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Analysis of a Stokes interface problem in multi-subdomains

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ABSTRACT

This paper studies a stationary Stokes problem with piecewise constant viscosities in multi-subdomains. For the variational formulation of this problem, we establish its validity of inf–sup stability with the constant depending on the second smallest viscosity.

1. Introduction

We consider the Stokes interface problem in multi-subdomains as follows. Find a velocity $\boldsymbol{u}$ and a pressure p such that

$$\begin{cases} -\text{div}\ (\nu\nabla \boldsymbol{u} - p\ \boldsymbol{I}) = \boldsymbol{f} & \text{in } \Omega, \\ \text{div}\ \boldsymbol{u} = 0 & \text{in } \Omega, \\ \boldsymbol{u} = 0 & \text{on } \Gamma_D, \end{cases} \tag{1}$$

with a piecewise constant viscosity ν. ν equals to ν_i in the subdomain Ω_i, where

$$0 < \nu_m \leq \nu_{m-1} \leq \cdots \leq \nu_1 = 1 \tag{2}$$

and m is the total number of the subdomains. $\Omega \subset \boldsymbol{R}^d$ $(d = 2, 3)$ represents a domain with a Lipschitz continuous boundary $\partial\Omega$. The subdomains $\{\Omega_i\}$ are assumed to be Lipschitz domains such that $\cap_{i=1}^m \Omega_i = \varnothing$ and $\bar{\Omega} = \cup_{i=1}^m \bar{\Omega}_i$. $\boldsymbol{n}$ is a normal vector to the boundary. $\boldsymbol{I}$ is the $d \times d$ identity matrix.

This problem comes from multi-phase incompressible flows, where fluids with different density and viscosity are involved. The density and viscosity are then discontinuous across the interface between different fluids.

The main work of the paper is the analysis of the inf–sup property for the variational Stokes interface problem in multi-subdomains. For the Stokes interface problem in two subdomains, Olshanskii and

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Reusken [1] proved an inf–sup result that the constants are uniform with respect to the jump in the viscosity coefficient. Based on the stable results, lots of research for Stokes interface problem goes well, such as a cut finite element method [2], hybridizable discontinuous Galerkin method [3], unfitted stabilized Nitsches finite element method [4] and Xfem extended finite element method [5]. This paper extends the theoretical analysis to the multi-subdomains case. The extension is nontrivial. It proves that the Stokes interface problem is almost well-posed, for constants are depending on Ω and possibly on the second smallest viscosity. This keeps line with the result in [1], for $m = 2$ subdomains where $\nu_{m-1} = 1$.

The rest of the paper is organized as follows. The variational formulation of the Stokes interface problem is described in next section. The main result is presented in Section 3, which shows that the validity of inf–sup stability for multi-subdomains with the constant $\hat{c}$ depending on, ν_{m-1}, the second smallest viscosity constant.

2. Notations and variational form

We use $\boldsymbol{V} = H_0^1(\Omega)^d$ for the velocity space. Considering the scalar product $(\nu\nabla\cdot, \nabla\cdot)$ on $\boldsymbol{V}$, we use the induced norm $\|\boldsymbol{u}\|_V := (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{u})^{\frac{1}{2}}$ *for any* $\boldsymbol{u} \in \boldsymbol{V}$.

The pressure space is defined by

$$M = \left\{ q \in L^2(\Omega) : \int_\Omega \nu^{-1} q \,\mathrm{d}\,x = 0 \right\}.$$

The scalar product and the induced norm in M are denoted by

$$(p, q)_M := \int_\Omega \nu^{-1} pq \,\mathrm{d}\,x = (\nu^{-1}p, q) \quad \forall\, p, q \in M,$$

and $\|p\|_M := (p, p)_M^{\frac{1}{2}}$.

The variational problem of (1) reads as follows: find $(\boldsymbol{u}, p) \in \boldsymbol{V} \times M$ such that

$$\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = f(\boldsymbol{v}) \quad \forall\, (\boldsymbol{v}, q) \in \boldsymbol{V} \times M. \tag{3}$$

Here the bilinear and linear forms are defined by

$$\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q)) = (\nu\nabla\boldsymbol{u}, \nabla\boldsymbol{v}) - (p, \mathrm{div}\ \boldsymbol{v}) + (q, \mathrm{div}\ \boldsymbol{u}) \tag{4}$$

and $f(\boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v})$.

