Unconditional point-wise error estimate of a conservative difference scheme for the two-dimensional space-fractional nonlinear Schrödinger equation

Dongdong Hu, Yongzhong Song, Wenjun Cai, Yushun Wang*

Jiangsu Key Laboratory for NSLSCS,

Jiangsu Collaborative Innovation Center of Biomedical Functional Materials,

School of Mathematical Sciences, Nanjing Normal University, Nanjing 210023, China

Abstract

In this paper we intend to construct a novel conservative difference scheme for the two-dimensional space-fractional nonlinear Schrödinger(2D SFNS) equation with the hyper-singular integral fractional Laplacian, where the temporal direction is discretized by the modified Crank-Nicolson method, and the spatial variable is approximated by the new fractional central difference method. The mass-energy conservative laws and the unconditional convergence in L^2 and L^∞ norm are rigorously proved for the proposed scheme. For 1D SFNS equation, the convergence relies strongly on the boundness of the numerical solution of the proposed scheme. However, we can not obtain the unconditional maximum boundness of the numerical solution by using the similar process for the 2D SFNS equation. The main contribution of this paper is that we first obtain the unconditional point-wise error estimate based on the popular "cut-off" function and the "lifting" technique for the 2D SFNS equation. Further, we reveal that the spatial discretization generates a block-Toeplitz coefficient matrix, and it will be ill-conditioned as the spatial grid mesh number M and the fractional order α increase. Thus, we exploit an efficient linearized iteration algorithm for the nonlinear system, such that it can be efficiently solved by the Krylov subspace solver with a suitable preconditioner, where the 2D fast Fourier transform is used in the solver to accelerate the matrix-vector product. Extensive numerical results are reported to confirm the theoretical analysis and the high efficiency of the proposed algorithm.

AMS subject classification: 65M06, 65M12, 65T50, 65F08, 35L05, 35R11

Keywords: Fractional nonlinear Schrödinger equation; Integral fractional Laplacian; Structure-preserving scheme; Krylov subspace solver; Fast algorithm; Unconditional point-wise error estimate

1 Introduction

Consider the two-dimensional space-fractional nonlinear Schrödinger equation

$$\mathbf{i}u_t - (-\Delta)^{\frac{\alpha}{2}}u + \beta|u|^2u = 0, \quad \mathbf{i}^2 = -1 \quad \boldsymbol{x} \in \mathbb{R}^2, \quad t > 0, \tag{1.1}$$

$$u(\boldsymbol{x}, 0) = \varphi(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathbb{R}^2, \tag{1.2}$$

where $1 < \alpha \leq 2$, β is a given real constant. We assume that u, φ are spatially compactly supported on the bounded domain Ω. Besides, we postulate that the functions u, φ are smooth enough, so that our proposed scheme can attain the desired convergence order. The system (1.1) conserves two important physical quantities, the mass and the energy, i.e.

$$\mathcal{M}(t) = \int_{\mathbb{R}^2} \Big|u(\boldsymbol{x}, t)\Big|^2 d\boldsymbol{x}, \qquad \mathcal{M}(t) = \mathcal{M}(0), \tag{1.3}$$

*Correspondence author. Email: wangyushun@njnu.edu.cn.

and

$$\mathcal{E}(t)=\int_{\mathbb{R}^2}\Big[\bar{u}(-\Delta)^{\frac{\alpha}{2}}u(\boldsymbol{x},t)-\frac{\beta}{2}\Big|u(\boldsymbol{x},t)\Big|^4\Big]d\boldsymbol{x},\qquad \mathcal{E}(t)=\mathcal{E}(0). \tag{1.4}$$

where $\bar{u}$ is the complex conjugate of u.

The hypersingular integral fractional Laplacian (IFL) is defined as [15, 16]

$$(-\Delta)^{\frac{\alpha}{2}}u(\boldsymbol{x})=c_{2,\alpha}\text{P.V.}\int_{\mathbb{R}^2}\frac{u(\boldsymbol{x})-u(\boldsymbol{y})}{|\boldsymbol{x}-\boldsymbol{y}|^{2+\alpha}}d\boldsymbol{y},\quad c_{2,\alpha}=\frac{2^\alpha\Gamma(\frac{\alpha+2}{2})}{\pi|\Gamma(-\alpha/2)|}. \tag{1.5}$$

Where P.V. stands for Cauchy principle value, Γ is the Gamma function, $|\boldsymbol{x}-\boldsymbol{y}|$ denotes the Euclidean distance between points $\boldsymbol{x}$ and $\boldsymbol{y}$. From a probabilistic point of view, the fractional Laplacian represents the infinitesimal generator of a symmetric α-stable Lévy process [1]. It is also equivalent to and can be directly derived from the Fourier representation [2, 3]

$$(-\Delta)^{\frac{\alpha}{2}}u(\boldsymbol{x})=\int_{\mathbb{R}^2}\big|\boldsymbol{k}\big|^\alpha\mathcal{F}[u,\boldsymbol{k}]e^{\mathbf{i}\boldsymbol{kx}}d\boldsymbol{k}, \tag{1.6}$$

where $|\boldsymbol{k}|^2=k_1^2+k_2^2$, $\boldsymbol{kx}=k_1x+k_2y$, and the Fourier transform is defined by

$$\mathcal{F}[u,\boldsymbol{k}]=\frac{1}{(2\pi)^2}\int_{\mathbb{R}^2}u(\boldsymbol{x},t)e^{-\mathbf{i}\boldsymbol{kx}}d\boldsymbol{x}.$$

2 Preliminaries

2.1 Spatial discretization

To discretize the IFL, we adopt a simple-and-easy fractional central difference scheme [4]

$$(-\Delta)^{\frac{\alpha}{2}}u(x,y)=\frac{1}{h^\alpha}\sum_{i,j\in\mathbb{Z}}g_{i,j}^{(\alpha)}u(x-ih,y-jh),$$

where h is spatial step size, $\mathbb{Z}$ is the set of all integer number, $g_{i,j}^{(\alpha)}$ are Fourier expansion coefficients of generating function $\varrho(x,y)=\Big[4\sin^2(\frac{x}{2})+4\sin^2(\frac{y}{2})\Big]^{\alpha/2}$, which can be calculated as

$$g_{i,j}^{(\alpha)}=\frac{1}{(2\pi)^2}\iint_{[-\pi,\pi]^2}\varrho(x,y)e^{-\mathbf{i}(ix+jy)}dxdy. \tag{2.1}$$

Denote

$$W^{n+\alpha,1}(\mathbb{R}^2)=\Big\{u\Big|u\in L^1(\mathbb{R}^2),\ \int_{\mathbb{R}^2}(1+|k|)^{n+\alpha}|\hat{u}(k_1,k_2)|dk_1dk_2<\infty\Big\}. \tag{2.2}$$

We have the following consequence.

Lemma 2.1 ([4])**.** *Suppose $u\in W^{2+\alpha,1}(\mathbb{R}^2)$, for a fixed $h\to 0$, we can obtain a second-order central difference approximation for the two-dimensional integral fractional Laplacian by*

$$(-\Delta)^{\frac{\alpha}{2}}u(x,y)=(-\Delta_h)^{\frac{\alpha}{2}}u(x,y)+\mathcal{O}(h^2). \tag{2.3}$$

2.2 Fractional Sobolev norm

In this section, we will introduce the fractional Sobolev norm and relevant lemmas. Denote $h\mathbb{Z}=\{x_i=ih,y_j=jh,i,j\in\mathbb{Z}\}$ be the infinite grid. For any grid functions $u=\{u_{i,j}\}$, $v=\{v_{i,j}\}$ on $h\mathbb{Z}$, the discrete inner product and the associated L_h^2 norm are defined as

$$(u,v)_h=h^2\sum_{i,j\in\mathbb{Z}}u_{i,j}v_{i,j}^*,\qquad \Big\|u\Big\|_{L_h^2}=\sqrt{(u,u)_h},\qquad \Big\|u\Big\|_\infty=\max_{i,j\in\mathbb{Z}}\Big|u_{i,j}\Big|,$$

Set $L_h^2 := \left\{v \Big| v = \left\{v_{i,j}\right\},\ \left\|v\right\|_{L_h^2} < \infty\right\}$, for $u \in L_h^2$, we define its semi-discrete Fourier transform $\hat{v}(k_1, k_2)$ by

$$\hat{v}(k_1, k_2) = \mathcal{F}\{v; k_1, k_2\} = h^2 \sum_{i,j \in \mathbb{Z}} v_{i,j} e^{-\mathbf{i}k_1 x_i - \mathbf{i}k_2 y_j}.$$

Moreover, we have the inversion formula

$$v_{i,j} = \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{v}(k_1, k_2) e^{\mathbf{i}k_1 x_i + \mathbf{i}k_2 y_j} dk_1 dk_2, \tag{2.4}$$

and the Parseval's identity

$$(u, v)_h = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{u}(k_1, k_2) \bar{\hat{v}}(k_1, k_2) dk_1 dk_2.$$

For a positive constant $\alpha \in (1, 2]$, we define the fractional Sobolev norms as

$$\left|u\right|^2_{H^{\frac{\alpha}{2}}} = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\left|k_1\right|^2 + \left|k_2\right|^2\right)^{\frac{\alpha}{2}} \left|\hat{u}(k_1, k_2)\right|^2 dk_1 dk_2, \tag{2.5}$$

$$\left|u\right|^2_{H^{\alpha}} = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left(\left|k_1\right|^2 + \left|k_2\right|^2\right)^{\alpha} \left|\hat{u}(k_1, k_2)\right|^2 dk_1 dk_2. \tag{2.6}$$

Therefore

$$\left\|u\right\|^2_{H^s} = \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left[1 + \left(\left|k_1\right|^2 + \left|k_2\right|^2\right)^s\right] \left|\hat{u}(k_1, k_2)\right|^2 dk_1 dk_2 = \left\|u\right\|^2_{L_h^2} + \left|u\right|^2_{H^s}. \tag{2.7}$$

Set $H^\alpha = \left\{u \Big| u = \left\{u_{i,j}\right\},\ \left\|u\right\|_{H^\alpha} < \infty\right\}$, we have the following essential consequences.

Lemma 2.2 (Discrete uniform Sobolev inequality)**.** *For any $1 < \alpha \leq 2$ and $u \in H^\alpha$, there exists a positive constant $\mathcal{C}_0$ such that*

$$\left\|u\right\|_\infty \leq \mathcal{C}_0 \left\|u\right\|_{H^\alpha}, \tag{2.8}$$

where $\mathcal{C}_0 = \frac{1}{4\pi^2}\left(\int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \frac{1}{\left[1+\left(|k_1|^2+|k_2|^2\right)^\alpha\right]} dk_1 dk_2\right)^{\frac{1}{2}}$, which is independent of h.

