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FINITE DIFFERENCE METHOD FOR GENERALIZED ZAKHAROV EQUATIONS

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ABSTRACT. A conservative difference scheme is presented for the initial-boundary value problem for generalized Zakharov equations. The scheme can be implicit or semiexplicit depending on the choice of a parameter. On the basis of a priori estimates and an inequality about norms, convergence of the difference solution is proved in order $O(h^2+\tau^2)$, which is better than previous results.

INTRODUCTION

The Zakharov equations [20]

$$iE_t + E_{xx} - NE = 0, \tag{1.1}$$

$$\frac{1}{\lambda^2}N_{tt} - (N + |E|^2)_{xx} = 0 \tag{1.2}$$

describe the propagation of Langmuir waves in plasmas. Here the complex unknown function E is the slowly varying envelope of the highly oscillatory electric field, and the unknown real function N denotes the fluctuation of the ion density about its equilibrium value.

The global existence of a weak solution for the Zakharov equations in one dimension is proved in [19], and existence and uniqueness of a smooth solution for the equations are obtained provided smooth initial data are prescribed.

Numerical methods for the Zakharov equations are studied only in [5, 9, 10, and 15]. A spectral method is used to compute solitary waves and the collision of two solitary waves in [15]. In [9, 10], Glassey considered an energy-preserving implicit difference scheme for the equations and proved its convergence in order $O(h+\tau)$. In [5], we propose a new conservative difference scheme which involves a parameter θ, $0 \le \theta \le \frac{1}{2}$; when $\theta = \frac{1}{2}$, the new scheme is identical to Glassey's scheme. For $\theta = 0$, the new scheme is semiexplicit, explicit in N, but implicit in E. Numerical experiments demonstrate that the new scheme with $\theta = 0$ is more accurate and efficient compared to $\theta = \frac{1}{2}$. Convergence of these schemes is proved in order $O(h+\tau)$ in [5, 9, and 10], while the order of the truncation errors is $O(h^2+\tau^2)$.

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For suitable initial data, the solution of the initial value problem for (1.1)–(1.2) converges as $\lambda \to \infty$ to a solution of the cubic nonlinear Schrödinger equation

$$iE_t + E_{xx} + |E|^2 E = 0 \tag{1.3}$$

(see [1, 17]).

The generalized nonlinear Schrödinger equation

$$iE_t + E_{xx} + f(|E|^2) \cdot E = 0 \tag{1.4}$$

has been considered in physics (see, for example, [2, 3, and 11]). Here, $f(s) = s^p (p > 0)$, $f(s) = 1 - e^{-s}$, $f(s) = \frac{s}{1+s}$ or $f(s) = \ln(1+s)$. Existence and uniqueness of the generalized solution for the equation (1.4) have been obtained and numerical methods for (1.4) have been studied (see [4, 6, and 18]).

The generalized Zakharov equations are also considered in [21]. In the present paper, we consider the following initial-boundary value problem of generalized Zakharov equations in one dimension:

$$iE_t + E_{xx} = Nf(|E|^2)E, \qquad x_L < x < x_R, \; T \geq t > 0, \tag{1.5}$$

$$N_{tt} - N_{xx} = \frac{\partial^2}{\partial x^2}(F(|E|^2)), \tag{1.6}$$

where

$$f \in C^\infty(R^+), \quad F(s) = \int_0^s f(\tau)\, d\tau; \tag{1.7}$$

$$E(x, 0) = E^0(x), \quad N(x, 0) = N^0(x), \quad N_t(x, 0) = N^1(x), \tag{1.8}$$

$$E|_{x=x_L} = E|_{x=x_R} = 0, \quad N|_{x=x_L} = N|_{x=x_R} = 0, \quad u|_{x=x_L} = u|_{x=x_R} = 0, \tag{1.9}$$

and the potential function u is given by

$$u_{xx} = N_t. \tag{1.10}$$

Moreover, we supplement (1.5)–(1.10) by imposing the compatibility condition

$$\int_{x_L}^{x_R} N^1(x)\, dx = 0. \tag{1.11}$$

We propose a conservative difference scheme with parameter θ of the generalized Zakharov equations. The difference scheme conserves two conservation laws that the differential equations possess. For $\theta = 0$, the scheme is semiexplicit. We will prove the convergence of the difference solution for all $\theta \in [0, \frac{1}{2}]$ in order $O(h^2 + \tau^2)$, which is consistent with the order of the truncation error of the difference scheme. This improves the results of order $O(h + \tau)$ which were given in [5, 9].

In §2, we describe the difference scheme and its basic properties. Some a priori estimates and the proof of convergence of the difference scheme are given in §3. Long and highly technical proofs of two lemmas in §3 are placed in the Supplement section at the end of this issue.

2. Finite difference scheme

We consider a finite difference method for the problem (1.5)–(1.11). As usual, the following notations are used:

$$x_j = x_L + jh, \quad t^n = n \cdot \tau, \quad 0 \le j \le J = \left[\frac{x_R - x_L}{h}\right], \quad n = 0, 1, 2, \dots, \left[\frac{T}{\tau}\right],$$
$$E(j, n) \equiv E(x_j, t^n), \quad N(j, n) \equiv N(x_j, t^n),$$
$$E_j^n \sim E(j, n), \quad N_j^n \sim N(j, n),$$
$$(W_j^n)_x = \frac{1}{h}(W_{j+1}^n - W_j^n), \quad (W_j^n)_{\bar{x}} = \frac{1}{h}(W_j^n - W_{j-1}^n),$$
$$(W_j^n)_t = \frac{1}{\tau}(W_j^{n+1} - W_j^n), \quad (W_j^n)_{\bar{t}} = \frac{1}{\tau}(W_j^n - W_j^{n-1}),$$
$$\|W^n\|_2^2 = h\sum_{j=1}^{J} |W_j^n|^2, \quad \|W^n\|_\infty = \sup_{1 \le j \le J} |W_j^n|,$$
$$r = \frac{\tau}{h}, \quad \beta = \frac{\tau}{h^2},$$

and in this paper C denotes a general constant, which may have different values in different occurrences. Thus, our scheme is written as

(2.1)
$$i(E_j^{n+1})_{\bar{t}} + \frac{1}{2}((E_j^{n+1})_{x\bar{x}} + (E_j^n)_{x\bar{x}}) = \frac{1}{4}(N_j^{n+1} + N_j^n)(E_j^{n+1} + E_j^n) \cdot \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}, \quad 1 \le j \le J,$$

(2.2)
$$(N_j^n)_{t\bar{t}} - (1 - 2\theta)(N_j^n)_{x\bar{x}} - \theta((N_j^{n+1})_{x\bar{x}} + (N_j^{n-1})_{x\bar{x}}) = (F(|E_j^n|^2))_{x\bar{x}}, \quad 0 \le \theta \le \frac{1}{2}.$$

The initial data and boundary conditions are approximated as

(2.3)
$$E_j^0 = E^0(x_j), \quad N_j^0 = N^0(x_j),$$
$$N_j^1 = N_j^0 + \tau N'(x_j) \quad \text{or}$$

(2.4)
$$N_j^1 = N_j^0 + \tau N^1(x_j) + \frac{\tau^2}{2}[(N_j^0)_{x\bar{x}} + (F(|E_j^0|^2))_{x\bar{x}}],$$

(2.5)
$$E_0^n = E_J^n = 0, \quad N_0^n = N_J^n = 0, \quad u_0^{n+\frac{1}{2}} = u_J^{n+\frac{1}{2}} = 0,$$

where the difference scheme (2.2) is used to approximate N_{tt}^0 in (2.4). We also define $\{u_j^{n+\frac{1}{2}}\}$, as R. T. Glassey did in [9], by

(2.6)
$$(u_j^{n+\frac{1}{2}})_{x\bar{x}} = (N_j^n)_t, \quad j = 1, 2, \dots, J-1.$$

We note that the equations (2.1) are implicit, and a tridiagonal system of equations is involved. For $\theta = \frac{1}{2}$, the equations (2.2) are also implicit, and another tridiagonal system of equations needs to be solved. However, when $\theta = 0$, then the scheme (2.2) for N is explicit, and no tridiagonal system needs to be solved.

