

## INF-SUP STABILITY OF GEOMETRICALLY UNFITTED STOKES FINITE ELEMENTS

JOHNNY GUZMÁN AND MAXIM OLSHANSKII

**ABSTRACT.** This paper shows an inf-sup stability property for several well-known 2D and 3D Stokes elements on triangulations which are not fitted to a given smooth or polygonal domain. The property implies stability and optimal error estimates for a class of unfitted finite element methods for the Stokes and Stokes interface problems, such as Nitsche-XFEM or cutFEM. The error analysis is presented for the Stokes problem. All assumptions made in the paper are satisfied once the background mesh is shape-regular and fine enough.

### 1. INTRODUCTION

Unfitted finite element (FE) methods incorporate geometrical information about the domain where the problem is posed without fitting the mesh to lower dimensional structures such as physical boundaries or internal interfaces. This is opposite to fitted discretizations such as (isoparametric) traditional FE and isogeometric analysis. The advantage of the unfitted approach is a relative ease of handling propagating interface and geometries defined implicitly, i.e., when a surface parametrization is not readily available. Prominent classes of unfitted FE are given by XFEM [24] and cutFEM [14] also known as Nitsche-XFEM methods or traceFEM in the case of embedded surfaces. In cutFEM, one considers background mesh and FE spaces not tailored to the problem geometry, while numerical integration in FE bilinear forms is performed over the physical domains  $\Omega$  and/or  $\partial\Omega$  which cut through the background mesh in an arbitrary way. Effectively, this leads to traces of the ambient FE spaces on the physical domain, where the original problem is posed, and integration over arbitrary cut simplexes.

The idea of unfitted FE can be followed back at least to the works of Barrett and Elliott [2–4], where a cut FE method was studied for the planar elliptic problems and elliptic interface problems. Over the last few decades, unfitted FE methods emerged in a powerful discretization approach that has been applied to the wide range of problems, including problems with interfaces, fluid equations, PDEs posed on surfaces, surface-bulk coupled problems, equations posed on evolving domains, etc.; see, e.g., [5, 10, 15, 21, 22, 27, 34, 35, 39, 40, 42]. Among important enabling techniques used in unfitted FEM are the Nitsche method for enforcing essential

---

Received by the editor May 31, 2016, and, in revised form, December 3, 2016, April 11, 2017, and April 18, 2017.

2010 *Mathematics Subject Classification.* Primary 65N30, 65N12, 76D07, 65N85.

*Key words and phrases.* XFEM, cutFEM, Stokes problem, LBB condition, finite elements.

The first author was partially supported by NSF through the Division of Mathematical Sciences grant 1318108.

The second author was partially supported by NSF through the Division of Mathematical Sciences grants 1522252, 1717516.

boundary and interface conditions [29], ghost penalty stabilization [13], and the properties of trace FE spaces on embedded surfaces [38]. We note that many of these developments are accomplished with rigorous stability and convergence analysis of the unfitted FE, which demonstrate both utility and reliability of the approach.

One important application of unfitted FE methods is the numerical simulation of fluid problems with evolving interfaces as occurs in fluid-structure interaction problems and two-phase flows. If the fluid is treated as incompressible, then the prototypical model suitable for numerical analysis is the stationary (interface) Stokes problem. This paper addresses the question of numerical stability of a certain class of geometrically unfitted Stokes finite elements. Unfitted FE methods for the Stokes problem received recently a closer attention in the literature. In [16] optimal order convergence results were shown for the unfitted inf-sup stable velocity-pressure 2D FE with Nitsche treatment of the boundary conditions and ghost-penalty stabilization for triangles cut by  $\partial\Omega$ . This analysis was extended to the Stokes interface problem and  $P_1$ iso $P_2 - P_1$  elements in [31]. Optimal order convergence in the energy norm for  $P_1^{\text{bubble}} - P_1$  unfitted FE using slightly different pressure stabilization over cut triangles was shown for the Stokes interface problem in [17]. In [32] the  $P_2 - P_1$  elements were analysed for the Stokes interface problem, when the pressure element is enriched to allow for the jump over unfitted interface, while the velocity element is globally continuous. Globally stabilized unfitted Stokes finite elements,  $P_1 - P_1$  and  $P_1 - P_1^{\text{disc}}$ , were studied in [17, 37, 45]. Other related work on geometrically unfitted FE for the Stokes problem can be found in [1, 28, 33, 41].

The analysis of inf-sup stable unfitted Stokes elements, however, is not a straightforward extension of the standard results for saddle point problems. In particular, it essentially relies on a certain uniform stability property of the finite element velocity-pressure pair. This property can be found as an assumption (explicitly or implicitly made) in [16, 17, 31]. Loosely speaking the following condition on FE velocity-pressure spaces is required: Assume a family of shape-regular triangulations  $\{\mathcal{T}_h\}_{h>0}$  of  $\mathbb{R}^2$ , and let  $\Omega \in \mathbb{R}^2$  be a bounded domain with smooth boundary. Consider the family of domains  $\Omega_h$ , where each  $\Omega_h$  consists of all triangles from  $\mathcal{T}_h$  which are *strictly inside*  $\Omega$ . Then one requires that the LBB constants (optimal constants from the FE velocity-pressure inf-sup stability condition) for the domains  $\Omega_h$  are uniformly in  $h$  bounded away from zero. In the same way the property is formulated in 3D. In section 2 we discuss what sort of difficulties one encounters trying to employ common techniques to verify this property.

Recently, in [32] the required uniform stability condition was proved for  $P_2 - P_1$ , the lowest order Taylor-Hood element. In this paper, we show the uniform inf-sup stability result for a wider class of elements, including  $P_{k+1} - P_k$ ,  $k \geq 1$ , and  $P_{k+d} - P_k^{\text{disc}}$  for  $k \geq 0$ ,  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , and several other elements; see section 6. Following [32] we employ the argument from [44]. This helps us to formulate more local conditions on FE spaces which are sufficient for the uniform inf-sup stability, but easier to check. Further we show that this condition is satisfied by a number of popular LBB-stable FE pairs.

This paper also applies the acquired uniform stability result to show the optimal order error estimates of the unfitted FE method for the Stokes problem. The analysis improves over the available in the literature by eliminating certain assumptions on how the surface  $\partial\Omega$  (or an interface in the two-phase fluid case) intersects

the background mesh. Instead, we impose certain assumptions, which are always satisfied once the background mesh is shape-regular and the mesh size is not too coarse with respect to the problem geometry; see section 4 for the assumptions and further discussion in Remark 1.

The remainder of the paper is organized as follows. In section 2 we define the problem of interest and formulate the central question we address in this paper about uniform inf-sup stability. Section 3 collects necessary preliminaries and auxiliary results. Here we present the unfitted finite element method for the Stokes problem. Further we formulate assumptions sufficient for the main uniform stability result. Section 5 shows how the well-posedness and optimal order error estimates for the unfitted FE method follow from our assumptions. In section 6 we give the examples of velocity and pressure spaces satisfying the assumptions.

## 2. PROBLEM SETTING

Consider the Stokes problem posed on a bounded domain with Lipschitz boundary  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ ,

$$(2.1) \quad \begin{cases} -\Delta u + \nabla p = f & \text{in } \Omega, \\ \operatorname{div} u = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Vector function  $u \in [H_0^1(\Omega)]^d$  and  $p \in L^2(\Omega)/\mathbb{R}$  are the weak solution to (2.1), having the physical meaning of fluid velocity and normalized kinematic pressure.

Assume there is a domain  $S \supset \Omega$ , and we let  $\{\mathcal{T}_h\}_{h>0}$  be an admissible family of triangulations of  $S$ . We are interested in a finite element method for (2.1) using spaces of piecewise polynomial functions with respect to  $\mathcal{T}_h$ . Note that we make no assumption on how  $\Omega$  overlaps with  $\mathcal{T}_h$ , i.e.,  $\partial\Omega$  may cut through tetrahedra or triangles from  $\mathcal{T}_h$  in an arbitrary way.

In the next section we give details of the finite element method. Now we formulate the stability condition, which is crucial for the analysis of this method (and likely many other unfitted FE methods for (2.1)). Consider the set of all strictly internal simplexes and define the corresponding subdomain of  $\Omega$ :

$$\mathcal{T}_h^i := \{T \in \mathcal{T}_h : T \subset \Omega\}, \quad \Omega_h^i := \operatorname{Int}\left(\bigcup_{T \in \mathcal{T}_h^i} \bar{T}\right).$$

For background finite element velocity and pressure spaces  $V_h$  and  $Q_h$ , consider their restrictions on  $\Omega_h^i$ , that is,  $V_h^i = V_h \cap [H_0^1(\Omega_h^i)]^d$  and  $Q_h^i = Q_h \cap L_0^2(\Omega_h^i)$ ,  $L_0^2(\Omega_h^i) := \{q \in L^2(\Omega_h^i) : \int_{\Omega_h^i} q \, dx = 0\}$ , and define

$$\theta_h := \inf_{q \in Q_h^i} \sup_{v \in V_h^i} \frac{\int_{\Omega_h^i} q \operatorname{div} v \, dx}{\|v\|_{H^1(\Omega_h^i)} \|q\|_{L^2(\Omega_h^i)}}.$$

We are interested in the following condition:

$$(2.2) \quad 0 < \inf_{h < h_0} \theta_h,$$

for some positive  $h_0$ .

Note that standard arguments based on the Nečas inequality and Fortin's projection operator (cf. [9]) cannot be applied in a straightforward way to yield (2.2)

for inf-sup stable elements (e.g., for Taylor-Hood element). For reference purpose recall the Nečas inequality:

$$(2.3) \quad C_N(\Omega_h^i) \|q\|_{L^2(\Omega_h^i)} \leq \sup_{v \in [H_0^1(\Omega_h^i)]^d} \frac{\int_{\Omega_h^i} q \operatorname{div} v \, dx}{\|v\|_{H^1(\Omega_h^i)}} \quad \forall q \in L_0^2(\Omega_h^i).$$

Since  $\Omega_h^i$  is Lipschitz for any given  $\mathcal{T}_h$ , the inequality holds with some domain dependent constant  $C_N(\Omega_h^i) > 0$ ; see, e.g., [11, 25]. However, we are not aware of a result in the literature which implies that  $C_N(\Omega_h^i)$  are uniformly bounded from below by a positive constant independent of  $h$ . For example, the well-known argument for proving (2.3) is based on the decomposition of a Lipschitz domain into a finite number of strictly star-shaped domains (see Lemma II.1.3 in [25]) and applying the result of Bogovskii [11] in each of the star domains. However, the number of the star domains in the decomposition of  $\Omega_h^i$  may infinitely grow for  $h \rightarrow 0$  even if  $\partial\Omega$  is smooth and  $\{\mathcal{T}_h\}_{h>0}$  is shape-regular, which would drive the lower bound for  $C_N(\Omega_h^i)$  to zero. Alternatively, the recent analysis from [7] provides a lower bound for  $C_N(\Omega_h^i)$  if there exist diffeomorphisms  $\Phi_h : \Omega_h^i \rightarrow \Omega$  with uniformly bounded  $W^{1,\infty}(\Omega_h^i)$  norms. We do not see how to construct such diffeomorphisms (note that  $\partial\Omega_h^i$  is not necessarily a graph of a function in the natural coordinates of  $\partial\Omega$ ). Additional difficulty stems from the observation that  $\mathcal{T}_h^i$  does not necessarily inherit a macro-element structure that  $\mathcal{T}_h$  may possess. This said, we shall look for a different approach to verify (2.2).

