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ANALYSIS OF AN XFEM DISCRETIZATION FOR STOKES INTERFACE PROBLEMS*

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Abstract. We consider a stationary Stokes interface problem. In the discretization the interface is not aligned with the triangulation. For the discretization we use the P_1 extended finite element space (P_1-XFEM) for the pressure and the standard conforming P_2 finite element space for the velocity. Since this pair is not necessarily LBB stable, a consistent stabilization term, known from the literature, is added. For the discrete bilinear form an inf-sup stability result is derived, which is uniform with respect to h (mesh size parameter), the viscosity quotient μ_1/μ_2, and the position of the interface in the triangulation. Based on this, discretization error bounds are derived. An optimal preconditioner for the stiffness matrix corresponding to this pair P_1-XFE for pressure and P_2-FE for velocity is presented. The preconditioner has block diagonal form, with a multigrid preconditioner for the velocity block and a new Schur complement preconditioner. Optimality of this block preconditioner is proved. Results of numerical experiments illustrate properties of the discretization method and of a preconditioned MINRES solver.

Key words. Stokes equations, interface problem, extended finite element space, preconditioning, Schur complement

AMS subject classifications. 65N15, 65N22, 65N30, 65F10

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1. Introduction.

In this paper we treat the following Stokes problem on a bounded polygonal domain Ω in d-dimensional Euclidean space ($d = 2, 3$): Find a velocity u and a pressure p such that

$$\begin{aligned} -\mathrm{div}\,(\mu(x)D(u)) + \nabla p &= f \quad \text{in } \Omega, \\ \mathrm{div}\, u &= 0 \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{1.1}$$

with $D(u) := \nabla u + (\nabla u)^T$ and a *piecewise constant viscosity* $\mu = \mu_i > 0$ in Ω_i. The subdomains $\Omega_1,\ \Omega_2$ are assumed to be Lipschitz domains such that $\Omega_1 \cap \Omega_2 = \emptyset$ and $\overline{\Omega} = \overline{\Omega}_1 \cup \overline{\Omega}_2$. By Γ we denote the interface between the subdomains, $\Gamma = \partial\Omega_1 \cap \partial\Omega_2$. For a corresponding weak formulation we introduce the spaces $V := H_0^1(\Omega)^d$ and

$$L^2_\mu(\Omega) := \left\{ p \in L^2(\Omega) \;\middle|\; \int_\Omega \mu^{-1} p(x)\, dx = 0 \right\}. \tag{1.2}$$

The scaling with μ in the Gauge condition in (1.2) is convenient for obtaining estimates that are uniform w.r.t. the jump in the viscosity; cf. [16]. The variational problem reads as follows: given $f \in V'$ find $(u, p) \in V \times L^2_\mu(\Omega)$ such that

$$\left\{ \begin{aligned} \tfrac{1}{2}(\mu D(u), D(v))_{0,\Omega} - (\mathrm{div}\, v, p)_{0,\Omega} &= f(v) \quad &&\text{for all } \ v \in V, \\ (\mathrm{div}\, u, q)_{0,\Omega} &= 0 &&\text{for all } \ q \in L^2_\mu(\Omega). \end{aligned} \right. \tag{1.3}$$

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Here $(\cdot,\cdot)_{0,\Omega}$ denotes the L^2 scalar product on Ω. This is a well-posed weak formulation [9].

An important motivation for considering this type of Stokes equations comes from two-phase incompressible flows. Often such problems are modeled by Navier–Stokes equations with discontinuous density and viscosity coefficients. The effect of interface tension can be taken into account by using a special localized force term at the interface [12]. If in such a setting one has highly viscous flows, then the Stokes equations with discontinuous viscosity are a reasonable model problem for method development and analysis. A well-known technique for capturing the unknown interface is based on the level set method; cf. [22, 5, 17] and the references therein. If the level set method is used, then typically in the discretization of the flow equations the interface is *not* aligned with the grid. This causes difficulties w.r.t. an accurate discretization of the flow variables. Recently, extended finite element techniques (XFEM, also called cut finite element methods) have been developed to obtain accurate finite element discretizations; cf., for example, [8, 13, 12]. Concerning theoretical analysis of XFEM applied to such Stokes interface problems little is known. In fact, the only two papers with rigorous analysis of XFEM applied to Stokes interface problems we know of are [13, 4]. In these papers XFE-spaces are used for both the pressure and velocity spaces; weak continuity of the velocity across the interface is enforced using a Nitsche method. Hansbo, Larson, and Zahedi [13] use the isoP_2-P_1 pair as underlying spaces, and to avoid instabilities due to "small cuts" the ghost penalty stabilization [3] is applied in a neighborhood of the interface. Cattaneo et al. [4] consider the P_1bubble-P_1 pair and apply the Brezzi–Pitkäranta stabilization [2] in the vicinity of the interface. They also consider the case of an underlying unstable P_1-P_1 pair and apply the Brezzi–Pitkäranta stabilization on the entire domain. Both in [13] and [4], for the discrete bilinear forms inf-sup stability results are derived which are uniform w.r.t. h (mesh size parameter), the position of the interface in the triangulation, and, in the case of [13], also w.r.t. the viscosity quotient μ_1/μ_2. Based on this, optimal discretization error bounds are derived. Furthermore, in [13] a uniform (w.r.t. the location of the interface) condition number bound for the stiffness matrix is derived with the help of a further stabilization of the velocity space. In [4] results on conditioning of the Schur complement are given.

In this paper we analyze an XFEM that differs from the ones considered in [13, 4]. In the discretization that we consider, the pressure variable is approximated in a conforming P_1-XFE space (as in [13, 4]), but the velocity is approximated in the standard conforming P_2-FE space. In the discretization we use the same ghost penalty stabilization technique as in [13]. For the discrete bilinear form we derive an inf-sup stability result. Similar to [13], a key property of this result is that the stability constant is uniform w.r.t. h, the viscosity quotient μ_1/μ_2, and the position of the interface in the triangulation. Based on this result and interpolation error estimates, discretization error bounds are derived. Due to the use of the standard P_2-FE velocity space the error bound is not optimal if the normal derivative of the velocity is discontinuous across the interface (which typically occurs if $\mu_1 \neq \mu_2$). However, the uniform stability result also holds if the P_2-FE velocity space is replaced by a larger conforming P_2-XFE space with better approximation properties; cf. Remark 1 below. The reason why we consider the standard P_2-FE velocity space is that in realistic two-phase flow applications, with small viscosity jumps, the pair P_1-XFE for pressure and P_2-FE for the velocity has shown to work satisfactorily [19, 7]. It turns out that the poor *asymptotic* approximation quality of the velocity in the P_2-FE space does not dominate the total error on the meshes used in practice. We will illustrate this in a numerical experiment in section 8.

Apart from the *different spaces* considered in this paper (compared to [13, 4]) we mention the following other two main new contributions of this paper. The first one is related to the linear algebra part. In [13] a condition number bound of the form $c(\mu_{\max}/\mu_{\min})^2 h^{-2}$ is derived for the stiffness matrix. In [4] an h-independent bound on the condition number of the Schur complement is derived. The dependence of this bound on the viscosity ratio is not studied. In both papers the topic of how to construct a good preconditioner for the stiffness matrix is *not* addressed. In this paper we derive an *optimal preconditioner* for the stiffness matrix corresponding to the pair P_1-XFE for pressure and P_2-FE for velocity. The preconditioner has block diagonal form, with a multigrid preconditioner for the velocity block and a new Schur complement preconditioner. Optimality of this Schur complement preconditioner w.r.t. h, μ, and how the interface intersects the triangulation is proved. This optimality is illustrated with results of numerical experiments with a preconditioned MINRES solver.

The other new contribution is a certain *uniform LBB stability result*. In our analysis and also in the papers [13, 4], an LBB stability result is needed that has a certain uniformity property w.r.t. the varying (for $h \downarrow 0$) subdomain consisting of the triangulations that are strictly contained in a physical subdomain (cf. section 4 for precise explanation). In [4] such a uniform LBB stability result is introduced as an assumption. A similar result for the isoP_2-P_1 pair is implicitly used in [13] to prove stability. In this paper, for the P_2-P_1 Hood–Taylor pair, we prove such a uniform LBB stability result. We expect that the analysis that we use can be extended to other LBB stable pairs, in particular the ones used in [13, 4].

2. The XFEM space of piecewise linears. We assume a family $\{\mathcal{T}_h\}_{h>0}$ of shape regular quasi-uniform triangulations of the domain Ω, consisting of simplices. The triangulations are *not* fitted to the interface Γ. We assume that the triangulation is sufficiently fine such that the interface is resolved. In particular the following *generic intersection assumption* should be satisfied: if $\Gamma \cap T \neq \emptyset$ for a $T \in \mathcal{T}_h$, then $\Gamma \cap \partial T$ consists of exactly two points (if $d = 2$) or of a closed curve (if $d = 3$). We introduce the subdomains $\Omega_{i,h} := \{\, T \in \mathcal{T}_h \mid T \subset \Omega_i \;\text{ or }\; \text{meas}_{d-1}(T \cap \Gamma) > 0 \,\}$, $i = 1, 2$, and the corresponding standard linear finite element spaces

$$Q_{i,h} := \{\, v_h \in C(\Omega_{i,h}) \mid v_{h|T} \in \mathcal{P}_1 \;\text{ for all } T \in \Omega_{i,h} \,\},\; i = 1, 2.$$

We use the same notation $\Omega_{i,h}$ for the set of tetrahedra as well as for the subdomain of Ω which is formed by these tetrahedra, as its meaning is clear from the context. For the stabilization procedure that is introduced below we need a further partitioning of $\Omega_{i,h}$. Define $\omega_{i,h} := \{\, T \in \Omega_{i,h} \mid \text{meas}_{d-1}(T \cap \Gamma) = 0 \,\}$, $i = 1, 2$, and $\mathcal{T}_h^\Gamma := \mathcal{T}_h \backslash (\omega_{1,h} \cup \omega_{2,h}) = \{\, T \in \mathcal{T}_h \mid \text{meas}_{d-1}(T \cap \Gamma) > 0 \,\}$. Note that $\mathcal{T}_h = \omega_{1,h} \cup \omega_{2,h} \cup \mathcal{T}_h^\Gamma$ holds and forms a disjoint union. Corresponding sets of faces (needed in the stabilization procedure) are given by $\mathcal{F}_i = \{\, F \subset \partial T \mid T \in \mathcal{T}_h^\Gamma, \;\; F \not\subset \partial\Omega_{i,h} \,\}$, $i = 1, 2$, and $\mathcal{F}_h := \mathcal{F}_1 \cup \mathcal{F}_2$. For each $F \in \mathcal{F}_h$ a fixed orientation of its normal is chosen and the unit normal with that orientation is denoted by n_F. These definitions are illustrated in Figure 1.

A given $p_h = (p_{1,h}, p_{2,h}) \in Q_{1,h} \times Q_{2,h}$ has two values, $p_{1,h}(x)$ and $p_{2,h}(x)$ for $x \in \mathcal{T}_h^\Gamma$. We define a uni-valued function $p_h^\Gamma \in C(\Omega_1 \cup \Omega_2)$ by

$$p_h^\Gamma(x) = p_{i,h}(x) \quad \text{for} \;\; x \in \Omega_i.$$

Using the generic intersection assumption we obtain that the mapping $p_h \mapsto p_h^\Gamma$ is

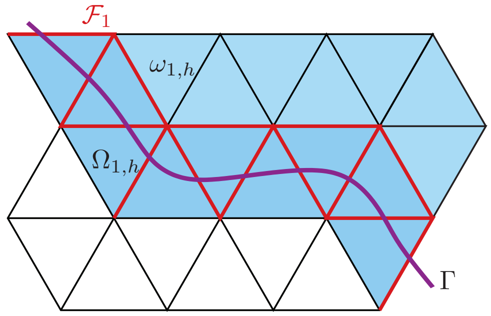

FIG. 1. *Set of faces $\mathcal{F}_1$ (in red) and subdomains $\omega_{1,h}$ (light-blue) and $\Omega_{1,h}$ (light- and darker blue triangles) for a two-dimensional example.*

bijective. On $Q_{1,h} \times Q_{2,h}$ we use a norm denoted by $\|p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} := \|p_{1,h}\|^2_{0,\Omega_{1,h}} + \|p_{2,h}\|^2_{0,\Omega_{2,h}}$. The XFEM space of piecewise linears is defined by

$$Q_h^\Gamma := (Q_{1,h} \times Q_{2,h})/\mathbb{R} = \{\, p_h \in Q_{1,h} \times Q_{2,h} \mid (\mu^{-1} p_h^\Gamma, 1)_{0,\Omega} = 0 \,\}. \tag{2.1}$$

The space $\{\, p_h^\Gamma \mid p_h \in Q_h^\Gamma \,\}$ is a subspace of the pressure space $L^2_\mu(\Omega)$, cf. (1.2). Note the subtle, but important, notational difference between $p_h \in Q_h^\Gamma$ and p_h^Γ. The former is a pair which corresponds to a multivalued function in $\mathcal{T}_h^\Gamma$, whereas the latter is a uni-valued function. In the analysis we need the following decomposition of this XFEM space into two orthogonal subspaces. We introduce the piecewise constant function $\bar{p}_\mu \in Q_h^\Gamma$:

$$\bar{p}_\mu := \begin{cases} \mu_1 |\Omega_1|^{-1} & \text{in } \Omega_1, \\ -\mu_2 |\Omega_2|^{-1} & \text{in } \Omega_2. \end{cases} \tag{2.2}$$

Using the one-dimensional subspace $M_0 := \operatorname{span}\{\bar{p}_\mu\} \subset Q_h^\Gamma$, the XFEM space is decomposed as $Q_h^\Gamma = M_0 \oplus M_0^\perp$ with $M_0^\perp := \{\, p_h \in Q_h^\Gamma \mid (p_h^\Gamma, \bar{p}_\mu^\Gamma)_{0,\Omega} = 0 \,\}$. We derive an elementary property as follows.

