$$\mathcal{A}\delta_{\hat{t}} v_j^n = \delta_x^2 \phi_j^{\bar{n}}, 1 \le j \le J-1, n \ge 1, \tag{74}$$

$$\mathcal{A}\delta_{\hat{t}} \phi_j^n = \Delta_j^n, 1 \le j \le J-1, n \ge 1, \tag{75}$$

$$u_j^0 = u_0(x_j), v_j^0 = v_0(x_j), \phi_j^0 = \phi_0(x_j), 0 \le j \le J, \tag{76}$$

$$u_0^n = u_J^n = 0, v_0^n = v_J^n = 0, \phi_0^n = \phi_J^n = 0, n \ge 0, \tag{77}$$

where

$$\Delta_j^n = \mathcal{A}v_j^{\bar{n}} + \mathcal{A}\left(\frac{F(v_j^{n+1}) - F(v_j^{n-1})}{v_j^{n+1} - v_j^{n-1}}\right) - \alpha\delta_x^2 v_j^{\bar{n}} + \omega\mathcal{A}(|u_j^n|^2).$$

It should be pointed out that Scheme B is a three-level implicit difference scheme, which needs a two-level scheme (such as Scheme A) or Taylor's Theorem to compute u^1, v^1 and ϕ^1. In this article, we calculate u^1, v^1, and ϕ^1 by means of Scheme A. Similar to the numerical analysis of Scheme A, Scheme B has the analogous properties. Due to space limitations, we omit the details of the proof.

Theorem 6.1. *Scheme B satisfies the following two discrete conservative laws*

$$Q^n = \frac{1}{2}(\|\mathbf{u}^{n+1}\|^2 + \|\mathbf{u}^n\|^2) = C, \tag{78}$$

$$\begin{aligned} E^n = {} & \frac{1}{2}(\|\mathbf{v}^{n+1}\|^2 + \|\mathbf{v}^n\|^2) - \frac{1}{2}(\langle\mathcal{N}\delta_x^2\phi^{n+1}, \phi^{n+1}\rangle + \langle\mathcal{N}\delta_x^2\phi^n, \phi^n\rangle) \\ & - \omega(\langle\mathcal{N}\delta_x^2\mathbf{u}^{n+1}, \mathbf{u}^{n+1}\rangle + \langle\mathcal{N}\delta_x^2\mathbf{u}^n, \mathbf{u}^n\rangle) - \frac{\alpha}{2}(\langle\mathcal{N}\delta_x^2\mathbf{v}^{n+1}, \mathbf{v}^{n+1}\rangle + \langle\mathcal{N}\delta_x^2\mathbf{v}^n, \mathbf{v}^n\rangle) \\ & + \langle F(\mathbf{v}^{n+1}) + F(\mathbf{v}^n), I\rangle + \omega(\langle\mathbf{v}^{n+1}, |\mathbf{u}^n|^2\rangle + \langle\mathbf{v}^n, |\mathbf{u}^{n+1}|^2\rangle) = C. \end{aligned} \tag{79}$$

Proof. Similar to the proof of Theorem 3.1. ■

Theorem 6.2. *Suppose that $\mathbf{u}^n, \mathbf{v}^n \in \mathbb{V}_h$ are the solutions of Scheme B, then the following estimates hold*

$$\|\mathbf{v}^n\| \le C, \|\delta_x\mathbf{v}^n\| \le C, \|\mathbf{v}^n\|_\infty \le C, \|\phi^n\| \le C,$$
$$\|\delta_x\phi^n\| \le C, \|\phi^n\|_\infty \le C, \|\delta_x\mathbf{u}^n\| \le C, \|\mathbf{u}^n\|_\infty \le C.$$

Proof. From Theorem 6.1, Sobolev's inequality and Lemma 2.1, we have the conclusion immediately. ■

Theorem 6.3. *The difference solutions of the Scheme B exist.*

Proof. Similar to the proof of Theorem 4.1, we can verify the existence of the difference solutions of Scheme B by means of Lemma 4.1. ■

