



# A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity

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## ABSTRACT

In this note we propose a finite element method for incompressible (or compressible) elasticity problems with discontinuous modulus of elasticity (or, if compressible, Poisson's ratio). The problem is written on mixed form using  $P^1$ -continuous displacements and elementwise  $P^0$  pressures, leading to the possibility of eliminating the pressure beforehand in the compressible case. In the incompressible case, the method is augmented by a stabilization term, penalizing the pressure jumps. We show a priori error estimates under certain regularity hypothesis. In particular we prove that if the exact solution is sufficiently smooth in each subdomain then the convergence order is optimal.

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## 1. Introduction

Hansbo and Hansbo [6,7] proposed an unfitted finite element method for elliptic interface methods using Nitsche's method [10] to allow for discontinuous or weakly discontinuous solutions. In particular, [7] deals with the elasticity problem, however avoiding the incompressible case. In this paper we aim to fill this gap.

We note that the equations of incompressible elasticity written on mixed form with the pressure as an auxiliary variable is form-identical to the Stokes equations of creeping viscous flow, and thus this paper serves as a basis for computations involving also two- (or multi-) fluid problems, which has been an important field of research during the last 10 years. Techniques such as front tracking and front capturing have been introduced in order to keep track of the front. Both these techniques have qualities and drawbacks. Typically a front tracking algorithm will give a sharp resolution of the front and since the mesh follows the interface it is also well suited to solve the fluid mechanics equations with discontinuous coefficients that the two-fluid problem gives rise to; on the other hand it breaks down if topological changes occur in the exact solution. This is essentially the reason why front capturing method like volume of fluid or level set methods were introduced: they handle topological change automatically. However, since the position of

the front is known only through the intermediate of the interface capturing function the mesh is not fitted to the discontinuities of the coefficients in the exact solution. It is well known that for elliptic problems this leads to loss of convergence and pollution effects perturbing the solution close to the front. This paper thus addresses also the latter question in the setting of fluid mechanics.

We consider the space of piecewise affine, continuous, functions for the approximation of the displacements and the space of piecewise constant functions for the pressure. The pressure jumps between adjacent elements are penalized to enhance stability. Across the interface we enrich the finite element space so that both the displacements and the pressures are allowed to be discontinuous. The solutions in the two subdomains are then coupled using Nitsche's method. We only consider the two dimensional case, but the same analysis carries over to three space dimensions under minor modifications.

We show that our formulation satisfies the inf-sup condition for stabilized methods related to the non-mixed constant-strain method. This stability estimate is then used in combination with approximation estimates using suitable extension operators to prove optimal a priori estimates in energy and  $L_2$ -norms.

## 2. The elasticity problem on mixed form

Let  $\Omega$  be a bounded domain in  $\mathbb{R}^2$ , with convex polygonal boundary  $\partial\Omega$  and an internal smooth boundary  $\Gamma$ , with bounded

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measure, dividing  $\Omega$  into two open sets  $\Omega_1$  and  $\Omega_2$ . The mixed form of the elasticity equations that we propose as a model problem is given by seeking the displacements  $\mathbf{u}$  and the pressure  $p$  such that

$$\begin{aligned} -\nabla \cdot (2\mu_i \boldsymbol{\varepsilon}(\mathbf{u})) + \nabla p &= \mathbf{f} \quad \text{in } \Omega_i, \quad i = 1, 2, \\ \nabla \cdot \mathbf{u} + \frac{1}{\lambda_i} p &= 0 \quad \text{in } \Omega_i, \quad i = 1, 2, \\ \mathbf{u} &= 0 \quad \text{on } \partial\Omega, \end{aligned} \quad (1)$$

where, in terms of Young's modulus  $E_i$  and Poisson's ratio  $\nu_i$ , we have  $\mu_i = E_i/(2(1+\nu_i))$  and  $\lambda_i = E_i \nu_i/((1+\nu_i)(1-2\nu_i))$ . (Strictly speaking,  $p$  can only be identified as the pressure in the incompressible case  $\nu_i \rightarrow 1/2$ , but for simplicity we shall omit this distinction.) Here  $\mathbf{f} \in [L^2(\Omega)]^2$ ,  $\boldsymbol{\varepsilon}(\mathbf{u})$  is the symmetric part of the displacement gradient and  $\Gamma$  denotes the interface between  $\Omega_1$  and  $\Omega_2$ , with  $\Omega = \Omega_1 \cup \Omega_2$ . The Lamé parameters  $\mu_i$  and  $\lambda_i$  are constant in each subdomain  $\Omega_i$ . For any sufficiently regular function  $x$  in  $\Omega_1 \cup \Omega_2$  we define the jump of  $x$  across  $\Gamma$  by  $\llbracket x \rrbracket = x_1 - x_2$ , where  $x_i = x|_{\Omega_i}$ . We also define the limits in domain  $\Omega_1$  or  $\Omega_2$  on  $\Gamma$  by  $p(\mathbf{x})|_{\Gamma^\pm} = \lim_{\epsilon \rightarrow \pm 0} p(\mathbf{x} + \epsilon \mathbf{n})$  where  $\mathbf{x} \in \Gamma$  and where  $\mathbf{n}$  is the outward pointing normal to  $\Omega_1$  on  $\Gamma$ . Let  $(u, v)_X$  denote the  $L^2$ -scalar product on  $X$ ,

$$(u, v)_X := \int_X u v \, dx,$$

with associated norm  $\|u\|_{0,X} := (u, u)_X^{1/2}$ . The Sobolev-norms associated with the spaces  $H^k(X)$  will be denoted by  $\|u\|_{k,X}$ . Define the max and min of the shear moduli by  $\mu_{\max} := \max_{\Omega} \mu$  and  $\mu_{\min} := \min_{\Omega} \mu$ . Formally we obtain the weak formulation by multiplying (1) by a function  $(\mathbf{v}, q) \in [H_0^1(\Omega)]^2 \times L^2(\Omega) = V \times Q$  and integrating by parts, leading to the problem of finding  $(\mathbf{u}, p) \in V \times Q$  such that

$$\begin{aligned} \sum_{i=1}^2 ((2\mu_i \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\Omega_i} - (p, \nabla \cdot \mathbf{v})_{\Omega_i} + (q, \nabla \cdot \mathbf{u})_{\Omega_i} + (\lambda_i^{-1} p, q)_{\Omega_i}) \\ + \int_{\Gamma} \llbracket p \mathbf{n} - 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n} \rrbracket \cdot \mathbf{v} \, ds = (\mathbf{f}, \mathbf{v})_{\Omega}, \quad \text{for all } (\mathbf{v}, q) \in V \times Q, \end{aligned} \quad (2)$$

where

$$(\boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\Omega_i} := \int_{\Omega_i} \boldsymbol{\varepsilon}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v}) \, dx.$$

Taking into account the following interface conditions

$$\llbracket p \mathbf{n} - 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n} \rrbracket \cdot \mathbf{n} = -\sigma \kappa \quad (3)$$

and

$$\llbracket 2\mu \boldsymbol{\varepsilon}(\mathbf{u}) \cdot \mathbf{n} \rrbracket \cdot \mathbf{t} = 0, \quad (4)$$

where  $\mathbf{t} = (-n_y, n_x)$  and  $\sigma \kappa$  denotes the surface tension force term, with  $\kappa$  the interface curvature, we arrive at the following well posed formulation: Find  $(u, p) \in V \times Q$  such that

$$B[(\mathbf{u}, p), (\mathbf{v}, q)] := (\mathbf{f}, \mathbf{v})_{\Omega} + \int_{\Gamma} \sigma \kappa \mathbf{v} \cdot \mathbf{n} \, ds, \quad \text{for all } (\mathbf{v}, q) \in V \times Q, \quad (5)$$

where

$$\begin{aligned} B[(\mathbf{u}, p), (\mathbf{v}, q)] &= \sum_{i=1}^2 ((2\mu_i \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\varepsilon}(\mathbf{v}))_{\Omega_i} - (p, \nabla \cdot \mathbf{v})_{\Omega_i} + (q, \nabla \cdot \mathbf{u})_{\Omega_i} \\ &\quad + (\lambda_i^{-1} p, q)_{\Omega_i}). \end{aligned}$$

For the analysis we will only consider the incompressible case to reduce the technical detail. When both materials are incompressible the pressure space must be chosen as  $L_0^2(\Omega)$ . The extension to the

compressible case is straightforward combining the results below with those of [7]. Moreover for the convergence analysis we will assume that  $\mathbf{u}$  is in  $H^2$  on each subdomain and  $p$  in  $H^1$  in each subdomain. For a regularity analysis in the compressible case we refer to [9] and in the incompressible case to [11]. In Section 5.2 we comment briefly on the case when one of the  $\lambda_i$  is finite but the other not.

### 3. The finite element formulation

In a standard finite element method the jump in normal derivative resulting from the continuity of the traction, when  $\mu_1 \neq \mu_2$  or  $\lambda_1 \neq \lambda_2$ , can be taken into account by letting  $\Gamma$  coincide with mesh-lines. However, for optimization or inverse identification purposes (or in general for two-fluid problems) where the interface will have to move, this implies remeshing during the simulation, at least close to the interface. Instead we propose to solve (1) approximately using piecewise linear displacements and piecewise constant pressures on a family of quasiuniform, conforming triangulations  $\mathcal{T}_h$  of  $\Omega$  which are independent of the location of the interface. To avoid loss in convergence order we relax the continuity requirement over  $\Gamma$  and allow the approximation to be discontinuous inside elements which intersect the interface. In the analysis we let  $C$  denote a generic constant that may change at each occurrence and that is independent of  $h$  and of the interface position.

We will use the following notation for mesh related quantities. Let  $h_K$  be the diameter of  $K$  and  $h = \max_{K \in \mathcal{T}_h} h_K$ . For any element  $K$ , let  $K_i = K \cap \Omega_i$  denote the part of  $K$  in  $\Omega_i$ . By  $G_h := \{K \in \mathcal{T}_h : K \cap \Gamma \neq \emptyset\}$  we denote the set of elements that are intersected by the interface. For an element  $K \in G_h$ , let  $\Gamma_K := \Gamma \cap K$  be the part of  $\Gamma$  intersecting  $K$ .

