UNIFORM ERROR BOUNDS OF A FINITE DIFFERENCE METHOD FOR THE KLEIN-GORDON-ZAKHAROV SYSTEM IN THE SUBSONIC LIMIT REGIME

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ABSTRACT. We establish uniform error bounds of a finite difference method for the Klein-Gordon-Zakharov (KGZ) system with a dimensionless parameter $\varepsilon \in (0, 1]$, which is inversely proportional to the acoustic speed. In the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$, the solution propagates highly oscillatory waves in time and/or rapid outgoing initial layers in space due to the singular perturbation in the Zakharov equation and/or the incompatibility of the initial data. Specifically, the solution propagates waves with $O(\varepsilon)$-wavelength in time and $O(1)$-wavelength in space as well as outgoing initial layers in space at speed $O(1/\varepsilon)$. This high oscillation in time and rapid outgoing waves in space of the solution cause significant burdens in designing numerical methods and establishing error estimates for KGZ system. By applying an asymptotic consistent formulation, we propose a uniformly accurate finite difference method and rigorously establish two independent error bounds at $O(h^2 + \tau^2/\varepsilon)$ and $O(h^2 + \tau + \varepsilon)$ with h mesh size and τ time step. Thus we obtain a uniform error bound at $O(h^2 + \tau)$ for $0 < \varepsilon \le 1$. The main techniques in the analysis include the energy method, cut-off of the nonlinearity to bound the numerical solution, the integral approximation of the oscillatory term, and ε-dependent error bounds between the solutions of KGZ system and its limiting model when $\varepsilon \to 0^+$. Finally, numerical results are reported to confirm our error bounds.

1. INTRODUCTION

We study the Klein-Gordon-Zakharov (KGZ) system which describes the interaction between a Langmuir wave and an ion acoustic wave in plasma [24]:

$$
(1.1) \quad
\begin{aligned}
&\partial_{tt}E(\mathbf{x}, t) - 3v_0^2 \Delta E(\mathbf{x}, t) + \omega_p^2 E(\mathbf{x}, t) + \omega_p^2 N(\mathbf{x}, t)E(\mathbf{x}, t) = 0, \\
&\partial_{tt}N(\mathbf{x}, t) - c_s^2 \Delta N(\mathbf{x}, t) - \frac{n_0 \varepsilon_0}{4mN_0}\Delta |E|^2(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,
\end{aligned}
$$

where t is time, $\mathbf{x} \in \mathbb{R}^d$ ($d = 1, 2, 3$) is the spatial coordinate, $E(\mathbf{x}, t)$ and $N(\mathbf{x}, t)$ are real-valued functions representing the fast time scale component of the electric field raised by electrons and the ion density fluctuation from the constant equilibrium, respectively. Here v_0 is the electron thermal velocity, ω_p is the electron plasma

frequency, c_s is the ion-acoustic speed, n_0 is plasma charge number, ε_0 is vacuum dielectric constant, m is ion mass and N_0 is electron density. It can be derived from the Euler equations for the electrons and ions, coupled with the Maxwell equation for the electron field by negeleting the magnetic effect and further assuming that ions move much slower than electrons (cf. [12, 16, 33, 39] for physical and formal derivations and [35] for mathematical justifications).

For scaling the KGZ system (1.1), we introduce

$$(1.2) \qquad \widetilde{t} = t/t_s, \quad \widetilde{\mathbf{x}} = \mathbf{x}/x_s, \quad \widetilde{E}(\widetilde{\mathbf{x}}, \widetilde{t}) = E(\mathbf{x}, t)/E_s, \quad \widetilde{N}(\widetilde{\mathbf{x}}, \widetilde{t}) = N(\mathbf{x}, t)/N_s,$$

where $t_s = \frac{1}{\omega_p}$, $x_s = \frac{\sqrt{3}v_0}{\omega_p}$, $E_s = 2c_s\sqrt{\frac{mN_0}{n_0\varepsilon_0}}$ and $N_s = 1$ are the dimensionless time, length, electric field and ion density unit, respectively. Plugging (1.2) into (1.1) and removing all "$\sim$" followed by replacing $N(\mathbf{x}, t)$ and $E(\mathbf{x}, t)$ by $N^\varepsilon(\mathbf{x}, t)$ and $E^\varepsilon(\mathbf{x}, t)$, respectively, we get the following dimensionless KGZ system as,

$$(1.3) \qquad \begin{aligned} &\partial_{tt}E^\varepsilon(\mathbf{x}, t) - \Delta E^\varepsilon(\mathbf{x}, t) + E^\varepsilon(\mathbf{x}, t) + N^\varepsilon(\mathbf{x}, t)E^\varepsilon(\mathbf{x}, t) = 0, \\ &\varepsilon^2\partial_{tt}N^\varepsilon(\mathbf{x}, t) - \Delta N^\varepsilon(\mathbf{x}, t) - \Delta|E^\varepsilon|^2(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \end{aligned}$$

where the dimensionless parameter $\varepsilon := \frac{\sqrt{3}v_0}{c_s} > 0$ is inversely proportional to the speed of sound. Here we consider the case when the thermal electron velocity is much smaller than the ion-acoustic speed, i.e., $3v_0^2 \ll c_s^2$, which gives $0 < \varepsilon \ll 1$, i.e., the KGZ system in the subsonic limit regime. To study the dynamics of the KGZ system (1.3), the initial data is usually given as

$$(1.4) \qquad \begin{aligned} E^\varepsilon(\mathbf{x}, 0) &= E_0(\mathbf{x}), &\partial_t E^\varepsilon(\mathbf{x}, 0) &= E_1(\mathbf{x}), \\ N^\varepsilon(\mathbf{x}, 0) &= N_0^\varepsilon(\mathbf{x}), &\partial_t N^\varepsilon(\mathbf{x}, 0) &= N_1^\varepsilon(\mathbf{x}). \end{aligned}$$

As it is known, (1.3) is time symmetric or time reversible and conserves the total *energy* (or Hamiltonian) [24, 25]

$$(1.5) \qquad \begin{aligned} \mathcal{H}^\varepsilon(t) &:= \int_{\mathbb{R}^d} \left[|\partial_t E^\varepsilon|^2 + |\nabla E^\varepsilon|^2 + |E^\varepsilon|^2 + \frac{1}{2}|\nabla\varphi^\varepsilon|^2 + \frac{1}{2}|N^\varepsilon|^2 + N^\varepsilon|E^\varepsilon|^2 \right] d\mathbf{x} \\ &\equiv \mathcal{H}^\varepsilon(0), \qquad t \geq 0, \end{aligned}$$

where φ^ε is defined by $\Delta\varphi^\varepsilon = \varepsilon\partial_t N^\varepsilon$ with $\lim_{|\mathbf{x}|\to\infty}\varphi^\varepsilon = 0$.

There have been extensive studies for the KGZ system in the literature for $\varepsilon = 1$, i.e., $O(1)$-acoustic-speed regime. Along the analytical part, for the derivation of the KGZ system from two-fluid Euler-Maxwell system, we refer to [12, 35]; and for the well-posedness of the Cauchy problem, we refer to [27, 29, 30, 37]. Along the numerical part, we refer to [38] for the finite difference method and [7, 10] for the exponential-wave-integrator Fourier pseudospectral method. However, in the subsonic limit regime, the analysis and efficient computation of the KGZ system are rather complicated [12, 24] due to the high oscillation in time and/or rapid outgoing waves in space of the solution as $\varepsilon \to 0^+$.

Based on the results in [14, 26], in the subsonic limit, i.e., $\varepsilon \to 0^+$, the KGZ system collapses to the Klein-Gordon (KG) equation. Formally we have $E^\varepsilon \to E_k$, where $E_k := E_k(\mathbf{x}, t)$ is the solution of the KG equation [14, 26]:

$$(1.6) \qquad \begin{aligned} &\partial_{tt}E_k(\mathbf{x}, t) - \Delta E_k(\mathbf{x}, t) + E_k(\mathbf{x}, t) - E_k(\mathbf{x}, t)^3 = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ &E_k(\mathbf{x}, 0) = E_0(\mathbf{x}), \quad \partial_t E_k(\mathbf{x}, 0) = E_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d. \end{aligned}$$

The KG equation (1.6) conserves the *energy*

$$\mathcal{H}(t) := \int_{\mathbb{R}^d} \left[|\partial_t E_k|^2 + |\nabla E_k|^2 + |E_k|^2 - \frac{1}{2}|E_k|^4 \right] d\mathbf{x} \equiv \mathcal{H}(0), \quad t \geq 0.$$

Different convergence rates can be obtained due to the incompatibility of the initial data $(E_0, E_1, N_0^\varepsilon, N_1^\varepsilon)$ for (1.3) with respect to (1.6), which can be characterized as

$$(1.7) \quad N_0^\varepsilon(\mathbf{x}) = -E_0(\mathbf{x})^2 + \varepsilon^\alpha \omega_0(\mathbf{x}), \quad N_1^\varepsilon(\mathbf{x}) = -2E_0(\mathbf{x})E_1(\mathbf{x}) + \varepsilon^\beta \omega_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d,$$

where $\alpha \geq 0$ and $\beta \geq -1$ are given parameters describing the incompatibility of the initial data of the KGZ system (1.3)–(1.4) with respect to that of the KG equation (1.6) in the subsonic limit regime such that the energy (1.5) is bounded, and $\omega_0(\mathbf{x})$ and $\omega_1(\mathbf{x})$ are given functions which are all independent of ε. Similar to the properties of the solutions of the Zakharov system in the subsonic limit regime [25, 28, 31], when $0 < \varepsilon \ll 1$, the solution of the KGZ system propagates waves with wavelength $O(\varepsilon)$ and $O(1)$ in time and space, respectively (cf. Fig. 1.1(a)), and/or rapid outgoing initial layers at speed $O(1/\varepsilon)$ in space (cf. Fig. 1.1(b)). More precisely, when $\alpha \geq 2$ and $\beta \geq 1$, the leading oscillation comes from the $\varepsilon^2 \partial_{tt}$ term; and otherwise from the incompatibility of the initial data.
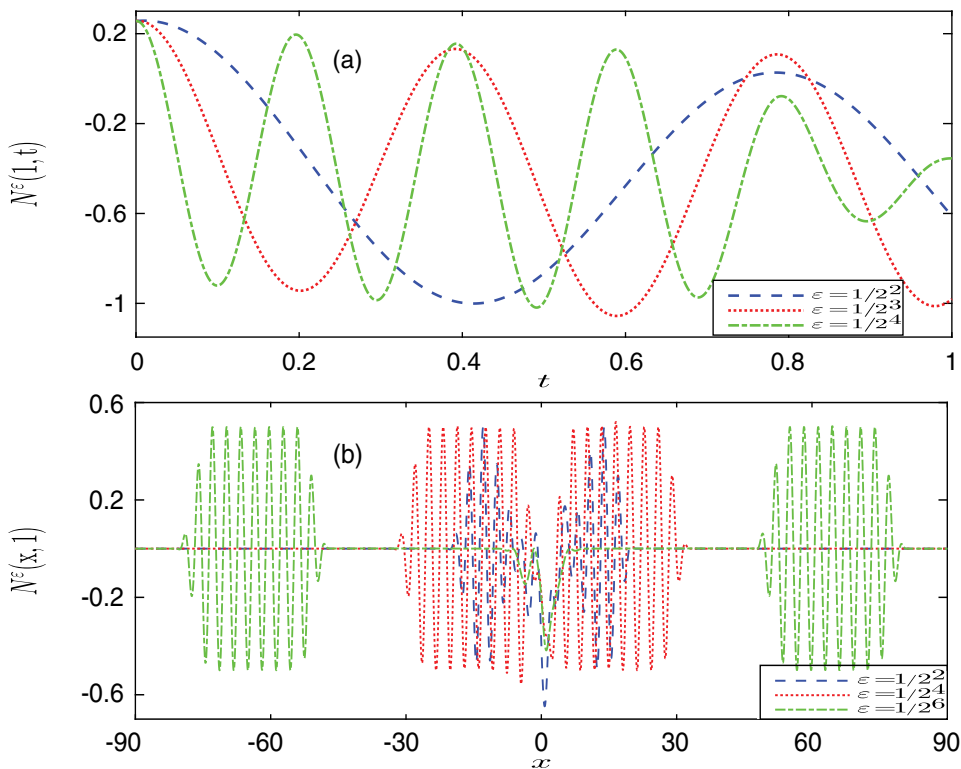

FIGURE 1.1. The temporal oscillation (a) and rapid outgoing wave in space (b) of the KGZ system (1.3) for $d = 1$.

To illustrate the temporal oscillation and rapid outgoing wave phenomena, Figure 1.1 shows the solutions $N^\varepsilon(x, 1)$, $N^\varepsilon(1, t)$ of the KGZ system (1.3) for $d = 1$

and the initial data

(1.8)
$$E_0(x) = \operatorname{sech}(x) \cos\left(\frac{x}{2}\right), \quad E_1(x) = \frac{1}{2}\psi\left((x+10)/5\right)\psi\left((10-x)/5\right)\sin\left(\frac{x}{2}\right),$$
$$\omega_0(x) = \psi\left((x+18)/10\right)\psi\left((18-x)/9\right)\sin\left(2x+\frac{\pi}{6}\right), \ \omega_1(x) = e^{-\frac{x^2}{3}}\sin(2x),$$

with

(1.9)
$$\psi(x) = \frac{\varphi(x)}{\varphi(x)+\varphi(1-x)}, \quad \varphi(x) = e^{-1/x}\chi_{(0,\infty)},$$

and χ being the characteristic function, $\alpha = \beta = 0$ in (1.7) for different ε, which was obtained numerically by the exponential-wave-integrator sine pseudospectral method on a bounded interval $[-200, 200]$ with the homogenous Dirichlet boundary condition [7].