Theorem 6.4. *Suppose that $u(x,t), v(x,t), \phi(x,t) \in C^{6,3}$ are the exact solutions of problem (3)–(7), then the difference solutions of Scheme B converge to the exact solutions with order $O(\tau^2 + h^4)$ in the L_2-norm.*

Proof. As its proof parallels that of Theorem 5.1, we omit the proof details. ■

Theorem 6.5. *Suppose that the conditions of Theorem 6.4 are satisfied, then the difference solutions of Scheme B depend continuously on the initial values.*

Proof. The proof procedure is quite similar to the method which we have utilized in Theorem 5.2. ■

VII. COMPUTATIONAL METHODS

A. Computational Methods of Scheme A

Equations (10)–(12) are nonlinear equations, and we construct an iteration scheme as follows

$$(i+6r)u_{j-1}^{n+1(s+1)} + (10i-12r)u_j^{n+1(s+1)} + (i+6r)u_{j+1}^{n+1(s+1)} = C_j^{n+1(s)}, \tag{80}$$

$$av_{j-1}^{n+1(s+1)} + c\phi_{j-1}^{n+1(s+1)} + bv_j^{n+1(s+1)} + d\phi_j^{n+1(s+1)} + av_{j+1}^{n+1(s+1)} + c\phi_{j+1}^{n+1(s+1)} = D_j^{n+1(s)}, \tag{81}$$

$$ev_{j-1}^{n+1(s+1)} + a\phi_{j-1}^{n+1(s+1)} + gv_j^{n+1(s+1)} + b\phi_j^{n+1(s+1)} + ev_{j+1}^{n+1(s+1)} + a\phi_{j+1}^{n+1(s+1)} = E_j^{n+1(s)}, \tag{82}$$

where $a = 1$, $b = 10$, $c = -6r$, $d = 12r$, $e = 6r\alpha - \frac{\tau}{2}$, $g = -12r\alpha - 5\tau$,

$$C_j^{n+1(s)} = 3\tau\mathcal{A}((u_j^{n+1(s)} + u_j^n)(v_j^{n+1(s)} + v_j^n)) + 12i\mathcal{A}u_j^n - 6\tau\delta_x^2 u_j^n, \tag{83}$$

$$D_j^{n+1(s)} = 12\mathcal{A}(v_j^n) + 6\tau\delta_x^2\phi_j^n, \tag{84}$$

$$E_j^{n+1(s)} = 6\tau\mathcal{A}(v_j^n) - 6\alpha\tau\delta_x^2 v_j^n + 12\mathcal{A}\phi_j^n + 12\tau\mathcal{A}\left(\frac{F(v_j^{n+1(s)}) - F(v_j^n)}{v_j^{n+1(s)} - v_j^n}\right)$$
$$+6\tau\omega\mathcal{A}(|u_j^{n+1(s)}|^2 + |u_j^n|^2). \tag{85}$$

The iteration initial value are given by $u_j^{n+1(0)} = u_j^n$, $v_j^{n+1(0)} = v_j^n$ and $\phi_j^{n+1(0)} = \phi_j^n$.

Equation (80) is a linear equation, which can be solved by means of Thomas' method. To solve (81)–(82) efficiently, we write (81)–(82) in matrix forms as follows

$$\begin{pmatrix} B & A & & & & \\ A & B & A & & & \\ & A & B & A & & \\ & & \ddots & \ddots & \ddots & \\ & & & A & B & A \\ & & & & A & B \end{pmatrix} \begin{pmatrix} X_1^{n+1(s+1)} \\ X_2^{n+1(s+1)} \\ \vdots \\ X_{J-2}^{n+1(s+1)} \\ X_{J-1}^{n+1(s+1)} \end{pmatrix} = \begin{pmatrix} K_1^{n(s)} \\ K_2^{n(s)} \\ \vdots \\ K_{J-2}^{n(s)} \\ K_{J-1}^{n(s)} \end{pmatrix}, \tag{86}$$

where $A = \begin{pmatrix} a & c \\ e & a \end{pmatrix}$, $B = \begin{pmatrix} b & d \\ g & b \end{pmatrix}$, $X_j^{n+1(s+1)} = \begin{pmatrix} v_j^{n+1(s+1)} \\ \phi_j^{n+1(s+1)} \end{pmatrix}$, $K_j^{n(s)} = \begin{pmatrix} D_j^{n+1(s)} \\ E_j^{n+1(s)} \end{pmatrix}$.