Theorem 1. *The difference problem* (2.1)–(2.6) *possesses the following invariants*:

$$\|E^n\|_2^2 = \text{Const}$$

and

$$\begin{aligned} H_h^{n+\frac{1}{2}} &= \|E_x^{n+1}\|_2^2 + \|E_x^n\|_2^2 + \|u_x^{n+\frac{1}{2}}\|_2^2 + (1-2\theta)h\sum_{j=1}^{J} N_j^{n+1}N_j^n \\ &\quad + \theta(\|N^{n+1}\|_2^2 + \|N_j^n\|_2^2) + \frac{1}{2}h\sum_{j=1}^{J}[F((|E_j^{n+1}|^2) + F(|E_j^n|^2))(N_j^{n+1} + N_j^n)] \\ &= \text{Const.} \end{aligned}$$

Proof. Computing the inner product of (2.1) with $(E_j^{n+1} + E_j^n)$ implies

(2.7)
$$\begin{aligned} &i((E_j^{n+1})_{\bar t},\ E_j^{n+1} + E_j^n) + \frac{1}{2}((E_j^{n+1})_{x\bar x} + (E_j^n)_{x\bar x},\ E_j^{n+1} + E_j^n) \\ &\quad = \frac{1}{4}\left((N_j^{n+1} + N_j^n)(E_j^{n+1} + E_j^n)\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2},\ E_j^{n+1} + E_j^n\right), \end{aligned}$$

where

$$\begin{aligned} &((E_j^{n+1})_{\bar t},\ E_j^{n+1} + E_j^n) \\ &\quad = \frac{1}{\tau}(E_j^{n+1} - E_j^n,\ E_j^{n+1} + E_j^n) \\ &\quad = \frac{1}{\tau}(\|E^{n+1}\|_2^2 - \|E^n\|_2^2) + h\sum_{j=1}^{J} E_j^{n+1}\cdot\overline{E_j^n} - h\sum_{j=1}^{J} E_j^n\cdot\overline{E_j^{n+1}}, \\ &\quad -((E_j^{n+1})_{x\bar x} + (E_j^n)_{x\bar x},\ E_j^{n+1} + E_j^n) \\ &\quad = ((E_j^{n+1})_x + (E_j^n)_x,\ (E_j^{n+1})_x + (E_j^n)_x) \\ &\quad = \|E_x^{n+1}\|_2^2 + \|E_x^n\|_2^2 + h\sum_{j=1}^{J}(E_j^{n+1})_x\cdot(\overline{E_j^n})_x + h\sum_{j=1}^{J}(E_j^n)_x\cdot(\overline{E_j^{n+1}})_x, \\ &\left((N_j^{n+1} + N_j^n)(E_j^{n+1} + E_j^n)\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2},\ E_j^{n+1} + E_j^n\right) \\ &\quad = h\sum_{j=1}^{J}(N_j^{n+1} + N_j^n)\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}\cdot|E_j^{n+1} + E_j^n|^2. \end{aligned}$$

Thus, we take the imaginary part of (2.7) and use the formulae derived above to get

$$\frac{1}{\tau}(\|E^{n+1}\|_2^2 - \|E^n\|_2^2) = 0,$$

i.e.,

$$\|E^n\|_2^2 = \|E^0\|_2^2 = \text{Const}. \tag{2.8}$$

Computing the inner product of (2.1) with $\tau(E_j^{n+1})_{\bar{t}}$ and taking the real part, we have

$$(2.9)\qquad -\frac{1}{2}(\|E_x^{n+1}\|_2^2 - \|E_x^n\|_2^2) = \frac{1}{4}h\sum_{j=1}^{J}(N_j^{n+1} + N_j^n)[F(|E_j^{n+1}|^2) - F(|E_j^n|^2)].$$

Next, we compute the inner product of (2.2) with $(u_j^{n+\frac{1}{2}} + u_j^{n-\frac{1}{2}})$, and by using (2.6) we obtain

$$\begin{aligned}((N_j^n)_{t\bar{t}}, u_j^{n+\frac{1}{2}} + u_j^{n-\frac{1}{2}}) &- (1-2\theta)(N_j^n, (N_j^n)_t + (N_j^{n-1})_t)\\ &- \theta(N_j^{n+1} + N_j^{n-1}, (N_j^n)_t + (N_j^{n-1})_t)\\ &= (F(|E_j^n|^2), (N_j^n)_t + (N_j^{n-1})_t),\end{aligned}$$

which is equivalent to
(2.10)

$$\begin{aligned}\|u_x^{n+\frac{1}{2}}\|_2^2 + \|u_x^{n-\frac{1}{2}}\|_2^2 &+ (1-2\theta)h\sum_{j=1}^{J} N_j^{n+1}\cdot N_j^n - (1-2\theta)h\sum_{j=1}^{J} N_j^n\cdot N_j^{n-1}\\ &+ \theta(\|N^{n+1}\|_2^2 - \|N^{n-1}\|_2^2) + h\sum_{j=1}^{J} F(|E_j^n|^2)(N_j^{n+1} - N_j^{n-1}) = 0,\end{aligned}$$

where (2.6) is used to reduce the first term.

It follows from (2.9) that

$$(2.11)\qquad \begin{aligned}-\|E_x^{n+1}\|_2^2 + \|E_x^{n-1}\|_2^2 &= \frac{1}{2}h\sum_{j=1}^{J}(F(|E_j^{n+1}|^2) - F(|E_j^n|^2))(N_j^{n+1} + N_j^n)\\ &\quad - \frac{1}{2}h\sum_{j=1}^{J}(F(|E_j^n|^2) + F(|E_j^{n-1}|^2))(N_j^n + N_j^{n-1})\\ &\quad - h\sum_{j=1}^{J}(F(|E_j^n|^2))(N_j^{n+1} - N_j^{n-1}).\end{aligned}$$

Combining (2.10) and (2.11) yields

$$H_h^{n+\frac{1}{2}} = H_h^{n-\frac{1}{2}} = \text{Const.}\quad \square$$

Theorem 2. *Assume* $E(x, t) \in C^5$, $N(x, t) \in C^5$ *for the solution of problem* (1.5)–(1.11) *and* $f(s) \in C^2(R^+)$. *Then the difference scheme* (2.1)–(2.2) *possesses truncation errors of order* $O(h^2 + \tau^2)$.

Proof. By Taylor's expansion, we have

$$E(j, n+1) + E(j, n) = \left[2E + \frac{\tau^2}{4}E_{tt} + O(\tau^3)\right]\Big|_{x=x_j,\, t=t^{n+\frac{1}{2}}},$$

$$(E(j, n+1))_{\bar{t}} = \left[E_t + \frac{\tau^2}{24}E_{ttt} + O(\tau^3)\right]\Big|_{x=x_j,\, t=t^{n+\frac{1}{2}}},$$

$$(N(j,n))_{x\bar{x}} = \left[N_{xx} + \frac{h^2}{12}N_{xxxx} + O(h^3)\right]\Big|_{x=x_j,\,t=t^n},$$