Proof. From the inversion of the Fourier transformation (2.4) and Cauchy-Buniakowsky-Schwarz inequality, we obtain

$$\begin{aligned}
\left\|u\right\|_\infty &= \max_{i,j \in \mathbb{Z}} \left| \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \hat{u}(k_1, k_2) e^{\mathbf{i}k_1 x_i + \mathbf{i}k_2 y_j} dk_1 dk_2 \right| \\
&\leq \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left|\hat{u}(k_1, k_2)\right| dk_1 dk_2 \\
&= \frac{1}{4\pi^2} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \left|\hat{u}(k_1, k_2)\right| \frac{\left[1 + \left(|k_1|^2 + |k_2|^2\right)^\alpha\right]^{\frac{1}{2}}}{\left[1 + \left(|k_1|^2 + |k_2|^2\right)^\alpha\right]^{\frac{1}{2}}} dk_1 dk_2 \\
&\leq \mathcal{C}_0 \left\|u\right\|_{H^\alpha}.
\end{aligned} \tag{2.9}$$

This completes the proof. $\square$

Lemma 2.3 (Fractional semi-norm equivalence [4])**.** *Supposed u is a well-defined grid function. For $u \in H^\alpha$, we have*

$$\left(\frac{2}{\pi}\right)^\alpha \left|u\right|^2_{H^{\frac{\alpha}{2}}} \leq \left((-\Delta_h)^{\frac{\alpha}{2}} u, u\right)_h \leq \left|u\right|^2_{H^{\frac{\alpha}{2}}}. \tag{2.10}$$

$$\left(\frac{2}{\pi}\right)^{2\alpha} \left|u\right|^2_{H^{\alpha}} \leq \left((-\Delta_h)^{\frac{\alpha}{2}} u, (-\Delta_h)^\alpha u\right)_h \leq \left|u\right|^2_{H^{\alpha}}. \tag{2.11}$$

3 Conservative difference scheme

In practical calculation, it needs to restrict the original problem on a bounded domain with homogeneous Dirichlet boundary condition due to the solution of (1.1)-(1.2) is fast decaying [17]. We can select a large domain $\Omega = (a,b)\times(c,d)$ such that

$$u(x,y,t) = 0, \quad (x,y)\in\mathbb{R}^2\setminus\Omega, \quad t\in[0,T], \tag{3.1}$$

where $a,c \ll 0$ and $b,d\gg 0$.

We choose the positive integers N, M_x, M_y, let $\tau := \frac{T}{N}$ and $h = \frac{b-a}{M_x} = \frac{d-c}{M_y}$ be the time step and mesh sizes, respectively. We define a partition of $(0,T]\times(a,b)\times(c,d)$ by $\Omega_\tau\times\Omega_h$ with the grid

$$\Omega_\tau = \Big\{t_n = n\tau \Big|\ n = 1,2\cdots,N\Big\}, \quad \Omega_h = \Big\{(x_i,y_j) = (L+ih, L+jh)\Big|\ i,j = 1,2\cdots,M-1\Big\}.$$

Let $\Psi_h^n = \Big\{w_{i,j}^n = w(x_i,y_j,t_n), w_{i,j}^n = 0, \text{if } i,j = 0,M\Big\}$. For any $w,v\in\Psi_h^n$, we introduce the following notations

$$\delta_t w_{i,j}^n = \frac{w_{i,j}^{n+1}-w_{i,j}^n}{\tau}, \qquad w_{i,j}^{n+1/2} = \frac{w_{i,j}^{n+1}+w_{i,j}^n}{2}, \qquad \Big(w^n, v^n\Big) = h^2\sum_{i=1}^{M_x-1}\sum_{j=1}^{M_y-1} w_{i,j}^n\bar{v}_{i,j}^n,$$

$$\Big\|w^n\Big\|_\infty = \max_{(x_i,y_j)\in\Omega_h}\Big|u_{i,j}^n\Big|, \qquad \Big\|w^n\Big\| = \sqrt{(w^n,w^n)}, \qquad \Big\|w^n\Big\|_p^p = h^2\sum_{i=1}^{M_x-1}\sum_{i=1}^{M_y-1}\Big|w_{i,j}^n\Big|^p.$$

Under above boundary constraint (3.1), for $u\in\Psi_h^n$, the centred difference scheme Lemma (2.1) can be simplified as

$$(-\Delta_h)^{\frac{\alpha}{2}} u_{i,j}^n = \frac{1}{h^\alpha}\sum_{p=i-M_x}^{i}\sum_{q=j-M_y}^{j} g_{p,q}^{(\alpha)} u_{i-p,j-q}^n, \tag{3.2}$$

the coefficients $g_{i,j}^{(\alpha)}$ can be calculated by

$$\begin{aligned} g_{i,j}^{(\alpha)} &= \frac{1}{4\pi^2}\iint_{[-\pi,\pi]^2}\varrho(x,y)e^{-\mathbf{i}(ix+jy)}dxdy = \frac{1}{4\pi^2}\iint_{[0,2\pi]^2}\varrho(x,y)e^{-\mathbf{i}(ix+jy)}dxdy \\ &\approx \frac{1}{M^2}\sum_{p=0}^{K-1}\sum_{q=0}^{K-1}\varrho(p\delta,q\delta)e^{-\mathbf{i}(ip+jq)\delta}, \quad \delta = 2\pi/K, \quad K\gg M_x, M_y, \end{aligned} \tag{3.3}$$

where the numerical trapezoidal quadrature formula is used. With the expression above, the coefficients $g_{i,j}^{(\alpha)}$ can be computed efficiently by the built-in function `fft2` in Matlab[1].

Let $u^n, U^n\in\Psi_h^n$ be the exact solution and numerical approximation, respectively. Considering the 2D SFNS equation (1.1) at (x_i,y_j,t_n) by using the modified Crank-Nicolson method in time and the fractional centred difference scheme (3.2) to discrete the 2D IFL

$$\mathbf{i}\delta_t u^n - (-\Delta_h)^{\frac{\alpha}{2}}u^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(|u^{n+1}|^2+|u^n|^2\Big)u^{n+\frac{1}{2}} = \mathcal{R}^n, \ 0\le n\le N-1, \tag{3.4}$$

$$u_{i,j}^0 = \varphi(x_i,y_j), \quad 0\le i\le M_x, \quad 0\le j\le M_y, \tag{3.5}$$

$$u_{i,j}^n = 0, \quad i,j = 0,M, \quad 1\le n\le N, \tag{3.6}$$

where

$$u^n = (u_1^n, u_2^n,\cdots,u_{M_y-1}^n)^T, \quad u_j^n = \big(u_{1,j}^n, u_{2,j}^n,\cdots,u_{M_x-1,j}^n\big),$$
$$U^n = (U_1^n, U_2^n,\cdots,U_{M_y-1}^n)^T, \quad U_j^n = \big(U_{1,j}^n, U_{2,j}^n,\cdots,U_{M_x-1,j}^n\big).$$

[1]Matlab® is a popular computing environment produced and distributed by MathWorks.

Replacing u by U, omitting the error terms, the conservative difference scheme reads

$$\mathbf{i}\delta_t U^n - (-\Delta_h)^{\frac{\alpha}{2}} U^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big) U^{n+\frac{1}{2}} = 0, \quad 0 \le n \le N-1, \tag{3.7}$$

$$U^0 = \varphi(x_i, y_j), \quad 0 \le i \le M_x, 0 \le j \le M_y, \tag{3.8}$$

$$U^n_{i,j} = 0, \quad i, j = 0, M, \quad 1 \le n \le N. \tag{3.9}$$

For convenience, we reformulate (3.7) as the matrix form

$$\Big(\mathbf{i}I - \frac{\tau}{2}\mathbb{M}\Big) U^{n+1} = \Big(\mathbf{i}I + \frac{\tau}{2}\mathbb{M}\Big) U^n - \frac{\beta\tau}{4}\Big(|U^{n+1}|^2 + |U^n|^2\Big)\Big(U^{n+1} + U^n\Big), \tag{3.10}$$

where

$$\mathbb{M} = \begin{pmatrix} \mathbb{M}_0 & \mathbb{M}_1 & \mathbb{M}_2 & \cdots & \mathbb{M}_{M-2} \\ \mathbb{M}_1 & \mathbb{M}_0 & \mathbb{M}_1 & \ddots & \mathbb{M}_{M-3} \\ \vdots & \mathbb{M}_1 & \mathbb{M}_0 & \ddots & \vdots \\ \mathbb{M}_{M-3} & \ddots & \ddots & \ddots & \mathbb{M}_1 \\ \mathbb{M}_{M-2} & \mathbb{M}_{M-3} & \cdots & \cdots & \mathbb{M}_0 \end{pmatrix}, \tag{3.11}$$

and each block $\mathbb{M}_j$ is a symmetric Toeplitz matrix, defined as

$$M_j = \begin{pmatrix} g^{(\alpha)}_{0,j} & g^{(\alpha)}_{1,j} & g^{(\alpha)}_{2,j} & \cdots & g^{(\alpha)}_{M_x-2,j} \\ g^{(\alpha)}_{1,j} & g^{(\alpha)}_{0,j} & g^{(\alpha)}_{1,j} & \ddots & g^{(\alpha)}_{M_x-3,j} \\ \vdots & g^{(\alpha)}_{1,j} & g^{(\alpha)}_{0,j} & \ddots & \vdots \\ g^{(\alpha)}_{M_x-3,j} & \ddots & \ddots & \ddots & g^{(\alpha)}_{1,j} \\ g^{(\alpha)}_{M_x-2,j} & g^{(\alpha)}_{M_x-3,j} & \cdots & \cdots & g^{(\alpha)}_{0,j} \end{pmatrix}. \tag{3.12}$$

4 Theoretical analysis

In this section, we give some useful lemmas, then discuss the conservative laws and the convergence analysis in details.

4.1 Auxiliary lemmas

Lemma 4.1. *Suppose $u(x,y,t) \in C^3\Big(W^{2+\alpha,1}(\Omega), [0,T]\Big)$, then we have*

$$|(\mathcal{R})^n_{i,j}| \le \mathcal{C}_R(\tau^2 + h^2), \quad |\delta_t(\mathcal{R})^n_{i,j}| \le \mathcal{C}_{\widehat{R}}(\tau^2 + h^2), \quad 1 \le i \le M_x - 1, 1 \le j \le M_y - 1, \quad 0 \le n \le N-1,$$

where $\mathcal{C}_R$ and $\mathcal{C}_{\widehat{R}}$ are two positive constants which are independent of τ and h.