We make the following assumptions regarding the mesh and the interface (from [6]).

- A1: We assume that the triangulation is non-degenerate, i.e. there exists  $C > 0$  such that

$$h_K / \rho_K \leq C \quad \forall K \in \mathcal{T}_h,$$

where  $h_K$  is the diameter of  $K$  and  $\rho_K$  is the diameter of the largest ball contained in  $K$ .

- A2: We assume that  $\Gamma$  intersects the boundary  $\partial K$  of an element  $K$  in  $G_h$  exactly twice and each (open) edge at most once.
- A3: Let  $\Gamma_{K,h}$  be the straight line segment connecting the points of intersection between  $\Gamma$  and  $\partial K$ . We assume that  $\Gamma_K$  is a function of length on  $\Gamma_{K,h}$ ; in local coordinates:

$$\Gamma_{K,h} = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = 0\}$$

and

$$\Gamma_K = \{(\xi, \eta) : 0 < \xi < |\Gamma_{K,h}|, \eta = \delta(\xi)\}.$$

Since the curvature of  $\Gamma$  is bounded the assumptions A2 and A3 are always fulfilled on sufficiently fine meshes. These assumptions essentially demand that the interface is well resolved by the mesh.

To obtain the finite element formulation corresponding to (5) we replace the test function  $\mathbf{v}$  by  $\tilde{\mathbf{v}}$  such that  $\tilde{\mathbf{v}}|_{\Omega_i} = \mathbf{v}_i$  with  $\mathbf{v}_i \in [H^1(\Omega_i)]^2$  and  $\mathbf{v}_i = 0$  on the boundary  $\partial\Omega$  thus allowing for a discontinuity over the interface  $\Gamma$ .

For the average across  $\Gamma$  of any sufficiently smooth function  $x$  discontinuous across  $\Gamma$  we will use the notation  $\{x\} = \kappa_1 x_1 + \kappa_2 x_2$  and  $\langle x \rangle = \kappa_2 x_1 + \kappa_1 x_2$  with  $\kappa_i|_{\Gamma_K} = |K_i|/|K|$ . Note that  $\kappa_1 + \kappa_2 = 1$ . We have

$$x_1 = \{x\} + \kappa_2 \llbracket x \rrbracket \quad \text{and} \quad x_2 = \{x\} - \kappa_1 \llbracket x \rrbracket, \quad (6)$$

and therefore

$$[ab] = \{a\}[b] + [a]\langle b \rangle. \quad (7)$$

We note that since  $\nabla p - \nabla \cdot (2\mu\epsilon(\mathbf{u})) \in L^2(\Omega)$  we have  $(p\mathbf{n} - 2\mu\epsilon(\mathbf{u}) \cdot \mathbf{n}) \in H^{-\frac{1}{2}}(\Gamma)$ . Integrating by parts and using (7) we now obtain for the interface term

$$\begin{aligned} \int_{\Gamma} [(p\mathbf{n} - 2\mu\epsilon(\mathbf{u}) \cdot \mathbf{n}) \tilde{\mathbf{v}}] ds &= \int_{\Gamma} [p\mathbf{n} - 2\mu\epsilon(\mathbf{u}) \cdot \mathbf{n}] [\tilde{\mathbf{v}}] ds \\ &\quad + \int_{\Gamma} [p\mathbf{n} - 2\mu\epsilon(\mathbf{u}) \cdot \mathbf{n}] \langle \tilde{\mathbf{v}} \rangle ds. \end{aligned} \quad (8)$$

Since  $\mathbf{v}_i \in [H^1(\Omega_i)]^2$  the quantities in (8) are well defined if the interface integrals are interpreted as duality pairings  $H^{-\frac{1}{2}}(\Gamma)/H^{\frac{1}{2}}(\Gamma)$ .

Using now the interface conditions (3) and (4), we have

$$\int_{\Gamma} [p\mathbf{n} - 2\mu\epsilon(\mathbf{u}) \cdot \mathbf{n}] \langle \tilde{\mathbf{v}} \rangle ds = - \int_{\Gamma} \sigma \kappa \langle \tilde{\mathbf{v}} \cdot \mathbf{n} \rangle ds. \quad (9)$$

After these preliminary considerations we are now ready to propose a finite element discretization of the problem, with this aim we introduce two conforming triangulations  $\mathcal{T}_1$  and  $\mathcal{T}_2$  such that the union of  $\mathcal{T}_1$  and  $\mathcal{T}_2$  gives a conforming triangulation of all  $\Omega$  and for every triangle  $K \in \mathcal{T}_1 \cap \mathcal{T}_2$  we have  $K \cap \Gamma \neq \emptyset$ . We also denote the set of interior faces of  $\mathcal{T}_i$  by  $\mathcal{F}_i$ . Associated with  $\mathcal{T}_1$  and  $\mathcal{T}_2$  we have the finite element spaces

$$\begin{aligned} V_i^h &= \{ \mathbf{v} \in [C^0(\Omega_i)]^2 : \mathbf{v}|_K \in [P^1(K)]^2, \forall K \in \mathcal{T}_i, \mathbf{v}|_{\partial\Omega} = 0 \}, \\ Q_i^h &= \{ p \in L^2_0(\Omega_i) : p|_K \in P^0(K), \forall K \in \mathcal{T}_i \}. \end{aligned}$$

Moreover we denote  $V_h = V_1^h \times V_2^h$  and  $Q_h = Q_1^h \times Q_2^h$ . The finite element discretization now takes the form: find  $(\mathbf{u}_{i,h}, p_{i,h}) \in V_i^h \times Q_i^h$ ,  $i = 1, 2$  such that  $\mathbf{u}_h = (\mathbf{u}_{1,h}, \mathbf{u}_{2,h})$ ,  $p_h = (p_{1,h}, p_{2,h})$  satisfy

$$B_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] + J(p_h, q_h) = (\mathbf{f}, \mathbf{v}_h)_{\Omega} + \int_{\Gamma} \sigma \kappa \langle \mathbf{v}_h \cdot \mathbf{n} \rangle ds, \quad (10)$$

for all  $(\mathbf{v}_h, q_h) \in V_h \times Q_h$ , where

$$\begin{aligned} B_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] &= a_h(\mathbf{u}_h, \mathbf{v}_h) + b_h(p_h, \mathbf{v}_h) - b_h(q_h, \mathbf{u}_h), \\ a_h(\mathbf{u}_h, \mathbf{v}_h) &= \int_{\Omega} 2\mu\epsilon(\mathbf{u}_h) : \epsilon(\mathbf{v}_h) dx - \int_{\Gamma} \{2\mu\epsilon(\mathbf{u}_h) \cdot \mathbf{n}\} [\mathbf{v}_h] ds \\ &\quad - \int_{\Gamma} \{2\mu\epsilon(\mathbf{v}_h) \cdot \mathbf{n}\} [\mathbf{u}_h] ds + \sum_{K \in \mathcal{G}_h} \int_K \gamma_u \mu_{\max} h_K^{-1} [\mathbf{u}_h] [\mathbf{v}_h] ds, \\ b_h(p_h, \mathbf{v}_h) &= - \int_{\Omega} p_h \nabla \cdot \mathbf{v}_h dx + \int_{\Gamma} \{p_h\} [\mathbf{v}_h \cdot \mathbf{n}] ds, \end{aligned}$$

and

$$J(p_h, q_h) = \sum_{F \in \mathcal{F}_1} \int_F \frac{\gamma_1}{\mu_1} h_F [p_{1,h}] [q_{1,h}] ds + \sum_{F \in \mathcal{F}_2} \int_F \frac{\gamma_2}{\mu_2} h_F [p_{2,h}] [q_{2,h}] ds$$

where  $h_F$  is the mean of the meshsizes of the elements sharing an element face  $F$ ,  $[p]$  denotes the jump of  $p$  across  $F$ , and  $(\gamma_u, \gamma_1, \gamma_2)$  are constants, the size of which will be discussed in the following.

**Remark 1.** Note that the displacements and the pressures are double valued in the overlap region. We illustrate this in a one dimensional case in Fig. 1. The pressure on either side is stabilized over the whole interior face in the overlapping domain, i.e., both over the part interior to  $\Omega_i$  and the part exterior to  $\Omega_i$ .

The formulation (10) satisfies the following consistency relation

**Lemma 2** (Galerkin orthogonality). *Let  $(\mathbf{u}_h, p_h)$  be the solution of the finite element formulation (10) and  $(\mathbf{u}, p) \in [H^2(\Omega_1) \cup H^2(\Omega_2)]^2 \times H^1(\Omega_1) \cup H^1(\Omega_2)$  be the solution of (5). Then*

$$B_h[(\mathbf{u}_h - \mathbf{u}, p_h - p), (\mathbf{v}_h, q_h)] = -J(p_h, q_h) \quad \forall (\mathbf{v}_h, q_h) \in V_h \times Q_h. \quad (11)$$

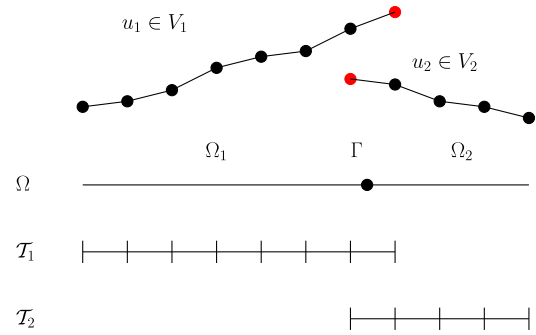


Fig. 1. Illustration of the overlapping displacement spaces in a one-dimensional model case.