The highly temporal oscillatory nature in the solution of the KGZ system (1.3) brings significant numerical difficulties, especially in the subsonic limit regime, i.e., $0 < \varepsilon \ll 1$. To the best of our knowledge, there are few results concerning error estimates of different numerical methods for KGZ system with respect to mesh size h and time step τ as well as the parameter $0 < \varepsilon \leq 1$. Recently, a conservative finite difference method (FDM) was proposed and analyzed in the subsonic limit regime [32], where it was proved that in order to obtain "correct" oscillatory solutions, the FDM requests the meshing strategy (or ε-scalability) $h = O(\varepsilon^{1/2})$ and $\tau = O(\varepsilon^{3/2})$. The reason is due to that $N^\varepsilon(\mathbf{x}, t)$ does not converge as $\varepsilon \to 0^+$ when $\alpha = 0$ or $\beta = -1$ [25, 31, 34] (cf. Figure 1.1).

The main aim of this paper is to propose and analyze a finite difference method for the KGZ system, which is uniformly accurate in both space and time for $0 < \varepsilon \leq 1$. The key points in designing the uniformly accurate finite difference method include: (i) reformulating the KGZ system into an asymptotic consistent formulation and (ii) using an integral approximation of the oscillatory term. To establish the error bounds, we apply the energy method, cut-off technique for treating the nonlinearity and the inverse estimates to bound the numerical solution, and the limiting equation via a nonlinear Klein-Gordon equation with an oscillatory potential. The error bounds of our new numerical method significantly relax the meshing strategy of the standard FDM for the KGZ system in the subsonic limit regime [32].

The rest of the paper is organized as follows. In Section 2, we introduce an asymptotic consistent formulation of the KGZ system, present a finite difference method and state our main results. Section 3 is devoted to the details of the error analysis. Numerical results are reported in Section 4 to confirm our error bounds. Finally some conclusions are drawn in Section 5. Throughout the paper, we adopt the standard Sobolev spaces as well as the corresponding norms and denote $A \lesssim B$ to represent that there exists a generic constant $C > 0$ independent of ε, τ, h, such that $|A| \leq C B$.

2. A finite difference method and its error bounds

In this section, we present a uniformly accurate finite difference method based on an asymptotic consistent formulation of the KGZ system and give its uniform error bounds.

2.1. **An asymptotic consistent formulation.** Following [8], we introduce

$$(2.1) \qquad F^\varepsilon(\mathbf{x}, t) = N^\varepsilon(\mathbf{x}, t) + |E^\varepsilon(\mathbf{x}, t)|^2 - G^\varepsilon(\mathbf{x}, t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t \geq 0,$$

where $G^\varepsilon(\mathbf{x}, t)$ represents initial layer caused by the incompatibility of the initial data (1.7), which is the solution of the linear wave equation

$$(2.2) \qquad \partial_{tt} G^\varepsilon(\mathbf{x}, t) - \frac{1}{\varepsilon^2} \Delta G^\varepsilon(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$
$$G^\varepsilon(\mathbf{x}, 0) = \varepsilon^\alpha \omega_0(\mathbf{x}), \quad \partial_t G^\varepsilon(\mathbf{x}, 0) = \varepsilon^\beta \omega_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Substituting (2.1) into the KGZ system (1.3), we can reformulate it into an asymptotic consistent formulation

$$\partial_{tt} E^\varepsilon(\mathbf{x}, t) - \Delta E^\varepsilon(\mathbf{x}, t) + \left[1 - E^\varepsilon(\mathbf{x}, t)^2 + F^\varepsilon(\mathbf{x}, t) + G^\varepsilon(\mathbf{x}, t)\right] E^\varepsilon(\mathbf{x}, t) = 0,$$
$$(2.3) \quad \varepsilon^2 \partial_{tt} F^\varepsilon(\mathbf{x}, t) - \Delta F^\varepsilon(\mathbf{x}, t) - \varepsilon^2 \partial_{tt} |E^\varepsilon(\mathbf{x}, t)|^2 = 0, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$
$$E^\varepsilon(\mathbf{x}, 0) = E_0(\mathbf{x}), \quad \partial_t E^\varepsilon(\mathbf{x}, 0) = E_1(\mathbf{x}), \quad F^\varepsilon(\mathbf{x}, 0) = 0, \quad \partial_t F^\varepsilon(\mathbf{x}, 0) = 0.$$

In the subsonic limit regime, i.e., $\varepsilon \to 0^+$, formally we have $E^\varepsilon(\mathbf{x}, t) \to E_{\mathrm{k}}(\mathbf{x}, t)$ and $F^\varepsilon(\mathbf{x}, t) \to 0$, where $E_{\mathrm{k}}(\mathbf{x}, t)$ is the solution of the KG equation (1.6). Moreover, as $\varepsilon \to 0^+$, formally we can also get $E^\varepsilon(\mathbf{x}, t) \to \widetilde{E}^\varepsilon(\mathbf{x}, t)$, where $\widetilde{E}^\varepsilon := \widetilde{E}^\varepsilon(\mathbf{x}, t)$ is the solution of the Klein-Gordon equation with an oscillatory potential $G^\varepsilon(\mathbf{x}, t)$ (KGE-OP)

$$(2.4) \qquad \partial_{tt} \widetilde{E}^\varepsilon(\mathbf{x}, t) - \Delta \widetilde{E}^\varepsilon(\mathbf{x}, t) + \left[1 - \widetilde{E}^\varepsilon(\mathbf{x}, t)^2 + G^\varepsilon(\mathbf{x}, t)\right] \widetilde{E}^\varepsilon(\mathbf{x}, t) = 0,$$
$$\widetilde{E}^\varepsilon(\mathbf{x}, 0) = E_0(\mathbf{x}), \quad \partial_t \widetilde{E}^\varepsilon(\mathbf{x}, 0) = E_1(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^d.$$

Inspired by the convergence of the Zakharov system to the nonlinear Schrödinger equation in the subsonic limit [28] and the analysis of the KGZ system converging to the KG equation [14], we can obtain the following result concerning the convergence from the KGZ system (2.3) to the KGE-OP (2.4),

$$(2.5) \qquad \|F^\varepsilon\|_{L^2} + \|F^\varepsilon\|_{L^\infty} + \|E^\varepsilon(\cdot, t) - \widetilde{E}^\varepsilon(\cdot, t)\|_{H^1} \leq C_T \varepsilon, \quad 0 \leq t \leq T,$$

where $0 < T < T^*$ with $T^* > 0$ being the maximum common existence time of the solutions of the KGZ system (2.3) and the KGE-OP (2.4) and C_T is a positive constant independent of ε. To illustrate this, Figure 2.2 depicts the convergence behavior between the solutions of the KGZ system (2.3) and the KGE-OP (2.4), where $\eta_{\mathrm{F}}^\varepsilon(t) := \|F^\varepsilon(\cdot, t)\|_{L^2}, \eta_\infty^\varepsilon(t) := \frac{1}{\varepsilon} \|F^\varepsilon(\cdot, t)\|_{L^\infty} + \|\partial_t F^\varepsilon(\cdot, t)\|_{L^\infty} + \|\partial_{tt} F^\varepsilon(\cdot, t)\|_{L^\infty}$ and $\eta_{\mathrm{e}}^\varepsilon(t) := \|E^\varepsilon(\cdot, t) - \widetilde{E}^\varepsilon(\cdot, t)\|_{H^1}$ for different ε with the same initial data as in (1.8) for $d = 1$ and $\alpha = 0$, $\beta = -1$.

2.2. **A uniformly accurate finite difference method.** For simplicity of notation, we will only present the numerical method for the KGZ system in one spatial dimension, and extensions to higher dimensions are straightforward. Practically, similar to most works for computation of the Zakharov-type system [8, 9, 13, 22, 32], (2.3) is truncated on a bounded interval $\Omega = (a, b)$ with the homogeneous Dirichlet boundary condition:

$$\partial_{tt} E^\varepsilon(x, t) - \partial_{xx} E^\varepsilon(x, t) + \left[1 - E^\varepsilon(x, t)^2 + F^\varepsilon(x, t) + G^\varepsilon\right] E^\varepsilon(x, t) = 0,$$
$$(2.6) \quad \varepsilon^2 \partial_{tt} F^\varepsilon(x, t) - \partial_{xx} F^\varepsilon(x, t) - \varepsilon^2 \partial_{tt} |E^\varepsilon(x, t)|^2 = 0, \quad x \in \Omega, \quad t > 0,$$
$$E^\varepsilon(x, 0) = E_0(x), \quad \partial_t E^\varepsilon(x, 0) = E_1(x), \quad F^\varepsilon(x, 0) = 0, \quad \partial_t F^\varepsilon(x, 0) = 0,$$
$$E^\varepsilon(a, t) = E^\varepsilon(b, t) = 0, \quad F^\varepsilon(a, t) = F^\varepsilon(b, t) = 0, \quad t \geq 0,$$
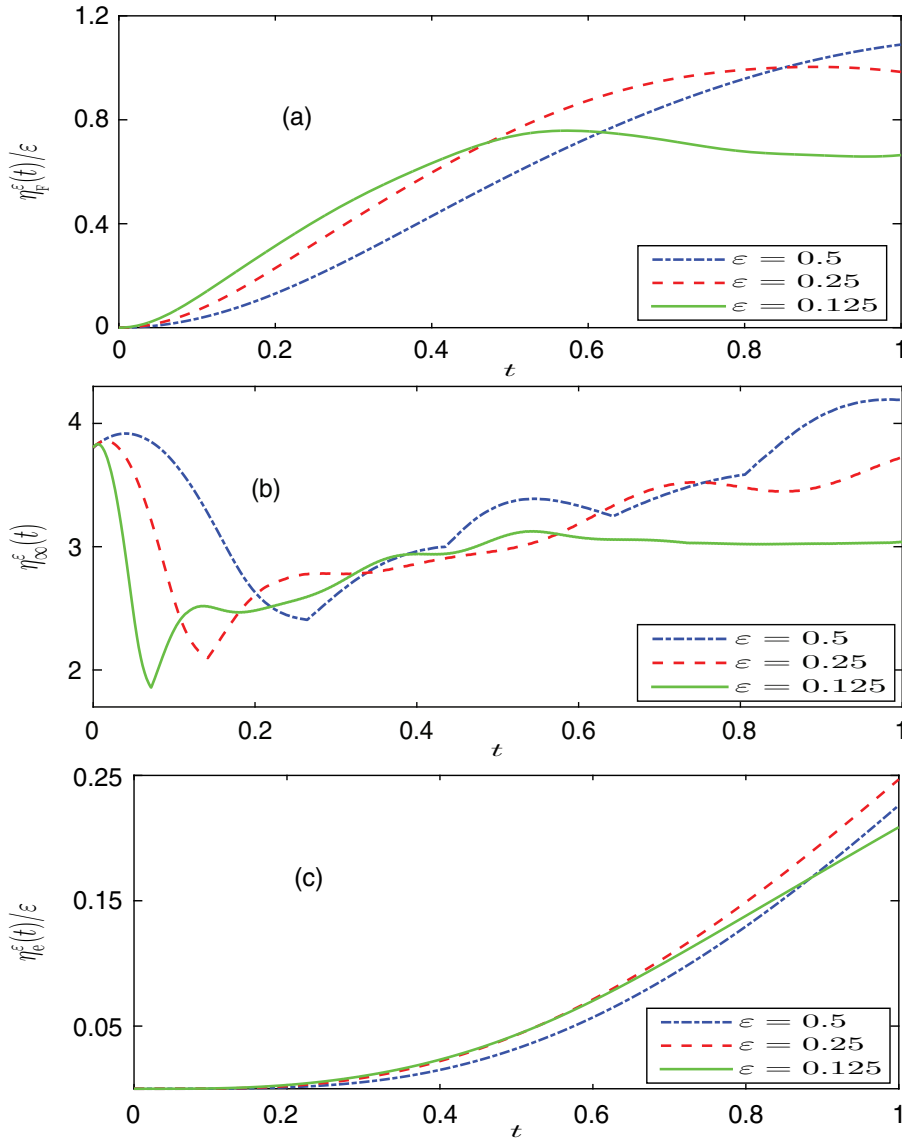
FIGURE 2.2. Time evolution of $\eta_F^\varepsilon(t)$, $\eta_\infty^\varepsilon(t)$ and $\eta_e^\varepsilon(t)$.

where $G^\varepsilon := G^\varepsilon(x,t)$ is defined as the solution of (2.2) with homogeneous Dirichlet boundary condition for $d = 1$,

$$
\begin{aligned}
&\partial_{tt} G^\varepsilon(x,t) - \frac{1}{\varepsilon^2} \partial_{xx} G^\varepsilon(x,t) = 0, \quad x \in \Omega, \quad t > 0, \\
&G^\varepsilon(x,0) = \varepsilon^\alpha \omega_0(x), \quad \partial_t G^\varepsilon(x,0) = \varepsilon^\beta \omega_1(x), \quad x \in \overline{\Omega}, \\
&G^\varepsilon(a,t) = G^\varepsilon(b,t) = 0, \quad t \geq 0.
\end{aligned}
$$
(2.7)

As $\varepsilon \to 0$, formally we have $E^\varepsilon(x,t) \to \widetilde{E}^\varepsilon(x,t)$ and $F^\varepsilon(x,t) \to 0$, where $\widetilde{E}^\varepsilon(x,t)$ is the solution of the KGE-OP,

(2.8)
$$\partial_{tt}\widetilde{E}^\varepsilon(x,t) - \partial_{xx}\widetilde{E}^\varepsilon(x,t) + \left[1 - \widetilde{E}^\varepsilon(x,t)^2 + G^\varepsilon(x,t)\right]\widetilde{E}^\varepsilon(x,t) = 0,$$
$$\widetilde{E}^\varepsilon(x,0) = E_0(x), \quad \partial_t\widetilde{E}^\varepsilon(x,0) = E_1(x), \quad x \in \overline{\Omega},$$
$$\widetilde{E}^\varepsilon(a,t) = \widetilde{E}^\varepsilon(b,t) = 0, \quad t \geq 0.$$

We remark here that our numerical method and its error bounds can be easily extended to the case when the homogeneous Dirichlet boundary condition is replaced by the period boundary condition.