We end this section noting that the finite element method and the analysis of the paper can be easily extended to the Stokes *interface* problem, a prototypical model of two-phase incompressible fluid flow. However, we are not adding these extra details to the present report.

### 3. FINITE ELEMENT METHOD

**3.1. Preliminaries.** We adopt the convention that elements  $T$  and element edges (also faces in 3D) are open sets. We use the over-line symbol to refer to their closure. For each simplex  $T \in \mathcal{T}_h$ , let  $h_T$  denote its diameter and define the global parameter of the triangulation by  $h = \max_T h_T$ . We assume that  $\mathcal{T}_h$  is shape regular, i.e., there exists  $\kappa > 0$  such that for every  $T \in \mathcal{T}_h$  the radius  $\rho_T$  of its inscribed sphere satisfies

$$(3.1) \quad \rho_T > h_T/\kappa.$$

The set of elements cutting the interface  $\Gamma \equiv \partial\Omega$ , and restricted to  $\Omega$  are also of interest. They are defined by:

$$\begin{aligned} \mathcal{T}_h^\Gamma &:= \{T \in \mathcal{T}_h : \operatorname{meas}_2(T \cap \Gamma) > 0\}, \\ \mathcal{T}_h^e &:= \{T \in \mathcal{T}_h : T \in \mathcal{T}_h^i \text{ or } T \in \mathcal{T}_h^\Gamma\}. \end{aligned}$$

In particular, for  $T \in \mathcal{T}_h^\Gamma$  we denote  $T_\Gamma = \overline{T} \cap \Gamma$ . Observe that the definition of  $\mathcal{T}_h^\Gamma$  guarantees that  $\sum_{T \in \mathcal{T}_h^\Gamma} |T_\Gamma| = |\Gamma|$ . Under these definitions we define the  $h$ -dependent domains

$$\Omega_h^\Gamma := \operatorname{Int}\left(\bigcup_{T \in \mathcal{T}_h^\Gamma} \overline{T}\right), \quad \Omega_h^e := \operatorname{Int}\left(\bigcup_{T \in \mathcal{T}_h^e} \overline{T}\right).$$

Note that  $\overline{\Omega_h^e} = \overline{\Omega_h^i} \cup \overline{\Omega_h^\Gamma}$  and that  $\Omega_h^i \subset \Omega \subset \Omega_h^e$ . For these domains define sets of faces:

$$\begin{aligned}\mathcal{F}_h^i &:= \{F : F \text{ is an interior face of } \mathcal{T}_h^i\}, \\ \mathcal{F}_h^\Gamma &:= \{F : F \text{ is a face of } \mathcal{T}_h^\Gamma, F \not\subset \partial\Omega_h^e\}, \\ \mathcal{F}_h^e &:= \{F : F \text{ is an interior face of } \mathcal{T}_h^e\}.\end{aligned}$$

Now we can define finite element spaces. A space of continuous functions on  $\Omega_h^e$  which are polynomials of degree  $k$  on each  $T \in \mathcal{T}_h^e$  is denoted by  $W_h^k$ . The spaces of discontinuous and continuous pressure spaces are given by

$$\begin{aligned}Q_h^{\text{disc}} &= \{q \in L^2(\Omega_h^e) : q|_T \in P^{k_p}(T) \quad \forall T \in \mathcal{T}_h^e\}, \\ Q_h^{\text{cont}} &= Q_h^{\text{disc}} \cap H^1(\Omega_h^e).\end{aligned}$$

Throughout this paper we will consider either  $Q_h = Q_h^{\text{disc}}$  (for  $k_p \geq 0$ ) or  $Q_h = Q_h^{\text{cont}}$  (for  $k_p \geq 1$ ). We will denote the finite element velocity space by  $V_h \subset [H^1(\Omega_h^e)]^d$ , and we will assume

$$(W_h^{k_u})^d \subset V_h \subset (W_h^s)^d$$

for some integer  $s \geq k_u \geq 1$ . In section 4, we introduce a more technical assumption 3 that our pair of spaces  $\{Q_h, V_h\}$  has to satisfy. Then, later we give examples of pairs that satisfy all necessary assumptions. For example, if  $Q_h = Q_h^{\text{disc}}$ , then  $V_h$  can be the space of continuous piecewise polynomials of degree  $k_p + d$ ; and if  $Q_h = Q_h^{\text{cont}}$ , then  $V_h$  can be the space of continuous piecewise polynomials of degree  $k_p + 1$ . We give more examples of spaces satisfying our assumptions in section 6.

**3.2. Finite element method.** We will use the notation  $(v, w) = \int_\Omega vw \, dx$ . Introduce the mesh-dependent bilinear forms

$$a_h(u_h, v_h) := (\nabla u_h, \nabla v_h) + s_h(u_h, v_h) + \mathbf{j}_h(u_h, v_h) + \eta j_h(u_h, v_h),$$

with

$$\begin{aligned}s_h(u, v) &= - \int_\Gamma \{(n \cdot \nabla u) \cdot v + (n \cdot \nabla v) \cdot u\} ds, \\ j_h(u, v) &= \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{h_T} \int_{T_\Gamma} u \cdot v ds, \\ \mathbf{j}_h(u, v) &= \sum_{F \in \mathcal{F}_h^\Gamma} \sum_{\ell=1}^s h_F^{2\ell-1} \int_F [\partial_n^\ell u] [\partial_n^\ell v] ds,\end{aligned}$$

and

$$b_h(p_h, v_h) := -(p_h, \operatorname{div} v_h) + r_h(p_h, v_h),$$

with

$$r_h(p, v) = \int_\Gamma p v \cdot n \, ds.$$

Here and further  $\partial_n^\ell q$  on face  $F$  denotes the derivative of order  $\ell$  of  $q$  in direction  $n$ , where  $n$  is normal to  $F$ ; and  $[\phi]$  denotes the jump of a quantity  $\phi$  over a face  $F$ .

We can now define the numerical method: Find  $u_h \in V_h$  and  $p_h \in Q_h$  such that

$$(3.2) \quad \begin{cases} a_h(u_h, v_h) + b_h(p_h, v_h) = (f, v_h), \\ b_h(q_h, u_h) - J_h(p_h, q_h) = 0 \end{cases}$$

for all  $v_h \in V_h$ ,  $q_h \in Q_h$ , where

$$J_h(q, p) = \sum_{F \in \mathcal{F}_h^\Gamma} \sum_{\ell=0}^{k_p} h_F^{1+2\ell} \int_F [\partial_n^\ell q] [\partial_n^\ell p] ds.$$

The unfitted FE method in (3.2) was introduced in [16].

Pressure solutions to both (2.1) and (3.2) are defined up to an additive constant. It is convenient to assume that the restriction of  $p_h$  on  $\Omega_h^i$  is from  $L_0^2(\Omega_h^i)$ . We shall fix one particular  $p$  solving (2.1) later.

Before proceeding with the analysis, we briefly discuss the role of different terms in the finite element formulation (3.2). First note that all volume integrals in (3.2) are computed over physical domains  $\Omega$  and  $\Gamma$  rather than computational domain  $\Omega_h^e$ . The gradient and div-terms appear due to the integration by parts in a standard weak formulation of the Stokes problem. Since finite element velocity trial and test functions do not satisfy homogenous Dirichlet conditions strongly on  $\Gamma$ , the integration by parts brings the  $s_h$  and  $r_h$  terms to the formulation. The  $-\int_\Gamma (n \cdot \nabla v) \cdot u ds$  integral in  $s_h$  is added to make formulation symmetric. It vanishes for  $u$ , the Stokes equations solution. The same is true for the  $r_h$  term in the continuity equation in (3.2). The penalty term  $j_h(u_h, v_h)$  weakly enforces the Dirichlet boundary conditions for  $u_h$ , as common for the Nitsche method, with a parameter  $\eta = O(1)$ . The terms  $\mathbf{j}_h(u_h, v_h)$  and  $J_h(p_h, q_h)$  are added for the numerical stability of the method: we need  $\mathbf{j}_h(u_h, v_h)$  to gain control over normal velocity derivatives in  $s_h$ , and we need  $J_h$  for pressure stability over cut triangles. In practice, both  $\mathbf{j}_h$  and  $J_h$  can be scaled by additional stabilization parameters of  $O(1)$  order; we omit this detail here.

We note that the unfitted FEM analyzed in the paper is closely related to the extended finite element method (XFEM). Indeed, the trace space of background finite element functions on the domain  $\Omega$  can be alternatively described as a FE space spanned over nodal shape functions from  $\Omega$  and further enriched by certain degrees of freedom tailored to  $\partial\Omega$ . Hence the results of this paper can be as well considered as the analysis of a certain class of XFEM methods for the Stokes problem.

The next section proves the key result for getting numerical stability and optimal order error estimates for the unfitted finite element method (3.2).

#### 4. STABILITY

We need to define some norms and semi-norms. First we define the mesh-dependent norm for the velocity

$$\|u\|_{V_h}^2 = |u|_{H^1(\Omega)}^2 + j_h(u, u) + \mathbf{j}_h(u, u).$$

Note that due to the boundary term  $j_h$ , the functional  $\|u\|_{V_h}$  defines a norm on  $V_h$  equivalent to the  $H^1(\Omega)$  norm,  $\|u\|_{H^1(\Omega)} \lesssim \|u\|_{V_h} \lesssim h_{\min}^{-1} \|u\|_{H^1(\Omega)}$ ,  $h_{\min} = \min_{T \in \mathcal{T}_h^e} h_T$ . We need a set of all tetrahedra intersected by  $\Gamma$  together with all tetrahedra from  $\Omega$  touching those:

$$\tilde{\mathcal{T}}_h^\Gamma = \{T : T \in \mathcal{T}_h^\Gamma \text{ or } T \subset \Omega, \overline{T} \cap \overline{\Omega_h^\Gamma} \neq \emptyset\},$$

and also

$$\tilde{\Omega}_h^\Gamma := \text{Int} \left( \bigcup_{T \in \tilde{\mathcal{T}}_h^\Gamma} \overline{T} \right).$$

For a generic set of tetrahedra  $\mathcal{T} \subset \mathcal{T}_h$  denote  $\omega(\mathcal{T}) \subset \mathcal{T}_h$  the set of all tetrahedra having at least one vertex in  $\mathcal{T}$ . We need the following assumptions on how well the geometry is resolved by the mesh.

**Assumption 1.** For any  $T \in \mathcal{T}_h^\Gamma$  we assume that the set  $W(T) = \mathcal{T}_h^i \cap \omega(\mathcal{T})$  is not empty.

