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# Numerical Simulation for the Initial-boundary Value Problem of the Klein-Gordon-Zakharov Equations

**Juan Chen**[1], **Lu-ming Zhang**[2]

[1]Department of Mathematics, Changshu Institute of Technology, Changshu 215500, China
(E-mail: cjwan061414@163.com)
[2] Department of Mathematics, Nanjing university of Aeronautics and Astronautics, Nanjing 210016, China

**Abstract** In this paper, a new conservative finite difference scheme with a parameter $\theta$ is proposed for the initial-boundary problem of the Klein-Gordon-Zakharov (KGZ) equations. Convergence of the numerical solutions are proved with order $O(h^2 + \tau^2)$ in the energy norm. Numerical results show that the scheme is accurate and efficient.

**Keywords** KGZ equations, conservative difference scheme, priori estimates, convergence
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## 1 Introduction

In [7] Dendy introduced a system of equations (KGZ) to model the interaction of the Langmuir wave and the ion acoustic wave in a plasma. Let $U(x,t)$ be the complex function of the fast time scale component of electric field raised by electrons, and $N(x,t)$ be the real function of the deviation of ion density from its equilibrium, the one-dimensional system then takes the form

$$U_{tt} - U_{xx} + U + NU + |U|^2U = 0, \tag{1.1}$$

$$N_{tt} - N_{xx} = (|U|^2)_{xx}, \tag{1.2}$$

In [16], well-posedness for the Cauchy problem of the KGZ equations in three space dimensions is proved. In [1, 9], non-perturbative solution and global smooth solution are obtained. Two finite difference schemes for the KGZ equations were considered in [19]. As coupled equations, the KGZ equations are similar to the Zakharov equations for which many conservative schemes have been presented ([5–6, 10–11, 17]). Actually, it can also be found that the Equation (1.2) has similar shape to the Klein-Gordon equation. It is known that many numerical methods have been considered for the Klein-Gordon Equation ([8, 12–13, 18]). These numerical methods give us many help to propose a new finite difference scheme for the KGZ equations.

In this paper, we consider the initial-boundary value problem of KGZ equations (1.1) and (1.2) with the following initial-boundary value conditions.

$$U|_{t=0} = U_0(x), \quad U_t|_{t=0} = U_1(x), \quad N|_{t=0} = N_0(x), \quad N_t|_{t=0} = N_1(x), \tag{1.3}$$

$$U|_{x=x_L} = U|_{x=x_R} = N|_{x=x_L} = N|_{x=x_R} = 0, \tag{1.4}$$

where $U_0(x)$, $U_1(x)$, $N_0(x)$ and $N_1(x)$ are known smooth functions.

The initial-boundary value problem (1.1)–(1.4) possesses the following conservative quantity:

$$\int_{x_L}^{x_R} [|U_t|^2 + |U_x|^2 + |U|^2 + N|U|^2 + \frac{1}{2}|V|^2 + \frac{1}{2}|N|^2 + \frac{1}{2}|U|^4] = \text{const}, \tag{1.5}$$

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where the function $V$ is defined as

$$V = -f_x, \qquad f_{xx} = N_t. \tag{1.6}$$

We propose a conservative difference scheme by introducing a parameter $\theta$, $0 \le \theta \le 1$. The scheme conserves the invariant (1.5) for any $\theta$, $0 \le \theta \le 1$. The truncation errors of the scheme are $O(h^2 + \tau^2)$. Convergence of the difference solutions is proved in energy norm on the basis of the priori estimates. Numerical results support the theoretical statements for the difference solutions and demonstrate that the difference scheme with $\theta = 0$ is more accurate and efficient.

The paper is organized as follows. A energy conservative difference scheme is given in Section 2. Some priori estimates for difference solutions are obtained in Section 3. Convergence and stability for the scheme are proved in Section 4. Finally, in Section 5 some numerical results are discussed.

## 2 Finite Difference Scheme and Its Conservative Law

We consider a finite difference method for the problem given in (1.1)–(1.4). As usual, the following notations are used:

$$x_j = x_L + jh, \quad 0 \le j \le J = \left[\frac{x_R - x_L}{h}\right], \quad t^n = n\tau, \qquad n = 0, 1, 2, \cdots, \left[\frac{T}{\tau}\right],$$
$$(w_j^n)_x = \frac{w_{j+1}^n - w_j^n}{h}, \quad (w_j^n)_{\overline{x}} = \frac{w_j^n - w_{j-1}^n}{h}, \quad (w_j^n)_t = \frac{w_j^{n+1} - w_j^n}{\tau}, \quad (w_j^n)_{\overline{t}} = \frac{w_j^n - w_j^{n-1}}{\tau},$$
$$(w^n, u^n) = h\sum_{j=0}^{J} w_j^n \overline{u}_j^n, \qquad \|w^n\|_p^p = h\sum_{j=0}^{J} |w_j^n|^p, \qquad \|w^n\|_\infty = \sup_{0\le j\le J} |w_j^n|.$$

where $h$ and $\tau$ are step size of space and time respectively. Let $C$ be a general positive constant which may have different values on different occasions. For briefness, we omit subscript 2 of $\|w^n\|_2$. Now, we consider the finite difference simulations for equations (1.1) and (1.2) as follows:

$$(U_j^n)_{t\overline{t}} - \frac{\theta}{2}(U_j^{n+1} + U_j^{n-1})_{x\overline{x}} - (1-\theta)(U_j^n)_{x\overline{x}} + \frac{1}{2}(U_j^{n+1} + U_j^{n-1})$$
$$+ \frac{1}{2}N_j^n(U_j^{n+1} + U_j^{n-1}) + \frac{1}{4}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1}) = 0, \tag{2.1}$$
$$(N_j^n)_{t\overline{t}} - \frac{1}{2}(N_j^{n+1} + N_j^{n-1})_{x\overline{x}} = (|U_j^n|^2)_{x\overline{x}}, \tag{2.2}$$

where $0 \le \theta \le 1$ is a constant. In addition, the initial and boundary conditions (1.3) and (1.4) are respectively approximated as

$$U_j^0 = U_0(x_j), \qquad N_j^0 = N_0(x_j), \qquad U_j^1 - U_j^{-1} = 2\tau U_1(x_j), \tag{2.3}$$
$$U_0^n = U_J^n = 0, \qquad N_0^n = N_J^n = 0, \qquad N_j^1 - N_j^{-1} = 2\tau N_1(x_j). \tag{2.4}$$