and

$$\begin{aligned}
&\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} \\
&\quad = \{[F(|E(j,n+\tfrac12)|^2) + F'(|E(j,n+\tfrac12)|^2)(|E(j,n+1)|^2 - |E(j,n+\tfrac12)|^2) \\
&\qquad + \tfrac12 F''(|E(j,n+\tfrac12)|^2)(|E(j,n+1)|^2 - |E(j,n+\tfrac12)|^2)^2 \\
&\qquad\qquad + \tfrac{1}{3!}F'''(|E(j,n+\tfrac12)|^2)(|E(j,n+1)|^2 - |E(j,n+\tfrac12)|^2)^3 + \cdots] \\
&\qquad - [F(|E(j,n+\tfrac12)|^2) + F'(|E(j,n+\tfrac12)|^2)(|E(j,n)|^2 - |E(j,n+\tfrac12)|^2) \\
&\qquad + \tfrac12 F''(|E(j,n+\tfrac12)|^2)(|E(j,n)|^2 - |E(j,n+\tfrac12)|^2)^2 \\
&\qquad + \tfrac{1}{3!}F'''(|E(j,n+\tfrac12)|^2)(|E(j,n)|^2 - |E(j,n+\tfrac12)|^2)^3 + \cdots]\}/ \\
&\qquad\qquad\qquad [|E(j,n+1)|^2 - |E(j,n)|^2] \\
&\quad = F'(|E(j,n+\tfrac12)|^2) + \tfrac12 F''(|E(j,n+\tfrac12)|^2) \\
&\qquad \cdot (|E(j,n+1)|^2 + |E(j,n)|^2 - 2|E(j,n+\tfrac12)|^2) + \tfrac16 F'''(|E(j,n+\tfrac12)|^2) \\
&\qquad \cdot [(|E(j,n+1)|^2 - |E(j,n+\tfrac12)|^2)^2 \\
&\qquad + (|E(j,n+1)|^2 - |E(j,n+\tfrac12)|^2)\cdot(|E(j,n)|^2 - |E(j,n+\tfrac12)|^2) \\
&\qquad\qquad + (|E(j,n)|^2 - |E(j,n+\tfrac12)|^2)^2] + O(\tau^3) \\
&\quad = F'(|E(j,n+\tfrac12)|^2) + \tfrac18 F''(|E(j,n+\tfrac12)|^2)\cdot\tau^2\cdot(|E(j,n+\tfrac12)|^2)_{tt} \\
&\qquad + \tfrac16 F'''(|E(j,n+\tfrac12)|^2) \\
&\qquad \cdot (|E(j,n+1)|^4 + |E(j,n)|^4 + 3|E(j,n+\tfrac12)|^4 \\
&\qquad + |E(j,n+1)|^2\cdot|E(j,n)|^2 - 3|E(j,n+1)|^2|E(j,n+\tfrac12)|^2 \\
&\qquad\qquad - 3|E(j,n)|^2\cdot|E(j,n+\tfrac12)|^2) + O(\tau^3) \\
&\quad = f(|E(j,n+\tfrac12)|^2) + \frac{\tau^3}{8}f'(|E(j,n+\tfrac12)|^2)\cdot(|E(j,n+\tfrac12)|^2)_{tt} \\
&\qquad + \tfrac16 f''(|E(j,n+\tfrac12)|^2)\cdot\frac{\tau^2}{2}[(|E(j,n+\tfrac12)|^2)_t]^2 + O(\tau^3).
\end{aligned}$$

Using equalities derived above, we obtain from the difference schemes (2.1) and (2.2)

(2.12)
$$\begin{aligned}
iE_t + E_{xx} = {} & Nf(|E|^2)\cdot E \\
& + \left[-\frac{i\tau^2}{24}E_{ttt} - \frac{\tau^2}{8}E_{xxtt} - \frac{h^2}{12}E_{xxxx}\right. \\
& \quad + \frac{\tau^2}{8}Ef(|E|^2)N_{tt} + \frac{\tau^2}{8}Nf(|E|^2)E_{tt} + \frac{\tau^2}{8}NEf'(|E|^2)(|E|^2)_{tt} \\
& \qquad \left. + \frac{\tau^2}{12}NEf''(|E|^2)((|E|^2)_t)^2\right] \\
& + O(h^3 + \tau^3)
\end{aligned}$$

and

$$(2.13)\qquad \begin{aligned} N_{tt} - N_{xx} &= (F(|E|^2))_{xx} \\ &+ \left[-\frac{\tau^2}{12}N_{tttt} + \theta\cdot\tau^2 N_{xxtt} + \frac{h^2}{12}N_{xxxx} + \frac{h^4}{12}(F(|E|^2))_{xxxx}\right] \\ &+ O(h^3+\tau^3). \end{aligned}$$

Thus, the truncation errors are $O(h^2+\tau^2)$.

In [5], we have used the conservative scheme to compute solitary waves and the interaction of two colliding solitary waves for the Zakharov equations (1.1) and (1.2). Numerical experiments demonstrate that the semiexplicit scheme with $\theta = 0$ is more efficient and accurate than the implicit scheme with $\theta = \frac{1}{2}$. For example, if we require that computational errors be less than 0.1 during the entire time period of the solitary wave, then the step sizes $h = \tau = 0.25$ can be taken for the scheme with $\theta = 0$, but the step sizes $h = \tau = 0.1$ are needed for the scheme with $\theta = \frac{1}{2}$. To achieve the same accuracy, the scheme with $\theta = \frac{1}{2}$ takes a longer CPU time than the scheme with $\theta = 0$, and the ratio of the CPU times used by the two schemes is

$$R_t = \frac{534.94}{78.26} \approx 6.8.$$

3. Convergence of the difference scheme

In this section, we consider the convergence of the difference scheme (2.1)–(2.6).

We define the errors by

$$(3.1)\qquad e_j^n = E(j,n) - E_j^n \quad\text{and}\quad \eta_j^n = N(j,n) - N_j^n.$$

Let

$$(3.2)\qquad (U_j^{n+\frac{1}{2}})_{x\bar{x}} = (\eta_j^{n+1})_{\bar{t}}, \quad U_0^{n+\frac{1}{2}} = U_J^{n+\frac{1}{2}} = 0.$$

Then the error equations are obtained as follows:

(3.3)

$$\begin{aligned} &i(e_j^{n+1})_{\bar{t}} + \frac{1}{2}[(e_j^{n+1})_{x\bar{x}} + (e_j^n)_{x\bar{x}}] \\ &\quad= R^E + \frac{1}{4}[N(j,n) + N(j,n+1)]\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} \\ &\qquad\cdot[E(j,n+1) + E(j,n)] \\ &\qquad - \frac{1}{4}(N_j^n + N_j^{n+1})\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1} + E_j^n), \end{aligned}$$

$$(3.4)\qquad \begin{aligned} &(\eta_j^n)_{t\bar{t}} - (1-2\theta)(\eta_j^n)_{x\bar{x}} - \theta[(\eta_j^{n-1})_{x\bar{x}} + (\eta_j^{n-1})_{x\bar{x}}] \\ &\quad= R^N + [F(|E(j,n)|)^2 - F(|E_j^n|^2)]_{x\bar{x}}, \end{aligned}$$

where

$$(3.5)\quad \begin{aligned} R^E = \Big[& -\frac{i\tau^2}{24}E_{ttt} - \frac{\tau^2}{8}E_{xxtt} - \frac{h^2}{12}E_{xxxx} \\ & + \frac{\tau^2}{8}Ef(|E|^2)N_{tt} + \frac{\tau^2}{8}Nf(|E|^2)E_{tt} + \frac{\tau^2}{8}NEf'(|E|^2)(|E|^2)_{tt} \\ & + \frac{\tau^2}{12}NEf''(|E|^2)((|E|^2)_t)^2\Big]\Big|_{x=x_j,\, t=t^{n+\frac{1}{2}}} \\ & + O(h^3+\tau^3), \end{aligned}$$

$$(3.6)\quad \begin{aligned} R^N = & \left[-\frac{\tau^2}{12}N_{tttt} + \theta\cdot\tau^2 N_{xxtt} + \frac{h^2}{12}N_{xxxx} + \frac{h^4}{12}(F(|E|^2))_{xxxx}\right]\Big|_{x=x_j,\, t=t^n} \\ & + O(h^3+\tau^3), \end{aligned}$$

in view of the formulae (2.12) and (2.13).

