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Explicit Multisymplectic Fourier Pseudospectral Scheme for the Klein–Gordon–Zakharov Equations *

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Applying the Fourier pseudospectral method to space derivatives and the symplectic Euler rule to time derivatives in the multisymplectic form of the Klein–Gordon–Zakharov equations, we derive an explicit multisymplectic scheme. The semi-discrete energy and momentum conservation laws are given. Some numerical experiments are carried out to show the accuracy of the numerical solutions. The performance of the scheme in preserving the global energy and momentum conservation laws are also checked.

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It is well known that the Klein–Gordon–Zakharov (KGZ) equations are

$$c^{-2}\phi_{tt} - \Delta\phi + c^2\phi + \phi\psi = 0, \quad \lambda^{-2}\psi_{tt} - \Delta\psi = \Delta|\phi|^2, \tag{1}$$

where ϕ is the electric field, ψ is the density fluctuation of ions, c^2 is the plasma frequency and λ is the ion sound speed. The equation, which describes the interaction between Langmuir waves and ion sound waves, plays an important role in the investigation of the dynamics of strong Langmuir turbulence in the plasma physics. In this Letter, we consider the following one-dimensional KGZ equation

$$\phi_{tt} - \phi_{xx} + \phi + \phi\psi = 0, \qquad \psi_{tt} - \psi_{xx} = |\phi|^2_{xx}. \tag{2}$$

There have been some works[1−3] on the analytic solutions for the KGZ Eq. (2). However, few numerical methods have been proposed for Eq. (2). In Ref. [4] Wang *et al.* derived an explicit and an implicit conservative difference method for the KGZ equation. The multisympletic numerical integrators were proposed in Refs. [5,6]. Now, the method has been successfully applied to many important physical and mathematical models.[7−19] In Ref. [17] Wang presented an implicit multisymplectic Fourier pseudospectral scheme for the KGZ Eq. (2). As we know, implicit methods result in a huge computational cost in solving systems of nonlinear equations at each time step. The present work focuses on constructing an explicit scheme, which can also preserve the multisymplectic conservation law, for the KGZ Eq. (2). In addition, we will check whether the derived scheme could provide accurate numerical solutions and preserve the discrete global energy and momentum conservation laws well.

In order to establish the new scheme for the KGZ Eq. (2), we prescribe the initial and periodic boundary conditions as

$$\begin{aligned}&\phi|_{t=0} = \phi_0(x), \quad \phi_t|_{t=0} = \phi_1(x), \quad \psi|_{t=0} = \psi_0(x),\\ &\psi_t|_{t=0} = \psi_1(x), \quad \phi(x_L, t) = \phi(x_R, t),\\ &\psi(x_L, t) = \psi(x_R, t).\end{aligned}$$

To avoid complex computations, we set $\phi(x,t) = u(x,t) + kv(x,t)$, where $u(x,t)$ and $v(x,t)$ are real functions, and $k^2 = -1$. Introducing new variables $u_x = p$, $v_x = q$, $u_t = r$, $v_t = s$, $f_x = g$ and $\psi_t = f_{xx}$, Eq. (2) can be written as the following first-order system

$$\begin{aligned}&-r_t + p_x = u + u\psi, \qquad -s_t + q_x = v + v\psi,\\ &u_t = r, \quad v_t = s, \quad -u_x = -p, \quad -v_x = -q,\\ &\frac{1}{2}f_t = \frac{1}{2}\psi + \frac{1}{2}(u^2+v^2), \qquad -\frac{1}{2}\psi_t + \frac{1}{2}g_x = 0,\\ &-\frac{1}{2}f_x = -\frac{1}{2}g,\end{aligned} \tag{3}$$

