Multi-symplectic structure, Inverse problem of the calculus of variations, Degasperis-Procesi equation

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1 Multi-symplectic structure

Various important PDEs can be written in the multi-symplectic formulation

$$Mz_t + Kz_x = \nabla_z S(z), \tag{1.1}$$

where M and K are skew-symmetric matrices, S is a given smooth function of z. Equations of this form have property that symplecticity is conserved

$$\omega_t + \kappa_x = 0 \quad \text{with} \quad \omega = \frac{1}{2}dz \wedge Mdz, \quad \kappa = \frac{1}{2}dz \wedge Kdz,$$

ans when S does not depend explicitly on t and x, energy and momentum are conserved

$$E_t + F_x = 0, \quad E(z) = S(z) - \frac{1}{2}z^T Kz_x, \quad F(z) = \frac{1}{2}z^T Kz_t,$$

$$I_t + G_x = 0, \quad G(z) = S(z) - \frac{1}{2}z^T Mz_t, \quad I(z) = \frac{1}{2}z^T Mz_x.$$

For example,

Sine-Gordon equation:

$$u_{tt} - u_{xx} + \sin(u) = 0, \tag{1.2}$$

$$M = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

$$S(z) = \frac{1}{2}v^2 - \frac{1}{2}w^2 - \cos(u).$$

KdV equation:

$$u_t + uu_x + u_{xxx} = 0, \tag{1.3}$$

$$M = \begin{pmatrix} 0 & 1/2 & 0 & 0 \\ -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$S(z) = \frac{1}{2}v^2 - uw + \frac{1}{6}u^3, \;\; z = (\psi, u, v, w)^T.$$

Camassa-Holm equation:

$$u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1.4}$$

$$M = \begin{pmatrix} 0 & 1/2 & 0 & 0 & -1/2 \\ -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$S(z) = -wu - u^3/2 - u\nu^2/2 + \nu v, \;\; z = (u, \psi, w, v, \nu)^T.$$

2 Multi-symplectic scheme

When designing a geometric integrator for (1.1), the principle requirement will be that the discretization conserves symplecticity. It is not possible in general to exactly conserve energy and momentum as well in a uniform discretization, but how closely energy and momentum are conserved will be a property of interest. Bridges and Reich define a numerical scheme as a multi-symplectic scheme if the scheme preserves a discrete multi-symplectic conservation law. There are two conventional multi-symplectic schemes,

Preissmann scheme:

$$M\delta_t^+ z_{i+1/2}^j + K\delta_x^+ z_i^{j+1/2} = \nabla_z S(z_{i+1/2}^{j+1/2}), \tag{2.1}$$

where

$$z_i^{j+1/2} = \frac{1}{2}(z_i^j + z_i^{j+1}), \quad z_{i+1/2}^j = \frac{1}{2}(z_i^j + z_{i+1}^j), \quad z_{i+1/2}^{j+1/2} = \frac{1}{2}(z_i^{j+1/2} + z_{i+1/2}^j),$$

Euler-box scheme:

$$M_+\delta_t^+ z_i^j + M_-\delta_t^- z_i^j + K_+\delta_x^+ z_i^j + K_-\delta_x^- z_i^j = \nabla_z S(z_i^j), \tag{2.2}$$

where

$$\begin{aligned} &M = M_+ + M_-, \quad M_+^T = -M_-, \\ &K = K_+ + K_-, \quad K_+^T = -K_-. \end{aligned}$$

There are also some other multi-symplectic schemes, for example, multi-symplectic variational integrators, multi-symplectic Runge-Kutta schemes, multi-symplectic Fourier spectral or pseudospectral methods, etc.

3 The inverse problem of the calculus of variations

The inverse problem of the calculus of variations, i.e., the existence and formulation of variational principles for systems of nonlinear partial differential equations.

3.1 Potential operator

Let N be an operator on the Banach space E. We shall introduce the concept of a Gâteau derivative of the operator N. If for some $u \in E$,

$$\lim_{\epsilon \to 0} \frac{N(u + \epsilon\psi) - N(u)}{\epsilon} = VN(u, \psi)$$

exists, then the operator $VN(u, \psi)$ is called the Gâteau derivative. The Gâteau derivative is, in general, not linear in ψ. A sufficient condition for linearity in ψ is that the operator $VN(u,\psi)$ be uniformly continuous in ψ. A linear uniformly continuous Gâteau differential is a Fréchet differential. We shall give an operational formula for the Fréchet derivative,

$$N_u^{'}\psi = \lim_{\epsilon\to 0}\frac{N(u+\epsilon\psi)-N(u)}{\epsilon} = \frac{\partial}{\partial\epsilon}N(u+\epsilon\psi)|_{\epsilon=0}, \tag{3.1}$$

$N_u^{'}$ is the Fréchet derivative of the operator N, while $N_u^{'}\psi$ is the Fréchet derivative of the operator in the direction ψ.

If $N : E \to R$ is a functional, and $E = \mathcal{F}(R^3)$ is a functional space on R^3, then the above definition can be written as follows,

$$\delta N = \lim_{\epsilon\to 0}\frac{N(u+\epsilon\delta u)-N(u)}{\epsilon} = \int \frac{\delta N}{\delta u}\delta u d^3x.$$

The operator $N : E \to R$, is a potential operator if there exists a functional (potential) F on E, s.t.,

$$\delta F = \lim_{\epsilon\to 0}\frac{F(u+\epsilon\delta u)-F(u)}{\epsilon} = \int N(u)\delta u d^3x, \tag{3.2}$$

and then the potential is given by

$$F = \int u\int_0^1 N(\lambda u)d\lambda d^3x. \tag{3.3}$$

According to the notation above, $N = \dfrac{\delta F}{\delta u}$, and the inverse problem for the PDEs is to find the functional F, s.t., $N = 0$ is the Euler-Lagrange equation equivalent to the original PDEs.

If we write the functional $F(u)$ in the following form

$$F(u) = \int L(u)d^3x,$$

where u may be a vector of functions, then the function $L(u)$ is commonly called the Lagrangian.

3.1.1 Symmetry condition

To determine the potentialness of an operator N, one must first compute the Fréchet derivative using (3.1), then check the symmetry requirement. That is

$$\int \psi N_u^{'}(\phi)d^3x = \int \phi N_u^{'}(\psi)d^3x,$$

which requires that

$$N_u^{'} = \tilde{N}_u^{'}, \tag{3.4}$$

where $\tilde{N}_u^{'}$ is the adjoint operator of $N_u^{'}$.