The coefficient matrix of (86) is a block tridiagonal matrix, so (86) can be realized by Thomas' method [19] efficiently. Additionally, the iteration scheme (86) continues until following condition is satisfied

$$\max_{1\le j\le J-1} |v_j^{n+1(s+1)} - v_j^{n+1(s)}| \le 10^{-12}.$$

B. Computational Methods of Scheme B

Equation (73) is linear and can be solved by Thomas' method efficiently. To solve (74) and (75), we construct an iteration scheme, which is identical to the form of (81)–(82). It is different from (81) to (82) in that $c = -12r$, $d = 24r$, $e = 12r\alpha - \tau$, $g = -24r\alpha - 10\tau$, and

$$D_j^{n+1(s)} = 12\mathcal{A}(v_j^{n-1}) + 12\tau\delta_x^2\phi_j^{n-1}, \tag{87}$$

$$E_j^{n+1(s)} = 12\mathcal{A}(\phi_j^{n-1}) + 24\tau\mathcal{A}\left(\frac{F(v_j^{n+1(s)}) - F(v_j^{n-1})}{v_j^{n+1(s)} - v_j^{n-1}}\right) + 24\tau\omega\mathcal{A}(|u_j^n|^2)$$
$$+12\tau\mathcal{A}(v_j^{n-1}) - 12\alpha\tau\delta_x^2 v_j^{n-1}. \tag{88}$$

The iteration initial values are given by $v_j^{n+1(0)} = 2v_j^n - v_j^{n-1}$ and $\phi_j^{n+1(0)} = 2\phi_j^n - \phi_j^{n-1}$.

VIII. NUMERICAL EXPERIMENTS

In this section, we will carry out some numerical experiments to verify the performance of our schemes. If $f(v) = \theta v^2$, some discussions of analytical solutions of SBq are given in [8–10]. Whereas problem (3)–(7) has many different forms of solitary wave solutions with respect to α, θ, and ω, we list two kinds of analytical solutions as follows for convenience of numerical comparison.

Case 1. If $(1 + \frac{3\theta-\alpha}{2})M^2 + 2(3\theta - \alpha)\delta = 1$ and $3\alpha \neq \theta$ and $(\alpha + 1)M^2 + 4\alpha\delta \neq 1$,

$$u(x,t) = \pm 6b_1\sqrt{\frac{\theta-\alpha}{\omega}}\sec h(\mu\xi)\tan h(\mu\xi)\exp\left(i\left(\frac{M}{2}x + \delta t\right)\right),$$
$$v(x,t) = -6b_1 sech^2(\mu\xi). \tag{89}$$

Case 2. If $3\alpha = \theta$ and $(\alpha + 1)M^2 + 4\alpha\delta \neq 1$,

$$u(x,t) = \pm\sqrt{\frac{6\alpha b_1}{\theta\omega}(d_1 - 4\alpha b_1)}\sec h(\mu\xi)\exp\left(i\left(\frac{M}{2}x + \delta t\right)\right),$$
$$v(x,t) = -2b_1 sech^2(\mu\xi), \tag{90}$$

where $b_1 = \delta + \frac{M^2}{4} > 0$, $d_1 = 1 - M^2$, $\mu = \sqrt{b_1}$, $\xi = x - Mt$, and M, δ are free parameters. We could select the value of M, δ at will according to the conditions of case 1 or case 2.

