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A new unfitted stabilized Nitsche's finite element method for Stokes interface problems

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ABSTRACT

In this paper, we introduce and analyze a new stabilized finite element method based on combining Nitsche's method with a ghost penalty method for two-phase Stokes flows involving two different kinematic viscosities by using the lowest equal order velocity–pressure pairs. The interface between two-phase flows does not need to align with the mesh and interface conditions are imposed weakly using a Nitsche type approach. This method has some prominent features: parameter-free, avoiding calculation of higher order derivatives or edge data structures and stabilization being completely local behavior about pressure. We prove that the method is inf–sup stable and obtain optimal order a priori error estimates. We also show that the estimate for the condition number of the stiffness matrix is independent of the location of the interface. Finally, we present some numerical examples to support our theoretical results.

1. Introduction

As two-phase flows are widely encountered in engineering and scientific applications, numerical simulation approaches for such problems have attracted much attention. In many numerical simulations, two-phase flows are modeled by Navier–Stokes equations with discontinuous density and viscosity coefficients. Stokes interface problems can be considered as a reasonable first step for the study of low Reynolds number two-phase flows.

Due to the discontinuity of the coefficients across the interface, it is well known that standard finite element methods, as well as finite difference methods, cannot obtain optimal a priori error estimates unless the mesh is fitted to the interface. However, repeated remeshing of the domain to obtain fitted meshes is very costly for many time dependent problems in which the interface moves with time. Henceforth, it is attractive to develop numerical methods based on unfitted mesh of the domain.

LeVeque and Li in [1] proposed an immersed interface method (IIM) using unfitted grid based on finite difference method to solve the elliptic interface problems. Subsequently, the IIM was extended to solve Stokes problems and Navier–Stokes problems with interface [2,3]. In the construction of the formulation using IIM, the jump condition was properly incorporated into the finite difference scheme to obtain a modified scheme. The resulting linear systems from these methods are nonsymmetric and indefinite. On the other hand, for finite element methods, Li, Lin and Wu proposed an immersed finite element method (IFEM) on Cartesian grid to solve elliptic interface problems in [4]. The linear systems arising from the IFEM

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are symmetric and positive definite. However, whether this method can be used to solve Stokes interface problems or not is still open.

In recent years, a novel extended finite element method based on Nitsche's method (Nitsche-XFEM) for avoiding fitted grid has become a popular approach. Nitsche's method was proposed originally for the weak enforcement of essential boundary conditions in [5], see also a recent work [6]. Later, this method was modified to enforce the jump conditions at the interface by A. Hansbo and P. Hansbo in [7] to solve elliptic interface problems. This method was also used to solve the compressible and incompressible elasticity problems with interface in [8,9] and the contrast problems in [10]. However, these methods suffer from ill-conditioning just as XFEM [11]. In [12], Burman proposed a ghost penalty method to ensure well conditioned system matrices. For the Stokes interface problems, some similar strategies have recently been presented. For instance, two modifications of the Mini element method in [13] and the classical Brezzi–Pitkäranta stabilized method in [14] were proposed to solve the saddle point interface problems in [15]. In [16], a Nitsche formulation for Stokes interface problems based on P_1 iso P_2/P_1 elements was proposed, and optimal order a priori error estimates were established.

For solving Stokes equations much attention has been attracted to use the equal order finite element pairs due to their convenience and small computational cost. However, it is well known that they do not satisfy the inf–sup condition [17]. In [18], a pressure projection method was presented to stabilize these element pairs. Later, Li and He also obtained this stabilized method based on two local Gauss integrations for solving Stokes/Navier–Stokes equations in [19,20]. In this paper, we consider a new stabilized mixed finite element method for Stokes interface problems on unfitted grid. This finite element method enforces the interface jump conditions weakly in Nitsche's method. The class of unfitted stabilized Nitsche's finite element method constructed in this paper has a similar structure of the stabilized finite element methods in [15,21]. However, rather than requiring calculation of higher order derivatives or edge based data structures, we use a local L^2 pressure projection to construct our scheme. Moreover, our proposed method can easily be extended to finite elements of high order. Using this approach, we show that the new method is inf–sup stable and optimally convergent. To guarantee well conditioned system matrix, we also introduce a ghost penalty term for velocity, and prove that the condition number of the stiffness matrix is $\mathcal{O}(h^{-2})$.

The outline of this paper is as follows. In Section 2, we introduce Stokes interface problems and the weak formulation. In Section 3, the approximation properties of the finite element spaces are presented. In Section 4, we prove the inf–sup stability. The optimal a priori error estimates are shown in Section 5. In Section 6, the condition number of the stiffness matrix is proved to be independent of the position of the interface relative to the mesh. In Section 7, we provide some numerical examples to verify our theoretical analysis. Finally, conclusions are summarized in Section 8.

2. Stokes interface problems

Consider the following Stokes interface problems (see [22]): find a velocity $\boldsymbol{u}$ and a pressure p such that

$$\begin{aligned} &-\mathrm{div}(\nu\nabla\boldsymbol{u})+\nabla p=\boldsymbol{f} \quad \text{in } \Omega_1\cup\Omega_2,\\ &\mathrm{div}\,\boldsymbol{u}=0 \quad \text{in } \Omega_1\cup\Omega_2,\\ &[\![\boldsymbol{u}]\!]=0 \quad \text{on } \Gamma,\\ &[\![\boldsymbol{\sigma}(\boldsymbol{u},p)\cdot\boldsymbol{n}]\!]=\boldsymbol{g} \quad \text{on } \Gamma,\\ &\boldsymbol{u}=0 \quad \text{on } \partial\Omega, \end{aligned} \tag{2.1}$$

with a piecewise constant viscosity:

$$\nu=\begin{cases}\nu_1>0 & \text{in } \Omega_1,\\ \nu_2>0 & \text{in } \Omega_2,\end{cases} \tag{2.2}$$

on a bounded polygonal domain $\Omega\subset\mathbb{R}^2$. The subdomains Ω_1 and Ω_2 are assumed to be Lipschitz domain and satisfy $\overline{\Omega}=\overline{\Omega}_1\cup\overline{\Omega}_2$ and $\Omega_1\cap\Omega_2=\emptyset$. The interface between the subdomains is denoted by $\Gamma=\partial\Omega_1\cap\partial\Omega_2$, which is assumed to have at least C^2-smooth, see Fig. 1. $\boldsymbol{\sigma}(\boldsymbol{u},p)=-p\boldsymbol{I}+\nu\nabla\boldsymbol{u}$ is the stress tensor where $\boldsymbol{I}$ is identity tensor and $\boldsymbol{n}$ is a unit normal to Γ pointing from Ω_1 to Ω_2, $\boldsymbol{f}$ and $\boldsymbol{g}$ are given functions. The jump across the interface is denoted by $[\![\boldsymbol{u}]\!]=(\boldsymbol{u}_1-\boldsymbol{u}_2)|_\Gamma$ with $\boldsymbol{u}_i=\boldsymbol{u}|_{\Omega_i}$, $i=1,2$. For interface conditions, we refer to [23] for details.

For a bounded domain O we will employ the standard Sobolev space $H^m(O)$ with norm $\|\cdot\|_{m,O}$ and space $H_0^1(O)$ with zero trace on ∂O. When $m=0$ we write $L^2(O)$ instead of $H^0(O)$ with inner product $(\cdot,\cdot)$ and $L_0^2(O)$ denote the space of all square integrable functions with vanishing mean. Spaces consisting of vector-valued functions will be denoted in boldface. For a bounded open set $O=O_1\cup O_2$, where O_i are open mutually disjoint components of O, we denote by $H^m(O_1\cup O_2)$ the Sobolev space of functions in O such that $u|_{O_i}\in H^m(O_i)$ with broken norm

$$\|\cdot\|_{m,O_1\cup O_2}=\left(\sum_{i=1}^{2}\|\cdot\|_{m,O_i}^2\right)^{1/2}.$$

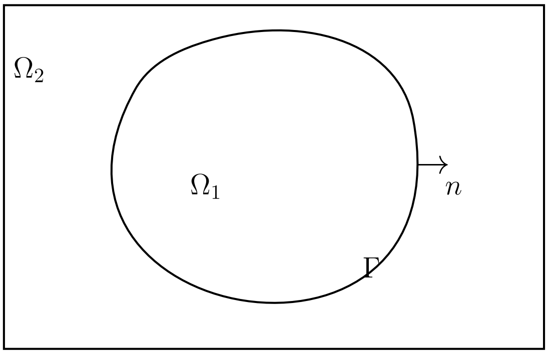

Fig. 1. Domain Ω, its subdomains Ω_1, Ω_2 and interface Γ.

