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Conservative difference methods for the Klein–Gordon–Zakharov equations☆

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Abstract

Firstly an implicit conservative finite difference scheme is presented for the initial-boundary problem of the one space dimensional Klein–Gordon–Zakharov (KGZ) equations. The existence of the difference solution is proved by Leray–Schauder fixed point theorem. It is proved by the discrete energy method that the scheme is uniquely solvable, unconditionally stable and second order convergent for U in l_∞ norm, and for N in l_2 norm on the basis of the priori estimates. Then an explicit difference scheme is proposed for the KGZ equations, on the basis of priori estimates and two important inequalities about norms, convergence of the difference solutions is proved. Because it is explicit and not coupled it can be computed by a parallel method. Numerical experiments with the two schemes are done for several test cases. Computational results demonstrate that the two schemes are accurate and efficient.

MSC: 65M06; 65M99

Keywords: Klein–Gordon–Zakharov equation; Finite difference scheme; Energy conservation; Solvability; Convergence; Stability

1. Introduction

As already pointed out by Dendy [5], Klein–Gordon–Zakharov (KGZ) equations

$$u_{tt} - u_{xx} + u + mu + |u|^2 u = 0, \tag{1.1}$$

$$m_{tt} - m_{xx} = (|u|^2)_{xx}, \tag{1.2}$$

are a classical model described the interaction of the Langmuir wave and the ion acoustic wave in a plasma. The equations are apparently coupled equations by two functions $u(x,t)$ and $m(x,t)$ to be solved. The function $u(x,t)$ denotes the fast time scale component of electric field raised by electrons and the function $u(x,t)$ denotes the deviation of ion density from its equilibrium. Here $u(x,t)$ is a complex function, $m(x,t)$ is a real function. Well-posedness in [13] and global smooth solutions in [2,15,1] is proved for the KGZ equations.

KGZ equations have similar shape to Zakharov equations and Klein–Gordon–Schrödinger equations. Many finite difference schemes have been presented for the Zakharov [14,7,8,4,3] and the Klein–Gordon–Schrödinger [17,18].

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But numerical studies for the KGZ equations are few. As we all know, the shape of Eqs. (1.1) and (1.2) are all similar to Klein–Gordon equation which has been solved by finite difference methods [10,6,16,9]. Thus, the methods of establishing difference schemes solved Klein–Gordon equation are helpful for us.

In this paper, we investigate the KGZ equations on $\{x_L < x < x_R, 0 < t \leqslant T\}$ and consider the numerical solution of initial-boundary value problem with the following initial and boundary value conditions:

$$u|_{t=0} = u_0(x), \quad u_t|_{t=0} = u_1(x), \quad m|_{t=0} = m_0(x), \quad m_t|_{t=0} = m_1(x), \tag{1.3}$$

$$u|_{x=x_L} = u|_{x=x_R} = 0, \quad m|_{x=x_L} = m|_{x=x_R} = 0, \tag{1.4}$$

where $u_0(x)$, $u_1(x)$, $m_0(x)$ and $m_1(x)$ are known smooth functions. The problem (1.1)–(1.4) possesses conservative quantity

$$\int_{x_L}^{x_R} \left[|u_t|^2 + |u_x|^2 + |u|^2 + m|u|^2 + \frac{1}{2}|v|^2 + \frac{1}{2}|m|^2 + \frac{1}{2}|u|^4 \right] = \text{const}, \tag{1.5}$$

where v is given by

$$v = -w_x, \quad w_{xx} = m_t. \tag{1.6}$$

The paper is organized as follows. In Section 2, we present an implicit conservative difference scheme for the initial-boundary problem (1.1)–(1.4), and discuss its discrete conservative law. In Section 3, we prove the existence of difference solution by Leray–Schauder fixed point theorem. In Section 4, we firstly give some priori estimates and then prove by discrete energy method that the difference scheme is uniquely solvable, unconditionally stable and second order convergent. In Section 5, we construct another conservative scheme which is explicit and discuss its discrete conservative law. In Section 6, we make some priori estimates of the explicit scheme proposed in Section 5, and introduce two important inequalities about norms and the convergence of the explicit scheme. In Section 7, various numerical results are provided to demonstrate the theoretical results.

2. Finite difference scheme and its conservative law

In this section, we describe the numerical method for the initial-boundary problem (1.1)–(1.4). For convenience, we introduce the following notations:

$$x_j = x_L + jh, \quad t^n = n\tau, \quad 0 \leqslant j \leqslant J, \quad n = 0, 1, 2, \ldots, K,$$

$$(U_j^n)_x = \frac{U_{j+1}^n - U_j^n}{h}, \quad (U_j^n)_{\bar{x}} = \frac{U_j^n - U_{j-1}^n}{h},$$

$$(U_j^n)_t = \frac{U_j^{n+1} - U_j^n}{\tau}, \quad (U_j^n)_{\bar{t}} = \frac{U_j^n - U_j^{n-1}}{\tau},$$

$$\langle U^n, V^n \rangle = h \sum_{j=1}^{J} U_j^n \overline{V}_j^n, \quad \|U^n\|_\infty = \sup_{1 \leqslant j \leqslant J} |U_j^n|, \quad \|U^n\|_p^p = h \sum_{j=1}^{J} |U_j^n|^p,$$

where $J = [(x_R - x_L)/h]$, $K = [T/\tau]$, h and τ are step size of space and time, respectively. In this paper, C denotes a general positive constant which may have different values in different place, and $U_j^n \sim u(x_j, t_n)$, $N_j^n \sim m(x_j, t_n)$, $u_j^n \equiv u(x_j, t_n)$, $m_j^n \equiv m(x_j, t_n)$, respectively.

Thus, the finite difference scheme we will consider for Eqs. (1.1) and (1.2) is written as

$$\begin{aligned} &(U_j^n)_{t\bar{t}} - \tfrac{1}{2}(U_j^{n+1} + U_j^{n-1})_{x\bar{x}} + \tfrac{1}{2}(U_j^{n+1} + U_j^{n-1}) + \tfrac{1}{4}(N_j^{n+1} + N_j^{n-1})(U_j^{n+1} + U_j^{n-1}) \\ &\quad + \tfrac{1}{4}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1}) = 0, \quad 1 \leqslant j \leqslant J-1, \quad 1 \leqslant n \leqslant K-1, \end{aligned} \tag{2.1}$$

$$(N_j^n)_{t\bar{t}} - \tfrac{1}{2}(N_j^{n+1} + N_j^{n-1})_{x\bar{x}} = \tfrac{1}{2}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)_{x\bar{x}}, \quad 1 \leqslant j \leqslant J-1, \quad 1 \leqslant n \leqslant K-1, \tag{2.2}$$

The initial and boundary conditions (1.3) and (1.4) are, respectively, approximated as

$$U_j^0 = u_0(x_j), \quad N_j^0 = m_0(x_j), \quad U_j^1 - U_j^0 = \tau u_1(x_j), \tag{2.3}$$

$$U_0^n = U_J^n = 0, \quad N_0^n = U_J^n = 0, \quad N_j^1 - N_j^0 = \tau m_1(x_j). \tag{2.4}$$

We also define $\{V_j^n\}$ by

$$(V_j^n)_{x\bar{x}} = (N_j^n)_t, \quad j = 1, 2, \ldots, J-1, \quad V_0^n = V_J^n = 0. \tag{2.5}$$

In order to prove that the scheme (2.1)–(2.4) can simulate conservative law (1.5) very well, we should introduce the following lemma.

Lemma 2.1. *For any two mesh functions* $\{w_j \mid j = 0, 1, \ldots, J\}$ *and* $\{v_j \mid j = 0, 1, \ldots, J\}$, *there is the identity*

$$h \sum_{j=1}^{J-1} w_j (v_j)_{x\bar{x}} = -h \sum_{j=0}^{J-1} (w_j)_x (v_j)_x - w_0 (v_0)_x + w_J (v_J)_{\bar{x}}. \tag{2.6}$$

It can be proved directly by simple computation.

Now, we give the discrete conservative law of the difference (2.1)–(2.4).

Theorem 2.1. *The difference scheme* (2.1)–(2.4) *admits the following invariant*

$$E^n = E^{n-1} = \cdots = E^0, \tag{2.7}$$

where

$$E^n = \|U_t^n\|_2^2 + \frac{1}{2}(\|U_x^{n+1}\|_2^2 + \|U_x^n\|_2^2) + \frac{1}{2}(\|U^{n+1}\|_2^2 + \|U^n\|_2^2) + \frac{1}{2}\|V_x^n\|_2^2$$
$$+ \frac{1}{4}(\|N^{n+1}\|_2^2 + \|N^n\|_2^2) + \frac{1}{4}(\|U^{n+1}\|_4^4 + \|U^n\|_4^4) + \frac{1}{2} h \sum_{j=1}^{J} [N_j^{n+1}|U_j^{n+1}|^2 + N_j^n |U_j^n|^2]$$

is called discrete energy.

Proof. Computing the inner product of difference Eq. (2.1) with $U^{n+1} - U^{n-1}$ and taking the real part, we have

$$\|U_t^n\|_2^2 - \|U_t^{n-1}\|_2^2 + \frac{1}{2}(\|U_x^{n+1}\|_2^2 - \|U_x^{n-1}\|_2^2) + \frac{1}{2}(\|U^{n+1}\|_2^2 - \|U^{n-1}\|_2^2)$$
$$+ \frac{1}{4}(\|U^{n+1}\|_4^4 - \|U^{n-1}\|_4^4) + \frac{1}{4} h \sum_{j=1}^{J} (N_j^{n+1} + N_j^{n-1})(|U_j^{n+1}|^2 - |U_j^{n-1}|^2) = 0. \tag{2.8}$$

Next, computing the inner product of (2.2) with $\frac{1}{2}(V^n + V^{n-1})$ and using Eq. (2.5), we obtain

$$A - B = Q,$$

and computing

$$A = \frac{1}{2} h \sum_{j=1}^{J} (N_j^n)_{t\bar{t}} (V_j^n + V_j^{n-1}) = \frac{h}{2\tau} \sum_{j=1}^{J} [(V_j^n)_{x\bar{x}} - (V_j^{n-1})_{x\bar{x}}](V_j^n + V_j^{n-1})$$
$$= -\frac{h}{2\tau} \sum_{j=1}^{J} [(V_j^n)_x - (V_j^{n-1})_x][(V_j^n)_x + (V_j^{n-1})_x] = -\frac{1}{2\tau}(\|V_x^n\|_2^2 - \|V_x^{n-1}\|_2^2),$$

$$B = -\frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}+N_j^{n-1})_x(V_j^n+V_j^{n-1})_x = \frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}+N_j^{n-1})(V_j^n+V_j^{n-1})_{x\bar{x}}$$

$$= \frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}+N_j^{n-1})(N_j^n+N_j^{n-1})_t = \frac{1}{4\tau}(\|N^{n+1}\|_2^2 - \|N^{n-1}\|_2^2),$$

$$Q = -\frac{1}{4}h\sum_{j=1}^{J}(V_j^n+V_j^{n-1})_x(|U_j^{n+1}|^2+|U_j^{n-1}|^2)_x = \frac{1}{4}h\sum_{j=1}^{J}(V_j^n+V_j^{n-1})_{x\bar{x}}(|U_j^{n+1}|^2+|U_j^{n-1}|^2)$$

$$= \frac{1}{4\tau}h\sum_{j=1}^{J}(N_j^{n+1}-N_j^{n-1})(|U_j^{n+1}|^2+|U_j^{n-1}|^2).$$