To quantify the error in the numerical solutions, we define the following error functions

$$E^u(h,\tau) = \sqrt{h\sum_{j=1}^{J-1}(|u_j^n - U_j^n|)^2}, E^v(h,\tau) = \sqrt{h\sum_{j=1}^{J-1}(v_j^n - V_j^n)^2}.$$

Furthermore, to investigate the stability of our scheme, we slightly perturb the initial function as follows

$$\tilde{\varepsilon}_0(x) = \tilde{\delta}_0(x) = \tilde{\sigma}_0(x) = \kappa\sin(x),$$

and the perturbed error functions are defined as follows

$$E^{\tilde{u}}(h,\tau) = \sqrt{h\sum_{j=1}^{J-1}(|\tilde{u}_j^n - U_j^n|)^2}, E^{\tilde{v}}(h,\tau) = \sqrt{h\sum_{j=1}^{J-1}(\tilde{v}_j^n - V_j^n)^2},$$

where κ is a constant.

TABLE I. Errors and convergence rates of Scheme A for Example 2 at $t = 10$ with $\tau = h^2$.

h	$E^u(h,\tau)$	$\frac{E^u(h,\tau)}{E^u\left(\frac{h}{2},\frac{\tau}{4}\right)}$	$E^v(h,\tau)$	$\frac{E^v(h,\tau)}{E^v\left(\frac{h}{2},\frac{\tau}{4}\right)}$
0.4	2.4000e-3		1.1944e-3	
0.2	1.5094e-4	15.900	7.4265e-5	16.083
0.1	9.4175e-6	16.028	4.6374e-6	16.014
0.05	6.2613e-7	15.041	3.1710e-7	14.624
0.025	4.3642e-8	14.347	1.9856e-8	15.970

TABLE II. Perturbed errors of Scheme A for Example 2 at $t = 10$ with $\tau = h^2$.

	$\kappa = 0.01$		$\kappa = 0.001$		$\kappa = 0.0001$	
h	$E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$	$E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$	$E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$
0.4	1.6745e-2	1.0487e-2	2.5511e-3	1.7551e-3	2.3610e-3	1.2268e-3
0.2	1.7239e-2	1.0502e-2	1.6740e-3	1.0653e-3	1.9526e-4	1.4237e-4
0.1	1.7278e-2	1.0493e-2	1.7057e-3	1.0465e-3	1.6835e-4	1.0774e-4
0.05	1.7280e-2	1.0492e-2	1.7082e-3	1.0454e-3	1.7058e-4	1.0458e-4
0.025	1.7280e-2	1.0492e-2	1.7082e-3	1.0454e-3	1.7062e-4	1.0450e-4

We choose a sufficiently long interval $x \in [-40, 40]$ for the computation so that the zeros of the boundary conditions do not introduce a significant error relative to the whole space problem [11]. We provide two different types of exact solutions related to case 1 and case 2, respectively.

Example 1. Let $\alpha = 1$, $\theta = \frac{4}{3}$, $\omega = \frac{1}{18}$, and to satisfy the condition of case 1, set $M = \frac{1}{\sqrt{5}}$ and $\delta = \frac{1}{12}$.

Example 2. Let $\alpha = \frac{1}{2}$, $\theta = \frac{3}{2}$, $\omega = \frac{1}{12}$, and to satisfy the condition of case 2, we set $M = \frac{\sqrt{3}}{3}$ and $\delta = \frac{1}{5}$.

A. Numerical Results of Scheme A

Table I lists some errors and the convergence rates of Scheme A an $t = 10$. As we see from Table I, the convergence order in the L_2 norm almost equals to 4 in the spatial direction, which is consistent with the conclusion in Theorem 5.1.

Table II lists the perturbed errors of Scheme A for different perturbed initial values with $\tau = h^2$. It can be seen from Table II that the solutions of Scheme A at $t = 10$ have subtle change while the initial function is slightly perturbed, which implies the stability of Scheme A.

B. Numerical Results of Scheme B

Table III lists the errors and the convergence rates of Scheme B ant $t = 10$. It can be seen from the Table III that the convergence order almost equals to 4 in the spatial direction.