Choose a mesh size $h := \Delta x = \frac{b-a}{M}$ with M being a positive integer and a time step $\tau := \Delta t > 0$ and denote the grid points and time steps as

$$x_j := a + jh, \quad j = 0, 1, \ldots, M; \quad t_k := k\tau, \quad k = 0, 1, 2, \ldots.$$

Define the index sets

$$\mathcal{T}_M = \{j \mid j = 1, 2, \ldots, M-1\}, \quad \mathcal{T}_M^0 = \{j \mid j = 0, 1, \ldots, M\}.$$

Let $E_j^{\varepsilon,k}$ and $F_j^{\varepsilon,k}$ be the approximations of $E^\varepsilon(x_j, t_k)$ and $F^\varepsilon(x_j, t_k)$, respectively, and denote $E^{\varepsilon,k} = (E_0^{\varepsilon,k}, E_1^{\varepsilon,k}, \ldots, E_M^{\varepsilon,k})^T$, $F^{\varepsilon,k} = (F_0^{\varepsilon,k}, F_1^{\varepsilon,k}, \ldots, F_M^{\varepsilon,k})^T \in \mathbb{R}^{M+1}$ as the numerical solution vectors at $t = t_k$. The finite difference operators are the standard notations as follows:

$$\delta_x^+ E_j^k = \frac{E_{j+1}^k - E_j^k}{h}, \quad \delta_t^+ E_j^k = \frac{E_j^{k+1} - E_j^k}{\tau}, \quad \delta_t^c E_j^k = \frac{E_j^{k+1} - E_j^{k-1}}{2\tau},$$
$$\delta_t^2 E_j^k = \frac{E_j^{k+1} - 2E_j^k + E_j^{k-1}}{\tau^2}, \quad \delta_x^2 E_j^k = \frac{E_{j+1}^k - 2E_j^k + E_{j-1}^k}{h^2}.$$

In this paper, we consider the finite difference discretization of (2.6) as follows,

(2.9a)
$$\delta_t^2 E_j^{\varepsilon,k} = \left(\delta_x^2 - 1 + |E_j^{\varepsilon,k}|^2 - F_j^{\varepsilon,k} - H_j^{\varepsilon,k}\right)\frac{E_j^{\varepsilon,k+1} + E_j^{\varepsilon,k-1}}{2},$$

(2.9b)
$$\varepsilon^2 \delta_t^2 F_j^{\varepsilon,k} = \frac{1}{2}\delta_x^2(F_j^{\varepsilon,k+1} + F_j^{\varepsilon,k-1}) + \varepsilon^2 \delta_t^2 |E_j^{\varepsilon,k}|^2, \quad j \in \mathcal{T}_M, \quad k \geq 1,$$

where we apply an average of the oscillatory potential G^ε over the interval $[t_{k-1}, t_{k+1}]$:

(2.10)
$$H_j^{\varepsilon,k} = \int_{-1}^1 (1 - |s|)G^\varepsilon(x_j, t_k + s\tau)ds, \quad j \in \mathcal{T}_M, \quad k \geq 1.$$

Meanwhile, the boundary and initial conditions are discretized as

(2.11)
$$E_0^{\varepsilon,k} = E_M^{\varepsilon,k} = 0, \quad F_0^{\varepsilon,k} = F_M^{\varepsilon,k} = 0, \quad k \geq 0,$$
$$E_j^{\varepsilon,0} = E_0(x_j), \quad F_j^{\varepsilon,0} = 0, \quad j \in \mathcal{T}_M^0.$$

Next we consider the value of the first step $E_j^{\varepsilon,1}$ and $F_j^{\varepsilon,1}$. By Taylor expansion, we get $E_j^{\varepsilon,1}$ as

(2.12) $\quad E_j^{\varepsilon,1} = E_0(x_j) + \tau E_1(x_j) + \frac{\tau^2}{2}\partial_{tt}E^\varepsilon(x_j,0), \; F_j^{\varepsilon,1} = \frac{\tau^2}{2}\partial_{tt}F^\varepsilon(x_j,0), \; j \in \mathcal{T}_M,$

where by (2.6),

$$\partial_{tt}E^\varepsilon(x,0) = E_0''(x) - E_0(x) - N_0^\varepsilon(x)E_0(x),$$
$$\partial_{tt}F^\varepsilon(x,0) = 2E_1(x)^2 + 2E_0(x)\partial_{tt}E^\varepsilon(x,0).$$

In practical computation, $H_j^{\varepsilon,k}$ in (2.10) can be obtained by solving the wave equation (2.7) via the sine pseudospectral discretization in space followed by integrating in time in phase space *exactly* [8] as

$$H_j^{\varepsilon,k} = \varepsilon^\alpha \sum_{l=1}^{M-1} \widetilde{(\omega_0)}_l \sin\left(\frac{jl\pi}{M}\right) \int_{-1}^1 (1-|s|)\cos\left(\theta_l(t_k+s\tau)\right)ds$$

$$+ \varepsilon^\beta \sum_{l=1}^{M-1} \frac{\widetilde{(\omega_1)}_l}{\theta_l} \sin\left(\frac{jl\pi}{M}\right) \int_{-1}^1 (1-|s|)\sin\left(\theta_l(t_k+s\tau)\right)ds$$

$$= 2\sum_{l=1}^{M-1} \sin\left(\frac{jl\pi}{M}\right)\left[\varepsilon^\alpha \widetilde{(\omega_0)}_l \cos\left(\theta_l t_k\right) + \varepsilon^\beta \frac{\widetilde{(\omega_1)}_l}{\theta_l}\sin\left(\theta_l t_k\right)\right] \int_0^1 \cos\left(\tau\theta_l s\right)(1-s)ds$$

$$= \frac{4}{\tau^2}\sum_{l=1}^{M-1} \frac{1}{\theta_l^2}\sin\left(\frac{jl\pi}{M}\right)\sin^2\left(\frac{\theta_l\tau}{2}\right)\left[\varepsilon^\alpha \widetilde{(\omega_0)}_l \cos\left(\theta_l t_k\right) + \varepsilon^\beta \frac{\widetilde{(\omega_1)}_l}{\theta_l}\sin\left(\theta_l t_k\right)\right],$$

where for $l \in \mathcal{T}_M$,

$$\theta_l = \frac{l\pi}{\varepsilon(b-a)}, \quad \widetilde{(\omega_0)}_l = \frac{2}{M}\sum_{j=1}^{M-1} \omega_0(x_j)\sin\left(\frac{jl\pi}{M}\right), \quad \widetilde{(\omega_1)}_l = \frac{2}{M}\sum_{j=1}^{M-1} \omega_1(x_j)\sin\left(\frac{jl\pi}{M}\right).$$

We remark here that our numerical method (2.9) is unconditionally stable due to the fact that it is implicit.

2.3. **Main results.** For simplicity of notation, we denote

$$\alpha^* := \min\{1, \alpha, 1+\beta\} \in [0,1].$$

Let $T^* > 0$ be the maximum common existence time of solutions to the KGZ system (2.6) and the KGE-OP (2.8). Then for $0 < T < T^*$, according to the known results in [1, 25, 28, 31], we can assume the exact solution $(E^\varepsilon(x,t), F^\varepsilon(x,t))$ of the KGZ system (2.6) and the solution $\widetilde{E}^\varepsilon(x,t)$ of the KGE-OP (2.8) are smooth enough and satisfy

$$\|E^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\partial_t E^\varepsilon\|_{W^{2,\infty}(\Omega)} + \|\partial_t^2 E^\varepsilon\|_{W^{2,\infty}(\Omega)} + \varepsilon\|\partial_t^3 E^\varepsilon\|_{W^{2,\infty}(\Omega)} \lesssim 1,$$

(A) $\|\widetilde{E}^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\partial_t \widetilde{E}^\varepsilon\|_{W^{2,\infty}(\Omega)} + \|\partial_t^2 \widetilde{E}^\varepsilon\|_{W^{2,\infty}(\Omega)} \lesssim 1, \quad \|\partial_t^3 \widetilde{E}^\varepsilon\|_{L^\infty(\Omega)} \lesssim \frac{1}{\varepsilon^{1-\alpha^*}},$

$$\|F^\varepsilon\|_{W^{4,\infty}(\Omega)} \lesssim \varepsilon, \quad \|\partial_t F^\varepsilon\|_{W^{4,\infty}(\Omega)} + \|\partial_{tt} F^\varepsilon\|_{W^{2,\infty}(\Omega)} + \varepsilon\|\partial_t^3 F^\varepsilon\|_{W^{2,\infty}(\Omega)} \lesssim 1.$$

Furthermore, we assume that the initial data satisfies

(B) $\qquad \|\mathrm{E}_0\|_{\mathrm{W}^{5,\infty}(\Omega)} + \|\mathrm{E}_1\|_{\mathrm{W}^{5,\infty}(\Omega)} + \|\omega_0\|_{\mathrm{W}^{3,\infty}(\Omega)} + \|\omega_1\|_{\mathrm{W}^{3,\infty}(\Omega)} \lesssim 1.$

It can be concluded from (2.2) and assumption (B) that

(2.13) $\qquad \|\partial_t^m G^\varepsilon\|_{W^{3,\infty}(\Omega)} \lesssim \varepsilon^{\alpha^*-m}, \qquad m = 0,1,2,3.$

To measure the error between the exact solution and the numerical solution of the KGZ system, we introduce some notation. Denote

$$X_M = \{v = (v_j)_{j\in\mathcal{T}_M^0} \mid v_0 = v_M = 0\} \subseteq \mathbb{R}^{M+1}.$$

The norms and inner products over X_M are defined as

$$\|u\|^2 = h \sum_{j=1}^{M-1} |u_j|^2, \quad \|u\|_4^4 = h \sum_{j=1}^{M-1} |u_j|^4, \quad \|\delta_x^+ u\|^2 = h \sum_{j=0}^{M-1} |\delta_x^+ u_j|^2,$$

$$\|u\|_\infty = \sup_{j \in \mathcal{T}_M^0} |u_j|, \quad (u,v) = h \sum_{j=1}^{M-1} u_j v_j, \quad \langle \delta_x^+ u, \delta_x^+ v \rangle = h \sum_{j=0}^{M-1} \delta_x^+ u_j \delta_x^+ v_j.$$

Then it is easy to get

$$(2.14) \quad (-\delta_x^2 u, v) = \langle \delta_x^+ u, \delta_x^+ v \rangle, \quad ((-\delta_x^2)^{-1} u, v) = (u, (-\delta_x^2)^{-1} v), \quad u, v \in X_M.$$

Define the error functions $e^{\varepsilon,k}$, $f^{\varepsilon,k}$ as

$$e_j^{\varepsilon,k} = E^\varepsilon(x_j, t_k) - E_j^{\varepsilon,k}, \quad f_j^{\varepsilon,k} = F^\varepsilon(x_j, t_k) - F_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad 0 \le k \le \frac{T}{\tau}.$$

Then we have the following error estimates for the finite difference discretization (2.9) with (2.10)–(2.12).

Theorem 2.1. *Under assumptions* (A) *and* (B), *there exist* $h_0 > 0$ *and* $\tau_0 > 0$ *sufficiently small and independent of* ε *such that, when* $0 < h \le h_0$, $0 < \tau \le \tau_0$, *the scheme* (2.9) *with* (2.10)–(2.12) *satisfies the following error estimates.*

$$(2.15) \qquad \|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad 0 \le k \le \frac{T}{\tau}, \quad \varepsilon \in (0,1],$$

$$(2.16) \qquad \|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau \varepsilon^{\alpha^*} + \varepsilon.$$

Thus by taking the minimum, we have the uniform ε-*independent error bound for* $\varepsilon \in (0,1]$ *and* $0 \le k \le \frac{T}{\tau}$,

$$(2.17) \quad \|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\| + \|f^{\varepsilon,k}\| \lesssim h^2 + \max_{0 < \varepsilon \le 1} \min \left\{ \frac{\tau^2}{\varepsilon}, \tau^2 + \tau \varepsilon^{\alpha^*} + \varepsilon \right\} \lesssim h^2 + \tau.$$

3. Error analysis

To prove Theorem 2.1, we will get the error bound (2.15) by using the energy method and (2.16) via the limiting equation KGE-OP (2.8), which can be displayed in the following diagram [3–5, 8, 15, 21]:

$$(E^{\varepsilon,k}, F^{\varepsilon,k}) \xrightarrow{O(h^2 + \tau^2 + \tau \varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})} (\widetilde{E}^\varepsilon, 0)$$

$$O(h^2 + \tau^2/\varepsilon) \searrow \qquad \downarrow O(\varepsilon)$$

$$(E^\varepsilon, F^\varepsilon).$$

To simplify notation, for a function $V(x,t)$, and a grid function $V^k \in X_M$ ($k \ge 0$), we denote for $k \ge 1$,

$$(\llbracket V \rrbracket)(x, t_k) = \frac{V(x, t_{k+1}) + V(x, t_{k-1})}{2}, \quad x \in \overline{\Omega}; \quad \llbracket V \rrbracket_j^k = \frac{V_j^{k+1} + V_j^{k-1}}{2}, \quad j \in \mathcal{T}_M^0.$$

3.1. **Solvability of the scheme** (2.9)–(2.11).