We note that the assumption can be weakened by allowing in  $W(T)$  neighbors of  $T$  of degree  $L$ , with some finite and mesh independent  $L \geq 2$ .

Given  $T \in \mathcal{T}_h^\Gamma$  we associate an arbitrary but fixed  $K_T \in W(T)$ , which can be reached from  $T$  by crossing faces in  $\mathcal{F}_h^\Gamma$ . More precisely, there exist simplices  $T = K_1, K_2, \dots, K_M = K_T$  with  $K_j \in \mathcal{T}_h^\Gamma$  for  $j < M$ . The number  $M$  is uniformly bounded and only depends on the shape regularity of the mesh. Note that by (3.1) there exists a constant  $c$  only depending on the shape regularity constant  $\kappa$  such that  $\frac{1}{c}h_T \leq h_{K_T} \leq ch_T$ . For  $T \in \mathcal{T}_h^i$  we define  $K_T = T$ .

**Assumption 2.** Let  $F \in \mathcal{F}_h^\Gamma$  with  $F = \partial T_1 \cap \partial T_2$ . We assume  $K_{T_2}$  can be reached from  $K_{T_1}$  by crossing a finite, independent of  $h$ , number of faces of tetrahedra from  $\mathcal{T}_h^i$ .

We recall that we assume that  $\Omega$  is Lipschitz.

*Remark 1.* One can check that the Assumptions 1 and 2 are satisfied if  $h$  is sufficiently small and the minimal angle condition (3.1) holds. This is an improvement of the available analysis of unfitted finite elements which commonly imposes a further restriction on how interface intersects  $\mathcal{T}_h$ . In 2D this extra assumption is formulated as follows:  $\partial\Omega$  does not intersect any edge from  $\mathcal{F}_h^e$  more than one time; see, e.g., [29]. An analogous restriction was commonly assumed in 3D. One easily builds an example showing that this extra assumption is not necessarily true for arbitrary fine mesh and smooth  $\partial\Omega$ , while enforcing it by “eliminating” ineligible elements introduces  $O(h^2)$  geometrical error diminishing possible benefits of using higher order elements. We do not need this extra assumption.

Assumptions 1 and 2 also allow local mesh refinement.

We will make use of the following well-known scaled trace inequality:

$$(4.1) \quad \|v\|_{L^2(\partial T)} \leq C(h_T^{-\frac{1}{2}}\|v\|_{L^2(T)} + h_T^{\frac{1}{2}}\|\nabla v\|_{L^2(T)}) \quad \forall v \in H^1(T).$$

We will also need a local trace inequality for parts of  $\Gamma$ . We give the proof of the result only assuming that the boundary is Lipschitz in the appendix. Under various stronger assumptions the following result was proved in [18, 29, 30, 40].

**Lemma 1.** Under the assumption that  $\Omega$  is Lipschitz we have the following inequality for every  $T \in \mathcal{T}_h^\Gamma$ :

$$(4.2) \quad \|v\|_{L^2(T \cap \Gamma)} \leq C(h_T^{-\frac{1}{2}}\|v\|_{L^2(T)} + h_T^{\frac{1}{2}}\|\nabla v\|_{L^2(T)}) \quad \forall v \in H^1(T),$$

with a constant  $C$  independent of  $v$ ,  $T$ , how  $\Gamma$  intersects  $T$ , and  $h < h_0$  for some arbitrary but fixed  $h_0$ .

One can show the following stability result.

**Lemma 2.** For  $\eta$  sufficiently large and  $h \leq h_0$  for sufficiently small  $h_0$ , there exists a mesh-independent constant  $c_0 > 0$  such that

$$(4.3) \quad c_0\|v_h\|_{V_h}^2 \leq a_h(v_h, v_h) \quad \forall v_h \in V_h.$$

*Proof.* To show (4.3) we need the following estimate (see Lemma 5.1 in [36]): For any  $T_1, T_2$  from  $\mathcal{T}_h^\Gamma$  sharing a face  $F = \overline{T_1} \cap \overline{T_2}$  it holds that

$$(4.4) \quad \|q\|_{L^2(T_1)}^2 \leq C \left( \|q\|_{L^2(T_2)}^2 + \sum_{\ell=0}^m h_F^{1+2\ell} \int_F [\partial_n^\ell q]^2 ds \right) \quad \forall q \in P_m(T_1) \times P_m(T_2),$$

with a constant  $C$  depending only on the shape regularity of  $\mathcal{T}_h$  and polynomial degree  $m$ . Thanks to FE inverse inequality, (4.4) and Poincaré inequality, we have for any  $T_1, T_2$  from  $\tilde{\mathcal{T}}_h^\Gamma$  sharing a face  $F = \overline{T_1} \cap \overline{T_2}$  the following estimate:

$$(4.5) \quad \begin{aligned} \|\nabla v_h\|_{L^2(T_1)}^2 &\leq C h_F^{-2} \|v_h - \alpha\|_{L^2(T_1)}^2 \\ &\leq C \left( h_{T_2}^{-2} \|v_h - \alpha\|_{L^2(T_2)}^2 + \sum_{\ell=1}^s h_F^{-1+2\ell} \int_F [\partial_n^\ell v_h]^2 \right) \\ &\leq C \left( \|\nabla v_h\|_{L^2(T_2)}^2 + \sum_{\ell=1}^s h_F^{-1+2\ell} \int_F [\partial_n^\ell v_h]^2 \right) \quad \forall v_h \in V_h, \end{aligned}$$

where we take  $\alpha = |T_2|^{-1} \int_{T_2} v_h ds$ . This inequality is also found in Proposition 5.1 in [36] in the case  $V_h = [W_h^1]^d$ . Thanks to Assumption 1 the estimate (4.5) implies

$$(4.6) \quad \|\nabla v_h\|_{L^2(\Omega_h^e)}^2 \leq C (\|\nabla v_h\|_{L^2(\Omega)}^2 + \mathbf{j}_h(v_h, v_h)).$$

Further, one uses the Cauchy-Schwarz inequality, trace inequality (4.2), and the FE inverse inequality to estimate

$$\begin{aligned} |s_h(v_h, v_h)| &= \left| \int_\Gamma (n \cdot \nabla v_h) \cdot v_h ds \right| \leq \sum_{T \in \mathcal{T}_h^\Gamma} \|\nabla v_h\|_{L^2(T_\Gamma)} \|v_h\|_{L^2(T_\Gamma)} \\ &\leq \frac{1}{2\eta} \sum_{T \in \mathcal{T}_h^\Gamma} h_T \|\nabla v_h\|_{L^2(T_\Gamma)}^2 + \frac{\eta}{2} j_h(v_h, v_h) \\ &\leq \frac{C}{2\eta} \sum_{T \in \mathcal{T}_h^\Gamma} (\|\nabla v_h\|_{L^2(T)}^2 + h_T^2 \|\nabla^2 v_h\|_{L^2(T)}^2) + \frac{\eta}{2} j_h(v_h, v_h) \\ &\leq \frac{C}{2\eta} \|\nabla v_h\|_{L^2(\Omega_h^e)}^2 + \frac{\eta}{2} j_h(v_h, v_h). \end{aligned}$$

Combining this with (4.6) and choosing  $\eta$  sufficiently large, but independent of  $h$ , proves the lemma.  $\square$

We need to define the scaled semi-norms for the pressure:

$$\begin{aligned} |p|_{H_{h,i}^1}^2 &= \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla p\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_h^i} h_F \|[p]\|_{L^2(F)}^2, \\ |p|_{H_{h,e}^1}^2 &= \sum_{T \in \mathcal{T}_h^e} h_T^2 \|\nabla p\|_{L^2(T)}^2 + \sum_{F \in \mathcal{F}_h^e} h_F \|[p]\|_{L^2(F)}^2. \end{aligned}$$

**Assumption 3.** Assume that there exists a constant  $\beta > 0$  independent of  $h$  and only depending on polynomial degree of finite element spaces and the shape regularity of  $\mathcal{T}_h$  such that

$$(4.7) \quad \beta |q|_{H_{h,i}^1} \leq \sup_{v \in V_h^i} \frac{\int_{\Omega_h^i} q \operatorname{div} v dx}{\|v\|_{H^1(\Omega_h^i)}} \quad \forall q \in Q_h,$$

where  $V_h^i = V_h \cap [H_0^1(\Omega_h^i)]^d$ .



We also need the following extension result. A proof of this result is given in the appendix.

**Lemma 3.** *For every  $q \in Q_h$  there exists a  $E_h q \in Q_h^{\text{disc}}$  such that*

$$E_h q = q \text{ on } \Omega_h^i$$

and

$$(4.8) \quad |E_h q|_{H_{h,e}^1} \leq C |q|_{H_{h,i}^1}.$$

Using the degrees of freedom of piecewise linear functions one can show the following result.

**Lemma 4.** *For every  $v_h \in W_h^1$  there exists a unique decomposition*

$$(4.9) \quad v_h = \pi_1 v_h + \pi_2 v_h,$$

where  $\pi_i v_h \in W_h^1$  for  $i = 1, 2$ ,  $\pi_2 v_h$  is supported in  $\tilde{\Omega}_h^\Gamma$  and such that

$$(4.10) \quad \pi_2 v_h = v_h \quad \text{on } \Omega_h^\Gamma$$

and

$$(4.11) \quad \sum_{T \in \tilde{\mathcal{T}}_h^\Gamma} \frac{1}{h_T^2} \|\pi_2 v_h\|_{L^2(T)}^2 \leq C \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{h_T^2} \|v_h\|_{L^2(T)}^2.$$

The constant  $C$  is independent of  $v_h$  and  $h$  and only depends on the shape regularity of the mesh. In particular, note that this implies  $\pi_1 v_h \in V_h^i$ .

