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Convergence analysis of a symmetric exponential integrator Fourier pseudo-spectral scheme for the Klein-Gordon-Dirac equation

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Convergence analysis of a symmetric exponential integrator Fourier pseudo-spectral scheme for the Klein-Gordon-Dirac equation☆

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Abstract

Recently, an exponential integrator Fourier pseudo-spectral (EIFP) scheme for the Klein-Gordon-Dirac (KGD) equation in the nonrelativistic limit regime has been proposed [Journal of Scientific Computing (2019) 79:1907-1935]. The scheme is fully explicit and numerical experiments show that it is very efficient due to the fast Fourier transform (FFT). However, the authors did not give a strict convergence analysis and error estimate for the scheme. In addition, there are two shortcomings. Firstly, the scheme did not satisfy time symmetry which is an important characteristic of the exact solution. Secondly, the scheme needs to calculate the derivative of the wave function, which is very complicated and not practical. In this paper, By setting two-level format for Klein-Gordon part and three-level format for Dirac part, respectively, we proposed a new EIFP scheme for the KGD equation with periodic boundary conditions. The new scheme is time symmetric and does not need to calculate the derivative of the unknown wave function. By using the standard energy method and the mathematical induction, we make a rigorously convergence analysis and establish error estimates without any CFL condition restrictions on the grid ratio. The convergence rate of the proposed scheme is proved to be at the second-order in time and spectral-order in space, respectively, in a generic H^m-norm. The numerical experiments are carried out to confirm our theoretical analysis. Because that our error estimates are given under the general H^m-norm, the conclusion can easily be extended to two- and three-dimensional problems without the stability (or CFL) condition under sufficient regular conditions.

Keywords: Klein-Gordon-Dirac equation, Exponential integrator, Periodic boundary condition, Fourier pseudo-spectral method, Optimal error estimates, Convergence analysis.

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1. Introduction

In quantum electrodynamics and/or particle physics, the Klein-Gordon-Dirac (KGD) equation (1.1) describes the nuclear force between nucleons through the Yukawa potential [3, 22, 26, 39, 46], and is a fundamental model to describe the dynamics of a complex-valued Dirac vector field interacting with a neutral real-valued meson scalar field. In this paper, we consider the numerical schemes for the dimensionless KGD equation in d-dimensions ($d = 1, 2, 3$) as follows

$$\begin{aligned} &\partial_{tt}u - \triangle u + u = g\Phi^*\beta\Phi, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ &i\partial_t\Phi = -i\sum_{j=1}^{d}\alpha_j\partial_j\Phi + \omega\beta\Phi + gu\beta\Phi, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \end{aligned} \tag{1.1}$$

which describes a complex-valued Dirac vector field $\Phi := \Phi(t, \mathbf{x}) = (\phi_1(t, \mathbf{x}), \phi_2(t, \mathbf{x}), \phi_3(t, \mathbf{x}), \phi_4(t, \mathbf{x}))^T \in \mathbb{C}^4$ interacting with a neutral real-valued scalar meson field $u := u(t, \mathbf{x}) \in \mathbb{R}$ through the Yukawa interaction with a coupling constant $0 < g \in \mathbb{R}$, $\omega > 0$ is the ratio of the mass of the meson to the mass of the electron, $i = \sqrt{-1}$, t is time, $\mathbf{x} \in \mathbb{R}^d$ is the spatial coordinate. In addition, $\alpha_1, \alpha_2, \alpha_3$ and β are the 4×4 matrices given as

$$\alpha_j = \begin{pmatrix} \mathbf{0} & \sigma_j \\ \sigma_j & \mathbf{0} \end{pmatrix}, \quad j = 1, 2, 3, \quad \beta = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix},$$

where I is the 2×2 identity matrix, σ_1, σ_2 and σ_3 are the Pauli matrices defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

To study the KGD equation (1.1), the initial data is usually assigned as

$$u(0, \mathbf{x}) = u^0(\mathbf{x}), \quad \partial_t u(0, \mathbf{x}) = \gamma(\mathbf{x}), \quad \Phi(0, \mathbf{x}) = \Phi^0(\mathbf{x}) = \left(\phi_1^0(\mathbf{x}), \phi_2^0(\mathbf{x}), \phi_3^0(\mathbf{x}), \phi_4^0(\mathbf{x})\right)^T, \tag{1.2}$$

where the functions $u^0(\mathbf{x}), \gamma(\mathbf{x}) \in \mathbb{R}$ and $\Phi^0(\mathbf{x}) \in \mathbb{C}^4$. The KGD equation (1.1) is dispersive, time symmetric or time reversible and conserves the total mass as

$$M(t) = \|\Phi(t, \cdot)\|^2 := \int_{\mathbb{R}^d} |\Phi(t, \mathbf{x})|^2 \mathrm{d}\mathbf{x} \equiv \|\Phi(0, \cdot)\|^2 \equiv \|\Phi^0\|^2, \quad t \geq 0, \tag{1.3}$$

and the energy as

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_{\mathbb{R}^d} \left(|\partial_t u|^2 + |\nabla u|^2 + |u|^2\right) \mathrm{d}\mathbf{x} \\ &+ \int_{\mathbb{R}^d} \left(i\Phi^* \sum_{j=1}^{d} \alpha_j\partial_j\Phi - \omega\Phi^*\beta\Phi - gu\Phi^*\beta\Phi\right)\mathrm{d}\mathbf{x} \equiv E(0), \quad t \geq 0. \end{aligned} \tag{1.4}$$

In lower dimensions ($d = 1, 2$), the KGD equation (1.1) can be reduced to a simplified form with Dirac vector fields of two components [9, 11] as

$$\begin{aligned} &\partial_{tt}u - \triangle u + u = g\Phi^*\sigma_3\Phi, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \\ &i\partial_t\Phi = -i\sum_{j=1}^{d}\sigma_j\partial_j\Phi + \omega\sigma_3\Phi + gu\sigma_3\Phi, \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0, \end{aligned} \tag{1.5}$$

where $u := u(t,\mathbf{x}) \in \mathbb{R}$ and $\Phi := \Phi(t,\mathbf{x}) = (\phi_1(t,\mathbf{x}), \phi_2(t,\mathbf{x}))^T \in \mathbb{C}^2$. Because of its simplicity, the two-component form (1.5) is widely used in one dimension and two dimensions. Analogously to the four-component case (1.1), the KGD equation (1.5) is dispersive, time symmetric or time reversible and conserves the total mass as

$$M(t) = \|\Phi(t,\cdot)\|^2 := \int_{\mathbb{R}^d} |\Phi(t,\mathbf{x})|^2 \mathrm{d}\mathbf{x} \equiv \|\Phi(0,\cdot)\|^2 \equiv \|\Phi^0\|^2, \quad t \geq 0, \tag{1.6}$$

and the energy as

$$\begin{aligned} E(t) :=& \frac{1}{2}\int_{\mathbb{R}^d} \left[|\partial_t u|^2 + |\nabla u|^2 + |u|^2\right] \mathrm{d}\mathbf{x} \\ &+ \int_{\mathbb{R}^d} \left[i\Phi^* \sum_{j=1}^{d} \sigma_j \partial_j \Phi - \omega \Phi^* \sigma_3 \Phi\right] \mathrm{d}\mathbf{x} - \int_{\mathbb{R}^d} g u \Phi^* \sigma_3 \Phi \mathrm{d}\mathbf{x} \equiv E(0), \quad t \geq 0. \end{aligned} \tag{1.7}$$

The KGD system (1.1) (or (1.5)) has been extensively studied theoretically in the literature, such as the existences of bound state solutions and the local and global well-posedness of the Cauchy problem, and we refer to [2, 4, 5, 16, 20, 21, 38, 44] and references therein for more details. For the numerical part, many efficient and accurate numerical methods have been proposed and analyzed, such as the finite difference time domain (FDTD) methods and spectral methods for efficient computations of wave propagation in classical/relativistic quantum physics, i.e., dispersive waves in the Gross-Pitaevskii equation [6, 7], the Klein-Gordon equation [12, 30, 43], the Dirac equation [9, 10, 23, 29, 35, 37, 51], the Klein-Gordon-Schrödinger equations [13, 18, 19, 40, 50], Schrödinger-KDV equation [41], the Klein-Gordon-Zakharov equations [15] and the Maxwell-Dirac systems [29], etc. Some authors developed multi-symplectic methods [28].

However, the numerical approaches for the KGD equation (1.1) (or (1.5)) proposed in the literature are limited [14, 36, 52, 53]. In the literature [52], the authors studied finite difference schemes and proved that the difference solution converges to the exact solution with second order in the l^2-norm. In [14], the authors proved that the difference solution converges to the exact solution with second order in the l^∞-norm for the one-dimensional KGD equation in the nonrelativistic limit regime. These methods mentioned above are either nonlinear implicit which need higher computational cost or suffer CFL condition restrictions. In addition, the generalization of this conclusion to higher dimensions also requires some stability constraints. For more details, see Remark 3.10 of [52] and Remark 3.1 of [14].

In a recent paper [53], the authors proposed the exponential integrator Fourier pseudo-spectral method for solving the KGD equation (1.1) in the nonrelativistic limit regime. The discretization is fully explicit and remains valid in one, two and three dimensions. Numerical experiments show that it is very efficient due to the fast Fourier transform (FFT). However, the authors did not give a strict convergence analysis and error estimate for the scheme. In addition, there are two shortcomings. Firstly, the scheme did not satisfy time symmetry which is an important characteristic of the exact solution. Secondly, the scheme needs to calculate the derivative of the wave function, which is very complicated and not practical.

In this paper, we propose a new exponential integrator Fourier pseudo-spectral scheme for the KGD equation. The new scheme is time symmetric, fully explicit and does not need to calculate the derivative of the wave function Φ. By using the standard energy method and the mathematical induction, we make a rigorously convergence analysis and establish error estimates without any

CFL condition restrictions on the grid ratio. The convergence rate of the proposed scheme is proved to be at the second-order in time and spectral-order in space, respectively, in a generic H^m-norm. Numerical experiment shows that the new scheme also conserves the total mass and energy very well in the discrete level very well.

This paper is organized as follows. In Section 2, we propose the new exponential integrator Fourier pseudo-spectral scheme for KGD equation (1.1) and introduce some important lemmas. In Section 3, the convergence rate of the proposed scheme at the order $O(h^{m_0} + \tau^2)$ in the H^m-norm is proved without any restrictions on the grid ratio. Numerical experiments are conducted to support the theoretical analysis in Section 4. Finally, we draw a conclusion in the last section. Throughout this paper, we adopt the notation $p \lesssim q$ to represent that there exists a generic constant $C > 0$, which is independent of the mesh size h and time step τ such that $|p| \le Cq$.

2. Exponential integrator Fourier pseudo-spectral method and their analysis

For simplicity, here we only present the scheme in one dimension in (1.5). Extension to high-dimension in (1.1) is straightforward, and the error estimate results remain valid without any modification.

We truncate the whole space problem onto an interval $\Omega = (a, b)$ with periodic boundary conditions and consider a finite time interval $[0, T]$ where $0 < T < T^*$ with T^* being the maximal existence time of the solution. So the problem can be represented as follows:

$$\begin{aligned}
&\partial_{tt}u - \partial_{xx}u + u = g\Phi^*\sigma_3\Phi, \quad x \in \Omega, \quad t > 0,\\
&i\partial_t\Phi = -i\sigma_1\partial_x\Phi + \omega\sigma_3\Phi + gu\sigma_3\Phi, \quad x \in \Omega, \quad t > 0,\\
&u(0, x) = u^0(x), \quad \partial_t u(0, x) = \gamma(x), \quad \Phi(0, x) = \Phi^0(x), \quad x \in \bar{\Omega},\\
&u(t, a) = u(t, b), \quad \partial_x u(t, a) = \partial_x u(t, b), \quad \Phi(t, a) = \Phi(t, b), \quad t \ge 0.
\end{aligned} \tag{2.8}$$

In this paper, for the convenience of analysis, we denote $\Omega_T = [0, T] \times \Omega$ and

$$F(\Phi) = g\Phi^*\sigma_3\Phi, \quad G(u, \Phi) = gu\sigma_3\Phi. \tag{2.9}$$

For integer $m > 0$, $\Omega = [a, b]$, we denote by $H^m(\Omega)$ the standard Sobolev space with norm

$$\|f\|_m^2 = \sum_{l=-\infty}^{\infty}(1 + |\mu_l|^2)^m \left|\hat{f}_l\right|^2, \quad \text{for} \quad f(x) = \sum_{l=-\infty}^{\infty} \hat{f}_l e^{i\mu_l(x-a)}, \quad \mu_l = \frac{2\pi l}{b - a}. \tag{2.10}$$

In order to analyze the Dirac part of (2.8) whose solution is two-component, we define

$$\|U\|_m^2 = \|U_1\|_m^2 + \|U_2\|_m^2, \quad U = (U_1, U_2)^T \in [H^m(\Omega)]^2, \tag{2.11}$$

and then we give the norm

$$\|U\|_m^2 = \sum_{l=-\infty}^{\infty}(1 + |\mu_l|^2)^m \left|\hat{U}_l\right|^2, \quad \text{for} \quad U(x) = \sum_{l=-\infty}^{\infty} \hat{U}_l e^{i\mu_l(x-a)}, \quad \mu_l = \frac{2\pi l}{b - a}, \tag{2.12}$$

where $\left|\widetilde{U}_l\right|^2 = \left|(\widetilde{U_1})_l\right|^2 + \left|(\widetilde{U_2})_l\right|^2$.