**Proof.** Using the formulation (10) we may write

$$a_h(\mathbf{u}, \mathbf{v}_h) + b_h(p, \mathbf{v}_h) - b_h(q_h, \mathbf{u}) = (\mathbf{f}, \mathbf{v}_h) + \int_{\Gamma} \sigma \kappa \langle \mathbf{v}_h \cdot \mathbf{n} \rangle ds. \quad (12)$$

The claim follows by noting that  $b_h(q_h, \mathbf{u}) = 0$  and by integrating by parts in the remaining two terms on the left hand side and using equality (9) and (1).  $\square$

A FE basis for  $V_h$  is easily obtained from a standard FE basis on the mesh by the introduction of new basis functions for the elements that intersect  $\Gamma$ . For piecewise linears, the standard interior nodal basis functions in  $\Omega$  may be partitioned into the two sets  $\{\psi_i^j\}_{j=1}^{N_i}$  of basis functions with support entirely in  $\Omega_i$ ,  $i = 1, 2$ , and the set  $\{\psi_r^k\}_{k=1}^M$  of basis functions which are non-zero on  $\Gamma$ . For each of the latter, let  $\psi_{r,i}^j := \psi_r^j|_{\Omega_i}$ . Then  $\{\psi_i^j\}_{j=1}^{N_i} \cup \{\psi_{r,i}^j\}_{k=1}^M$  is a basis for  $V_i^h$ . As a consequence, the number of basis functions on each element that intersects  $\Gamma$  is doubled.

#### 4. Approximation properties

The solution of (1) loses regularity at the interface. Indeed the pressure and normal component of the deformation tensor may jump. We introduce the broken Sobolev spaces

$$H^k(\Omega_1 \cup \Omega_2) := \{ v : v|_{\Omega_i} \in H^k(\Omega_i), i = 1, 2 \}$$

with the corresponding norms

$$\|u\|_{k, \Omega_1 \cup \Omega_2}^2 := \sum_{i=1}^2 \|u\|_{k, \Omega_i}^2.$$

We need to show that our approximating spaces  $V_h$  and  $Q_h$  have optimal approximation properties on  $H^2(\Omega_1 \cup \Omega_2)$  and  $H^1(\Omega_1 \cup \Omega_2)$ , respectively. This follows from some minor modifications of the analysis in [6]. Let  $\mathcal{G}_h$  denote the set of elements that are intersected by the interface. First define the following norms on the trace of a function on  $\Gamma$ .

$$\begin{aligned} \|\mathbf{v}\|_{1/2, h, \Gamma}^2 &:= \sum_{K \in \mathcal{G}_h} h_K^{-1} \|\mathbf{v}\|_{0, \Gamma_K}^2, \\ \|\mathbf{v}\|_{-1/2, h, \Gamma}^2 &:= \sum_{K \in \mathcal{G}_h} h_K^1 \|\mathbf{v}\|_{0, \Gamma_K}^2. \end{aligned}$$

We note for future reference that

$$(\mathbf{u}, \mathbf{v})_{\Gamma} \leq \|\mathbf{u}\|_{1/2, h, \Gamma} \|\mathbf{v}\|_{-1/2, h, \Gamma}. \quad (13)$$

For the analysis we will use the following mesh dependent norms defined for functions in  $[H^2(\Omega_1 \cup \Omega_2)]^2$  or  $[H^2(\Omega_1 \cup \Omega_2)]^2 \times H^1(\Omega)$ :

$$\begin{aligned} \|\mathbf{v}\|^2 &:= \|\mathbf{v}\|_{0, \Omega}^2 + \|\mu^{1/2} \nabla \mathbf{v}\|_{0, \Omega_1 \cup \Omega_2}^2 + \|\{2\mu\epsilon(\mathbf{v}) \cdot \mathbf{n}\}\|_{-1/2, h, \Gamma}^2 \\ &\quad + \|[\![\mathbf{v}]\!]\|_{1/2, h, \Gamma}^2 \end{aligned}$$

and

$$|||(\mathbf{u}, p)|||^2 := |||\mathbf{u}|||^2 + \frac{1}{\mu_{\max}} \|p\|_{0,\Omega}^2$$

and on  $V_h \times Q_h$  the norm

$$|||(\mathbf{u}_h, p_h)|||^2 := |||\mathbf{u}_h|||^2 + \frac{1}{\mu_{\max}} \|p_h\|_{0,\Omega}^2 + J(p_h, p_h).$$

To show that functions in the space  $(V_h, Q_h)$  approximate functions  $\mathbf{v} \in [H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)]^2$  and  $q \in L_0^2(\Omega) \cap H^1(\Omega_1 \cup \Omega_2)$  to the order  $h$  in the norm  $|||\cdot|||$  and  $\|\cdot\|_{0,\Omega}$ , respectively, we construct an interpolant of  $(\mathbf{v}, q)$  by standard interpolants of  $[H^2]^2 \times H^1$ -extensions of  $(\mathbf{v}_1, q_1)$  and  $(\mathbf{v}_2, q_2)$  as follows. Choose extension operators  $E_i^k : H^k(\Omega_i) \rightarrow H^k(\Omega)$  such that  $(E_i^2 \mathbf{v}, E_i^1 q)|_{\Omega_i} = (\mathbf{v}, q)$  and

$$\|E_i^k \mathbf{w}\|_{s,\Omega} \leq C \|\mathbf{w}\|_{s,\Omega_i}, \quad \forall \mathbf{w} \in H^s(\Omega_i), \quad s = 0, \dots, k, \quad (14)$$

cf. Dautray and Lions [5]. Let  $I_h$  be the standard interpolant and  $C_h$  be the local projection on the piecewise constants. We define

$$(I_h^* \mathbf{v}, C_h^* q) := ((I_{h,1}^* \mathbf{v}_1, I_{h,2}^* \mathbf{v}_2), (C_{h,1}^* q_1, C_{h,2}^* q_2)), \quad (15)$$

where  $I_{h,i}^* \mathbf{v}_i := (I_h E_i^2 \mathbf{v}_i)|_{\Omega_i}$  and  $C_{h,i}^* q_i := (C_h E_i^1 q_i)|_{\Omega_i}$ . The following theorem is valid.

**Theorem 3.** Let  $(I_h^*, C_h^*)$  be a pair of interpolation operators defined as in (15). Then

$$|||\mathbf{v} - I_h^* \mathbf{v}||| \leq Ch \|\mathbf{v}\|_{2,\Omega_1 \cup \Omega_2}$$

and

$$\|q - C_h^* q\|_{0,\Omega} \leq Ch \|q\|_{1,\Omega_1 \cup \Omega_2}$$

for all  $\mathbf{v} \in [H_0^1(\Omega) \cap H^2(\Omega_1 \cup \Omega_2)]^2$  and  $p \in L_0^2(\Omega) \cap H^1(\Omega_1 \cup \Omega_2)$ .

For the proof of this we need the following variant of a trace inequality on a reference element that we recall from [6] and state here without proof.

**Lemma 4.** Map a triangle  $K \in G_h$  onto the unit reference triangle  $\tilde{K}$  by an affine map and denote by  $\tilde{\Gamma}_K$  the corresponding image of  $\Gamma_K$ . Under the assumptions A1–A3 of Section 3 there exists a constant  $C$ , depending on  $\Gamma$  but independent of the mesh, such that

$$\|\mathbf{w}\|_{0,\tilde{\Gamma}_K}^2 \leq C \|\mathbf{w}\|_{0,\tilde{K}}^2 \|\mathbf{w}\|_{1,\tilde{K}}, \quad \forall \mathbf{w} \in H^1(\tilde{K}). \quad (16)$$

**Proof (Theorem 3).** Recall that  $K_i = K \cap \Omega_i$  and let  $\mathbf{v}_i^* = E_i^2 \mathbf{v}_i$  denote the extension of  $\mathbf{v}_i$  to  $\Omega$  and similarly  $q_i^* = E_i^1 q_i$  denote the extension of  $q_i$  to  $\Omega$ . By standard interpolation estimates for  $I_h$  and  $C_h$  respectively we now obtain

$$\begin{aligned} \|\nabla(\mathbf{v}_i - I_{h,i}^* \mathbf{v}_i)\|_{0,K_i}^2 &= \|\nabla(\mathbf{v}_i^* - I_{h,i}^* \mathbf{v}_i^*)\|_{0,K_i}^2 \leq \|\nabla(\mathbf{v}_i^* - I_{h,i}^* \mathbf{v}_i^*)\|_{0,K}^2 \\ &\leq Ch_K^2 \|\mathbf{v}_i^*\|_{2,K}^2 \end{aligned}$$

and in the same fashion

$$\|q_i - C_{h,i}^* q_i\|_{0,K_i}^2 \leq Ch_K^2 \|q_i^*\|_{1,K}^2.$$

Summing over all triangles we obtain using (14)

$$\|\nabla(\mathbf{v}_i - I_{h,i}^* \mathbf{v}_i)\|_{0,\Omega_i}^2 \leq Ch^2 \sum_{K \cap \Omega_i \neq \emptyset} \|\mathbf{v}_i^*\|_{2,K}^2 \leq Ch^2 \|\mathbf{v}_i\|_{2,\Omega_i}^2 \quad (17)$$

and

$$\|q_i - C_{h,i}^* q_i\|_{0,\Omega_i}^2 \leq Ch^2 \|q_i\|_{1,\Omega_i}^2. \quad (18)$$

We turn now to the jumps on the interface. Since the mesh is non-degenerate, it follows from Lemma 4, scaled by the map from the reference triangle, that for  $s \in \mathbb{R}$

$$h_K^{-s} \|\mathbf{w}\|_{0,\Gamma_K}^2 \leq C \left( h_K^{-1-s} \|\mathbf{w}\|_{0,K}^2 + h_K^{1-s} \|\mathbf{w}\|_{1,K}^2 \right), \quad \forall \mathbf{w} \in H^1(K).$$

Hence it follows, using once again standard interpolation estimates, that

$$\begin{aligned} h_K^{-1} \|\llbracket \mathbf{v} - I_h^* \mathbf{v} \rrbracket\|_{0,\Gamma_K}^2 &\leq Ch_K^{-1} \sum_i \|\mathbf{v}_i - I_{h,i}^* \mathbf{v}_i\|_{0,\Gamma_K}^2 \\ &= Ch_K^{-1} \sum_i \|\mathbf{v}_i^* - I_h \mathbf{v}_i^*\|_{0,\Gamma_K}^2 \\ &\leq C \sum_i (h_K^{-2} \|\mathbf{v}_i^* - I_h \mathbf{v}_i^*\|_{0,K}^2 + \|\mathbf{v}_i^* - I_h \mathbf{v}_i^*\|_{1,K}^2) \\ &\leq Ch_K^2 \sum_i \|\mathbf{v}_i^*\|_{2,K}^2 \end{aligned}$$

and summing the contributions from  $K \in G_h$ , we get from (14) that

$$\|\llbracket \mathbf{v} - I_h^* \mathbf{v} \rrbracket\|_{1/2,h,\Gamma} \leq Ch \sum_i \|\mathbf{v}_i^*\|_{2,K} \leq Ch \|\mathbf{v}\|_{2,\Omega_1 \cup \Omega_2} \quad (19)$$

and

$$\|\llbracket q - C_h^* q \rrbracket\|_{-1/2,h,\Gamma} \leq Ch \sum_i \|q_i^*\|_{1,K} \leq Ch \|q\|_{1,\Omega_1 \cup \Omega_2}. \quad (20)$$