LEMMA 2.1. *$p_h \in M_0^\perp$ has the property $(p_{i,h}, 1)_{0,\Omega_i} = 0$ for $i = 1, 2$.*

Proof. From $p_h \in Q_h^\Gamma$ it follows that $(\mu^{-1} p_h^\Gamma, 1)_{0,\Omega} = 0$, and hence $\mu_1^{-1}(p_{1,h}, 1)_{0,\Omega_1} + \mu_2^{-1}(p_{2,h}, 1)_{0,\Omega_2} = 0$ holds. From $(p_h^\Gamma, \bar{p}_\mu^\Gamma)_{0,\Omega} = 0$ we get $\mu_1 |\Omega_1|^{-1}(p_{1,h}, 1)_{0,\Omega_1} - \mu_2 |\Omega_2|^{-1}(p_{2,h}, 1)_{0,\Omega_2} = 0$. These two relations imply $(p_{i,h}, 1)_{0,\Omega_i} = 0$ for $i = 1, 2$. ∎

For the stabilization we introduce the bilinear form

$$\begin{aligned} j(p_h, q_h) &:= \sum_{i=1}^{2} j_i(p_{i,h}, q_{i,h}), \quad p_h, q_h \in Q_{1,h} \times Q_{2,h}, \\ \text{with } j_i(p_{i,h}, q_{i,h}) &:= \mu_i^{-1} \sum_{F \in \mathcal{F}_i} h_F^3 ([\nabla p_{i,h} \cdot n_F], [\nabla q_{i,h} \cdot n_F])_{0,F}, \end{aligned} \tag{2.3}$$

which is also referred to as a ghost penalty term; cf. [3]. Here $[\nabla p_{i,h} \cdot n_F]$ denotes the jump of the normal component of the piecewise constant function $\nabla p_{i,h}$ across the face F. All constants used in the results below are *independent of h and μ and of how the interface Γ intersects the triangulation $\mathcal{T}_h$.*

LEMMA 2.2. *The following holds (see* [13, *Lemma* 3.8]*):*

$$\mu_i^{-1} \|p_{i,h}\|^2_{0,\Omega_{i,h}} \le c\big(\mu_i^{-1} \|p_{i,h}\|^2_{0,\omega_{i,h}} + j_i(p_{i,h}, p_{i,h})\big) \quad \textit{for all} \;\; p_{i,h} \in Q_{i,h}, \;\; i = 1, 2.$$

Proof. Note that

$$\|p_{i,h}\|_{0,\Omega_{i,h}}^2 = \|p_{i,h}\|_{0,\omega_{i,h}}^2 + \sum_{T\in\Omega_{i,h}\setminus\omega_{i,h}} \|p_{i,h}\|_{0,T}^2,$$

and hence we only have to treat $\|p_{i,h}\|_{0,T}$, $T \in \Omega_{i,h} \setminus \omega_{i,h}$. We write $p = p_{i,h}$, which is a piecewise linear function on $\Omega_{i,h}$. Take $T_0 = T \in \Omega_{i,h} \setminus \omega_{i,h}$ and $x \in T_0$. There is a sequence of simplices $T_1, \ldots, T_k$ with faces $F_j = \bar{T}_j \cap \bar{T}_{j-1} \in \mathcal{F}_i$, $j = 1, \ldots, k$, and $T_k \in \omega_{i,h}$. The number k is uniformly bounded (often, $k = 1$ holds). The barycenter of F_j is denoted by m_j. With an appropriate orientation of the jump operator $[\cdot]_F$ we have the relations $\sum_{j=1}^k [\nabla p]_{F_j} = \nabla p_{|T_k} - \nabla p_{|T_0}$,

$$\sum_{j=1}^k [\nabla p]_{F_j} \cdot m_j = p(m_1) - p(m_k) + \nabla p_{|T_k} \cdot m_k - \nabla p_{|T_0} \cdot m_1.$$

Using these, for $x \in T_0$ one obtains

$$p(x) = p(m_1) + \nabla p_{|T_0} \cdot (x - m_1) = p(m_k) + \nabla p_{|T_k} \cdot (x - m_k) + \sum_{j=1}^k [\nabla p]_{F_j} \cdot (m_j - x).$$

Because the tangential component of ∇p is continuous along the faces we have $[\nabla p]_{F_j} = [\nabla p \cdot n_{F_j}]_{F_j} n_{F_j}$. Using an inverse inequality $\|\nabla p\|_{0,T_k} \le c h_{F_k}^{-1} \|p\|_{0,T_k}$, the estimate $\|x - m_j\| \le c h_{F_j}$, and $|T_0| \sim |T_j|$, $j = 1, \ldots, k$, we get

$$\begin{aligned} \|p\|_{0,T_0}^2 &\le c \left(\|p\|_{0,T_k}^2 + \sum_{j=1}^k h_{F_j}^2 \frac{|T_0|}{|F_j|} \|[\nabla p \cdot n_{F_j}]\|_{0,F_j}^2 \right) \\ &\le c \left(\|p\|_{0,T_k}^2 + \sum_{j=1}^k h_{F_j}^3 \|[\nabla p \cdot n_{F_j}]\|_{0,F_j}^2 \right). \end{aligned}$$

We sum over $T_0 = T \in \Omega_{i,h} \setminus \omega_{i,h}$ and use a finite overlap argument, resulting in

$$\begin{aligned} \sum_{T\in\Omega_{i,h}\setminus\omega_{i,h}} \|p_{i,h}\|_{0,T}^2 &\le c \left(\|p_{i,h}\|_{0,\omega_{i,h}}^2 + \sum_{F\in\mathcal{F}_i} h_F^3 \|[\nabla p_{i,h} \cdot n_F]\|_{0,F}^2 \right) \\ &\le c\big(\|p_{i,h}\|_{0,\omega_{i,h}}^2 + \mu_i j_i(p_{i,h}, p_{i,h})\big), \end{aligned}$$

which completes the proof. ∎

3. Discrete problem. We introduce the usual bilinear forms

$$a(u, v) := \frac{1}{2} \int_\Omega \mu D(u) : D(v)\, dx, \quad b(v, p) = -(\operatorname{div} v, p)_{0,\Omega},$$

with $D(v) := \nabla v + (\nabla v)^T$. For discretization of the pressure we use the XFEM space Q_h^Γ. Note that for $p_h \in Q_h^\Gamma$ we have $p_h^\Gamma \in L_\mu^2(\Omega)$. For the velocity discretization we use the standard conforming P_2-space

$$V_h := \{\, v_h \in C(\Omega)^d \mid v_{h|T} \in \mathcal{P}_2^d \ \text{ for all } T \in \mathcal{T}_h,\ v_{|\partial\Omega} = 0 \,\} \subset H_0^1(\Omega)^d.$$

The discretization of (1.3) that we consider is as follows: determine $(u_h, p_h) \in V_h \times Q_h^\Gamma$ such that

$$
\begin{aligned}
&k\big((u_h,p_h),(v_h,q_h)\big) = f(v_h) \quad \text{for all } (v_h,q_h) \in V_h \times Q_h^\Gamma,\\
&k\big((u_h,p_h),(v_h,q_h)\big) := a(u_h,v_h) + b(v_h,p_h^\Gamma) - b(u_h,q_h^\Gamma) + \varepsilon_p j(p_h,q_h),
\end{aligned}
\tag{3.1}
$$

with a (sufficiently large) stabilization parameter $\varepsilon_p \geq 0$. Note that the XFEM space Q_h^Γ, which is used in the analysis below, will be replaced by $Q_h^{\Gamma_h}$ for the numerical experiments in section 8 due to implementation reasons, where Γ_h is a piecewise planar approximation of Γ.

4. A uniform inf-sup. In this section we prove an inf-sup result for the P_2-P_1 Hood–Taylor pair that is *uniform w.r.t. a variation of the domain* (as explained below). We need such a result in our stability analysis in section 5. Closely related uniform inf-sup results are used in other recent analyses of unfitted finite element methods. For example, in [4, Theorem 1] a uniform (w.r.t. domain variation) inf-sup *assumption* for the P_1bubble-P_1 pair is used. A similar assumption (for the isoP_2-P_1 pair) is required for the stability proof in [13] (but not explicitly mentioned there). We expect that the technique that we use for the P_2-P_1 pair in this section is also applicable to other LBB-stable mixed FE pairs.

We start with formulating this uniform inf-sup result. For this we consider a situation with a subdomain Ω_i, $i = 1, 2$, as in the previous section, where part of its boundary (or the whole boundary), which is denoted by Γ, is not aligned with the finite element triangulation. The results derived in this section hold for both Ω_1 and Ω_2. To simplify notation, we will consequently drop the subdomain index i for the remainder of this section and write Ω, Ω_h, and ω_h instead of Ω_i, $\Omega_{i,h}$, and $\omega_{i,h}$, respectively. *Note that the domain ω_h varies with h.* In this section we do *not* assume that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform but will use its shape regularity. We assume that each $T \in \omega_h$ has at least one vertex in the interior of ω_h.

Let $V_{h,0}(\omega_h) \subset V_h$ be the space of continuous piecewise quadratics on ω_h that are zero on $\partial\omega_h$ and let $Q_h(\omega_h)$ be the space of continuous piecewise linears on ω_h. On the subdomain ω_h, which is Lipschitz, the following LBB inf-sup property holds: there exists $c_{LBB}(\omega_h) > 0$ such that

$$
\sup_{v \in V_{h,0}(\omega_h)} \frac{(\operatorname{div} v, p)_{0,\omega_h}}{\|v\|_{1,\omega_h}} \geq c_{LBB}(\omega_h)\|p\|_{0,\omega_h} \quad \text{for all } p \in Q_h(\omega_h) \text{ with } (p,1)_{0,\omega_h} = 0.
\tag{4.1}
$$

Here and in the remainder, $\|\cdot\|_{1,\omega}$ denotes the Sobolev H^1-norm on the Lipschitz domain ω. The main result of this section is formulated in the following theorem.

THEOREM 4.1. *For $c_{LBB}(\omega_h)$ as in (4.1) the following holds:*

$$
\inf_{h>0} c_{LBB}(\omega_h) > 0.
\tag{4.2}
$$

We outline the main idea of the proof. We need the inf-sup property for the pair $H_0^1(\Omega)^d \times L_0^2(\Omega)$,

$$
\exists\, \beta > 0: \quad \sup_{v \in H_0^1(\Omega)^d} \frac{(\operatorname{div} v, p)_{0,\Omega}}{\|v\|_{1,\Omega}} \geq \beta\, \|p\|_{0,\Omega} \quad \text{for all } p \in L^2(\Omega) \text{ with } (p,1)_{0,\Omega} = 0.
\tag{4.3}
$$

In the analysis we need the following scaled norms for $q_h \in Q_h(\tau)$ with $\tau \subset \mathcal{T}_h$ a subset of simplices:

$$\|q_h\|_{0,h^{-1},\tau}^2 := \sum_{T\in\tau} h_T^{-2}\,\|q_h\|_{0,T}^2, \qquad |q_h|_{1,h,\tau}^2 := \sum_{T\in\tau} h_T^2\,\|\nabla q_h\|_{0,T}^2.$$

We use the so-called weak inf-sup property for the P_2-P_1 pair on ω_h. There exists a constant $\hat\beta > 0$, depending only on the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$, such that

$$\sup_{v_h\in V_{h,0}(\omega_h)} \frac{(\operatorname{div} v_h, q_h)_{0,\omega_h}}{\|v_h\|_{1,\omega_h}} \geq \hat\beta\,|q_h|_{1,h,\omega_h} \quad \text{for all } q_h \in Q_h(\omega_h). \tag{4.4}$$

This result is proved in Lemma 4.23 in [6]. (The result is essentially Proposition 1 in [1].) Take $q_h \in Q_h(\omega_h)$ with $(q_h, 1)_{0,\omega_h} = 0$. We introduce a suitable extension (Lemma 4.2 below) $q_h^e \in Q_h(\Omega_h)$ such that certain norms of q_h^e can be controlled by the corresponding norms of q_h. We shift q_h^e by a constant and apply the result (4.3), which yields a "suitable" $v \in H_0^1(\Omega_1)^d$. Of this v, extended by zero, we take the Scott–Zhang interpolation $w_h = I_{SZ}v \in Q_{h,0}(\Omega_h)^d$. Finally, we use a unique decomposition $w_h = v_h + r_h$ with $v_h \in Q_{h,0}(\omega_h)^d \subset V_{h,0}(\omega_h)$ and a $r_h \in Q_{h,0}(\Omega_h)^d$ which has nonzero nodal values only on $\partial\omega_h$. It turns out that norms of both v_h and r_h can be controlled by the corresponding norm of w_h. This v_h can be used in a perturbation argument as in [23] ("Verfürth's trick"), where we use (4.4). All details are given in the proof of Theorem 4.1 below.

All constants hidden in the $\lesssim$ and $\sim$ notation below depend only on the shape regularity of $\{\mathcal{T}_h\}_{h>0}$.

Define $\hat\Gamma_h := \partial\omega_h$ and $\mathcal{V}(\hat\Gamma_h)$ the set of its vertices. We need suitable extension operators, which are treated in the next lemma.

Lemma 4.2. *There exists a linear extension operator $E_h : Q_h(\omega_h) \to Q_h(\Omega_h)$ with $(E_h\, q_h)_{|\omega_h} = q_{h|\omega_h}$ such that*

$$\|E_h\, q_h\|_{0,\Omega_h} \lesssim \|q_h\|_{0,\omega_h},$$
$$|E_h\, q_h|_{1,h,\Omega_h} \lesssim |q_h|_{1,h,\omega_h}$$

for all $q_h \in Q_h(\omega_h)$.