or

$$\boldsymbol{M}z_t + \boldsymbol{K}z_x = \nabla_z S(z), \tag{4}$$

where $z = (u, v, r, s, p, q, \psi, f, g)^T$, $S(z) = (u^2 + v^2)/2 + (u^2+v^2)\psi/2 + (r^2+s^2)/2 - (p^2+q^2)/2 + \psi^2/4 - g^2/4$, $\boldsymbol{M} = [m_{i,j}] \in \mathcal{R}^{9\times 9}$ is skew-symmetric matrices with elements $m_{1,3} = -m_{3,1} = -1$, $m_{2,4} = -m_{4,2} = -1$, $m_{7,8} = -m_{8,7} = \frac{1}{2}$ and $\boldsymbol{K} = [k_{i,j}] \in \mathcal{R}^{9\times 9}$ is skew-symmetric matrices with elements $k_{1,5} = -k_{5,1} = 1$, $k_{2,6} = -k_{6,2} = 1$, $k_{8,9} = -k_{9,8} = \frac{1}{2}$. The system (4) satisfies the multisymplectic conservation law (MCL)

$$\partial_t\omega + \partial_x\kappa = 0, \quad \omega = \frac{1}{2}dz \wedge \boldsymbol{M}dz, \quad \kappa = \frac{1}{2}dz \wedge \boldsymbol{K}dz. \tag{5}$$

As usual, we introduce some notations: $x_j = x_L + hj$, $t_n = n\tau$, $j = 0, 1, \ldots, N-1$; $n = 0, 1, 2, \ldots$, where $h = (x_R - x_L)/N$, and τ is temporal step. Denote u_j^n as the approximation of the value of $u(x_j, t_n)$. It is noticed that the first-order partial differential operator ∂_x yields the Fourier spectral differential matrix

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D_1, in the case that the space direction is discretized by a Fourier spectral method. Here D_1 is a skew-symmetric matrix with elements

$$(D_1)_{i,j} = \begin{cases} \frac{1}{2}\mu(-1)^{i+j}\cot(\mu\frac{x_i - x_j}{2}), & i \neq j, \\ 0, & i = j. \end{cases}$$

where $\mu = 2\pi/(x_R - x_L)$. For details, see Refs. [19,18] Applying the Fourier pseudospectral method to the multisymplectic system (3) in space direction, one obtains

$$\begin{aligned}
&-\frac{dr_i}{dt} + (D_1\boldsymbol{p})_i = u_i + u_i\psi_i,\\
&-\frac{ds_i}{dt} + (D_1\boldsymbol{q})_i = v_i + v_i\psi_i,\\
&\frac{du_i}{dt} = r_i, \quad \frac{dv_i}{dt} = s_i, \quad -(D_1\boldsymbol{u})_i = -p_i,\\
&-(D_1\boldsymbol{v})_i = -q_i,\\
&\frac{1}{2}\frac{df_i}{dt} = \frac{1}{2}\psi_i + \frac{1}{2}(u_i^2 + v_i^2), \qquad -\frac{1}{2}\frac{d\psi_i}{dt} + \frac{1}{2}(D_1\boldsymbol{g})_i = 0,\\
&-\frac{1}{2}(D_1\boldsymbol{f})_i = -\frac{1}{2}g_i,
\end{aligned} \tag{6}$$

where $\boldsymbol{u} = (u_0, u_1, \ldots u_{N-1})^T$, $\boldsymbol{p} = (p_0, p_1, \ldots p_{N-1})^T$, $\boldsymbol{q} = (q_0, q_1, \ldots q_{N-1})^T$, $\boldsymbol{v} = (v_0, v_1, \ldots v_{N-1})^T$, $\boldsymbol{g} = (g_0, g_1, \ldots g_{N-1})^T$ and $\boldsymbol{f} = (f_0, f_1, \ldots f_{N-1})^T$. Equation (6) can be rewritten in the compact form

$$\boldsymbol{M}\frac{dz_i}{dt} + \boldsymbol{K}\sum_{j=0}^{N-1}(D_1)_{i,j}z_j = \nabla_z S(z_i). \tag{7}$$