Next, we consider the mean viscous stress at the interface. Lemma 4 applied to  $\mathbf{n} \cdot \nabla \mathbf{w}$  and scaling gives

$$h_K \|\mathbf{n} \cdot \nabla \mathbf{w}\|_{0,\Gamma_K}^2 \leq C (\|\mathbf{w}\|_{1,K}^2 + h_K^2 \|\mathbf{w}\|_{2,K}^2), \quad \forall \mathbf{w} \in [H^2(K)]^2.$$

Using this result applied to  $\mathbf{w} = \mathbf{v}_i^* - I_h \mathbf{v}_i^*$  and again standard interpolation estimates, it follows that

$$\begin{aligned} h_K \|\{\mathbf{n} \cdot \nabla(\mathbf{v} - I_h^* \mathbf{v})\}\|_{0,\Gamma_K}^2 &\leq Ch_K \sum_i \|\nabla(\mathbf{v}_i - I_{h,i}^* \mathbf{v}_i) \cdot \mathbf{n}\|_{0,\Gamma_K}^2 \\ &= Ch_K \sum_i \|\nabla(\mathbf{v}_i^* - I_h \mathbf{v}_i^*) \cdot \mathbf{n}\|_{0,\Gamma_K}^2 \\ &\leq C \sum_i (h_K \|\mathbf{v}_i^* - I_h \mathbf{v}_i^*\|_{1,K}^2 + h_K^2 \|\mathbf{v}_i^* - I_h \mathbf{v}_i^*\|_{2,K}^2) \\ &\leq Ch_K^2 \sum_i \|\mathbf{v}_i^*\|_{2,K}^2. \end{aligned}$$

Summing again the contributions from  $K \in G_h$ , we deduce from (14) that

$$\|\{\mathbf{n} \cdot \nabla(\mathbf{v} - I_h^* \mathbf{v})\}\|_{-1/2,h,\Gamma} \leq Ch \|\mathbf{v}\|_{2,\Omega_i}. \quad (21)$$

The claim now follows from Eqs. (17), (18), (19) and (21).  $\square$

**Lemma 5.** Let  $q \in H^1(\Omega_1 \cup \Omega_2)$ . Then there holds

$$J(C_h^* q^*, C_h^* q^*) \leq Ch^2 \|q\|_{H^1(\Omega_1 \cup \Omega_2)}^2.$$

**Proof.** This is easily proved using the standard trace inequality in the following fashion

$$h_K \| [q_i^* - C_{h,i}^* q_i^*] \|_{0,F}^2 \leq C (\|q_i^* - C_{h,i}^* q_i^*\|_{0,K}^2 + h_K^2 \|q_i^*\|_{1,K}^2) \leq Ch_K^2 \sum \|q_i^*\|_{1,K}^2.$$

This leads to

$$\begin{aligned} J(C_h^* q^*, C_h^* q^*) &\leq \sum_{K \in \mathcal{T}_1} Ch_K^2 \sum \|q_i^*\|_{1,K}^2 + \sum_{K \in \mathcal{T}_2} Ch_K^2 \sum \|q_2^*\|_{1,K}^2 \\ &\leq Ch^2 (\|q\|_{1,\Omega_1}^2 + \|q\|_{1,\Omega_2}^2). \quad \square \end{aligned}$$

## 5. The inf-sup condition

We need to show that the formulation satisfies an inf-sup condition with regard to the norm  $|||(\cdot, \cdot)|||_h$  in the case  $\lambda_i \rightarrow \infty$ . We remark that if  $\lambda_i$  remain bounded from above, the pressure does not need stabilization and can be eliminated elementwise, recovering

the constant-strain finite element method. In this case there is no stability problem in the pressure, and the results of [7] can be directly applied in the compressible subdomain. We shall therefore in the following concentrate on the case when both materials are assumed incompressible.

Observe that the pressure stabilization in  $J(\cdot, \cdot)$  acts over the whole interior faces in  $G_h$ , i.e. not only the part of the faces in the computational domain  $\Omega_i$ . This is of fundamental importance for the below analysis, indeed this “ghost penalty” is what gives us stability of the interface pressure.

Here we prove the inf-sup condition only in the particular case of the mixed stabilized  $P_1 \times P_0$  formulation for convenience. However, the general framework holds also when using for instance the pressure stabilisations proposed in [4] or [1] and we expect the inf-sup condition to hold uniformly in these cases as well, provided the stabilization operator acts in all elements of  $\mathcal{T}_1$  and  $\mathcal{T}_2$ .

We need the following inverse inequality, for a proof of which we refer to [6].

**Lemma 6.** For  $\mathbf{v} \in V_h$ , the following inverse inequality holds:

$$\|\{\mathbf{n} \cdot \nabla \mathbf{v}\}\|_{-1/2, h, \Gamma}^2 \leq C_I \|\nabla \mathbf{v}\|_{0, \Omega_1 \cup \Omega_2}^2.$$

### 5.1. Fortin interpolant

The key result for the inf-sup condition is the construction of the Fortin interpolant. The upshot in the case of Nitsche XFEM for Stokes' problem is the need to construct a Fortin interpolant with some additional orthogonality on the interface. The interpolant defined on mesh  $\mathcal{T}_i$  will be denoted by  $\pi_h^i$  and the global interpolant by  $\pi_h$ . Note that the global interpolant is not continuous across the interface. Let us consider the construction of the Fortin interpolant on the space  $V_1^h$  the construction on  $V_2^h$  then follows by symmetry. Let  $\mathcal{T}_\Gamma$  be the subset of triangles of  $\mathcal{T}_1$  such that

$$\mathcal{T}_\Gamma = \{K \in \mathcal{T}_1 : K \cap K' \neq \emptyset, K' \in \mathcal{T}_1 \cap \mathcal{T}_2\}.$$

That is all the elements cut by the interface and the elements sharing a face or a vertex with them. This is a subset of the mesh of width approximately  $2h$ . Let us now regroup the elements in  $\mathcal{T}_\Gamma$  in patches  $\{\mathcal{P}_j\}$ , cutting up the interface zone in macroelements, each containing a “sufficient” amount of basis functions to construct a patch function  $0 \leq \varphi_j \leq 1$  associated to each  $\mathcal{P}_j$  that is zero on the interior patch boundary (the boundary faces of  $\mathcal{P}_j$  with non-zero intersection with  $\Omega_i$ ) and takes the value 1 on at least one face cut by the interface. The  $\mathcal{P}_j$  can be constructed so that

- $\exists c_1, c_2 > 0$  such that  $c_1 h \leq \text{diam}(\mathcal{P}_j) \leq c_2 h$ .
- $\exists c_1, c_2 > 0$   $c_1 h \leq \int_{\Gamma \cap \mathcal{P}_j} \varphi_j \, ds \leq c_2 h$ .
- $\exists c_1, c_2 > 0$   $c_1 h^{-1} \leq |\nabla \varphi_j| \leq c_2 h^{-1}$ .

The boundary zone  $\mathcal{T}_\Gamma$ , with a patch  $\mathcal{P}_j$  is illustrated in Fig. 2. We want to prove that we can construct an interpolant  $\pi_h : [H^1(\Omega)]^2 \rightarrow V^h$  such that

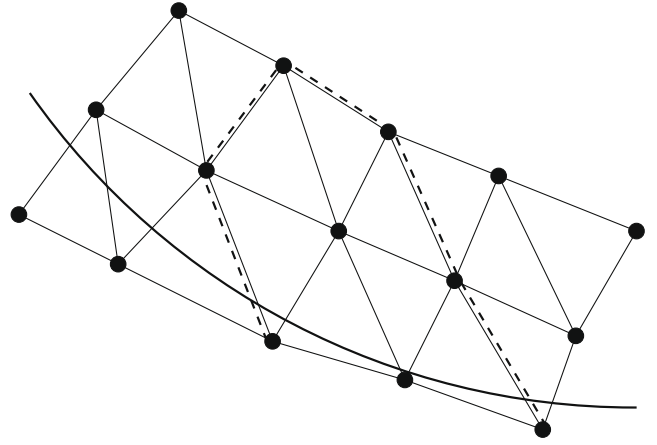
$$\|\mathbf{v} - \pi_h \mathbf{v}\|_{0, \Omega} + h \|\nabla(\mathbf{v} - \pi_h \mathbf{v})\|_{0, \Omega_1 \cup \Omega_2} \leq ch \|\nabla \mathbf{v}\|_{0, \Omega}$$

and

$$\int_{\Gamma \cap \mathcal{P}_j} (\mathbf{v} - \pi_h^k \mathbf{v}) \, ds = 0, \quad \text{for } k = 1, 2 \quad (22)$$

for all  $\mathcal{P}_j$ .

