The Convergence of the Bilinear and Linear Immersed Finite Element Solutions to Interface Problems

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This article analyzes the error in both the bilinear and linear immersed finite element (IFE) solutions for second-order elliptic boundary problems with discontinuous coefficients. The discontinuity in the coefficients is supposed to happen across general curves, but the mesh of the IFE methods can be allowed not to align with the curve of discontinuity. It has been shown that the bilinear and linear IFE solutions converge to the exact solution under the usual assumptions about the meshes and regularity. Numer Methods Partial Differential Eq 28: 312–330, 2012

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I. INTRODUCTION

In this article we analyze the error in bilinear and linear immersed finite element (IFE) solutions for the following interface problem:

$$-\nabla \cdot (\beta \nabla u) = f, \quad (x, y) \in \Omega, \tag{1.1}$$

$$u|_{\partial\Omega} = 0, \tag{1.2}$$

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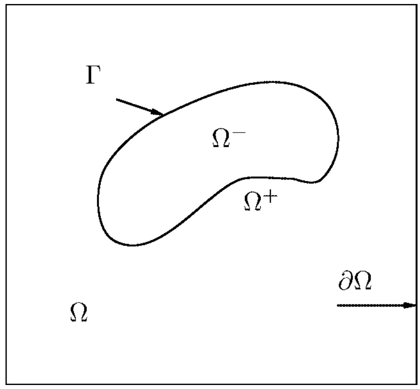

FIG. 1. The domain Ω of the interface problem is separated by the interface Γ across which the coefficient has a jump discontinuity.

together with the jump conditions on the interface Γ:

$$[u]|_\Gamma = 0, \tag{1.3}$$

$$\left[\beta \frac{\partial u}{\partial n}\right]\bigg|_\Gamma = 0. \tag{1.4}$$

Without loss of generality, we consider the case in which $\Omega \subset I\!R^2$ is a rectangular domain, the interface Γ is a curve separating Ω into two sub-domains Ω^-, Ω^+ such that $\bar{\Omega} = \overline{\Omega^-} \cup \overline{\Omega^+} \cup \Gamma$, and the coefficient $\beta(x, y)$ is a positive piecewise constant function defined by

$$\beta(x, y) = \begin{cases} \beta^-, & (x, y) \in \Omega^-, \\ \beta^+, & (x, y) \in \Omega^+, \end{cases}$$

see Fig. 1 for an illustration. The involved bilinear and linear IFE spaces were introduced in [1,2] and the approximation capability of the two IFE spaces were discussed in [3,4].

In contrast to the conventional finite element methods which usually require the mesh to be aligned with the interface, see for example [5–7] and reference therein, the basic idea of IFE methods is to use a finite element space whose mesh can be independent of the interface, such as a Cartisian mesh. This feature becomes clearly advantageous when the interfaces are moving in a simulation or a physical model requires structured meshes for an interface problem [8–11]. Since 1998, several types of IFE spaces have been developed, such as 1D linear IFE [12], 2D linear IFE [1,4,13], 3D linear IFE [14], bilinear IFE [2], quadratic IFEs [15], and 1D IFEs of arbitrary order [16]. Error analysis in [3,4,16,17] show that these IFE spaces have the approximation capability similar to that of standard finite element spaces using polynomials of same degree. Based on these results, the IFE spaces have been employed in Galerkin methods, finite volume methods, and discontinuous Galerkin methods [1, 2, 12, 14–16, 18–21] to solve interface problems. The usage of IFE methods with structured meshes for solving some engineering problems can be seen in [8–10, 22].

From the numerical experiments published up to now, we have observed that all of the IFE methods can generate approximate solutions to interface problems with the same optimal convergence rates as the corresponding standard finite element spaces using polynomials with the same degrees. For example, the IFE Galerkin methods with the bilinear or linear IFE spaces have the $O(h^2)$ convergence rate for L^2 norm and $O(h)$ convergence rate for H^1 norm [1–4]. However, to our knowledge, the convergence analysis for IFE methods has been carried out only for 1D problems [16, 17] where the IFE methods are conforming methods. The analysis for 2D and 3D IFE methods is more complicated because of the discontinuity in the functions of the involved IFE spaces. As the 2D and 3D interface problems are much more important from the point of view of applications, the theoretical analysis on the convergence of the 2D and 3D IFE methods demands immediate attentions.

In this article, we will analyze the convergence of the bilinear and linear Galerkin IFE solutions for the interface problem of the popular second-order elliptic equation with a piecewise constant coefficient. We would like to point out that the locations of the interface in different interface elements are usually not the same. This essentially leads to different IFE spaces on the reference element. Therefore, the constants C in the error bounds obtained on interface elements by the standard scaling argument depends on the location of the interface. It is not clear how these constants C behave when the mesh parameter h tends to zero; hence the error bounds obtained this way is invalid for us to draw the convergence conclusion. Therefore, the core effort of our analysis is to derive error bounds in which the constants C are independent of the the interfaces and the mesh parameter h.

The rest of this article is organized as follows. In Section II, we introduce some preliminaries and notations. In Section III, we recall the definitions of the bilinear and 2D linear IFE spaces used in the finite element methods to be analyzed. In Section IV, we will present some properties of the two IFE spaces. In Section V, we derive the bound in the H^1 norm errors of the bilinear and linear IFE methods which leads to the convergence of these IFE methods.

II. SOME PRELIMINARIES AND NOTATIONS

For any subset Λ of Ω, we let

$$\Lambda^s = \Lambda \cap \Omega^s, \quad s = -, +$$

and let

$$\begin{aligned}
\mathrm{PW}_p^m(\Lambda) &= \left\{u \mid u|_{\Lambda^s} \in W_p^m(\Lambda^s), s = -, +\right\}, p \geq 1, m = 0, 1, 2,\\
\mathrm{PH}_{\mathrm{int}}^2(\Lambda) &= \left\{u \in C(\Lambda) \mid u|_{\Lambda^s} \in H^2(\Lambda^s), s = -, +, \left[\beta \frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_{\Gamma \cap \Lambda} = 0\right\},\\
\mathrm{PC}_{\mathrm{int}}^m(\Lambda) &= \left\{u \in C(\bar{\Lambda}) \mid u|_{\overline{\Lambda^s}} \in C^m(\overline{\Lambda^s}), s = -, +, \left[\beta \frac{\partial u}{\partial \mathbf{n}_\Gamma}\right]_{\Gamma \cap \Lambda} = 0\right\},
\end{aligned}$$

where $W_p^m(\Lambda)$ is the standard Soblev space defined on a set Λ equipped with the norm $\|\cdot\|_{m,p,\Lambda}$ and seminorm $|\cdot|_{m,p,\Lambda}$. As usual, we let $\mathrm{PH}^m(\Lambda) = \mathrm{PW}_2^m(\Lambda)$. Obviously, we have $\mathrm{PC}_{\mathrm{int}}^2(\Lambda) \subset \mathrm{PH}_{\mathrm{int}}^2(\Lambda)$. Also, for any function $u \in \mathrm{PW}_p^m(\Lambda)$, we let

$$\|u\|_{m,p,\Lambda}^2 = \|u\|_{m,p,\Lambda^-}^2 + \|u\|_{m,p,\Lambda^+}^2, \tag{2.1}$$

and the seminorm of $\mathrm{PW}_p^m(\Lambda)$ can be defined accordingly by

$$|u|_{m,p,\Lambda}^2 = |u|_{m,p,\Lambda^-}^2 + |u|_{m,p,\Lambda^+}^2. \tag{2.2}$$

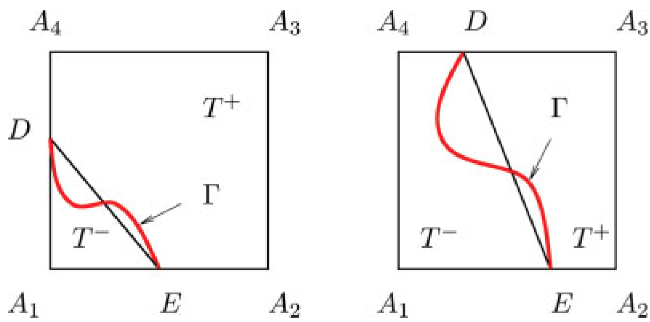

FIG. 2. Two typical rectangular interface elements. The element on the left is of Type I whereas the one on the right is of Type II. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

When $p = 2$, we will drop p from the notation of the norms, for example, we will use $\|u\|_{m,\Lambda} = \|u\|_{m,2,\Lambda}$.