Choose the mesh size $h := \Delta x = (b-a)/M$ with M a positive even integer, time step $\tau := \Delta t > 0$ and denote grid points as $x_j := a + jh$ for $j = 0, 1, \cdots, M$, $t_n := n\tau$ for $n = 0, 1, \cdots, N$ with $N = \frac{T}{\tau}$. Let

$$
\begin{aligned}
&Z_M := Y_M \times Y_M, \\
&Y_M := \text{span}\left\{ \phi_l(x) = \mathrm{e}^{i\mu_l(x-a)} \middle| \mu_l = \frac{2\pi l}{b-a}, \quad l = -\frac{M}{2}, \cdots, \frac{M}{2} - 1 \right\}, \\
&\bar{X}_M := \text{span}\left\{ U = (U_0, \cdots, U_M) \mid U_j = \mathbb{R}, \quad j = 0, \cdots, M, U_0 = U_M \right\}, \\
&X_M := \text{span}\left\{ U = (U_0, \cdots, U_M) \mid u_j = \mathbb{C}^2, \quad j = 0, \cdots, M, \quad U_0 = U_M \right\}.
\end{aligned}
\tag{2.13}
$$

For any periodic function $U(x)$ on $[a, b]$ and vector $U \in \bar{X}_M$ (or $U \in X_M$), define $P_M : L^2(a, b) \to Y_M$ (or $L^2(a, b) \to Z_M$) as the standard projection operator, $I_M : C(a, b) \to Y_M$ (or $I_M : C(a, b) \to Z_M$) and $I_M : \bar{X}_M \to Y_M$ (or $X_M \to Z_M$) as the trigonometric interpolation operators [45], i.e.

$$
(P_M U)(x) = \sum_{l=-M/2}^{M/2-1} \widehat{U}_l \mathrm{e}^{i\mu_l(x-a)}, \quad (I_M U)(x) = \sum_{l=-M/2}^{M/2-1} \widetilde{U}_l \mathrm{e}^{i\mu_l(x-a)}, \quad a \le x \le b, \tag{2.14}
$$

with $l = -M/2, \cdots, M/2 - 1$ and the Fourier and discrete Fourier transform coefficients are defined as

$$
\widehat{U}_l = \frac{1}{b-a} \int_a^b U(x) \mathrm{e}^{-i\mu_l(x-a)} dx, \quad \widetilde{U}_l = \frac{1}{M} \sum_{j=0}^{M-1} U_j \mathrm{e}^{-i\mu_l(x_j-a)}, \tag{2.15}
$$

respectively, where U_j is interpreted as $U(x_j)$ for the periodic function $U(x)$. The Fourier spectral method for discretizing the KGD equation (2.8) is given in the following way: Find $u_M(t, x) \in Y_M$ and $\Phi_M(t, x) \in Z_M$, i.e.

$$
\begin{aligned}
u_M(t, x) &= \sum_{l=-M/2}^{M/2-1} \widehat{(u_M)}_l(t) \mathrm{e}^{i\mu_l(x-a)}, \quad a \le x \le b, \quad t \ge 0, \\
\Phi_M(t, x) &= \sum_{l=-M/2}^{M/2-1} \widehat{(\Phi_M)}_l(t) \mathrm{e}^{i\mu_l(x-a)}, \quad a \le x \le b, \quad t \ge 0,
\end{aligned}
\tag{2.16}
$$

such that

$$
\begin{aligned}
&\partial_{tt} u_M(t, x) - \partial_{xx} u_M(t, x) + u_M(t, x) = P_M\Big[g\Phi_M^*(t, x)\sigma_3\Phi_M(t, x)\Big], \\
&i\partial_t \Phi_M(t, x) = \Big[-i\sigma_1\partial_x + \omega\sigma_3\Big]\Phi_M(t, x) + P_M\Big[g u_M(t, x)\sigma_3\Phi_M(t, x)\Big], \\
&a \le x \le b, \quad t \ge 0.
\end{aligned}
\tag{2.17}
$$

Plugging (2.16) into (2.17) and noticing the orthogonality of the Fourier functions $\mathrm{e}^{i\mu_l(x-a)}$ for $l = -M/2, \cdots, M/2 - 1$, we find

$$
\begin{aligned}
&\frac{\mathrm{d}^2}{\mathrm{d}t^2} \widehat{(u_M)}_l(t) + \left(\mu_l^2 + 1\right) \widehat{(u_M)}_l(t) = \widehat{(F_M)}_l(t), \quad t \ge 0. \\
&i \frac{\mathrm{d}}{\mathrm{d}t} \widehat{(\Phi_M)}_l(t) = \left[\mu_l\sigma_1 + \omega\sigma_3\right] \widehat{(\Phi_M)}_l(t) + \widehat{(G_M)}_l(t),
\end{aligned}
\tag{2.18}
$$

where

$$F_M(t,x) = F\Big(\Phi_M(t,x)\Big), \quad G_M(t,x) = G\Big(u_M(t,x), \Phi_M(t,x)\Big). \tag{2.19}$$

We rewrite the above ODEs as

$$\begin{aligned} &\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{(u_M)}_l(t) + \beta_l^2 \widehat{(u_M)}_l(t) = \widehat{(F_M)}_l(t), \\ &\frac{\mathrm{d}}{\mathrm{d}t}\widehat{(\Phi_M)}_l(t) = -iM_l\widehat{(\Phi_M)}_l(t) - i\widehat{(G_M)}_l(t), \end{aligned} \tag{2.20}$$

where

$$\beta_l = \sqrt{1+\mu_l^2}, \quad M_l = \mu_l\sigma_1 + \omega\sigma_3 = \begin{pmatrix} \omega & \mu_l \\ \mu_l & -\omega \end{pmatrix}, \tag{2.21}$$

where $M_l = Q_l D_l Q_l^*$ with

$$Q_l = \frac{1}{\sqrt{2\delta_l}}\begin{pmatrix} \sqrt{\delta_l+\omega} & -\sqrt{\delta_l-\omega} \\ \sqrt{\delta_l-\omega} & \sqrt{\delta_l+\omega} \end{pmatrix}, \quad D_l = \begin{pmatrix} \delta_l & 0 \\ 0 & -\delta_l \end{pmatrix}, \quad \delta_l = \sqrt{\omega^2+\mu_l^2}.$$

Using the variation-of-constants formula as in the exponential integrator [24, 27] and appropriate variable substitution, the general solution of the ODEs (2.20) satisfies the formulation

$$\begin{aligned} \widehat{(u_M)}_l(t_n+s) =& \cos(\beta_l s)\widehat{(u_M)}_l(t_n) + \frac{\sin(\beta_l s)}{\beta_l}\widehat{(\dot u_M)}_l(t_n) \\ &+ \frac{s}{\beta_l}\int_0^1 \sin((1-z)\beta_l s)\widehat{(F_M)}_l(t_n+zs)\mathrm{d}z, \end{aligned} \tag{2.22}$$

$$\begin{aligned} \widehat{(\dot u_M)}_l(t_n+s) =& -\beta_l\sin(\beta_l s)\widehat{(u_M)}_l(t_n) + \cos(\beta_l s)\widehat{(\dot u_M)}_l(t_n) \\ &+ s\int_0^1 \cos((1-z)\beta_l s)\widehat{(F_M)}_l(t_n+zs)\mathrm{d}z, \end{aligned} \tag{2.23}$$

$$\widehat{(\Phi_M)}_l(t_n+s) = e^{-isM_l}\widehat{(\Phi_M)}_l(t_n) - is\int_0^1 e^{-is(1-z)M_l}\widehat{(G_M)}_l(t_n+zs)\mathrm{d}z, \tag{2.24}$$

where $\dot u_M(t_n,x) = \partial_t u_M(t_n,x)$. Taking $s=\tau$ in (2.22) and (2.23) and taking $s = 2\tau$ with n replaced by $n-1$ in (2.24), we have

$$\begin{aligned} \widehat{(u_M)}_l(t_{n+1}) =& \cos(\beta_l \tau)\widehat{(u_M)}_l(t_n) + \frac{\sin(\beta_l \tau)}{\beta_l}\widehat{(\dot u_M)}_l(t_n) \\ &+ \frac{\tau}{\beta_l}\int_0^1 \sin((1-z)\beta_l \tau)\widehat{(F_M)}_l(t_n+z\tau)\mathrm{d}z, \\ \widehat{(\dot u_M)}_l(t_{n+1}) =& -\beta_l\sin(\beta_l \tau)\widehat{(u_M)}_l(t_n) + \cos(\beta_l \tau)\widehat{(\dot u_M)}_l(t_n) \\ &+ \tau\int_0^1 \cos((1-z)\beta_l \tau)\widehat{(F_M)}_l(t_n+z\tau)\mathrm{d}z, \\ \widehat{(\Phi_M)}_l(t_{n+1}) =& e^{-2i\tau M_l}\widehat{(\Phi_M)}_l(t_{n-1}) - 2i\tau\int_0^1 e^{-2i\tau(1-z)M_l}\widehat{(G_M)}_l(t_{n-1}+2z\tau)\mathrm{d}z. \end{aligned} \tag{2.25}$$

In order to design an explicit symmetric scheme, we approximate the integrals by the following Deuflhard-type quadratures

$$\begin{aligned}
&\int_0^1 \sin((1-z)\beta_l\tau)\widehat{(F_M)}_l(t_n+z\tau)\mathrm{d}z \approx \frac{1}{2}\sin(\beta_l\tau)\widehat{(F_M)}_l(t_n),\\
&\int_0^1 \cos((1-z)\beta_l\tau)\widehat{(F_M)}_l(t_n+z\tau)\mathrm{d}u \approx \frac{1}{2}\Big(\cos(\beta_l\tau)\widehat{(F_M)}_l(t_n)+\widehat{(F_M)}(t_{n+1})\Big),\\
&\int_0^1 e^{-2i\tau(1-z)M_l}\widehat{G(\Phi_M)}_l(t_{n-1}+2z\tau)\mathrm{d}z \approx e^{-i\tau M_l}\widehat{(G_M)}_l(t_n).
\end{aligned} \tag{2.26}$$

To calculate the value of the first step, we take $s=\tau$ in (2.24) and obtain

$$\widehat{(\Phi_M)}_l(\tau) = e^{-i\tau M_l}\widehat{(\Phi_M)}_l(0) - i\tau\int_0^1 e^{-i\tau(1-z)M_l}\widehat{(G_M)}_l(z\tau)\mathrm{d}z, \tag{2.27}$$

then we approximate the integrals by the following quadratures

$$\int_0^1 e^{-i\tau(1-z)M_l}\widehat{(G_M)}_l(z\tau)\mathrm{d}z \approx e^{-i\tau M_l}\widehat{(G_M)}_l(0). \tag{2.28}$$

In practice, the integrals defined in the first expression of (2.15) for computing the Fourier transform coefficients are not suitable, and they are usually replaced by the interpolations as defined in the second expression of (2.15). Let u_j^n, $\dot{u}_j^n$ and Φ_j^n ($n=0,1,\cdots,j=0,\cdots,M$) be the approximations to $u(t_n,x_j)$ and $\partial_t u(t_n,x_j)$ and $\Phi(t_n,x_j)$, respectively. Choose $u_j^0=u^0(x_j)$, $\dot{u}_j^0=\gamma(x_j)$ and $\Phi_j^0=\Phi^0(x_j)$, then for $n=0,1,\cdots$, a symmetric exponential integrator Fourier pseudo-spectral (SEIFP) discretization for the KGD equation (2.8) is,

$$\begin{aligned}
&u_j^{n+1} = \sum_{l=-M/2}^{M/2-1}\widetilde{(u^{n+1})}_l \mathrm{e}^{2ijl\pi/M}, \quad \dot{u}_j^{n+1} = \sum_{l=-M/2}^{M/2-1}\widetilde{(\dot{u}^{n+1})}_l \mathrm{e}^{2ijl\pi/M},\\
&\Phi_j^{n+1} = \sum_{l=-M/2}^{M/2-1}\widetilde{(\Phi^{n+1})}_l \mathrm{e}^{2ijl\pi/M},
\end{aligned} \tag{2.29}$$

where

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = \cos(\beta_l\tau)\widetilde{(u^n)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\dot{u}^n)}_l + \tau\frac{\sin(\beta_l\tau)}{2\beta_l}\widetilde{F}_l^n, \quad n\ge 0,\\
&\widetilde{(\dot{u}^{n+1})}_l = -\beta_l\sin(\beta_l\tau)\widetilde{(u^n)}_l + \cos(\beta_l\tau)\widetilde{(\dot{u}^n)}_l + \frac{\tau}{2}\Big(\cos(\beta_l\tau)\widetilde{F}_l^n + \widetilde{F}_l^{n+1}\Big), \quad n\ge 0,\\
&\widetilde{(\Phi^{n+1})}_l = e^{-2i\tau M_l}\widetilde{(\Phi^{n-1})}_l - 2i\tau e^{-i\tau M_l}\widetilde{G}_l^n, \quad n\ge 1,
\end{aligned} \tag{2.30}$$

and the $\widetilde{(\Phi^1)}_l$ is given by

$$\widetilde{(\Phi^1)}_l = e^{-i\tau M_l}\widetilde{(\Phi^0)}_l - i\tau e^{-i\tau M_l}\widetilde{G}_l^0, \tag{2.31}$$

with the vector F^n, F^{n+1} and G^n defined as

$$F_j^n = F(\Phi_j^n), \quad F_j^{n+1} = F(\Phi_j^{n+1}), \quad G_j^n = G(u_j^n, \Phi_j^n). \tag{2.32}$$

The SEIFP scheme (2.29)-(2.31) is an explicit scheme. In fact, the practical computations of the KGD (2.8) are decoupled at each time step, i.e., one first updates the Dirac field Φ^{n+1}, and then updates the Klein-Gordon field $(u^{n+1}, \dot{u}^{n+1})$. The memory cost is $O(M)$ and the computational load per time step is $O(M\ln M)$ thanks to the FFT. Similar to the Von Neumann analysis in [9, 12], we also have the following lemma with the proof omitted for brevity.

Lemma 2.1. *The SEIFP scheme* (2.29)-(2.31) *for the KGD equation* (2.8) *with $g = 0$ is unconditionally stable for any $\tau, h > 0$.*

Remark 2.1. *Through simple calculations, we come to the conclusion that the scheme SEIFP is equivalent to the two-step scheme whose coefficients satisfy*

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = 2\cos(\beta_l\tau)\widetilde{(u^n)}_l - \widetilde{(u^{n-1})}_l + \tau\frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{F_l^n}, \quad n \geq 1,\\
&\widetilde{(\dot{u}^{n+1})}_l = -2\beta_l\sin(\beta_l\tau)\widetilde{(u^n)}_l + \widetilde{(\dot{u}^{n-1})}_l + \frac{\tau}{2}\Big(2\cos(\beta_l\tau)\widetilde{F_l^n} + \widetilde{F_l^{n+1}} + \widetilde{F_l^{n-1}}\Big), \quad n \geq 1,\\
&\widetilde{(\Phi^{n+1})}_l = e^{-2i\tau M_l}\widetilde{(\Phi^{n-1})}_l - 2i\tau e^{-i\tau M_l}\widetilde{G_l^n}, \quad n \geq 1,
\end{aligned} \tag{2.33}$$

with the value of the first step

$$\begin{aligned}
&\widetilde{(u^1)}_l = \cos(\beta_l\tau)\widetilde{(u^0)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\dot{u}^0)}_l + \tau\frac{\sin(\beta_l\tau)}{2\beta_l}\widetilde{F_l^0}, \quad n \geq 0,\\
&\widetilde{(\dot{u}^1)}_l = -\beta_l\sin(\beta_l\tau)\widetilde{(u^0)}_l + \cos(\beta_l\tau)\widetilde{(\dot{u}^0)}_l + \frac{\tau}{2}\Big(\cos(\beta_l\tau)\widetilde{F_l^0} + \widetilde{F_l^1}\Big), \quad n \geq 0,\\
&\widetilde{(\Phi^1)}_l = e^{-i\tau M_l}\widetilde{(\Phi^0)}_l - i\tau e^{-i\tau M_l}\widetilde{G_l^0}.
\end{aligned}$$

Exchanging $n+1$ with $n-1$ and replacing τ by $-\tau$ in the coefficients (2.33)*, it remains the same. So the scheme SEIFP is time symmetric.*

Remark 2.2. *When we do not need to approximate the derivative of u, the scheme SEIFP for the KGD equation* (2.8) *is often simplified to to the following scheme*

$$u_j^{n+1} = \sum_{l=-M/2}^{M/2-1}\widetilde{(u^{n+1})}_l e^{2ijl\pi/M}, \quad \Phi_j^{n+1} = \sum_{l=-M/2}^{M/2-1}\widetilde{(\Phi^{n+1})}_l e^{2ijl\pi/M}, \tag{2.34}$$

where

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = 2\cos(\beta_l\tau)\widetilde{(u^n)}_l - \widetilde{(u^{n-1})}_l + \tau\frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{F_l^n},\\
&\widetilde{(\Phi^{n+1})}_l = e^{-2i\tau M_l}\widetilde{(\Phi^{n-1})}_l - 2i\tau e^{-i\tau M_l}\widetilde{G_l^n},
\end{aligned} \tag{2.35}$$

with the value of the first step calculated as follows

$$\begin{aligned}
&\widetilde{(u^1)}_l = \cos(\beta_l\tau)\widetilde{(u^0)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\dot{u}^0)}_l + \tau^2\frac{\sin(\beta_l\tau)}{2\beta_l\tau}\widetilde{F_l^0},\\
&\widetilde{(\Phi^1)}_l = e^{-i\tau M_l}\widetilde{(\Phi^0)}_l - i\tau e^{-i\tau M_l}\widetilde{G_l^0}.
\end{aligned} \tag{2.36}$$

The equivalent form is very concise and practical.