From (2.3) and (2.1); (2.4) and (2.2) we obtain

$$\frac{2U_j^1 - 2U_j^0 - 2\tau U_1(x_j)}{\tau^2} - \theta(U_j^1 - \tau U_1(x_j))_{x\overline{x}} - (1-\theta)(U_j^0)_{x\overline{x}} + (U_j^1 - \tau U_1(x_j))$$
$$+ N_j^0(U_j^1 - \tau U_1(x_j)) + \frac{1}{2}(|U_j^1|^2 + |U_j^1 - 2\tau U_1(x_j)|^2)(U_j^1 - \tau U_1(x_j)) = 0, \tag{2.5}$$
$$\frac{2N_j^1 - 2N_j^0 - 2\tau N_1(x_j)}{\tau^2} - (N_j^1 - \tau N_1(x_j))_{x\overline{x}} = (|U_j^0|^2)_{x\overline{x}}. \tag{2.6}$$

We also define $\{f_j^n\}$ by

$$(f_j^n)_{x\overline{x}} = (N_j^n)_t, \quad j = 1, 2, \cdots, J-1, \qquad f_0^n = f_J^n = 0. \tag{2.7}$$

By the following lemma, we may prove that the scheme given in (2.1)–(2.7) can conserve the invariant (1.5).

**Lemma 2.1.** *For any two mesh functions $\{w_j\}$ and $\{v_j\}$, $j = 0, 1, \cdots, J$, there is the identity*

$$h\sum_{j=1}^{J-1} w_j(v_j)_{x\overline{x}} = -h\sum_{j=0}^{J-1}(w_j)_x(v_j)_x - w_0(v_0)_x + w_J(v_J)_{\overline{x}}. \tag{2.8}$$

It can be proved directly by simple computation.

**Theorem 2.1.** *For the difference scheme (2.1)–(2.7), there is the discrete conservative law*

$$E^n = E^{n-1} = \cdots = E^0 = \text{const}, \tag{2.9}$$

*where*

$$\begin{aligned} E^n =& \|U_t^n\|^2 + \frac{1}{2}(\|U^{n+1}\|^2 + \|U^n\|^2) + \frac{\theta}{2}(\|U_x^{n+1}\|^2 + \|U_x^n\|^2) \\ &+ (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(U_j^n)_x(\overline{U}_j^{n+1})_x\Big\} + \frac{1}{4}(\|N^{n+1}\|^2 + \|N^n\|^2) + \frac{1}{4}(\|U^{n+1}\|_4^4 + \|U^n\|_4^4) \\ &+ \frac{1}{2}\|f_x^n\|^2 + \frac{1}{2}h\sum_{j=1}^{J-1}(N_j^{n+1}|U_j^n|^2 + N_j^n|U_j^{n+1}|^2). \end{aligned} \tag{2.10}$$

*Proof.* Computing the inner product of (2.1) with $(U^{n+1} - U^{n-1})$ and taking the real part, we have

$$\begin{aligned} &\|U_t^n\|^2 - \|U_t^{n-1}\|^2 + \frac{1}{2}(\|U^{n+1}\|^2 - \|U^{n-1}\|^2) + \frac{\theta}{2}(\|U_x^{n+1}\|^2 \\ &- \|U_x^{n-1}\|^2) + (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}[(U_j^n)_x(\overline{U}_j^{n+1})_x - (U_j^{n-1})_x(\overline{U}_j^n)_x]\Big\} \\ &+ \frac{1}{4}(\|U^{n+1}\|_4^4 - \|U^{n-1}\|_4^4) + \frac{1}{2}h\sum_{j=1}^{J-1}N_j^n(|U_j^{n+1}|^2 - |U_j^{n-1}|^2) = 0. \end{aligned} \tag{2.11}$$

Next, computing the inner product of (2.2) with $\frac{1}{2}(f^n + f^{n-1})$ and using Equation (2.7), we obtain

$$\frac{1}{2}(\|f_x^n\|^2 - \|f_x^{n-1}\|^2) + \frac{1}{4}(\|N^{n+1}\|^2 - \|N^{n-1}\|^2) + \frac{1}{2}h\sum_{j=1}^{J-1}(N_j^{n+1} - N_j^{n-1})|U_j^n|^2 = 0. \tag{2.12}$$

In the computation of (2.11) and (2.12), we have used the boundary conditions and the Lemma 2.1. The result (2.9) follows from (2.11) and (2.12).

## 3 Some Priori Estimates for Difference Solutions

We firstly introduce some auxiliary lemmas.