3.2 Examples

3.2.1 Sine-Gordon equation

Consider the sine-Gordon equation (1.2)

$$N(u) = u_{tt} - u_{xx} + \sin(u) = 0.$$

We first check the symmetry condition (3.4),

$$N_u^{'}\delta u = \frac{\partial}{\partial \epsilon}(N(u + \epsilon \delta u)) \mid_{\epsilon=0} = \delta u_{tt} - \delta u_{xx} + \cos(u)\delta u,$$

$$\tilde{N}_u^{'}\delta u = \delta u_{tt} - \delta u_{xx} + \cos(u)\delta u.$$

Thus the operator N is symmetric. Using the formula (3.3), we can get the potential $F(u)$

$$\begin{aligned} F(u) &= \int u \int_0^1 N(\lambda u) d\lambda dx dt \\ &= \int u \int_0^1 \lambda u_{tt} - \lambda u_{xx} + \sin(\lambda u) d\lambda dx dt \\ &= \int u \Big[\frac{1}{2}\lambda^2 u_{tt} - \frac{1}{2}\lambda^2 u_{xx} - \frac{1}{u}\cos(\lambda u)\Big]_0^1 dx dt \\ &= \int (\frac{1}{2} u u_{tt} - \frac{1}{2} u u_{xx} - \cos(u) + 1) dx dt. \end{aligned}$$

Using the integrations by parts, we can derive the common potential

$$F(u) = \int (\frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \cos(u) + 1) dx dt,$$

and the Lagrangian

$$L(u, u_t, u_x) = \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \cos(u) + 1. \tag{3.5}$$

The Euler-Lagrange equation of $F(u)$ is given by

$$\frac{\partial L}{\partial u} - \frac{\partial}{\partial t}\frac{\partial L}{\partial u_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial u_x} = 0$$

i.e.,

$$u_{tt} - u_{xx} + \sin(u) = 0,$$

which is nothing other than the sine-Gordon equation (1.2).

3.2.2 KdV equation

Consider the KdV equation (1.3)

$$N(u) = u_t + uu_x + u_{xxx} = 0.$$

Similar procedures as Example 1, we can obtain

$$N_u^{'}\delta u = \delta u_t + (u\delta u)_x + \delta u_{xxx}, \tag{3.6}$$

$$\tilde{N}_u^{'}\delta u = -\delta u_t - u\delta u_x - \delta u_{xxx}. \tag{3.7}$$

Obviously $N_u^{'} \neq \tilde{N}_u^{'}$ and no potential exists. However, we will deduce an alternate potential for it using a simple transformation in the next section.

3.3 The alternate potential

As we have already indicated in the above, the set of potential operators is quit small relative to the set of nonlinear equations. Next, we will provide another means of formulating a variational principle, the alternate potential.

Note that the Fréchet derivative and its adjoint, Eqs (3.6) and (3.7) have the following similarity: with a sign change, they are nearly identical. For even derivative, the sign does not change. We thus propose the following transformation,

$$u = \psi_x,$$

Then the KdV equation becomes

$$N(\psi) = \psi_{xt} + \psi_x\psi_{xx} + \psi_{xxxx} = 0. \tag{3.8}$$

The equation for ψ is now fourth order, and the Fréchet derivative is give by

$$N_\psi^{'}\delta\psi = \delta\psi_{xt} + \delta\psi_x\psi_{xx} + \psi_x\delta\psi_{xx} + \delta\psi_{xxxx} = \tilde{N}_u^{'}\delta\psi.$$

Thus (3.8) is a potential operator and we can compute its potential

$$F(\psi) = \int \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 - \frac{1}{2}\psi_{xx}^2 \, dxdt,$$

and the Lagrangian

$$L(\psi_t, \psi_x, \psi_{xx}) = \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 - \frac{1}{2}\psi_{xx}^2. \tag{3.9}$$

The Euler-Lagrange equation of $F(\psi)$ is

$$\begin{aligned} -\frac{\partial}{\partial t}\frac{\partial L}{\partial \psi_t} - \frac{\partial}{\partial x}\frac{\partial L}{\partial \psi_x} + \frac{\partial}{\partial_{xx}}\frac{\partial L}{\partial \psi_{xx}} &= -\frac{1}{2}\psi_{xt} - \frac{1}{2}\psi_{xt} - \psi_x\psi_{xx} - \psi_{xxxx} \\ &= -u_t - uu_x - u_{xxx} = 0, \end{aligned}$$

which is the KdV equation (1.3) after applying -1 on both sides of the equation.

4 Derivation of the multi-symplectic structure

In this section, we will give an approach to derive the multi-symplectic struction (1.1). We first introduce some notations of the multi-symplectic geometry.

4.1 Notations and formulations

Let X be an $n+1$ dimensional manifold (which in applications is usually spacetime) and let Y be a N dimensional fiber bundle over X. Coordinates on X are denoted $x^\mu, \mu = 1, 2, \cdots, n, 0$, and fiber coordinates on Y are denoted by $y^A, A = 1, 2, \cdots, N$. These induce coordinates y^A_μ on the fibers of $J^k(Y)$ which is the k^{th} jet bundle of Y.

Remark 4.1. $y^A_\mu = y^A_{\mu_1\cdots\mu_s} = \partial_{\mu_1\cdots\mu_s} y^A,\ 1 \leq s \leq k.$

Consider the Lagrangian density $\mathcal{L} : J^k(Y) \to \Lambda^{n+1}(X)$, where $\Lambda^{n+1}(X)$ is the bundle of $(n+1)$-forms on X. In coordinates, we write

$$\mathcal{L}(\gamma) = L(x^\mu, y^A, y^A_\mu) d^{n+1}x,$$

where $\gamma \in J^k(Y)_y$.

The covariant Legendre transformation is given in coordinates by

$$\begin{cases} p_A^{\mu_1\cdots\mu_k} = \dfrac{\partial L}{\partial y^A_{\mu_1\cdots\mu_k}}, \\ \quad\vdots \\ p_A^{\mu_1\cdots\mu_s} = \dfrac{\partial L}{\partial y^A_{\mu_1\cdots\mu_s}} - D_\nu p_A^{\mu_1\cdots\mu_s\nu}, \\ \quad\vdots \\ p_A^{\mu_1} = \dfrac{\partial L}{\partial y^A_{\mu_1}} - D_\nu p_A^{\mu_1\nu}, \\ H = L - p_A^{\mu_1} y^A_{\mu_1} - \cdots - p_A^{\mu_1\cdots\mu_k} y^A_{\mu_1\cdots\mu_k}. \end{cases} \tag{4.1}$$

Notice that formally, the last of these equations defines the (negative of the) energy while the others are reminiscent of the usual relation $p_i = \dfrac{\partial L}{\partial \dot{q}_i}$ from classical mechanics.

Finally, we can obtain the multi-symplectic structure (1.1) from the De Donder-Weyl equations

$$\begin{cases} \dfrac{\partial p_A^{\mu_1\cdots\mu_s\nu}}{\partial x^\nu} = \dfrac{\partial H}{\partial y^A_{\mu_1\cdots\mu_s}}, \\ \dfrac{\partial y^A_{\mu_1\cdots\mu_s}}{\partial x^\nu} = -\dfrac{\partial H}{\partial p_A^{\mu_1\cdots\mu_s\nu}}, \quad 1 \leq s \leq k-1. \end{cases} \tag{4.2}$$

4.2 Examples

4.2.1 Sine-Gordon equation

The Lagrangian of the sine-Gordon equation is defined by (3.5). Taking the covariant Legendre transformation yields

$$\begin{cases} p^x = \dfrac{\partial L}{\partial u_x} = u_x, \\ p^t = \dfrac{\partial L}{\partial u_t} = -u_t, \end{cases} \Longrightarrow \begin{cases} u_x = p^x, \\ u_t = -p^t. \end{cases}$$

The Hamiltonian is given by

$$\begin{aligned} H(u, p^t, p^x) &= L - p^t u_t - p^x u_x \\ &= \frac{1}{2}u_x^2 - \frac{1}{2}u_t^2 - \cos(u) + 1 - p^t u_t - p^x u_x \\ &= \frac{1}{2}(p^x)^2 - \frac{1}{2}(p^t)^2 - \cos(u) + 1 + (p^t)^2 - (p^x)^2 \\ &= \frac{1}{2}(p^t)^2 - \frac{1}{2}(p^x)^2 - \cos(u) + 1. \end{aligned}$$

The De Donder-Weyl equations satisfy

$$\begin{cases} \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial H}{\partial u}, \\ -\dfrac{\partial u}{\partial t} = \dfrac{\partial H}{\partial p^t}, \\ -\dfrac{\partial u}{\partial x} = \dfrac{\partial H}{\partial p^x}. \end{cases}$$

We can arrange the three equations into the multi-symplectic structure (1.1) with the matrices M and K defined as follows

$$M = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} u \\ p^t \\ p^x \end{pmatrix},$$

which is a little different from the regular multi-symplectic structure. But if we let $v = -p^t, w = p^x$, the matrices become the same as the regular ones.