Thus

$$\frac{1}{2}(\|V_x^n\|_2^2 - \|V_x^{n-1}\|_2^2) + \frac{1}{4}(\|N^{n+1}\|_2^2 - \|N^{n-1}\|_2^2) + \frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}-N_j^{n-1})(|U_j^{n+1}|^2+|U_j^{n-1}|^2) = 0. \quad (2.9)$$

Adding (2.9) to (2.8), and noticing

$$\frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}-N_j^{n-1})(|U_j^{n+1}|^2+|U_j^{n-1}|^2) + \frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1}+N_j^{n-1})(|U_j^{n+1}|^2-|U_j^{n-1}|^2)$$

$$= \frac{1}{2}h\sum_{j=1}^{J}[N_j^{n+1}|U_j^{n+1}|^2 - N_j^{n-1}|U_j^{n-1}|^2], \quad (2.10)$$

we have

$$\|U_t^n\|_2^2 - \|U_t^{n-1}\|_2^2 + \frac{1}{2}(\|U_x^{n+1}\|_2^2 - \|U_x^{n-1}\|_2^2) + \frac{1}{2}(\|U^{n+1}\|_2^2 - \|U^{n-1}\|_2^2) + \frac{1}{4}(\|U^{n+1}\|_4^4 - \|U^{n-1}\|_4^4)$$

$$+ \frac{1}{2}(\|V_x^n\|_2^2 - \|V_x^{n-1}\|_2^2) + \frac{1}{4}(\|N^{n+1}\|_2^2 - \|N^{n-1}\|_2^2) + \frac{1}{2}h\sum_{j=1}^{J}[N_j^{n+1}|U_j^{n+1}|^2 - N_j^{n-1}|U_j^{n-1}|^2] = 0.$$

Thus

$$E^n = E^{n-1} = \cdots = E^0.$$

This completes the proof of Theorem 2.1. □

3. The existence of difference solutions

In this section we will discuss the solvability of the difference system (2.1)–(2.4), i.e.,

Theorem 3.1. *The solution of difference scheme* (2.1)–(2.4) *exist.*

Proof. Now, we prove the existence of difference solutions (U^{n+1}, N^{n+1}) for the finite difference system (2.1)–(2.4). For any mesh function (ϕ, ψ) defined on $[x_L, x_R]$, and $\phi|_{x=x_L} = \phi|_{x=x_R} = \psi|_{x=x_L} = \psi|_{x=x_R} = 0$, we define a mesh function (Φ, Ψ) as follows:

$$[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})] - \frac{\tau^2}{2}(\phi_j + U_j^{n-1})_{x\bar{x}} + \frac{\tau^2}{2}(\phi_j + U_j^{n-1}) + \frac{\tau^2}{4}(\psi_j + N_j^{n-1})(\phi_j + U_j^{n-1})$$
$$+ \frac{\tau^2}{4}(|\phi_j|^2 + |U_j^{n-1}|^2)(\phi_j + U_j^{n-1}) = 0, \quad 1 \leqslant j \leqslant J-1, \ 1 \leqslant n \leqslant K-1, \tag{3.1}$$

$$[(\Psi_j - N_j^n) - (N_j^n - N_j^{n-1})] - \frac{\tau^2}{2}(\psi_j + N_j^{n-1})_{x\bar{x}} = \frac{\tau^2}{2}(|\phi_j|^2 + |U_j^{n-1}|^2)_{x\bar{x}},$$
$$1 \leqslant j \leqslant J-1, \ 1 \leqslant n \leqslant K-1, \tag{3.2}$$

where $\Phi|_{x=x_L} = \Phi|_{x=x_R} = \Psi|_{x=x_L} = \Psi|_{x=x_R} = 0$. It defines a mapping $(\Phi, \Psi) = T(\phi, \psi)$ of $H^1 \times H^1$ into itself. Obviously, the mapping $T(\phi, \psi)$ is continuous for any $(\phi, \psi) \in H^1 \times H^1$. In order to obtain the existence of the solutions for the finite difference system (2.1)–(2.4), it is sufficient to prove the uniform boundedness for all the possible fixed point (Φ, Ψ) for the mapping λT with respect to the parameter $0 \leqslant \lambda \leqslant 1$ by Leray–Schauder fixed point theorem. Then the fixed point (Φ, Ψ) of the mapping λT satisfies

$$[(\Phi_j - U_j^n) - (U_j^n - U_j^{n-1})] - \frac{\lambda\tau^2}{2}(\Phi_j + U_j^{n-1})_{x\bar{x}} + \frac{\lambda\tau^2}{2}(\Phi_j + U_j^{n-1})$$
$$+ \frac{\lambda\tau^2}{4}(\Psi_j + N_j^{n-1})(\Phi_j + U_j^{n-1}) + \frac{\lambda\tau^2}{4}(|\Phi_j|^2 + |U_j^{n-1}|^2)(\Phi_j + U_j^{n-1}) = 0,$$
$$1 \leqslant j \leqslant J-1, \ 1 \leqslant n \leqslant K-1, \tag{3.3}$$

$$[(\Psi_j - N_j^n) - (N_j^n - N_j^{n-1})] - \frac{\lambda\tau^2}{2}(\Psi_j + N_j^{n-1})_{x\bar{x}} = \frac{\lambda\tau^2}{2}(|\Phi_j|^2 + |U_j^{n-1}|^2)_{x\bar{x}},$$
$$1 \leqslant j \leqslant J-1, \ 1 \leqslant n \leqslant K-1. \tag{3.4}$$

Computing the inner product of difference equation (3.3) with $(\Phi - U^{n-1}) = [(\Phi - U^n) + (U^n - U^{n-1})]$, and then taking the real part in the resulting formula, we obtain

$$\|\Phi - U^n\|_2^2 - \|U^n - U^{n-1}\|_2^2 + \frac{\lambda\tau^2}{2}(\|\Phi_x\|_2^2 - \|U_x^{n-1}\|_2^2) + \frac{\lambda\tau^2}{2}(\|\Phi\|_2^2 - \|U^{n-1}\|_2^2)$$
$$+ \frac{\lambda\tau^2}{4}h\sum_{j=1}^{J-1}(\Psi_j + N_j^{n-1})(|\Phi_j|^2 - |U_j^{n-1}|^2) + \frac{\lambda\tau^2}{4}(\|\Phi\|_4^4 - \|U^{n-1}\|_4^4) = 0. \tag{3.5}$$

We define $\{W_j\}$ by $(W_j)_{x\bar{x}} = (\Psi_j - N_j^n)/\tau$, and computing the inner product of difference equation (3.4) with $\frac{1}{2}(W + V^{n-1})$, we obtain

$$A' - B' = Q', \tag{3.6}$$

and computing

$$A' = \frac{1}{2}h\sum_{j=1}^{J}[(\Psi_j - N_j^n) - (N_j^n - N_j^{n-1})](W_j + V_j^{n-1}) = \frac{h\tau}{2}\sum_{j=1}^{J}[(W_j)_{x\bar{x}} - (V_j^{n-1})_{x\bar{x}}](W_j + V_j^{n-1})$$
$$= -\frac{\tau}{2}[\|W_x\|_2^2 - \|V_x^{n-1}\|_2^2], \tag{3.7}$$

$$B' = -\frac{\lambda\tau^2}{4}h\sum_{j=1}^{J}(\Psi_j + N_j^{n-1})_x(W_j + V_j^{n-1})_x = \frac{\lambda\tau^2}{4}h\sum_{j=1}^{J}(\Psi_j + N_j^{n-1})(\Psi_j + V_j^{n-1})_{x\bar{x}}$$

$$= \frac{\lambda\tau}{4}h\sum_{j=1}^{J}(\Psi_j + N_j^{n-1})(\Psi_j - N_j^{n-1}) = \frac{\lambda\tau}{4}(\|\Psi\|_2^2 - \|N^{n-1}\|_2^2), \tag{3.8}$$

$$Q' = -\frac{\lambda\tau^2}{4}h\sum_{j=1}^{J}(W_j + V_j^{n-1})_x(|\Phi_j|^2 + |U_j^{n-1}|^2)_x = \frac{\lambda\tau^2}{4}h\sum_{j=1}^{J}(W_j + V_j^{n-1})_{x\bar{x}}(|\Phi_j|^2 + |U_j^{n-1}|^2)$$

$$= \frac{\lambda\tau}{4}h\sum_{j=1}^{J}(\Psi_j - N_j^{n-1})(|\Phi_j|^2 + |U_j^{n-1}|^2). \tag{3.9}$$

Substituting (3.7)–(3.9) into (3.6), we obtain

$$\frac{1}{2}(\|W_x\|_2^2 - \|V_x^{n-1}\|_2^2) + \frac{\lambda}{4}(\|\Psi\|_2^2 - \|N^{n-1}\|_2^2) = -\frac{\lambda}{4}h\sum_{j=1}^{J}(\Psi_j - N_j^{n-1})(|\Phi_j|^2 + |U_j^{n-1}|^2). \tag{3.10}$$

Therefore, it follows from (3.5) and (3.10) that

$$\frac{1}{\tau^2}\|\Phi - U^n\|_2^2 + \frac{\lambda}{2}\|\Phi_x\|_2^2 + \frac{\lambda}{2}\|\Phi\|_2^2 + \frac{\lambda}{2}h\sum_{j=1}^{J-1}\Psi_j|\Phi_j|^2 + \frac{\lambda}{4}\|\Phi\|_4^4 + \frac{1}{2}\|W_x\|_2^2 + \frac{\lambda}{4}\|\Psi\|_2^2$$

$$= \frac{1}{\tau^2}\|U^n - U^{n-1}\|_2^2 + \frac{\lambda}{2}\|U_x^{n-1}\|_2^2 + \frac{\lambda}{2}\|U^{n-1}\|_2^2$$

$$+ \frac{\lambda}{2}h\sum_{j=1}^{J-1}N_j^{n-1}|U_j^{n-1}|^2 + \frac{\lambda}{4}\|U^{n-1}\|_4^4 + \frac{1}{2}\|V_x^{n-1}\|_2^2 + \frac{\lambda}{4}\|N^{n-1}\|_2^2. \tag{3.11}$$

Noticing

$$\left|\frac{\lambda}{2}h\sum_{j=1}^{J}\Psi_j|\Phi_j|^2\right| \leqslant \frac{\lambda}{4}h\sum_{j=1}^{J}((\Psi_j)^2 + |\Phi_j|^4|) \leqslant \frac{\lambda}{4}(\|\Psi\|_2^2 + \|\Phi\|_4^4), \tag{3.12}$$

$$\left|\frac{\lambda}{2}h\sum_{j=1}^{J}N_j^{n-1}|U_j^{n-1}|^2\right| \leqslant \frac{\lambda}{4}(\|N^{n-1}\|_2^2 + \|U^{n-1}\|_4^4), \tag{3.13}$$

then from (3.11), we get

$$\frac{1}{\tau^2}\|\Phi - U^n\|_2^2 + \frac{\lambda}{2}\|\Phi_x\|_2^2 + \frac{\lambda}{2}\|\Phi\|_2^2 + \frac{1}{2}\|W_x\|_2^2$$

$$\leqslant \|U_t^{n-1}\|_2^2 + \frac{\lambda}{2}\|U_x^{n-1}\|_2^2 + \frac{\lambda}{2}\|U^{n-1}\|_2^2 + \frac{\lambda}{2}\|U^{n-1}\|_4^4 + \frac{1}{2}\|V_x^{n-1}\|_2^2 + \frac{\lambda}{2}\|N^{n-1}\|_2^2. \tag{3.14}$$