3. Analysis

To obtain the inf–sup result, we firstly recall the Nečas inequality

$$c(\Omega)\|p\| \le \|\nabla p\|_{-1} := \sup_{\boldsymbol{u}\in\boldsymbol{V}} \frac{(\mathrm{div}\ \boldsymbol{u}, p)}{\nabla\boldsymbol{u}} \qquad \forall p \in L_0^2(\Omega),$$

and its equivalent form: for any $p \in L^2(\Omega)$ such that $(p, 1) = 0$ there exists $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$\|p\|^2 = (\mathrm{div}\ \boldsymbol{u}, p) \quad \text{and} \quad c(\Omega)\|\nabla\boldsymbol{u}\| \le \|p\|.$$

Due to the pressure space M differs the classical one $L_0^2(\Omega)$, it needs to extend the Nečas inequality to the new formulation corresponding to the norm $\|\cdot\|_V$ and $\|\cdot\|_M$. The proof of our main result relies on the $(\cdot,\cdot)_M$-orthogonal decomposition

$$M = M_0 \oplus M_0^\perp,$$

where M_0 is the piecewise constant subspace of M,

$$M_0 = \left\{ p \in M : \ p = \sum_{i=1}^{m} \alpha_i \chi_i \quad \text{and} \quad \sum_{i=1}^{m} \alpha_i \nu_i^{-1} |\Omega_i| = 0 \right\}. \tag{5}$$

Here the α_i is the real number and the χ_i is the characteristic function of the subdomain Ω_i defined by

$$\chi_i = \begin{cases} 1, & \text{in } x \in \Omega_i, \\ 0, & \text{in } x \in \Omega / \Omega_i. \end{cases}$$

Lemma 3.1. *Let $M_0^\perp \subset M$ be the $(\cdot,\cdot)_M$-orthogonal compliment of M_0, then*

$$M_0^\perp = \left\{ p \in M : \ \int_{\Omega_i} p \; d\,x = 0 \; for \; i = 1, \dots, m \right\}.$$

Proof. For any $p \in M_0^\perp \subset M$ and any $q = \sum_{i=1}^{m} \alpha_i \chi_i \in M_0 \subset M$, we have $(p,\, 1)_M = (q,\, 1)_M = 0$. That is,

$$\sum_{i=1}^{m-1} \int_{\Omega_i} \nu_i^{-1} p \; \mathrm{d}\,x + \nu_m^{-1} \int_{\Omega_m} p \; \mathrm{d}\,x = 0 \tag{6}$$

and

$$\sum_{i=1}^{m-1} \nu_i^{-1} \alpha_i |\Omega_i| + \nu_m^{-1} \alpha_m |\Omega_m| = 0. \tag{7}$$

By the definition of $M_0^\perp$, we have

$$0 = (p,\, q)_M = \sum_{i=1}^{m-1} \int_{\Omega_i} \alpha_i \nu_i^{-1} p \; \mathrm{d}\,x + \nu_m^{-1} \alpha_m \int_{\Omega_m} p \; \mathrm{d}\,x. \tag{8}$$

Multiplying (6) by α_m and subtracting (8) gives

$$\sum_{i=1}^{m-1} \nu_i^{-1} (\alpha_i - \alpha_m) \int_{\Omega_i} p \; \mathrm{d}\,x = 0,$$

which, together with (7), implies

$$0 = \sum_{i=1}^{m-1} \left(\alpha_i + \sum_{j=1}^{m-1} \frac{\nu_m |\Omega_j|}{\nu_j |\Omega_m|} \alpha_j \right) \nu_i^{-1} \int_{\Omega_i} p \; \mathrm{d}\,x. \tag{9}$$

Let $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_{m-1})^t$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_{m-1})^t$ with $\gamma_j = \frac{\nu_m |\Omega_j|}{\nu_j |\Omega_m|}$ and $\boldsymbol{\beta} = (\beta_1, \dots, \beta_{m-1})^t$ with $\beta_i = (\alpha_i + \sum_{j=1}^{m-1} \gamma_j \alpha_j)$. Then $\boldsymbol{\beta} = A\boldsymbol{\alpha}$ with $A = I + \boldsymbol{d}\boldsymbol{\gamma}^t$ and $\boldsymbol{d} = (1, \dots, 1)^t$. By the Sherman–Morrison formula, A is invertible. Hence, $\boldsymbol{\beta}$ is arbitrary. Thus the matrix–vector representation of (9)

$$0 = \beta^t \cdot \left(\nu_1^{-1} \int_{\Omega_1} p \; \mathrm{d}\,x, \dots, \nu_{m-1}^{-1} \int_{\Omega_{m-1}} p \; \mathrm{d}\,x \right)^t$$

implies $\int_{\Omega_i} p \; \mathrm{d}\,x = 0$ for $i = 1, \dots, m$. $\square$

Next, we consider the components p_0 and $p_0^\perp$ from the decomposition $p = p_0 + p_0^\perp$ for any $p \in M$, respectively.