4.2.2 KdV equation

Taking the covariant Legendre transformation of Lagrangian (3.9) of the KdV equation yields

$$\begin{cases} p^{xx} = \dfrac{\partial L}{\partial \psi_{xx}} = -\psi_{xx}, \Longrightarrow \psi_{xx} = -p^{xx}, \\ p^t = \dfrac{\partial L}{\partial \psi_t} = \dfrac{1}{2}\psi_x, \\ p^x = \dfrac{\partial L}{\partial \psi_x} - D_x p^{xx} = \dfrac{1}{2}\psi_t + \dfrac{1}{2}\psi_x^2 + \psi_{xxx}. \end{cases} \tag{4.3}$$

The Hamiltonian is given by

$$
\begin{aligned}
H(\psi,\psi_t,\psi_x,p^t,p^x,p^{xx}) &= L - p^t\psi_t - p^x\psi_x - p_{xx}\psi_{xx} \\
&= \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 - \frac{1}{2}\psi_{xx}^2 - p^t\psi_t - p^x\psi_x - p^{xx}\psi_{xx} \\
&= \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 - \frac{1}{2}(p^{xx})^2 - p^t\psi_t - p^x\psi_x + (p^{xx})^2 \\
&= \frac{1}{2}\psi_t\psi_x + \frac{1}{6}\psi_x^3 + \frac{1}{2}(p^{xx})^2 - p^t\psi_t - p^x\psi_x.
\end{aligned}
$$

The De Donder-Weyl equations yield

$$
\begin{cases}
\dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial H}{\partial \psi}, & -\dfrac{\partial \psi}{\partial t} = \dfrac{\partial H}{\partial p^t}, \\
0 = \dfrac{\partial H}{\partial \psi_t}, & -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial H}{\partial p^x}, \\
\dfrac{\partial p^{xx}}{\partial x} = \dfrac{\partial H}{\partial \psi_x}, & -\dfrac{\partial \psi_x}{\partial x} = \dfrac{\partial H}{\partial p^{xx}}.
\end{cases}
$$

We can also arrange the equations into the multi-symplectic structure (1.1) with the matrices M and K defined as follows

$$
M = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \quad
K = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0
\end{pmatrix},
$$

$$
z = (\psi,\psi_t,\psi_x,p^t,p^x,p^{xx})^T.
$$

The variables of this structure are two more than the regular one. Note that ψ_t and ψ_x in (4.3) can be expressed as functions with respect to p^t and p^x as follows

$$
\begin{cases}
\psi_x = 2p^t, \\
\psi_t = 2p^x - (2p^t)^2 + 2p^{xx}_x.
\end{cases} \tag{4.4}
$$

Then the Hamiltonian may be changed as

$$
\begin{aligned}
\tilde{H}(\psi,p^t,p^x,p^{xx}) &= L - p^t\psi_t - p^x\psi_x - p^{xx}\psi_{xx} \\
&= \frac{1}{2}(2p^x - (2p^t)^2 + 2p^{xx}_x)(2p^t) + \frac{1}{6}(2p^t)^3 - \frac{1}{2}(p^{xx})^2 \\
&\quad - p^t(2p^x - (2p^t)^2 + 2p^{xx}_x) - p^x(2p^t) + (p^{xx})^2 \\
&= -2p^tp^x + \frac{1}{2}(p^{xx})^2 + \frac{4}{3}(p^t)^3.
\end{aligned}
$$

Then the De Donder-Weyl equations yield

$$\begin{cases} \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial \tilde{H}}{\partial \psi}, \\ \qquad\quad -\dfrac{\partial \psi}{\partial t} = \dfrac{\partial \tilde{H}}{\partial p^t}, \\ \qquad\quad -\dfrac{\partial \psi}{\partial x} = \dfrac{\partial \tilde{H}}{\partial p^x}, \\ \qquad\qquad\quad 0 = \dfrac{\partial \tilde{H}}{\partial p^{xx}}. \Longrightarrow \text{Wrong!} \end{cases}$$

The reason why the De Donder-Weyl equations do not work here may be that we use some transformations after the Legendre transform. But we notice that the De Donder-Weyl equations actually come from the Legendre transformation except the first equation. Thus we can write all the equations directly in the following form

$$\begin{cases} \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial \tilde{H}}{\partial \psi}, \\ \psi_{xx} = -p^{xx}, \\ \psi_x = 2p^t, \\ \psi_t = 2p^x - (2p^t)^2 + 2p^{xx}_x. \end{cases} \tag{4.5}$$

The gradient of the Hamiltonian is

$$\nabla \tilde{H} = \begin{pmatrix} 0 \\ -2p^x + 4(p^t)^2 \\ -2p^t \\ p^{xx} \end{pmatrix}.$$

According to the order of gradient, we can arrange (4.5) into the following form

$$\begin{cases} \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = 0, \\ -\dfrac{\partial \psi}{\partial t} + 2\dfrac{\partial p^{xx}}{\partial x} = -2p^x + 4(p^t)^2, \\ -\dfrac{\partial \psi}{\partial x} = -2p^t, \\ -2\dfrac{\partial p^t}{\partial x} = p^{xx}, \end{cases} \tag{4.6}$$

and the multi-symplectic matrices M and K are as follows

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{pmatrix}, \quad z = \begin{pmatrix} \psi \\ p^t \\ p^x \\ p^{xx} \end{pmatrix}.$$

Let $u = 2p^t, v = -p^{xx}, w = p^x$, and arranging the order of variables as $z = (\psi, u, v, w)^T$, yield exactly the regular multi-symplectic structure of the KdV equation. Here we can conjecture that the structure matrices derived from the variation approach are the the same as the regular ones after some transformations.

