Blow-up Criterion for an Incompressible Navier-Stokes/Allen-Cahn System with Different Densities

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Abstract

This paper is concerned with a coupled Navier-Stokes/Allen-Cahn system describing a diffuse interface model for two-phase flow of viscous incompressible fluids with different densities in a bounded domain $\Omega \subset \mathbb{R}^N (N = 2, 3)$. We prove the existence and uniqueness of local strong solution to the initial boundary value problem. Moreover, we establish a criterion for possible break down of such solutions at finite time in terms of the temporal integral of both the maximum norm of the deformation tensor of velocity gradient and the square of maximum norm of gradient of phase field variable in 2D. In 3D, the temporal integral of the square of maximum norm of velocity is also needed. Here, we suppose the initial density function ρ_0 has a positive lower bound.

Keywords: Diffuse interface model; Nonhomogeneous Navier-Stokes equations; Allen-Cahn equation; Local strong solutions; Blow-up criterion

1 Introduction

In this paper, we investigate a diffusive interface model, which describes the motion of a mixture of two viscous incompressible fluids with different densities. This thermodynamically and mechanically consistent model has many interesting features, thus representing an important development in fluid mechanics. In fact, this model describes two-phase mixture of fluids undergoing phase transitions, where sharp interfaces are replaced by narrow transition layers. The latter feature has the advantage to deal with interfaces that merge, reconnect and hit conditions. This is in contrast to sharp interface models which usually fail in these situations. A phase field variable χ is introduced and a mixing energy is defined in terms of χ and its spatial gradient. The model consists of Navier-Stokes equations governing the fluid velocity coupled with a convective Allen-Cahn equation for the change of the concentration caused by diffusion. It is evident that, the change of the concentration is effected by the velocity of the fluids, and the velocity of the fluids is also related with the concentration because of the surface tension.

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Actually, the phase field variable χ defined by concentration difference can also be assumed to satisfy different variants of Cahn-Hilliard or other types of dynamics, see [5, 14, 27].

In this paper, we are interested in the following coupled Navier-Stokes/Allen-Cahn system for viscous incompressible fluids with different densities

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla p = \mathrm{div}(2\eta(\chi) Du) - \delta \mathrm{div}(\nabla\chi \otimes \nabla\chi), \\ \mathrm{div} u = 0, \\ (\rho\chi)_t + \mathrm{div}(\rho u \chi) = -\mu, \\ \rho\mu = -\delta\Delta\chi + \rho\dfrac{\partial f}{\partial \chi} \end{cases} \tag{1.1}$$

for $(x,t) \in \Omega \times (0,+\infty)$, where Ω is a bounded domain in $\mathbb{R}^N (N = 2,3)$ with smooth boundary $\partial\Omega$, $\rho \geq 0$ is the total density, u denotes the mean velocity of the fluid mixture, $Du = \frac{1}{2}(\nabla u + \nabla u^T)$, p is the pressure, χ represents the concentration difference of the two fluids, μ is the chemical potential, $\eta(\chi) > 0$ is the viscosity of the mixture, the free energy density satisfies double-well structure $f(\chi) = \frac{1}{\delta}(\frac{\chi^4}{4} - \frac{\chi^2}{2})$, positive constant δ denotes the width of the interface. The usual Kronecker product is denoted by $\otimes$, i.e. $(a \otimes b)_{ij} = a_i b_j$ for $a, b \in \mathbb{R}^N$. The equations $(1.1)_{1-3}$ are nonhomogeneous incompressible Navier-Stokes equations, which have an extra term $\nabla\chi \otimes \nabla\chi$ describing capillary effect related to the free energy

$$F(\rho,\chi) = \int_\Omega \left(\rho f(\chi) + \frac{\delta}{2}|\nabla\chi|^2 \right) \mathrm{d}x.$$

The system (1.1) is a highly nonlinear system coupling hyperbolic equations with parabolic equations. Here, we point out some special cases of this coupled system:

(i) When the densities of the two fluids are the same or at lest very close ("matched densities"), the total density ρ is assumed to be constant, then (1.1) reduces to an incompressible Navier-Stokes/Allen-Cahn system. From another point of view, it is also closely related to liquid crystal model, Magnetohydrodynamics (MHD) equations, and viscoelastic system with infinite Weissenberg number, see [38].

(ii) When χ is a constant, the system (1.1) becomes a nonhomogeneous incompressible Navier-Stokes equations. It has been paid many attentions, see Antontsev and Kazhikov [4], Kazhikov [21], Simon [30], Lions [23], Choe and Kim [10], and the references therein.

(iii) When ρ and χ are constants, the system (1.1) reduces to classical incompressible Navier-Stokes equation, which is the fundamental equation to describe Newtonian fluids. It has attracted great interests, see Lions [23] and Feireisl [34] for survey of important developments.

(iv) When ρ is a constant and $u = 0$, the system (1.1) turns out to be the Allen-Cahn equation, which was originally introduced by Allen and Cahn [2] to describe the motion of antiphase boundaries in crystalline solids. This type of equation has been extensively studied, see [11, 19, 31] for example.

The diffuse interface models for two-phase flow of incompressible viscous fluids with "matched densities" have been extensively studied. For incompressible Navier-Stokes/Allen-Cahn system, Xu et al. [38] discussed the axisymmetric solutions in 3D. They prove the global regularity of the constructed solutions in both large viscosity and small initial data cases. Zhao et al. [40] investigated the vanishing viscosity limit. They proved that the solutions of the Navier-Stokes/Allen-Cahn system converge to that of the Euler/Allen-Cahn system in a proper small time interval. Gal and Grasselli [17] showed the existence of the trajectory attractor. For another type of diffuse interface model – Navier-Stokes/Cahn-Hilliard system, Boyer [6] studied the existence of global weak and strong solutions in 2D, the existence of unique strong solution in 3D and the stability of the stationary solutions. For the studies on well-posedness, asymptotic behavior, attractor, etc, see [1, 17, 18] and the references cited therein. Moreover, for numerical simulations, such as jet pinching-off and drop formation, we refer the readers to [7, 24, 35, 39].

It is evident that, the densities in two fluids are often quite different. Therefore, the investigations on the phase field models for two-phase flow with non-matched densities are significant. To our knowledge, there are only a few theoretical results available to compressible models. For compressible Navier-Stokes/Allen-Cahn system, Feireisl et al. [15] proved the existence of weak solutions in 3D. In [13], we obtained the global well-posedness in 1D with constant mobility. We prove the existence of the initial boundary value problem in various regularity classes, as well as uniqueness for strong solutions. For compressible Navier-Stokes/Cahn-Hilliard system, Abels and Feireisl [3] derived the existence of weak solutions.

In this paper, we investigate the Navier-Stokes/Allen-Cahn system for two fluids with non-matched densities, but the velocity u satisfies the divergence-free condition $\mathrm{div} u = 0$, i.e. the fluids are incompressible and with different densities. We are interested in the existence of unique local strong solution and the main mechanism for possible breakdown of such a local strong solution.

We supplement the system (1.1) with the following initial conditions

$$(\rho, u, \chi)\Big|_{t=0} = (\rho_0, u_0, \chi_0), \qquad x \in \Omega, \tag{1.2}$$

the usual no-slip boundary condition on the velocity and Neumann boundary condition on the phase field variable

$$\Big(u, \frac{\partial \chi}{\partial \mathbf{n}}\Big)\Big|_{\partial\Omega} = (0, 0), \qquad t \geq 0, \tag{1.3}$$

where $\mathbf{n}$ is the unit outward normal vector of $\partial\Omega$.

Notations For $p \geq 1$, denote $L^p = L^p(\Omega)$ as the L^p space with the norm $\|\cdot\|_{L^p}$. For $k \geq 1$ and $p \geq 1$, denote $W^{k,p} = W^{k,p}(\Omega)$ for a Sobolev space, whose norm is denoted by $\|\cdot\|_{W^{k,p}}$, and specially $H^k = W^{k,2}(\Omega)$.

Definition 1.1 *For $T > 0$, (ρ, u, p, χ, μ) is called a strong solution of the coupled Navier-Stokes/Allen-Cahn system* (1.1) *in $\Omega \times (0, T]$, if*

$$\begin{aligned}
&\rho \in L^\infty(0, T; W^{2,6}), \quad \rho_t \in L^\infty(0, T; W^{1,6}), \quad 0 < c^{-1} \leq \rho \leq c,\\
&u \in L^\infty\big(0, T; H^2 \cap H_0^1\big) \cap L^2(0, T; W^{2,6}), \quad u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; H_0^1),\\
&p \in L^\infty(0, T; H^1) \cap L^2(0, T; W^{1,6}),
\end{aligned}$$

$$\chi \in L^\infty(0,T;H^3)\cap L^2(0,T;H^4),\ \chi_t \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2),\ \chi_{tt}\in L^2(0,T;L^2),$$
$$\mu \in L^\infty(0,T;H^1)\cap L^2(0,T;H^2),\quad \mu_t \in L^2(0,T;L^2),$$

and $(\rho,\ u,\ p,\ \chi,\ \mu)$ *satisfies* (1.1) *a.e. in* $\Omega\times(0,T]$.

Our first result is the existence and uniqueness of local strong solution.

Theorem 1.1 *Assume that* $\rho_0 \in W^{2,6}(\Omega)$ *satisfies* $0 < c_0^{-1} \le \rho_0 \le c_0$ *for some constant* c_0, $u_0 \in H^2(\Omega)\cap H_0^1(\Omega)$, $\chi_0 \in H^3(\Omega)$ *and* $u_0|_{\partial\Omega} = 0$, $\mathrm{div}u_0 = 0$ *for* $x \in \Omega$. *Then there exist a time* $T_* > 0$, *a constant* $c = c(c_0, T_*)$ *and a unique strong solution* (ρ,u,p,χ,μ) *of the problem* (1.1)–(1.3) *in* $\Omega\times(0,T_*]$.

Noticing that if the density ρ do not appear in Allen-Cahn equation $(1.1)_{4,5}$ and the viscous coefficient is a constant, the system (1.1) reduces to the Ginzburg-Landau approximation model of nonhomogeneous incompressible nematic liquid crystals. Global existence of weak solutions to this type of model have been proved in [20, 28]. Wen and Ding [36] establish the global existence and uniqueness of solution with small initial data to original nonhomogeneous incompressible nematic liquid crystals in 2D. The system (1.1) becomes nontrivial because of the appearance of the density ρ. Within our knowledge, there are no results for this type of model even for local existence or global solutions with small initial data in 2D. The existence of global solutions is closely related to the estimate for $\|\nabla\rho\|_{L^\infty(Q_T)}$, which is the main difficulty we can not handle.

Besides the global existence, another interesting question is the main mechanism of possible break down of local strong solutions. For such question, the pioneering work is obtained by Beale, Kato and Majda [9], they proved that the maximum norm of the vorticity $\nabla\times u$ controls the breakdown of smooth solutions of the 3D Euler equations. Later, Ponce [29] derived the same blow-up criterion when the vorticity was substituted by the deformation tensor Du, that is, a solution remains smooth if

$$\int_0^T \|Du\|_{L^\infty}\mathrm{d}t$$

remains bounded. The works on blow-up criterion for incompressible Navier-Stokes equation, we refer the readers to Serrin [32] and Struwe [33] for example. Recently, for nonhomogeneous incompressible Navier-Stokes equations, i.e. the velocity is divergence-free, but the density is not assumed to be a constant, Kim [22] established a weak Serrin class blow-up criterion.

Motivated by these works, we will establish in this paper the blow-up criterion of breakdown of local strong solutions in finite time.

Theorem 1.2 *Let* (ρ,u,p,χ,μ) *be a strong solution of the initial boundary value problem* (1.1)–(1.3). *If* $0 < T_* < +\infty$ *is the maximum time of existence, then*

$$\int_0^{T_*}\left(\|Du\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}^2\right)\mathrm{d}t = +\infty, \qquad \text{if } N = 2, \tag{1.4}$$

$$\int_0^{T_*}\left(\|Du\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\nabla\chi\|_{L^\infty}^2\right)\mathrm{d}t = +\infty, \quad \text{if } N = 3. \tag{1.5}$$

Remark (i) To our knowledge, there are only a few theoretical investigations on Navier-Stokes/Allen-Cahn system for two-phase flow with different densities. For compressible fluids, the first result addressing solvability is due to Able and Feireisl [3], in which the authors proved the existence of global weak solutions in 3D, but not uniqueness. In another paper [13], we proved global well-posedness in 1D with constant viscosity coefficients. This paper is concerned with incompressible fluids with different densities, and the viscosity coefficient depends on phase variable χ, which is of interest from the physical point of view.
(ii) We should point out that, because of the appearance of the density ρ in Allen-Cahn equation $(1.1)_{4,5}$, we can not handle the vacuum state. Moreover, our proof strongly depends on the divergence-free condition, which ensure the solutions being away from vacuum. So the conclusions in this paper can not be extended to corresponding compressible system directly.
(iii) It is a very interesting question to ask whether there exists global weak or strong solutions to the initial boundary value problem of (1.1)–(1.3) in dimensions at least two.

Since the constant δ play no role in the analysis, we assume henceforth that $\delta = 1$. Throughout this paper, we assume that $\eta(s) \in C^1(\mathbb{R})$ and there exist positive constants $\underline{\eta}$, $\overline{\eta}$ and $\tilde{\eta}$, such that

$$0 < \underline{\eta} \leq \eta(s) \leq \overline{\eta}, \qquad |\eta^{'}(s)| \leq \tilde{\eta}. \tag{1.6}$$

The structure of this article is as follows. In Section 2, inspired by the works on liquid crystals [36], we prove the local existence of unique strong solution by using the technique of iteration. Firstly, we introduce an auxiliary problem for nonhomogeneous incompressible Navier-Stokes equations. The proof of the existence and uniqueness of solution to this auxiliary problem is similar to that in [36]. If the initial density is more regular, the density is also regular too, see Lemma 2.2. This is an important character of the density not only in the proof of local strong solution, but also in the discussion of blow-up criterion. Basing on these results and classical theory, we construct the approximate solutions and begin to do iterate. At last, in terms of the estimates for the approximate solutions, we derive the desired local strong solution by taking limits. Then we establish the blow-up criterion of local strong solutions by contradiction in Section 3.

2 Local strong solutions

Let T be a fixed time with $0 < T < 1$. Denote

$$V_{T,K_1} = \left\{ v \,\middle|\, v(x,0) = u_0(x),\ v\big|_{\partial\Omega} = 0,\ \|v\|_V \leq K_1 \right\},$$
$$\Phi_{T,K_2} = \left\{ \varphi \,\middle|\, \varphi(x,0) = \chi_0(x),\ \frac{\partial \varphi}{\partial \mathbf{n}}\Big|_{\partial\Omega} = 0,\ \|\varphi\|_\Phi \leq K_2 \right\},$$

where

$$\|v\|_V = \|v_t\|_{L^\infty(0,T;L^2)} + \|v_t\|_{L^2(0,T;H^1)} + \|v\|_{L^\infty(0,T;H^2)} + \|v\|_{L^2(0,T;W^{2,6})},$$
$$\|\varphi\|_\Phi = \|\varphi_t\|_{L^\infty(0,T;H^1)} + \|\varphi_t\|_{L^2(0,T;H^2)} + \|\varphi\|_{L^\infty(0,T;H^3)} + \|\varphi\|_{L^2(0,T;H^4)},$$

and the constants $K_1, K_2 > 1$ will be determined later.

Firstly, before proving local existence of strong solutions, we consider the following auxiliary problem with $(v,\varphi)\in V_{T,K_1}\times\Phi_{T,K_2}$

$$\begin{cases}\rho_t+(v\cdot\nabla)\rho=0,\\ \rho u_t+\nabla p=\mathrm{div}(2\eta(\varphi)Du)+\rho f_1+f_2,\\ \mathrm{div}u=0\end{cases}\tag{2.1}$$

subject to the initial and boundary conditions

$$(\rho,u)\big|_{t=0}=(\rho_0,u_0),\quad x\in\Omega,\qquad u\big|_{\partial\Omega}=0,\quad t\geq 0.\tag{2.2}$$

There exists a unique strong solution to the problem (2.1)–(2.2).