*Proof.* For a set of tetrahedra  $\tau$ ,  $V(\tau)$  denotes the set of all vertices of tetrahedra from  $\tau$ . For  $v_h \in W_h^1$ , one defines  $\pi_2 v_h(x) = v_h(x)$  for all  $x \in V(\mathcal{T}_h^\Gamma)$  and  $\pi_2 v_h(x) = 0$  for all  $x \in V(\mathcal{T}_h \setminus \mathcal{T}_h^\Gamma)$ . It is clear that  $\pi_2 v_h = v_h$  on  $\Omega_h^\Gamma$  and  $\pi_1 v_h \in V_h^i$ . For any  $T \in \tilde{\mathcal{T}}_h^\Gamma$  let  $\tilde{\omega}(T) = \omega(T) \cap \mathcal{T}_h^\Gamma$ . Thanks to the shape regularity assumption, we have for any  $\tilde{T} \in \tilde{\mathcal{T}}_h^\Gamma$ :

$$\begin{aligned} h_{\tilde{T}}^{-2} \|\pi_2 v_h\|_{L^2(\tilde{T})}^2 &\leq C h_{\tilde{T}}^{-2} |\tilde{T}| \sum_{x \in V(\tilde{T})} |\pi_2 v_h(x)|^2 = h_{\tilde{T}}^{-2} |\tilde{T}| \sum_{x \in V(\tilde{T}) \cap V(\mathcal{T}_h^\Gamma)} |\pi_2 v_h(x)|^2 \\ &= h_{\tilde{T}}^{-2} |\tilde{T}| \sum_{x \in V(\tilde{T}) \cap V(\mathcal{T}_h^\Gamma)} |v_h(x)|^2 \leq C \sum_{T \in \tilde{\omega}(\tilde{T})} h_T^{-2} |T| \sum_{x \in V(T)} |v_h(x)|^2 \\ &\leq C \sum_{T \in \tilde{\omega}(\tilde{T})} h_T^{-2} \|v_h\|_{L^2(T)}^2. \end{aligned}$$

Summing over all  $\tilde{T} \in \tilde{\mathcal{T}}_h^\Gamma$  and using shape regularity again we prove the result in (4.11).  $\square$

The following theorem shows the LBB stability result for the internal domain  $\Omega_h^i$  and so proves the key uniform bound (2.2). Note again that  $\Omega_h^i$  is not an  $O(h^2)$  approximation of a smooth domain and there is no uniform in  $h$  result concerning decomposition of  $\Omega_h^i$  into a union of a finite number of star-shaped domains. The latter is a standard assumption for proving the differential counterpart of this finite element condition; see, e.g., [25]. This result is crucial for the stability and convergence analysis of the unfitted FE method (3.2). For the lowest order Taylor-Hood element, the proof of the following result is found in [32]. We follow a similar argument, but extend the result so it can be applied to higher order elements in two and three dimensions.

**Theorem 1.** *Suppose Assumptions 1–3 hold. Then, there exists a constant  $\theta > 0$  and a constant  $h_0$  such that for all  $q \in Q_h$  with  $\int_{\Omega_h^i} q dx = 0$  we have the following result for  $h \leq h_0$ :*

$$(4.12) \quad \theta \|q\|_{L^2(\Omega_h^i)} \leq \sup_{v \in V_h^i} \frac{\int_{\Omega_h^i} q \operatorname{div} v \, dx}{\|v\|_{H^1(\Omega_h^i)}}.$$

The constant  $\theta > 0$  is independent of  $q$  and  $h$ .

*Proof.* Let  $\psi = E_h q$  given by Lemma 3 and let  $c_h = \frac{1}{|\Omega|} \int_{\Omega} E_h q$ . Using that  $\Gamma$  is Lipschitz, there exists a  $v \in [H_0^1(\Omega)]^2$  with the following properties (cf. [11, 25]):

$$(4.13) \quad \operatorname{div} v = \psi - c_h \quad \text{on } \Omega$$

and

$$(4.14) \quad \|v\|_{H^1(\Omega)} \leq C \|\psi - c_h\|_{L^2(\Omega)}.$$

Extend  $v$  by zero to all of  $\Omega_h^e$ . Let  $v_h \in W_h^1$  be the Scott-Zhang interpolant of  $v$  and  $v_h|_{\partial\Omega_h^e} = 0$ . We will write  $(v, w)_e = \int_{\Omega_h^e} v w dx$ . With the help of (4.13) and the decomposition (4.9), we obtain

$$(4.15) \quad \|\psi - c_h\|_{L^2(\Omega)}^2 = (\operatorname{div} v, \psi) = (\operatorname{div} \pi_1 v_h, \psi)_e + (\operatorname{div}(v - v_h), \psi)_e + (\operatorname{div} \pi_2 v_h, \psi)_e.$$

Integration by parts over each  $T \in \mathcal{T}_h^e$  gives

$$(\operatorname{div}(v - v_h), \psi)_e = - \sum_{T \in \mathcal{T}_h^e} \int_T (v - v_h) \cdot \nabla \psi \, dx - \sum_{F \in \mathcal{F}_h^e} \int_F [\psi] (v - v_h) \cdot n \, ds.$$

We proceed by applying the Cauchy-Schwarz inequality, elementwise trace inequality, and the definition of the  $H_{h,e}^1$  norm. This gives the bound

$$(\operatorname{div}(v - v_h), \psi)_e \leq C \left( \sum_{T \in \mathcal{T}_h^e} \left( \frac{1}{h_T^2} \|v - v_h\|_{L^2(T)}^2 + \|\nabla(v - v_h)\|_{L^2(T)}^2 \right) \right)^{1/2} |\psi|_{H_{h,e}^1}.$$

Using the approximation properties of the Scott-Zhang interpolant, (4.14) and (4.8), we have

$$(4.16) \quad (\operatorname{div}(v - v_h), \psi)_e \leq C \|\psi - c_h\|_{L^2(\Omega)} |\psi|_{H_{h,e}^1} \leq C \|\psi - c_h\|_{L^2(\Omega)} |q|_{H_{h,i}^1}.$$

In a similar fashion, but now using inverse FE estimates instead of approximation results, and recalling that  $\operatorname{supp}(\pi_2 v_h) \subset \tilde{\Omega}_h^\Gamma$ , we show

$$(4.17) \quad \begin{aligned} (\operatorname{div} \pi_2 v_h, \psi)_e &\leq C \left( \sum_{T \in \tilde{\mathcal{T}}_h^\Gamma} \frac{1}{h_T^2} \|\pi_2 v_h\|_{L^2(T)}^2 \right)^{1/2} |\psi|_{H_{h,e}^1} \\ &\leq C \left( \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{h_T^2} \|v_h\|_{L^2(T)}^2 \right)^{1/2} |\psi|_{H_{h,e}^1}. \end{aligned}$$

Note the following Friedrich's-type FE inequality:

$$\begin{aligned} h_T^{-2} \|v_h\|_{L^2(T)}^2 + h_T^{-1} \|v_h\|_{L^2(\partial T)}^2 \\ \leq C (\|\nabla v_h\|_{L^2(T)}^2 + h_T^{-1} \|v_h\|_{L^2(F)}^2) \quad \forall T \in \mathcal{T}_h, F \text{ is a face of } T. \end{aligned}$$

We apply the above inequality elementwise and use  $v_h = 0$  on  $\partial\Omega_h^e$  to show that

$$(4.18) \quad \left( \sum_{T \in \mathcal{T}_h^\Gamma} \frac{1}{h_T^2} \|v_h\|_{L^2(T)}^2 \right)^{1/2} \leq C \|\nabla v_h\|_{L^2(\Omega_h^\Gamma)} \leq C \|\nabla v\|_{L^2(\Omega_h^e)} = C \|\nabla v\|_{L^2(\Omega)}.$$

In the last inequality we used the stability of the Scott-Zhang interpolant. Hence, using (4.14) we get from (4.17)–(4.18) the estimate

$$(4.19) \quad (\operatorname{div} \pi_2 v_h, \psi)_e \leq C \|\psi - c_h\|_{L^2(\Omega)} |q|_{H_{h,i}^1}.$$

The last term on the right-hand side of (4.15) we handle as follows:

$$(\operatorname{div} \pi_1 v_h, \psi)_e = (\operatorname{div} \pi_1 v_h, \psi)_{L^2(\Omega_h^i)} \leq \|\pi_1 v_h\|_{H^1(\Omega_h^i)} \sup_{w \in V_h^i} \frac{(\operatorname{div} w, q)_{L^2(\Omega_h^i)}}{\|w\|_{H^1(\Omega_h^i)}}.$$

Now we bound  $\|\pi_1 v_h\|_{H^1(\Omega)}$ :

$$\|\pi_1 v_h\|_{H^1(\Omega)} \leq (\|\pi_2 v_h\|_{H^1(\tilde{\Omega}_h^\Gamma)} + \|v_h\|_{H^1(\Omega)}).$$

Using inverse estimates, (4.11) and (4.18) we get

$$\|\pi_2 v_h\|_{H^1(\tilde{\Omega}_h^\Gamma)} \leq C \|v\|_{H^1(\Omega_h^e)} = C \|v\|_{H^1(\Omega)}.$$

Hence, the stability of the Scott-Zhang interpolant and (4.14) imply

$$(4.20) \quad \|\pi_1 v_h\|_{H^1(\Omega)} \leq C \|\psi - c_h\|_{L^2(\Omega)}.$$

Therefore, we get from (4.16), (4.19), (4.20), and (4.15), the upper bound

$$\|\psi - c_h\|_{L^2(\Omega)} \leq C \left( \sup_{w \in V_h^i} \frac{(\operatorname{div} w, q)_{L^2(\Omega_h^i)}}{\|w\|_{H^1(\Omega_h^i)}} + |q|_{H_{h,i}^1} \right).$$

Using Assumption 3 we get

$$(4.21) \quad \|\psi - c_h\|_{L^2(\Omega)} \leq C \sup_{w \in V_h^i} \frac{(\operatorname{div} w, q)_{L^2(\Omega_h^i)}}{\|w\|_{H^1(\Omega_h^i)}}.$$

Finally, note that

$$\|c_h\|_{L^2(\Omega)} \leq |\Omega|^{1/d} |c_h| = |\Omega|^{-1+1/d} \left| \int_{\Omega} \psi dx \right| = |\Omega|^{-1+1/d} \left| \int_{\Omega \setminus \Omega_h^i} \psi dx \right|.$$

The last equality holds since  $\psi = q$  in  $\Omega_h^i$  and  $\int_{\Omega_h^i} q dx = 0$ . After applying Cauchy-Schwarz inequality and using that  $|\Omega \setminus \Omega_h^i|^{1/d} \leq h^{1/d}$  we have that

$$\|c_h\|_{L^2(\Omega)} \leq Ch^{1/d} \|\psi\|_{L^2(\Omega)}.$$

Hence, using the triangle inequality in (4.21) and assuming  $h$  is sufficiently small, we have

$$\|\psi\|_{L^2(\Omega)} \leq C \sup_{w \in V_h^i} \frac{(\operatorname{div} w, q)_{L^2(\Omega_h^i)}}{\|w\|_{H^1(\Omega_h^i)}}.$$

We note that the constant  $C$  is independent of  $h$  and  $q$ . The result now follows after noting that  $\|q\|_{L^2(\Omega_h^i)} \leq \|\psi\|_{L^2(\Omega)}$  and letting  $\theta = \frac{1}{C}$ .  $\square$

**Corollary 1.** *If Assumptions 1–3 hold true, the following stability condition is satisfied by the  $b_h$  and  $J_h$  forms of the finite element method (3.2),*

$$(4.22) \quad c_b \|q\|_{L^2(\Omega)} \leq \sup_{v \in V_h} \frac{b_h(v, q)}{\|v\|_{V_h}} + J_h^{\frac{1}{2}}(q, q) \quad \forall q \in Q_h, \text{ s.t. } q|_{\Omega_h^i} \in L_0^2(\Omega_h^i).$$

The constant  $c_b > 0$  is independent of  $q$  and  $h$ .