4.2.3 KP equation

The KP equation can be written as

$$(2u_t + 6uu_x + u_{xxx})_x + \sigma u_{yy} = 0. \tag{4.7}$$

To put the KP equation in the variation frame work, we follow the discussion in [1], and let $u = \psi_{xx}$, then ψ satisfies equation,

$$2\psi_{xxxt} + 6\psi_{xx}\psi_{xxxx} + 6\psi_{xxx}^2 + \psi_{xxxxxx} + \sigma\psi_{xxyy} = N(\psi) = 0. \tag{4.8}$$

We can check the symmetry requirement and find $N'_\psi = \tilde{N}'_\psi$. Hence

$$L = \psi_{xx}\psi_{xt} - \frac{1}{2}\psi_{xxx}^2 + \frac{\sigma}{2}\psi_{xy}^2 + \psi_{xx}^3.$$

Let $v = \psi_x, u = \psi_{xx}, w = \psi_{xy}, p = \psi_{xt}$, taking the covariant Legendre transformation of the Lagrangian L,

$$\begin{cases} p^{xxx} = -\psi_{xxx}, \\ p^{xy} = \sigma\psi_{xy}, \\ p^{xt} = \psi_{xx}, \\ p^{xx} = \psi_{xt} + 3\psi_{xx}^2 + \psi_{xxxx}, \\ p^{x} = -2\psi_{xxt} - 6\psi_{xx}\psi_{xxx} - \sigma\psi_{xyy} - \psi_{xxxxx}. \end{cases} \tag{4.9}$$

According to the covariant De Donder-Weyl equations, we can write the KP equation in the matrix form with M, K and L as follows,

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$z = (\psi, v, u, w, p, p^x, p^{xx}, p^{xy}, p^{xt}, p^{xxx})^T \in R^{10},$$

$$H(z) = up + \frac{1}{2}(p^{xxx})^2 + \frac{\sigma}{2}w^2 + u^3 - p^x v - p^{xx}u - p^{xt}p - p^{xy}w.$$

These are the multi-symplectic structures derived in [1]. Here we can further eliminate some variables in (4.9) and obtain more compact equations,

$$\begin{cases} p^{xxx} = -\dfrac{\partial p^{xt}}{\partial x}, \\ p^{xy} = \sigma\dfrac{\partial v}{\partial y}, \\ p^{xt} = \dfrac{\partial v}{\partial x}, \\ p^{xx} = \dfrac{\partial v}{\partial t} + 3(p^{xt})^2 - \dfrac{\partial p^{xxx}}{\partial x}, \\ p^{x} = -\dfrac{\partial p^{xt}}{\partial t} - \dfrac{\partial p^{xy}}{\partial y} - \dfrac{\partial p^{xx}}{\partial x}, \end{cases} \tag{4.10}$$

and the new Hamiltonian is

$$\tilde{H}(\psi, v, p^x, p^{xx}, p^{xy}, p^{xt}, p^{xxx}) = \frac{1}{2}(p^{xxx})^2 + (p^{xt})^3 - p^x v - p^{xx}p^{xt} - \frac{1}{2\sigma}(p^{xy})^2.$$

Obviously, the number of variables is three less than the one in [1].

The gradient of the new Hamiltonian is given by

$$\nabla\tilde{H} = \begin{pmatrix} 0 \\ -p^x \\ -v \\ -p^{xt} \\ -\frac{1}{\sigma}p^{xy} \\ 3(p^{xt})^2 - p^{xx} \\ p^{xxx} \end{pmatrix}.$$

Adding the first two equations of the De Donder-Wely equations, we can arrange (4.10) as follows

$$\begin{cases} \dfrac{\partial p^x}{\partial x} = 0, \\ \dfrac{\partial p^{xt}}{\partial t} + \dfrac{\partial p^{xx}}{\partial x} + \dfrac{\partial p^{xy}}{\partial y} = -p^x, \\ -\dfrac{\partial \psi}{\partial x} = -v, \\ -\dfrac{\partial v}{\partial x} = -p^{xt}, \\ -\dfrac{\partial v}{\partial y} = -\dfrac{1}{\sigma}p^{xy}, \\ -\dfrac{\partial v}{\partial t} + \dfrac{\partial p^{xxx}}{\partial x} = 3(p^{xt})^2 - p^{xx}, \\ -\dfrac{\partial p^{xt}}{\partial x} = p^{xxx}, \end{cases} \tag{4.11}$$

which can be written in the multi-symplectic structure with the matrices defined by,

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$z = (\psi, v, p^x, p^{xx}, p^{xy}, p^{xt}, p^{xxx})^T.$$

The Preissmann scheme for the KP equation:

Using the implicit point scheme to discretize both the time direction and spatial direction, yields

$$\begin{aligned}
&\delta_x^+ \delta_t \delta_y p^x = 0, \\
&\delta_t^+ \delta_x \delta_y p^{xt} + \delta_x^+ \delta_t \delta_y p^{xx} + \delta_y^+ \delta_t \delta_x p^{xy} = -\delta_t \delta_x \delta_y p^x, \\
&\delta_x^+ \delta_t \delta_y \psi = \delta_t \delta_x \delta_y v, \\
&\delta_x^+ \delta_t \delta_y v = \delta_t \delta_x \delta_y p^{xt}, \\
&\delta_y^+ \delta_t \delta_x v = \frac{1}{\sigma} \delta_t \delta_x \delta_y p^{xy}, \\
-&\delta_t^+ \delta_x \delta_y v + \delta_x^+ \delta_t \delta_y p^{xxx} = 3(\delta_t \delta_x \delta_y p^{xt})^2 - \delta_t \delta_x \delta_y p^{xx}, \\
-&\delta_x^+ \delta_t \delta_y p^{xt} = \delta_t \delta_x \delta_y p^{xxx}.
\end{aligned}$$

Eliminating the intermediate variables, we obtian

$$2\delta_t^+ \delta_x^+ \delta_t \delta_x^3 \delta_y^3 u_{ij}^k + 3(\delta_x^+)^2 \delta_t \delta_x \delta_y^2 (\delta_t \delta_x \delta_y u_{ij}^k)^2 + (\delta_x^+)^4 \delta_t^2 \delta_y^3 u_{ij}^k + \sigma (\delta_y^+)^2 \delta_t^2 \delta_x^4 \delta_y u_{ij}^k = 0, \tag{4.12}$$

which is still the 45 points scheme derived in [1] from the more complex multi-symplectic stucture. But actually, we can omit the operators δ_t, δ_y, and obtain a simpler scheme

$$2\delta_t^+ \delta_x^+ \delta_x^3 \delta_y^2 u_{ij}^k + 3(\delta_x^+)^2 \delta_x \delta_y (\delta_t \delta_x \delta_y u_{ij}^k)^2 + (\delta_x^+)^4 \delta_t \delta_y^2 u_{ij}^k + \sigma (\delta_y^+)^2 \delta_t \delta_x^4 u_{ij}^k = 0,$$

which is equivalent to (4.12) according to the explanation in [2].

Remark 4.2. *Here for simplicity, $\psi, v, p^x, p^{xx}, p^{xy}, p^{xt}, p^{xxx}$ denote the discrete points valued on (x_i, y_j, t_k).*

4.2.4 Camassa-Holm equation

Cohen et. al. [3] have given the multi-structure of the Camassa-Holm equation as we mentioned above. But it isn't derived from the variational approach. Here we use the variational approach to re-construct the multi-symplectic structure.