Lemma 3.1 (Solvability of the difference equations). *For any given E^k, F^k, E^{k-1}, $F^{k-1} \in X_M$, denote $C^k = \|F^k\| + \|H^{\varepsilon,k}\| + \|E^k\|^2 + \|\delta_x^+ E^k\|^2$, there exists $\tau_s > 0$ which depends on C^k such that when $\tau < \tau_s$, there exists a unique solution of the discretization* (2.9).

Proof. We first prove the existence of a solution for the first equation $(2.9a)$. For any E^k, E^{k-1}, $F^k \in X_M$, we rewrite $(2.9a)$ as

$$(3.1) \qquad \llbracket E \rrbracket^k = E^k + \frac{\tau^2}{2} U^k(\llbracket E \rrbracket^k),$$

where $U^k : X_M \to X_M$ defined as

$$(U^k(u))_j = (\delta_x^2 - 1 + |E_j^k|^2 - F_j^k - H_j^{\varepsilon,k}) u_j, \quad j \in \mathcal{T}_M, \quad k \geq 1.$$

Denote the map $V^k : X_M \to X_M$ as

$$V^k(u) = u - E^k - \frac{\tau^2}{2} U^k(u), \quad u \in X_M,$$

and it is obvious that V^k is continuous from X_M to X_M. Moreover,

$$(V^k(u), u) = \|u\|^2 - (E^k, u) + \frac{\tau^2}{2}(\|\delta_x^+ u\|^2 + \|u\|^2) + \frac{\tau^2}{2}(F^k + H^{\varepsilon,k} - |E^k|^2, u^2).$$

Applying the Cauchy inequality, the Sobolev inequality [40], and Young's inequality, we can obtain

$$
\begin{aligned}
|(F^k + H^{\varepsilon,k} - |E^k|^2, u^2)| &\leq \|u\|_4^2 (\|F^k\| + \|H^{\varepsilon,k}\| + \|E^k\|_4^2) \\
&\leq C \|\delta_x^+ u\|^{1/2} \|u\|^{3/2} (\|F^k\| + \|H^{\varepsilon,k}\| + \|\delta_x^+ E^k\|^{1/2} \|E^k\|^{3/2}) \\
&\leq C C^k \|\delta_x^+ u\|^{1/2} \|u\|^{3/2} \\
(3.2) \qquad &\leq \|\delta_x^+ u\|^2 + (CC^k)^{4/3} \|u\|^2.
\end{aligned}
$$

Hence

$$(V^k(u), u) \geq \left[\left(1 - \frac{\tau^2}{2}(CC^k)^{4/3}\right) \|u\| - \|E^k\| \right] \|u\|.$$

Setting $\tau_s = (CC^k)^{-2/3}$, when $\tau < \tau_s$, one has

$$\lim_{\|u\| \to \infty} \frac{(V^k(u), u)}{\|u\|} = \infty,$$

which implies that there exists a solution u^* such that $V^k(u^*) = 0$ by applying the Brouwer fixed point theorem [4, 23]. Thus $(2.9a)$ is solvable.

Next we prove the uniqueness for $(2.9a)$. Suppose there exist two solutions $E^{(1)}$, $E^{(2)} \in X_M$ satisfying the equation $(2.9a)$, i.e., for $j \in \mathcal{T}_M$,

$$(3.3) \qquad \frac{E_j^{(1)} - 2E_j^k + E_j^{k-1}}{\tau^2} = \left[\delta_x^2 - 1 + |E_j^k|^2 - F_j^k - H_j^{\varepsilon,k} \right] \frac{E_j^{(1)} + E_j^{k-1}}{2},$$

$$(3.4) \qquad \frac{E_j^{(2)} - 2E_j^k + E_j^{k-1}}{\tau^2} = \left[\delta_x^2 - 1 + |E_j^k|^2 - F_j^k - H_j^{\varepsilon,k} \right] \frac{E_j^{(2)} + E_j^{k-1}}{2}.$$

Denote $u = E^{(1)} - E^{(2)} \in X_M$ and subtract (3.4) from (3.3), we get

$$(3.5) \qquad u_j = \frac{\tau^2}{2} \left[\delta_x^2 - 1 + |E_j^k|^2 - F_j^k - H_j^{\varepsilon,k} \right] u_j,$$

which implies that

$$\|u\|^2 + \frac{\tau^2}{2}(\|\delta_x^+ u\|^2 + \|u\|^2) = \frac{\tau^2}{2}\left(|E^k|^2 - F^k - H^{\varepsilon,k}, u^2\right).$$

Noticing (3.2), when $\tau < \tau_s$, one obtains

$$\|u\|^2 + \frac{\tau^2}{2}(\|\delta_x^+ u\|^2 + \|u\|^2) \leq \frac{\tau^2}{2}(\|\delta_x^+ u\|^2 + (CC^k)^{4/3}\|u\|^2) \leq \frac{\tau^2}{2}\|\delta_x^+ u\|^2 + \frac{1}{2}\|u\|^2,$$

which yields that $u = 0$. Thus the solution of (2.9a) is unique.

Finally we prove (2.9b) is uniquely solvable. Multiplying (2.9b) by τ^2, we see that the coefficient matrix for the unknown $F^{k+1} \in X_M$, of order $(M-1)\times(M-1)$, is

$$A^k = \begin{pmatrix} \varepsilon^2 + \frac{\tau^2}{h^2} & -\frac{\tau^2}{2h^2} & 0 & \cdots & 0 \\ -\frac{\tau^2}{2h^2} & \varepsilon^2 + \frac{\tau^2}{h^2} & -\frac{\tau^2}{2h^2} & \cdots & 0 \\ 0 & -\frac{\tau^2}{2h^2} & \varepsilon^2 + \frac{\tau^2}{h^2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \varepsilon^2 + \frac{\tau^2}{h^2} \end{pmatrix},$$

which is a symmetric positive definite matrix and it is invertible by Gerschgorin [20] for any τ, h, $\varepsilon > 0$. The proof is completed. □

Combining assumption (A), (B), and (2.17), and using induction technique, we can get that there exists a constant $C > 0$ independent of ε, when $h < h_0$, $\tau < \tau_0$, the solution of the difference equation (2.9) satisfies

$$\|F^{\varepsilon,k}\| + \|H^{\varepsilon,k}\| + \|E^{\varepsilon,k}\|^2 + \|\delta_x^+ E^{\varepsilon,k}\|^2 \leq C.$$

Thus by applying Lemma 3.1, we can get the solvability of (2.9) for any $k \geq 1$.

Corollary 3.1. *There exists $h_0 > 0$, $\tau_0 > 0$ independent of ε, such that the difference discretization* (2.9) *with* (2.11)–(2.12) *is uniquely solvable for $1 \leq k \leq \frac{T}{\tau} - 1$.*

3.2. **An error bound via the energy method.** To bound the numerical solution, following the idea in [2–4, 6, 36], we truncate the nonlinearity to a global Lipschitz function with compact support, then the error can be achieved if the numerical solution is close to the bounded exact solution. Choose a smooth function $\rho(s) \in C^\infty(\mathbb{R})$ such that

$$\rho(s) = \begin{cases} 1, & |s| \leq 1, \\ \in [0,1], & |s| \leq 2, \\ 0, & |s| \geq 2, \end{cases}$$

and set

$$M_0 = \max\left\{ \sup_{\varepsilon \in (0,1]} \|E^\varepsilon\|_{L^\infty(\Omega_T)},\ \sup_{\varepsilon \in (0,1]} \|\widetilde{E}^\varepsilon\|_{L^\infty(\Omega_T)} \right\},$$

where $\Omega_T = \Omega \times [0,T]$, which is well defined by assumption (A). For $s, y_1, y_2 \in \mathbb{R}$, define

(3.6) $$\rho_B(s) = s^2\rho(s/B), \quad B = M_0 + 1$$

and

$$g(y_1, y_2) = \frac{1}{2}\int_0^1 \rho'_B(sy_1 + (1-s)y_2)ds.$$

Then $\rho_B(s)$ is globally Lipschitz and there exists $C_B > 0$, such that

(3.7) $|\rho_B(s_1) - \rho_B(s_2)| \le C_B |s_1 - s_2| \quad \forall s_1, s_2 \in \mathbb{R}.$

Set $\hat{E}^{\varepsilon,0} = E^{\varepsilon,0}$, $\hat{F}^{\varepsilon,0} = F^{\varepsilon,0}$, $\hat{E}^{\varepsilon,1} = E^{\varepsilon,1}$, $\hat{F}^{\varepsilon,1} = F^{\varepsilon,1}$, and define $\hat{E}^{\varepsilon,k}$, $\hat{F}^{\varepsilon,k} \in X_M$ for $k \ge 1$ as follows:

$$\delta_t^2 \hat{E}_j^{\varepsilon,k} = (\delta_x^2 - 1 - H_j^{\varepsilon,k})[\![\hat{E}^\varepsilon]\!]_j^k + \left(\rho_B(\hat{E}_j^{\varepsilon,k}) - \hat{F}_j^{\varepsilon,k}\right) g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}),$$

(3.8)

$$\varepsilon^2 \delta_t^2 \hat{F}_j^{\varepsilon,k} = \frac{1}{2} \delta_x^2 (\hat{F}_j^{\varepsilon,k+1} + \hat{F}_j^{\varepsilon,k-1}) + \varepsilon^2 \delta_t^2 \rho_B(\hat{E}_j^{\varepsilon,k}).$$

Here $(\hat{E}^{\varepsilon,k}, \hat{F}^{\varepsilon,k})$ can be viewed as another approximation of $(E^\varepsilon(x_j, t_k), F^\varepsilon(x_j, t_k))$. Applying similar techniques in the proof for Lemma 3.1, we can get that (3.8) is uniquely solvable when τ, h are sufficiently small.

Define the error function $\hat{e}^{\varepsilon,k}$, $\hat{f}^{\varepsilon,k} \in X_M$ as

$$\hat{e}_j^{\varepsilon,k} = E^\varepsilon(x_j, t_k) - \hat{E}_j^{\varepsilon,k}, \quad \hat{f}_j^{\varepsilon,k} = F^\varepsilon(x_j, t_k) - \hat{F}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M^0, \quad k \ge 0.$$

Regarding the error bounds on $(\hat{e}^{\varepsilon,k}, \hat{f}^{\varepsilon,k})$, we have the following estimates.

Theorem 3.2. *Under assumption* (A)*, there exists $\tau_1 > 0$ sufficiently small, when $0 < \tau \le \tau_1$, the scheme* (3.8) *satisfies the following error estimates:*

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad 0 \le k \le \frac{T}{\tau}, \quad 0 < \varepsilon \le 1.$$

In order to prove it, we introduce the local truncation error $\hat{\xi}_j^{\varepsilon,k}$, $\hat{\eta}_j^{\varepsilon,k} \in X_M$ as

(3.9)

$$\hat{\xi}_j^{\varepsilon,k} = \delta_t^2 E^\varepsilon(x_j, t_k) - (\delta_x^2 - 1 - H_j^{\varepsilon,k})([\![E^\varepsilon]\!])(x_j, t_k)$$
$$\quad - \left[\rho_B(E^\varepsilon(x_j, t_k)) - F^\varepsilon(x_j, t_k)\right] g\left(E^\varepsilon(x_j, t_{k+1}), E^\varepsilon(x_j, t_{k-1})\right)$$
$$\quad = \delta_t^2 E^\varepsilon(x_j, t_k) - \left[\delta_x^2 - 1 + |E^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} - F^\varepsilon(x_j, t_k)\right]([\![E^\varepsilon]\!])(x_j, t_k),$$
$$\hat{\eta}_j^{\varepsilon,k} = \varepsilon^2 \delta_t^2 F^\varepsilon(x_j, t_k) - \delta_x^2([\![F^\varepsilon]\!])(x_j, t_k) - \varepsilon^2 \delta_t^2 \rho_B(E^\varepsilon(x_j, t_k))$$
$$\quad = \varepsilon^2 \delta_t^2 F^\varepsilon(x_j, t_k) - \delta_x^2([\![F^\varepsilon]\!])(x_j, t_k) - \varepsilon^2 \delta_t^2 |E^\varepsilon(x_j, t_k)|^2, \quad k \ge 1.$$

For the local truncation error, we have the following error bounds.