*Proof.* Fix some  $q \in Q_h$ , such that  $q|_{\Omega_h^i} \in L_0^2(\Omega_h^i)$ . Using (4.4), Assumption 1 and the finite overlap argument, one shows that

$$\|q\|_{L^2(\Omega)}^2 \leq c (\|q\|_{L^2(\Omega_h^i)}^2 + J_h(q, q)).$$

Thanks to the uniform inf-sup property from Theorem 1 there exists  $v \in V_h$  with  $\text{supp}(v) \subset \Omega_h^i$  such that

$$(4.23) \quad \|q\|_{L^2(\Omega)}^2 \leq c \left( \frac{(\text{div } v, q)^2}{\|v\|_{H^1(\Omega_h^i)}^2} + J_h(q, q) \right).$$

Using  $v = 0$  in  $\Omega_h^\Gamma$  and applying the FE inverse inequalities we show

$$\begin{aligned} \mathbf{j}_h(v, v) &= \sum_{F \in \mathcal{F}_h^\Gamma, \text{ s.t. } F \subset \partial\Omega_h^i} \sum_{\ell=1}^s h_F^{2\ell-1} \int_F [\partial_n^\ell v]^2 \\ &\leq C \sum_{T \in \tilde{\mathcal{T}}_h^\Gamma \cap \mathcal{T}_h^i} \|\nabla v\|_{L^2(T)}^2 ds \leq C \|v\|_{H^1(\Omega_h^i)}^2. \end{aligned}$$

This estimate and  $j_h(v, v) = 0$  for  $v \in V_h$  with  $\text{supp}(v) \subset \Omega_h^i$  imply the uniform equivalence  $\|v\|_{V_h} \simeq \|v\|_{H^1(\Omega_h^i)}$ . Using this in (4.23) yields

$$\|q\|_{L^2(\Omega)}^2 \leq c \left( \frac{(\text{div } v, q)^2}{\|v\|_{V_h}^2} + J_h(q, q) \right).$$

Finally, we note that  $(\text{div } v, q) = b_h(v, q)$  if  $\text{supp}(v) \subset \Omega_h^i$ . This completes the proof.  $\square$

## 5. WELL POSEDNESS AND ERROR ESTIMATES

One easily verifies that  $a_h$  is continuous:

$$a_h(u, v) \leq C_a \|u\|_{V_h} \|v\|_{V_h} \quad \forall u, v \in V_h,$$

with some  $C_a > 0$  independent of  $h$  and the position of  $\Gamma$ . The continuity and coercivity of the  $a_h(u, v)$  form (Lemma 2) and the inf-sup stability of the  $b_h(v, q)$  form (Corollary 1) readily imply the stability for the bilinear form of the finite element method (3.2) with respect to the product norm,

$$(5.1) \quad C_s \|u_h, p_h\| \leq \sup_{\{v, q\} \in V_h \times Q_h} \frac{A_h(u_h, p_h; v, q)}{\|v, q\|} \quad \forall \{u_h, p_h\} \in V_h \times Q_h,$$

with some  $C_s > 0$  independent of  $h$  and the position of  $\Gamma$  and

$$A_h(u, p; v, q) := a_h(u, v) + b_h(v, p) + b_h(u, q) - J_h(p, q),$$

$$\|v, q\| := \left( \|v\|_{V_h}^2 + \|q\|_{L^2(\Omega)}^2 + J_h(q, q) \right)^{\frac{1}{2}}.$$

The proof of (5.1) extends standard arguments (cf., e.g., [23]) for  $J_h \neq 0$ . For completeness we sketch the proof here. For given  $\{u_h, p_h\} \in V_h \times Q_h$ , thanks to (4.22), one can find  $z \in V_h$  such that  $\|z\|_{V_h} = \|p_h\|_{L^2(\Omega)}$  and

$$\begin{aligned} c_b \|p_h\|_{L^2(\Omega)}^2 &\leq b_h(z, p_h) + J_h^{\frac{1}{2}}(p_h, p_h) \|p_h\|_{L^2(\Omega)} \\ &= A_h(u_h, p_h; z, 0) - a_h(u_h, z) + J_h^{\frac{1}{2}}(p_h, p_h) \|p_h\|_{L^2(\Omega)} \\ &\leq A_h(u_h, p_h; z, 0) + \frac{C_a^2}{c_b} \|u_h\|_{V_h}^2 + \frac{c_b}{4} \|p_h\|_{L^2(\Omega)}^2 + \frac{1}{c_b} J_h(p_h, p_h) \\ &\quad + \frac{c_b}{4} \|p_h\|_{L^2(\Omega)}^2. \end{aligned}$$

Combining this inequality with

$$J_h(p_h, p_h) + c_0 \|u_h\|_{V_h}^2 \leq A_h(u_h, p_h; u_h, -p_h),$$

we get

$$c \|u_h, p_h\|^2 \leq A_h(u_h, p_h; u_h + \alpha z, -p_h),$$

for a suitable  $\alpha > 0$  and a constant  $c > 0$  depending only on  $c_b$ ,  $C_a$ , and  $c_0$ . Inequality (5.1) follows by noting  $\|u_h, p_h\| \geq \frac{1}{1+\alpha} \|v, q\|$ , with  $v = u_h - \alpha z$ ,  $q = -p_h$ .

One verifies that  $A_h$  is continuous

$$(5.2) \quad A_h(u, p; v, q) \leq C_c \|u, p\| \|v, q\| \quad \forall \{u, p\}, \{v, q\} \in V_h \times Q_h,$$

with some  $C_c > 0$  independent of  $h$  and the position of  $\Gamma$ . Note also that  $A_h$  is symmetric. Therefore, by the Banach–Nečas–Babuška theorem (see, e.g., Theorem 2.6 in [23]) the problem (3.2) is well-posed and its solution satisfies the stability bound

$$\|u_h, p_h\| \leq C_s^{-1} \|f\|_{V_h'}.$$

Further, in this section, we assume that the solution to the Stokes problem is sufficiently smooth, i.e.,  $u \in H^{s+1}(\Omega)$  and  $p \in H^{k_p+1}(\Omega)$ . Since we are assuming that  $\Gamma$  is Lipschitz, there exist extensions of  $u$  and  $p$ , which we also denote by  $u, p$ , such that  $u \in H^{s+1}(S)$  and  $p \in H^{k_p+1}(S)$  (see [43]). We let  $I_h u$  be the Scott–Zhang interpolant of  $u$  onto  $\left[W_h^{k_u}\right]^d$ . We also let  $I_h p$  be the Scott–Zhang interpolant of  $p$  in the case  $Q_h = Q_h^{\text{cont}}$  and the  $L^2$  projection onto discontinuous piecewise polynomials of degree  $k_p$  if  $Q_h = Q_h^{\text{disc}}$ . For the pressure interpolant we can always assume  $(I_h p)|_{\Omega_h^i} \in L_0^2(\Omega_h^i)$  by choosing a suitable additive constant in the definition of  $p$ . Applying trace inequalities (4.1) and (4.2), standard approximation properties of  $I_h$ , and extension results, one obtains the approximation property in the product norm:

$$(5.3)$$

$$\begin{aligned} &\|u - I_h u, p - I_h p\| \\ &\leq C \left( h^{\min\{k_u, k_p+1\}} (\|u\|_{H^{k_u+1}(\Omega)} + \|p\|_{H^{k_p+1}(\Omega)}) + h^{k_u} \sum_{\ell=k_u+1}^{s+1} h^{\ell-k_u-1} \|u\|_{H^\ell(\Omega)} \right). \end{aligned}$$

We also have the following continuity result and approximation results:

$$(5.4) \quad \begin{aligned} A_h(u - I_h u, p - I_h p; v, q) &\leq C \|u - I_h u, p - I_h p\| \|v, q\| \\ &\quad + |s_h(u - I_h u, v)| + |r_h(p - I_h p, v)|, \end{aligned}$$

$$(5.5) \quad \begin{aligned} &|s_h(u - I_h u, v)| + |r_h(p - I_h p, v)| \\ &\leq C h^{\min\{k_u, k_p+1\}} (\|u\|_{H^{k_u+1}(\Omega)} + \|p\|_{H^{k_p+1}(\Omega)}) \|v\|_{V_h}, \end{aligned}$$

for all  $\{v, q\} \in V_h \times Q_h$ . Here we used (4.2), (4.1), and (4.6).

Denote by  $e_u = u - u_h$  and  $e_p = p - p_h$  the finite element error functions. Note that for  $u \in H^{s+1}(S)$  and  $p \in H^{k_p+1}(S)$  the jumps of derivatives in bilinear forms  $\mathbf{j}_h$  and  $J_h$  vanish. This and the boundary condition  $u|_\Gamma = 0$  imply  $\mathbf{j}_h(u, v_h) = j_h(u, v_h) = J_h(p, q_h) = 0$  and  $s_h(u, v_h) = -\int_\Gamma (n \cdot \nabla u) \cdot v_h$ . Hence, it is easy to see that the method (3.2) is consistent for  $u$  and  $p$  sufficiently smooth as stated above, i.e., (3.2) is satisfied with  $u_h$  replaced by  $u$ . Therefore, the Galerkin orthogonality holds:

$$(5.6) \quad A_h(e_u, e_p; v_h, q_h) = 0$$

for all  $v_h \in V_h$  and  $q_h \in Q_h$ .

The optimal order error estimate in the energy norm is given in the next theorem.

**Theorem 2.** *For sufficiently smooth  $u, p$  solving (2.1) and  $u_h, p_h$  solving (3.2), the error estimate holds:*

$$\begin{aligned} |u - u_h|_{H^1(\Omega)} + \|p - p_h\|_{L^2(\Omega)} &\leq \|u - u_h, p - p_h\| \\ &\leq C \left( h^{\min\{k_u, k_p+1\}} (\|u\|_{H^{k_u+1}(\Omega)} + \|p\|_{H^{k_p+1}(\Omega)}) + h^{k_u} \sum_{\ell=k_u+1}^{s+1} h^{\ell-k_u-1} \|u\|_{H^\ell(\Omega)} \right), \end{aligned}$$

with a constant  $C$  independent of  $h$  and the position of  $\Gamma$  with respect to the triangulation  $\mathcal{T}_h$ .

*Proof.* The results follows from the inf-sup stability (5.1), continuity (5.4), Galerkin orthogonality (5.6), and approximation properties (5.3), (5.5), by standard arguments; see, for example, section 2.3 in [23].  $\square$

Using the Aubin-Nitsche duality argument one shows the optimal order error estimate for the velocity in  $L^2(\Omega)$ -norm. Consider the dual adjoint problem. Let  $w \in [H_0^1(\Omega)]^d$  and  $r \in L_0^2(\Omega)$  be the solution to the problem

$$(5.7) \quad \begin{cases} -\Delta w - \nabla r = e_u & \text{in } \Omega, \\ \operatorname{div} w = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

We assume that  $\Omega$  is such that (5.7) is  $H^2$ -regular, i.e., for  $e_u \in [L^2(\Omega)]^d$  it holds that  $w \in [H^2(\Omega)]^d$  and  $r \in H^1(\Omega)$  and

$$\|w\|_{H^2(\Omega)} + \|r\|_{H^1(\Omega)} \leq C(\Omega) \|e_u\|_{L^2(\Omega)}.$$

By the standard arguments (section 2.3 in [23]) the results in (5.1), (5.2), (5.6), (5.3), and the above regularity assumption lead to the following theorem.