We will use $\mathcal{T}_h$ to denote the collection of all elements in a mesh with parameter h. We note that when h is small enough, most of elements in $\mathcal{T}_h$ are noninterface elements not intersecting with the interface Γ. Only those elements in the vicinity of Γ have the possibility to be cut through by Γ and become the so-called interface elements. We will use $\mathcal{T}_{\text{int}}$ to denote the collection of all interface elements of $\mathcal{T}_h$ and let $\Omega_{\text{int}} = \cup_{T \in \mathcal{T}_{\text{int}}} T$.

In all the discussions from now on, we assume that the usual hypothesis (H_1)–(H_5) used in [3] hold as follows:

- (H_1): An interface Γ will not intersect an edge of any element at more than two points unless this edge is part of Γ.
- (H_2): If Γ intersects the boundary of an element at two points, then these two points must be on different edges of this element.
- (H_3): The family of meshes $\mathcal{T}_h$ with $h > 0$ is regular. (See Definition 3.4.1 of [23]).
- (H_4): The interface curve Γ is defined by a piecewise C^2 function, and the mesh $\mathcal{T}_h$ is formed such that the subset of Γ in any interface element is C^2.
- (H_5): The interface Γ is smooth enough so that $\text{PC}^3_{\text{int}}(T)$ is dense in $\text{PH}^2_{\text{int}}(T)$ for any interface element $T \in \mathcal{T}_h$.

III. BILINEAR AND 2D LINEAR IMMERSED FINITE ELEMENT SPACES

In this section, we recall the definitions of the bilinear and 2D linear IFE spaces discussed in [1–4]. First, we recall the definition of the bilinear IFE space discussed in [2, 3]. On each of the noninterface element T, we let the local finite element space $S_h(T)$ be $S_h^{\text{non}}(T)$ spanned by the four standard bilinear nodal basis functions $\psi_i(x, y), i = 1, 2, 3, 4$ on T. To describe the local IFE space on an interface element $T \in \mathcal{T}_{\text{int}}$, we assume that the vertices of T are $A_i, i = 1, 2, 3, 4$, with $A_i = (x_i, y_i)^t$. Without loss of generality, we assume that ∂T intersects with Γ at two points $D = (x_D, y_D)^T$ and $E = (x_E, y_E)^T$. There are two types of rectangle interface elements. Type I are those for which the interface intersects with two of its adjacent edges; Type II are those for which the interface intersects with two of its opposite edges, see the sketch in Fig. 2.

As the line $\overline{DE}$ separates T into two subsets T^- and T^+, we define a bilinear immersed function as follows:

$$\phi(x,y) = \begin{cases} \phi^-(x,y) = a^-x + b^-y + c^- + d^-xy, & (x,y) \in T^-, \\ \phi^+(x,y) = a^+x + b^+y + c^+ + d^+xy, & (x,y) \in T^+, \\ \phi^-(D) = \phi^+(D), \ \ \phi^-(E) = \phi^+(E), \\ \phi^-\left(\dfrac{D+E}{2}\right) = \phi^+\left(\dfrac{D+E}{2}\right), \\ \int_{\overline{DE}} \left(\beta^- \dfrac{\partial \phi^-}{\partial \mathbf{n}_{\overline{DE}}} - \beta^+ \dfrac{\partial \phi^+}{\partial \mathbf{n}_{\overline{DE}}}\right) ds = 0, \end{cases} \tag{3.1}$$

where $\mathbf{n}_{\overline{DE}}$ is the unit vector perpendicular to the line $\overline{DE}$.

However, when $\overline{DE}$ is vertical or horizontal, $\phi^-(D) = \phi^+(D)$ and $\phi^-(E) = \phi^+(E)$ automatically imply $\phi^-(\frac{D+E}{2}) = \phi^+(\frac{D+E}{2})$. To handle this special situation, we need to propose one more condition to uniquely determine the piecewise bilinear polynomial. In fact, by direct verification, we can show the following. For general cases in which $\overline{DE}$ is neither vertical nor horizontal, $\phi^-(\frac{D+E}{2}) = \phi^+(\frac{D+E}{2})$ and $d^- = d^+$ are equivalent to each other based on $\phi^-(D) = \phi^+(D)$ and $\phi^-(E) = \phi^+(E)$. Also, $d^- = d^+$ is one of the key tools for the interpolation error estimate, see [3]. Therefore, we use $d^- = d^+$ to replace $\phi^-(\frac{D+E}{2}) = \phi^+(\frac{D+E}{2})$ for the definition of the bilinear IFE function:

$$\phi(x,y) = \begin{cases} \phi^-(x,y) = a^-x + b^-y + c^- + d^-xy, & (x,y) \in T^-, \\ \phi^+(x,y) = a^+x + b^+y + c^+ + d^+xy, & (x,y) \in T^+, \\ \phi^-(D) = \phi^+(D), \ \ \phi^-(E) = \phi^+(E), d^- = d^+, \\ \int_{\overline{DE}} \left(\beta^- \dfrac{\partial \phi^-}{\partial \mathbf{n}_{\overline{DE}}} - \beta^+ \dfrac{\partial \phi^+}{\partial \mathbf{n}_{\overline{DE}}}\right) ds = 0. \end{cases} . \tag{3.2}$$

Now, we let $\phi_i(X)$ be the bilinear IFE function described by (3.2) such that

$$\phi_i(x_j, y_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j \end{cases}$$

for $1 \le i, j \le 4$, and we call them the bilinear IFE nodal basis functions on an interface element T. We then let $S_h^{\text{int}}(T) = \text{span}\{\phi_i, i = 1, 2, 3, 4\}$.

In summery, for each element $T \in \mathcal{T}_h$, we let

$$S_h(T) = \begin{cases} S_h^{\text{non}}(T), & \text{if } T \text{ is a noninterface element}, \\ S_h^{\text{int}}(T), & \text{if } T \text{ is an interface element}. \end{cases}$$

Let $\mathcal{N}_h = \{X_i\}_{i=1}^{N_n}$ be the set of nodes in $\mathcal{T}_h$ and let $\Phi_i(X)$ be a piecewise bilinear function such that $\Phi_i|_T \in S_h(T)$ and

$$\Phi_i(X_j) = \delta_{ij}, \quad \forall X_j \in \mathcal{N}_h.$$

Finally, the bilinear IFE space on the whole domain Ω is

$$S_h(\Omega) = \text{span}\{\Phi_i(x), 1 \le i \le N_n\},$$

and we will also use the space $S_{h,0}(\Omega) \subset S_h(\Omega)$ that consists of functions of $S_h(\Omega)$ vanishing on $\mathcal{N}_h \cap \partial\Omega$.