It follows from (3.14) that

$$\|\Phi\|_2^2 \leqslant \tau^2(\|U_t^{n-1}\|_2^2 + \lambda\|U_x^{n-1}\|_2^2 + \lambda\|U^{n-1}\|_2^2 + \lambda\|U^{n-1}\|_4^4 + \|V_x^{n-1}\|_2^2 + \lambda\|N^{n-1}\|_2^2) + \|U^n\|^2$$

$$= C(\|U_t^{n-1}\|_2^2, \|U^n\|^2, \|U_x^{n-1}\|_2^2, \|U^{n-1}\|_2^2, \|U^{n-1}\|_4^4, \|V_x^{n-1}\|_2^2, \|N^{n-1}\|_2^2). \tag{3.15}$$

According to the definition of the norm $\|\cdot\|_p$, we can obtain the following estimates

$$\|\Phi^n\|_4 \leqslant \|\Phi^n\|_2 \leqslant C. \tag{3.16}$$

Using Young's inequality $ab \leqslant \frac{1}{4}a^2 + b^2$, we have

$$\left|\frac{\lambda}{2}h\sum_{j=1}^{J}\Psi_j|\Phi_j|^2\right| \leqslant \frac{\lambda}{8}h\sum_{j=1}^{J}((\Psi_j)^2 + 4|\Phi_j|^4|) \leqslant \frac{\lambda}{8}(\|\Psi\|_2^2 + 4\|\Phi\|_4^4). \tag{3.17}$$

It can be concluded from (3.11), (3.13)–(3.17), that

$$\|\Psi\|_2^2 \leqslant C(\|U_t^{n-1}\|_2^2, \|U^n\|^2, \|U_x^{n-1}\|_2^2, \|U^{n-1}\|_2^2, \|U^{n-1}\|_4^4, \|V_x^{n-1}\|_2^2, \|N^{n-1}\|_2^2). \tag{3.18}$$

It follows from (3.15) and (3.18) that

$$\begin{aligned}\|(\Phi, \Psi)\|^2 &= \|\Phi\|_2^2 + \|\Psi\|_2^2 \\ &\leqslant C(\|U_t^{n-1}\|_2^2, \|U^n\|^2, \|U_x^{n-1}\|_2^2, \|U^{n-1}\|_2^2, \|U^{n-1}\|_4^4, \|V_x^{n-1}\|_2^2, \|N^{n-1}\|_2^2).\end{aligned} \tag{3.19}$$

This means that $\|(\Phi, \Psi)\|$ is uniformly bounded with the parameter $0 \leqslant \lambda \leqslant 1$. Thus, the solution of the finite difference systems (2.1)–(2.4) exists.

The uniqueness of the difference solution will be proved in Section 4 based on some priori estimates. □

4. Stability and convergence of the scheme

In this section we will give some priori estimates, and then discuss the uniqueness, convergence and the stability of the difference solution.

First, some lemmas are introduced.

Lemma 4.1 (*Zhou [19,20], Discrete Sobolev's inequality*). *Suppose that $\{u_j\}$ are mesh functions. Given $\varepsilon > 0$, there exists a constant C dependent on ε such that*

$$\|\delta^k u_h\|_p \leqslant \varepsilon\|\delta^n u_h\|_2 + C\|u_h\|_2, \tag{4.1}$$

where $2 \leqslant p \leqslant \infty,\ 0 \leqslant k < n$.

Lemma 4.2 (*Zhou [19,20], Gronwall's inequality*). *Suppose that discrete function $w(n)$ satisfies the recurrence formula*

$$w_n - w_{n-1} \leqslant A\tau w_n + B\tau w_{n-1} + C_n\tau,$$

where A, B and $C_n (n = 1, \ldots, N)$ are nonnegative constants. Then

$$\max_{1 \leqslant n \leqslant N} |w_n| \leqslant \left(w_0 + \tau\sum_{k=1}^{N} C_k\right) \mathrm{e}^{2(A+B)T}, \tag{4.2}$$

where τ is small, such that $(A + B)\tau \leqslant (N - 1)/2N (N > 1)$.

Theorem 4.1. *If $U_0(x) \in H^1$, $U_1(x) \in L^2$, $N_0(x) \in H^1$, $N_1(x) \in L^2$, then the following estimates hold*:

$$\begin{aligned}&\|U_t^n\|_2 \leqslant C, \quad \|U_x^n\|_2 \leqslant C, \quad \|U^n\|_2 \leqslant C, \quad \|U^n\|_\infty \leqslant C, \\ &\|V_x^n\|_2 \leqslant C, \quad \|N^n\|_2 \leqslant C, \quad \|U^n\|_4 \leqslant C.\end{aligned} \tag{4.3}$$

Proof. According to Theorem 2.1 and noticing

$$\left|\frac{1}{2}h\sum_{j=1}^{J} N_j^{n+1}|U_j^{n+1}|^2\right| \leqslant \frac{1}{4}h\sum_{j=1}^{J}((N_j^{n+1})^2 + |U_j^{n+1}|^4|) \leqslant \frac{1}{4}(\|N^{n+1}\|_2^2 + \|U^{n+1}\|_4^4),$$

and

$$\left|\frac{1}{2}h\sum_{j=1}^{J}N_j^n|U_j^n|^2\right|\leqslant\frac{1}{4}(\|N^n\|_2^2+\|U^n\|_4^4),$$

we have

$$\|U_t^n\|_2^2+\tfrac{1}{2}(\|U_x^{n+1}\|_2^2+\|U_x^n\|_2^2)+\tfrac{1}{2}(\|U^{n+1}\|_2^2+\|U^n\|_2^2)+\tfrac{1}{2}\|V_x^n\|_2^2\leqslant C.$$

Then,

$$\|U_t^n\|_2\leqslant C,\quad \|U_x^n\|_2\leqslant C,\quad \|U^n\|_2\leqslant C,\quad \|V_x^n\|_2\leqslant C.$$

According to Lemma 4.1 and the definition of the norm $\|\cdot\|_p$, we can obtain the following estimates:

$$\|U^n\|_\infty\leqslant C,\quad \|U^n\|_4\leqslant\|U^n\|_2\leqslant C. \tag{4.4}$$

Using Young's inequality $ab\leqslant\frac{1}{4}a^2+b^2$, we have

$$\left|h\sum_{j=1}^{J}N_j^{n+1}|U_j^{n+1}|^2\right|\leqslant h\sum_{j=1}^{J}\left[\frac{1}{4}(N_j^{n+1})^2+|U_j^{n+1}|^4\right]=\frac{1}{4}\|N^{n+1}\|_2^2+\|U^{n+1}\|_4^4,$$

$$\left|h\sum_{j=1}^{J}N_j^{n}|U_j^{n}|^2\right|\leqslant h\sum_{j=1}^{J}\left[\frac{1}{4}(N_j^{n})^2+|U_j^{n}|^4\right]=\frac{1}{4}\|N^{n}\|_2^2+\|U^{n}\|_4^4.$$

Then according to Lemma 4.1 and (4.4), we have

$$\|N^n\|_2\leqslant C.$$

This completes the proof of Theorem 4.1. □

Secondly, we prove the uniqueness of the difference solution, i.e.,

Theorem 4.2. *The difference solution of system* (2.1)–(2.4) *is unique when τ is suitably small.*

Proof. Let $\varepsilon_j^n=\widetilde{U}_j^n-U_j^n$, $\xi_j^n=\widetilde{N}_j^n-N_j^n$, $(\zeta_j^n)_{x\bar{x}}=(\xi_j^n)_t$, where $(\widetilde{U}^n,\widetilde{N}^n)$ and (U^n,N^n) are two solutions of the difference (2.1)–(2.4) with initial value $(\widetilde{U}^0,\widetilde{N}^0),(\widetilde{U}^1,\widetilde{N}^1)$ and $(U^0,N^0),(U^1,N^1)$ respectively, and satisfy the following inequalities:

$$\|\widetilde{U}^0\|_\infty\leqslant M,\quad \|\widetilde{U}^1\|_\infty\leqslant M,\quad \|U^0\|_\infty\leqslant M,\quad \|U^1\|_\infty\leqslant M.$$

$$\|\widetilde{N}^0\|_2\leqslant M,\quad \|\widetilde{N}^1\|_2\leqslant M,\quad \|N^0\|_2\leqslant M,\quad \|N^1\|_2\leqslant M.$$

According to Theorem 4.1, there exists a constant $C(M)$, such that

$$\|\widetilde{U}^n\|_\infty\leqslant C(M),\quad \|U^n\|_\infty\leqslant C(M),\quad \|\widetilde{N}^n\|_2\leqslant C(M),\quad \|N^n\|_2\leqslant C(M).$$

Then (ε^n,ξ^n) satisfies the following homogeneous system

$$\begin{aligned}&(\varepsilon_j^n)_{t\bar{t}}-\tfrac{1}{2}(\varepsilon_j^{n+1}+\varepsilon_j^{n-1})_{x\bar{x}}+\tfrac{1}{2}(\varepsilon_j^{n+1}+\varepsilon_j^{n-1})+\tfrac{1}{4}(\widetilde{N}_j^{n+1}+\widetilde{N}_j^{n-1})(\widetilde{U}_j^{n+1}+\widetilde{U}_j^{n-1})\\&\quad-\tfrac{1}{4}(N_j^{n+1}+N_j^{n-1})(U_j^{n+1}+U_j^{n-1})+\tfrac{1}{4}(|\widetilde{U}_j^{n+1}|^2+|\widetilde{U}_j^{n-1}|^2)(\widetilde{U}_j^{n+1}+\widetilde{U}_j^{n-1})\\&\quad-\tfrac{1}{4}(|U_j^{n+1}|^2+|U_j^{n-1}|^2)(U_j^{n+1}+U_j^{n-1})=0,\quad 1\leqslant j\leqslant J-1,\ 1\leqslant n\leqslant K-1,\end{aligned} \tag{4.5}$$

$$(\xi_j^n)_{t\bar{t}} - \frac{1}{2}(\xi_j^{n+1} + \xi_j^{n-1})_{x\bar{x}} = \frac{1}{2}(|\widetilde{U}_j^{n+1}|^2 + |\widetilde{U}_j^{n-1}|^2)_{x\bar{x}} - \frac{1}{2}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)_{x\bar{x}},$$
$$1 \leqslant j \leqslant J-1, \quad 1 \leqslant n \leqslant K-1, \tag{4.6}$$

$$\varepsilon_j^0 = 0, \quad \xi_j^0 = 0, \quad \varepsilon_j^1 - \varepsilon_j^0 = 0, \tag{4.7}$$

$$\varepsilon_0^n = \varepsilon_J^n = 0, \quad \xi_0^n = \xi_J^n = 0, \quad \xi_j^1 - \xi_j^0 = 0. \tag{4.8}$$

