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Local energy-preserving and momentum-preserving algorithms for coupled nonlinear Schrödinger system

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ABSTRACT

In this paper, local energy and momentum conservation laws are proposed for the coupled nonlinear Schrödinger system. The two local conservation laws are more essential than global conservation laws since they are independent of the boundary conditions. Based on the rule that numerical algorithms should conserve the intrinsic properties of the original problems as much as possible, we propose local energy-preserving and momentum-preserving algorithms for the problem. The proposed algorithms conserve the local energy and momentum conservation laws in any local time–space region, respectively. With periodic boundary conditions, we prove the proposed algorithms admit the charge, global energy and global momentum conservation laws. Numerical experiments are conducted to show the performance of the proposed methods. Numerical results verify the theoretical analysis.

1. Introduction

The nonlinear Schrödinger equation arises as model equation with second-order dispersion and cubic nonlinearity for describing the dynamics of slowing vary wave packets in nonlinear optics and fluid dynamics. It has been studied by many authors analytically and numerically [1–4]. If there are two or more modes, the coupled nonlinear Schrödinger system would be the relevant model. The system of two coupled nonlinear Schrödinger (CNLS) system

$$\begin{aligned} &i\phi_t + \phi_{xx} + (|\phi|^2 + \beta|\psi|^2)\phi = 0, \\ &i\psi_t + \psi_{xx} + (|\psi|^2 + \beta|\phi|^2)\psi = 0, \end{aligned} \tag{1}$$

where ϕ and ψ represent the complex amplitudes of two wave packets and β is a real-valued cross-phase modulation coefficient, was first derived by Benney and Newell [5] for two interacting nonlinear wave packets in a dispersive and conservative system. In recent years, the system has attracted a great deal of attentions. With initial conditions

$$\phi(x,0) = \phi_0(x), \quad \psi(x,0) = \psi_0(x), \tag{2}$$

and suitable boundary conditions, such as periodic boundary conditions

$$\phi(x_L,t) = \phi(x_R,t), \quad \psi(x_L,t) = \psi(x_R,t), \tag{3}$$

the CNLS system admits following conservation laws

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- Charge conservation law

$$\mathcal{Q}(t) = \int_{x_L}^{x_R} [|\phi(x,t)|^2 + |\psi(x,t)|^2] dx = \mathcal{Q}(0); \tag{4}$$

- Global energy conservation law (GECL)

$$\mathcal{E}(t) = \int_{x_L}^{x_R} \left[\frac{1}{4}(|\phi|^4 + |\psi|^4) - \frac{1}{2}(|\phi_x|^2 + |\psi_x|^2) + \frac{\beta}{2}|\phi|^2|\psi|^2\right] dx = \mathcal{E}(0); \tag{5}$$

- Global momentum conservation law (GMCL)

$$\mathcal{M}(t) = \int_{x_L}^{x_R} \Im(\phi\bar{\phi}_x + \psi\bar{\psi}_x) dx = \mathcal{M}(0), \tag{6}$$

where $\Im$ stands for the imaginary part.

The CNLS system has been studied numerically in recent years by using various methods. Ismail and Taha [6–8] respectively studied Crank–Nicolson, highly accurate and linearly implicit schemes, which all conserve the charge conservation law, for the CNLS system. In [9–12], the authors discussed the multisymplectic Preissman scheme for the CNLS system. Wang et al. [13] proved the multisymplectic Preissman scheme conserved the charge conservation law and also analyzed some other properties of the scheme. Chen et al. [14] and Ma et al. [15] studied the multisymplectic splitting methods for the problem. Wang and Li [16] simulated the CNLS system by using semi-explicit multisymplectic Euler-box schemes. In [17], a self-adjoint multisymplectic composition scheme was derived to investigate the strongly CNLS system. All methods in [6–15] preserve charge conservation law, however, they don't possess energy and momentum conservation laws. In Refs. [18–21], some methods conserved charge and energy conservation laws were proposed for simulating the CNLS system.

It should be noted that the energy-preserving property (5) has its own constrain. The reason is that the energy conservation law is defined on the global times level and it inevitably depends on the boundary conditions. In other words, if the boundary conditions are not suitable, the energy-preserving algorithms can't be applied to the problem. Up to now, all the energy-preserving methods proposed in Refs. [18–21] and so on are constructed with suitable boundary conditions, such as periodic or homogeneous boundary conditions. However, the boundary conditions can't be suitable in many cases.

If we set complex functions $\phi(x,t) = u(x,t) + iv(x,t)$ and $\psi(x,t) = p(x,t) + iq(x,t)$, where $u(x,t), v(x,t), p(x,t)$ and $q(x,t)$ are real-valued functions, the CNLS system (1) can be rewritten into a combination of ordinary differential equations (ODEs)

$$\begin{cases} -u_t = f + [u^2 + v^2 + \beta(p^2 + q^2)]v, \\ v_t = g + [u^2 + v^2 + \beta(p^2 + q^2)]u, \end{cases} \quad \begin{cases} b_x = f, \\ v_x = b, \end{cases} \quad \begin{cases} a_x = g, \\ u_x = a, \end{cases} \tag{7a}$$

$$\begin{cases} -p_t = m + [p^2 + q^2 + \beta(u^2 + v^2)]q, \\ q_t = n + [p^2 + q^2 + \beta(u^2 + v^2)]p, \end{cases} \quad \begin{cases} d_x = m, \\ q_x = d, \end{cases} \quad \begin{cases} c_x = n, \\ p_x = c, \end{cases} \tag{7b}$$

Proposition 1. *The CNLS system (7) admits the local energy conservation law (LECL)*

$$\partial_t\left[\frac{1}{4}(|\phi|^4 + |\psi|^4) + \frac{\beta}{2}|\phi|^2|\psi|^2 - \frac{1}{2}(|\phi_x|^2 + |\psi_x|^2)\right] + \partial_x(bv_t + au_t + dq_t + cp_t) = 0, \tag{8}$$

and the local momentum conservation law (LMCL)

$$\partial_t(v_x u - u_x v + q_x p - p_x q) + \partial_x[vu_t - uv_t + qp_t - pq_t + a^2 + b^2 + c^2 + d^2 + \frac{1}{2}(|\phi|^4 + |\psi|^4) + \beta|\phi|^2|\psi|^2] = 0. \tag{9}$$

Proof. In Eq. (7a), multiplying $-u_t = f + [u^2 + v^2 + \beta(p^2 + q^2)]v$ by v_t and $v_t = g + [u^2 + v^2 + \beta(p^2 + q^2)]u$ by u_t gives

$$\begin{aligned} 0 &= gu_t + fv_t + (|\phi|^2 + \beta|\psi|^2)(uu_t + vv_t) = a_x u_t + b_x v_t + \frac{1}{2}(|\phi|^2 + \beta|\psi|^2)\partial_t|\phi|^2 \\ &= \partial_x(au_t + bv_t) - au_{xt} - bv_{xt} + \frac{1}{2}(|\phi|^2 + \beta|\psi|^2)\partial_t|\phi|^2 \\ &= \partial_x(au_t + bv_t) - \frac{1}{2}\partial_t(a^2 + b^2) + \frac{1}{2}|\phi|^2\partial_t|\phi|^2 + \frac{\beta}{2}|\psi|^2\partial_t|\phi|^2 \\ &= \partial_x(au_t + bv_t) + \partial_t\left(\frac{1}{4}|\phi|^4 - \frac{1}{2}|\phi_x|^2\right) + \frac{\beta}{2}|\psi|^2\partial_t|\phi|^2. \end{aligned} \tag{10}$$

Similarly, it follows from Eq. (7b) that

$$\partial_x(cp_t + dq_t) + \partial_t\left(\frac{1}{4}|\psi|^4 - \frac{1}{2}|\psi_x|^2\right) + \frac{\beta}{2}|\phi|^2\partial_t|\psi|^2 = 0. \tag{11}$$

Adding Eq. (10) on Eq. (11) and noting

$$\frac{\beta}{2}|\psi|^2\partial_t|\phi|^2 + \frac{\beta}{2}|\phi|^2\partial_t|\psi|^2 = \frac{\beta}{2}\partial_t(|\phi|^2|\psi|^2),$$

we obtain LECL (8). The LMCL (9) can be also proved easily. Here, we ellipsis the details of proof. □

If the boundary conditions are suitable, such as periodic boundary conditions (3), integrating LECL (8) and LMCL (9) over the space domain $[x_L, x_R]$, we obtain the GECL and GMCL. They are just (5) and (6). It is worth to note that the LECL and LMCL are local. It means that they are independent of the boundary conditions. That is to say the local conservation laws (8) and (9) are preserved in any local time–space region. Therefore, the local conservation laws are more essential than global conservation laws in physics. Based on the rule that numerical algorithms should preserve the intrinsic properties of the original problems as much as possible, we expect the constructed algorithms for the CNLS system can preserve the LECL or LMCL, not only preserve GECL or GMCL. Actually, if the algorithms are local energy-preserving or local momentum-preserving algorithms, summing the LECL or LMCL over the space index, we will obtain the GECL or GMCL. That is to say, the local energy-preserving or local momentum-preserving algorithm includes global energy-preserving or global momentum-preserving algorithm. Therefore, whether and how to construct new algorithms preserved the LECL or LMCL, are worth to study. For the CNLS problem, if the idea is feasible, we can extend the applying scope of the traditional global energy-preserving and momentum-preserving algorithms.

The aim of this paper is to construct some new algorithms, which preserve the LECL or LMCL, for the CNLS system (1). Then, some numerical experiments are conducted to exhibit the performance and efficiency of the proposed algorithms.

A plan of the paper is as follows: some operator definitions and its properties are given in Section 2. In Section 3, two local energy-preserving algorithms are proposed for the CNLS system. Then, their conservative properties are discussed. In Section 4, we derive a local momentum-preserving algorithm for the problem and also prove some conservative properties of the method. Stability analysis is provided in Section 5. Numerical experiments are presented in Section 6. Finally, we make conclusions.

2. Operator definitions and properties

As usual, we introduce some notations: $x_j = jh, t_k = k\tau, \quad j = 1, 2, \cdots, N+1; k = 0, 1, 2, \cdots$, where $h = (x_R - x_L)/N$, τ are spatial length and temporal step span. The approximation of the value of the function $u(x,t)$ at the node (x_j, t_k) is denoted by u_j^k. We also need to define vector norms

$$\|u^k\| = \left(h\sum_{j=1}^{N}|u_j^k|^2\right)^{1/2}, \quad \|u^k\|_4 = \left(h\sum_{j=1}^{N}|u_j^k|^4\right)^{1/4}.$$

In order to derive the algorithms conveniently, we also give some operators definition. Define the finite difference operators

$$\delta_t f_j^k = \frac{f_j^{k+1} - f_j^k}{\tau}, \quad \delta_x f_j^k = \frac{f_{j+1}^k - f_j^k}{h},$$

and averaging operators

$$A_t f_j^k = \frac{f_j^{k+1} + f_j^k}{2}, \quad A_x f_j^k = \frac{f_{j+1}^k + f_j^k}{2}.$$

The operators have the following properties:

- Commutative law

$$\delta_t\delta_x f_j^k = \delta_x\delta_t f_j^k, \quad A_tA_x f_j^k = A_xA_t f_j^k, \quad \delta_tA_x f_j^k = A_x\delta_t f_j^k, \quad A_t\delta_x f_j^k = \delta_xA_t f_j^k.$$

- Discrete Leibnite rule

$$\delta_x(f\cdot g)_j^k = \left(af_{j+1}^k + (1-a)f_j^k\right)\cdot\delta_x g_j^k + \delta_x f_j^k\cdot\left((1-a)g_{j+1}^k + ag_j^k\right), \quad \forall 0 \leqslant a \leqslant 1.$$

Especially, we have

$$
\begin{aligned}
&a=0, \quad \delta_x(f\cdot g)_j^k = f_j^k\cdot\delta_x g_j^k + \delta_x f_j^k\cdot g_{j+1}^k,\\
&a=\frac{1}{2}, \quad \delta_x(f\cdot g)_j^k = A_x f_j^k\cdot\delta_x g_j^k + \delta_x f_j^k\cdot A_x g_j^k,\\
&a=1, \quad \delta_x(f\cdot g)_j^k = f_{j+1}^k\cdot\delta_x g_j^k + \delta_x f_j^k\cdot g_j^k.
\end{aligned}
\tag{12}
$$

Taking $f=g$ gives

$$
\delta_x\left[\frac{1}{2}\left(f_j^k\right)^2\right] = \delta_x f_j^k\cdot A_x f_j^k \quad \textit{and} \quad \delta_x\left(\frac{1}{2}f_{j-1}^k\cdot f_j^k\right) = f_j^k\cdot A_x\delta_x f_{j-1}^k.
\tag{13}
$$

Similarly, we can obtain a series of analogous discrete Leibnite rules in the time direction.

3. Local energy-preserving algorithms

In this section, we will use the concatenating method to construct local energy-preserving algorithms for the CNLS system (1). The idea is to use the Runge–Kutta method to deal with time and space derivatives separately.