**Lemma 3.1 (Discrete Sobolev's inequality)[20].** *For any discrete function $u_h = \{u_j | j = 0, 1, \cdots, J\}$ on the finite interval $[0, l]$ and for any given $\varepsilon > 0$, there exists a constant $C$ dependent only on $\varepsilon$ such that*

$$\|\delta^k u_h\|_p \le \varepsilon\|\delta^n u_h\|_2 + C\|u_h\|_2, \tag{3.1}$$

*where* $2 \le p \le \infty,\ 0 \le k < n.$

**Lemma 3.2 (Gronwall's inequality)[20].** *Suppose that the non-negative mesh functions* $\{w(n), \rho(n), n = 1, 2, \cdots, N, N\tau = T\}$ *satisfy the inequality:*

$$w(n) \le \rho(n) + \tau\sum_{l=1}^{n} B_l w(l), \tag{3.2}$$

*where* $B_l\,(l = 1, 2, \cdots, N)$ *are non-negative constant. Then for any* $0 \le n \le N$*, there is*

$$w(n) \le \rho(n)\exp\Big(n\tau\sum_{l=1}^{n} B_l\Big).$$

**Lemma 3.3.** *Let* $\gamma = \frac{\tau}{h} < \sqrt{\frac{1}{1-\theta}},\ 0 \le \theta \le 1$. *If we define* $\beta = \frac{1+\gamma^2(1-\theta)}{1-\gamma^2(1-\theta)} > 1$*, then the following inequality holds:*

$$R_\tau \le \beta Q_\tau, \tag{3.3}$$

*where*

$$Q_\tau = \|U_t^n\|^2 + (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J}(U_j^n)_x(\overline{U}_j^{n+1})_x\Big\},$$

$$R_\tau = \|U_t^n\|^2 + \frac{1-\theta}{2}(\|U_x^n\|^2 + \|U_x^{n+1}\|^2).$$

*Proof.* Let $DU_j^n = (U_j^n)_x$, $D^2U_j^n = (U_j^n)_{x\overline{x}}$, and note $\sum\limits_{j=1}^{J-1} DU_j^n DV_j^n = -\sum\limits_{j=1}^{J-1} U_j^n D^2V_j^n$, we have

$$\begin{aligned}
Q_\tau =&\frac{h}{\tau^2}\sum_{j=1}^{J-1}|U_j^{n+1}-U_j^n|^2 - \frac{1}{2}(1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(U_j^n D^2\overline{U}_j^{n+1} + \overline{U}_j^{n+1}D^2U_j^n)\Big\}\\
=&\frac{h}{\tau^2}\sum_{j=1}^{J-1}(U_j^{n+1}-U_j^n)(\overline{U}_j^{n+1}-\overline{U}_j^n) - \frac{1}{2}(1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(\overline{U}_j^n D^2U_j^{n+1} + \overline{U}_j^{n+1}D^2U_j^n)\Big\}\\
=&\frac{h}{\tau^2}\mathrm{Re}\Big\{\sum_{j=1}^{J-1}(U_j^n, U_j^{n+1})Q_D(\overline{U}_j^n, \overline{U}_j^{n+1})^T\Big\},
\end{aligned} \tag{3.4}$$

where

$$Q_D = \begin{bmatrix} 1, & -1-\frac{\tau^2}{2}(1-\theta)D^2 \\ -1-\frac{\tau^2}{2}(1-\theta)D^2, & 1 \end{bmatrix}.$$

Assume that $(Y_1, Y_2)^T$ is an eigenfunction associated with the eigenvalue $\lambda$ of $Q_D$, then

$$Y_1 - Y_2 - \frac{\tau^2}{2}(1-\theta)D^2Y_2 = \lambda Y_1, \tag{3.5}$$

$$-Y_1 + Y_2 - \frac{\tau^2}{2}(1-\theta)D^2Y_1 = \lambda Y_2. \tag{3.6}$$

By adding and subtracting these equations, we can obtain

$$-\frac{\tau^2}{2}(1-\theta)D^2(Y_1+Y_2)=\lambda(Y_1+Y_2), \tag{3.7}$$

$$\Big[2+\frac{\tau^2}{2}(1-\theta)D^2\Big](Y_1-Y_2)=\lambda(Y_1-Y_2). \tag{3.8}$$

If we look for an eigenfunction with $Y_1=Y_2=Y$, then (3.8) always holds and (3.7) implies that $Y$ is an eigenfunction of the operator $-\frac{\tau^2}{2}(1-\theta)D^2$ with eigenvalue $-\frac{\tau^2}{2}(1-\theta)\mu^2$, where $\mu^2$ is the eigenvalue of $D^2$. On the other hand, if we seek an eigenfunction with $Y_1=-Y_2=Y$, then (3.7) holds and (3.8) implies that $\lambda$ is of the form $2+\frac{\tau^2}{2}(1-\theta)\mu^2$. Similarly, we have

$$R_\tau=\frac{h}{\tau^2}\mathrm{Re}\Big\{\sum_{j=1}^{J-1}(U_j^n,U_j^{n+1})R_D(\overline{U}_j^n,\overline{U}_j^{n+1})^T\Big\}, \tag{3.9}$$

where

$$R_D=\begin{bmatrix} 1-\dfrac{\tau^2}{2}(1-\theta)D^2, & -1\\ -1, & 1-\dfrac{\tau^2}{2}(1-\theta)D^2\end{bmatrix}.$$

Then the eigenvalues and eigenfunction of $R_D$ are

$$\Big(-\frac{\tau^2}{2}(1-\theta)\mu^2,(Y,Y)^T\Big),\qquad \Big(2-\frac{\tau^2}{2}(1-\theta)\mu^2,(Y,-Y)^T\Big).$$

Since $R_D,Q_D$ have a common set of eigenfunction, the inequality $R_\tau\le\beta Q_\tau$ is equivalent to

$$\lambda(R_D)\le\beta\lambda(Q_D), \tag{3.10}$$

for the corresponding eigenvalues.