Proof. The proof is very similar with [7], here we omit it for brevity. □

Lemma 4.2 ([5]). *For time sequences $w = \{w^0, w^1, \cdots, w^n, w^{n+1}\}$ and $g = \{g^0, g^1, \cdots, g^n, g^{n+1}\}$, there is*

$$| \, 2\tau \sum_{k=0}^{n} g^k \delta_t w^k \, | \le \tau \sum_{k=1}^{n} | \, w^k \, |^2 + \tau \sum_{k=0}^{n-1} | \, \delta_t g^k \, |^2 + \frac{1}{2} | \, w^{n+1} \, |^2 + 2 \, | \, g^n \, |^2 + | \, w^0 \, |^2 + | \, g^0 \, |^2 .$$

Lemma 4.3 (Gronwall inequality I [8]). *Suppose that the discrete grid function $\{w^n \mid n = 0, 1, 2, \cdots, N; \; N\tau = T\}$ satisfies the following inequality*

$$w^n - w^{n-1} \le A\tau w^n + B\tau w^{n-1} + C_n\tau,$$

where A, B and C_n are non-negative constants, then

$$\max_{1\leq n\leq N} \mid w^n \mid \leq \Big(w^0 + \tau\sum_{k=1}^{N} C_k\Big)e^{2(A+B)T},$$

where τ is sufficiently small, such that $(A+B)\tau \leq \frac{N-1}{2N} < \frac{1}{2}$ $(N>1)$.

Lemma 4.4 (Gronwall inequality II [8])**.** *Suppose that the discrete grid function $\{w^n \mid n = 0,1,2,\cdots,N;\ N\tau = T\}$ satisfies the following inequality*

$$w^n \leq A + \tau\sum_{k=1}^{n} B_k w^k,$$

where A and B_k $(k=0,1,2,\cdots,N)$ are non-negative constants, then

$$\max_{1\leq n\leq N} \mid w^n \mid \leq A\exp(2\tau\sum_{k=1}^{N} B_k),$$

where τ is sufficiently small, such that $\tau \max_{1\leq n\leq N} B_k \leq 1/2$.

Lemma 4.5 ([6])**.** *For any grid function $U^n \in \Psi_h^n$, the inequality*

$$\Big\|U^n\Big\|_\infty \leq \mathcal{C}_{Inv}h^{-1}\Big\|U^n\Big\|. \tag{4.1}$$

holds.

Lemma 4.6. *For any grid function $U^n, V^n \in \Psi_h^n$, there exits a linear operator $\mathcal{L}^\alpha$, such that*

$$\Big((-\Delta_h)^{\alpha/2}U^n, V^n\Big) = \Big(\mathcal{L}^\alpha U^n, \mathcal{L}^\alpha V^n\Big), \tag{4.2}$$

together with Lemma 2.3, it implies

$$\mathcal{C}_{\alpha/2}\Big|U^n\Big|_{H^{\alpha/2}} \leq \Big\|\mathcal{L}^\alpha U^n\Big\| \leq \Big|U\Big|_{H^{\alpha/2}}, \quad \mathcal{C}_\alpha\Big|U^n\Big|_{H^\alpha} \leq \Big\|(-\Delta_h)^{\alpha/2}U^n\Big\| \leq \Big|U^n\Big|_{H^\alpha}, \tag{4.3}$$

where $\mathcal{C}_{\alpha/2} = (\frac{2}{\pi})^{\alpha/2}$ and $\mathcal{C}_\alpha = (\frac{2}{\pi})^\alpha$.

Proof. Notice that we can rewrite the inner product $\Big((-\Delta_h)^{\frac{\alpha}{2}}U^n, V^n\Big)$ in the matrix form as

$$\Big((-\Delta_h)^{\frac{\alpha}{2}}U^n, V^n\Big) = (\mathbb{M}U^n, V^n).$$

From Lemma 2.3 and the symmetric property form (3.11), we find the differentiation matrix $\mathbb{M}$ is a nonnegative definite and therefore can be decomposed as $\mathbb{M} = L^\alpha L^\alpha$ with L^α being a symmetric matrix. Consequently,

$$(\mathbb{M}U^n, V^n) = (L^\alpha L^\alpha U^n, V^n) = (L^\alpha U, L^\alpha V^n).$$

Denote the linear operator $\mathcal{L}^\alpha$ such that $\mathcal{L}^\alpha V^n = L^\alpha V^n$. We complete the proof. □

Lemma 4.7. *For any grid function $U^n \in \Psi_h^n$, we have*

$$\begin{aligned} &Im\Big((-\Delta_h)^{\alpha/2}U^{n+1/2}, U^{n+1/2}\Big) = 0, \\ &Re\Big((-\Delta_h)^{\alpha/2}U^{n+1/2}, \delta_t u^n\Big) = \frac{1}{2\tau}\Big(\Big\|\mathcal{L}^\alpha U^{n+1}\Big\|^2 - \Big\|\mathcal{L}^\alpha U^n\Big\|^2\Big). \end{aligned} \tag{4.4}$$

Proof. By admitting Lemma 4.3, the consequences are straightforward, here we omit it for brevity. □

Lemma 4.8 (Discrete mass and energy conservation). *The difference scheme (3.7) satisfies the mass and energy conservation law in discrete sense*

$$\delta_t \mathcal{M}^n = 0, \quad \mathcal{M}^n = \Big\|U^n\Big\|^2, \quad 0 \le n \le N-1 \tag{4.5}$$

$$\delta_t \mathcal{E}^n = 0, \quad \mathcal{E}^n = \Big\|\mathcal{L}^\alpha U^n\Big\|^2 - \frac{\beta}{2}\Big\|U^n\Big\|_4^4, \quad 0 \le n \le N-1. \tag{4.6}$$

Proof. Computing the discrete inner product of (3.7) with $U^{n+1/2}$ and $\delta_t U^n$, we then taking respectively the imaginary and real part to arrive at (4.5) and (4.6). This ends the proof. □

4.2 Convergence analysis

In this part, we carry out discussing the convergence of the scheme (3.7) by introducing a "cut-off" function to truncate the nonlinearity to a golbal Lipschitz function with compact support [9–12], which can be achieve by assuming the exact solution is bounded. Following such idea, it only can obtain the conditional convergence with the grid restriction $\tau \lesssim h$, we refer the reader to the associated works [13, 14]. Here, we apply the same idea as well as the "lifting" technique, i.e., taking the L^2-norm of both side of the error equations to derive the unconditional convergence consequence in L^2-norm. Further, by virtue of the maximum boundness of the numerical solution, we obtain the unconditional point-wise error estimate for the proposed scheme.

Denote

$$M_1 = \max_{0\le n\le N}\Big\{\Big\|u^n\Big\|_\infty\Big\}, \tag{4.7}$$

and we choose a smooth function $\sigma(\rho) \in C^\infty([0,\infty))$ defined as

$$\sigma(\rho) = \begin{cases} 1, & 0 \le |\rho| \le 1, \\ \exp\left(1 + \frac{1}{(|s|-1)^2-1}\right), & 1 \le |\rho| < 2, \\ 0, & |\rho| \ge 2. \end{cases}$$

Denote $M_0 = (\epsilon + M_1)^2 > 0$, where ϵ is a given positive constant, and define

$$F_{M_0}(\rho) = \sigma(\frac{\rho}{M_0})\rho, \quad 0 \le \rho < \infty, \tag{4.8}$$

then $F_{M_0}(\rho) \in C^\infty([0,\infty))$ is global Lipschitz and [9–11]

$$(1). \ \ |F_{M_0}(\rho_1) - F_{M_0}(\rho_2)| \le \mathcal{C}_L|\sqrt{\rho_1} - \sqrt{\rho_2}|, \quad 0 \le \rho_1, \rho_2 < \infty. \tag{4.9}$$

$$(2). \ \ |F_{M_0}(\rho)| \le \mathcal{C}_B, \quad 0 \le \rho < \infty. \tag{4.10}$$

In our convergence analysis, we take $\epsilon = \max\Big\{1, \mathcal{C}_{IV}\Big\}$, where $\mathcal{C}_{IV} = 2\mathcal{C}_0 T\sqrt{\mathcal{C}_I + \mathcal{C}_{III}^2}$, and

$$\mathcal{C}_{III} = \mathcal{C}_\alpha^{-1}\Big(2\sqrt{\mathcal{C}_I} + \sqrt{2\mathcal{C}_I\mathcal{C}_{II}} + \sqrt{(b-a)(d-c)}\mathcal{C}_R\Big),$$

$$\mathcal{C}_{II} = \Big(\beta^2\mathcal{C}_B^2 + \frac{\beta^2 M_1^2 \mathcal{C}_L^2}{2}\Big), \quad \mathcal{C}_I = \frac{\mathcal{C}_R^2(b-a)(d-c)T}{2}\exp\Big((1 + \frac{\beta^2 M_1^2\mathcal{C}_L^2}{4})T\Big),$$

which are positive constants independent of τ, h.

Theorem 4.1. *Suppose that $u(x,y,t) \in C^3\Big(W^{2+\alpha,1}(\Omega), [0,T]\Big)$ is the solution of (1.1). Let U^n be the solution of the numerical scheme (3.7). For sufficiently small steps τ, h, the scheme (3.7) is unconditionally convergent in L^2-norm, and we have*

$$\Big\|u^n - U^n\Big\| \le \sqrt{\mathcal{C}_I}(\tau^2 + h^2), \quad \Big\|U^n\Big\|_\infty \le \sqrt{M_0}, \quad \Big\|u^n - U^n\Big\|_\infty \le \epsilon, \quad 1 \le n \le N, \tag{4.11}$$

where $\mathcal{C}_I = \frac{\mathcal{C}_R^2(b-a)(d-c)T}{2}\exp\Big((1 + \frac{\beta^2 M_1^2\mathcal{C}_L^2}{4})T\Big)$.

Proof. The proof is mainly divided into two step:

Step 1: We establish the boundness of numerical solution U^n in L^∞-norm when $\tau \lesssim h$.