Proof. For $T \in \mathcal{T}_h$ let $\mathcal{V}(T)$ denote the set of its $d+1$ vertices and define

$$\Omega_T := \{\tilde T \in \mathcal{T}_h :\ \mathcal{V}(\tilde T) \cap \mathcal{V}(T) \neq \emptyset\}.$$

For $T \in \mathcal{T}_h^\Gamma$ let $\gamma_T \subset \hat\Gamma_h$ be the smallest set of connected $(d-1)$-simplices such that $\Omega_T \cap \hat\Gamma_h \subset \gamma_T$. For each vertex $v_j \in \mathcal{T}_h^\Gamma$ we define a *connected vertex* $v_j^c \in \hat\Gamma_h$ by $v_j^c := v_j$ if $v_j \in \hat\Gamma_h$; otherwise $v_j^c := v_k$ for a fixed vertex $v_k \in \hat\Gamma_h$ with $v_j, v_k \in \mathcal{V}(\tilde T)$ for some $\tilde T \in \mathcal{T}_h^\Gamma$. Note that $v_j^c \in \gamma_T$ holds for all $v_j \in \mathcal{V}(T)$.

Let $q_h \in Q_h(\omega_h)$ and define $(E_h\, q_h)_{|\omega_h} := q_{h|\omega_h}$. On $T \in \mathcal{T}_h^\Gamma = \Omega_h \setminus \omega_h$ the extension E_h is defined by $(E_h q_h)(v_j) := q_h(v_j^c)$ for all $v_j \in \mathcal{V}(T)$. Then

$$\begin{aligned}
\|\nabla(E_h\, q_h)\|_{0,T}^2 &\sim \sum_{v_j,v_k\in\mathcal{V}(T)} \left(\frac{(E_h q_h)(v_j) - (E_h q_h)(v_k)}{h_T}\right)^2 h_T^d \\
&= \sum_{v_j,v_k\in\mathcal{V}(T)} \left(\frac{q_h(v_j^c) - q_h(v_k^c)}{h_T}\right)^2 h_T^d \\
&\lesssim h_T \|\nabla q_h\|_{0,\gamma_T}^2 \lesssim \|\nabla q_h\|_{0,\omega_T}^2,
\end{aligned}$$

where $\omega_T \subset \omega_h$ is the set of all $\tilde{T} \in \omega_h$ which have a face in γ_T. With the finite overlap property we conclude $|E_h\, q_h|_{1,h,\Omega_h} \lesssim |q_h|_{1,h,\omega_h}$. A similar argument can be applied for the estimate w.r.t. the scaled L^2 norm. ∎

Let $\mathcal{T}_h^{\Gamma,e} \subset \Omega_h$ be the set of all $T \in \Omega_h$ with at least one vertex in $\mathcal{V}(\hat{\Gamma}_h)$. Note that $\mathcal{T}_h^{\Gamma} \subset \mathcal{T}_h^{\Gamma,e}$. Furthermore, we define

$$\begin{aligned} Q_{h,0}(\mathcal{T}_h^{\Gamma}) &:= \{p \in C(\mathcal{T}_h^{\Gamma}) :\ p_{|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h^{\Gamma},\ p = 0 \text{ on } \partial\Omega_h\}, \\ Q_{h,0}(\mathcal{T}_h^{\Gamma,e}) &:= \{p \in C(\mathcal{T}_h^{\Gamma,e}) :\ p_{|T} \in \mathcal{P}_1 \text{ for all } T \in \mathcal{T}_h^{\Gamma,e},\ p = 0 \text{ on } \partial\mathcal{T}_h^{\Gamma,e}\}. \end{aligned}$$

Note that $p_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma})$ as well as $p_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma,e})$ is completely determined by its values at the vertices on $\hat{\Gamma}_h$. The following lemma and corollary give norm equivalences for such functions.

LEMMA 4.3. *The following holds for all $q_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma})$:*

$$\|q_h\|^2_{0,h^{-1},\mathcal{T}_h^{\Gamma}} \sim \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{d-2} \sum_{x_i\in\mathcal{V}(T)\cap\mathcal{V}(\hat{\Gamma}_h)} q_h(x_i)^2 \sim |q_h|^2_{1,\mathcal{T}_h^{\Gamma}}. \tag{4.5}$$

Proof. Take $q_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma})$. Then

$$\begin{aligned} \|q_h\|^2_{0,h^{-1},\mathcal{T}_h^{\Gamma}} &= \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{-2} \int_T q_h^2\, dx \\ &\sim \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{-2}\, |T| \sum_{x_i\in\mathcal{V}(T)} q_h(x_i)^2 = \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{d-2} \sum_{x_i\in\mathcal{V}(T)\cap\mathcal{V}(\hat{\Gamma}_h)} q_h(x_i)^2. \end{aligned}$$

For each vertex $x_i \in \mathcal{V}(T) \cap \mathcal{V}(\hat{\Gamma}_h)$ there exists another vertex $\tilde{x}_i \in \mathcal{V}(T)$ with $\tilde{x}_i \notin \mathcal{V}(\hat{\Gamma}_h)$, i.e., $q_h(\tilde{x}_i) = 0$. Hence,

$$(\nabla q_{h|T})^2 \gtrsim \left(\frac{q_h(x_i) - q_h(\tilde{x}_i)}{h_T}\right)^2 = h_T^{-2}\, q_h(x_i)^2,$$

and thus

$$\begin{aligned} \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{d-2} \sum_{x_i\in\mathcal{V}(T)\cap\mathcal{V}(\hat{\Gamma}_h)} q_h(x_i)^2 &\lesssim \sum_{T\in\mathcal{T}_h^{\Gamma}} h_T^{d} (\nabla q_{h|T})^2 \\ &\sim \sum_{T\in\mathcal{T}_h^{\Gamma}} \int_T (\nabla q_h)^2\, dx = |q_h|^2_{1,\mathcal{T}_h^{\Gamma}}. \end{aligned}$$

From the inverse inequality $\|\nabla q_h\|^2_{0,T} \lesssim h_T^{-2} \|q_h\|^2_{0,T}$ we obtain $|q_h|^2_{1,\mathcal{T}_h^{\Gamma}} \lesssim \|q_h\|^2_{0,h^{-1},\mathcal{T}_h^{\Gamma}}$. This completes the proof. ∎

COROLLARY 4.4. *From Lemma* 4.3 *it follows that the results in* (4.5) *also hold for $q_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma,e})$ with $\mathcal{T}_h^{\Gamma}$ replaced by $\mathcal{T}_h^{\Gamma,e}$. Furthermore it follows that for $q_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma,e})$ we have*

$$\|q_h\|_{0,h^{-1},\mathcal{T}_h^{\Gamma,e}} \sim \|q_h\|_{0,h^{-1},\mathcal{T}_h^{\Gamma}} \sim |q_h|_{1,\mathcal{T}_h^{\Gamma}} \sim |q_h|_{1,\mathcal{T}_h^{\Gamma,e}}. \tag{4.6}$$

Proof of Theorem 4.1. For the inf-sup constant $c_{LBB}(\omega_h)$ in (4.1) we have to study $\liminf_{h\to 0} c_{LBB}(\omega_h)$.

Take $0 < h \le h_0$ (with h_0 specified below) and $q_h \in Q_h(\omega_h)$ with $(q_h, 1)_{0,\omega_h} = 0$, $\|q_h\|_{0,\omega_h} = 1$. Define $q_h^e := E_h\, q_h \in Q_h(\Omega_h)$ and let $c_h \in \mathbb{R}$ such that $(q_h^e + c_h, 1)_{0,\Omega} = 0$. Note that $c_h = -|\Omega|^{-1}(q_h^e, 1)_{0,\Omega\setminus\omega_h}$, and hence

$$|c_h| \le |\Omega|^{-1}\, |\Omega \setminus \omega_h|^{\frac{1}{2}}\, \|q_h^e\|_{0,\Omega_h} \le ch^{\frac{1}{2}}\|q_h\|_{0,\omega_h} = ch^{\frac{1}{2}},$$

where the constant $c > 0$ depends only on Ω and the shape-regularity of $\{\mathcal{T}_h\}_{h>0}$. Take $h_0 > 0$ sufficiently small such that $ch_0^{\frac{1}{2}} \le \frac{1}{2}|\Omega|^{-\frac{1}{2}}$. Thus we get

$$\|q_h^e + c_h\|_{0,\Omega} \ge \|q_h^e\|_{0,\Omega} - |c_h||\Omega|^{\frac{1}{2}} \ge \|q_h\|_{0,\omega_h} - |c_h||\Omega|^{\frac{1}{2}} \ge 1 - ch^{\frac{1}{2}}|\Omega|^{\frac{1}{2}} \ge \frac{1}{2}. \tag{4.7}$$

Using (4.3) for $p = q_h^e + c_h \in L^2(\Omega)/\mathbb{R}$, it follows that there exists $v \in H_0^1(\Omega)^d$ with $\|v\|_{1,\Omega} = 1$ such that

$$(\operatorname{div} v, q_h^e + c_h)_{0,\Omega} = (\operatorname{div} v, q_h^e)_{0,\Omega} \ge \frac{1}{2}\beta\|q_h^e + c_h\|_{0,\Omega} \ge \frac{1}{4}\beta. \tag{4.8}$$

Extending v by zero outside Ω we obtain $v \in H_0^1(\Omega_h)^d$. Let $w_h = I_{SZ}\, v \in Q_{h,0}(\Omega_h)^d$ be the componentwise Scott–Zhang interpolation of $v \in H_0^1(\Omega_h)^d$; cf. [20]. Here $Q_{h,0}(\Omega_h)$ denotes the set of all functions from $Q_h(\Omega_h)$ which are vanishing on the boundary. For the Scott–Zhang interpolation w_h the following holds:

$$\|v - w_h\|_{0,h^{-1},\Omega_h} \le \hat{c}_1\|v\|_{1,\Omega_h} = \hat{c}_1\|v\|_{1,\Omega} = \hat{c}_1, \tag{4.9}$$

$$\|w_h\|_{1,\Omega_h} \le \hat{c}_2\|v\|_{1,\Omega_h} = \hat{c}_2\|v\|_{1,\Omega} = \hat{c}_2 \tag{4.10}$$

with constants $\hat{c}_1, \hat{c}_2 > 0$ only depending on the shape regularity of $\{\mathcal{T}_h\}_{h>0}$. We can uniquely decompose $w_h = v_h + r_h$ with $v_h \in Q_{h,0}(\omega_h)^d \subset V_{h,0}(\omega_h)$ and $r_h \in Q_{h,0}(\mathcal{T}_h^{\Gamma,e})^d$. Note that w_h and r_h coincide on $\mathcal{T}_h^{\Gamma}$. Using this, (4.10), and Corollary 4.4 we get

$$\begin{aligned}\|v_h\|_{1,\omega_h} &\le \|w_h\|_{1,\omega_h} + \|r_h\|_{1,\omega_h} \le \hat{c}_2 + \|r_h\|_{1,\omega_h\cap\mathcal{T}_h^{\Gamma,e}} \\ &\lesssim \hat{c}_2 + \|r_h\|_{0,h^{-1},\mathcal{T}_h^{\Gamma,e}} \sim \hat{c}_2 + |r_h|_{1,\mathcal{T}_h^{\Gamma}} = \hat{c}_2 + |w_h|_{1,\mathcal{T}_h^{\Gamma}} \\ &\le \hat{c}_2 + \|w_h\|_{1,\Omega_h} \lesssim \hat{c}_2.\end{aligned} \tag{4.11}$$

Furthermore, again with Corollary 4.4, we have

$$\|r_h\|_{0,h^{-1},\mathcal{T}_h^{\Gamma,e}} \sim |r_h|_{1,\mathcal{T}_h^{\Gamma}} = |w_h|_{1,\mathcal{T}_h^{\Gamma}} \le \hat{c}_2. \tag{4.12}$$

We introduce the notation $\xi_h := c_{LBB}(\omega_h) = \sup_{\hat{v}\in V_{h,0}(\omega_h)} \frac{(\operatorname{div}\hat{v}_h, q_h)_{0,\omega_h}}{\|\hat{v}_h\|_{1,\omega_h}}$. From (4.4) we have $|q_h|_{1,h,\omega_h} \le \frac{\xi_h}{\beta}$. Using (4.11) and $v_h = 0$ on $\Omega_h \setminus \omega_h$ we get

$$\begin{aligned}\xi_h &\ge \frac{(\operatorname{div} v_h, q_h)_{0,\omega_h}}{\|v_h\|_{1,\omega_h}} \gtrsim (\operatorname{div} v_h, q_h)_{0,\omega_h} \\ &= (\operatorname{div} v_h, q_h^e)_{0,\Omega_h} \ge \frac{1}{4}\beta + (\operatorname{div}(v_h - v), q_h^e)_{0,\Omega_h},\end{aligned} \tag{4.13}$$

where in the last inequality we used (4.8). Since $v_h - v = 0$ on $\partial\Omega_h$ we have

$$\begin{aligned}|(\operatorname{div}(v_h - v), q_h^e)_{0,\Omega_h}| &= |(v_h - v, \nabla q_h^e)_{0,\Omega_h}| \\ &\le |q_h^e|_{1,h,\Omega_h}\|v_h - v\|_{0,h^{-1},\Omega_h} \\ &\lesssim |q_h|_{1,h,\omega_h}\left(\|v_h - w_h\|_{0,h^{-1},\Omega_h} + \|w_h - v\|_{0,h^{-1},\Omega_h}\right),\end{aligned}$$

using Lemma 4.2 in the last inequality. Due to (4.9) and (4.12) we conclude that

$$|(\operatorname{div}(v_h - v), q_h^e)_{0,\Omega_h}| \lesssim \frac{\xi_h}{\hat{\beta}}\left(\|r_h\|_{0,h^{-1},\Omega_h} + \hat{c}_1\right) \lesssim \frac{\xi_h}{\hat{\beta}}(\hat{c}_2 + \hat{c}_1).$$

Hence, in (4.13) we get

$$\xi_h \gtrsim \frac{1}{4}\beta - \frac{\xi_h}{\hat{\beta}},$$

which implies $\xi_h \geq \xi_0 > 0$ with ξ_0 only depending on β, $\hat{\beta}$, and the shape regularity of $\{\mathcal{T}_h\}_{h>0}$. This completes the proof. ∎

5. Stability analysis. In this section we derive a discrete inf-sup result for the bilinear form $k(\cdot,\cdot)$ w.r.t. the space $V_h \times Q_h^\Gamma$; cf. Theorem 5.4. Such a result also holds if we replace V_h by a larger H^1-conforming space $\tilde{V}_h \supset V_h$; cf. Remark 1. The analysis is along the same lines as in [13, 16].