Lemma 1 (Sobolev estimate [8]). *Suppose* $W \in L_q(R^n)$, $D^m W \in L_r(R^n)$, $1 \le q, r < \infty$. *Then for* $0 \le j \le m$, $\frac{j}{m} \le \alpha \le 1$, *we have*

$$\|D^j W\|_{L_p} \le C\|D^m W\|_{L_r}^{\alpha} \cdot \|W\|_{L_q}^{1-\alpha},$$

where $\frac{1}{p} = \frac{j}{n} + \alpha(\frac{1}{r} - \frac{m}{n}) + (1-\alpha)\frac{1}{q}$.

Lemma 2. *Let* $r = \frac{\tau}{h} < \sqrt{\frac{1}{1-2\theta}}$, $0 \le \theta \le \frac{1}{2}$. *If we define* $C_1 = \frac{2+(1-2\theta)r^2}{2-(1-2\theta)r^2}$, *then the following inequality holds*:

$$R_\tau \le C_1 Q_\tau,$$

where

$$R_\tau = \|u_x^{n+\frac{1}{2}}\|_2^2 + \tfrac{1}{2}(1-2\theta)(\|N^{n+1}\|_2^2 + \|N^n\|_2^2),$$

$$Q_\tau = \|u_x^{n+\frac{1}{2}}\|_2^2 + (1-2\theta)h\sum_{j=1}^{J} N_j^{n+1}\cdot N_j^n.$$

Proof of Lemma 2. Let $(W_j^n)_t = u_j^{n+\frac{1}{2}}$ and $(W_j^0)_{x\bar{x}} = N_j^0$; then $W_0^n = W_J^n = 0$ and $N_j^n = (W_j^n)_{x\bar{x}}$. Thus, we have

$$Q_\tau = h\sum_{j=1}^{J}[(W_j^n)_{xt}]^2 + (1-2\theta)h\sum_{j=1}^{J}(W_j^{n+1})_{x\bar{x}}\cdot(W_j^n)_{x\bar{x}},$$

$$R_\tau = h\sum_{j=1}^{J}[(W_j^n)_{xt}]^2 + \tfrac{1}{2}(1-2\theta)h\sum_{j=1}^{J}[(N_j^{n+1})^2 + (N_j^n)^2].$$

We use the following notation:

$$DW_j^n \equiv (W_j^n)_x, \quad D^2W_j^n \equiv (W_j^n)_{x\bar{x}},$$

$$Q_D \equiv \begin{bmatrix} -\tau^{-2}D^2, & \tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4 \\ \tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4, & -\tau^{-2}D^2 \end{bmatrix}$$

and

$$R_D \equiv \begin{bmatrix} -\tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4, & \tau^{-2}D^2 \\ \tau^{-2}D^2, & -\tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4 \end{bmatrix}.$$

It is easily verified that

$$\begin{aligned} Q_\tau &= h\sum_{j=1}^{J}(W_j^{n+1}, W_j^n) \\ &\quad \cdot \begin{bmatrix} -\tau^{-2}D^2, & \tau^{-2}D^2 + \frac{1}{2}(1-2\theta)D^4 \\ \tau^{-2}D^2 & +\frac{1}{2}(1-2\theta)D^4, & -\tau^{-2}D^2 \end{bmatrix} \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix} \\ &= h\sum_{j=1}^{J}(W_j^{n+1}, W_j^n)\cdot Q_D \cdot \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix}, \end{aligned}$$

and

$$R_\tau = h\sum_{j=1}^{J}(W_j^{n+1}, W_j^n)\cdot R_D \cdot \begin{pmatrix} W_j^{n+1} \\ W_j^n \end{pmatrix}.$$

Assume that (Y_1, Y_2) is an eigenfunction associated with an eigenvalue λ of Q_D; then

$$\begin{aligned} -\tau^{-2}D^2Y_1 + \tau^{-2}D^2Y_2 + \tfrac{1}{2}(1-2\theta)D^4Y_2 &= \lambda Y_1, \\ \tau^{-2}D^2Y_1 + \tfrac{1}{2}(1-2\theta)D^4Y_1 - \tau^{-2}D^2Y_2 &= \lambda Y_2. \end{aligned}$$

By adding and subtracting these equations, we obtain

$$\tfrac{1}{2}(1-2\theta)D^4(Y_1+Y_2) = \lambda(Y_1+Y_2), \tag{3.7}$$

$$-2\tau^{-2}D^2(Y_1-Y_2) - \tfrac{1}{2}(1-2\theta)D^4(Y_1-Y_2) = \lambda(Y_1-Y_2). \tag{3.8}$$

If we look for an eigenfunction with $Y_1 = Y_2 = Y$, then (3.8) always holds and (3.7) implies that Y is an eigenfunction of the operator $\frac{1}{2}(1-2\theta)D^4$ with eigenvalue $\frac{1}{2}(1-2\theta)\mu_4$, where μ_4 is the eigenvalue of D^4. This provides J eigenvalues of Q_D. On the other hand, if we seek an eigenfunction with $Y_1 = -Y_2 = Y$, then (3.7) holds and (3.8) implies that the eigenvalue λ is of the form $-2\tau^2\mu_2 - \frac{1}{2}(1-2\theta)\mu_4$ with μ_2 an eigenvalue of D^2.

For an eigenvalue of R_D, we have

$$\begin{aligned} -\tau^{-2}D^2Y_1 + \tfrac{1}{2}(1-2\theta)D^4Y_1 + \tau^{-2}D^2Y_2 &= \lambda Y_1, \\ \tau^{-2}D^2Y_1 - \tau^{-2}D^2Y_2 + \tfrac{1}{2}(1-2\theta)D^4Y_2 &= \lambda Y_2. \end{aligned}$$

A similar argument yields that the eigenvalues and eigenfunctions of R_D are

$$\{\tfrac{1}{2}(1-2\theta)\mu_4, (Y, Y)\}, \{-2\tau^{-2}\mu_2 + \tfrac{1}{2}(1-2\theta)\mu_4, (Y, -Y)\}.$$

Since R_D and Q_D have a common set of eigenfunctions, the inequality $R_\tau \le CQ_\tau$ is equivalent to

$$\lambda(R_D) \le C\lambda(Q_D) \tag{3.9}$$

for the corresponding eigenvalues.

It follows from Fourier analysis that the eigenvalues of the operators D^2 and D^4 are

$$\mu_2 = 2h^{-2}\left(\cos\frac{2\pi jh}{x_R - x_L} - 1\right) \quad \text{and} \quad \mu_4 = \mu_2^2, \quad j = 1, 2, \dots, J.$$

Thus, we have

$$\lambda_j^R = -2\tau^{-2}\cdot 2h^{-2}\left(\cos\frac{2\pi jh}{x_R-x_L}-1\right)+(1-2\theta)2h^{-4}\left(\cos\frac{2\pi jh}{x_R-x_L}-1\right)^2,$$
$$\lambda_j^Q = -2\tau^{-2}\cdot 2h^{-2}\left(\cos\frac{2\pi jh}{x_R-x_L}-1\right)+(1-2\theta)2h^{-4}\left(\cos\frac{2\pi jh}{x_R-x_L}-1\right)^2.$$

Substituting these eigenvalues into (3.9) yields

$$4\tau^{-2}\cdot h^{-2}\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)+2(1-2\theta)h^{-4}\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)^2$$
$$\leq C\left[4\tau^{-2}h^{-2}\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)-2(1-2\theta)h^{-4}\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)^2\right],$$

i.e.,

(3.10)
$$2+(1-2\theta)r^2\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)\leq\left[2-(1-2\theta)r^2\left(1-\cos\frac{2\pi jh}{x_R-x_L}\right)\right].$$

The inequality (3.10) holds with $C_1=\frac{2+(1-2\theta)r^2}{2-(1-2\theta)r^2}$, provided $r<\sqrt{\frac{1}{1-2\theta}}$. This completes the proof. □