3.1. Local energy-preserving scheme I (LEP-I)

In ODEs (7), discreting the time derivatives by using mid-point rule and space derivatives by using leap-frog rule, we obtain scheme

$$
\begin{cases}
-\delta_t u_j^k = A_t f_j^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t v_j^k,\\
\delta_t v_j^k = A_t g_j^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t u_j^k,
\end{cases}
\quad
\begin{cases}
\delta_x b_j^k = f_j^k,\\
\delta_x v_j^k = b_{j+1}^k,
\end{cases}
\quad
\begin{cases}
\delta_x a_j^k = g_j^k,\\
\delta_x u_j^k = a_{j+1}^k,
\end{cases}
\tag{14a}
$$

$$
\begin{cases}
-\delta_t p_j^k = A_t m_j^k + A_t\left[\left|\psi_j^k\right|^2 + \beta\left|\phi_j^k\right|^2\right]\cdot A_t q_j^k,\\
\delta_t q_j^k = A_t n_j^k + A_t\left[\left|\psi_j^k\right|^2 + \beta\left|\phi_j^k\right|^2\right]\cdot A_t p_j^k,
\end{cases}
\quad
\begin{cases}
\delta_x d_j^k = m_j^k,\\
\delta_x q_j^k = d_{j+1}^k,
\end{cases}
\quad
\begin{cases}
\delta_x c_j^k = n_j^k,\\
\delta_x p_j^k = c_{j+1}^k,
\end{cases}
\tag{14b}
$$

where $\left|\phi_j^k\right|^2 = \left(u_j^k\right)^2 + \left(v_j^k\right)^2$ and $\left|\psi_j^k\right|^2 = \left(p_j^k\right)^2 + \left(q_j^k\right)^2$. Eliminating the auxiliary variables, we obtain a scheme

$$
\begin{cases}
-\delta_t u_j^k = A_t\delta_x^2 v_{j-1}^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t v_j^k,\\
\delta_t v_j^k = A_t\delta_x^2 u_{j-1}^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t u_j^k
\end{cases}
\tag{15a}
$$

$$
\begin{cases}
-\delta_t p_j^k = A_t\delta_x^2 q_{j-1}^k + A_t\left[\left|\psi_j^k\right|^2 + \beta\left|\phi_j^k\right|^2\right]\cdot A_t q_j^k,\\
\delta_t q_j^k = A_t\delta_x^2 p_{j-1}^k + A_t\left[\left|\psi_j^k\right|^2 + \beta\left|\phi_j^k\right|^2\right]\cdot A_t p_j^k
\end{cases}
\tag{15b}
$$

or an equivalent scheme

$$
\begin{cases}
i\delta_t\phi_j^k + A_t\delta_x^2\phi_{j-1}^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t\phi_j^k = 0,\\
i\delta_t\psi_j^k + A_t\delta_x^2\psi_{j-1}^k + A_t\left[\left|\psi_j^k\right|^2 + \beta\left|\phi_j^k\right|^2\right]\cdot A_t\psi_j^k = 0.
\end{cases}
\tag{16}
$$

Next, we analyze some conservative properties of above algorithm.

Theorem 1. *The algorithm (15) or (16) satisfies the discrete local energy conservation law*

$$
\begin{aligned}
\mathcal{E}(x_j,t_k) = {} & \delta_t\left[\frac{1}{4}\left(\left|\phi_j^k\right|^4 + \left|\psi_j^k\right|^4\right) + \frac{\beta}{2}\left|\phi_j^k\right|^2\left|\psi_j^k\right|^2 - \frac{1}{2}\left(\left|\delta_x\phi_j^k\right|^2 + \left|\delta_x\psi_j^k\right|^2\right)\right]\\
& + \delta_x\left(A_t b_j^k\cdot\delta_t v_j^k + A_t a_j^k\cdot\delta_t u_j^k + A_t d_j^k\cdot\delta_t q_j^k + A_t c_j^k\cdot\delta_t p_j^k\right) = 0.
\end{aligned}
\tag{17}
$$

In other words, the algorithms is a local energy-preserving algorithm.

Proof. In Eq. (14a), multiplying the term $-\delta_t u_j^k = A_t f_j^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t v_j^k$ by $\delta_t v_j^k$ and the term $\delta_t v_j^k = A_t g_j^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot A_t u_j^k$ by $\delta_t u_j^k$, and then adding them together gives

$$A_t f_j^k \cdot \delta_t v_j^k + A_t g_j^k \cdot \delta_t u_j^k + A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot\left(A_t u_j^k \cdot \delta_t u_j^k + A_t v_j^k \cdot \delta_t v_j^k\right) = 0. \tag{18}$$

By using the Leibnite rule (12), we obtain the term

$$\begin{aligned}
A_t f_j^k \cdot \delta_t v_j^k + A_t g_j^k \cdot \delta_t u_j^k &= A_t\delta_x b_j^k \cdot \delta_t v_j^k + A_t\delta_x a_j^k \cdot \delta_t u_j^k \\
&= \delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k\right) - A_t b_{j+1}^k \cdot \delta_x\delta_t v_j^k - A_t a_{j+1}^k \cdot \delta_x\delta_t u_j^k \\
&= \delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k\right) - A_t\delta_x v_j^k \cdot \delta_x\delta_t v_j^k - A_t\delta_x u_j^k \cdot \delta_x\delta_t u_j^k \\
&= \delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k\right) - \frac{1}{2}\delta_t\left[\left(\delta_x v_j^k\right)^2 + \left(\delta_x u_j^k\right)^2\right] \\
&= \delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k\right) - \frac{1}{2}\delta_t|\delta_x\phi_j^k|^2,
\end{aligned} \tag{19}$$

and the term

$$A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\cdot\left(A_t u_j^k \cdot \delta_t u_j^k + A_t v_j^k \cdot \delta_t v_j^k\right) = \frac{1}{2}\delta_t\left|\phi_j^k\right|^2 \cdot A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right] = \frac{1}{4}\delta_t\left|\phi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|\phi_j^k\right|^2 \cdot A_t|\psi_j^k|^2. \tag{20}$$

Therefore, the Eq. (18) can be rewritten into

$$\delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k\right) - \frac{1}{2}\delta_t|\delta_x\phi_j^k|^2 + \frac{1}{4}\delta_t\left|\phi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|\phi_j^k\right|^2 \cdot A_t\left|\psi_j^k\right|^2 = 0. \tag{21}$$

Similarly, from Eq. (14b), we have

$$\delta_x\left(A_t d_j^k \cdot \delta_t q_j^k + A_t c_j^k \cdot \delta_t p_j^k\right) - \frac{1}{2}\delta_t\left|\delta_x\psi_j^k\right|^2 + \frac{1}{4}\delta_t\left|\psi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|\psi_j^k\right|^2 \cdot A_t\left|\phi_j^k\right|^2 = 0. \tag{22}$$

Summing Eqs. (21) and (22) and noting

$$\frac{\beta}{2}\delta_t\left|\phi_j^k\right|^2 \cdot A_t\left|\psi_j^k\right|^2 + \frac{\beta}{2}\delta_t\left|\psi_j^k\right|^2 \cdot A_t\left|\phi_j^k\right|^2 = \frac{\beta}{2}\delta_t\left(\left|\phi_j^k\right|^2\left|\psi_j^k\right|^2\right),$$

we complete the proof. □

Corollary 1. *For the periodic conditions (3), the LEP-I algorithm (15) or (16) possesses the global energy conservation law*

$$\mathcal{E}^{k+1} = \mathcal{E}^k = \cdots = \mathcal{E}^1 = \mathcal{E}^0, \tag{23}$$

where

$$\mathcal{E}^k = \frac{1}{4}\left(\left\|\phi_j^k\right\|_4^4 + \left\|\psi_j^k\right\|_4^4\right) - \frac{1}{2}\left(\left\|\delta_x\phi_j^k\right\|^2 + \left\|\delta_x\psi_j^k\right\|^2\right) + \frac{\beta}{2}h\sum_{j=1}^N\left|\phi_j^k\right|^2\left|\psi_j^k\right|^2.$$

Proof. Summing the discrete local energy conservation law (17) over all space index j, we have

$$\begin{aligned}
&\delta_t\sum_{j=1}^N\left[\frac{1}{4}\left(\left|\phi_j^k\right|^4 + \left|\psi_j^k\right|^4\right) + \frac{\beta}{2}\left|\phi_j^k\right|^2\left|\psi_j^k\right|^2 - \frac{1}{2}\left(\left|\delta_x\phi_j^k\right|^2 + \left|\delta_x\psi_j^k\right|^2\right)\right] \\
&\quad + \sum_{j=1}^N\delta_x\left(A_t b_j^k \cdot \delta_t v_j^k + A_t a_j^k \cdot \delta_t u_j^k + A_t d_j^k \cdot \delta_t q_j^k + A_t c_j^k \cdot \delta_t p_j^k\right) = 0.
\end{aligned}$$

The last term in left-hand in above equation vanishes because of the periodic boundary conditions. Therefore, we have

$$\delta_t\sum_{j=1}^N\left[\frac{1}{4}\left(\left|\phi_j^k\right|^4 + \left|\psi_j^k\right|^4\right) + \frac{\beta}{2}\left|\phi_j^k\right|^2\left|\psi_j^k\right|^2 - \frac{1}{2}\left(\left|\delta_x\phi_j^k\right|^2 + \left|\delta_x\psi_j^k\right|^2\right)\right] = 0,$$

which implies global energy conservation law (23). □

Remark 1.

1. The local energy conservation law (17) is conserved exactly in any local region. The values of variables a_j^k, b_j^k, c_j^k and d_j^k are determined by Eq. (14).
2. It is evident that Corollary 1 is consistent with the global energy conservation law (5). The LEP-I algorithm can preserve the global energy conservation law exactly.

In general, the charge conservation law (4), which is a quadratic invariant, plays an important part in quantum physics. Therefore, to discuss whether it can be conserved exactly is necessary.

Theorem 2. *With periodic boundary conditions (3), the LEP-I algorithm (16) preserves the charge exactly, namely,*

$$\mathcal{Q}^{k+1} = \|\phi^{k+1}\|^2 + \|\psi^{k+1}\|^2 = \cdots = \|\phi^0\|^2 + \|\psi^0\|^2 = \mathcal{Q}^0, \tag{24}$$

that is, the algorithm is unitary. Thus the method is unconditionally stable with respect to the initial value.

Proof. Multiplying the first line of (16) by $2hA_t\overline{\phi_j^k}$ and summing it over space index j, we have

$$i\sum_{j=1}^{N}\delta_t\phi_j^k\cdot 2hA_t\overline{\phi_j^k} + \sum_{j=1}^{N}A_t\delta_x^2\phi_{j-1}^k\cdot 2hA_t\overline{\phi_j^k} + \sum_{j=1}^{N}A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\left(A_t\phi_j^k\cdot 2hA_t\overline{\phi_j^k}\right) = 0. \tag{25}$$

The first term

$$i\sum_{j=1}^{N}\delta_t\phi_j^k\cdot 2hA_t\overline{\phi_j^k} = \frac{ih}{\tau}\sum_{j=1}^{N}\left(\left|\phi_j^{k+1}\right|^2 - \left|\phi_j^k\right|^2 + 2i\Im\left(\phi_j^{k+1}\overline{\phi_j^k}\right)\right) = \frac{i}{\tau}(\|\phi^{k+1}\|^2 - \|\phi^k\|^2) - \frac{2h}{\tau}\sum_{j=1}^{N}\Im\left(\phi_j^{k+1}\overline{\phi_j^k}\right). \tag{26}$$

The second term

$$\begin{aligned}\sum_{j=1}^{N}A_t\delta_x^2\phi_{j-1}^k\cdot 2hA_t\overline{\phi_j^k} &= \frac{2}{h}\left(\sum_{j=1}^{N}A_t\phi_{j-1}^k\cdot A_t\overline{\phi_j^k} + 2\sum_{j=1}^{N}\left|A_t\phi_j^k\right|^2 + \sum_{j=1}^{N}A_t\phi_{j+1}^k\cdot A_t\overline{\phi_j^k}\right)\\ &= \frac{2}{h}\left[2\Re\left(\sum_{j=1}^{N}A_t\phi_{j-1}^k\cdot A_t\overline{\phi_j^k}\right) + 2\sum_{j=1}^{N}\left|A_t\phi_j^k\right|^2\right],\end{aligned} \tag{27}$$

where $\Re$ stands for the real part of complex function, is a real function. The last term

$$\sum_{j=1}^{N}A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\left(A_t\phi_j^k\cdot 2hA_t\overline{\phi_j^k}\right) = 2h\sum_{j=1}^{N}A_t\left[\left|\phi_j^k\right|^2 + \beta\left|\psi_j^k\right|^2\right]\left|A_t\phi_j^k\right|^2, \tag{28}$$

is also a real function. Therefore, the imaginary part of (25) implies

$$\|\phi^{k+1}\|^2 = \|\phi^k\|^2. \tag{29}$$

Similarly, multiplying the second line of (16) with $2hA_t\overline{\psi_j^k}$ and summing it over space index j gives

$$\|\psi^{k+1}\|^2 = \|\psi^k\|^2. \tag{30}$$

Summing (29) and (30) yields (24). □

Remark 2. From Eqs. (29) and (30), we find that the algorithm (16) also preserves charge conservation laws $\|\phi^{k+1}\|^2 = \|\phi^k\|^2$ and $\|\psi^{k+1}\|^2 = \|\psi^k\|^2$.

