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Fourth-order compact and energy conservative difference schemes for the nonlinear Schrödinger equation in two dimensions ☆

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A B S T R A C T

In this paper, a fourth-order compact and energy conservative difference scheme is proposed for solving the two-dimensional nonlinear Schrödinger equation with periodic boundary condition and initial condition, and the optimal convergent rate, without any restriction on the grid ratio, at the order of $O(h^4+\tau^2)$ in the discrete L^2-norm with time step τ and mesh size h is obtained. Besides the standard techniques of the energy method, a new technique and some important lemmas are proposed to prove the high order convergence. In order to avoid the outer iteration in implementation, a linearized compact and energy conservative difference scheme is derived. Numerical examples are given to support the theoretical analysis.

1. Introduction

The nonlinear Schrödinger (NLS) equation is a famous equation used widely in many fields of physics, such as plasma physics, quantum physics and nonlinear optics. In this paper, we consider the following cubic NLS equation in two dimensions

$$i\frac{\partial u}{\partial t}+\Delta u+\beta|u|^2u=0,\quad (x,y)\in\mathcal{R}\times\mathcal{R},\ 0<t\leqslant T, \tag{1.1}$$

subject to (l_1,l_2)-periodic boundary condition

$$u(x,y,t)=u(x+l_1,y,t),\ u(x,y,t)=u(x,y+l_2,t),\quad (x,y)\in\mathcal{R}\times\mathcal{R},\ 0<t\leqslant T \tag{1.2}$$

and initial condition

$$u(x,y,0)=\varphi(x,y),\quad (x,y)\in\mathcal{R}\times\mathcal{R}, \tag{1.3}$$

where $\Delta=\frac{\partial}{\partial x^2}+\frac{\partial}{\partial y^2}$ is the Laplacian operator, $\beta\neq 0$ is a given real constant, $\varphi(x,y)$ is a given (l_1,l_2)-periodic complex-valued function. The NLS equation (1.1) is focusing for $\beta>0$, and defocusing for $\beta<0$, which is a generic model for the slowly

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varying envelop of a wave-train in conservative, dispersive, mildly nonlinear wave phenomena. It is also obtained as the subsonic limit of the Zakharov model for Langmuir waves in plasma physics [55]. It is possible for solutions of the two-dimensional NLS equation to develop singularities at some finite time [29]. Periodic boundary condition is used for certain problems in a very large or even infinite domain consisting of (infinitely) many small cells where particles pass through the boundary of each cell in their individual periodic pattern. So the domain's boundary condition will be neglected, and one can solve the problem numerically only on a central cell and its surrounding cells.

Extensive mathematical and numerical studies have been carried out for the NLS equation in the literature. Along the mathematical front, for the derivation, well-posedness and dynamical properties of the NLS equation, we refer to [5,14,40,48] and the references therein. In fact, it is easy to show that the periodic-initial value problem (1.1)–(1.3) conserves the total *mass*

$$Q(u(\cdot,\cdot,t)) := \int_\Omega |u(x,y,t)|^2 dxdy \equiv Q(u(\cdot,\cdot,0)) = Q(\varphi), \quad t \in \mathcal{R}^+ \tag{1.4}$$

and the *energy*

$$E(u(\cdot,\cdot,t)) := \int_\Omega \left[|\nabla u(x,y,t)|^2 - \frac{\beta}{2}|u(x,y,t)|^4\right] dxdy \equiv E(u(\cdot,\cdot,0)) = E(\varphi), \quad t \in \mathcal{R}^+, \tag{1.5}$$

where $\Omega = [0,l_1] \times [0,l_2]$. These conservative laws imply that, $||u(\cdot,\cdot,t)||_{H_p^1} < \infty$ for the defocusing case or the focusing case with $||\varphi||^2 \leqslant \frac{2-\widetilde{\varepsilon}}{4(1+\varepsilon)\beta}$ where ε and $\widetilde{\varepsilon}$ are two positive numbers which can be arbitrary small. The theory on existence of global solution of the NLS equation [14,48] gives that, if $||u(\cdot,\cdot,t)||_{H_p^1} < \infty, u(x,y,t) \in L^\infty(\mathcal{R}^+, H_p^1(\Omega))$.

Along the numerical front, different efficient and accurate numerical methods including the time-splitting pseudospectral method [6,8,9,50], finite difference method [16,21,26,47,52,53,56], finite element method [1,2,27,34], discontinuous Galerkin method [33,54], meshless methods [23,24] and Runge–Kutta or Crank–Nicolson pseudospectral method [13,46,25] have been developed for the NLS equation. Of course, each method has its advantages and disadvantages. For numerical comparisons between different numerical methods for the NLS equation, we refer to [7,16,41,51] and the references therein.

Error estimates for different numerical methods of NLS equation in one dimension have been established in the literature. For the analysis of splitting error of the time-splitting or split-step method for the NLS equation, we refer to [11,20,39,44,50] and the references therein. For the error estimates of the implicit Runge–Kutta finite element method for NLS, we refer to [2,45]. Error bounds, without any restriction on the grid ratio, of conservative finite difference (CNFD) method for NLS equation in one dimension was established in [15,16,28,52]. In fact, their proofs for CNFD scheme rely strongly on not only the conservative property of the method but also the discrete version of the Sobolev inequality in one dimension

$$||f||_{L^\infty} \leqslant C||f||_{H^1}, \ \ \forall f \in H_0^1(\Omega) \quad \text{or} \quad f \in H_p^1(\Omega) \quad \text{with} \quad \Omega \subset \mathcal{R}. \tag{1.6}$$

which immediately imply *a priori* uniform bound for $||f||_{L^\infty}$. However, the extension of the discrete version of the above Sobolev inequality is no longer valid in two dimensions. Thus the techniques used in [15,16,28] for obtaining l^∞ error bounds of the CNFD scheme for the NLS equation only work for conservative schemes in one dimension and they cannot be extended to high dimensions. Due to the difficulty in obtaining the *a priori* uniform estimate of the numerical solution, few error estimates are available in the literature of finite difference methods for the NLS equation in two dimensions. Gao and Xie [26] proposed a fourth-order alternating direction implicit compact finite difference scheme for two-dimensional Schrödinger equations, and also used induction argument to prove that their scheme was conditionally convergent in the discrete L^2-norm. In their analysis, a serious restriction on the grid ratio was necessary. Bao and Cai [3] used different techniques to establish optimal but conditional error bounds of CNFD scheme and semi-implicit finite difference (SIFD) method for the homogeneous initial-boundary value problem of the Gross-Pitaevskii equation with angular momentum rotation in two and three dimensions.

Recently, there has been growing interest in high-order compact methods for solving partial differential equations [10,17–19,21,22,26,31,32,37,38,42,43,49,53]. It was shown that the high-order difference methods play an important role in the simulation of high frequency wave phenomena. However, because the discretization of nonlinear term in compact scheme is more complicated than that in second-order one, *a priori* estimate in the discrete L^∞-norm is hard to be obtained, so the unconditional convergence of any compact difference scheme for nonlinear equation is difficult to be proved. In fact, for any compact difference scheme of nonlinear equations, especially those in two or three dimensions, there are very few results on unconditional convergence. In [52], the error estimates of two compact difference schemes for one-dimensional nonlinear Schrödinger equation were established without any restrictions on the mesh ratios, but the analysis method used there cannot be extended directly to high dimensions. In this paper, we will give a compact and conservative difference scheme for solving the periodic-initial value problem of the NLS equation in two dimensions, and prove that the proposed scheme is convergent at the order of $O(h^4 + \tau^2)$ in the discrete L^2-norm without any restriction on the mesh ratio.

The remainder of this paper is arranged as follows. In Section 2, some notations are given and a difference scheme is proposed. In Section 3, some auxiliary lemmas are introduced or proved. In Section 4, solvability, discrete conservative laws of the proposed scheme are discussed and *a priori* estimate is obtained, then the convergence is proved based on the estimation. Lastly, numerical experiments are presented in Section 5 and some remarks are given in the concluding section.

2. Notations and compact finite difference scheme

Numerically we solve the periodic-initial value problem (1.1)–(1.3) on a finite domain $\Omega \times [0, T]$. Before giving the conservative compact difference scheme, some notations are firstly introduced.

For a positive integer N, let time-step $\tau = T/N,\ t_n = n\tau,\ 0 \leqslant n \leqslant N$, denote $\Omega_\tau = \{t_n = n\tau |\ n = 0, 1, 2, \ldots, N-1\}$, $\Omega'_\tau = \{t_n = n\tau |\ n = 1, 2, \ldots, N-1\}$ and $\overline{\Omega}_\tau = \{t_n = n\tau |\ n = 0, 1, \ldots, N\}$. Given a temporal discrete function $\{u^n | t_n \in \overline{\Omega}_\tau\}$, we denote $u^{n+\frac{1}{2}} = (u^{n+1} + u^n)/2,\ \mathcal{A}_\tau u^n = (u^{n+1} + u^{n-1})/2,\ \delta_t^+ u^n = (u^{n+1} - u^n)/\tau,\ \delta_t u^n = (u^{n+1} - u^{n-1})/2\tau$.

For two positive integers J and K, let space-steps $h_1 = l_1/J,\ h_2 = l_2/K,\ h = \max\{h_1, h_2\},\ x_j = jh_1,\ 0 \leqslant j \leqslant J-1,\ y_k = kh_2$, $0 \leqslant k \leqslant K-1$, and the grid $\Omega_h = \{(x_j, y_k) |\ j = 0, 1, \ldots, J-1;\ k = 0, 1, \ldots, K-1\}$. To approximate the periodic boundary conditions, let $x_{-1} = -h_1,\ x_J = Jh_1,\ y_{-1} = -h_2,\ y_K = Kh_2$ and the extended discrete grid $\Omega_h^E = \{(x_j, y_k) |\ j = -1, 0, 1, \ldots, J;$ $k = -1, 0, 1, \ldots, K\}$. Given a grid function $u = \{u_{j,k} | (x_j, y_k) \in \Omega_h^E\}$, denote

$$\delta_x^+ u_{j,k} = (u_{j+1,k} - u_{j,k})/h_1, \quad \delta_x^- u_j = (u_{j,k} - u_{j-1,k})/h_1, \quad \mathcal{A}_{h_1} u_{j,k} = u_{j,k} + \frac{h_1^2}{12} \delta_x^+ \delta_x^- u_{j,k} = \frac{1}{12}(u_{j-1,k} + 10u_{j,k} + u_{j+1,k}).$$

Notations $\delta_y^- u_{j,k}, \delta_y^+ u_{j,k}, \mathcal{A}_{h_2} u_{j,k}$ are defined similarly.

