A STABLE FINITE ELEMENT FOR THE STOKES EQUATIONS

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ABSTRACT - We present in this paper a new velocity-pressure finite element for the computation of Stokes flow. We discretize the velocity field with continuous piecewise linear functions enriched by bubble functions, and the pressure by piecewise linear functions. We show that this element satisfies the usual inf-sup condition and converges with first order for both velocities and pressure. Finally we relate this element to families of higer order elements and to the popular Taylor-Hood element.

1. Introduction.

We consider approximations of the Stokes problem for a viscous incompressible flow. In its simplest form we have to solve

$$
(1.1)\qquad \begin{cases} -\Delta \underset{\sim}{u} + \underset{\sim}{\mathrm{grad}}\, p = \underset{\sim}{f} & \text{in } \Omega, \\ \mathrm{div}\, \underset{\sim}{u} = 0 & \text{in } \Omega, \\ \underset{\sim}{u} = 0 & \text{on } \partial\Omega. \end{cases}
$$

It is well known that the variational formulation of this problem

$$
(1.2)\qquad \sum_{i,j=1}^{2} \int_\Omega \varepsilon_{ij}(\underset{\sim}{u})\, \varepsilon_{ij}(\underset{\sim}{v})\, d\underset{\sim}{x} - \int p\, \mathrm{div}\, \underset{\sim}{v}\, d\underset{\sim}{x} = \int_\Omega \underset{\sim}{f} \cdot \underset{\sim}{v}\, d\underset{\sim}{x} \quad \forall \underset{\sim}{v} \in (H_0^1(\Omega))^2,
$$

$$
\int_\Omega q\, \mathrm{div}\, \underset{\sim}{u}\, d\underset{\sim}{x} = 0 \quad \forall q \in L^2(\Omega)/\mathbf{R}
$$

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is equivalent to a saddle-point problem

$$\inf_{\underset{\sim}{v}} \sup_{q} \left\{ \frac{1}{2} \int_{\Omega} |\underset{\sim}{\varepsilon}(\underset{\sim}{v})|^2 \, d\underset{\sim}{x} - \int_{\Omega} q \operatorname{div} \underset{\sim}{v} \, d\underset{\sim}{x} - \int_{\Omega} \underset{\sim}{f} \cdot \underset{\sim}{v} \, d\underset{\sim}{x} \right\}, \tag{1.3}$$

and that the approximation of this problem is studied in the framework of mixed methods. (Here $\varepsilon_{ij}(\underset{\sim}{u})$ denotes $(\partial_i u_j + \partial_j u_i)/2$). If we look for a discretization by finite elements of (1.2), that is, if we determine $\underset{\sim}{u}_h$, p_h in finite dimensional subspaces V_h, Q_h of $(H_0^1(\Omega))^2$ and $L^2(\Omega)$ respectively from the equations

$$\begin{aligned} &\sum_{i,j=1}^{2} \int_{\Omega} \varepsilon_{ij}(\underset{\sim}{u}_h) \, \varepsilon_{ij}(\underset{\sim}{v}) \, d\underset{\sim}{x} - \int_{\Omega} p_h \operatorname{div} \underset{\sim}{v} \, d\underset{\sim}{x} = \int_{\Omega} \underset{\sim}{f} \cdot \underset{\sim}{v} \, d\underset{\sim}{x} \quad \forall v \in V_h \\ &\int_{\Omega} q \operatorname{div} p_h \, d\underset{\sim}{x} = 0 \quad \forall q \in Q_h \end{aligned} \tag{1.4}$$

then we have to choose V_h and Q_h properly so that the inf-sup condition of the theory of mixed methods is satisfied, that is, so that

$$\inf_{q_h \in Q_h} \sup_{\underset{\sim}{v}_h \in V_h} \frac{\int_{\Omega} q_h \operatorname{div} \underset{\sim}{v}_h \, d\underset{\sim}{x}}{\|\underset{\sim}{v}_h\|_1 \, \|q_h\|_{0/\mathbf{R}}} =: k_h \geqq k_0 > 0. \tag{1.5}$$