Figures 1 and 3 plot the exact solutions and the numerical solutions (via Scheme A) of Example 1 and Example 2 at $t = 10$ with $h = \tau = 0.1$, while Figures 2 and 4 simulate the numerical solutions (via Scheme A) of Example 1 and Example 2 from $t = 0$ to $t = 10$ with $h = \tau = 0.1$. Figures 1–4 show that the numerical solutions of Example 1 and Example 2 fit very well to the exact solutions.

TABLE III. Errors and convergence rates of Scheme B for Example 2 at $t = 10$ with $\tau = h^2$.

h	$E^u(h,\tau)$	$\frac{E^u(h,\tau)}{E^u\left(\frac{h}{2},\frac{\tau}{4}\right)}$	$E^v(h,\tau)$	$\frac{E^v(h,\tau)}{E^v\left(\frac{h}{2},\frac{\tau}{4}\right)}$
0.4	6.2677e-3		5.0790e-3	
0.2	3.9363e-4	15.923	3.1747e-4	15.998
0.1	2.4617e-5	15.990	1.9835e-5	16.006
0.05	1.5391e-6	15.994	1.2396e-6	16.001
0.025	9.8212e-8	15.671	7.7481e-8	15.999

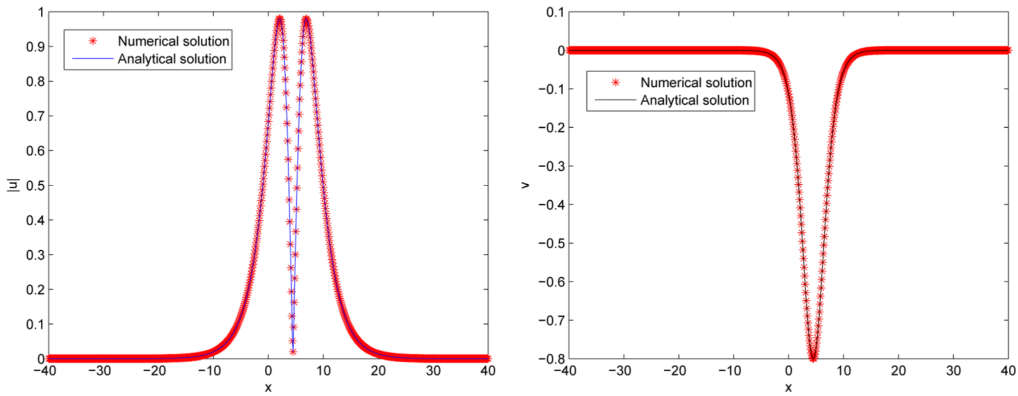

FIG. 1. Scheme A: numerical solutions of Example 1 (left $|u|$ and right v) and its exact solutions at $t = 10$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

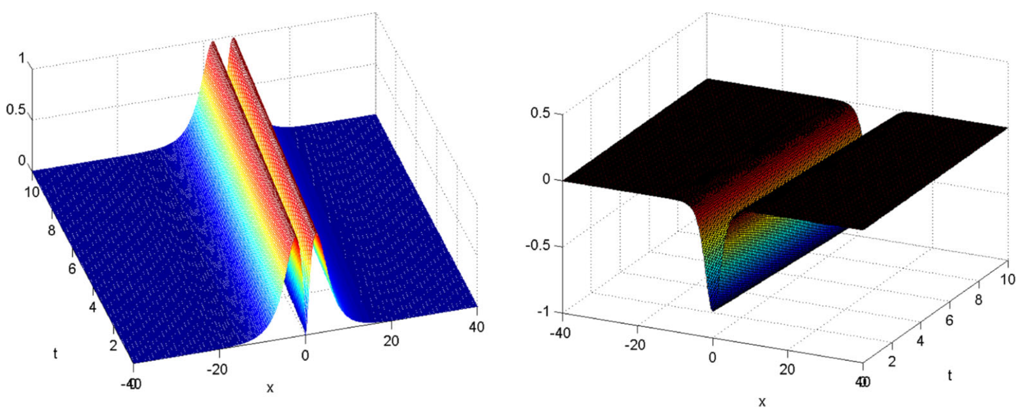

FIG. 2. Scheme A: numerical solutions of Example 1 (left $|u|$ and right v) for $t \in [0, 10]$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

Taking $h = \tau = 0.1$, Figure 5 plots the two discrete conservative laws of Theorem 3.1 from $t = 0$ to $t = 10$. As can be seen from Figure 5, Scheme A can simulate the conservation quantities very well.