3.2. Local energy-preserving scheme II (LEP-II)

In ODEs (7), applying mid-point rule to both time and space derivatives gives a scheme.

$$\begin{cases} -\delta_t A_x u_j^k = A_t A_x f_j^k + A_t\left[\left|A_x\phi_j^k\right|^2 + \beta\left|A_x\psi_j^k\right|^2\right]\cdot A_t A_x v_j^k,\\ \delta_t A_x v_j^k = A_t A_x g_j^k + A_t\left[\left|A_x\phi_j^k\right|^2 + \beta\left|A_x\psi_j^k\right|^2\right]\cdot A_t A_x u_j^k,\end{cases} \quad \begin{cases}\delta_x b_j^k = A_x f_j^k,\\ \delta_x v_j^k = A_x b_j^k,\end{cases} \quad \begin{cases}\delta_x a_j^k = A_x g_j^k,\\ \delta_x u_j^k = A_x a_j^k,\end{cases} \tag{31a}$$

$$\begin{cases} -\delta_t A_x p_j^k = A_t A_x m_j^k + A_t\left[\left|A_x\psi_j^k\right|^2 + \beta\left|A_x\phi_j^k\right|^2\right]\cdot A_t A_x q_j^k,\\ \delta_t A_x q_j^k = A_t A_x n_j^k + A_t\left[\left|A_x\psi_j^k\right|^2 + \beta\left|A_x\phi_j^k\right|^2\right]\cdot A_t A_x p_j^k,\end{cases} \quad \begin{cases}\delta_x d_j^k = A_x m_j^k,\\ \delta_x q_j^k = A_x d_j^k,\end{cases} \quad \begin{cases}\delta_x c_j^k = A_x n_j^k,\\ \delta_x p_j^k = A_x c_j^k,\end{cases} \tag{31b}$$

where $\left|A_x\phi_j^k\right|^2 = \left(A_xu_j^k\right)^2 + \left(A_x v_j^k\right)^2$ and $\left|A_x\psi_j^k\right|^2 = \left(A_xp_j^k\right)^2 + \left(A_xq_j^k\right)^2$. Eliminating the auxiliary variables, we have a scheme

$$\begin{cases} -\delta_t A_x^2 u_{j-1}^k = A_t\delta_x^2 v_{j-1}^k + A_x\left[A_t\left(\left|A_x\phi_{j-1}^k\right|^2 + \beta\left|A_x\psi_{j-1}^k\right|^2\right)\cdot A_tA_x v_{j-1}^k\right], \\ \delta_t A_x^2 v_{j-1}^k = A_t\delta_x^2 u_{j-1}^k + A_x\left[A_t\left(\left|A_x\phi_{j-1}^k\right|^2 + \beta\left|A_x\psi_{j-1}^k\right|^2\right)\cdot A_tA_x u_{j-1}^k\right], \end{cases} \tag{32a}$$

$$\begin{cases} -\delta_t A_x^2 p_{j-1}^k = A_t\delta_x^2 q_{j-1}^k + A_x\left[A_t\left(\left|A_x\psi_{j-1}^k\right|^2 + \beta\left|A_x\phi_{j-1}^k\right|^2\right)\cdot A_tA_x q_{j-1}^k\right], \\ \delta_t A_x^2 q_{j-1}^k = A_t\delta_x^2 p_{j-1}^k + A_x\left[A_t\left(\left|A_x\psi_{j-1}^k\right|^2 + \beta\left|A_x\phi_{j-1}^k\right|^2\right)\cdot A_tA_x p_{j-1}^k\right], \end{cases} \tag{32b}$$

or an equivalent scheme

$$\begin{cases} i\delta_t A_x^2\phi_{j-1}^k + A_t\delta_x^2\phi_{j-1}^k + A_x\left[A_t\left(\left|A_x\phi_{j-1}^k\right|^2 + \beta\left|A_x\psi_{j-1}^k\right|^2\right)\cdot A_tA_x\phi_{j-1}^k\right] = 0, \\ i\delta_t A_x^2\psi_{j-1}^k + A_t\delta_x^2\psi_{j-1}^k + A_x\left[A_t\left(\left|A_x\psi_{j-1}^k\right|^2 + \beta\left|A_x\phi_{j-1}^k\right|^2\right)\cdot A_tA_x\psi_{j-1}^k\right] = 0. \end{cases} \tag{33}$$

The LEP-II algorithm (32) or (33) has following conservative properties.

Theorem 3. *The algorithm (32) or (33) is a local energy-preserving algorithm, which admits the discrete local energy conservation law*

$$\begin{aligned} \mathcal{E}(x_j,t_k) = {} & \delta_t\left[\frac{1}{4}\left(\left|A_x\phi_j^k\right|^4 + \left|A_x\psi_j^k\right|^4\right) + \frac{\beta}{2}\left|A_x\phi_j^k\right|^2\left|A_x\psi_j^k\right|^2 - \frac{1}{2}\left(\left|\delta_x\phi_j^k\right|^2 + \left|\delta_x\psi_j^k\right|^2\right)\right] \\ & + \delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k + A_td_j^k\cdot\delta_tq_j^k + A_tc_j^k\cdot\delta_tp_j^k\right) = 0. \end{aligned} \tag{34}$$

Proof. In Eq. (31a), multiplying the term $-\delta_tA_xu_j^k = A_tA_xf_j^k + A_t\left[\left|A_x\phi_j^k\right|^2 + \beta\left|A_x\psi_j^k\right|^2\right]\cdot A_tA_x v_j^k$ by $\delta_tA_x v_j^k$ and the term $\delta_tA_x v_j^k = A_tA_xg_j^k + A_t\left[\left|A_x\phi_j^k\right|^2 + A_x\beta\left|\psi_j^k\right|^2\right]\cdot A_tA_xu_j^k$ by $\delta_tA_xu_j^k$, and then summing them together reads

$$A_tA_xf_j^k\cdot\delta_tA_x v_j^k + A_tA_xg_j^k\cdot\delta_tA_xu_j^k + A_t\left[\left|A_x\phi_j^k\right|^2 + \beta\left|A_x\psi_j^k\right|^2\right]\cdot\left(A_tA_xu_j^k\cdot\delta_tA_xu_j^k + A_tA_x v_j^k\cdot\delta_tA_x v_j^k\right) = 0. \tag{35}$$

In (35), by using the Leibnite rule (12), the terms

$$\begin{aligned} A_tA_xf_j^k\cdot\delta_tA_x v_j^k + A_tA_xg_j^k\cdot\delta_tA_xu_j^k &= A_t\delta_xb_j^k\cdot\delta_tA_x v_j^k + A_t\delta_xa_j^k\cdot\delta_tA_xu_j^k \\ &= \delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k\right) - A_tA_xb_j^k\cdot\delta_x\delta_t v_j^k - A_tA_xa_j^k\cdot\delta_x\delta_tu_j^k \\ &= \delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k\right) - A_t\delta_x v_j^k\cdot\delta_x\delta_t v_j^k - A_t\delta_xu_{j+1}^k\cdot\delta_x\delta_tu_j^k \\ &= \delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k\right) - \frac{1}{2}\delta_t\left[\left(\delta_x v_j^k\right)^2 + \left(\delta_xu_j^k\right)^2\right] \\ &= \delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k\right) - \frac{1}{2}\delta_t|\delta_x\phi_j^k|^2, \end{aligned} \tag{36}$$

and the term

$$\begin{aligned} A_t\left[\left|A_x\phi_j^k\right|^2 + \beta\left|A_x\psi_j^k\right|^2\right]\cdot\left(A_tA_xu_j^k\cdot\delta_tA_xu_j^k + A_t v_j^k\cdot\delta_t v_j^k\right) &= \frac{1}{2}\delta_t\left|A_x\phi_j^k\right|^2\cdot\left[A_t\left|A_x\phi_j^k\right|^2 + \beta A_t\left|A_x\psi_j^k\right|^2\right] \\ &= \frac{1}{4}\delta_t\left|A_x\phi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|A_x\phi_j^k\right|^2\cdot A_t\left|A_x\psi_j^k\right|^2. \end{aligned} \tag{37}$$

Combining (36) and (37), the Eq. (35) reads

$$\delta_x\left(A_tb_j^k\cdot\delta_t v_j^k + A_ta_j^k\cdot\delta_tu_j^k\right) - \frac{1}{2}\delta_t\left|\delta_x\phi_j^k\right|^2 + \frac{1}{4}\delta_t\left|A_x\phi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|A_x\phi_j^k\right|^2\cdot A_t|A_x\psi_j^k|^2 = 0. \tag{38}$$

Similarly, from Eq. (31b), we obtain

$$\delta_x\left(A_td_j^k\cdot\delta_tq_j^k + A_tc_j^k\cdot\delta_tp_j^k\right) - \frac{1}{2}\delta_t\left|\delta_x\psi_j^k\right|^2 + \frac{1}{4}\delta_t\left|A_x\psi_j^k\right|^4 + \frac{\beta}{2}\delta_t\left|A_x\psi_j^k\right|^2\cdot A_t\left|A_x\phi_j^k\right|^2 = 0. \tag{39}$$

Summing Eqs. (38) and (39) and noting

$$\frac{\beta}{2}\delta_t\left|A_x\phi_j^k\right|^2\cdot A_t\left|A_x\psi_j^k\right|^2+\frac{\beta}{2}\delta_t\left|A_x\psi_j^k\right|^2\cdot A_t\left|A_x\phi_j^k\right|^2=\frac{\beta}{2}\delta_t\left(\left|A_x\phi_j^k\right|^2\left|A_x\psi_j^k\right|^2\right),$$

we complete the proof. □

Corollary 2. *For the periodic conditions (3), the LEP-II algorithm (32) or (33) satisfies the global energy conservation law*

$$\mathcal{E}^{k+1}=\mathcal{E}^k=\cdots=\mathcal{E}^1=\mathcal{E}^0, \tag{40}$$

where

$$\mathcal{E}^k=\frac{1}{4}\left(\left\|A_x\phi_j^k\right\|_4^4+\left\|A_x\psi_j^k\right\|_4^4\right)-\frac{1}{2}\left(\left\|\delta_x\phi_j^k\right\|^2+\left\|\delta_x\psi_j^k\right\|^2\right)+\frac{\beta}{2}h\sum_{j=1}^N\left|A_x\phi_j^k\right|^2\left|A_x\psi_j^k\right|^2.$$

Summing the discrete local energy conservation law (34) over all space index j and noting the the periodic boundary conditions (3), the conclusion can be approved. Obviously, The Corollary 2 is consistent with the global energy conservation law (5). Now, we discuss whether the LEP-II algorithm preserves the charge conservation law.

Theorem 4. *With periodic boundary conditions (3), the scheme (33) preserves the charge exactly, that is*

$$\mathcal{Q}^{k+1}=\|A_x\phi^{k+1}\|^2+\|A_x\psi^{k+1}\|^2=\cdots=\|A_x\phi^0\|^2+\|A_x\psi^0\|^2=\mathcal{Q}^0. \tag{41}$$

Proof. Multiplying the first line of (33) by $2hA_t\overline{\phi_j^k}$ and summing it over space index j, we have

$$i\sum_{j=1}^N\delta_tA_x^2\phi_{j-1}^k\cdot 2hA_t\overline{\phi_j^k}+\sum_{j=1}^NA_t\delta_x^2\phi_{j-1}^k\cdot 2hA_t\overline{\phi_j^k}+\sum_{j=1}^NA_x\left[\theta_{j-1}^k\cdot\left(A_tA_x\phi_j^k\right)\right]\cdot 2hA_t\overline{\phi_j^k}=0. \tag{42}$$

where $\theta_j^k=A_t\left(\left|A_x\phi_j^k\right|^2+\beta\left|A_x\psi_j^k\right|^2\right)$. The first term becomes

$$\begin{aligned}&\frac{ih}{4\tau}\sum_{j=1}^N\Big(\phi_{j+1}^{k+1}\overline{\phi_j^{k+1}}+2\left|\phi_j^{k+1}\right|^2+\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}-\phi_{j+1}^k\overline{\phi_j^{k+1}}-2\phi_j^k\overline{\phi_j^{k+1}}-\phi_{j-1}^k\overline{\phi_j^{k+1}}+\phi_{j+1}^{k+1}\overline{\phi_j^k}+2\phi_{j+1}^k\overline{\phi_j^k}\\&\quad+\phi_{j-1}^{k+1}\overline{\phi_j^k}-\phi_{j+1}^k\overline{\phi_j^k}-2\left|\phi_j^k\right|^2-\phi_{j-1}^k\overline{\phi_j^k}\Big).\end{aligned} \tag{43}$$