The variational formulation of Eqs. (2.1) is to seek $(\boldsymbol{u}, p) \in \boldsymbol{H}_0^1(\Omega) \times L_0^2(\Omega)$ such that

$$\begin{cases} a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{v}, p) = F(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega), \\ b(\boldsymbol{u}, q) = 0 & \forall q \in L_0^2(\Omega), \end{cases} \tag{2.3}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = (\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{v})$, $b(\boldsymbol{v}, q) = -(\operatorname{div} \boldsymbol{v}, q)$ and $F(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}) + \langle \boldsymbol{g}, \boldsymbol{v} \rangle_\Gamma$ with $\langle \boldsymbol{g}, \boldsymbol{v} \rangle_\Gamma = \int_\Gamma \boldsymbol{g}\boldsymbol{v} ds$.

We have the following regularity theorem for the weak solutions $(\boldsymbol{u}, p)$ of the variational problem (2.3), see Theorem 1.1 in [24].

Lemma 2.1. *Assume that $\boldsymbol{f} \in \boldsymbol{L}^2(\Omega)$ and $\boldsymbol{g} \in \boldsymbol{H}^{\frac{1}{2}}(\Gamma)$, then the variational problem (2.3) has a unique solution $(\boldsymbol{u}, p) \in (\boldsymbol{H}^2(\Omega_1 \cup \Omega_2) \cap \boldsymbol{H}_0^1(\Omega)) \times (H^1(\Omega_1 \cup \Omega_2) \cap L_0^2(\Omega))$ which satisfies the estimate*

$$\|\boldsymbol{u}\|_{1,\Omega} + \|\boldsymbol{u}\|_{2,\Omega_1 \cup \Omega_2} + \|p\|_{0,\Omega} + \|p\|_{1,\Omega_1 \cup \Omega_2} \le C \left(\|\boldsymbol{f}\|_{0,\Omega} + \|\boldsymbol{g}\|_{\frac{1}{2},\Gamma} \right),$$

here and below, C denotes a generic constant independent of mesh size.

The key idea of our method is to replace the strong interface conditions $(2.1)_3$ and $(2.1)_4$ by a weak formulation [5,7]. This approach is analogous to that of discontinuous Galerkin methods for elliptic equations [25].

Let $\mathcal{T}_h$ be a conforming triangulation of Ω, generated independently of the location of the interface Γ. We will use the following notation for mesh quantities. Let h_K be the diameter of K and $h = \max_{K \in \mathcal{T}_h} h_K$. By $G_h := \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$ we denote the set of elements that are intersected by the interface. For an element $K \in G_h$, let $\Gamma_K := \Gamma \cap K$ be the part of Γ in K. We define meshes on the subdomains $\Omega_i,\ i = 1, 2$, as follows:

$$\mathcal{T}_{h,i} = \{K \in \mathcal{T}_h;\ \exists x, y \in K \cap \Omega_i,\ \text{and } x \neq y\}.$$

Denote

$$\Omega_{h,i} = \bigcup_{K \in \mathcal{T}_{h,i}} K, \quad i = 1, 2.$$

We make the following assumptions (see [7]):

- A1: We assume that the triangulation $\mathcal{T}_h$ is non-degenerate and quasi-uniform, i.e., there exist two positive constants C_1 and C_2, such that

  $$h_K/\rho_K \le C_1, \qquad h/h_K \le C_2, \quad \forall K \in \mathcal{T}_h,$$

  where ρ_K is the diameter of the largest ball contained in K.
- A2: We assume that Γ intersects each element boundary ∂K exactly twice and each (open) edge at most once.
- A3: Let $\Gamma_{K,h}$ be the straight line segment connecting the points of intersection between Γ and ∂K. We assume that Γ_K is a function of length on $\Gamma_{K,h}$, in local coordinate:

  $$\Gamma_{K,h} = \{(\xi, \eta);\ 0 < \xi < |\Gamma_{K,h}|,\ \eta = 0\}, \qquad \Gamma_K = \{(\xi, \eta);\ 0 < \xi < |\Gamma_{K,h}|,\ \eta = \delta(\xi)\}.$$

Since the curvature of Γ is bounded, the assumptions A2 and A3 are always fulfilled on sufficiently fine meshes. Thus the assumptions essentially demand that the interface is well resolved by the mesh.

For $i = 1, 2$, denote by $\boldsymbol{V}_{h,i}$ the space of vector valued continuous piecewise linear polynomials on $\mathcal{T}_{h,i}$, that is

$$\boldsymbol{V}_{h,i} = \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega_{h,i});\ \boldsymbol{v}|_K \text{ is linear},\ \forall K \in \mathcal{T}_{h,i},\ \boldsymbol{v}|_{\partial\Omega} = 0\},$$

and define our velocity space as $\boldsymbol{V}_h = \boldsymbol{V}_{h,1} \times \boldsymbol{V}_{h,2}$.

Similarly, denote

$$Q_{h,i} = \{q \in H^1(\Omega_{h,i});\ q|_K \text{ is linear},\ \forall K \in \mathcal{T}_{h,i}\},$$

and define our pressure space as $Q_h = Q_{h,1} \times Q_{h,2}$ with $\int_{\Omega\setminus G_h} q_h dx = 0$, for all $q_h \in Q_h$.

Remark 2.1. We would like to point out that both the finite element spaces $\boldsymbol{V}_h$ and Q_h are double valued on overlapping domain G_h.

We denote by $P_k(D)$ the space of polynomials of degree k on D. For defining a ghost penalty, we decompose the interface zone of the triangulation $\mathcal{T}_{h,i}$ in $\mathcal{N}_{l,i}$ patches $\mathcal{P}_{l,i}$ with diameters $h_{\mathcal{P}_{l,i}} = \mathcal{O}(h)$ consisting of a moderate number of elements, in such a way that every interface element is included in one $\mathcal{P}_{l,i}$, for details one can refer to [12]. Let $J_{l,i}$ be the L^2-projection onto $P_1(\mathcal{P}_{l,i})$ and $\Pi_{K,i}$ be the L^2-projection onto $P_0(K)$ for each $K \in \mathcal{T}_{h,i}$. Denote $\boldsymbol{J}_{l,i} = (J_{l,i}, J_{l,i})$. For any element $K \in \mathcal{T}_h$, let $K_i = K \cap \Omega_i$ denote the part of K in Ω_i. For any function ϕ discontinuous across interface Γ, we define the weighted average $\{\phi\}_w = w_1\phi_1 + w_2\phi_2$ and $\{\phi\}^w = w_2\phi_1 + w_1\phi_2$ with $w_i = \frac{|K_i|}{|K|}$. Recalling the definition of $[\![\phi]\!]$, we have $[\![\phi\psi]\!] = \{\phi\}_w[\![\psi]\!] + [\![\phi]\!]\{\psi\}^w$.

Now we propose our stabilized Nitsche method: find $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ such that

$$A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h) = L_h(\boldsymbol{v}_h), \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h, \tag{2.4}$$

here $A_h(\cdot, \cdot; \cdot, \cdot)$ is a bilinear form defined by

$$A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) + b_h(\boldsymbol{u}_h, q_h),$$

where

$$\begin{aligned} a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) \ &= \int_{\Omega_1\cup\Omega_2} \nu\nabla\boldsymbol{u}_h \cdot \nabla\boldsymbol{v}_h dx - \int_\Gamma \{\nu\nabla\boldsymbol{u}_h \cdot \boldsymbol{n}\}_w[\![\boldsymbol{v}_h]\!]ds \\ &\quad - \int_\Gamma \{\nu\nabla\boldsymbol{v}_h \cdot \boldsymbol{n}\}_w[\![\boldsymbol{u}_h]\!]ds + \sum_{K\in G_h}\int_{\Gamma_K} \lambda h_K^{-1}[\![\boldsymbol{u}_h]\!][\![\boldsymbol{v}_h]\!]ds, \\ b_h(\boldsymbol{v}_h, p_h) &= -\int_{\Omega_1\cup\Omega_2} p_h \operatorname{div} \boldsymbol{v}_h dx + \int_\Gamma \{p_h\}_w[\![\boldsymbol{v}_h]\!] \cdot \boldsymbol{n} ds, \\ G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) &= \sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}} h_{\mathcal{P}_{l,i}}^{-2} \int_{\mathcal{P}_{l,i}} (\boldsymbol{u}_{h,i} - \boldsymbol{J}_{l,i}\boldsymbol{u}_{h,i})(\boldsymbol{v}_{h,i} - \boldsymbol{J}_{l,i}\boldsymbol{v}_{h,i})dx, \\ G_2(p_h, q_h) &= \sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}} \int_K (p_{h,i} - \Pi_{K,i}p_{h,i})(q_{h,i} - \Pi_{K,i}q_{h,i})dx, \end{aligned}$$

with λ a parameter that will be specified below, and

$$L_h(\boldsymbol{v}_h) = \int_{\Omega_1\cup\Omega_2} \boldsymbol{f} \cdot \boldsymbol{v}_h dx + \int_\Gamma \boldsymbol{g}\{\boldsymbol{v}_h\}^w ds.$$

Remark 2.2. The bilinear form $a_h(\cdot, \cdot)$ is a standard Nitsche's method that are used to enforce weakly the jump conditions at the interface.