Taking the wedge product of dz_i with the variational equation associated with Eq. (7) and noting $dz_i \wedge S_{zz}(z_i)dz_i = 0$, we obtain the semi-discrete system (7) which possesses N semi-discrete MCLs

$$\frac{d}{dt}\omega_i + \sum_{j=0}^{N-1}(D_1)_{i,j}\kappa_{i,j} = 0, \qquad i = 0, 1, 2, \ldots, N-1, \tag{8}$$

where $\omega_i = \frac{1}{2}(dz_i \wedge \boldsymbol{M}dz_i), \kappa_{i,j} = dz_i \wedge \boldsymbol{K}dz_j$. Since the symmetry of $\kappa_{i,j}$ and skew-symmetric of D_1, summing Eq. (7) over the spatial index leads to total symplecticity $\frac{d}{dt}\sum_{i=0}^{N-1}\omega_i = 0$. Discretizing Eq. (7) with respect to time by the symplectic Euler rule yields

$$\boldsymbol{M}_+\delta_t^+ dz_i^n + \boldsymbol{M}_-\delta_t^- dz_i^n + \boldsymbol{K}\sum_{j=0}^{N-1}(D_1)_{i,j}z_j^n = \nabla_z S(z_i^n), \tag{9}$$

where δ_t^+ and δ_t^- are, respectively, the forward and backward difference operators, and $\boldsymbol{M}_+$ and $\boldsymbol{M}_-$ satisfies $\boldsymbol{M} = \boldsymbol{M}_+^T + \boldsymbol{M}_-$, $\boldsymbol{M}_+^T = -\boldsymbol{M}_-$. Taking the wedge product of dz_i^n with the variational equation associated with Eq. (9) leads to N full-discrete MCLs

$$\frac{\omega_i^{n+1} - \omega_i^n}{\tau} + \sum_{j=0}^{N-1}(D_1)_{i,j}\kappa_{i,j}^n = 0, \ i = 0, 1, 2, \ldots, N-1, \tag{10}$$

where $\omega_i^n = dz_i^{n-1} \wedge \boldsymbol{M}_+ dz_i^n, \kappa_{i,j}^n = dz_i^n \wedge \boldsymbol{K}dz_j^n$. Here we take $\boldsymbol{M}_+$ as the upper triangle matrix and $\boldsymbol{M}_-$ as the lower triangle matrix. Substituting the two splitting matrices into Eq. (9), one can obtain

$$\begin{aligned}
&-\delta_t^+ r_i^n + (D_1\boldsymbol{p}^n)_i = u_i^n + u_i\psi_i^n,\\
&-\delta_t^+ s_i^n + (D_1\boldsymbol{q}^n)_i = v_i^n + v_i^n\psi_i^n,\\
&\delta_t^- u_i^n = r_i^n, \quad \delta_t^- v_i^n = s_i^n, \quad -(D_1\boldsymbol{u}^n)_i = -p_i^n,\\
&-(D_1\boldsymbol{v}^n)_i = -q_i^n,\\
&\frac{1}{2}\delta_t^+ f_i^n = \frac{1}{2}\psi_i^n + \frac{1}{2}((u_i^n)^2 + (v_i^n)^2),\\
&-\frac{1}{2}\delta_t^-\psi_i^n + \frac{1}{2}(D_1\boldsymbol{g}^n)_i = 0, \quad -\frac{1}{2}(D_1\boldsymbol{f}^n)_i = -\frac{1}{2}g_i^n.
\end{aligned} \tag{11}$$

Eliminating the auxiliary variables yields a fully explicit scheme

$$\begin{aligned}
&-\delta_t^+\delta_t^- u_i^n + (D_1^2\boldsymbol{u}^n)_i = u_i^n + u_i^n\psi_i^n,\\
&-\delta_t^+\delta_t^- v_i^n + (D_1^2\boldsymbol{v}^n)_i = v_i^n + v_i^n\psi_i^n,\\
&-\delta_t^+\delta_t^-\psi_i^n = (D_1^2\Psi^n)_i + (D_1^2[(\boldsymbol{u}^n)^2 + (\boldsymbol{v}^n)^2])_i,
\end{aligned} \tag{12}$$

where $(\boldsymbol{u}^n)^2 = [(u_1^n)^2, (u_2^n)^2, \ldots, (u_{N-1}^n)^2]^T$ and $(\boldsymbol{v}^n)^2 = [(v_1^n)^2, (v_2^n)^2, \ldots, (v_{N-1}^n)^2]^T$.