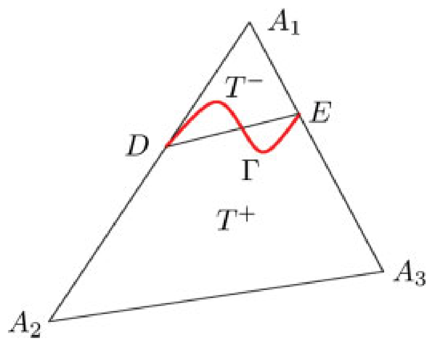

FIG. 3. A typical triangular interface element. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

Second, we recall the definition of the 2D linear IFE space discussed in [1,4]. On each of the noninterface element T, we let the local finite element space $S_h(T)$ be $S_h^{\text{non}}(T)$ spanned by the three standard bilinear nodal basis functions $\psi_i(x,y), i = 1,2,3$ on T. For an interface element T with vertices $A_i = (x_i, y_i), i = 1,2,3$, without loss of generality, we assume that ∂T intersects with Γ at two points $D = (x_D, y_D)^T$ and $E = (x_E, y_E)^T$. There are only one type of triangle interface elements, see the sketch in Fig. 3.

With the same idea as above, we define a 2D linear immersed function as follows:

$$\phi(x,y) = \begin{cases} \phi^-(x,y) = a^-x + b^-y + c^-, & (x,y) \in T^-, \\ \phi^+(x,y) = a^+x + b^+y + c^+, & (x,y) \in T^+, \\ \phi^-(D) = \phi^+(D), \ \ \phi^-(E) = \phi^+(E), & \\ \int_{\overline{DE}} \left(\beta^- \dfrac{\partial \phi^-}{\partial \mathbf{n}_{\overline{DE}}} - \beta^+ \dfrac{\partial \phi^+}{\partial \mathbf{n}_{\overline{DE}}} \right) ds = 0, & \end{cases} \tag{3.3}$$

where $\mathbf{n}_{\overline{DE}}$ is the unit vector perpendicular to the line $\overline{DE}$. We let $\phi_i(X)$ be the bilinear IFE function described by (3.3) such that

$$\phi_i(x_j, y_j) = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j, \end{cases}$$

for $1 \leq i, j \leq 3$, and we call them the 2D linear IFE nodal basis functions on an interface element T. We then let $S_h^{\text{int}}(T) = \text{span}\{\phi_i, i = 1,2,3\}$. Then we use the same way as in bilinear IFE space to define $S_h(T)$, $\Phi_i(X)$, $S_h(\Omega)$ and $S_{h,0}(\Omega)$ for the 2D linear IFE space.

IV. PROPERTIES OF THE BILINEAR AND 2D LINEAR IFE SPACES

We refer the readers to [1–4] for numerous features of the bilinear and 2D linear IFE spaces, including the existence and uniqueness of the bilinear and 2D linear IFE local nodal basis functions $\phi_i(X)$ and the following approximation capability:

$$\|u - I_h u\|_{0,\Omega} + h\|u - I_h u\|_1 \leq Ch^2 \|u\|_{2,\Omega}, \quad \forall u \in \text{PH}^2_{\text{int}}(\Omega). \tag{4.1}$$

where $I_h u \in S_h(\Omega)$ is the interpolation of u.

We now present additional properties of the bilinear and 2D linear IFE spaces. We will discuss both Type I and Type II interface elements configured as in Fig. 2 for the bilinear IFE space and the only type of interface elements configured as in Fig. 3 for the 2D linear IFE space. Unless otherwise specified, all the generic constants C in the presentation below are independent of the interface Γ and mesh parameter h. Also, T stands for either a rectangular or triangular element. Without loss of generality, we assume that $A_1 \in T^-$, $A_2, A_3, A_4 \in T^+$ for a type I rectangular interface element, $A_1, A_4 \in T^-$, $A_2, A_3 \in T^+$ for a Type II rectangular interface element, and $A_1 \in T^-$, $A_2, A_3 \in T^+$ for a triangular interface element.

For the bilinear IFE space, some discussions involve the reference element $\hat{T} = \Box \hat{A}_1 \hat{A}_2 \hat{A}_3 \hat{A}_4$ with

$$\hat{A}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \hat{A}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{A}_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \hat{A}_4 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For any function u defined on a rectangular element $T = \Box A_1 A_2 A_3 A_4$, we let $\hat{u}$ be the corresponding function on $\hat{T}$ induced by u with

$$\hat{u}(\hat{X}) = u(F(\hat{X})),$$

where $F : \hat{T} \longrightarrow T$ is the affine mapping defined by

$$X = F(\hat{X}) = \begin{bmatrix} x_2 - x_1 & x_4 - x_1 \\ y_4 - y_1 & y_3 - y_1 \end{bmatrix} \hat{X} + A_1.$$

Under this affine mapping, points $D \in T$ and $E \in T$ are mapped to

$$\hat{D} = \begin{pmatrix} 0 \\ \hat{b} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix}, \quad 0 \le \hat{a}, \hat{b} \le 1. \tag{4.2}$$

for Type I interface elements and

$$\hat{D} = \begin{pmatrix} \hat{b} \\ 1 \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix}, \quad 0 \le \hat{a}, \hat{b} \le 1. \tag{4.3}$$

for Type II interface elements.

For the 2D linear IFE space, the involved reference element is $\hat{T} = \triangle \hat{A}_1 \hat{A}_2 \hat{A}_3$ with

$$\hat{A}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \hat{A}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{A}_3 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

For any function u defined on a triangular element $T = \triangle A_1 A_2 A_3$, we let $\hat{u}$ be the corresponding function on $\hat{T}$ induced by u with

$$\hat{u}(\hat{X}) = u(F(\hat{X})),$$

where $F : \hat{T} \longrightarrow T$ is the affine mapping defined by

$$X = F(\hat{X}) = \begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix} \hat{X} + A_1.$$

Under this affine mapping, points $D \in T$ and $E \in T$ are mapped to

$$\hat{D} = \begin{pmatrix} 0 \\ \hat{b} \end{pmatrix}, \quad \hat{E} = \begin{pmatrix} \hat{a} \\ 0 \end{pmatrix}, \quad 0 \le \hat{a}, \hat{b} \le 1. \tag{4.4}$$

Let J_F be the Jacobian of an affine mapping F.

Lemma 4.1. *On each interface element T, we have following results:*

1. *If $T = \square A_1A_2A_3A_4$ is a rectangular interface element of Type I, then every $\tilde{u}_h \in S_h^{\text{int}}(T)$ can be uniquely represented as follows:*

$$\tilde{u}_h(X) = \begin{cases} \tilde{u}_h^-(X) = \tilde{u}_h(A_1)\psi_1(X) + c_2\psi_2(X) + c_3\psi_3(X) + c_4\psi_4(X), & X \in T^-, \\ \tilde{u}_h^+(X) = c_1\psi_1(X) + \tilde{u}_h(A_2)\psi_2(X) + \tilde{u}_h(A_3)\psi_3(X) + \tilde{u}_h(A_4)\psi_4(X), & X \in T^+, \end{cases} \tag{4.5}$$

2. *If $T = \square A_1A_2A_3A_4$ is a rectangular interface element of Type II, then every $\tilde{u}_h \in S_h^{\text{int}}(T)$ can be uniquely represented as follows:*

$$\tilde{u}_h(X) = \begin{cases} \tilde{u}_h^-(X) = \tilde{u}_h(A_1)\psi_1(X) + c_2\psi_2(X) + c_3\psi_3(X) + \tilde{u}_h(A_4)\psi_4(X), & X \in T^-, \\ \tilde{u}_h^+(X) = c_1\psi_1(X) + \tilde{u}_h(A_2)\psi_2(X) + \tilde{u}_h(A_3)\psi_3(X) + c_4\psi_4(X), & X \in T^+, \end{cases} \tag{4.6}$$