Assumption 1. Assume that for any $p_0 \in M_0$, there exists $\bar{\boldsymbol{u}} \in H_0^1(\Omega)^d$ such that

$$\|p_0\|_M^2 = (\text{div } \bar{\boldsymbol{u}}, p_0) \quad \text{and} \quad \|\bar{\boldsymbol{u}}\|_V \le \hat{c}\|p_0\|_M, \tag{10}$$

where $\hat{c}$ is a positive constant depending on Ω and possibly on ν_{m-1} in (2).

Remark. Assumption 1 with $\hat{c}$ independent of ν was proved in [1] for $m = 2$ subdomains where $\nu_{m-1} = 1$. Following the idea of [1], we establish the validity of (10) for multi-subdomains with the constant $\hat{c}$ depending on, ν_{m-1}, the second smallest viscosity constant.

Lemma 3.2. *There exists a positive constant c independent of ν such that (10) is valid with $\hat{c} = c\nu_{m-1}^{-\frac{1}{2}}$.*

Proof. For any $p_0 \in M_0$, let $\tilde{p}_0 := \nu^{-1}p_0$. It is clear that $\tilde{p}_0 \in L_0^2(\Omega)$. Hence, by the Nečas inequality there exist $\bar{\boldsymbol{u}} \in H_0^1(\Omega)^d$ and a positive constant c independent of ν such that

$$\text{div } \bar{\boldsymbol{u}} = \tilde{p}_0 \quad \text{and} \quad \|\nabla \bar{\boldsymbol{u}}\| \le c_{(\Omega)}\|\tilde{p}_0\|.$$

The direct computation leads to

$$(\text{div } \bar{\boldsymbol{u}}, p_0) = (\tilde{p}_0, \nu\tilde{p}_0) = \|p_0\|_M^2.$$

Let $p_0 = \sum_{i=1}^m \alpha_i \chi_i \in M_0$, then $\alpha_m \nu_m^{-1} = -\sum_{i=1}^{m-1} \alpha_i \nu_i^{-1} \frac{|\Omega_i|}{|\Omega_m|}$. By the Cauchy–Schwarz inequality, we have

$$\alpha_m^2 \nu_m^{-2} |\Omega_m| \le \left(\sum_{i=1}^{m-1} \alpha_i^2 \nu_i^{-2} |\Omega_i|\right) \left(\sum_{i=1}^{m-1} \frac{|\Omega_i|}{|\Omega_m|}\right),$$

which implies

$$\begin{aligned}
\|\tilde{p}_0\|^2 &= \alpha_m^2 \nu_m^{-2}|\Omega_m| + \sum_{i=1}^{m-1} \alpha_i^2 \nu_i^{-2} |\Omega_i| \\
&\le \left(1 + \sum_{i=1}^{m-1} \frac{|\Omega_i|}{|\Omega_m|}\right) \sum_{i=1}^{m-1} \alpha_i^2 \nu_i^{-2}|\Omega_i| \\
&\le \left(1 + \sum_{i=1}^{m-1} \frac{|\Omega_i|}{|\Omega_m|}\right) \nu_{m-1}^{-1} \sum_{i=1}^{m-1} \alpha_i^2 \nu_i^{-1}|\Omega_i| \\
&\le \tilde{c}_{(\Omega)} \nu_{m-1}^{-1} \|p_0\|_M^2.
\end{aligned}$$

Therefore,

$$\|\bar{\boldsymbol{u}}\|_V \le \|\nabla \bar{\boldsymbol{u}}\| \le c_{(\Omega)}\|\tilde{p}_0\| \le c_{(\Omega)} \tilde{c}_{(\Omega)}^{\frac{1}{2}} \nu_{m-1}^{-\frac{1}{2}} \|p_0\|_M.$$

So $\hat{c} = c_{(\Omega)} \tilde{c}_{(\Omega)}^{\frac{1}{2}} \nu_{m-1}^{-\frac{1}{2}}$. $\square$