Let $u = \psi_x$, the Camassa-Holm equation (1.4) can be written in the following form

$$\psi_{xt} - \psi_{xxxt} + 3\psi_x\psi_{xx} - 2\psi_{xx}\psi_{xxx} - \psi_x\psi_{xxxx} = 0, \tag{4.13}$$

for which we can check the symmetry requirement directly and find that $N'_\psi = \tilde{N}'_\psi$ (see Appendix). Thus we can construct the Lagrangian by (3.3),

$$L = -\frac{1}{2}(\psi_t\psi_x + \psi_{xt}\psi_{xx} + \psi_x^3 + \psi_x\psi_{xx}^2). \tag{4.14}$$

Taking the covariant Legendre transformation of Lagrangian (4.14) yields

$$\begin{cases} p^{xx} = \dfrac{\partial L}{\partial \psi_{xx}} = -\dfrac{1}{2}\psi_{xt} - \psi_x\psi_{xx}, \\ p^{xt} = \dfrac{\partial L}{\partial \psi_{xt}} = -\dfrac{1}{2}\psi_{xx}, \\ p^{x} = \dfrac{\partial L}{\partial \psi_{x}} - D_x p^{xx} - D_t p^{xt} = -\dfrac{1}{2}\psi_t - \dfrac{3}{2}\psi_x^2 - \dfrac{1}{2}\psi_{xx}^2 - \dfrac{\partial p^{xx}}{\partial x} - \dfrac{\partial p^{xt}}{\partial t}, \\ p^{t} = \dfrac{\partial L}{\partial \psi_{t}} = -\dfrac{1}{2}\psi_x, \end{cases} \tag{4.15}$$

We can use these equations to eliminate the variables ψ_t, ψ_x and ψ_{xx}, then

$$\begin{cases} \dfrac{\partial p^t}{\partial t} = p^{xx} + 4p^t p^{xt}, \\ \dfrac{\partial p^t}{\partial x} = p^{xt}, \\ \dfrac{1}{2}\dfrac{\partial \psi}{\partial t} + \dfrac{\partial p^{xx}}{\partial x} + \dfrac{\partial p^{xt}}{\partial t} = -p^x - 6(p^t)^2 - 2(p^{xt})^2, \\ \dfrac{1}{2}\dfrac{\partial \psi}{\partial x} = -p^t. \end{cases} \tag{4.16}$$

The new Hamiltonian is given by

$$\tilde{H}(\psi, p^t, p^x, p^{xt}, p^{xx}) = 4(p^t)^3 + 2p^t p^x + 4p^t(p^{xt})^2 + 2p^{xx}p^{xt},$$

and its gradient is

$$\nabla\tilde{H} = \begin{pmatrix} 0 \\ 12(p^t)^2 + 2p^x + 4(p^{xt})^2 \\ 2p^t \\ 8p^t p^{xt} + 2p^{xx} \\ 2p^{xt} \end{pmatrix}.$$

Adding the first equation of the De Donder-Wely equations

$$\frac{\partial p^t}{\partial t} + \frac{\partial p^x}{\partial x} = \frac{\partial \tilde{H}}{\partial \psi} = 0,$$

and arranging the order of equations (4.16), yield

$$\begin{cases} \dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = 0, \\ -\dfrac{\partial \psi}{\partial t} - 2\dfrac{\partial p^{xx}}{\partial x} - 2\dfrac{\partial p^{xt}}{\partial t} = 2p^x + 12(p^t)^2 + 4(p^{xt})^2, \\ -\dfrac{\partial \psi}{\partial x} = 2p^t, \\ 2\dfrac{\partial p^t}{\partial t} = 2p^{xx} + 8p^t p^{xt}, \\ 2\dfrac{\partial p^t}{\partial x} = 2p^{xt}, \end{cases} \tag{4.17}$$

which can be written in the matrix form with M and K defined by

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad K = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix}.$$

Let $u = -2p^t, \nu = -2p^{xt}, v = -p^{xx}, w = p^x$, and arranging the order of variables as $z = (u, \psi, w, v, \nu)^T$, yield exactly the regular multi-symplectic structure of the Camassa-Holm equation.

5 Degasperis-Procesi equation

5.1 Lagrangian density for the DP equation

Camassa-Holm equation and Degasperis-Procesi equation are both belonging to the one-parameter family of partial differential equations:

$$u_t - u_{xxt} + (b+1)uu_x = bu_x u_{xx} + uu_{xxx}. \tag{5.1}$$

In the particular case $b = 2$ this becomes the Camassa-Holm equation while $b = 3$ leads to the Degasperis-Procesi equation. We check the operator N for the Degasperis-Procesi equation and find that $N_u' \neq \tilde{N}_u'$ which means N is not a potential operator. Furthermore, it is also not an alternate potential operator under the transformation $u = \psi_x$ which can be easily seen by changing some coefficients in Appendix. Thus we can not derive the

Lagrangian by the method above. In order to give the Lagrangian of the Degasperis-Procesi equation, we first introduce the quantity

$$m = u - u_{xx},$$

which is just the Helmholtz operator acting on u, so that the family of equation (5.1) may be written in the form

$$m_t + um_x + bu_x m = 0, \tag{5.2}$$

which is more conveniently rewritten as the system

$$p_t + (up)_x = 0, \quad m = -p^b. \tag{5.3}$$

With the further rewriting of equation (5.3), for $b \neq 0$ we introduce a potential η such that

$$\eta_x = p, \quad \eta_t = -pu. \tag{5.4}$$

Then the system of equations (5.3) are equivalent to

$$\eta_t + u\eta_x = 0, \quad u - u_{xx} = -\eta_x^b, \tag{5.5}$$

which can be written in a single equation

$$\frac{\eta_t}{\eta_x} - \big(\frac{\eta_t}{\eta_x}\big)_{xx} - \eta_x^b = 0. \tag{5.6}$$

Then in terms of η we find that (apart from the particular cases $b = 0, 1$) the equation (5.1) may be derived from the following action [4]

$$S \equiv \int\int L dxdt = \int\int \Big(\frac{1}{2}\frac{\eta_t}{\eta_x}[(\log \eta_x)_{xx} + 1] + \frac{\eta_x^b}{b-1}\Big) dxdt. \tag{5.7}$$

5.2 Bi-hamiltonian structure

Starting from the Lagrangian we can apply a Legendre transformation in the usual way. The conjugate momentum is

$$\zeta \equiv \frac{\partial \mathcal{L}}{\partial \eta_t} = \frac{1}{2\eta}[(\log \eta_x)_{xx} + 1],$$

and (for $b \neq 1$) the Hamiltonian is

$$H = \int (\zeta\eta_t - \mathcal{L})dx = -\frac{1}{b-1}\int \eta_x^b dx = \frac{1}{b-1}\int m dx. \tag{5.8}$$

Having applied the Legendre transformation we then find that for any b the PDE (5.2) can be written in Hamiltonian form as

$$m_t = \hat{B}\frac{\delta H}{\delta m}, \tag{5.9}$$

with the operator

$$\hat{B} = -(bm\partial_x + m_x)(\partial_x - \partial_x^3)^{-1}(bm\partial_x + (b-1)m_x) \tag{5.10}$$

is skew-symmetric and satisfies the Jacobi identity (for a proof see [5]).