Theorem 1. Semi-discrete Eq. (6) or (7) has an exact semi-discrete energy conservation law

$$\begin{aligned}
&\partial_t E_i - \frac{1}{2}\sum_{j=0}^{N-1}(D_1)_{i,j}\langle\partial_t z_i, \boldsymbol{K}z_j\rangle = 0,\\
&E_i = S(z_i) - \frac{1}{2}\langle z_i, \boldsymbol{K}(D_1\boldsymbol{z})_i\rangle.
\end{aligned} \tag{13}$$

Proof. The semi-discrete Eq. (7) can be written as

$$M\partial_t z_i + \boldsymbol{K}(D_1\boldsymbol{z})_i = \nabla_z S(z_i), \tag{14}$$

where $(D_1\boldsymbol{z})_i = ((D_1\boldsymbol{u})_i, (D_1\boldsymbol{v})_i, (D_1\boldsymbol{r})_i, (D_1\boldsymbol{s})_i, (D_1\boldsymbol{p})_i, (D_1\boldsymbol{q})_i, (D_1\Psi)_i, (D_1\boldsymbol{f})_i, (D_1\boldsymbol{g})_i)^T$. Taking the inner product of Eq. (14) with $\partial_t z_i$ gives

$$\langle\partial_t z_i, \boldsymbol{K}(D_1\boldsymbol{z})_i\rangle = \partial_t S(z_i). \tag{15}$$

Note

$$\begin{aligned}
&\langle\partial_t z_i, \boldsymbol{K}(D_1\boldsymbol{z})_i\rangle\\
=&\frac{1}{2}[\partial_t\langle z_i, \boldsymbol{K}(D_1\boldsymbol{z})_i\rangle - \langle z_i, \boldsymbol{K}\partial_t(D_1\boldsymbol{z})_i\rangle\\
&+ \sum_{j=0}^{N-1}(D_1)_{i,j}\langle\partial_t z_i, \boldsymbol{K}z_j\rangle - \langle\partial_t(D_1\boldsymbol{z})_i, \boldsymbol{K}z_i\rangle]\\
=&\frac{1}{2}\partial_t\langle z_i, \boldsymbol{K}(D_1\boldsymbol{z})_i\rangle + \frac{1}{2}\sum_{j=0}^{N-1}(D_1)_{i,j}\langle\partial_t z_i, \boldsymbol{K}z_j\rangle.
\end{aligned} \tag{16}$$

Combining Eqs. (15) and (16) yields (13).

It is worth mentioning that the discrete energy conservation law (13) is local since it is independent of the boundary conditions. Under periodic boundary conditions, summing (13) over index i leads to the semi-discrete global energy conservation law $\frac{d}{dt}\sum_{i=0}^{N-1} E_i = 0$. Therefore, the semi-discrete global energy is

$$GE = \sum_{i=0}^{N-1} E_i = \sum_{i=0}^{N-1}(\frac{1}{2}|\phi_i|^2 + \frac{1}{2}|\phi_i|^2\psi_i + \frac{1}{2}|\partial_t\phi_i|^2 + \frac{1}{4}\psi_i^2 - \frac{1}{2}u_i(D_1\boldsymbol{p})_i - \frac{1}{2}v_i(D_1\boldsymbol{q})_i - \frac{1}{4}f_i(D_1\boldsymbol{g})_i).$$

Theorem 2. Applying a symplectic Euler discretization in time to Eq. (4) yields an exact semi-discrete momentum conservation law

$$\partial_x G^n + \delta_t^+ I^n = 0, \tag{17}$$

with $G^n = S(z^n) - \langle \delta_t^- z^n, \boldsymbol{M}_+ z^n\rangle$, $I^n = -\langle \partial_x z^{n-1}, \boldsymbol{M}_+ z^n\rangle$.