Condition (1.5) expresses a compatibility between V_h and Q_h and can be verified only for quite special choices. Many popular finite element methods use discontinuous approximation of the pressure, i.e., $Q_h \not\subset C^0(\Omega)$, (see, e. g., Crouzeix-Raviart [4] Fortin [6]). However one of the most popular approximation schemes, introduced by Taylor and Hood, uses piecewise quadratic velocities and piecewise continuous pressures. Bercovier and Pironneau [1] have shown the convergence of this approximation, although not with optimal order. Verfürth [9] has recently completed the proof to show that optimal order convergence indeed holds. However this line of analysis is quite intricate and cannot be easily extended to other elements.

We show here how it is possible to build elements satisfying (1.5) by a very simple strategy, whenever the pressure field is continuous.

2. The mini element.

Since we know that the continuous problem satisfies the inf-sup condition

$$(2.1) \qquad \inf_{q \in L^2} \sup_{\underset{\sim}{v} \in (H_0^1)^2} \frac{\int_\Omega q \operatorname{div} v \, dx}{\|v\|_1 \, \|q\|_{0/\mathbf{R}}} \geqq k > 0,$$

condition (1.5) can be verified by constructing an operator Π_h: $(H_0^1(\Omega))^2 \rightarrow V_h$ such that

$$(2.2) \qquad \int_\Omega q_h \operatorname{div} (\Pi_h \underset{\sim}{v} - \underset{\sim}{v}) \, dx = 0 \quad \forall q_h \in Q_h \quad \forall \underset{\sim}{v} \in (H_0^1)^2,$$

and

$$(2.3) \qquad \|\Pi_h \underset{\sim}{v}\|_1 \leqq c \, \|\underset{\sim}{v}\|_1 \quad \forall \underset{\sim}{v} \in (H_0^1)^2$$

with c independent of h (cf. Fortin [5]).
If the pressure q_h is continuous, we may integrate (2.2) by parts to get

$$(2.4) \qquad \int_\Omega (\underset{\sim}{v} - \Pi_h \underset{\sim}{v}) \cdot \underset{\sim}{\operatorname{grad}} \, q_h \, \underset{\sim}{dx} = 0 \quad \forall q_h \in Q_h.$$

Hence, if q_h is a polynomial of degree k, on each element T, (2.4) follows from the more general condition

$$(2.5) \qquad \int_T (\underset{\sim}{v} - \Pi_h \underset{\sim}{v}) \cdot \underset{\sim}{\phi}_h \, dx = 0 \quad \forall \underset{\sim}{\phi}_h \in (P_{k-1}(T))^2 \quad \forall T.$$

It is possible to insure (2.5) by including in the velocity space, as necessary, internal degrees of freedom in each element, i e., so called bubble shape functions; The simplest example is the following, which we call MINI.

For the sake of simplicity we suppose that Ω is a convex polygon and we consider a partition $\mathcal{T}_h$ of Ω into triangular elements with the usual minimum angle condition. We define for $k \geqq 1$

$$(2.6) \qquad \begin{aligned} &M_0^k(\mathcal{T}_h) = \{ v \mid v \in C^0(\Omega), \quad v_{|T} \in P_k(T) \quad \forall T \in \mathcal{T}_h \}, \\ &\mathring{M}_0^k(\mathcal{T}_h) = M_0^k(\mathcal{T}_h) \cap H_0^1(\Omega) \end{aligned}$$

and for $k \geqq 3$,

$$B^k(\mathcal{T}_h) = \{v \mid v_{|T} \in P_k(T) \cap H_0^1(T) \quad \forall T \in \mathcal{T}_h\}. \tag{2.7}$$

(For $k=3$, the functions of B^k are those of the form $\alpha(T)\,\lambda_1\lambda_2\lambda_3 =: \alpha(T)\,\phi_T^0$ on each triangle, T, where λ_i are the barycentric coordinates on T and $\alpha(T) \in \mathbf{R}$).