The idea is to start from the global Clément interpolant of  $\mathbf{v}$  and then construct a correction using the space  $\Phi = \{\varphi_j\}$  so that the condition (22) is satisfied. Let  $I_h \mathbf{v} \in [C^0(\bar{\Omega})]^2$  denote the global Clément interpolant of  $\mathbf{v}$ . In each patch  $\mathcal{P}_j$  there is exactly one (vector



**Fig. 2.** The interface zone  $\mathcal{T}_\Gamma$ , the thick line corresponds to the interface  $\Gamma$  and the dotted thick line outlines a patch  $\mathcal{P}_j$ .

valued) degree of freedom  $\mathbf{y}_j$  associated to the basis function  $\varphi_j$ . We now fix the  $\mathbf{y}_j$  by requiring

$$\int_{\Gamma \cap \mathcal{P}_j} \varphi_j \mathbf{y}_j \, ds = \int_{\Gamma \cap \mathcal{P}_j} (\mathbf{v} - I_h \mathbf{v}) \, ds, \quad (23)$$

where

$$\varphi_j = \begin{bmatrix} \varphi_j & 0 \\ 0 & \varphi_j \end{bmatrix}.$$

Finally we set  $\pi_h^1 \mathbf{v} = I_h \mathbf{v}|_{\mathcal{T}_1} + \sum_j \varphi_j \mathbf{y}_j$ . Clearly, by construction (22) holds. The approximability is a consequence of the stability of the projection. Since  $\pi_h^1 \mathbf{v} = I_h \mathbf{v}$ , except in the interface zone, we only need to prove stability for the interpolant in  $\mathcal{T}_\Gamma$ . This is where the position of the interface could upset things. The fact that we have chosen the patches to include the neighbours seems to work around this problem. Consider, for some  $K \in \mathcal{P}_j$

$$\begin{aligned} \|\nabla \pi_h^1 \mathbf{v}\|_{0, K \cap \Omega_1} &\leq \|\nabla I_h \mathbf{v}\|_{0, K \cap \Omega_1, 0} + \|\nabla(\varphi_j \mathbf{y}_j)\|_{K \cap \Omega_1} \\ &\leq c \|\nabla \mathbf{v}\|_{0, K} + ch^{-1} |\mathbf{y}_j| \text{meas}(K)^{\frac{1}{2}} \leq c(\|\nabla \mathbf{v}\|_{0, K} + |\mathbf{y}_j|). \end{aligned}$$

It remains to show that  $|\mathbf{y}_j|$  is bounded by the norm of  $\nabla \mathbf{v}$  over some patch of diameter  $h$ . Solving for  $\mathbf{y}_j$  in (23) we have

$$\mathbf{y}_j = \frac{\int_{\Gamma \cap \mathcal{P}_j} (\mathbf{v} - I_h \mathbf{v}) \, ds}{\int_{\Gamma \cap \mathcal{P}_j} \varphi_j \, ds}$$

and thus

$$\begin{aligned} |\mathbf{y}_j| &\leq \left| \frac{\int_{\Gamma \cap \mathcal{P}_j} (\mathbf{v} - I_h \mathbf{v}) \, ds}{\int_{\Gamma \cap \mathcal{P}_j} \varphi_j \, ds} \right| \leq ch^{-1} \left| \int_{\Gamma \cap \mathcal{P}_j} (\mathbf{v} - I_h \mathbf{v}) \, ds \right| \\ &\leq ch^{-\frac{1}{2}} \|\mathbf{v} - I_h \mathbf{v}\|_{0, \Gamma \cap \mathcal{P}_j} \leq c(h^{-1} \|\mathbf{v} - I_h \mathbf{v}\|_{0, \mathcal{P}_j} + \|\nabla(\mathbf{v} - I_h \mathbf{v})\|_{0, \mathcal{P}_j}) \\ &\leq c \|\nabla \mathbf{v}\|_{0, \Delta_{\mathcal{P}_j}}, \end{aligned}$$

where  $\Delta_{\mathcal{P}_j}$  denotes the patch consisting of the elements in  $\mathcal{P}_j$  and those sharing a node with an element in  $\mathcal{P}_j$ .

The approximability in  $L^2$  is now easily obtained, noting that for  $K \in \mathcal{P}_j$

$$\|\mathbf{v} - \pi_h^1 \mathbf{v}\|_{0, K} \leq \|\mathbf{v} - I_h \mathbf{v}\|_{0, K} + \|\varphi_j \mathbf{y}_j\|_{0, K} \leq \|\mathbf{v} - I_h \mathbf{v}\|_{0, K} + ch_K |\mathbf{y}_j|$$

and applying the previous result. Using arguments similar to that of Section 4 one may readily prove the stability estimate

$$\|\pi_h \mathbf{v}\| \leq c \|\mathbf{v}\|_{1, \Omega}, \quad \forall \mathbf{v} \in [H^1(\Omega)]^2. \quad (24)$$



We will now use the proposed interpolant to prove the inf-sup condition for our formulation. Let us start with a preliminary lemma

**Lemma 7.** Let  $\mathbf{v} \in [H^1(\Omega)]^2$ . Then there holds, for  $i = 1, 2$ ,

$$|(\llbracket p_h \rrbracket, \mathbf{n} \cdot (\mathbf{v} - \pi_h^i \mathbf{v}))_T| \leq C J(p_h, p_h)^{\frac{1}{2}} \|\mu_{\max}^{\frac{1}{2}} \nabla \mathbf{v}\|_{0,\Omega}.$$

**Proof.** Using the orthogonality on the patches  $\mathcal{P}_k$ , we may write, with  $\xi_k \in \mathbb{R}$ ,

$$\begin{aligned} |(\llbracket p_h \rrbracket, \mathbf{n} \cdot (\mathbf{v} - \pi_h^i \mathbf{v}))_T| &\leq \sum_k |(\llbracket p_h \rrbracket, \mathbf{n} \cdot (\mathbf{v} - \pi_h^i \mathbf{v}))_{\mathcal{P}_k \cap T}| \\ &\leq \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap T} \| \mathbf{v} - \pi_h^i \mathbf{v} \|_{0,\mathcal{P}_k \cap T} \\ &\leq C \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap T} h^{\frac{1}{2}} \| \nabla \mathbf{v} \|_{0,\mathcal{P}_k} \\ &\leq C \left( h \sum_k \| \llbracket p_h \rrbracket - \xi_k \|_{0,\mathcal{P}_k \cap T}^2 \right)^{\frac{1}{2}} \| \nabla \mathbf{v} \|_{0,\Omega}. \end{aligned}$$

We now choose  $\xi_k$  as the arithmetic mean of all the jumps in the patch  $\mathcal{P}_k$ ,

$$\xi_k = \frac{1}{N_k} \sum_{K \in \mathcal{P}_k: K \cap \Gamma \neq \emptyset} \llbracket p_h \rrbracket_{K \cap \Gamma},$$

where  $N_k$  denotes the number of triangles in  $\mathcal{P}_k \cap \mathcal{T}_T$ . It then follows, since  $N_k$  is bounded from above by a moderate constant, uniformly in  $h$  and  $k$  and using the relation  $(a - b)^2 \leq 2a^2 + 2b^2$ , that for each  $K \in \mathcal{P}_k$  such that  $K \cap \Gamma \neq \emptyset$

$$\begin{aligned} \| \llbracket p_h \rrbracket - \xi_k \|_{K \cap \Gamma}^2 &\leq 2 \| N_k^{-1} \sum_{K' \in \mathcal{K}'} (p_{1,h}|_K - p_{1,h}|_{K'}) \|_{0,K \cap \Gamma}^2 \\ &\quad + 2 \| N_k^{-1} \sum_{K' \in \mathcal{K}'} (p_{2,h}|_K - p_{2,h}|_{K'}) \|_{0,K \cap \Gamma}^2 \\ &\leq C \sum_{F \in \mathcal{E}_k} \left( \int_F [p_{1,h}]^2 ds + \int_F [p_{2,h}]^2 ds \right) \end{aligned}$$

where  $\mathcal{K}' := \{K' \in \mathcal{P}_j : K' \cap \Gamma \neq \emptyset; K' \neq K\}$ ,  $\mathcal{E}_k := \{F \in \mathcal{P}_k : F \cap \Gamma \neq \emptyset\}$  and where the constant  $C$  depends only on the  $N_k$ . We conclude by summing over all the patches  $\mathcal{P}_j$ .  $\square$

We will now prove the main result of this section:

**Theorem 8.** Let  $(\mathbf{u}_h, p_h) \in V_h \times Q_h$ . Then

$$c_s \| |(\mathbf{u}_h, p_h)| | \|_h \leq \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, q_h)] + J(p_h, q_h)}{\| |(\mathbf{v}_h, q_h)| \|_h}. \quad (25)$$

**Proof.** We start by choosing  $(\mathbf{v}_h, q_h) = (\mathbf{u}_h, p_h)$  to obtain

$$B_h[(\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)] + J(p_h, p_h) = a_h(\mathbf{u}_h, \mathbf{u}_h) + J(p_h, p_h).$$

We will need a version of Korn's inequality for functions in  $V_h$ . From [3] (Eq. (1.19)), using  $V_h \subset [H^1(\Omega_1 \cup \Omega_2)]^d$  and  $\mathbf{v}|_{\partial\Omega} = 0$  for  $\mathbf{v} \in V_h$ , we have:

$$\| \boldsymbol{\varepsilon}(\mathbf{v}) \|_{0,\Omega_1 \cup \Omega_2}^2 + \| [\mathbf{v}] \|_{1/2,h,\Gamma}^2 \geq C \| \mathbf{v} \|_{1,\Omega_1 \cup \Omega_2}^2 \quad \forall \mathbf{v} \in V_h,$$

while from [2], using  $\mathbf{v}|_{\partial\Omega} = 0$ ,  $\mathbf{v}|_{\Gamma_K} \in L^2(\Gamma_K)$  for  $\mathbf{v} \in V_h$  and a standard error estimate, we have

$$\| \mathbf{v} \|_{1,\Omega_1 \cup \Omega_2}^2 + \| [\mathbf{v}] \|_{1/2,h,\Gamma}^2 \geq C \| \mathbf{v} \|_{0,\Omega}^2 \quad \forall \mathbf{v} \in V_h$$

and thus

$$\| \boldsymbol{\varepsilon}(\mathbf{v}) \|_{0,\Omega_1 \cup \Omega_2}^2 + \| [\mathbf{v}] \|_{1/2,h,\Gamma}^2 \geq C_K \| \mathbf{v} \|_{1,\Omega_1 \cup \Omega_2}^2 \quad \forall \mathbf{v} \in V_h. \quad (26)$$