Remark 2.3. The terms $G_1(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and $G_2(p_h, q_h)$ in Eq. (2.4) are the stabilization terms. If interface does not cut through any element $K \in \mathcal{T}_h$, the stabilization term $G_2(p_h, q_h)$ is sufficient to prove the inf–sup stability of the method (see [18]). In contrast, if the interface does not align with the mesh, to prove the inf–sup condition and control the condition number of the stiffness matrix independently of the location of the interface, both the stabilization terms $G_1(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and $G_2(p_h, q_h)$ are needed. Both of them defined on G_h which are called ghost penalty (see [12]) are computed on the whole interface element. Similar to stabilized methods proposed in [16,15], the presented method is parameter free. However, our method avoids using higher order derivatives or edge data structures. Moreover, this method can be extended to finite elements of high order (see [26] for details).

Since the stabilization terms $G_1(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and $G_2(p_h, q_h)$ are not the residual of the Stokes equations, the finite element formulation (2.4) is not consistent. However, we have the following weak consistent result.

Lemma 2.2. *Let $(\boldsymbol{u}, p) \in (\boldsymbol{H}^2(\Omega_1 \cup \Omega_2) \cap \boldsymbol{H}_0^1(\Omega)) \times (H^1(\Omega_1 \cup \Omega_2) \cap L_0^2(\Omega))$ be the solution of (2.3) and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ be the solution of the finite element formulation (2.4), then*

$$A_h(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h; \boldsymbol{v}_h, q_h) = G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h), \quad \forall(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h. \tag{2.5}$$

Proof. Multiplying $(2.1)_1$ and $(2.1)_2$ by test functions $\boldsymbol{v}_h$ and q_h, respectively, then using integration by parts and noting that the interface conditions $(2.1)_3$ and $(2.1)_4$, we have

$$a_h(\boldsymbol{u},\boldsymbol{v}_h)+b_h(\boldsymbol{v}_h,p)+b_h(\boldsymbol{u},q_h)=L_h(\boldsymbol{v}_h). \tag{2.6}$$

Subtracting (2.4) from (2.6), we get (2.5). □

3. Approximation properties

First of all, we define the following mesh dependent norms:

$$\|\boldsymbol{v}\|_h := (\|\nabla\boldsymbol{v}\|^2_{0,\Omega_1\cup\Omega_2}+\|[\![\boldsymbol{v}]\!]\|^2_{1/2,h,\Gamma})^{1/2},$$
$$\|\boldsymbol{v}\|_\star := (\|\nabla\boldsymbol{v}\|^2_{0,\Omega_{h,1}\cup\Omega_{h,2}}+\|[\![\boldsymbol{v}]\!]\|^2_{1/2,h,\Gamma})^{1/2},$$

and

$$|\!|\!|(\boldsymbol{v},q)|\!|\!|_h := (\|\boldsymbol{v}\|_h^2+\|q\|^2_{0,\Omega_1\cup\Omega_2})^{1/2},$$
$$|\!|\!|(\boldsymbol{v},q)|\!|\!|_\star := (\|\boldsymbol{v}\|_\star^2+\|q\|^2_{0,\Omega_{h,1}\cup\Omega_{h,2}})^{1/2},$$

where

$$\|\phi\|_{1/2,h,\Gamma} := \left(\sum_{K\in G_h} h_K^{-1}\|\phi\|^2_{0,\Gamma_K}\right)^{1/2},$$

$$\|\phi\|_{-1/2,h,\Gamma} := \left(\sum_{K\in G_h} h_K\|\phi\|^2_{0,\Gamma_K}\right)^{1/2}.$$

It is easy to show that

$$\langle\phi,\psi\rangle_\Gamma \le \|\phi\|_{1/2,h,\Gamma}\|\psi\|_{-1/2,h,\Gamma}. \tag{3.1}$$

For the following analysis we need the famous Stein's extension theorem (see [27,28]).

Lemma 3.1. *There exist two extension operators $E_i : H^k(\Omega_i)\to H^k(\Omega)$ for all non-negative integers k such that*

$$(E_iv_i)|_{\Omega_i}=v_i \quad \text{and} \quad \|E_iv_i\|_{H^k(\Omega)}\le C\|v_i\|_{H^k(\Omega_i)},$$

for all $v_i\in H^k(\Omega_i)$, $i=1,2$.

To define the interpolation operators, we let $\widetilde{\boldsymbol{V}}_h$ and $\widetilde{Q}_h$ be the space of vector and scalar valued continuous piecewise linear polynomials on the triangulation $\mathcal{T}_h$, respectively. Let $\boldsymbol{I}_h : \boldsymbol{H}^2(\Omega)\cap\boldsymbol{H}_0^1(\Omega)\to\widetilde{\boldsymbol{V}}_h$ be the standard nodal interpolation operator and $\rho_h : H^1(\Omega)\to\widetilde{Q}_h$ be the L^2-projection. For $\boldsymbol{v}_i=\boldsymbol{v}|_{\Omega_i}$ and $q_i=q|_{\Omega_i}$, we define

$$(\boldsymbol{I}_h^\star\boldsymbol{v},\rho_h^\star q) := ((\boldsymbol{I}^\star_{h,1}\boldsymbol{v}_1,\boldsymbol{I}^\star_{h,2}\boldsymbol{v}_2),(\rho^\star_{h,1}q_1,\rho^\star_{h,2}q_2)), \tag{3.2}$$

where $\boldsymbol{I}^\star_{h,i}\boldsymbol{v}_i=(\boldsymbol{I}_h\boldsymbol{E}_i\boldsymbol{v}_i)|_{\Omega_i}$ and $\rho^\star_{h,i}q_i=(\rho_hE_iq_i)|_{\Omega_i}$.

The following interpolation error estimates are valid.

Theorem 3.1. *Let $(\boldsymbol{I}_h^\star,\rho_h^\star)$ be the interpolation operators defined in (3.2). Then we have*

$$\|\boldsymbol{v}-\boldsymbol{I}_h^\star\boldsymbol{v}\|_h\le Ch\|\boldsymbol{v}\|_{2,\Omega_1\cup\Omega_2}, \tag{3.3}$$

and

$$\|q-\rho_h^\star q\|_{0,\Omega_1\cup\Omega_2}\le Ch\|q\|_{1,\Omega_1\cup\Omega_2}, \tag{3.4}$$

for all $\boldsymbol{v}\in\boldsymbol{H}^2(\Omega_1\cup\Omega_2)\cap\boldsymbol{H}_0^1(\Omega)$ and $q\in H^1(\Omega_1\cup\Omega_2)\cap L_0^2(\Omega)$.

Proof. Let $\boldsymbol{v}_i=\boldsymbol{v}|_{\Omega_i}$ and $K_i=K\cap\Omega_i$ with $K\in\mathcal{T}_{h,i}$, $i=1,2$. By the standard interpolation estimate and the continuity of the extension operator Lemma 3.1, we have

$$\begin{aligned}\|\nabla(\boldsymbol{v}_i-\boldsymbol{I}_h^\star\boldsymbol{v}_i)\|^2_{0,\Omega_i} &= \sum_{K\in\mathcal{T}_{h,i}}\|\nabla(\boldsymbol{v}_i-\boldsymbol{I}^\star_{h,i}\boldsymbol{v}_i)\|^2_{0,K_i}\\ &\le \sum_{K\in\mathcal{T}_{h,i}}\|\nabla(\boldsymbol{E}_i\boldsymbol{v}_i-\boldsymbol{I}_h\boldsymbol{E}_i\boldsymbol{v}_i)\|^2_{0,K}\\ &\le Ch^2\|\boldsymbol{E}_i\boldsymbol{v}_i\|^2_{2,\Omega}\le Ch^2\|\boldsymbol{v}_i\|^2_{2,\Omega_i}.\end{aligned} \tag{3.5}$$

Next we consider the jump on the interface. The following interface trace inequality holds (see [7]):

$$\|\phi\|_{0,\Gamma_K}^2 \le C(h_K^{-1}\|\phi\|_{0,K}^2 + h_K\|\phi\|_{1,K}^2), \quad \forall \phi \in H^1(K). \tag{3.6}$$