Lemma 3. *Assume* $E^0(x)\in H_0^1$, $N^0(x)\in L_2$, $N^1(x)\in L_2$; $|F(s)|\leq C_1s^\alpha+C_2$, $0\leq\alpha<\frac{3}{2}$, *for* $s\geq 0$, *and* $r=\frac{\tau}{h}<\sqrt{\frac{1}{1-2\theta}}$, $0\leq\theta\leq\frac{1}{2}$. *Then we have the estimates*

$$\|E^n\|_2\leq C,\quad \|E_x^n\|_2\leq C,\quad \|E^n\|_\infty\leq C,$$
$$\|N^n\|_2\leq C,\quad \|u_x^{n+\frac{1}{2}}\|_2\leq C,\quad \|u^{n+\frac{1}{2}}\|_\infty\leq C,\quad 0\leq n\leq\frac{T}{\tau}.$$

Proof. It follows from Theorem 1 that

$$\|E^n\|_2\leq C,$$

and

$$\|E_x^{n+1}\|_2^2+\|E_x^n\|_2^2+\|u_x^{n+\frac{1}{2}}\|_2^2+(1-2\theta)h\sum_{j=1}^J N_j^{n+1}\cdot N_j^n$$
$$+\theta(\|N^{n+1}\|_2^2+\|N^n\|_2^2)+\frac{1}{2}h\sum_{j=1}^J[F(|E_j^{n+1}|^2)+F(|E_j^n|^2)](N_j^{n+1}+N_j^n)$$
$$=\text{Const}.$$

Using Lemma 2, we have

(3.11)
$$\|E_x^{n+1}\|_2^2+\|E_x^n\|_2^2+\frac{1}{C_1}\|u_x^{n+\frac{1}{2}}\|_2^2+\left(\frac{1}{2}\frac{1-2\theta}{C_1}+\theta\right)(\|N^{n+1}\|_2^2-\|N^n\|_2^2)$$
$$+\frac{1}{2}h\sum_{j=1}^J[F(|E_j^{n+1}|^2)+F(|E_j^n|^2)](N_j^{n+1}+N_j^n)\leq C.$$

The last term $\mathscr{L}$ on the left of the inequality (3.11) is estimated by

$$
\begin{aligned}
|\mathscr{L}| &\leq \frac{1}{2}h\sum_{j=1}^{J}|[F(|E_j^{n+1}|^2)\cdot N_j^{n+1} + F(|E_j^n|^2)\cdot N_j^{n+1} \\
&\qquad\qquad + F(|E_j^{n+1}|^2)N_j^n + F(|E_j^n|^2)N_j^n]| \\
&\leq \frac{\varepsilon_1}{2}h\sum_{j=1}^{J}[(N_j^{n+1})^2 + (N_j^n)^2] \\
&\quad + \frac{1}{2\varepsilon_1}h\sum_{j=1}^{J}[(F(|E_j^{n+1}|^2))^2 + (F(|E_j^n|^2))^2]
\end{aligned}
\tag{3.12}
$$

for any $\varepsilon_1 > 0$. By Lemma 1 and the Interpolation Lemma [14], we get

$$
\begin{aligned}
&h\sum_{j=1}^{J}[(F(|E_j^{n+1}|^2))^2 + (F(|E_j^n|^2))^2] \\
&\quad\leq Ch\sum_{j=1}^{J}(|E_j^{n+1}|^{4\alpha}| + |E_j^n|^{4\alpha} + |E_j^{n+1}|^{2\alpha} + |E_j^n|^{2\alpha}) + C \\
&\quad= C + Ch\sum_{j=1}^{J}(|E_j^{n+1}|^{6-2\delta} + |E_j^n|^{6-2\delta} + |E_j^{n+1}|^{3-\delta} + |E_j^n|^{3-\delta}) \\
&\quad\leq C + C[\|E_x^{n+1}\|_2^{2-\delta}\cdot\|E^{n+1}\|^{4-\delta} + \|E_x^n\|_2^{2-\delta}\cdot\|E^n\|_2^{4-\delta} \\
&\qquad + \|E_x^{n+1}\|_2^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|E^{n+1}\|_2^{\frac{5}{2}-\frac{\delta}{2}} + \|E_x^n\|^{\frac{1}{2}-\frac{\delta}{2}}\cdot\|E^n\|_2^{\frac{5}{2}-\frac{\delta}{2}}],
\end{aligned}
$$

where $\alpha = \frac{3}{2} - \frac{\delta}{2}$, $3 \geq \delta > 0$.

Using the inequality

$$
a\cdot b \leq \frac{1}{p}(\varepsilon_2 a)^p + \frac{1}{p'}\left(\frac{1}{\varepsilon_2}b\right)^{p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,\, a,\, b,\, p,\, p',\, \varepsilon_2 > 0,
$$

we have

$$
\begin{aligned}
&h\sum_{j=1}^{J}[(F(|E_j^{n+1}|^2))^2 + (F(|E_j^n|^2))^2] \\
&\qquad \leq C + \varepsilon_2(\|E_x^{n+1}\|_2^2 + \|E_x^n\|_2^2) + \frac{C}{\varepsilon_2}.
\end{aligned}
\tag{3.13}
$$

Substituting (3.12) and (3.13) into (3.11) and choosing $\varepsilon_1 = \varepsilon_2 = \frac{1-2\theta}{2C_1} + \theta$, we get the following inequality:

$$
\begin{aligned}
&\frac{1}{2}\|E_x^{n+1}\|_2^2 + \frac{1}{2}\|E_x^n\|_2^2 + \frac{1}{C_1}\|u_x^{n+\frac{1}{2}}\|_2^2 \\
&\quad + \frac{1}{2}\left(\frac{1-2\theta}{2C_1} + \theta\right)(\|N^{n+1}\|_2^2 + \|N^n\|_2^2) \leq C,
\end{aligned}
$$

from which the following estimates are obtained:

$$
\|E_x^n\|_2^2 \leq C, \quad \|N^n\|_2^2 \leq C, \quad \|u_x^{n+\frac{1}{2}}\|_2^2 \leq C.
$$

It follows from the discrete Sobolev inequality in one dimension that

$$\|E^n\|_\infty \le C, \quad \|u^{n+\frac{1}{2}}\|_\infty \le C. \quad \square$$

The proofs of the following two lemmas are given in the Supplement section.

Lemma 4. *Assume that the function* $F(s)$ *is analytic in* R^+, $E^0(x) \in H_0^1$, $N^0(x) \in L_2$, $N^1(x) \in L_2$; $f(s) \in C^\infty(R^+)$, $|F(s)| \le C_1 s^\alpha + C_2$, $0 \le \alpha < \frac{3}{2}$, *for* $s \ge 0$, *and* $r = \frac{\tau}{h} < \sqrt{\frac{1}{1-2\theta}}$, $0 \le \theta \le \frac{1}{2}$. *Suppose that the solution of problem* (1.5)–(1.11) *satisfies* $E(x,t) \in C^5$, $N(x,t) \in C^5$. *Then we have the following estimates*:

$$|P_1^{n+\frac{1}{2}} - P_n^{n+\frac{1}{2}}| \le C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2),$$

where

$$P_1^{n+\frac{1}{2}} = \mathrm{Re}\Bigg((N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} \cdot (E(j,n+1)+E(j,n)) - (N_j^n + N_j^{n+1})\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1}+E_j^n),\ e_j^{n+1} - e_j^n\Bigg),$$

$$P_2^{n+\frac{1}{2}} = h\sum_{j=1}^{J}[F(|E(j,n+1)|^2) - F(|E(j,n)|^2) - F(|E_j^{n+1}|^2) + F(|E(j,n)|^2)](\eta_j^{n+1} + \eta_j^n).$$