In the analysis of this paper, the following lemmas are useful.

Lemma 2.2. *[45] For any $\mu \geq 0$, $k \geq 0$ and $U \in [H_p^{\mu+k}(\Omega)]$ (or $U \in [H_p^{\mu+k}(\Omega)]^2$), we have*

$$\|U - P_M(U)\|_\mu \leq Ch^k \|U\|_{\mu+k}, \quad \|P_M(U)\|_{\mu+k} \leq C \|u\|_{\mu+k}. \tag{2.37}$$

Moreover, if $\mu + k > 1/2$, we have

$$\|U - I_M(U)\|_\mu \leq Ch^k \|U\|_{\mu+k}, \quad \|I_M(U)\|_{\mu+k} \leq C \|U\|_{\mu+k}. \tag{2.38}$$

Here $C > 0$ is a generic constant which is independent of h and U.

Lemma 2.3. *For $\mu > 1/2$, $u \in H^\mu(\Omega)$ and $U, W \in [H^\mu(\Omega)]^2$, we have the following results*

$$\begin{aligned} &\|uW\|_\mu \leq C_\mu \|u\|_\mu \|W\|_\mu, \quad \|u\sigma_3 V\|_\mu \leq C_\mu \|u\|_\mu \|W\|_\mu, \\ &\|U^* W\|_\mu \leq C_\mu \|U\|_\mu \|W\|_\mu, \quad \|U^*\sigma_3 W\|_\mu \leq C_\mu \|U\|_\mu \|W\|_\mu. \end{aligned} \tag{2.39}$$

Proof. Using the bilinear estimate [1]

$$\|fg\|_\mu \leq C_\mu \|f\|_\mu \|g\|_\mu, \quad \mu > 1/2, \quad f, g \in H^\mu(\Omega), \tag{2.40}$$

we obtain

$$\|uW\|_\mu^2 = \|uW_1\|_\mu^2 + \|uW_2\|_\mu^2 \leq C_{\mu 1}^2 \|u\|_\mu^2 \|W_1\|_\mu^2 + C_{\mu 2}^2 \|u\|_\mu^2 \|W_2\|_\mu^2 \leq C_\mu \|u\|_\mu \|W\|_\mu, \tag{2.41}$$

where $C_\mu = \max\{C_{\mu 1}, C_{\mu 2}\}$. Other results can be similarly proved. □

Lemma 2.4. *For $\mu > 1/2$, $u_1, u_2 \in H^\mu(\Omega)$, $\Phi_1, \Phi_2 \in [H^\mu(\Omega)]^2$ and $F(\Phi)$, $G(u, \Phi)$ defined in* (2.9)*, we have the following results*

$$\begin{aligned} &\|F(\Phi_1) - F(\Phi_2)\|_\mu \leq |g| \Big(\|\Phi_1\|_\mu + \|\Phi_2\|_\mu \Big) \|\Phi_1 - \Phi_2\|_\mu, \\ &\|G(u_1, \Phi_1) - G(u_2, \Phi_2)\|_\mu \leq |g| \Big(\|u_1\|_\mu \|\Phi_1 - \Phi_2\|_\mu + \|\Phi_2\|_\mu \|u_1 - u_2\|_\mu \Big), \\ &\|G(u_1, \Phi_1) - G(u_2, \Phi_2)\|_\mu \leq |g| \Big(\|u_2\|_\mu \|\Phi_1 - \Phi_2\|_\mu + \|\Phi_1\|_\mu \|u_1 - u_2\|_\mu \Big). \end{aligned} \tag{2.42}$$

Proof. By direct calculation, we obtain

$$F(\Phi_1) - F(\Phi_2) = g(\Phi_1^*\sigma_3\Phi_1) - g(\Phi_2^*\sigma_3\Phi_2) = g\Phi_1^*\sigma_3(\Phi_1 - \Phi_2) + g(\Phi_1 - \Phi_2)^*\sigma_3\Phi_2. \tag{2.43}$$

Using the Lemma 2.3 in (2.43) gives the first inequality. Other inequalities can be similarly proved □

Lemma 2.5. *For $F(\Phi)$ and $G(u, \Phi)$ defined in* (2.9) *and u, Φ satisfying the necessary regularity, then at any $t = t_n$, we have the following results*

$$\begin{aligned} &\|F(\Phi)\|_\mu \leq |g| \cdot \|\Phi\|_\mu^2, \quad \|\partial_t F(\Phi)\|_\mu \leq 2\,|g| \cdot \|\Phi\|_\mu \|\partial_t \Phi\|_\mu, \\ &\|\partial_{tt} F(\Phi)\|_\mu \leq 2\,|g| \cdot \Big(\|\partial_t \Phi\|_\mu^2 + \|\Phi\|_\mu \|\partial_{tt}\Phi\|_\mu \Big), \\ &\|G(u, \Phi)\|_\mu \leq |g| \cdot \|u\|_\mu \|\Phi\|_\mu, \quad \|\partial_t G(u, \Phi)\|_\mu \leq |g| \cdot \Big(\|u\|_\mu \|\partial_t \Phi\|_\mu + \|\Phi\|_\mu \|\partial_t u\|_\mu \Big), \\ &\|\partial_{tt} G(u, \Phi)\|_\mu \leq |g| \cdot \Big(\|\partial_{tt} u\|_\mu \|\Phi\|_\mu + 2\,\|\partial_t u\|_\mu \|\partial_t \Phi\|_\mu + \|u\|_\mu \|\partial_{tt}\Phi\|_\mu \Big), \end{aligned} \tag{2.44}$$

where $\Phi := \Phi(t_n, x)$ and $u := u(t_n, x)$, respectively.

Proof. It is easy to prove this conclusion using Lemma 2.3. □

Lemma 2.6. *For β_l and M_l defined in* (2.21)*, denote $|M_l|$ as the spectral norm of M_l, we have the following results*

$$|M_l| = \sqrt{\omega^2 + |\mu_l|^2}, \quad \left|e^{-ik\tau M_l}\right| = 1, \tag{2.45}$$

Proof. The proof is obvious and omitted here. □

3. Error Estimates

In this section, we present the rigorous error estimate results of the new scheme SEIFP for solving the KGD equation (2.8). Motivated by the analytical results, we make the following assumptions on the exact solution u and Φ of the KGD equation (2.8):

$$\begin{aligned} u \in & C\Big([0,T]; H_p^{m+m_0}(\Omega)\Big) \cap C^1\Big([0,T]; H_p^{m+m_0-1}(\Omega)\Big) \cap C^2\Big([0,T]; H_p^{m-1}(\Omega)\Big), \\ \Phi \in & C\Big([0,T]; [H_p^{m+m_0}(\Omega)]^2\Big) \cap C^1\Big([0,T]; [H_p^{m}(\Omega)]^2\Big) \cap C^2\Big([0,T]; [H_p^{m-1}(\Omega)]^2\Big), \end{aligned} \tag{A}$$

where $m \geq 3/2$, $m_0 \geq 1$ and

$$H_p^m(\Omega) = \Big\{u \in H^m(\Omega), \quad \partial_x^l(a) = \partial_x^l(b), \quad l = 0, \cdots, m-1\Big\},$$

and here the boundary values are understood in the trace sense. In the subsequent discussion, we will omit Ω when referring to the space norm taken on Ω. We denote

$$M_u := \|u(t,x)\|_{L^\infty([0,T];H^m(\Omega))}, \quad M_u' := \|\partial_t u(t,x)\|_{L^\infty([0,T];H^{m-1}(\Omega))}, \quad M_\Phi := \|\Phi(t,x)\|_{L^\infty([0,T];[H^{m-1}(\Omega)]^2)}.$$

Define the error function as:

$$\bar{e}^n := u(t_n, x) - I_M u^n, \quad \bar{\bar{e}}^n := \partial_t u(t_n, x) - I_M \dot{u}^n, \quad e^n := \Phi(t_n, x) - I_M \Phi^n, \tag{3.46}$$

where $n = 0, 1, \cdots, N$ with $\bar{e}^n \in Y_M$, $\bar{\bar{e}}^n \in Y_M$ and $e^n \in Z_M$ being the numerical approximations obtained from the scheme (2.29)-(2.31), then we could prove the following error estimates.

Theorem 3.1. *Under the assumption (A), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small such that when $0 < h \leq h_0$ and $0 < \tau \leq \tau_0$, we have the following error estimate for the scheme* (2.29)-(2.31) *:*

$$\begin{aligned} &\|u(t_n,x) - (I_M u^n)(x)\|_m \lesssim h^{m_0} + \tau^2, \quad \|(I_M u^n)(x)\|_m \leq 1 + M_u, \\ &\|\partial_t u(t_n,x) - (I_M \dot{u}^n)(x)\|_{m-1} \lesssim h^{m_0} + \tau^2, \quad \|(I_M \dot{u}^n)(x)\|_{m-1} \leq 1 + M_u', \\ &\|\Phi(t_n,x) - (I_M \Phi^n)(x)\|_{m-1} \lesssim h^{m_0} + \tau^2, \quad \|(I_M \Phi^n)(x)\|_{m-1} \leq 1 + M_\Phi, \quad 0 \leq n \leq N, \end{aligned} \tag{3.47}$$

where $M_u := \|u(t,x)\|_{L^\infty([0,T];H^m(\Omega))}$, $M_u' := \|\partial_t u(t,x)\|_{L^\infty([0,T];H^{m-1}(\Omega))}$ and $M_\Phi := \|\Phi(t,x)\|_{L^\infty([0,T];H^{m-1}(\Omega))}$.

Proof. In order to proceed to the proof, we first define the projected error

$$\begin{aligned} &\bar{e}_M^n(x) = P_M u(t_n,x) - (I_M u^n)(x), \quad \bar{\bar{e}}_M^n(x) = P_M \partial_t u(t_n,x) - (I_M \dot{u}^n)(x), \\ &e_M^n(x) = P_M \Phi(t_n,x) - (I_M \Phi^n)(x). \end{aligned} \tag{3.48}$$

Then by triangle inequality and using Lemma 2.2 under the assumption (A), we have

$$\begin{aligned}
\|u(t_n,x)-(I_Mu^n)(x)\|_m &\le \left\|\bar{e}_M^n\right\|_m + \|u(t_n,x)-P_Mu(t_n,x)\|_m \\
&\le \left\|\bar{e}_M^n\right\|_m + Ch^{m_0}\|u(t_n,x)\|_{m+m_0} \lesssim \left\|\bar{e}_M^n\right\|_m + h^{m_0}, \\
\|\partial_t u(t_n,x)-(I_M\dot{u}^n)(x)\|_{m-1} &\le \left\|\bar{\bar{e}}_M^n\right\|_{m-1} + \|\partial_t u(t_n,x)-P_M\partial_t u(t_n,x)\|_{m-1} \\
&\le \left\|\bar{\bar{e}}_M^n\right\|_{m-1} + Ch^{m_0}\|\partial_t u(t_n,x)\|_{m+m_0-1} \lesssim \left\|\bar{\bar{e}}_M^n\right\|_{m-1} + h^{m_0}, \\
\|\Phi(t_n,x)-(I_M\Phi^n)(x)\|_{m-1} &\le \left\|e_M^n\right\|_{m-1} + \|\Phi(t_n,x)-P_M\Phi(t_n,x)\|_{m-1} \\
&\le \left\|e_M^n\right\|_{m-1} + Ch^{m_0}\|\Phi(t_n,x)\|_{m+m_0-1} \lesssim \left\|e_M^n\right\|_{m-1} + h^{m_0}.
\end{aligned} \tag{3.49}$$

Obviously it is sufficient to work out the corresponding estimate for $\bar{e}_M^n$, $\bar{\bar{e}}_M^n$ and e_M^n to prove (3.47). Since the calculations of the first step are different from the others, we investigate the first step separately

Lemma 3.1. *Under the assumption (A), there exist constants $h_0 > 0$ and $\tau_0 > 0$ sufficiently small such that when $0 < h \le h_0$ and $0 < \tau \le \tau_0$, we have the following error estimate for the scheme* (2.29) *with* (2.31) *when $n = 0$:*

$$\begin{aligned}
&\left\|u(\tau,x)-(I_Mu^1)(x)\right\|_m \lesssim h^{m_0}+\tau^2, \quad \left\|(I_Mu^1)(x)\right\|_m \le 1+M_u, \\
&\left\|\partial_t u(\tau,x)-(I_M\dot{u}^1)(x)\right\|_{m-1} \lesssim h^{m_0}+\tau^2, \quad \left\|(I_M\dot{u}^1)(x)\right\|_{m-1} \le 1+M_u', \\
&\left\|\Phi(\tau,x)-(I_M\Phi^1)(x)\right\|_{m-1} \lesssim h^{m_0}+\tau^2, \quad \left\|(I_M\Phi^1)(x)\right\|_{m-1} \le 1+M_\Phi,
\end{aligned} \tag{3.50}$$

where $M_u := \|u(t,x)\|_{L^\infty([0,T];H^m(\Omega))}$, $M_u' := \|\partial_t u(t,x)\|_{L^\infty([0,T];H^{m-1}(\Omega))}$ and $M_\Phi := \|\Phi(t,x)\|_{L^\infty([0,T];H^{m-1}(\Omega))}$.