Then, using again the standard interpolation estimate and the continuity of the extension operator Lemma 3.1, we have

$$\begin{aligned}
\|[\![\boldsymbol{v} - \boldsymbol{I}_h^\star \boldsymbol{v}]\!]\|_{1/2,h,\Gamma}^2 &= \sum_{K\in G_h} h_K^{-1}\|[\![\boldsymbol{v} - \boldsymbol{I}_h^\star \boldsymbol{v}]\!]\|_{0,\Gamma_K}^2 \\
&\le \sum_{i=1}^2 \sum_{K\in G_h} h_K^{-1}\|\boldsymbol{v}_i - \boldsymbol{I}_h^\star \boldsymbol{v}_i\|_{0,\Gamma_K}^2 \\
&\le \sum_{i=1}^2 \sum_{K\in G_h} C(h_K^{-2}\|\boldsymbol{E}_i\boldsymbol{v}_i - \boldsymbol{I}_h\boldsymbol{E}_i\boldsymbol{v}_i\|_{0,K}^2 + \|\boldsymbol{E}_i\boldsymbol{v}_i - \boldsymbol{I}_h\boldsymbol{E}_i\boldsymbol{v}_i\|_{1,K}^2) \\
&\le \sum_{i=1}^2 \sum_{K\in G_h} Ch_K^2\|\boldsymbol{E}_i\boldsymbol{v}_i\|_{2,K}^2 \\
&\le Ch^2 \sum_{i=1}^2 \|\boldsymbol{E}_i\boldsymbol{v}_i\|_{2,\Omega}^2 \\
&\le Ch^2 \sum_{i=1}^2 \|\boldsymbol{v}_i\|_{2,\Omega_i}^2.
\end{aligned} \tag{3.7}$$

Combining (3.5) and (3.7) yields (3.3). Inequality (3.4) can be shown similarly. This completes the proof. □

4. Stability analysis

Since the functions in $\boldsymbol{V}_h$ are piecewise linear, the following inverse inequality holds (see [7]).

Lemma 4.1. *For any $\boldsymbol{v}_h \in \boldsymbol{V}_h$, the following inverse inequality holds*

$$\|\{\nabla \boldsymbol{v}_h \cdot \boldsymbol{n}\}_w\|_{-1/2,h,\Gamma} \le C\|\nabla \boldsymbol{v}_h\|_{0,\Omega_1\cup\Omega_2}. \tag{4.1}$$

The following lemma shows that the bilinear form $a_h(\cdot,\cdot) + G_1(\cdot,\cdot)$ is continuous and coercive.

Lemma 4.2. *For all $\boldsymbol{u}_h, \boldsymbol{v}_h \in \boldsymbol{V}_h$, we have*

$$a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) \le C\|\boldsymbol{u}_h\|_\star\|\boldsymbol{v}_h\|_\star, \tag{4.2}$$

$$\|\boldsymbol{v}_h\|_\star^2 \le C(a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) + G_1(\boldsymbol{v}_h, \boldsymbol{v}_h)). \tag{4.3}$$

Proof. Continuity (4.2) follows from the Cauchy–Schwarz inequality and the inverse inequality (4.1). To prove coercivity, using (3.1) and the inverse inequality (4.1), for sufficiently large λ, we have

$$\begin{aligned}
a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) &= \|\nu^{1/2}\nabla \boldsymbol{v}_h\|_{0,\Omega_1\cup\Omega_2}^2 - 2([\![\boldsymbol{v}_h]\!], \{\nu\nabla \boldsymbol{v}_h \cdot \boldsymbol{n}\}_w)_\Gamma + \|\lambda^{1/2}[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma}^2 \\
&\ge \|\nu^{1/2}\nabla \boldsymbol{v}_h\|_{0,\Omega_1\cup\Omega_2}^2 - 2\|[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma}\|\{\nu\nabla \boldsymbol{v}_h \cdot \boldsymbol{n}\}_w\|_{-1/2,h,\Gamma} + \|\lambda^{1/2}[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma}^2 \\
&\ge C(\|\nabla \boldsymbol{v}_h\|_{0,\Omega_1\cup\Omega_2}^2 + \|[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma}^2).
\end{aligned} \tag{4.4}$$

For the ghost penalty, we have the following inequality (see Lemma 4.2 in [12]):

$$\|\nabla \boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \le C(\|\nabla \boldsymbol{v}_h\|_{0,\Omega_1\cup\Omega_2} + G_1(\boldsymbol{v}_h, \boldsymbol{v}_h)). \tag{4.5}$$

The coercivity (4.3) follows immediately from (4.4) and (4.5). □

Lemma 4.3. *Let $q_h = (q_{h,1}, q_{h,2}) \in Q_h$, then for $i = 1, 2$, there exists a constant C such that*

$$\|q_{h,i}\|_{0,\Omega_{h,i}}^2 \le C(\|q_{h,i}\|_{0,\Omega_{h,i}\setminus G_h}^2 + G_2(q_{h,i}, q_{h,i})). \tag{4.6}$$

Proof. For any $K \in G_h$, there exists an element $K' \in \mathcal{T}_{h,i}\setminus G_h$ such that K' shares an edge or a vertex with K. Since Γ is C^2-smooth, such an element K' always exists on sufficiently fine meshes. The extension to K of $q_{h,i}|_{K'}$ is still denoted $q_{h,i}|_{K'}$.

Let $\boldsymbol{x}_0 \in K \cap K'$, since $q_{h,i}$ is a continuous piecewise linear polynomial on $\mathcal{T}_{h,i}$, for any given $\boldsymbol{x} \in K$, expressing $q_{h,i}|_K$ and $q_{h,i}|_{K'}$ Taylor expansion around $\boldsymbol{x}_0$, respectively, we obtain

$$q_{h,i}|_K(\boldsymbol{x}) = q_{h,i}|_K(\boldsymbol{x}_0) + \nabla q_{h,i}|_K \cdot (\boldsymbol{x} - \boldsymbol{x}_0), \tag{4.7}$$

and

$$q_{h,i}|_{K'}(\boldsymbol{x}) = q_{h,i}|_{K'}(\boldsymbol{x}_0) + \nabla q_{h,i}|_{K'} \cdot (\boldsymbol{x} - \boldsymbol{x}_0). \tag{4.8}$$

Subtracting (4.8) from (4.7), we have

$$q_{h,i}|_K = q_{h,i}|_{K'} + [\![\nabla q_{h,i}]\!] \cdot (\boldsymbol{x} - \boldsymbol{x}_0). \tag{4.9}$$

Taking squares on both sides of (4.9), integrating over K and applying the Cauchy–Schwarz inequality give that

$$\|q_{h,i}\|_{0,K}^2 \le C(\|q_{h,i}|_{K'}\|_{0,K}^2 + h^2\|[\![\nabla q_{h,i}]\!]\|_{0,K}^2). \tag{4.10}$$

By the quasi-uniform property of triangulation $\mathcal{T}_{h,i}$, since $q_{h,i}|_{K'}$ is a linear polynomial on $K \cup K'$, we have $\|q_{h,i}|_{K'}\|_{0,K} \le C\|q_{h,i}|_{K'}\|_{0,K'}$, then summing over $K \in G_h$ yields

$$\|q_{h,i}\|_{0,G_h}^2 \le C\left(\|q_{h,i}\|_{0,\Omega_{h,i}\setminus G_h}^2 + \sum_{K\in G_h} h^2\|\nabla q_{h,i}\|_{0,K}^2\right), \tag{4.11}$$

hence

$$\|q_{h,i}\|_{0,\Omega_{h,i}}^2 \le C\left(\|q_{h,i}\|_{0,\Omega_{h,i}\setminus G_h}^2 + \sum_{K\in G_h} h^2\|\nabla q_{h,i}\|_{0,K}^2\right). \tag{4.12}$$

Since $\Pi_{K,i}q_{h,i}$ is a constant on K, using inverse inequality yields (4.6). The proof is completed. $\square$

Using Verfürth's trick [29], we can prove the following weak inf–sup condition. To this end, we first introduce an interpolation operator proposed by Becker et al. in [9] (also see [21]). For $i = 1, 2$, let $G_{h,i}$ be all the elements cut by the interface and the elements sharing an edge or a vertex with them in the triangulation $\mathcal{T}_{h,i}$. We regroup the elements in $G_{h,i}$ into patches $\{P_j^i\}$ in such a way that each patch can be associated with a patch function ϕ_j^i, and there exist constants c_1 and c_2 such that

- $c_1 h \le \operatorname{diam}(P_j^i) \le c_2 h$,
- $c_1 h \le \int_{\Gamma\cap P_j^i} \phi_j^i ds \le c_2 h$,
- $c_1 h^{-1} \le |\nabla \phi_j^i| \le c_2 h^{-1}$.

Next, for $i = 1, 2$, we can define Becker–Burman–Hansbo (BBH) interpolation operators $\pi_h^i : \boldsymbol{H}^1(\Omega_{h,i}) \to \boldsymbol{V}_{h,i}$, see [9] for details. From the definition, we can get

$$\int_{\Gamma\cap P_j^i} (\boldsymbol{v} - \pi_h^i \boldsymbol{v}) ds = 0, \tag{4.13}$$

for all P_j^i and

$$\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{0,\Omega} + h\|\boldsymbol{v} - \pi_h \boldsymbol{v}\|_{1,\Omega_{h,1}\cup\Omega_{h,2}} \le Ch\|\nabla \boldsymbol{v}\|_{0,\Omega} \tag{4.14}$$

where π_h is defined by $\pi_h|_{\Omega_{h,i}} = \pi_h^i$.