See also the proof on discrete spaces of [8]. Applying now (26) and (13) and Lemma 6, we get

$$\begin{aligned} a_h(\mathbf{u}_h, \mathbf{u}_h) &\geq \| \mu^{1/2} \boldsymbol{\varepsilon}(\mathbf{u}_h) \|_{0,\Omega_1 \cup \Omega_2}^2 - 2 \| \{ 2\mu \mathbf{n} \cdot \boldsymbol{\varepsilon}(\mathbf{u}_h) \} \|_{-1/2,h,\Gamma} \| [\mathbf{u}_h] \|_{1/2,h,\Gamma} \\ &\quad + \gamma_u \| \mu_{\max}^{1/2} [\mathbf{u}_h] \|_{1/2,h,\Gamma}^2 + \sum_{i=1}^2 \sum_{F \in \mathcal{F}_i} \frac{\gamma_i}{\mu_i} h_K \| [p_{i,h}] \|_{0,F}^2 \\ &\geq \frac{C_K}{2} \mu_{\min} \| \mathbf{u}_h \|_{1,\Omega_1 \cup \Omega_2}^2 + \left( \frac{\mu_{\min}}{2} - \frac{8C_I \mu_{\max}}{\epsilon} \right) \| \boldsymbol{\varepsilon}(\mathbf{u}_h) \|_{0,\Omega_1 \cup \Omega_2}^2 \\ &\quad + \frac{1}{\epsilon \mu_{\max}} \| \{ 2\mu \mathbf{n} \cdot \boldsymbol{\varepsilon}(\mathbf{u}_h) \} \|_{-1/2,h,\Gamma}^2 \\ &\quad + \left( \gamma_u \mu_{\max} - \epsilon \mu_{\max} - \frac{1}{2} \mu_{\max} \right) \| [\mathbf{u}_h] \|_{1/2,h,\Gamma}^2 \\ &\quad + \sum_{i=1}^2 \sum_{F \in \mathcal{F}_i} \frac{\gamma_i}{\mu_i} h_K \| [p_{i,h}] \|_{0,F}^2. \end{aligned} \quad (27)$$

Taking  $\epsilon = 16C_I \frac{\mu_{\max}}{\mu_{\min}}$ ,

$$c \| |(\mathbf{u}_h, p_h)| |^2 \leq B_h[(\mathbf{u}_h, p_h), (\mathbf{u}_h, p_h)]$$

for  $\gamma_u > \epsilon + 1/2$ .

By the surjectivity of the divergence operator from  $[H_0^1(\Omega)]^2$  to  $L_0^2(\Omega)$  there exists  $\mathbf{v}_p \in [H_0^1(\Omega)]^2$  such that  $\mu_{\max}^{-1} p_h = -\nabla \cdot \mathbf{v}_p$  and  $\| \mathbf{v}_p \|_{1,\Omega} \leq C \mu_{\max}^{-1} \| p_h \|_{0,\Omega}$ . Taking now  $(\mathbf{w}_h, q_h) = (\pi_h \mathbf{v}_p, 0)$  where  $\pi_h$  is the interpolation operator introduced in the previous section we obtain

$$B_h[(\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)] = a_h(\mathbf{u}_h, \pi_h \mathbf{v}_p) + b_h(p_h, \pi_h \mathbf{v}_p - \mathbf{v}_p) + \mu_{\max}^{-1} \| p_h \|_{0,\Omega}^2.$$

Using the Cauchy–Schwarz inequality we have

$$\begin{aligned} a_h(\mathbf{u}_h, \pi_h \mathbf{v}_p) &\geq - \| |\mathbf{u}_h| | \| \| |\pi_h \mathbf{v}_p| | \| \geq -C \| |\mathbf{u}_h| | \| \mu_{\max}^{\frac{1}{2}} \| \nabla \mathbf{v}_p \|_{0,\Omega} \\ &\geq -C \| |\mathbf{u}_h| | \| \mu_{\max}^{-\frac{1}{2}} \| p_h \|_{0,\Omega}. \end{aligned}$$

Using integration by parts in the term  $b_h(p_h, \pi_h \mathbf{v}_p - \mathbf{v}_p)$  we have

$$\begin{aligned} b_h(p_h, \pi_h \mathbf{v}_p - \mathbf{v}_p) &= \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_{\partial K \cap \Omega_i} [p_h] \mathbf{n} \cdot (\mathbf{v}_p - \pi_h \mathbf{v}_p) ds \\ &\quad + \sum_i \int_{\mathcal{P}_i \cap \Gamma} (\llbracket p_h \rrbracket \mathbf{n} \cdot (\mathbf{v}_p - \pi_h \mathbf{v}_p)) - \{ p_h \} \llbracket \mathbf{n} \cdot (\mathbf{v}_p - \pi_h \mathbf{v}_p) \rrbracket ds \\ &= \sum_{i=1}^2 \sum_{K \in \mathcal{T}_h} \frac{1}{2} \int_{\partial K \cap \Omega_i} [p_h] \mathbf{n} \cdot (\mathbf{v}_p - \pi_h \mathbf{v}_p) ds \\ &\quad + \sum_i \int_{\mathcal{P}_i \cap \Gamma} \llbracket p_h \rrbracket \mathbf{n} \cdot \langle \mathbf{v}_p - \pi_h \mathbf{v}_p \rangle ds = I + II. \end{aligned}$$

Treating these term separately we first note that by using the approximation properties of  $\pi_h$  and  $H^1$ -stability of  $\mathbf{v}_p$  we have

$$I \geq -C J(p_h, p_h)^{\frac{1}{2}} \| \mu_{\max}^{-\frac{1}{2}} p_h \|_{0,\Omega}.$$

For the second term we use Lemma 7 which gives

$$II \geq - \left| \sum_i \int_{\mathcal{P}_i \cap \Gamma} \llbracket p_h \rrbracket \mathbf{n} \cdot \langle \mathbf{v}_p - \pi_h \mathbf{v}_p \rangle ds \right| \geq -C J(p_h, p_h)^{\frac{1}{2}} \| \mu_{\max}^{-\frac{1}{2}} p_h \|_{0,\Omega}.$$

Note that the different weights here are of no importance since the orthogonality property is satisfied on each side separately and since the weights are upper bounded by 1. We conclude that

$$B_h[(\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)] \geq \| \mu_{\max}^{-\frac{1}{2}} p_h \|_{0,\Omega}^2 - C \left( J(p_h, p_h)^{\frac{1}{2}} + \| |\mathbf{u}_h| | \| \right) \| \mu_{\max}^{-\frac{1}{2}} p_h \|_{0,\Omega}. \quad (28)$$

Using now an arithmetic–geometric inequality we get

$$B_h[(\mathbf{u}_h, p_h), (\pi_h \mathbf{v}_p, 0)] \geq \frac{1}{2} \| \mu_{\max}^{-\frac{1}{2}} p_h \|_{0,\Omega}^2 - C J(p_h, p_h) - C \| |\mathbf{u}_h| |^2 \quad (29)$$

for  $\epsilon < C^{-1}$  there holds

$$C \| |(\mathbf{u}_h, p_h)| |^2 \leq B_h[(\mathbf{u}_h, p_h), (\mathbf{u}_h + \epsilon \pi_h \mathbf{v}_p, p_h)] + J(p_h, p_h).$$

The claim now follows noting that there exists  $c_s$  such that

$$2c_s |||(\mathbf{u}_h + \epsilon \pi_h \mathbf{v}_p, p_h)|||_h \leq |||(\mathbf{u}_h, p_h)|||_h,$$

which is a consequence of the stability of  $\pi_h \mathbf{v}_p$ .  $\square$

**Remark 9.** Note that both the parameter  $\gamma_u$  and the pressure stabilization parameters  $\gamma_1, \gamma_2$  must be larger than certain moderate mesh dependent constants and,  $\gamma_u$  also depends on the ratio  $\frac{\mu_{\max}}{\mu_{\min}}$ .

## 5.2. A sketch of the case of one compressible and one incompressible material

Assume that  $\lambda_1$  is bounded but not  $\lambda_2$ . We will then work in the norm

$$|||(\mathbf{u}_h, p_h)|||_h^2 = |||\mathbf{u}_h|||^2 + \frac{1}{\mu_{\max}} \|p_h\|_{0,\Omega_2}^2 + \lambda_1^{-1} \|p_h\|_{0,\Omega_1}^2 + J(p_h, p_h). \quad (30)$$

For simplicity we assume that the operator  $J(\cdot, \cdot)$  acts in the whole domain. Including the term  $\lambda_1^{-1}(p_h, q_h)_{\Omega_1}$  in the definition of  $B_h[\cdot, \cdot]$  in (10), coercivity now gives us

$$c(|||\mathbf{u}_h|||^2 + \lambda_1^{-1} \|p_h\|_{0,\Omega_1}^2 + J(p_h, p_h)) \leq (B_h[(\mathbf{u}_h, p_h), (\mathbf{v}_h, p_h)] + J(p_h, p_h)). \quad (31)$$

However, the pressure is no longer in  $L_0^2(\Omega)$ . On the other hand we can repeat the above inf-sup analysis using  $\mathbf{v}_p$  such that  $\nabla \cdot \mathbf{v}_p = \mu_{\max}^{-1}(p_h - \bar{p}_h)$ , where  $\bar{p}_h$  denotes the average of  $p_h$  over  $\Omega$ . Clearly

$$\mu_{\max}^{-1} \|p_h - \bar{p}_h\|_{0,\Omega}^2 = (p_h, \nabla \cdot \mathbf{v}_p)_{\Omega}$$

and using the same arguments as in the previous section, together with coercivity (31) we obtain inf-sup control of the quantity

$$|||\mathbf{u}_h||| + \lambda_1^{-1} \|p_{1,h}\|_{0,\Omega_1} + \mu_{\max}^{-\frac{1}{2}} \|p_h - \bar{p}_h\|_{0,\Omega} + J(p_h, p_h). \quad (32)$$

We will now show that the control of (32) is sufficient to bound the pressure part of (30) By the decomposition  $p_{i,h} = (p_{i,h} - \bar{p}_{i,h}) + \bar{p}_{i,h}$  and the orthogonality

$$(p_{i,h} - \bar{p}_{i,h}, \bar{p}_{i,h} - \bar{p}_h)_{\Omega_i} = 0,$$

there holds

$$\|p_h - \bar{p}_h\|_{0,\Omega}^2 = \sum_{i=1}^2 \left( \|p_{i,h} - \bar{p}_{i,h}\|_{0,\Omega_i}^2 + \|\bar{p}_h - \bar{p}_{i,h}\|_{0,\Omega_i}^2 \right). \quad (33)$$

On the other hand, observing that for  $i = 1, 2$ ,

$$\|p_{i,h}\|_{0,\Omega_i}^2 = \|\bar{p}_{i,h}\|_{0,\Omega_i}^2 + \|p_{i,h} - \bar{p}_{i,h}\|_{0,\Omega_i}^2$$

it follows from (33) and (32) that we only need to bound  $\|\bar{p}_{2,h}\|_{0,\Omega_2}^2$ . Noting that

$$\begin{aligned} \|\bar{p}_{2,h}\|_{0,\Omega_2} &\leq \|\bar{p}_h\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2} \\ &\leq \|\bar{p}_{1,h}\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{1,h}\|_{0,\Omega_2} + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2} \\ &= (\text{meas}(\Omega_2)/\text{meas}(\Omega_1))^{\frac{1}{2}} (\|\bar{p}_{1,h}\|_{0,\Omega_1} + \|\bar{p}_h - \bar{p}_{1,h}\|_{0,\Omega_1}) \\ &\quad + \|\bar{p}_h - \bar{p}_{2,h}\|_{0,\Omega_2}. \end{aligned} \quad (34)$$

the required bound on (30) now follows by combining the inf-sup control of the quantity in (32) with the upper bound (34).