Noting

$$\sum_{j=1}^N\phi_{j+1}^{k+1}\overline{\phi_j^{k+1}}+\sum_{j=1}^N\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}=\sum_{j=1}^N\phi_j^{k+1}\overline{\phi_{j-1}^{k+1}}+\sum_{j=1}^N\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}=2\sum_{j=1}^N\Re\left(\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}\right),$$

and

$$\begin{aligned}&-\sum_{j=1}^N\phi_{j+1}^k\overline{\phi_j^{k+1}}+\sum_{j=1}^N\phi_{j-1}^{k+1}\overline{\phi_j^k}=2i\sum_{j=1}^N\Im(\phi_{j-1}^{k+1}\overline{\phi_j^k}),-2\sum_{j=1}^N\phi_j^k\overline{\phi_j^{k+1}}+2\sum_{j=1}^N\phi_j^{k+1}\overline{\phi_j^k}=4i\sum_{j=1}^N\Im\left(\phi_j^{k+1}\overline{\phi_j^k}\right),\\&-\sum_{j=1}^N\phi_{j-1}^k\overline{\phi_j^{k+1}}+\sum_{j=1}^N\phi_{j+1}^{k+1}\overline{\phi_j^k}=2i\sum_{j=1}^N\Im\left(\phi_{j-1}^k\overline{\phi_j^{k+1}}\right),-\sum_{j=1}^N\phi_{j+1}^k\overline{\phi_j^k}-\sum_{j=1}^N\phi_{j-1}^k\overline{\phi_j^k}=-2\sum_{j=1}^N\Re\left(\phi_{j-1}^k\overline{\phi_j^k}\right).\end{aligned}$$

Therefore, the imaginary part of (43) is

$$\frac{h}{4\tau}\left[\sum_{j=1}^N2\left|\phi_j^{k+1}\right|^2+\sum_{j=1}^N2\Re\left(\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}\right)-\sum_{j=1}^N2\left|\phi_j^k\right|^2-\sum_{j=1}^N2\Re\left(\phi_{j-1}^k\overline{\phi_j^k}\right)\right]. \tag{44}$$

The second term is a real function, since it is the same as (27). The last term

$$\begin{aligned}\textstyle\sum_{j=1}^NA_x\left[\theta_{j-1}^k\cdot\left(A_tA_x\phi_j^k\right)\right]\cdot 2hA_t\overline{\phi_j^k}&=\frac{h}{2}\left[\sum_{j=1}^N\theta_j^kA_t\phi_{j+1}^k\cdot A_t\overline{\phi_j^k}+\sum_{j=1}^N\theta_{j-1}^kA_t\phi_{j-1}^k\cdot A_t\overline{\phi_j^k}+\sum_{j=1}^N\theta_j^k\left|A_t\phi_j^k\right|^2+\sum_{j=1}^N\theta_{j-1}^k\left|A_t\phi_j^k\right|^2\right]\\&=\frac{h}{2}\left[2\sum_{j=1}^N\Re\left(\theta_{j-1}^kA_t\phi_{j-1}^k\cdot A_t\overline{\phi_j^k}\right)+\sum_{j=1}^N\theta_j^k\left|A_t\phi_j^k\right|^2+\sum_{j=1}^N\theta_{j-1}^k\left|A_t\phi_j^k\right|^2\right],\end{aligned} \tag{45}$$

is also a real function. Therefore, taking the imaginary part of (42) gives

$$2h\sum_{j=1}^N\left|\phi_j^{k+1}\right|^2+2h\sum_{j=1}^N\Re\left(\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}\right)=2h\sum_{j=1}^N\left|\phi_j^k\right|^2+2h\sum_{j=1}^N\Re\left(\phi_{j-1}^k\overline{\phi_j^k}\right). \tag{46}$$

Eq. (46) is equivalent to

$$h\sum_{j=1}^{N}\left|\phi_{j-1}^{k+1}\right|^2+h\sum_{j=1}^{N}\left|\phi_{j}^{k+1}\right|^2+2h\sum_{j=1}^{N}\Re\left(\phi_{j-1}^{k+1}\overline{\phi_{j}^{k+1}}\right)=h\sum_{j=1}^{N}\left|\phi_{j-1}^{k}\right|^2+h\sum_{j=1}^{N}\left|\phi_{j}^{k}\right|^2+2h\sum_{j=1}^{N}\Re\left(\phi_{j-1}^{k}\overline{\phi_{j}^{k}}\right), \tag{47}$$

which implies

$$\|A_x\phi^{k+1}\|^2=\|A_x\phi^{k}\|^2. \tag{48}$$

Similarly, multiplying the second line of (33) by $2hA_t\overline{\psi_j^k}$ and summing it over all space index j leads to

$$\|A_x\psi^{k+1}\|^2=\|A_x\psi^{k}\|^2. \tag{49}$$

Eqs. (48) and (49) implies the charge conservation law (41). □

Remark 3. The algorithm (33) doesn't satisfy $\|\phi^{k+1}\|^2+\|\psi^{k+1}\|^2=\|\phi^0\|^2+\|\psi^0\|^2$. That is to say the conservation law is not preserved at the mesh points. This means that the discrete global norm conservation of the algorithm is only in the sense of that corresponding discretization.

4. Local momentum-preserving algorithm I (LMP-I)

In the previous section, we proposed two algorithms, which all preserve local energy conservation law, for the CNLS system. With periodic boundary conditions, they conserve charge and global energy conservation laws. Up to now, all the conservative methods in the literatures [18–21] for the CNLS system (1) are only charge-preserving and global energy-preserving methods. However, as we know, the CNLS system also admits local momentum conservation law (9) and global momentum conservation law (6). Momentum conservation law is also an important invariant in physics. But there is little momentum-preserving method in literatures. Therefore, to construct algorithms, which possess the momentum conservation law, are interesting. Next, we will propose an algorithm which can preserve the local momentum conservation law (9) and global momentum conservation law (6), simultaneously.

In ODEs (7), discreting the time and space derivatives both by using mid-point rule, we obtain a scheme

$$\begin{cases}-\delta_tA_xu_j^k=A_tA_xf_j^k+A_x\left[\left|A_t\phi_j^k\right|^2+\beta\left|A_t\psi_j^k\right|^2\right]\cdot A_tA_xv_j^k,\\ \delta_tA_xv_j^k=A_tA_xg_j^k+A_x\left[\left|A_t\phi_j^k\right|^2+A_t\beta\left|\psi_j^k\right|^2\right]\cdot A_tA_xu_j^k,\end{cases}\quad\begin{cases}\delta_xb_j^k=A_xf_j^k,\\ \delta_xv_j^k=A_xb_j^k,\end{cases}\quad\begin{cases}\delta_xa_j^k=A_xg_j^k,\\ \delta_xu_j^k=A_xa_j^k,\end{cases} \tag{50a}$$

$$\begin{cases}-\delta_tA_xp_j^k=A_tA_xm_j^k+A_x\left[\left|A_t\psi_j^k\right|^2+\beta\left|A_t\phi_j^k\right|^2\right]\cdot A_tA_xq_j^k,\\ \delta_tA_xq_j^k=A_tA_xn_j^k+A_x\left[\left|A_t\psi_j^k\right|^2+\beta\left|A_t\phi_j^k\right|^2\right]\cdot A_tA_xp_j^k,\end{cases}\quad\begin{cases}\delta_xd_j^k=A_xm_j^k,\\ \delta_xq_j^k=A_xd_j^k,\end{cases}\quad\begin{cases}\delta_xc_j^k=A_xn_j^k,\\ \delta_xp_j^k=A_xc_j^k,\end{cases} \tag{50b}$$

where $\left|A_t\phi_j^k\right|^2=\left(A_tu_j^k\right)^2+\left(A_tv_j^k\right)^2$ and $\left|A_t\psi_j^k\right|^2=\left(A_tp_j^k\right)^2+\left(A_tq_j^k\right)^2$. Eliminating the auxiliary variables, we have an equivalent scheme

$$\begin{cases}-\delta_tA_x^2u_{j-1}^k=A_t\delta_x^2v_{j-1}^k+A_x\left[A_x\left(\left|A_t\phi_{j-1}^k\right|^2+\beta\left|A_t\psi_{j-1}^k\right|^2\right)\cdot A_tA_xv_{j-1}^k\right],\\ \delta_tA_x^2v_{j-1}^k=A_t\delta_x^2u_{j-1}^k+A_x\left[A_x\left(\left|A_t\phi_{j-1}^k\right|^2+\beta\left|A_t\psi_{j-1}^k\right|^2\right)\cdot A_tA_xu_{j-1}^k\right],\end{cases} \tag{51a}$$

$$\begin{cases}-\delta_tA_x^2p_{j-1}^k=A_t\delta_x^2q_{j-1}^k+A_x\left[A_x\left(\left|A_t\psi_{j-1}^k\right|^2+\beta\left|A_t\phi_{j-1}^k\right|^2\right)\cdot A_tA_xq_{j-1}^k\right],\\ \delta_tA_x^2q_{j-1}^k=A_t\delta_x^2p_{j-1}^k+A_x\left[A_x\left(\left|A_t\psi_{j-1}^k\right|^2+\beta\left|A_t\phi_{j-1}^k\right|^2\right)\cdot A_tA_xp_{j-1}^k\right],\end{cases} \tag{51b}$$

or a scheme

$$\begin{cases}i\delta_tA_x^2\phi_{j-1}^k+A_t\delta_x^2\phi_{j-1}^k+A_x\left[A_x\left(\left|A_t\phi_{j-1}^k\right|^2+\beta\left|A_t\psi_{j-1}^k\right|^2\right)\cdot A_tA_x\phi_{j-1}^k\right]=0,\\ i\delta_tA_x^2\psi_{j-1}^k+A_t\delta_x^2\psi_{j-1}^k+A_x\left[A_x\left(\left|A_t\psi_{j-1}^k\right|^2+\beta\left|A_t\phi_{j-1}^k\right|^2\right)\cdot A_tA_x\psi_{j-1}^k\right]=0.\end{cases} \tag{52}$$

Now, we discuss some conservative properties of above algorithm.

Theorem 5. *The algorithm (51) or (52) is a local momentum-preserving algorithm, which satisfies the discrete local momentum conservation law*

$$\begin{aligned}\mathcal{M}(x_j,t_k) &= \delta_t\left[A_x u_j^k\cdot\delta_x v_j^k - A_x v_j^k\cdot\delta_x u_j^k + A_x p_j^k\cdot\delta_x q_j^k - A_x q_j^k\cdot\delta_x p_j^k\right]\\ &\quad+\delta_x\Bigg[A_t v_j^k\cdot\delta_t u_j^k - A_t u_j^k\cdot\delta_t v_j^k + A_t q_j^k\cdot\delta_t p_j^k - A_t p_j^k\cdot\delta_t q_j^k + \left(\left(A_t a_j^k\right)^2+\left(A_t b_j^k\right)^2+\left(A_t c_j^k\right)^2+\left(A_t d_j^k\right)^2\right)\\ &\quad+\frac{1}{2}\left(\left|A_t\phi_j^k\right|^4+\left|A_t\psi_j^k\right|^4\right)+\beta\left|A_t\phi_j^k\right|^2\left|A_t\psi_j^k\right|^2\Bigg]=0.\end{aligned} \tag{53}$$

Proof. In Eq. (50a), multiplying $-\delta_t A_x u_j^k = A_t A_x f_j^k + A_x\left[\left|A_t\phi_j^k\right|^2+\beta\left|A_t\psi_j^k\right|^2\right]\cdot A_t A_x v_j^k$ by $\delta_x A_t v_j^k$ and $\delta_t A_x v_j^k = A_t A_x g_j^k + A_x\left[\left|A_t\phi_j^k\right|^2 + A_t\beta\left|\psi_j^k\right|^2\right]\cdot A_t A_x u_j^k$ by $\delta_x A_t u_j^k$, and then adding them, we have

$$\begin{aligned}&\delta_t A_x u_j^k\cdot\delta_x A_t v_j^k - \delta_t A_x v_j^k\cdot\delta_x A_t u_j^k + A_t A_x f_j^k\cdot\delta_x A_t v_j^k + A_t A_x g_j^k\cdot\delta_x A_t u_j^k\\ &\quad+A_x\left[\left|A_t\phi_j^k\right|^2+\beta\left|A_t\psi_j^k\right|^2\right]\left(A_t A_x u_j^k\cdot\delta_x A_t u_j^k + A_t A_x v_j^k\cdot\delta_x A_t v_j^k\right)=0.\end{aligned} \tag{54}$$

