Attmpt of constructing convex splitting

consider the dissipative system:

$$\dot{y} = D\nabla H(y)$$

Where, D is a negative operator, H is the Lyapunov functional.

The classical convex splitting method for the above system reads

$$\frac{y^{n+1} - y^n}{\Delta t} = D\nabla H_+(y^{n+1}) - D\nabla H_-(y^n)$$

Being different from the traditional fully-implicit method, the above convex splitting method can be viewed as

$$\min_{y\in\mathbb{R}^d} \quad \frac{1}{2\Delta t} y^\top y - D\nabla H_+(y) - \frac{1}{\Delta t} y^\top y^n + D\nabla H_-(y^n)^\top y$$

usually, the convex part and the concave part are treated separately, and we consider the additive Runge-Kutta method to the above system

$$\begin{aligned} y_i &= y^0 + h\sum_{j=0}^{m} a_{ij} k_j - h\sum_{j=0}^{i-1} \widehat{a}_{ij} \hat{k}_j \\ k_i &= D\nabla H_+(y_i) \\ \hat{k}_i &= D\nabla H_-(y_i) \\ y^1 &= y^0 + h\sum_{i=0}^{s} b_i k_i + h\sum_{i=0}^{s} \hat{b}_i \hat{k}_i, \end{aligned}$$

where $m \leq s$ and we emphasize that the concave part should be treated explicitly, which ensures that， balabala

in order to construct energy stable numerical schemes, the coefficients $\{a_{ij}\}$ and $\{b_i\}$ should be selected technique such that the dissipative law of the original system being inherited under discrete level, to be clear,

$$H(y^1) - H(y^0) \leq 0$$

Second order ADRK

We first consider the Runge-Kutta method with the buthcher tabular

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \widehat{\mathbf{A}} = \begin{pmatrix} 0 & 0 \\ \hat{c} & 0 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} e & f \end{pmatrix}, \quad \widehat{\mathbf{b}} = \begin{pmatrix} \hat{e} & \widehat{f} \end{pmatrix}$$

the second order conditions of such ADRK reads

$$\begin{aligned} &e + f = 1 \quad \hat{e} + \widehat{f} = 1 \\ &e(a+b) + f(c+d) = 1/2 \quad \hat{e}(a+b) + \widehat{f}(c+d) = 1/2 \\ &\hat{c} f = 1/2 \quad \hat{c}\widehat{f} = 1/2 \end{aligned}$$

$$f = \widehat{f}, \quad e = \hat{e} = 1 - f, \quad \hat{c} = \frac{1}{2f}$$

here, we additionally assume that $c = \hat{c}$, and $b = 0$ then, we can get the following Runge-Kutta method

$$A = \begin{pmatrix} a & 0 \\ \frac{1}{2f} & (1-\frac{1}{f})a \end{pmatrix}, \quad \widehat{A} = \begin{pmatrix} 0 & 0 \\ \frac{1}{2f} & 0 \end{pmatrix}, \quad \mathbf{b} = \widehat{\mathbf{b}} = \begin{pmatrix} 1-f & f \end{pmatrix}$$

for convenience, we assume that $D = -1$, then, the above 2-stage ADRK method reads

$$\begin{cases} y_1 = y^0 - hak_1 \\ y_2 = y^0 - h\frac{1}{2f}k_1 - ha(1-\frac{1}{f})k_2 + h\frac{1}{2f}\hat{k}_1 \\ y^1 = y^0 + h(1-f)k_1 + hfk_2 + h(1-f)\hat{k}_1 + hf\hat{k}_2 \\ k_i = \nabla H_+(y_i) \\ \hat{k}_i = \nabla H_-(y_i) \end{cases}$$

Energy Stable CSRK

Now, we denote $\mathbf{v}$ to represent

$$\mathbf{v} = \begin{pmatrix} \nabla H_-(y^0) \\ \nabla H_+(y_1) \\ \nabla H_-(y_1) \\ \nabla H_+(y_2) \\ \nabla H_-(y_2) \\ \nabla H_+(y^1) \end{pmatrix}$$

Then, we have

$$H(y^1) - H(y^0) = H(y^1) - H(y_2) + H(y_2) - H(y_1) + H(y_1) - H(y^0)$$

calculating directly to see

$$H(y_1) - H(y^0) \leq (\nabla H_+(y_1) - \nabla H_-(y^0), y_1 - y^0)$$

one can see that

$$\begin{gathered} \nabla H_+(y_1) - \nabla H_-(y^0) = [(\mathbf{e_1^+} - \mathbf{e}^-) \otimes I_d]\mathbf{v} \\ y_1 - y^0 = -hak_1 = -ha[\mathbf{e}_1^+ \otimes I_d]\mathbf{v} \end{gathered}$$

similarly, we have

$$\begin{aligned} &H(y_2) - H(y_1) \leq (\nabla H_+(y_2) - \nabla H_-(y_1), y_2 - y_1) \\ &\nabla H_+(y_2) - \nabla H_-(y_1) = [(\mathbf{e}_2^+ - \mathbf{e}_1^-) \otimes I_d]\mathbf{v} \\ &y_2 - y_1 = h\left[\left((a - \frac{1}{2f})\mathbf{e}_1^+ - a(1 - \frac{1}{f})\mathbf{e}_2^+ + \frac{1}{2f}\mathbf{e}_2^-\right) \otimes I_d\right]\mathbf{v} \end{aligned}$$

and

$$\begin{aligned} &H(y^1) - H(y_2) \leq (\nabla H_+(y^1) - \nabla H_-(y_2), y^1 - y_2) \\ &\nabla H_+(y^1) - \nabla H_-(y_2) = [(\mathbf{e}^+ - \mathbf{e}_2^-) \otimes I_d]\mathbf{v} \\ &y^1 - y_2 = h\left[\left((a - \frac{1}{2f})\mathbf{e}_1^+ - a(1 - \frac{1}{f})\mathbf{e}_2^+ + \frac{1}{2f}\mathbf{e}_2^-\right) \otimes I_d\right]\mathbf{v} \end{aligned}$$

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