Computing the inner product of difference Eq. (4.5) with $\varepsilon^{n+1} - \varepsilon^{n-1}$, then taking the real part of the result, we have

$$\|\varepsilon_t^n\|_2^2 - \|\varepsilon_t^{n-1}\|_2^2 + \frac{1}{2}(\|\varepsilon_x^{n+1}\|_2^2 - \|\varepsilon_x^{n-1}\|_2^2) + \frac{1}{2}(\|\varepsilon^{n+1}\|_2^2 - \|\varepsilon^{n-1}\|_2^2) + A'' + B'' = 0, \tag{4.9}$$

where

$$\begin{aligned}
A'' &= \frac{1}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}[(\widetilde{N}_j^{n+1} + \widetilde{N}_j^{n-1})(\widetilde{U}_j^{n+1} + \widetilde{U}_j^{n-1}) - (N_j^{n+1} + N_j^{n-1})(U_j^{n+1} + U_j^{n-1})](\overline{\varepsilon_j^{n+1}} - \overline{\varepsilon_j^{n-1}})\right\} \\
&= \frac{1}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}[(\xi_j^{n+1} + \xi_j^{n-1})(\widetilde{U}_j^{n+1} + \widetilde{U}_j^{n-1}) + (N_j^{n+1} + N_j^{n-1})(\varepsilon_j^{n+1} + \varepsilon_j^{n-1})](\overline{\varepsilon_j^{n+1}} - \overline{\varepsilon_j^{n-1}})\right\} \\
&= \frac{\tau}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}(\xi_j^{n+1} + \xi_j^{n-1})(\widetilde{U}_j^{n+1} + \widetilde{U}_j^{n-1})(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t\right\} \\
&\quad + \frac{\tau}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}(N_j^{n+1} + N_j^{n-1})(\varepsilon_j^{n+1} + \varepsilon_j^{n-1})(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t\right\} \\
&\leqslant \frac{\tau}{4}(\|\widetilde{U}^{n+1}\|_\infty + \|\widetilde{U}^{n-1}\|_\infty)\sum_{j=1}^{J}|(\xi_j^{n+1} + \xi_j^{n-1})(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t| \\
&\quad + \frac{\tau}{4}\sum_{j=1}^{J}|(N_j^{n+1} + N_j^{n-1})(\varepsilon_j^{n+1} + \varepsilon_j^{n-1})||(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t| \\
&\leqslant C\tau(\|\xi^{n+1}\|_2^2 + \|\xi^{n-1}\|_2^2 + \|\varepsilon_t^n\|_2^2 + \|\varepsilon_t^{n-1}\|_2^2) + C\tau(\|(N^{n+1} + N^{n-1})(\varepsilon^{n+1} + \varepsilon^{n-1})\|_2^2 + \|\varepsilon_t^n\|_2^2 \\
&\quad + \|\varepsilon_t^{n-1}\|_2^2).
\end{aligned} \tag{4.10}$$

$$\begin{aligned}
B'' &= \frac{1}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}[(|\widetilde{U}_j^{n+1}|^2 + |\widetilde{U}_j^{n-1}|^2)(\widetilde{U}_j^{n+1} + \widetilde{U}_j^{n-1}) - (|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1})](\overline{\varepsilon_j^{n+1}} - \overline{\varepsilon_j^{n-1}})\right\} \\
&\leqslant \frac{\tau}{4}\mathrm{Re}\left\{h\sum_{j=1}^{J}|\widetilde{U}_j^{n+1}\overline{\varepsilon_j^{n+1}} + \widetilde{U}_j^{n-1}\overline{\varepsilon_j^{n-1}} + \varepsilon_j^{n+1}\overline{\widetilde{U}_j^{n+1}} + \varepsilon_j^{n-1}\overline{\widetilde{U}_j^{n-1}}|(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t\right\} \\
&\quad + \frac{\tau}{4}\mathrm{Re}\left\{h\sum_{j=1}^{J}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)\cdot|\varepsilon_j^{n+1} + \varepsilon_j^{n-1}|\cdot|(\overline{\varepsilon_j^{n}} + \overline{\varepsilon_j^{n-1}})_t|\right\} \\
&\leqslant C\tau(\|\varepsilon^{n+1}\|_2^2 + \|\varepsilon^{n-1}\|_2^2 + \|\varepsilon_t^n\|_2^2 + \|\varepsilon_t^{n-1}\|_2^2).
\end{aligned} \tag{4.11}$$

According to Lemma 4.1 and Theorem 4.1, we have

$$\begin{aligned}\|(N^{n+1}+N^{n-1})(\varepsilon^{n+1}+\varepsilon^{n-1})\|_2 &\leqslant C(\|N^{n+1}\|_2+N^{n-1}\|_2)\|\varepsilon^{n+1}+\varepsilon^{n-1}\|_\infty\\ &\leqslant C(\|\varepsilon_x^{n+1}+\varepsilon_x^{n-1}\|_2+\|\varepsilon^{n+1}+\varepsilon^{n-1}\|_2)\\ &\leqslant C(\|\varepsilon_x^{n+1}\|_2+\|\varepsilon_x^{n-1}\|_2+\|\varepsilon^{n+1}\|_2+\|\varepsilon^{n-1}\|_2).\end{aligned} \tag{4.12}$$

Substituting (4.12) into (4.10), we have

$$A''\leqslant C\tau(\|\xi^{n+1}\|_2^2+\|\xi^{n-1}\|_2^2+\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n-1}\|_2^2+\|\varepsilon^{n+1}\|_2^2+\|\varepsilon^{n-1}\|_2^2+\|\varepsilon_t^{n}\|_2^2+\|\varepsilon_t^{n-1}\|_2^2). \tag{4.13}$$

Substituting (4.11), (4.13) into (4.9), we have

$$\begin{aligned}&\|\varepsilon_t^n\|_2^2-\|\varepsilon_t^{n-1}\|_2^2+\tfrac{1}{2}(\|\varepsilon_x^{n+1}\|_2^2-\|\varepsilon_x^{n-1}\|_2^2)+\tfrac{1}{2}(\|\varepsilon^{n+1}\|_2^2-\|\varepsilon^{n-1}\|_2^2)\\ &\quad\leqslant C\tau(\|\xi^{n+1}\|_2^2+\|\xi^{n-1}\|_2^2+\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n-1}\|_2^2+\|\varepsilon^{n+1}\|_2^2+\|\varepsilon^{n-1}\|_2^2+\|\varepsilon_t^{n}\|_2^2+\|\varepsilon_t^{n-1}\|_2^2).\end{aligned} \tag{4.14}$$

Computing the inner product of difference Eq. (4.5) with $\frac{1}{2}(\zeta^n-\zeta^{n-1})$, we have

$$\frac{1}{2\tau}(\|\zeta_x^n\|_2^2-\|\zeta_x^{n-1}\|_2^2)+\frac{1}{4\tau}(\|\xi^{n+1}\|_2^2-\|\xi^{n-1}\|_2^2)+Q''=0, \tag{4.15}$$

where

$$\begin{aligned}Q''&=\frac{1}{4}\sum_{j=1}^{J}[(|\widetilde{U}_j^{n+1}|^2+|\widetilde{U}_j^{n-1}|^2)-(|U_j^{n+1}|^2+|U_j^{n-1}|^2)]_{x\bar{x}}(\zeta_j^n-\zeta_j^{n-1})\\ &=\frac{1}{4h}h\sum_{j=1}^{J}(\widetilde{U}_{j+1}^{n+1}\overline{\varepsilon}_{j+1}^{n+1}+\widetilde{U}_{j+1}^{n-1}\overline{\varepsilon}_{j+1}^{n-1}+\varepsilon_{j+1}^{n+1}\overline{U}_{j+1}^{n+1}+\varepsilon_{j+1}^{n-1}\overline{U}_{j+1}^{n-1}-\widetilde{U}_j^{n+1}\overline{\varepsilon}_j^{n+1}\\ &\quad+\widetilde{U}_j^{n-1}\overline{\varepsilon}_j^{n-1}+\varepsilon_j^{n+1}\overline{U}_j^{n+1}+\varepsilon_j^{n-1}\overline{U}_j^{n-1})(\zeta_j^n+\zeta_j^{n-1})_x\\ &=\frac{1}{4}h\sum_{j=1}^{J}(\widetilde{U}_{j+1}^{n+1}(\overline{\varepsilon}_j^{n+1})_x+\widetilde{U}_{j+1}^{n-1}(\overline{\varepsilon}_j^{n-1})_x+\varepsilon_{j+1}^{n+1}(\overline{U}_j^{n+1})_x\\ &\quad+\varepsilon_{j+1}^{n-1}(\overline{U}_j^{n-1})_x+(\widetilde{U}_j^{n+1})_x\overline{\varepsilon}_j^{n+1}+(\widetilde{U}_j^{n-1})_x\overline{\varepsilon}_j^{n-1}+(\varepsilon_j^{n+1})_x\overline{U}_j^{n+1}+(\varepsilon_j^{n-1})_x\overline{U}_j^{n-1})(\zeta_j^n+\zeta_j^{n-1})_x\\ &\leqslant C(\|\zeta_x^n\|_2^2+\|\zeta_x^{n-1}\|_2^2+\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n-1}\|_2^2+\|\varepsilon^{n}\|_2^2+\|\varepsilon^{n-1}\|_2^2+\|\varepsilon^{n+1}U_x^{n+1}\|_2^2+\|\varepsilon^{n-1}U_x^{n-1}\|_2^2).\end{aligned}$$

According to Theorem 4.1 and Lemma 4.1, we have

$$\|\varepsilon^n U_x^n\|_2\leqslant C\|\varepsilon^n\|_\infty\|U_x^n\|_2\leqslant C\|\varepsilon_x^n\|_2+C\|\varepsilon^n\|_2.$$

Then we can obtain

$$Q''\leqslant C(\|\zeta_x^n\|_2^2+\|\zeta_x^{n-1}\|_2^2+\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n-1}\|_2^2+\|\varepsilon^{n}\|_2^2+\|\varepsilon^{n-1}\|_2^2). \tag{4.16}$$

It follows from (4.15) and (4.16) that

$$\begin{aligned}&\tfrac{1}{2}(\|\zeta_x^n\|_2^2-\|\zeta_x^{n-1}\|_2^2)+\tfrac{1}{4}(\|\xi^{n+1}\|_2^2-\|\xi^{n-1}\|_2^2)\\ &\quad\leqslant C\tau(\|\zeta_x^n\|_2^2+\|\zeta_x^{n-1}\|_2^2+\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n-1}\|_2^2+\|\varepsilon^{n}\|_2^2+\|\varepsilon^{n-1}\|_2^2).\end{aligned} \tag{4.17}$$

Adding (4.17) to (4.14) and let

$$S^n=\|\varepsilon_t^n\|_2^2+\tfrac{1}{2}(\|\varepsilon_x^{n+1}\|_2^2+\|\varepsilon_x^{n}\|_2^2)+\tfrac{1}{2}(\|\varepsilon^{n+1}\|_2^2+\|\varepsilon^{n}\|_2^2)+\tfrac{1}{2}\|\zeta_x^n\|_2^2+\tfrac{1}{4}(\|\xi^{n+1}\|_2^2+\|\xi^{n}\|_2^2), \tag{4.18}$$

the following inequality holds

$$S^n - S^{n-1} \leqslant C\tau(S^n + S^{n-1}). \tag{4.19}$$

If τ is small, such that $2C\tau \leqslant (N-1)/2N (N>1)$, then it follows from the Lemma 4.2 that

$$S^n \leqslant S^0 \exp(CT). \tag{4.20}$$

According to (4.7), (4.8) and (4.18), we know $S^0 = 0$, thus we get

$$S^n \leqslant 0. \tag{4.21}$$

From formula (4.18) and (4.21), using Lemma 4.1, we get $\|\varepsilon^n\|_\infty = 0$, $\|\xi^n\|_2 = 0$, i.e.,

$$\widetilde{U}^n = U^n, \quad \widetilde{N}^n = N^n.$$

Therefore the proof of Theorem 4.2 is completed. □

Thirdly, we prove the convergence of the difference scheme. We define the truncation errors as follows:

$$\begin{aligned} r_j^n = (u_j^n)_{t\bar{t}} - \tfrac{1}{2}(u_j^{n+1} + u_j^{n-1})_{x\bar{x}} + \tfrac{1}{2}(u_j^{n+1} + u_j^{n-1}) + \tfrac{1}{4}(m_j^{n+1} + m_j^{n-1})(u_j^{n+1} + u_j^{n-1}) \\ + \tfrac{1}{4}(|u_j^{n+1}|^2 + |u_j^{n-1}|^2)(u_j^{n+1} + u_j^{n-1}), \end{aligned} \tag{4.22}$$

$$\sigma_j^n = (m_j^n)_{t\bar{t}} - \tfrac{1}{2}(m_j^{n+1}) + m_j^{n-1}))_{x\bar{x}} - \tfrac{1}{2}(|u_j^{n+1}|^2 + |u_j^{n-1}|^2)_{x\bar{x}}. \tag{4.23}$$

Then from Taylor's expansion, it can be easily obtained that

Lemma 4.3. *Assume that the conditions of Theorem* 4.1 *are satisfied and* $u(x.t) \in C_{x,t}^{4,4}$, $m(x,t) \in C_{x,t}^{4,4}$. *Then the truncation errors of the difference schemes* (2.1)–(2.4) *satisfy*

$$|r_j^n| + |\sigma_j^n| = \mathrm{O}(\tau^2 + h^2),$$

as $\tau \to 0,\ h \to 0$.