By using the Leibnite rule, we have the terms

$$\begin{aligned}\delta_t A_x u_j^k\cdot\delta_x A_t v_j^k - \delta_t A_x v_j^k\cdot\delta_x A_t u_j^k &= \frac{1}{2}\delta_t A_x u_j^k\cdot\delta_x A_t v_j^k - \frac{1}{2}\delta_t A_x v_j^k\cdot\delta_x A_t u_j^k + \frac{1}{2}\delta_t A_x u_j^k\cdot\delta_x A_t v_j^k - \frac{1}{2}\delta_t A_x v_j^k\cdot\delta_x A_t u_j^k\\ &= \frac{1}{2}\delta_t\left(A_x u_j^k\cdot\delta_x v_j^k - A_x v_j^k\cdot\delta_x u_j^k\right) - \frac{1}{2}A_t A_x u_j^k\cdot\delta_t\delta_x v_j^k + \frac{1}{2}A_t A_x v_j^k\cdot\delta_t\delta_x u_j^k + \frac{1}{2}\delta_t A_x u_j^k\\ &\quad\cdot\delta_x A_t v_j^k - \frac{1}{2}\delta_t A_x v_j^k\cdot\delta_x A_t u_j^k\\ &= \frac{1}{2}\delta_t\left(A_x u_j^k\cdot\delta_x v_j^k - A_x v_j^k\cdot\delta_x u_j^k\right) + \frac{1}{2}\delta_x\left(A_t v_j^k\cdot\delta_t u_j^k - A_t u_j^k\cdot\delta_t v_j^k\right),\end{aligned} \tag{55}$$

the terms

$$A_t A_x f_j^k\cdot\delta_x A_t v_j^k + A_t A_x g_j^k\cdot\delta_x A_t u_j^k = A_t\delta_x b_j^k\cdot A_x A_t v_j^k + A_t\delta_x a_j^k\cdot A_x A_t a_j^k = \frac{1}{2}\delta_x[(A_x b_j^k)^2 + (A_x a_j^k)^2]. \tag{56}$$

and the last term

$$\begin{aligned}&A_x\left[\left|A_t\phi_j^k\right|^2+\beta\left|A_t\psi_j^k\right|^2\right]\left(A_t A_x u_j^k\cdot\delta_x A_t u_j^k + A_t A_x v_j^k\cdot\delta_x A_t v_j^k\right) = A_x\left[\left|A_t\phi_j^k\right|^2+\beta\left|A_t\psi_j^k\right|^2\right]\cdot\frac{1}{2}\delta_x\left|A_t\phi_j^k\right|^2\\ &\quad=\frac{1}{4}\delta_x\left|A_t\phi_j^k\right|^4+\frac{\beta}{2}A_x\left|A_t\psi_j^k\right|^2\cdot\delta_x\left|A_t\phi_j^k\right|^2.\end{aligned} \tag{57}$$

Therefore, the Eq. (54) can be reduced to

$$\delta_x\left[A_t v_j^k\cdot\delta_t u_j^k - A_t u_j^k\cdot\delta_t v_j^k + \left(A_x a_j^k\right)^2 + \left(A_x b_j^k\right)^2 + \frac{1}{2}\left|A_t\phi_j^k\right|^4\right] + \beta A_x\left|A_t\psi_j^k\right|^2\cdot\delta_x\left|A_t\phi_j^k\right|^2 + \delta_t\left(A_x u_j^k\cdot\delta_x v_j^k - A_x v_j^k\cdot\delta_x u_j^k\right)=0. \tag{58}$$

Similarly, from Eq. (50b), we have

$$\delta_x\left[A_t q_j^k\cdot\delta_t p_j^k - A_t p_j^k\cdot\delta_t q_j^k + \left(A_x c_j^k\right)^2 + \left(A_x d_j^k\right)^2 + \frac{1}{2}\left|A_t\psi_j^k\right|^4\right] + \beta A_x\left|A_t\phi_j^k\right|^2\cdot\delta_x\left|A_t\psi_j^k\right|^2 + \delta_t\left(A_x p_j^k\cdot\delta_x q_j^k - A_x q_j^k\cdot\delta_x p_j^k\right)=0. \tag{59}$$

Adding (58) and (59), we complete the proof. □

Corollary 3. *For the periodic boundary conditions (3), the local momentum-preserving algorithm (51) or (52) possesses the discrete global momentum conservation law*

$$\mathcal{M}^{k+1} = \mathcal{M}^k = \cdots = \mathcal{M}^1 = \mathcal{M}^0, \tag{60}$$

where

$$\mathcal{M}^k = h\sum_{j=1}^{N}\left[A_x u_j^k\cdot\delta_x v_j^k - A_x v_j^k\cdot\delta_x u_j^k + A_x p_j^k\cdot\delta_x q_j^k - A_x q_j^k\cdot\delta_x p_j^k\right].$$

To prove the Corollary 3, we only need to sum the discrete conservation law (53) over all space index j and use the periodic boundary conditions. Next, we present the discrete charge conservation law of the LMP-I algorithm (52).

Theorem 6. *With periodic boundary conditions (3), the algorithm (52) preserves the charge exactly, that is,*

$$\mathcal{Q}^{k+1} = \|A_x\phi^{k+1}\|^2 + \|A_x\psi^{k+1}\|^2 = \cdots = \|A_x\phi^0\|^2 + \|A_x\psi^0\|^2 = \mathcal{Q}^0. \tag{61}$$

Proof. To prove this property, we may multiply the first line of algorithm (33) by $2hA_t\overline{\phi_j^k}$ and sum over all space index j. Then, we have

$$i\sum_{j=1}^N \delta_t A_x^2\phi_{j-1}^k \cdot 2hA_t\overline{\phi_j^k} + \sum_{j=1}^N A_t\delta_x^2\phi_{j-1}^k \cdot 2hA_t\overline{\phi_j^k} + \sum_{j=1}^N A_x\left[\theta_{j-1}^k \cdot A_tA_x\phi_{j-1}^k\right] \cdot 2hA_t\overline{\phi_j^k} = 0, \tag{62}$$

where $\theta_j^k = A_x\left(\left|A_t\phi_j^k\right|^2 + \beta\left|A_t\psi_j^k\right|^2\right)$. The first term in Eq. (62) is the same as the one in (42) and its imaginary part is

$$\frac{h}{4\tau}\left[\sum_{j=1}^N 2\left|\phi_j^{k+1}\right|^2 + \sum_{j=1}^N 2\Re\left(\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}\right) - \sum_{j=1}^N 2\left|\phi_j^k\right|^2 - \sum_{j=1}^N 2\Re\left(\phi_{j-1}^k\overline{\phi_j^k}\right)\right]. \tag{63}$$

The second term is also a real function (see Eq. (27)). The last term

$$\begin{aligned}
\sum\nolimits_{j=1}^N A_x\left[\theta_{j-1}^k \cdot A_tA_x\phi_{j-1}^k\right] \cdot 2hA_t\overline{\phi_j^k} &= \frac{h}{2}\left[\sum_{j=1}^N \theta_j^k A_t\phi_{j+1}^k \cdot A_t\overline{\phi_j^k} + \sum_{j=1}^N \theta_{j-1}^k A_t\phi_{j-1}^k \cdot A_t\overline{\phi_j^k} + \sum_{j=1}^N \theta_{j-1}^k \cdot \left|A_t\phi_j^k\right|^2 + \sum_{j=1}^N \theta_j^k \cdot \left|A_t\phi_j^k\right|^2\right] \\
&= \frac{h}{2}\left[\sum_{j=1}^N \theta_{j-1}^k A_t\phi_j^k \cdot A_t\overline{\phi_{j-1}^k} + \sum_{j=1}^N \theta_{j-1}^k A_t\phi_{j-1}^k \cdot A_t\overline{\phi_j^k} + \sum_{j=1}^N \theta_{j-1}^k \cdot \left|A_t\phi_j^k\right|^2 + \sum_{j=1}^N \theta_j^k \cdot \left|A_t\phi_j^k\right|^2\right] \\
&= \frac{h}{2}\left[\sum_{j=1}^N 2\Re\left(\theta_{j-1}^k A_t\phi_j^k \cdot A_t\overline{\phi_{j-1}^k}\right) + \sum_{j=1}^N \theta_{j-1}^k \cdot \left|A_t\phi_j^k\right|^2 + \sum_{j=1}^N \theta_j^k \cdot \left|A_t\phi_j^k\right|^2\right]
\end{aligned} \tag{64}$$

is also a real function. Therefore, the imaginary part of Eq. (62) implies

$$2h\sum_{j=1}^N\left|\phi_j^{k+1}\right|^2 + 2h\sum_{j=1}^N\Re\left(\phi_{j-1}^{k+1}\overline{\phi_j^{k+1}}\right) = 2h\sum_{j=1}^N\left|\phi_j^k\right|^2 + 2h\sum_{j=1}^N\Re\left(\phi_{j-1}^k\overline{\phi_j^k}\right). \tag{65}$$

It is the same as (46) and can be rewritten into

$$\|A_x\phi^{k+1}\|^2 = \|A_x\phi^k\|^2. \tag{66}$$

Similarly, we have

$$\|A_x\psi^{k+1}\|^2 = \|A_x\psi^k\|^2. \tag{67}$$

Adding (66) and (67) reads the discrete charge conservation law (61). □

5. Stability analysis

In this section, we first present some priori estimates which imply the proposed local energy-preserving algorithms LEP-I and LEP-II are unconditionally stable. Second, we display the linear stability analysis for the local momentum-preserving algorithm LMP-I.

5.1. Nonlinear stability analysis for LEP-I and LEP-II algorithms

Lemma 1 (Discrete Sobolev inequality [22]). *Suppose that* $\{u_j^k | j = 1, 2, \cdots, N\}$ *are discrete functions on the finite interval* $[x_L, x_R]$. *For any given* $\varepsilon > 0$, *there exists a constant* $C_0, C(\varepsilon)$ *independent of functions* u_j^k $(j = 1, 2, \cdots, N)$ *and* h *such that*

$$\|u^k\|_\infty \leqslant C_0\sqrt{\|u^k\|}\sqrt{\|\delta_x u^k\| + \|u^k\|},$$

or

$$\|u^k\|_\infty \leqslant \varepsilon\|\delta_x u^k\| + C\|u^k\|.$$

For the difference solutions of the LEP-I algorithm, we have the following priori estimates.

Theorem 7. *Assume $\phi_0(x), \psi_0(x) \in H^1$, then the following estimates hold:*

$$\|\phi^k\| \leqslant C, \quad \|\psi^k\| \leqslant C, \quad \|\delta_x\phi^k\| \leqslant C, \quad \|\delta_x\psi^k\| \leqslant C, \quad \|\phi^k\|_\infty \leqslant C, \quad \|\psi^k\|_\infty \leqslant C.$$

Proof. It follows from (24) that

$$\|\phi^k\| \leqslant C, \quad \|\psi^k\| \leqslant C. \tag{68}$$

By using Lemma 1 and (68), we have

$$\|\phi^k\|_4^4 \leqslant \|\phi^k\|^2\|\phi^k\|_\infty^2 \leqslant C\|\phi^k\|_\infty^2 \leqslant \varepsilon\|\delta_x\phi^k\|^2 + C, \tag{69}$$

$$\|\psi^k\|_4^4 \leqslant \|\psi^k\|^2\|\psi^k\|_\infty^2 \leqslant C\|\psi^k\|_\infty^2 \leqslant \varepsilon\|\delta_x\psi^k\|^2 + C, \tag{70}$$

$$h\sum_{j=1}^{N}|\phi_j^k|^2|\psi_j^k|^2 \leqslant \frac{1}{2}\left(\|\phi^k\|_4^4 + \|\psi^k\|_4^4\right) \leqslant \frac{\varepsilon}{2}(\|\delta_x\phi^k\|^2 + \|\delta_x\psi^k\|^2) + C. \tag{71}$$

Combining inequalities (69)–(71), the discrete energy conservation law (23) implies

$$\frac{1}{2}(\|\delta_x\phi^k\|^2 + \|\delta_x\psi^k\|^2) = \frac{1}{4}\left(\|\phi^k\|_4^4 + \|\psi^k\|_4^4\right) + \frac{\beta}{2}h\sum_{j=1}^{N}|\phi_j^k|^2|\psi_j^k|^2 - \mathcal{E}^0 \leqslant \frac{\varepsilon}{4}(1+\beta)(\|\delta_x\phi^k\|^2 + \|\delta_x\psi^k\|^2) + C. \tag{72}$$

Taking sufficient small ε such that $\varepsilon(1+\beta) < 2$ in (72), we have

$$\|\delta_x\phi^k\| \leqslant C, \quad \|\delta_x\psi^k\| \leqslant C. \tag{73}$$

It follows from Lemma 1, inequalities (68) and (73) that

$$\|\phi^k\|_\infty \leqslant C, \quad \|\psi^k\|_\infty \leqslant C.$$

□

Corollary 4. *Theorem 7 implies the local energy-preserving algorithm LEP-I is unconditionally stable.*

For the local energy-preserving algorithm LEP-II, we can also prove it is unconditionally stable by using the similar method.

Theorem 8. *Assume the initial conditions $\phi_0(x), \psi_0(x) \in H^1$, then the local energy-preserving algorithm LEP-II is unconditionally stable.*

5.2. Linear stability analysis for LMP-I algorithm

For the local momentum-preserving algorithm LMP-I (52), it is difficult to implement the nonlinear stability analysis. Therefore, we will use the Von Neumann stability analysis for the linearized difference scheme. To apply von Neumann stability analysis, we consider the linearized form of the CNLS system Eq. (1) and this can be assumed of the real system

$$Z_t + BZ_{xx} + \widetilde{F}Z = 0, \tag{74}$$

where

$$Z = \begin{bmatrix} u \\ v \\ p \\ q \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad \widetilde{F} = \begin{bmatrix} 0 & \bar{z}_1 & 0 & 0 \\ -\bar{z}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{z}_2 \\ 0 & 0 & -\bar{z}_2 & 0 \end{bmatrix},$$

and

$$\bar{z}_1 = \max_{x\in[x_L,x_R]}(u^2 + v^2 + \beta(p^2+q^2)), \quad \bar{z}_2 = \max_{x\in[x_L,x_R]}(p^2 + q^2 + \beta(u^2+v^2)).$$

The linearized version of the proposed local momentum-preserving algorithm LMP-I (52) applied to Eq. (74) is

$$A_x^2\frac{Z_{j-1}^{k+1} - Z_{j-1}^k}{\tau} + B\delta_x^2\frac{Z_{j-1}^{k+1} + Z_{j-1}^k}{2} + \widetilde{F}A_x^2\frac{Z_{j-1}^{k+1} + Z_{j-1}^k}{2} = 0. \tag{75}$$

To apply von Neumann stability analysis for the linearized difference scheme (75), we assume that

$$Z_j^k = G^k \rho e^{i\zeta jh}, \tag{76}$$

where $i = \sqrt{-1}, \zeta \in \mathbb{R}$, $\rho \in \mathbb{R}^4$ is a constant vector and $G \in \mathbb{R}^{4\times 4}$ is the amplification matrix.