Lemma 4.4. *For any $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$, it holds*

$$\|\pi_h \boldsymbol{v}\|_\star \le C\|\nabla \boldsymbol{v}\|_{0,\Omega}. \tag{4.15}$$

Proof. By definition, we have $\|\pi_h \boldsymbol{v}\|_\star^2 = \|\nabla \pi_h \boldsymbol{v}\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}^2 + \|[\![\pi_h \boldsymbol{v}]\!]\|_{1/2,h,\Gamma}^2$. Employing (4.14) implies $\|\nabla \pi_h \boldsymbol{v}\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}^2 \le \|\nabla(\boldsymbol{v} - \pi_h \boldsymbol{v})\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}^2 + \|\nabla \boldsymbol{v}\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}^2 \le C\|\nabla \boldsymbol{v}\|_{0,\Omega}^2$. Since $[\![\boldsymbol{v}]\!] = 0$, we have $\|[\![\pi_h \boldsymbol{v}]\!]\|_{1/2,h,\Gamma}^2 = \|[\![\pi_h \boldsymbol{v} - \boldsymbol{v}]\!]\|_{1/2,h,\Gamma}^2$. Using the trace inequality (3.6) and (4.14) again yields $\|[\![\pi_h \boldsymbol{v}]\!]\|_{1/2,h,\Gamma}^2 \le C\|\nabla \boldsymbol{v}\|_{0,\Omega}^2$. This completes the proof. $\square$

Lemma 4.5. *For any $q_h \in Q_h$, we have*

$$\sup_{\boldsymbol{v}_h\in \boldsymbol{V}_h} \frac{b_h(\boldsymbol{v}_h, q_h)}{\|\boldsymbol{v}_h\|_\star} \ge C(\|q_h\|_{0,\Omega_1\cup\Omega_2} - G_2(q_h, q_h)^{1/2}). \tag{4.16}$$

Proof. Let $K_i = K \cap \Omega_i$ with $K \in \mathcal{T}_h$. For $q_h = (q_{h,1}, q_{h,2}) \in Q_h$, we know $q_h \in L^2(\Omega)$, then there exists a function $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ such that $\operatorname{div} \boldsymbol{v} = -q_h$ and $\|\boldsymbol{v}\|_{1,\Omega} \le C\|q_h\|_{0,\Omega_1\cup\Omega_2}$ (see [17]). Then by the definition of $b_h(\cdot, \cdot)$ and $[\![\boldsymbol{v}]\!] = 0$, we have $b_h(\boldsymbol{v}, q_h) = \|q_h\|_{0,\Omega_1\cup\Omega_2}^2$.

Note that

$$b_h(\boldsymbol{v}_h, q_h) = b_h(\boldsymbol{v}_h - \boldsymbol{v}, q_h) + b_h(\boldsymbol{v}, q_h) = b_h(\boldsymbol{v}_h - \boldsymbol{v}, q_h) + \|q_h\|^2_{0,\Omega_1\cup\Omega_2}. \tag{4.17}$$

We estimate the term $b_h(\boldsymbol{v}_h - \boldsymbol{v}, q_h)$ in the following. Taking $\boldsymbol{v}_h = \pi_h\boldsymbol{v}$, using integration by parts yields

$$b_h(\boldsymbol{v}_h - \boldsymbol{v}, q_h) = \sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}}\int_{K_i}(\pi_h\boldsymbol{v} - \boldsymbol{v})\nabla q_{h,i}dx - \int_\Gamma \{\pi_h\boldsymbol{v} - \boldsymbol{v}\}^w \cdot \boldsymbol{n}[\![q_h]\!]ds = I_1 + I_2. \tag{4.18}$$

By the interpolation estimate (4.14) and the standard inverse inequality (see [30]), noting that $\Pi_{K,i}q_{h,i}$ is a piecewise constant on $\mathcal{T}_{h,i}$, we have

$$\begin{aligned}
I_1 &\geq -\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}}\left|\int_{K_i}(\pi_h\boldsymbol{v} - \boldsymbol{v})\nabla q_{h,i}dx\right| \\
&\geq -\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}}\left|\int_{K_i}(\pi_h\boldsymbol{v} - \boldsymbol{v})\nabla (q_{h,i} - \Pi_{K,i}q_{h,i})dx\right| \\
&\geq -\sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}}\|\pi_h\boldsymbol{v} - \boldsymbol{v}\|_{0,K}\|\nabla(q_{h,i} - \Pi_{K,i}q_{h,i})\|_{0,K} \\
&\geq -Ch\|\nabla\boldsymbol{v}\|_{0,\Omega}h^{-1}G_2(q_h, q_h)^{1/2} \\
&= -C\|\nabla\boldsymbol{v}\|_{0,\Omega}G_2(q_h, q_h)^{1/2}.
\end{aligned} \tag{4.19}$$

For the second term we use (4.13) to obtain

$$I_2 = -\sum_{P_j^1}\int_{\Gamma\cap P_j^1} w_2(\pi_h\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{n}(q_{h,1} - \bar{q}_{h,1})ds + \sum_{P_j^2}\int_{\Gamma\cap P_j^2} w_1(\pi_h\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{n}(q_{h,2} - \bar{q}_{h,2})ds, \tag{4.20}$$

where $\bar{q}_{h,i}$ is the mean of $q_{h,i}$ on the patch P_j^i.

It suffices to estimate the first term in (4.20), since the second term can be estimated similarly.

$$\sum_{P_j^1}\int_{\Gamma\cap P_j^1} w_2(\pi_h\boldsymbol{v} - \boldsymbol{v})\cdot\boldsymbol{n}(q_{h,1} - \bar{q}_{h,1})ds \leq C\|\pi_h\boldsymbol{v} - \boldsymbol{v}\|_{1/2,h,\Gamma}\sum_{P_j^1}\|q_{h,1} - \bar{q}_{h,1}\|_{-1/2,h,\Gamma\cap P_j^1}. \tag{4.21}$$

Using once again the trace inequality (3.6), we have

$$\begin{aligned}
\|\pi_h\boldsymbol{v} - \boldsymbol{v}\|_{1/2,h,\Gamma} &\leq C\sum_{K\in G_h} h^{-1/2}\|\pi_h\boldsymbol{v} - \boldsymbol{v}\|_{0,\Gamma_K} \\
&\leq C\sum_{K\in G_h}(h^{-1}\|\pi_h\boldsymbol{v} - \boldsymbol{v}\|_{0,K} + \|\nabla(\pi_h\boldsymbol{v} - \boldsymbol{v})\|_{0,K}) \\
&\leq C\|\nabla\boldsymbol{v}\|_{0,\Omega},
\end{aligned} \tag{4.22}$$

and

$$\begin{aligned}
\sum_{P_j^1}\|q_{h,1} - \bar{q}_{h,1}\|_{-1/2,h,\Gamma\cap P_j^1} &\leq C\sum_{P_j} h^{1/2}\|q_{h,1} - \bar{q}_{h,1}\|_{0,\Gamma\cap P_j^1} \\
&\leq C\sum_{P_j^1}(\|q_{h,1} - \bar{q}_{h,1}\|_{0,P_j^1} + h\|\nabla(q_{h,1} - \bar{q}_{h,1})\|_{0,P_j^1}) \\
&\leq C\sum_{K\in\mathcal{T}_{h,1}} h\|\nabla q_{h,1}\|_{0,K} \\
&\leq C\sum_{K\in\mathcal{T}_{h,1}} h\|\nabla(q_{h,1} - \Pi_{K,1}q_{h,1})\|_{0,K} \\
&\leq CG_2(q_h, q_h)^{1/2}.
\end{aligned} \tag{4.23}$$

Thus

$$I_2 \geq -C\|\nabla\boldsymbol{v}\|_{0,\Omega}G_2(q_h, q_h)^{1/2}. \tag{4.24}$$

Since $\boldsymbol{v} \in \boldsymbol{H}^1(\Omega)$ and $\|\nabla \boldsymbol{v}\|_{0,\Omega} \leq C\|q_h\|_{0,\Omega_1\cup\Omega_2}$, combining (4.17), (4.19) and (4.24) and applying the stability property of the interpolation operator Lemma 4.4 yield (4.16). This completes the proof. □

Theorem 4.1. *For all $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$, the following inf–sup condition holds*

$$\sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times Q_h} \frac{A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h)}{|||(\boldsymbol{v}_h, q_h)|||_\star} \geq C|||(\boldsymbol{u}_h, p_h)|||_\star. \tag{4.25}$$