Lemma 2.1 *Assume that $\rho_0\in H^1(\Omega)\cap L^\infty(\Omega)$, $\rho_0\geq 0$, $u_0\in H^2(\Omega)\cap H_0^1(\Omega)$, $f_i\in L^2(0,T;H^1(\Omega))$, $f_{it}\in L^2(Q_T)$ $(i=1,2)$, and the following compatible conditions are valid*

$$\mathrm{div}(2\eta(\chi_0)Du_0)-\nabla p_0(x)+f_2(x,0)=\rho_0^{1/2}g(x)\quad\text{and}\quad \mathrm{div}u_0(x)=0,\quad\text{in }\Omega,$$

for $(p_0,g)\in H^1(\Omega)\times L^2(\Omega)$. Then for any $T>0$, the problem (2.1)–(2.2) admits a unique solution (ρ,u,p) such that

$$\begin{aligned}&\rho\in L^\infty(0,T;H^1)\cap L^\infty(Q_T),\quad \rho_t\in L^\infty(0,T;L^2),\\ &u\in L^\infty(0,T;H^2\cap H_0^1)\cap L^2(0,T;W^{2,6}),\\ &u_t\in L^2(0,T;H_0^1),\qquad \sqrt{\rho}u_t\in L^\infty(0,T;L^2),\\ &p\in L^\infty(0,T;H^1)\cap L^2(0,T;W^{1,6}).\end{aligned}$$

When $\mathrm{div}(2\eta(\varphi)Du)$ is substituted by Δu in (2.1), this result has been obtained by Wen and Ding [36]. The treatment of the coefficient $\eta(\varphi)$ is similar to the proof of step 3 in this section, so we omit the proof of this lemma here.

Lemma 2.2 *In addition to the conditions in Lemma 2.1, if $\rho_0\in W^{2,6}(\Omega)$, $0<c_0^{-1}\leq\rho\leq c_0$ for some constant c_0, then we also have*

$$\rho\in L^\infty(0,T;W^{2,6}),\quad \rho_t\in L^\infty(0,T;W^{1,6}),\quad 0<c^{-1}\leq\rho\leq c.$$

Proof. The existence and uniqueness of strong solutions to the hyperbolic equation $(2.1)_1$ is well known. Moreover, from [10] the solution satisfies the following estimates

$$0<c^{-1}\leq\rho\leq c,\quad\text{in }Q_T,\tag{2.3}$$

$$\sup_{0\leq t\leq T}\left(\|\rho\|_{H^1}+K_1^{-1}\|\rho_t\|_{L^2}\right)\leq c\exp\{cK_1T^{1/2}\}.\tag{2.4}$$

Differentiating $(2.1)_1$ with respect to x, multiplying by $r|\nabla\rho|^{r-2}\nabla\rho$ $(1\leq r<+\infty)$ and integrating the result with respect to x over Ω, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\rho|^r\mathrm{d}x=-\int_\Omega(v\cdot\nabla)(|\nabla\rho|^r)\mathrm{d}x-r\int_\Omega|\nabla\rho|^{r-2}\nabla\rho\cdot\nabla(v\cdot\nabla)\rho\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_\Omega \mathrm{div} v |\nabla\rho|^r \mathrm{d}x - r\int_\Omega |\nabla\rho|^{r-2}\nabla\rho\cdot\nabla(v\cdot\nabla)\rho\,\mathrm{d}x \\
&\le (1+r)\|\nabla v\|_{L^\infty}\int_\Omega |\nabla\rho|^r \mathrm{d}x.
\end{aligned}
$$

From which we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\rho\|_{L^r} \le \frac{r+1}{r}\|\nabla v\|_{L^\infty}\|\nabla\rho\|_{L^r}.
$$

Then Gronwall's inequality implies

$$
\|\nabla\rho\|_{L^r} \le \|\nabla\rho_0\|_{L^r}\exp\left\{\frac{r+1}{r}\int_0^T \|\nabla v\|_{W^{1,6}}\mathrm{d}t\right\}.
$$

Sending $r\to+\infty$, recalling $\rho_0\in W^{2,6}(\Omega)$ and using Hölder's inequality yield

$$
\sup_{0\le t\le T}\|\nabla\rho\|_{L^\infty} \le c\exp\{cK_1T^{1/2}\}. \tag{2.5}
$$

Differentiating $(2.1)_1$ with respect to x twice and multiplying the above equation by $l|\nabla^2\rho|^{l-2}\nabla^2\rho$ $(2\le l\le 6)$, and integrating the result over Ω, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\nabla^2\rho|^l\mathrm{d}x =& -l\int_\Omega |\nabla^2\rho|^{l-2}\nabla^2\rho : \nabla^2(v\cdot\nabla)\rho\mathrm{d}x + \int_\Omega \mathrm{div}v|\nabla^2\rho|^l\mathrm{d}x \\
& - 2l\int_\Omega |\nabla^2\rho|^{l-2}\nabla^2\rho : \nabla(v\cdot\nabla)\nabla\rho\mathrm{d}x \\
\le& c\|\nabla^2 v\|_{L^6}\|\nabla\rho\|_{L^\infty}\left(\int_\Omega |\nabla^2\rho|^{\frac{6}{5}(l-1)}\mathrm{d}x\right)^{5/6} + c\|\nabla v\|_{L^\infty}\int_\Omega |\nabla^2\rho|^l\mathrm{d}x \\
\le& c\|\nabla^2 v\|_{L^6}\|\nabla\rho\|_{L^\infty}\left(\int_\Omega |\nabla^2\rho|^l\mathrm{d}x\right)^{(l-1)/l} + c\|\nabla v\|_{L^\infty}\int_\Omega |\nabla^2\rho|^l\mathrm{d}x \\
\le& c\|v\|_{W^{2,6}}(\|\nabla\rho\|_{L^\infty}+1)\int_\Omega |\nabla^2\rho|^l\mathrm{d}x + c\|v\|_{W^{2,6}}\|\nabla\rho\|_{L^\infty},
\end{aligned}
$$

where we have used Hölder's inequality for $2\le l\le 6$ in the third step. Applying Gronwall's inequality, we obtain

$$
\int_\Omega |\nabla^2\rho|^l\mathrm{d}x \le c\left(1+K_1T^{1/2}\exp\{cK_1T^{1/2}\}\right)\exp\left\{cK_1T^{1/2}(\exp\{cK_1T^{1/2}\}+1)\right\}. \tag{2.6}
$$

Furthermore, differentiating $(2.1)_1$ with respect to x, we derive that

$$
\begin{aligned}
\|\nabla\rho_t\|_{L^l} &\le c\|\nabla v\|_{L^l}\|\nabla\rho\|_{L^\infty} + c\|v\|_{L^\infty}\|\nabla^2\rho\|_{L^l} \le c\|v\|_{H^2}(\|\nabla\rho\|_{L^\infty}+\|\nabla^2\rho\|_{L^l}) \\
&\le cK_1\left(1+K_1T^{1/2}\exp\{cK_1T^{1/2}\}\right)\exp\left\{cK_1T^{1/2}(\exp\{cK_1T^{1/2}\}+1)\right\}.
\end{aligned} \tag{2.7}
$$

Then Lemma 2.2 follows from (2.3), (2.4), (2.6) and (2.7). □

Next, we consider the following linearized problem

$$
\begin{cases}
\rho_t + (v\cdot\nabla)\rho = 0,\\
\rho u_t + \nabla p = \mathrm{div}(2\eta(\varphi)Du) - \rho(v\cdot\nabla)v - \mathrm{div}(\nabla\chi\otimes\nabla\chi),\\
\mathrm{div}u = 0,\\
\chi_t = \dfrac{1}{\rho^2}\Delta\chi - (v\cdot\nabla)\varphi - \dfrac{1}{\rho}(\varphi^3-\varphi)
\end{cases}
\tag{2.8}
$$

with the initial boundary conditions (1.2) and (1.3), where $(v,\varphi)\in V_{T,K_1}\times\Phi_{T,K_2}$. Recalling Lemma 2.2 and the definition of V_{T,K_1}, Φ_{T,K_2}, we have $-(v\cdot\nabla)\varphi - \frac{1}{\rho}(\varphi^3-\varphi)\in W_2^{2,1}(Q_T)$. It follows from classical arguments [26] that the equation $(2.8)_4$ subject to the corresponding initial boundary value conditions admits a unique solution such that

$$
\chi\in W_2^{4,2}(Q_T)\cap L^\infty(0,T;H^3),\quad \chi_t\in L^\infty(0,T;H^1).
$$

Moreover, by Lemma 2.1 the problem $(2.8)_{1-3}$ with the corresponding initial boundary value conditions has a unique solution (ρ,u,p) and the regularities like that in Lemma 2.1–2.2.

Therefore, we have a solution (ρ^1,u^1,p^1,χ^1) of the problem (2.8) with (v,φ) replaced by (u^0,χ^0), where $(u^0,\chi^0)\in V_{T,K_1}\times\Phi_{T,K_2}$. Suppose $(u^{k-1},\chi^{k-1})\in V_{T,K_1}\times\Phi_{T,K_2}$ for $k\geq 1$, then we can construct an approximate solution (ρ^k,u^k,p^k,χ^k) satisfying the following problem

$$
\begin{cases}
\rho^k_t + (u^{k-1}\cdot\nabla)\rho^k = 0,\\
\rho^k u^k_t + \nabla p^k = \mathrm{div}(2\eta(\chi^{k-1})Du^k) - \rho^k(u^{k-1}\cdot\nabla)u^{k-1} - \mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k),\\
\mathrm{div}u^k = 0,\\
\chi^k_t = \dfrac{1}{(\rho^k)^2}\triangle\chi^k - (u^{k-1}\cdot\nabla)\chi^{k-1} - \dfrac{1}{\rho^k}(\chi^{k-1})^3 + \dfrac{1}{\rho^k}\chi^{k-1}
\end{cases}
\tag{2.9}
$$

supplemented with initial and boundary conditions

$$
\begin{aligned}
&(\rho^k,u^k,\chi^k)\big|_{t=0} = (\rho_0,u_0,\chi_0),\quad x\in\Omega,\\
&\Big(u^k,\frac{\partial\chi^k}{\partial\mathbf{n}}\Big)\Big|_{\partial\Omega} = (0,0),\qquad t\geq 0.
\end{aligned}
$$

In what follows, we prove Theorem 1.1 by iteration. Throughout this paper, we denote by $A\lesssim B$ if there exists a constant C such that $A\leq CB$. Moreover, in step 1-3, we denote by C a constant whose value may be different from line to line but independent of K_1 and K_2.
Step 1: It holds that

$$
0 < C^{-1}\leq\rho^k\leq C, \tag{2.10}
$$

$$
\|\rho^k\|_{L^\infty(0,T;W^{2,6})}\leq C, \tag{2.11}
$$

$$
\|\rho^k_t\|_{L^\infty(0,T;W^{1,6})}\leq CK_1. \tag{2.12}
$$

Taking $0<T<T_1 := \dfrac{1}{K_1^2}$, then the estimates (2.10)–(2.12) are the directly deductions of (2.3), (2.4), (2.6) and (2.7).

Step 2: We will prove

$$\|\chi^k\|_\Phi = \|\chi_t^k\|_{L^\infty(0,T;H^1)} + \|\chi_t^k\|_{L^2(0,T;H^2)} + \|\chi^k\|_{L^\infty(0,T;H^3)} + \|\chi^k\|_{L^2(0,T;H^4)} \le K_2. \tag{2.13}$$

We rewrite the equation $(2.9)_4$ as follows

$$(\rho^k)^2\chi_t^k - \Delta\chi^k = -(\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1} - \rho^k(\chi^{k-1})^3 + \rho^k\chi^{k-1}. \tag{2.14}$$

Differentiating (2.14) with respect to t, multiplying the result by χ_t^k, then integrating over Ω yield

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\rho^k\chi_t^k|^2\mathrm{d}x + \int_\Omega |\nabla\chi_t^k|^2\mathrm{d}x \\
=&-\int_\Omega \rho^k\rho_t^k|\chi_t^k|^2\mathrm{d}x - \int_\Omega 2\rho^k\rho_t^k(u^{k-1}\cdot\nabla)\chi^{k-1}\chi_t^k\mathrm{d}x - \int_\Omega(\rho^k)^2(u_t^{k-1}\cdot\nabla)\chi^{k-1}\chi_t^k\mathrm{d}x \\
&-\int_\Omega(\rho^k)^2(u^{k-1}\cdot\nabla)\chi_t^{k-1}\chi_t^k\mathrm{d}x - \int_\Omega \rho_t^k(\chi^{k-1})^3\chi_t^k\mathrm{d}x - \int_\Omega 3\rho^k(\chi^{k-1})^2\chi_t^{k-1}\chi_t^k\mathrm{d}x \\
&+\int_\Omega \rho_t^k\chi^{k-1}\chi_t^k\mathrm{d}x + \int_\Omega \rho^k\chi_t^{k-1}\chi_t^k\mathrm{d}x \\
\lesssim&\|\rho_t^k\|_{L^\infty}\|\rho^k\chi_t^k\|_{L^2}^2 + \|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^6}\|\nabla\chi^{k-1}\|_{L^3}\|\rho^k\chi_t^k\|_{L^2} + \|u_t^{k-1}\|_{L^2}\|\nabla\chi^{k-1}\|_{L^\infty}\|\rho^k\chi_t^k\|_{L^2} \\
&+\|u^{k-1}\|_{L^\infty}\|\nabla\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} + \|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^6}^3\|\rho^k\chi_t^k\|_{L^2} + \|\chi^{k-1}\|_{L^\infty}^2\|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} \\
&+\|\rho_t^k\|_{L^\infty}\|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} + \|\chi_t^{k-1}\|_{L^2}\|\rho^k\chi_t^k\|_{L^2} \\
\lesssim&(\|\rho_t^k\|_{W^{1,6}}+1)\int_\Omega|\rho^k\chi_t^k|^2\mathrm{d}x + \|\rho_t^k\|_{W^{1,6}}^2\|u^{k-1}\|_{H^1}^2\|\nabla\chi^{k-1}\|_{H^1}^2 + \|u_t^{k-1}\|_{L^2}^2\|\nabla\chi^{k-1}\|_{H^2}^2 \\
&+\|u^{k-1}\|_{H^2}^2\|\nabla\chi_t^{k-1}\|_{L^2}^2 + \|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{H^1}^6 + \|\chi^{k-1}\|_{H^2}^4\|\chi_t^{k-1}\|_{L^2}^2 \\
&+\|\rho_t^k\|_{W^{1,6}}^2\|\chi_t^{k-1}\|_{L^2}^2 + \|\chi_t^{k-1}\|_{L^2}^2 \\
\lesssim&(K_1+1)\int_\Omega|\rho^k\chi_t^k|^2\mathrm{d}x + K_1^4K_2^2 + K_1^2K_2^2 + K_1^2K_2^6 + K_2^6 + K_2^2.
\end{aligned}$$

Taking $T_2 := \min\left\{T_1, \dfrac{1}{K_1^4K_2^2 + K_1^2K_2^6}\right\}$ and using Gronwall's inequality, for any $0 < T < T_2$ we have

$$\sup_{0\le t\le T}\int_\Omega|\rho^k\chi_t^k|^2\mathrm{d}x + \int_0^T\int_\Omega|\nabla\chi_t^k|^2\mathrm{d}x\mathrm{d}t \le C. \tag{2.15}$$