It follows from Fourier analysis that the eigenvalues of the operator $D^2$ is

$$\mu^2=2h^{-2}(\cos 2\pi jh-1),\qquad j=1,2,\cdots,J-1.$$

Thus, the following inequality can be obtained

$$2-\frac{\tau^2}{2}(1-\theta)\cdot 2h^{-2}(\cos 2\pi jh-1)\le\beta\Big(2+\frac{\tau^2}{2}(1-\theta)\cdot 2h^{-2}(\cos 2\pi jh-1)\Big),$$

i.e.,

$$1+\gamma^2(1-\theta)\sin^2\pi jh\le\beta(1-\gamma^2(1-\theta)\sin^2\pi jh).$$

The inequality holds with $\beta=\frac{1+\gamma^2(1-\theta)}{1-\gamma^2(1-\theta)}$, provided $\gamma<\sqrt{\frac{1}{1-\theta}}$. This completes the proof. □

**Theorem 3.1.** *Assume $U_0(x)\in H^1,U_1(x)\in L^2,\ N_0(x)\in H^1,\ N_1(x)\in L^2$, and suppose that the conditions of Lemma 3.3 are satisfied, then the following estimates hold:*

$$\begin{aligned}&\|U_t^n\|\le C,\quad \|U_x^n\|\le C,\quad \|U^n\|\le C,\quad \|U^n\|_\infty\le C,\\ &\|f_x^n\|\le C,\quad \|N^n\|\le C,\quad \|U^n\|_4\le C.\end{aligned} \tag{3.11}$$

*Proof.* It follows from (2.9) and Lemma 3.3 that

$$\begin{aligned}&\frac{1}{\beta}\Big[\|U_t^n\|^2+\frac12(1-\theta)(\|U_x^{n+1}\|^2+\|U_x^n\|^2)\Big]+\frac12(\|U^{n+1}\|^2+\|U^n\|^2)\\ &+\frac{\theta}{2}(\|U_x^{n+1}\|^2+\|U_x^n\|^2)+\frac14(\|N^{n+1}\|^2+\|N^n\|^2)+\frac12\|f_x^n\|^2\\ &+\frac14(\|U^{n+1}\|_4^4+\|U^n\|_4^4)+\frac12 h\sum_{j=1}^{J-1}(N_j^{n+1}|U_j^n|^2+N_j^n|U_j^{n+1}|^2)\le C.\end{aligned} \tag{3.12}$$

Since
$$\Big|\frac{1}{2}h\sum_{j=1}^{J-1}N_j^n|U_j^{n+1}|^2\Big| \le \frac{1}{4}h\sum_{j=1}^{J-1}((N_j^n)^2+|U_j^{n+1}|^4) \le \frac{1}{4}(\|N^n\|^2+\|U^{n+1}\|_4^4)$$
and
$$\Big|\frac{1}{2}h\sum_{j=1}^{J-1}N_j^{n+1}|U_j^n|^2\Big| \le \frac{1}{4}(\|N^{n+1}\|^2+\|U^n\|_4^4),$$
then, we get
$$\frac{1}{\beta}\Big[\|U_t^n\|^2+\frac{1}{2}(1-\theta)(\|U_x^{n+1}\|^2+\|U_x^n\|^2)\Big]+\frac{1}{2}(\|U^{n+1}\|^2+\|U^n\|^2)$$
$$+\frac{\theta}{2}(\|U_x^{n+1}\|^2+\|U_x^n\|^2)+\frac{1}{2}\|f_x^n\|^2\le C.$$

Therefore
$$\|U_t^n\|\le C,\qquad \|U_x^n\|\le C,\qquad \|U^n\|\le C,\qquad \|f_x^n\|\le C.$$
According to Lemma 3.1, the following estimates are obtained:
$$\|U^n\|_\infty\le C,\qquad \|U^n\|_4\le C.$$
Using Young's inequality $ab\le\frac{1}{4}a^2+b^2$, we have
$$\Big|h\sum_{j=1}^{J-1}N_j^n|U_j^{n+1}|^2\Big|\le h\sum_{j=1}^{J-1}\Big[\frac{1}{4}(N_j^n)^2+|U_j^{n+1}|^4\Big]=\frac{1}{4}\|N^n\|^2+\|U^{n+1}\|_4^4,$$
$$\Big|h\sum_{j=1}^{J-1}N_j^{n+1}|U_j^n|^2\Big|\le h\sum_{j=1}^{J-1}\Big[\frac{1}{4}(N_j^{n+1})^2+|U_j^n|^4\Big]=\frac{1}{4}\|N^{n+1}\|^2+\|U^n\|_4^4.$$
Thus, it follows from (3.12) that
$$\|N^n\|\le C.$$
□

## 4 Convergence and Stability of Difference Solutions

In order to state the main theorem, we define the truncation errors by
$$\begin{aligned}r_j^n =&(U(x_j,t^n))_{t\bar t}-\frac{\theta}{2}(U(x_j,t^{n+1})+U(x_j,t^{n-1}))_{x\bar x}-(1-\theta)(U(x_j,t^n))_{x\bar x}\\ &+\frac{1}{2}[U(x_j,t^{n+1})+U(x_j,t^{n-1})]+\frac{1}{2}N(x_j,t^n)[U(x_j,t^{n+1})+U(x_j,t^{n-1})]\\ &+\frac{1}{4}[|U(x_j,t^{n+1})|^2+|U(x_j,t^{n-1})|^2][U(x_j,t^{n+1})+U(x_j,t^{n-1})],\end{aligned}\tag{4.1}$$
$$\sigma_j^n =(N(x_j,t^n))_{t\bar t}-\frac{1}{2}(N(x_j,t^{n+1})+N(x_j,t^{n-1}))_{x\bar x}-(|U(x_j,t^n)|^2)_{x\bar x}.\tag{4.2}$$

**Theorem 4.1.** *Assume that the conditions of Theorem 3.1 are satisfied and $U(x.t)\in C^{4,4}$, $N(x,t)\in C^{4,4}$. Then the truncation errors of the difference schemes (2.1)–(2.7) satisfy: $|r_j^n|+|\sigma_j^n|=O(\tau^2+h^2)$ as $\tau\to 0,\ h\to 0$.*