3. *If $T = \triangle A_1A_2A_3$ is a triangular interface element, then every $\tilde{u}_h \in S_h^{\text{int}}(T)$ can be uniquely represented as follows:*

$$\tilde{u}_h(X) = \begin{cases} \tilde{u}_h^-(X) = \tilde{u}_h(A_1)\psi_1(X) + c_2\psi_2(X) + c_3\psi_3(X), & X \in T^-, \\ \tilde{u}_h^+(X) = c_1\psi_1(X) + \tilde{u}_h(A_2)\psi_2(X) + \tilde{u}_h(A_3)\psi_3(X), & X \in T^+, \end{cases} \tag{4.7}$$

4. *There exist constants C_1 and C_2 such that*

$$C_1\|\vec{u}^+\| \le \|\vec{u}^-\| \le C_2\|\vec{u}^+\|, \tag{4.8}$$

where

$$\vec{u}^- = \begin{bmatrix} \tilde{u}_h(A_1) \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}, \quad \vec{u}^+ = \begin{bmatrix} c_1 \\ \tilde{u}_h(A_2) \\ \tilde{u}_h(A_3) \\ \tilde{u}_h(A_4) \end{bmatrix}. \tag{4.9}$$

for rectangular interface elements of Type I,

$$\vec{u}^- = \begin{bmatrix} \tilde{u}_h(A_1) \\ c_2 \\ c_3 \\ \tilde{u}_h(A_4) \end{bmatrix}, \quad \vec{u}^+ = \begin{bmatrix} c_1 \\ \tilde{u}_h(A_2) \\ \tilde{u}_h(A_3) \\ c_4 \end{bmatrix}. \tag{4.10}$$

for rectangular interface elements of Type II, and

$$\vec{u}^- = \begin{bmatrix} \tilde{u}_h(A_1) \\ c_2 \\ c_3 \end{bmatrix}, \quad \vec{u}^+ = \begin{bmatrix} c_1 \\ \tilde{u}_h(A_2) \\ \tilde{u}_h(A_3) \end{bmatrix}. \tag{4.11}$$

for triangular interface elements.

5. There exist constants C_3 and C_4 independent of interface and partition such that

$$C_3\|\vec{w}\| \le \|\vec{u}\| \le C_4\|\vec{w}\|, \tag{4.12}$$

where

$$\vec{u} = \begin{bmatrix} \tilde{u}_h(A_1) \\ \tilde{u}_h(A_2) \\ \tilde{u}_h(A_3) \\ \tilde{u}_h(A_4) \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix},$$

for rectangular interface elements and

$$\vec{u} = \begin{bmatrix} \tilde{u}_h(A_1) \\ \tilde{u}_h(A_2) \\ \tilde{u}_h(A_3) \end{bmatrix}, \quad \vec{w} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix},$$

for triangular interface elements.

Proof. Without loss of generality, we only need to discuss on the reference interface element. For a local interface element $[0,h]\times[0,h]$, if we define $\hat{a} = \frac{a}{h}$ and $\hat{b} = \frac{b}{h}$, then we can get the same conclusions. First, we discuss rectangular interface element of Type I. Applying jump conditions specified in (3.2) to $\hat{\tilde{u}}_h$ in (4.5), we obtain a linear system about coefficients in (4.5) which can be written as

$$A_l\vec{C} = A_r\vec{u}, \;\; \vec{C} = (c_1, c_2, c_3, c_4)^t, \;\; \vec{u} = (\tilde{u}_h(A_1), \tilde{u}_h(A_2), \tilde{u}_h(A_3), \tilde{u}_h(A_4))^t.$$

where

$$A_l = \begin{pmatrix} 1-\hat{b} & 0 & 0 & -\hat{b} \\ 1-\hat{a} & -\hat{a} & 0 & 0 \\ 1 & 1 & -1 & 1 \\ -2\hat{a}-2\hat{b}+\hat{a}^2+\hat{b}^2 & R(-2\hat{b}+\hat{a}^2+\hat{b}^2) & -R(\hat{a}^2+\hat{b}^2) & R(-2\hat{a}+\hat{a}^2+\hat{b}^2) \end{pmatrix},$$

$$A_r = \begin{pmatrix} 1-\hat{b} & 0 & 0 & -\hat{b} \\ 1-\hat{a} & -\hat{a} & 0 & 0 \\ 1 & 1 & -1 & 1 \\ R(-2\hat{a}-2\hat{b}+\hat{a}^2+\hat{b}^2) & -2\hat{b}+\hat{a}^2+\hat{b}^2 & -(\hat{a}^2+\hat{b}^2) & -2\hat{a}+\hat{a}^2+\hat{b}^2 \end{pmatrix}.$$

Here $R = \beta^-/\beta^+$. Direct calculations give us

$$\begin{aligned} \det(A_l) &= -2R\hat{a}^2 + 2R\hat{a}^2\hat{b} - 2R\hat{b}^2 + 2R\hat{a}\hat{b}^2 - R\hat{a}^3\hat{b} - R\hat{a}\hat{b}^3 - 2\hat{a}^2\hat{b} - 2\hat{a}\hat{b}^2 + \hat{a}^3\hat{b} + \hat{a}\hat{b}^3 \\ &= -2R\hat{a}^2(1-\hat{b}) - 2R\hat{b}^2(1-\hat{a}) - R\hat{a}^3\hat{b} - R\hat{a}\hat{b}^3 - \hat{a}^2\hat{b}(2-\hat{a}) - \hat{a}\hat{b}^2(2-\hat{b}) \\ &< 0, \end{aligned}$$

which shows that the matrix A_l is nonsingular for all $\hat{a}, \hat{b} \in [0,1]$. Hence $\vec{C}$ can be uniquely determined by $\vec{u}$ and this leads to the unique representation of $\tilde{u}_h \in S_h^{\text{int}}(T)$ in (4.5).

Rearrange the linear system above we can have

$$A_- \vec{u}^- = A_+ \vec{u}^+.$$

where

$$A_- = \begin{pmatrix} 1-\hat{b} & 0 & 0 & \hat{b} \\ 1-\hat{a} & \hat{a} & 0 & 0 \\ 1 & -1 & 1 & -1 \\ R(-2\hat{a}-2\hat{b}+\hat{a}^2+\hat{b}^2) & -R(-2\hat{b}+\hat{a}^2+\hat{b}^2) & R(\hat{a}^2+\hat{b}^2) & -R(-2\hat{a}+\hat{a}^2+\hat{b}^2) \end{pmatrix},$$

$$A_+ = \begin{pmatrix} 1-\hat{b} & 0 & 0 & \hat{b} \\ 1-\hat{a} & \hat{a} & 0 & 0 \\ 1 & -1 & 1 & -1 \\ -2\hat{a}-2\hat{b}+\hat{a}^2+\hat{b}^2 & -(-2\hat{b}+\hat{a}^2+\hat{b}^2) & \hat{a}^2+\hat{b}^2 & -(-2\hat{a}+\hat{a}^2+\hat{b}^2) \end{pmatrix}.$$

By direct calculations, we can show that A_-^{-1} and A_+^{-1} exist such that the entries of $A_-^{-1}A_+$ and $A_+^{-1}A_-$ are either polynomials of $\hat{a}$ and $\hat{b}$ or linear combination of functions in following forms:

$$\frac{\hat{a}^{r_a}\hat{b}^{r_b}}{\hat{a}^2+\hat{b}^2}, r_a \geq 0, r_b \geq 0, r_a + r_b \geq 2.$$

It can be shown that all of these functions of $\hat{a}$ and $\hat{b}$ are bounded by a constant independent of $\hat{a}$ and $\hat{b}$. Therefore, there exits a constant C such that $\|A_-^{-1}A_+\| \leq C,\ \|A_+^{-1}A_-\| \leq C,\ \forall \hat{a}, \hat{b} \in [0, 1]$ and this leads to (4.8).