Choose $\psi^0 = U^0 \in \Psi_h^n$, and define $\psi^n \in \Psi_h^n$ $(n = 0, 1, \cdots)$, we consider the following difference scheme

$$\mathbf{i}\delta_t\psi^n - (-\Delta_h)^{\alpha/2}\psi^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(F_{M_0}\big(|\psi^{n+1}|^2\big) + F_{M_0}\big(|\psi^n|^2\big)\Big)\psi^{n+\frac{1}{2}} = 0. \tag{4.12}$$

Noticing the definition of function F_{M_0} and the fact $|u^n|^2 \le M_0$, the exact solution equation (3.5) can be reformulated as

$$\mathbf{i}\delta_t u^n - (-\Delta_h)^{\alpha/2}u^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(F_{M_0}\big(|u^{n+1}|^2\big) + F_{M_0}\big(|u^n|^2\big)\Big)u^{n+\frac{1}{2}} = \mathcal{R}^n. \tag{4.13}$$

In fact, ψ^n can be viewed as another approximation of u^n. Define the error function

$$\hat{\varepsilon}^n = u^n - \psi^n. \tag{4.14}$$

Subtracting (4.13) form (4.12), we obtain

$$\mathbf{i}\delta_t\hat{\varepsilon}^n - (-\Delta_h)^{\alpha/2}\hat{\varepsilon}^{n+\frac{1}{2}} + \hat{\xi}^n = \mathcal{R}^n, \tag{4.15}$$

where

$$\begin{aligned}
\hat{\xi}^n &= \frac{\beta}{2}\Big(F_{M_0}(|u^{n+1}|^2) + F_{M_0}(|u^n|^2)\Big)u^{n+\frac{1}{2}} - \frac{\beta}{2}\Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\psi^{n+\frac{1}{2}}\\
&= \frac{\beta}{2}\Big(F_{M_0}(|u^{n+1}|^2) + F_{M_0}(|u^n|^2)\Big)u^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\hat{\varepsilon}^{n+\frac{1}{2}}\\
&\quad - \frac{\beta}{2}\Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)u^{n+\frac{1}{2}}\\
&= \frac{\beta}{2}\Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\hat{\varepsilon}^{n+\frac{1}{2}} + \frac{\beta}{2}u^{n+\frac{1}{2}}\Big[\Big(F_{M_0}(|u^{n+1}|^2) + F_{M_0}(|u^n|^2)\Big)\\
&\quad - \Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\Big]\\
&= \hat{\xi}_1^n + \hat{\xi}_2^n.
\end{aligned} \tag{4.16}$$

It follows from (4.9), (4.16) and assumption on exact solution that

$$\Big\|\hat{\xi}_1^n\Big\|^2 = \Big\|\frac{\beta}{2}\Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\hat{\varepsilon}^{n+\frac{1}{2}}\Big\|^2 \le \frac{\beta^2\mathcal{C}_B^2}{2}\Big(\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 + \Big\|\hat{\varepsilon}^n\Big\|^2\Big), \tag{4.17}$$

and

$$\begin{aligned}
\Big\|\hat{\xi}_2^n\Big\|^2 &= \Big\|\frac{\beta}{2}U^{n+\frac{1}{2}}\Big[\Big(F_{M_0}(|u^{n+1}|^2) + F_{M_0}(|u^n|^2)\Big) - \Big(F_{M_0}(|\psi^{n+1}|^2) + F_{M_0}(|\psi^n|^2)\Big)\Big]\Big\|^2\\
&\le \frac{\beta^2 M_1^2\mathcal{C}_L^2}{4}\Big(\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 + \Big\|\hat{\varepsilon}^n\Big\|^2\Big).
\end{aligned} \tag{4.18}$$

Computing the discrete inner product of (4.15) with $\hat{\varepsilon}^{n+1/2}$, and taking imaginary part, by using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 - \Big\|\hat{\varepsilon}^n\Big\|^2 &= -\tau\mathrm{Im}\Big(\hat{\xi}^n, \hat{\varepsilon}^{n+1/2}\Big) + \tau\mathrm{Im}\Big(\mathcal{R}^n, \hat{\varepsilon}^{n+1/2}\Big)\\
&\le \frac{\tau}{2}\Big(\Big\|\hat{\xi}_2^n\Big\|^2 + \Big\|\mathcal{R}^n\Big\|^2 + \Big\|\hat{\varepsilon}^n\Big\|^2 + \Big\|\hat{\varepsilon}^{n+1}\Big\|^2\Big)\\
&\le \frac{\tau}{2}\Big\|\mathcal{R}^n\Big\|^2 + \frac{\tau}{2}\Big(1 + \frac{\beta^2 M_1^2\mathcal{C}_L^2}{4}\Big)\Big(\Big\|\hat{\varepsilon}^n\Big\|^2 + \Big\|\hat{\varepsilon}^{n+1}\Big\|^2\Big).
\end{aligned} \tag{4.19}$$

Follows from the Gronwall inequality (Lemma 4.3), we derive

$$\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 \le \Big(\Big\|\hat{\varepsilon}^0\Big\|^2 + \frac{\tau}{2}\sum_{k=1}^{N}\Big\|\mathcal{R}^k\Big\|^2\Big)\exp\Big(\big(1 + \frac{\beta^2 M_1^2\mathcal{C}_L^2}{4}\big)T\Big) \le \mathcal{C}_I(\tau^2 + h^2)^2, \tag{4.20}$$

where $\mathcal{C}_I = \frac{\mathcal{C}_R^2(b-a)(d-c)T}{2}\exp\Big((1+\frac{\beta^2 M_1^2\mathcal{C}_L^2}{4})T\Big)$.

From the inverse inequality (Lemma 4.5), we have

$$\Big\|\hat{\varepsilon}^{n+1}\Big\|_\infty \le \mathcal{C}_{Inv}h^{-1}\Big\|\hat{\varepsilon}^{n+1}\Big\| \le \mathcal{C}_{Inv}\sqrt{\mathcal{C}_I}h^{-1}(\tau^2+h^2). \tag{4.21}$$

Hence, if τ and h are sufficiently small, and $\tau \lesssim h$, we have

$$\Big\|\hat{\varepsilon}^{n+1}\Big\|_\infty \le \mathcal{C}_{Inv}\sqrt{\mathcal{C}_I}h \le 1, \quad \Big\|\psi^{n+1}\Big\|_\infty \le \Big\|\hat{\varepsilon}^{n+1}\Big\|_\infty + \Big\|U^{n+1}\Big\|_\infty \le 1+M_1 \le \sqrt{M_0}. \tag{4.22}$$

Step 2: We obtain the unconditional boundedness of numerical solution U^n in L^∞-norm by the "liftting" technique.

We reformulate the error equation (4.14) as

$$(-\Delta_h)^{\alpha/2}\hat{\varepsilon}^{n+\frac{1}{2}} = \mathbf{i}\delta_t\hat{\varepsilon}^n + \hat{\xi}^n - \mathcal{R}^n, \tag{4.23}$$

By taking the discrete L^2-norm of both sides of above equations

$$\Big\|(-\Delta_h)^{\alpha/2}\hat{\varepsilon}^{n+\frac{1}{2}}\Big\| \le \Big\|\delta_t\hat{\varepsilon}^n\Big\| + \Big\|\hat{\xi}^n\Big\| + \Big\|\mathcal{R}^n\Big\|. \tag{4.24}$$

According to (4.17,4.18,4.16), we have

$$\Big\|\hat{\xi}\Big\|^2 \le 2\Big(\Big\|\hat{\xi}_1\Big\|^2 + \Big\|\hat{\xi}_2\Big\|^2\Big) \le \mathcal{C}_{II}\Big(\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 + \Big\|\hat{\varepsilon}^n\Big\|^2\Big), \tag{4.25}$$

where $\mathcal{C}_{II} = \Big(\beta^2\mathcal{C}_B^2 + \frac{\beta^2 M_1^2\mathcal{C}_L^2}{2}\Big)$.

By assuming τ is sufficiently small and no more than 1, we derive from Lemma 2.3 that

$$\begin{aligned}\Big|\hat{\varepsilon}^{n+\frac{1}{2}}\Big|_{H^\alpha} &\le \mathcal{C}_\alpha^{-1}\Big\|(-\Delta_h)_2^{\alpha/2}\hat{\varepsilon}^{n+\frac{1}{2}}\Big\| \le \mathcal{C}_\alpha^{-1}\Big(\Big\|\delta_t\hat{\varepsilon}^n\Big\| + \Big\|\hat{\xi}^n\Big\| + \Big\|\mathcal{R}^n\Big\|\Big)\\ &\le \mathcal{C}_{III}\tau^{-1}(\tau^2+h^2).\end{aligned} \tag{4.26}$$

where $\mathcal{C}_{III} = \mathcal{C}_\alpha^{-1}\Big(2\sqrt{\mathcal{C}_I} + \sqrt{2\mathcal{C}_I\mathcal{C}_{II}} + \sqrt{(b-a)(d-c)}\mathcal{C}_R\Big)$.

With the help of Lemma 2.2 and (4.20), we have

$$\begin{aligned}\Big\|\hat{\varepsilon}^{n+1}\Big\|_\infty - \Big\|\hat{\varepsilon}^n\Big\|_\infty &\le 2\Big\|\hat{\varepsilon}^{n+\frac{1}{2}}\Big\|_\infty \le 2\mathcal{C}_0\Big(\Big\|\hat{\varepsilon}^{n+1}\Big\|^2 + \Big|\hat{\varepsilon}^n\Big|_{H^\alpha}^2\Big)^{1/2}\\ &\le 2\mathcal{C}_0\sqrt{\mathcal{C}_I + \mathcal{C}_{III}^2}\tau^{-1}(\tau^2+h^2).\end{aligned} \tag{4.27}$$

By recursion, we derive

$$\Big\|\hat{\varepsilon}^{n+1}\Big\|_\infty - \Big\|\hat{\varepsilon}^0\Big\|_\infty \le \mathcal{C}_{IV}\tau^{-2}(\tau^2+h^2), \tag{4.28}$$

where $\mathcal{C}_{IV} = 2\mathcal{C}_0 T\sqrt{\mathcal{C}_I + \mathcal{C}_{III}^2}$.

Thus, if τ, h are sufficiently small, and assuming $h \lesssim \tau$, we obtain

$$\|\hat{\varepsilon}^{n+1}\|_\infty \le \mathcal{C}_{IV}, \quad \|\psi^{n+1}\|_\infty \le \|\hat{\varepsilon}^{n+1}\|_\infty + \|u^{n+1}\|_\infty \le \mathcal{C}_{IV} + M_1 \le \sqrt{M_0}. \tag{4.29}$$

Together with (4.22) and (4.29), for any small enough τ, h, it is always true to have

$$\|\psi^n\|_\infty \le \sqrt{M_0}. \tag{4.30}$$

Recalling the definition of the function F_{M_0}, one can easily obtain

$$F_{M_0}(|\psi^n|^2) = |\psi^n|^2. \tag{4.31}$$

Hence, scheme (4.12) is equivalent to scheme (3.7), then we have

$$U^n = \psi^n, \quad \varepsilon^n = \hat{\varepsilon}^n, \quad 0 \le n \le N, \tag{4.32}$$

where $\varepsilon^n = u^n - U^n$.