A grid function $u = \{u_{j,k} | (x_j, y_k) \in \Omega_h^E\}$ is called *periodic* if

$$(x-\textit{periodic}) \quad u_{-1,k} = u_{J-1,k}, \quad u_{0,k} = u_{J,k}, \quad k = -1, 0, 1, \ldots, K; \quad (y-\textit{periodic}) \quad u_{j,-1} = u_{j,K-1}, \quad u_{j,0} = u_{j,K}, \quad j = -1, 0, 1, \ldots, J.$$

Let $\mathcal{V}_h = \{u | u = \{u_{j,k} | (x_j, y_k) \in \Omega_h^E\}$ and u is *periodic*$\}$ be the space of periodic grid functions on Ω_h^E. For any two grid functions $u, v \in \mathcal{V}_h$, define the discrete inner product as

$$(u, v) = h_1 h_2 \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} u_{j,k} \overline{v}_{j,k}.$$

The discrete L^2-norm (or l^2-norm) of v and its difference quotients are defined, respectively, as

$$||v|| = \sqrt{(v, v)}, \quad ||\delta_x^+ v|| = \sqrt{(\delta_x^+ v, \delta_x^+ v)}, \quad ||\delta_y^+ v|| = \sqrt{(\delta_y^+ v, \delta_y^+ v)}, \quad |v|_1 = ||\nabla_h v|| = \sqrt{||\delta_x^+ v||^2 + ||\delta_y^+ v||^2},$$

We also define the discrete L^p $(1 \leqslant p < \infty)$ norm as

$$||v||_p = \sqrt[p]{h_1 h_2 \sum_{j=0}^{J-1} \sum_{k=0}^{K-1} |v_{j,k}|^p}.$$

When $p = \infty$, we denote

$$||v||_\infty = \max_{(x_j, y_k) \in \Omega_h} |v_{j,k}|$$

as the discrete maximum norm (or l^∞-norm). It should be pointed out that the discrete norm $||\delta_x^+ v||, ||\delta_y^+ v||, |v|_1$ defined above are just only semi-norms.

For simplicity, we denote $u_{j,k}^n$ and $U_{j,k}^n$ as the exact value and the approximation of $u(x, y, t)$ at the point (x_j, y_k, t_n), respectively. Let C denote a positive constant independent of discretization parameters, but may have different values at different occurrences.

2.1. Some properties of circulant matrix

A matrix in the form of

$$A = \begin{pmatrix} a_0 & a_1 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & \cdots & a_{n-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}$$

is called a circulant matrix [30]. Because the matrix A is determined by the entries in the first row, the matrix can be denoted as

$$A = C(a_0, a_1, \ldots, a_{n-1}).$$

Now we list some useful lemmas as follows:

Lemma 2.1 [30]. *If a circulant matrix A is invertible, then its inverse matrix A^{-1} is circulant.*

Lemma 2.2 [30]. *For a real circulant matrix $A = C(a_0, a_1, \ldots, a_{n-1})$, all eigenvalues of A are given by*

$$f(\varepsilon_k), \quad k = 0, 1, 2, \ldots, n-1, \tag{2.1}$$

where $f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_{n-1}x^{n-1}$, and $\varepsilon_k = cos(\frac{2k\pi}{n}) + isin(\frac{2k\pi}{n})$.

2.2. Compact approximation of second derivative

The key point of the high-order compact (HOC) method is to discretize the derivative with the fewest nodes to get the expected accuracy. In order to achieve higher accuracy, we adopt implicit compact scheme which equals a combination of the nodal derivatives to a combination of the nodal values of the function. Next, by Taylor formula, expanding all the nodal derivatives and nodal values of the function at the same node, we can determine the coefficients of the combinations under the requirement of accuracy order. The nodal derivatives are implicitly evaluated herein by solving linear algebraic equations in strip. The stencil and the weighted factors are restricted to be symmetric so that the resulting schemes are non-dissipative.

For the discretization of the second order derivatives w_{xx} and w_{yy} of the (l_1, l_2)-periodic function $w = w(x, y)$, we have the formula [36,38]:

$$\mathcal{A}_{h_1} w_{xx}(x_j, y_k) = \delta_x^+ \delta_x^- w(x_j, y_k) + O(h_1^4) \Rightarrow w_{xx}(x_j, y_k) = \mathcal{A}_{h_1}^{-1} \delta_x^+ \delta_x^- w(x_j, y_k) + O(h_1^4), \tag{2.2}$$

$$\mathcal{A}_{h_2} w_{yy}(x_j, y_k) = \delta_y^+ \delta_y^- w(x_j, y_k) + O(h_2^4) \Rightarrow w_{yy}(x_j, y_k) = \mathcal{A}_{h_2}^{-1} \delta_y^+ \delta_y^- w(x_j, y_k) + O(h_2^4). \tag{2.3}$$

Omitting the small terms $O(h_1^4)$ and $O(h_2^4)$, we obtain the approximation of the w_{xx} and w_{yy} as

$$\mathcal{A}_{h_1} w_{xx}(x_j, y_k) \approx \delta_x^+ \delta_x^- W_{jk} \Rightarrow w_{xx}(x_j, y_k) \approx \mathcal{A}_{h_1}^{-1} \delta_x^+ \delta_x^- W_{jk}, \tag{2.4}$$

$$\mathcal{A}_{h_2} w_{yy}(x_j, y_k) \approx \delta_y^+ \delta_y^- W_{jk} \Rightarrow w_{yy}(x_j, y_k) \approx \mathcal{A}_{h_2}^{-1} \delta_y^+ \delta_y^- W_{jk}, \tag{2.5}$$

where W_{jk} is the approximation of $w(x_j, y_k)$. The corresponding matrix form of 2.4,2.5 is

$$A_1(\Pi_{h_1} w_{xx}) \approx \delta_x^+ \delta_x^- W \Rightarrow \Pi_{h_1} w_{xx} \approx A_1^{-1} \delta_x^+ \delta_x^- W,$$
$$A_2(\Pi_{h_2} w_{yy}) \approx \delta_y^+ \delta_y^- W \Rightarrow \Pi_{h_2} w_{yy} \approx A_2^{-1} \delta_y^+ \delta_y^- W,$$

where

$$\Pi_{h_1} w_{xx} = \big(w_{xx}(x_0, y_k), w_{xx}(x_1, y_k), \ldots, w_{xx}(x_{J-1}, y_k)\big), \quad k = 0, 1, \ldots, K-1,$$
$$\Pi_{h_2} w_{yy} = \big(w_{yy}(x_j, y_0), w_{yy}(x_j, y_1), \ldots, w_{yy}(x_j, y_{K-1})\big), \quad j = 0, 1, \ldots, J-1,$$
$$W = \big(W_0, W_1, \ldots, W_{J-1}\big)$$

and

$$A_1 = C\left(\frac{5}{6}, \frac{1}{12}, 0, \ldots, 0, \frac{1}{12}\right), \quad A_2 = C\left(\frac{5}{6}, \frac{1}{12}, 0, \ldots, 0, \frac{1}{12}\right) \tag{2.6}$$

are $J \times J$ order matrix and $K \times K$ order matrix, respectively.

2.3. Fourth order compact finite difference scheme

Imposing the compact difference/Crank–Nicolson discretization on the spatial/temporal direction of the NLS equations (1.1)–(1.3) gives

$$i\delta_t^+ u_{j,k}^n + \mathcal{A}_{h_1}^{-1} \delta_x^+ \delta_x^- u_{j,k}^{n+\frac{1}{2}} + \mathcal{A}_{h_2}^{-1} \delta_y^+ \delta_y^- u_{j,k}^{n+\frac{1}{2}} + \frac{\beta}{2}\left(\left|u_{j,k}^n\right|^2 + \left|u_{j,k}^{n+1}\right|^2\right) u_{j,k}^{n+\frac{1}{2}} = r_{j,k}^n, \quad (x_j, y_k) \in \Omega_h, \quad t_n \in \Omega_\tau, \tag{2.7}$$

$$u^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{2.8}$$

$$u_{j,k}^0 = \varphi(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E. \tag{2.9}$$

where $u_{j,k}^n = u(x_j, y_k, t_n)$ and $r_{j,k}^n$ is the truncation error.

On the truncation error $r_{j,k}^n$, using Taylor's expansion and noticing 2.2,2.3, we obtain.

Lemma 2.3. *Suppose that $\varphi(x, y) \in H^1(\Omega)$ and $u(x, y, t) \in C^{6,6,4}(\Omega \times (0, T])$, there is*

$$\left|r_{j,k}^n\right| \leqslant C(h^4 + \tau^2), \quad (x_j, y_k) \in \Omega_h, \quad 0 \leqslant n \leqslant N-1. \tag{2.10}$$

Omitting the small term $r^n_{j,k}$ in (2.7) yields the following difference scheme

$$i\delta_t^+ U^n_{j,k} + \mathcal{A}_{h_1}^{-1}\delta_x^+\delta_x^- U^{n+\frac{1}{2}}_{j,k} + \mathcal{A}_{h_2}^{-1}\delta_y^+\delta_y^- U^{n+\frac{1}{2}}_{j,k} + \frac{\beta}{2}\left(\left|U^n_{j,k}\right|^2 + \left|U^{n+1}_{j,k}\right|^2\right)U^{n+\frac{1}{2}}_{j,k} = 0, \quad (x_j, y_k) \in \Omega_h, \quad t_n \in \Omega_\tau, \tag{2.11}$$

$$U^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{2.12}$$

$$U^0_{j,k} = \varphi(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E. \tag{2.13}$$

i.e.,

$$\mathcal{A}_{h_1}\mathcal{A}_{h_2}\left[i\delta_t^+ U^n_{j,k} + \frac{\beta}{2}\left(\left|U^n_{j,k}\right|^2 + \left|U^{n+1}_{j,k}\right|^2\right)U^{n+\frac{1}{2}}_{j,k}\right] + \mathcal{A}_{h_2}\delta_x^+\delta_x^- U^{n+\frac{1}{2}}_{j,k} + \mathcal{A}_{h_1}\delta_y^+\delta_y^- U^{n+\frac{1}{2}}_{j,k} = 0, \quad (x_j, y_k) \in \Omega_h, \quad t_n \in \Omega_\tau, \tag{2.14}$$

$$U^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{2.15}$$

$$U^0_{j,k} = \varphi(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E. \tag{2.16}$$

3. Some useful lemmas

Lemma 2.2 gives that the eigenvalues of the $J \times J$ order circulant matrix A_1 are in the form of

$$\lambda_j = \frac{5}{6} + \frac{1}{6}cos\left(\frac{2j\pi}{J}\right), \quad j = 0, 1, 2, \ldots, J-1. \tag{3.1}$$

This implies

$$2/3 \leqslant \lambda_j \leqslant 1. \tag{3.2}$$

This indicates that the matrix A_1 is positive definite.