Similar arguments can be applied to obtain results for Type II rectangular and triangular interface elements. We only need to deal with different matrices A_l, A_r, A_-, and A_+. We can still show that A_l is nonsingular for the other two kinds of interface elements. For rectangular interface elements of Type II, we can see that the entries of $A_-^{-1}A_+$ and $A_+^{-1}A_-$ are either polynomials of $\hat{a}$ and $\hat{b}$ or linear combination of functions in following forms:

$$\frac{\hat{a}^{r_a}\hat{b}^{r_b}}{(\hat{a}-\hat{b})^2+1}, r_a \geq 0, r_b \geq 0, r_a + r_b \geq 0.$$

For the triangular interface elements, we can see that the entries of $A_-^{-1}A_+$ and $A_+^{-1}A_-$ are either polynomials of $\hat{a}$ and $\hat{b}$ or linear combination of functions in following forms:

$$\frac{\hat{a}^{r_a}\hat{b}^{r_b}}{\hat{a}^2+\hat{b}^2}, r_a \geq 0, r_b \geq 0, r_a + r_b \geq 2.$$

Again, these functions of $\hat{a}$ and $\hat{b}$ are bounded by a constant independent of $\hat{a}$ and $\hat{b}$. Therefore, there exits a constant C such that $\|A_-^{-1}A_+\| \leq C,\ \|A_+^{-1}A_-\| \leq C,\ \forall \hat{a}, \hat{b} \in [0, 1]$ and this leads to (4.8).

Finally similar argument can be used to prove (4.12). ■

The result in the next lemma indicates that the bilinear and 2D linear IFE functions also satisfy an inverse inequality. We start from introducing an auxiliary set in each interface element. For a rectangular interface element T of Type I, without loss of generality, we assume that T is of Type

I with the configuration in Fig. 2. Let $T_{1/2} \subset T$ be the half of T which is formed by the diagonal line and not cut by $\overline{DE}$. For example, $T_{1/2} \subset T^+$ in Fig. 2. For a rectangular interface element T of Type II, without loss of generality, we assume that T is of Type II with the configuration in Fig. 2. By the two bisectors of its four edges, T is divided into four squares. The line $\overline{DE}$ can only passes at most three of them. Let $T_{1/4} \subset T$ be the square which is not cut by $\overline{DE}$. For example, $T_{1/4} \subset T^+$ in Fig. 2. For each element $T = \Box A_1A_2A_3A_4$, we let h_x and h_y be the side length in x-direction and y-direction separately. Define $h_T = \max\{h_x, h_y\}$. For a triangular interface element T, without loss of generality, we assume that T has the configuration in Fig. 3. By the three bisectors of its angles, T is divided into six sub-triangles.

The line $\overline{DE}$ can only passes at most four of them and at least two adjoint sub-triangles are not cut by $\overline{DE}$. Let $T_{1/3} \subset T$ be the union of two of these adjoint sub-triangles. For example, $T_{1/3} \subset T^+$ in Fig. 3. By the shape regular assumption, there exists a C independent of h, T and interfaces such that $|T_{1/3}| \geq C|T|$, $\forall T \in \mathcal{T}_h$. Also, for each element $T = \triangle A_1A_2A_3$, we let $h_T = \mathrm{diam}(T)$.

In the following discussion, we will use the notation $T_{1/t} \subset T^s, s = +$ or $-$ with $t = 2$ for rectangular interface elements of Type I, $t = 4$ for rectangular interface elements of Type II, and $t = 3$ for triangular interface elements.

Lemma 4.2. *There exists a constant C such that for all $\tilde{u}_h \in S_h^{\mathrm{int}}(T)$, we have*

$$\begin{cases} \|\tilde{u}_h\|_{\infty,T^s} \leq \dfrac{C}{h}\|\tilde{u}_h\|_{0,T}, \\ \|\tilde{u}_{hx}\|_{\infty,T^s} \leq \dfrac{C}{h}\|\tilde{u}_{hx}\|_{0,T}, \quad \|\tilde{u}_{hy}\|_{\infty,T^s} \leq \dfrac{C}{h}\|\tilde{u}_{hy}\|_{0,T}, \end{cases} \tag{4.13}$$

provided that $T_{1/t} \subset T^s, s = +$ or $-, t = 2, 4, 3$.

Proof. For each $\tilde{u}_h \in S_h^{\mathrm{int}}(T)$, its restriction on T^s is just a bilinear or linear polynomial, we denote it by $\tilde{u}_h^s$. Let $\tilde{u}_{h,\mathrm{ext}}^s$ be the extension of $\tilde{u}_h^s$ to the whole interface element T. Then, we can obtain the first inequality in (4.13) as follows:

$$\begin{aligned} \|\tilde{u}_h\|_{\infty,T^s} &= \left\|\tilde{u}_h^s\right\|_{\infty,T^s} = \left\|\widehat{\tilde{u}_h^s}\right\|_{\infty,\hat{T}^s} \leq \left\|\widehat{\tilde{u}_{h,ext}^s}\right\|_{\infty,\hat{T}} \leq C\left\|\widehat{\tilde{u}_{h,ext}^s}\right\|_{0,\hat{T}_{1/t}} = C\left\|\widehat{\tilde{u}_h^s}\right\|_{0,\hat{T}_{1/t}} \\ &\leq C|J_F|^{-1/2}\left\|\tilde{u}_h^s\right\|_{0,T_{1/t}} \leq Ch^{-1}\|\tilde{u}_h\|_{0,T_{1/t}} \leq Ch^{-1}\|\tilde{u}_h\|_{0,T}. \end{aligned}$$

The other inequalities can be shown similarly. ■

We now consider the relationship between $S_h^{\mathrm{non}}(T)$ and $S_h^{\mathrm{int}}(T)$. Let $d = 4$ for rectangular elements and $d = 3$ for triangular elements. Consider the following mappings:

$$\bar{I}_h : C(T) \longrightarrow S_h^{\mathrm{non}}(T), \bar{I}_h\phi(X) = \sum_{i=1}^{d} \phi(A_i)\psi_i(X),$$

$$\tilde{I}_h : C(T) \longrightarrow S_h^{\mathrm{int}}(T), \tilde{I}_h\phi(X) = \sum_{i=1}^{d} \phi(A_i)\phi_i(X).$$

By direct verification, we can easily see that

$$\bar{I}_h\tilde{I}_h\bar{u}_h = \bar{u}_h, \quad \forall \bar{u}_h \in S_h^{\mathrm{non}}(T), \quad \tilde{I}_h\bar{I}_h\tilde{u}_h = \tilde{u}_h, \quad \forall \tilde{u}_h \in S_h^{\mathrm{int}}(T),$$

and

$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{k,T}$$
$$= \|\bar{u}_h - \tilde{I}_h\bar{u}_h\|_{k,T}, k = 0, 1,\ \forall \bar{u}_h \in S_h^{\text{non}}(T), \tilde{u}_h \in S_h^{\text{int}}(T) \text{ such that } \bar{u}_h = \bar{I}_h\tilde{u}_h \text{ or } \tilde{u}_h = \tilde{I}_h\bar{u}_h.$$

In fact, our interpolation operator I_h for the bilinear and 2D linear IFE spaces [3, 4] can be locally defined by

$$I_h = \begin{cases} \bar{I}_h, & \text{on noninterface elements,} \\ \tilde{I}_h, & \text{on interface elements.} \end{cases}$$

Lemma 4.3. *There exists a constant C such that*

$$\|\bar{I}_h\tilde{u}_h\|_{0,T} \le C\|\tilde{u}_h\|_{0,T}, \quad \forall \tilde{u}_h \in S_h^{\text{int}}(T). \tag{4.14}$$