Differentiating $(2.9)_4$ with respect to t, multiplying by $\Delta\chi_t^k$ and differentiating the result over Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\chi_t^k|^2\mathrm{d}x + \int_\Omega\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x \\
=&\int_\Omega\frac{2\rho_t^k}{(\rho^k)^3}\triangle\chi^k\Delta\chi_t^k\mathrm{d}x + \int_\Omega(u_t^{k-1}\cdot\nabla)\chi^{k-1}\Delta\chi_t^k\mathrm{d}x + \int_\Omega(u^{k-1}\cdot\nabla)\chi_t^{k-1}\Delta\chi_t^k\mathrm{d}x \\
&-\int_\Omega\frac{\rho_t^k}{(\rho^k)^2}(\chi^{k-1})^3\Delta\chi_t^k\mathrm{d}x + \int_\Omega\frac{3}{\rho^k}(\chi^{k-1})^2\chi_t^{k-1}\Delta\chi_t^k\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\int_\Omega \frac{\rho_t^k}{(\rho^k)^2}\chi^{k-1}\Delta\chi_t^k \mathrm{d}x-\int_\Omega \frac{1}{\rho^k}\chi_t^{k-1}\Delta\chi_t^k \mathrm{d}x\\
\lesssim&\|\rho_t^k\|_{L^\infty}\|\Delta\chi^k\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|u_t^{k-1}\|_{L^2}\|\nabla\chi^{k-1}\|_{L^\infty}\|\Delta\chi_t^k\|_{L^2}+\|u^{k-1}\|_{L^\infty}\|\nabla\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}\\
&+\|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^6}^3\|\Delta\chi_t^k\|_{L^2}+\|\chi^{k-1}\|_{L^\infty}^2\|\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}\\
&+\|\rho_t^k\|_{L^\infty}\|\chi^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}+\|\chi_t^{k-1}\|_{L^2}\|\Delta\chi_t^k\|_{L^2}\\
\lesssim&\frac{1}{2}\int_\Omega\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x+\|\rho_t^k\|_{W^{1,6}}^2\|\Delta\chi^k\|_{L^2}^2+\|u_t^{k-1}\|_{L^2}^2\|\nabla\chi^{k-1}\|_{H^2}^2+\|u^{k-1}\|_{H^2}^2\|\nabla\chi_t^{k-1}\|_{L^2}^2\\
&+\|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{H^1}^6+\|\chi^{k-1}\|_{H^2}^4\|\chi_t^{k-1}\|_{L^2}^2+\|\rho_t^k\|_{W^{1,6}}^2\|\chi^{k-1}\|_{L^2}^2+\|\chi_t^{k-1}\|_{L^2}^2.
\end{aligned}
$$

It follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\chi_t^k|^2\mathrm{d}x+\int_\Omega\frac{1}{(\rho^k)^2}|\Delta\chi_t^k|^2\mathrm{d}x\lesssim K_1^2\|\Delta\chi^k\|_{L^2}^2+K_1^2K_2^6+K_2^6+K_1^2K_2^2+K_2^2. \tag{2.16}
$$

From the equation (2.14) and the estimate (2.15), we see that

$$
\begin{aligned}
\|\Delta\chi^k\|_{L^2}^2\lesssim&\|\chi_t^k\|_{L^2}^2+\|u^{k-1}\|_{L^6}^2\|\nabla\chi^{k-1}\|_{L^3}^2+\|\chi^{k-1}\|_{L^6}^6+\|\chi^{k-1}\|_{L^2}^2\\
\lesssim&\|\chi_t^k\|_{L^2}^2+\|u^{k-1}\|_{H^1}^2\|\nabla\chi^{k-1}\|_{H^1}^2+\|\chi^{k-1}\|_{H^1}^6+\|\chi^{k-1}\|_{L^2}^2\\
\lesssim&1+K_1^2K_2^2+K_2^6+K_2^2.
\end{aligned}\tag{2.17}
$$

Substituting (2.17) into (2.16) and applying Gronwall's inequality, for any $0<T<T_2$ we obtain

$$
\|\chi_t^k\|_{L^\infty(0,T;H^1)}\le C. \tag{2.18}
$$

By (2.15)–(2.18) and the elliptic estimates for Neumann boundary value problem

$$
\|\nabla^2\chi_t^k\|_{L^2}\lesssim\|\Delta\chi_t^k\|_{L^2}+\|\nabla\chi_t^k\|_{L^2},
$$

for any $0<T<T_2$ we get

$$
\|\chi_t^k\|_{L^2(0,T;H^2)}\le C. \tag{2.19}
$$

On the other hand, for any $0<T<T_2$, using the elliptic estimates for the equation (2.14) gives

$$
\begin{aligned}
\|\chi^k\|_{H^3}^2\lesssim&\|(\rho^k)^2\chi_t^k\|_{H^1}^2+\|(\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1}\|_{H^1}^2+\|\rho^k(\chi^{k-1})^3\|_{H^1}^2+\|\rho^k\chi^{k-1}\|_{H^1}^2+\|\chi_0\|_{H^3}^2\\
:=&\sum_{i=1}^4 I_i+\|\chi_0\|_{H^3}^2.
\end{aligned}
$$

In what follows, we deal with the terms on the right hand side one by one. By (2.18) we get

$$
I_1\lesssim\|(\rho^k)^2\chi_t^k\|_{L^2}^2+\|\nabla\left((\rho^k)^2\chi_t^k\right)\|_{L^2}^2\lesssim\|\chi_t^k\|_{L^2}^2+\|\nabla\chi_t^k\|_{L^2}^2+\|\nabla\rho^k\|_{L^\infty}^2\|\chi_t^k\|_{L^2}^2\le C.
$$

For I_2, we have

$$
I_2\lesssim\int_\Omega|u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x+\int_\Omega|\nabla u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x+\int_\Omega|u^{k-1}|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x
$$

$$+\|\nabla\rho^k\|_{L^\infty}^2\int_\Omega |u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x := \sum_{i=1}^4 J_i,$$

where

$$\begin{aligned}
J_1+J_4 \lesssim & \int_\Omega |u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x \lesssim \int_\Omega |u^{k-1}-u_0|^2|\nabla\chi^{k-1}|^2\mathrm{d}x + \int_\Omega |\nabla\chi^{k-1}|^2\mathrm{d}x \\
\lesssim & \|\nabla\chi^{k-1}\|_{L^\infty}^2\int_\Omega \left|\int_0^t u_t^{k-1}(x,s)\mathrm{d}s\right|^2\mathrm{d}x + \int_\Omega |\nabla(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \\
\lesssim & K_2^2T\int_0^T\int_\Omega |u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + K_2^{-2}\int_\Omega |\nabla^2(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + K_2^2\int_\Omega |\chi^{k-1}-\chi_0|^2\mathrm{d}x + 1 \\
\lesssim & K_1^2K_2^2T^2 + K_2^2\int_\Omega\left|\int_0^t \chi_t^{k-1}(x,s)\mathrm{d}s\right|^2\mathrm{d}x + 1 \\
\lesssim & K_1^2K_2^2T^2 + K_2^2T\int_0^T\int_\Omega |\chi_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + 1 \lesssim K_1^2K_2^2T^2 + K_2^4T^2 + 1,
\end{aligned}$$

$$\begin{aligned}
J_2 = & \int_\Omega |\nabla u^{k-1}|^2|\nabla\chi^{k-1}|^2\mathrm{d}x \lesssim K_2^2\int_\Omega |\nabla(u^{k-1}-u_0)|^2\mathrm{d}x + \int_\Omega |\nabla(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \\
\lesssim & K_1^{-2}\int_\Omega |\nabla^2(u^{k-1}-u_0)|^2\mathrm{d}x + K_1^2K_2^2\int_\Omega |u^{k-1}-u_0|^2\mathrm{d}x + K_2^4T^2 + 1 \\
\lesssim & K_1^2K_2^2T\int_0^T\int_\Omega |u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t + K_2^4T^2 + 1 \lesssim K_1^4K_2^2T^2 + K_2^4T^2 + 1,
\end{aligned}$$

and

$$\begin{aligned}
J_3 = & \int_\Omega |u^{k-1}|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x \lesssim \int_\Omega |u^{k-1}-u_0|^2|\nabla^2\chi^{k-1}|^2\mathrm{d}x + \int_\Omega |\nabla^2(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + 1 \\
\lesssim & \|u^{k-1}-u_0\|_{L^6}^2\|\nabla^2\chi^{k-1}\|_{L^3}^2 + K_2^{-2}\int_\Omega |\nabla^3(\chi^{k-1}-\chi_0)|^2\mathrm{d}x + K_2^4\int_\Omega |\chi^{k-1}-\chi_0|^2\mathrm{d}x + 1 \\
\lesssim & K_2^2\int_\Omega |\nabla(u^{k-1}-u_0)|^2\mathrm{d}x + K_2^6T^2 + 1 \lesssim K_1^4K_2^2T^2 + K_2^6T^2 + 1.
\end{aligned}$$

Similarly, we deduce that

$$\begin{aligned}
I_3 = & \|\rho^k(\chi^{k-1})^3\|_{H^1}^2 \lesssim \|\rho^k(\chi^{k-1})^3\|_{L^2}^2 + \|3\rho^k(\chi^{k-1})^2\nabla\chi^{k-1}\|_{L^2}^2 + \|\nabla\rho^k(\chi^{k-1})^3\|_{L^2}^2 \\
\lesssim & \|\chi^{k-1}\|_{L^6}^6 + \|\chi^{k-1}\|_{L^6}^4\|\nabla\chi^{k-1}\|_{L^6}^2 + \|\nabla\rho^k\|_{L^\infty}^2\|\chi^{k-1}\|_{L^6}^6 \\
\lesssim & \|\chi^{k-1}\|_{H^1}^6 + \|\chi^{k-1}\|_{H^1}^4\|\nabla^2\chi^{k-1}\|_{L^2}^2 \\
\lesssim & (K_2^2T^2 + K_2^4T^2 + 1)^3 + K_2^4T^2 + K_2^6T^2 + K_2^8T^2 + K_2^{12}T^2 + 1,
\end{aligned}$$

where we have used

$$\begin{aligned}
& \|\nabla\chi^{k-1}\|_{L^2}^4\|\nabla^2\chi^{k-1}\|_{L^2}^2 \\
\lesssim & \|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2\|\nabla\chi^{k-1}\|_{L^2}^2\|\nabla^2\chi^{k-1}\|_{L^2}^2 + \|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2\|\nabla^2\chi^{k-1}\|_{L^2}^2 + \|\nabla^2\chi^{k-1}\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&\lesssim K_2^4\|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2+K_2^2\|\nabla(\chi^{k-1}-\chi_0)\|_{L^2}^2+\|\nabla^2(\chi^{k-1}-\chi_0)\|_{L^2}^2+1\\
&\lesssim K_2^{-2}\|\nabla^2(\chi^{k-1}-\chi_0)\|_{L^2}^2+K_2^{-2}\|\nabla^3(\chi^{k-1}-\chi_0)\|_{L^2}^2+(K_2^{10}+K_2^6+K_2^4)\|\chi^{k-1}-\chi_0\|_{L^2}^2+1\\
&\lesssim (K_2^{10}+K_2^6+K_2^4)T\int_0^T\int_\Omega|\chi_t^{k-1}|^2\mathrm{d}x\mathrm{d}t+1\lesssim (K_2^{10}+K_2^6+K_2^4)K_2^2T^2+1.
\end{aligned}
$$

At last,

$$
\begin{aligned}
I_4=&\|\rho^k\chi^{k-1}\|_{H^1}^2\lesssim\|\chi^{k-1}\|_{L^2}^2+\|\nabla\chi^{k-1}\|_{L^2}^2+\|\nabla\rho^k\|_{L^\infty}^2\|\chi^{k-1}\|_{L^2}^2\\
\lesssim&\|\chi^{k-1}\|_{H^1}^2\lesssim K_2^2T^2+K_2^4T^2+1.
\end{aligned}
$$

Putting all the above estimates together, for any $0<T<T_2$ we obtain

$$
\|\chi^k\|_{L^\infty(0,T;H^3)}\le C. \tag{2.20}
$$

From the equation (2.14) we can also derive that

$$
\begin{aligned}
\|\nabla^4\chi\|_{L^2}^2\lesssim&\|\nabla^2\left((\rho^k)^2\chi_t^k\right)\|_{L^2}^2+\|\nabla^2\left((\rho^k)^2(u^{k-1}\cdot\nabla)\chi^{k-1}\right)\|_{L^2}^2\\
&+\|\nabla^2\left(\rho^k(\chi^{k-1})^3\right)\|_{L^2}^2+\|\nabla^2\left(\rho^k\chi^{k-1}\right)\|_{L^2}^2+1\\
\lesssim&\|\chi_t^k\|_{H^2}^2+K_1^2K_2^2+K_2^6+K_2^2+1.
\end{aligned}
$$

From which and (2.19), for any $0<T<T_2$ we get

$$
\|\chi^k\|_{L^2(0,T;H^4)}\le C. \tag{2.21}
$$

Then (2.13) follows from (2.18), (2.19), (2.20) and (2.21) by choosing $K_2\ge C$.

Step 3: We prove that

$$
\|u^k\|_V=\|u_t^k\|_{L^\infty(0,T;L^2)}+\|u_t^k\|_{L^2(0,T;H^1)}+\|u^k\|_{L^\infty(0,T;H^2)}+\|u^k\|_{L^2(0,T;W^{2,6})}\le K_1, \tag{2.22}
$$

$$
\|p^k\|_{L^\infty(0,T;H^1)}+\|p^k\|_{L^2(0,T;W^{1,6})}\le C. \tag{2.23}
$$

Differentiating $(2.9)_2$ with respect to t, multiplying the result by u_t^k, and integrating over Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^k|u_t^k|^2\mathrm{d}x+\int_\Omega 2\eta(\chi^{k-1})|Du_t^k|^2\mathrm{d}x\\
=&-\frac{1}{2}\int_\Omega\rho_t^k|u_t^k|^2\mathrm{d}x-\int_\Omega 2\eta'(\chi^{k-1})\chi_t^{k-1}Du^k:Du_t^k\mathrm{d}x-\int_\Omega\rho_t^k(u^{k-1}\cdot\nabla)u^{k-1}\cdot u_t^k\mathrm{d}x\\
&-\int_\Omega\rho^k(u_t^{k-1}\cdot\nabla)u^{k-1}\cdot u_t^k\mathrm{d}x-\int_\Omega\rho^k(u^{k-1}\cdot\nabla)u_t^{k-1}\cdot u_t^k\mathrm{d}x\\
&-\int_\Omega\mathrm{div}(\nabla\chi_t^k\otimes\nabla\chi^k)\cdot u_t^k\mathrm{d}x-\int_\Omega\mathrm{div}(\nabla\chi^k\otimes\nabla\chi_t^k)\cdot u_t^k\mathrm{d}x\\
\lesssim&\|\rho_t^k\|_{L^\infty}\|u_t^k\|_{L^2}^2+\|\chi_t^{k-1}\|_{L^6}\|Du^k\|_{L^3}\|Du_t^k\|_{L^2}+\|\rho_t^k\|_{L^\infty}\|u^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^2}\|u_t^k\|_{L^2}\\
&+\|u_t^{k-1}\|_{L^6}\|\nabla u^{k-1}\|_{L^3}\|u_t^k\|_{L^2}+\|u^{k-1}\|_{L^\infty}\|\nabla u_t^{k-1}\|_{L^2}\|u_t^k\|_{L^2}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\nabla^2\chi_t^k\|_{L^2}\|\nabla\chi^k\|_{L^\infty}\|u_t^k\|_{L^2}+\|\nabla\chi_t^k\|_{L^6}\|\nabla^2\chi^k\|_{L^3}\|u_t^k\|_{L^2}\\
\lesssim&(\|\rho_t^k\|_{W^{1,6}}+K_1^4+K_2^4)\int_\Omega\rho^k|u_t^k|^2\mathrm{d}x+\underline{\eta}\int_\Omega|Du_t^k|^2\mathrm{d}x+\|\chi_t^{k-1}\|_{H^1}^2\|u^k\|_{H^2}^2\\
&+\|\rho_t^k\|_{W^{1,6}}^2\|u^{k-1}\|_{H^2}^2\|\nabla u^{k-1}\|_{L^2}^2+K_1^{-4}\|u_t^{k-1}\|_{H^1}^2\|u^{k-1}\|_{H^2}^2+K_2^{-4}\|\nabla^2\chi_t^k\|_{L^2}^2\|\nabla\chi^k\|_{H^2}^2.
\end{aligned}
$$