Note that the constant still degenerates for large  $\lambda_1$ . This reflects the instability of the problem due to the average value of  $p_h$  that has to be fixed by the small  $L^2$ -coercivity. The same holds for the case where  $\lambda_1$  is moderate on a very small subdomain  $\Omega_1$ . The quantities  $\mu_{\max}^{-1} \|p_{1,h} - \bar{p}_{1,h}\|_{0,\Omega_1}^2$ ,  $\mu_{\max}^{-1} \|p_{2,h} - \bar{p}_{2,h}\|_{0,\Omega_2}^2$  and  $\mu_{\max}^{-1} \|p_h - \bar{p}_h\|_{0,\Omega}^2$ , however can be controlled independently of  $\lambda_1$ ,

provided the pressure jumps are penalized. Finally, note that  $\bar{p}_h \rightarrow 0$  as  $\lambda_1 \rightarrow \infty$ .

## 6. A priori error estimates

**Proposition 10.** Assume that the solution  $(\mathbf{u}, p)$  to problem (2) resides in  $[H^2(\Omega_1 \cup \Omega_2) \cap H^1(\Omega)]^2 \times H^1(\Omega_1 \cup \Omega_2) \cap L_0^2(\Omega)$ ; then the finite element solution (10) satisfies the error estimate

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \leq c_{\mu} h \left( \|\mu_{\max}^{\frac{1}{2}} \mathbf{u}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right),$$

where  $c_{\mu}$  is independent of  $h$  but not of the ratio  $\left(\frac{\mu_{\max}}{\mu_{\min}}\right)$ .

**Proof.** In view of Theorem 3 we only need to show the inequality for  $|||(I_h^* \mathbf{u} - \mathbf{u}_h, C_h^* p - p_h)|||$ . First we set  $\eta_h = \mathbf{u}_h - I_h^* \mathbf{u}$  and  $\zeta_h = p_h - C_h^* p$  and note that

$$|||(\mathbf{u} - \mathbf{u}_h, p - p_h)||| \leq |||(\mathbf{u} - I_h^* \mathbf{u}, p - C_h^* p)||| + |||(\eta_h, \zeta_h)|||_h.$$

By Theorem 8 and using Galerkin orthogonality we obtain,

$$\begin{aligned} |||(\eta_h, \zeta_h)|||_h &\leq \frac{1}{c_s} \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\eta_h, \zeta_h), (\mathbf{v}_h, q_h)] + J(\zeta_h, q_h)}{|||(\mathbf{v}_h, q_h)|||_h} \\ &\leq \frac{1}{c_s} \sup_{(\mathbf{v}_h, q_h) \in V_h \times Q_h} \frac{B_h[(\mathbf{u} - I_h^* \mathbf{u}, p - C_h^* p), (\mathbf{v}_h, q_h)] - J(C_h^* p, q_h)}{|||(\mathbf{v}_h, q_h)|||_h}. \end{aligned}$$

It remains to use interpolation estimates to bound the terms on the right hand side. We will make repeated use of inequality (13), Theorem 3 and Lemma 4 below. Treating  $B_h[(\mathbf{u} - I_h^* \mathbf{u}, p - C_h^* p), (\mathbf{v}_h, q_h)]$  termwise we obtain

$$\begin{aligned} a(\mathbf{u} - I_h^* \mathbf{u}, \mathbf{v}_h) &= (2\mu \varepsilon(\mathbf{u} - I_h^* \mathbf{u}), \varepsilon(\mathbf{v}_h))_{0,\Omega_1 \cup \Omega_2} - (\{2\mu \varepsilon(\mathbf{u} - I_h^* \mathbf{u})\}, [\mathbf{v}_h])_r \\ &\quad - (\{2\mu \varepsilon(\mathbf{v}_h)\}, [\mathbf{u} - I_h^* \mathbf{u}])_r + (\gamma_u \mu_{\max} h^{-1} [\mathbf{u} - I_h^* \mathbf{u}], [\mathbf{v}_h])_r \\ &\leq |||\mathbf{u} - I_h^* \mathbf{u}||| |||\mathbf{v}_h||| \leq ch \|\mu_{\max}^{\frac{1}{2}} \mathbf{u}\|_{2,\Omega_1 \cup \Omega_2} |||(\mathbf{v}_h, q_h)|||_h, \end{aligned}$$

$$\begin{aligned} b(p - C_h^* p, \mathbf{v}_h) &= -(p - C_h^* p, \nabla \cdot \mathbf{v}_h)_{\Omega_1 \cup \Omega_2} + (p - C_h^* p, [\mathbf{v}_h])_r \\ &\leq \left( \|\mu_{\min}^{-\frac{1}{2}} (p - C_h^* p)\|_{0,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} h(p - C_h^* p)|_{r^+}\|_{0,r} \right. \\ &\quad \left. + \|\mu_{\min}^{-\frac{1}{2}} h(p - C_h^* p)|_{r^-}\|_{0,r} \right) \times \left( \frac{\mu_{\max}}{\mu_{\min}} \right) |||(\mathbf{v}_h, q_h)|||_h \\ &\leq c \left( \frac{\mu_{\max}}{\mu_{\min}} \right) h \|\mu_{\min}^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} |||(\mathbf{v}_h, q_h)|||_h, \end{aligned}$$

$$\begin{aligned} b(q_h, \mathbf{u} - I_h^* \mathbf{u}) &= (q_h, \nabla \cdot (\mathbf{u} - I_h^* \mathbf{u}))_{\Omega_1 \cup \Omega_2} + (\{q_h\}, [\mathbf{u} - I_h^* \mathbf{u}])_r \\ &\leq \left( \frac{\mu_{\max}}{\mu_{\min}} \right) |||(\mathbf{u} - I_h^* \mathbf{u}, 0)||| |||(\mathbf{v}_h, q_h)|||_h \\ &\quad + \frac{1}{\gamma_u} |||(\mathbf{u} - I_h^* \mathbf{u}, 0)||| |||(\mathbf{v}_h, q_h)|||_h \\ &\leq c \left( \frac{\mu_{\max}}{\mu_{\min}} \right) h \|\mu_{\max}^{\frac{1}{2}} \mathbf{u}\|_{2,\Omega_1 \cup \Omega_2} |||(\mathbf{v}_h, q_h)|||_h, \end{aligned}$$

and for the weakly consistent term we have

$$\begin{aligned} J(C_h^* p, q_h) &\leq J_2(p^* - C_h^* p^*, p^* - C_h^* p^*)^{1/2} J_2(q_h, q_h)^{1/2} \\ &\leq c \left( \sum_{K \in \mathcal{T}} \|h(p^* - C_h^* p^*)\|_{\partial K} \right)^{1/2} J(q_h, q_h) \\ &\leq ch \|\mu^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} |||(\mathbf{v}_h, q_h)|||_h. \quad \square \end{aligned}$$

Using the Aubin–Nitsche duality argument we prove the following  $L_2(\Omega)$ -estimate for the displacements.

**Proposition 11.** Under the same assumptions as Proposition 10 there holds

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \leq c_\mu h^2 \left( \|\mu_{\max}^{\frac{1}{2}} \mathbf{u}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right),$$

where  $c_\mu$  is independent of  $h$  but not of the ratio  $\left(\frac{\mu_{\max}}{\mu_{\min}}\right)$ .

**Proof.** Let  $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2)$  and  $r = (r_1, r_2)$  be the solution of the problem

$$\begin{aligned} -\nabla \cdot (2\mu_i \varepsilon(\mathbf{w})) + \nabla r &= \psi \quad \text{in } \Omega_i, \quad i = 1, 2, \\ \nabla \cdot \mathbf{w} &= 0 \quad \text{in } \Omega_i, \quad i = 1, 2, \\ \mathbf{w} &= 0 \quad \text{on } \partial\Omega, \\ \llbracket r \mathbf{n} - 2\mu \varepsilon(\mathbf{w}) \cdot \mathbf{n} \rrbracket &= 0 \quad \text{on } \Gamma, \\ \llbracket \mathbf{w} \rrbracket &= 0 \quad \text{on } \Gamma. \end{aligned} \quad (35)$$

We assume that the solution enjoys the additional regularity

$$\|\mu_{\max}^{\frac{1}{2}} \mathbf{w}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} r\|_{1,\Omega_1 \cup \Omega_2} \leq c \|\psi\|_{0,\Omega}. \quad (36)$$