Lemma 3.3. *For any $p_0^\perp \in M_0^\perp$, there exist $\tilde{\boldsymbol{u}} \in H_0^1(\Omega)^d$ and a positive constant c independent of ν such that*

$$\|p_0^\perp\|_M^2 = (\text{div } \tilde{\boldsymbol{u}}, p_0^\perp) \quad \textit{and} \quad \|\tilde{\boldsymbol{u}}\|_V \le c\|p_0^\perp\|_M. \tag{11}$$

Proof. The lemma may be proved in the same fashion as that of Theorem 1 in [1]. For the convenience of readers, we briefly state here. For $i = 1, 2, \ldots, m$, there exist $\{\boldsymbol{u}_i\} \subset \boldsymbol{V}$ with $\boldsymbol{u}_i = 0$ on Ω/Ω_i and positive constants $\{c_i\}$ independent of ν such that

$$\|\nu_i^{-\frac{1}{2}} p_0^\perp\|_{\Omega_i}^2 = (\mathrm{div}\ \boldsymbol{u}_i, p_0^\perp)_{\Omega_i} \quad \text{and} \quad c_i\|\nu_i^{\frac{1}{2}} \nabla \boldsymbol{u}_i\| \le \|\nu_i^{-\frac{1}{2}} p_0^\perp\|_{\Omega_i}. \tag{12}$$

Let $\tilde{\boldsymbol{u}} := \sum_{i=1}^m \boldsymbol{u}_i \in \boldsymbol{V}$, taking sum of (12) over $i = 1, 2, \ldots, m$ gives

$$\|p_0^\perp\|_M^2 = (\mathrm{div}\ \tilde{\boldsymbol{u}}, p_0^\perp) \quad \text{and} \quad c\|\tilde{\boldsymbol{u}}\|_V \le \|p_0^\perp\|_M, \tag{13}$$

with $c := \min_{1\le i\le m}\{c_i\}$. □

Theorem 3.1. *Under Assumption 1, there exists a positive constant c independent of ν such that*

$$\sup_{0\ne \boldsymbol{u}\in \boldsymbol{V}} \frac{(\mathrm{div}\ \boldsymbol{u}, p)}{\|\boldsymbol{u}\|_V} \ge c\hat{c}^{-2}\|p\|_M \quad \forall p \in M. \tag{14}$$

Proof. For any $p \in M$, $p = p_0 + p_0^\perp$, where $p_0 \in M_0$ and $p_0^\perp \in M_0^\perp$. By Assumption 1 and Lemma 3.2, let $\bar{\boldsymbol{u}} \in \boldsymbol{V}$ and $\tilde{\boldsymbol{u}} \in \boldsymbol{V}$ be the functions satisfy (10) and (11), respectively. Note that $(\mathrm{div}\ \tilde{\boldsymbol{u}}, p_0) = 0$. Then it follows from the Cauchy–Schwarz and the Young inequalities that for any positive constant α

$$\begin{aligned}(\mathrm{div}\ (\bar{\boldsymbol{u}} + \alpha\tilde{\boldsymbol{u}}), p) &= (\mathrm{div}\ \bar{\boldsymbol{u}}, p_0) + \alpha(\mathrm{div}\ \tilde{\boldsymbol{u}}, p_0^\perp) + (\mathrm{div}\ \bar{\boldsymbol{u}}, p_0^\perp) + \alpha(\mathrm{div}\ \tilde{\boldsymbol{u}}, p_0) \\ &\ge \|p_0\|_M^2 + \alpha\|p_0^\perp\|_M^2 - \hat{c}d^{\frac{1}{2}}\|p_0\|_M\|p_0^\perp\|_M \\ &\ge \frac{1}{2}\|p_0\|_M^2 + \left(\alpha - \frac{\hat{c}^2 d}{2}\right)\|p_0^\perp\|_M^2.\end{aligned}$$

Choosing $\alpha = \frac{1}{2}(1 + \hat{c}^2 d)$ gives

$$(\mathrm{div}\ (\bar{\boldsymbol{u}} + \alpha\tilde{\boldsymbol{u}}), p) \ge \frac{1}{2}\|p\|_M^2.$$

By (10) and (11), we have

$$\|\bar{\boldsymbol{u}} + \alpha\tilde{\boldsymbol{u}}\|_V \le \hat{c}\|p_0\|_M + c\alpha\|p_0^\perp\|_M \le c\hat{c}^2\|p\|_M.$$

Now, (14) is a direct consequence of the above two inequalities. This completes the proof of the theorem. □

Finally, we can get the following inf–sup stability for the bilinear form (4).