For the real, positive definite, symmetric and circulant matrices A_1 and A_2, we denote

$$M_1 = A_1^{-1}, \quad M_2 = A_2^{-1}. \tag{3.3}$$

Then from Lemma 2.1 and the knowledge of matrices, we know that M_1 and M_2 are also real, positive definite, symmetric and circulant.

Properties of *Kronecker* product give that the matrices

$$M = M_1 \otimes M_2, \quad H_1 = I_2 \otimes M_1, \quad H_2 = M_2 \otimes I_1 \tag{3.4}$$

are three $JK \times JK$ order circulant matrices which are real, positive definite and symmetric. Here I_1 and I_2 are $J \times J$ order and $K \times K$ order unitary matrices, respectively

Thus, we can introduce the following discrete norm

$$|||w||| = [(H_1 w, w) + (H_2 w, w)]^{\frac{1}{2}}, \quad w \in \mathcal{V}_h. \tag{3.5}$$

On the relation between $||| \cdot |||$ and $|| \cdot ||$, there is the following lemma:

Lemma 3.1. *The discrete norms $||| \cdot |||$ and $|| \cdot ||$ are equivalent, in fact, for any grid function $w \in \mathcal{V}_h$, there is*

$$c_1||w|| \leqslant |||w||| \leqslant c_2||w||, \tag{3.6}$$

where $c_1 = \sqrt{2}$, $c_2 = \sqrt{3}$.

Proof. It follows from (3.2), (3.3) and (3.4) that the eigenvalues of $H_s,\ s = 1, 2$ satisfy

$$1 \leqslant \mu_{s,l} \leqslant \frac{3}{2}, \quad s = 1, 2;\ l = 1, 2, \ldots, JK. \tag{3.7}$$

This gives the spectral radius $\rho(H_s) \leqslant \frac{3}{2},\ s = 1, 2$, and consequently

$$||H_s|| = \rho(H_s) \leqslant \frac{3}{2}. \tag{3.8}$$

The properties of H_1 and H_2 give

$$2||u||^2 \leqslant (H_1 u, u) + (H_2 u, u) \leqslant (||H_1|| + ||H_2||)||u||^2 \leqslant 3||u||^2. \tag{3.9}$$

This, together with (3.5), gives (3.6). □

The following lemmas will be used frequently in this paper.

Lemma 3.2. *For any a x-periodic grid function w, a y-periodic grid function v and a grid function $u \in \mathcal{V}_h$, there are*

$$\delta_x^+(M_1 w) = M_1\delta_x^+ w, \quad w = (w_0, w_1, \ldots, w_{J-1}), \tag{3.10}$$
$$\delta_y^+(M_2 v) = M_2\delta_y^+ v, \quad v = (v_0, v_1, \ldots, v_{K-1}), \tag{3.11}$$
$$\delta_x^+(H_1 u) = H_1\delta_x^+ u, \quad \delta_y^+(H_2 u) = H_2\delta_y^+ u. \tag{3.12}$$

Proof. For the kth entry of the vector $M_1 w$, there is

$$(M_1 w)_k = \sum_{j=0}^{J-1} m_j w_{j+k}, \tag{3.13}$$

where the x-periodicity of w was used. This gives

$$\delta_x^+(M_1 w)_k = \frac{1}{h_1}\left(\sum_{j=0}^{J-1} m_j w_{j+k} - \sum_{j=0}^{J-1} m_j w_{j+k-1}\right) = \sum_{j=0}^{J-1} m_j (w_{j+k} - w_{j+k-1})/h_1. \tag{3.14}$$

For the kth element of the vector $M_1\delta_x^+ w$, there is

$$(M_1\delta_x^+ w)_k = \sum_{j=0}^{J-1} m_j (w_{j+k} - w_{j+k-1})/h_1. \tag{3.15}$$

Then (3.10) is gotten from (3.14) and (3.15). Similarly, (3.11) can be proved. And then (3.12) can be proved consequently based on (3.10) and (3.11). □

Lemma 3.3. *For any two grid functions $u, v \in \mathcal{V}_h$, there are*

$$\left(\delta_x^+\delta_x^- u, v\right) = -\left(\delta_x^+ u, \delta_x^+ v\right), \tag{3.16}$$
$$\left(\delta_y^+\delta_y^- u, v\right) = -\left(\delta_y^+ u, \delta_y^+ v\right). \tag{3.17}$$

Proof. Direct calculation and the periodicity of the two grid functions yield (3.16) and (3.17). □

Lemma 3.4. *For the approximation $u^n \in \mathcal{V}_h,\ n = 0, 1, \ldots, N-1$, there are*

$$\mathrm{Im}\left(H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right) = 0, \tag{3.18}$$
$$\mathrm{Re}\left(H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}}, \delta_t^+ u^n\right) = \frac{1}{2\tau}\left(\left|\left|\left|\nabla_h u^{n+1}\right|\right|\right|^2 - |||\nabla_h u^n|||^2\right), \tag{3.19}$$

where "Im(s)" and "Re(s)" mean taking the imaginary part and the real part of a complex number s, respectively.

Proof. Lemma 3.2, Lemma 3.3, definition (3.5) together with the symmetry of H_1 give

$$\begin{aligned}\mathrm{Im}\left(H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right) &= \mathrm{Im}\left(\delta_x^+\delta_x^- u^{n+\frac{1}{2}}, H_1 u^{n+\frac{1}{2}}\right) = \mathrm{Im}\left(\delta_x^+ u^{n+\frac{1}{2}}, \delta_x^+ H_1 u^{n+\frac{1}{2}}\right) = \mathrm{Im}\left(\delta_x^+ u^{n+\frac{1}{2}}, H_1\delta_x^+ u^{n+\frac{1}{2}}\right)\\ &= \mathrm{Im}\left(H_1\delta_x^+ u^{n+\frac{1}{2}}, \delta_x^+ u^{n+\frac{1}{2}}\right).\end{aligned} \tag{3.20}$$

Similarly,

$$\mathrm{Im}\left(H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right) = \mathrm{Im}\left(H_2\delta_y^+ u^{n+\frac{1}{2}}, \delta_y^+ u^{n+\frac{1}{2}}\right). \tag{3.21}$$

It follows from (3.20), (3.21) and the definition (3.5) that

$$\mathrm{Im}\left(H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right) = \mathrm{Im}\left|\left|\left|\nabla_h u^{n+\frac{1}{2}}\right|\right|\right|^2 = 0. \tag{3.22}$$

Also using Lemma 3.2, Lemma 3.3, definition (3.5) and the symmetry of $H_s,\ s = 1, 2$, we obtain

$$\begin{aligned}\mathrm{Re}\left(H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}}, \delta_t^+ u^n\right) &= \frac{1}{2}\mathrm{Re}\left(\delta_x^+ u^{n+1} + \delta_x^+ u^n, H_1\delta_x^+ u^{n+1} - H_1\delta_x^+ u^n\right) + \frac{1}{2}\mathrm{Re}\left(\delta_y^+ u^{n+1} + \delta_y^+ u^n, H_2\delta_y^+ u^{n+1} - H_2\delta_y^+ u^n\right)\\ &= \frac{1}{2\tau}\left(\left|\left|\left|\nabla_h u^{n+1}\right|\right|\right|^2 - |||\nabla_h u^n|||^2\right).\end{aligned} \tag{3.23}$$

This completes the proof. □

Lemma 3.5 [38]. *For time sequences $w = \{w^0, w^1, \ldots, w^n, w^{n+1}\}$ and $g = \{g^0, g^1, \ldots, g^{n-1}, g^n\}$, there is*

$$\left|2\tau\sum_{l=0}^{n} g^l \delta_t w^l\right| \leqslant |w^0|^2 + \tau\sum_{l=1}^{n}|w^l|^2 + |w^{n+1}|^2 + |g^0|^2 + \tau\sum_{l=0}^{n-1}|\delta_t g^l|^2 + |g^n|^2. \tag{3.24}$$

In numerical analysis, discrete version of interpolation inequalities and Sobolev embedding theorems shall be used repeatedly.