Proof. Denote the true solution of the KGD equation (2.8) as Fourier series

$$u(t,x) = \sum_{l=-\infty}^{\infty} \widehat{u}_l(t)\mathrm{e}^{i\mu_l(x-a)}, \quad \Phi(t,x) = \sum_{l=-\infty}^{\infty} \widehat{\Phi}_l(t)\mathrm{e}^{i\mu_l(x-a)}. \tag{3.51}$$

For the first step, by plugging (3.51) into (2.30) and (2.31) with $n = 0$, we obtain the local truncation errors

$$\begin{aligned}
\widehat{u}_l(\tau) =& \cos(\beta_l\tau)\widehat{u}_l(0) + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{\dot{u}}_l(0) + \frac{\tau\sin(\tau\beta_l)}{2\beta_l}\widehat{F}_l(0) + \widehat{\xi}_l^1, \\
\widehat{\dot{u}}_l(\tau) =& -\beta_l\sin(\beta_l\tau)\widehat{u}_l(0) + \cos(\beta_l\tau)\widehat{\dot{u}}_l(0) + \frac{\tau}{2}\Big(\cos(\beta_l\tau)\widehat{F}_l(0) + \widehat{F}_l(\tau)\Big) + \widehat{\dot{\xi}}_l^1, \\
\widehat{\Phi}_l(\tau) =& e^{-i\tau M_l}\widehat{\Phi}_l(0) - i\tau e^{-i\tau M_l}\widehat{G}_l(0) + \widehat{\xi}_l^1,
\end{aligned} \tag{3.52}$$

where

$$F(t,x) = F\Big(\Phi(t,x)\Big), \quad G(t,x) = G\Big(u(t,x),\Phi(t,x)\Big). \tag{3.53}$$

On the other hand, in the Fourier frequency space, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\widehat{u}_l(t) + \beta_l^2\widehat{u}_l(t) = \widehat{F}_l(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\widehat{\Phi}_l(t) = -iM_l\widehat{\Phi}_l(t) - i\widehat{G}_l(t). \tag{3.54}$$

where β_l and M_l have been given in (2.21). By using the variation-of-constant formula in (3.54), we obtain the following relation

$$
\begin{aligned}
\widehat{u}_l(\tau) &= \cos(\beta_l\tau)\widehat{u}_l(0) + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{\dot{u}}_l(0) + \tau\int_0^1 \frac{\sin((1-u)\beta_l\tau)}{\beta_l}\widehat{F}_l(z\tau)\mathrm{d}z,\\
\widehat{\dot{u}}_l(\tau) &= -\beta_l\sin(\beta_l\tau)\widehat{u}_l(0) + \cos(\beta_l\tau)\widehat{\dot{u}}_l(0) + \tau\int_0^1 \cos((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\mathrm{d}z,\\
\widehat{\Phi}_l(\tau) &= e^{-i\tau M_l}\widehat{\Phi}_l(0) - i\tau\int_0^1 e^{-i\tau(1-z)M_l}\widehat{G}_l(z\tau)\mathrm{d}z.
\end{aligned}
\tag{3.55}
$$

Subtracting (3.55) from (3.52), we have

$$
\begin{aligned}
\widehat{\xi}_l^1 &= \frac{\tau}{\beta_l}\left(\int_0^1 \sin((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\mathrm{d}z - \frac{\sin(\beta_l\tau)}{2}\widehat{F}_l(0)\right),\\
\widehat{\dot{\xi}}_l^1 &= \tau\left(\int_0^1 \cos((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\mathrm{d}z - \frac{1}{2}\Big(\cos(\beta_l\tau)\widehat{F}_l(0) + \widehat{F}_l(\tau)\Big)\right),\\
\widehat{\xi}_l^1 &= -i\tau\left(\int_0^1 e^{-i\tau(1-z)M_l}\widehat{G}_l(z\tau)\mathrm{d}z - e^{-i\tau M_l}\widehat{G}_l(0)\right).
\end{aligned}
\tag{3.56}
$$

Applying the following quadrature errors

$$
\begin{aligned}
\int_0^1 f(z)\mathrm{d}z &= \frac{1}{2}\Big(f(0)+f(1)\Big) - \frac{1}{2}\int_0^1 z(1-z)f''(z)\mathrm{d}z,\\
\int_0^1 f(z)\mathrm{d}z &= f(0) + \int_0^1 (1-z)f'(z)\mathrm{d}z,
\end{aligned}
$$

to the above formulas (3.56), we get

$$
\begin{aligned}
\widehat{\xi}_l^1 &= -\frac{\tau}{2\beta_l}\int_0^1 z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l^0(z\tau)\mathrm{d}z, \quad \widehat{\dot{\xi}}_l^1 = -\frac{\tau}{2}\int_0^1 z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l^0(z\tau)\mathrm{d}z\\
\widehat{\xi}_l^1 &= -i\tau\int_0^1 (1-z)\frac{\mathrm{d}}{\mathrm{d}z}R_l^0(z\tau)\mathrm{d}z.
\end{aligned}
\tag{3.57}
$$

where

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l^0(z\tau) =&\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\sin((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\Big),\\
=&-\beta_l^2\tau^2\sin((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\\
&-\beta_l\tau\cos((1-z)\beta_l\tau)\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(z\tau)+\sin((1-z)\beta_l\tau)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(z\tau),\\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l^0(z\tau) =&\frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\cos((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\Big)\\
=&-\beta_l^2\tau^2\cos((1-z)\beta_l\tau)\widehat{F}_l(z\tau)\\
&+\beta_l\tau\sin((1-\theta_1)\beta_l\tau)\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(z\tau)+\cos((1-\theta_1)\beta_l\tau)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(z\tau),\\
\frac{\mathrm{d}}{\mathrm{d}z}R_l^0(z\tau) =&\frac{\mathrm{d}}{\mathrm{d}z}\Big(e^{-i\tau(1-z)M_l}\widehat{G}_l(z\tau)\Big)\\
=&e^{-i\tau M_l}\Big[i\tau M_l e^{izM_l}\widehat{G}_l(z\tau)+\frac{\mathrm{d}}{\mathrm{d}z}\widehat{G}_l(z\tau)\Big].
\end{aligned}
\tag{3.58}
$$

Applying the Hölder's inequality, we get

$$
\begin{aligned}
\Big|\widehat{\bar{\xi}_l^1}\Big| &\le \frac{\tau}{2\beta_l}\left(\int_0^1\Big[z(1-z)\Big]^2\mathrm{d}z\right)^{1/2}\left(\int_0^1\left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l^0(z\tau)\right|^2\mathrm{d}z\right)^{1/2},\\
\Big|\widehat{\dot{\bar{\xi}}_l^1}\Big| &\le \frac{\tau}{2}\left(\int_0^1\Big[z(1-z)\Big]^2\mathrm{d}z\right)^{1/2}\left(\int_0^1\left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l^0(z\tau)\right|^2\mathrm{d}z\right)^{1/2},\\
\Big|\widehat{\xi_l^1}\Big| &\le \frac{\tau}{2}\left(\int_0^1(1-z)^2\mathrm{d}z\right)^{1/2}\left(\int_0^1\left|\frac{\mathrm{d}}{\mathrm{d}z}R_l^0(z\tau)\right|^2\mathrm{d}z\right)^{1/2}.
\end{aligned}
\tag{3.59}
$$

We define the local truncation error functions as

$$
\bar{\xi}^1(x)=\sum_{l=-M/2}^{M/2-1}\widehat{\bar{\xi}_l^1}\mathrm{e}^{i\mu_l(x-a)},\quad \dot{\bar{\xi}}^1(x)=\sum_{l=-M/2}^{M/2-1}\widehat{\dot{\bar{\xi}}_l^1}\mathrm{e}^{i\mu_l(x-a)},\quad \xi^1(x)=\sum_{l=-M/2}^{M/2-1}\widehat{\xi_l^1}\mathrm{e}^{i\mu_l(x-a)}. \tag{3.60}
$$

From the definition of norm (2.11) and (2.12), we have the estimates on the local errors as

$$
\begin{aligned}
\left\|\bar{\xi}^1\right\|_m^2 &= \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^m \left|\widehat{\bar{\xi}_l^1}\right| \\
&\lesssim \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^m \Big((1+|\mu_l|^2)\tau^4 \int_0^1 \left|\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z \\
&\qquad + \tau^2 \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z + (1+|\mu_l|^2)^{-1} \int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z \Big) \\
&\lesssim \tau^2 \int_0^1 \Big(\tau^4 \|F(z\tau,x)\|_{m+1}^2 + \tau^2 \left\|\frac{\mathrm{d}}{\mathrm{d}z}F(z\tau,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}z^2}F(z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
&= \tau^5 \int_0^\tau \Big(\|F(w,x)\|_{m+1}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}F(w,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}w^2}F(w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w, \\
\left\|\bar{\xi}^1\right\|_{m-1}^2 &= \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \left|\widehat{\bar{\xi}_l^1}\right| \\
&\lesssim \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \Big((1+|\mu_l|^2)^2\tau^4 \int_0^1 \left|\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z \\
&\qquad + \tau^2 \int_0^1 (1+|\mu_l|^2) \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z + \int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(z\tau)\right|^2 \mathrm{d}z \Big) \\
&\lesssim \tau^2 \int_0^1 \Big(\tau^4 \|F(z\tau,x)\|_{m+1}^2 + \tau^2 \left\|\frac{\mathrm{d}}{\mathrm{d}z}F(z\tau,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}z^2}F(z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
&= \tau^5 \int_0^\tau \Big(\|F(w,x)\|_{m+1}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}F(w,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}w^2}F(w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w, \\
\left\|\xi^1\right\|_{m-1}^2 &= \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \left|\widehat{\xi_l^1}\right| \\
&\lesssim \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \Big((1+|\mu_l|^2)\tau^2 \int_0^1 \left|\widehat{G}_l(z\tau)\right|^2 \mathrm{d}z + \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{G}_l(z\tau)\right|^2 \Big)\mathrm{d}z \\
&\lesssim \tau^2 \int_0^1 \Big(\tau^2 \|G(z\tau,x)\|_m^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}z}G(z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
&= \tau^3 \int_0^\tau \Big(\|G(w,x)\|_m^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}G(w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w.
\end{aligned}
\tag{3.61}
$$

From Lemma 2.5 and the assumption (A), we have

$$\left\|\bar{\xi}^1\right\|_m^2 \lesssim \tau^6, \quad \left\|\bar{\dot{\xi}}^1\right\|_{m-1}^2 \lesssim \tau^6, \quad \left\|\xi^1\right\|_{m-1}^2 \lesssim \tau^4. \tag{3.62}$$

Subtracting the the coefficients of scheme (2.30) and (2.31) from (3.52), we get the error iteration

$$\begin{aligned}
&\widehat{(\bar{e}_M^1)}_l = \cos(\beta_l\tau)\widehat{(\bar{e}_M^0)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\bar{\dot{e}}_M^0)}_l + \widehat{\bar{\eta}_l^1} + \widehat{\bar{\xi}_l^1}, \\
&\widehat{(\bar{\dot{e}}_M^1)}_l = -\beta_l\sin(\beta_l\tau)\widehat{(\bar{e}_M^0)}_l + \cos(\beta_l\tau)\widehat{(\bar{\dot{e}}_M^0)}_l + \widehat{\bar{\dot{\eta}}_l^1} + \widehat{\bar{\dot{\xi}}_l^1}, \\
&\widehat{(e_M^1)}_l = e^{-i\tau M_l}\widehat{(e_M^0)}_l + \widehat{\eta_l^1} + \widehat{\xi_l^1},
\end{aligned} \tag{3.63}$$

where $\widehat{\bar{\xi}_l^1}$, $\widehat{\bar{\dot{\xi}}_l^1}$ and $\widehat{\xi_l^1}$ are defined in (3.52) and

$$\begin{aligned}
&\widehat{\bar{\eta}_l^1} = \frac{\tau\sin(\beta_l\tau)}{2\beta_l}\left(\widehat{F}_l(0) - \widetilde{F}_l^0\right), \quad \widehat{\bar{\dot{\eta}}_l^1} = \frac{\tau}{2}\Big(\cos(\beta_l\tau)(\widehat{F}_l(0) - \widetilde{F}_l^0) + (\widehat{F}_l(\tau) - \widetilde{F}_l^1)\Big), \\
&\widehat{\eta_l^1} = -i\tau e^{-i\tau M_l}\Big(\widehat{G}_l(0) - \widetilde{G_l^0}\Big),
\end{aligned} \tag{3.64}$$

From the definition (3.48), the projected error can be expressed as

$$\begin{aligned}
&\bar{e}_M^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\bar{e}_M^1)}_l e^{i\mu_l(x-a)}, \quad \bar{\dot{e}}_M^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(\bar{\dot{e}}_M^1)}_l e^{i\mu_l(x-a)}, \\
&e_M^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{(e_M^1)}_l e^{i\mu_l(x-a)}.
\end{aligned} \tag{3.65}$$

We define the error functions of nonlinear terms as

$$\bar{\eta}^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\bar{\eta}_l^1} e^{i\mu_l(x-a)}, \quad \bar{\dot{\eta}}^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\bar{\dot{\eta}}_l^1} e^{i\mu_l(x-a)}, \quad \eta^1(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\eta_l^1} e^{i\mu_l(x-a)}. \tag{3.66}$$

By the definition of norm (2.11) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\|\bar{\eta}^1\right\|_m \le \frac{\tau}{2}\left\|P_M F(0,x) - (I_M F^0)(x)\right\|_{m-1}, \\
&\left\|\bar{\dot{\eta}}^1\right\|_{m-1} \le \frac{\tau}{2}\Big(\left\|P_M F(0,x) - (I_M F^0)(x)\right\|_{m-1} + \left\|P_M F(\tau,x) - (I_M F^1)(x)\right\|_{m-1}\Big), \\
&\left\|\eta^1\right\|_{m-1} \le \tau\left\|P_M G(0,x) - (I_M G^0)(x)\right\|_{m-1}.
\end{aligned} \tag{3.67}$$

From Lemma 2.2, we obtain

$$\begin{aligned}
&\left\|u(0,x) - (I_M u^0)(x)\right\|_m \le Ch^{m_0}\left\|u(0,x)\right\|_{m+m_0} \lesssim h^{m_0}, \\
&\left\|\partial_t u(0,x) - (I_M \dot{u}^0)(x)\right\|_{m-1} \le Ch^{m_0}\left\|\partial_t u(0,x)\right\|_{m+m_0-1} \lesssim h^{m_0}, \\
&\left\|\Phi(0,x) - (I_M \Phi^0)(x)\right\|_{m-1} \le Ch^{m_0}\left\|\Phi(0,x)\right\|_{m+m_0-1} \lesssim h^{m_0}.
\end{aligned} \tag{3.68}$$

From the Lemma 2.2 and the triangle inequality, we have

$$\begin{aligned}
&\left\|\bar{e}_M^0\right\|_m = \left\|u(0,x) - (I_M u^0)(x)\right\|_m + \|u(0,x) - (P_M u)(0,x)\|_m \lesssim h^{m_0},\\
&\left\|\bar{e}_M^0\right\|_{m-1} = \left\|\partial_t u(0,x) - (I_M \dot{u}^0)(x)\right\|_{m-1} + \|\partial_t u(0,x) - (P_M \dot{u})(0,x)\|_{m-1} \lesssim h^{m_0},\\
&\left\|e_M^0\right\|_{m-1} = \left\|\Phi(0,x) - (I_M \Phi^0)(x)\right\|_{m-1} + \|\Phi(0,x) - (P_M \Phi)(0,x)\|_{m-1} \lesssim h^{m_0}.
\end{aligned} \tag{3.69}$$

According to the triangle inequality and Lemma 2.2, we obtain

$$\begin{aligned}
&\left\|(I_M u^0)(x)\right\|_m \le C\,\|u(0,x)\|_m \lesssim 1, \quad \left\|(I_M \dot{u}^0)(x)\right\|_{m-1} \le C\,\|\dot{u}(0,x)\|_{m-1} \lesssim 1.\\
&\left\|(I_M \Phi^0)(x)\right\|_{m-1} \le C\,\|\Phi(\tau,x)\|_{m-1} \lesssim 1.
\end{aligned} \tag{3.70}$$

From Lemma 2.2, Lemma 2.4, (3.70) and the triangle inequality, we have

$$\begin{aligned}
&\left\|P_M F(0,x) - I_M F^0\right\|_{m-1}\\
\lesssim&\left\|I_M F(\Phi(0,x)) - I_M F(I_M \Phi^0)\right\|_{m-1} + Ch^{m_0}\left\|F(\Phi(0,x))\right\|_{m+m_0-1}\\
\lesssim&\left\|F(\Phi(0,x)) - F(I_M \Phi^0)\right\|_{m-1} + h^{m_0}\\
\lesssim&\left\|\Phi(0,x) - I_M \Phi^0\right\|_{m-1}\left(\|\Phi(0,x)\|_{m-1} + \left\|I_M \Phi^0\right\|_{m-1}\right) + h^{m_0} \lesssim h^{m_0}.
\end{aligned} \tag{3.71}$$