We will use the fact that the Taylor–Hood P_2-P_1 pair is uniformly stable on the subdomains $\omega_{i,h}$; cf. Theorem 4.1. In the next three lemmas we derive lower bounds for $\sup_{v_h \in V_h} \frac{b(v_h, p_h^\Gamma)}{\|\mu^{\frac{1}{2}}\nabla v_h\|_0}$. We first consider $\hat{p}_h \in M_0$ (Lemma 5.1) and then $\tilde{p}_h \in M_0^\perp$ (Lemma 5.2), and then we combine these results to obtain an estimate for $p_h \in Q_h^\Gamma$ (Lemma 5.3). The constants used in the estimates are independent of h, μ and of how the interface Γ intersects the triangulation.

LEMMA 5.1. *There exist $h_0 > 0$ and $c > 0$ such that for all $h \leq h_0$*

$$\sup_{v_h \in V_h} \frac{b(v_h, \hat{p}_h^\Gamma)}{\|\mu^{\frac{1}{2}}\nabla v_h\|_0} \geq c\|\mu^{-\frac{1}{2}}\hat{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \quad \textit{for all } \hat{p}_h \in M_0.$$

Proof. It suffices to consider $\hat{p}_h = \bar{p}_\mu$ as in (2.2). Define $\bar{p} := \mu^{-1}\bar{p}_\mu = (|\Omega_1|^{-1}, -|\Omega_2|^{-1}) \in Q_{1,h} \times Q_{2,h}$. The relation $\|\mu^{-\frac{1}{2}}\bar{p}_\mu^\Gamma\|_{0,\Omega} = C(\mu,\Omega)^{\frac{1}{2}}\|\bar{p}^\Gamma\|_{0,\Omega}$ holds with

$$C(\mu,\Omega) = \frac{\mu_1|\Omega_1|^{-1} + \mu_2|\Omega_2|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} \geq \mu_{\max}\min_{i=1,2}\frac{|\Omega_i|^{-1}}{|\Omega_1|^{-1} + |\Omega_2|^{-1}} = c\,\mu_{\max}$$

with $\mu_{\max} = \max\{\mu_1, \mu_2\}$. For $v_h \in V_h$ we have $0 = \int_\Omega \operatorname{div} v_h\,dx = \int_{\Omega_1}\operatorname{div} v_h\,dx + \int_{\Omega_2}\operatorname{div} v_h\,dx$, and using this one derives the relation

$$b(v_h, \bar{p}_\mu^\Gamma) = C(\mu,\Omega)b(v_h, \bar{p}^\Gamma), \quad v_h \in V_h. \tag{5.1}$$

Let $q_h \in C(\Omega)$ be the continuous piecewise linear nodal interpolation of $\bar{p}^\Gamma$. Then $\|q_h - \bar{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$ holds. Define $\alpha = \frac{1}{|\Omega|}(q_h, 1)_{0,\Omega}$ and $q_h^* = q_h - \alpha$, and hence $(q_h^*, 1)_{0,\Omega} = 0$. Note that $|\alpha| = \frac{1}{|\Omega|}|(q_h,1)_{0,\Omega}| = \frac{1}{|\Omega|}|(q_h - \bar{p}^\Gamma, 1)_{0,\Omega}| \leq c\|q_h - \bar{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$ holds. This implies $\|q_h^* - \bar{p}^\Gamma\|_{0,\Omega} \leq ch^{\frac{1}{2}}$. From the LBB stability of the standard P_2-P_1 Taylor–Hood pair on Ω it follows that there exists $\hat{v}_h \in V_h$ with $\|\hat{v}_h\|_1 = 1$ and $c > 0$ such that $b(\hat{v}_h, q_h^*) \geq c\|q_h^*\|_{0,\Omega}$ holds. Using this we obtain, with suitable constants $c > 0$,

$$\begin{aligned} b(\hat{v}_h, \bar{p}^\Gamma) &\geq b(\hat{v}_h, q_h^*) - d^{\frac{1}{2}}\|\hat{v}_h\|_1\|q_h^* - \bar{p}^\Gamma\|_{0,\Omega} \\ &\geq c\|q_h^*\|_{0,\Omega} - ch^{\frac{1}{2}} \geq c\|\bar{p}^\Gamma\|_{0,\Omega} - ch^{\frac{1}{2}} \geq c\|\bar{p}^\Gamma\|_{0,\Omega}, \end{aligned}$$

provided h is sufficiently small. Combining this with the result in (5.1) yields

$$b(\hat{v}_h, \bar{p}_\mu^\Gamma) = C(\mu,\Omega) b(v_h, \bar{p}^\Gamma) \geq c\, C(\mu,\Omega) \|\bar{p}^\Gamma\|_{0,\Omega}$$
$$= c\, C(\mu,\Omega)^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega} \geq c \mu_{\max}^{\frac{1}{2}} \|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega}.$$

Finally, note that $\|\mu^{-\frac{1}{2}} \bar{p}_\mu\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \leq (1+ch)\|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega} \leq c\|\mu^{-\frac{1}{2}} \bar{p}_\mu^\Gamma\|_{0,\Omega}$ and $\|\mu^{\frac{1}{2}} \nabla \hat{v}_h\|_0 \leq \mu_{\max}^{\frac{1}{2}} \|\hat{v}_h\|_1 = \mu_{\max}^{\frac{1}{2}}$ hold. ∎

LEMMA 5.2. *There exist $h_0 > 0$ and $c_1, c_2 > 0$ such that for all $h \leq h_0$*

$$\sup_{v_h \in V_h(\omega_{1,h}\cup\omega_{2,h})} \frac{b(v_h, \tilde{p}_h^\Gamma)}{\|\mu^{\frac{1}{2}} \nabla v_h\|_0} \geq c_1 \|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2 \frac{j(\tilde{p}_h, \tilde{p}_h)}{\|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}}$$

for all $\tilde{p}_h \in M_0^\perp \setminus \{0\}$ with $V_h(\omega_{1,h}\cup\omega_{2,h}) := \{\, v_h \in V_h \mid \operatorname{supp}(v_h) \subset \bar{\omega}_{1,h} \cup \bar{\omega}_{2,h} \,\}$.

Proof. Take $\tilde{p}_h = (\tilde{p}_{1,h}, \tilde{p}_{2,h}) \in M_0^\perp$, $\tilde{p}_h \neq 0$. Define $\alpha_i = \frac{1}{|\omega_{i,h}|}(\tilde{p}_{i,h}, 1)_{0,\omega_{i,h}}$ and $p_{i,h}^* = \tilde{p}_{i,h} - \alpha_i$, and hence $(p_{i,h}^*, 1)_{0,\omega_{i,h}} = 0$. Using (4.2) it follows that there exist $\tilde{v}_{i,h} \in V_h$ with $\operatorname{supp}(\tilde{v}_{i,h}) \subset \bar{\omega}_{i,h}$, $\|\tilde{v}_{i,h}\|_1 = \|p_{i,h}^*\|_{0,\omega_{i,h}}$, and a constant $c > 0$ such that $b(\tilde{v}_{i,h}, p_{i,h}^*) \geq c\|p_{i,h}^*\|_{0,\omega_{i,h}}^2$ (with $p_{i,h}^*$ extended by zero outside $\Omega_{i,h}$). Using that $\tilde{v}_{i,h} = 0$ on $\partial\omega_{i,h}$ and $\tilde{p}_{i,h} - p_{i,h}^* = \alpha_i$ is constant we get that $b(\tilde{v}_{i,h}, p_{i,h}^*) = b(\tilde{v}_{i,h}, \tilde{p}_{i,h})$ holds. Since $p_h^* = (p_{1,h}^*, p_{2,h}^*) \in Q_{1,h} \times Q_{2,h}$ we can apply Lemma 2.2 and thus get, with constant $c_1, c_2 > 0$,

$$\begin{aligned} b(\mu_i^{-1} \tilde{v}_{i,h}, \tilde{p}_{i,h}) &= b(\mu_i^{-1} \tilde{v}_{i,h}, p_{i,h}^*) \geq c \mu_i^{-1} \|p_{i,h}^*\|_{0,\omega_{i,h}}^2 \\ &\geq c_1 \mu_i^{-1} \|p_{i,h}^*\|_{0,\Omega_{i,h}}^2 - c_2 j(p_h^*, p_h^*). \end{aligned} \tag{5.2}$$

Since $j(\tilde{p}_h, \tilde{p}_h)$ depends only on $\nabla \tilde{p}_h$ we have $j(p_h^*, p_h^*) = j(\tilde{p}_h, \tilde{p}_h)$. From Lemma 2.1 we get $(\tilde{p}_{i,h}, 1)_{0,\Omega_i} = 0$. Using this we obtain

$$\begin{aligned} |\alpha_i| &= \frac{1}{|\omega_{i,h}|} |(\tilde{p}_{i,h}, 1)_{0,\omega_{i,h}}| = \frac{1}{|\omega_{i,h}|} \Big| \int_{\Omega_i \setminus \omega_{i,h}} \tilde{p}_{i,h}\, dx \Big| \\ &\leq \frac{1}{|\omega_{i,h}|} |\Omega_i \setminus \omega_{i,h}|^{\frac{1}{2}} \|\tilde{p}_{i,h}\|_{0,\Omega_i} \leq c h^{\frac{1}{2}} \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}}. \end{aligned} \tag{5.3}$$

Thus, for h sufficiently small there exists $c > 0$ such that

$$\|p_{i,h}^*\|_{0,\Omega_{i,h}} \geq \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}} - c|\alpha_i| \geq \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}} (1 - c h^{\frac{1}{2}}) \geq c \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}}.$$

Using this in (5.2) we get

$$b(\mu_i^{-1} \tilde{v}_{i,h}, \tilde{p}_{i,h}) \geq c_1 \|\mu_i^{-\frac{1}{2}} \tilde{p}_{i,h}\|_{0,\Omega_{i,h}}^2 - c_2 j(\tilde{p}_h, \tilde{p}_h),$$

and thus, with $\tilde{v}_h := \mu_1^{-1} \tilde{v}_{1,h} + \mu_2^{-1} \tilde{v}_{2,h} \in V_h(\omega_{1,h} \cup \omega_{2,h})$,

$$\begin{aligned} b(\tilde{v}_h, \tilde{p}_h^\Gamma) &= b(\mu_1^{-1} \tilde{v}_{1,h}, \tilde{p}_{1,h}) + b(\mu_2^{-1} \tilde{v}_{2,h}, \tilde{p}_{2,h}) \\ &\geq c_1 \|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 - c_2 j(\tilde{p}_h, \tilde{p}_h). \end{aligned} \tag{5.4}$$

Using (5.3) we get $\|\tilde{v}_{i,h}\|_1 = \|\tilde{p}_{i,h}^*\|_{0,\omega_{i,h}} \leq \|\tilde{p}_{i,h}\|_{0,\omega_{i,h}} + c|\alpha_i| \leq c\|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}}$, and thus

$$\|\mu^{\frac{1}{2}} \nabla \tilde{v}_h\|_0^2 = \sum_{i=1}^2 \mu_i^{-1} \|\nabla \tilde{v}_{i,h}\|_0^2 \leq c \sum_{i=1}^2 \|\mu_i^{-\frac{1}{2}} \tilde{p}_{i,h}\|_{0,\Omega_{i,h}}^2 = c\|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2$$

holds. Combining this with the estimate in (5.4) completes the proof. ∎

LEMMA 5.3. *There exist $h_0 > 0$ and $c_1, c_2 > 0$ such that for all $h \le h_0$,*

$$\sup_{v_h \in V_h} \frac{b(v_h, p_h^\Gamma)}{\|\mu^{\frac12}\nabla v_h\|_0} \ge c_1 \|\mu^{-\frac12} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2 \frac{j(p_h,p_h)}{\|\mu^{-\frac12} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}} \quad \text{for all } p_h \in Q_h^\Gamma\backslash\{0\}.$$

Proof. Take $p_h = (p_{1,h}, p_{2,h}) \in Q_h^\Gamma \setminus \{0\}$. We use the decomposition $p_h = \hat{p}_h + \tilde{p}_h$, $\hat{p}_h \in M_0$, $\tilde{p}_h \in M_0^\perp$. From the lemmas above it follows that there exist $\hat{v}_h \in V_h$, $\tilde{v}_h \in V_h(\omega_{1,h} \cup \omega_{1,h})$ with $\|\mu^{\frac12}\nabla \hat{v}_h\|_0 = \|\mu^{-\frac12}\hat{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$, $\|\mu^{\frac12}\nabla \tilde{v}_h\|_0 = \|\mu^{-\frac12}\tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$ such that

$$b(\hat{v}_h, \hat{p}_h^\Gamma) \ge c_1 \|\mu^{-\frac12}\hat{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}, \quad b(\tilde{v}_h, \tilde{p}_h^\Gamma) \ge c_2 \|\mu^{-\frac12}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_3 j(\tilde{p}_h, \tilde{p}_h)$$

with $c_j > 0$, $j = 1,2,3$. Note that $\tilde{v}_h = 0$ on $\partial\omega_{i,h}$ and $\hat{p}_h^\Gamma$ is constant on $\omega_{i,h}$, and hence $b(\tilde{v}_h, \hat{p}_h^\Gamma) = -\sum_{i=1}^2 (\operatorname{div} \tilde{v}_h, \hat{p}_h^\Gamma)_{0,\omega_{i,h}} = 0$ holds. Take $v_h := \hat{v}_h + \gamma \tilde{v}_h \in V_h$ with $\gamma > 0$. We then get

$$\begin{aligned} b(v_h, p_h^\Gamma) &= b(\hat{v}_h, \hat{p}_h^\Gamma) + \gamma b(\tilde{v}_h, \tilde{p}_h^\Gamma) + b(\hat{v}_h, \tilde{p}_h^\Gamma) \\ &\ge c_1 \|\mu^{-\frac12}\hat{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \gamma c_2 \|\mu^{-\frac12}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} - \gamma c_3 j(\tilde{p}_h, \tilde{p}_h) + b(\hat{v}_h, \tilde{p}_h^\Gamma). \end{aligned}$$