The MINI finite element uses the finite element spaces

$$V_h = (\overset{\circ}{M}{}_0^1)^2 \oplus (B^3)^2 \tag{2.8}$$

$$Q_h = M_0^1. \tag{2.9}$$

In this case condition (2.5) becomes

$$\int_T (\underset{\sim}{v} - \Pi_h \underset{\sim}{v})\, dx = 0 \quad \forall T,\ \forall v \in (H_0^1)^2. \tag{2.10}$$

For this choice of spaces we now construct Π_h: $(H_0^1)^2 \rightarrow V_h$ and verify the conditions (2.3) and (2.10) which imply (1.5). First let $\tilde{\Pi}_h$: $(H_0^1)^2 \rightarrow (\overset{\circ}{M}{}_0^1)^2$ satisfy

$$\sum_T h_T^{2r-2} \|\tilde{\Pi}_h \underset{\sim}{v} - \underset{\sim}{v}\|^2_{r,T} \leqq C\, \|\underset{\sim}{v}\|^2_{1,\Omega} \ (h_T = \operatorname{diam} T), \quad r=0,1. \tag{2.11}$$

Such an operator is constructed for example by Clement [3]. (For smooth $\underset{\sim}{v}$, $\tilde{\Pi}_h \underset{\sim}{v}$ is close to the piecewise linear interpolant v^I, but is defined via local averages rather then point values, which are not defined for general $\underset{\sim}{v} \in (\overset{\circ}{H}{}^1)^2$). To ensure (2.10) we perturb $\tilde{\Pi}_h v$ by the appropriate multiple of the bubble function on each triangle. More precisely we set

$$\Pi_h \underset{\sim}{v} = \tilde{\Pi}_h \underset{\sim}{v} + \underset{\sim}{\alpha}(T)\phi_T^0 \quad \text{on } T, \tag{2.12}$$

with $\underset{\sim}{\alpha}(T)$ given by:

$$\underset{\sim}{\alpha}(T) = \int_T \phi_T^0\, d\underset{\sim}{x} = \int_T (\tilde{\Pi}_h \underset{\sim}{v} - \underset{\sim}{v})\, d\underset{\sim}{x} \tag{2.13}$$

We now verify (2.3). Clearly

$$\|\Pi_h \underset{\sim}{v}\|_{1,T} \leqq \|\tilde{\Pi}_h \underset{\sim}{v}\|_{1,T} + \|\underset{\sim}{\alpha}(T)\,\phi_T^0\|_{1,T}. \tag{2.14}$$

By a simple scaling argument we obtain

(2.15) $$\|\underset{\sim}{\alpha}(T)\,\phi_T{}^0\|_{1,T} \leqq c\,|\underset{\sim}{\alpha}(T)|$$

(2.16) $$|\underset{\sim}{\alpha}(T)| \leqq ch_T{}^{-1}\|\Pi_h\underset{\sim}{v}-\underset{\sim}{v}\|_{0,T}.$$

Using (2.14)-(2.16), summing over T, and using (2.11) we obtain (2.3). We have therefore proved that the MINI element (2.8), (2.9) satisfies the inf-sup condition (1.5). Hence by well-known arguments we have [2].

(2.17) $$\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_1+\|p+p_h\|_{0/\mathbf{R}} \leqq C\inf\{\|\underset{\sim}{u}-\underset{\sim}{v}\|_1+\|p-q\|_{0/\mathbf{R}}\} \leqq Ch\,\|f\|_0$$

where the infimum extends over $v \in V_h$ and $q \in Q_h$, and the constant C is independent of h, and we have used the H^2 regularity for the Stokes problem [8]. Moreover applying the usual Aubin-Nitsche duality argument one can easily prove

$$\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_0+\|p-p_h\|_{-1/\mathbf{R}} \leqq Ch\,(\|\underset{\sim}{u}-\underset{\sim}{u}_h\|_1+\|p-p_h\|_{0/\mathbf{R}}) \leqq Ch^2\,\|f\|_0.$$

3. Possible extensions and remarks.

The element of the previous section can obviously be embedded in a whole family of elements. For instance we may choose, for $k \geqq 1$.