For the integrable case $b = 2$, the Camassa-Holm equation, there are two local Hamiltonian structures, give by

$$\begin{cases} m_t = B_0 \dfrac{\delta H_0}{\delta m}, & B_0 = -\partial_x(1-\partial_x^2), \ H_0 = \dfrac{1}{2}\displaystyle\int (u^3 + uu_x^2)dx, \\ m_t = B_1 \dfrac{\delta H_1}{\delta m}, & B_1 = -(m\partial_x + \partial_x m), \ H_1 = \dfrac{1}{2}\displaystyle\int (u^2 + u_x^2)dx, \end{cases}$$

and a nonlocal Hamiltonian structure obtained by applying the recursion operator $R = B_1 B_0^{-1}$ to B_1, i.e.,

$$B_2 \equiv B_1 B_0^{-1} B_1 = \hat{B}|_{b=2}.$$

For the integrable case $b = 3$, the Degasperis-Procesi equation, there is only one local Hamiltonian structures, and the operator (5.10) gives the second Hamiltonian structure, viz

$$B_0 = -\partial_x(1-\partial_x^2)(4-\partial_x^2), \quad H_0 = \frac{1}{6}\int u^3 dx, \quad B_1 = \hat{B}|b = 3.$$

Remark 5.1.

$$\frac{\delta H_1}{\delta m} = \frac{\delta H_1}{\delta u}\frac{\delta u}{\delta m} = (1-\partial_x^2)^{-1}(u - u_{xx}) = u,$$

$$B_1 u = -(m\partial_x + \partial_x m)u = -mu_x - (mu)_x = -2mu_x - m_x u.$$

5.3 Peakon solution

The N-peakon solutions of (5.1) take the following form

$$u(x,t) = \sum_{j=1}^{N} p_j(t) e^{-|x-q_j(t)|}, \tag{5.11}$$

where the positions $q_j(t)$ of the peaks and their momenta $p_j(t)$ satisfy an associated dynamical system written as

$$\begin{aligned} \frac{dq_j}{dt} &= \sum_{k=1}^{N} p_k g(q_j - q_k), \\ \frac{dp_j}{dt} &= -(b-1)\sum_{k=1}^{N} p_j p_k g'(q_j - q_k), \end{aligned} \tag{5.12}$$

where $g(x) = e^{-|x|}$. Clearly, this takes the canonical Hamiltonian form only for $b = 2$. However, for $b \neq 1$, the system has a Poisson structure

$$\frac{dq_j}{dt} = \{q_j, h\}, \quad \frac{dp_j}{dt} = \{p_j, h\}, \quad h = \sum_{j=1}^{N} p_j,$$

with the Poisson bracket defined as

$$\begin{aligned}
&\{p_j, p_k\} = -(b-1)g'(q_j - q_k)p_j p_k, \\
&\{q_j, p_k\} = p_k g(q_j - q_k),
\end{aligned}$$

Case N=1:

$$\begin{cases} \dfrac{dq}{dt} = p, \\ \dfrac{dp}{dt} = 0, \end{cases} \Longrightarrow \begin{cases} q = ct, \\ p = c, \end{cases}$$

$$u(x,t) = ce^{-|x-ct|}.$$

Case N=2:

$$\begin{cases} \dfrac{dq_1}{dt} = p_1 + p_2 e^{-|q_1 - q_2|}, \\ \dfrac{dq_2}{dt} = p_2 + p_1 e^{-|q_2 - q_1|}, \\ \dfrac{dp_1}{dt} = -2p_1 p_2 \operatorname{sgn}(q_1 - q_2) e^{-|q_1 - q_2|}, \\ \dfrac{dp_2}{dt} = -2p_2 p_1 \operatorname{sgn}(q_2 - q_1) e^{-|q_2 - q_1|}. \end{cases}$$

5.4 Multi-symplectic structures for the DP equation

5.4.1 Structure 1

The Lagrangian in (5.7) is

$$L(\eta_t, \eta_x, \eta_{xx}, \eta_{xxx}) = \frac{1}{2}\eta_x^{-2}\eta_t\eta_{xxx} - \frac{1}{2}\eta_x^{-3}\eta_t\eta_{xx}^2 + \frac{1}{2}\eta_x^{-1}\eta_t + \frac{\eta_x^b}{b-1}.$$

Taking the covariant Legendre transformation yields

$$\begin{cases} p^{xxx} = \dfrac{\partial L}{\partial \eta_{xxx}} = \dfrac{1}{2}\eta_x^{-2}\eta_t, \qquad \Longrightarrow \boxed{\eta_t = 2\eta_x^2 p^{xxx}} \\ p^{xx} = \dfrac{\partial L}{\partial \eta_{xx}} - D_x p^{xxx} = -\dfrac{1}{2}\eta_x^{-2}\eta_{tx}, \\ p^x = \dfrac{\partial L}{\partial \eta_x} - D_x p^{xx} = -\dfrac{1}{2}\eta_x^{-2}(\eta_t - \eta_{txx}) - \eta_x^{-3}(\eta_t\eta_{xxx} + \eta_{tx}\eta_{xx}) + \dfrac{3}{2}\eta_x^{-4}\eta_t\eta_{xx}^2 + \dfrac{b}{b-1}\eta_x^{b-1}, \\ p^t = \dfrac{\partial L}{\partial \eta_t} = \dfrac{1}{2}\eta_x^{-2}\eta_{xxx} - \dfrac{1}{2}\eta_x^{-3}\eta_{xx}^2 + \dfrac{1}{2}\eta_x^{-1}. \qquad \Longrightarrow \boxed{\eta_{xxx} = 2\eta_x^2 p^t + \eta_x^{-1}\eta_{xx}^2 - \eta_x} \end{cases}$$

$$
\begin{aligned}
H(\eta,\eta_x,\eta_{xx},p^t,p^x,p^{xx},p^{xxx}) &= L - p^t\eta_t - p^x\eta_x - p^{xx}\eta_{xx} - p^{xxx}\eta_{xxx}\\
&= \frac{\eta_x^b}{b-1} - p^x\eta_x - p^{xx}\eta_{xx} - 2\eta_x^2 p^t p^{xxx} - \eta_x^{-1}\eta_{xx}^2 p^{xxx} + \eta_x p^{xxx}.
\end{aligned}
$$

The De Donder-Wely equations:

$$
\begin{cases}
\dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial H}{\partial \eta}, & -\dfrac{\partial \eta}{\partial t} = \dfrac{\partial H}{\partial p^t},\\
\dfrac{\partial p^{xx}}{\partial x} = \dfrac{\partial H}{\partial \eta_x}, & -\dfrac{\partial \eta}{\partial x} = \dfrac{\partial H}{\partial p^x},\\
\dfrac{\partial p^{xxx}}{\partial x} = \dfrac{\partial H}{\partial \eta_{xx}}, & -\dfrac{\partial \eta_x}{\partial x} = \dfrac{\partial H}{\partial p^{xx}},\\
 & -\dfrac{\partial \eta_{xx}}{\partial x} = \dfrac{\partial H}{\partial p^{xxx}},
\end{cases}
$$

Let $p=\eta_x, w=\eta_{xx}$, then the De Donder-Wely equations can be rewritten into the matrix form with

$$
M = \begin{pmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}, \qquad
K = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 1\\
0 & 0 & 0 & 0 & 0 & 0 & 0\\
-1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -1 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

$$
\nabla_z H(z) = \begin{pmatrix}
0\\
\dfrac{b}{b-1}p^{b-1} - p^x - 4pp^t p^{xxx} + p^{-2}w^2 p^{xxx} + p^{xxx}\\
-p^{xx} - 2p^{-1}p^{xxx}w\\
-2p^2 p^{xxx}\\
-p\\
-w\\
-2p^2p^t - p^{-1}w^2 + p
\end{pmatrix}, z = \begin{pmatrix}
\eta\\ p\\ w\\ p^t\\ p^x\\ p^{xx}\\ p^{xxx}
\end{pmatrix}.
$$