Here and below, (x) is omitted without affecting understanding. Similarly, we obtain

$$\begin{aligned}
&\left\|P_M G(0,x) - I_M G^0\right\|_{m-1}\\
\lesssim&\left\|I_M G\big(u(0,x), \Phi(0,x)\big) - I_M G\big(I_M u^0, I_M \Phi^0\big)\right\|_{m-1} + Ch^{m_0}\left\|G\big(u(0,x), \Phi(0,x)\big)\right\|_{m+m_0-1}\\
\lesssim&\left\|G\big(u(0,x), \Phi(0,x)\big) - G\big(I_M u^0, I_M \Phi^0\big)\right\|_{m-1} + h^{m_0}\\
\lesssim&\Big(\left\|\Phi(0,x) - I_M \Phi^0\right\|_{m-1}\|u(0,x)\|_{m-1} + \left\|u(0,x) - I_M u^0\right\|_{m-1}\left\|I_M \Phi^0\right\|_{m-1}\Big) + h^{m_0}\\
\lesssim&\left(\left\|\bar{e}_M^0\right\|_{m-1} + \left\|e_M^0\right\|_{m-1}\right) + h^{m_0} \lesssim \left(\left\|\bar{e}_M^0\right\|_m + \left\|e_M^0\right\|_{m-1}\right) + h^{m_0} \lesssim h^{m_0}.
\end{aligned} \tag{3.72}$$

Inserting (3.71) and (3.72) into (3.67), when $\tau \le 1$, we have

$$\left\|\bar{\eta}^1\right\|_m \lesssim \tau h^{m_0} \lesssim h^{m_0}. \quad \left\|\eta^1\right\|_{m-1} \lesssim \tau h^{m_0} \lesssim h^{m_0}. \tag{3.73}$$

Using Cauchy-Schwarz inequality and (3.63), (3.78), (3.69) and (3.62), we have

$$\begin{aligned}
&\left\|\bar{e}_M^1\right\|_m \le \left\|e_M^0\right\|_m + \left\|\dot{e}_M^0\right\|_{m-1} + \left\|\eta^1\right\|_m + \left\|\xi^1\right\|_m \lesssim \tau^2 + h^{m_0},\\
&\left\|e_M^1\right\|_{m-1} \le \left\|e_M^0\right\|_{m-1} + \left\|\eta^1\right\|_{m-1} + \left\|\xi^1\right\|_{m-1} \lesssim \tau^2 + h^{m_0}.
\end{aligned} \tag{3.74}$$

From Lemma 2.2 and the triangle inequality, it is easy to get

$$\begin{aligned}
&\left\|u(\tau,x) - I_M u^1\right\|_m \le \left\|\bar{e}_M^1\right\|_m + h^{m_0} \lesssim \tau^2 + h^{m_0}.\\
&\left\|\Phi(\tau,x) - I_M \Phi^1\right\|_{m-1} \le \left\|e_M^1\right\|_{m-1} + h^{m_0} \lesssim \tau^2 + h^{m_0}.
\end{aligned} \tag{3.75}$$

There exist sufficiently small $\tau_1 > 0$ and $h_1 > 0$, when $0 < \tau \le \tau_1$ and $0 < h \le h_1$, we have

$$\begin{aligned}
&\left\|(I_M u^1)(x)\right\|_m \le \|u(\tau,x)\|_m + \left\|u(\tau,x) - (I_M u^1)(x)\right\|_m \le 1 + M_u,\\
&\left\|(I_M \Phi^1)(x)\right\|_{m-1} \le \|\Phi(\tau,x)\|_{m-1} + \left\|\Phi(\tau,x) - (I_M \Phi^1)(x)\right\|_{m-1} \le 1 + M_\Phi.
\end{aligned} \tag{3.76}$$

Similarly, using Lemma 2.2, Lemma 2.4 and (3.76), we have

$$
\begin{aligned}
&\left\|P_M F(\tau,x) - I_M F^1\right\|_{m-1} \\
\leq &\left\|I_M F(\Phi(\tau,x)) - I_M F(I_M\Phi^1)\right\|_{m-1} + Ch^{m_0}\left\|F(\Phi(\tau,x))\right\|_{m+m_0-1}, \\
\lesssim &\left\|F(\Phi(\tau,x)) - F(I_M\Phi^1)\right\|_{m-1} + h^{m_0}, \\
\lesssim &\left\|\Phi(\tau,x) - I_M\Phi^1\right\|_{m-1}\left(\|\Phi(\tau,x)\|_{m-1} + \left\|I_M\Phi^1\right\|_{m-1}\right) + h^{m_0} \lesssim h^{m_0}.
\end{aligned}
\tag{3.77}
$$

From (3.67) and (3.77), we have

$$
\left\|\bar{\eta}^1\right\|_{m-1} \lesssim \tau h^{m_0} \lesssim h^{m_0}. \tag{3.78}
$$

From the triangle inequality, (3.63) and the definition (2.11), we obtain

$$
\left\|\bar{e}_M^1\right\|_{m-1} \leq \left\|\bar{e}_M^0\right\|_m + \left\|\bar{e}_M^0\right\|_{m-1} + \left\|\bar{\eta}^1\right\|_{m-1} + \left\|\bar{\xi}^1\right\|_{m-1} \lesssim \tau^2 + h^{m_0}. \tag{3.79}
$$

From the triangle inequality and (3.79), we get

$$
\left\|\partial_t u(\tau,x) - (I_M\dot{u}^1)(x)\right\|_{m-1} \leq \|\partial_t u(\tau,x) - P_M\partial_t u(\tau,x)\|_{m-1} + \left\|\bar{e}_M^1\right\|_{m-1} \lesssim \tau^2 + h^{m_0}. \tag{3.80}
$$

There exist sufficiently small $\tau_2 > 0$ and $h_2 > 0$, when $0 < \tau \leq \tau_2$ and $0 < h \leq h_2$, we have

$$
\left\|(I_M\dot{u}^1)(x)\right\|_{m-1} \leq \|\partial_t u(\tau,x)\|_{m-1} + \left\|\partial_t u(\tau,x) - (I_M\dot{u}^1)(x)\right\|_{m-1} \leq 1 + M_u'. \tag{3.81}
$$

The theorem has been proved by taking $h_0 = \min\{h_1, h_2\}$ and $\tau_0 = \min\{1, \tau_1, \tau_2\}$. □

In order to prove Theorem 3.1, we need the following steps to prepare.

Step1 : Estimates on local truncation errors $\bar{\xi}_l^{n+1}$, $\dot{\bar{\xi}}_l^{n+1}$ and ξ_l^{n+1}. By plugging (3.51) into (2.30), we define the Fourier transform coefficient $\widehat{\bar{\xi}_l^{n+1}}$, $\widehat{\dot{\bar{\xi}}_l^{n+1}}$ and $\widehat{\xi_l^{n+1}}$ ($1 \leq n \leq T/\tau - 1$, $l = -M/2, \cdots, M/2-1$) of local truncation errors as

$$
\begin{aligned}
\widehat{u_l}(t_{n+1}) =& \cos(\beta_l\tau)\widehat{u_l}(t_n) + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{\dot{u}_l}(t_n) + \frac{\tau\sin(\beta_l\tau)}{2\beta_l}\widehat{F}_l(t_n) + \widehat{\bar{\xi}_l^{n+1}}, \\
\widehat{\dot{u}_l}(t_{n+1}) =& -\beta_l\sin(\beta_l\tau)\widehat{u_l}(t_n) + \cos(\beta_l\tau)\widehat{\dot{u}_l}(t_n) + \frac{\tau}{2}\Big(\cos(\beta\tau)\widehat{F}_l(t_n) + \widehat{F}_l(t_{n+1})\Big) + \widehat{\dot{\bar{\xi}}_l^{n+1}}, \\
\widehat{\Phi}_l(t_{n+1}) =& e^{-2i\tau M_l}\widehat{\Phi}_l(t_{n-1}) - 2i\tau e^{-i\tau M_l}\widehat{G}_l(t_n) + \widehat{\xi_l^{n+1}}.
\end{aligned}
\tag{3.82}
$$

By using the variation-of-constant formula in (3.54), we obtain the following relation

$$
\begin{aligned}
\widehat{u_l}(t_{n+1}) =& \cos(\beta_l\tau)\widehat{u_l}(t_n) + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{\dot{u}_l}(t_n) + \tau\int_0^1 \frac{\sin((1-u)\beta_l\tau)}{\beta_l}\widehat{F}_l(t_n + z\tau)\mathrm{d}z, \\
\widehat{\dot{u}_l}(t_{n+1}) =& -\beta_l\sin(\beta_l\tau)\widehat{u_l}(t_n) + \cos(\beta_l\tau)\widehat{\dot{u}_l}(t_n) + \tau\int_0^1 \cos((1-z)\beta_l\tau)\widehat{F}_l(t_n + z\tau)\mathrm{d}z, \\
\widehat{\Phi}_l(t_{n+1}) =& e^{-2i\tau M_l}\widehat{\Phi}_l(t_{n-1}) - 2i\tau\int_0^1 e^{-2i\tau(1-z)M_l}\widehat{G}_l(t_{n-1} + 2z\tau)\mathrm{d}z.
\end{aligned}
\tag{3.83}
$$

Subtract (3.82) from the formula (3.83), we get

$$
\begin{aligned}
\widehat{\xi_l^{n+1}} &= \frac{\tau}{\beta_l}\left(\int_0^1 \sin((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\mathrm{d}z - \frac{1}{2}\sin(\beta_l\tau)\widehat{F}_l(t_n)\right),\\
\widehat{\bar{\xi}_l^{n+1}} &= \tau\left(\int_0^1 \cos((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\mathrm{d}z - \frac{1}{2}\Big(\cos(\beta_l\tau)\widehat{F}_l(t_n)+\widehat{F}_l(t_{n+1})\Big)\right),\\
\widehat{\varsigma_l^{n+1}} &= -2i\tau\left(\int_0^1 e^{-2i(1-z)M_l}\widehat{G}_l(t_{n-1}+2z\tau)\mathrm{d}z - e^{-i\tau M_l}\widehat{G}_l(t_n)\right).
\end{aligned}
\tag{3.84}
$$

Applying the following quadrature errors

$$
\begin{aligned}
\int_0^1 f(z)\mathrm{d}z &= \frac{1}{2}\Big(f(0)+f(1)\Big) - \frac{1}{2}\int_0^1 z(1-z)f''(z)\mathrm{d}z,\\
\int_0^1 f(z)\mathrm{d}z &= f(\frac{1}{2}) + \frac{1}{8}\int_0^1 (1-|1-2z|)^2 f''(z)\mathrm{d}z,
\end{aligned}
$$

to the above formulas (3.56), we get

$$
\begin{aligned}
&\widehat{\xi_l^{n+1}} = -\frac{\tau}{2\beta_l}\int_0^1 z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l(t_n+z\tau)\mathrm{d}z, \quad \widehat{\bar{\xi}_l^{n+1}} = -\frac{\tau}{2}\int_0^1 z(1-z)\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l(t_n+z\tau)\mathrm{d}z,\\
&\widehat{\varsigma_l^{n+1}} = -\frac{i\tau}{8}\int_0^1 (1-|1-2z|)^2\frac{\mathrm{d}^2}{\mathrm{d}z^2}R_l(t_n+z\tau)\mathrm{d}z,
\end{aligned}
\tag{3.85}
$$

where

$$
\begin{aligned}
\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l(t_n+z\tau) =& \frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\sin((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\Big)\\
=& -\beta_l^2\tau^2\sin((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\\
& -\beta_l\tau\cos((1-z)\beta_l\tau)\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(t_n+z\tau) + \sin((1-z)\beta_l\tau)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(t_n+z\tau),\\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l(t_n+z\tau) =& \frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(\cos((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\Big)\\
=& -\beta_l^2\tau^2\cos((1-z)\beta_l\tau)\widehat{F}_l(t_n+z\tau)\\
& +\beta_l\tau\sin((1-z)\beta_l\tau)\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(t_n+z\tau) + \cos((1-\theta_1)\beta_l\tau)\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(t_n+z\tau),\\
\frac{\mathrm{d}^2}{\mathrm{d}z^2}R_l(t_n+z\tau) =& \frac{\mathrm{d}^2}{\mathrm{d}z^2}\Big(e^{-i\tau(1-z)M_l}\widehat{G}_l(t_n+z\tau)\Big),\\
=& e^{-i\tau(1-z)M_l}\Big[-\tau^2M_l^2\widehat{G}_l(z\tau) + i\tau M_l\frac{\mathrm{d}}{\mathrm{d}z}\widehat{G}_l(t_n+z\tau) + \frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{G}_l(t_n+z\tau)\Big].
\end{aligned}
\tag{3.86}
$$

Applying the Hölder's inequality, we get

$$
\begin{aligned}
\left|\widehat{\bar{\xi}_l^{n+1}}\right| &\le \frac{\tau}{2\beta_l}\left(\int_0^1 \left[z(1-z)\right]^2 \mathrm{d}z\right)^{1/2}\left(\int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}P_l(t_n+z\tau)\right|^2 \mathrm{d}z\right)^{1/2}, \\
\left|\widehat{\dot{\bar{\xi}}_l^{n+1}}\right| &\le \frac{\tau}{2}\left(\int_0^1 \left[z(1-z)\right]^2 \mathrm{d}z\right)^{1/2}\left(\int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}Q_l(t_n+z\tau)\right|^2 \mathrm{d}z\right)^{1/2}, \qquad (3.87)\\
\left|\widehat{\xi_l^{n+1}}\right| &\le \frac{\tau}{8}\left(\int_0^1 (1-|1-2z|)^4 \mathrm{d}z\right)^{1/2}\left(\int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}R_l(t_n+z\tau)\right|^2 \mathrm{d}z\right)^{1/2}.
\end{aligned}
$$

We define the local truncation error functions as

$$
\begin{aligned}
&\bar{\xi}^{n+1}(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\bar{\xi}_l^{n+1}} \mathrm{e}^{i\mu_l(x-a)}, \quad \dot{\bar{\xi}}^{n+1}(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\dot{\bar{\xi}}_l^{n+1}} \mathrm{e}^{i\mu_l(x-a)}, \\
&\xi^{n+1}(x) = \sum_{l=-M/2}^{M/2-1} \widehat{\xi_l^{n+1}} \mathrm{e}^{i\mu_l(x-a)}.
\end{aligned} \qquad (3.88)
$$