Since $\hat{p}_h$ is constant on $\Omega_{i,h}$ we have $j(\tilde{p}_h, \tilde{p}_h) = j(p_h, p_h)$. Furthermore,

$$\begin{aligned} |b(\hat{v}_h, \tilde{p}_h^\Gamma)| &\le d^{\frac12} \|\mu^{\frac12}\nabla \hat{v}_h\|_0 \|\mu^{-\frac12}\tilde{p}_h^\Gamma\|_{0,\Omega} \le d^{\frac12} \|\mu^{-\frac12}\hat{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \|\mu^{-\frac12}\tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\ &\le \frac12 c_1 \|\mu^{-\frac12}\hat{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \frac12 d c_1^{-1} \|\mu^{-\frac12}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}}. \end{aligned}$$

For $\gamma = \frac{c_1^2 + d}{2c_1c_2}$ we thus get, with a suitable constant c,

$$b(v_h, p_h^\Gamma) \ge \frac12 c_1 \big(\|\mu^{-\frac12}\hat{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|\mu^{-\frac12}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} \big) - c j(p_h, p_h),$$

and then by combining this with $\|\mu^{-\frac12}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} \le 2(\|\mu^{-\frac12}\hat{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} + \|\mu^{-\frac12}\tilde{p}_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}})$ we obtain

$$\frac{b(v_h, p_h^\Gamma)}{\|\mu^{-\frac12}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}} \ge \frac14 c_1 \|\mu^{-\frac12}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c \frac{j(p_h,p_h)}{\|\mu^{-\frac12}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}}. \tag{5.5}$$

From $0 = (\hat{p}_h^\Gamma, \tilde{p}_h^\Gamma)_{0,\Omega} = \sum_{i=1}^2 (\hat{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_i}$ we obtain

$$\begin{aligned} \left| \sum_{i=1}^2 (\hat{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_{i,h}} \right| &= \left| \sum_{i=1}^2 (\hat{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_{i,h}\backslash\Omega_i} \right| \le c h^{\frac12} \sum_{i=1}^2 \|\hat{p}_{i,h}\|_{0,\Omega_{i,h}} \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}} \\ &\le c h^{\frac12} \sum_{i=1}^2 \big(\|\hat{p}_{i,h}\|^2_{0,\Omega_{i,h}} + \|\tilde{p}_{i,h}\|^2_{0,\Omega_{i,h}} \big). \end{aligned}$$

Using this we get

$$
\begin{aligned}
&\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 \\
&\quad = \sum_{i=1}^{2} \mu_i^{-1} \|\hat{p}_{i,h} + \tilde{p}_{i,h}\|_{0,\Omega_{i,h}}^2 \\
&\quad = \sum_{i=1}^{2} \mu_i^{-1} \big(\|\hat{p}_{i,h}\|_{0,\Omega_{i,h}}^2 + \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}}^2 + 2(\hat{p}_{i,h}, \tilde{p}_{i,h})_{0,\Omega_{i,h}} \big) \\
&\quad \geq (1 - ch^{\frac{1}{2}}) \sum_{i=1}^{2} \mu_i^{-1} \big(\|\hat{p}_{i,h}\|_{0,\Omega_{i,h}}^2 + \|\tilde{p}_{i,h}\|_{0,\Omega_{i,h}}^2 \big) \\
&\quad = (1 - ch^{\frac{1}{2}}) \big(\|\mu^{-\frac{1}{2}} \hat{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + \|\mu^{-\frac{1}{2}} \tilde{p}_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 \big) \\
&\quad = (1 - ch^{\frac{1}{2}}) \big(\|\mu^{\frac{1}{2}} \nabla \hat{v}_h\|_0^2 + \|\mu^{\frac{1}{2}} \nabla \tilde{v}_h\|_0^2 \big).
\end{aligned}
$$

Hence, for h sufficiently small there exists $c > 0$ such that

$$
\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \geq c \big(\|\mu^{\frac{1}{2}} \nabla \hat{v}_h\|_0^2 + \|\mu^{\frac{1}{2}} \nabla \tilde{v}_h\|_0^2 \big)^{\frac{1}{2}} \geq \frac{1}{2} \min\{1, \gamma^{-1}\} c \|\mu^{\frac{1}{2}} \nabla v_h\|_0,
$$

and combining this with (5.5) completes the proof. ∎

For the main result in the next theorem we introduce a mesh- and μ-dependent norm on $V_h \times Q_h^\Gamma$:

$$
\|(u_h, p_h)\|_h^2 := \|\mu^{\frac{1}{2}} D(u_h)\|_0^2 + \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(p_h, p_h). \tag{5.6}
$$

From Korn's inequality it follows that this defines a norm on $V_h \times Q_h^\Gamma$.

THEOREM 5.4. *There exist constants $h_0 > 0$, $\epsilon_0 > 0$ and $c_s > 0$ such that for all $h \leq h_0$, $\varepsilon_p \geq \epsilon_0$ the following holds:*

$$
\sup_{(v_h, q_h) \in V_h \times Q_h^\Gamma} \frac{k\big((u_h, p_h), (v_h, q_h)\big)}{\|(v_h, q_h)\|_h} \geq c_s \|(u_h, p_h)\|_h \quad \text{for all} \;\; (u_h, p_h) \in V_h \times Q_h^\Gamma.
$$

The constants are independent of μ and of how the interface Γ intersects the triangulation.

Proof. Take $(u_h, p_h) \in V_h \times Q_h^\Gamma$. From Lemma 5.3 it follows that there exists, for $h_0 > 0$ sufficiently small, $w_h \in V_h$ with $\|\mu^{\frac{1}{2}} \nabla w_h\|_0 = \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$ and

$$
b(-w_h, p_h) \geq c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 - c_2 j(p_h, p_h).
$$

Take $(v_h, q_h) = (u_h - \alpha w_h, p_h)$ with $\alpha > 0$. Note that $\|\mu^{\frac{1}{2}} D(v)\|_0 \leq c \|\mu^{\frac{1}{2}} \nabla v\|_0$ for $v \in H^1(\Omega)$ holds. We then obtain, with suitable strictly positive constants,

$$
\begin{aligned}
&k\big((u_h, p_h), (v_h, q_h)\big) \\
&\quad = a(u_h, u_h) - \alpha a(u_h, w_h) + \alpha b(-w_h, p_h) + \varepsilon_p j(p_h, p_h) \\
&\quad \geq \|\mu^{\frac{1}{2}} D(u_h)\|_0^2 - \tilde{c}\alpha \|\mu^{\frac{1}{2}} D(u_h)\|_0 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \\
&\qquad + \alpha c_1 \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + (\varepsilon_p - \alpha c_2) j(p_h, p_h) \\
&\quad \geq \frac{1}{2} \|\mu^{\frac{1}{2}} D(u_h)\|_0^2 + \alpha \left(c_1 - \frac{1}{2} \tilde{c}^2 \alpha \right) \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + (\varepsilon_p - \alpha c_2) j(p_h, p_h).
\end{aligned}
$$

We take α such that $c_1 - \frac{1}{2}\tilde{c}^2\alpha = \frac{1}{2}c_1$ holds, and ε_p such that $\varepsilon_p - \alpha c_2 \geq 1$. Thus we obtain, with suitable $c > 0$,

$$k\big((u_h, p_h), (v_h, q_h)\big) \geq c|\!|\!|(u_h, p_h)|\!|\!|_h^2.$$

Combining this with

$$\begin{aligned}|\!|\!|(v_h, q_h)|\!|\!|_h^2 &= \|\mu^{\frac{1}{2}} D(u_h - \alpha w_h)\|_0^2 + \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(p_h, p_h)\\ &\leq 2\|\mu^{\frac{1}{2}} D(u_h)\|_0^2 + (c\alpha^2 + 1)\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(p_h, p_h) \leq c|\!|\!|(u_h, p_h)|\!|\!|_h^2\end{aligned}$$

completes the proof. ∎

6. Discretization error analysis. We introduce the space $Q_{\text{reg}} = H^2(\Omega_{1,h}) \times H^2(\Omega_{2,h})$. The norm in (5.6) is well-defined also for $(u, p) \in H^1(\Omega)^d \times Q_{\text{reg}}$. Let $\mathcal{E}_i : H^2(\Omega_i) \to H^2(\Omega_{i,h})$ be a bounded extension operator. Hence, there is a constant c, independent of h, such that $\|\mathcal{E}_i p\|_{2,\Omega_{i,h}} \leq c\|p\|_{2,\Omega_i}$ for all $p \in H^2(\Omega_i)$. For $p \in H^2(\Omega_1 \cup \Omega_2)$ we define $\mathcal{E}p := (\mathcal{E}_1 p_{|\Omega_1}, \mathcal{E}_2 p_{|\Omega_2}) \in Q_{\text{reg}}$. Note that for such extensions the stabilization term vanishes: $j(\mathcal{E}p, q_h) = 0$ for all $p \in H^2(\Omega_1 \cup \Omega_2)$ and $q_h \in Q_h^\Gamma$, i.e., we have a *consistent stabilization.* Based on this observation we obtain the following Cea-estimate.

THEOREM 6.1. *Assume that the solution (u, p) of (1.3) has the regularity property $p \in H^2(\Omega_1 \cup \Omega_2)$. Let $h_0 > 0$ and ε_p be as in Theorem 5.4. Take $h \leq h_0$ and let $(u_h, p_h) \in V_h \times Q_h^\Gamma$ be the solution of the discretization (3.1). There exists a constant $c > 0$, independent of h and μ and of how the interface Γ intersects the triangulation, such that*

$$|\!|\!|(u - u_h, \mathcal{E}p - p_h)|\!|\!|_h \leq c \min_{(v_h, q_h)\in V_h\times Q_h^\Gamma} |\!|\!|(u - v_h, \mathcal{E}p - q_h)|\!|\!|_h.$$

Proof. For $A \in \mathbb{R}^{d\times d}$ we have $\operatorname{tr}(A)^2 = \frac{1}{4}\operatorname{tr}(A + A^T)^2 \leq \frac{d}{4}\operatorname{tr}((A + A^T)^2)$, and thus for $w \in C^1(\Omega)^d$ we get $|\operatorname{div} w|^2 = |\operatorname{tr} \nabla w|^2 \leq \frac{d}{4}\operatorname{tr}((\nabla w + (\nabla w)^T)^2) = \frac{d}{4} D(w) : D(w)$. Hence, for $(w, q) \in H^1(\Omega)^d \times (Q_{\text{reg}} + Q_h^\Gamma)$ the estimate

$$|b(w, q^\Gamma)| \leq \|\mu^{\frac{1}{2}} \operatorname{div} w\|_0 \|\mu^{-\frac{1}{2}} q^\Gamma\|_{0,\Omega} \leq \frac{1}{2}\sqrt{d}\|\mu^{\frac{1}{2}} D(w)\|_0 \|\mu^{-\frac{1}{2}} q\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} \tag{6.1}$$

holds. From this, the definition of the bilinear form $k(\cdot, \cdot)$ and the Cauchy–Schwarz inequality one obtains boundedness w.r.t. $|\!|\!|\cdot|\!|\!|_h$:

$$\big|k\big((w, r), (v, q)\big)\big| \leq c|\!|\!|(w, r)|\!|\!|_h |\!|\!|(v, q)|\!|\!|_h \quad \text{for all } (w, r), (v, q) \in H^1(\Omega)^d \times (Q_{\text{reg}} + Q_h^\Gamma)$$

with c depending only on ε_p and d. For $p \in H^2(\Omega_1 \cup \Omega_2)$ we have $j(\mathcal{E}p, q_h) = 0$ for all $q_h \in Q_h^\Gamma$. Using this and the conformity property, i.e., $V_h \subset H_0^1(\Omega)^2$, $q_h^\Gamma \in L_\mu^2(\Omega)$ for $q_h \in Q_h^\Gamma$, we obtain consistency,

$$k\big((u, \mathcal{E}p), (v_h, q_h)\big) = k\big((u_h, p_h), (v_h, q_h)\big) \quad \text{for all } \ (v_h, q_h) \in V_h \times Q_h^\Gamma,$$

and hence $k(U - W_h, R_h) = k(U_h - W_h, R_h)$ holds for all $R_h \in V_h \times Q_h^\Gamma$ with $U = (u, \mathcal{E}p)$, $U_h = (u_h, p_h)$ and $W_h \in V_h \times Q_h^\Gamma$. The proof is easily completed using the standard Cea-argument; cf., for example, Lemma 2.28 in [6]. ∎

Remark 1. The results in Theorems 5.4 and 6.1 also hold if instead of V_h one takes a larger velocity space $\tilde{V}_h \supset V_h$, which is conforming, i.e, $\tilde{V}_h \subset H_0^1(\Omega)^d$ holds.