Lemma 3.6. *For any grid function u defined on Ω_h^E, it holds that*

$$||u||_4^4 \leqslant ||u||^2\left(2||\nabla_h u|| + \frac{1}{l}||u||\right)^2, \tag{3.25}$$

where $l = \min\{l_1,\ l_2\}$. Generally, for any $p = 2m,\ m \in \mathbb{Z}^+$, there is

$$||u||_p \leqslant ||u||^{\frac{2}{p}}\left(\frac{p}{2}||\nabla_h u|| + \frac{1}{l}||u||\right)^{1-\frac{2}{p}}. \tag{3.26}$$

Proof. For any $m, s = 0, 1, \ldots, M_1$, and $m > s$, using mean value theorem, we have

$$\begin{aligned}|u_{m,k}|^2 - |u_{s,k}|^2 &= \sum_{j=s}^{m-1}\left(|u_{j+1,k}|^2 - |u_{j,k}|^2\right) = \sum_{j=s}^{m-1}\left(|u_{j+1,k}| - |u_{j,k}|\right)\left(|u_{j+1,k}| + |u_{j,k}|\right),\\ &\leqslant \sum_{j=s}^{m-1}|u_{j+1,k} - u_{j,k}|\left(|u_{j+1,k}| + |u_{j,k}|\right) = h_1\sum_{j=s}^{m-1}|\delta_x^+ u_{j,k}|\left(|u_{j+1,k}| + |u_{j,k}|\right) \leqslant 2\left(h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}}.\end{aligned}$$

It is easy to verify the above inequality also holds for $m < s$. Thus we have

$$|u_{m,k}|^2 \leqslant 2\left(h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}} + |u_{s,k}|^2, \quad \forall 0 \leqslant m,\ s \leqslant J-1$$

Multiplying the above inequality by h_1 and summing up for s from 0 to $J-1$, we have

$$l_1|u_{j,k}|^2 \leqslant 2\left(h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}} + h_1\sum_{j=0}^{J-1}|u_{j,k}|^2.$$

Dividing the result by l_1 and noticing that the above inequality holds for $\forall 0 \leqslant m \leqslant J-1$, we have

$$\max_{0\leqslant m\leqslant J-1}|u_{m,k}|^2 \leqslant 2\left(h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}} + \frac{1}{l_1}h_1\sum_{j=0}^{J-1}|u_{j,k}|^2.$$

Multiplying the above inequality by h_2 and summing over k, applying Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}h_2\sum_{k=0}^{K-1}\max_{0\leqslant j\leqslant J-1}|u_{j,k}|^2 &\leqslant 2h_2\sum_{k=0}^{K-1}\left(h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}} + \frac{1}{l_1}||u||^2\\ &\leqslant 2\left(h_2\sum_{k=0}^{K-1}h_1\sum_{j=0}^{J-1}|u_{j,k}|^2\right)^{\frac{1}{2}}\left(h_2\sum_{k=0}^{K-1}h_1\sum_{j=0}^{J-1}|\delta_x^+ u_{j,k}|^2\right)^{\frac{1}{2}} + \frac{1}{l_1}||u||^2 = 2||u||\cdot||\delta_x^+ u|| + \frac{1}{l_1}||u||^2.\end{aligned} \tag{3.27}$$

On the other hand, similar to the analysis above, we have

$$h_1\sum_{j=0}^{J-1}\max_{0\leqslant k\leqslant K-1}|u_{j,k}|^2 \leqslant 2||u||\cdot||\delta_y^+ u|| + \frac{1}{l_2}||u||^2. \tag{3.28}$$

Now we give the estimate of $||u||_4$. Firstly,

$$\begin{aligned}h_1h_2\sum_{j=0}^{J-1}\sum_{k=0}^{K-1}|u_{j,k}|^4 &\leqslant \left(h_1\sum_{j=0}^{J-1}\max_{0\leqslant k\leqslant K-1}|u_{j,k}|^2\right)\left(h_2\sum_{k=0}^{K-1}\max_{0\leqslant j\leqslant J-1}|u_{j,k}|^2\right) \leqslant ||u||^2\cdot\left(2||\delta_x^+ u|| + \frac{1}{l_1}||u||\right)\cdot\left(2||\delta_y^+ u|| + \frac{1}{l_2}||u||\right)\\ &\leqslant ||u||^2\cdot\left(2||\nabla_h u|| + \frac{1}{l}||u||\right)^2.\end{aligned} \tag{3.29}$$

This completes the proof of (3.25). By similar method, we can give the proof of the inequality (3.26).

A general case of the inequality (3.26), i.e.,

$$\|u\|_p \leqslant \|u\|^{\frac{2}{p}}\left(C_p\|\nabla_h u\| + \frac{1}{l}\|u\|\right)^{1-\frac{2}{p}}, \quad 2 \leqslant p < \infty,$$

where $C_p = \max\{2\sqrt{2},\ p/\sqrt{2}\},\ l = \min\{l_1,\ l_2\}$, was proved in [57], but the coefficient C_p is bigger than $p/2$. □

4. Numerical analysis of the compact scheme

In this section, based on the preparation given in Section 2 and Section 3, we analyze the finite difference scheme (2.14)–(2.16).

Using *Kronecker* product [35], the matrix form of (2.7)–(2.9) can be written as

$$i\delta_t^+ u^n + H_1\delta_x^+\delta_x^- u^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- u^{n+\frac{1}{2}} + \frac{\beta}{2}\left(|u^n|^2 + |u^{n+1}|^2\right)u^{n+\frac{1}{2}} = r^n, \quad t_n \in \Omega_\tau, \tag{4.1}$$

$$u^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{4.2}$$

$$u_{j,k}^0 = \varphi(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E, \tag{4.3}$$

where $H_1 = I_2 \otimes M_1,\ H_2 = I_1 \otimes M_2$. Correspondingly, the matrix form of the difference scheme (2.11)–(2.13) can be written as

$$i\delta_t^+ U^n + H_1\delta_x^+\delta_x^- U^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- U^{n+\frac{1}{2}} + \frac{\beta}{2}\left(\left|U^n\right|^2 + \left|U^{n+1}\right|^2\right)U^{n+\frac{1}{2}} = 0, \quad t_n \in \Omega_\tau, \tag{4.4}$$

$$U^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{4.5}$$

$$U_{j,k}^0 = \varphi(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E, \tag{4.6}$$

Remark 4.1. Though (4.4)–(4.6) and (2.14)–(2.16) are two different forms of the same scheme (2.11)–(2.13), the latter is suitable for computing in implementation but not suitable for analysis, because the nonlinear terms distribute in different grids via a certain portion. Thus, in numerical analysis, we will use the former, i.e. the matrix form of (4.4)–(4.6).

4.1. Existence

Lemma 4.1 [12]. *Let $(H, \langle\cdot,\cdot\rangle)$ be a finite-dimensional inner product space, $||\cdot||$ be the associated norm, and $g: H \to H$ be continuous. Assume, moreover, that*

$$\exists\alpha > 0, \quad \forall z \in H, \quad ||z|| = \alpha, \quad \mathrm{Re}\langle g(z), z\rangle > 0.$$

Then, there exists a $z^* \in H$ such that $g(z^*) = 0$ and $||z^*|| \leqslant \alpha$.

Theorem 4.1. *The finite difference scheme (2.14)–(2.16) is solvable.*

Proof. For a fixed n, (4.4) can be written as

$$U^{n+\frac{1}{2}} = U^n + i\frac{\tau}{2}\left[H_1\delta_x^+\delta_x^- U^{n+\frac{1}{2}} + H_2\delta_y^+\delta_y^- U^{n+\frac{1}{2}} + \frac{\beta}{2}\left(\left|U^n\right|^2 + \left|U^{n+1}\right|^2\right)U^{n+\frac{1}{2}}\right], \tag{4.7}$$

$$U^n \in \mathcal{V}_h, \quad U^{n+1} \in \mathcal{V}_h. \tag{4.8}$$

We define a mapping $\mathcal{F}: \mathcal{V}_h \to \mathcal{V}_h$ as follows

$$\mathcal{F}w = w - U^n - i\frac{\tau}{2}\left[H_1\delta_x^+\delta_x^- w + H_2\delta_y^+\delta_y^- w + \frac{\beta}{2}\left(\left|U^n\right|^2 + \left|2w - U^n\right|^2\right)w\right], \tag{4.9}$$

$$U^n \in \mathcal{V}_h, \quad w \in \mathcal{V}_h, \tag{4.10}$$

which is obviously continue. Computing the inner product of (4.9) with w and taking the real part yield

$$\begin{aligned}\mathrm{Re}(\mathcal{F}w, w) &= ||w||^2 - \mathrm{Re}(U^n, w) + \frac{\tau}{2}\mathrm{Im}\left(|||\nabla_h w|||^2 - \frac{\beta}{2}h_1h_2\sum_{j=0}^{J-1}\sum_{k=0}^{K-1}\left(|U_{j,k}^n|^2 + |2w_{j,k} - U_{j,k}^n|^2\right)|w_{j,k}|^2\right)\\ &= ||w||^2 - \mathrm{Re}(U^n, w) \geqslant ||w||^2 - ||w||||U^n|| = ||w||(||w|| - ||U^n||),\end{aligned} \tag{4.11}$$

where Lemma 3.4 was used. It follows from (4.11) that, if $||w|| = ||U^n|| + 1$, there is $\mathrm{Re}(\mathcal{F}w, w) \geqslant 0$. Thus, the existence of $U^{n+\frac{1}{2}}$ follows from Lemma 4.1 and consequently the existence of U^{n+1} is obtained. □

4.2. Conservation

Lemma 4.2. *The scheme (2.14)–(2.16) is conservative in the sense*

$$Q^n = Q^0, \quad t_n \in \overline{\Omega}_\tau, \tag{4.12}$$

$$E^n = E^0, \quad t_n \in \overline{\Omega}_\tau, \tag{4.13}$$

where

$$Q^n = ||U^n||^2, \quad E^n = |||\nabla_h U^n|||^2 - \frac{\beta}{2}||U^n||_4^4,$$

are the called discrete total mass and discrete total energy, respectively.

Proof. Computing the discrete inner product of (4.4) with $U^{n+\frac{1}{2}}$, then taking the imaginary part, we obtain

$$\delta_t^+ ||U^n||^2 = 0, \quad t_n \in \Omega_\tau, \tag{4.14}$$

where Lemma 3.4 was used. This gives (4.12).