Theorem 3.2. *Under Assumption 1, there exists a positive constant c independent of ν such that*

$$\sup_{0\ne(\boldsymbol{v},q)\in \boldsymbol{V}\times M} \frac{\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q))}{(\|\boldsymbol{v}\|_V^2 + \|q\|_M^2)^{\frac{1}{2}}} \ge c\hat{c}^{-4}(\|\boldsymbol{u}\|_V^2 + \|p\|_M^2)^{\frac{1}{2}} \quad \forall(\boldsymbol{u}, p) \in \boldsymbol{V} \times M.$$

Proof. It follows from Theorem 3.1 that for any $p \in M$ there exist $\boldsymbol{w} \in \boldsymbol{V}$ and a positive constant c independent of ν such that

$$\|\boldsymbol{w}\|_V = \|p\|_M \quad \text{and} \quad (\mathrm{div}\ \boldsymbol{w}, p) \ge c\hat{c}^{-2}\|p\|_M^2. \tag{15}$$

Setting $(\boldsymbol{v}, q) = (\boldsymbol{u}, p)$ gives

$$\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{u}, p)) = \|\boldsymbol{u}\|_V^2. \tag{16}$$

Setting $(\boldsymbol{v}, q) = (-\boldsymbol{w}, 0)$ and following from (15), the Cauchy–Schwarz and Young inequalities gives

$$\begin{aligned} \mathcal{L}((\boldsymbol{u}, p), (-\boldsymbol{w}, 0)) &= -(\nu \nabla \boldsymbol{u}, \nabla \boldsymbol{w}) + (p, \operatorname{div} \boldsymbol{w}) \\ &\geq -\|\boldsymbol{u}\|_V \|\boldsymbol{w}\|_V + c\hat{c}^{-2}\|p\|_M^2 \\ &\geq \frac{c\hat{c}^{-2}}{2}\|p\|_M^2 - \frac{1}{2c\hat{c}^{-2}}\|\boldsymbol{u}\|_V^2. \end{aligned} \tag{17}$$

Multiplying (17) by $c\hat{c}^{-2}$ and adding to (16) gives

$$\begin{aligned} \mathcal{L}((\boldsymbol{u}, p), (-c\hat{c}^{-2}\boldsymbol{w} + \boldsymbol{u}, p)) &\geq \frac{1}{2}\|\boldsymbol{u}\|_V^2 + \frac{c\hat{c}^{-4}}{2}\|p\|_M^2 \\ &\geq min\left\{\frac{1}{2}, \frac{c\hat{c}^{-4}}{2}\right\}(\|\boldsymbol{u}\|_V^2 + \|p\|_M^2). \end{aligned} \tag{18}$$

By (15) and the triangle inequality,

$$\begin{aligned} \|\boldsymbol{u} - c\hat{c}^{-2}\boldsymbol{w}\|_V^2 + \|p\|_M^2 &\leq 2\|\boldsymbol{u}\|_V^2 + (2c\hat{c}^{-4} + 1)\|p\|_M^2 \\ &\leq (2c\hat{c}^{-4} + 3)(\|\boldsymbol{u}\|_V^2 + \|p\|_M^2). \end{aligned} \tag{19}$$

Note $\hat{c}^{-2}$ is as small as ν_{m-1} by Lemma 3.2. Hence,

$$\begin{aligned} \sup_{0 \neq (\boldsymbol{v}, q) \in \boldsymbol{V} \times M} \frac{\mathcal{L}((\boldsymbol{u}, p), (\boldsymbol{v}, q))}{(\|\boldsymbol{v}\|_V^2 + \|q\|_M^2)^{\frac{1}{2}}} &\geq \frac{min\left\{\frac{1}{2}, \frac{c\hat{c}^{-4}}{2}\right\}}{(2c\hat{c}^{-4} + 3)^{\frac{1}{2}}}(\|\boldsymbol{u}\|_V^2 + \|p\|_M^2)^{\frac{1}{2}} \\ &\geq c\hat{c}^{-4}(\|\boldsymbol{u}\|_V^2 + \|p\|_M^2)^{\frac{1}{2}}. \quad \square \end{aligned}$$

4. Conclusion

This paper extends the existed analysis of Stokes interface problem in two subdomains to the case in multi-subdomains. The extension is nontrivial. It establishes the validity of inf–sup stability for multi-subdomains with the constant $\hat{c}$ depending on, ν_{m-1}, the second smallest viscosity constant. For the problem in two subdomains, $\hat{c}$ independent of ν was proved in [1], where $\nu_{m-1} = 1$. This keeps line with our results.

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