Therefore, the consequence (4.11) is straightforward. This completes the proof. □

Theorem 4.2. *Suppose that $u(x,y,t) \in C^3\Big(W^{2+\alpha,1}(\Omega),[0,T]\Big)$ is the solution of (1.1). Let U^n be the solution of the numerical scheme (3.7). For sufficiently small steps τ, h, the scheme (3.7) is unconditionally convergent in L^∞-norm, and we have*

$$\Big\|u^n - U^n\Big\|_\infty \le \mathcal{C}_{VII}(\tau^2 + h^2), \quad 1 \le n \le N,$$

where $\mathcal{C}_{VII}$ is a positive constant which is independent of τ and h.

Proof. Subtracting (3.7) from (3.5) and denoting $\varepsilon^n = u^n - U^n$, we obtain the error equations

$$\mathbf{i}\delta_t \varepsilon^n - (-\Delta_h)^{\alpha/2}\varepsilon^{n+\frac{1}{2}} + \xi^n = \mathcal{R}^n, \tag{4.33}$$

where

$$\begin{aligned}
\xi^n &= \frac{\beta}{2}\Big(|u^{n+1}|^2 + |u^n|^2\Big)u^{n+\frac{1}{2}} - \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big)U^{n+\frac{1}{2}} \\
&= \frac{\beta}{2}\Big(|u^{n+1}|^2 + |u^n|^2\Big)u^{n+\frac{1}{2}} + \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big)\varepsilon^{n+\frac{1}{2}} - \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big)u^{n+\frac{1}{2}} \\
&= \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big)\varepsilon^{n+\frac{1}{2}} + \frac{\beta}{2}u^{n+\frac{1}{2}}\Big[\Big(|u^{n+1}|^2 + |u^n|^2\Big) - \Big(|U^{n+1}|^2 + |U^n|^2\Big)\Big] \\
&= \xi_1^n + \xi_2^n.
\end{aligned} \tag{4.34}$$

in which

$$\begin{aligned}
\xi_1^n &= \frac{\beta}{2}\Big(|U^{n+1}|^2 + |U^n|^2\Big)\varepsilon^{n+\frac{1}{2}} \\
&= \frac{\beta}{2}\Big(|u^{n+1}|^2 + |\varepsilon^{n+1}|^2 + |u^n|^2 + |\varepsilon^n|^2 - 2\mathrm{Re}(u^n\bar{\varepsilon}^n) - 2\mathrm{Re}(u^{n+1}\bar{\varepsilon}^{n+1})\Big)\varepsilon^{n+\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\xi_2^n &= \frac{\beta}{2}u^{n+\frac{1}{2}}\Big[\Big(|u^{n+1}|^2 + |u^n|^2\Big) - \Big(|U^{n+1}|^2 + |U^n|^2\Big)\Big] \\
&= \frac{\beta}{2}u^{n+\frac{1}{2}}\Big(\bar{\varepsilon}^{n+1}u^{n+1} + \bar{u}^{n+1}\varepsilon^{n+1} - \bar{\varepsilon}^{n+1}\varepsilon^{n+1} + \bar{\varepsilon}^n u^n + \bar{u}^n\varepsilon^n - \bar{\varepsilon}^n\varepsilon^n\Big).
\end{aligned}$$

Based on the boundedness consequences (4.11) and the assumption on the exact solution, we have

$$\Big\|\xi_1^n\Big\|^2 \le \frac{\beta^2(M_1+\epsilon)^4}{2}\Big(\Big\|\varepsilon^{n+1}\Big\|^2 + \Big\|\varepsilon^n\Big\|^2\Big), \tag{4.35}$$

$$\Big\|\xi_2^n\Big\|^2 \le \frac{\beta^2 M_1^2(2M_1+\epsilon)^2}{2}\Big(\Big\|\varepsilon^{n+1}\Big\|^2 + \Big\|\varepsilon^n\Big\|^2\Big), \tag{4.36}$$

$$\Big\|(-\Delta_h)_2^{\frac{\alpha}{2}}\xi_1^n\Big\|^2 \le \frac{\beta^2(M_1+\epsilon)^4}{2}\Big(\Big\|(-\Delta_h)_2^{\frac{\alpha}{4}}\varepsilon^{n+1}\Big\|^2 + \Big\|(-\Delta_h)_2^{\frac{\alpha}{4}}\varepsilon^n\Big\|^2\Big), \tag{4.37}$$

$$\Big\|(-\Delta_h)_2^{\frac{\alpha}{2}}\xi_2^n\Big\|^2 \le \frac{\beta^2 M_1^2(2M_1+\epsilon)^2}{2}\Big(\Big\|(-\Delta_h)_2^{\frac{\alpha}{4}}\varepsilon^{n+1}\Big\|^2 + \Big\|(-\Delta_h)_2^{\frac{\alpha}{4}}\varepsilon^n\Big\|^2\Big). \tag{4.38}$$

Thus, we obtain

$$\Big\|\xi^n\Big\|^2 \le 2\Big(\Big\|\xi_1^n\Big\|^2 + \Big\|\xi_2^n\Big\|^2\Big) \le \mathcal{C}_{pV}\Big(\Big\|\varepsilon^{n+1}\Big\|^2 + \Big\|\varepsilon^n\Big\|^2\Big), \tag{4.39}$$

$$\begin{aligned}
\Big\|(-\Delta_h)^{\frac{\alpha}{2}}\xi^n\Big\|^2 &\le 2\Big(\Big\|(-\Delta_h)^{\frac{\alpha}{2}}\xi_1^n\Big\|^2 + \Big\|(-\Delta_h)_2^{\frac{\alpha}{2}}\xi_2^n\Big\|^2\Big) \\
&\le \mathcal{C}_V\Big(\Big\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\Big\|^2 + \Big\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^n\Big\|^2\Big),
\end{aligned} \tag{4.40}$$

where $\mathcal{C}_V = \beta^2(M_1+\epsilon)^4 + \beta^2 M_1^2(2M_1+\epsilon)^2$.

Computing the discrete inner product of (4.23) with $2\tau\delta_t(-\Delta_h)^{\alpha/2}\varepsilon^n$, and taking the real part, we obtain

$$\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2 - \left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n}\right\|^2 = 2\tau\mathrm{Re}\Big(\xi^n, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^n\Big) - 2\tau\mathrm{Re}\Big(\mathcal{R}^n, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^n\Big). \tag{4.41}$$

where

$$\begin{aligned}
2\tau\mathrm{Re}\Big(\xi^n, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^n\Big) &= 2\tau\mathrm{Re}\Big((-\Delta_h)^{\alpha/2}\xi^n, \delta_t\varepsilon^n\Big)\\
&= 2\tau\mathrm{Re}\Big((-\Delta_h)^{\alpha/2}\xi^n, -\mathbf{i}(-\Delta_h)^{\alpha/2}\varepsilon^{n+1/2} + \mathbf{i}\xi^n - \mathbf{i}\mathcal{R}^n\Big) \qquad (4.42)\\
&= 2\tau\mathrm{Im}\Big((-\Delta_h)^{\alpha/2}\xi^n, (-\Delta_h)^{\alpha/2}\varepsilon^{n+1/2}\Big) + 2\tau\mathrm{Im}\Big((-\Delta_h)^{\alpha/2}\xi^n, \mathcal{R}^n\Big),
\end{aligned}$$

In virtue of (4.40) and the Cauchy-Schwarz inequality

$$\begin{aligned}
&2\tau\mathrm{Im}\Big((-\Delta_h)^{\alpha/2}\xi^n, (-\Delta_h)^{\alpha/2}\varepsilon^{n+1/2}\Big) \le 4\tau\Big(\left\|(-\Delta_h)^{\frac{\alpha}{2}}\xi^n\right\|^2 + \left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1/2}\right\|^2\Big)\\
&\quad\le 4\mathcal{C}_{IV}\tau\Big(\left\|(-\Delta_h)^{\alpha/2}\varepsilon^n\right\|^2 + \left\|(-\Delta_h)^{\alpha/2}\varepsilon^{n+1}\right\|^2\Big) + 2\tau\Big(\left\|(-\Delta_h)^{\alpha/2}\varepsilon^{n+1}\right\|^2 + \left\|(-\Delta_h)^{\alpha/2}\varepsilon^n\right\|^2\Big)\\
&\quad\le (4\mathcal{C}_V + 2)\tau\Big(\left\|(-\Delta_h)^{\alpha/2}\varepsilon^{n+1}\right\|^2 + \left\|(-\Delta_h)^{\alpha/2}\varepsilon^n\right\|^2\Big). \qquad (4.43)
\end{aligned}$$

and

$$\begin{aligned}
2\tau\mathrm{Im}\Big((-\Delta_h)^{\alpha/2}\xi^n, \mathcal{R}^n\Big) &\le 4\tau\Big(\left\|(-\Delta_h)^{\alpha/2}\xi^n\right\|^2 + \left\|\mathcal{R}^n\right\|^2\Big) \qquad (4.44)\\
&\le 4\mathcal{C}_V\tau\Big(\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2 + \left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^n\right\|^2\Big) + 4\mathcal{C}_R^2(b-a)(d-c)\tau(\tau^2+h^2)^2.
\end{aligned}$$

Substituting (4.43) and (4.44) into (4.41), we have

$$\begin{aligned}
\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2 - \left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n}\right\|^2 &\le (8\mathcal{C}_{IV}+2)\tau\Big(\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2 + \left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^n\right\|^2\Big) \qquad (4.45)\\
&\quad + 2\tau\mathrm{Re}(\mathcal{R}^n, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^n) + 4\mathcal{C}_R^2(b-a)(d-c)\tau(\tau^2+h^2)^2,
\end{aligned}$$

Summing up the superscript n from 0 to n_0 and then replacing n_0 by n, we arrive at

$$\begin{aligned}
\left\|(-\Delta_h)^{\frac{\alpha}{2}}_{2}\varepsilon^{n+1}\right\|^2 &\le 4(4\mathcal{C}_V+1)\tau\sum_{k=0}^{n+1}\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^k\right\|^2 + 2\tau\sum_{k=0}^{n}\mathrm{Re}\Big(\mathcal{R}^k, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^k\Big) \qquad (4.46)\\
&\quad + 4\mathcal{C}_R^2(b-a)(d-c)T(\tau^2+h^2)^2,
\end{aligned}$$

where

$$\begin{aligned}
2\tau\sum_{k=0}^{n}\mathrm{Re}\Big(\mathcal{R}^k, \delta_t(-\Delta_h)^{\alpha/2}\varepsilon^k\Big) &\le \tau\sum_{k=0}^{n}\left\|(-\Delta_h)^{\alpha/2}\varepsilon^k\right\|^2 + \frac{1}{2}\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2\\
&\quad + (\mathcal{C}_{\widehat{R}}^2 + 2\mathcal{C}_R^2)T(b-a)(d-c)(\tau^2+h^2)^2, \qquad (4.47)
\end{aligned}$$

Substituting (4.47) into (4.46) and recalling the Gronwall (Lemma 4.3) inequality, we derive

$$\begin{aligned}
\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^{n+1}\right\|^2 &\le 16(2\mathcal{C}_V+1)\tau\sum_{k=0}^{n+1}\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^k\right\|^2 + 2(5\mathcal{C}_{\widehat{R}}^2 + 2\mathcal{C}_R^2)T(b-a)(d-c)(\tau^2+h^2)^2\\
&\le \mathcal{C}_{VI}(\tau^2+h^2)^2, \qquad (4.48)
\end{aligned}$$

where $\mathcal{C}_{VI} = 2(5\mathcal{C}_{\widehat{R}}^2 + 2\mathcal{C}_R^2)T(b-a)(d-c)\exp\Big(32(2\mathcal{C}_V+1)T\Big)$.