Computing the discrete inner product of (4.4) with $\delta_t^+ U^n$, then taking the real part, we obtain

$$\frac{1}{2\tau}\left[\left(|||\nabla_h U^{n+1}|||^2 - \frac{\beta}{2}||U^{n+1}||_4^4\right) - \left(|||\nabla_h U^n|||^2 - \frac{\beta}{2}||U^n||_4^4\right)\right] = 0, \quad t_n \in \Omega_\tau, \tag{4.15}$$

where Lemma 3.4 was used. This gives (4.13). $\square$

4.3. A priori estimate

Lemma 4.3. *Assume that one of the following two conditions is satisfied:*

$$\text{(a)}\ \varphi \in H_p^1(\Omega), \quad \beta < 0, \tag{4.16}$$

$$\text{(b)}\ \varphi \in H_p^1(\Omega), \quad ||\varphi||^2 \leqslant \frac{2-\widetilde{\varepsilon}}{4(1+\varepsilon)\beta}, \quad \beta > 0, \tag{4.17}$$

where ε and $\widetilde{\varepsilon}$ are two positive numbers which can be arbitrary small, then there are some estimates for the solution of the difference scheme (2.14)–(2.16):

$$||U^n|| = ||\phi||, \quad ||\nabla_h U^n|| \leqslant C, \quad t_n \in \overline{\Omega}_\tau. \tag{4.18}$$

Proof. It follows from (4.12) that

$$||U^n|| \leqslant C, \quad t_n \in \overline{\Omega}_\tau. \tag{4.19}$$

Firstly, under the condition (a), we obtain directly from (4.13) that

$$|||\nabla_h U^n||| \leqslant C, \quad t_n \in \overline{\Omega}_\tau.$$

This, together with Lemma 3.1, gives

$$||\nabla_h U^n|| \leqslant C, \quad t_n \in \overline{\Omega}_\tau. \tag{4.20}$$

Secondly, under the condition (b), using Lemma 3.6 and Lemma 4.2, we obtain

$$\begin{aligned} ||U^n||_4^4 &\leqslant ||U^n||^2\left(2||\nabla_h U^n|| + \frac{1}{l}||U^n||\right)^2 \leqslant 4(1+\varepsilon)||U^n||^2||\nabla_h U^n||^2 + (1+\varepsilon^{-1})\frac{1}{l^2}||U^n||^4 \\ &= 4(1+\varepsilon)||\varphi||^2||\nabla_h U^n||^2 + (1+\varepsilon^{-1})\frac{1}{l^2}||U^n||^4 \leqslant \frac{2}{\beta}||\nabla_h U^n||^2 - \widetilde{\varepsilon}||\nabla_h U^n||^2 + (1+\varepsilon^{-1})\frac{1}{l^2}||U^n||^4. \end{aligned} \tag{4.21}$$

where the inequality $(a+b)^2 \leqslant (1+\varepsilon)a^2 + (1+\frac{1}{\varepsilon})b^2$ was used, ε and $\widetilde{\varepsilon}$ are two positive numbers which can be arbitrary small. This, together with Lemma 3.1 and Lemma 4.2, gives

$$||\nabla_h U^n|| \leqslant C. \tag{4.22}$$

This completes the proof. $\square$

4.4. Convergence

On the global error estimate of the difference scheme (2.14)–(2.16), we have the following theorem:

Theorem 4.2. *Under assumptions* (a) *and* (b), *and suppose that* $\varphi(x,y) \in H^1(\Omega), u(x,y,t) \in C^{6,6,4}(\Omega \times (0,T])$, *then the difference solution of the scheme* (2.14)–(2.16) *converges, without any restriction on the grid ratio, to the solution of the periodic-initial value problem* (1.1)–(1.3) *with order* $O(h^4 + \tau^2)$ *in the discrete* L^2*-norm.*

Proof. Denote

$$e^n_{j,k} = u^n_{j,k} - U^n_{j,k}, (x_j, y_k) \in \Omega^E_h, \quad t_n \in \overline{\Omega}_\tau.$$

Subtracting (4.4)–(4.6) from (4.1)–(4.3) yields the following error equation

$$i\delta_t^+ e^n + H_1 \delta_x^+ \delta_x^- e^{n+\frac{1}{2}} + H_2 \delta_y^+ \delta_y^- e^{n+\frac{1}{2}} + \beta g^{n+\frac{1}{2}} = r^n, \quad t_n \in \Omega_\tau, \tag{4.23}$$

$$e^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{4.24}$$

$$e^0_{j,k} = 0, \quad (x_j, y_k) \in \Omega^E_h, \tag{4.25}$$

where

$$g^{n+\frac{1}{2}}_{j,k} = \frac{1}{2}\left(\left|u^n_{j,k}\right|^2 + \left|u^{n+1}_{j,k}\right|^2\right) u^{n+\frac{1}{2}}_{j,k} - \frac{1}{2}\left(\left|U^n_{j,k}\right|^2 + \left|U^{n+1}_{j,k}\right|^2\right) U^{n+\frac{1}{2}}_{j,k}.$$

Noticing

$$\begin{aligned} g^{n+\frac{1}{2}}_{j,k} &= \frac{1}{2}\left(\left|u^n_{j,k}\right|^2 + \left|u^{n+1}_{j,k}\right|^2\right) u^{n+\frac{1}{2}}_{j,k} - \frac{1}{2}\left(\left|U^n_{j,k}\right|^2 + \left|U^{n+1}_{j,k}\right|^2\right) U^{n+\frac{1}{2}}_{j,k} \\ &= \frac{1}{2}\left(u^n_{j,k}\overline{e}^n_{j,k} + u^{n+1}_{j,k}\overline{e}^{n+1}_{j,k} + e^n_{j,k}\overline{U}^n_{j,k} + e^{n+1}_{j,k}\overline{U}^{n+1}_{j,k}\right) u^{n+\frac{1}{2}}_{j,k} + \frac{1}{2}\left(\left|U^n_{j,k}\right|^2 + \left|U^{n+1}_{j,k}\right|^2\right) e^{n+\frac{1}{2}}_{j,k} = \phi^{n+\frac{1}{2}}_{j,k} + \psi^{n+\frac{1}{2}}_{j,k}, \end{aligned} \tag{4.26}$$

where

$$\begin{aligned} \phi^{n+\frac{1}{2}}_{j,k} &= \frac{1}{2}\left(u^n_{j,k}\overline{e}^n_{j,k} + u^{n+1}_{j,k}\overline{e}^{n+1}_{j,k} + e^n_{j,k}\overline{U}^n_{j,k} + e^{n+1}_{j,k}\overline{U}^{n+1}_{j,k}\right) u^{n+\frac{1}{2}}_{j,k} \\ &= \frac{1}{2}\left(u^n_{j,k}\overline{e}^n_{j,k} + u^{n+1}_{j,k}\overline{e}^{n+1}_{j,k} + e^n_{j,k}(\overline{u}^n_{j,k} - \overline{e}^n_{j,k}) + e^{n+1}_{j,k}(\overline{u}^{n+1}_{j,k} - \overline{e}^{n+1}_{j,k})\right) u^{n+\frac{1}{2}}_{j,k} \\ &= \frac{1}{2}\left(u^n_{j,k}\overline{e}^n_{j,k} + \overline{u}^n_{j,k}e^n_{j,k} + u^{n+1}_{j,k}\overline{e}^{n+1}_{j,k} + \overline{u}^{n+1}_{j,k}e^{n+1}_{j,k} - |e^n_{j,k}|^2 - |e^{n+1}_{j,k}|^2\right) u^{n+\frac{1}{2}}_{j,k}, \end{aligned} \tag{4.27}$$

$$\psi^{n+\frac{1}{2}}_{j,k} = \frac{1}{2}\left(|u^n_{j,k}|^2 - 2\mathrm{Re}(u^n_{j,k}\overline{e}^n_{j,k}) + |e^n_{j,k}|^2 + |u^{n+1}_{j,k}|^2 - 2\mathrm{Re}(u^{n+1}_{j,k}\overline{e}^{n+1}_{j,k}) + |e^{n+1}_{j,k}|^2\right) e^{n+\frac{1}{2}}_{j,k}. \tag{4.28}$$

Computing the inner product of (4.23) with $2e^{n+\frac{1}{2}}$ and then taking the imaginary part, we obtain

$$\delta_t^+ ||e^n||^2 + 2\beta \mathrm{Im}(g^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}) = 2\mathrm{Im}(r^n, e^{n+\frac{1}{2}}), \tag{4.29}$$

where Lemma 3.4 was used. Applying Cauchy–Schwartz inequality on the last two terms of (4.29) gives

$$\left|2\beta \mathrm{Im}\left(g^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)\right| = \left|2\beta \mathrm{Im}\left(\phi^{n+\frac{1}{2}}, e^{n+\frac{1}{2}}\right)\right| \leqslant C\left(1 + ||e^n||_4^4 + ||e^{n+1}||_4^4\right)\left(||e^n||^2 + ||e^{n+1}||^2\right) \tag{4.30}$$

and

$$2\left|\left(r^n, e^{n+\frac{1}{2}}\right)\right| \leqslant ||r^n||^2 + ||e^n||^2 + ||e^{n+1}||^2. \tag{4.31}$$

It follows from (4.21) and (4.22) that

$$||U^n||_4^4 \leqslant C. \tag{4.32}$$

This, together with

$$||e^n||_4^4 = ||u^n - U^n||_4^4 \leqslant 8(||u^n||_4^4 + ||U^n||_4^4), \tag{4.33}$$

gives

$$||e^n||_4^4 \leqslant C. \tag{4.34}$$

Substituting (4.30), (4.31) and (4.34) into (4.29) yields

$$\delta_t^+||e^n||^2 \leqslant C\left(||e^n||^2 + ||e^{n+1}||^2 + ||r^n||^2\right). \tag{4.35}$$

This, together with *Gronwall* inequality and Lemma 2.3, gives that, for a sufficiently small τ, there is

$$||e^n||^2 \leqslant C(h^4 + \tau^2), \quad t_n \in \overline{\Omega}_\tau. \tag{4.36}$$

This completes the proof. □

Remark 4.2. Obviously, the two-level compact difference scheme (2.14)–(2.16) is fully nonlinear and implicit, to obtain the solution $U_{j,k}^{n+1}$ in the time level $n+1$, an outer nonlinear iteration for $U_{j,k}^{n+1}$ need be done and the iterative values of $U_{j,k}^{n+1}$ are solved by an inner linear system. Therefore, number of operation for the nonlinear scheme may be large. In fact, if the fully nonlinear system is not solved numerically to extremely high accuracy, e.g. at machine accuracy, then the mass and energy of the numerical solution obtained in practical computation are no longer conserved. This motivates us also consider the following three-level linearized scheme which uses Crank–Nicolson/leap-frog schemes for discretizing linear/nonlinear terms, respectively, as

$$\mathcal{A}_{h_2}\mathcal{A}_{h_1}\left[i\delta_t U_{j,k}^n + \beta\left|U_{j,k}^n\right|^2\mathcal{A}_\tau U_{j,k}^n\right] + \mathcal{A}_{h_2}\delta_x^+\delta_x^+\mathcal{A}_\tau U_{j,k}^n + \mathcal{A}_{h_1}\delta_y^+\delta_y^+\mathcal{A}_\tau U_{j,k}^n = 0, \quad t_n \in \Omega_\tau', \tag{4.37}$$

$$U^n \in \mathcal{V}_h, \quad t_n \in \overline{\Omega}_\tau, \tag{4.38}$$

$$U_{j,k}^0 = \varphi(x_j, y_k), U_{j,k}^1 = \varphi(x_j, y_k) + \tau\varphi_1(x_j, y_k), \quad (x_j, y_k) \in \Omega_h^E, \tag{4.39}$$

where

$$\varphi_1(x_j, y_k) = i[\Delta\varphi(x_j, y_k) + \beta|\varphi(x_j, y_k)|^2\varphi(x_j, y_k)].$$

By similar discussion of the two-level scheme (2.14)–(2.16), we can prove that the linearized scheme (4.37)–(4.39) conserves discrete total mass and discrete total energy, is unconditionally convergent with order $O(h^4 + \tau^2)$ in the discrete L^2-norm.