Proof. Let

$$\bar{u}_h = \bar{I}_h\tilde{u}_h = \sum_{i=1}^{d} \tilde{u}_h(A_i)\psi_i(X).$$

By the shape regular assumption, there exist two positive constants C_1 and C_2 independent of h and interfaces such that $C_1h_T^2 \le |J_F| \le C_2h_T^2$. Then

$$\|\bar{u}_h\|_{0,T}^2 = \int_T \bar{u}_h^2(X)dX = \int_{\hat{T}} \widehat{\bar{u}_h}^2(\hat{X})|J_F|d\hat{X}$$
$$\le Ch_T^2 \int_{\hat{T}} \widehat{\bar{u}_h}^2(\hat{X})d\hat{X} \le Ch_T^2\|\vec{u}\|^2 \le Ch_T^2(\|\vec{u}^-\|^2 + \|\vec{u}^+\|^2).$$

Since $T_{1/t} \subset T^s, s = +$ or $-, t = 2, 4, 3$, then using representation (4.5)–(4.7) for $\tilde{u}_h$, we have

$$\|\tilde{u}_h\|_{0,T}^2 = \left\|\tilde{u}_h^-\right\|_{0,T^-}^2 + \left\|\tilde{u}_h^+\right\|_{0,T^+}^2 \ge \left\|\tilde{u}_h^s\right\|_{0,T^s}^2 = \int_{T^s} \left[\tilde{u}_h^s(X)\right]^2 dX = \int_{\hat{T}^s} \widehat{\tilde{u}_h^s}^2(\hat{X})|J_F|d\hat{X}$$
$$\ge Ch_T^2 \int_{\hat{T}^s} \widehat{\tilde{u}_h^s}^2(\hat{X})d\hat{X} \ge Ch_T^2 \int_{\hat{T}_{1/t}} \widehat{\tilde{u}_h^s}^2(\hat{X})d\hat{X} = Ch_T^2(\vec{u}^s)^t\hat{A}_{1/2}\vec{u}^s \ge Ch_T^2\|\vec{u}^s\|^2,$$

where we have used the fact that matrix $\hat{A}_{1/2} = (\int_{\hat{T}_{1/t}} \hat{\psi}_i\hat{\psi}_j d\hat{X})_{i,j=1}^d$ is positive definite. Combining these two inequalities and applying (4.8) leads to (4.14). ∎

Lemma 4.4. *There exits a constant C such that*

$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,T} \le Ch\|\tilde{u}_h\|_{1,T}, \quad \forall \tilde{u}_h \in S_h^{\text{int}}(T). \tag{4.15}$$

Proof. Let $\hat{\bar{I}}_h$ denote the interpolation $\bar{I}_h$ defined on $\hat{T}$. Applying the result in Lemma 4.3 on $\hat{T}$, we have

$$\left\|\widehat{\tilde{u}_h} - \widehat{\bar{I}_h\tilde{u}_h}_{0,\hat{T}}\right\| = \left\|\widehat{\tilde{u}_h} - c - \left(\hat{\bar{I}}_h\widehat{\tilde{u}}_h - \hat{\bar{I}}_h c\right)_{0,\hat{T}}\right\| \le C\left\|\widehat{\tilde{u}}_h - c_{0,\hat{T}}\right\| \le C\left\|\hat{\tilde{u}}_h - c_{1,\hat{T}}\right\|,$$

for any constant c and this leads to

$$\left\|\widehat{\tilde{u}}_h - \widehat{I_h \tilde{u}_h}_{0,\hat{T}}\right\| \le C \inf_c \left\|\widehat{\tilde{u}}_h - c_{1,\hat{T}}\right\| \le C\left|\widehat{\tilde{u}}_h\right|_{1,\hat{T}}.$$

Then (4.15) follows from the standard scaling argument. ∎

Lemma 4.5. *Let $\mathbf{n}(\overline{DE}) = (\overline{n}_x, \overline{n}_y)^T$ be the unit normal vector of $\overline{DE}$. Let $X_{\overline{DE}} \in \overline{DE}$. Then for any function $u(x, y)$ satisfying*

$$[u]|_{\overline{DE}} = 0, \ \left[\beta \frac{\partial u}{\partial n}\right]|_{\overline{DE}} = 0,$$

we have

$$\nabla u^+(X_{\overline{DE}}) = N^-_{\overline{DE}} \nabla u^-(X_{\overline{DE}}), \quad N^-_{\overline{DE}} = \begin{pmatrix} \overline{n}_y^2 + \rho \overline{n}_x^2 & (\rho - 1)\overline{n}_x\overline{n}_y \\ (\rho - 1)\overline{n}_x\overline{n}_y & \overline{n}_x^2 + \rho\overline{n}_y^2 \end{pmatrix}, \tag{4.16}$$

$$\nabla u^-(X_{\overline{DE}}) = N^+_{\overline{DE}} \nabla u^+(X_{\overline{DE}}), \quad N^+_{\overline{DE}} = \begin{pmatrix} \overline{n}_y^2 + \tilde{\rho} \overline{n}_x^2 & (\tilde{\rho} - 1)\overline{n}_x\overline{n}_y \\ (\tilde{\rho} - 1)\overline{n}_x\overline{n}_y & \overline{n}_x^2 + \tilde{\rho}\overline{n}_y^2 \end{pmatrix}. \tag{4.17}$$

Proof. The proof for these results can be found in [3,4]. ∎

For interface elements as configured in Figs. 2 and 3, we have the following lemma.

Lemma 4.6. *Assume f is a continuous piecewise bilinear or 2D linear function and its two pieces on $\overline{A_1A_2}$ is separated by $E \in \overline{A_1A_2}$. If $f(A_1) = f(A_2) = 0$ and $\|A_1A_2\| = h$, then*

$$|f(E)| \le \frac{1}{2}|[\nabla f(E)] \cdot \vec{v}_{\overline{A_1A_2}}|h, \tag{4.18}$$

where $\vec{v}_{\overline{A_1A_2}}$ is the unit vector pointing from A_1 to A_2.

Proof. For any continuous piecewise linear function f on $\overline{A_1A_2}$ with $f(A_1) = f(A_2) = 0$, we can use the Taylor expansion to obtain the following:

$$0 = f(A_1) = f(E^-) + \nabla f(E^-) \cdot (A_1 - E) = f(E) + \nabla f(E^-) \cdot (A_1 - E),$$
$$0 = f(A_2) = f(E^+) + \nabla f(E^+) \cdot (A_2 - E) = f(E) + \nabla f(E^+) \cdot (A_2 - E).$$

Hence

$$\begin{aligned} f(E) &= \frac{1}{2}[-\nabla f(E^-) \cdot (A_1 - E) - \nabla f(E^+) \cdot (A_2 - E)] \\ &= \frac{1}{2}[\nabla f(E^-) \cdot \vec{v}_{\overline{A_1A_2}}\|E - A_1\| - \nabla f(E^+) \cdot \vec{v}_{\overline{A_1A_2}}\|A_2 - E\|]. \end{aligned}$$

As $f(A_1) = f(A_2) = 0$ and f is continuous, we have

$$\nabla f(E^-) \cdot \vec{v}_{\overline{A_1A_2}} > 0, \nabla f(E^+) \cdot \vec{v}_{\overline{A_1A_2}} < 0 \text{ if } f(E) > 0,$$
$$\nabla f(E^-) \cdot \vec{v}_{\overline{A_1A_2}} < 0, \nabla f(E^+) \cdot \vec{v}_{\overline{A_1A_2}} > 0 \text{ if } f(E) < 0.$$

Because $\|E - A_1\| \le h$ and $\|A_2 - E\| \le h$, we get

$$f(E) \le \frac{1}{2}[\nabla f(E^-) \cdot \vec{v}_{\overline{A_1A_2}} h - \nabla f(E^+) \cdot \vec{v}_{\overline{A_1A_2}} h] \le \frac{1}{2}|[\nabla f(E)] \cdot \vec{v}_{\overline{A_1A_2}}| h \text{ if } f(E) > 0,$$

$$f(E) \ge \frac{1}{2}[\nabla f(E^-) \cdot \vec{v}_{\overline{A_1A_2}} h - \nabla f(E^+) \cdot \vec{v}_{\overline{A_1A_2}} h] \ge -\frac{1}{2}|[\nabla f(E)] \cdot \vec{v}_{\overline{A_1A_2}}| h \text{ if } f(E) < 0,$$

which lead to (4.18). ∎

We derive an estimate for the difference $\tilde{u}_h - \bar{I}_h \tilde{u}_h$ on ∂T in the following lemma.