An obvious possibility is to extend the velocity space by additional basis functions to account for the kink of u at the interface. In [14] a kink enrichment is presented, which leads to an XFEM space $\tilde{V}_h = V_h \oplus \mathrm{span}\{v_j \cdot \Psi^\Gamma \mid j \in \mathcal{J}_\Gamma\}$. Here v_j, $j \in \mathcal{J}_\Gamma$, denote basis functions with $\mathrm{supp}\, v_j \cap \Gamma \neq \emptyset$ and Ψ^Γ is a special enrichment function with a kink at Γ, which has a support only on tetrahedra cut by the interface. Theorem 5.4 also holds for the pair $\tilde{V}_h \times Q_h^\Gamma$. It is clear that $\tilde{V}_h$ has better approximation properties for functions with kinks than the standard space V_h, but it is not known whether an optimal approximation result $\inf_{v_h \in \tilde{V}_h} \|u - v_h\|_1 \leq ch^2$ holds for this space. The results on conditioning of the stiffness matrix, derived for the pair $V_h \times Q_h^\Gamma$ in the next section, do not hold for the pair $\tilde{V}_h \times Q_h^\Gamma$.

Bounds for the approximation error

$$
\begin{aligned}
(6.2) \qquad & \min_{(v_h,q_h) \in V_h \times Q_h^\Gamma} |\!|\!|(u - v_h, \mathcal{E}p - q_h)|\!|\!|_h^2 \\
&= \min_{(v_h,q_h) \in V_h \times Q_h^\Gamma} \Big(\|\mu^{\frac{1}{2}} D(u - v_h)\|_0^2 + \|\mu^{-\frac{1}{2}}(\mathcal{E}p - q_h)\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 \\
&\qquad + j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) \Big)
\end{aligned}
$$

can be derived using standard interpolation error bounds. We first consider the terms related to the pressure approximation.

LEMMA 6.2. *There exists a constant c such that for all $p \in H^2(\Omega_1 \cup \Omega_2)$ the following holds:*

$$
(6.3) \quad \min_{q_h Q_h^\Gamma} \Big(\|\mu^{-\frac{1}{2}}(\mathcal{E}p - q_h)\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 + j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) \Big) \leq ch^4 \|\mu^{-\frac{1}{2}} p\|_{2,\Omega_1\cup\Omega_2}^2.
$$

Proof. Take $p \in H^2(\Omega_1 \cup \Omega_2)$. For $\mathcal{E}p = (\hat{p}_1, \hat{p}_2) \in Q_{\mathrm{reg}}$ let $I_h \hat{p}_i$ be the standard nodal interpolation on the vertices of $\Omega_{i,h}$. Hence,

$$
(6.4) \qquad \|\hat{p}_i - I_h \hat{p}_i\|_{\ell,\Omega_{i,h}} \leq ch^{2-\ell} \|\hat{p}_i\|_{2,\Omega_{i,h}} \leq ch^{2-\ell} \|p\|_{2,\Omega_i}, \quad \ell = 0, 1,
$$

holds. For $q = (q_1, q_2) \in Q_{\mathrm{reg}} + Q_h^\Gamma$ and $F \in \mathcal{F}_i$ with $F = T_1 \cap T_2$ and $T_1, T_2 \in \Omega_{i,h}$, we have

$$
\|[\nabla q_i \cdot n_F]\|_F^2 \leq \sum_{j=1}^{2} \|\nabla q_i\|_{\partial T_j}^2 \leq c \sum_{j=1}^{2} \big(h^{-1} \|\nabla q_i\|_{0,T_j}^2 + h \|\nabla^2 q_i\|_{0,T_j}^2 \big).
$$

Using this we get

$$
\begin{aligned}
j(q, q) &= \sum_{i=1}^{2} \sum_{F \in \mathcal{F}_i} \mu_i^{-1} h_F^3 \|[\nabla q_i \cdot n_F]\|_F^2 \\
&\leq c \sum_{i=1}^{2} \mu_i^{-1} \left(h^2 \|\nabla q_i\|_{0,\Omega_{i,h}}^2 + h^4 \sum_{T \in \Omega_{i,h}} \|\nabla^2 q_i\|_{0,T}^2 \right).
\end{aligned}
$$

We take $q = \mathcal{E}p - q_h$, $q_h = (q_{1,h}, q_{2,h}) \in Q_h^\Gamma$, and noting that $\nabla^2 q_{i,h|T} = 0$ we thus obtain

$$
\begin{aligned}
j(\mathcal{E}p - q_h, \mathcal{E}p - q_h) &\leq c \sum_{i=1}^{2} \mu_i^{-1} \big(h^2 \|\nabla(\hat{p}_i - q_{i,h})\|_{0,\Omega_{i,h}}^2 + h^4 \|\nabla^2 \hat{p}_i\|_{0,\Omega_{i,h}}^2 \big) \\
&\leq ch^2 \sum_{i=1}^{2} \mu_i^{-1} \|\nabla(\hat{p}_i - q_{i,h})\|_{0,\Omega_{i,h}}^2 + ch^4 \|\mu^{-\frac{1}{2}} p\|_{2,\Omega_1\cup\Omega_2}^2.
\end{aligned}
$$

We take $q_h = (I_h \hat{p}_1, I_h \hat{p}_2)$, and using the interpolation error bounds in (6.4) we obtain the bound in (6.3). ∎

For the velocity term in (6.2) we obviously also have the optimal error bound

$$\min_{v_h \in V_h} \|\mu^{\frac{1}{2}} D(u - v_h)\|_0^2 \leq c\mu_{\max} h^4 \|u\|_{3,\Omega}^2 \quad \text{for} \quad u \in H^3(\Omega) \tag{6.5}$$

with $\mu_{\max} = \max\{\mu_1, \mu_2\}$. In our applications, however, we typically do not have the regularity property $u \in H^3(\Omega)$. The velocity u is smooth in the interior of Ω_i but has a discontinuity in its first derivative across the interface Γ. Hence, globally, the best one can have is an asymptotic error bound of the form $\|u - v_h\|_1^2 \leq ch$. To improve on this one might use an XFEM velocity space, too, for example, $\tilde{V}_h$ as explained in Remark 1. It turns out, however, that in many applications the suboptimal velocity approximation using standard P_2 finite elements does not dominate the total error for realistic mesh sizes. This is illustrated by the numerical example in section 8.3.

As far as we know, rigorous regularity results for the Stokes interface problem (1.1), e.g., $u \in H^2(\Omega_1 \cup \Omega_2)$, $p \in H^1(\Omega_1 \cup \Omega_2)$ are not known in the literature.

Using a duality argument one can derive an L^2 error bound along the same lines as for the standard Stokes equation.

7. Schur complement preconditioner.

We introduce a matrix-vector representation of the discrete problem (3.1). In V_h we use the standard nodal basis denoted by $(\psi_j)_{1\leq j\leq m}$, i.e., $V_h \ni u_h = \sum_{j=1}^m x_j \psi_j$. The vector representation of u_h is denoted by $\mathbf{x} = (x_1, \ldots, x_m)^T \in \mathbb{R}^m$. In $Q_{i,h}$ we have a standard nodal basis denoted by $(\phi_{i,j})_{1\leq j \leq n_i}$, $i = 1, 2$, i.e., $Q_{1,h} \times Q_{2,h} \ni p_h = (p_{1,h}, p_{2,h}) = \left(\sum_{j=1}^{n_1} y_{1,j}\phi_{1,j}, \sum_{j=1}^{n_2} y_{2,j}\phi_{2,j}\right)$. The vector representation of p_h is denoted by $\mathbf{y} = (y_{1,1}, \ldots, y_{1,n_1}, y_{2,1}, \ldots, y_{2,n_2})^T \in \mathbb{R}^{n_1+n_2}$. Using the quasi-uniformity of the triangulation we conclude that there are strictly positive constants c_i, independent of h, such that

$$c_1 h^d \|\mathbf{y}\|^2 \leq \|p_{1,h}\|_{0,\Omega_{1,h}}^2 + \|p_{2,h}\|_{0,\Omega_{2,h}}^2 = \|p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 \leq c_2 h^d \|\mathbf{y}\|^2 \tag{7.1}$$

for all $p_h \in Q_{1,h} \times Q_{2,h}$. Here, $\|\cdot\|$ denotes the Euclidean vector norm. We use $\langle \cdot, \cdot \rangle$ to denote the Euclidean scalar product. The bilinear forms $a(\cdot,\cdot)$, $b(\cdot,\cdot)$, $j(\cdot,\cdot)$ have corresponding matrix representations, denoted by $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{(n_1+n_2)\times m}$, $J \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$, respectively. The matrix A is symmetric positive definite. The matrix J is symmetric positive semi-definite. Define $\mathbf{1} := (1, \ldots, 1)^T \in \mathbb{R}^{n_1+n_2}$. From $b(u_h, 1) = 0$ for all $u_h \in V_h$ and $j(1, q_h) = 0$ for all $q_h \in Q_{1,h} \times Q_{2,h}$ it follows that $B^T\mathbf{1} = J\mathbf{1} = 0$ holds.

Finally, we introduce two mass matrices in the pressure space:

$$\begin{aligned} M &= \operatorname{blockdiag}(M_1, M_2), \quad (M_i)_{k,l} := (\mu_i^{-1}\phi_{i,k}, \phi_{i,l})_{0,\Omega_{i,h}}, \quad 1 \leq k, l \leq n_i, \quad i = 1, 2, \\ \hat{M} &= \operatorname{blockdiag}(\hat{M}_1, \hat{M}_2), \quad (\hat{M}_i)_{k,l} := (\mu_i^{-1}\phi_{i,k}, \phi_{i,l})_{0,\Omega_i}, \quad 1 \leq k, l \leq n_i, \quad i = 1, 2. \end{aligned}$$

For these mass matrices we have the relations

$$\begin{aligned} \langle M\mathbf{y}, \mathbf{y}\rangle &= \|\mu_1^{-\frac{1}{2}} p_{1,h}\|_{0,\Omega_{1,h}}^2 + \|\mu_2^{-\frac{1}{2}} p_{2,h}\|_{0,\Omega_{2,h}}^2 = \|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2, \\ \langle \hat{M}\mathbf{y}, \mathbf{y}\rangle &= \|\mu_1^{-\frac{1}{2}} p_{1,h}\|_{0,\Omega_1}^2 + \|\mu_2^{-\frac{1}{2}} p_{2,h}\|_{0,\Omega_2}^2 = \sum_{i=1}^2 \|\mu_i^{-\frac{1}{2}} p_h^\Gamma\|_{0,\Omega_i}^2 = \|\mu^{-\frac{1}{2}} p_h^\Gamma\|_{0,\Omega}^2. \end{aligned}$$

The matrix-vector representation of the discrete problem (3.1) is as follows. First note that $(\mu^{-1}p_h^\Gamma, 1)_{0,\Omega} = 0$ iff $\langle \hat{M}\mathbf{y}, \mathbf{1}\rangle = 0$. The discrete problem is given by the following: determine $(\mathbf{x}, \mathbf{y})$ with $\langle \hat{M}\mathbf{y}, \mathbf{1}\rangle = 0$ such that

$$\begin{pmatrix} A & B^T \\ -B & \varepsilon_p J \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}, \quad b_j = (f, \psi_j)_{0,\Omega},\ 1 \le j \le m.$$

For the iterative solution of this system it is convenient to use the following equivalent, symmetric formulation: determine $(\mathbf{x}, \mathbf{y})$ with $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$ such that

$$K \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{b} \\ 0 \end{pmatrix}, \quad K := \begin{pmatrix} A & B^T \\ B & -\varepsilon_p J \end{pmatrix}. \tag{7.2}$$

Note that K has a one-dimensional kernel, spanned by $(0\ \mathbf{1})^T$. The Schur complement of K is denoted by $S = BA^{-1}B^T + \varepsilon_p J$. We consider the block diagonal preconditioner,

$$Q = \begin{pmatrix} Q_A & 0 \\ 0 & Q_S \end{pmatrix}, \quad Q_S = \hat{M} + \varepsilon_p J, \quad Q_A \text{ symmetric positive definite.} \tag{7.3}$$

In our applications, cf. section 8, we use for Q_A a symmetric multigrid iteration applied to A. The symmetric positive definite Schur complement preconditioner $Q_S = \hat{M} + \varepsilon_p J$ is analyzed in section 7.1.

When solving the linear system (7.2) we have to satisfy the consistency condition $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$. The following lemma shows that for a Krylov subspace method applied to the preconditioned matrix $Q^{-1}K$ this condition is automatically satisfied. We use the properties $J\mathbf{1} = 0$, and hence $Q_S\mathbf{1} = \hat{M}\mathbf{1}$, i.e., $Q_S^{-1}\hat{M}\mathbf{1} = \mathbf{1}$.

Lemma 7.1. *Define $Y = \{(\mathbf{x}\ \mathbf{y})^T \in \mathbb{R}^{m+n_1+n_2} \mid \mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}\}$. Then $Q^{-1}K : Y \to Y$ is a bijection.*

Proof. The space Y forms a direct sum with the kernel $\mathrm{span}\{(0\ \mathbf{1})^T\}$ of the matrix K. Hence, the range of $K : Y \to \mathbb{R}^{m+n_1+n_2}$ has co-dimension 1. For $(\mathbf{x}\ \mathbf{y})^T$ define $(\tilde{\mathbf{x}}\ \tilde{\mathbf{y}})^T = Q^{-1}K(\mathbf{x}\ \mathbf{y})^T$. For $\tilde{\mathbf{y}}$ we have

$$\langle \hat{M}\tilde{\mathbf{y}}, \mathbf{1}\rangle = \langle B\mathbf{x} - \varepsilon_p J\mathbf{y}, Q_S^{-1}\hat{M}\mathbf{1}\rangle = \langle B\mathbf{x} - \varepsilon_p J\mathbf{y}, \mathbf{1}\rangle = \langle \mathbf{x}, B^T\mathbf{1}\rangle - \varepsilon_p\langle y, J\mathbf{1}\rangle = 0.$$

Hence, $(\tilde{\mathbf{x}}\ \tilde{\mathbf{y}})^T \in Y$ holds. ∎

As we will see in the next section, the matrices M and $\hat{M} + \varepsilon_p J$ are spectrally equivalent. If we would use $Q_S = M$ as the Schur complement preconditioner, it is not clear how to satisfy the consistency condition $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$. This is the reason why besides the mass matrix M we also need the mass matrix $\hat{M}$.