Lemma 5. *Assume* $E^0(x) \in H_0^1$, $N^0(x) \in L_2$, $N^1(x) \in L_2$; $f(s) \in C^\infty(R^+)$, $|F(s)| \le C_1 s^\alpha + C_2$, $0 \le \alpha < \frac{3}{2}$, *for* $s \ge 0$, *and* $r = \frac{\tau}{h} < \sqrt{\frac{1}{1-2\theta}}$, $0 \le \theta \le \frac{1}{2}$. *Suppose that the solution of problem* (1.5)–(1.11) *satisfies* $E(x,t) \in C^5$, $N(x,t) \in C^5$. *Then we have the following estimates*:

$$|(R^E, e_j^{n+1} - e_j^n)| \le C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).$$

Theorem 3. *Assume that the function* $F(s)$ *is analytic in* R^+, $E^0(x) \in H_0^1$, $N^0(x) \in L_2$, $N^1(x) \in L_2$; $f(s) \in C^\infty(R^+)$, $|F(s)| \le C_1 s^\alpha + C_2$, $0 \le \alpha < \frac{3}{2}$, *for* $s \ge 0$, *and* $r = \frac{\tau}{h} < \sqrt{\frac{1}{1-2\theta}}$, $0 \le \theta \le \frac{1}{2}$. *Suppose that the solution of problem* (1.5)–(1.11) *satisfies* $E(x,t) \in C^5$, $N(x,t) \in C^5$. *Then the solution of the difference problem* (2.1)–(2.6) *converges to the solution of problem* (1.5)–(1.11) *with order* $O(h^2+\tau^2)$ *in the* L_∞ *norm for* E, *and in the* L_2 *norm for* N.

Proof. First, we derive an estimate of e_j^n. Computing the inner product of (3.3) with $(e_j^{n+1} + e_j^n)$ and taking the imaginary part, we have

$$\frac{1}{\tau}(\|e^{n+1}\|_2^2 - \|e^n\|_2^2) = \mathrm{Im}(R^E, e_j^{n+1} + e_j^n) + P_3, \tag{3.14}$$

where

$$\begin{aligned}
P_3 = \operatorname{Im}\Bigg(&\frac{1}{4}(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2} \\
&\cdot(E(j,n+1)+E(j,n)) \\
&-\frac{1}{4}(N_j^n+N_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n),\, e_j^{n+1}+e_j^n\Bigg) \\
= \operatorname{Im}&\left(\frac{1}{4}(\eta_j^n+\eta_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n),\, e_j^{n+1}+e_j^n\right) \\
+\operatorname{Im}&\Bigg(\frac{1}{4}(N(j,n)+N(j,n+1)) \\
&\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right) \\
&\cdot(E_j^{n+1}+E_j^n),\, e_j^{n+1}+e_j^n\Bigg).
\end{aligned}$$

Using the inequalities (4.2), (4.3) in the Supplement, and Lemma 3, we obtain

$$|P_3| \le C\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2). \tag{3.15}$$

It is easy to obtain the estimate

$$\begin{aligned}
|\operatorname{Im}(R^E, e_j^{n+1}+e_j^n)| &= |\operatorname{Im}(O(h^2+\tau^2), e_j^{n+1}+e_j^n)| \\
&\le C((h^2+\tau^2)^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2).
\end{aligned} \tag{3.16}$$

Thus, it follows from (3.14), (3.15), and (3.16) that

$$\begin{aligned}
\|e^{n+1}\|_2^2-\|e^n\|_2^2 \le C\tau(\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2) \\
+C\tau(h^2+\tau^2)^2.
\end{aligned} \tag{3.17}$$

Computing the inner product of (3.4) with $U_j^{n+\frac12}+U_j^{n-\frac12}$ and using (3.2), we have

(3.18)

$$\begin{aligned}
&((U_j^{n+\frac12})_{x\bar{x}\bar{t}}, U_j^{n+\frac12}+U_j^{n-\frac12})-(1-2\theta)(\eta_j^n, (U_j^{n+\frac12}+U_j^{n-\frac12})_{x\bar{x}}) \\
&\quad -\theta(\eta_j^{n+1}+\eta_j^{n-1}, (U_j^{n+\frac12}+U_j^{n-\frac12})_{x\bar{x}}) \\
&\quad = (R^N, U_j^{n+\frac12}+U_j^{n-\frac12})+(F(|E(j,n)|^2)-F(|E_j^n|^2), (U_j^{n+\frac12}+U_j^{n-\frac12})_{x\bar{x}}),
\end{aligned}$$

where

$$\begin{aligned}
-((U_j^{n+\frac12})_{x\bar{x}\bar{t}}, U_j^{n+\frac12}+U_j^{n-\frac12}) &= \frac{1}{\tau}((U_j^{n+\frac12}-U_j^{n-\frac12})_x, (U_j^{n+\frac12}+U_j^{n-\frac12})_x) \\
&= \frac{1}{\tau}(\|U_j^{n+\frac12}\|_2^2-\|(U_j^{n-\frac12})_x\|_2^2),
\end{aligned}$$

$$\begin{aligned}\tau(\eta_j^n, (U_j^{n+\frac12} + U_j^{n-\frac12})_{x\bar{x}}) &= \tau(\eta_j^n, (\eta_j^{n+1})_{\bar{t}} + (\eta_j^n)_{\bar{t}}) \\ &= h\sum_{j=1}^{J}(\eta_j^{n+1}\eta_j^n - \eta_j^n\eta_j^{n-1}),\end{aligned}$$

$$\tau(\eta_j^{n+1} + \eta_j^{n-1}, (U_j^{n+\frac12} + U_j^{n-\frac12})_{x\bar{x}}) = \|\eta^{n+1}\|_2^2 - \|\eta^{n-1}\|_2^2,$$

$$\begin{aligned}&\tau(F(|E(j, n)|^2) - F(|E_j^n|^2), (U_j^{n+\frac12} + U_j^{n-\frac12})_{x\bar{x}}) \\ &\quad = h\sum_{j=1}^{J}[F(|E(j, n)|^2) - F(|E_j^n|^2)](\eta_j^{n+1} - \eta_j^{n-1}) \\ &\quad = h\sum_{j=1}^{J}[F(|E(j, n+1)|^2) - F(|E_j^{n+1}|^2)](\eta_j^{n+1} + \eta_j^n) \\ &\qquad - h\sum_{j=1}^{J}[F(|E(j, n)|^2) - F(|E_j^n|^2)](\eta_j^n + \eta_j^{n-1}) \\ &\qquad - h\sum_{j=1}^{J}[F(|E(j, n+1)|^2) - F(|E(j, n)|^2) \\ &\qquad\qquad - F(|E_j^{n+1}|^2) + F(|E_j^n|^2)](\eta_j^{n+1} + \eta_j^n),\end{aligned}$$

$$|\tau(R^N, U_j^{n+\frac12} + U_j^{n-\frac12})| \le C\tau(h^2+\tau^2)^2 + C\tau(\|U^{n+\frac12}\|_2^2 + \|U^{n-\frac12}\|_2^2).$$

Thus,

(3.19)

$$\begin{aligned}&\|U_x^{n+\frac12}\|_2^2 - \|U_x^{n-\frac12}\|_2^2 + (1-2\theta)h\sum_{j=1}^{J}\eta_j^{n+1}\eta_j^n - (1-2\theta)h\sum_{j=1}^{J}\eta_j^n\eta_j^{n-1} \\ &\quad + \theta(\|\eta^{n+1}\|_2^2 - \|\eta^{n-1}\|_2^2) + h\sum_{j=1}^{J}[F(|E(j, n+1)|^2) - F(|E_j^{n+1}|^2)](\eta_j^{n+1} + \eta_j^n) \\ &\quad - h\sum_{j=1}^{J}[F(|E(j, n)|^2) - F(|E_j^n|^2)](\eta_j^n + \eta_j^{n-1}) \\ &\le P_2^{n+\frac12} + C\tau(h^2+\tau^2)^2 + C\tau(\|U^{n+\frac12}\|_2^2 + \|U_x^{n-\frac12}\|_2^2).\end{aligned}$$