Figure 6 plots the discrete conservative laws of Scheme B (in Theorem 6.1) form $t = 0$ to $t = 10$ with $h = \tau = 0.1$. It can be seen from Figure 6 that Scheme B conserves the conservation quantities very well.

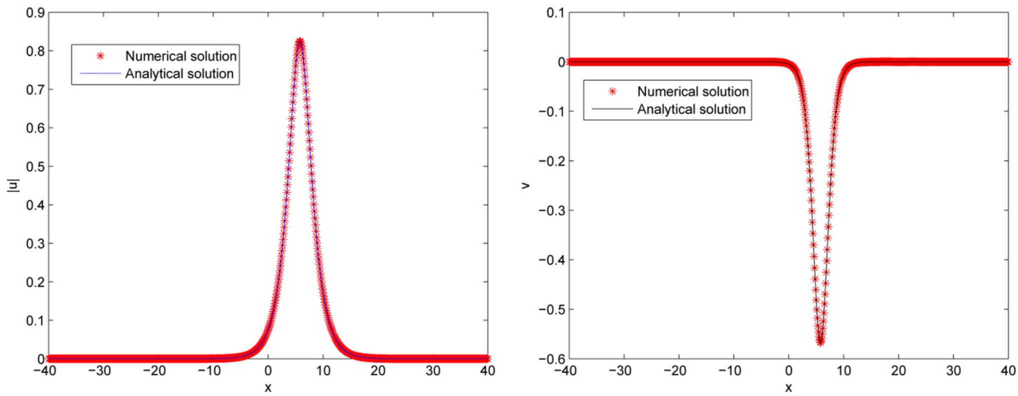

FIG. 3. Scheme A: numerical solutions of Example 2 (left $|u|$ and right v) and its exact solutions at $t = 10$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

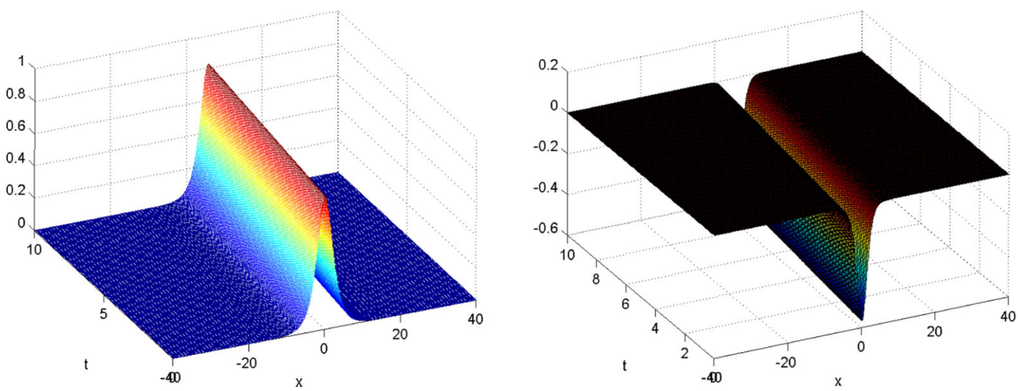

FIG. 4. Scheme A: numerical solutions of Example 2 (left $|u|$ and right v) for $t \in [0, 10]$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