*Proof.* By Taylor's expansion, we have
$$(U(x_j,t^n))_{t\bar t}=U_{tt}(x_j,t^n)+\frac{\tau^2}{12}U_{tttt}(x_j,t^n)+O(\tau^4),$$
$$(U(x_j,t^n))_{x\bar x}=U_{xx}(x_j,t^n)+\frac{h^2}{12}U_{xxxx}(x_j,t^n)+O(h^4),$$
$$|U(x_j,t^{n+1})|^2+|U(x_j,t^{n-1})|^2=2|U(x_j,t^n)|^2+\tau^2\frac{\partial^2}{\partial t^2}|U(x_j,t^n)|^2+O(\tau^4).$$

Then $r_j^n$ can be written as

$$\begin{aligned}r_j^n =&U_{tt}(x_j,t^n)+\frac{\tau^2}{12}U_{tttt}(x_j,t^n)+O(\tau^4)-L_1-L_2+U(x_j,t^n)+\frac{\tau^2}{2}U_{tt}(x_j,t^n)\\&+O(\tau^4)+N(x_j,t^n)U(x_j,t^n)+\frac{\tau^2}{2}N(x_j,t^n)U_{tt}(x_j,t^n)+O(\tau^4)+L_3,\end{aligned}$$

where

$$\begin{aligned}L_1 =& \frac{\theta}{2}[U_{xx}(x_j,t^{n+1})+U_{xx}(x_j,t^{n-1})]+\frac{\theta}{24}h^2[U_{xxxx}(x_j,t^{n+1})+U_{xxxx}(x_j,t^{n-1})]+O(h^4)\\ =& \theta U_{xx}(x_j,t^n)+\frac{\theta}{2}\tau^2 U_{xxtt}(x_j,t^n)+\frac{\theta}{12}h^2U_{xxxx}(x_j,t^n)+O(h^2\tau^2+h^4+\tau^4),\\ L_2 =& (1-\theta)[U_{xx}(x_j,t^n)+\frac{h^2}{12}U_{xxxx}(x_j,t^n)]+O(h^4),\\ L_3 =& |U(x_j,t^n)|^2U(x_j,t^n)+\frac{\tau^2}{2}\Big[|U(x_j,t^n)|^2U_{tt}(x_j,t^n)+\frac{\partial^2}{\partial t^2}|U(x_j,t^n)|^2U(x_j,t^n)\Big]+O(\tau^4).\end{aligned}$$

Thus, according to (1.1) we obtain

$$\begin{aligned}r_j^n =&\frac{\tau^2}{12}U_{tttt}(x_j,t^n)-\frac{\theta}{2}\tau^2U_{xxtt}(x_j,t^n)-\frac{h^2}{12}U_{xxxx}(x_j,t^n)\\&+\frac{\tau^2}{2}U_{tt}(x_j,t^n)+\frac{\tau^2}{2}N(x_j,t^n)U_{tt}(x_j,t^n)+\frac{\tau^2}{2}\Big[|U(x_j,t^n)|^2U_{tt}(x_j,t^n)\\&+\frac{\partial^2}{\partial t^2}|U(x_j,t^n)|^2U(x_j,t^n)\Big]+O(h^4+h^2\tau^2+\tau^4)=O(h^2+\tau^2).\end{aligned}$$

As for $\sigma_j^n$, we use Taylor's theorem again to get

$$\sigma_j^n = N_{tt}(x_j,t^n)+O(\tau^2)-\frac{1}{2}[N_{xx}(x_j,t^{n+1})+N_{xx}(x_j,t^{n-1})+O(h^2)]-\Big[\frac{\partial^2}{\partial x^2}|U(x_j,t^n)|^2+O(h^2)\Big].$$

Since

$$N_{xx}(x_j,t^n)-\frac{1}{2}[N_{xx}(x_j,t^{n+1})+N_{xx}(x_j,t^{n-1})]=O(\tau^2),$$

then, the result $\sigma_j^n=O(h^2+\tau^2)$ can be obtained by Equation (1.2). This completes the proof.

Now, we are going to analyze convergence of the difference scheme (2.1)–(2.7).

Define:

$$e_j^n=U(x_j,t^n)-U_j^n,\qquad \eta_j^n=N(x_j,t^n)-N_j^n, \tag{4.3}$$

$$\frac{1}{h^2}(F_{j+1}^n-2F_j^n+F_{j-1}^n)=\frac{1}{\tau}(\eta_j^{n+1}-\eta_j^n),\qquad j=1,2,\cdots,J-1. \tag{4.4}$$

$$F_0^n=F_J^n=0. \tag{4.5}$$

**Theorem 4.2.** *Assume that the conditions of Theorem 4.1 are satisfied, then the solution of the difference problem (2.1)–(2.7) converges to the solution of the problem stated in (1.1)–(1.4) with order $O(\tau^2+h^2)$ in the norm $L_\infty$ norm for $U^n$, and in the $L_2$ norm for $N^n$.*

*Proof.* Subtracting (2.1) from (4.1), we obtain:

$$\begin{aligned}r_j^n =&(e_j^n)_{t\bar t}-\frac{\theta}{2}(e_j^{n+1}+e_j^{n-1})_{x\bar x}-(1-\theta)(e_j^n)_{x\bar x}+\frac{1}{2}(e_j^{n+1}+e_j^{n-1})\\&+\frac{1}{2}\eta_j^n(U(x_j,t^{n+1})+U(x_j,t^{n-1}))+\frac{1}{2}N_j^n(e_j^{n+1}+e_j^{n-1})+Q',\end{aligned}$$

i.e.