From the definition of norm (2.11) and (2.12), we have the estimates on the local errors as

$$
\begin{aligned}
\left\|\bar{\xi}^{n+1}\right\|_m^2 =& \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^m \left|\widehat{\bar{\xi}_l^{n+1}}\right| \\
\lesssim& \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^m \Big((1+|\mu_l|^2)\tau^4 \int_0^1 \left|\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z \\
&\qquad + \tau^2 \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z + (1+|\mu_l|^2)^{-1} \int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z\Big) \\
\lesssim& \tau^2 \int_0^1 \Big(\tau^4 \left\|F(t_n+z\tau,x)\right\|_{m+1}^2 + \tau^2 \left\|\frac{\mathrm{d}}{\mathrm{d}z}F(t_n+z\tau,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}z^2}F(t_n+z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
=& \tau^5 \int_0^\tau \Big(\left\|F(t_n+w,x)\right\|_{m+1}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}F(t_n+w,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}w^2}F(t_n+w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w, \\
\left\|\dot{\bar{\xi}}^{n+1}\right\|_{m-1}^2 =& \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \left|\widehat{\dot{\bar{\xi}}_l^{n+1}}\right| \\
\lesssim& \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \Big((1+|\mu_l|^2)^2\tau^4 \int_0^1 \left|\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z \\
&\qquad + \tau^2 \int_0^1 (1+|\mu_l|^2) \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z + \int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{F}_l(t_n+z\tau)\right|^2 \mathrm{d}z\Big) \\
\lesssim& \tau^2 \int_0^1 \Big(\tau^4 \left\|F(t_n+z\tau,x)\right\|_{m+1}^2 + \tau^2 \left\|\frac{\mathrm{d}}{\mathrm{d}z}F(t_n+z\tau,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}z^2}F(t_n+z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
=& \tau^5 \int_0^\tau \Big(\left\|F(t_n+w,x)\right\|_{m+1}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}F(t_n+w,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}w^2}F(t_n+w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w, \\
\left\|\xi^{n+1}\right\|_{m-1}^2 =& \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \left|\widehat{\xi_l^{n+1}}\right| \\
\lesssim& \tau^2 \sum_{l=-M/2}^{M/2-1} (1+|\mu_l|^2)^{m-1} \Big((1+|\mu_l|^2)^2\tau^4 \int_0^1 \left|\widehat{G}_l(t_n+z\tau)\right|^2 \mathrm{d}z \\
&\qquad + (1+|\mu_l|^2)\tau^2 \int_0^1 \left|\frac{\mathrm{d}}{\mathrm{d}z}\widehat{G}_l(t_n+z\tau)\right|^2 \mathrm{d}z + \int_0^1 \left|\frac{\mathrm{d}^2}{\mathrm{d}z^2}\widehat{G}_l(t_n+z\tau)\right|^2 \mathrm{d}z\Big) \\
\lesssim& \tau^2 \int_0^1 \Big(\tau^4 \left\|G(t_n+z\tau,x)\right\|_{m+1}^2 + \tau^2 \left\|\frac{\mathrm{d}}{\mathrm{d}z}G(t_n+z\tau,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}z^2}G(t_n+z\tau,x)\right\|_{m-1}^2 \Big)\mathrm{d}z \\
=& \tau^5 \int_0^\tau \Big(\left\|G(t_n+w,x)\right\|_{m+1}^2 + \left\|\frac{\mathrm{d}}{\mathrm{d}w}G(t_n+w,x)\right\|_m^2 + \left\|\frac{\mathrm{d}^2}{\mathrm{d}w^2}G(w,x)\right\|_{m-1}^2 \Big)\mathrm{d}w.
\end{aligned}
\tag{3.89}
$$

By the assumption (A), we get form (3.89) that

$$\left\|\bar{\xi}^{n+1}\right\|_m^2 \lesssim \tau^6, \quad \left\|\bar{\dot{\xi}}^{n+1}\right\|_{m-1}^2 \lesssim \tau^6, \quad \left\|\xi^{n+1}\right\|_{m-1}^2 \lesssim \tau^6. \tag{3.90}$$

Step2 : Estimates on nonlinear errors $\bar{\eta}^{n+1}$, $\bar{\dot{\eta}}^{n+1}$ and η^{n+1}. Subtracting the coefficients (2.30) from (3.82), we get the error iteration

$$\begin{aligned}
\widehat{(\bar{e}_M^{n+1})}_l &= \cos(\beta_l\tau)\widehat{(\bar{e}_M^n)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\bar{\dot{e}}_M^n)}_l + \widehat{\bar{\eta}_l^{n+1}} + \widehat{\bar{\xi}_l^{n+1}},\\
\widehat{(\bar{\dot{e}}_M^{n+1})}_l &= -\beta_l\sin(\beta_l\tau)\widehat{(\bar{e}_M^n)}_l + \cos(\beta_l\tau)\widehat{(\bar{\dot{e}}_M^n)}_l + \widehat{\bar{\dot{\eta}}_l^{n+1}} + \widehat{\bar{\dot{\xi}}_l^{n+1}},\\
\widehat{(e_M^{n+1})}_l &= e^{-2i\tau M_l}\widehat{(e_M^{n-1})}_l + \widehat{\eta_l^{n+1}} + \widehat{\xi_l^{n+1}},
\end{aligned} \tag{3.91}$$

where $\widehat{\bar{\xi}_l^{n+1}}$, $\widehat{\bar{\dot{\xi}}_l^{n+1}}$ and $\widehat{\xi_l^{n+1}}$ are defined in (3.82) and

$$\begin{aligned}
\widehat{\bar{\eta}_l^{n+1}} =& \frac{\tau\sin(\beta_l\tau)}{2\beta_l}\Big(\widehat{F}_l(t_n) - \widetilde{F}_l^n\Big), \quad \widehat{\bar{\dot{\eta}}_l^{n+1}} = \frac{\tau}{2}\Big(\cos(\beta_l\tau)(\widehat{F}_l(t_n) - \widetilde{F}_l^n) + (\widehat{F}_l(t_{n+1}) - \widetilde{F}_l^{n+1})\Big),\\
\widehat{\eta_l^{n+1}} =& -2i\tau e^{-i\tau M_l}\Big(\widehat{G}_l(t_n) - \widetilde{G}_l^n\Big).
\end{aligned} \tag{3.92}$$

Using triangle inequality in (3.92) results in

$$\begin{aligned}
\left|\widehat{\bar{\eta}_l^{n+1}}\right| \le& \frac{\tau}{2\beta_l}\left|\widehat{F}_l(t_n) - \widetilde{F}_l^n\right|, \quad \left|\widehat{\bar{\dot{\eta}}_l^{n+1}}\right| \le \frac{\tau}{2}\Big(\left|\widehat{F}_l(t_n) - \widetilde{F}_l^n\right| + \left|\widehat{F}_l(t_{n+1}) - \widetilde{F}_l^{n+1}\right|\Big),\\
\left|\widehat{\eta_l^{n+1}}\right| \le& \tau\left|\widehat{G}_l(t_n) - \widetilde{G}_l^n\right|.
\end{aligned} \tag{3.93}$$

Here we define nonlinear error functions as

$$\begin{aligned}
\bar{\eta}^{n+1}(x) &= \sum_{l=-M/2}^{M/2-1}\widehat{\bar{\eta}_l^{n+1}}\mathrm{e}^{i\mu_l(x-a)}, \quad \bar{\dot{\eta}}^{n+1}(x) = \sum_{l=-M/2}^{M/2-1}\widehat{\bar{\dot{\eta}}_l^{n+1}}\mathrm{e}^{i\mu_l(x-a)},\\
\eta^{n+1}(x) &= \sum_{l=-M/2}^{M/2-1}\widehat{\eta_l^{n+1}}\mathrm{e}^{i\mu_l(x-a)}, \quad n = 1, 2, \cdots.
\end{aligned} \tag{3.94}$$

By the definition (2.11), (2.12) and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
&\left\|\bar{\eta}^{n+1}\right\|_m \le \frac{\tau}{2}\left\|P_M F(t_n, x) - (I_M F^n)(x)\right\|_{m-1},\\
&\left\|\bar{\dot{\eta}}^{n+1}\right\|_{m-1} \le \frac{\tau}{2}\Big(\left\|P_M F(t_n, x) - (I_M F^n)(x)\right\|_{m-1} + \left\|P_M F(t_{n+1}, x) - (I_M F^{n+1})(x)\right\|_{m-1}\Big),\\
&\left\|\eta^{n+1}\right\|_{m-1} \le \tau\left\|P_M G(t_n, x) - (I_M G^n)(x)\right\|_{m-1}.
\end{aligned} \tag{3.95}$$

Step3 : Error equations on $\bar{e}_M^n$, $\bar{\dot{e}}_M^n$ and e_M^n. It follows from (3.91) that

$$\begin{aligned}
\left|\widehat{(\bar{e}_M^{n+1})}_l\right|^2 &\lesssim (1+\tau)\left|\cos(\beta_l\tau)\widehat{(\bar{e}_M^n)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widehat{(\bar{\dot{e}}_M^n)}_l\right|^2 + (1+\frac{1}{\tau})\Big(\left|\widehat{\bar{\eta}_l^{n+1}}\right|^2 + \left|\widehat{\bar{\xi}_l^{n+1}}\right|^2\Big),\\
\left|\widehat{(\bar{\dot{e}}_M^{n+1})}_l\right| &\lesssim (1+\tau)\left|-\beta_l\sin(\beta_l\tau)\widehat{(\bar{e}_M^n)}_l + \cos(\beta_l\tau)\widehat{(\bar{\dot{e}}_M^n)}_l\right|^2 + (1+\frac{1}{\tau})\Big(\left|\widehat{\bar{\dot{\eta}}_l^{n+1}}\right|^2 + \left|\widehat{\bar{\dot{\xi}}_l^{n+1}}\right|^2\Big).
\end{aligned}$$

By direct calculation, we obtain

$$\beta_l^2 \left|\widehat{(\bar{e}_M^{n+1})}_l\right|^2 + \left|\widehat{(\bar{\dot{e}}_M^{n+1})}_l\right|^2 \lesssim (1+\tau)\left(\beta_l^2 \left|\widehat{(\bar{e}_M^{n})}_l\right|^2 + \left|\widehat{(\bar{\dot{e}}_M^{n})}_l\right|^2\right) + (1+\frac{1}{\tau})\left(\beta_l^2 \left|\widehat{\bar{\eta}_l^{n+1}}\right|^2 + \beta_l^2 \left|\widehat{\bar{\xi}_l^{n+1}}\right|^2 + \left|\widehat{\bar{\dot{\eta}}_l^{n+1}}\right|^2 + \left|\widehat{\bar{\dot{\xi}}_l^{n+1}}\right|^2\right). \tag{3.96}$$

From the definition (2.11), it is easy to get

$$\left\|\bar{e}_M^{n+1}\right\|_m^2 + \left\|\bar{\dot{e}}_M^{n+1}\right\|_{m-1}^2 \le (1+\tau)\left(\left\|\bar{e}_M^{n}\right\|_m^2 + \left\|\bar{\dot{e}}_M^{n}\right\|_{m-1}^2\right) + (1+\frac{1}{\tau})\left(\left\|\bar{\eta}_l^{n+1}\right\|_m + \left\|\bar{\xi}_l^{n+1}\right\|_m + \left\|\bar{\dot{\eta}}^{n+1}\right\|_{m-1} + \left\|\bar{\dot{\xi}}_l^{n+1}\right\|_{m-1}^2\right), \tag{3.97}$$

Multiplying both sides of the last expression in (3.91) from left by $\left(\widehat{(e_M^{n+1})_l} + \mathrm{e}^{-2i\tau M_1}\widehat{(e_M^{n-1})_l}\right)$, taking the real parts and using the Cauchy inequality, we obtain

$$\left|\widehat{(e_M^{n+1})}_l\right|^2 - \left|\widehat{(e_M^{n-1})}_l\right|^2 \le \tau\left(\left|\widehat{(e_M^{n+1})}_l\right|^2 + \left|\widehat{(e_M^{n-1})}_l\right|^2\right) + \frac{1}{\tau}\left(\left|\widehat{\eta_l^{n+1}}\right|^2 + \left|\widehat{\xi_l^{n+1}}\right|^2\right). \tag{3.98}$$

From the definition (2.12), we obtain

$$\left\|e_M^{n+1}\right\|_{m-1}^2 - \left\|e_M^{n-1}\right\|_{m-1}^2 \le \tau\Big(\left\|e_M^{n+1}\right\|_{m-1}^2 + \left\|e_M^{n-1}\right\|_{m-1}^2\Big) + \frac{1}{\tau}\Big(\left\|\eta_l^{n+1}\right\|_{m-1}^2 + \left\|\xi_l^{n+1}\right\|_{m-1}^2\Big), \tag{3.99}$$

We adapt the mathematical induction to prove Theorem 3.1. Now we assume that (3.47) is valid for all $0 \le n \le k$, then we need to show that it is still valid when $n = k+1$. Under the assumption for $1 \le n \le k$, using triangle inequality, Lemma 2.4 and Lemma 2.2, we obtain for $1 \le n \le k$

$$\begin{aligned}
&\|P_M F(t_n,x) - I_M F^n\|_{m-1} \\
&\le \left\|I_M F(\Phi(t_n,x)) - I_M F(I_M\Phi^n)\right\|_{m-1} + Ch^{m_0}\left\|F(\Phi(t_n,x))\right\|_{m+m_0-1}, \\
&\lesssim \left\|F(\Phi(t_n,x)) - F(I_M\Phi^n)\right\|_{m-1} + h^{m_0}, \\
&\lesssim \|\Phi(t_n,x) - I_M\Phi^n\|_{m-1}\left(\|\Phi(t_n,x)\|_{m-1} + \|I_M\Phi^n\|_{m-1}\right) + h^{m_0} \lesssim \left\|e_M^n\right\|_{m-1} + h^{m_0}.
\end{aligned} \tag{3.100}$$

In the similar way, we get for $1 \le n \le k$

$$\begin{aligned}
&\|P_M G(t_n,x) - I_M G^n\|_{m-1} \\
&\lesssim \left\|I_M G\big(u(t_n,x),\Phi(t_n,x)\big) - I_M G\big(I_M u^n, I_M\Phi^n\big)\right\|_{m-1} + Ch^{m_0}\left\|G\big(u(t_n,x),\Phi(t_n,x)\big)\right\|_{m+m_0-1} \\
&\lesssim \left\|G\big(u(t_n,x),\Phi(t_n,x)\big) - G\big(I_M u^n, I_M\Phi^n\big)\right\|_{m-1} + h^{m_0} \\
&\lesssim \Big(\|u(t_n,x) - I_M u^n\|_{m-1}\|I_M\Phi^n\|_{m-1} + \|\Phi(t_n,x) - I_M\Phi^n\|_{m-1}\|u(t_n,x)\|_{m-1}\Big) + h^{m_0} \\
&\lesssim \left(\left\|\bar{e}_M^n\right\|_{m-1} + \left\|e_M^n\right\|_{m-1}\right) + h^{m_0} \lesssim \left(\left\|\bar{e}_M^n\right\|_{m} + \left\|e_M^n\right\|_{m-1}\right) + h^{m_0}.
\end{aligned} \tag{3.101}$$

Plugging (3.100) and (3.101) into (3.95) gives for $1 \le n \le k$

$$\left\|\bar{\eta}^{n+1}\right\|_m \lesssim \tau\Big(\left\|e_M^n\right\|_{m-1} + h^{m_0}\Big), \quad \left\|\eta^{n+1}\right\|_{m-1} \lesssim \tau\Big(\left\|\bar{e}_M^n\right\|_m + \left\|e_M^n\right\|_{m-1} + h^{m_0}\Big). \tag{3.102}$$

From the third formula of (2.30), we get

$$\left|\widetilde{(\Phi^{n+1})_l}\right| \leq \left|\widetilde{(\Phi^{n-1})_l}\right| + 2\tau \left|\widetilde{G_l^n}\right|. \tag{3.103}$$