Remark 4.3. Utilizing the proposed numerical analysis techniques together with induction argument and 'cut off' technique used in [3,4,34], we can prove the convergence of the compact difference scheme for generalized NLS equation in two or three dimensions.

5. Numerical examples

In this section, some numerical tests are given to support our theoretical analysis on convergence and stability. All experiments were carried out via FORTRAN on a PC with 512 RAM. In implementation, we use algebra multi-grid method to compute the algebra equations gotten from using the proposed schemes to solve the examples, and choose the outer iteration tolerance as 10^{-8}.

Example 5.1 [54]. We consider NLS equation in two dimensions

$$iu_t + \Delta u + \beta|u|^2u = 0, \tag{5.1}$$

which generates a progressive plane wave solution

$$u(x, y, t) = A\exp(i(k_1x + k_2y - \omega t)),$$

where

$$\omega = k_1^2 + k_2^2 - \beta|A|^2.$$

Our numerical experiments are conducted in finite domain $[0, 2\pi) \times [0, 2\pi)$ with $A = 1,\ k_1 = k_2 = 1$ and $h_1 = h_2 = h$. Initial data are evaluated by taking $t = 0$ from the exact solution and $(2\pi, 2\pi)$-periodic condition is used.

At first, numerical accuracy of the proposed compact scheme is examined with $\beta = -2$. We compute the maximum norm errors $e(\tau, h) = ||u^N - U^N||_\infty$ of the numerical solution at $t = 1$. Suppose that

$$e(\tau, h) = O(\tau^\alpha + h^\gamma)$$

where α and γ are two positive constants. In Table 1, in order to test $\gamma = 4$, the time step τ is fixed as $\tau = 0.00001$ and the numerical solution is approximated on the halving space grids with the coarsest grid $h = \pi/2$. Thus, we have $e(\tau, h) = C(u)h^\gamma$, and

$$\gamma \approx \log_2(e(\tau, h/2)/e(\tau, h)).$$

Table 1
Errors computed by the proposed scheme with $\beta = -2,\ \tau = 0.00001$ at $t = 1$.

h	$e(\tau, h)$	γ
$\pi/2$	6.2781E−02	–
$\pi/4$	3.7771E−03	4.05
$\pi/8$	2.3298E−04	4.02

Table 2
Errors computed by the proposed scheme with $\beta = -2,\ h = \pi/20$ at $t = 1$.

τ	$e(\tau, h)$	α
0.2	1.6188E−02	–
0.1	4.6880E−03	1.79
0.05	1.2596E−04	1.90
0.025	3.2500E−05	1.95

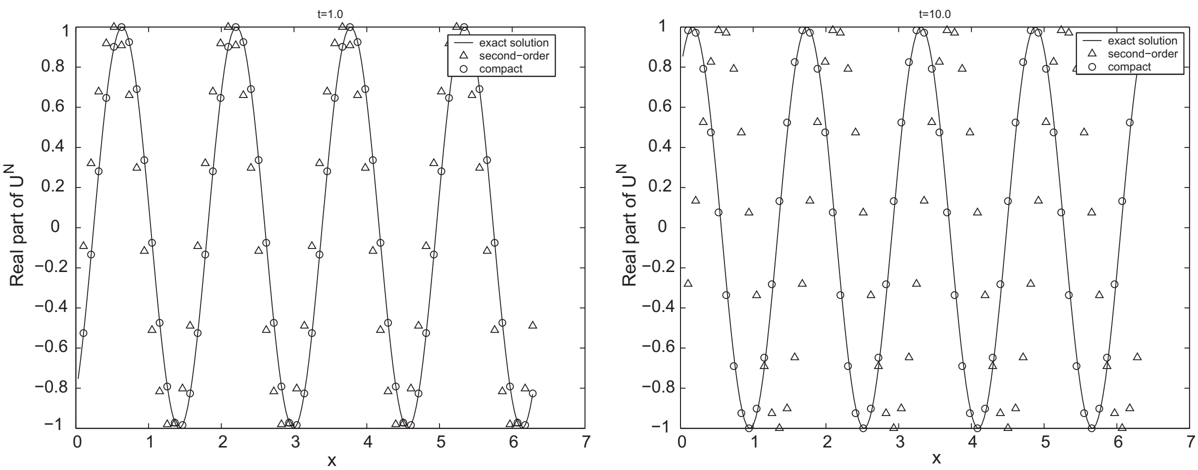

Fig. 1. Numerical simulation of wave $u(x, \pi, 1)$ (left) and $u(x, \pi, 10)$ (right) with $\beta = -2,\ h = \pi/30,\ \tau = 0.001,\ A = 1,\ k_1 = k_2 = 4$.

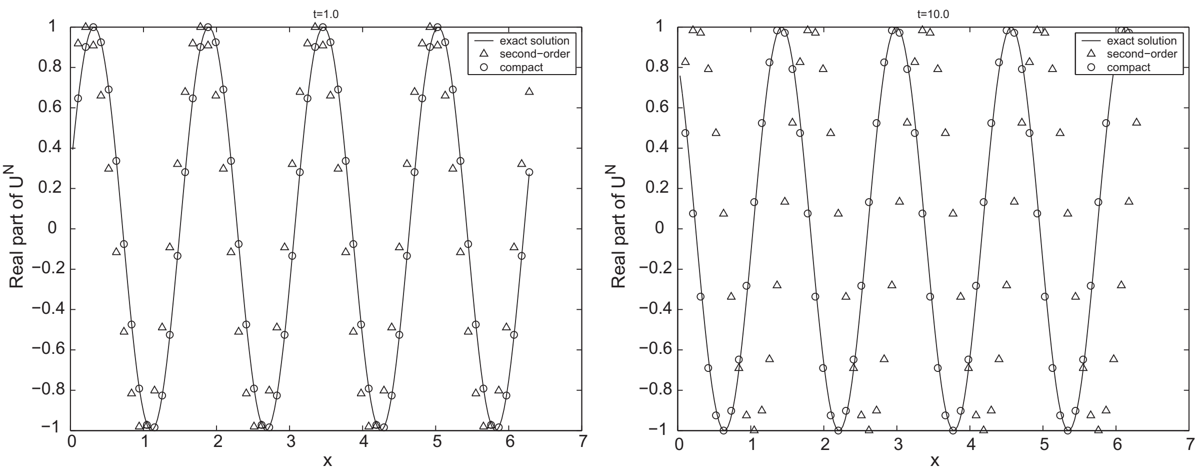

Fig. 2. Numerical simulation of wave $u(x, \pi/10, 1)$ (left) and $u(x, \pi/10, 10)$ (right) with $\beta = -2,\ h = \pi/30,\ \tau = 0.001,\ A = 1,\ k_1 = k_2 = 4$.

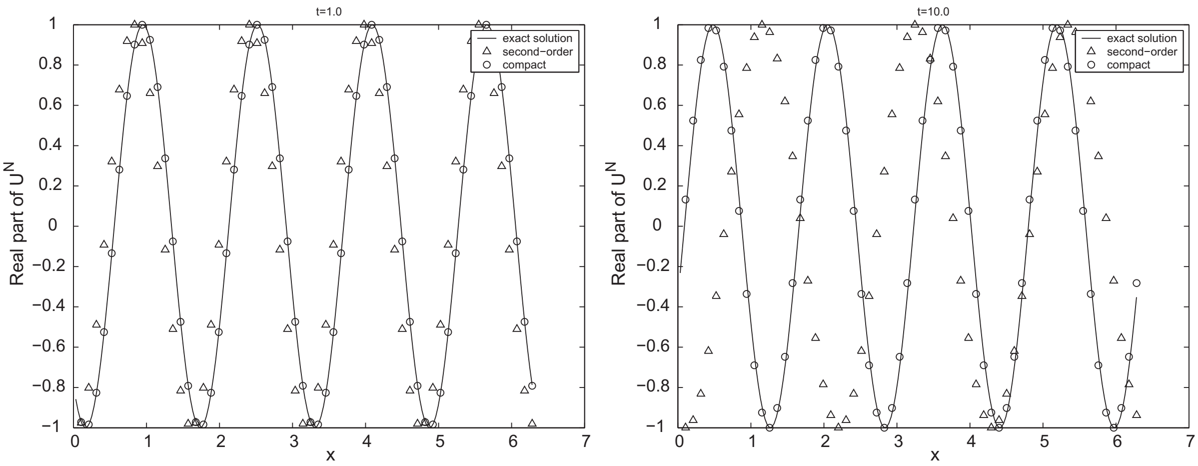

Fig. 3. Numerical simulation of wave $u(x, 2\pi/5, 1)$ (left) and $u(x, 2\pi/5, 10)$ (right) with $\beta = -2,\ h = \pi/30,\ \tau = 0.001,\ A = 1,\ k_1 = k_2 = 4$.