Substituting (76) into the linear scheme (75), we get after some manipulation.

$$\mu(G-I) + \gamma r B(G+I) + \frac{\tau}{4}\gamma\widetilde{F}(G+I) = 0, \tag{77}$$

where $\mu = \cos^2(\zeta h/2), \gamma = -2\sin^2(\zeta h/2)$ and $r = \tau/h^2$. Simplification of (77) leads to

$$\left(\mu I + \gamma r B + \frac{\tau}{4}\gamma\widetilde{F}\right)G = \mu I - \gamma r B - \frac{\tau}{4}\gamma\widetilde{F}. \tag{78}$$

Recalling the skew symmetric property of matrices B and $\widetilde{F}$, it is easy to see that matrix $\mu I + \gamma r B + \frac{\tau}{4}\gamma\widetilde{F}$ is nonsingular and shares the same set of eigenvalues $\{\mu + iw, \mu - iw, \mu + iw, \mu - iw\}$ with matrix $\mu I - \gamma r B - \frac{\tau}{4}\gamma\widetilde{F}$. Thus, the maximal module of the eigenvalues of G is one. Actually, the modulus of all of these eigenvalues is equal to 1. This means that our scheme is unconditionally stable in the linear sense according to von Neumann stability analysis.

6. Numerical experiments

In this section, we conduct some numerical experiments to show the performance of the proposed algorithms: (i) to simulate the interaction of two solitary waves; (ii) to verify excellent preservation of local energy and global energy of the two local energy-preserving algorithms LEP-I and LEP-II; (iii) to exhibit the good preservation in local momentum and global momentum of local momentum-preserving algorithm LMP-I; (iv) to present the good ability of the three algorithms in preserving the charge conservation law; (v) to make comparison with existing methods.

For the local energy-preserving algorithms LEP-I and LEP-II, the discrete local energy $\mathcal{E}(x_j, t_k)$ at any local point (x_j, t_k) can be calculated by Eqs. (17) and (34), respectively. In the following numerical tests, the local energy $\mathcal{E}(0, t_k)$ is monitored. In order to show the discrete local energy at $t_k = k\tau$ level, we define

$$\mathcal{E}^k_{local} = \max_{1\leqslant j\leqslant N}|\mathcal{E}(x_j, t_k)|.$$

The discrete local momentum $\mathcal{M}(x_j, t_k)$ at any local point (x_j, t_k) can be obtained by Eq. (53). Similarly, the local momentum $\mathcal{M}(0, t_k)$ will be scaled throughout computations. Here, we define

$$\mathcal{M}^k_{local} = \max_{1\leqslant j\leqslant N}|\mathcal{M}(x_j, t_k)|$$

as the discrete local momentum at $t_k = k\tau$. The fully implicit algorithms (16), (33) and (52) can be written as the nonlinear systems $F(U^{k+1}, U^k) = 0$, where

$$U^k = \left[u_1^k, \cdots, u_N^k, v_1^k, \cdots, v_N^k, p_1^k, \cdots, p_N^k, q_1^k, \cdots, q_N^k\right]^T.$$

In each time step, we use Newton iteration technique with the vector $U^{k+1,(0)} = U^k$ as the initial iteration vector to obtain the vector U^{k+1}. We conduct the iterations (repeating the calculations for the same time step $k+1$ with increasing value of the superscript n) until convergence, i.e., when the following criteria is satisfied

$$\max_{1\leqslant j\leqslant 4N}\left|U_j^{k+1,(n+1)} - U_j^{k+1,(n)}\right| < 10^{-13},$$

and

$$|F(U^{k+1,(n+1)}, U^k)| < 10^{-13}.$$

To simulate the collision of two solitary waves, we consider the following initial condition

$$\begin{aligned} \phi(x,0) &= \sqrt{2}r_1 sech(r_1 x + D_0/2)e^{iV_1 x/4}, \\ \psi(x,0) &= \sqrt{2}r_2 sech(r_2 x - D_0/2)e^{-iV_2 x/4} \end{aligned} \tag{79}$$

where $-40 \leqslant x \leqslant 40$ and $D_0 = 20$. The boundary conditions are periodic. In following computations, we take time step $\tau = 0.1$ and $h = 80/199$.

6.1. Collision of solitary waves with equal amplitudes

In this section, we investigate the role of nonlinear coupling parameter β. For this case, we choose $r_1 = r_2 = 1$ and $V_1 = V_2 = 1$ in initial condition (79).

Firstly, we take $\beta = 2/3$. The simulations of collision are displayed in Fig. 1. One can see that the collision takes place at about $t = 18$ and then the two solitary waves are seem to be trapped. The phenomenon indicates the collision is inelastic. Fig. 2 shows the changes in charge $\mathcal{Q}$, global energy $\mathcal{E}$, local energy $\mathcal{E}(0, t_k)$ and local energy $\mathcal{E}_{local}$ against time. From the left

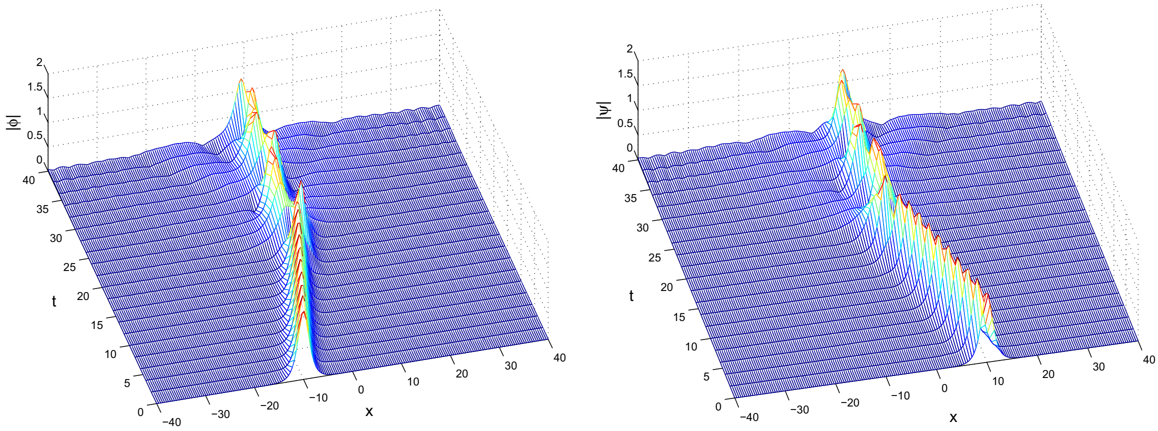

Fig. 1. Collision of two solitons with $\beta = 2/3, r_1 = r_2 = 1$ and $V_1 = V_2 = 1$.

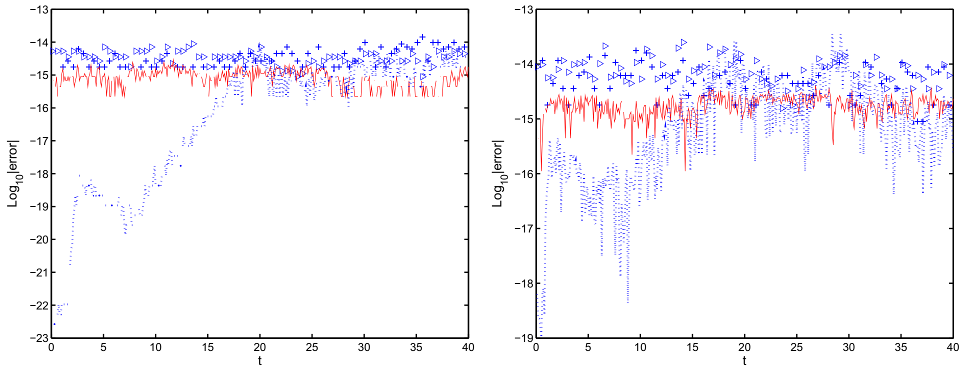

Fig. 2. The errors in invariants obtained by LEP-I (left) and LEP-II (right) algorithms. + symbols: charge $\mathcal{Q}$; Read solid line: global energy $\mathcal{E}$; Dotted line: local energy $\mathcal{E}(0, t_k)$; ▹ symbols: local energy $\mathcal{E}_{local}$.

graph, one can see that the errors in charge, global energy and local energy $\mathcal{E}_{local}$ are all within the roundoff error of machine. Moveover, as time evolves, the errors oscillate near zero but don't exhibit any growth for the duration of the simulation. Therefore, the LEP-I method preserves the charge, global energy and local energy exactly. The dotted line in the left graph shows variation of the errors in local energy $\mathcal{E}(0, t_k)$. It is clear the error increases to 10^{-16} from $t = 0$ to $t = 18$ and then it oscillates within a very small interval near zero in the scale of 10^{-16}. The reason for it increases from 10^{-23} to 10^{-16} at the beginning time is that the values of ϕ and ψ are almost zeros, and then the interaction takes place at $t = 18$. The errors in local energy $\mathcal{E}(0, t_k)$ are always within the roundoff error of machine, which indicates $\mathcal{E}(0, t_k)$ is preserved exactly. From the right graph, obviously, the LEP-II algorithm also conserves the discrete conservation laws exactly. Fig. 3 is plotted for the local momentum-preserving algorithm LMP-I, which displays the errors in charge $\mathcal{Q}$, global momentum $\mathcal{M}$, local momentum $\mathcal{M}(0, t_k)$ and local momentum $\mathcal{M}_{local}$ against time. From the figure, it is clear the LMP-I conserves the charge, global momentum, local momentum $\mathcal{M}(0, t_k)$ and local momentum $\mathcal{M}_{local}$ exactly. Numerical results verify theoretical analysis.

Secondly, we choose $\beta = 1$. The interaction of the two solitary waves are shown in Fig. 4. The two waves emerge without any changes in their shapes. This phenomenon indicates that the interaction is elastic. Actually, the case of $\beta = 1$ gives Manakov's equation which is completely integrable. Fig. 5 represents the changes in charge, global energy, local energy $\mathcal{E}(0, t_k)$ and local energy $\mathcal{E}_{local}$ as time evolves for the two local energy-preserving algorithms LEP-I (left) and LEP-II (right). It is clear that the two methods preserve the charge, global energy, the local energy exactly. Fig. 6 exhibits the abilities of the local momentum-preserving method LMP-I in conserving the charge, global momentum and local momentum $\mathcal{M}(0, t_k)$ and global momentum $\mathcal{M}_{local}$. They are all preserved exactly.

Finally, we discuss the case of $\beta = 2$. In such case, two stable solitary waves are generated as shown in Fig. 7. From Fig. 8, we see LEP-I and LEP-II methods show excellent performance in preserving charge, global energy, local energy $\mathcal{E}(0, t_k)$ and local energy $\mathcal{E}_{local}$. The LMP-I preserves the charge, global momentum and local momentum $\mathcal{M}(0, t_k)$ and global momentum $\mathcal{M}_{local}$ exactly, which can be seen in Fig. 9.

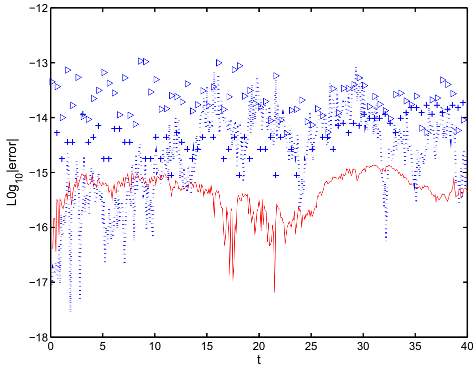

Fig. 3. The errors in invariants obtained by LMP-I algorithm. + symbols: charge $\mathcal{Q}$; Read solid line: global momentum $\mathcal{M}$; Dotted line: local momentum $\mathcal{M}(0, t_k)$; ▹ symbols: local momentum $\mathcal{M}_{local}$.

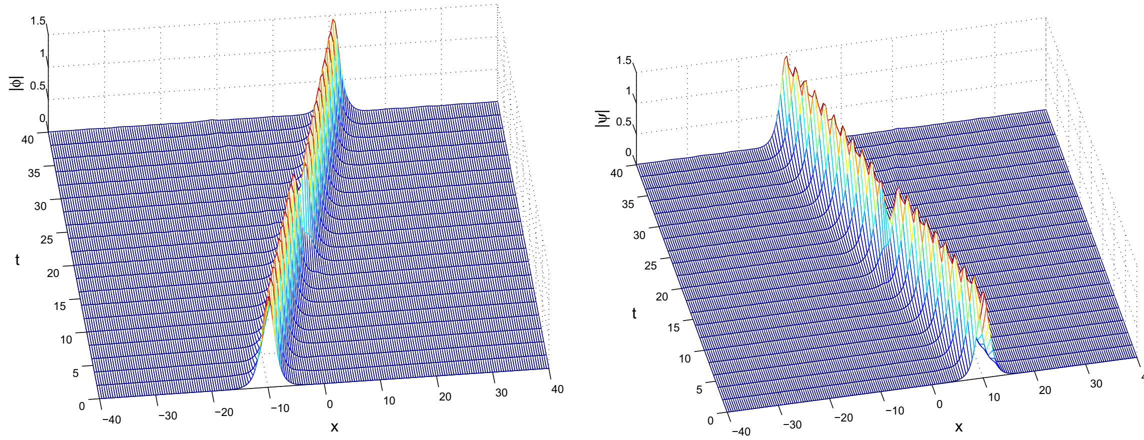

Fig. 4. Collision of two solitons with $\beta = 1, r_1 = r_2 = 1$ and $V_1 = V_2 = 1$.