**Theorem 3.** *For sufficiently smooth  $u, p$  solving (2.1) and  $u_h, p_h$  solving (3.2), the error estimate holds:*

$$|u - u_h|_{L^2(\Omega)} \leq Ch \|u - u_h, p - p_h\|,$$

with a constant  $C$  independent of  $h$  and the position of  $\Gamma$  with respect to the triangulation  $\mathcal{T}_h$ .

## 6. EXAMPLE OF SPACES SATISFYING ASSUMPTION 3

**6.1. Generalized Taylor-Hood elements.** Consider  $Q_h = Q_h^{\text{cont}} := W_h^k$  and  $V_h = [W_h^{k+1}]^d$ ,  $k \geq 1$ . In this case, the proof of estimate (4.7) from Assumption 3 is given in section 8 of [9] for  $d = 2$  (two-dimensional case). In three-dimensional case and  $k = 1$ , the result can be found in Lemma 4.23 in [23]. Below we extend the proof for all  $k \geq 1$  in 3D. We require each  $T \in \mathcal{T}_h^i$  to have at least three edges in the interior of  $\Omega_h^i$ . Note that the proof in [9] for  $d = 2$  does not need a similar assumption. For any edge from the set of internal edges of  $\mathcal{T}_h^i$ ,  $E \in \mathcal{E}_h^i$ , we denote a unit tangent vector be  $t_E$  (any of two, but fixed),  $x_E$  is the midpoint of  $E$ , and  $\omega(E)$  is a set of tetrahedra sharing  $E$ . Also we denote by  $\phi_E \in W_h^2(\Omega_h^i)$  a piecewise quadratic function such that  $\phi_E(x_E) = 1$  and  $\phi_E(x) = 0$ , where  $x$  is any vertex or a midpoint of any other edge from  $\mathcal{E}_h^i$ . For  $p \in Q_h$  we set

$$v(x) = - \sum_{E \in \mathcal{E}_h^i} h_E^2 \phi_E(x) [t_E \cdot \nabla p(x)] t_E.$$

Since the pressure tangential derivative  $t_E \cdot \nabla p$  is continuous across faces  $F$  that contain  $E$ , it is easy to see that  $v \in V_h^i$ . We compute

$$\begin{aligned} \int_{\Omega_h^i} \operatorname{div} v p \, dx &= - \int_{\Omega_h^i} v \cdot \nabla p \, dx = \sum_{E \in \mathcal{E}_h^i} h_E^2 \int_{\omega(E)} \phi_E |t_E \cdot \nabla p|^2 \, dx \\ &\geq c \sum_{E \in \mathcal{E}_h^i} h_E^2 \int_{\omega(E)} |t_E \cdot \nabla p|^2 \, dx. \end{aligned}$$

The constant  $c > 0$  in the last inequality depends only on the polynomial degree  $k$  and shape regularity condition (3.1). From the condition (3.1) we also infer  $h_E \simeq h_T$  for  $T \in \omega(E)$ . This gives, after rearranging terms, the estimate

$$\begin{aligned} \int_{\Omega_h^i} \operatorname{div} v p \, dx &\geq c \sum_{T \in \mathcal{T}_h^i} \sum_{E \in \bar{T} \cap \Omega_h^i} h_T^2 \int_T |t_E \cdot \nabla p|^2 \, dx \\ &\geq c \sum_{T \in \mathcal{T}_h^i} h_T^2 \int_T |\nabla p|^2 \, dx = c \|p\|_{H_{h,i}^1}^2. \end{aligned}$$

For the last inequality we used the assumption that at least three edges of the tetrahedra are internal and we apply the shape regularity condition one more time. Due to the finite element inverse inequalities and the obvious estimate

$|\phi_E| + h_T |\nabla \phi_E| \leq c$  on  $T$ , with  $E \in \overline{T}$ , we have

$$\begin{aligned} \|v\|_{H^1(\Omega_h^i)}^2 &\leq c \int_{\Omega_h^i} |\nabla v|^2 dx \leq \sum_{T \in \mathcal{T}_h^i} \sum_{E \in \overline{T} \cap \Omega_h^i} h_T^4 \int_T (|\nabla \phi_E|^2 |\nabla p|^2 + |\phi_E|^2 |\nabla^2 p|^2) dx \\ &\leq \sum_{T \in \mathcal{T}_h^i} \int_T h_T^2 (|\nabla p|^2 + h_T^2 |\nabla^2 p|^2) dx \leq c \|p\|_{H_{h,i}^1}^2. \end{aligned}$$

This shows (4.7).

**6.2. Bercovier-Pironneau element.** This is a “cheap” version of the lowest order Taylor-Hood element. In 2D the element is defined in [6], the 3D version can be found, e.g., in [23]. To define the velocity space, one refines each triangle of  $\mathcal{T}_h$  by connecting midpoints on the edges in 2D, while in 3D one divides a tetrahedron into six tetrahedra by the same procedure. Then the velocity space consists of piecewise linear continuous function with respect to the refined triangulation,  $V_h = [W_{h/2}^1]^d$ , and  $Q_h = Q_h^{\text{cont}} := W_h^1$ . For this element, one shows (4.7) following the lines of the proof of Theorem 8.1 in [9] for  $k = 1$  in 2D or the arguments from the section 6.1 with obvious modifications: For example, in the 3D case one substitutes “edge-bubbles”  $\phi_E$  by their  $P_1$ iso $P_2$  counterparts.

**6.3.  $P_{k+2} - P_k^{\text{disc}}$  (for  $d = 2$ ) and  $P_{k+3} - P_k^{\text{disc}}$  (for  $d = 3$ ) elements.** We only consider the two-dimensional case  $d = 2$  as the case  $d = 3$  is similar. We let  $Q_h = Q_h^{\text{disc}}$  be the space of piecewise polynomial functions of degree  $k$  and let  $V_h = [W_h^{k+2}]^2$ . The canonical degrees of freedom of a function  $m \in P_{k+2}(T)$  are given by

$$\begin{aligned} \int_T m s dx &\quad \text{for all } s \in P_{k-1}(T), \\ \int_E m q dx &\quad \text{for all edges } E \text{ of } T, q \in P_k(E), \\ m(x) &\quad \text{for all the vertices } x \text{ of } T. \end{aligned}$$

To show Assumption 3 holds in this case, take  $q \in Q_h^{\text{disc}}$ . We can choose  $v \in V_h^i$  (using the degrees of freedom above) such that

$$\int_T v \cdot w dx = -h_T^2 \int_T \nabla q \cdot w dx \quad \text{for all } w \in P_{k-1}(T),$$

and for all  $T \in \mathcal{T}_h^i$ . Also for every interior edges  $E$  of  $\Omega_h^i$ ,

$$\int_E r v \cdot n^+ ds = h_E \int_E r (q^+ n^+ + q^- n^-) \cdot n^+ dx \quad \text{for all } r \in P_k(E),$$

where  $E = \partial T^+ \cap \partial T^-$  and  $T^+, T^- \in \mathcal{T}_h^i$ . Also,  $n^\pm$  is the outward pointing unit normal of  $T^\pm$ .

To pin down  $v \in V_h^i$ , we make  $v$  vanish on all vertices and have tangential components vanish on all edges. Finally, we make  $v \equiv 0$  on  $\partial \Omega_h^i$ .

Using elementwise integration by parts, we get

$$\int_{\Omega_h^i} \text{div } v q dx = \sum_{T \in \mathcal{T}_h^i} - \int_T v \cdot \nabla q dx + \int_{\partial T} q v \cdot n ds.$$



From the construction of  $v$ , we see that

$$\int_{\Omega_h^i} \operatorname{div} v q \, dx = \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2 + \sum_{E \in \mathcal{E}_h^i} h_F \| [q] \|_{L^2(F)}^2 = |q|_{H_h^i}^2.$$

It is not difficult to show, using a scaling argument that  $\|v\|_{H^1(\Omega_h^i)} \leq C|q|_{H_h^i}$ . From this we see that Assumption 3 holds.

**6.4. Bernardi-Raugel element.** In a similar way we can show that the Bernardi-Raugel spaces satisfy Assumption 3. The space of Bernardi-Raugel elements consists of piecewise constant pressure and for the velocity one takes  $P^1$  continuous functions enriched with the normal components of the velocity as a degree of freedom at barycenter face nodes [8].

**6.5. Mini-element.** Let  $\Omega \subset \mathbb{R}^2$ . With  $k_p = 1$ ,  $Q_h = Q_h^{\text{cont}}$  and  $V_h = [W_h^1]^2 + \{v : v|_T \in b_T c_T, \text{ where } c_T \in [\mathbf{P}^0(T)]^2 \text{ for all } T \in \mathcal{T}_h^e\}$ . Here  $b_T$  is the cubic bubble.

To prove (4.7) we consider an arbitrary  $q \in Q_h$ . A simple argument gives

$$\sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2 \leq C \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\sqrt{b_T} \nabla q\|_{L^2(T)}^2.$$

Integration by parts gives  $\|\sqrt{b_T} \nabla q\|_{L^2(T)}^2 = - \int_T \operatorname{div}(b_T \nabla q) q \, dx$ . If we define  $w_h \in V_h^i$  in the following way  $w_h|_T := -h_T^2 b_T \nabla q|_T$ , then we have

$$\sum_{T \in \mathcal{T}_h^i} h_T^2 \|\sqrt{b_T} \nabla q\|_{L^2(T)}^2 = \int_{\Omega_h^i} \operatorname{div} w_h q \, dx.$$

Hence, we get

$$\sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2 \leq C \sup_{v \in V_h^i} \frac{(\operatorname{div} v, q)}{\|v\|_{H^1(\Omega_h^i)}} \|w_h\|_{H^1(\Omega_h^i)}.$$

Now, using Poincaré's inequality

$$\|w_h\|_{H^1(\Omega_h^i)}^2 \leq C \sum_{T \in \mathcal{T}_h^i} \|\nabla w_h\|_{L^2(T)}^2 \leq C \sum_{T \in \mathcal{T}_h^i} h_T^4 \|\nabla b_T\|_{L^\infty(T)}^2 \|\nabla q\|_{L^2(T)}^2.$$

Since  $h_T^2 \|\nabla b_T\|_{L^\infty(T)}^2 \leq C$ , we get

$$\|w_h\|_{H^1(\Omega_h^i)}^2 \leq \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2.$$

The result now follows.

**6.6. Generalized conforming Crouzeix-Raviart element.** This element is defined by  $Q_h = Q_h^{\text{disc}}$  with  $k_p = k \geq 1$  for  $d = 2$  or  $k \geq 2$  for  $d = 3$ . We define the velocity space to be  $V_h = [W_h^{k+1}]^d + \{v : v|_T \in b_T \nabla \mathbf{P}^k(T) \text{ for all } T \in \mathcal{T}_h^e\}$ , where  $b_T$  is cubic bubble in two dimensions or quartic bubble in three dimensions. This  $P_{k+1}^{\text{bubble}} - P_k^{\text{disc}}$  space was first introduced in [20]. The proof of (4.7) in this case will be similar to that of mini-element. We leave the details to the reader.