We define

$$e_j^n = u_j^n - U_j^n, \quad \eta_j^n = m_j^n - N_j^n, \quad (\vartheta_j^n)_{x\bar{x}} = (\eta_j^n)_t, \quad j = 1, 2, \ldots, J-1, \tag{4.24}$$

the following theorem on convergence of the (2.1)–(2.4) can be proved.

Theorem 4.3. *Assume that the conditions of Lemma* 4.3 *are satisfied. Then the solution of the difference problem* (2.1)–(2.4) *converges to the solution of the problem* (1.1)–(1.4) *with order* $\mathrm{O}(\tau^2 + h^2)$ *in the* l_∞ *norm for* U^n, *and in the* l_2 *norm for* N^n.

Proof. Subtracting (2.1) and (2.2) from (4.22) and (4.23), respectively, we obtain

$$\begin{aligned} &(e_j^n)_{t\bar{t}} - \tfrac{1}{2}(e_j^{n+1} + e_j^{n-1})_{x\bar{x}} + \tfrac{1}{2}(e_j^{n+1} + e_j^{n-1}) + \tfrac{1}{4}(m_j^{n+1} + m_j^{n-1})(u_j^{n+1} + u_j^{n-1}) \\ &\quad - \tfrac{1}{4}(N_j^{n+1} + N_j^{n-1})(U_j^{n+1} + U_j^{n-1}) + \tfrac{1}{4}(|u_j^{n+1}|^2 + |u_j^{n-1}|^2)(u_j^{n+1} + u_j^{n-1}) \\ &\quad - \tfrac{1}{4}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1}) = r_j^n, \quad 1 \leqslant j \leqslant J-1, \quad 1 \leqslant n \leqslant K-1, \end{aligned} \tag{4.25}$$

$$\begin{aligned} &(\eta_j^n)_{t\bar{t}} - \tfrac{1}{2}(\eta_j^{n+1} + \eta_j^{n-1})_{x\bar{x}} - \tfrac{1}{2}(|u_j^{n+1}|^2 + |u_j^{n-1}|^2)_{x\bar{x}} + \tfrac{1}{2}(|U_j^{n+1}|^2 + |U_j^{n-1}|^2)_{x\bar{x}} = \sigma_j^n, \\ &\quad 1 \leqslant j \leqslant J-1, \quad 1 \leqslant n \leqslant K-1. \end{aligned} \tag{4.26}$$

Computing the inner product of (4.25) with $e^{n+1} - e^{n-1}$ and taking the real part, we have

$$\|e_t^n\|_2^2 - \|e_t^{n-1}\|_2^2 + \tfrac{1}{2}(\|e_x^{n+1}\|_2^2 - \|e_x^{n-1}\|_2^2) + \tfrac{1}{2}(\|e^{n+1}\|_2^2 - \|e^{n-1}\|_2^2) = L_1 - L_2 - L_3, \tag{4.27}$$

where

$$L_1 = \mathrm{Re}\left\{h\sum_{j=1}^{J} r_j^n\,(\overline{e}_j^{n+1} - \overline{e}_j^{n-1})\right\} = \mathrm{Re}\left\{h\tau\sum_{j=1}^{J} r_j^n(\overline{e}_j^n + \overline{e}_j^{n-1})_t\right\} \leqslant C\tau(\|r^n\|_2^2 + \|e_t^n\|_2^2 + \|e_t^{n-1}\|_2^2), \tag{4.28}$$

$$\begin{aligned} L_2 &= \frac{1}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}[m_j^{n+1} + m_j^{n-1})(u_j^{n+1} + u_j^{n-1}) - (N_j^{n+1} + N_j^{n-1})(U_j^{n+1} + U_j^{n-1})](\overline{e_j^{n+1}} - \overline{e_j^{n-1}})\right\} \\ &\leqslant C\tau(\|\eta^{n+1}\|_2^2 + \|\eta^{n-1}\|_2^2 + \|e_t^n\|_2^2 + \|e_t^{n-1}\|_2^2), \end{aligned} \tag{4.29}$$

$$\begin{aligned} L_3 &= \frac{1}{4}\mathrm{Re}\left\{\sum_{j=1}^{J}[(|u_j^{n+1}|^2 + |u_j^{n-1}|^2)(u_j^{n+1} + u_j^{n-1}) - (|U_j^{n+1}|^2 + |U_j^{n-1}|^2)(U_j^{n+1} + U_j^{n-1})](\overline{e_j^{n+1}} - \overline{e_j^{n-1}})\right\} \\ &\leqslant C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^{n-1}\|_2^2 + \|e^{n+1}\|_2^2 + \|e^{n-1}\|_2^2 + \|e_t^n\|_2^2 + \|e_t^{n-1}\|_2^2), \end{aligned} \tag{4.30}$$

where the inequalities (4.29) and (4.30) can be proved similarly as the inequalities (4.10) and (4.11).

Substituting (4.28)–(4.30) into (4.27), we have

$$\begin{aligned} &\|e_t^n\|_2^2 - \|e_t^{n-1}\|_2^2 + \tfrac{1}{2}(\|e_x^{n+1}\|_2^2 - \|e_x^{n-1}\|_2^2) + \tfrac{1}{2}(\|e^{n+1}\|_2^2 - \|e^{n-1}\|_2^2) \\ &\quad\leqslant C\tau(\|r^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^{n-1}\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^{n-1}\|_2^2 \\ &\qquad + \|e^{n+1}\|_2^2 + \|e^{n-1}\|_2^2 + \|e_t^n\|_2^2 + \|e_t^{n-1}\|_2^2). \end{aligned} \tag{4.31}$$

Computing the inner product of (4.26) with $\frac{1}{2}(\vartheta^n + \vartheta^{n-1})$ we obtain

$$\frac{1}{2\tau}(\|\vartheta_x^n\|_2^2 - \|\vartheta_x^{n-1}\|_2^2) + \frac{1}{4\tau}(\|\eta^{n+1}\|_2^2 - \|\eta^{n-1}\|_2^2) + L_4 + L_5 = 0, \tag{4.32}$$

where

$$\begin{aligned} L_4 &= \frac{1}{4}\sum_{j=1}^{J}[(|u_j^{n+1}|^2 + |u_j^{n-1}|^2) - (|U_j^{n+1}|^2 + |U_j^{n-1}|^2)]_{x\bar{x}}(\vartheta_j^n - \vartheta_j^{n-1}) \\ &\leqslant C(\|\vartheta_x^n\|_2^2 + \|\vartheta_x^{n-1}\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^{n-1}\|_2^2 + \|e^n\|_2^2 + \|e^{n-1}\|_2^2). \end{aligned} \tag{4.33}$$

(It can be proved similarly as the estimate of Q'')

$$L_5 = \left\langle \sigma^n, \frac{1}{2}(\vartheta^n + \vartheta^{n-1})\right\rangle = \frac{1}{2}h\sum_{j=1}^{J}\sigma_j^n(\vartheta_j^n + \vartheta_j^{n-1}) \leqslant C(\|\sigma^n\|_2^2 + \|\vartheta_x^n\|_2^2 + \|\vartheta_x^{n-1}\|_2^2). \tag{4.34}$$

(The inequality

$$|\vartheta_j^n| = \left|\sum_{k=1}^{j}(\vartheta_k^n - \vartheta_{k-1}^n)\right| = \left|h\sum_{k=1}^{j}(F_{k-1}^n)_x\right| \leqslant C\|\vartheta_x^n\|_2$$

is used in the proof of (4.34).)

Substituting (4.33) and (4.34) into (4.32), we obtain

$$\frac{1}{2}(\|\vartheta_x^n\|_2^2-\|\vartheta_x^{n-1}\|_2^2)+\frac{1}{4}(\|\eta^{n+1}\|_2^2-\|\eta^{n-1}\|_2^2)\leqslant C\tau(\|\sigma^n\|_2^2+\|\vartheta_x^n\|_2^2+\|\vartheta_x^{n-1}\|_2^2+\|e_x^{n+1}\|_2^2+\|e_x^{n-1}\|_2^2 + \|e^{n+1}\|_2^2+\|e^{n-1}\|_2^2). \tag{4.35}$$

Adding (4.35) to (4.31) and defining

$$G^n=\|e_t^n\|_2^2+\frac{1}{2}(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2)+\frac{1}{2}(\|e^{n+1}\|_2^2+\|e^n\|_2^2)+\frac{1}{2}\|\vartheta_x^n\|_2^2+\frac{1}{4}(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2),$$

the following inequality holds

$$G^n-G^{n-1}\leqslant C\tau(\|r^n\|_2^2+\|\sigma^n\|_2^2)+C\tau(G^n+G^{n-1}). \tag{4.36}$$

It follows from (4.36) and Lemma 4.2 that

$$G^n\leqslant(G^0+C(h^2+\tau^2)^2)\exp(CT)\leqslant C(G^0+(h^2+\tau^2)^2),\quad 1\leqslant n\leqslant N. \tag{4.37}$$

It is clear that $e^0,\ e^1,\ \eta^0$ and η^1 are of second-order accuracy and it follows from [7] that $\|\vartheta_x^0\|_2=\mathrm{O}(h^2+\tau^2)$. Thus, $G^0=\mathrm{O}(h^2+\tau^2)^2$. It follows from (4.37) that

$$\|e_x^n\|_2\leqslant\mathrm{O}(h^2+\tau^2),\quad \|e^n\|_2\leqslant\mathrm{O}(h^2+\tau^2),$$
$$\|e_t^n\|_2\leqslant\mathrm{O}(h^2+\tau^2),\quad \|\eta^n\|_2\leqslant\mathrm{O}(h^2+\tau^2).$$

Using Lemma 4.1, we obtain

$$\|e^n\|_\infty\leqslant\mathrm{O}(h^2+\tau^2).$$

This completes the proof. □

Finally, We can similarly prove that the solutions of the problem (2.1)–(2.4) are unconditionally stable, i.e.,

Theorem 4.4. *Under the conditions of Theorem* 4.3, *the solution of the difference scheme* (2.1)–(2.4) *is unconditionally stable for initial data.*

5. Another explicit conservative difference scheme

In this section, we will propose an explicit difference scheme for the problem (1.1)–(1.4), and discuss the discrete conservative law of the scheme.