Choosing  $\psi = \mathbf{u} - \mathbf{u}_h$ , we may write

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}^2 = a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w}) - (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), r)_{0,\Omega}$$

and proceed using Galerkin orthogonality and the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_1 \cup \Omega_2}^2 &= a_h(\mathbf{u} - \mathbf{u}_h, \mathbf{w} - I_h^* \mathbf{w}) - (\nabla \cdot (\mathbf{u} - \mathbf{u}_h), (r - C_h^* r))_{0,\Omega_1 \cup \Omega_2} \\ &\quad + (\nabla \cdot I_h^* \mathbf{w}, p - p_h)_{0,\Omega_1 \cup \Omega_2} + J(-p_h, C_h^* r) \\ &\leq \|(\mathbf{u} - \mathbf{u}_h, 0)\|_h \|(\mathbf{w} - I_h^* \mathbf{w}, 0)\|_h \\ &\quad + \|\nabla \cdot (\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega_1 \cup \Omega_2} \|r - C_h^* r\|_{0,\Omega_1 \cup \Omega_2} \\ &\quad + \|I_h^* \mathbf{w} - \mathbf{w}\|_{1,\Omega_1 \cup \Omega_2} \|p - p_h\|_{0,\Omega_1 \cup \Omega_2} + J(-p_h, C_h^* r^*) \\ &\leq c \left( \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_h + J(p_h, p_h)^{\frac{1}{2}} \right) \\ &\quad \times \left( \|(\mathbf{w} - I_h^* \mathbf{w}, r - C_h^* r)\|_h + J(C_h^* r^*, C_h^* r^*)^{\frac{1}{2}} \right). \end{aligned}$$

As a consequence of Proposition 10, Theorem 3, Lemma 5 and the regularity hypothesis (36) we may conclude that

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_1 \cup \Omega_2}^2 &\leq Ch \left( \|\mu_{\max}^{\frac{1}{2}} \mathbf{w}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} r\|_{1,\Omega_1 \cup \Omega_2} \right) \|(\mathbf{u} - \mathbf{u}_h, p - p_h)\|_h \\ &\leq Ch^2 \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega_1 \cup \Omega_2} \left( \|\mu_{\max}^{\frac{1}{2}} \mathbf{u}\|_{2,\Omega_1 \cup \Omega_2} + \|\mu_{\min}^{-\frac{1}{2}} p\|_{1,\Omega_1 \cup \Omega_2} \right), \end{aligned}$$

and the result follows.  $\square$

## 7. Numerical results

We remark that in order to simplify the implementation, the interface is defined by the zero isoline of a level set function; this isoline is then approximated by linear interpolation on the current computational mesh. The interface is thus represented by a chain of straight segments with lengths depending on the local meshsize. In our numerical experiments, we have noted that the discrete half norm convergence on the interface is not in the asymptotic regime on meshes where the  $L_2$  norm of the error clearly is; the approximation of the interface may be the cause of this phenomenon.

### 7.1. An artificial interface in an incompressible medium

Our first example shows that we have optimal convergence in the case of an artificial interface in an incompressible solid with constant material parameters. The problem, often used in the context of Stokes problem of creeping flow, is posed on the unit square with exact solution given by  $\mathbf{u} = (20xy^3, 5x^4 - 5y^4)$  and  $p = 60x^2y - 20y^3 - 5$ . Choosing  $\gamma_{1,2} = 1$  and  $\gamma_\mu = 10$ , we obtain the optimal convergence shown in Fig. 3, first order for the pressure and second order for the displacements. The position of the artificial interface

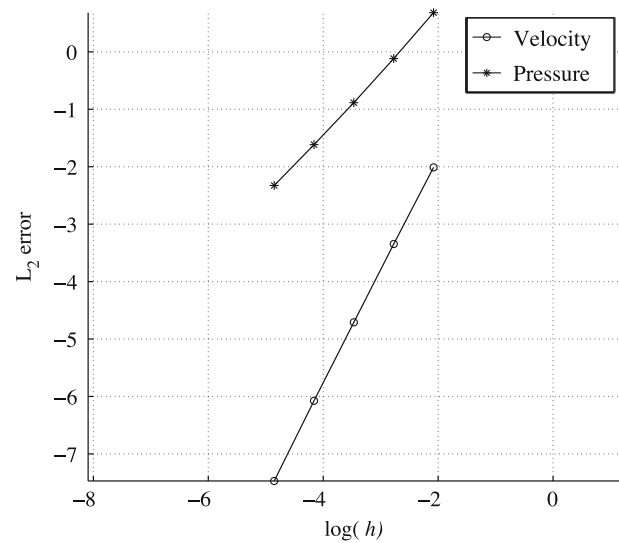


Fig. 3. Convergence for an artificial interface.

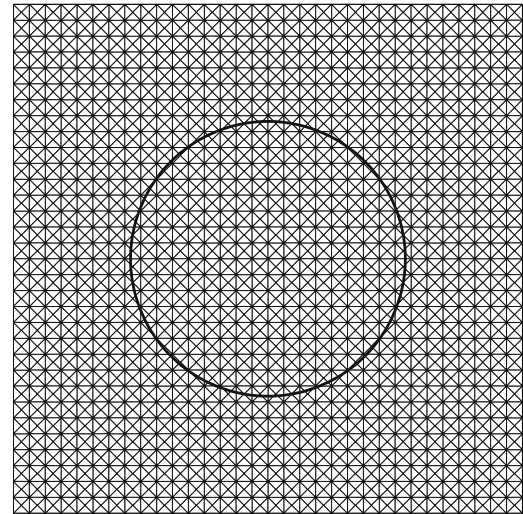


Fig. 4. Position of the artificial interface.

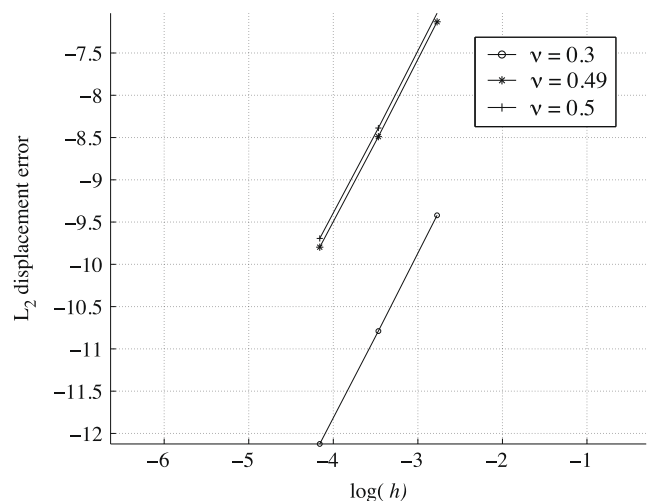


Fig. 5. Convergence for the elasticity problem.



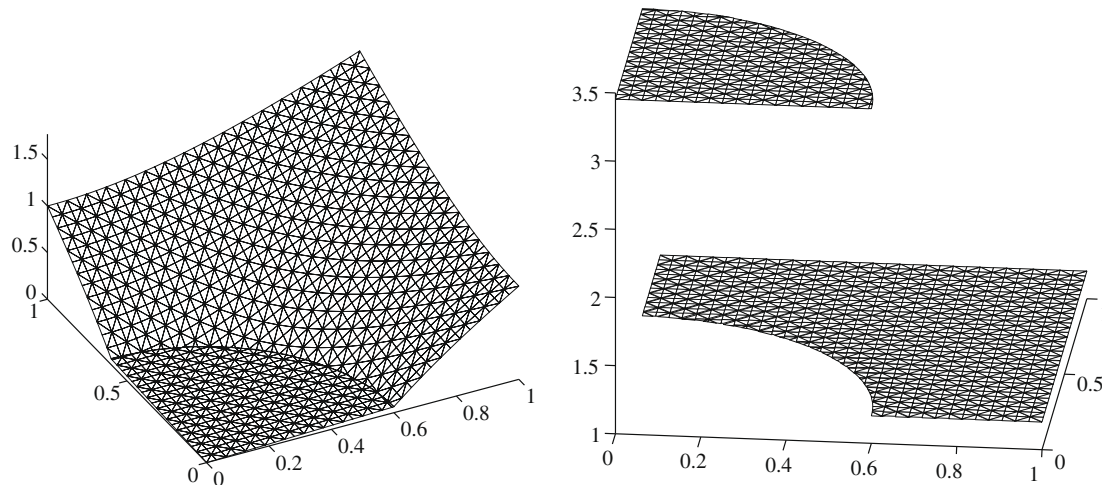


Fig. 6. Elevation of  $|u|$  and  $p$  for  $v_1 = 0.49$ .

is shown in Fig. 4 on one of the computational meshes in the sequence.

## 7.2. An elastic interface problem with $v_1 \rightarrow 1/2$

We have considered a radially symmetric problem with different material properties in concentric discs around the origin. The inner disc has material parameters  $E_1$ ,  $\nu_1$ , and the outer  $E_2$ ,  $\nu_2$ . At any point, the displacement vector can be written  $\mathbf{u} = (u_r, u_\theta)$ , where  $u_r$  is the radial component of the displacement and  $u_\theta$  is the circumferential component. The material is subjected to a boundary displacement  $\mathbf{u} = \mathbf{x}$  (in Cartesian coordinates), and the exact solution to the problem is given by (cf. [12])

$$u_r(r) = \begin{cases} \left( \left(1 - \frac{b^2}{a^2}\right)c + \frac{b^2}{a^2} \right) r, & 0 \leq r \leq a, \\ \left( r - \frac{b^2}{r} \right) c + \frac{b^2}{r}, & a \leq r \leq b, \end{cases}$$

$$u_\theta = 0,$$

with

$$c = \frac{(\lambda_1 + \mu_1 + \mu_2)b^2}{(\lambda_2 + \mu_2)a^2 + (\lambda_1 + \mu_1)(b^2 - a^2) + \mu_2b^2}.$$

When  $\nu_1 = 1/2$ , the last relation degenerates to  $c = 1/(b^2 - a^2)$ . We have set  $E_1 = E_2 = 1$ ,  $\nu_2 = 0.25$  and set  $\nu_1 \in \{0.3, 0.49, 0.5\}$ . The problem was solved on a quarter of a disc with symmetry boundary conditions on the vertical and horizontal boundaries and with the given boundary condition on the circumference. The stability parameters were chosen as in the previous example.

In Fig. 5 we show the convergence in displacements for the different  $\nu_1$ . We obtain second order convergence in all cases, though the error on a given mesh increases with  $\nu_1$ . We also show, in Fig. 6, an elevation of the length of the displacement vectors and the  $p$  variable for  $\nu_1 = 0.49$ .

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