$$
\begin{aligned}
&(e_j^n)_{t\bar{t}} - \frac{\theta}{2}(e_j^{n+1} + e_j^{n-1})_{x\overline{x}} - (1-\theta)(e_j^n)_{x\overline{x}} + \frac{1}{2}(e_j^{n+1} + e_j^{n-1}) \\
=& r_j^n - \frac{1}{2}\eta_j^n(U(x_j, t^{n+1}) + U(x_j, t^{n-1})) - \frac{1}{2}N_j^n(e_j^{n+1} + e_j^{n-1}) - Q',
\end{aligned} \tag{4.6}
$$

where

$$
\begin{aligned}
Q' =& \frac{1}{4}(|U(x_j, t^{n+1})|^2 + |U(x_j, t^{n-1})|^2)(U(x_j, t^{n+1}) + U(x_j, t^{n-1})) \\
&- \frac{1}{4}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1}).
\end{aligned} \tag{4.7}
$$

Computing the inner product of (4.6) with $e^{n+1} - e^{n-1}$ and taking the real part, we have:

$$
\begin{aligned}
&\|e_t^n\|^2 - \|e_t^{n-1}\|^2 + \frac{\theta}{2}(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) \\
&+ \frac{1}{2}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) + (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(e_j^n)_x(\overline{e}_j^{n+1} - \overline{e}_j^{n-1})_x\Big\} \\
\leq& C\tau(\|r^n\|^2 + \|\eta^n\|^2 + \|N^n(e^{n+1} + e^{n-1})\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2 + \|e^{n+1}\|^2 + \|e^{n-1}\|^2).
\end{aligned} \tag{4.8}
$$

Using Theorem 3.1 and Lemma 3.1, we get

$$
\begin{aligned}
\|N^n(e^{n+1} + e^{n-1})\| \leq& C\|N^n\|\|e^{n+1} + e^{n-1}\|_\infty \\
\leq& C\|e_x^{n+1} + e_x^{n-1}\| + C\|e^{n+1} + e^{n-1}\| \\
\leq& C(\|e_x^{n+1}\| + \|e_x^{n-1}\| + \|e^{n+1}\| + \|e^{n-1}\|).
\end{aligned} \tag{4.9}
$$

Since

$$
\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(e_j^n)_x(\overline{e}_j^{n+1} - \overline{e}_j^{n-1})_x\Big\} = \mathrm{Re}\Big\{h\sum_{j=1}^{J-1}[(e_j^n)_x(\overline{e}_j^{n+1})_x - (e_j^{n-1})_x(\overline{e}_j^n)_x]\Big\}, \tag{4.10}
$$

It follows from Lemma 3.3 that

$$
\|e_t^n\|^2+\|e_t^{n-1}\|^2 \leq \beta\Big\{\|e_t^n\|^2+\|e_t^{n-1}\|^2+(1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}[(e_j^n)_x(\overline{e}_j^{n+1})_x+(e_j^{n-1})_x(\overline{e}_j^n)_x]\Big\}\Big\}. \tag{4.11}
$$

Substituting (4.9), (4.10), (4.11) into (4.8), we have

$$
\begin{aligned}
&\|e_t^n\|^2 - \|e_t^{n-1}\|^2 + \frac{\theta}{2}(\|e_x^{n+1}\|^2 - \|e_x^{n-1}\|^2) + \frac{1}{2}(\|e^{n+1}\|^2 - \|e^{n-1}\|^2) \\
&+ (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}[(e_j^n)_x(\overline{e}_j^{n+1})_x - (e_j^{n-1})_x(\overline{e}_j^n)_x]\Big\} \\
\leq& C\tau\Big\{\|r^n\|^2 + \|\eta^n\|^2 + \|e_x^{n+1}\|^2 + \|e_x^{n-1}\|^2 + \|e_t^n\|^2 + \|e_t^{n-1}\|^2 + \|e^{n+1}\|^2 \\
&+ \|e^{n-1}\|^2 + (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}[(e_j^n)_x(\overline{e}_j^{n+1})_x + (e_j^{n-1})_x(\overline{e}_j^n)_x]\Big\}\Big\}.
\end{aligned} \tag{4.12}
$$

Next, subtracting (2.2) from (4.2), we obtain

$$
(\eta_j^n)_{t\bar{t}} - \frac{1}{2}(\eta_j^{n+1} + \eta_j^{n-1})_{x\overline{x}} = \sigma_j^n + (|U(x_j, t^n)|^2 - |U_j^n|^2)_{x\overline{x}}. \tag{4.13}
$$

Computing the inner product of (4.13) with $\frac{1}{2}(F^n + F^{n-1})$, we have

$$
-\frac{1}{2\tau}(\|F_x^n\|^2 - \|F_x^{n-1}\|^2) - \frac{1}{4\tau}(\|\eta^{n+1}\|^2 - \|\eta^{n-1}\|^2)
$$

$$=\Big(\sigma^n, \frac{1}{2}(F^n + F^{n-1})\Big) + \Big([U(x_j, t^n)\overline{e}_j^n + e_j^n\overline{U}_j^n]_{x\overline{x}}, \frac{1}{2}(F^n + F^{n-1})\Big). \tag{4.14}$$

The right hand side of (4.14) may be estimated as follows:

$$\begin{aligned}
&\Big([U(x_j, t^n)\overline{e}_j^n + e_j^n\overline{U}_j^n]_{x\overline{x}}, \frac{1}{2}(F^n + F^{n-1})\Big)\\
\le&\frac{1}{2}\Big|h\sum_{j=1}^{J-1}[U(x_j, t^n)\overline{e}_j^n + e_j^n\overline{U}_j^n]_x(F_j^n + F_j^{n-1})_x\Big|\\
\le&\frac{1}{2}h\sum_{j=1}^{J-1}|U(x_{j+1}, t^n)(\overline{e}_j^n)_x + (U(x_j, t^n))_x\overline{e}_j^n + e_{j+1}^n(\overline{U}_j^n)_x + (e_j^n)_x\overline{U}_j^n|\cdot|(F_j^n + F_j^{n-1})_x|\\
\le&C(\|F_x^n\|^2 + \|F_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e^n\|^2 + \|e^nU_x^n\|^2).
\end{aligned}$$