By virtue of (4.3) is therefore

$$\left|\varepsilon^n\right|_{H^\alpha}^2 \le \mathcal{C}_\alpha^{-2}\left\|(-\Delta_h)^{\frac{\alpha}{2}}\varepsilon^n\right\|^2 \le \mathcal{C}_\alpha^{-2}\mathcal{C}_{VI}\Big(\tau^2+h^2\Big)^2, \tag{4.49}$$

With the help of the discrete uniform Sobolev inequality (Lemma 2.2), we obtain the error estimate in L^∞-norm, it follows that

$$\left\|\varepsilon^n\right\|_\infty \le \mathcal{C}_0\left\|\varepsilon^n\right\|_{H^\alpha} = \mathcal{C}_0\sqrt{\left|\varepsilon^{n+1}\right|^2_{H^\alpha} + \left\|\varepsilon^{n+1}\right\|^2} \le \mathcal{C}_{VII}(\tau^2+h^2), \tag{4.50}$$

where $\mathcal{C}_{VII} = \mathcal{C}_0\sqrt{\mathcal{C}_\alpha^{-2}\mathcal{C}_{VI} + \mathcal{C}_I}$.

This completes the proof. □

Remark 4.1. *It is worth to noting that the proof of the existence and uniqueness of the numerical scheme (3.7) is similar to Ref. [13], where the key property (4.2) is used.*

5 Numerical experiments

In this section, we pay attentions to the fast implementation for the proposed scheme. Later on, we report some numerical results by employing the fast solvers to the 2D SFNS equation.

5.1 Fast implementation

In practice, we solve the nonlinear equations (3.10) by the following iterative scheme

$$\left(\mathbf{i}I - \frac{\tau}{2}\mathbb{M}\right)U^{n+1,s+1} = \left(\mathbf{i}I + \frac{\tau}{2}\mathbb{M}\right)U^n - \frac{\beta\tau}{4}\left(|U^{n+1,s}|^2 + |U^n|^2\right)\left(U^{n+1,s} + U^n\right), \tag{5.1}$$

where

$$U^{n+1,0} = \begin{cases} U^n, & n = 0, \\ 2U^n - U^{n-1}, & n \ge 1. \end{cases}$$

Then $U^{n+1,s}$ numerically converges to the solution U^{n+1} if there satisfies $\left\|U^{n+1,s+1} - U^{n+1,s}\right\| \le TOL$ for the given stopping criteria TOL.

Denote $A = \left(\mathbf{i}I - \frac{\tau}{2}\mathbb{M}\right)$ and let b represents the right-hand side of (5.1), then in each iteration one has to solve a linear algebraic system like $Ax = b$ for the unknowns. In the practical calculation, we employ the Krylov subspace method, more specifically, the conjugated gradient method (CG) as the linear solver. In view of the coefficient matrix A enjoys a block Toeplitz structure as (3.11), the multiplication AU can be fast computed by the 2D FFT algorithm [4] in Matlab codes (see Appendix), i.e., $AU = \texttt{toepblockmultip}(\mathbf{ac}, U)$, where $\mathbf{ac}$ is the first column vector of matrix A. Thus, the function `toepblockmultip` can be used in the CG solver to accelerate the matrix-vector product. In Fig. 1, we plot the spectrum distributions of the coefficient matrix A which are obtained by setting the computing domain as $\Omega = [-1,1]^2$ with $\tau = h = 1/32$. Obviously, the spectra of the matrix A are not grouped around 1, and the matrix A will be ill-conditioned. It is no doubt that this fact will cause the CG method to slow down. It is necessary for us to consider a suitable preconditioner P for the linear system $Ax = b$, then we turn to solve the preconditioned system

$$P^{-1}Ax = P^{-1}b. \tag{5.2}$$

In the current paper, we choose the block circulant preconditioner P [4] which implemented in Matlab codes (see Appendix)

$$pc = \texttt{precondvec}(\mathbf{ac}), \tag{5.3}$$

where pc is the first column of the preconditioner P.

In Fig. 2, we plot the spectrum distributions of the matrix $P^{-1}A$ for different α, The eigenvalues of $P^{-1}A$ are well grouped around 1 expect for a few outliers, and the vast majority of the eigenvalues are well separated away from 0. This fact exactly confirms that the preconditioner is effective for the preconditioned system (5.2).

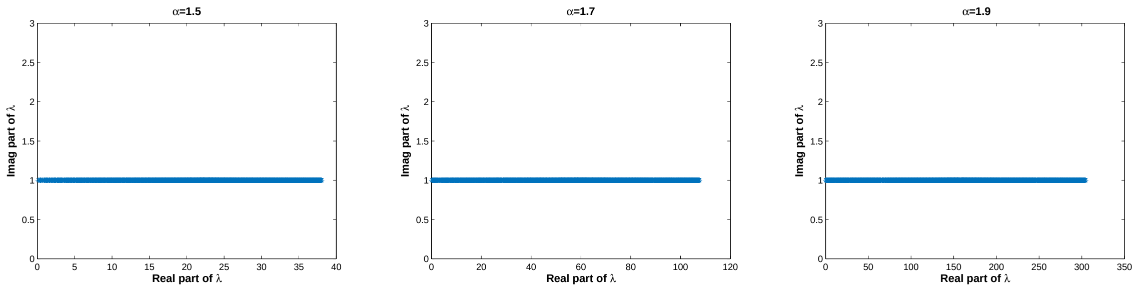

Fig. 1: Spectrum distributions of the coefficient matrix A for different α.

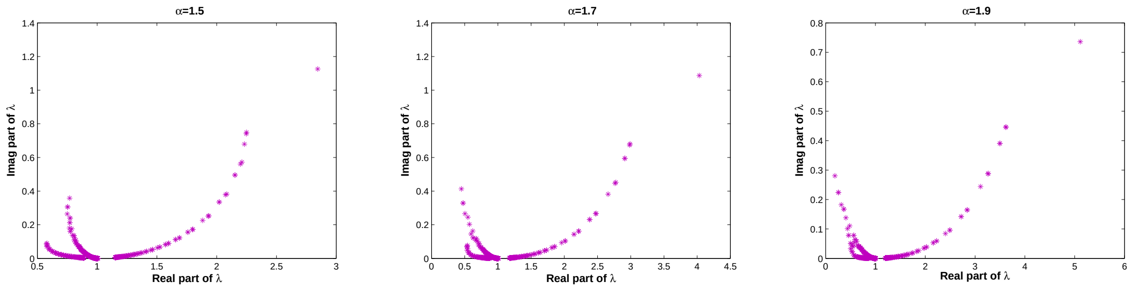

Fig. 2: Spectrum distributions of the matrix $P^{-1}A$ for different α.

5.2 Numerical examples

In this part, we carry out the simulations by using the Matlab R2014a software on a Lenovo computer, Intel(R) Core(TM) i7-9750H, 2.6 GHz CPU machine with 16 GB RAM. In order to illustrating the efficiency of the proposed fast solvers, we employ the proposed solvers to the 2D SFNS equation for comparisons. For simplicity, we denote the solvers as abbreviations in the following

- -Backslash- the linear system of (5.1) solved by the built-in function "\" in Matlab.
- -CGfast- the linear system of (5.1) solved by CG method and the 2D FFT algorithm.
- -PCGfast- the linear system of (5.1) solved by CG method with preconditioner and the 2D FFT algorithm.

We test the convergence order with $\tau = h$ by

$$Rate = \frac{\log\left(\left\|\varepsilon(h_1,\tau)\right\|_* \Big/ \left\|\varepsilon(h_2,\tau)\right\|_*\right)}{\log(h_1/h_2)},$$

where $\|\varepsilon(h,\tau)\|_*$ can be the norm $\|u^N - U^N\|$ and $\|u^N - U^N\|_\infty$ with steps τ and h.

Define the relative errors of mass and energy as

$$RM^n = \frac{\left|\mathcal{M}^n - \mathcal{M}^0\right|}{\mathcal{M}^0}, \qquad RE^n = \frac{\left|\mathcal{E}^n - \mathcal{E}^0\right|}{\mathcal{E}^0}.$$

Example 5.1 (**Numerical accuracy**). Consider the following equation with a source term

$$\mathbf{i}u_t - (-\Delta)^{\frac{\alpha}{2}}u + 2|u|^2u = f(x,y,t), \quad (x,y,t) \in (-1,1)\times(-1,1)\times(0,T] \tag{5.4}$$

The initial date, and the source term $f(x,y,t)$ are determined by the exact solution

$$u(x,y,t) = \mathbf{i}\exp(-t)(1-x^2)^2(1-y^2)^2. \tag{5.5}$$

We note that the solution satisfies homogeneous Dirichlet boundary condition

$$u|_{\partial\Omega} = 0, \quad \Omega = [-1, 1] \times [-1, 1]. \tag{5.6}$$

In practice, the function f is numerically prepared with a fine mesh size h/κ $(\kappa > 1, \kappa \in \mathbb{N}_+)$, i.e., computing $f(x, y, t) \approx \mathbf{i}u_t - (-\Delta_{h/\kappa})^{\frac{\alpha}{2}} u + 2|u|^2 u$ with u.