By using (1.6), it follows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^k|u_t^k|^2\mathrm{d}x+\int_\Omega|Du_t^k|^2\mathrm{d}x\\
\lesssim&(K_1^4+K_2^4+1)\int_\Omega\rho^k|u_t^k|^2\mathrm{d}x+K_2^2\|u^k\|_{H^2}^2+K_1^6+K_1^{-2}\|u_t^{k-1}\|_{H^1}^2+K_2^{-2}\|\chi_t^k\|_{H^2}^2.
\end{aligned}\tag{2.24}
$$

It remains for us to deal with the term $\|u^k\|_{H^2}^2$. We rewrite the equation $(2.9)_2$ as

$$
-\mathrm{div}(2\eta(\chi^{k-1})Du^k)+\nabla p=-\rho^ku_t^k-\rho^k(u^{k-1}\cdot\nabla)u^{k-1}-\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k).
$$

It follows from (1.6) and the estimates for the stationary Stokes equation [16] that

$$
\|u^k\|_{H^2}^2+\|p^k\|_{H^1}^2\lesssim\|\rho^ku_t^k\|_{L^2}^2+\|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^2}^2+\|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^2}^2,
$$

where

$$
\begin{aligned}
&\|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^2}^2\lesssim\int_\Omega|u^{k-1}-u_0|^2|\nabla u^{k-1}|^2\mathrm{d}x+\int_\Omega|\nabla(u^{k-1}-u_0)|^2\mathrm{d}x+1\\
\lesssim&\int_\Omega\left|\int_0^tu_t^{k-1}(x,s)\mathrm{d}s\right|^2|\nabla u^{k-1}|^2\mathrm{d}x+K_1^{-2}\int_\Omega|\nabla^2(u^{k-1}-u_0)|^2\mathrm{d}x+K_1^2\int_\Omega|u^{k-1}-u_0|^2\mathrm{d}x+1\\
\lesssim&T\int_0^t\int_\Omega|u_t^{k-1}(x,s)|^2|\nabla u^{k-1}(x,t)|^2\mathrm{d}x\mathrm{d}s+K_1^2T\int_0^T\int_\Omega|u_t^{k-1}|^2\mathrm{d}x\mathrm{d}t+1\\
\lesssim&T\int_0^T\|u_t^{k-1}(\cdot,s)\|_{L^6}^2\|\nabla u^{k-1}\|_{L^3}^2\mathrm{d}s+K_1^4T^2+1\\
\lesssim&T\|\nabla u^{k-1}\|_{H^1}^2\int_0^T\|u_t^{k-1}(\cdot,s)\|_{H^1}^2\mathrm{d}s+K_1^4T^2+1\lesssim K_1^4T+K_1^4T^2+1.
\end{aligned}
$$

From (2.13), we can deduce that

$$
\begin{aligned}
\|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^2}^2\lesssim&\int_\Omega|\nabla(\chi^k-\chi_0)|^2|\nabla^2\chi^k|^2\mathrm{d}x+\int_\Omega|\nabla^2(\chi^k-\chi_0)|^2\mathrm{d}x+1\\
\lesssim&K^{12}T^2+K_2^6T^2+1.
\end{aligned}
$$

Therefore, for any $0<T<T_2$ we obtain

$$
\|u^k\|_{L^\infty(0,T;H^2)}^2+\|p^k\|_{L^\infty(0,T;H^1)}^2\lesssim\|\sqrt{\rho^k}u_t^k\|_{L^2}^2+1.\tag{2.25}
$$

Substituting (2.25) into (2.24) gives

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho^k|u_t^k|^2\mathrm{d}x+\int_\Omega|Du_t^k|^2\mathrm{d}x
$$

$$\lesssim (K_1^4+K_2^4+1)\int_\Omega \rho^k|u_t^k|^2\mathrm{d}x + K_2^2+K_1^6+K_1^{-2}\|u_t^{k-1}\|_{H^1}^2+K_2^{-2}\|\chi_t^k\|_{H^2}^2.$$

Taking $T_3 := \min\left\{T_2, \dfrac{1}{K_1^6}\right\}$, applying Gronwall's inequality and recalling the equation $(2.9)_2$, for any $0<T<T_3$ we get

$$\sup_{0\le t\le T}\int_\Omega \rho^k|u_t^k|^2\mathrm{d}x + \int_0^T\int_\Omega |Du_t^k|^2\mathrm{d}x\mathrm{d}t \le C.$$

The well-known Korn's inequality [8, 12] implies that, for bounded connected open domain $\Omega\subset\mathbb{R}^d$ $(N=2,3)$, there exists a (generic) positive constant C_Ω such that

$$\|\nabla v\|_{L^2(\Omega)} \le C_\Omega(\|Dv\|_{L^2(\Omega)}+\|v\|_{L^2(\Omega)}),\quad \forall\ v\in \left(H^1(\Omega)\right)^N. \tag{2.26}$$

Hence, for any $0<T<T_3$ we have

$$\|u_t^k\|_{L^\infty(0,T;L^2)}+\|u_t^k\|_{L^2(0,T;H^1)}\le C. \tag{2.27}$$

Moreover, the estimates for the stationary Stokes equation [16] also implies that

$$\begin{aligned}
\|u^k\|_{W^{2,6}}+\|p^k\|_{W^{1,6}} &\lesssim \|\rho^k u_t^k\|_{L^6}+\|\rho^k(u^{k-1}\cdot\nabla)u^{k-1}\|_{L^6}+\|\mathrm{div}(\nabla\chi^k\otimes\nabla\chi^k)\|_{L^6}\\
&\lesssim \|u_t^k\|_{L^6}+\|u^{k-1}\|_{L^\infty}\|\nabla u^{k-1}\|_{L^6}+\|\nabla\chi^k\|_{L^\infty}\|\nabla^2\chi^k\|_{L^6}\\
&\lesssim \|u_t^k\|_{H^1}+\|u^{k-1}\|_{H^2}^2+\|\chi^k\|_{H^3}^2\lesssim \|u_t^k\|_{H^1}+K_1^2+1.
\end{aligned}$$

Then for any $0<T<T_3$, it holds that

$$\|u^k\|_{L^2(0,T;W^{2,6})}+\|p^k\|_{L^2(0,T;W^{1,6})}\le C. \tag{2.28}$$

Here, we have normalized p as $\int_\Omega p(x,t)\mathrm{d}x=0$. Choosing $K_1\ge C$, then (2.22) and (2.23) follows from (2.25), (2.27) and (2.28).

Step 4: Taking limits.

Here, we denote by $\tilde{C}$ a constant whose value may be different from line to line depending on K_1, K_2 and other known constants. Denote $\overline{\rho}^{k+1}=\rho^{k+1}-\rho^k$, $\overline{u}^{k+1}=u^{k+1}-u^k$, $\overline{p}^{k+1}=p^{k+1}-p^k$, $\overline{\chi}^{k+1}=\chi^{k+1}-\chi^k$. Then from (2.9) we have the following system

$$\begin{cases}
\overline{\rho}_t^{k+1}+(u^k\cdot\nabla)\overline{\rho}^{k+1}=-(\overline{u}^k\cdot\nabla)\rho^k,\\
\rho^{k+1}\overline{u}_t^{k+1}+\nabla\overline{p}^{k+1}=\mathrm{div}(2\eta(\chi^k)D\overline{u}^{k+1})+\mathrm{div}(2\eta'(\theta\overline{\chi}^k)\overline{\chi}^kDu^k)-\overline{\rho}^{k+1}u_t^k-\overline{\rho}^{k+1}(u^k\cdot\nabla)u^k\\
\qquad\qquad -\rho^k(\overline{u}^k\cdot\nabla)u^k-\rho^k(u^{k-1}\cdot\nabla)\overline{u}^k-\mathrm{div}(\nabla\overline{\chi}^{k+1}\otimes\nabla\chi^{k+1})-\mathrm{div}(\nabla\chi^k\otimes\nabla\overline{\chi}^{k+1}),\\
\mathrm{div}\,\overline{u}^{k+1}=0,\\
\overline{\chi}_t^{k+1}=\dfrac{1}{(\rho^{k+1})^2}\Delta\overline{\chi}^{k+1}-\overline{\rho}^{k+1}\dfrac{\rho^{k+1}+\rho^k}{(\rho^{k+1}\rho^k)^2}\Delta\chi^k-(\overline{u}^k\cdot\nabla)\chi^k-(u^{k-1}\cdot\nabla)\overline{\chi}^k\\
\qquad +\overline{\rho}^{k+1}\dfrac{(\chi^{k-1})^3-\chi^{k-1}}{\rho^{k+1}\rho^k}-\dfrac{(\chi^k)^2+\chi^k\chi^{k-1}+(\chi^{k-1})^2-1}{\rho^{k+1}}\overline{\chi}^k,
\end{cases} \tag{2.29}$$

where $0 < \theta < 1$ is a constant. The above system is supplemented with the initial boundary conditions

$$
\begin{aligned}
&\left(\overline{\rho}^{k+1}, \overline{u}^{k+1}, \overline{\chi}^{k+1}\right)\Big|_{t=0} = (0,0,0), \quad x \in \Omega,\\
&\left(\overline{u}^{k+1}, \frac{\partial \overline{\chi}^{k+1}}{\partial \mathbf{n}}\right)\Big|_{\partial\Omega} = (0,0), \qquad t \geq 0.
\end{aligned}
$$

Multiplying $(2.29)_1$ by $\overline{\rho}^{k+1}$ and integrating the result over Ω yield

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\overline{\rho}^{k+1}|^2 \mathrm{d}x =& -2\int_\Omega (\overline{u}^k \cdot \nabla)\rho^k \overline{\rho}^{k+1}\mathrm{d}x \lesssim \|\overline{u}^k\|_{L^2}\|\nabla\rho^k\|_{L^\infty}\|\overline{\rho}^{k+1}\|_{L^2}\\
\lesssim& \|\overline{u}^k\|_{L^2}\|\nabla\rho^k\|_{W^{1,6}}\|\overline{\rho}^{k+1}\|_{L^2} \lesssim \|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2 + \|\overline{\rho}^{k+1}\|_{L^2}^2. \qquad (2.30)
\end{aligned}
$$

Multiplying $(2.29)_2$ by $\overline{u}^{k+1}$ and integrating over Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega 2\eta(\chi^k)|D\overline{u}^{k+1}|^2\mathrm{d}x\\
=&\frac{1}{2}\int_\Omega \rho_t^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x - 2\int_\Omega \eta'(\theta\overline{\chi}^k)\overline{\chi}^k Du^k : D\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \overline{\rho}^{k+1}u_t^k\cdot\overline{u}^{k+1}\mathrm{d}x\\
&-\int_\Omega \overline{\rho}^{k+1}(u^k\cdot\nabla)u^k\cdot\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \rho^k(\overline{u}^k\cdot\nabla)u^k\cdot\overline{u}^{k+1}\mathrm{d}x - \int_\Omega \rho^k(u^{k-1}\cdot\nabla)\overline{u}^k\cdot\overline{u}^{k+1}\mathrm{d}x\\
&+\int_\Omega (\nabla\overline{\chi}^{k+1}\otimes\nabla\chi^{k+1}) : \nabla\overline{u}^{k+1}\mathrm{d}x + \int_\Omega (\nabla\chi^k\otimes\nabla\overline{\chi}^{k+1}) : \nabla\overline{u}^{k+1}\mathrm{d}x\\
\lesssim&\|\rho_t^{k+1}\|_{L^\infty}\|\overline{u}^{k+1}\|_{L^2}^2 + \|\overline{\chi}^k\|_{L^3}\|Du^k\|_{L^6}\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{\rho}^{k+1}\|_{L^2}\|u_t^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^6}\\
&+\|\overline{\rho}^{k+1}\|_{L^2}\|u^k\|_{L^\infty}\|\nabla u^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^6} + \|\overline{u}^k\|_{L^6}\|\nabla u^k\|_{L^3}\|\overline{u}^{k+1}\|_{L^2} + \|u^{k-1}\|_{L^\infty}\|\nabla\overline{u}^k\|_{L^2}\|\overline{u}^{k+1}\|_{L^2}\\
&+\|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\chi^{k+1}\|_{L^\infty}\|\nabla\overline{u}^{k+1}\|_{L^2} + \|\nabla\chi^k\|_{L^\infty}\|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\overline{u}^{k+1}\|_{L^2}.
\end{aligned}
$$

Together with Sobolev embedding theorem and Korn's inequality (2.26), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega 2\eta(\chi^k)|D\overline{u}^{k+1}|^2\mathrm{d}x\\
\lesssim&\|\rho_t^{k+1}\|_{W^{1,6}}\|\overline{u}^{k+1}\|_{L^2}^2 + \left(\|\overline{\chi}^k\|_{L^2} + \|\nabla\overline{\chi}^k\|_{L^2}\right)\|u^k\|_{H^2}\|D\overline{u}^{k+1}\|_{L^2}\\
&+\left(\|\overline{\rho}^{k+1}\|_{L^2}\|u_t^k\|_{H^1} + \|\overline{\rho}^{k+1}\|_{L^2}\|u^k\|_{H^2}\|\nabla u^k\|_{H^1}\right)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right)\\
&+\left(\|\nabla u^k\|_{H^1}\|\overline{u}^{k+1}\|_{L^2} + \|u^{k-1}\|_{H^2}\|\overline{u}^{k+1}\|_{L^2}\right)\left(\|D\overline{u}^k\|_{L^2} + \|\overline{u}^k\|_{L^2}\right)\\
&+\left(\|\nabla\overline{\chi}^{k+1}\|_{L^2}\|\nabla\chi^{k+1}\|_{H^2} + \|\nabla\chi^k\|_{H^2}\|\nabla\overline{\chi}^{k+1}\|_{L^2}\right)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right)\\
\lesssim&\|\overline{u}^{k+1}\|_{L^2}^2 + \left(\|\overline{\chi}^k\|_{L^2} + \|\nabla\overline{\chi}^k\|_{L^2}\right)\|D\overline{u}^{k+1}\|_{L^2} + \left(\|D\overline{u}^k\|_{L^2} + \|\overline{u}^k\|_{L^2}\right)\|\overline{u}^{k+1}\|_{L^2}\\
&+\Big(\|\overline{\rho}^{k+1}\|_{L^2}\left(\|u_t^k\|_{H^1} + 1\right) + \|\nabla\overline{\chi}^{k+1}\|_{L^2}\Big)\left(\|D\overline{u}^{k+1}\|_{L^2} + \|\overline{u}^{k+1}\|_{L^2}\right).
\end{aligned}
$$

By using Cauchy inequality and (1.6), it follows that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^{k+1}|\overline{u}^{k+1}|^2\mathrm{d}x + \int_\Omega |D\overline{u}^{k+1}|^2\mathrm{d}x \lesssim \|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2 + (\|u_t^k\|_{H^1}^2 + 1)\|\overline{\rho}^{k+1}\|_{L^2}^2 + \|\nabla\overline{\chi}^{k+1}\|_{L^2}^2
$$

$$
+\|\rho^k\overline{\chi}^k\|_{L^2}^2+\|\nabla\overline{\chi}^k\|_{L^2}^2+\|\sqrt{\rho^k}\overline{u}^k\|_{L^2}^2+\varepsilon\|D\overline{u}^k\|_{L^2}^2. \tag{2.31}
$$

Multiplying $(2.29)_4$ by $\Delta\overline{\chi}^{k+1}$ and integrating over Ω, we get