Proof. By definition

$$A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) + b_h(\boldsymbol{u}_h, q_h).$$

Setting $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h, -p_h)$, using the coercivity (4.3) of the bilinear form $a_h(\boldsymbol{v}_h, \boldsymbol{v}_h) + G_1(\boldsymbol{v}_h, \boldsymbol{v}_h)$ yields

$$\begin{aligned} A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h) &= a_h(\boldsymbol{u}_h, \boldsymbol{u}_h) + G_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + G_2(p_h, p_h) \\ &\geq C(\|\boldsymbol{u}_h\|_\star^2 + G_2(p_h, p_h)). \end{aligned} \tag{4.26}$$

Choosing $\boldsymbol{w} \in \boldsymbol{V}_h$ be a function for which the supremum of Lemma 4.5 is obtained, we scale $\boldsymbol{w}$ so that $\|\boldsymbol{w}\|_\star = \|p_h\|_{0,\Omega_1\cup\Omega_2}$. Letting $(\boldsymbol{v}_h, q_h) = (-\boldsymbol{w}, 0)$, using the weak inf–sup condition of Lemma 4.5, the continuity (4.2) of the bilinear form $a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h)$ and the Cauchy–Schwarz inequality, we get

$$\begin{aligned} A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h) &= -a_h(\boldsymbol{u}_h, \boldsymbol{w}) - b_h(\boldsymbol{w}, p_h) - G_1(\boldsymbol{u}_h, \boldsymbol{w}) \\ &\geq C\left(\|p_h\|_{0,\Omega_1\cup\Omega_2} - G_2(p_h, p_h)^{1/2} - \|\boldsymbol{u}_h\|_\star\right)\|p_h\|_{0,\Omega_1\cup\Omega_2} \\ &\geq C_1\|p_h\|_{0,\Omega_1\cup\Omega_2}^2 - C_2G_2(p_h, p_h) - C_3\|\boldsymbol{u}_h\|_\star^2, \end{aligned} \tag{4.27}$$

where $C_i,\ i = 1, 2, 3$ are positive constants.

Next, by Lemma 4.3, taking $(\boldsymbol{v}_h, q_h) = (\boldsymbol{u}_h - \delta\boldsymbol{w}, -p_h)$, choosing δ sufficiently small, combining (4.26) and (4.27) gives

$$\begin{aligned} A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h) &= (A_h(\boldsymbol{u}_h, p_h; \boldsymbol{u}_h, -p_h) + G_1(\boldsymbol{u}_h, \boldsymbol{u}_h) + G_2(p_h, p_h)) \\ &\quad + \delta(A_h(\boldsymbol{u}_h, p_h; -\boldsymbol{w}, 0) - G_1(\boldsymbol{u}_h, \boldsymbol{w})) \\ &\geq C(\|\boldsymbol{u}_h\|_\star^2 + \|p_h\|_{0,\Omega_1\cup\Omega_2}^2 + G_2(p_h, p_h)) \\ &\geq C(\|\boldsymbol{u}_h\|_\star^2 + \|p_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}^2) \\ &= C|||(\boldsymbol{u}_h, p_h)|||_\star^2. \end{aligned} \tag{4.28}$$

On the other hand we have

$$|||(\boldsymbol{v}_h, q_h)|||_\star \leq |||(\boldsymbol{u}_h, p_h)|||_\star + \delta\|\boldsymbol{w}\|_\star \leq C|||(\boldsymbol{u}_h, p_h)|||_\star,$$

which combining with (4.28) proves the stability estimate (4.25). □

5. A priori error estimates

Now we show the following a priori estimates for the error in the discrete solution.

Theorem 5.1. *Let $(\boldsymbol{u}, p)$ be the solution of the Stokes interface problems (2.3), and let $(\boldsymbol{u}_h, p_h) \in \boldsymbol{V}_h \times Q_h$ solve the stabilized Nitsche finite element problems (2.4). Furthermore assume $(\boldsymbol{u}, p) \in \boldsymbol{H}^2(\Omega_1 \cup \Omega_2) \times H^1(\Omega_1 \cup \Omega_2)$, then*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)|||_h \leq Ch(\|\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|p\|_{1,\Omega_1\cup\Omega_2}). \tag{5.1}$$

Proof. Note that

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)|||_h \leq |||(\boldsymbol{u} - \boldsymbol{I}_h^\star\boldsymbol{u}, p - \rho_h^\star p)|||_h + |||(\boldsymbol{u}_h - \boldsymbol{I}_h^\star\boldsymbol{u}, p_h - \rho_h^\star p)|||_\star. \tag{5.2}$$

With the help of Theorem 3.1, we only need to consider the term $|||(\boldsymbol{u}_h - \boldsymbol{I}_h^\star\boldsymbol{u}, p_h - \rho_h^\star p)|||_\star$. By applying the inf–sup condition Theorem 4.1 and the weak consistency relation Lemma 2.2, we have

$$\begin{aligned} &|||(\boldsymbol{u}_h - \boldsymbol{I}_h^\star\boldsymbol{u}, p_h - \rho_h^\star p)|||_\star \\ &\quad\leq C \sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times Q_h} \frac{A_h(\boldsymbol{u}_h - \boldsymbol{I}_h^\star\boldsymbol{u}, p_h - \rho_h^\star p; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h - \boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) - G_2(p_h - \rho_h^\star p, q_h)}{|||(\boldsymbol{v}_h, q_h)|||_\star} \\ &\quad= C \sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times Q_h} \frac{A_h(\boldsymbol{u} - \boldsymbol{I}_h^\star\boldsymbol{u}, p - \rho_h^\star p; \boldsymbol{v}_h, q_h) - G_1(\boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) + G_2(\rho_h^\star p, q_h)}{|||(\boldsymbol{v}_h, q_h)|||_\star}. \end{aligned} \tag{5.3}$$

Using the Cauchy–Schwarz inequality, applying Theorem 3.1 and the trace inequality (3.6), we can show that

$$\begin{aligned} A_h(\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u}, p-\rho_h^\star p; \boldsymbol{v}_h, q_h) &= a_h(\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p-\rho_h^\star p) + b_h(\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u}, q_h) \\ &\le C\big(\|\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u}\|_h + \|\{\nu\nabla(\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u})\cdot\boldsymbol{n}\}_w\|_{-1/2,h,\Gamma} \\ &\quad + \|p-\rho_h^\star p\|_{0,\Omega_1\cup\Omega_2} + \|\{p-\rho_h^\star p\}_w\|_{-1/2,h,\Gamma}\big)|||(\boldsymbol{v}_h, q_h)|||_\star \\ &\le Ch(\|\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|p\|_{1,\Omega_1\cup\Omega_2})|||(\boldsymbol{v}_h, q_h)|||_\star. \end{aligned} \tag{5.4}$$

For the weak consistent term we have

$$-G_1(\boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) + G_2(\rho_h^\star p, q_h) = G_1(\boldsymbol{E}\boldsymbol{u}-\boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) - G_1(\boldsymbol{E}\boldsymbol{u}, \boldsymbol{v}_h) + G_2(\rho_h^\star p - Ep, q_h) + G_2(Ep, q_h).$$

According to the definitions of $G_1(\cdot,\cdot)$ and $G_2(\cdot,\cdot)$, applying the interpolation estimate, the inverse inequality and Lemma 3.1 yield

$$\begin{aligned} -G_1(\boldsymbol{I}_h^\star\boldsymbol{u}, \boldsymbol{v}_h) + G_2(\rho_h^\star p, q_h) &\le \sum_{i=1}^{2}\sum_{l=1}^{\mathcal{N}_{l,i}} h_{\mathcal{P}_{l,i}}^{-2}\big(\|(\boldsymbol{E}_i\boldsymbol{u}_i - \boldsymbol{I}_{h,i}^\star\boldsymbol{u}_i) - \boldsymbol{J}_{l,i}(\boldsymbol{E}_i\boldsymbol{u}_i - \boldsymbol{I}_{h,i}^\star\boldsymbol{u}_i)\|_{0,\mathcal{P}_{l,i}} + \|\boldsymbol{E}_i\boldsymbol{u}_i - \boldsymbol{J}_{l,i}\boldsymbol{E}_i\boldsymbol{u}_i\|_{0,\mathcal{P}_{l,i}}\big) \\ &\quad \times \|\boldsymbol{v}_{h,i} - \boldsymbol{J}_{l,i}\boldsymbol{v}_{h,i}\|_{0,\mathcal{P}_{l,i}} + \sum_{i=1}^{2}\sum_{K\in\mathcal{T}_{h,i}}\big(\|(E_ip_i - \rho_{h,i}^\star p_i) - \Pi_{K,i}(E_ip_i - \rho_{h,i}^\star p_i)\|_{0,K} \\ &\quad + \|E_ip_i - \Pi_{K,i}E_ip_i\|_{0,K}\big)\|q_{h,i} - \Pi_{K,i}q_{h,i}\|_{0,K} \\ &\le C\sum_{i=1}^{2}(h\|\boldsymbol{E}_i\boldsymbol{u}_i\|_{2,\Omega}\|\nabla\boldsymbol{v}_{h,i}\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + h^2\|E_ip_i\|_{1,\Omega}\|\nabla q_{h,i}\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}) \\ &\le Ch(\|\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|p\|_{1,\Omega_1\cup\Omega_2})|||(\boldsymbol{v}_h, q_h)|||_\star. \end{aligned} \tag{5.5}$$

The error estimate (5.1) follows from combining (5.3)–(5.5) and using the triangle inequality (5.2). The proof is completed. □

By the Aubin–Nitsche duality argument as standard finite element theory (see [30]), an L^2 estimate for the velocity can be proven. We only state it as follows.