Proof. Apply a symplectic Euler discretization to Eq. (4) in time to get the semi-discrete equation

$$\boldsymbol{M}_+\delta_t^+ z^n + \boldsymbol{M}_-\delta_t^- z^n + \boldsymbol{K}\partial_x z^n = \nabla_z S(z^n). \tag{18}$$

Taking the inner product with $\partial_x z^n$ yields

$$\begin{aligned}&\langle \partial_x z^n, \boldsymbol{M}_+\delta_t^+ z^n\rangle + \langle \partial_x z^n, \boldsymbol{M}_-\delta_t^- z^n\rangle \\ =&\langle \partial_x z^n, \nabla_z S(z^n)\rangle = \partial_x S(z^n).\end{aligned} \tag{19}$$

Then adding and subtracting like terms and using properties of the inner product gives

$$\begin{aligned}&\partial_x S(z^n) + \partial_x\langle \delta_t^- z^n, \boldsymbol{M}_+ z^n\rangle \\ =&\langle \partial_t z^n, \boldsymbol{M}_+\delta_t^+ z^n\rangle + \langle \delta_t^-\partial_t z^n, \boldsymbol{M}_+ z^n\rangle \\ =&\delta_t^+\langle \partial_x z^{n-1}, \boldsymbol{M}_+ z^n\rangle.\end{aligned} \tag{20}$$

Thus, Eq. (17) is proved.

Under periodic boundary conditions, we have the semi-discrete global momentum conservation law $\delta_t^+\int_a^b I^n dx = 0$. The semi-discrete global momentum can be written as

$$GM = \int_a^b I^n dx = \int_a^b (-u_x^{n-1}u_t^n - v_x^{n-1}v_t^n + \frac{1}{2}\psi_x^{n-1}f^n)dx.$$

In the following, we will carry out some numerical experiments to show the performance of the scheme (12). The KGZ Eq. (2) admits the following solitary wave solution[17]

$$\begin{aligned}\phi(x,t,\nu,x_0) =& \frac{1}{\sqrt{2\nu^2+1}}\mathrm{sech}(\sqrt{\nu^2+0.5}(x-x_0) \\ &+ \nu t)\exp(k(\nu(x-x_0) + \sqrt{\nu^2+0.5}t)), \\ \psi(x,t,\nu,x_0) =& -\mathrm{sech}^2(\sqrt{\nu^2+0.5}(x-x_0)+\nu t),\end{aligned} \tag{21}$$

where $k = \sqrt{-1}$, x_0 and ν are constants, respecting the initial phase and the propagating velocity of a single soliton, respectively. The amplitude of the ϕ component is $\frac{1}{\sqrt{2\nu^2+1}}$, while the amplitude of the ψ component is -1. Accuracy of migration of single soliton ϕ at time $t = n\tau$ can be measured by

$$\begin{aligned}\| L(\phi) \|_\infty =& \max_{0\le i\le N-1} | \phi(x_i, n\tau) - \phi_i^n |, \\ \| L(\phi) \|_2 =& (h\sum_{i=0}^{N-1} | \phi(x_i, n\tau) - \phi_i^n |^2)^{1/2}.\end{aligned}$$