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x+\int_\Omega\frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x\\
\lesssim&\|\overline{\rho}^{k+1}\|_{L^2}\|\Delta\chi^k\|_{L^\infty}\|\Delta\overline{\chi}^{k+1}\|_{L^2}+\|\overline{\rho}^{k+1}\|_{L^2}\left(\|\chi^{k-1}\|_{L^\infty}^3+\|\chi^{k-1}\|_{L^\infty}\right)\|\Delta\overline{\chi}^{k+1}\|_{L^2}\\
&+\|\overline{u}^k\|_{L^2}\|\nabla\chi^k\|_{L^\infty}\|\Delta\overline{\chi}^{k+1}\|_{L^2}+\|u^{k-1}\|_{L^\infty}\|\nabla\overline{\chi}^k\|_{L^2}\|\Delta\overline{\chi}^{k+1}\|_{L^2}\\
&+\left(\|\chi^k\|_{L^\infty}^2+\|\chi^k\|_{L^\infty}\|\chi^{k-1}\|_{L^\infty}+\|\chi^{k-1}\|_{L^\infty}^2+1\right)\|\overline{\chi}^k\|_{L^2}\|\Delta\overline{\chi}^{k+1}\|_{L^2}\\
\lesssim&\frac{1}{2}\int_\Omega\frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x+\|\overline{\rho}^{k+1}\|_{L^2}^2\|\Delta\chi^k\|_{H^2}^2+\|\overline{\rho}^{k+1}\|_{L^2}^2\left(\|\chi^{k-1}\|_{H^2}^6+\|\chi^{k-1}\|_{H^2}^2\right)\\
&+\|\overline{u}^k\|_{L^2}^2\|\nabla\chi^k\|_{H^2}^2+\|u^{k-1}\|_{H^2}^2\|\nabla\overline{\chi}^k\|_{L^2}^2+\left(\|\chi^k\|_{H^2}^4+\|\chi^k\|_{H^2}^2\|\chi^{k-1}\|_{H^2}^2+\|\chi^{k-1}\|_{H^2}^4+1\right)\|\overline{\chi}^k\|_{L^2}^2.
\end{aligned}
$$

From which we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x+\int_\Omega\frac{1}{(\rho^{k+1})^2}|\Delta\overline{\chi}^{k+1}|^2\mathrm{d}x\\
\lesssim&\left(\|\chi^k\|_{H^4}^2+1\right)\|\overline{\rho}^{k+1}\|_{L^2}^2+\|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2+\|\nabla\overline{\chi}^k\|_{L^2}^2+\|\rho^k\overline{\chi}^k\|_{L^2}^2.
\end{aligned}\tag{2.32}
$$

Multiplying $(2.29)_4$ by $(\rho^{k+1})^2\overline{\chi}^{k+1}$ and integrating over Ω, we can deduce that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\rho^{k+1}\overline{\chi}^{k+1}|^2\mathrm{d}x+\int_\Omega|\nabla\overline{\chi}^{k+1}|^2\mathrm{d}x\\
\lesssim&\|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2+\left(\|\chi^k\|_{H^4}^2+1\right)\|\overline{\rho}^{k+1}\|_{L^2}^2+\|\sqrt{\rho^k}\,\overline{u}^k\|_{L^2}^2+\|\nabla\overline{\chi}^k\|_{L^2}^2+\|\rho^k\overline{\chi}^k\|_{L^2}^2.
\end{aligned}\tag{2.33}
$$

Putting (2.30)–(2.33) together gives

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\overline{\rho}^{k+1}\|_{L^2}^2+\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2+\|\nabla\overline{\chi}^{k+1}\|_{L^2}^2+\|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2\right)\\
&+\|D\overline{u}^{k+1}\|_{L^2}^2+\left\|(\rho^{k+1})^{-1}\Delta\overline{\chi}^{k+1}\right\|_{L^2}^2\\
\lesssim&\left(\|u_t^k\|_{H^1}^2+\|\chi^k\|_{H^4}^2+1\right)\left(\|\overline{\rho}^{k+1}\|_{L^2}^2+\|\sqrt{\rho^{k+1}}\overline{u}^{k+1}\|_{L^2}^2+\|\nabla\overline{\chi}^{k+1}\|_{L^2}^2+\|\rho^{k+1}\overline{\chi}^{k+1}\|_{L^2}^2\right)\\
&+\left(\|\sqrt{\rho^k}\overline{u}^k\|_{L^2}^2+\|\nabla\overline{\chi}^k\|_{L^2}^2+\|\rho^k\overline{\chi}^k\|_{L^2}^2\right)+\varepsilon\|D\overline{u}^k\|_{L^2}^2.
\end{aligned}
$$

Set

$$
\begin{aligned}
&A^k(t)=\|\overline{\rho}^k(\cdot,t)\|_{L^2}^2+\|\sqrt{\rho^k}\,\overline{u}^k(\cdot,t)\|_{L^2}^2+\|\nabla\overline{\chi}^k(\cdot,t)\|_{L^2}^2+\|\rho^k\overline{\chi}^k(\cdot,t)\|_{L^2}^2,\\
&B^k(t)=\|D\overline{u}^k(\cdot,t)\|_{L^2}^2+\left\|(\rho^k)^{-1}\Delta\overline{\chi}^k(\cdot,t)\right\|_{L^2}^2\\
&c^k(t)=\|u_t^k(\cdot,t)\|_{H^1}^2+\|\chi^k(\cdot,t)\|_{H^4}^2+\|\chi_t^k(\cdot,t)\|_{H^2}^2.
\end{aligned}
$$

Then we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}A^{k+1}(t)+B^{k+1}(t)\le\tilde{C}\big(c^k(t)+1\big)A^{k+1}(t)+\tilde{C}A^k(t)+\varepsilon B^k(t),
$$

where $\int_0^T c^k(t)\mathrm{d}t \le \tilde{C}$. By using Gronwall's inequality, we get

$$\sup_{0\le t\le T} A^{k+1}(t) \le \left(\tilde{C}\int_0^T A^k(t)\mathrm{d}t + \varepsilon\int_0^T B^k(t)\mathrm{d}t\right)\exp\left\{\tilde{C}(1+T)\right\}.$$

Hence

$$\begin{aligned}
&\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t \\
\le &\left(\tilde{C}\,T\sup_{0\le t\le T} A^k(t) + \varepsilon\int_0^T B^k(t)\mathrm{d}t\right)\left(\tilde{C}(1+T)\exp\{\tilde{C}(1+T)\}+1\right).
\end{aligned}$$

Recalling $0<T<1$, choosing $T_4 := \left\{T_3,\ \dfrac{1}{4\tilde{C}(2\tilde{C}\exp\{2\tilde{C}\}+1)}\right\}$ and $\varepsilon = \dfrac{1}{4(2\tilde{C}\exp\{2\tilde{C}\}+1)}$, then for any $0<T<T_4$ and $k\ge 1$, we have

$$\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t \le \frac{1}{4}\left(\sup_{0\le t\le T} A^k(t) + \int_0^T B^k(t)\mathrm{d}t\right).$$

By iteration, we derive that

$$\sup_{0\le t\le T} A^{k+1}(t) + \int_0^T B^{k+1}(t)\mathrm{d}t \le \frac{1}{4^{k-1}}\left(\sup_{0\le t\le T} A^2(t) + \int_0^T B^2(t)\mathrm{d}t\right).$$

Together with Korn's inequality, we have

$$\begin{aligned}
&\left\|\overline{\rho}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{u}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{\chi}^{k+1}\right\|_{L^\infty(0,T;H^1)} \\
&\quad + \left\|\overline{u}^{k+1}\right\|_{L^2(0,T;H^1)} + \left\|\overline{\chi}^{k+1}\right\|_{L^2(0,T;H^2)} \le \frac{1}{2^{k-1}}\tilde{C}.
\end{aligned}$$

Hence, we get

$$\begin{aligned}
&\sum_{k=2}^{\infty}\left\|\overline{\rho}^k\right\|_{L^\infty(0,T;L^2)} < \infty, \\
&\sum_{k=2}^{\infty}\left(\left\|\overline{u}^{k+1}\right\|_{L^\infty(0,T;L^2)} + \left\|\overline{u}^{k+1}\right\|_{L^2(0,T;H^1)}\right) < \infty, \\
&\sum_{k=2}^{\infty}\left(\left\|\overline{\chi}^{k+1}\right\|_{L^\infty(0,T;H^1)} + \left\|\overline{\chi}^{k+1}\right\|_{L^2(0,T;H^2)}\right) < \infty.
\end{aligned}$$

Therefore, as $k\to\infty$ we have

$$\begin{aligned}
&\rho^k \to \rho^1 + \sum_{k=2}^{\infty}\overline{\rho}^k, \qquad \text{in } L^\infty(0,T;L^2), \\
&u^k \to u^1 + \sum_{k=2}^{\infty}\overline{u}^k, \qquad \text{in } L^\infty(0,T;L^2)\cap L^2(0,T;H^1),
\end{aligned} \tag{2.34}$$

$$\chi^k \to \chi^1 + \sum_{k=2}^{\infty} \overline{\chi}^k, \qquad \text{in } L^\infty(0,T;H^1) \cap L^2(0,T;H^2). \tag{2.35}$$

By (2.10)–(2.13), (2.22), (2.23), after taking possible subsequences (denoted by itself for convenience), sending $k \to \infty$, we have

$$\begin{aligned}
&\rho^k \to \rho, \quad \text{strongly in } C(0,T;H^1),\\
&(\nabla\rho^k,\, \rho_t^k) \rightharpoonup (\nabla\rho,\, \rho_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&(\nabla^2\rho^k,\, \nabla\rho_t^k) \rightharpoonup (\nabla^2\rho,\, \nabla\rho_t), \quad \text{weakly in } L^2(0,T;L^6),\\
&u^k \to u, \quad \text{strongly in } C(0,T;H^1),\\
&(\nabla u^k,\, \nabla^2 u^k,\, u_t^k) \rightharpoonup (\nabla u,\, \nabla^2 u,\, u_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&\nabla^2 u^k \rightharpoonup \nabla^2 u, \quad \text{weakly in } L^2(0,T;L^6),\\
&\nabla u_t^k \rightharpoonup \nabla u_t, \quad \text{weakly in } L^2(0,T;L^2),\\
&(p^k,\, \nabla p^k) \rightharpoonup (p,\, \nabla p), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&\nabla p^k \rightharpoonup \nabla p, \quad \text{weakly in } L^2(0,T;L^6),\\
&\chi^k \to \chi, \quad \text{strongly in } C(0,T;H^2),\\
&(\nabla\chi^k,\, \nabla^2\chi^k,\, \nabla^3\chi^k,\, \chi_t^k,\, \nabla\chi_t^k) \rightharpoonup (\nabla\chi,\, \nabla^2\chi,\, \nabla^3\chi,\, \chi_t,\, \nabla\chi_t), \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2),\\
&(\nabla^4\chi^k,\, \nabla^2\chi_t) \rightharpoonup (\nabla^4\chi,\, \nabla^2\chi_t), \quad \text{weakly in } L^2(0,T;L^2).
\end{aligned}$$

By lower semi-continuity, we derive

$$\|\rho\|_{L^\infty(0,T;W^{2,6})} + \|\rho_t\|_{L^\infty(0,T;W^{1,6})} + \|u\|_V + \|p\|_{L^\infty(0,T;H^1)} + \|p\|_{L^2(0,T;W^{1,6})} + \|\chi\|_\Phi \le \tilde{C}.$$

By the uniqueness of the limits, we get $\rho = \rho^1 + \sum_{k=2}^\infty \overline{\rho}^k$, $u = u^1 + \sum_{k=2}^\infty \overline{u}^k$, $\chi = \chi^1 + \sum_{k=2}^\infty \overline{\chi}^k$. On the other hand, (2.34) and (2.35) also imply

$$\begin{aligned}
&u^{k-1} \to u, \qquad \text{in } L^\infty(0,T;L^2) \cap L^2(0,T;H^1),\\
&\chi^{k-1} \to \chi, \qquad \text{in } L^\infty(0,T;H^1) \cap L^2(0,T;H^2),
\end{aligned}$$

as $k \to \infty$.

Taking limits in (2.9), we see that (ρ, u, p, χ) is accurately a solution of the problem (1.1) with the regularities like in Theorem 1.1. The uniqueness of the solution can be obtained by the standard energy method similar to step 4. Therefore, the proof of Theorem 1.1 is complete. □

3 Blow-up criterion

Let $0 < T_* < \infty$ be the maximum time for the existence of strong solution (ρ, u, p, χ, μ) to the problem (1.1)–(1.3). In other words, (ρ, u, p, χ, μ) is a strong solution of (1.1)–(1.3) in $\Omega \times (0,T]$ for any $0 < T < T_*$, but not a strong solution in $\Omega \times (0, T_*]$. We prove (1.4) and (1.5) by contradiction: if not, i.e.

$$\int_0^{T_*} \left(\|Du\|_{L^\infty} + \|\nabla\chi\|_{L^\infty}^2 \right) \mathrm{d}t \le M_0 < +\infty, \quad \text{if } N = 2, \tag{3.1}$$

$$\int_0^{T_*} \left(\|Du\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\nabla\chi\|_{L^\infty}^2\right) \mathrm{d}t \le M_0 < +\infty, \quad \text{if } N = 3 \tag{3.2}$$

holds for some constant $M_0 > 0$, then there exists a positive constant C depending only on ρ_0, u_0, χ_0, T_* and M_0 such that

$$\sup_{0\le t<T_*} \Big(\|\rho\|_{W^{2,6}}^2 + \|\rho_t\|_{W^{1,6}}^2 + \|u\|_{H^2}^2 + \|u_t\|_{L^2}^2 + \|p\|_{H^1}^2 + \|\chi\|_{H^3}^2 + \|\chi_t\|_{H^1}^2 + \|\mu\|_{H^1}^2\Big)$$
$$+ \int_0^{T_*} \Big(\|u\|_{W^{2,6}}^2 + \|u_t\|_{H^1}^2 + \|p\|_{W^{1,6}}^2 + \|\chi\|_{H^4}^2 + \|\chi_t\|_{H^2}^2 + \|\mu\|_{H^2}^2 + \|\mu_t\|_{L^2}^2\Big)\mathrm{d}t \le C. \tag{3.3}$$

In terms of the a priori estimates (3.3), we can prove that T_* is not the maximum time, which is the desired contradiction.

Firstly, we deal with the bounds of ρ and $\nabla\rho$, which is very important in the proof of the following estimates.

Lemma 3.1 *Let $0 < T_* < +\infty$ be the maximum time of a strong solution (ρ, u, p, χ, μ) to the problem* (1.1)–(1.3). *If* (3.1) *and* (3.2) *hold, then*

$$0 < C^{-1} \le \rho(x,t) \le C, \quad (x,t) \in Q_{T_*}, \tag{3.4}$$

$$\sup_{0\le t<T_*} \|\nabla\rho\|_{L^\infty} \le C, \tag{3.5}$$

where $Q_{T_*} = \Omega \times (0, T_*)$.

Proof. For any $1 \le r < +\infty$, multiplying $(1.1)_1$ by $r\rho^{r-1}$, integrating the result with respect to x over Ω and by using $(1.1)_3$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho^r \mathrm{d}x = -\int_\Omega u\cdot\nabla(\rho^r)\mathrm{d}x = \int_\Omega \rho^r \mathrm{div} u\, \mathrm{d}x = 0.$$

From which we have

$$\|\rho(\cdot,t)\|_{L^r} = \|\rho_0\|_{L^r}, \qquad 0 \le t < T_*.$$

Sending $r \to +\infty$ and recalling $0 < c_0^{-1} \le \rho_0 \le c_0$, we obtain (3.4). The proof of (3.5) is similar to (2.5). Then lemma 3.1 follows. □

Next lemma is concerned with basic energy estimate.