Next we will show the elimination process since we would follow the same order to obtain the final scheme in the discrete situation. Firstly, we write this system componentwise

$$
\begin{cases}
p_t^t + p_x^x = 0, & (5.13)\\
p_x^{xx} = \dfrac{b}{b-1}p^{b-1} - p^x - 4pp^t p^{xxx} + p^{-2}w^2p^{xxx} + p^{xxx}, & (5.14)\\
p_x^{xxx} = -p^{xx} - 2p^{-1}p^{xxx}w, & (5.15)\\
-\eta_t = -2p^2p^{xxx}, & (5.16)\\
-\eta_x = -p, & (5.17)\\
-p_x = -w, & (5.18)\\
-w_x = -2p^2p^t - p^{-1}w^2 + p. & (5.19)
\end{cases}
$$

$$\begin{cases} (5.18),(5.19) \Longrightarrow p^t = \frac{1}{2}p^{-2}p_{xx} - \frac{1}{2}p^{-3}p_x^2 + \frac{1}{2}p^{-1} & (5.20) \\ (5.20) \Longrightarrow p_t^t = \frac{1}{2}p^{-2}(p_{xxt} - p_t) - p^{-3}(p_t p_{xx} + p_x p_{xt}) \\ \qquad\qquad + \frac{3}{2}p^{-4}p_t p_x^2, & (5.21) \\ (5.15),(5.16),(5.17) \Longrightarrow p_x^{xx} = p^{-3}p_x p_t - \frac{1}{2}p^{-2}p_{tx}, & (5.22) \\ \underbrace{(5.16),(5.17),(5.18),(5.20),(5.22)} \\ \qquad\qquad \Downarrow \\ p_x^x = \frac{1}{2}p^{-2}(p_{txx} - p_t) - p^{-3}(2p_t p_{xx} + 2p_x p_{tx} + \eta_t p_{xxx} - \eta_t p_x) \\ \qquad + 3p^{-4}(\frac{3}{2}p_x^2 p_t + 2\eta_t p_x p_{xx}) - 6p^{-5}p_x^3\eta_t + bp^{b-2}p_x, & (5.23) \\ \underbrace{(5.13),(5.21),(5.23)} \\ \qquad \Downarrow \times(-p^2) \\ 0 = \underbrace{p_t - p_{txx}}_{①} \underbrace{+p^{-1}(3p_t p_{xx} + 3p_x p_{tx} + \eta_t p_{xxx} - \eta_t p_x)}_{②} \\ \quad \underbrace{-6p^{-2}(p_x^2 p_t + \eta_t p_x p_{xx})}_{③} \underbrace{+6p^{-3}p_x^3\eta_t}_{④} \underbrace{-bp^b p_x}_{⑤}, & (5.24) \end{cases}$$

Using the potential equation (5.4), we can obtain

①: $p_t + p_{xxx}u + 3p_{xx}u_x + 3p_x u_{xx} + pu_{xxx}$

②: $p^{-1}(-6p_x p_{xx}u - 3pp_{xx}u_x - 6p_x^2 u_x + pp_x u - 3pp_x u_{xx} - pp_{xxx}u)$

③: $-6p^{-2}(-p_x^3 u - pp_x^2 u_x - pp_x p_{xx} u)$

④: $-6p^{-2}p_x^3 u$

⑤: $-bp^b p_x$

Then we recast the terms with respect to the derivatives of u

$$\begin{cases} u & : \quad p_x u \\ u_x & : \quad 0 \\ u_{xx} & : \quad 0 \\ u_{xxx} & : \quad pu_{xxx} \end{cases}$$

The Euler-Lagrangian equation yields

$$p_t + p_x u + pu_{xxx} - bp^b p_x = 0,$$

$$
\begin{aligned}
&\Rightarrow p_t + p_x u + p u_x - p u_x + p u_{xxx} - b p^b p_x = 0,\\
&\Rightarrow p_t + (pu)_x - p u_x + p u_{xxx} - b p^b p_x = 0,\\
&\Rightarrow p_t + (pu)_x - p m_x - b p^b p_x = 0,
\end{aligned}
$$

Since $m = -p^b$, then

$$
m_x = -b p^{b-1} p_x, \Longrightarrow -p m_x - b p^b p_x = b p^b p_x - b p^b p_x = 0.
$$

Thus the final equation is equal to (5.3).

5.4.2 Structure 2

Using integrations by parts, the Lagrangian density (5.7) may be changed as

$$
S \equiv \int\int L dxdt = \int\int \Big(\frac{1}{2}\frac{\eta_t}{\eta_x} - \frac{1}{2}(\frac{\eta_t}{\eta_x})_x(\log \eta_x)_x + \frac{\eta_x^b}{b-1}\Big) dxdt, \quad (5.25)
$$

and the Lagrangian is

$$
L(\eta_t, \eta_x, \eta_{tx}, \eta_{xx}) = \frac{1}{2}\eta_x^{-1}\eta_t - \frac{1}{2}\eta_x^{-2}\eta_{tx}\eta_{xx} + \frac{1}{2}\eta_x^{-3}\eta_t\eta_{xx}^2 + \frac{\eta_x^b}{b-1}.
$$

Taking the covariant Legendre transformation yields

$$
\begin{aligned}
&\left.\begin{aligned}
p^{xx} &= \frac{\partial L}{\partial \eta_{xx}} = -\frac{1}{2}\eta_x^{-2}\eta_{tx} + \eta_x^{-3}\eta_t\eta_{xx},\\
p^{tx} &= \frac{\partial L}{\partial \eta_{tx}} = -\frac{1}{2}\eta_x^{-2}\eta_{xx},
\end{aligned}\right\} \Longrightarrow
\begin{aligned}
&\eta_{xx} = -2\eta_x^2 p^{tx},\\
&\eta_{tx} = -2\eta_x^2 p^{xx} - 4\eta_x\eta_t p^{tx},
\end{aligned}\\
&p^x = \frac{\partial L}{\partial \eta_x} - D_x p^{xx}\\
&\quad = -\frac{1}{2}\eta_x^{-2}(\eta_t - \eta_{txx}) - \eta_x^{-3}(\eta_{tx}\eta_{xx} + \eta_t\eta_{xxx}) - \frac{1}{2}\eta_x^{-4}\eta_t\eta_{xx}^2 + \frac{b}{b-1}\eta_x^{b-1},\\
&p^t = \frac{\partial L}{\partial \eta_t} - D_x p^{tx} = \frac{1}{2}\eta_x^{-1} + \frac{1}{2}\eta_x^{-2}\eta_{xxx} - \frac{1}{2}\eta_x^{-3}\eta_{xx}^2.
\end{aligned}
$$

The Hamiltonian can then be written as

$$
\begin{aligned}
H(\eta, \eta_t, \eta_x, p^t, p^x, p^{tx}, p^{xx}) &= L - p^t\eta_t - p^x\eta_x - p^{tx}\eta_{tx} - p^{xx}\eta_{xx}\\
&= \frac{1}{2}\eta_x^{-1}\eta_t + 2\eta_x\eta_t(p^{tx})^2 + \frac{\eta_x^b}{b-1} - p^t\eta_t - p^x\eta_x + 2\eta_x^2 p^{tx} p^{xx}.
\end{aligned}
$$

The De Donder-Wely equations:

$$
\begin{cases}
\dfrac{\partial p^t}{\partial t} + \dfrac{\partial p^x}{\partial x} = \dfrac{\partial H}{\partial \eta}, & -\dfrac{\partial \eta}{\partial t} = \dfrac{\partial H}{\partial p^t}, \\
\dfrac{\partial p^{tx}}{\partial x} = \dfrac{\partial H}{\partial \eta_t}, & -\dfrac{\partial \eta}{\partial x} = \dfrac{\partial H}{\partial p^x}, \\
\dfrac{\partial p^{xx}}{\partial x} = \dfrac{\partial H}{\partial \eta_x}, & -\dfrac{\partial \eta_t}{\partial x} = \dfrac{\partial H}{\partial p^{tx}}, \\
 & -\dfrac{\partial \eta_x}{\partial x} = \dfrac{\partial H}{\partial p^{xx}}.
\end{cases}
$$