Lemma 3.3. *Under assumption* (A)*, we have for $j \in \mathcal{T}_M$,*

$$|\hat{\xi}_j^{\varepsilon,k}| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \ |\hat{\eta}_j^{\varepsilon,k}| \lesssim h^2 + \tau^2, \ 1 \le k \le \frac{T}{\tau} - 1; \ |\delta_t^c \hat{\eta}_j^{\varepsilon,k}| \lesssim h^2 + \frac{\tau^2}{\varepsilon}, \ 2 \le k \le \frac{T}{\tau} - 2.$$

Proof. By (2.6) and Taylor expansion, we have

$$\delta_t^2 E^\varepsilon(x_j, t_k) = \sum_{m=\pm 1} \int_0^1 (1-s) \partial_{tt} E^\varepsilon(x_j, t_k + ms\tau) ds$$

$$= \int_{-1}^1 (1 - |s|) \left(\partial_{xx} E^\varepsilon - E^\varepsilon + (E^\varepsilon)^3 - E^\varepsilon F^\varepsilon - E^\varepsilon G^\varepsilon\right)(x_j, t_k + s\tau) ds$$

$$= \partial_{xx} E^\varepsilon(x_j, t_k) - E^\varepsilon(x_j, t_k) + E^\varepsilon(x_j, t_k)^3 - E^\varepsilon(x_j, t_k) F^\varepsilon(x_j, t_k)$$

$$\quad + \frac{\tau^2}{6} \int_{-1}^1 (1 - |s|)^3 \partial_{tt} \left(\partial_{xx} E^\varepsilon - E^\varepsilon + (E^\varepsilon)^3 - E^\varepsilon F^\varepsilon\right)(x_j, t_k + s\tau) ds$$

$$\quad - \int_{-1}^1 (1 - |s|) E^\varepsilon(x_j, t_k + s\tau) G^\varepsilon(x_j, t_k + s\tau) ds.$$

Similarly, by Taylor expansion, one can easily get that

$$\left[\delta_x^2 - 1 + |E^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} - F^\varepsilon(x_j, t_k)\right] ([E^\varepsilon])(x_j, t_k)$$

$$= \partial_{xx} E^\varepsilon(x_j, t_k) + \left[|E^\varepsilon(x_j, t_k)|^2 - 1 - H_j^{\varepsilon,k} - F^\varepsilon(x_j, t_k)\right] E^\varepsilon(x_j, t_k)$$

$$+ \frac{h^2}{6} \int_{-1}^1 (1 - |s|)^3 ([\partial_x^4 E^\varepsilon])(x_j + sh, t_k) ds$$

$$+ \frac{\tau^2}{2} \int_{-1}^1 (1 - |s|) \partial_x^2 \partial_t^2 E^\varepsilon(x_j, t_k + s\tau) ds$$

$$+ \frac{\tau^2}{2} \left[|E^\varepsilon(x_j, t_k)|^2 - 1 - H_j^{\varepsilon,k} - F^\varepsilon(x_j, t_k)\right] \int_{-1}^1 (1 - |s|) \partial_{tt} E^\varepsilon(x_j, t_k + s\tau) ds.$$

Note that by (2.10), we have

$$\int_{-1}^1 (1 - |s|) E^\varepsilon(x_j, t_k + s\tau) G^\varepsilon(x_j, t_k + s\tau) \, ds - E^\varepsilon(x_j, t_k) H_j^{\varepsilon,k}$$

$$= \tau E_t^\varepsilon(x_j, t_k) \int_0^1 s(1 - s) \left[G^\varepsilon(x_j, t_k + s\tau) - G^\varepsilon(x_j, t_k - s\tau)\right] ds + A_1$$

$$(3.10) \qquad = \tau^2 E_t^\varepsilon(x_j, t_k) \int_0^1 s(1 - s) \int_{-s}^s \partial_t G^\varepsilon(x_j, t_k + \theta\tau) \, d\theta ds + A_1,$$

where

$$A_1 = \tau^2 \int_{-1}^1 (1 - |s|) G^\varepsilon(x_j, t_k + s\tau) \int_0^s (s - \theta) \partial_{tt} E^\varepsilon(x_j, t_k + \theta\tau) d\theta ds.$$

Accordingly, by assumption (A) and (2.13), we deduce that

$$|\hat{\xi}_j^{\varepsilon,k}| \lesssim h^2 \|\partial_x^4 E^\varepsilon\|_{L^\infty} + \tau^2 \Big[\|\partial_t^2 E^\varepsilon\|_{L^\infty}(1 + \|G^\varepsilon\|_{L^\infty} + \|F^\varepsilon\|_{L^\infty} + \|E^\varepsilon\|_{L^\infty}^2)$$

$$+ \|\partial_x^2 \partial_t^2 E^\varepsilon\|_{L^\infty} + \|\partial_t E^\varepsilon\|_{L^\infty}(\|\partial_t G^\varepsilon\|_{L^\infty} + \|\partial_t F^\varepsilon\|_{L^\infty})$$

$$+ \|E^\varepsilon\|_{L^\infty}(\|\partial_t E^\varepsilon\|_{L^\infty}^2 + \|\partial_t^2 F^\varepsilon\|_{L^\infty})\Big]$$

$$\lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Similar expansion gives

$$\hat{\eta}_j^{\varepsilon,k} = \frac{\varepsilon^2 \tau^2}{6} \int_{-1}^1 (1 - |s|)^3 \left[\partial_t^4 F^\varepsilon(x_j, t_k + s\tau) - \partial_t^4 |E^\varepsilon|^2(x_j, t_k + s\tau)\right] ds$$

$$- \frac{\tau^2}{2} \int_{-1}^1 (1 - |s|) \partial_x^2 F_{tt}^\varepsilon(x_j, t_k + s\tau) ds - \frac{h^2}{6} \int_{-1}^1 (1 - |s|)^3 ([\partial_x^4 F^\varepsilon])(x_j + sh, t_k) ds,$$

which implies

$$|\hat{\eta}_j^{\varepsilon,k}| \lesssim h^2 \|\partial_x^4 F^\varepsilon\|_{L^\infty} + \tau^2 \left(\|\partial_x^2 \partial_t^2 F^\varepsilon\|_{L^\infty} + \varepsilon^2 \|\partial_t^4 F^\varepsilon\|_{L^\infty} + \varepsilon^2 \|\partial_t^4 |E^\varepsilon|^2\|_{L^\infty}\right)$$

$$\lesssim h^2 + \tau^2, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Applying δ_t^c to $\hat{\eta}_j^{\varepsilon,k}$, one can deduce that

$$|\delta_t^c \hat{\eta}_j^{\varepsilon,k}| \lesssim h^2 \|\partial_x^4 \partial_t F^\varepsilon\|_{L^\infty} + \tau^2 (\|\partial_x^2 \partial_t^3 F^\varepsilon\|_{L^\infty} + \varepsilon^2 \|\partial_t^5 F^\varepsilon\|_{L^\infty} + \varepsilon^2 \|\partial_t^5 |E^\varepsilon|^2\|_{L^\infty})$$

$$\lesssim h^2 + \frac{\tau^2}{\varepsilon}, \quad j \in \mathcal{T}_M, \quad 2 \le k \le \frac{T}{\tau} - 2.$$

Thus the proof is completed. $\hfill\square$

For the initial step, we have the following estimates.

Lemma 3.4. *Under assumption* (A), *the initial and first step errors of the discretization* (2.12) *satisfy*

$$\hat{e}_j^{\varepsilon,0} = \hat{f}_j^{\varepsilon,0} = 0, \quad |\hat{e}_j^{\varepsilon,1}| + |\delta_x^+ \hat{e}_j^{\varepsilon,1}| + |\hat{f}_j^{\varepsilon,1}| \lesssim \frac{\tau^3}{\varepsilon}, \quad |\delta_t^+ \hat{e}_j^{\varepsilon,0}| + |\delta_t^+ \hat{f}_j^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon}.$$

Proof. By the definition of $\hat{E}_j^{\varepsilon,1}$, one can derive that

$$|\hat{e}_j^{\varepsilon,1}| = \frac{\tau^3}{2} \left| \int_0^1 (1-s)^2 \partial_t^3 E^\varepsilon(x_j, s\tau) ds \right| \lesssim \tau^3 \|\partial_t^3 E^\varepsilon\|_{L^\infty} \lesssim \frac{\tau^3}{\varepsilon},$$

which implies that $|\delta_t^+ \hat{e}_j^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon}$. Similarly, $|\delta_x^+ \hat{e}_j^{\varepsilon,1}| \lesssim \tau^3 \|\partial_x \partial_t^3 E^\varepsilon\|_{L^\infty} \lesssim \frac{\tau^3}{\varepsilon}$. It follows from (2.12) and assumption (A) that

$$|\hat{f}_j^{\varepsilon,1}| = \frac{\tau^3}{2} \left| \int_0^1 (1-s)^2 \partial_t^3 F^\varepsilon(x_j, s\tau) ds \right| \lesssim \tau^3 \|\partial_t^3 F^\varepsilon\|_{L^\infty} \lesssim \frac{\tau^3}{\varepsilon}.$$

Recalling that $\hat{f}_j^{\varepsilon,0} = 0$, we can get $|\delta_t^+ \hat{f}_j^{\varepsilon,0}| \lesssim \frac{\tau^2}{\varepsilon}$, which completes the proof. $\hfill\square$

Subtracting (3.8) from (3.9), we have the error equations

$$(3.11) \quad \begin{aligned} \delta_t^2 \hat{e}_j^{\varepsilon,k} &= (\delta_x^2 - 1 - H_j^{\varepsilon,k}) \frac{\hat{e}_j^{\varepsilon,k+1} + \hat{e}_j^{\varepsilon,k-1}}{2} + r_j^k + \hat{\xi}_j^{\varepsilon,k}, \\ \varepsilon^2 \delta_t^2 \hat{f}_j^{\varepsilon,k} &= \frac{1}{2} \delta_x^2 (\hat{f}_j^{\varepsilon,k+1} + \hat{f}_j^{\varepsilon,k-1}) + \varepsilon^2 \delta_t^2 p_j^k + \hat{\eta}_j^{\varepsilon,k}, \ j \in \mathcal{T}_M, \ 1 \le k < \frac{T}{\tau}, \end{aligned}$$

where

$$(3.12) \quad \begin{aligned} r_j^k &= \left(|E^\varepsilon|^2 - F^\varepsilon \right) (\![E^\varepsilon]\!)(x_j, t_k) - \left(\rho_B(\hat{E}_j^{\varepsilon,k}) - \hat{F}_j^{\varepsilon,k} \right) g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}), \\ p_j^k &= |E^\varepsilon(x_j, t_k)|^2 - \rho_B(\hat{E}_j^{\varepsilon,k}). \end{aligned}$$

By the property of ρ_B (cf. (3.7)), one can easily get that

$$(3.13) \quad |p_j^k| = |\rho_B(E^\varepsilon(x_j, t_k)) - \rho_B(\hat{E}_j^{\varepsilon,k})| \le C_B |\hat{e}_j^{\varepsilon,k}|, \quad j \in \mathcal{T}_M, \quad 0 \le k \le \frac{T}{\tau}.$$

By the definition of $g(\cdot, \cdot)$, and noticing that

$$(\![E^\varepsilon]\!)(x_j, t_k) = g\left(E^\varepsilon(x_j, t_{k+1}), E^\varepsilon(x_j, t_{k-1}) \right),$$

it is known from [13] that for $j \in \mathcal{T}_M$, $1 \le k \le \frac{T}{\tau} - 1$,

$$(3.14) \quad \left| g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}) \right| \lesssim 1, \quad \left| (\![E^\varepsilon]\!)(x_j, t_k) - g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}) \right| \lesssim \sum_{l=k\pm 1} |\hat{e}_j^{\varepsilon,l}|.$$

Proof of Theorem 3.2. Multiplying both sides of the first equation of (3.11) by $2\tau\delta_t^c\hat{e}_j^{\varepsilon,k}$, summing together for $j \in \mathcal{T}_M$, we obtain for $1 \le k \le \frac{T}{\tau} - 1$,

$$(3.15)$$

$$\|\delta_t^+\hat{e}^{\varepsilon,k}\|^2 - \|\delta_t^+\hat{e}^{\varepsilon,k-1}\|^2 + \frac{1}{2}(\|\delta_x^+\hat{e}^{\varepsilon,k+1}\|^2 - \|\delta_x^+\hat{e}^{\varepsilon,k-1}\|^2 + \|\hat{e}^{\varepsilon,k+1}\|^2 - \|\hat{e}^{\varepsilon,k-1}\|^2)$$

$$= (-H^{\varepsilon,k}[\![\hat{e}^\varepsilon]\!]^k + r^k + \hat{\xi}^{\varepsilon,k}, \hat{e}^{\varepsilon,k+1} - \hat{e}^{\varepsilon,k-1}).$$

For analyzing the second equation of (3.11), we introduce $\hat{u}^{\varepsilon,k+1/2} \in X_M$ by

$$-\delta_x^2\hat{u}_j^{\varepsilon,k+1/2} = \delta_t^+(\hat{f}_j^{\varepsilon,k} - p_j^k).$$

Multiplying both sides of the second equation of (3.11) by $\tau(\hat{u}_j^{\varepsilon,k+1/2} + \hat{u}_j^{\varepsilon,k-1/2})$, summing together for $j \in \mathcal{T}_M$, we have

$$(3.16)$$

$$\varepsilon^2(\|\delta_x^+\hat{u}^{\varepsilon,k+1/2}\|^2 - \|\delta_x^+\hat{u}^{\varepsilon,k-1/2}\|^2) + \frac{1}{2}(\|\hat{f}^{\varepsilon,k+1}\|^2 - \|\hat{f}^{\varepsilon,k-1}\|^2)$$

$$= ([\![\hat{f}^\varepsilon]\!]^k, p^{k+1} - p^{k-1}) + \tau(\hat{\eta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+1/2} + \hat{u}^{\varepsilon,k-1/2}), \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Introduce a discrete "energy"

$$(3.17)$$

$$\mathcal{A}^k = \|\delta_t^+\hat{e}^{\varepsilon,k}\|^2 + \frac{1}{2}(\|\hat{e}^{\varepsilon,k}\|^2 + \|\hat{e}^{\varepsilon,k+1}\|^2 + \|\delta_x^+\hat{e}^{\varepsilon,k}\|^2 + \|\delta_x^+\hat{e}^{\varepsilon,k+1}\|^2)$$

$$+ \varepsilon^2\|\delta_x^+\hat{u}^{\varepsilon,k+1/2}\|^2 + \frac{1}{2}(\|\hat{f}^{\varepsilon,k+1}\|^2 + \|\hat{f}^{\varepsilon,k}\|^2), \quad 0 \le k \le \frac{T}{\tau} - 1.$$

Combining (3.15) and (3.16), we get for $1 \le k \le \frac{T}{\tau} - 1$,

$$(3.18)$$

$$\mathcal{A}^k - \mathcal{A}^{k-1} = (-H^{\varepsilon,k}[\![\hat{e}^\varepsilon]\!]^k + r^k + \hat{\xi}^{\varepsilon,k}, \hat{e}^{\varepsilon,k+1} - \hat{e}^{\varepsilon,k-1})$$

$$+ ([\![\hat{f}^\varepsilon]\!]^k, p^{k+1} - p^{k-1}) + \tau(\hat{\eta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+1/2} + \hat{u}^{\varepsilon,k-1/2}).$$