Using Theorem 3.1 and Lemma 3.1,

$$\|e^nU_x^n\| \le C\|e^n\|_\infty\|U_x^n\| \le C\|e_x^n\| + C\|e^n\|.$$

Then we can obtain

$$\Big([U(x_j, t^n)\overline{e}_j^n + e_j^n\overline{U}_j^n]_{x\overline{x}}, \frac{1}{2}(F^n + F^{n-1})\Big) \le C(\|F_x^n\|^2 + \|F_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e^n\|^2). \tag{4.15}$$

Since

$$|F_j^n| = \Big|\sum_{k=1}^{j}(F_k^n - F_{k-1}^n)\Big| = \Big|h\sum_{k=1}^{j}(F_{k-1}^n)_x\Big| \le C\|F_x^n\|,$$

then, we have

$$\Big(\sigma^n, \frac{1}{2}(F^n + F^{n-1})\Big) = \frac{1}{2}h\sum_{j=1}^{J-1}\sigma_j^n(F_j^n + F_j^{n-1}) \le C(\|\sigma^n\|^2 + \|F_x^n\|^2 + \|F_x^{n-1}\|^2). \tag{4.16}$$

According to (4.14)–(4.16), we obtain

$$\begin{aligned}
&\|F_x^n\|^2 - \|F_x^{n-1}\|^2 + \frac{1}{2}(\|\eta^{n+1}\|^2 - \|\eta^{n-1}\|^2)\\
\le&C\tau(\|\sigma^n\|^2 + \|F_x^n\|^2 + \|F_x^{n-1}\|^2 + \|e_x^n\|^2 + \|e^n\|^2).
\end{aligned} \tag{4.17}$$

Now, add (4.12) to (4.17) and let

$$\begin{aligned}
B^n =&\|e_t^n\|^2 + \frac{\theta}{2}(\|e_x^{n+1}\|^2 + \|e_x^n\|^2) + (1-\theta)\mathrm{Re}\Big\{h\sum_{j=1}^{J-1}(e_j^n)_x(\overline{e}_j^{n+1})_x\Big\}\\
&+\frac{1}{2}(\|e^{n+1}\|^2 + \|e^n\|^2) + \|F_x^n\|^2 + \frac{1}{2}(\|\eta^{n+1}\|^2 + \|\eta^n\|^2).
\end{aligned} \tag{4.18}$$

According to Theorem 4.1, the following inequality holds

$$\begin{aligned}
B^n - B^{n-1} \le&C\tau(\|r^n\|^2 + \|\sigma^n\|^2) + C\tau(B^n + B^{n-1})\\
\le&C\tau(h^2 + \tau^2)^2 + C\tau(B^n + B^{n-1}).
\end{aligned} \tag{4.19}$$

Summing (4.19) up for $n$ and using Lemma 3.2, we have

$$B^N \le (B^0 + C(h^2 + \tau^2)^2)\exp(CN\tau) \le C(B^0 + (h^2 + \tau^2)^2). \tag{4.20}$$

It follows from (4.18) and Lemma 3.3 that

$$\begin{aligned}
&\frac{1}{\beta}\|e_t^N\|^2 + \Big(\frac{1-\theta}{2\beta} + \frac{\theta}{2}\Big)(\|e_x^{N+1}\|^2 + \|e_x^N\|^2) + \frac{1}{2}(\|e^{N+1}\|^2 + \|e^N\|^2)\\
&+\|F_x^N\|^2 + \frac{1}{2}(\|\eta^{N+1}\|^2 + \|\eta^N\|^2) \le C(B^0 + (h^2 + \tau^2)^2).
\end{aligned} \tag{4.21}$$

It is clear that $e^0$, $e^1$, $\eta^0$ and $\eta^1$ are second order in precision and it follows from [10] that $\|F_x^0\| = O(h^2+\tau^2)$. Thus $B^0 = O(h^2+\tau^2)^2$. According to (4.21), the following four inequalities can be obtained:

$$\|e_x^N\| \le O(h^2+\tau^2), \qquad \|e^N\| \le O(h^2+\tau^2),$$
$$\|e_t^N\| \le O(h^2+\tau^2), \qquad \|\eta^N\| \le O(h^2+\tau^2).$$

Using Lemma 3.2, we get

$$\|e^N\|_\infty \le O(h^2+\tau^2).$$

This completes the proof. □

## 5 Numerical Experiments

In this section, we present numerical evidence to confirm that the scheme described in this paper produces second order accuracy in the $l_\infty$ norm for $U^n$ and in the $l_2$ norm for $N^n$. We consider the equations and initial-boundary conditions as follows:

$$U_{tt} - U_{xx} + U + NU + |U|^2U = 0, \qquad -20 < x < 20, \quad 0 \le t \le T,$$
$$N_{tt} - N_{xx} = (|U|^2)_{xx}, \qquad -20 < x < 20, \quad 0 \le t \le T,$$
$$U(-20,t) = U(20,t) = 0, \qquad N(-20,t) = N(20,t) = 0, \qquad 0 \le t \le T,$$
$$U(x,0) = \frac{\sqrt{10}-\sqrt{2}}{2}\mathrm{sech}\Big(\sqrt{\frac{1+\sqrt{5}}{2}}x\Big)\cdot \exp\Big[i\Big(\sqrt{\frac{2}{1+\sqrt{5}}}x\Big)\Big], \qquad -20 \le x \le 20,$$
$$N(x,0) = -2\mathrm{sech}^2\Big(\sqrt{\frac{1+\sqrt{5}}{2}}x\Big), \qquad -20 \le x \le 20.$$