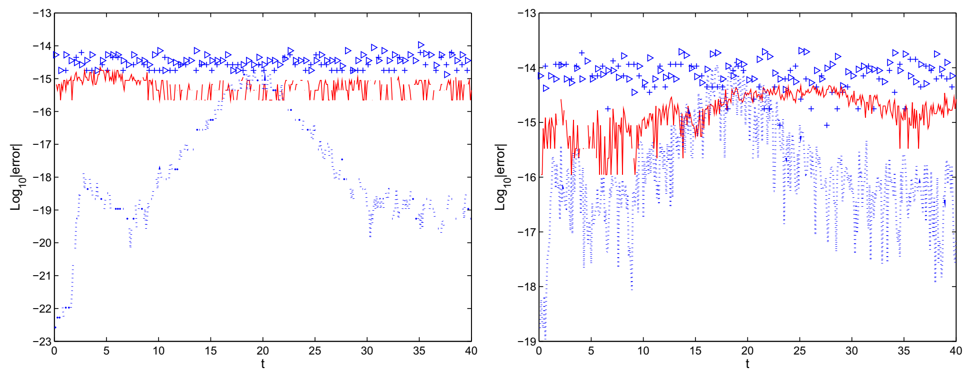

Fig. 5. The errors in invariants obtained by LEP-I (left) and LEP-II (right) algorithms. + symbols: charge $\mathcal{Q}$; Read solid line: global energy $\mathcal{E}$; Dotted line: local energy $\mathcal{E}(0, t_k)$; ▹ symbols: local energy $\mathcal{E}_{local}$.

Additionally, as the value of β is increased, more and more solitary waves will be formed after solitary waves collision. The results have been reported in Ref. [17]. Actually, for these cases, the three proposed methods also preserve the conservation laws exactly. The corresponding results are not given here.

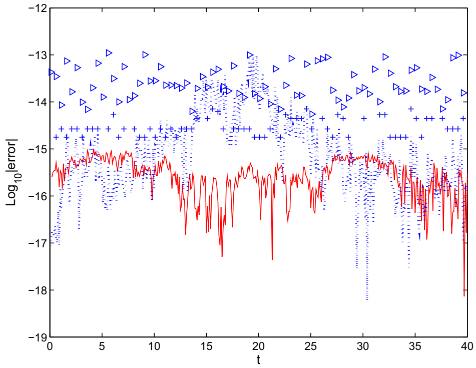

Fig. 6. The errors in invariants obtained by LMP-I algorithm. + symbols: charge $\mathcal{Q}$; Read solid line: global momentum $\mathcal{M}$; Dotted line: local momentum $\mathcal{M}(0, t_k)$; $\triangleright$ symbols: local momentum $\mathcal{M}_{local}$.

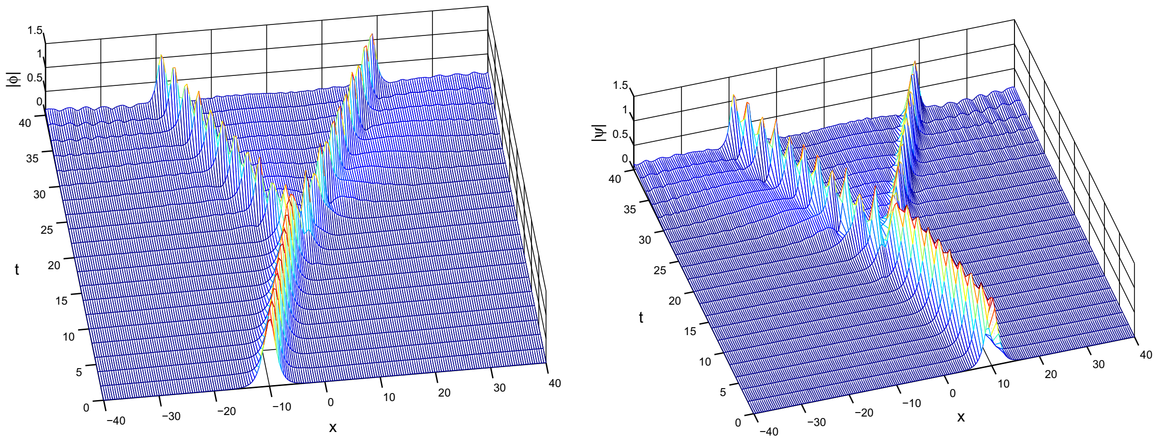

Fig. 7. Collision of two solitons with $\beta = 2, r_1 = r_2 = 1$ and $V_1 = V_2 = 1$.

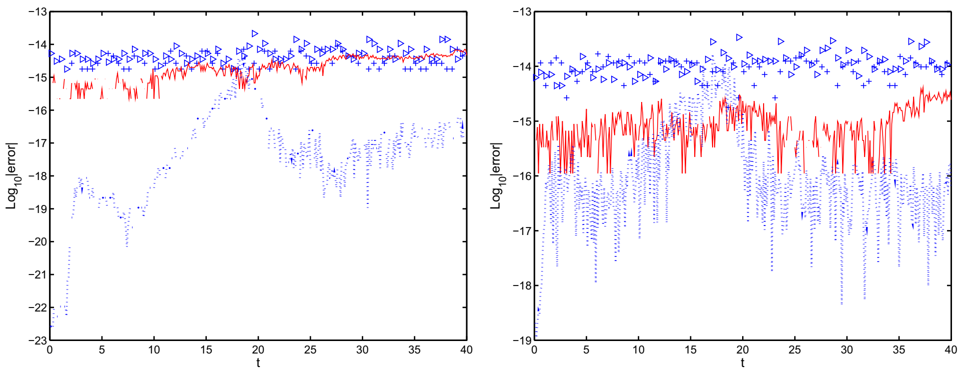

Fig. 8. The errors in invariants obtained by LEP-I (left) and LEP-II (right) algorithms. + symbols: charge $\mathcal{Q}$; Read solid line: global energy $\mathcal{E}$; Dotted line: local energy $\mathcal{E}(0, t_k)$; $\triangleright$ symbols: local energy $\mathcal{E}_{local}$.

6.2. Collision of solitary waves with different amplitudes

Here, we will discuss the collision of the two solitary waves with different amplitudes. We take $\beta = 2/3$ in CNLS system (1) and $r_1 = 1.2, r_2 = 1, V_1 = V_2 = 1$ in initial condition (79). The evolution of the shapes of ϕ and ψ are depicted in Fig. 10. The conservative properties of the two local energy-preserving methods LEP-I and LEP-II are shown in Fig. 11. Fig. 12 exhibits

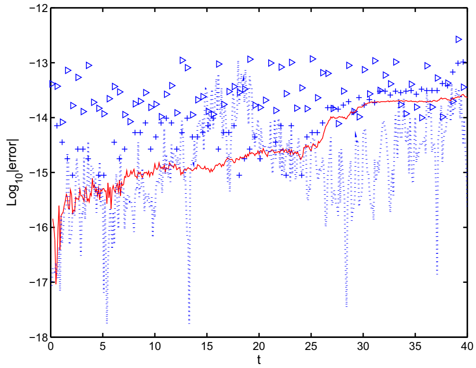

Fig. 9. The errors in invariants obtained by LMP-I algorithm. + symbols: charge $\mathcal{Q}$; Read solid line: global momentum $\mathcal{M}$; Dotted line: local momentum $\mathcal{M}(0, t_k)$; $\triangleright$ symbols: local momentum $\mathcal{M}_{local}$.

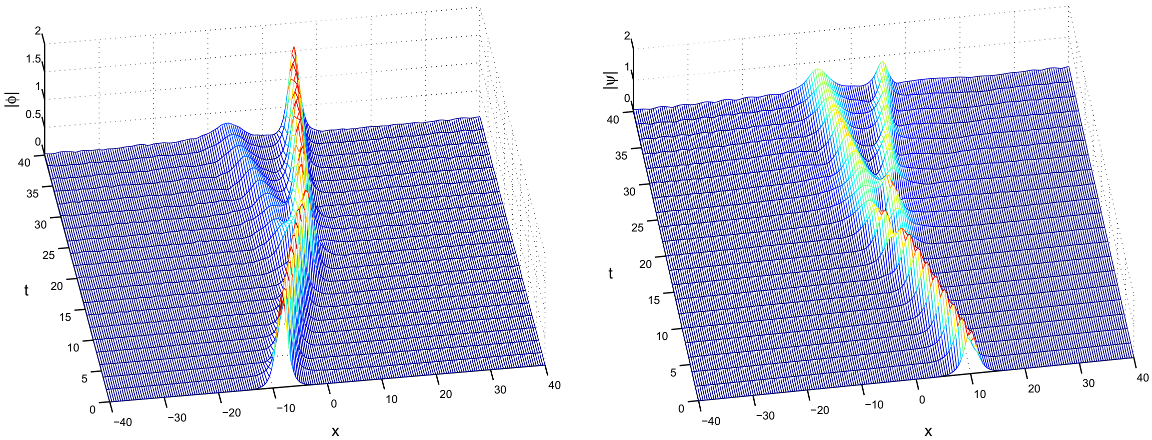

Fig. 10. Collision of two solitons with $\beta = 2/3, r_1 = 1.2, r_2 = 1$ and $V_1 = V_2 = 1$.

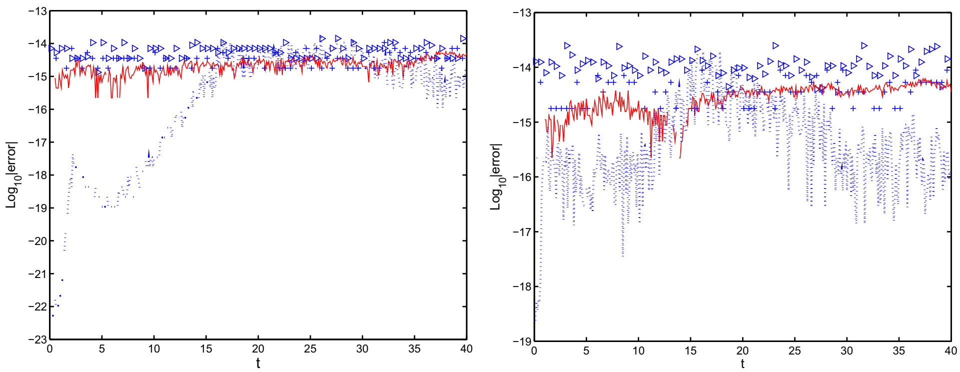

Fig. 11. The errors in invariants obtained by LEP-I (left) and LEP-II (right) algorithms. + symbols: charge $\mathcal{Q}$; Read solid line: global energy $\mathcal{E}$; Dotted line: local energy $\mathcal{E}(0, t_k)$; $\triangleright$ symbols: local energy $\mathcal{E}_{local}$.

the performance of the local momentum-preserving method LEP-I in preserving charge, global momentum and local momentum. From the Figs. 11 and 12, one can see the proposed algorithms all preserve the conservative invariants exactly.

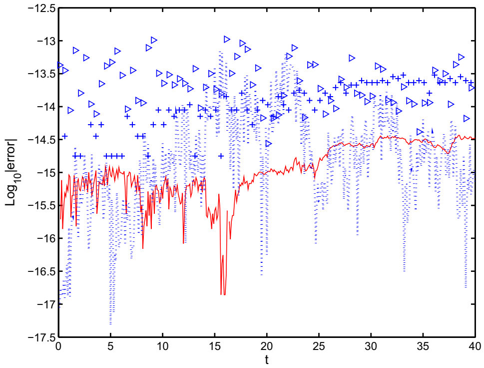

Fig. 12. The errors in invariants obtained by LMP-I algorithm. + symbols: charge $\mathcal{Q}$; Read solid line: global momentum $\mathcal{M}$; Dotted line: local momentum $\mathcal{M}(0, t_k)$; $\triangleright$ symbols: local momentum $\mathcal{M}_{local}$.