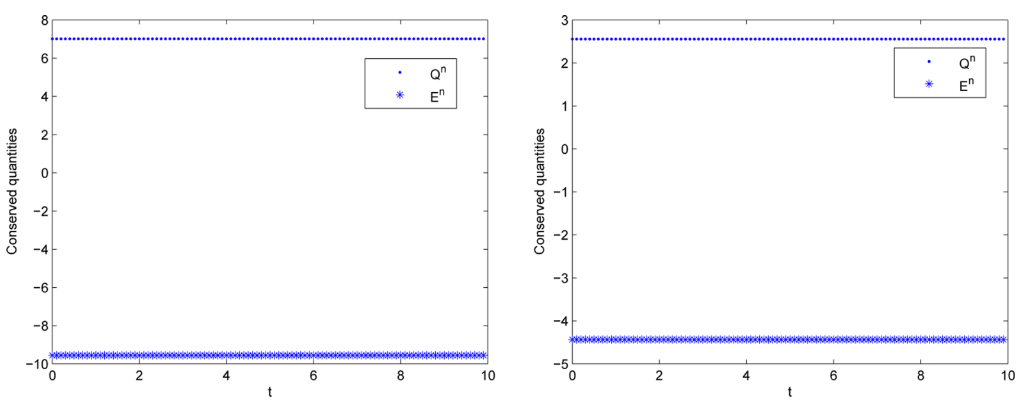

FIG. 5. Scheme A: the conserved quantity curves of Example 1 (left) and Example 2 (right) from $t = 0$ to $t = 10$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

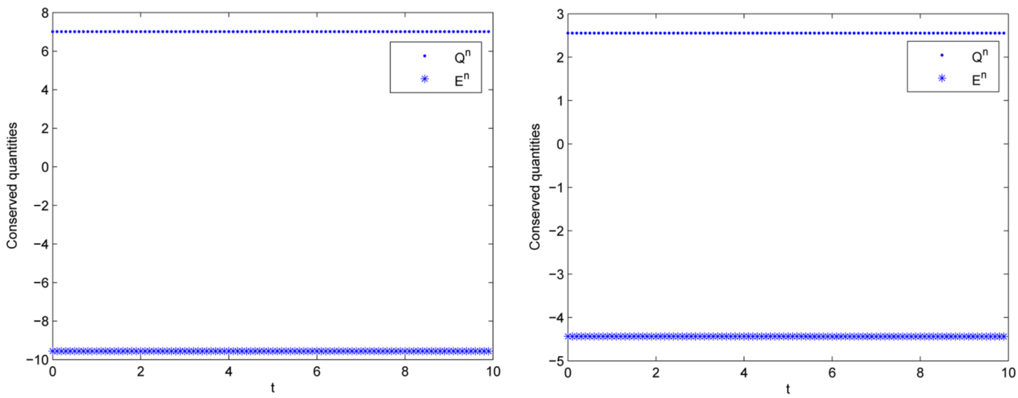

FIG. 6. Scheme B: the conserved quantity curves of Example 1 (left) and Example 2 (right) from $t = 0$ to $t = 10$ with $h = \tau = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

TABLE IV. Perturbed errors of Scheme B for Example 2 at $t = 10$ with $\tau = h^2$.

h	$\kappa = 0.01$ $E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$	$\kappa = 0.001$ $E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$	$\kappa = 0.0001$ $E^{\tilde{u}}(h,\tau)$	$E^{\tilde{v}}(h,\tau)$
0.4	1.9465e-2	1.2647e-2	6.8169e-3	5.4937e-3	6.3051e-3	5.1133e-3
0.2	1.7366e-2	1.0550e-2	1.8316e-3	1.1702e-3	4.6067e-4	3.6049e-4
0.1	1.7285e-2	1.0496e-2	1.7136e-3	1.0507e-3	1.7746e-4	1.1156e-4
0.05	1.7281e-2	1.0493e-2	1.7085e-3	1.0457e-3	1.7096e-4	1.0484e-4
0.025	1.7280e-2	1.0492e-2	1.7082e-3	1.0454e-3	1.7065e-4	1.0452e-4

TABLE V. The L_∞-norm errors of Example 1 at $t = 1$ for different mesh sizes.