The explicit scheme is written as follows:

$$(U_j^n)_{t\bar{t}}-(U_j^n)_{x\bar{x}}+\frac{1}{2}(U_j^{n+1}+U_j^{n-1})+\frac{1}{2}(N_j^n)(U_j^{n+1}+U_j^{n-1})+\frac{1}{4}(|U_j^{n+1}|^2+|U_j^{n-1}|^2)(U_j^{n+1}+U_j^{n-1})=0,$$
$$1\leqslant j\leqslant J-1,\quad 1\leqslant n\leqslant K-1, \tag{5.1}$$

$$(N_j^n)_{t\bar{t}}-(N_j^n)_{x\bar{x}}=(|U_j^n|^2)_{x\bar{x}},\quad 1\leqslant j\leqslant J-1,\quad 1\leqslant n\leqslant K-1. \tag{5.2}$$

The initial and boundary conditions (1.3) and (1.4) are, respectively, approximated as

$$U_j^0=u_0(x_j),\quad N_j^0=m_0(x_j),\quad U_j^1-U_j^{-1}=2\tau u_1(x_j), \tag{5.3}$$

$$U_0^n=U_J^n=0,\quad N_0^n=U_J^n=0,\quad N_j^1-N_j^{-1}=2\tau m_1(x_j). \tag{5.4}$$

In (5.1) and (5.2), let $n=0$. Then, eliminating U^{-1} and N^{-1} from (5.3) and (5.4), we obtain

$$\frac{2U_j^1-2u_0(x_j)-2\tau u_1(x_j)}{\tau^2}-(U_j^0)_{x\bar{x}}+(U_j^1-\tau u_1(x_j))+N_j^0(U_j^1-\tau u_1(x_j))$$
$$+\frac{1}{2}(|U_j^1|^2+|U_j^1-2\tau u_1(x_j)|^2)(U_j^1-\tau u_1(x_j))=0, \tag{5.3$'$}$$

$$\frac{2N_j^1-2m_0(x_j)-2\tau m_1(x_j)}{\tau^2}-(N_j^0)_{x\bar{x}}=(|U_j^0|^2)_{x\bar{x}}. \tag{5.4$'$}$$

We also define $\{V_j^n\}$ by

$$(V_j^n)_{x\bar{x}}=(N_j^n)_t,\quad j=1,2,\ldots,J-1,\quad V_0^n=V_J^n=0. \tag{5.5}$$

Now, we give the discrete conservative law of the difference (5.1)–(5.4).

Theorem 5.1. *The difference scheme* (5.1)–(5.4) *admits the following invariant*

$$\widetilde{E}^n=\widetilde{E}^{n-1}=\cdots=\widetilde{E}^0, \tag{5.6}$$

where

$$\widetilde{E}^n=\|U_t^n\|_2^2+\mathrm{Re}\left\{h\sum_{j=1}^J(\overline{U_j^{n+1}})_x(U_j^n)_x\right\}+\frac{1}{2}(\|U^{n+1}\|_2^2+\|U^n\|_2^2)+\frac{1}{2}\|V_x^n\|_2^2$$
$$+\frac{1}{2}h\sum_{j=1}^J N_j^{n+1}N_j^n+\frac{1}{4}(\|U^{n+1}\|_4^4+\|U^n\|_4^4)+\frac{1}{2}h\sum_{j=1}^J[N_j^n|U_j^{n+1}|^2+N_j^{n+1}|U_j^n|^2]$$

is called discrete energy.

Proof. Computing the inner product of difference Eq. (5.1) with $U^{n+1}-U^{n-1}$ and taking the real part, we have

$$\|U_t^n\|_2^2-\|U_t^{n-1}\|_2^2+\mathrm{Re}\left\{h\sum_{j=1}^J(\overline{U_j^{n+1}})_x(U_j^n)_x-h\sum_{j=1}^J(\overline{U_j^n})_x(U_j^{n-1})_x\right\}+\frac{1}{2}(\|U^{n+1}\|_2^2-\|U^{n-1}\|_2^2)$$
$$+\frac{1}{4}(\|U^{n+1}\|_4^4-\|U^{n-1}\|_4^4)+\frac{1}{2}h\sum_{j=1}^J N_j^n(|U_j^{n+1}|^2-|U_j^{n-1}|^2)=0. \tag{5.7}$$

Next, computing the inner product of (5.2) with $\frac{1}{2}(V^n+V^{n-1})$ and using Eq. (5.5), we obtain

$$A'''-B'''=Q''',$$

where

$$A'''=\frac{1}{2}h\sum_{j=1}^J(N_j^n)_{t\bar{t}}(V_j^n+V_j^{n-1})=\frac{h}{2\tau}\sum_{j=1}^J[(V_j^n)_{x\bar{x}}-(V_j^{n-1})_{x\bar{x}}](V_j^n+V_j^{n-1})$$
$$=-\frac{h}{2\tau}\sum_{j=1}^J[(V_j^n)_x-(V_j^{n-1})_x][(V_j^n)_x+(V_j^{n-1})_x]=-\frac{1}{2\tau}(\|V_x^n\|_2^2-\|V_x^{n-1}\|_2^2),$$

$$B''' = -\frac{1}{2}h\sum_{j=1}^{J}(N_j^n)_x(V_j^n + V_j^{n-1})_x = \frac{1}{2}h\sum_{j=1}^{J}(N_j^n)(V_j^n + V_j^{n-1})_{x\bar{x}}$$

$$= \frac{1}{2}h\sum_{j=1}^{J}(N_j^n)(N_j^n + N_j^{n-1})_t = \frac{1}{2\tau}h\sum_{j=1}^{J}N_j^n N_j^{n+1} - \frac{1}{2\tau}h\sum_{j=1}^{J}N_j^n N_j^{n+1},$$

$$Q''' = -\frac{1}{2}h\sum_{j=1}^{J}(V_j^n + V_j^{n-1})_x|U_j^n|^2 = \frac{1}{2}h\sum_{j=1}^{J}(V_j^n + V_j^{n-1})_{x\bar{x}}|U_j^n|^2 = \frac{1}{2\tau}h\sum_{j=1}^{J}(N_j^{n+1} - N_j^{n-1})|U_j^n|^2.$$

Thus

$$\frac{1}{2}(\|V_x^n\|_2^2 - \|V_x^{n-1}\|_2^2) + \frac{h}{2}\sum_{j=1}^{J}N_j^n N_j^{n+1} - \frac{h}{2}\sum_{j=1}^{J}N_j^n N_j^{n+1} + \frac{h}{2}\sum_{j=1}^{J}(N_j^{n+1} - N_j^{n-1})|U_j^n|^2 = 0. \tag{5.8}$$

Adding (5.8) to (5.7), and noting

$$\frac{1}{2}h\sum_{j=1}^{J}N_j^n(|U_j^{n+1}|^2 - |U_j^{n-1}|^2) + \frac{h}{2}\sum_{j=1}^{J}(N_j^{n+1} - N_j^{n-1})|U_j^n|^2$$

$$= \frac{1}{2}h\sum_{j=1}^{J}[N_j^n|U_j^{n+1}|^2 + N_j^{n+1}|U_j^n|^2] - \frac{1}{2}h\sum_{j=1}^{J}[N_j^{n-1}|U_j^n|^2 + N_j^n|U_j^{n-1}|^2], \tag{5.9}$$

we obtain the result of Theorem 5.1

$$\widetilde{E}^n = \widetilde{E}^{n-1} = \cdots = \widetilde{E}^0.$$

This completes the proof. □

6. Stability and convergence of the explicit scheme

In this section, we will first introduce two important inequalities, then estimate the difference solution and prove the stability and convergence of the explicit scheme.

Lemma 6.1. *Let* $\gamma = \tau/h < \sqrt{1/(1-2\theta)}$, $0 \leqslant \theta \leqslant \frac{1}{2}$. *If we define* $\beta = (1+\gamma^2(1-2\theta))/(1-\gamma^2(1-2\theta)) > 1$, *then the following inequality holds*:

$$R_\tau \leqslant \beta Q_\tau \tag{6.1}$$

where

$$Q_\tau = \|U_t^n\|^2 + (1-2\theta)\mathrm{Re}\left\{h\sum_{j=1}^{J}(U_j^n)_x(\overline{U_j^{n+1}})_x\right\},$$

$$R_\tau = \|U_t^n\|^2 + \tfrac{1}{2}(1-2\theta)(\|U_x^{n+1}\|^2 + \|U_x^n\|^2),$$

and U_j^n *are complex mesh functions.*

Proof. *Case* 1: When U_j^n are real mesh functions, the Lemma 6.1 is proved by Luming Zhang in paper [17].

Case 2: When U_j^n are complex mesh functions, we can obtain

$$\begin{aligned}
&\|U_t^n\|^2+(1-2\theta)\mathrm{Re}\left\{h\sum_{j=1}^{J}(U_j^n)_x(\overline{U_j^{n+1}})_x\right\}\\
&\quad=\|\mathrm{Re}\{U_t^n\}\|^2+(1-2\theta)h\sum_{j=1}^{J}\mathrm{Re}\{(U_j^n)_x\}\mathrm{Re}\{(U_j^{n+1})_x\}\\
&\qquad+\|\mathrm{Im}\{U_t^n\}\|^2+(1-2\theta)h\sum_{j=1}^{J}\mathrm{Im}\{(U_j^n)_x\}\mathrm{Im}\{(U_j^{n+1})_x\}\\
&\quad\geqslant\frac{1}{\beta}[\|\mathrm{Re}\{U_t^n\}\|^2+\tfrac{1}{2}(1-2\theta)\|\mathrm{Re}\{U_x^n\}\|^2+\|\mathrm{Re}\{U_j^{n+1}\}\|^2]\\
&\qquad+\frac{1}{\beta}[\|\mathrm{Im}\{U_t^n\}\|^2+\tfrac{1}{2}(1-2\theta)\|\mathrm{Im}\{U_x^n\}\|^2+\|\mathrm{Im}\{U_j^{n+1}\}\|^2]\\
&\quad=\frac{1}{\beta}[\|U_t^n\|^2+\tfrac{1}{2}(1-2\theta)\|U_x^n\|^2+\|U_x^{n+1}\|^2]. \qquad (6.2)
\end{aligned}$$

This completes the proof. □

Lemma 6.2 (*Chang and Jiang [4], Chang and Guo [3]*). *Let* $\gamma=\tau/h<\sqrt{1/(1-2\theta)}$, $0\leqslant\theta\leqslant\frac{1}{2}$. *If we define* $\beta=(1+\gamma^2(1-2\theta))/(1-\gamma^2(1-2\theta))>1$, *then the following inequality holds*:

$$\widetilde{R}_\tau\leqslant\beta\widetilde{Q}_\tau,, \qquad (6.3)$$

where

$$\widetilde{Q}_\tau=\|V_t^n\|^2+(1-2\theta)h\sum_{j=1}^{J}N_j^n\cdot N_j^{n+1},$$

$$\widetilde{R}_\tau=\|V_t^n\|^2+\tfrac{1}{2}(1-2\theta)(\|N^{n+1}\|^2+\|N^n\|^2),$$

and N_j^n, V_j^n *are real mesh functions.*

Theorem 6.1. *Let* $\gamma=\tau/h<1$, $0\leqslant\theta\leqslant\frac{1}{2}$. *If* $U_0(x)\in H^1$, $U_1(x)\in L^2$, $N_0(x)\in H^1$, $N_1(x)\in L^2$, *then the following estimates hold*:

$$\begin{aligned}
&\|U_t^n\|_2\leqslant C,\quad \|U_x^n\|_2\leqslant C,\quad \|U^n\|_2\leqslant C,\quad \|U^n\|_\infty\leqslant C,\\
&\|V_x^n\|_2\leqslant C,\quad \|N^n\|_2\leqslant C,\quad \|U^n\|_4\leqslant C. \qquad (6.4)
\end{aligned}$$

Proof (*By mathematics induction method*). Obviously, from the conditions of Theorem 6.1 and (5.3), (5.4) we have

$$\begin{aligned}
&\|U_t^{-1}\|_2\leqslant C,\quad \|U_t^0\|_2\leqslant C,\quad \|U_x^0\|_2\leqslant C,\quad \|U^0\|_2\leqslant C,\\
&\|U^0\|_\infty\leqslant C,\quad \|V_x^0\|_2\leqslant C,\quad \|N^0\|_2\leqslant C,\quad \|U^0\|_4\leqslant C, \qquad (6.5)
\end{aligned}$$

i.e., Theorem 6.1 holds when $n=0$.