7.1. Analysis of the preconditioner.

We analyze the quality of the block diagonal preconditioner Q given in (7.3).

We start with a main result, which shows that the weighted mass matrix M is uniformly spectrally equivalent to the Schur complement.

Theorem 7.2. *Take $\varepsilon_p > 0$. There exist constants $c_1, c_2 > 0$, independent of h, μ and of how Γ intersects the triangulation, such that with $S = BA^{-1}B^T + \varepsilon_p J$ we have*

$$c_1\langle M\mathbf{y}, \mathbf{y}\rangle \le \langle S\mathbf{y}, \mathbf{y}\rangle \le c_2\langle M\mathbf{y}, \mathbf{y}\rangle \quad \text{for all } \mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}. \tag{7.4}$$

Proof. Take $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$. We use the relation

$$\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} = \max_{\mathbf{x}\in\mathbb{R}^m} \frac{\langle B\mathbf{x}, \mathbf{y}\rangle}{\langle A\mathbf{x}, \mathbf{x}\rangle^{\frac{1}{2}}} = \max_{u_h\in V_h} \frac{b(u_h, p_h^\Gamma)}{a(u_h, u_h)^{\frac{1}{2}}}. \tag{7.5}$$

We use the estimate (6.1) and thus get

$$\begin{aligned}\max_{u_h\in V_h} \frac{b(u_h, p_h^\Gamma)}{a(u_h, u_h)^{\frac{1}{2}}} &\le c \max_{u_h\in V_h} \frac{\|\mu^{\frac{1}{2}} D(u_h)\|_{0,\Omega}\|\mu^{-\frac{1}{2}} p_h^\Gamma\|_{0,\Omega}}{\|\mu^{\frac{1}{2}} D(u_h)\|_{0,\Omega}} \\ &= c\|\mu^{-\frac{1}{2}} p_h^\Gamma\|_{0,\Omega} \le c\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} = c\langle M\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}}.\end{aligned}$$

Hence $\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle \le c\langle M\mathbf{y}, \mathbf{y}\rangle$ holds. Using an inverse inequality we get

$$\begin{aligned}\langle J\mathbf{y}, \mathbf{y}\rangle = j(p_h, p_h) &= \sum_{i=1}^{2} \mu_i^{-1} \sum_{F\in\mathcal{F}_i} h_F^3\|[\nabla p_{i,h}\cdot n_F]\|_{0,F}^2 \\ &\le c\sum_{i=1}^{2} \mu_i^{-1} \sum_{T\in\Omega_{i,h}} h_T^3\|\nabla p_{i,h}\|_{0,\partial T}^2 \le c\sum_{i=1}^{2} \mu_i^{-1} \sum_{T\in\Omega_{i,h}} h_T^2\|\nabla p_{i,h}\|_{0,T}^2 \\ &\le c\sum_{i=1}^{2} \mu_i^{-1} \sum_{T\in\Omega_{i,h}} \|p_{i,h}\|_{0,T}^2 = c\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}^2 = c\langle M\mathbf{y}, \mathbf{y}\rangle.\end{aligned} \tag{7.6}$$

Hence, $\langle S\mathbf{y}, \mathbf{y}\rangle = \langle (BA^{-1}B^T + \varepsilon_p J)\mathbf{y}, \mathbf{y}\rangle \le c(1+\varepsilon_p)\langle M\mathbf{y}, \mathbf{y}\rangle$ holds for all $\varepsilon_p \ge 0$, which proves the second inequality in (7.4).

Using (7.5) and Lemma 5.3 we get, with suitable constants c_1, c_2,

$$\begin{aligned}\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} &\ge c_1\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}} - c_2\frac{j(p_h, p_h)}{\|\mu^{-\frac{1}{2}} p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}} \\ &= c_1\langle M\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} - c_2\frac{\langle J\mathbf{y}, \mathbf{y}\rangle}{\langle M\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}}}.\end{aligned}$$

This yields $\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}}\langle M\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} + c_2\langle J\mathbf{y}, \mathbf{y}\rangle \ge c_1\langle M\mathbf{y}, \mathbf{y}\rangle$.

Using $\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}}\langle M\mathbf{y}, \mathbf{y}\rangle^{\frac{1}{2}} \le \frac{1}{2}c_1^{-1}\langle BA^{-1}B^T\mathbf{y}, \mathbf{y}\rangle + \frac{1}{2}c_1\langle M\mathbf{y}, \mathbf{y}\rangle$ we thus get

$$\langle S\mathbf{y}, \mathbf{y}\rangle \ge c_1^2 \min\left\{1, \frac{\varepsilon_p}{2c_1c_2}\right\}\langle M\mathbf{y}, \mathbf{y}\rangle,$$

which proves the first inequality in (7.4). ∎

As can be seen from the proof, the constants c_i in (7.4) depend on the value of the stabilization parameter ε_p.

As noted at the end of the previous section, in view of the consistency condition $\mathbf{y} \in \mathbf{1}^{\perp_{\hat{M}}}$, it is more convenient to use the matrix $Q_S = \hat{M} + \varepsilon_p J$ instead of M as a preconditioner for the Schur complement S. In the next lemma we show that these two are uniformly spectrally equivalent.

LEMMA 7.3. *Take $\varepsilon_p > 0$. There exist constants $c_1, c_2 > 0$, independent of h, μ and of how Γ intersects the triangulation, such that*

$$c_1\langle M\mathbf{y},\mathbf{y}\rangle \le \langle(\hat{M}+\varepsilon_p J)\mathbf{y},\mathbf{y}\rangle \le c_2\langle M\mathbf{y},\mathbf{y}\rangle \quad \textit{for all} \ \ \mathbf{y}\in\mathbb{R}^{n_1+n_2}. \tag{7.7}$$

Proof. From $\|\mu^{-\frac{1}{2}}p_h^\Gamma\|_{0,\Omega} \le \|\mu^{-\frac{1}{2}}p_h\|_{0,\Omega_{1,h}\cup\Omega_{2,h}}$ we obtain $\langle\hat{M}\mathbf{y},\mathbf{y}\rangle \le \langle M\mathbf{y},\mathbf{y}\rangle$. Combining this with the result in (7.6) proves the second inequality in (7.7). Using Lemma 2.2 we get

$$\begin{aligned}\langle M\mathbf{y},\mathbf{y}\rangle = \|\mu^{-\frac{1}{2}}p_h\|^2_{0,\Omega_{1,h}\cup\Omega_{2,h}} &\le c\big(\|\mu^{-\frac{1}{2}}p_h^\Gamma\|^2_{0,\omega_{1,h}\cup\omega_{2,h}} + j(p_h,p_h)\big)\\ &\le c\big(\|\mu^{-\frac{1}{2}}p_h^\Gamma\|^2_{0,\Omega} + \varepsilon_p j(p_h,p_h)\big) = c\big((\hat{M}+\varepsilon_p J)\mathbf{y},\mathbf{y}\rangle,\end{aligned}$$

and thus the first inequality in (7.7) holds, too. ∎

The results above yield that the spectral condition number of $Q_S^{-1}S$ is uniformly bounded on $\mathbf{1}^{\perp_{\hat{M}}}$. Finally, we show that linear systems with matrix Q_S can be solved (approximately) with low computational costs. In [18] it is proved that for $\mu_1 = \mu_2 = 1$ the *diagonally scaled* matrix $\hat{D}^{-1}\hat{M}$ with $\hat{D} := \mathrm{diag}(\hat{M})$ is uniformly (w.r.t. h and w.r.t. the position of the interface in the grid) well-conditioned. Due to the possibly small support of some extended basis functions, without the diagonal scaling the condition number of the mass matrix $\hat{M}$ is not uniformly bounded. Here, we have to study the conditioning of $Q_S = \hat{M} + \varepsilon_p J$. We benefit from the stabilizing term $\varepsilon_p J$, and a conditioning result is easily obtained, as shown in the following lemma.

LEMMA 7.4. *Take $\varepsilon_p > 0$. Define $D := \mathrm{diag}(\hat{M}+\varepsilon_p J)$. There exist constants $c_1, c_2 > 0$, independent of h, μ and of how Γ intersects the triangulation, such that*

$$c_1\langle D\mathbf{y},\mathbf{y}\rangle \le \langle(\hat{M}+\varepsilon_p J)\mathbf{y},\mathbf{y}\rangle \le c_2\langle D\mathbf{y},\mathbf{y}\rangle \quad \textit{for all} \ \ \mathbf{y}\in\mathbb{R}^{n_1+n_2}. \tag{7.8}$$

Proof. By $A \sim B$ we denote uniform spectral equivalence of the symmetric positive definite matrices A and B. Define $D_M := \mathrm{diag}(M)$. If in (7.7) for $\mathbf{y}$ we take the standard basis vectors, we obtain $D \sim D_M$. From the definition of M and the result in (7.1) it follows that $D_M \sim M$. Thus we get $D \sim M$. Using (7.7) we conclude $D \sim \hat{M} + \varepsilon_p J$. ∎

As a direct consequence of this lemma, we can see that the Jacobi method applied to $\hat{M}+\varepsilon_p J$ is a good preconditioner for the Schur coplement. Now we apply a standard analysis as in, e.g., [21, 15], to derive results on the spectrum of the preconditioned matrix $Q^{-1}K$. From the results in Theorem 7.2 and Lemma 7.3 it follows that there are constants $\gamma_S > 0$ and Γ_S, independent of h, μ and of how the interface intersects the triangulation, such that for the Schur complement preconditioner Q_S as in (7.3), with a fixed $\varepsilon_p > 0$, we have the following spectral equivalence:

$$\gamma_S\langle Q_S\mathbf{y},\mathbf{y}\rangle \le \langle S\mathbf{y},\mathbf{y}\rangle \le \Gamma_S\langle Q_S\mathbf{y},\mathbf{y}\rangle \quad \text{for all} \ \ \mathbf{y}\in\mathbf{1}^{\perp_{\hat{M}}}. \tag{7.9}$$

For Q_A we take a symmetric multigrid preconditioner. Thus there exists $\gamma_A > 0$ independent of h and of how the interface intersects the triangulation such that

$$\gamma_A\langle Q_A\mathbf{x},\mathbf{x}\rangle \le \langle A\mathbf{x},\mathbf{x}\rangle \le \langle Q_A\mathbf{x},\mathbf{x}\rangle \quad \text{for all} \ \ \mathbf{x}\in\mathbb{R}^m. \tag{7.10}$$

In the upper bound in (7.10) we have a constant 1, because the iteration matrix of a symmetric multigrid method for the diffusion equation is positive definite. The spectral constant γ_A may depend on the quotient μ_1/μ_2.

COROLLARY 7.5. *All nonzero eigenvalues of $Q^{-1}K$ lie in the union of the intervals*

$$[\gamma_A,1]\cup\left[\frac{1}{2}\left(\gamma_A+\sqrt{\gamma_A^2+4\gamma_A\gamma_S}\right),\ \frac{1}{2}\left(1+\sqrt{1+4\Gamma_S}\right)\right]$$
$$\cup\left[\frac{1}{2}\left(1-\sqrt{(1+4\Gamma_S}\right),\ \frac{1}{2}\left(\gamma_A-\sqrt{\gamma_A^2+4\gamma_A\gamma_S}\right)\right].$$

Proof. The proof follows from (7.9), (7.10), and the analysis in [15, Lemma 5.14]. ∎

This shows that Q is an optimal preconditioner for K. Systems with the Schur complement preconditioner Q_S can be solved (approximately) with acceptable computational costs, cf. Lemma 7.4.

8. Numerical experiments.

8.1. The sliver experiment. In our first experiment we want to investigate the influence of the parameter ε_p on the stability of the resulting discretizations. To this end, we introduce the so-called sliver experiment. Praxis has shown that in unstabilized discretizations the stability problems seem to arise from those XFEM functions which have a tiny support. In this experiment we deliberately create such functions and repeatedly shrink their support by defining a sequence of planar interfaces $\Gamma_k := \{(x,y,z)\in\Omega \mid z=0.1\cdot 2^{-k}\}$, $k\ge 0$, approaching the x-y-plane. We choose a uniform grid of the domain $\Omega=(-1,1)^3$, consisting of $4\times4\times4$ equally sized cubes. Each of these cubes is then subdivided into six tetrahedra. We take $\mu_1=\mu_2=1$. As a measure of stability, we want to estimate

$$\inf_{(u_h,p_h)\in V_h\times Q_h^\Gamma}\ \sup_{(v_h,q_h)\in V_h\times Q_h^\Gamma}\frac{k\big((u_h,p_h),(v_h,q_h)\big)}{\|(u_h,p_h)\|\|(v_h,q_h)\|},\qquad (8.1)$$
$$\text{where } \|(u_h,p_h)\|^2=\|u_h\|_1^2+\|p_h\|_0^2.$$

Using Lemma 7.3 and the coercivity of the bilinear form a, it can be shown that this can be estimated by the smallest nonzero eigenvalue of the following matrix:

$$\begin{pmatrix}A+M_v & 0\\ 0 & \hat{M}+\varepsilon J\end{pmatrix}^{-1}\begin{pmatrix}A & B^T\\ -B & \varepsilon_p J\end{pmatrix},\qquad (8.2)$$

where M_v is mass matrix in the velocity space. We denote this smallest nonzero eigenvalue by C_{stab}.