## APPENDIX A. PROOF OF LEMMA 1

First we state a result found, for example, in [12, 26], which requires that  $\Gamma$  is Lipschitz.

**Proposition 1.** *There exists a constant  $C$  such that*

$$\|v\|_{L^2(\Gamma)}^2 \leq C \|v\|_{H^1(\Omega)} \|v\|_{L^2(\Omega)} \quad \text{for all } v \in H^1(\Omega).$$

For the moment we assume the following result.

**Lemma 5.** *Let  $T \in \mathcal{T}_h$ . There exists an extension operator  $R_T : H^1(T) \rightarrow H^1(R^d)$  such that*

$$(A.1) \quad R_T v = v \quad \text{on } T,$$

$$(A.2) \quad \|R_T v\|_{L^2(R^d)} + h_T \|\nabla R_T v\|_{L^2(R^d)} \leq C (\|v\|_{L^2(T)} + h_T \|\nabla v\|_{L^2(T)}),$$

where the constant  $C$  is independent of  $T$  and  $v$ .

**A.1. Proof of Lemma 1.** Let  $T \in \mathcal{T}_h$  and let  $v \in H^1(T)$ . Then, we have, using Proposition 1,

$$\begin{aligned} \|v\|_{L^2(T \cap \Gamma)} &\leq \|R_T v\|_{L^2(\Gamma)} \leq C \|R_T v\|_{L^2(R^d)}^{1/2} \|R_T v\|_{H^1(R^d)}^{1/2} \\ &\leq C \left( \|R_T v\|_{L^2(R^d)}^{1/2} \|\nabla R_T v\|_{L^2(R^d)}^{1/2} + \|R_T v\|_{L^2(R^d)} \right). \end{aligned}$$

We apply the arithmetic-geometric mean inequality and use  $h_T \leq h_0$  to get

$$\|v\|_{L^2(T)} \leq C (h_T^{-1/2} \|R_T v\|_{L^2(R^d)} + h_T^{1/2} \|\nabla R_T v\|_{L^2(R^d)}).$$

The result now follows after applying Lemma 5.

**A.2. Proof of Lemma 5.** We will denote the reference tetrahedra of unit size with a vertex at the origin  $\hat{T}$ . Then, we know ([43]) there exists an extension operator from  $R : H^1(\hat{T}) \rightarrow H_0^1(B_2)$  such that

$$(A.3) \quad R\hat{v} = \hat{v} \quad \text{on } \hat{T},$$

$$(A.4) \quad \|R\hat{v}\|_{H^1(B_2)} \leq C \|\hat{v}\|_{H^1(\hat{T})}.$$

Here  $B_2$  is the ball with radius 2 centered at the origin.

Let  $F_T : \hat{T} \rightarrow T$  be the onto affine mapping and has the form  $F_T(\hat{x}) = B\hat{x} + b$ . For any  $v \in H^1(T)$  we can define  $\hat{v} \in H^1(\hat{T})$  in the following way:  $\hat{v}(\hat{x}) = v(F_T(\hat{x}))$ .

Our desired extension will be given by

$$(R_T v)(x) = (R\hat{v})(F_T^{-1}(x))$$

For notational convenience we use  $w = R_T v$ . Then, we see that  $\hat{w} = R\hat{v}$ . Using a change of variables formula we get

$$\|\nabla w\|_{L^2(R^d)}^2 = \int_{F(B_2)} |\nabla w(x)|^2 dx = \int_{B_2} |B^{-t} \nabla \hat{w}(\hat{x})|^2 |\det B| d\hat{x}.$$

Using that the mesh is shape regular we have (see [19])  $|B_{ij}| \leq C h_T$ ,  $|B_{ij}^{-1}| \leq C h_T^{-1}$ . Therefore, we obtain

$$\int_{B_2} |B^{-t} \nabla \hat{w}(\hat{x})|^2 |\det B| d\hat{x} \leq C h_T^{d-2} \|\nabla \hat{w}\|_{L^2(B_2)}^2 = C h_T^{d-2} \|\nabla R\hat{v}\|_{L^2(B_2)}^2.$$

Using (A.4) we obtain

$$\|\nabla w\|_{L^2(R^d)}^2 \leq C h_T^{d-2} (\|\hat{v}\|_{L^2(\hat{T})}^2 + \|\nabla \hat{v}\|_{L^2(\hat{T})}^2).$$

It is standard to show, again using a change of variable formula, and the bounds for  $B$  and  $B^{-1}$  above that

$$h_T^{d-2}(\|\hat{v}\|_{L^2(\hat{T})}^2 + \|\nabla \hat{v}\|_{L^2(\hat{T})}^2) \leq C(h_T^{-2}\|v\|_{L^2(T)}^2 + \|\nabla v\|_{L^2(T)}^2).$$

Therefore, we have shown

$$h_T \|\nabla R_T v\|_{L^2(R^d)} \leq (\|v\|_{L^2(T)} + h_T \|\nabla v\|_{L^2(T)}).$$

The bound for  $\|R_T v\|_{L^2(R^d)}$  follows a similar argument.

## APPENDIX B. PROOF OF LEMMA 3

Let  $q \in Q_h$ . Note that the extension  $E_h q$  does not have to be continuous even if  $Q_h = Q_h^{\text{cont}}$ . Now for every  $T \in \mathcal{T}_h$  we let  $q_T^{\text{ext}} \in P^{k_p}(\mathbb{R}^d)$  be the natural extension of  $q_T \equiv q|_T$  to the entire  $\mathbb{R}^d$ .

For  $T \in \mathcal{T}_h^\Gamma$ , we define  $E_h q|_T = q_{K_T}^{\text{ext}}|_T$ , where  $K_T \in \Omega_h^i$  is given by Assumption 1 (see the remark right below the assumption). Since  $K_T \in W(T) \Rightarrow \text{dist}(K_T, T) \leq Ch_T$ , it follows that

$$\|\nabla E_h q\|_{L^2(T)} = \|\nabla q_{K_T}^{\text{ext}}\|_{L^2(T)} \leq C \|\nabla q\|_{L^2(K_T)}.$$

Hence, we have

$$(B.1) \quad \sum_{T \in \mathcal{T}_h^\Gamma} h_T^2 \|\nabla E_h q\|_{L^2(T)}^2 \leq C \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2.$$

To bound the face terms, we let  $F \in \mathcal{F}_h^\Gamma$ , where  $F = \partial T \cap \partial \tilde{T}$ . If we use the notation  $K = K_T$  and  $\tilde{K} = K_{\tilde{T}}$  belonging to  $\mathcal{T}_h^i$  we have  $E_h q|_T = q_K^{\text{ext}}|_T$  and  $E_h q|_{\tilde{T}} = q_{\tilde{K}}^{\text{ext}}|_{\tilde{T}}$ . Now due to the assumption 2 there exists a sequence of tetrahedra  $K = K_1, K_2, \dots, K_M = \tilde{K}$  all belonging to  $\mathcal{T}_h^i$  where  $K_i, K_{i+1}$  share a common face which we denote by  $F_i$  and the number  $M$  is bounded and only depends on the shape regularity of the mesh.

First using inverse estimates we get

$$h_F^{1/2} \|[E_h q]\|_{L^2(F)} = h_F^{1/2} \|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(F)} \leq C \|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(T)}.$$

It is easy to see that since  $K_1$  and  $T$  belong to the same patch  $W(T)$  that

$$\|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(T)} \leq C \|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)}.$$

Thanks to the triangle inequality we get

$$\|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)} \leq \|q_{K_1} - q_{K_2}^{\text{ext}}\|_{L^2(K_1)} + \|q_{K_2}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)}.$$

Using the equivalence of norms in the finite dimensional case we obtain

$$\|q_{K_1} - q_{K_2}^{\text{ext}}\|_{L^2(K_1)} \leq C \left( h_{F_1}^{1/2} \|[q]\|_{L^2(F_1)} + h_{K_1} \|\nabla(q_{K_1} - q_{K_2}^{\text{ext}})\|_{L^2(K_1)} \right).$$

We also have

$$\|q_{K_2}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)} \leq C \|q_{K_2}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_2)}.$$

So we get,

$$\begin{aligned} \|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)} &\leq C \left( h_{F_1}^{1/2} \|[q]\|_{L^2(F_1)} + h_{K_1} \|\nabla(q_{K_1} - q_{K_2}^{\text{ext}})\|_{L^2(K_1)} \right) \\ &\quad + \|q_{K_2}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_2)}. \end{aligned}$$

If we continue this we will get

$$\|q_{K_1}^{\text{ext}} - q_{K_M}^{\text{ext}}\|_{L^2(K_1)} \leq C \left( \sum_{j=1}^{M-1} h_{F_j}^{1/2} \|[q]\|_{L^2(F_j)} + \sum_{j=1}^{M-1} h_{K_j} \|\nabla(q_{K_j} - q_{K_{j+1}}^{\text{ext}})\|_{L^2(K_j)} \right).$$

Again, we see that

$$\sum_{j=1}^{M-1} h_{K_j} \|\nabla(q_{K_j} - q_{K_{j+1}}^{\text{ext}})\|_{L^2(K_j)} \leq C \sum_{j=1}^M h_{K_j} \|\nabla q\|_{L^2(K_j)}.$$

Hence, we get

$$h_F^{1/2} \|[E_h q]\|_{L^2(F)} \leq C \left( \sum_{j=1}^{M-1} h_{F_j}^{1/2} \|[q]\|_{L^2(F_j)} + \sum_{j=1}^M h_{K_j} \|\nabla q\|_{L^2(K_j)} \right).$$

If we now sum over  $F \in \mathcal{F}_h^\Gamma$  we get

$$\sum_{F \in \mathcal{F}_h^\Gamma} h_F \|[E_h q]\|_{L^2(F)}^2 \leq C \sum_{T \in \mathcal{T}_h^i} h_T^2 \|\nabla q\|_{L^2(T)}^2 + C \sum_{F \in \mathcal{F}_h} h_F \|[q]\|_{L^2(F)}^2.$$

The result now follows by combining this inequality with (B.1).

#### ACKNOWLEDGMENT

We would like to thank the anonymous referees for valuable suggestions, which stimulated us to weaken assumptions in section 4 and lead to a better presentation.