Lemma 4.7. *There exits a constant C such that*

$$\|\tilde{u}_h - \bar{I}_h \tilde{u}_h\|_{0,\partial T} \le C h^{1/2} |\tilde{u}_h|_{1,T}, \quad \forall \tilde{u}_h \in S_h^{\text{int}}(T). \tag{4.19}$$

Proof. Without loss of generality, we consider those interface elements as configured in Figs. 2 and 3. First we discuss interface elements of Type I. We note that $\tilde{u}_h$ is a C^0 piecewise bilinear function and $\bar{I}_h \tilde{u}_h$ is its bilinear interpolation. Hence

$$\|\tilde{u}_h - \bar{I}_h \tilde{u}_h\|_{0,\overline{A_2A_3}} = \|\tilde{u}_h - \bar{I}_h \tilde{u}_h\|_{0,\overline{A_3A_4}} = 0.$$

In addition, as $\tilde{u}_h(X) - \bar{I}_h \tilde{u}_h(X)$ is piecewise linear on $\overline{A_1A_2}$ and vanishes at A_1 and A_2, we can show that

$$\max_{X \in \overline{A_1A_2}} |\tilde{u}_h(X) - \bar{I}_h \tilde{u}_h(X)| \le |\tilde{u}_h(E) - \bar{I}_h \tilde{u}_h(E)|. \tag{4.20}$$

As $\bar{I}_h \tilde{u}_h$ is continuous on T, by applying (4.18) to the restriction of $f(X) = \tilde{u}_h(X) - \bar{I}_h \tilde{u}_h(X)$ on $\overline{A_1A_2}$, we have

$$\begin{aligned} |\tilde{u}_h(E) - \bar{I}_h \tilde{u}_h(E)| &\le \frac{1}{2}|[\nabla(\tilde{u}_h(E) - \bar{I}_h \tilde{u}_h(E))] \cdot \vec{v}_{\overline{A_1A_2}}| h_x \\ &= \frac{1}{2}|[\nabla \tilde{u}_h(E)] \cdot \vec{v}_{\overline{A_1A_2}}| h_x \\ &\le C \|\nabla \tilde{u}_h^+(E)\| h_x \\ &\le C |\tilde{u}_h|_{1,T}, \end{aligned} \tag{4.21}$$

where we have applied Lemma 4.5 and the inverse inequality (4.13) to obtain the last two inequalities above, respectively. Then, the inequalities (4.20) and (4.21) lead to

$$\begin{aligned} \|\tilde{u}_h - \bar{I}_h \tilde{u}_h\|_{0,\overline{A_1A_2}} &= \left[\int_{\overline{A_1A_2}} [\tilde{u}_h(x) - \bar{I}_h \tilde{u}_h(x)]^2 dx\right]^{1/2} \\ &\le \left[[\tilde{u}_h(E) - \bar{I}_h \tilde{u}_h(E)]^2 \int_{\overline{A_1A_2}} dx\right]^{1/2} \\ &\le h_x^{1/2} |\tilde{u}_h(E) - \bar{I}_h \tilde{u}_h(E)| \\ &\le C h^{1/2} |\tilde{u}_h|_{1,T}. \end{aligned}$$

Similar estimate can be derived for $\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_4A_1}}$ and these estimates yield (4.19) for interface elements of Type I.

Similarly, for interface elements of Type II, we have

$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_2A_3}} = \|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_4A_1}} = 0,$$
$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_1A_2}} \le Ch^{1/2}|\tilde{u}_h|_{1,T},$$
$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_3A_4}} \le Ch^{1/2}|\tilde{u}_h|_{1,T},$$

which yield (4.19) for rectangular interface elements of Type II.

Also, for triangular interface elements, we have

$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_3A_1}} = 0$$
$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_1A_2}} \le Ch^{1/2}|\tilde{u}_h|_{1,T}$$
$$\|\tilde{u}_h - \bar{I}_h\tilde{u}_h\|_{0,\overline{A_2A_3}} \le Ch^{1/2}|\tilde{u}_h|_{1,T},$$

which yield (4.19) for triangular interface elements. The proof for this lemma is then finished. ■

V. THE CONVERGENCE OF 2D LINEAR AND BILINEAR IFE SOLUTIONS

We now consider the error bound for the immersed finite element solution of the interface (1.1)–(1.4). Specifically, as the IFE space is nonconforming, the IFE method to be considered uses a discrete bilinear form as follows: find $u_h \in S_h(\Omega)$ such that

$$\sum_{T\in\mathcal{T}_h}\int_T \beta\nabla u_h\nabla v_h dX = \int_\Omega f v_h dX, \quad \forall v_h \in S_{h,0}(\Omega). \tag{5.1}$$

We refer the readers to [2–4] for numerical examples demonstrating the performance these methods using either the linear or bilinear IFE spaces.

Assuming the exact solution u of the interface problem (1.1)–(1.4) has the $\mathrm{PH}^2_{\mathrm{int}}$ regularity, then the general Berger-Scott-Strang lemma [24] implies that the error in the IFE solution generated by (5.1) have the following error bound:

$$|u-u_h|_{1,h} \le C\left(\inf_{v_h\in S_h(\Omega)}\|u-v_h\|_{1,h} + \sup_{v_h\in S_h(\Omega)}\frac{|a_h(u,v_h)-(f,v_h)|}{|v_h|_{1,h}}\right), \tag{5.2}$$

$$a_h(w,v) = \sum_{T\in\mathcal{T}_h}\int_T \beta\nabla w\nabla v dX, \tag{5.3}$$

$$\|v\|^2_{1,h} = \|v\|^2_{0,h} + |v|^2_{1,h}, \|v\|^2_{0,h} = \sum_{T\in\mathcal{T}_h}\|v\|^2_{0,T}, |v|^2_{1,h} = \sum_{T\in\mathcal{T}_h}|v|^2_{1,T}. \tag{5.4}$$

and $(w,v)_\Lambda$ denotes the L^2 inner product of w and v on a set Λ and we often omit its set symbol when $\Lambda = \Omega$.

By using (4.1), the approximation error term on the right-hand side of (5.2) has the following bound:

$$\inf_{v_h \in S_h(\Omega)} \|u - v_h\|_{1,h} \le \|u - I_h u\|_{1,h} \le Ch\|u\|_2. \tag{5.5}$$

Therefore, we will focus on the estimation of the consistency error term on the right hand of (5.2).

In the analysis of the IFE methods, we often need to estimate $\beta \frac{\partial v}{\partial n}$ on edges of elements for a $v \in \mathrm{PH}^2_{\mathrm{int}}(\Omega)$. The following lemmas indicates that $\beta \frac{\partial v}{\partial n}$ on edges of element is well defined.