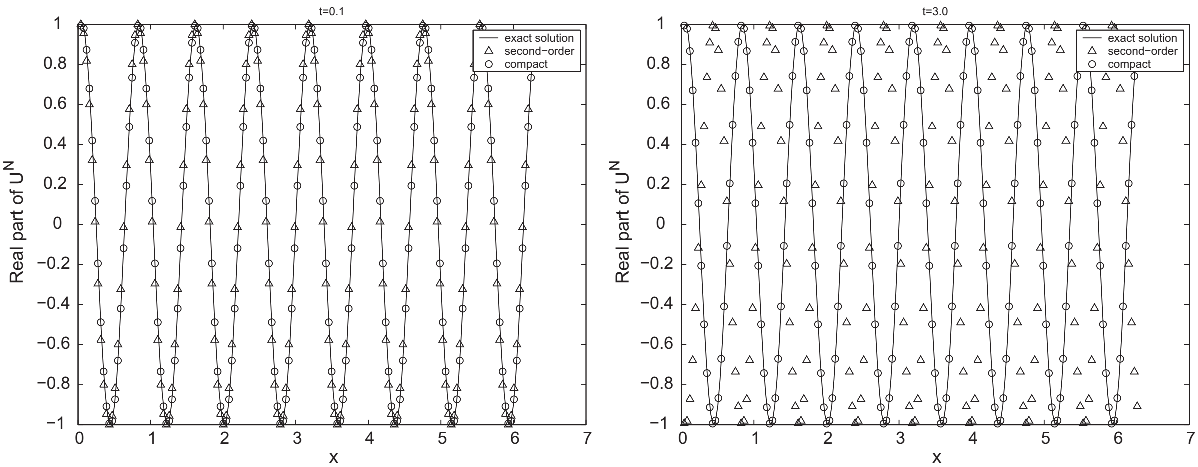

Fig. 4. Numerical simulation of wave $u(x, \pi, 0.1)$ (left) and $u(x, \pi, 3.0)$ (right) with $\beta = -2,\ h = \pi/80,\ \tau = 0.0001,\ A = 1,\ k_1 = k_2 = 8$.

Table 3
Comparison of the scheme (2.14)–(2.16) and the scheme (4.37)–(4.39) with $\beta = 2$ at $t = 1$.

h	τ	Scheme	$e(\tau, h)$	CPU
$\pi/2$	0.04	CFD1	5.4627E−02	7.36
		CFD2	8.3406E−02	2.06
$\pi/4$	0.01	CFD1	3.2455E−03	14.6
		CFD2	5.3715E−03	8.59
$\pi/8$	0.0025	CFD1	1.9988E−04	60.52
		CFD2	3.3414E−04	38.29
$\pi/16$	0.000625	CFD1	1.4549E−05	472.36
		CFD2	2.7462E−05	246.09

Table 1 suggests that $\gamma = 4$. Data in Table 2 are calculated in a similar way with a small space-step $h = \pi/20$, we have $e(\tau, h) = C(u)\tau^{\alpha}$, and

$$\alpha \approx \log_2(e(\tau/2, h)/e(\tau, h)).$$

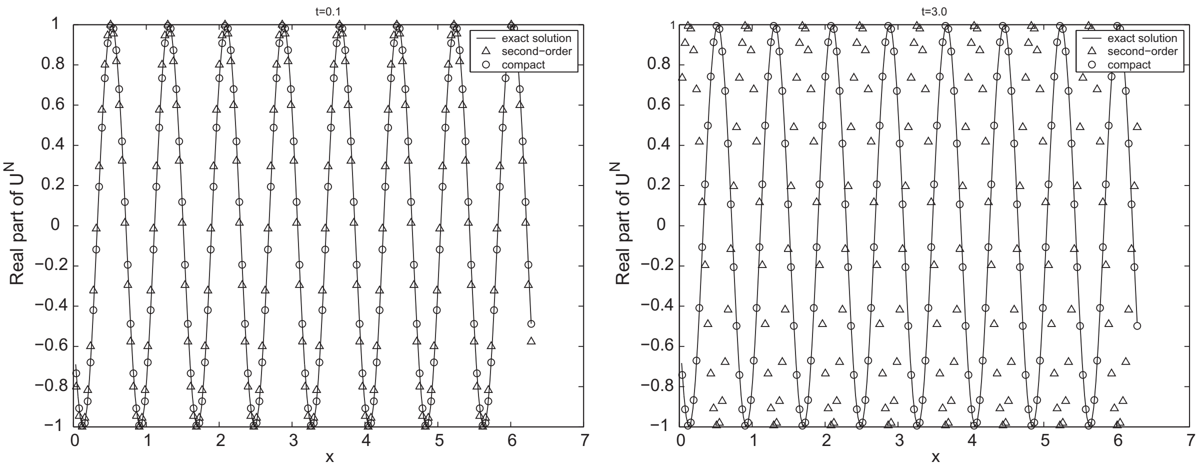

Fig. 5. Numerical simulation of wave $u(x,\pi/10,0.1)$ (left) and $u(x,\pi/10,3.0)$ (right) with $\beta=-2,\ h=\pi/80,\ \tau=0.0001,\ A=1,\ k_1=k_2=8$.

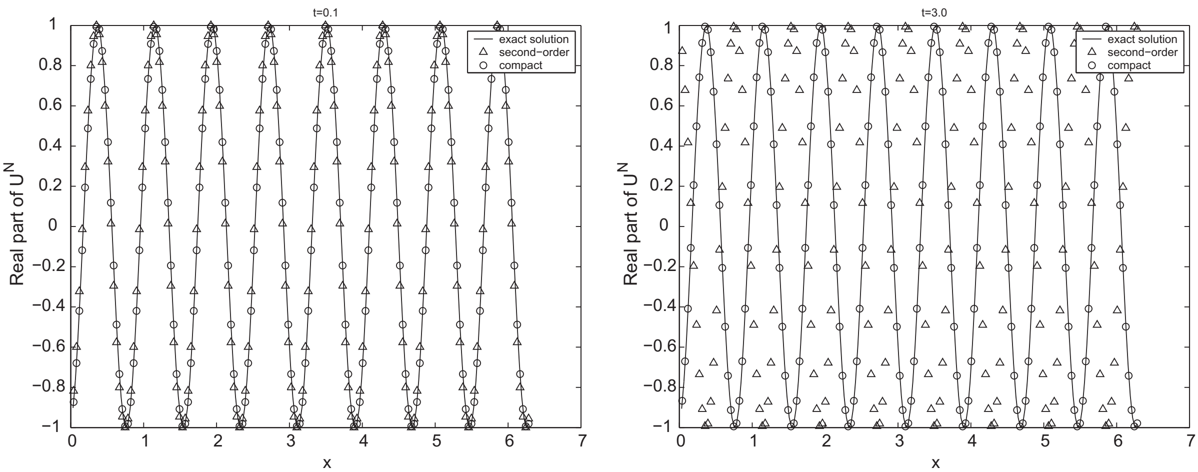

Fig. 6. Numerical simulation of wave $u(x,2\pi/5,0.1)$ (left) and $u(x,2\pi/5,3.0)$ (right) with $\beta=-2,\ h=\pi/80,\ \tau=0.0001,\ A=1,\ k_1=k_2=8$.

Table 2 suggests that $\alpha=2$. Experimentally, we conclude that $\alpha=2$ and $\gamma=4$, which confirm the theoretical accuracy in Theorem 4.2.

Secondly, we compare the compact method (2.14)–(2.16) with the following second-order scheme,

$$i\delta_t^+U_{j,k}^n+\delta_x^2U_{j,k}^{n+\frac{1}{2}}+\delta_y^2U_{j,k}^{n+\frac{1}{2}}+\frac{\beta}{2}\left(\left|U_{j,k}^n\right|^2+\left|U_{j,k}^{n+1}\right|^2\right)U_{j,k}^{n+\frac{1}{2}}=0,\quad (x_j,y_k)\in\Omega_h,\ t_n\in\Omega_\tau, \tag{5.2}$$

$$U^n\in\mathcal{V}_h,\quad t_n\in\overline{\Omega}_\tau, \tag{5.3}$$

$$U_{j,k}^0=\varphi(x_j,y_k),\quad (x_j,y_k)\in\Omega_h^E. \tag{5.4}$$

Since the compact scheme adds only a little computational cost to the second-order one, simulations presented here are focused on the long-time behavior in capturing high-frequency waves. For our comparisons, the real parts of numerical solutions and the exact solution at $y=\pi,\ \pi/10,2\pi/5$, are plotted in figures. Given the spacing $h=\pi/30$ and time-step $\tau=1$, we compute the numerical approximations of wave $u(x,y,t)$ with $A=1,\ k_1=k_2=4$ by the two methods, at $T=1$ and $T=10$ respectively, as shown in Figs. 1–3. Observation shows that, when the time T is small, two approaches approximate the solution well; however, the error of the compact solution is smaller when the simulating time T becomes large. Similar phenomena are seen again in Figs. 4–6, where a high-frequency wave $u(x,y,t)$ with $A=1,\ k_1=k_2=8$ is approximated. From Figs. 1–6, one can find that

(1) The compact scheme is more accurate than the second-order one in simulation of the long-time behavior of high-frequency waves;
(2) The phase error makes a dominant contribution to the total error;
(3) The phenomenon of 'wave-shift' occurs in simulation by using the second-order scheme.

Thirdly, we compare the nonlinear compact difference scheme (2.14)–(2.16) with the linearized compact difference scheme (4.37)–(4.39). Here, we mainly consider the accuracy and the efficiency of them, and the numerical results are listed in Table 3. For simplicity, we denote the scheme (2.14)–(2.16) and the scheme (4.37)–(4.39) as CFD1 and CFD2, respectively. From the results in Table 3 and other numerical results not shown here for brevity, following observations can be drawn:

(1) Though the nonlinear compact difference scheme (2.14)–(2.16) is more accurate than the linearized compact difference scheme (4.37)–(4.39), they have the same convergence order, i.e. $O(h^4 + \tau^2)$.
(2) Though the two compact difference scheme have the same convergence order, the linearized one is more efficient than the nonlinear one.