We employ the PCGfast , CGfast method and the Backslash method for solving Ex. 5.1 in $t \in [0, 1]$ with the stopping criteria $TOL = 1e - 10$, the errors in L^2-norm, $L^\infty-$norm and the associated CPU time costs for different time steps and fractional orders are plotted in Tab. 1. Although, the PCGfast method and the CGfast both can fast solve Ex. 5.1, it is obvious to find that the CPU time requirement is much fewer than the CGfast method. Moreover, when we use the Backslash method for Ex. 5.1 with $\tau = h = 1/64, 1/128$, the current computer will not work due to the out of memory. this exactly confirms that the PCGfast method is more suited than the CGfast method and the Backslash method, even if the coefficient matrix become ill-conditioned. Subsequently, the PCGfast method is utilized to solve Ex. 5.1 in $t \in [0, 1]$, and the stopping criteria $TOL = 1e - 10$. The numerical results are listed in Tab. 2, which verifies that our proposed scheme enjoys the second-order convergence in L^2- and $L^\infty-$norm in time and space, respectively. For testing the conservative laws, we turn to solve Ex. 5.1 without the source term function (i.e. $f(x, y, t) = 0$) with $\tau = h = 1/5$ in $t \in [0, 100]$ with the stopping criteria $TOL = 1e - 14$, the relative errors of mass and energy are listed in Tabs. 3,4, which shows that the proposed scheme possesses the great capacity of preserving the discrete mass and energy.

Table. 1: Comparisons of CPU time of the PCGfast method, the CGfast method and the Backslash method for Ex. 5.1 .

$\tau = h$	α	Errors		CPU time		
		$\left\|u^n - U^n\right\|$	$\left\|u^n - U^n\right\|_\infty$	PCGfast	CGfast	Backslash
1/32	1.3	3.4370e-04	3.0061e-04	3.41s	4.69s	331.36s
	1.5	4.7353e-04	5.9198e-04	3.95s	7.45s	333.96s
	1.7	6.4318e-04	6.3004e-04	5.27s	11.15s	331.90s
	1.9	8.6626e-04	8.4077e-04	5.82s	15.37s	343.44s
	2.0	9.6453e-04	9.2922e-04	6.07s	17.55s	343.55s
1/64	1.3	7.1388e-05	6.1496e-05	34.97s	63.59s	out of memory
	1.5	1.0814e-04	1.4599e-04	42.36s	105.82s	out of memory
	1.7	1.4998e-04	1.5463e-04	52.19s	172.92s	out of memory
	1.9	2.1080e-04	2.0031e-04	68.66s	265.63s	out of memory
	2.0	2.4079e-04	2.2959e-04	72.88s	316.70s	out of memory
1/128	1.3	1.2853e-05	1.2727e-05	249.20s	516.05s	out of memory
	1.5	2.1576e-05	3.2816e-05	318.01s	964.69s	out of memory
	1.7	3.3784e-05	3.2096e-05	401.57s	1671.40s	out of memory
	1.9	5.1105e-05	5.2712e-05	540.66s	2931.00s	out of memory
	2.0	6.0165e-05	5.8918e-05	620.18s	9295.10s	out of memory

Example 5.2. Consider the 2D SFNS equation (1.1) and taking $\beta = 1$, The initial date is chosen as the following solitary wave

$$u(x, y, 0) = \frac{2}{\sqrt{\pi}} \exp(-(x^2 + y^2)), \quad (x, y) \in (-10, 10) \times (-10, 10). \tag{5.7}$$

Since the initial value $u(x, y, 0)$ exponentially decays to zero when (x, y) is away from the origin, we impose the homogeneous boundary condition in the computation.

Table. 2: Errors and corresponding spatial and temporal observation orders of the PCGfast with $\tau = h$ for EX. 5.1.

α	$\tau = h$	$\|u^n - U^n\|$	Rate	$\|u^n - U^n\|_\infty$	Rate	CPU time
1.2	1/8	5.7800e-03	-	4.6540e-03	-	0.23s
	1/16	1.3304e-03	2.1193	1.0201e-03	2.1898	1.95s
	1/32	3.0098e-04	2.1440	2.2735e-04	2.1657	5.21s
	1/64	5.9241e-05	2.3450	4.5191e-05	2.3308	45.48s
1.4	1/8	7.6029e-03	-	8.6786e-03	-	0.19s
	1/16	1.7654e-03	2.1066	1.9351e-03	2.1651	1.95s
	1/32	4.0706e-04	2.1166	5.1270e-04	1.9162	6.79s
	1/64	9.0194e-05	2.1741	1.4019e-04	1.8707	81.68s
1.6	1/8	1.0215e-02	-	1.0610e-02	-	0.21s
	1/16	2.3572e-03	2.1155	2.2141e-03	2.2607	2.49s
	1/32	5.4315e-04	2.1177	5.4177e-04	2.0310	9.39s
	1/64	1.2363e-04	2.1353	1.2128e-04	2.1594	130.65s
1.8	1/8	1.3302e-02	-	1.3100e-02	-	0.23s
	1/16	3.1636e-03	2.0720	3.1320e-03	2.0644	3.01s
	1/32	7.5372e-04	2.0695	8.3538e-04	1.9066	12.93s
	1/64	1.7935e-04	2.0713	2.1643e-04	1.9485	204.50s

Table. 3: Errors of the discrete energy $\mathcal{E}$ at different time with $\tau = h = 1/5$ for EX. 5.1.

t	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
20	1.2119e-13	2.2040e-13	3.5018e-13	5.3037e-13	8.1772e-13
40	2.4127e-13	4.3977e-13	6.9402e-13	1.0620e-12	1.6353e-12
60	3.6227e-13	6.5667e-13	1.0403e-12	1.5933e-12	2.4547e-12
80	4.9049e-13	8.7429e-13	1.3900e-12	2.1253e-12	3.2779e-12
100	6.1741e-13	1.0934e-12	1.7368e-12	2.6566e-12	4.0941e-12

Table. 4: Errors of the discrete mass $\mathcal{M}$ at different time with $\tau = h = 1/5$ for EX. 5.1.

t	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
20	1.7700e-13	2.8978e-13	4.4661e-13	6.6562e-13	1.0181e-12
40	3.5315e-13	5.7872e-13	8.8767e-13	1.3336e-12	2.0352e-12
60	5.3132e-13	8.6380e-13	1.3297e-12	2.0011e-12	3.0538e-12
80	7.1387e-13	1.1514e-12	1.7765e-12	2.6679e-12	4.0753e-12
100	8.9826e-13	1.4403e-12	2.2193e-12	3.3352e-12	5.0909e-12

In this simulation, we apply the PCGfast method for solving Ex. 5.2 with $\tau = h = 1/10$ and the stopping criteria $TOL = 1e-14$in $t \in [0, 1]$, the numerical solutions $|U^n|$ and the associated contour graph at $t = 1$ are shown in Fig. 3. It is obvious to find that the solitary wave propagates to the bound as a standard circle, and the expansion and propagation of solitons become fast as the fractional order α increases. In Figs. 4,5, we validate the conservative prosperities of the proposed scheme, i.e. the mass and the energy conservation. By setting $\tau = h = 1/5$, we find the discrete energy is depending on the values of fractional order α, but the discrete mass dose not change with α. Also, the relative errors of the mass and energy uniformly reach to the machine accuracy with long time evolution, which illustrates that the proposed scheme successfully preserves the discrete invariants.

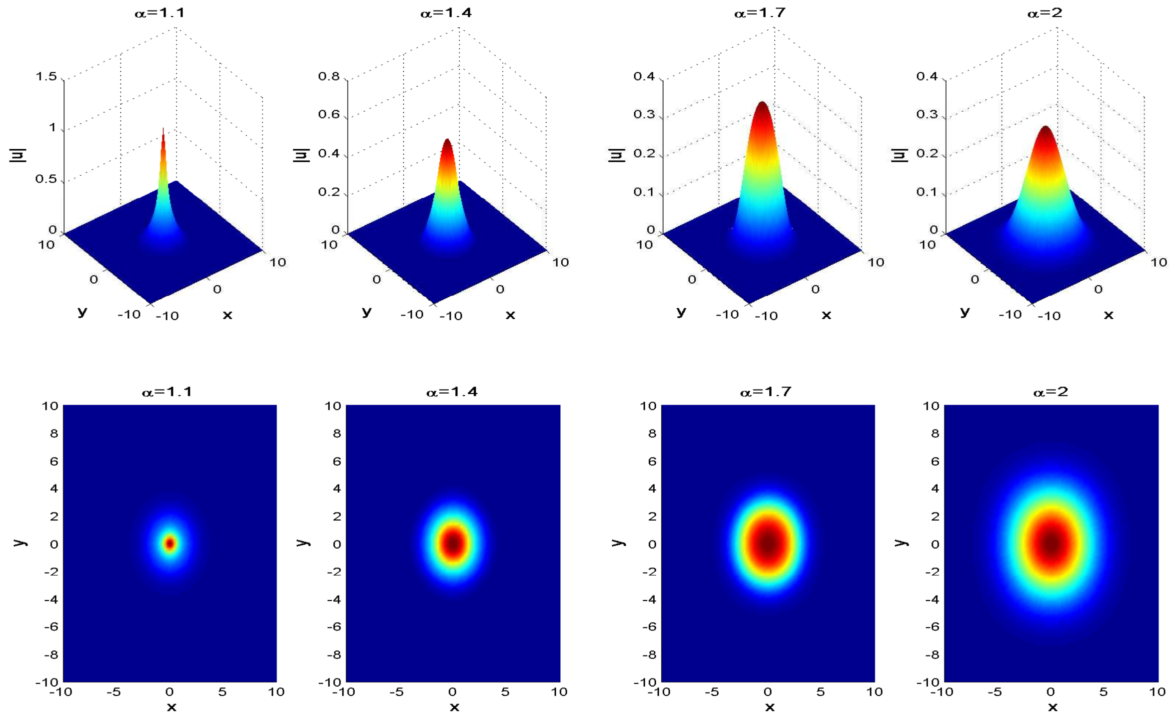

Fig. 3: Profile (up) and contour plot (down) of numerical solution with $\tau = h = 1/5$ at $t = 1$.