Lemma 3.2 *Under the same assumptions in Lemma* 3.1*, we have*

$$\sup_{0\le t<T_*} \int_\Omega \left(\frac{|u|^2}{2} + \frac{|\nabla\chi|^2}{2} + \frac{\rho(\chi^4 - 2\chi^2)}{4}\right) \mathrm{d}x + \int_0^{T_*}\int_\Omega \left(2\eta(\chi)|Du|^2 + \mu^2\right) \mathrm{d}x\mathrm{d}t = E_0, \tag{3.6}$$

where

$$E_0 := \int_\Omega \left(\frac{\rho_0|u_0|^2}{2} + \frac{|\nabla\chi_0|^2}{2} + \frac{\rho_0(\chi_0^4 - 2\chi_0^2)}{4}\right) \mathrm{d}x.$$

Furthermore,

$$\sup_{0\le t<T_*} \int_\Omega \chi^4 \mathrm{d}x + \int_0^{T_*}\int_\Omega \left(|\nabla u|^2 + |\nabla^2\chi|^2 + \chi_t^2\right) \mathrm{d}x\mathrm{d}t \le C. \tag{3.7}$$

Proof. Multiplying $(1.1)_2$ by u, integrating the result over Ω and recalling $(1.1)_{1,3}$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\rho|u|^2}{2}\mathrm{d}x + \int_\Omega 2\eta(\chi)|Du|^2\mathrm{d}x = -\int_\Omega u\cdot\nabla\chi\Delta\chi\mathrm{d}x. \tag{3.8}$$

Multiplying $(1.1)_4$ by μ and integrating over Ω, by using $(1.1)_{1,5}$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{|\nabla\chi|^2}{2} + \frac{\rho(\chi^4-2\chi^2)}{4}\right)\mathrm{d}x + \int_\Omega \mu^2\mathrm{d}x = \int_\Omega u\cdot\nabla\chi\Delta\chi\mathrm{d}x. \tag{3.9}$$

Adding (3.8) to (3.9), then integrating the resulting equation with respect to t from 0 to T_*, we obtain (3.6).

From (3.6), by using Cauchy inequality and (3.4) we have

$$\int_\Omega \rho\chi^4\mathrm{d}x \lesssim \int_\Omega \rho\chi^2\mathrm{d}x + 1 \lesssim \frac{1}{2}\int_\Omega \rho\chi^4\mathrm{d}x + 1.$$

It follows that

$$\int_\Omega \rho\chi^4\mathrm{d}x \le C. \tag{3.10}$$

Applying the standard H^2-estimate to the equation $(1.1)_5$ yields

$$\begin{aligned}\|\nabla^2\chi\|_{L^2}^2 &\lesssim \|\Delta\chi\|_{L^2}^2 + \|\nabla\chi\|_{L^2}^2 \lesssim \|\rho\mu\|_{L^2}^2 + \|\rho(\chi^3-\chi)\|_{L^2}^2 + \|\nabla\chi\|_{L^2}^2 \\ &\lesssim \|\mu\|_{L^2}^2 + \|\chi\|_{L^6}^6 + \|\chi\|_{L^2}^2 + \|\nabla\chi\|_{L^2}^2 \lesssim \|\mu\|_{L^2}^2 + \|\chi\|_{H^1}^6 + \|\chi\|_{L^2}^2 + \|\nabla\chi\|_{L^2}^2 \\ &\lesssim \|\mu\|_{L^2}^2 + 1,\end{aligned} \tag{3.11}$$

where we have used (3.4), (3.6) and (3.10). On the other hand, from the equation $(1.1)_4$ we get

$$\|\rho\chi_t\|_{L^2}^2 \lesssim \|\rho u\cdot\nabla\chi\|_{L^2}^2 + \|\mu\|_{L^2}^2 \lesssim \|\nabla\chi\|_{L^\infty}^2\|u\|_{L^2}^2 + \|\mu\|_{L^2}^2 \lesssim \|\nabla\chi\|_{L^\infty}^2 + \|\mu\|_{L^2}^2. \tag{3.12}$$

Integrating (3.11) and (3.12) with respect to t over $(0,T_*)$, by using (3.1), (3.2), (3.6) and Korn's inequality (2.26), we derive (3.7). Then Lemma 3.2 follows. □

The following estimate is crucial in the proof of the forthcoming lemmas.

Lemma 3.3 *We assume that the hypotheses in Lemma* 3.1 *hold. Then for any* $2 \le r \le 6$, *there holds*

$$\sup_{0\le t<T_*}\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x + \int_0^{T_*}\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x\mathrm{d}t \le C. \tag{3.13}$$

Proof. From $(1.1)_{4,5}$ we have

$$\rho\chi_t + \rho u\cdot\nabla\chi = \frac{1}{\rho}\Delta\chi - (\chi^3-\chi). \tag{3.14}$$

Differentiating the above equation with respect to x yields

$$\rho\nabla\chi_t + \chi_t\nabla\rho + \rho(u\cdot\nabla)\nabla\chi + \nabla\rho(u\cdot\nabla)\chi + \rho\nabla(u\cdot\nabla)\chi = \nabla\left(\frac{1}{\rho}\Delta\chi\right) - (3\chi^2-1)\nabla\chi. \quad (3.15)$$

Multiplying (3.15) by $r|\nabla\chi|^{r-2}\nabla\chi$ ($2\le r\le 6$) and integrating the result over Ω, we get

$$\begin{aligned}
&\int_\Omega \rho(|\nabla\chi|^r)_t \mathrm{d}x + r\int_\Omega |\nabla\chi|^{r-2}\chi_t\nabla\chi\cdot\nabla\rho\,\mathrm{d}x + \int_\Omega \rho(u\cdot\nabla)(|\nabla\chi|^r)\mathrm{d}x\\
&+ r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\rho(u\cdot\nabla)\chi\,\mathrm{d}x + r\int_\Omega \rho|\nabla\chi|^{r-2}\nabla\chi\cdot\nabla(u\cdot\nabla)\chi\,\mathrm{d}x\\
=&r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\left(\frac{1}{\rho}\Delta\chi\right)\mathrm{d}x - r\int_\Omega(3\chi^2-1)|\nabla\chi|^r\mathrm{d}x.
\end{aligned} \quad (3.16)$$

We calculate the first term on the right hand side

$$\begin{aligned}
&r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\left(\frac{1}{\rho}\Delta\chi\right)\mathrm{d}x\\
=&r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\mathrm{div}\left(\frac{1}{\rho}\nabla^2\chi\right)\mathrm{d}x + r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\left(\frac{1}{\rho}\right)\Delta\chi\,\mathrm{d}x\\
&- r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\nabla\left(\frac{1}{\rho}\right) : \nabla^2\chi\,\mathrm{d}x\\
=&-r\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x - r(r-2)\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}\big|\nabla|\nabla\chi|\big|^2\mathrm{d}x\\
&+ r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\left(\frac{1}{\rho}\right)\Delta\chi\,\mathrm{d}x - r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\nabla\left(\frac{1}{\rho}\right) : \nabla^2\chi\,\mathrm{d}x.
\end{aligned} \quad (3.17)$$

Putting (3.17) into (3.16), integrating by parts and using $(1.1)_1$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x + r(r-1)\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x\\
=&-r\int_\Omega |\nabla\chi|^{r-2}\chi_t\nabla\chi\cdot\nabla\rho\,\mathrm{d}x - r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\rho(u\cdot\nabla)\chi\,\mathrm{d}x\\
&- r\int_\Omega \rho|\nabla\chi|^{r-2}\nabla\chi\cdot\nabla(u\cdot\nabla)\chi\,\mathrm{d}x + r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\cdot\nabla\left(\frac{1}{\rho}\right)\Delta\chi\,\mathrm{d}x\\
&- r\int_\Omega |\nabla\chi|^{r-2}\nabla\chi\otimes\nabla\left(\frac{1}{\rho}\right) : \nabla^2\chi\,\mathrm{d}x - r\int_\Omega(3\chi^2-1)|\nabla\chi|^r\mathrm{d}x := \sum_{i=1}^{6} I_i.
\end{aligned} \quad (3.18)$$

In what follows, we estimate I_i ($i=1,2,3,4,5,6$) one by one in dimension three for example.

$$\begin{aligned}
I_1 \lesssim&\|\chi_t\|_{L^2}\left(\int_\Omega (|\nabla\chi|^{r-1})^2\mathrm{d}x\right)^{1/2} \lesssim \|\chi_t\|_{L^2}\left(\int_\Omega \left(\big|\nabla(|\nabla\chi|^{r-1})\big|^{\frac{6}{5}} + |\nabla\chi|^{\frac{6}{5}(r-1)}\right)\mathrm{d}x\right)^{5/6}\\
\lesssim&\|\chi_t\|_{L^2}\left(\int_\Omega |\nabla\chi|^{\frac{6}{5}(r-2)}|\nabla^2\chi|^{\frac{6}{5}}\mathrm{d}x\right)^{5/6} + \|\chi_t\|_{L^2}^2\left(\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x + 1\right)
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \|\chi_t\|_{L^2}\left(\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x\right)^{1/2}\left(\int_\Omega \rho|\nabla\chi|^{\frac{3}{2}(r-2)}\mathrm{d}x\right)^{1/3}+\|\chi_t\|_{L^2}^2\left(\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x+1\right)\\
&\lesssim \varepsilon\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x+\|\chi_t\|_{L^2}^2\left(\int_\Omega \rho|\nabla\chi|^{\frac{3}{2}(r-2)}\mathrm{d}x\right)^{2/3}+\|\chi_t\|_{L^2}^2\left(\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x+1\right)\\
&\lesssim \varepsilon\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x+\|\chi_t\|_{L^2}^2\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x+\|\chi_t\|_{L^2}^2,
\end{aligned}
$$

where we have used Sobolev embedding theorem, Poincaré's inequality, Hölder's inequality, Cauchy inequality and ε is a sufficiently small constant to be determined later. Observing the second term on the right hand side, it should be satisfied that $0\le \frac{6}{5}(r-1)\le r$, $0\le \frac{3}{2}(r-2)\le r$, i.e. $2\le r\le 6$ should be satisfied. Similarly,

$$
\begin{aligned}
I_2 &\lesssim \|u\|_{L^6}\left(\int_\Omega |\nabla\chi|^{\frac{6r}{5}}\mathrm{d}x\right)^{5/6}\lesssim \|\nabla u\|_{L^2}\left(\int_\Omega |\nabla\chi|^r\mathrm{d}x\right)^{1/2}\left(\int_\Omega |\nabla\chi|^{\frac{3r}{2}}\mathrm{d}x\right)^{1/3}\\
&\lesssim \|\nabla u\|_{L^2}^2\int_\Omega |\nabla\chi|^r\mathrm{d}x+\left(\int_\Omega (|\nabla\chi|^r)^{\frac{3}{2}}\mathrm{d}x\right)^{2/3}\\
&\lesssim \left(\|\nabla u\|_{L^2}^2+1\right)\int_\Omega |\nabla\chi|^r\mathrm{d}x+\int_\Omega \big|\nabla(|\nabla\chi|^r)\big|\mathrm{d}x\\
&\lesssim \left(\|\nabla u\|_{L^2}^2+1\right)\int_\Omega |\nabla\chi|^r\mathrm{d}x+\int_\Omega |\nabla\chi|^{r-1}|\nabla^2\chi|\mathrm{d}x\\
&\lesssim \varepsilon\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x+\left(\|\nabla u\|_{L^2}^2+1\right)\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x.
\end{aligned}
$$

Moreover, by using Cauchy inequality, we have

$$
\begin{aligned}
&I_3=-r\int_\Omega \rho|\nabla\chi|^{r-2}\nabla\chi\cdot D(u\cdot\nabla)\chi\mathrm{d}x\lesssim \|Du\|_{L^\infty}\int_\Omega |\nabla\chi|^r\mathrm{d}x,\\
&I_4+I_5\lesssim \int_\Omega |\nabla\chi|^{r-1}|\nabla^2\chi|\mathrm{d}x\le \varepsilon\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x+\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x,\\
&I_6\lesssim \left(\|\chi\|_{L^\infty}^2+1\right)\int_\Omega |\nabla\chi|^r\mathrm{d}x\lesssim \left(\|\nabla^2\chi\|_{L^2}^2+1\right)\int_\Omega |\nabla\chi|^r\mathrm{d}x.
\end{aligned}
$$

Putting all these estimates into (3.18) and choosing $\varepsilon>0$ small enough, we obtain

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x+\int_\Omega \frac{1}{\rho}|\nabla\chi|^{r-2}|\nabla^2\chi|^2\mathrm{d}x\\
&\lesssim \left(\|\chi_t\|_{L^2}^2+\|\nabla u\|_{L^2}^2+\|Du\|_{L^\infty}+\|\nabla^2\chi\|_{L^2}^2+1\right)\int_\Omega \rho|\nabla\chi|^r\mathrm{d}x+\|\chi_t\|_{L^2}^2.
\end{aligned}
$$

Applying Gronwall's inequality and using (3.1), (3.2), (3.6), (3.7), we derive (3.13). The case of dimension two is similar. Therefore, the proof of this lemma is complete. □

Then we continue to do some estimates for χ and u.

Lemma 3.4 *Suppose that the assumptions in Lemma* 3.1 *are satisfied, we have*

$$
\begin{aligned}
&\sup_{0\le t<T_*}\int_\Omega \left(|\nabla u|^2+|\chi_t|^2+|\Delta\chi|^2+|\mu|^2\right)\mathrm{d}x\\
&+\int_0^{T_*}\int_\Omega \left(|u_t|^2+|\nabla\chi_t|^2+|\nabla^2 u|^2+|\nabla^3\chi|^2\right)\mathrm{d}x\mathrm{d}t\le C.
\end{aligned}
\tag{3.19}
$$

Proof. Multiplying $(1.1)_2$ by u_t and integrating over Ω yield

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x\\
=&\int_\Omega \eta'(\chi)\chi_t|Du|^2\mathrm{d}x-\int_\Omega \rho(u\cdot\nabla)u\cdot u_t\,\mathrm{d}x-\int_\Omega \mathrm{div}(\nabla\chi\otimes\nabla\chi)\cdot u_t\,\mathrm{d}x.
\end{aligned}
\tag{3.20}
$$

If $N=2$, by Nirenberg's interpolation inequality, we have

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x\\
\lesssim&\|\chi_t\|_{L^2}\|Du\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^4}\|\nabla u\|_{L^4}\|u_t\|_{L^2}+\big\||\nabla\chi||\nabla^2\chi|\big\|_{L^2}\|u_t\|_{L^2}\\
\lesssim&\|Du\|_{L^\infty}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\|Du\|_{L^\infty}\|\chi_t\|_{L^2}^2+\frac12\int_\Omega \rho|u_t|^2\mathrm{d}x\\
&+\|u\|_{L^2}\|\nabla u\|_{L^2}(\|\nabla u\|_{L^2}^2+\|\nabla u\|_{L^2}\|\nabla^2 u\|_{L^2})+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x.
\end{aligned}
$$

From which we get

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x\lesssim&\|Du\|_{L^\infty}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\|Du\|_{L^\infty}\|\chi_t\|_{L^2}^2+\|\nabla u\|_{L^2}^3\\
&+\|\nabla u\|_{L^2}^2\|\nabla^2 u\|_{L^2}+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x.
\end{aligned}
\tag{3.21}
$$

It follows from [16] that

$$
\begin{aligned}
\|\nabla^2 u\|_{L^2}\lesssim&\|\rho u_t\|_{L^2}+\|\rho u\cdot\nabla u\|_{L^2}+\|\mathrm{div}(\nabla\chi\otimes\nabla\chi)\|_{L^2}\\
\lesssim&\|\rho u_t\|_{L^2}+\|u\|_{L^4}\|\nabla u\|_{L^4}+\|\nabla\chi\nabla^2\chi\|_{L^2}\\
\lesssim&\|\rho u_t\|_{L^2}+\|u\|_{L^2}^{1/2}\|\nabla u\|_{L^2}^{1/2}(\|\nabla u\|_{L^2}+\|\nabla u\|_{L^2}^{1/2}\|\nabla^2 u\|_{L^2}^{1/2})+\|\nabla\chi\nabla^2\chi\|_{L^2}.
\end{aligned}
$$

Together with Cauchy inequality, we get

$$
\|\nabla^2 u\|_{L^2}^2\lesssim\int_\Omega \rho|u_t|^2\mathrm{d}x+\|\nabla u\|_{L^2}^4+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x+1
\tag{3.22}
$$

Moreover, from the equations $(1.1)_{4,5}$, Sobolev embedding theorem, the estimate (3.13) for $r=6$ and the fact $\|\chi\|_{L^\infty}\le C\|\chi\|_{W^{1,6}}\le C$, we see that

$$
\|\chi_t\|_{L^2}^2\lesssim\|u\|_{L^3}^2\|\nabla\chi\|_{L^6}^2+\|\Delta\chi\|_{L^2}^2+\|\chi^3-\chi\|_{L^2}^2\lesssim\|\nabla u\|_{L^2}^2+\|\Delta\chi\|_{L^2}^2+1
\tag{3.23}
$$