Assume that Theorem 6.1 holds when $n=k$, i.e.,

$$\begin{aligned}
&\|U_t^{k-1}\|_2\leqslant C,\quad \|U_x^k\|_2\leqslant C,\quad \|U^k\|_2\leqslant C,\quad \|U^k\|_\infty\leqslant C,\\
&\|V_x^k\|_2\leqslant C,\quad \|N^k\|_2\leqslant C,\quad \|U^k\|_4\leqslant C. \qquad (6.6)
\end{aligned}$$

According to Theorem 5.1 and noticing

$$\left|\frac{1}{2}h\sum_{j=1}^{J} N_j^n|U_j^{n+1}|^2\right| \leqslant \frac{1}{4}h\sum_{j=1}^{J}((N_j^n)^2+|U_j^{n+1}|^4|) \leqslant \frac{1}{4}(2\|N^n\|_2^2+\frac{1}{2}\|U^{n+1}\|_4^4),$$

and

$$\left|\frac{1}{2}h\sum_{j=1}^{J} N_j^{n+1}|U_j^n|^2\right| \leqslant \frac{1}{4}\left(\frac{1}{2\beta}\|N^{n+1}\|_2^2+2\beta\|U^n\|_4^4\right),$$

we have

$$\frac{1}{\beta}(\|U_t^k\|_2^2+\tfrac{1}{2}(\|U_x^{k+1}\|^2+\|U_x^{k+1}\|^2))+\frac{1}{2}(\|U^{k+1}\|_2^2+\|U^k\|_2^2)+\frac{1}{2\beta}\left(\|V_x^k\|_2^2\right.$$
$$\left.+\frac{1}{4}(\|N^{k+1}\|^2+\|N^k\|^2)\right)+\frac{1}{8}(\|U^{k+1}\|_4^4+\|U^k\|_4^4)\leqslant C+\frac{1}{2}\|N^k\|^2+\frac{\beta}{2}\|U^k\|^2. \tag{6.7}$$

According to (6.6), (6.7) and Lemma 4.1, we obtain

$$\begin{aligned}&\|U_t^k\|_2\leqslant C, \quad \|U_x^{k+1}\|_2\leqslant C, \quad \|U^{k+1}\|_2\leqslant C, \quad \|U^{k+1}\|_\infty\leqslant C,\\ &\|V_x^{k+1}\|_2\leqslant C, \quad \|N^{k+1}\|_2\leqslant C, \quad \|U^{k+1}\|_4\leqslant C.\end{aligned} \tag{6.8}$$

Then for any $n\in\{0,1,2,\ldots,N\}$,

$$\begin{aligned}&\|U_t^n\|_2\leqslant C, \quad \|U_x^n\|_2\leqslant C, \quad \|U^n\|_2\leqslant C, \quad \|U^n\|_\infty\leqslant C,\\ &\|V_x^n\|_2\leqslant C, \quad \|N^n\|_2\leqslant C, \quad \|U^n\|_4\leqslant C.\end{aligned} \tag{6.9}$$

This completes the proof of Theorem 6.1. □

Theorem 6.2. *Assume that the conditions of Theorem* 6.1 *are satisfied. Then the solution of the difference problem* (5.1)–(5.4) *conditionally converges to the solution of the problem* (1.1)–(1.4) *with order* $O(\tau^2+h^2)$ *in the* l_∞ *norm for* U^n, *and in the* l_2 *norm for* N^n.

The theorem can be proved in the same way as that used to prove Theorem 4.3. However, some modifications must be made to estimate the term associate with $\mathrm{Re}\{h\sum_{j=1}^{J}(e_j^n)_x(\overline{e_j^{n+1}})_x\}$ and $h\sum_{j=1}^{J}\eta_j^n\cdot\eta_j^{n+1}$, which are one of the major contributions of this paper. The $\mathrm{Re}\{h\sum_{j=1}^{J}(e_j^n)_x(\overline{e_j^{n+1}})_x\}$ and $h\sum_{j=1}^{J}\eta_j^n\cdot\eta_j^{n+1}$ can be estimated by using the two inequalities in Lemmas 6.1 and 6.2 of this section, and the rest of the proof will proceed similarly as the Theorem 4.3. The details are omitted.

Theorem 6.3. *Under the conditions of Theorem* 6.1, *the solution of the difference scheme* (5.1)–(5.4) *is conditionally stable for initial data.*

7. Numerical experiments

In this section we will report some results of numerical computations using the two finite difference schemes proposed in the previous sections. Therefore, we should first rewrite the (2.1)–(2.4) and the (5.1)–(5.4) as two systems which are suitable for computing. For convenience, we denote (2.1)–(2.4) and (5.1)–(5.4) as Schemes I and II, respectively.

Scheme I: System (2.1) and (2.3) of difference equations is a system of transcendental equations and can be written into a tri-diagonal system, which can be solved by means of the Thomas-iteration method. The formula of iteration for solving (2.1) and (2.3) is

$$-\frac{1}{2}r^2U_{j-1}^{n+1(s+1)}+\left(1+r^2+\frac{\tau^2}{2}+\frac{1}{4}\tau^2(N_j^{n+1(s)}+N_j^{n-1})\right)U_j^{n+1(s+1)}-\frac{1}{2}r^2U_{j+1}^{n+1(s+1)}=p_j^{n+1(s)},$$
$$1\leqslant j\leqslant J-1,\quad n\geqslant 1, \tag{7.1}$$

$$U_0^{n+1(s+1)}=U_J^{n+1(s+1)}=0,\quad n\geqslant 0, \tag{7.2}$$

$$U_j^1=U_j^0+\tau u_1(x_j),\quad U_j^0=u_0(x_j), \tag{7.3}$$

where

$$p_j^{n+1(s)}=\frac{1}{2}\tau^2(U_j^{n-1})_{x\bar{x}}+2U_j^n-\left(1+\frac{\tau^2}{2}+\frac{\tau^2}{2}(N_j^{n+1(s)}+N_j^{n-1})\right)U_j^{n-1}$$
$$-\frac{\tau^2}{4}(|U_j^{n+1(s)}|^2+|U_j^{n-1}|^2)(U_j^{n+1(s)}+U_j^{n-1}),$$
$$1\leqslant j\leqslant J-1,\quad n\geqslant 1, \tag{7.4}$$

System (2.2) and (2.4) of difference equations also can be written as a tri-diagonal system, which can be solved by the Thomas-iteration method too. The formula of iteration for solving the system (2.2) and (2.4) is

$$-\tfrac{1}{2}r^2N_{j-1}^{n+1(s+1)}+(1+r^2)N_j^{n+1(s+1)}-\tfrac{1}{2}r^2N_{j+1}^{n+1(s+1)}=q_j^{n(s)},\quad 1\leqslant j\leqslant J-1,\quad n\geqslant 1, \tag{7.5}$$

$$N_0^{n+1}=N_J^{n+1}=0,\quad n\geqslant 0, \tag{7.6}$$

$$N_j^1=N_j^0+\tau m_1(x_j),\quad N_j^0=m_0(x_j), \tag{7.7}$$

where

$$q_j^{n(s)}=\frac{\tau^2}{2}(|U_j^{n+1(s)}|^2+|U_j^{n-1}|^2)_{x\bar{x}}+\frac{\tau^2}{2}(N_j^{n-1})_{x\bar{x}}+2N_j^n-N_j^{n-1},\quad 1\leqslant j\leqslant J-1,\quad n\geqslant 1. \tag{7.8}$$

Scheme II: Scheme (5.1)–(5.4) is an explicit scheme which can be rewritten as follows:

$$U_j^{n+1(s+1)}=\left[1+\frac{\tau^2}{2}(1+N_j^n+\frac{1}{2}(|U_j^{n+1(s)}|^2+|U_j^{n-1}|^2))\right]^{-1}\left[2U_j^n-U_j^{n-1}-\frac{\tau^2}{2}(1+N_j^n\right.$$
$$\left.+\frac{1}{2}(|U_j^{n+1(s)}|^2+|U_j^{n-1}|^2))U_j^{n-1}+r^2(U_{j+1}^n-2U_j^n+U_{j-1}^n)\right],\quad 1\leqslant j\leqslant J-1,\quad n\geqslant 1, \tag{7.9}$$

$$N_j^{n+1}=2N_j^n-N_j^{n-1}+r^2(N_{j+1}^n-2N_j^n+N_{j-1}^n+|U_{j+1}^n|^2-2|U_j^n|^2+|U_{j-1}^n|^2),$$
$$1\leqslant j\leqslant J-1,\quad n\geqslant 1, \tag{7.10}$$

$$U_0^n=U_J^n=0,\quad N_0^{n+1}=N_J^{n+1}=0,\quad n\geqslant 0, \tag{7.11}$$

$$U_j^1=U_j^0+\tau u_1(x_j),\quad U_j^0=u_0(x_j),\quad N_j^1=N_j^0+\tau m_1(x_j),\quad N_j^0=m_0(x_j). \tag{7.12}$$

In the implementation, we control the $U_j^{n(s+1)}-U_j^{n(s)}\leqslant 10^{-8}$, $N_j^{n(s+1)}-N_j^{n(s)}\leqslant 10^{-8}$ and let $U_j^{n+1(0)}=U_j^n$.