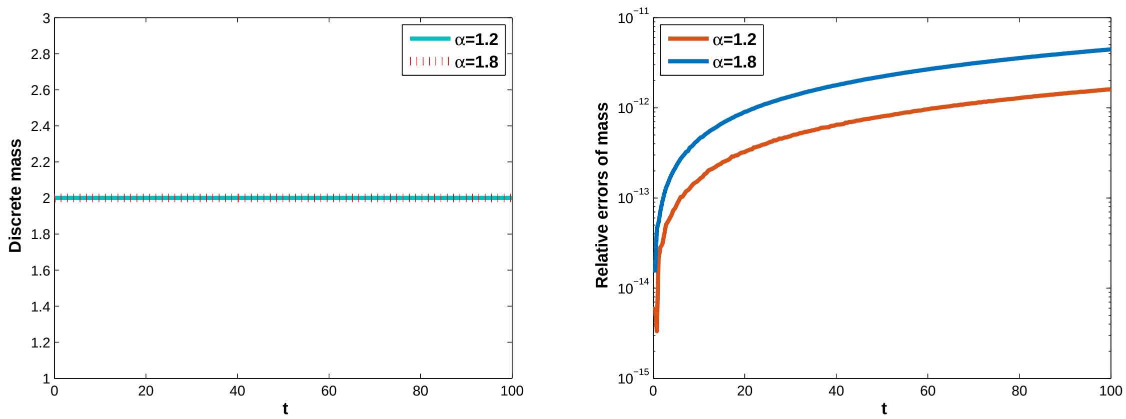

Fig. 4: Time evolution of thep discrete mass (Left) and the relative errors of mass (Right) with $\tau = h = 1/5$.

6 Conclusion

In this paper, we develop a conservative difference scheme for the 2D SFNS equation, the rigorous theoretical analysis are provided, including the conservative laws and the unconditional second-order convergence in L^2 and L^∞ norm. The unconditional point-wise error estimate is first obtained for the 2D SFNS equation. Further, the fast solvers are provided to accelerate the proposed scheme to obtain the numerical solutions. Extensive numerical results are reported to verify the correctness of the theoretical results. Our numerical algorithms and the unconditional convergence analysis can be extended to the 2D strongly coupled SFNS equation, the 2D SFNS equation with wave operator, and the 2D space-fractional Kelin-Gordon-Schrödinger equation, and so on, which will be our future work.

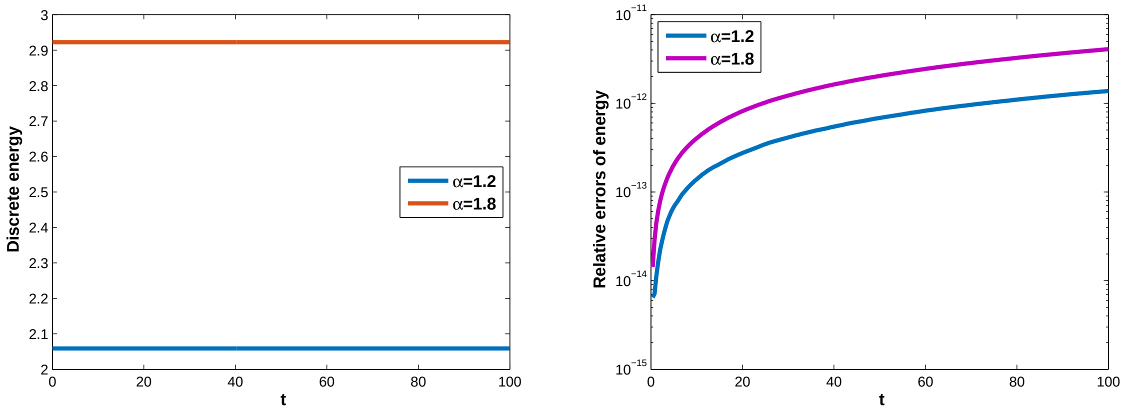

Fig. 5: Time evolution of the discrete energy (Left) and the relative errors of energy (Right) with $\tau = h = 1/5$.

7 Acknowledgements

The authors would like to express the thanks to the referees for their valuable comments and suggestions. This work is supported by the National Key Research and Development Project of China (Grant No. 2018YFC1504205), the National Natural Science Foundation of China (Grant Nos. 11771213, 61872422, 11971242), the Key Project of Jiangsu University Natural Science Foundation (Grant No. 18KJA110003), the Postgraduate Research & Practice Innovation Program of Jiangsu Province (Grant No. KYCX20_1165), and the Priority Academic Program Development of Jiangsu Higher Education Institutions.

8 Appendix: Matlab code

The purpose of this appendix is to briefly describe the Matlab code, which plays an essential role in our algorithm implementations, and has the following format:

$$\mathtt{A} * \mathtt{U} = \mathtt{toepblockmultip}(\mathbf{ac}, \mathtt{U});$$

$$\mathtt{pc} = \mathtt{precondvec}(\mathtt{ac});$$

where **ac** is the first column of matrix A, and the matrix A possesses a block Toeplitz structure as the form of (3.11). The function $\mathtt{toepblockmultip}(\mathbf{ac}, \mathtt{U})$ can reduce the multiplications of $A * U$ from $\mathcal{O}(M^4)$ to $\mathcal{O}\big(M^2 \log(M)\big)$ and the memory requirements from $\mathcal{O}(M^4)$ to $\mathcal{O}(M^2)$, where M is the spatial grid mesh number. The function `precondvec(ac)` is to derive the first column of the preconditioner matrix.

```
function AU = toepblockmultip( gcoe,x )
    % Fast compute A*U by the two-dimensional FFT algorithm
    % U: is a column vector; x: is a matrix form of U
    % ac: is the first column of matrix A; gcoe: is a matrix form of ac
    mx = size( gcoe,1 ); m = mx^2;
    U = [ reshape( x,mx,mx );zeros( mx ) ];
    v = [ reshape( U,2 * m,1 );zeros( 2 * m,1 ) ];
    Cc = [ gcoe;zeros( 1,mx ); gcoe( end:-1:2,1:end ) ];
    cc = reshape( Cc,2 * m,1 );
    tt = reshape( Cc( 1:end,end:-1:2 ),2 * m - 2 * mx,1 );
    ac = [ cc;zeros( 2 * mx,1 );tt ];
    Ac = reshape( ac,2*mx,2*mx );
    V =  reshape( v,2*mx,2*mx );
    YY = ifft2( fft2( Ac ).*fft2( V ) );
    yy = reshape( YY,4*m,1 );
```

```
    y = ( yy( 1:2 * m ) );
    w = reshape( y,2 * mx,mx );
    AU = reshape( w( 1:mx,1:mx ),m,1 );
end
```

```
function pc=precondvec(gcoe)
    n=size( gcoe,1 );   pc=zeros(n,n);  m = n+1;
    pc( 1,1 ) = gcoe( 1,1 );
    for j = 1:m-2
        jj = j+1;
       pc( 1,jj ) = ( ( n-j ) * n * gcoe( 1,jj ) + j * n * gcoe( 1,n-jj+2 ) ) / ( n^2 );
    end
    for k = 1:m-2
        kk = k+1;
       pc( kk,1 ) = ( n * ( n-k ) * gcoe( kk,1 ) + n * k * gcoe( n-kk+2,1 ) ) / ( n^2 );
    end
    for j=1:m-2
        jj = j+1;
        for k = 1:m-2
            kk = k+1;
           pc( kk,jj ) = ( ( n-j ) * ( n-k ) * gcoe( kk,jj ) + j * ( n-k ) * gcoe( ...
               kk,n-jj+2 )+ ( n-j ) * k * gcoe( n-kk+2,jj ) + j * k * gcoe( ...
               n-kk+2,n-jj+2 ) ) / ( n^2 );
        end
    end
end
```

References

[1] Du Q., Gunzburger M., Lehoucq R., Zhou K., Analysis and approximation of nonlocal diffusion problems with volume constraints, *SIAM Review*, **54** (2012), 667-696.

[2] Radulescu V., Bisci G., Servadei R., Variational methods for nonlocal fractional problems, Cambridge university press, Cambridge, 2016.

[3] Valdinoci E., From the long jump random walk to the fractional Laplacian, *Bol. Soc. Esp. Mat. Apl. Sema*, **49** (2009), 33-44.

[4] Hao Z., Zhang Z., Du R., Fractional centered difference scheme for high-dimensional integral fractional Laplacian, *J. Comput. Phys.,* 2020, doi: 10.1016/j.jcp.2020.109851.

[5] Hu D., Cai W., Song Y., Wang Y., A fourth-order dissipation-preserving algorithm with fast implementation for space fractional nonlinear damped wave equations, *Commun. Nonlinear Sci. Numer. Simul.,* **91** (2020), doi: 10.1016/j.cnsns.2020.105432.

[6] Sun W., Wang J.. Optimal error analysis of Crank-Nicolson schemes for a coupled nonlinear Schrödinger system in 3D, *J. Comput. Appl. Math.,* **317** (2017), 685-699.

[7] Wang P., Huang C., Zhao L., Point-wise error estimate of a conservative difference scheme for the fractional Schrödinger equation, *J. Comput. Appl. Math.,* **306** (2016), 231C247.

[8] Zhou Y., Application of discrete functional analysis to the finite difierence methods, International Academic Publishers, Beijing, 1990.

[9] Bao W., Cai Y., Optimal error estimates of finite difference methods for the Gross-Pitaevskii equation with angular momentum rotation, Math. of Comput., **82** (2011), 99-128.

[10] Bao W., Cai Y., Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator, *SIAM J. Numer. Anal.,* **50** (2012), 492C521.

[11] Wang T., Jiang J., Xue, X., Unconditional and optimal H 1 error estimate of a Crank-Nicolson finite difference scheme for the nonlinear Schrödinger equation, J. Math. Anal. Appl., **459** (2018), 945C958.

[12] Thomee V., Galerkin Finite Element Methods for Parabolic Problems, Springer, Berlin, 1997.

[13] Wang P., Huang C., An energy conservative difference scheme for the nonlinear fractional Schrödinger equations, *J. Comput. Phys.,* **293** (2015), 238-251.

[14] Yin B., Wang J., Liu Y., Li H., A structure preserving difference scheme with fast algorithms for high dimensional nonlinear space-fractional Schrödinger equations, *J. Comput. Phys.,* (2020), doi: 10.1016/j.jcp.2020.109869.

[15] Lischkea A., Panga G., Guliana M., Song F., Glusab C., Zheng X., Mao Z., Cai W., Meerschaert M., Ainsworth M., Karniadakis G., What is the fractional Laplacian? A comparative review with new results, *J. Comput. Phys.,* **404** (2020), doi: 10.1016/j.jcp.2019.109009.

[16] Pozrikidis C., The Fractional Laplacian, CRC Press, Boca Raton, 2016.

[17] Felmer P., Quaas A., Tan J., Positive solutions of the nonlinear Schrödinger equation with the fractional Laplacian, Proceedings of the Royal Society of Edinburgh, 2012, 142.