Lemma 5.1. *We have the following trace inequality on $T \in \mathcal{T}_h$:*

$$\left\| \beta \frac{\partial v}{\partial n} \right\|^2_{-1/2, E_i(\partial T)} \le C\left(\frac{1}{h_T} |v|^2_{1,T} + h_T |v|^2_{2,T} \right), \quad \forall v \in \mathrm{PH}^2_{\mathrm{int}}(\Omega), \ \ 1 \le i \le d. \tag{5.6}$$

Proof. Without loss generality, we assume that $T \in \mathcal{T}_{\mathrm{int}}$. For any $v \in \mathrm{PH}^2_{\mathrm{int}}(\Omega)$, we let

$$\vec{q} = \beta \begin{bmatrix} v_x \\ v_y \end{bmatrix}, w = \beta(v_{xx} + v_{yy}).$$

Then for any $\phi \in C_0^\infty(T)$, we can easily see that

$$\int_T w\phi dX = -\int_T \vec{q} \cdot \nabla \phi dX.$$

This implies that $\vec{q} \in H(\mathrm{div}, T)$ and $\mathrm{div}(\vec{q}) = \beta(v_{xx} + v_{yy})$. On the reference element, we recall the following standard inequality for functions in $H(\mathrm{div}, \hat{T})$ [25]:

$$\|\hat{\vec{q}} \cdot \mathbf{n}\|^2_{-1/2, E_i(\partial \hat{T})} \le C\big(\|\hat{\vec{q}}\|^2_{0,\hat{T}} + \|\mathrm{div}(\hat{\vec{q}})\|^2_{0,\hat{T}}\big).$$

Then, we can obtain (5.6) by using the above trace inequality on the reference element and the usual scaling procedure. ∎

Furthermore, we note that, for a $v \in \mathrm{PH}^2_{\mathrm{int}}(\Omega)$, $\beta \frac{\partial v}{\partial n}$ is in fact a piecewise $H^{1/2}$ function on each edge of an interface element. Hence, we can estimate $\beta \frac{\partial v}{\partial n}$ on edges of interface elements in the L^2 norm. To be specific, for each element $T = \square A_1 A_2 A_3 A_4 \in \mathcal{T}_h$, we let

$$E_1(\partial T) = \overline{A_1 A_2}, \ E_2(\partial T) = \overline{A_2 A_3}, \ E_3(\partial T) = \overline{A_3 A_4}, \ E_4(\partial T) = \overline{A_4 A_1}, \tag{5.7}$$

and for each element $T = \triangle A_1 A_2 A_3 \in \mathcal{T}_h$, we let

$$E_1(\partial T) = \overline{A_1 A_2}, \ E_2(\partial T) = \overline{A_2 A_3}, \ E_3(\partial T) = \overline{A_3 A_1}. \tag{5.8}$$

Lemma 5.2. *There exists a constant C such that for h small enough, we have*

$$\left(\sum_{T \in \mathcal{T}_{\mathrm{int}}} \sum_{i=1}^{d} \left\| \beta \frac{\partial v}{\partial n} \right\|^2_{0, E_i(\partial T)} \right)^{1/2} \le C\|v\|_{2,\Omega}, \quad \forall v \in \mathrm{PH}^2_{\mathrm{int}}(\Omega). \tag{5.9}$$

Proof. The results follow the following estimates.

$$
\begin{aligned}
&\left(\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left\|\beta\frac{\partial v}{\partial n}\right\|_{0,E_i(\partial T)}^2\right)^{1/2}\\
&=\left[\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left(\left\|\beta\frac{\partial v}{\partial n}\right\|_{0,E_i(\partial T)^+}^2+\left\|\beta\frac{\partial v}{\partial n}\right\|_{0,E_i(\partial T)^-}^2\right)\right]^{1/2}\\
&\le\left[\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left(\left\|\beta\frac{\partial v}{\partial n}\right\|_{1,T^+}^2+\left\|\beta\frac{\partial v}{\partial n}\right\|_{1,T^-}^2\right)\right]^{1/2}\\
&\le C\left[\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left(\|v\|_{2,T^+}^2+\|v\|_{2,T^-}^2\right)\right]^{1/2}\\
&\le C\|v\|_{2,\Omega}.
\end{aligned}
$$

■

Now, we are ready to derive an error bound in the discrete H^1 norm for the bilinear IFE solution.

Theorem 5.1. *There exists a constant C such that*

$$\|u-u_h\|_{1,h}\le Ch^{1/2}\|u\|_{2,\Omega}. \tag{5.10}$$

Proof. By direct calculations, we have

$$|a_h(u,v_h)-(f,v_h)|=\sum_{T\in\mathcal{T}_h}\left(\beta\frac{\partial u}{\partial n},v_h\right)_{\partial T}.$$

As $\bar{I}_h v_h$ and $\beta\frac{\partial u}{\partial n}$ are continuous and the unit normal vectors of an edge in its two neighbor elements have opposite directions, then we can show

$$a_h(u,v_h)-(f,v_h)=\left|\sum_{T\in\mathcal{T}_h}\sum_{i=1}^{d}\left(\beta\frac{\partial u}{\partial n},v_h-\bar{I}_h v_h\right)_{E_i(\partial T)}\right|.$$

Then, by Lemma 4.7 and Lemma 5.2, we have

$$
\begin{aligned}
|a_h(u,v_h)-(f,v_h)|&\le\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left\|\beta\frac{\partial u}{\partial n}\right\|_{0,E_i(\partial T)}\|v_h-\bar{I}_h v_h\|_{0,E_i(\partial T)}\\
&\le Ch^{1/2}\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left\|\beta\frac{\partial u}{\partial n}\right\|_{0,E_i(\partial T)}|v_h|_{1,T}
\end{aligned}
$$

$$\leq Ch^{1/2}\left(\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}\left\|\beta\frac{\partial u}{\partial n}\right\|^2_{0,E_i(\partial T)}\right)^{1/2}\left(\sum_{T\in\mathcal{T}_{\text{int}}}\sum_{i=1}^{d}|v_h|^2_{1,T}\right)^{1/2}$$

$$\leq Ch^{1/2}|v_h|_{1,h}\|u\|_{2,\Omega}. \tag{5.11}$$

Finally, the result of this theorem follows from applying the Berger-Scott-Strang inequality (5.2), the bound for the interpolation error (5.5) and (5.11). ■

The error bound obtained in Theorem 5.1 clearly indicates that the linear and bilinear IFE solutions converge to the exact solution of the interface problem when the mesh size h tends to zero. On the other hand, we note that the error bound in this theorem has an $O(h^{1/2})$ convergence rate which is sub-optimal from the point of view of the approximation capability of the 2D linear and bilinear IFE spaces. Recall that the H^1-norm error bound for the interpolation in the 2D linear and bilinear IFE space is $O(h)$, see (4.1) and [3, 4]. In addition, all the numerical examples in [1–4] indicate that the 2D linear and bilinear IFE methods can generate approximated solutions to the interface problem with the optimal convergence rate.

The analysis to show that the linear and bilinear IFE methods have the optimal convergence rate is still elusive. A better error bound might be obtained by a more careful estimation of $|a_h(u, v_h) - (f, v_h)|$. In deriving its estimate in (5.11), we have used the L^2 norm for both $\beta\frac{\partial u}{\partial n}$ and $v_h - \bar{I}_h v_h$. On the other hand, we note that $\beta\frac{\partial u}{\partial n}$ is a piecewise $H^{1/2}$ function on edges of interface elements. This implies that one may use $H^{-1/2}$ norm to measure $v_h - \bar{I}_h v_h$ for a higher order bound. Again, the challenge is to make sure that the generic constant C in the related estimates is independent of the interface location. The other task is to derive the error estimate in the L^2 for IFE solutions. The Nitsche's technique can be applied if we can extend Lemma 4.7 such that it is true for every function $\tilde{u}_h \in \mathrm{PH}^2_{\text{int}}(T)$. All of these lead to future research topics.

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