Witch together with (3.21), (3.22), and by using Korn's inequality yield

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x+\int_\Omega |\nabla^2 u|^2\mathrm{d}x\\
\lesssim&(\|Du\|_{L^\infty}+\|\nabla u\|_{L^2}^2)\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\|Du\|_{L^\infty}\|\Delta\chi\|_{L^2}^2\\
&+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x+\|Du\|_{L^\infty}+\|\nabla u\|_{L^2}^2+1.
\end{aligned}\tag{3.24}
$$

If $N=3$, from (3.20) and by using (3.23), we get

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x\\
\lesssim&\|\chi_t\|_{L^2}\|Du\|_{L^\infty}\|Du\|_{L^2}+\|u\|_{L^\infty}\|\nabla u\|_{L^2}\|u_t\|_{L^2}+\big\||\nabla\chi||\nabla^2\chi|\big\|_{L^2}\|u_t\|_{L^2}\\
\lesssim&(\|Du\|_{L^\infty}+\|u\|_{L^\infty}^2)\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\|Du\|_{L^\infty}\|\Delta\chi\|_{L^2}^2+\frac{1}{2}\int_\Omega \rho|u_t|^2\mathrm{d}x\\
&+\|u\|_{L^\infty}^2\|u\|_{L^2}^2+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x+\|Du\|_{L^\infty}+1,
\end{aligned}
$$

where we have used Korn's inequality (2.26) in the last step. Then it follows that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\int_\Omega \rho|u_t|^2\mathrm{d}x\\
\lesssim&(\|Du\|_{L^\infty}+\|u\|_{L^\infty}^2)\int_\Omega \eta(\chi)|Du|^2\mathrm{d}x+\|Du\|_{L^\infty}\|\Delta\chi\|_{L^2}^2\\
&+\int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x+\|u\|_{L^\infty}^2+\|Du\|_{L^\infty}+1.
\end{aligned}\tag{3.25}
$$

Next, need to estimate the term $\|\Delta\chi\|_{L^2}^2$. We deal with the case $N=3$, the case of $N=2$ is similar. Multiplying (3.15) by $\nabla\chi_t$ and integrating the result over Ω yield

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{\rho}|\Delta\chi|^2\mathrm{d}x+\int_\Omega \rho|\nabla\chi_t|^2\mathrm{d}x\\
=&\frac{1}{2}\int_\Omega \left(\frac{1}{\rho}\right)_t|\Delta\chi|^2\mathrm{d}x-\int_\Omega \chi_t\nabla\rho\cdot\nabla\chi_t\mathrm{d}x-\int_\Omega \rho(u\cdot\nabla)\nabla\chi\cdot\nabla\chi_t\mathrm{d}x\\
&-\int_\Omega (u\cdot\nabla)\chi\nabla\rho\cdot\nabla\chi_t\mathrm{d}x-\int_\Omega \rho\nabla\chi_t\cdot\nabla(u\cdot\nabla)\chi\,\mathrm{d}x-\int_\Omega (3\chi^2-1)\nabla\chi\cdot\nabla\chi_t\mathrm{d}x\\
\lesssim&\|u\Delta\chi\|_{L^2}\|\Delta\chi\|_{L^2}+\|\chi_t\|_{L^2}\|\nabla\chi_t\|_{L^2}+\|u\nabla^2\chi\|_{L^2}\|\nabla\chi_t\|_{L^2}\\
&+\|u\|_{L^2}\|\nabla\chi\|_{L^\infty}\|\nabla\chi_t\|_{L^2}+\|\nabla\chi_t\|_{L^2}\|\nabla u\|_{L^2}\|\nabla\chi\|_{L^\infty}+\|3\chi^2-1\|_{L^\infty}\|\nabla\chi\|_{L^2}\|\nabla\chi_t\|_{L^2}\\
\lesssim&\|\Delta\chi\|_{L^2}^2+\varepsilon\|\nabla\chi_t\|_{L^2}^2+\int_\Omega |u|^2|\nabla^2\chi|^2\mathrm{d}x+\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1,
\end{aligned}\tag{3.26}
$$

where we have used (3.6), (3.7), (3.23) and (2.26) in the last step. In what follows, we estimate the first term on the right hand side. By Nirenberg's interpolation inequality

$$
\|\nabla^2\chi\|_{L^3}\lesssim\|\nabla\chi\|_{L^6}^{1/2}\|\nabla^3\chi\|_{L^2}^{1/2},
$$

and (3.13) (with $r=6$), we see that

$$\int_\Omega u^2|\nabla^2\chi|^2dx \lesssim \|u\|_{L^6}^2\|\nabla^2\chi\|_{L^3}^2 \lesssim \|\nabla u\|_{L^2}^2\|\nabla\chi\|_{L^6}\|\nabla^3\chi\|_{L^2} \lesssim \varepsilon\|\nabla^3\chi\|_{L^2}^2+\|\nabla u\|_{L^2}^4. \tag{3.27}$$

It remains for us to estimate $\|\nabla^3\chi\|_{L^2}^2$. Applying the standard H^3-estimate to the Neumann boundary value problem of the equation

$$\frac{1}{\rho}\nabla\Delta\chi = \rho\nabla\chi_t+\chi_t\nabla\rho+\rho(u\cdot\nabla)\nabla\chi+\nabla\rho(u\cdot\nabla)\chi+\rho\nabla(u\cdot\nabla)\chi-\nabla\left(\frac{1}{\rho}\right)\Delta\chi+(3\chi^2-1)\nabla\chi,$$

by using (3.6), (3.7), (3.23) and (3.27), we have

$$\begin{aligned}
\|\nabla^3\chi\|_{L^2}^2 \lesssim & \|\nabla\Delta\chi\|_{L^2}^2+\|\nabla\chi\|_{H^1}^2\\
\lesssim & \|\nabla\chi_t\|_{L^2}^2+\|\chi_t\|_{L^2}^2+\|(u\cdot\nabla)\nabla\chi\|_{L^2}^2+\|u\cdot\nabla\chi\|_{L^2}^2+\|\nabla(u\cdot\nabla)\chi\|_{L^2}^2\\
&+\|\Delta\chi\|_{L^2}^2+\|\chi^2\nabla\chi\|_{L^2}^2+\|\nabla\chi\|_{H^1}^2\\
\lesssim & \|\nabla\chi_t\|_{L^2}^2+\|\chi_t\|_{L^2}^2+\int_\Omega |u|^2|\nabla^2\chi|^2\mathrm{d}x+\|u\|_{L^3}^2\|\nabla\chi\|_{L^6}^2+\|\nabla u\|_{L^2}^2\|\nabla\chi\|_{L^\infty}^2\\
&+\|\Delta\chi\|_{L^2}^2+\|\chi\|_{L^4}^4\|\nabla\chi\|_{L^\infty}^2+\|\nabla^2\chi\|_{L^2}^2+\|\nabla\chi\|_{L^2}^2\\
\lesssim & \|\nabla\chi_t\|_{L^2}^2+\varepsilon\|\nabla^3\chi\|_{L^2}^2+(\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2)\|\nabla u\|_{L^2}^2+\|\Delta\chi\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1.
\end{aligned}$$

It follows that

$$\|\nabla^3\chi\|_{L^2}^2 \lesssim \|\nabla\chi_t\|_{L^2}^2+(\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2)\|\nabla u\|_{L^2}^2+\|\Delta\chi\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1. \tag{3.28}$$

Putting (3.28) into (3.27) yields

$$\int_\Omega u^2|\nabla^2\chi|^2\mathrm{d}x \lesssim \varepsilon\|\nabla\chi_t\|_{L^2}^2+(\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2)\|\nabla u\|_{L^2}^2+\|\Delta\chi\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1.$$

Substituting the above inequality into (3.26), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{\rho}|\Delta\chi|^2\mathrm{d}x+\int_\Omega\rho|\nabla\chi_t|^2\mathrm{d}x \lesssim & \int_\Omega\frac{1}{\rho}|\Delta\chi|^2\mathrm{d}x+(\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2)\int_\Omega\eta(\chi)|Du|^2\mathrm{d}x\\
&+\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1.
\end{aligned} \tag{3.29}$$

Putting (3.24), (3.25) and (3.29) together gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\eta(\chi)|Du|^2+\frac{1}{\rho}|\Delta\chi|^2\right)\mathrm{d}x+\int_\Omega\left(\rho|u_t|^2+|\nabla^2u|^2+\rho|\nabla\chi_t|^2\right)\mathrm{d}x\\
\lesssim & \left(\|Du\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+1\right)\int_\Omega\left(\eta(\chi)|Du|^2+\frac{1}{\rho}|\Delta\chi|^2\right)\mathrm{d}x\\
&+\|Du\|_{L^\infty}+\|\nabla u\|_{L^2}^2+\|\nabla\chi\|_{L^\infty}^2+\int_\Omega\frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x+1, \qquad \text{if } N=2
\end{aligned}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\eta(\chi)|Du|^2+\frac{1}{\rho}|\Delta\chi|^2\right)\mathrm{d}x+\int_\Omega\left(\rho|u_t|^2+\rho|\nabla\chi_t|^2\right)\mathrm{d}x$$

$$\lesssim \left(\|Du\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\chi\|_{L^\infty}^2 + 1\right) \int_\Omega \left(\eta(\chi)|Du|^2 + \frac{1}{\rho}|\Delta\chi|^2\right) \mathrm{d}x$$
$$+ \|Du\|_{L^\infty} + \|u\|_{L^\infty}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\chi\|_{L^\infty}^2 + \int_\Omega \frac{1}{\rho}|\nabla\chi|^2|\nabla^2\chi|^2\mathrm{d}x + 1, \qquad \text{if } N = 3.$$

Hence, applying Gronwall's inequality and using (3.1), (3.2), (3.6), (3.13) for $r = 4$, we obtain

$$\sup_{0\le t<T_*} \int_\Omega \left(\eta(\chi)|Du|^2 + \frac{1}{\rho}|\Delta\chi|^2\right) \mathrm{d}x + \int_0^{T_*} \int_\Omega \left(\rho|u_t|^2 + \rho|\nabla\chi_t|^2\right) \mathrm{d}x\mathrm{d}t \le C. \tag{3.30}$$

Moreover, from Korn's inequality (2.26), (3.22), (3.23), (3.28) and equation $(1.1)_4$, we can easily see that (3.19) holds. Then Lemma 3.4 is obtained. □

Lemma 3.5 *Under the assumptions in Lemma* 3.1*, we have the inequality*

$$\sup_{0\le t<T_*} \int_\Omega \left(\rho|u_t|^2 + \frac{1}{\rho}|\nabla\chi_t|^2\right) \mathrm{d}x + \int_0^{T_*} \int_\Omega (|\nabla u_t|^2 + \rho|\chi_{tt}|^2)\mathrm{d}x\mathrm{d}t \le C. \tag{3.31}$$

Proof. Differentiating $(1.1)_2$ with respect to t yields

$$\rho u_{tt} + \rho_t u_t + \rho u\cdot\nabla u_t + \rho u_t\cdot\nabla u + \rho_t u\cdot\nabla u + \nabla p_t = \mathrm{div}(2\eta(\chi)Du)_t - \mathrm{div}(\nabla\chi_t\otimes\nabla\chi + \nabla\chi\otimes\nabla\chi_t).$$

Multiplying the above equation by u_t, integrating the result over Ω, and recalling $(1.1)_{1,3}$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|u_t|^2\mathrm{d}x + \int_\Omega 2\eta(\chi)|Du_t|^2\mathrm{d}x \\
&= -\int_\Omega 2\eta'(\chi)\chi_t Du : Du_t\mathrm{d}x - \frac{1}{2}\int_\Omega \rho_t|u_t|^2\mathrm{d}x - \int_\Omega \rho(u\cdot\nabla)u_t\cdot u_t\mathrm{d}x - \int_\Omega \rho(u_t\cdot\nabla u)\cdot u_t\mathrm{d}x \\
&\quad - \int_\Omega \rho_t(u\cdot\nabla)u\cdot u_t\mathrm{d}x + \int_\Omega \nabla\chi_t\otimes\nabla\chi : \nabla u_t\mathrm{d}x + \int_\Omega \nabla\chi\otimes\nabla\chi_t : \nabla u_t\mathrm{d}x \\
&\lesssim \|\chi_t\|_{L^3}\|Du\|_{L^6}\|Du_t\|_{L^2} + \|u\|_{L^\infty}\|u_t\|_{L^2}^2 + \|u\|_{L^\infty}\|u_t\|_{L^2}\|Du_t\|_{L^2} + \|Du\|_{L^\infty}\|u_t\|_{L^2}^2 \\
&\quad + \|u\|_{L^\infty}^2\|\nabla u\|_{L^2}\|u_t\|_{L^2} + \|\nabla\chi\|_{L^\infty}\|\nabla\chi_t\|_{L^2}\|Du_t\|_{L^2} \\
&\lesssim \int_\Omega \eta(\chi)|Du_t|^2\mathrm{d}x + \left(\|u\|_{H^2}^2 + \|Du\|_{L^\infty} + 1\right)\int_\Omega \rho|u_t|^2\mathrm{d}x \\
&\quad + \left(\|u\|_{H^2}^2 + \|\nabla\chi\|_{L^\infty}^2\right)\|\nabla\chi_t\|_{L^2}^2 + \|u\|_{H^2}^2 + 1,
\end{aligned}$$

where we have used (3.6), (3.19) and Korn's inequality (2.26) in the last step. From which we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \rho|u_t|^2\mathrm{d}x + \int_\Omega \eta(\chi)|Du_t|^2\mathrm{d}x \\
&\lesssim \left(\|u\|_{H^2}^2 + \|Du\|_{L^\infty} + 1\right)\int_\Omega \rho|u_t|^2\mathrm{d}x + \left(\|u\|_{H^2}^2 + \|\nabla\chi\|_{L^\infty}^2\right)\|\nabla\chi_t\|_{L^2}^2 + \|u\|_{H^2}^2 + 1.
\end{aligned} \tag{3.32}$$

In the following, we deal with the term $\|\nabla\chi_t\|_{L^2}^2$. Differentiating (3.14) with respect to t gives

$$\rho\chi_{tt}+\rho_t\chi_t+\rho u\cdot\nabla\chi_t+\rho u_t\cdot\nabla\chi+\rho_t u\cdot\nabla\chi=\frac{1}{\rho}\Delta\chi_t+\left(\frac{1}{\rho}\right)_t\Delta\chi-(3\chi^2-1)\chi_t. \tag{3.33}$$