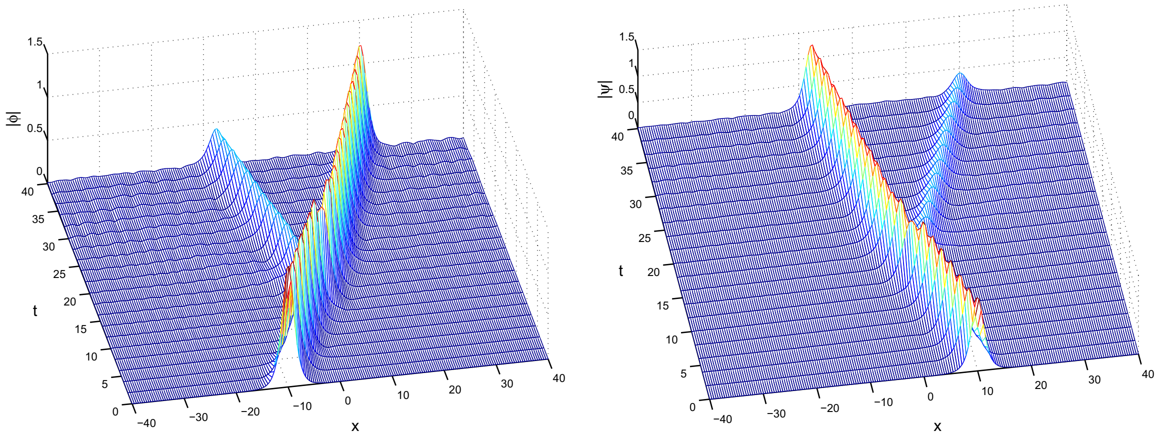

Fig. 13. Collision of two solitons with $\beta = 2/3, V_1 = 1.6, V_2 = 1$ and $r_1 = r_2 = 1$.

6.3. Collision of solitary waves with different velocities

In this part, we will show the case of interaction of two solitary waves with different velocities. We choose $\beta = 2/3$ in CNLS system (1) and $r_1 = r_2 = 1, V_1 = 1.6, V_2 = 1$ in initial condition (79). The simulations of this case are shown in Fig. 13. From the left figure, one can see that two solitary waves are generated after collision. Moreover, the solitary wave with large amplitude moves to right direction and the one with small amplitude moves in opposite direction. Furthermore,

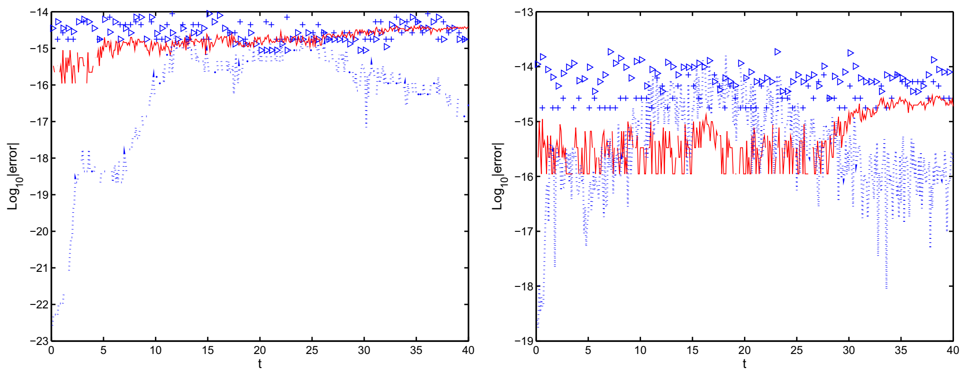

Fig. 14. The errors in invariants obtained by LEP-I (left) and LEP-II (right) algorithms. + symbols: charge $\mathcal{Q}$; Read solid line: global energy $\mathcal{E}$; Dotted line: local energy $\mathcal{E}(0, t_k)$; $\triangleright$ symbols: local energy $\mathcal{E}_{local}$.

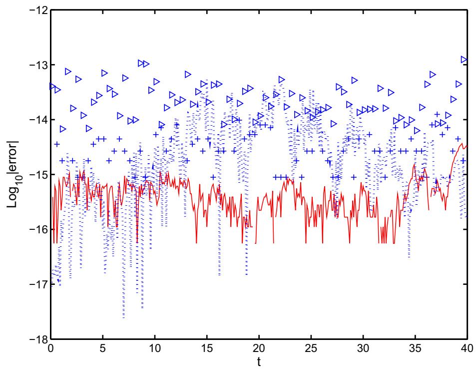

Fig. 15. The errors in invariants obtained by LMP-I algorithm. + symbols: charge $\mathcal{Q}$; Read solid line: global momentum $\mathcal{M}$; Dotted line: local momentum $\mathcal{M}(0,t_k)$; ▹ symbols: local momentum $\mathcal{M}_{local}$.

the velocity of the solitary wave with large amplitude has almost no change. Figs. 14 and 15 show the conservative properties of the local energy-preserving methods and local momentum-preserving method. From the figures, one finds that the conservative invariants are all conserved exactly.

6.4. Some numerical comparison

In this part, we will make some numerical comparisons between the proposed local energy-preserving methods and other methods, including existing energy-preserving methods [18–21] and multisymplectic methods [9,12–14,17].

First, we conduct some numerical comparisons with existing conservative methods. The proposed conservative methods in Refs. [18–20] and the present LEP-I method (16) are same. But we prove the LEP-I method is a local energy-preserving algorithm, i.e. it preserves the local energy conservation law in any time–space region. Therefore, we only make some numerical comparisons between LEP-I, LEP-II and the method proposed in Ref. [20]. We consider the CNLS system for the case of $\beta = 2/3$ with initial conditions (79) where $r_1 = r_2 = 1, V_1 = V_2 = 1$, $D_0 = 25$ and $-30 \leqslant x \leqslant 30$. The computational parameters are $\tau = 0.1$ and $h = 0.1$. Tables 1, 2 display the errors in charge and energy at different times ($|\mathcal{Q} - \mathcal{Q}^0|$ and $|\mathcal{E} - \mathcal{E}^0|$ of the method [20] are obtained by the given values in corresponding literature. See pages 614 and 620). It is clear that all the three methods preserve the charge conservation law exactly, however, the present LEP-I and LEP-II methods conserve the energy conservation law much more better than the energy-preserving method in Ref. [20]. From Table 2, one can see that the energy-preserving scheme [20] can't conserve the global energy to the machine accuracy. The reason is as follows: the scheme couples 3 time levels and its energy conservation law (See Ref. [20], page 609) requires 2 time level values. In order to implement the scheme, the vectors ϕ^1 and ψ^1 should be provided by another two-time level scheme. Here, the vectors ϕ^0, ψ^0, ϕ^1 and ψ^1 should satisfy its energy conservation law, if not the obtained $\psi^{k-1}, \psi^{k-1}, \psi^k$ and ψ^k can't satisfy the energy conservation law. In Ref. [20], the authors use Crank–Nicolson (C–N) scheme to obtain the vectors ϕ^1 and ψ^1. But C–N scheme doesn't conserve the energy conservation law of scheme [20].

Second, we make some comparisons of the present local energy- and momentum-preserving algorithms with the multisymplectic methods in Refs. [9,12–14,17]. Introducing the canonical momenta $u_x = a$, $v_x = b$, $p_x = c$, $q_x = d$, the CNLS system (1) can be written into the multisymplectic PDE

Table 1
$|\mathcal{Q} - \mathcal{Q}^0|$ obtained by various methods at different times.

Method\Time	$T = 12$	$T = 28$	$T = 40$
LEP-I	6.2172×10^{-15}	8.8818×10^{-16}	8.3920×10^{-15}
LEP-II	7.9936×10^{-15}	8.6578×10^{-15}	7.1054×10^{-15}
Ref. [20]	8.9103×10^{-15}	1.8803×10^{-14}	2.7601×10^{-15}

Table 2
$|\mathcal{E} - \mathcal{E}^0|$ obtained by various methods at different times.

Method\Time	$T = 12$	$T = 28$	$T = 40$
LEP-I	2.2204×10^{-16}	1.3323×10^{-15}	6.6613×10^{-16}
LEP-II	6.6613×10^{-15}	2.6645×10^{-15}	1.9984×10^{-15}
Ref. [20]	1.3611×10^{-7}	5.9283×10^{-8}	1.7257×10^{-4}

$$\mathbf{K}z_t + \mathbf{L}z_x = \nabla_z S(z),$$

where $\mathbf{K}$, $\mathbf{L}$ are skew symmetry matrices (for the elements of matrices, please refer to Ref. [9]), the state variable $z = [u, v, a, b, p, q, c, d]^T$ and Hamiltonian function

$$S(z) = \frac{1}{2}\left(a^2 + b^2 + c^2 + d^2 + \frac{1}{2}(u^2 + v^2)^2 + \frac{1}{2}(p^2 + q^2)^2 + \frac{1}{2}\beta(u^2 + v^2)(p^2 + q^2)\right). \tag{80}$$

Sun and Qin [9] first applied multisymplectic integrator to CNLS system (1) and derived a multisymplectic Preissman scheme

$$\begin{aligned} &i\delta_t\left(\phi_{j-\frac{1}{2}}^{k+\frac{1}{2}} + \phi_{j+\frac{1}{2}}^{k+\frac{1}{2}}\right) + \delta_x\phi_j^{k+\frac{1}{2}} + \left(|\phi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2 + \beta|\psi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|\right)\phi_{j+\frac{1}{2}}^{k+\frac{1}{2}} = 0,\\ &i\delta_t\left(\psi_{j-\frac{1}{2}}^{k+\frac{1}{2}} + \psi_{j+\frac{1}{2}}^{k+\frac{1}{2}}\right) + \delta_x\psi_j^{k+\frac{1}{2}} + \left(|\psi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|^2 + \beta|\phi_{j+\frac{1}{2}}^{k+\frac{1}{2}}|\right)\psi_{j+\frac{1}{2}}^{k+\frac{1}{2}} = 0. \end{aligned} \tag{81}$$

In Ref. [9], many numerical experiments are conducted to show the performance of the multisymplectic scheme (81) in preserving the discrete global energy. Unfortunately, as choosing $\tau = 0.02$ and $h = 0.2$, the errors in global energy obtained by the scheme (81) are in the scale of 10^{-2} throughout computations. It means the multisymplectic Preissman scheme (81) can't conserve the global energy exactly. Wang et al. [13] proved that the errors in discrete GECL of multisymplectic Preissman scheme (81) satisfied the following relation

$$\begin{aligned} \mathcal{E}^{k+1} - \mathcal{E}^k = &\frac{\tau^2}{8}h\sum_{j=1}^{N}\left[|\delta_t\phi_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2\left(|\phi_{j-\frac{1}{2}}^{k+1}|^2 - |\phi_{j-\frac{1}{2}}^{k}|^2\right) + |\delta_t\psi_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2\left(|\psi_{j-\frac{1}{2}}^{k+1}|^2 - |\psi_{j-\frac{1}{2}}^{k}|^2\right)\right]\\ &+\frac{\tau^2}{4}\beta h\sum_{j=1}^{N}\left[|\delta_t\psi_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2\left(|\phi_{j-\frac{1}{2}}^{k+1}|^2 - |\phi_{j-\frac{1}{2}}^{k}|^2\right) + |\delta_t\phi_{j-\frac{1}{2}}^{k+\frac{1}{2}}|^2\left(|\psi_{j-\frac{1}{2}}^{k+1}|^2 - |\psi_{j-\frac{1}{2}}^{k}|^2\right)\right]. \end{aligned}$$

Aydın and Karasözen [12] applied multisymplectic Preissman scheme to investigate the CNLS system with periodic plane wave solutions. They calculated the errors in local energy, local momentum, global energy and global momentum. The numerical results (see Ref. [12]) shown that the errors in global energy and momentum are in the scale of 10^{-3} and 10^{-4}. However, the changes in local energy and local momentum are very big.

Actually, in Ref. [23], Bridges and Reich have proved that if the covariant Hamiltonian function $S(z)$ is a quadratic function of z and the Preissman scheme is used, then both discrete energy and discrete momentum are conserved to the machine accuracy. When $S(z)$ is super-quadratic and the Preissman scheme is used, global momentum is still conserved to machine accuracy, however the LECL and LMCL are not conserved exactly. Since $S(z)$ (80) associated with CNLS system is a super-quadratic function of z, therefore, the multisymplectic Preissman scheme (81) can't conserve the discrete GECL, LECL and LMCL exactly.

In Ref. [17], Cai investigated the CNLS system by using explicit multisymplectic composition Euler-box schemes. As $\tau = 0.005$ and $h = 0.2$, the errors in discrete global energy are in the scale of 10^{-5}. Chen et al. [14] constructed a multisymplectic Fourier pseudospectral method for the system. The numerical results show that the errors in invariants are in the scale of 10^{-4}, which implies the method can't conserve the global energy and momentum.

By comparisons, it is easy to find the merits of the algorithms in this paper. These algorithms not only conserve the global structure (GECL or GMCL) of the CNLS system, but also possess the local structure (LECL or LMCL).

7. Conclusions

Generally, partial differential equations have different structures, such as symplectic and multisymplectic conservation laws, local energy and momentum conservation laws. All these conservation laws are local, that is to say they are independent of the boundary conditions. In this paper, based on the rule that numerical algorithms should preserve the intrinsic properties of the original problems as much as possible, we construct two local energy-preserving algorithms and one local momentum-preserving algorithm for CNLS system (1). The algorithms preserve the discrete local energy and local momentum conservation laws exactly. We say these methods are local energy-preserving and local momentum-preserving methods. With suitable boundary conditions, these algorithms will be energy-preserving and momentum-preserving methods. For example, with periodic boundary conditions, the local energy-preserving methods conserve the charge and global energy exactly. Numerical experiments are conducted to check the performance of the proposed algorithms. Numerical results verify the theoretical analysis.

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