Figure 2 shows the values of C_{stab} for the two choices $\varepsilon_p=10^{-5}$ and $\varepsilon_p=1$. Even though there are five orders of magnitude between them, the stability results are almost identical. The same holds for choices of ε_p in between those values, which we did not plot here for the sake of a better visualization. For the unstabilized discretization, we remark that due to numerical instabilities the computed values for C_{stab} might be inaccurate. However, the value of C_{stab} seems to deteriorate approximately as $\mathcal{O}(\delta^3)$, where δ is the distance of the interface to the x-y-plane. It appears that already "tiny amounts" of the stabilization suffice to restore the method's stability. Furthermore, variation of the parameter ε_p seems to have a very mild influence on the stability of the method.

8.2. Experiments with a smooth velocity solution. In this section we want to investigate the convergence properties of the method. To this end, we prescribe

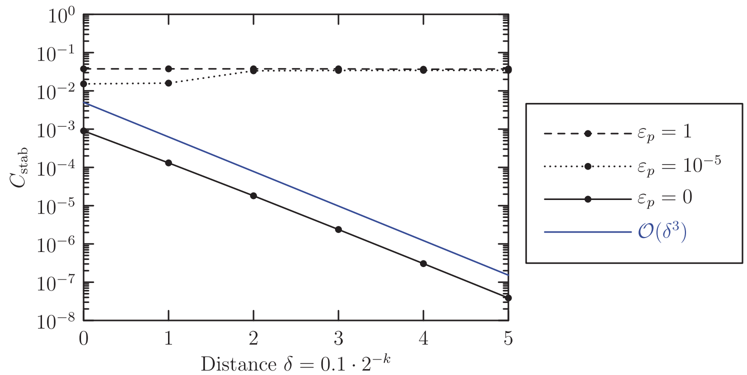

FIG. 2. *Values of the stability constant C_{stab} for different values of ε_p as the interface's distance to the x-y-plane is decreased.*

Dirichlet boundary conditions and an external force $f = f_\Omega + \hat{f}_\Gamma$, $\hat{f}_\Gamma(v) := \sigma \int_\Gamma v \cdot \mathbf{n}\, ds$ with $\sigma := 10$ for $v \in V$, such that the analytical solution is

$$u(x,y,z) = \alpha(r)\, e^{-r^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}, \quad \text{where } r = \sqrt{x^2+y^2+z^2},$$

$$\alpha(r) = \begin{cases} \mu_1^{-1} & \text{for } r < r_\Gamma, \\ \mu_2^{-1} + (\mu_1^{-1} - \mu_2^{-1}) e^{r^2 - r_\Gamma^2} & \text{for } r \geq r_\Gamma, \end{cases} \tag{8.3}$$

$$p(x,y,z) = x^3 + \begin{cases} \sigma & x \in \Omega_1, \\ 0 & \text{else}, \end{cases}$$

where the domain is $\Omega := (-1,1)^3$ and $\Omega_1 := S_{2/3}$ the sphere of radius $r_\Gamma := 2/3$ around the origin. Note that the function $\alpha(r)$ is continuous and has a kink at $r = r_\Gamma$ in the case of nonmatching viscosities μ_i. Note also that the velocity vectors are tangential to the interface, i.e., $u \cdot n_\Gamma = 0$ with n_Γ the outer normal to Ω_1, which is necessary for the assumption of a stationary interface.

For simplicity, as a first test case we choose $\mu_1 = \mu_2 = 1$, while a more realistic setting will be examined in section 8.3. For this choice we have $\alpha \equiv 1$ and thus the functions u, p can be ideally approximated by the ansatz spaces while not being a part of them. For the discretization of $\hat{f}_\Gamma$ we use $\hat{f}_{\Gamma_h}(v_h) := \sigma \int_{\Gamma_h} v_n \cdot \mathbf{n}_h\, ds$, which is second order accurate. Here Γ_h is a piecewise planar approximation to Γ with $\mathrm{dist}(x,\Gamma) \leq ch^2$ for all $x \in \Gamma_h$, cf. [11], which is also used in the construction of the XFEM space $Q_h^{\Gamma_h}$.

In a first step, we want to investigate the sensitivity of the discretization error w.r.t. ε_p. We therefore choose a fixed grid of $16 \times 16 \times 16$ cubes which are each subsequently subdivided into six tetrahedra. Afterward we change the value of ε_p and compute the discretization error. For the solution of the linear system of equations a preconditioned MINRES method was used, with the preconditioners defined as in the previous section. The MINRES iteration was stopped when the residual fell below the threshold of 10^{-9}.

Table 1 shows the resulting iteration counts and discretization errors for various values of ε_p. One can clearly see that for $\varepsilon_p < 1$, its magnitude is virtually insignif-

TABLE 1
Discretization errors and iteration counts for various values of ε_p. For $\varepsilon_p = 0$ the residual did not fall below 10^{-6}.

ε_p	$\|e_p^\Gamma\|_0$	$\|e_u\|_1$	Iterations
0	$1.82 \cdot 10^{-2}$	$4.90 \cdot 10^{-3}$	> 1000
10^{-5}	$9.64 \cdot 10^{-3}$	$4.87 \cdot 10^{-3}$	97
10^{-3}	$9.49 \cdot 10^{-3}$	$4.86 \cdot 10^{-3}$	96
10^{-1}	$9.76 \cdot 10^{-3}$	$4.97 \cdot 10^{-3}$	102
1	$1.33 \cdot 10^{-2}$	$6.41 \cdot 10^{-3}$	95
10	$3.01 \cdot 10^{-2}$	$1.43 \cdot 10^{-2}$	102
10^3	$9.34 \cdot 10^{-2}$	$4.61 \cdot 10^{-2}$	97

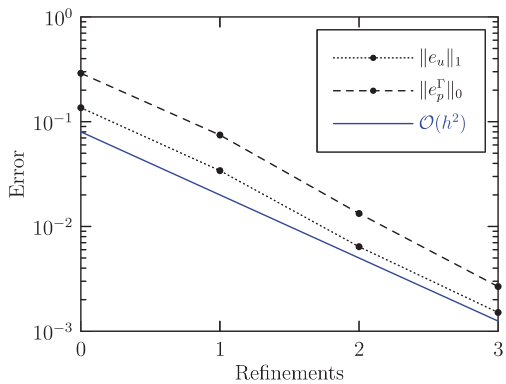

FIG. 3. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ using an artificial surface force term $\hat{f}_\Gamma$.*

icant for the properties of the resulting discretization. Note that the error in the pressure variable is only about half of the corresponding value for the unstabilized discretization. For $\varepsilon_p = 1$ the errors increase only very slightly. Also note that for $\varepsilon_p = 0$ the MINRES iteration did not converge to the target residual due to the poor stability properties. For all other choices of ε_p, the introduced preconditioners show to be effective with iteration counts around 100. We can therefore conclude that ε_p only has a very mild influence on the properties of the resulting discretization. Note that without using any Schur complement preconditioner (i.e., $Q_S = I$) the residual did not fall below 10^{-3} within 1000 MINRES iterations. This shows that the Schur complement preconditioner is crucial for the iterative solution of the linear systems.

After having established the discretization's small sensitivity w.r.t. ε_p, we want to inspect the convergence behavior w.r.t. h. To this end we choose a fixed value $\varepsilon_p = 1$ and start with a uniform grid of $4 \times 4 \times 4$ cubes which are subsequently each divided into six tetrahedra. We then perform uniform mesh refinements and look at the influence on the discretization error and the iteration counts.

Figure 3 shows the discretization errors for the different refinement levels. As predicted by the analysis, we have a second order convergence behavior. The iteration counts varied between values of 95 and 102, confirming the optimal behavior of the preconditioners introduced. Note that for the second order convergence behavior, the use of the P_1-XFE space for the pressure is essential. If one uses the standard P_1-FE space instead, the rate of convergence drops to $\mathcal{O}(h^{\frac{1}{2}})$; cf. [12].

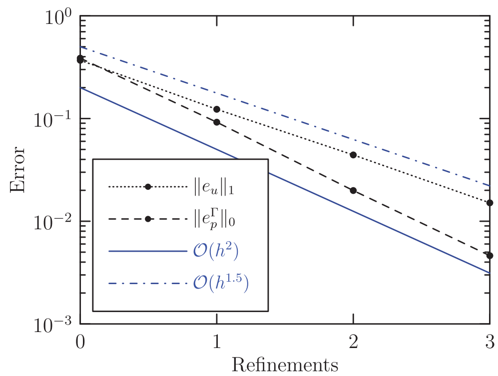

FIG. 4. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ using a surface tension force term f_Γ.*

8.3. Experiments with more realistic parameter settings. In two-phase flows the pressure jump at the interface is induced by surface tension. To incorporate the effect of surface tension we consider the same test case as in section 8.2, but we replace the artificial surface force $\hat{f}_\Gamma$ by the surface tension force $f_\Gamma(v) = \int_\Gamma \tau\kappa v \cdot \mathbf{n}\, ds$. Here $\tau > 0$ is a constant surface tension coefficient and $\kappa(x)$ denotes the local curvature of Γ.

Choosing $\tau = \frac{10}{3}$ we have $\tau\kappa = \tau\frac{2}{r_\Gamma} = 10 = \sigma$, and thus for the continuous setting both surface forces coincide, i.e., $f_\Gamma = \hat{f}_\Gamma$. This, however, does *not* hold for the discrete case, i.e., $f_{\Gamma_h} \neq \hat{f}_{\Gamma_h}$, which is due to the fact that for f_{Γ_h} the curvature has to be evaluated from the *approximate* interface Γ_h. For the discretization f_{Γ_h} of the surface tension force we use a Laplace–Beltrami technique, described in [11] and analyzed in [11, 10], which has a discretization order of 1.5. Due to the first Strang lemma the same convergence order is expected for the sum of the velocity error (in $\|\cdot\|_1$) and pressure error (in $\|\cdot\|_0$). We take $\varepsilon_p = 1$ and apply grid refinement as in the previous experiment. The error plot given in Figure 4 shows that the velocity error has an $\mathcal{O}(h^{\frac{3}{2}})$ behavior and the pressure error converges with second order.

Finally, we consider an experiment which mimics a two-phase flow water/air system with nonmatching viscosities. The solution is chosen as in (8.3) with $\mu_1 = 10^{-3}$, $\mu_2 = 10^{-1}$, and $\tau = 700$. These values for viscosity and surface tension coefficient τ correspond to the dimensionless formulation of the two-phase Stokes equations for an air bubble with radius $\frac{2}{3}\,mm$ in ambient water, assuming a characteristic length $L = 10^{-3}\,m$ and a characteristic velocity $U = 10^{-2}\,m/s$. Figure 5 shows a plot of the velocity and pressure along the x-axis on the finest grid (refinement level 3). Note the large scaling of the pressure and velocity solution, yielding $\|p\|_0 = 2.15 \cdot 10^3$ and $\|u\|_1 = 2.97 \cdot 10^3$, whereas in the previous examples both norms are of order 1. Due to the kink of the velocity at the interface (see u_2 in Figure 5), which is not aligned with the triangulation, for the standard velocity space without enrichment one expects a poor convergence order of 0.5. Figure 6 shows the convergence behavior for different grid refinement levels. We observe a convergence order of 1.5, showing that the surface tension discretization error dominates the error induced by the velocity kink. A reduced order of 0.5 is expected on fine enough grids, which, however, could not be tested in this experiment due to memory limitations. The results in this experiment are in accordance with our experience that for the simulation of realistic two-phase flows usually the pressure jump enrichment and the discretization of the surface ten-

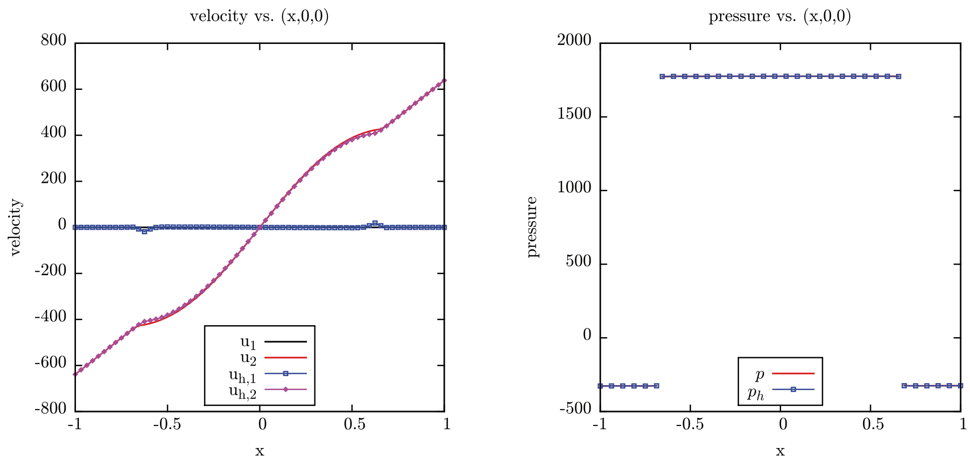

FIG. 5. *Exact and discrete velocity and pressure solutions along the x-axis for $\varepsilon_p = 1$ on grid refinement level* 3 *and a realistic parameter setting corresponding to an air bubble in water.*

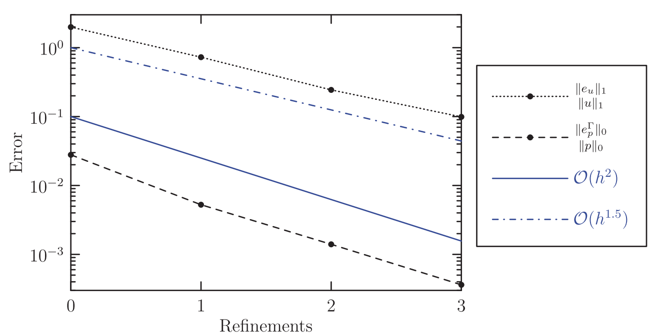

FIG. 6. *Discretization errors for different refinement levels of the mesh for $\varepsilon_p = 1$ and a realistic parameter setting corresponding to an air bubble in water.*

sion force are essential, whereas the velocity kink enrichment (often) seems to be of minor importance. A similar experience is reported in [19].

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