Multiplying (3.33) by χ_{tt}, integrating the result over Ω and noticing

$$\begin{aligned}\int_\Omega\frac{1}{\rho}\Delta\chi_t\chi_{tt}\mathrm{d}x&=-\int_\Omega\frac{1}{\rho}\nabla\chi_t\cdot\nabla\chi_{tt}\mathrm{d}x-\int_\Omega\nabla\left(\frac{1}{\rho}\right)\cdot\nabla\chi_t\,\chi_{tt}\mathrm{d}x\\&=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{\rho}|\nabla\chi_t|^2\mathrm{d}x+\frac{1}{2}\int_\Omega\left(\frac{1}{\rho}\right)_t|\nabla\chi_t|^2\mathrm{d}x-\int_\Omega\nabla\left(\frac{1}{\rho}\right)\cdot\nabla\chi_t\chi_{tt}\mathrm{d}x,\end{aligned}$$

we have

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{\rho}|\nabla\chi_t|^2\mathrm{d}x+\int_\Omega\rho|\chi_{tt}|^2\mathrm{d}x\\&\lesssim\int_\Omega|u||\chi_t||\chi_{tt}|\mathrm{d}x+\int_\Omega|u||\nabla\chi_t||\chi_{tt}|\mathrm{d}x+\int_\Omega|u_t||\nabla\chi||\chi_{tt}|\mathrm{d}x+\int_\Omega|u|^2|\nabla\chi||\chi_{tt}|\mathrm{d}x\\&\quad+\int_\Omega|u||\nabla\chi_t|^2\mathrm{d}x+\int_\Omega|\nabla\chi_t||\chi_{tt}|\mathrm{d}x+\int_\Omega|u||\Delta\chi||\chi_{tt}|\mathrm{d}x+\int_\Omega(|\chi|^2+1)|\chi_t||\chi_{tt}|\mathrm{d}x\\&\lesssim\|u\|_{L^\infty}\|\chi_t\|_{L^2}\|\chi_{tt}\|_{L^2}+\|u\|_{L^\infty}\|\nabla\chi_t\|_{L^2}\|\chi_{tt}\|_{L^2}+\|u_t\|_{L^6}\|\nabla\chi\|_{L^3}\|\chi_{tt}\|_{L^2}\\&\quad+\|u\|_{L^6}^2\|\nabla\chi\|_{L^6}\|\chi_{tt}\|_{L^2}+\|u\|_{L^\infty}\|\nabla\chi_t\|_{L^2}^2+\|\nabla\chi_t\|_{L^2}\|\chi_{tt}\|_{L^2}\\&\qquad+\|u\|_{L^\infty}\|\Delta\chi\|_{L^2}\|\chi_{tt}\|_{L^2}+\|\chi\|_{L^\infty}^2\|\chi_t\|_{L^2}\|\chi_{tt}\|_{L^2}+\|\chi_t\|_{L^2}\|\chi_{tt}\|_{L^2}\\&\lesssim\varepsilon\|\sqrt{\rho}\chi_{tt}\|_{L^2}^2+(\|u\|_{H^2}^2+1)\|\nabla\chi_t\|_{L^2}^2+\|u\|_{H^2}^2\|\chi_t\|_{L^2}^2+\|\nabla u_t\|_{L^2}^2\|\nabla\chi\|_{H^1}^2\\&\quad+\|\nabla u\|_{L^2}^4\|\nabla^2\chi\|_{L^2}^2+\|u\|_{H^2}^2\|\Delta\chi\|_{L^2}^2+\|\chi\|_{H^2}^4\|\chi_t\|_{L^2}^2+\|\chi_t\|_{L^2}^2\\&\lesssim\varepsilon\|\sqrt{\rho}\chi_{tt}\|_{L^2}^2+(\|u\|_{H^2}^2+1)\|\nabla\chi_t\|_{L^2}^2+\|u\|_{H^2}^2+\|\nabla u_t\|_{L^2}^2+1,\end{aligned}$$

where we have used (3.4), (3.5), (3.6) and (3.19). It follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{1}{\rho}|\nabla\chi_t|^2\mathrm{d}x+\int_\Omega\rho|\chi_{tt}|^2\mathrm{d}x\\&\le C(\|u\|_{H^2}^2+1)\int_\Omega\frac{1}{\rho}|\nabla\chi_t|^2\mathrm{d}x+C\|u\|_{H^2}^2+C\|Du_t\|_{L^2}^2+C\|u_t\|_{L^2}^2+C.\end{aligned}\tag{3.34}$$

Multiplying (3.34) by $\underline{\eta}/C$, adding the resulting inequality to (3.32) and by using (1.6) yield

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(\rho|u_t|^2+\frac{1}{\rho}|\nabla\chi_t|^2\right)\mathrm{d}x+\int_\Omega\left(|Du_t|^2+\rho|\chi_{tt}|^2\right)\mathrm{d}x\\&\lesssim\left(\|u\|_{H^2}^2+\|Du\|_{L^\infty}+\|\nabla\chi\|_{L^\infty}^2+1\right)\int_\Omega\left(\rho|u_t|^2+\frac{1}{\rho}|\nabla\chi_t|^2\right)\mathrm{d}x+\|u\|_{H^2}^2+1.\end{aligned}$$

From the equations $(1.1)_{2,4,5}$ and the assumptions on the initial data ρ_0, u_0, χ_0, applying Gronwall's inequality, by (3.1) and (3.19), we obtain(3.31). Thus we complete the proof of this lemma. □

In terms of the results for the stationary Stokes equation, we derive the higher order estimates for u and p.

Lemma 3.6 *If the assumptions in Lemma* 3.1 *are valid, there holds*

$$\sup_{0\le t<T_*}(\|u\|_{H^2}^2+\|p\|_{H^1}^2+\|\nabla^3\chi\|_{L^2}^2+\|\nabla\mu\|_{L^2}^2)+\int_0^{T_*}(\|u\|_{W^{2,6}}^2+\|p\|_{W^{1,6}}^2+\|\mu_t\|_{L^2}^2)\mathrm{d}t\le C. \tag{3.35}$$

Proof. From (3.19) and Nirenberg's interpolation inequality

$$\|\chi\|_{L^\infty}\lesssim\|\chi\|_{L^2}^{1/4}\|\nabla^2\chi\|_{L^2}^{3/4}+\|\chi\|_{L^2},$$

we have

$$\begin{aligned}\|\nabla^3\chi\|_{L^2}\lesssim&\|\nabla\chi_t\|_{L^2}+\|\chi_t\|_{L^2}+\|\nabla^2u\|_{L^2}+\|\nabla\chi\|_{L^2}^{1/4}\|\nabla^3\chi\|_{L^2}^{3/4}+\|\Delta\chi\|_{L^2}+1\\ \lesssim&\frac12\|\nabla^3\chi\|_{L^2}+\|\nabla\chi_t\|_{L^2}+\|\chi_t\|_{L^2}+\|\nabla^2u\|_{L^2}+\|\nabla\chi\|_{L^2}+\|\Delta\chi\|_{L^2}+1.\end{aligned}$$

By (3.6), (3.19), (3.31), we get

$$\|\nabla^3\chi\|_{L^2}\lesssim\|\nabla^2u\|_{L^2}+1. \tag{3.36}$$

On the other hand, $(1.1)_2$ can be rewritten as

$$-\mathrm{div}(2\eta(\chi)Du)+\nabla p=-\rho u\cdot\nabla u-\mathrm{div}(\nabla\chi\otimes\nabla\chi)-\rho u_t. \tag{3.37}$$

By the estimates for the stationary Stokes equations (see [16]), we have

$$\begin{aligned}\|u\|_{H^2}+\|p\|_{H^1}\lesssim&\|u\cdot\nabla u\|_{L^2}+\|\mathrm{div}(\nabla\chi\otimes\nabla\chi)\|_{L^2}+\|u_t\|_{L^2}\\ \lesssim&\|u\|_{L^6}\|\nabla u\|_{L^3}+\|\nabla\chi\|_{L^\infty}\|\nabla^2\chi\|_{L^2}+\|u_t\|_{L^2}\\ \lesssim&\|\nabla u\|_{L^2}^{3/2}\|\nabla^2u\|_{L^2}^{1/2}+\|\nabla\chi\|_{L^2}^{1/4}\|\nabla^3\chi\|_{L^2}^{3/4}\|\nabla^2\chi\|_{L^2}+\|u_t\|_{L^2}\\ \lesssim&\|\nabla^2u\|_{L^2}^{1/2}+\|\nabla^3\chi\|_{L^2}^{3/4}+1\lesssim\|\nabla^2u\|_{L^2}^{1/2}+\|\nabla^2u\|_{L^2}^{3/4}+1\\ \lesssim&\frac12\|\nabla^2u\|_{L^2}+1,\end{aligned} \tag{3.38}$$

where we have used Nirenberg's interpolation inequality, (3.4), (3.6), (3.19),and (3.31). From (3.36) and (3.38) we obtain

$$\sup_{0\le t<T_*}(\|u\|_{H^2}+\|p\|_{H^1}+\|\nabla^3\chi\|_{L^2})\le C. \tag{3.39}$$

The estimates for the stationary Stokes equation (3.37) (see [16]) and (3.39) also imply

$$\begin{aligned}\|u\|_{W^{2,6}}+\|p\|_{W^{1,6}}\lesssim&\|u\cdot\nabla u\|_{L^6}+\|\mathrm{div}(\nabla\chi\otimes\nabla\chi)\|_{L^6}+\|u_t\|_{L^6}\\ \lesssim&\|u\|_{L^\infty}\|\nabla u\|_{L^6}+\|\nabla\chi\|_{L^\infty}\|\nabla^2\chi\|_{L^6}+\|\nabla u_t\|_{L^2}\\ \lesssim&\|\nabla u_t\|_{L^2}+1.\end{aligned}$$

Here, we have normalized p as $\int_\Omega p(x,t)\mathrm{d}x=0$. From the above inequality and (3.31), we have

$$\int_0^{T_*}(\|u\|_{W^{2,6}}^2+\|p\|_{W^{1,6}}^2)\mathrm{d}t\lesssim\int_0^{T_*}(\|\nabla u_t\|_{L^2}^2+1)\mathrm{d}t\le C.$$

Recalling $(1.1)_{4,5}$ and by using (3.4), (3.5), (3.19), (3.31), (3.39), we get

$$\|\nabla\mu\|_{L^2} \lesssim \|\nabla^3\chi\|_{L^2} + \|\Delta\chi\|_{L^2} + (\|\chi\|_{L^\infty}^2 + 1)\|\nabla\chi\|_{L^2} \le C,$$

and

$$\begin{aligned}\|\mu_t\|_{L^2}^2 \lesssim & \|\chi_{tt}\|_{L^2}^2 + \|\chi_t(u\cdot\nabla)\rho\|_{L^2}^2 + \|u\cdot\nabla\chi_t\|_{L^2}^2 + \|u_t\cdot\nabla\chi\|_{L^2}^2 + \|(u\cdot\nabla)\rho(u\cdot\nabla)\chi\|_{L^2}^2 \\ \lesssim & \|\chi_{tt}\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\chi_t\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla\chi_t\|_{L^2}^2 + \|u_t\|_{L^6}^2\|\nabla\chi\|_{L^3}^2 + \|u\|_{L^\infty}^2\|\nabla\chi\|_{L^2}^2 \\ \lesssim & \|\chi_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + 1.\end{aligned}$$

Integrating the above inequality over $(0, T_*)$ and by using (3.31) again, we arrive at (3.35). The proof of this lemma is complete. □

At last, we deal with the higher order estimates for ρ, χ and μ.

Lemma 3.7 *Assume that the hypotheses in Lemma* 3.1 *hold, then we have*

$$\sup_{0\le t<T_*}(\|\rho\|_{W^{2,6}}^2 + \|\rho_t\|_{W^{1,6}}^2) + \int_0^{T_*}(\|\nabla^4\chi\|_{L^2}^2 + \|\nabla^2\chi_t\|_{L^2}^2 + \|\nabla^2\mu\|_{L^2}^2)\mathrm{d}t \le C. \tag{3.40}$$

Proof. In terms of (3.35), the proof of the estimates for ρ is similar to the proof of (2.6) and (2.7). Applying the standard H^2-estimate to the equation (3.33), we get

$$\begin{aligned}\|\nabla^2\chi_t\|_{L^2} \lesssim & \|\Delta\chi_t\|_{L^2} + \|\nabla\chi_t\|_{L^2} \\ \lesssim & \|\chi_{tt}\|_{L^2} + \|u\chi_t\|_{L^2} + \|u\cdot\nabla\chi_t\|_{L^2} + \|u_t\cdot\nabla\chi\|_{L^2} \\ & + \||u|^2\nabla\chi\|_{L^2} + \|u\Delta\chi\|_{L^2} + \|(\chi^2+1)\chi_t\|_{L^2} + \|\nabla\chi_t\|_{L^2} \\ \lesssim & \|\chi_{tt}\|_{L^2} + \|u\|_{L^6}\|\chi_t\|_{L^3} + \|u\|_{L^6}\|\nabla\chi_t\|_{L^3} + \|u_t\|_{L^6}\|\nabla\chi\|_{L^3} \\ & + \|u\|_{L^6}\|u\|_{L^3}\|\nabla\chi\|_{L^\infty} + \|u\|_{L^6}\|\Delta\chi\|_{L^3} + (\|\chi\|_{L^\infty}^2+1)\|\chi_t\|_{L^2} + \|\nabla\chi_t\|_{L^2} \\ \lesssim & \|\chi_{tt}\|_{L^2} + \|\nabla u\|_{L^2}\|\chi_t\|_{H^1} + \|\nabla u\|_{L^2}\|\nabla\chi_t\|_{L^2}^{1/2}\|\nabla^2\chi_t\|_{L^2}^{1/2} + \|\nabla u_t\|_{L^2}\|\nabla\chi\|_{H^1} \\ & + \|\nabla u\|_{L^2}\|u\|_{H^1}\|\nabla\chi\|_{L^\infty} + \|\nabla u\|_{L^2}\|\Delta\chi\|_{H^1} + (\|\chi\|_{H^2}^2+1)\|\chi_t\|_{L^2} + \|\nabla\chi_t\|_{L^2} \\ \lesssim & \frac{1}{2}\|\nabla^2\chi_t\|_{L^2} + \|\chi_{tt}\|_{L^2} + \|\nabla u_t\|_{L^2} + 1,\end{aligned}$$

where we have use Nirenberg's interpolation inequality, (3.4), (3.5), (3.31) and (3.35) in the last step. From the above inequality and (3.31) we have

$$\int_0^{T_*}\|\nabla^2\chi_t\|_{L^2}^2\mathrm{d}t \lesssim \int_0^{T_*}(\|\chi_{tt}\|_{L^2}^2 + \|\nabla u_t\|_{L^2}^2 + 1)\mathrm{d}t \le C. \tag{3.41}$$

Applying the standard L^2-estimate to the equation (3.14) and by using (3.4) and (3.5),

$$\|u\|_{L^\infty} \lesssim \|u\|_{H^2} \le C, \quad \|\chi\|_{L^\infty} \lesssim \|\chi\|_{H^2} \le C, \quad \|\nabla\chi\|_{L^\infty} \lesssim \|\nabla\chi\|_{H^2} \le C,$$

we have

$$\|\nabla^4\chi\|_{L^2} \lesssim \|\nabla^2(\rho^2\chi_t)\|_{L^2} + \|\nabla^2(\rho^2 u\cdot\nabla\chi)\|_{L^2} + \|\nabla^2(\rho(\chi^3-\chi))\|_{L^2} + 1$$

$$\begin{aligned}&\lesssim\|\nabla^2\chi_t\|_{L^2}+\|\nabla\chi_t\|_{L^2}+\|\chi_t\|_{L^2}+\|\chi_t\|_{L^\infty}\|\nabla^2\rho\|_{L^2}+\|\nabla^3\chi\|_{L^2}\\&\quad+\|\nabla^2\chi\|_{L^2}+\|\nabla u\|_{L^6}\|\nabla^2\chi\|_{L^3}+\|\nabla^2 u\|_{L^2}+\|\nabla u\|_{L^2}+\|\nabla^2\rho\|_{L^2}+1\\&\lesssim\|\nabla^2\chi_t\|_{L^2}+1,\end{aligned}\tag{3.42}$$

where we have used (3.31), (3.19) and the estimates for ρ in the last step. Furthermore, from $(1.1)_5$ we can deduce that

$$\|\nabla^2\mu\|_{L^2}\lesssim\|\nabla^4\chi\|_{L^2}+\|\nabla^3\chi\|_{L^2}+\|\nabla^2\rho\|_{L^2}\|\Delta\chi\|_{L^\infty}+\|\nabla^2\chi\|_{L^2}+1\lesssim\|\nabla^4\chi\|_{L^2}+1.\tag{3.43}$$

From (3.41), (3.42) and (3.43), we obtain (3.40). Therefore, Lemma 3.7 is established. □

From Lemma 3.1– Lemma 3.7 we obtain (3.3). Hence

$$(\rho,u,\chi)(x,T_*)=\lim_{t\to T_*}(\rho,u,\chi)(x,t)$$

exists. Moreover, $u(x,T_*)\big|_{\partial\Omega}=0$ and $\operatorname{div}u(x,T_*)=0$ for $x\in\Omega$. Thus, we can extend the strong solution to $T_*+\delta$ with some constant $\delta>0$, which contradicts the definition of T_*. Therefore, (3.1) and (3.2) are false. The proof of Theorem 1.2 is now complete. □

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