We now compute the inner product of (3.3) with $\tau(e_j^{n+1})_{\bar{t}}$ and take the real part. There results the equality

$$-\frac{1}{2}(\|e_x^{n+1}\|_2^2 - \|e_x^n\|_2^2) = \mathrm{Re}(R^E, e_j^{n+1} - e_j^n) + \tfrac14 P_1^{n+\frac12}.$$

Using Lemma 5, we have

$$\begin{aligned}2(\|e_x^{n+1}\|_2^2 - \|e_x^n\|_2^2) \le{}& -P_1^{n+\frac12} + C\tau(h^2+\tau^2)^2 \\ &+ C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\ &\qquad + \|\eta^{n+1}\|_2^2 + \|\eta^{n+1}\|_2^2).\end{aligned} \tag{3.20}$$

Multiplying (3.17) by a positive constant C_ε and adding the result to the sum of (3.19) and (3.20), we obtain

$$L^{n+\frac{1}{2}} \le L^{n-\frac{1}{2}} + C\tau(h^2+\tau^2)^2 + C\tau G^{n+\frac{1}{2}} - (P_1^{n+\frac{1}{2}} - P_2^{n+\frac{1}{2}}), \tag{3.21}$$

where

$$\begin{aligned} L^{n+\frac{1}{2}} = {} & C_\varepsilon \|e^{n+1}\|_2^2 + 2\|e_x^{n+1}\|_2^2 + \|U_x^{n+1}\|_2^2 \\ & + (1-2\theta)h\sum_{j=1}^{J} \eta_j^{n+1}\eta_j^n + \theta(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) \\ & + h\sum_{j=1}^{J}[F(|E(j,n+1)|^2) - F(|E_j^{n+1}|^2)](\eta_j^{n+1}+\eta_j^n), \end{aligned}$$

$$\begin{aligned} G^{n+\frac{1}{2}} = {} & \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 - \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\ & + \|U_x^{n+\frac{1}{2}}\|_2^2 + \|U_x^{n-\frac{1}{2}}\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2. \end{aligned}$$

It follows from Lemma 4 that

$$\begin{aligned} L^{n+\frac{1}{2}} & \le L^{n-\frac{1}{2}} + C\tau(h^2+\tau^2)^2 + C\tau G^{n+\frac{1}{2}} \\ & \le L^{-\frac{1}{2}} + C(h^2+\tau^2)^2 + C\tau\sum_{l=0}^{n} G^{l+\frac{1}{2}} \\ & \le C(h^2+\tau^2)^2 + C\tau\sum_{l=0}^{n} G^{l+\frac{1}{2}}. \end{aligned} \tag{3.22}$$

Lemma 2 yields

$$\begin{aligned} L^{n+\frac{1}{2}} \ge {} & C_\varepsilon\|e^{n+1}\|_2^2 + 2\|e_x^{n+1}\|_2^2 + \frac{1}{C_1}\|U_x^{n+\frac{1}{2}}\|_2^2 \\ & + \left(\frac{1-2\theta}{2C_1} + \theta\right)(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) \\ & + h\sum_{j=1}^{J}[F(|E(j,n+1)|^2) - F(|E_j^{n+1}|^2)](\eta_j^{n+1}+\eta_j^n), \end{aligned} \tag{3.23}$$

while

(3.24)

$$\begin{aligned} & \left| h\sum_{j=1}^{J}[F(|E(j,n+1)|^2) - F(|E_j^{n+1}|^2)](\eta_j^{n+1}+\eta_j^n) \right| \\ & \le \varepsilon_3(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{1}{\varepsilon_3}h\sum_{j=1}^{J}|F(|E(j,n+1)|^2) - F(|E_j^{n+1}|^2)|^2 \\ & \le \varepsilon_3(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{1}{\varepsilon_3}h\sum_{j=1}^{J}|f(\xi_3)|^2(|E(j,n+1)| + |E_j^{n+1}|)^2|e_j^{n+1}|^2 \\ & \le \varepsilon_3(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \frac{C_3}{\varepsilon_3}\|e^{n+1}\|_2^2. \end{aligned}$$

Substituting (3.24) into (3.23) and choosing $\varepsilon_3 = \frac{1}{2}(\frac{1-2\theta}{2C_1} + \theta)$ and $C_\varepsilon = \frac{2C_3}{\varepsilon_3}$, we have

$$
\begin{aligned}
L^{n+\frac{1}{2}} &\geq \frac{2C_3}{\frac{1-2\theta}{2C_1} + \theta}\|e^{n+1}\|_2^2 + 2\|e_x^{n+1}\|_2^2 + \frac{1}{C_1}\|U_x^{n+\frac{1}{2}}\|_2^2 \\
&\quad + \frac{1}{2}\left(\frac{1-2\theta}{2C_1} + \theta\right)(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) \\
&\geq C(\|e^{n+1}\|_2^2 + \|e_x^{n+1}\|_2^2 + \|U_x^{n+\frac{1}{2}}\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).
\end{aligned}
\tag{3.25}
$$

Thus, combining (3.25) and (3.22) yields

$$
\begin{aligned}
&\|e^{n+1}\|_2^2 + \|e_x^{n+1}\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2 + \|U_x^{n+\frac{1}{2}}\|_2^2 \\
&\leq C(h^2 + \tau^2)^2 + C\tau\sum_{l=0}^{n} G^{l+\frac{1}{2}},
\end{aligned}
$$

which is equivalent to

$$G^{n+\frac{1}{2}} \leq 2C(h^2 + \tau^2)^2 + 2C\tau\sum_{l=0}^{n} G^{l+\frac{1}{2}},$$

i.e.,

$$G^{n+\frac{1}{2}} \leq C(h^2 + \tau^2)^2 + C\tau\sum_{l=0}^{n-1} G^{l+\frac{1}{2}}.$$

Using the discrete Gronwall inequality [12], we obtain

$$G^{n+\frac{1}{2}} \leq C(h^2 + \tau^2)^2, \quad 0 \leq n \leq \frac{T}{\tau},$$

where C_T is a constant depending on T.

It follows from the definition of $G^{n+\frac{1}{2}}$ that

$$\|e^n\|_\infty \leq C(\|e^n\|_2 + \|e_x^n\|_2) \leq C(h^2 + \tau^2)$$

and

$$\|\eta^n\|_2 \leq C(h^2 + \tau^2).$$

This completes the proof. □

Finally, it is easy to verify that all lemmas and theorems in this paper hold for the periodic initial-value problem for the generalized Zakharov equations.

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Supplement to

FINITE DIFFERENCE METHOD FOR GENERALIZED ZAKHAROV EQUATIONS

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In this section, long and highly technical proofs of two Lemmas in Section 3 are given.