Let $v = \eta_t, p = \eta_x$, then the De Donder-Wely equations can be rewritten into the matrix form with

$$
M = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad K = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\nabla_z H(z) = \begin{pmatrix} 0 \\ \frac{1}{2}p^{-1} + 2p(p^{tx})^2 - p^t \\ -\frac{1}{2}p^{-2}v + 2v(p^{tx})^2 + \frac{b}{b-1}p^{b-1} - p^x + 4pp^{tx}p^{xx} \\ -v \\ -p \\ 4pvp^{tx} + 2p^2p^{xx} \\ 2p^2p^{tx} \end{pmatrix}, z = \begin{pmatrix} \eta \\ v \\ p \\ p^t \\ p^x \\ p^{tx} \\ p^{xx} \end{pmatrix}.
$$

Following the process above, we also write this system componentwise

$$
\begin{cases}
p^t_t + p^x_x = 0, \\
p^{tx}_x = \frac{1}{2}p^{-1} + 2p(p^{tx})^2 - p^t, \\
p^{xx}_x = -\frac{1}{2}p^{-2}v + 2v(p^{tx})^2 + \frac{b}{b-1}p^{b-1} - p^x + 4pp^{tx}p^{xx} \\
-\eta_t = -v \\
-\eta_x = -p, \\
-v_x = 4pvp^{tx} + 2p^2p^{xx}, \\
-p_x = 2p^2p^{tx}.
\end{cases}
$$

After tedious manipulation, we get the same equation as (5.24) except the difference between η_t and v. The further calculations are no different to that of the first structure.

5.5 Conjecture

In [3], Cohen et. al. give another multi-symplectic formulation according to the system

$$\begin{aligned} & u_t + uu_x + P_x = 0, \\ & P - P_{xx} = u^2 + \frac{1}{2}u_x^2, \\ & (u^2 + u_x^2)_t + (u(u^2 + u_x^2))_x = (u^3 - 2Pu)_x. \end{aligned} \tag{5.26}$$

The third equation is the transport equation which the energy density should satisfy. After denoting $u^2 + u_x^2$ by α, we can rewrite (5.26) as

$$\begin{aligned} & u_t + uu_x + P_x = 0, \\ & P - P_{xx} = \frac{1}{2}u^2 + \frac{1}{2}\alpha, \\ & \alpha_t + (u\alpha)_x = (u^3 - 2Pu)_x. \end{aligned} \tag{5.27}$$

which has the following multi-symplectic structure

$$M = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \end{pmatrix}$$

$$S(z) = -\gamma u + \frac{u^2\alpha}{2} - \frac{u^4}{4} + Pu^2 - \alpha w - P^2 + r^2, \ z = (u, \beta, w, \alpha, \phi, \gamma, P, r)^T.$$

The Euler-box scheme based on this structure can simulate the peakon solutions well while another structure mentioned above can not.

The DP equation can also be written in the system of equations [6]

$$\begin{aligned} & u_t + uu_x + P_x = 0, \\ & P - P_{xx} = \frac{3}{2}u^2, \\ & (u^3)_t + (\frac{3}{4}u^4 + P^2 - P_x^2)_x = 0. \end{aligned} \tag{5.28}$$

Thus we conject that whether there exists an "α" like above such that (5.28) has a multi-symplectic structure.

Appendix A

Camassa-Holm equation with the transformation $u = \psi_x$ becomes

$$\psi_{xt} - \psi_{xxxt} + 3\psi_x\psi_{xx} - 2\psi_{xx}\psi_{xxx} - \psi_x\psi_{xxxx} = 0,$$

and the Fréchet derivative is

$$\begin{aligned} N_\psi^{'}\delta\psi =& \delta\psi_{xt} - \delta\psi_{xxxt} + 3\delta\psi_x\psi_{xx} + 3\psi_x\delta\psi_{xx} \\ & - 2\delta\psi_{xx}\psi_{xxx} - 2\psi_{xx}\delta\psi_{xxx} - \delta\psi_x\psi_{xxxx} - \psi_x\delta\psi_{xxxx}, \end{aligned}$$

$$\begin{aligned} \tilde{N}_\psi^{'}\delta\psi =& \delta\psi_{xt} - \delta\psi_{xxxt} - 3(\psi_{xx}\delta\psi)_x + 3(\psi_x\delta\psi)_{xx} \\ & - 2(\delta\psi\psi_{xxx})_{xx} + 2(\delta\psi\psi_{xx})_{xxx} + (\delta\psi\psi_{xxxx})_x - (\delta\psi\psi_x)_{xxxx} \\ =& \delta\psi_{xt} - \delta\psi_{xxxt} - 3(\psi_{xxx}\delta\psi + \psi_{xx}\delta\psi_x) \\ & + 3(\psi_{xxx}\delta\psi + 2\psi_{xx}\delta\psi_x + \psi_x\delta\psi_{xx}) \\ & - 2(\delta\psi_{xx}\psi_{xxx} + 2\delta\psi_x\psi_{xxxx} + \delta\psi\psi_{xxxxx}) \\ & + 2(\delta\psi_{xxx}\psi_{xx} + 3\delta\psi_{xx}\psi_{xxx} + 3\delta\psi_x\psi_{xxxx} + \delta\psi\psi_{xxxxx}) \\ & + \delta\psi_x\psi_{xxxx} + \delta\psi\psi_{xxxxx} \\ & - (\delta\psi_{xxxx}\psi_x + 4\delta\psi_{xxx}\psi_{xx} + 6\delta\psi_{xx}\psi_{xxx} + 4\delta\psi_x\psi_{xxxx} + \delta\psi\psi_{xxxxx}). \end{aligned}$$

We can compare the coefficients of the terms $\delta\psi, \delta\psi_x, \delta\psi_{xx}, \delta\psi_{xxx}, \delta\psi_{xxxx}$

$$N_\psi^{'}\delta\psi : \begin{cases} \delta\psi : 0 \\ \delta\psi_x : 3\psi_{xx} - \psi_{xxxx} \\ \delta\psi_{xx} : 3\psi_x - 2\psi_{xxx} \\ \delta\psi_{xxx} : -2\psi_{xx} \\ \delta\psi_{xxxx} : -\psi_x \end{cases}$$

$$\tilde{N}_\psi^{'}\delta\psi : \begin{cases} \delta\psi : -3\psi_{xxx} + 3\psi_{xxx} - 2\psi_{xxxxx} + 2\psi_{xxxxx} + \psi_{xxxxx} - \psi_{xxxxx} = 0 \\ \delta\psi_x : -3\psi_{xx} + 6\psi_{xx} - 4\psi_{xxxx} + 6\psi_{xxxx} + \psi_{xxxx} - 4\psi_{xxxx} = 3\psi_{xx} - \psi_{xxxx} \\ \delta\psi_{xx} : 3\psi_x - 2\psi_{xxx} + 6\psi_{xxx} - 6\psi_{xxx} = 3\psi_x - 2\psi_{xxx} \\ \delta\psi_{xxx} : 2\psi_{xx} - 4\psi_{xx} = -2\psi_{xx} \\ \delta\psi_{xxxx} : -\psi_x \end{cases}$$

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