Theorem 5.2. *Under the same conditions in Theorem 5.1, the following L^2 error estimate for the velocity holds*

$$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega} \le Ch^2(\|\boldsymbol{u}\|_{2,\Omega_1\cup\Omega_2} + \|p\|_{1,\Omega_1\cup\Omega_2}).$$

6. Condition member of the system matrix

In this section we will show that the condition number of the stiffness matrix is $\mathcal{O}(h^{-2})$ independent of the location of the interface relative to the mesh. The analysis follows the approach of Ern and Guermond [31]. Let $\{\phi_i\}_{i=1}^N$ be a basis in $\boldsymbol{V}_h \times Q_h$ and $\mathcal{A}$ be the stiffness matrix associated with the bilinear form $A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h)$. Since $\mathcal{T}_{h,i}$ is quasi-uniform, we have the following estimate

$$C_1h|W|_N \le (\|\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}) \le C_2h|W|_N, \quad \forall\, \boldsymbol{w}_h = (\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h, \tag{6.1}$$

where $W = \{W_i\}_{i=1}^N$ denotes the expansion coefficients of $(\boldsymbol{v}_h, q_h)$ in the basis $\{\phi_i\}_{i=1}^N$ and $|W|_N$ denotes the Euclidean norm.

The condition number of the matrix $\mathcal{A}$ is defined by

$$\kappa(\mathcal{A}) := \|\mathcal{A}\|\,\|\mathcal{A}^{-1}\|, \tag{6.2}$$

where

$$\|\mathcal{A}\| = \sup_{U\in\mathbb{R}^N}\frac{|\mathcal{A}U|_N}{|U|_N}. \tag{6.3}$$

The condition number estimate follows from the following two lemmas.

Lemma 6.1. *For any $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$, the following estimate holds*

$$\|\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \le C|||(\boldsymbol{v}_h, q_h)|||_\star. \tag{6.4}$$

Proof. We claim

$$\|\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \le C(\|\nabla\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma}). \tag{6.5}$$

In fact, assume that the right hand side of (6.5) is zero. It then follows that

$$\|\nabla \boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} = 0.$$

This implies $\boldsymbol{v}_h$ is a piecewise constant in $\mathcal{T}_{h,1}$ and $\mathcal{T}_{h,2}$. Next, using that $\|[\![\boldsymbol{v}_h]\!]\|_{1/2,h,\Gamma} = 0$ and $\boldsymbol{v}_h = 0$ on $\partial\Omega$, it follows that $\boldsymbol{v}_h \equiv 0$. Thus, if the right-hand side of (6.5) is zero, then the left-hand side is also zero. Finally, finite dimensionality, together with scaling, deduces that (6.5) holds. Recalling the definition of $|\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_\star$, the lemma follows. □

Applying the standard inverse inequality and the trace inequality on each $\mathcal{T}_{h,i}$, we obtain the following inverse estimate.

Lemma 6.2. *For all $(\boldsymbol{v}_h, q_h) \in \boldsymbol{V}_h \times Q_h$, there exists a constant C such that*

$$|\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_\star \le Ch^{-1}(\|\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}). \tag{6.6}$$

Now, we show the condition number.

Theorem 6.1. *The condition number of the stiffness matrix satisfies*

$$\kappa(\mathcal{A}) \le Ch^{-2}. \tag{6.7}$$

Proof. Let $U = \{U_i\}_{i=1}^N$ and $V = \{V_i\}_{i=1}^N$ denote the expansion coefficients of $(\boldsymbol{u}_h, p_h)$ and $(\boldsymbol{v}_h, q_h)$ in the basis $\{\phi_i\}_{i=1}^N$, respectively. By definition,

$$|\mathcal{A}U|_N = \sup_{V\in\mathbb{R}^N} \frac{(\mathcal{A}U, V)_N}{|V|_N} = \sup_{V\in\mathbb{R}^N} \frac{A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h)}{|V|_N}. \tag{6.8}$$

Using Lemma 4.2, the Cauchy–Schwarz inequality, the inverse estimate (6.6) and (6.1), we can obtain

$$\begin{aligned} &A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h) \\ &\quad = a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) + b(\boldsymbol{v}_h, p_h) + b(\boldsymbol{u}_h, q_h) - G_2(p_h, q_h) \\ &\quad \le C(\|\boldsymbol{u}_h\|_\star \|\boldsymbol{v}_h\|_\star + \|\nabla \boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \|p_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|\nabla \boldsymbol{u}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \\ &\qquad + \|p_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}) \\ &\quad \le C|\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_\star |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_\star \\ &\quad \le Ch^{-2}(\|\boldsymbol{u}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|p_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}})(\|\boldsymbol{v}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|q_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}) \\ &\quad \le C|U|_N |V|_N. \end{aligned} \tag{6.9}$$

Then, we have

$$\|\mathcal{A}\| = \sup_{U\in\mathbb{R}^N} \frac{|\mathcal{A}U|_N}{|U|_N} \le C. \tag{6.10}$$

Next, by (6.1) and (6.4) and the inf–sup stability Theorem 4.1, we can deduce

$$\begin{aligned} h|U|_N &\le C(\|\boldsymbol{u}_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}} + \|p_h\|_{0,\Omega_{h,1}\cup\Omega_{h,2}}) \le C|\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_\star \\ &\le C \sup_{(\boldsymbol{v}_h,q_h)\in\boldsymbol{V}_h\times Q_h} \frac{A_h(\boldsymbol{u}_h, p_h; \boldsymbol{v}_h, q_h) + G_1(\boldsymbol{u}_h, \boldsymbol{v}_h) - G_2(p_h, q_h)}{|\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_\star}. \end{aligned} \tag{6.11}$$

By (6.9) and $\|(\boldsymbol{v}_h, q_h)\|_\star \ge Ch|V|_N$, we get

$$h|U|_N \le Ch^{-1}|\mathcal{A}U|_N.$$

Note that U is arbitrary, it follows that

$$\|\mathcal{A}^{-1}\| \le Ch^{-2}. \tag{6.12}$$

Substituting (6.10) and (6.12) into the definition of $\kappa(\mathcal{A})$ yields (6.7). It completes the proof. □

7. Numerical examples

This section is devoted to some numerical experiments to illustrate the accuracy and fidelity of the proposed method in two space dimensions. Firstly, we need to point out that in the following numerical examples, we have not integrated on the exact interface, but on a its approximation by linear interpolation on the current computational mesh and set the penalty parameter $\lambda = 10$. Next we introduce the method constructing the patch $\mathcal{P}_{l,i}$ in the ghost penalty $G_1(\boldsymbol{u}_h, \boldsymbol{v}_h)$. We denote $\mathcal{F}_i$ the set of the edges of the triangles in G_h, that have a nonempty intersection with Ω_i. For each $F \in \mathcal{F}_i$, there exists a pair of elements K and K' in $\mathcal{T}_{h,i}$ such that $F = K \cap K'$, then we set $\mathcal{P}_{l,i} = K \cup K'$. We denote by e^h_{uL2} and e^h_{uH1} the error for the velocity in the L^2 norm and H^1 norm, respectively, and e^h_{pL2} the error for the pressure in the L^2 norm.

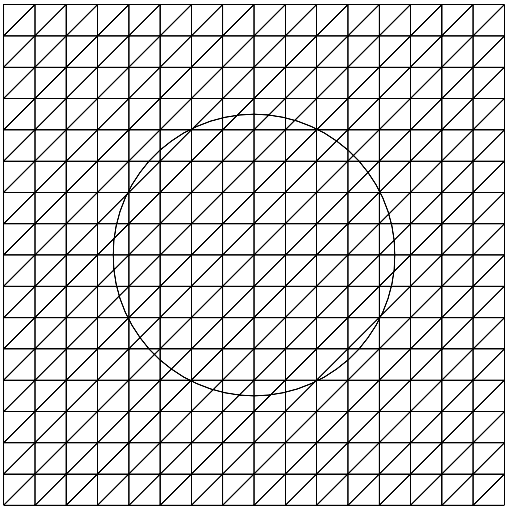

Fig. 2. Position of the interface.