(3.1) $$V_h=(\overset{\circ}{M}_0{}^k(\mathcal{T}_h))^2 \oplus (B^{k+2}(\mathcal{T}_h))^2$$

(3.2) $$Q_h=M_0{}^k(\mathcal{T}_h)$$

The second element ($k=2$) of the family would use P_2 elements enriched by 3 P_4-bubbles for velocities and P_2 continuous pressure. It must be remarked that the choice of Q_h is richer than necessary as far the order of convergence is concerned. We could then consider another family of elements

(3.3) $$V_h=(\overset{\circ}{M}_0{}^k(\mathcal{T}_h))^2 \oplus (B^{k+1}(\mathcal{T}_h))^2$$

(3.4) $$Q_h=M_0{}^{k-1}(\mathcal{T}_h)$$

this time for $k \geq 2$. The first member of this family can be seen as an enriched version of the Taylor Hood element. It must be noted that proving convergence is now much simpler than in the standard Taylor Hood.

Using continuous field can, in practice, be seen as an advantage, the number of degrees of freedom being smaller than for discontinuous pressure elements. For instance in the MINI element we have 3 d. o. f. per vertex plus 2 internal nodes in each element; these last nodes can easily be eliminated by the classical process of static condensation.

On the other hand, discontinuous pressures are apparently more adapted for the use of penalty methods. In such methods, problem (1.4) is usually perturbed for $\sigma>0$ small, into the following system

$$\sum_{i,j=1}^{2}\int_{\Omega}\varepsilon_{ij}(\underset{\sim}{u}_h^{\sigma})\,\varepsilon_{ij}(\underset{\sim}{v})\,d\underset{\sim}{x}-\int_{\Omega}p_h^{\sigma}\operatorname{div}\underset{\sim}{v}\,d\underset{\sim}{x}=\int_{\Omega}\underset{\sim}{f}\underset{\sim}{v}d\underset{\sim}{x}\quad\forall\underset{\sim}{v}\in V_h,\tag{3.5}$$

$$\int_{\Omega}q_h\operatorname{div}\underset{\sim}{u}_h^{\sigma}\,dx+\sigma\int_{\Omega}p_h^{\sigma}\,q_h\,dx=0\quad\forall q_h\in Q_h.\tag{3.6}$$

For discontinuous pressures, the inverse of the «mass» matrix arising from the term $\int_{\Omega}p_h^{\sigma}\,q_h\,d\underset{\sim}{x}$ is local and p_h can be eliminated from the system. For continuous pressures, this inverse is in general a full matrix and this elimination is virtually impossible. It must however be noted that one may replace (3.6) by

$$\int_{\Omega}q_h\operatorname{div}\underset{\sim}{u}_h^{\sigma}\,d\underset{\sim}{x}+\sigma\,(p_h^{\sigma},\,q_h)_h=0\quad q_h\in Q_h\tag{3.7}$$

where $(\cdot,\cdot)_h$ is *any* scalar product on Q_h; in particular this scalar product could be associated with a diagonal matrix so that the elimination of p_h^{σ} can be performed. It is easy to show that if the scalar product $(\cdot,\cdot)_h$ is «properly scaled», that is if

$$(q_h^1,\,q_h^2)_h\leqq c\,||q_h^1||_0\,||q_h^2||_0\quad\forall q_h^1,\,q_h^2\in Q_h\tag{3.8}$$

$$(1,1)_h\geqq c,\ c\text{ independent of }h,\tag{3.9}$$

then we have

$$||\underset{\sim}{u}_h-\underset{\sim}{u}_h^{\sigma}||_1+||p_h-p_h^{\sigma}||_{0/\mathbf{R}}\leqq c\sigma,\ c\text{ independent of }h.\tag{3.10}$$