Now we estimate the terms in (3.18), respectively. By the definition of r^k, it can be derived that

$$(3.19)$$

$$r_j^k = \left(|E^\varepsilon(x_j, t_k)|^2 - F^\varepsilon(x_j, t_k)\right)\left(([\![E^\varepsilon]\!])(x_j, t_k) - g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1})\right)$$

$$+ g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1})\left(p_j^k - \hat{f}_j^{\varepsilon,k}\right).$$

In view of assumption (A), (3.13), and (3.14), we can get that

$$(3.20) \qquad |r_j^k| \lesssim |\hat{e}_j^{\varepsilon,k+1}| + |\hat{e}_j^{\varepsilon,k}| + |\hat{e}_j^{\varepsilon,k-1}| + |\hat{f}_j^{\varepsilon,k}|.$$

This implies that

$$(-H^{\varepsilon,k}[\![\hat{e}^\varepsilon]\!]^k + r^k + \hat{\xi}^{\varepsilon,k}, \hat{e}^{\varepsilon,k+1} - \hat{e}^{\varepsilon,k-1})$$

$$= \tau(-H^{\varepsilon,k}[\![\hat{e}^\varepsilon]\!]^k + r^k + \hat{\xi}^{\varepsilon,k}, \delta_t^+\hat{e}^{\varepsilon,k} + \delta_t^+\hat{e}^{\varepsilon,k-1})$$

$$\lesssim \tau(1 + \|H^{\varepsilon,k}\|_\infty)(\|r^k\|^2 + \|\hat{\xi}^{\varepsilon,k}\|^2 + \sum_{l=k\pm1}\|\hat{e}^{\varepsilon,l}\|^2 + \sum_{l=k-1}^{k}\|\delta_t^+\hat{e}^{\varepsilon,l}\|^2)$$

$$(3.21) \quad \lesssim \tau(\|\hat{\xi}^{\varepsilon,k}\|^2 + \mathcal{A}^k + \mathcal{A}^{k-1}).$$

It can be easily seen from (3.14) and assumption (A) that

$$p^{k+1} - p^{k-1} = E^\varepsilon(x_j, t_{k+1})^2 - E^\varepsilon(x_j, t_{k-1})^2 - 4\tau g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1})\delta_t^c \hat{E}_j^{\varepsilon,k}$$

$$= 2(([E^\varepsilon])(x_j, t_k) - g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}))(E^\varepsilon(x_j, t_{k+1}) - E^\varepsilon(x_j, t_{k-1}))$$

$$+ 2g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1})(\hat{e}_j^{\varepsilon,k+1} - \hat{e}_j^{\varepsilon,k-1})$$

$$\lesssim \tau \|E_t^\varepsilon\|_{L^\infty}(|\hat{e}_j^{\varepsilon,k+1}| + |\hat{e}_j^{\varepsilon,k-1}|) + \tau(|\delta_t^+ \hat{e}_j^{\varepsilon,k}| + |\delta_t^+ \hat{e}_j^{\varepsilon,k-1}|),$$

which yields

$$([\![\hat{f}^\varepsilon]\!]^k, p^{k+1} - p^{k-1}) \lesssim \tau\Big[\sum_{l=k-1}^{k} \|\delta_t^+ \hat{e}^{\varepsilon,l}\|^2 + \sum_{l=k\pm 1} (\|\hat{e}^{\varepsilon,l}\|^2 + \|\hat{f}^{\varepsilon,l}\|^2)\Big]$$

$$(3.22) \qquad\qquad \lesssim \tau(\mathcal{A}^k + \mathcal{A}^{k-1}), \quad 1 \le k \le \frac{T}{\tau} - 1.$$

Hence it can be concluded from (3.18), (3.21), and (3.22) that

$$(3.23) \quad \mathcal{A}^k - \mathcal{A}^{k-1} - \tau(\hat{\eta}^{\varepsilon,k}, \hat{u}^{\varepsilon,k+1/2} + \hat{u}^{\varepsilon,k-1/2}) \lesssim \tau\left(\|\hat{\xi}^{\varepsilon,k}\|^2 + \mathcal{A}^k + \mathcal{A}^{k-1}\right).$$

Applying (2.14), the Sobolev inequality, and the Cauchy inequality, we obtain

$$-\frac{\mathcal{A}^k}{4} + \tau \sum_{l=1}^{k} \left(\hat{\eta}^{\varepsilon,l}, \hat{u}^{\varepsilon,l+1/2} + \hat{u}^{\varepsilon,l-1/2}\right)$$

$$= -\frac{\mathcal{A}^k}{4} + \sum_{l=1}^{k} \left((-\delta_x^2)^{-1}\hat{\eta}^{\varepsilon,l}, \hat{f}^{\varepsilon,l+1} - p^{l+1} - (\hat{f}^{\varepsilon,l-1} - p^{l-1})\right)$$

$$= -\frac{\mathcal{A}^k}{4} - 2\tau \sum_{l=2}^{k-1} \left(\delta_t^c(-\delta_x^2)^{-1}\hat{\eta}^{\varepsilon,l}, \hat{f}^{\varepsilon,l} - p^l\right)$$

$$+ \sum_{l=k}^{k+1} \left((-\delta_x^2)^{-1}\hat{\eta}^{\varepsilon,l-1}, \hat{f}^{\varepsilon,l} - p^l\right) - \sum_{l=0}^{1} \left((-\delta_x^2)^{-1}\hat{\eta}^{\varepsilon,l+1}, \hat{f}^{\varepsilon,l} - p^l\right)$$

$$(3.24) \qquad \lesssim \mathcal{A}^0 + \tau \sum_{l=2}^{k-1} (\|\delta_t^c \hat{\eta}^{\varepsilon,l}\|^2 + \mathcal{A}^l) + \sum_{l=1}^{2} \|\hat{\eta}^{\varepsilon,l}\|^2 + \sum_{l=k-1}^{k} \|\hat{\eta}^{\varepsilon,l}\|^2.$$

Summing the equation (3.23) together for $k = 1, 2, \ldots, m \le \frac{T}{\tau} - 1$, applying (3.24), we obtain that

$$(3.25)$$

$$\mathcal{A}^m \lesssim \mathcal{A}^0 + \tau \sum_{l=1}^{m} \mathcal{A}^l + \sum_{l=1}^{2} \|\hat{\eta}^{\varepsilon,l}\|^2 + \sum_{l=m-1}^{m} \|\hat{\eta}^{\varepsilon,l}\|^2 + \tau \sum_{l=1}^{m} \|\hat{\xi}^{\varepsilon,l}\|^2 + \tau \sum_{l=2}^{m-1} \|\delta_t^c \hat{\eta}^{\varepsilon,l}\|^2.$$

By Lemma 3.4 and the discrete Sobolev inequality, we deduce that

$$(3.26) \qquad \varepsilon\|\delta_x^+ \hat{u}^{\varepsilon,1/2}\| \lesssim \varepsilon\|\delta_t^+(\hat{f}^{\varepsilon,0} - p^0)\| \lesssim \varepsilon\|\delta_t^+ \hat{f}^{\varepsilon,0}\| + \varepsilon\|\delta_t^+ \hat{e}^{\varepsilon,0}\| \lesssim \tau^2,$$

which together with Lemma 3.4 yields that

$$\mathcal{A}^0 \lesssim \tau^4/\varepsilon^2.$$

Applying Lemma 3.3 and (3.25), it can be concluded that there exists $\tau_1 > 0$ such that when $\tau \leq \tau_1$, we have

$$\mathcal{A}^m \lesssim \left(h^2 + \frac{\tau^2}{\varepsilon} \right)^2 + \tau \sum_{i=1}^{m-1} \mathcal{A}^i.$$

Applying the discrete Gronwall inequality, for sufficiently small τ, we can conclude that

$$\mathcal{A}^m \lesssim \left(h^2 + \frac{\tau^2}{\varepsilon} \right)^2, \quad 0 \leq m \leq \frac{T}{\tau} - 1,$$

which completes the proof of Theorem 3.2 by recalling (3.17). $\square$

3.3. Another bound via the limiting equation (2.4).

Theorem 3.5. *Under assumptions* (A)–(B), *there exists* $\tau_2 > 0$ *sufficiently small, when* $0 < \tau \leq \tau_2$, *the scheme* (3.8) *satisfies the following error estimates:*

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon, \quad 0 \leq k \leq \frac{T}{\tau}.$$

Define another error function

$$\widetilde{e}_j^{\varepsilon,k} = \widetilde{E}^\varepsilon(x_j, t_k) - \hat{E}_j^{\varepsilon,k}, \quad \widetilde{f}_j^{\varepsilon,k} = -\hat{F}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \quad 0 \leq k \leq \frac{T}{\tau},$$

where $\widetilde{E}^\varepsilon(x, t)$ is the solution of the KGE-OP (2.8). The local truncation error $\widetilde{\xi}^{\varepsilon,k}$, $\widetilde{\eta}^{\varepsilon,k} \in X_M$ is defined as

(3.27)
$$\begin{aligned}
\widetilde{\xi}_j^{\varepsilon,k} &= \delta_t^2 \widetilde{E}^\varepsilon(x_j, t_k) - (\delta_x^2 - 1 - H_j^{\varepsilon,k})(\llbracket \widetilde{E}^\varepsilon \rrbracket)(x_j, t_k) \\
&\quad - \rho_B(\widetilde{E}^\varepsilon(x_j, t_k)) g(\widetilde{E}^\varepsilon(x_j, t_{k+1}), \widetilde{E}^\varepsilon(x_j, t_{k-1})) \\
&= \delta_t^2 \widetilde{E}^\varepsilon(x_j, t_k) - \left(\delta_x^2 - 1 + |\widetilde{E}^\varepsilon(x_j, t_k)|^2 - H_j^{\varepsilon,k} \right)(\llbracket \widetilde{E}^\varepsilon \rrbracket)(x_j, t_k), \\
\widetilde{\eta}_j^{\varepsilon,k} &= -\varepsilon^2 \delta_t^2 \rho_B(\widetilde{E}^\varepsilon(x_j, t_k)) = -\varepsilon^2 \delta_t^2 |\widetilde{E}^\varepsilon(x_j, t_k)|^2.
\end{aligned}$$

Lemma 3.6. *Under assumption* (A), *we can obtain the following error bounds:*

$$\|\widetilde{\xi}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*}, \quad \|\widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^2, \quad \|\delta_t^c \widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^{1+\alpha^*}.$$

Proof. Similar to the proof of Lemma 3.3, we can get that

$$\begin{aligned}
\widetilde{\xi}_j^{\varepsilon,k} = &-\frac{h^2}{6} \int_{-1}^1 (1 - |s|)^3 (\llbracket \partial_x^4 \widetilde{E}^\varepsilon \rrbracket)(x_j + sh, t_k)ds \\
&+ \frac{\tau^2}{6} \int_{-1}^1 (1 - |s|)^2 \partial_{tt} \left[\partial_{xx} \widetilde{E}^\varepsilon - \widetilde{E}^\varepsilon + |\widetilde{E}^\varepsilon|^3 \right](x_j, t_k + s\tau)ds \\
&- \frac{\tau^2}{2} \int_{-1}^1 (1 - |s|) \partial_x^2 \partial_t^2 \widetilde{E}^\varepsilon(x_j, t_k + s\tau)ds - A_2 \\
&- \frac{\tau^2}{2} \left(|\widetilde{E}^\varepsilon(x_j, t_k)|^2 - 1 - H_j^{\varepsilon,k} \right) \int_{-1}^1 (1 - |s|) \partial_{tt} \widetilde{E}^\varepsilon(x_j, t_k + s\tau)ds,
\end{aligned}$$

where

$$A_2 = \int_{-1}^{1} (1 - |s|)\widetilde{E}^{\varepsilon}(x_j, t_k + s\tau)G^{\varepsilon}(x_j, t_k + s\tau)\,ds - \widetilde{E}^{\varepsilon}(x_j, t_k)H_j^{\varepsilon,k}$$

$$= \tau \int_{-1}^{1} (1 - |s|)G^{\varepsilon}(x_j, t_k + s\tau) \int_0^s \partial_t \widetilde{E}^{\varepsilon}(x_j, t_k + \theta\tau)d\theta ds$$

$$\lesssim \tau \|G^{\varepsilon}\|_{L^{\infty}} \|\partial_t \widetilde{E}^{\varepsilon}\|_{L^{\infty}} \lesssim \tau\varepsilon^{\alpha^*}.$$

Hence we can conclude from assumption (A) that

$$\|\widetilde{\xi}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*}.$$

Note that by assumption (A), it is easy to get that

$$\partial_t^3|\widetilde{E}^{\varepsilon}|^2 = 6\partial_t\widetilde{E}^{\varepsilon}\partial_{tt}\widetilde{E}^{\varepsilon} + 2\widetilde{E}^{\varepsilon}\partial_t^3\widetilde{E}^{\varepsilon} \lesssim \varepsilon^{\alpha^*-1},$$

which indicates that

$$\|\widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^2, \quad \|\delta_t^c\widetilde{\eta}^{\varepsilon,k}\| \lesssim \varepsilon^{1+\alpha^*},$$

thus the proof is completed. $\qquad\square$

Analogous to Lemma 3.4, we have the error bounds for $\widetilde{e}^{\varepsilon,0}$, $\widetilde{f}^{\varepsilon,0}$, $\widetilde{e}^{\varepsilon,1}$, and $\widetilde{f}^{\varepsilon,1}$.