Similarly, we can define $\| L(\psi) \|_\infty$ and $\| L(\psi) \|_2$. The discrete global energy and momentum on the nth time level are calculated by

$$\begin{aligned}GE^n =& h\sum_{i=0}^{N-1}(\frac{1}{2}|\phi_i^n|^2 + \frac{1}{2}|\phi_i^n|^2\psi_i + \frac{1}{2}|\delta_t^-\phi_i|^2 + \frac{1}{4}\psi_i^2 \\ &- \frac{1}{2}u_i(D_1\boldsymbol{p}^n)_i - \frac{1}{2}v_i(D_1\boldsymbol{q}^n)_i - \frac{1}{4}f_i(D_1\boldsymbol{g}^n)_i), \\ GM^n =& h\sum_{i=0}^{N-1}((-D_1\boldsymbol{u}^{n-1})_i\delta_t^- u^n - (D_1\boldsymbol{v}^{n-1})_i\delta_t^- v^n \\ &+ \frac{1}{2}(D_1\Psi^{n-1})_i f_i^n),\end{aligned}$$

where $(D_1\boldsymbol{p}^n)_i$, $(D_1\boldsymbol{q}^n)_i$, $(D_1\boldsymbol{g}^n)_i$ and f_i^n are determined by Eq. (11). For our computations, we take

$$\begin{aligned}\phi_0 &= \phi(x,0,\nu,0), \quad \phi_1 = \phi_t(x,0,\nu,0)\,|_{t=0}, \\ \psi_0 &= \psi(x,0,\nu,0), \quad \psi_1 = \psi_t(x,0,\nu,0)\,|_{t=0}.\end{aligned}$$

as the initial values, where $-20 \le x \le 20$.

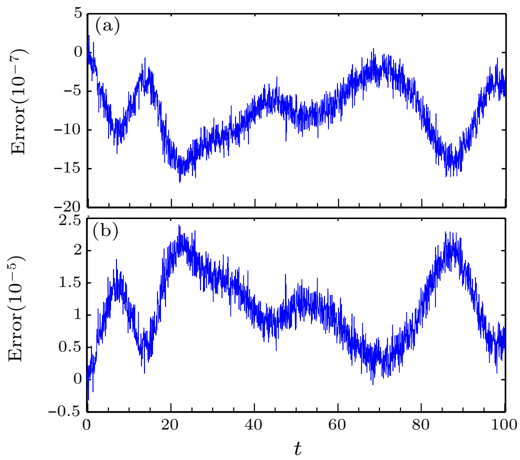

Fig. 1. The changes in discrete conservation laws for the single soliton with $\nu = 0.1$: (a) global energy, (b) global momentum.

Firstly, we choose $\nu = 0.1$ so that the amplitude of the soliton ϕ is 0.9804. The run of the algorithms is carried with $N = 200$ and temporal step $\tau = 0.001$ up to time $T = 100$. The obtained error norms, discrete global energy and momentum at different times are listed in Table 1. Obviously, at time $T = 100$, the

L_∞ and L_2 error norms of solution ϕ are about 10^{-6} and 2×10^{-6}. We also find that the error norms of solution ψ are always in the scale of 10^{-6} throughout computations. Therefore, in long time computations, the accuracies of solutions are satisfactory. From the table, one can see the global energy and momentum remain almost constant. Figure 1 displays the changes in the global energy and global momentum. We find the changes in global energy and momentum are in the scale of 10^{-7} and 10^{-5}. Moreover, the errors oscillate near zero but do not exhibit any growth for the duration of the simulation. Therefore, the present method preserves the conservation laws very well. Now, we choose $\nu=1$ and $\nu=0.5$, so that the amplitudes of the soliton ϕ are 0.3333 and 0.6667. The obtained error norms, discrete global energy and momentum at different times are also listed in Table 1. Again, the global energy and momentum remain almost constant against time and the accuracies of solutions ϕ and ψ are all satisfactory. Hence, the proposed scheme can also provide good numerical solutions for the single soliton moving with larger velocity ν.

Table 1. Global energy, global momentum and error norms for the soliton with $\nu=0.1$, $\nu=0.5$ and $\nu=1$ at different times.