We have that for $1 \leq n \leq k$, there exist $\tau_1 > 0$ sufficiently small such that, when $0 < \tau \leq \tau_1$,

$$\begin{aligned}
&\left\|I_M\Phi^{n+1}\right\|_{m-1} \leq \left\|I_M\Phi^{n-1}\right\|_{m-1} + 2\tau \left\|I_M G^n\right\|_{m-1} \\
\leq& 1 + M_\Phi + 2\tau \left\|I_M G(I_M u^n, I_M\Phi^n)\right\|_{m-1} \lesssim 1 + M_\Phi + 2\tau \left\|G(I_M u^n, I_M\Phi^n)\right\|_{m-1} \\
\lesssim& 1 + M_\Phi + 2\tau \left\|I_M\Phi^n\right\|_{m-1} \left\|I_M u^n\right\|_{m-1} \leq 1 + M_\Phi + (1 + M_u)(1 + M_\Phi) \lesssim 1.
\end{aligned} \tag{3.104}$$

Similarly, from the assumption (A), Lemma 2.2, Lemma 2.4 and (3.104), we obtain

$$\begin{aligned}
&\left\|P_M F(t_{n+1}, x) - I_M F^{n+1}\right\|_{m-1} \\
\leq& \left\|I_M F(\Phi(t_{n+1}, x)) - I_M F(I_M\Phi^{n+1})\right\|_{m-1} + Ch^{m_0} \left\|F(\Phi(t_{n+1}, x))\right\|_{m+m_0-1} \\
\lesssim& \left\|F(\Phi(t_{n+1}, x)) - F(I_M\Phi^{n+1})\right\|_{m-1} + h^{m_0} \\
\lesssim& \left\|e_M^{n+1}\right\|_{m-1} \left(\left\|\Phi(t_{n+1}, x)\right\|_{m-1} + \left\|I_M\Phi^{n+1}\right\|_{m-1}\right) + h^{m_0} \lesssim \left\|e_M^{n+1}\right\|_{m-1} + h^{m_0}.
\end{aligned} \tag{3.105}$$

Plugging (3.105) and (3.100) into (3.95) leads to

$$\left\|\bar{\eta}^{n+1}\right\|_{m-1} \lesssim \tau\Big(\left\|e_M^n\right\|_{m-1} + \left\|e_M^{n+1}\right\|_{m-1} + h^{m_0}\Big), \quad 1 \leq n \leq k. \tag{3.106}$$

Plugging (3.102) and (3.106) into (3.97) and (3.99), it is easy to get

$$\begin{aligned}
\left\|\bar{e}_M^{n+1}\right\|_m^2 + \left\|\bar{\dot{e}}_M^{n+1}\right\|_{m-1}^2 \lesssim& \left\|\bar{e}_M^n\right\|_m^2 + \left\|\bar{\dot{e}}_M^n\right\|_{m-1}^2 \\
&+ \tau \left(\left\|\bar{e}_M^n\right\|_m^2 + \left\|\bar{\dot{e}}_M^n\right\|_{m-1}^2 + \left\|e_M^n\right\|_{m-1}^2 + \left\|e_M^{n+1}\right\|_{m-1}^2\right) \\
&+ (1 + \frac{1}{\tau})\Big(\left\|\bar{\xi}_l^{n+1}\right\|_m + \left\|\bar{\dot{\xi}}_l^{n+1}\right\|_{m-1}^2 \Big) + \tau h^{2m_0}, \\
\left\|e_M^{n+1}\right\|_{m-1}^2 - \left\|e_M^{n-1}\right\|_{m-1}^2 \lesssim& \tau\Big(\left\|e_M^{n+1}\right\|_{m-1}^2 + \left\|e_M^{n-1}\right\|_{m-1}^2 + \left\|\bar{e}_M^n\right\|_{m-1} + \left\|e_M^n\right\|_{m-1} \Big) \\
&+ \tau h^{2m_0} + \frac{1}{\tau} \left\|\xi_l^{n+1}\right\|_m^2.
\end{aligned} \tag{3.107}$$

When $\tau \leq 1$, considering (3.90) and adding the two formulas of (3.107) together gives

$$\begin{aligned}
&\left\|\bar{e}_M^{n+1}\right\|_m^2 + \left\|\bar{\dot{e}}_M^{n+1}\right\|_{m-1}^2 - \left\|\bar{e}_M^n\right\|_m^2 - \left\|\bar{\dot{e}}_M^n\right\|_{m-1}^2 + \left\|e_M^{n+1}\right\|_{m-1}^2 - \left\|e_M^{n-1}\right\|_{m-1}^2 \\
\lesssim& \tau \left(\left\|\bar{e}_M^n\right\|_m^2 + \left\|\bar{\dot{e}}_M^n\right\|_{m-1}^2 + \left\|e_M^{n+1}\right\|_{m-1}^2 + \left\|e_M^n\right\|_{m-1}^2 + \left\|e_M^{n-1}\right\|_{m-1}^2\right) + \tau^5 + \tau h^{2m_0}.
\end{aligned} \tag{3.108}$$

Define the energy

$$\mathcal{E}^n = \left\|\bar{e}_M^n\right\|_m^2 + \left\|\bar{\dot{e}}_M^n\right\|_{m-1}^2 + \left\|e_M^n\right\|_{m-1}^2 + \left\|e_M^{n-1}\right\|_{m-1}^2. \tag{3.109}$$

From (3.108) and the definition of energy (3.109), we obtain

$$\mathcal{E}^{n+1} - \mathcal{E}^n \lesssim \tau\Big(\mathcal{E}^{n+1} + \mathcal{E}^n\Big) + \tau^5 + \tau h^{2m_0}, \quad 1 \leq n \leq k, \tag{3.110}$$

Summing up (3.110) for $n = 1, \cdots, k$, using (3.95), we derive

$$\mathcal{E}^{k+1} - \mathcal{E}^1 \lesssim \tau \sum_{j=1}^{k+1} \mathcal{E}^j + k\tau^5 + k\tau h^{2m_0}. \tag{3.111}$$

Noticing (3.69), (3.74) and (3.79), it is easy to get that

$$\mathcal{E}^1 = \left\|\bar{e}_M^1\right\|_m^2 + \left\|\bar{\dot{e}}_M^1\right\|_{m-1}^2 + \left\|e_M^1\right\|_m^2 + \left\|e_M^0\right\|_m^2 \lesssim \tau^4 + h^{2m_0}. \tag{3.112}$$

Using the discrete Gronwall's inequality, there exist $\tau_2 > 0$ and $h_2 > 0$ sufficiently small such that, when $0 < \tau \leq \tau_2$ and $0 < h \leq h_2$, we get

$$\mathcal{E}^{k+1} \lesssim h^{2m_0} + \tau^4. \tag{3.113}$$

From the definition (3.109), we obtain

$$\left\|\bar{e}_M^{k+1}\right\|_m \lesssim h^{m_0} + \tau^2, \quad \left\|\bar{\dot{e}}_M^{k+1}\right\|_{m-1} \lesssim h^{m_0} + \tau^2, \quad \left\|e_M^{k+1}\right\|_{m-1} \lesssim h^{m_0} + \tau^2. \tag{3.114}$$

According to the triangle inequality and lemma 2.2, we have

$$\begin{aligned}
&\left\|u(t_{k+1}, x) - (I_M u^{k+1})(x)\right\|_m \lesssim \left\|\bar{e}_M^{k+1}\right\|_m + h^{m_0} \leq h^{m_0} + \tau^2, \\
&\left\|\partial_t u(t_{k+1}, x) - (I_M \dot{u}^{k+1})(x)\right\|_m \lesssim \left\|\bar{\dot{e}}_M^{k+1}\right\|_m + h^{m_0} \leq h^{m_0} + \tau^2, \\
&\left\|\Phi(t_{k+1}, x) - (I_M \Phi^{k+1})(x)\right\|_m \lesssim \left\|e_M^{k+1}\right\|_m + h^{m_0} \leq h^{m_0} + \tau^2.
\end{aligned} \tag{3.115}$$

which immediately implies

$$\begin{aligned}
&\left\|(I_M u^{k+1})(x)\right\|_m \leq \|u(t_{k+1}, x)\|_m + \left\|\bar{e}_M^{k+1}(x)\right\|_m \leq M_u + C(h^{m_0} + \tau^2). \\
&\left\|(I_M \dot{u}^{k+1})(x)\right\|_{m-1} \leq \|\partial_t u(t_{k+1}, x)\|_{m-1} + \left\|\bar{\dot{e}}_M^{k+1}(x)\right\|_{m-1} \leq M_u' + C(h^{m_0} + \tau^2). \\
&\left\|(I_M \Phi^{k+1})(x)\right\|_{m-1} \leq \|\Phi(t_{k+1}, x)\|_{m-1} + \left\|e_M^{k+1}(x)\right\|_{m-1} \leq M_\Phi + C(h^{m_0} + \tau^2).
\end{aligned} \tag{3.116}$$

Thus there exist $h_3 > 0$ and $\tau_3 > 0$ sufficiently small such that when $0 < h \leq h_3$ and $0 < \tau \leq \tau_3$, we have

$$\left\|(I_M u^{k+1})(x)\right\|_m \leq 1 + M_u, \quad \left\|(I_M \dot{u}^{k+1})(x)\right\|_{m-1} \leq 1 + M_u', \quad \left\|(I_M \Phi^{k+1})(x)\right\|_{m-1} \leq 1 + M_\Phi, \quad . \tag{3.117}$$

Therefore, the proof of (3.32) is completed under the choice of $h_0 = \min\{h_2, h_3\}$ and $\tau_0 = \min\{1, \tau_1, \tau_2, \tau_3\}$. □

Remark 3.1. *An alternative EIFP scheme updates $\widetilde{(u^{n+1})}_l$, $\widetilde{(\dot{u}^{n+1})}_l$ and $\widetilde{(\Phi^{n+1})}_l$ can also be given as follows*

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = \cos(\beta_l\tau)\widetilde{(u^n)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\dot{u}^n)}_l + \frac{1-\cos(\beta_l\tau)}{\beta_l^2}\widetilde{F_l^n}, \\
&\widetilde{(\dot{u}^{n+1})}_l = -\beta_l \sin(\beta_l\tau)\widetilde{(u^n)}_l + \cos(\beta_l\tau)\widetilde{(\dot{u}^n)}_l - \frac{\sin(\beta_l\tau)}{2\beta_l}\Big(\widetilde{F_l^n} + \widetilde{F_l^{n+1}}\Big), \\
&\widetilde{(\Phi^{n+1})}_l = e^{-2i\tau M_l}\widetilde{(\Phi^{n-1})}_l + \Big(e^{-2i\tau M_l} - 1\Big)M_l^{-1}\widetilde{G_l^n}, \\
&\widetilde{(\Phi^1)}_l = e^{-i\tau M_l}\widetilde{(\Phi^0)}_l + \Big(e^{-i\tau M_l} - 1\Big)M_l^{-1}\widetilde{G_l^0}.
\end{aligned} \tag{3.118}$$

However this scheme is not time symmetric. Clearly, following the analogous lines as the proof of Theorem 3.1, we would get the same form of error bounds for the this scheme. Due to the space limitation of the paper, we will not discuss it here.

Remark 3.2. *The analysis results in this paper only apply to the periodic boundary value problem of the KGD equation* (1.1). *For Dirichlet initial-boundary value problem of the NLD equation, we can give symmetric exponential integrator sin pseudo-spectral method and the conclusion still holds with slightly modified regular conditions.*

4. Numerical results

In this section, we report numerical results to support our theoretical analysis on the new scheme. The computational interval $[a, b]$ is chosen large enough such that the periodic boundary conditions do not introduce a significant aliasing error relative to the problem in the whole space. Here we take $a = -16$ and $b = 16$. Let $u(t, x)$, $\partial_t u(t, x)$ and $\Phi(t, x)$ be the '*exact*' solution which is obtained numerically by using the scheme (2.29)-(2.31) with very fine mesh size and small time step, e.g. $h = 1/2^8$ and $\tau = 10^{-4}$. In order to quantify the convergence, we define the error functions, H^m-error as

$$\|e_u(t_n)\|_m = \|u(t_n, x) - I_M u^n\|_m\,, \quad \|\dot{e}_u(t_n)\|_{m-1} = \|\partial_t u(t_n, x) - I_M \dot{u}^n\|_{m-1}\,,$$
$$\|e_\Phi(t_n)\|_{m-1} = \|\Phi(t_n, x) - I_M \Phi^n\|_{m-1}\,, \quad m = 2, 3.$$

For the initial condition, here we take

$$u_0(x) = e^{-x^2/2}, \quad \gamma(x) = \frac{3}{2} e^{-(x-1)^2/2}, \quad \Phi_0(x) = \left(e^{-x^2/2}, e^{-(x-1)^2/2}\right)^T.$$

We test and study the temporal and spatial error separately. Table 1 shows the temporal errors of the numerical scheme (2.29)-(2.31) at $T = 1$ under different τ with a small mesh size $h = 1/2^6$ such that the discretization error in space is negligible. Tables 2 shows the spatial error of the scheme (2.29)-(2.31) at $T = 1$ under different h with a very small time step $\tau = 10^{-4}$ such that the discretization error in time is negligible.

Table 1: Temporal errors of the new scheme (2.29)-(2.31) with convergence order: $\|e_u(T)\|_m$, $\|\dot{e}_u(T)\|_{m-1}$ and $\|e_\Phi(T)\|_{m-1}$ with $m = 2, 3$ at $T = 1$ for different τ with $h = 1/2^6$.

	$\tau_0 = 0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
$\|e_u(T)\|_2$	3.29E − 3	8.07E − 4	2.00E − 4	5.02E − 5	1.25E − 5	3.12E − 6
order	−	2.02	2.00	2.00	2.00	2.00
$\|\dot{e}_u(T)\|_1$	4.32E − 3	1.04E − 3	2.57E − 4	6.41E − 5	1.60E − 5	4.00E − 6
order	−	2.06	2.01	2.00	2.00	2.00
$\|e_\Phi(T)\|_1$	1.93E − 2	4.74E − 3	1.18E − 3	2.95E − 4	7.36E − 5	1.84E − 5
order	−	2.03	2.00	2.01	2.00	2.00

	$\tau_0 = 0.01$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
$\|e_u(T)\|_3$	6.42E − 3	1.56E − 3	3.88E − 4	9.69E − 5	2.42E − 5	6.05E − 6
order	−	2.04	2.00	2.00	2.00	2.00
$\|\dot{e}_u(T)\|_2$	1.12E − 2	2.64E − 3	6.49E − 4	1.62E − 4	4.04E − 5	1.01E − 5
order	−	2.09	2.02	2.00	2.00	2.00
$\|e_\Phi(T)\|_2$	4.04E − 2	9.72E − 3	2.41E − 3	6.01E − 4	1.50E − 4	3.75E − 5
order	−	2.05	2.01	2.01	2.00	2.00

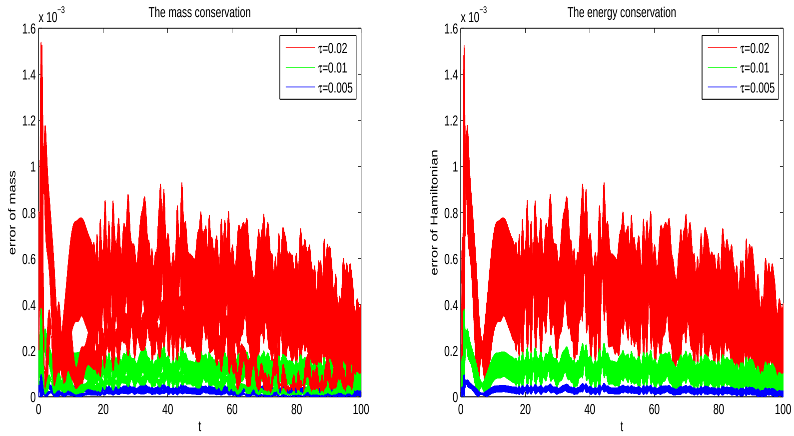

Figure 1: The conservation of total mass and energy

Table 2: Spatial errors of the new scheme (2.29)-(2.31) with convergence order: $\|e_u(T)\|_m$, $\|\dot{e}_u(T)\|_{m-1}$ and $\|e_\Phi(T)\|_{m-1}$ with $m = 2, 3$ at $T = 1$ for different h with $\tau = 10^{-4}$.