Lemma 3.7. *Under assumptions* (A) *and* (B), *the first step errors of the discretization* (2.12) *satisfy*

$$\widetilde{e}_j^{\varepsilon,0} = \widetilde{f}_j^{\varepsilon,0} = 0, \qquad |\widetilde{e}_j^{\varepsilon,1}| + |\delta_x^+\widetilde{e}_j^{\varepsilon,1}| \lesssim \tau^3 + \tau^2\varepsilon^{\alpha^*},$$

$$|\widetilde{f}_j^{\varepsilon,1}| + |\delta_t^+\widetilde{e}_j^{\varepsilon,0}| \lesssim \tau^2 + \tau\varepsilon^{\alpha^*}, \qquad |\delta_t^+\widetilde{f}_j^{\varepsilon,0}| \lesssim \tau.$$

Proof. It follows from (2.4) and (2.6) that $\partial_{tt}E^{\varepsilon}(x_j, 0) = \partial_{tt}\widetilde{E}^{\varepsilon}(x_j, 0)$. By (2.12) and assumption (B), one gets that

$$\widetilde{e}_j^{\varepsilon,1} = \frac{\tau^3}{2} \int_0^1 (1-s)^2 \partial_t^3 \widetilde{E}^{\varepsilon}(x_j, s\tau)ds$$

$$= \frac{\tau^3}{2} \int_0^1 (1-s)^2 \partial_t \left(\partial_{xx}\widetilde{E}^{\varepsilon} - \widetilde{E}^{\varepsilon} + |\widetilde{E}^{\varepsilon}|^3 - \widetilde{E}^{\varepsilon}G^{\varepsilon} \right)(x_j, s\tau)ds$$

$$= \frac{\tau^3}{2} \int_0^1 (1-s)^2 \partial_t \left(\partial_{xx}\widetilde{E}^{\varepsilon} - \widetilde{E}^{\varepsilon} + |\widetilde{E}^{\varepsilon}|^3 \right)(x_j, s\tau)ds$$

$$+ \frac{\tau^2}{2} E_0(x_j)\varepsilon^{\alpha}\omega_0(x_j) - \tau^2 \int_0^1 (1-s)\widetilde{E}^{\varepsilon}(x_j, s\tau)G^{\varepsilon}(x_j, s\tau)ds$$

$$\lesssim \tau^3 + \tau^2\varepsilon^{\alpha^*}.$$

Thus this gives that $|\delta_t^+\widetilde{e}_j^{\varepsilon,0}| \lesssim \tau^2 + \tau\varepsilon^{\alpha^*}$. Similar arguments can deduce that $|\delta_x^+\widetilde{e}_j^{\varepsilon,1}| \lesssim \tau^3 + \tau^2\varepsilon^{\alpha^*}$. By the definition, we have

$$|\widetilde{f}_j^{\varepsilon,1}| = |F_j^{\varepsilon,1}| \lesssim \tau^2|\partial_{tt}F^{\varepsilon}(x_j, 0)| \lesssim \tau^2.$$

The remaining conclusions are direct. $\qquad\square$

Proof of Theorem 3.5. Subtracting (3.8) from (3.27), one has the error equations

$$(3.28) \quad \begin{aligned} \delta_t^2 \widetilde{e}_j^{\varepsilon,k} &= (\delta_x^2 - 1 - H_j^{\varepsilon,k})[\![\widetilde{e}^{\varepsilon}]\!]_j^k + \widetilde{r}_j^k + \widetilde{\xi}_j^{\varepsilon,k}, \\ \varepsilon^2\delta_t^2\widetilde{f}_j^{\varepsilon,k} &= \delta_x^2[\![\widetilde{f}^{\varepsilon}]\!]_j^k + \varepsilon^2\delta_t^2\widetilde{p}_j^k + \widetilde{\eta}_j^{\varepsilon,k}, \quad j \in \mathcal{T}_M, \quad 1 \le k \le \frac{T}{\tau} - 1, \end{aligned}$$

where

$$\widetilde{r}_j^k = |\widetilde{E}^\varepsilon(x_j, t_k)|^2 ([\widetilde{E}^\varepsilon])(x_j, t_k) - \left(\rho_B(\hat{E}_j^{\varepsilon,k}) - [\![\hat{F}^\varepsilon]\!]_j^k\right) g(\hat{E}_j^{\varepsilon,k+1}, \hat{E}_j^{\varepsilon,k-1}),$$

$$\widetilde{p}_j^k = |\widetilde{E}^\varepsilon(x_j, t_k)|^2 - \rho_B(\hat{E}_j^{\varepsilon,k}),$$

Suppose $\widetilde{u}^{\varepsilon,k+\frac{1}{2}} \in X_M$ is the solution to the equation

$$-\delta_x^2 \widetilde{u}_j^{\varepsilon,k+\frac{1}{2}} = \delta_t^+(\widetilde{f}_j^{\varepsilon,k} - \widetilde{p}_j^k), \quad j \in \mathcal{T}_M, \quad 0 \le k \le \frac{T}{\tau} - 1.$$

Denote

$$\widetilde{\mathcal{A}}^k = \|\delta_t^+ \widetilde{e}^{\varepsilon,k}\|^2 + \frac{1}{2}\left(\|\widetilde{e}^{\varepsilon,k}\|^2 + \|\widetilde{e}^{\varepsilon,k+1}\|^2 + \|\delta_x^+ \widetilde{e}^{\varepsilon,k}\|^2 + \|\delta_x^+ \widetilde{e}^{\varepsilon,k+1}\|^2\right)$$

$$+ \varepsilon^2 \|\delta_x^+ \widetilde{u}^{\varepsilon,k+1/2}\|^2 + \frac{1}{2}\left(\|\widetilde{f}^{\varepsilon,k}\|^2 + \|\widetilde{f}^{\varepsilon,k+1}\|^2\right).$$

Applying the same approach as in the former part, there exists $\tau_2 > 0$ sufficiently small independent of ε such that

$$\widetilde{\mathcal{A}}^k \lesssim \widetilde{\mathcal{A}}^0 + \tau \sum_{l=1}^{k-1} \widetilde{\mathcal{A}}^l + \sum_{l=1}^{2} \|\widetilde{\eta}^{\varepsilon,l}\|^2 + \sum_{l=k-1}^{k} \|\widetilde{\eta}^{\varepsilon,l}\|^2 + \tau \sum_{l=1}^{k} \|\widetilde{\xi}^{\varepsilon,l}\|^2 + \tau \sum_{l=2}^{k-1} \|\delta_t \widetilde{\eta}^{\varepsilon,l}\|^2.$$

By Lemma 3.7 and the discrete Sobolev inequality, we deduce that

$$\varepsilon\|\delta_x^+ \widetilde{u}^{\varepsilon,1/2}\| \lesssim \varepsilon\|\delta_t^+ \widetilde{f}^{\varepsilon,0}\| + \varepsilon\|\delta_t^+ \widetilde{e}^{\varepsilon,0}\| \lesssim \varepsilon\tau,$$

which together with Lemma 3.7 yield that

$$\widetilde{\mathcal{A}}^0 \lesssim (\tau^2 + \tau\varepsilon^{\alpha^*})^2.$$

Applying Lemma 3.6, it can be concluded that when $0 < \tau \le \tau_2$,

$$\widetilde{\mathcal{A}}^k \lesssim (h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})^2 + \tau \sum_{i=1}^{k-1} \widetilde{\mathcal{A}}^i.$$

It follows from the discrete Gronwall inequality that

$$\widetilde{\mathcal{A}}^k \lesssim (h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*})^2,$$

implying that

$$\|\widetilde{e}^{\varepsilon,k}\| + \|\delta_x^+ \widetilde{e}^{\varepsilon,k}\| + \|\widetilde{f}^{\varepsilon,k}\| \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon^{1+\alpha^*}.$$

Using assumption (B) and the triangle inequality, we obtain that

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| \lesssim \|\widetilde{e}^{\varepsilon,k}\| + \|\delta_x^+ \widetilde{e}^{\varepsilon,k}\| + \|E^\varepsilon(\cdot, t_k) - \widetilde{E}^\varepsilon(\cdot, t_k)\|_{H^1}$$
$$\lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon,$$

$$\|\hat{f}^{\varepsilon,k}\| \lesssim \|\widetilde{f}^{\varepsilon,k}\| + \|F^\varepsilon(\cdot, t_k)\|_{L^2} \lesssim h^2 + \tau^2 + \tau\varepsilon^{\alpha^*} + \varepsilon,$$

which completes the proof of Theorem 3.5. $\qquad\qquad\qquad\square$

3.4. **Proof of Theorem 2.1.** Now we have proved the two types of estimates (2.15) and (2.16) for $(\hat{E}^{\varepsilon,k}, \hat{F}^{\varepsilon,k})$, which is the solution of the modified finite difference discretization (3.8) with (2.10)–(2.12). Hence we can get the uniform error bounds for $(\hat{E}^{\varepsilon,k}, \hat{F}^{\varepsilon,k})$:

$$\|\hat{e}^{\varepsilon,k}\| + \|\delta_x^+ \hat{e}^{\varepsilon,k}\| + \|\hat{f}^{\varepsilon,k}\| \lesssim h^2 + \tau,$$

which together with the inverse inequality [36] yields

$$\|\hat{E}^{\varepsilon,k}\|_\infty - \|E^\varepsilon(\cdot, t_k)\|_{L^\infty} \leq \|\hat{e}^{\varepsilon,k}\|_\infty \lesssim \|\delta_x^+ \hat{e}^{\varepsilon,k}\| \lesssim h^2 + \tau, \quad 0 \leq k \leq \frac{T}{\tau}.$$

Thus there exists $h_0 > 0$ and $\tau_3 > 0$ sufficiently small such that when $0 < h \leq h_0$ and $0 < \tau \leq \tau_3$,

$$\|\hat{E}^{\varepsilon,k}\|_\infty \leq 1 + \|E^\varepsilon(\cdot, t_k)\|_\infty \leq 1 + M_0, \quad 0 \leq k \leq \frac{T}{\tau}.$$

Set $\tau_0 = \min\{\tau_1, \tau_2, \tau_3\}$, when $0 < h \leq h_0$, $0 < \tau \leq \tau_0$, (3.8) collapses to (2.9), i.e., $(\hat{E}^{\varepsilon,k}, \hat{F}^{\varepsilon,k})$ are identical to $(E^{\varepsilon,k}, F^{\varepsilon,k})$, which completes the proof. $\square$

Remark 3.8. The error bounds in Theorem 2.1 are still valid in higher dimensions, e.g., $d = 2, 3$. The key point is the discrete Sobolev inequality in higher dimensions as [4, 36]

$$\|\psi_h\|_\infty \leq \frac{1}{C_d(h)} \|\psi_h\|_{H^1}, \quad \text{with} \quad C_d(h) \sim \begin{cases} \dfrac{1}{|\ln h|}, & d = 2, \\ h^{1/2}, & d = 3, \end{cases}$$

where ψ_h is a mesh function over Ω with homogeneous Dirichlet boundary condition. Thus by requiring an additional condition on the time step τ,

$$\tau = o(C_d(h)),$$

the same error bounds can be obtained.

4. NUMERICAL RESULTS

In this section, we present numerical results for the KGZ system (2.6) by the finite difference discretization (2.9) with (2.10)–(2.12). In our experiment, the initial condition is set as

$$E_0(x) = e^{-x^2} \sin x, \quad E_1(x) = \operatorname{sech}(x^2/2) \cos x,$$
$$\omega_0(x) = \operatorname{sech}(x^2) \cos(3x), \quad \omega_1(x) = \operatorname{sech}(x^2) \sin(4x),$$

and the parameters α and β are chosen as:

Case I. $\alpha = 1$ and $\beta = 0$;

Case II. $\alpha = 0$ and $\beta = -1$.

In practical computation, the truncated domain is set as $\Omega_\varepsilon = \left[-30 - \frac{1}{\varepsilon}, 30 + \frac{1}{\varepsilon}\right]$, which is large enough such that the homogeneous Dirichlet boundary condition does not introduce significant errors. Similar to the truncation for the Zakharov system, the bounded computational domain Ω_ε has to be chosen as ε-dependent due to the fact that the rapid outgoing waves are at wave speed $O\left(\frac{1}{\varepsilon}\right)$ and the homogeneous Dirichlet boundary condition is taken at the boundary. The computational ε-dependent domain can be fixed as ε-independent if one applies absorbing boundary condition (ABC) [17] or transport boundary condition (TBC) [18, 19], or perfected matched layer (PML) [11] for the wave-type equations in (2.6) and (2.2) during the truncation (cf. [8]).

TABLE 1. Spatial errors at time $t = 1$ for Case II, i.e., $\alpha = 0$, $\beta = -1$.