ν	Time	GE^n	GM^n	$\parallel L(\phi)\parallel_\infty$	$\parallel L(\phi)\parallel_2$	$\parallel L(\psi)\parallel_\infty$	$\parallel L(\phi)\parallel_2$
0.1	0	1.884384	0.378440	0.00	0.00	0.00	0.00
	40	1.884383	0.378426	5.57×10^{-7}	1.43×10^{-6}	2.72×10^{-6}	2.93×10^{-6}
	80	1.884383	0.378426	9.82×10^{-7}	1.97×10^{-6}	3.64×10^{-6}	3.69×10^{-6}
	100	1.884384	0.378435	1.03×10^{-6}	2.28×10^{-6}	3.26×10^{-6}	3.54×10^{-6}
0.5	0	1.785102	1.294887	0.00	0.00	0.00	0.00
	40	1.785095	1.294863	7.25×10^{-7}	1.26×10^{-6}	1.38×10^{-6}	1.97×10^{-6}
	80	1.785095	1.294862	1.41×10^{-6}	2.47×10^{-6}	2.69×10^{-6}	3.83×10^{-6}
	100	1.785090	1.294843	1.81×10^{-6}	3.16×10^{-6}	3.41×10^{-6}	4.93×10^{-6}
1	0	1.440395	1.306180	0.00	0.00	0.00	0.00
	40	1.440413	1.306223	2.09×10^{-6}	2.78×10^{-6}	5.77×10^{-6}	6.27×10^{-6}
	80	1.440392	1.306170	3.58×10^{-6}	5.01×10^{-6}	9.66×10^{-6}	1.10×10^{-5}
	100	1.440413	1.306222	4.37×10^{-6}	6.15×10^{-6}	1.18×10^{-5}	1.35×10^{-5}

Table 2. The convergence rate in time, $N=200$, $T=20$.

τ	0.1	0.05	0.01	0.005	0.0025
$\parallel L(\phi)\parallel_\infty$	2.6331×10^{-3}	6.6624×10^{-4}	2.6788×10^{-5}	6.7158×10^{-6}	1.7155×10^{-6}
Order		1.9826	1.9968	1.9960	1.9689
$\parallel L(\psi)\parallel_\infty$	4.9170×10^{-3}	1.2448×10^{-3}	4.9892×10^{-5}	1.2368×10^{-5}	3.1309×10^{-6}
Order		1.9819	1.9988	2.0122	1.9820

Table 3. The maximum errors of solutions ϕ and ψ, $\tau=0.0001$, $T=10$.

N	48	64	96	128	256
$\parallel L(\phi)\parallel_\infty$	2.4001×10^{-3}	1.0241×10^{-4}	1.3701×10^{-6}	4.5513×10^{-7}	7.4025×10^{-7}
$\parallel L(\psi)\parallel_\infty$	3.6102×10^{-3}	1.1215×10^{-4}	1.1862×10^{-6}	6.6063×10^{-8}	6.4079×10^{-8}

Secondly, to test the convergence rate in time, we calculate the problem with $\nu=0.1$, $N=200$ and various time step τ up to time $T=20$. The rate of convergence can be obtained using the formula Order $=\ln(L_\infty(\tau_2)/L_\infty(\tau_1))/\ln(\tau_2/\tau_1)$. The L_∞ error norms and convergence rates are presented in Table 2. From the table, one can see that the convergence rate in time is of order 2. To test the convergence in space, we conduct numerical experiments with $\nu=0.1$, $\tau=0.0001$ and various values of N up to $T=10$. Table 3 displays the L_∞ error norms. These results indicate the numerical solutions converge rapidly to the accurate solutions in space, which is indicative of exponential convergence.

In summary, we have proposed a fully explicit method for the KGZ Eq. (2). The semi-discrete conservation laws are discussed. Numerical experiments shows that the method has many advantages such as exponential convergence rate in space, good numerical solutions and good preservation of discrete conserved invariants in long-time computations.

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