	$h_0 = 4$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e_u(T)\|_2$	8.44E − 1	3.57E − 1	2.33E − 2	7.05E − 5	4.26E − 10
$\|\dot{e}_u(T)\|_1$	1.89E + 0	2.36E − 1	1.64E − 2	7.60E − 5	2.45E − 10
$\|e_\Phi(T)\|_1$	7.92E − 1	5.65E − 1	1.17E − 1	8.03E − 4	5.55E − 9

	$h_0 = 4$	$h_0/2$	$h_0/2^2$	$h_0/2^3$	$h_0/2^4$
$\|e_u(T)\|_3$	9.15E − 1	5.45E − 1	6.98E − 2	4.04E − 4	5.05E − 9
$\|\dot{e}_u(T)\|_2$	1.28E + 0	3.20E − 1	4.39E − 2	4.36E − 4	2.88E − 9
$\|e_\Phi(T)\|_2$	9.11E − 1	8.98E − 1	3.62E − 1	4.93E − 3	6.87E − 8

The values of total mass and energy in the discrete level by the new scheme (2.29)-(2.31) with $h = 0.1$ and $\tau = 0.02, 0.01, 0.005$ are listed in Figure 1.

Next we compare numerically our scheme SEIFP with the exponential integrator Fourier pseudo-spectral scheme of [53] which we denote as EIFP. Here we take the error functions

$$\|e(t_n)\|_m = \|e_u(t_n)\|_m + \|\dot{e}_u(t_n)\|_{m-1} + \|e_\Phi(t_n)\|_{m-1} .$$

It is found in Table 3 that for the same τ and h, The accuracy of the two schemes SEIFP and EIFP is comparable. In order to achieve the same accuracy, the scheme SEIFP requires less CUP time. Here we only show this result by selecting H^m norm with $m = 2$. The other situations with $m > 2$ are similar.

Table 3: Comparison of the efficiency of the new method: $\|e(\cdot, T)\|_1$ at $T = 5$ for different τ with $h = 2^{-6}$.

$\|e(\cdot,T)\|_1$		$\tau_0=0.1$	$\tau_0/2$	$\tau_0/2^2$	$\tau_0/2^3$	$\tau_0/2^4$	$\tau_0/2^5$
SEIFP	Error	1.02E − 3	2.44E − 4	6.05E − 5	1.51E − 5	3.78E − 6	9.44E − 7
	CPU time	3.00E − 2	5.30E − 2	1.11E − 1	2.17E − 1	4.29E − 1	8.69E − 1
EIFP	Error	1.11E − 3	2.56E − 4	6.22E − 5	1.54E − 5	3.99E − 6	9.76E − 7
	CPU time	4.03E − 2	9.55E − 2	2.04E − 1	3.09E − 1	6.33E − 1	1.66E + 0

Based on results from Tables 1, Table2 and Figure 1, we can draw the following observations: (i) The new scheme (2.29)-(2.31) have the second-order temporal accuracy and spectral-order accuracy in spatial discretization, respectively. This supports Theorem 3.1. (ii) The new scheme is very stable and allows use of large time steps and mesh size, do not suffers from a CFL-type stability condition. (iii) The scheme SEIFP conserves the discrete masses and energy very well, although it is not a conservative scheme.

Remark 4.1. *Why the symmetric scheme can conserve the discrete mass and energy very well over long time is an interesting topic. Existing literatures have show that in the framework of ordinary differential equations, symmetric methods can conserve the discrete energy (Hamiltonian). Similar results have been extended to numerical schemes for partial differential equations. However, regarding the long-term stability of the discrete mass, we have not found a rigorous theoretical analysis. Exploring the principle for this phenomenon is interesting and will be our future work.*

Remark 4.2. *Here we explain in detail how to calculate the error between the solutions obtained with different mesh sizes for example $h = 1/2^6$ and $h = 1/2^8$, respectively. We only show the Dirac part. We denote the two solution vectors as $\Phi^n = (\Phi_1^n, \Phi_2^n)$ and $\Psi^n = (\Psi_1^n, \Psi_2^n)$, respectively. Obviously Φ_1^n and Φ_2^n are $(b-a)/2^6$-dimensional while Ψ_1^n and Ψ_2^n are $(b-a)/2^8$-dimensional. We get the error $e^n = (e_1^n, e_2^n)$, where $e_1^n = \Phi_1^n - \bar{\Psi}_1^n$ and $e_2^n = \Phi_2^n - \bar{\Psi}_2^n$, here $\bar{\Psi}_1^n(i) = \Psi_1^n(4i-3)$ and $\bar{\Psi}_2^n(i) = \Psi_2^n(4i-3)$. Then we obtain $\|e_\Phi(t_n)\|_{m-1} = \|I_M e^n\|_{m-1}$. The way of calculating $\|e_u(t_n)\|_m$ and $\|\dot{e}_u(t_n)\|_{m-1}$ is completely similar.*

5. Conclusions and discussions

In this paper, we constructed a symmetric exponential integrator Fourier pseudo-spectral scheme for the KGD equation with periodic boundary conditions and analyze its convergence properties. The scheme was proved to converge unconditionally at the second-order in time and spectrally in space, respectively, in a generic norm by using the energy method, mathematical induction. The convergence rate of the new scheme, without any restrictions on the grid ratio, at the order of $O(h^{m_0} + \tau^2)$ in H^m-norm. Numerical experiments are conducted to confirm our error estimates of the proposed scheme.

We can extend the scheme (2.29)-(2.31) as follows

$$\begin{aligned}
&\widetilde{(u^{n+1})}_l = \cos(\beta_l\tau)\widetilde{(u^n)}_l + \frac{\sin(\beta_l\tau)}{\beta_l}\widetilde{(\dot{u}^n)}_l + \frac{\tau^2}{2}\bar{\chi}(\beta_l\tau)\widetilde{F_l^n},\\
&\widetilde{(\dot{u}^{n+1})}_l = -\beta_l\sin(\beta_l\tau)\widetilde{(u^n)}_l + \cos(\beta_l\tau)\widetilde{(\dot{u}^n)}_l - \frac{\tau}{2}\Big(\bar{\chi}_0(\beta_l\tau)\widetilde{F_l^n} + \bar{\chi}_1(\beta_l\tau)\widetilde{F_l^{n+1}}\Big),\\
&\widetilde{(\Phi^{n+1})}_l = e^{-2i\tau M_l}\widetilde{(\Phi^{n-1})}_l - 2i\tau\chi(-i\tau M_l)\widetilde{G_l^n},\\
&\widetilde{(\Phi^1)}_l = e^{-i\tau M_l}\widetilde{(\Phi^0)}_l - i\tau\chi_0(\beta_l\tau)\widetilde{G_l^0},
\end{aligned} \tag{5.119}$$

where $\bar{\chi}(\beta_l\tau)$, $\bar{\chi}_0(\beta_l\tau)$, $\bar{\chi}(\beta_l\tau)_1$ $\chi(\beta_l\tau)$ and $\chi_0(\beta_l\tau)$ are filter functions. Clearly, the scheme SEIFP belongs to the class of schemes (5.119). Through the flexible selection of the filter function, we can get more new scheme. Convergence analysis will be more complicated, this will be our future work. Higher order energy-preserving fourth-order exponential integrator Fourier spectral/pseudo-spectral method is also an interesting topic [33, 34].

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Dear Editor,

Thank you very much for giving us the opportunity to revise our manuscript. We are very grateful to the reviewers for their careful reading the manuscript and for their valuable comments. We have carefully taken their comments into consideration in preparing our revision. Below is our response to their comments.

For reviewer 1:

- *"1-) What are the applications of Klein-Gordon-Dirac equation in science, physics or engineering? This, should be more elaborated in the paper. "*

  Response: We think that your suggestion is reasonable. A brief description of the application in quantum electrodynamics and/or particle physics has been given in the introduction of the new version.

- *"2-) In abstract, "a exponential" should be "an exponential"*
  3-) In the first line after Section 3, correct "mew scheme"
  4-) In the proof of Lemma 3.1, after Eq.(3.56) remove one of "and"s
  5-) After Eq. (3.70), "Cauchy-Schwartz" must be "Cauchy-Schwarz"
  6-) After Eq. (3.114), correct "Gronwalls" as "Gronwall's""

  Response: Thank you for your reading the manuscript carefully. The corresponding changes have been made in the new version. In addition, we have gone through the calculations and derivation once again very carefully until we can't find any errors. It is worth noting that in the new version, the locations of the modified places have slightly changed.

- *"7-) In numerical results section, the authors take exact solution as numerical solution with fine mesh size and small time step. How do the authors compare the numerical solutions? For example, how do they compare the solutions obtained with $h = 1/2^6$ and $h = 1/2^8$ mesh sizes, respectively? Since their dimensions are mismatched (I mean solution vectors are not in same size). Did the authors interpolate a function to the fine solution and then compare this function with coarse mesh solution? This issue must be explained in the paper in details. "*

  Response: Thank you for your valuable suggestions. We think that your suggestions are very valuable and helpful to us. Here we explain in detail how to compare the solutions obtained with $h = 1/2^6$ and $h = 1/2^8$ mesh sizes, respectively. We only show the Dirac part. We denote the two solution vectors as $\Phi^n = (\Phi_1^n, \Phi_2^n)$ and $\Psi^n = (\Psi_1^n, \Psi_2^n)$, respectively. Obviously Φ_1^n and Φ_2^n are $(b-a)/2^6$-dimensional while Ψ_1^n and Ψ_2^n are $(b-a)/2^8$-dimensional. We get the error $e^n = (e_1^n, e_2^n)$, where $e_1^n = \Phi_1^n - \bar{\Psi}_1^n$ and $e_2^n = \Phi_2^n - \bar{\Psi}_2^n$, here $\bar{\Psi}_1^n(i) = \Psi_1^n(4i-3)$ and

$\bar{\Psi}_2^n(i) = \Psi_2^n(4i-3)$. Then we obtain $\|e_\Phi(t_n)\|_{m-1} = \|I_M e^n\|_{m-1}$. The way of calculating $\|e_u(t_n)\|_m$ and $\|\dot{e}_u(t_n)\|_{m-1}$ is completely similar. The corresponding explanation has been given in the new version. (see Remark 4.2)
We don't know if this problem has been explained clearly, if you have any questions, please feel free to ask us.

- *"8-) The authors claimed that their method is computationally is cheaper than the ref. [Journal of Scientific Computing (2019) 79:1907-1935]. Is it possible to compare the current method and the method in the mentioned ref. in the sense of CPU times and give the results in numerical section? "*

  Response: We think that this is a very good research idea. We have compare the current method and the method in the mentioned ref. in the sense of CPU times and give the results in numerical section of the new version.

- *"9-) Regarding the subject of this manuscript, its content should be updated by the following studies.*
  Application of a collocation method based on linear barycentric interpolation for solving 2D and 3D Klein-Gordon-Schrödinger (KGS) equations numerically, Engineering Computations, Vol. ahead-of-print No. ahead of- print. https://doi.org/10.1108/EC-06-2020-0312
  Two numerical meshless techniques based on radial basis functions (RBFs) and the method of generalized moving least squares (GMLS) for simulation of coupled Klein-Kordon-Schrödinger (KGS) equations, Computers and Mathematics with Applications Volume 71, Issue 4, February 2016, Pages 892-921.
  A Haar wavelet collocation method for coupled nonlinear Schrödinger-KdV equations, International Journal of Modern Physics C, Volume 27 Issue 09 Pages 1650103 "

  Response: Your suggestion is reasonable. The corresponding references have been cited in the new version.

For reviewer 2:

- *"As stated in Lemma 2.1, the SEIFP scheme is stable under the stability conditions $0 \lesssim 1$ and $h > 0$. The author is suggested to give a more accurate upper bound for τ. Otherwise, this kind of stability condition cannot provide any guidance for numerical experiments."*

  Response: Thank you for your reading the manuscript carefully. We think that your suggestion is very valuable and helpful to us. However, in the process of revising the manuscript, we reconsidered the concept of stability. After serious thinking, we decided to study linear stability as done in Remark 2.2 of [IMA Journal of Numerical Analysis 40(2):1266-1293]. We obtain that the SEIFP scheme for the KGD equation with $g = 0$ is unconditionally stable for any $\tau, h > 0$. (see Lemma 2.1 of the new version)

- *"In Theorem 2.1 the scheme is proved to be symmetric. However, the approach used is different from the standard strategy to prove the symmetry of a multistep method introduced in Chapter III.9 of the classic textbook*
  E. Hairer, S.P. Nørsett and G. Wanner, Solving Ordinary Differential Equations I, Springer, 2008.
  The author should list some references about this different approach."

  Response: Thanks for your valuable suggestions. As you said, our method of proving the time symmetry of the scheme is really not clear enough. In the new version, we converted the scheme SEIFP into an equivalent two-step scheme and then obtain the time symmetry of the scheme based on the standard strategy to prove the symmetry of a multi-step method introduced in Chapter III.9 of the above-mentioned classic textbook. (see Remark 2.1 of the new version)

- *"In the numerical part, the errors in the discrete mass and energy are displayed to be preserved to a magnitude less than* 10^{-3} *over long time with different time steps. Some theoretical discussions are suggested on the conservative properties of the new scheme. Is there any possible to construct an explicit but conservative scheme for this equation?"*

  Response: We think that your suggestion is reasonable. Some theoretical discussions on the conservative properties of the new scheme have been given in the new version. (see Remark 4.1 of the new version)

  As far as we know, there is no explicit energy-conserving scheme. In other words, the existing energy-conserving schemes are almost all implicit. Anyway, we think that your idea is very meaningful because the study of explicit energy-conserving scheme is indeed an interesting work.

- *"There are many typos in the manuscript. The author should pay much attention on the paper work. Some of them are listed below.*
  In Abstract, "a exponential integrator" → "an exponential integrator"; "propose the new" → "proposed a new";
  Line 29, Page 3, "In thai paper" → "In this paper"; Line 30, "full explicit" → "fully explicit";
  Line 35, Page 4, "also conserves" → "also conserve"; Line 39, "section 3" → "Section 3";
  The caption of Figure 1, "conversation" → "conservation"."

  Response: Thank you for your reading the manuscript carefully. We think that your suggestions are very valuable and helpful to us.

Based on your valuable suggestions, the corresponding changes have been made in the new version. In addition, we have gone through the calculations and derivation once again very carefully until we can't find any errors.

Again, we greatly appreciate the reviewers for all the helpful suggestions.

Sincerely,

Jiyong Li