Table 1
The numerical and exact solutions of $u(x,t)$ at $t=1$ (Scheme II)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(0,1)$	$(10,1)$
(0.1, 0.1)	−1.6367e − 006–1.0265e − 006i	0.3062–0.4770i	1.1882e − 005 + 7.7813e − 006i
(0.05, 0.05)	−1.6291e − 006–1.0267e − 006i	0.3061–0.4767i	1.1898e − 005 + 7.7710e − 006i
The exact solution	−1.6258e − 006–1.0267e − 006i	0.3060–0.4766i	1.1898e − 005 + 7.7666e − 006i

To test our two difference schemes, we consider the following simple initial and boundary value problem:

$$u_{tt}-u_{xx}+u+mu+|u|^2u=0,\quad -20<x<20,\ \ 0<t\leqslant 1,$$

$$m_{tt}-m_{xx}=(|u|^2)_{xx},\quad -20<x<20,\ \ 0<t\leqslant 1,$$

$$u(-20,t)=u(20,t)=0,\quad 0\leqslant t\leqslant 1,$$

$$m(-20,t)=m(20,t)=0,\quad 0\leqslant t\leqslant 1,$$

$$u_0(x)=\frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\cdot\exp\left[\mathrm{i}\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right],\quad -20\leqslant x\leqslant 20,$$

$$m_0(x)=-2\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right),\quad -20\leqslant x\leqslant 20,$$

$$u_1(x)=\frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\cdot\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\cdot\exp\left[\mathrm{i}\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right]$$
$$-\frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\cdot\exp\left[\mathrm{i}\left(\sqrt{\frac{2}{1+\sqrt{5}}}x\right)\right],\quad -20\leqslant x\leqslant 20,$$

$$m_1(x)=-4\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right)\cdot\tanh\left(\sqrt{\frac{1+\sqrt{5}}{2}}x\right),\quad -20\leqslant x\leqslant 20.$$

The exact solution of above problem, which has been derived in [12,11], will be used in our computation for comparison. The solution is

$$u(x,t)=\frac{\sqrt{10}-\sqrt{2}}{2}\,\mathrm{sech}\left(\sqrt{\frac{1+\sqrt{5}}{2}}x-t\right)\cdot\exp\left[\mathrm{i}\left(\sqrt{\frac{2}{1+\sqrt{5}}}x-t\right)\right],\tag{7.13}$$

$$m(x,t)=-2\,\mathrm{sech}^2\left(\sqrt{\frac{1+\sqrt{5}}{2}}x-t\right).\tag{7.14}$$

In order to show the convergence order and the unconditional stability of Scheme I, conditional stability of Scheme II , Let $h=\tau=0.1$, 0.05 and $h=0.05$, $\tau=0.1$, respectively. Tables 1 and 2 give the numerical and exact solutions

Table 2
The numerical and exact solutions of $m(x,t)$ at $t=1$ (Scheme II)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(-5,1)$	$(0,1)$	$(5,1)$	$(10,1)$
(0.1, 0.1)	−9.8991e − 012	−3.3103e − 006	−0.8403	−1.7706e − 004	−5.2944e − 010
(0.05, 0.05)	−9.7422e − 012	−3.2581e − 006	−0.8400	−1.7684e − 004	−5.2875e − 010
The exact solution	−9.6801e − 012	−3.2373e − 006	−0.8399	−1.7675e − 004	−5.2851e − 010

Table 3
The errors of numerical solution of $u(x,t)$ at $t=1$ (Scheme I)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(0,1)$	$(10,1)$
(0.1, 0.1)	−1.1249e − 008 + 3.7928e − 010i	−2.8104e − 004 − 3.3176e − 004i	−1.9035e − 008 + 1.4914e − 008i
(0.05, 0.05)	−2.9223e − 009 + 8.9837e − 012i	−7.4145e − 005 − 8.2816e − 005i	−4.8106e − 009 + 3.7165e − 009i

Table 4
The errors of numerical solution of $u(x,t)$ at $t=1$ (Scheme II)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(0,1)$	$(10,1)$
(0.1, 0.1)	−1.0777e − 008 + 3.7691e − 010i	−2.7989e − 004–3.2741e − 004i	−1.8232e − 008 + 1.4235e − 008i
(0.05, 0.05)	−2.9008e − 009 + 8.9557e − 012i	−7.4085e − 005–8.2227e − 005i	−4.7773e − 009 + 3.6890e − 009i

Table 5
The errors of numerical solution of $m(x,t)$ at $t=1$ (Scheme I)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(-5,1)$	$(0,1)$	$(5,1)$	$(10,1)$
(0.1, 0.1)	−2.2268e − 013	−7.3241e − 008	−2.6012e − 004	−3.1065e − 007	−9.1287e − 013
(0.05, 0.05)	−6.2031e − 014	−2.0752e − 008	−6.8993e − 005	−8.0310e − 008	−2.4103e − 013

Table 6
The errors of numerical solution of $m(x,t)$ at $t=1$ (Scheme II)

$(h,\tau)\backslash(x,t)$	$(-10,1)$	$(-5,1)$	$(0,1)$	$(5,1)$	$(10,1)$
(0.1, 0.1)	−2.1881e − 013	−7.3158e − 008	−2.5876e − 004	−3.0445e − 007	−9.1072e − 013
(0.05, 0.05)	−6.2018e − 014	−2.0738e − 008	−6.8959e − 005	−8.0272e − 008	−2.4015e − 013

at several fixed points. Tables 3–6 give the absolute errors of the numerical solutions U and N of Schemes I and II at some points when $t=1$, respectively. The curves of the solitary waves at various times computed by Scheme II with $h=\tau=0.1$ are given in Figs. 1 and 2, the wave at $t=T$ agrees with the one at $t=0$ quite well, this also demonstrates the accuracy of Scheme I. In fact the Scheme II has the similar results which can be shown by the numerical examples. The curves of the two scheme's errors in $\|\cdot\|_2$ norms are plotted in Figs. 3 and 4, and we can find from Tables 3 and 4 and Figs. 3 and 4 that Scheme II is more accurate than Scheme I. Tables 7 and 8 show the stability of the two scheme, it is easy to see that Scheme I is unconditionally stable but Scheme II is conditionally stable.

It can be seen that the numerical results demonstrate the efficiency and accuracy of the two difference schemes we have proposed.

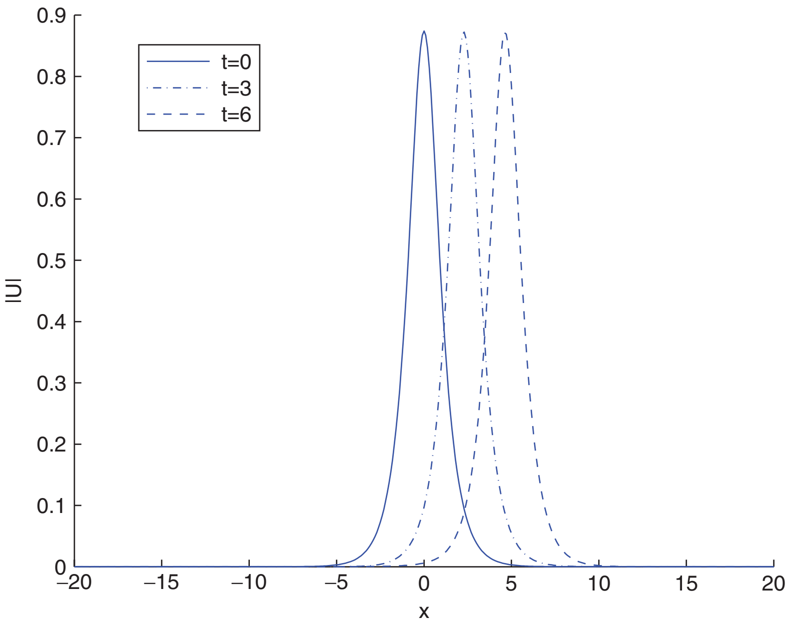

Fig. 1. $|U|$ Computed by Scheme I with $h = \tau = 0.1$.

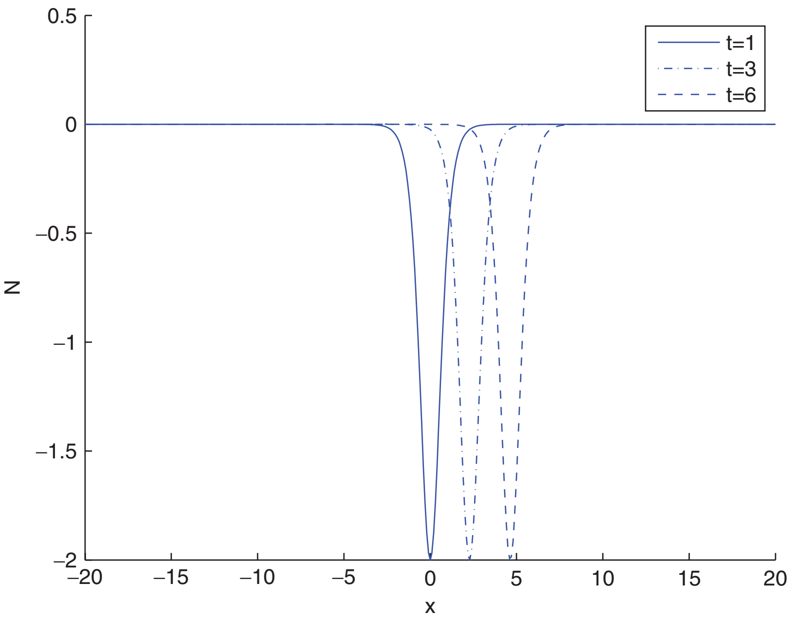

Fig. 2. N Computed by Scheme I with $h = \tau = 0.1$.

8. Conclusion

In this study, we constructed two finite difference schemes for KGZ equation, one (Scheme I) is implicit, and another one (Scheme II) is explicit, but both of them are conservative on discrete energy law. It is shown by the discrete energy method that Scheme I is uniquely solvable, unconditionally convergent and stable, and Scheme II is conditionally convergent and stable. Numerical results demonstrate both of the two schemes are viable.

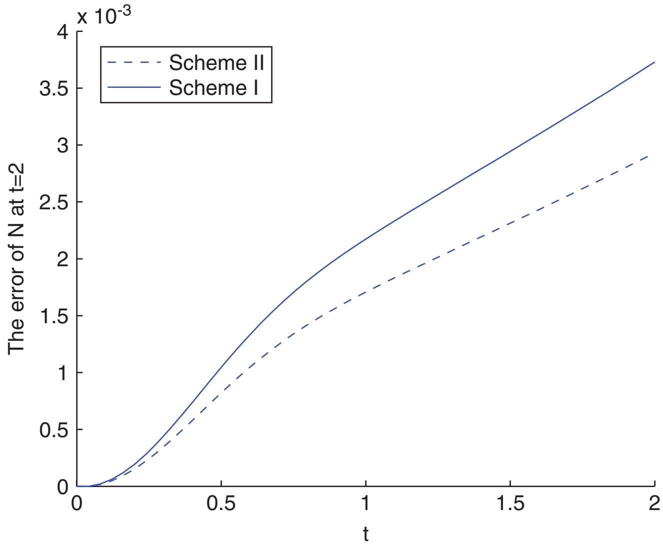

Fig. 3. Error of U in L_2 norm at $t=2$: a comparison for Schemes I and II with $h=\tau=0.04$.

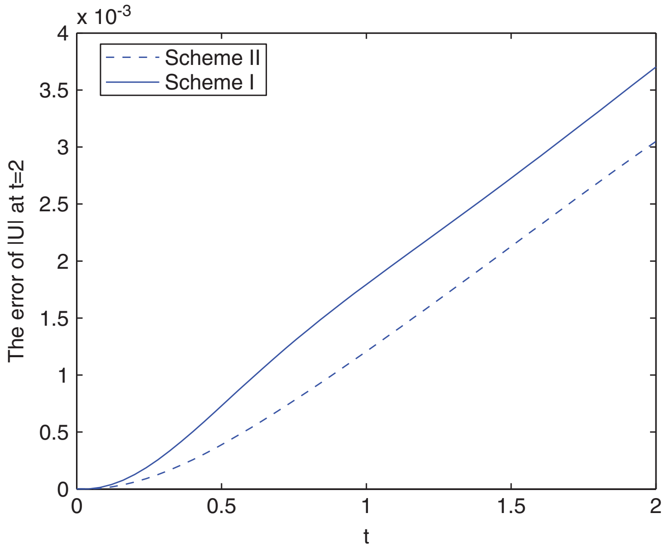

Fig. 4. Error of N in L_2 norm at $t=2$: a comparison for Schemes I and II with $h=\tau=0.04$.

Table 7
The errors of $|U|$ computed by Scheme II at various $r=h/\tau$

$(h,\tau)\backslash t$	1	2	4
(0.1, 0.1)	0.0048	0.0121	0.0196
(0.05, 0.1)	0.0189	1.1024e + 004	1.7165e + 004

Table 8
The errors of $|U|$ computed by Scheme I at various $r = h/\tau$

$(h, \tau)\backslash t$	1	2	4
(0.1, 0.1)	0.0061	0.0147	0.0236
(0.05, 0.1)	0.0041	0.0113	0.0187

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