(h,τ)	Scheme A $E_\infty(h,\tau)$	Scheme B $E_\infty(h,\tau)$	QBFEM [11] $E_\infty(h,\tau)$	FDM [14] $E_\infty(h,\tau)$
$\left(\frac{2}{5}, \frac{1}{100}\right)$	1.4619e-4	1.4846e-4	4.7862e-3	9.4495e-3
$\left(\frac{1}{5}, \frac{1}{200}\right)$	9.3002e-6	9.6988e-6	1.1744e-3	2.3771e-3

Table IV lists the perturbed errors of Scheme B for different perturbed initial values at $t = 10$ with $\tau = h^2$, which indicates that the smaller perturbed initial value is, the less perturbed error. Tables II and IV also show that the perturbed errors converge to a fixed value with the decrease of the spatial step h, which illustrates the convergence of our scheme to some extent.

C. NUMERICAL COMPARISON WITH OTHER SCHEMES

In this section, we will compare our schemes with previous numerical methods. We mainly compare the computational accuracy and efficiency of our schemes with QBFEM [11] and FDM [14]. The L_∞-norm errors of Example 1 and Example 2 are listed in Tables V and VI, respectively.

Remark. $E^u_\infty(h,\tau) = \max\limits_{0\le j\le J}(|U^n_j - u^n_j|)$, $E_\infty(h,\tau) = E^u_\infty(h,\tau) + E^v_\infty(h,\tau)$.

As can be seen from Tables V–VI that Scheme A and Scheme B are more accurate than QBFEM [11] and FDM [14]. Furthermore, the CPU time are reported in Tables VII–VIII.

TABLE VI. The L_∞-norm errors of Example 2 at $t = 1$ for different mesh sizes.

(h, τ)	Scheme A $E_\infty(h,\tau)$	Scheme B $E_\infty(h,\tau)$	QBFEM [11] $E_\infty(h,\tau)$	FDM [14] $E_\infty(h,\tau)$
$\left(\frac{2}{5}, \frac{1}{100}\right)$	2.0330e-4	2.0725e-4	8.6709e-3	9.0125e-3
$\left(\frac{1}{5}, \frac{1}{200}\right)$	1.2755e-5	1.3761e-5	2.5199e-3	2.2196e-3

TABLE VII. Comparison of the CPU time for Example 1 at $t = 1$.

(h, τ)		Scheme A	Scheme B	QBFEM [11]	FDM [14]
$\left(\frac{2}{5}, \frac{1}{100}\right)$	CPU time(s)	4.41	2.53	108.32	1.61
$\left(\frac{1}{5}, \frac{1}{200}\right)$	CPU time(s)	11.72	8.92	3240	5.87

TABLE VIII. Comparison of the CPU time for Example 2 at $t = 1$.

(h, τ)		Scheme A	Scheme B	QBFEM [11]	FDM [14]
$\left(\frac{2}{5}, \frac{1}{100}\right)$	CPU time(s)	4.42	2.98	107.84	1.77
$\left(\frac{1}{5}, \frac{1}{200}\right)$	CPU time(s)	16.76	8.65	3199	5.48

From Tables V to VIII, we can see that Scheme A is the most accurate of all the schemes mentioned. However, Scheme B is even better than Scheme A in the aspect of efficiency. Clearly, FDM [14] is the most time-saving method, while lowest in accuracy. QBFEM [11] is the most time-consuming method, which requires tremendous matrix multiplication operations in the process of solving ordinary differential equations by means of the standard fourth-order Runge-Kutta (RK4) method, and the computational complexity of QBFEM [11] is $O(J^2)$. Comparing with QBFEM [11], Scheme A and Scheme B or FDM [14] requires $O(J)$ operations.

It should be pointed that RK4 method is used to solve the semidiscrete system of SBq equation, thus, the iterating times of QBFEM [11] is constant at 4. Although Scheme A, Scheme B and FDM [14] are nonlinear schemes, we found that the maximum iterating times of it requires no more than three times.

IX. CONCLUSION

In this article, two conservative compact finite difference schemes for problem (3)–(7) are presented. The conservative properties, existence of the difference solutions, convergence and stability of our schemes are theoretically analyzed. The numerical results illustrate the conserved quantities, convergence rates, and stability of our schemes. The comparisons between our schemes with previous methods are conducted to show the accuracy and efficiency of our schemes.

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