Combining the above observations, we think that the linearized compact scheme is a better choice in practical computation.
Example 5.2 (*[29,54]*). We then consider the 2D NLS equation

$$iu_t + \Delta u + \beta |u|^2 u = 0, \tag{5.5}$$

with singular solution (when $\beta = 1$). Initial condition

$$u(x, y, 0) = (1 + \sin x)(2 + \sin y),$$

and $(2\pi, 2\pi)$-periodic condition are used in domain $[0, 2\pi) \times [0, 2\pi)$.

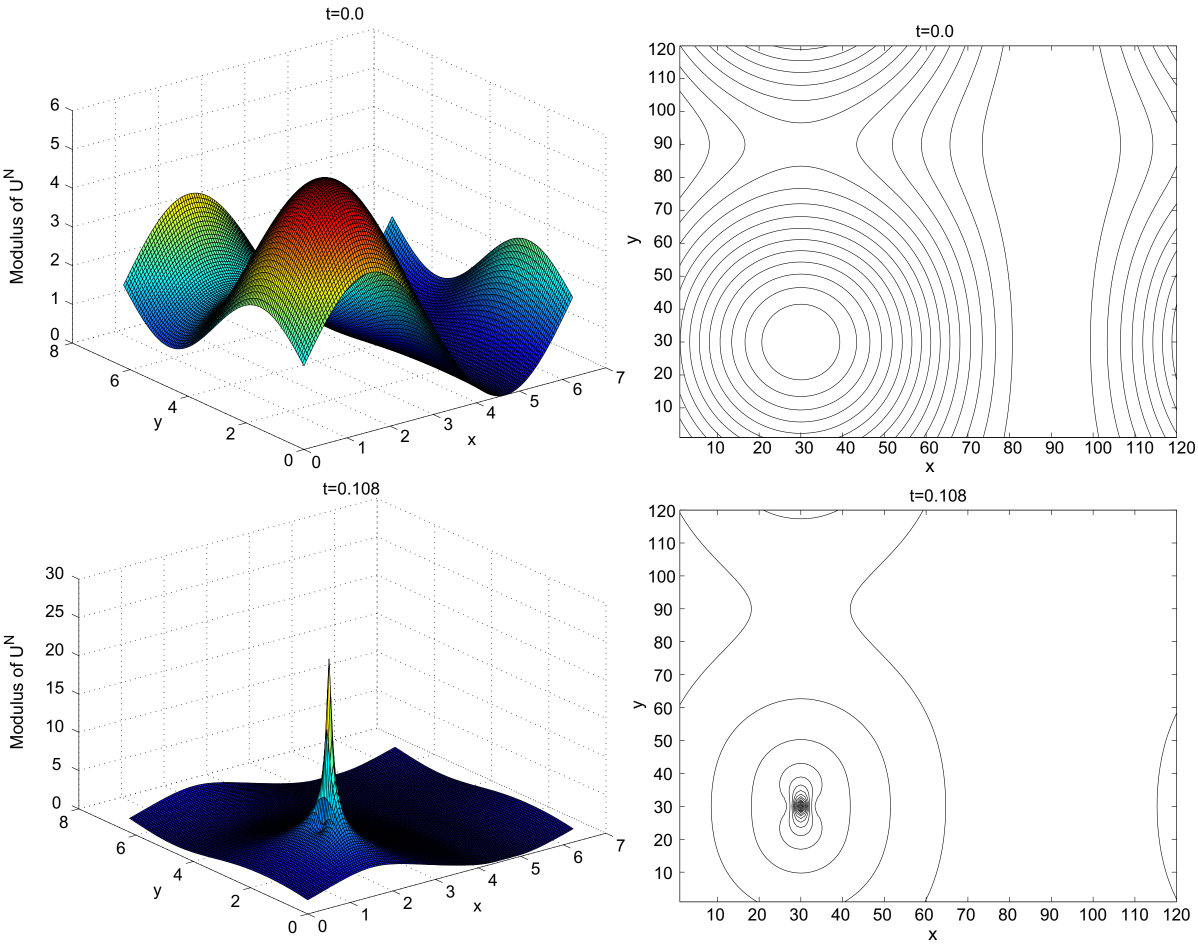

Fig. 7. Profiles and contours of modulus of initial data (top) and of singular solution at $t = 0.108$ (bottom) of Example 5.2 with $\beta = 1$, $h = \pi/60$, $\tau = 0.0001$.

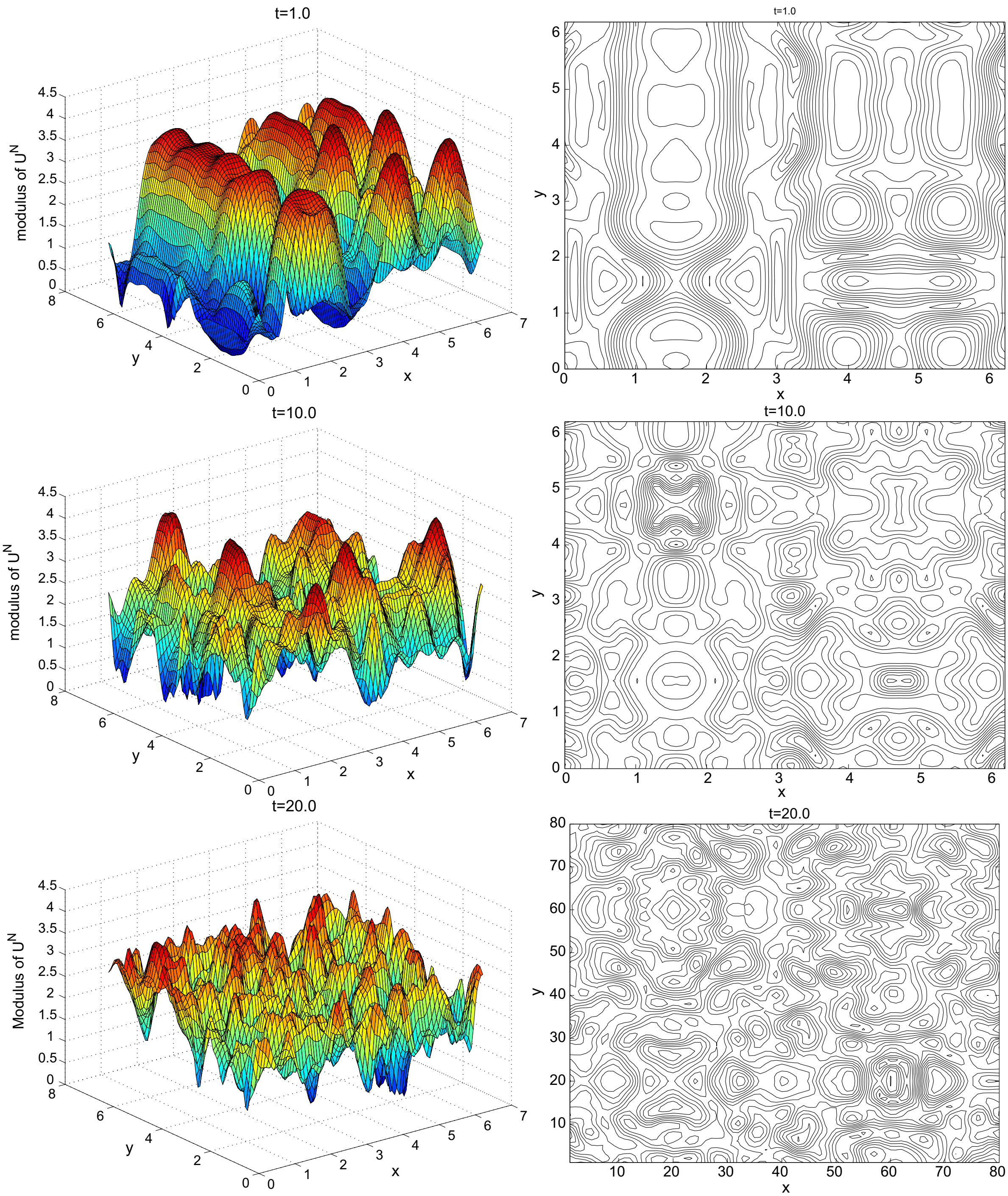

Fig. 8. Profiles and contours of modulus of initial data and numerical solution of Example 5.2 with $\beta = -1,\ h = \pi/40,\ \tau = 0.001$ at $t = 1.0$ (top), $t = 10.0$ (middle) and $t = 20.0$ (bottom), respectively.

Fig. 7 shows the singular solution of Example 5.2, which satisfies the result in [54]. Fig.8 shows the evolution of the waves of Example 5.2, from which we can see that the number of waves becomes more and more along with the evolution of the time.

6. Conclusion

In the literature, there is no result on unconditional convergence of compact difference scheme for solving NLS equation in two dimensions. In this paper, we propose a two-level compact difference scheme which is proved by introducing a new technique to conserve the discrete total mass and energy, then we introduce some important lemmas to prove that, without any restriction on the grid ratio, the proposed scheme is convergent with order $O(h^4 + \tau^2)$ in the discrete L^2-norm. Except for the nonlinear two-level scheme, we propose a linearized three-level compact one which also is unconditionally convergent with order $O(h^4 + \tau^2)$ in the discrete L^2-norm.

Further more, utilizing the proposed techniques together with the induction argument and '*cut off*' technique used in [3], one can prove the conditional forth-order convergence of the compact difference scheme for generalized NLS equation in two or three dimensions.

The techniques proposed in this paper can also be used to prove the convergence of any conservative compact schemes for solving the two-dimensional coupled nonlinear Schrödinger equations, Gross-Pitaevskii equation, Ginzburg–Landau equation, Klein–Gordon–Schrödinger equation and Zakharov equation.

Unconditional convergence in the discrete L_2-norm of compact difference solutions of the NLS equation in 3-dimensions is under way and will be presented in a separated report. Future works are planed to study the unconditional maximum norm convergence of high-order accurate difference schemes for solving the NLS equation in 2- and 3-dimensions.

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