Indeed, comparing (1.4) with (3.5), (3.7) and using the stability of the solution of (1.4) we get

$$||\underset{\sim}{u}_h-\underset{\sim}{u}_h^{\sigma}||_1+||p_h-p_h^{\sigma}||_{0/\mathbf{R}}\leqq c\,\sigma\sup_{q_h\in Q_h}\frac{(q_h,\,p_h^{\sigma})_h}{||q_h||_0}\leqq c\,\sigma\,||p_h^{\sigma}||_0\,;\tag{3.11}$$

we may now write p_h^{σ} as

$$p_h^{\sigma}=p^{\sigma}{}_{h,0}+\gamma 1\ \text{ with }\gamma\in\mathbf{R}\text{ and }\int_{\Omega}p^{\sigma}{}_{h,0}\,d\underset{\sim}{x}=0.\tag{3.12}$$

Equation (3.7) with $q_h=1$ yields then

$$(3.13) \qquad \sigma\,(p^\sigma_{h,0}+\gamma 1,\, 1)_h=0$$

so that, using (3.8) and (3.9) in (3.13) we obtain

$$(3.14) \qquad |\gamma|=|(p^\sigma_{h,1},\, 1)_h/(1,\, 1)_h\,|\leqq c\,\|p^\sigma_{h,0}\|_0$$

which joined to (3.12) gives

$$(3.15) \qquad \|p_h^\sigma\|_0\leqq c\,\|p_h^\sigma\|_{0/\mathbf{R}}.$$

Hence (3.11) becomes

$$(3.16) \qquad \|\underset{\sim}{u}_h-\underset{\sim}{u}_h^\sigma\|_1+\|p_h-p_h^\sigma\|_{0/\mathbf{R}}\leqq c\,\sigma\,\|p_h^\sigma\|_{0/\mathbf{R}}$$

Note now that (3.16) implies

$$(3.17) \qquad c\sigma\,\|p_h^\sigma\|_{0/\mathbf{R}}\geqq\|p_h-p_h^\sigma\|_{0/\mathbf{R}}\ \geqq\|p_h^\sigma\|_{0/\mathbf{R}}-\|p_h\|_{0/\mathbf{R}}$$

and hence, for σ small enough:

$$(3.18) \qquad \|p_h^\sigma\|_{0/\mathbf{R}}\leqq c\,\|p_h\|_{0/\mathbf{R}}\leqq \text{const}$$

which joined with (3.16) gives the result (3.10).

REMARK. If the scalar product $(\cdot,\cdot)_h$ is such that

$$(3.19) \qquad (q_h^0,\, 1)_h=0 \quad \forall q_h^0\in Q_{h/\mathbf{R}}$$

then it comes from (3.14) that $\gamma=0$, and then $p^\sigma_{h,0}=p_h^\sigma$: hence p_h^σ itself will have zero mean value and the previous proof can be simplified. In its turn (3.19) will be satisfied, for instance, if $(\cdot,\cdot)_h$ corresponds to a quadrature formula which is exact for functions of Q_h. This is the case with P_1-continuous pressure if the scalar product $\int_T p_h\, q_h\, d\underset{\sim}{x}$ is approximated by $\frac{\text{area}\,(T)}{3}\sum_{i=1}^{3} p_h\,(a_i)\, q_h\,(a_i)$ where a_i are the vertices of T.

REMARK. A disadvantage of the continuous pressure field is that, after the elimination of p_h in (3.5), (3.7), the resulting matrix in the u_h unknowns has a *larger bandwidth*. However we think that in the MINI element the total number of degrees of freedom is so small that this drawback is not serious. Suitable algorithms for numerical treatment of this type of discretizations can be found in [7].

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