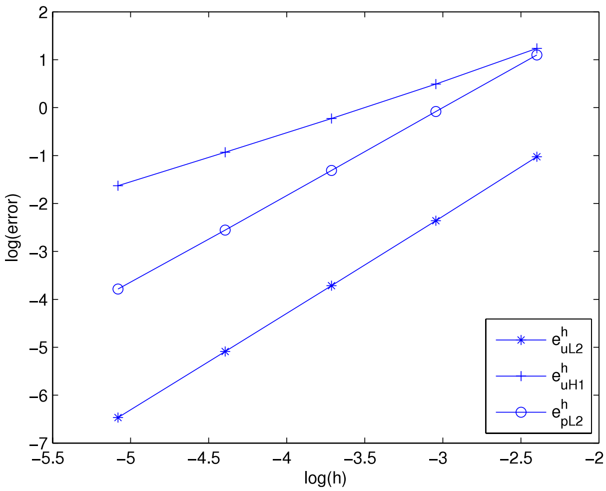

Fig. 3. Convergence rates of the error for the velocity and the pressure for example 7.1.

7.1. A noninterface problem: colliding flow

We consider a continuous problem with an artificial interface as proposed in [9]. Let $\Omega = [-1, 1] \times [-1, 1]$, the interface is a circle centered in $(0, 0)$ with the radius 0.53 and $\nu_1 = \nu_2 = 1$. The position of the artificial interface is shown in Fig. 2. We adapt the right hand side $\boldsymbol{f}$ and the Dirichlet boundary condition such that $(\boldsymbol{u}, p)$ is solution of the Stokes equations:

$$\boldsymbol{u} = (20xy^3, 5x^4 - 5y^4), \qquad p = 60x^2y - 20y^3.$$

The convergence for the velocity in the L^2 norm and H^1 norm and the pressure in the L^2 norm is given in Fig. 3. From Fig. 3, we see that the expected optimal convergence order for the velocity in the L^2 norm and the H^1 norm is obtained. For the pressure, we observe a higher order convergence than the expected one in the L^2 norm. These results confirm our theoretical results.

7.2. An interface problem

Next we consider a simple example where there exists a jump in the pressure coming from [16]. Let $\Omega = [0, 4] \times [-0.4, 0.6]$. The interface is the straight line $y = 0$. We choose the boundary data and the external volume forces $\boldsymbol{f}$ and $\boldsymbol{g}$ such that the exact solution of the Stokes interface problems is given by

$$\boldsymbol{u} = \left(\frac{x^2y}{\nu}, \frac{-xy^2}{\nu} \right),$$
$$p = 2xy + x^2 + 10\mathcal{X}(y)$$

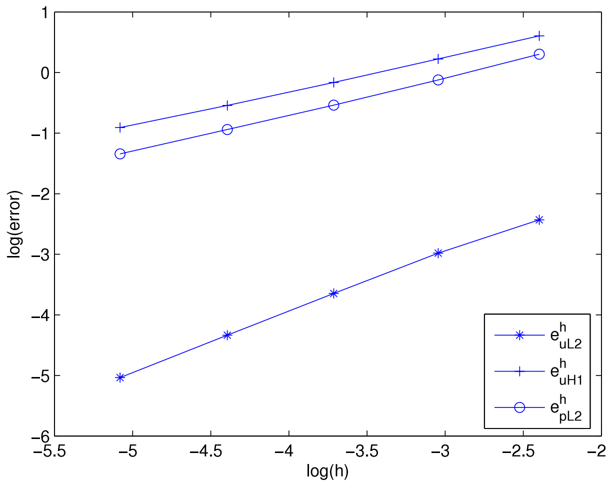

Fig. 4. Convergence rates of the error for the velocity and the pressure for example 7.2 using the standard stabilized method in [18].

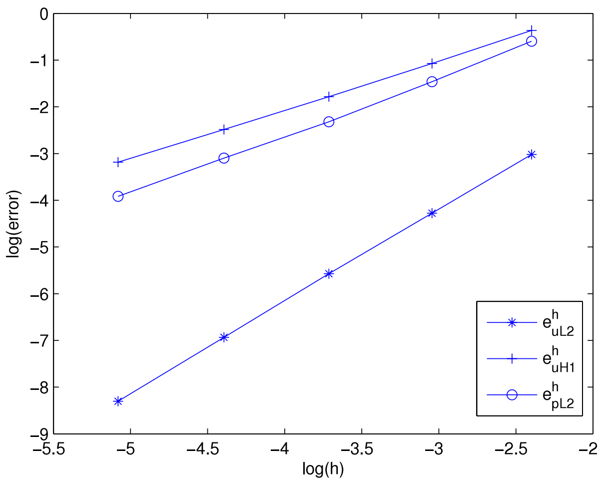

Fig. 5. Convergence rates of the error for the velocity and the pressure for example 7.2 using the proposed method.

where the viscosity

$$\nu = \begin{cases} 100 & y > 0, \\ 1 & y < 0, \end{cases} \quad \text{and} \quad \mathcal{X}(y) = \begin{cases} 0 & y > 0, \\ 1 & y < 0. \end{cases}$$

We perform numerical tests for the problem with two approaches: a standard stabilized finite element method proposed by Bochev, Dohrmann and Gunzburger (BDG method) in [18] and the presented method in this paper on the unfitted grid. The convergence results are given in Figs. 4 and 5. In Fig. 4 we observe that the convergence behavior of each variable for the standard BDG method on the unfitted grid is suboptimal, that is, the error for the pressure in the L^2 norm and the velocity in the H^1 norm is $\mathcal{O}(h^{1/2})$ and the velocity in the L^2 norm $\mathcal{O}(h)$. However, we see the optimal convergence in Fig. 5 using our proposed method, first order for the pressure in the L^2 norm and the velocity in the H^1 norm and second order for the velocity in the L^2 norm.

7.3. *Condition number test*

This numerical example demonstrates the condition number using the proposed method is independent of the location of the interface. We consider the example described in Section 7.1 with the radius r of the circle ranging from $0.5 + 10^{-3}$ to $0.5 + 10^{-12}$. Choosing the mesh size be $\frac{1}{64}$, we compute the condition numbers of the corresponding stiffness matrix which are given in Table 1. We can see that the condition numbers using the proposed method are bounded as expected. In contrast, the condition numbers grow significantly as $r \to 0.5$ when the ghost penalty is not included. In Fig. 6 we plot the condition numbers with varying mesh sizes when $r = 0.5 + 10^{-12}$. We can see that the condition number using the proposed method grows as $\mathcal{O}(h^{-2})$. This confirms the theoretical result in Section 6.

Table 1
Condition numbers with varying radius.

Case	$r = 0.5 + 1e - 3$	0.5+1e−6	0.5+1e−9	0.5+1e−12
With ghost penalty	5.8501e+05	5.6374e+05	5.6373e+05	5.6374e+05
Without ghost penalty	1.4481e+06	5.9915e+09	1.2682e+13	4.0485e+014

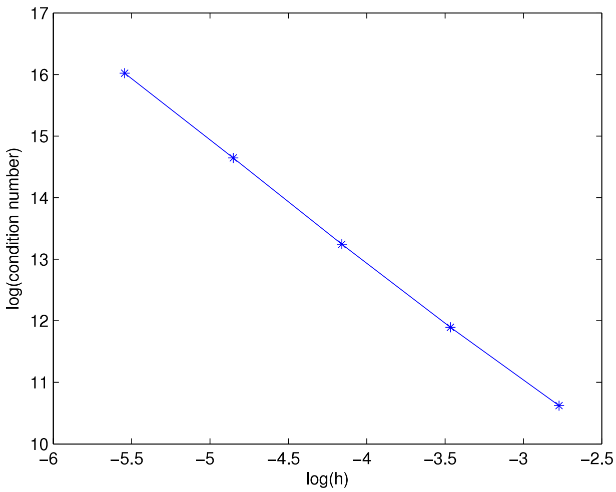

Fig. 6. Condition number for the example 7.1 with the radius $0.5 + 10^{-12}$.

8. Conclusions

We have proposed a new unfitted stabilized finite element method to solve the Stokes interface problems. The interface conditions are enforced weakly using Nitsche's method. We use the lowest equal order finite element spaces to approximate the velocity and the pressure. The optimal order error estimates are obtained. The condition number of the stiffness matrix is independent of the location of the interface. The theoretical predictions are confirmed by a series of numerical tests. In this paper, we only consider C^0 finite element method to solve the Stokes interface problems. In the future, we will address discontinuous Galerkin method to solve such problems. In addition, extending the proposed method to other models (such as two-phase Navier–Stokes flows and fluid–structure interaction problems) will also be one of our future subjects.

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