$e^\varepsilon(1)$	$h_0 = 0.2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$	$h_0/2^5$
$\varepsilon = 1$	1.57E-2	4.05E-3	1.02E-3	2.56E-4	6.39E-5	1.60E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2$	1.35E-2	3.48E-3	8.76E-4	2.19E-4	5.49E-5	1.37E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^2$	1.30E-2	3.35E-3	8.44E-4	2.11E-4	5.29E-5	1.32E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^3$	1.32E-2	3.42E-3	8.60E-4	2.15E-4	5.39E-5	1.35E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^4$	1.33E-2	3.43E-3	8.65E-4	2.17E-4	5.42E-5	1.36E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^5$	1.33E-2	3.44E-3	8.66E-4	2.17E-4	5.43E-5	1.36E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^6$	1.33E-2	3.44E-3	8.66E-4	2.17E-4	5.42E-5	1.36E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^7$	1.33E-2	3.44E-3	8.65E-4	2.17E-4	5.42E-5	1.36E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^8$	1.33E-2	3.43E-3	8.65E-4	2.17E-4	5.42E-5	1.36E-5
rate	-	1.95	1.99	2.00	2.00	2.00
$n^\varepsilon(1)$	$h_0 = 0.2$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$	$h_0/2^5$
$\varepsilon = 1$	1.91E-2	4.79E-3	1.20E-3	2.99E-4	7.49E-5	1.87E-5
rate	-	2.00	2.00	2.00	2.00	2.00
$\varepsilon = 1/2$	1.61E-2	3.98E-3	9.92E-4	2.48E-4	6.20E-5	1.55E-5
rate	-	2.02	2.00	2.00	2.00	2.00
$\varepsilon = 1/2^2$	6.59E-3	1.67E-3	4.18E-4	1.05E-4	2.62E-5	6.56E-6
rate	-	1.98	2.00	2.00	2.00	1.99
$\varepsilon = 1/2^3$	5.30E-3	1.35E-3	3.39E-4	8.49E-5	2.13E-5	5.33E-6
rate	-	1.97	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^4$	5.12E-3	1.30E-3	3.28E-4	8.20E-5	2.05E-5	5.15E-6
rate	-	1.97	1.99	2.00	2.00	1.99
$\varepsilon = 1/2^5$	5.06E-3	1.29E-3	3.23E-4	8.10E-5	2.03E-5	5.09E-6
rate	-	1.97	1.99	2.00	2.00	1.99
$\varepsilon = 1/2^6$	5.03E-3	1.28E-3	3.21E-4	8.05E-5	2.02E-5	5.08E-6
rate	-	1.97	1.99	2.00	2.00	1.99
$\varepsilon = 1/2^7$	5.01E-3	1.28E-3	3.21E-4	8.02E-5	2.01E-5	5.07E-6
rate	-	1.97	1.99	2.00	2.00	1.99
$\varepsilon = 1/2^8$	5.01E-3	1.27E-3	3.20E-4	8.01E-5	2.01E-5	5.07E-6
rate	-	1.97	1.99	2.00	1.99	1.99

To quantify the numerical errors, we introduce the error functions as follows,

$$e^\varepsilon(t_k) := \frac{\|e^{\varepsilon,k}\| + \|\delta_x^+ e^{\varepsilon,k}\|}{\|E^\varepsilon(\cdot, t_k)\|_{H^1}}, \quad n^\varepsilon(t_k) := \frac{\|n^{\varepsilon,k}\|}{\|N^\varepsilon(\cdot, t_k)\|_{L^2}},$$

where $e^{\varepsilon,k} = E^\varepsilon(\cdot, t_k) - E^{\varepsilon,k}$, $n^{\varepsilon,k} = N^\varepsilon(\cdot, t_k) - N^{\varepsilon,k}$. The "exact" solution is obtained by the EWI-SP method [4] with very small mesh size $h = 1/64$ and time

TABLE 2. Temporal errors at time $t = 1$ for Case I, i.e., $\alpha = 1$, $\beta = 0$.

$e^\varepsilon(1)$	$\tau_0 = 0.05$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon = 1$	6.34E-3	1.64E-3	4.16E-4	1.05E-4	2.62E-5	6.59E-6	1.67E-6	4.20E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2$	5.19E-3	1.35E-3	3.42E-4	8.61E-5	2.16E-5	5.43E-6	1.37E-6	3.45E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^2$	5.17E-3	1.34E-3	3.41E-4	8.58E-5	2.15E-5	5.41E-6	1.37E-6	3.45E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^3$	5.17E-3	1.34E-3	3.41E-4	8.58E-5	2.15E-5	5.41E-6	1.37E-6	3.45E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^4$	5.16E-3	1.34E-3	3.40E-4	8.56E-5	2.15E-5	5.40E-6	1.37E-6	3.45e-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^5$	5.14E-3	1.34E-3	3.39E-4	8.55E-5	2.15E-5	5.39E-6	1.36E-6	3.44E-7
rate	-	1.94	1.98	1.99	1.99	1.99	1.98	1.98
$\varepsilon = 1/2^6$	5.13E-3	1.33E-3	3.39E-4	8.54E-5	2.15E-5	5.39E-6	1.36E-6	3.45E-7
rate	-	1.95	1.97	1.99	1.99	1.99	1.98	1.98
$\varepsilon = 1/2^7$	5.00E-3	1.30E-3	3.29E-4	8.32E-5	2.09E-5	5.26E-6	1.33E-6	3.36E-7
rate	-	1.95	1.98	1.98	1.99	1.99	1.98	1.98
$\varepsilon = 1/2^8$	5.01E-3	1.30E-3	3.29E-4	8.29E-5	2.09E-5	5.25E-6	1.33E-6	3.36E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.98
$n^\varepsilon(1)$	$\tau_0 = 0.05$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon = 1$	6.59E-3	1.70E-3	4.29E-4	1.08E-4	2.70E-5	6.78E-6	1.71E-6	4.28E-7
rate	-	1.96	1.98	1.99	2.00	2.00	1.99	2.00
$\varepsilon = 1/2$	1.87E-2	4.85E-3	1.23E-3	3.08E-4	7.71E-5	1.93E-5	4.85E-6	1.21E-6
rate	-	1.95	1.98	1.99	2.00	2.00	1.99	2.00
$\varepsilon = 1/2^2$	1.50E-2	4.63E-3	1.24E-3	3.15E-4	7.90E-5	1.98E-5	4.96E-6	1.24E-6
rate	-	1.70	1.90	1.98	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^3$	8.65E-3	3.50E-3	1.44E-3	4.36E-4	1.13E-4	2.83E-5	7.09E-6	1.77E-6
rate	-	1.31	1.28	1.73	1.95	1.99	2.00	2.00
$\varepsilon = 1/2^4$	5.55E-3	1.96E-3	9.14E-4	4.87E-4	1.86E-4	5.02E-5	1.27E-5	3.18E-6
rate	-	1.50	1.10	0.91	1.39	1.89	1.99	2.00
$\varepsilon = 1/2^5$	4.97E-3	1.45E-3	5.53E-4	2.66E-4	1.56E-4	7.76E-5	2.38E-5	6.09E-6
rate	-	1.78	1.39	1.06	0.77	1.01	1.70	1.97
$\varepsilon = 1/2^6$	4.70E-3	1.30E-3	4.38E-4	1.81E-4	8.58E-5	4.86E-5	2.86E-5	1.12E-5
rate	-	1.85	1.57	1.27	1.08	0.82	0.76	1.36
$\varepsilon = 1/2^7$	4.19E-3	1.20E-3	3.57E-4	1.41E-4	6.30E-5	2.96E-5	1.55E-5	9.45E-6
rate	-	1.81	1.75	1.34	1.16	1.09	0.93	0.72
$\varepsilon = 1/2^8$	3.96E-3	1.12E-3	3.28E-4	1.12E-4	4.84E-5	2.26E-5	1.07E-5	5.30E-6
rate	-	1.83	1.77	1.55	1.21	1.10	1.08	1.01

step $\tau = 10^{-6}$. The errors are displayed at $t = 1$. For spatial error analysis, we set a time step $\tau = 10^{-5}$, such that the temporal error can be neglected; for temporal error analysis, the mesh size h is set as $h = 2.5 \times 10^{-4}$ such that the spatial error can be ignored.

TABLE 3. Temporal errors at time $t = 1$ for Case II, i.e., $\alpha = 0$, $\beta = -1$.

$e^\varepsilon(1)$	$\tau_0 = 0.05$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon = 1$	6.34E-3	1.64E-3	4.16E-4	1.05E-4	2.62E-5	6.59E-6	1.67E-6	4.20E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2$	5.48E-3	1.42E-3	3.62E-4	9.11E-5	2.29E-5	5.74E-6	1.45E-6	3.65E-7
rate	-	1.94	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^2$	5.16E-3	1.34E-3	3.41E-4	8.58E-5	2.16E-5	5.41E-6	1.37E-6	3.42E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^3$	5.20E-3	1.35E-3	3.44E-4	8.66E-5	2.18E-5	5.46E-6	1.38E-6	3.47E-7
rate	-	1.94	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^4$	5.47E-3	1.42E-3	3.59E-4	9.04E-5	2.27E-5	5.71E-6	1.44E-6	3.63E-7
rate	-	1.95	1.98	1.99	1.99	1.99	1.98	1.99
$\varepsilon = 1/2^5$	5.77E-3	1.63E-3	4.04E-4	1.00E-4	2.51E-5	6.30E-6	1.59E-6	4.01E-7
rate	-	1.83	2.01	2.01	2.00	1.99	1.99	1.99
$\varepsilon = 1/2^6$	5.52E-3	1.98E-3	5.67E-4	1.32E-4	3.15E-5	7.81E-6	1.96E-6	4.93E-7
rate	-	1.48	1.80	2.11	2.06	2.01	1.99	1.99
$\varepsilon = 1/2^7$	5.45E-3	1.92E-3	8.40E-4	2.40E-4	5.20E-5	1.18E-5	2.88E-6	7.26E-7
rate	-	1.51	1.19	1.81	2.21	2.13	2.04	1.99
$\varepsilon = 1/2^8$	5.45E-3	1.90E-3	8.42E-4	4.03E-4	1.13E-4	2.35E-5	5.17E-6	1.24E-6
rate	-	1.52	1.17	1.06	1.83	2.27	2.19	2.06
$n^\varepsilon(1)$	$\tau_0 = 0.05$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$	$\tau_0/2^6$	$\tau_0/2^7$
$\varepsilon = 1$	6.59E-3	1.70E-3	4.29E-4	1.08E-4	2.70E-5	6.78E-6	1.71E-6	4.28E-7
rate	-	1.96	1.98	1.99	2.00	2.00	1.99	2.00
$\varepsilon = 1/2$	1.33E-2	3.45E-3	8.73E-4	2.19E-4	5.49E-5	1.38E-5	3.46E-6	8.65E-7
rate	-	1.95	1.98	1.99	2.00	2.00	1.99	2.00
$\varepsilon = 1/2^2$	9.61E-3	2.97E-3	7.98E-4	2.03E-4	5.09E-5	1.28E-5	3.20E-6	8.00E-7
rate	-	1.70	1.89	1.98	1.99	2.00	2.00	2.00
$\varepsilon = 1/2^3$	5.55E-3	2.30E-3	9.03E-4	2.77E-4	7.21E-5	1.81E-5	4.54E-6	1.14E-6
rate	-	1.27	1.35	1.70	1.94	1.99	2.00	2.00
$\varepsilon = 1/2^4$	3.92E-3	1.35E-3	6.30E-4	3.13E-4	1.17E-4	3.23E-5	8.18E-6	2.05E-6
rate	-	1.54	1.10	1.01	1.43	1.85	1.98	2.00
$\varepsilon = 1/2^5$	3.91E-3	1.19E-3	4.18E-4	1.89E-4	1.07E-4	4.80E-5	1.53E-5	3.96E-6
rate	-	1.71	1.51	1.14	0.82	1.16	1.65	1.95
$\varepsilon = 1/2^6$	3.53E-3	1.37E-3	4.38E-4	1.49E-4	6.27E-5	3.45E-5	1.84E-5	7.03E-6
rate	-	1.37	1.64	1.55	1.25	0.86	0.91	1.39
$\varepsilon = 1/2^7$	3.31E-3	1.27E-3	5.49E-4	1.78E-4	5.74E-5	2.26E-5	1.11E-5	6.51E-6
rate	-	1.38	1.21	1.63	1.63	1.35	1.02	0.77
$\varepsilon = 1/2^8$	3.18E-3	1.20E-3	5.47E-4	2.52E-4	7.81E-5	2.34E-5	8.60E-6	3.80E-6
rate	-	1.41	1.13	1.12	1.69	1.74	1.45	1.18

Table 1 depicts the spatial errors for Case II initial data, which clearly demonstrates that our numerical method is uniformly second order accurate in h for all $\varepsilon \in (0, 1]$. The result for Case I initial data is similar, which is omitted here for brevity.

Tables 2 and 3 present the temporal errors for Cases I and II, respectively, from which we can conclude that the method is uniformly convergent in time for both initial data. Specifically, Table 2 shows the method is uniformly second order accurate for E^ε, while for N^ε, it is second order in time when $\tau \lesssim \varepsilon$ or $\varepsilon \lesssim \tau^2$ (cf. upper and lower triangle parts, respectively). There is a resonance regime when $\tau \sim \varepsilon$ where the convergence rate degenerates to the first order, which agrees with the analysis (2.15)–(2.16). For $\alpha = 0$, $\beta = -1$, the upper and lower triangle parts of Table 3 suggest that the method is second and first order in time when $\tau \lesssim \varepsilon$ and $\varepsilon \lesssim \tau$, respectively. Moreover, the upper triangle parts of Tables 2 and 3 show the order of the errors at $O(\tau^2/\varepsilon)$ for n^ε (cf. each column), which confirms our error analysis in Section 3.

5. Conclusion

We presented a uniformly accurate finite difference method and carried out its rigorous error bounds for the Klein-Gordon-Zakharov (KGZ) system in d ($d = 1, 2, 3$) dimensions, which involves a dimensionless parameter $\varepsilon \in (0, 1]$. When $0 < \varepsilon \ll 1$, i.e., in the subsonic limit regime, the solution of the KGZ system propagates highly oscillatory waves in time and/or rapid outgoing waves in space. Our method was designed by reformulating the KGZ system into an asymptotic consistent formulation followed by adopting an integral approximation for the oscillating term. By applying the energy method and the limiting equation, two independent error bounds were obtained, which depend explicitly on the parameter ε, mesh size h and time step τ. Thus it can be established that the method is uniformly convergent for $\varepsilon \in (0, 1]$ with quadratic and linear convergence in space and time, respectively. The error bound is confirmed by the numerical results, which also suggest that our estimates are sharp.

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