The exact solutions of the KGZ equations, which were given in [14–15], will be used in our computation for comparison. The solutions can be written as

$$U(x,t) = \frac{\sqrt{10}-\sqrt{2}}{2}\mathrm{sech}\Big(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\Big)\cdot \exp\Big[i\Big(\sqrt{\frac{2}{1+\sqrt{5}}}x - t\Big)\Big], \tag{5.1}$$

$$N(x,t) = -2\mathrm{sech}^2\Big(\sqrt{\frac{1+\sqrt{5}}{2}}x - t\Big). \tag{5.2}$$

**Table 1.** Errors for Schemes (2.1)–(2.7) with $\theta = 0$, when $T = 5$

| spatial step size | time step size | $\|\|e^n\|\|_\infty \times 10^2$ | $\|\|\eta^n\|\|_2 \times 10^2$ |
|---|---|---|---|
| $h = 0.1$ | $\tau = 0.1$ | 2.09 | 10.1 |
| | $\tau = 0.02$ | 1.02 | 3.25 |
| | $\tau = 0.005$ | 1.09 | 4.05 |
| $h = 0.05$ | $\tau = 0.05$ | 0.52 | 2.45 |
| | $\tau = 0.025$ | 0.29 | 1.16 |
| | $\tau = 0.005$ | 0.24 | 0.91 |
| $h = 0.025$ | $\tau = 0.025$ | 0.13 | 0.62 |
| | $\tau = 0.0125$ | 0.09 | 0.32 |
| | $\tau = 0.00625$ | 0.08 | 0.18 |
| $h = 0.0125$ | $\tau = 0.0125$ | 0.08 | 0.19 |
| | $\tau = 0.00625$ | 0.07 | 0.06 |
| $h = 0.00625$ | $\tau = 0.00625$ | 0.06 | 0.08 |

**Table 2.** Errors for Schemes (2.1)–(2.7) with $\theta = 0.5$, when $T = 5$

| spatial step size | time step size | $\|\|e^n\|\|_\infty \times 10^2$ | $\|\|\eta^n\|\|_2 \times 10^2$ |
|---|---|---|---|
| $h = 0.1$ | $\tau = 0.1$ | 6.68 | 24.9 |
| | $\tau = 0.05$ | 2.84 | 9.92 |
| | $\tau = 0.025$ | 1.89 | 6.49 |
| | $\tau = 0.005$ | 1.36 | 4.11 |
| $h = 0.05$ | $\tau = 0.05$ | 0.91 | 3.29 |
| | $\tau = 0.025$ | 0.42 | 1.37 |
| | $\tau = 0.005$ | 0.25 | 0.89 |
| $h = 0.025$ | $\tau = 0.025$ | 0.23 | 0.83 |
| | $\tau = 0.0125$ | 0.16 | 0.41 |
| | $\tau = 0.00625$ | 0.27 | 0.25 |
| $h = 0.0125$ | $\tau = 0.0125$ | 0.15 | 0.29 |
| | $\tau = 0.00625$ | 0.09 | 0.08 |
| $h = 0.00625$ | $\tau = 0.00625$ | 0.30 | 0.26 |

**Table 3.** Comparison for some $\theta$ values, when $T = 10$

| step size | $\theta$ | $\|\|e^n\|\|_\infty \times 10^2$ | $\|\|\eta^n\|\|_2 \times 10^2$ | $E^n$ |
|---|---|---|---|---|
| $h = \tau = 0.1$ | 0 | 4.29 | 17.3 | 6.191891 |
| | 0.2 | 5.15 | 20.3 | 6.195988 |
| | 0.5 | 6.68 | 24.9 | 6.202180 |
| | 0.8 | 8.40 | 29.9 | 6.208322 |
| | 1.0 | 9.64 | 33.3 | 6.212411 |
| $h = \tau = 0.05$ | 0 | 1.02 | 3.98 | 6.207482 |
| | 0.2 | 1.23 | 4.67 | 6.208568 |
| | 0.5 | 1.61 | 5.73 | 6.210093 |
| | 0.8 | 2.03 | 6.85 | 6.211666 |
| | 1.0 | 2.33 | 7.61 | 6.212672 |
| $h = \tau = 0.025$ | 0 | 0.25 | 1.01 | 6.211380 |
| | 0.2 | 0.31 | 1.16 | 6.211621 |
| | 0.5 | 0.40 | 1.44 | 6.212060 |
| | 0.8 | 0.50 | 1.68 | 6.212382 |
| | 1.0 | 0.57 | 1.85 | 6.212605 |
| $h = \tau = 0.0125$ | 0 | 0.15 | 0.36 | 6.212455 |
| | 0.2 | 0.13 | 0.38 | 6.212539 |
| | 0.5 | 0.29 | 0.58 | 6.212746 |
| | 0.8 | 0.12 | 0.39 | 6.212528 |
| | 1.0 | 0.13 | 0.42 | 6.212611 |
| $h = \tau = 0.00625$ | 0 | 0.09 | 0.22 | 6.209661 |
| | 0.2 | 0.37 | 0.52 | 6.210019 |
| | 0.5 | 0.58 | 0.50 | 6.209419 |
| | 0.8 | 0.83 | 0.98 | 6.210181 |
| | 1.0 | 0.38 | 0.55 | 6.209933 |

In numerical experiments, we choose different values of $h, \tau$ and $\theta$ to compute approximate solutions which compare with the exact ones. In Table 1 and 2, we give the errors for the parameter $\theta = 0, 0.5$ respectively. In Table 3, we make the comparison of $\tau = h$ for various values of $\theta$.

The numerical results displayed in Tables 1–3 demonstrate that the approximate solutions of our proposed scheme are accurate of order $O(h^2 + \tau^2)$. Furthermore, the numerical results of the discrete energy $E^n$ given in Table 3 illustrate that our scheme conserves the invariant (1.5) very well on which different values of $\theta$ have little impact. Therefore, it can be seen that our present method provides a good approximation for the solutions of the studied problem. In addition, it is worth pointing out that our difference scheme is more accurate and efficient when the parameter $\theta$ nearly equals to 0.

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