#### REFERENCES

- [1] S. Amdouni, K. Mansouri, Y. Renard, M. Arfaoui, and M. Moakher, *Numerical convergence and stability of mixed formulation with X-FEM cut-off*, European Journal of Computational Mechanics/Revue Européenne de Mécanique Numérique **21** (2012), no. 3-6, 160–173.
- [2] J. W. Barrett and C. M. Elliott, *A finite-element method for solving elliptic equations with Neumann data on a curved boundary using unfitted meshes*, IMA J. Numer. Anal. **4** (1984), no. 3, 309–325, DOI 10.1093/imanum/4.3.309. MR752608
- [3] J. W. Barrett and C. M. Elliott, *Finite element approximation of the Dirichlet problem using the boundary penalty method*, Numer. Math. **49** (1986), no. 4, 343–366, DOI 10.1007/BF01389536. MR853660
- [4] J. W. Barrett and C. M. Elliott, *Fitted and unfitted finite-element methods for elliptic equations with smooth interfaces*, IMA J. Numer. Anal. **7** (1987), no. 3, 283–300, DOI 10.1093/imanum/7.3.283. MR968524
- [5] R. Becker, E. Burman, and P. Hansbo, *A Nitsche extended finite element method for incompressible elasticity with discontinuous modulus of elasticity*, Comput. Methods Appl. Mech. Engrg. **198** (2009), no. 41-44, 3352–3360, DOI 10.1016/j.cma.2009.06.017. MR2571349
- [6] M. Bercovier and O. Pironneau, *Error estimates for finite element method solution of the Stokes problem in the primitive variables*, Numer. Math. **33** (1979), no. 2, 211–224, DOI 10.1007/BF01399555. MR549450
- [7] C. Bernardi, M. Costabel, M. Dauge, and V. Girault, *Continuity properties of the inf-sup constant for the divergence*, SIAM J. Math. Anal. **48** (2016), no. 2, 1250–1271, DOI 10.1137/15M1044989. MR3479696
- [8] C. Bernardi and G. Raugel, *Analysis of some finite elements for the Stokes problem*, Math. Comp. **44** (1985), no. 169, 71–79, DOI 10.2307/2007793. MR771031
- [9] D. Boffi, F. Brezzi, and M. Fortin, *Finite elements for the Stokes problem*, Mixed finite elements, compatibility conditions, and applications (D. Boffi and L. Gastaldi, eds.), Springer, 2008, Lecture Notes in Mathematics. Springer Verlag. Vol. 1939, pp. 45–100.

- [10] D. Boffi and L. Gastaldi, *A finite element approach for the immersed boundary method*, Comput. & Structures **81** (2003), no. 8-11, 491–501, DOI 10.1016/S0045-7949(02)00404-2. In honour of Klaus-Jürgen Bathe. MR2001876
- [11] M. E. Bogovskiĭ, *Solution of the first boundary value problem for an equation of continuity of an incompressible medium* (Russian), Dokl. Akad. Nauk SSSR **248** (1979), no. 5, 1037–1040. MR553920
- [12] S. C. Brenner and L. R. Scott, *The Mathematical Theory of Finite Element Methods*, 3rd ed., Texts in Applied Mathematics, vol. 15, Springer, New York, 2008. MR2373954
- [13] E. Burman, *Ghost penalty* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **348** (2010), no. 21-22, 1217–1220, DOI 10.1016/j.crma.2010.10.006. MR2738930
- [14] E. Burman, S. Claus, P. Hansbo, M. G. Larson, and A. Massing, *CutFEM: discretizing geometry and partial differential equations*, Internat. J. Numer. Methods Engrg. **104** (2015), no. 7, 472–501, DOI 10.1002/nme.4823. MR3416285
- [15] E. Burman and P. Hansbo, *Fictitious domain finite element methods using cut elements: II. A stabilized Nitsche method*, Appl. Numer. Math. **62** (2012), no. 4, 328–341, DOI 10.1016/j.apnum.2011.01.008. MR2899249
- [16] E. Burman and P. Hansbo, *Fictitious domain methods using cut elements: III. A stabilized Nitsche method for Stokes’ problem*, ESAIM Math. Model. Numer. Anal. **48** (2014), no. 3, 859–874, DOI 10.1051/m2an/2013123. MR3264337
- [17] L. Cattaneo, L. Formaggia, G. F. Iori, A. Scotti, and P. Zunino, *Stabilized extended finite elements for the approximation of saddle point problems with unfitted interfaces*, Calcolo **52** (2015), no. 2, 123–152, DOI 10.1007/s10092-014-0109-9. MR3348149
- [18] A. Y. Chernyshenko and M. A. Olshanskii, *Non-degenerate Eulerian finite element method for solving PDEs on surfaces*, Russian J. Numer. Anal. Math. Modelling **28** (2013), no. 2, 101–124, DOI 10.1515/rnam-2013-0007. MR3043557
- [19] P. G. Ciarlet, *The Finite Element Method for Elliptic Problems*, Classics in Applied Mathematics, vol. 40, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002. Reprint of the 1978 original [North-Holland, Amsterdam; MR0520174 (58 #25001)]. MR1930132
- [20] M. Crouzeix and P.-A. Raviart, *Conforming and nonconforming finite element methods for solving the stationary Stokes equations. I*, Rev. Française Automat. Informat. Recherche Opérationnelle Sér. Rouge **7** (1973), no. R-3, 33–75. MR0343661
- [21] K. Deckelnick, C. M. Elliott, and T. Ranner, *Unfitted finite element methods using bulk meshes for surface partial differential equations*, SIAM J. Numer. Anal. **52** (2014), no. 4, 2137–2162, DOI 10.1137/130948641. MR3249369
- [22] J. Dolbow and I. Harari, *An efficient finite element method for embedded interface problems*, Internat. J. Numer. Methods Engrg. **78** (2009), no. 2, 229–252, DOI 10.1002/nme.2486. MR2510520
- [23] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements*, Applied Mathematical Sciences, vol. 159, Springer-Verlag, New York, 2004. MR2050138
- [24] T.-P. Fries and T. Belytschko, *The extended/generalized finite element method: an overview of the method and its applications*, Internat. J. Numer. Methods Engrg. **84** (2010), no. 3, 253–304, DOI 10.1002/nme.2914. MR2732652
- [25] G. P. Galdi, *An Introduction to the Mathematical Theory of the Navier-Stokes Equations: Steady-state problems*, 2nd ed., Springer Monographs in Mathematics, Springer, New York, 2011. MR2808162
- [26] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Classics in Applied Mathematics, vol. 69, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011. Reprint of the 1985 original [MR0775683]; With a foreword by Susanne C. Brenner. MR3396210
- [27] S. Groß, M. A. Olshanskii, and A. Reusken, *A trace finite element method for a class of coupled bulk-interface transport problems*, ESAIM Math. Model. Numer. Anal. **49** (2015), no. 5, 1303–1330, DOI 10.1051/m2an/2015013. MR3423226
- [28] S. Groß and A. Reusken, *An extended pressure finite element space for two-phase incompressible flows with surface tension*, J. Comput. Phys. **224** (2007), no. 1, 40–58, DOI 10.1016/j.jcp.2006.12.021. MR2322259

- [29] A. Hansbo and P. Hansbo, *An unfitted finite element method, based on Nitsche's method, for elliptic interface problems*, Comput. Methods Appl. Mech. Engrg. **191** (2002), no. 47-48, 5537–5552, DOI 10.1016/S0045-7825(02)00524-8. MR1941489
- [30] A. Hansbo and P. Hansbo, *A finite element method for the simulation of strong and weak discontinuities in solid mechanics*, Comput. Methods Appl. Mech. Engrg. **193** (2004), no. 33-35, 3523–3540, DOI 10.1016/j.cma.2003.12.041. MR2075053
- [31] P. Hansbo, M. G. Larson, and S. Zahedi, *A cut finite element method for a Stokes interface problem*, Appl. Numer. Math. **85** (2014), 90–114, DOI 10.1016/j.apnum.2014.06.009. MR3239219
- [32] M. Kirchhart, S. Groß, and A. Reusken, *Analysis of an XFEM discretization for Stokes interface problems*, SIAM J. Sci. Comput. **38** (2016), no. 2, A1019–A1043, DOI 10.1137/15M1011779. MR3479700
- [33] G. Legrain, N. Moës, and A. Huerta, *Stability of incompressible formulations enriched with X-FEM*, Comput. Methods Appl. Mech. Engrg. **197** (2008), no. 21-24, 1835–1849, DOI 10.1016/j.cma.2007.08.032. MR2417162
- [34] Z. Li, *The immersed interface method using a finite element formulation*, Appl. Numer. Math. **27** (1998), no. 3, 253–267, DOI 10.1016/S0168-9274(98)00015-4. MR1634348
- [35] Z. Li, T. Lin, and X. Wu, *New Cartesian grid methods for interface problems using the finite element formulation*, Numer. Math. **96** (2003), no. 1, 61–98, DOI 10.1007/s00211-003-0473-x. MR2018791
- [36] A. Massing, M. G. Larson, A. Logg, and M. E. Rognes, *A stabilized Nitsche fictitious domain method for the Stokes problem*, J. Sci. Comput. **61** (2014), no. 3, 604–628, DOI 10.1007/s10915-014-9838-9. MR3268662
- [37] A. Massing, M. G. Larson, A. Logg, and M. E. Rognes, *A stabilized Nitsche overlapping mesh method for the Stokes problem*, Numer. Math. **128** (2014), no. 1, 73–101, DOI 10.1007/s00211-013-0603-z. MR3248049
- [38] M. A. Olshanskii, A. Reusken, and J. Grande, *A finite element method for elliptic equations on surfaces*, SIAM J. Numer. Anal. **47** (2009), no. 5, 3339–3358, DOI 10.1137/080717602. MR2551197
- [39] M. A. Olshanskii, A. Reusken, and X. Xu, *An Eulerian space-time finite element method for diffusion problems on evolving surfaces*, SIAM J. Numer. Anal. **52** (2014), no. 3, 1354–1377, DOI 10.1137/130918149. MR3215065
- [40] A. Reusken, *Analysis of trace finite element methods for surface partial differential equations*, IMA J. Numer. Anal. **35** (2015), no. 4, 1568–1590, DOI 10.1093/imanum/dru047. MR3407236
- [41] H. Sauerland and T.-P. Fries, *The extended finite element method for two-phase and free-surface flows: a systematic study*, J. Comput. Phys. **230** (2011), no. 9, 3369–3390, DOI 10.1016/j.jcp.2011.01.033. MR2780468
- [42] B. Schott and W. A. Wall, *A new face-oriented stabilized XFEM approach for 2D and 3D incompressible Navier-Stokes equations*, Comput. Methods Appl. Mech. Engrg. **276** (2014), 233–265, DOI 10.1016/j.cma.2014.02.014. MR3212334
- [43] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Mathematical Series, No. 30, Princeton University Press, Princeton, N.J., 1970. MR0290095
- [44] R. Verfürth, *Error estimates for a mixed finite element approximation of the Stokes equations*, RAIRO Anal. Numér. **18** (1984), no. 2, 175–182, DOI 10.1051/m2an/1984180201751. MR743884
- [45] Q. Wang and J. Chen, *A new unfitted stabilized Nitsche's finite element method for Stokes interface problems*, Comput. Math. Appl. **70** (2015), no. 5, 820–834, DOI 10.1016/j.camwa.2015.05.024. MR3378970

DIVISION OF APPLIED MATHEMATICS, BROWN UNIVERSITY, PROVIDENCE, RHODE ISLAND 02912  
 Email address: johnny-guzman@brown.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF HOUSTON, HOUSTON, TEXAS 77204  
 Email address: molshan@math.uh.edu