Proof of Lemma 4. Direct computation implies that

$$
\begin{aligned}
&P_1^{n+\frac{1}{2}} - P_2^{n+\frac{1}{2}} \\
=&\mathrm{Re}\left\{ h\sum_{j=1}^{J}\left[(N(j,n)+N(j,n+1))(F(|E(j,n+1)|^2)+F(|E(j,n)|^2)) \right.\right.\\
&-(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n))\\
&\cdot(\overline{E_j^{n+1}-E_j^n})-(N_j^n+N_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\\
&\left.\left.\cdot(\overline{E(j,n+1)-E(j,n)})+(N_j^n+N_j^{n+1})(F(|E_j^{n+1}|^2)-F(|E_j^n|^2))\right]\right\}\\
&-h\sum_{j=1}^{J}[F(|E(j,n+1)|^2)-F(|E(j,n)|^2)-F(|E_j^{n+1}|^2)+F(|E_j^n)|^2)]\\
&\cdot[N(j,n+1)+N(j,n)-N_j^{n+1}-N_j^n]\\
=&\mathrm{Re}\left\{ h\sum_{j=1}^{J}[(N_j^{n+1}+N_j^n)(\overline{E(j,n+1)-E(j,n)})-(N(j,n)+N(j,n+1))\right.\\
&\cdot(\overline{E_j^{n+1}-E_j^n})]\left[\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n))\right.\\
&\left.\left.-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\right]\right\}\\
=&\mathrm{Re}\left\{ h\sum_{j=1}^{J}[(N(j,n)+N(j,n+1))(\overline{e_j^{n+1}-e_j^n})-(\overline{E(j,n+1)-E(j,n)}\right.\\
&\cdot(\eta_j^{n+1}+\eta_j^n)]\left[\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}+e_j^n)\right.
\end{aligned}
$$

$$+\left(\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)$$

$$\cdot(E(j,n+1)+E(j,n))\Big]\Big\}. \tag{4.1}$$

Making Taylor's expansion we have

$$\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}=\sum_{l=1}^{\infty}\frac{1}{l!}F^{(l)}(|E_j^n|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}. \tag{4.2}$$

Using the formulae

$$a^l-b^l=(a-b)\sum_{k=0}^{l-1}a^{l-1-k}\cdot b^k \quad , \quad (a-b)^l=\sum_{k=0}^{l}(-1)^lC_l^k a^{l-k}\cdot b^k,$$

the estimates in Lemma 3 and $E(x,t)\in C^{(5)}, N(x,t)\in C^{(5)}$, we obtain

$$\left|\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right|$$

$$=\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)((|E(j,n+1)|^2-|E(j,n)|^2)^{l-1}-(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1})$$

$$+(F^{(l)}(|E(j,n)|^2)-F^{(l)}(|E_j^n|^2))(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}]\Bigg|$$

$$=\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)(|E(j,n+1)|^2-|E(j,n)|^2-|E_j^{n+1}|^2+|E_j^n|^2)$$

$$\cdot\sum_{k=0}^{l-2}(|E(j,n+1)|^2-|E(j,n)|^2)^{l-2-k}(|E_j^{n+1}|^2-|E_j^n|^2)^k$$

$$+F^{(l+1)}(\xi_1)(|E(j,n)|^2-|E_j^n|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}]\Bigg|$$

$$\le C(|e_j^n|+|e_j^{n+1}|), \tag{4.3}$$

where ξ_1 is located between $|E(j,n)|^2$ and $|E_j^n|^2$,

$$\left|\left(\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)_x\right|=\frac{1}{h}\left|\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}-\frac{F(|E_{j-1}^{n+1}|^2)-F(|E_{j-1}^n|^2)}{|E_{j-1}^{n+1}|^2-|E_{j-1}^n|^2}\right|$$

$$\le C(|(E_j^{n+1})_x|+|(E_j^n)_x|), \tag{4.4}$$

and

$$P_x\equiv\left|\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)_x\right|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)(|E(j,n+1)|^2-|E(j,n)|^2)^{l-1}$$

$$-F^{(l)}(|E(j-1,n)|^2)(|E(j-1,n+1)|^2-|E(j-1,n)|^2)^{l-1}$$

$$-F^{(l)}(|E_j^n)|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}+F^{(l)}(|E_{j-1}^n)|^2)(|E_{j-1}^{n+1}|^2-|E_{j-1}^n|^2)^{l-1}\Bigg|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}\Bigg\{\sum_{k=0}^{l-1}(-1)^kC_{l-1}^k[F^{(l)}(|E(j,n)|^2)|E(j,n+1)|^{2(l-1-k)}|E(j,n)|^{2k}$$

$$-F^{(l)}(|E(j-1,n)|^2)(|E(j-1,n+1)|^{2(l-1-k)}|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_j^n|^2)|E_j^{n+1}|^{2(l-1-k)}|E_j^n|^{2k}+F^{(l)}(|\dot{E}_{j-1}^n)|^2)|E_{j-1}^{n+1}|^{2(l-1-k)}|E_{j-1}^n|^{2k}]\Bigg\}\Bigg|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}\Bigg\{\sum_{k=0}^{l-1}(-1)^kC_{l-1}^k[(F^{(l)}(|E(j,n)|^2)-F^{(l)}(|E_j^n|^2))|E(j,n+1)|^{2(l-1-k)}|E(j,n)|^{2k}$$

$$+F^{(l)}(|E_j^n)|^2)(|E(j,n+1)|^{2(l-1-k)}-|E_j^{n+1}|^{2(l-1-k)})|E(j,n)|^{2k}$$

$$+F^{(l)}(|E_j^n)|^2)|E_j^{n+1}|^{2(l-1-k)}(|E(j,n)|^{2k}-|E_j^n|^{2k})$$

$$-(F^{(l)}(|E(j-1,n)|^2)-F^{(l)}(|E_{j-1}^n|^2))|E(j-1,n+1)|^{2(l-1-k)}|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_{j-1}^n|^2)(|E(j-1,n+1)|^{2(l-1-k)}-|E_{j-1}^{n+1}|^{2(l-1-k)})|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_{j-1}^n)|^2)|E_{j-1}^{n+1}|^{2(l-1-k)}(|E(j-1,n)|^{2k}-|E_{j-1}^n|^{2k})]\Bigg\}\Bigg|.$$

Making Taylor's expansion , using

$$a^{2l}-b^{2l}=(a^2-b^2)\sum_{m=0}^{l-1}a^{2(l-1-m)}\cdot b^{2m}$$

and

$$|E(j,n)|^2-|E_j^n|^2=\mathrm{Re}(e_j^n\cdot\overline{(E(j,n)+E_j^n)})$$

etc., we have

$$
\begin{aligned}
P_x = \Bigg| & \sum_{l=1}^{\infty} \frac{1}{l!} \Bigg\{ \sum_{k=0}^{l-1} (-1)^k C_{l-1}^k \cdot \mathrm{Re} \Bigg[\\
& \sum_{m=1}^{\infty} \frac{1}{m!} \frac{1}{h} \big(F^{(l+m)}(|E_j^n|^2)(e_j^n)^m (\overline{E(j,n) + E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& - F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n) + E_{j-1}^n})^m \\
& \cdot |E(j-1,n+1)|^{2(l-1-k)} |E(j-1,n)|^{2k} \big) \\
& + \frac{1}{h} \Bigg(F^{(l)}(|E_j^n|^2) |E(j,n)|^{2k} \overline{e_j^{n+1}} (E(j,n+1) + E_j^{n+1}) \sum_{m=0}^{l-k-2} |E(j,n+1)|^{2(l-k-2-m)} \\
& \cdot |E_j^{n+1}|^{2m} - F^{(l)}(|E_{j-1}^n|^2) |E(j-1,n)|^{2k} \overline{e_{j-1}^{n+1}} (E(j-1,n+1) + E_{j-1}^{n+1}) \\
& \cdot \sum_{m=0}^{l-k-2} |E(j-1,n+1)|^{2(l-k-2-m)} |E_{j-1}^{n+1}|^{2m} \Bigg) \\
& + \frac{1}{h} \Bigg(F^{(l)}(|E_j^n|^2) |E_j^{n-1}|^{2(l-1-k)} \overline{e_j^n} (E(j,n) + E_j^n) \sum_{m=0}^{k-1} |E(j,n)|^{2(k-1-m)} |E_j^n|^{2m} \\
& - F^{(l)}(|E_{j-1}^n|^2) |E_{j-1}^{n+1}|^{2(l-1-k)} \overline{e_{j-1}^n} (E(j-1,n) + E_{j-1}^n) \\
& \cdot \sum_{m=0}^{k-1} |E(j-1,n)|^{2(k-1-m)} |E_{j-1}^n|^{2m} \Bigg) \Bigg] \Bigg\} \Bigg| . \qquad (4.5)
\end{aligned}
$$

Now, we estimate the terms in (4.5):

$$
\begin{aligned}
& \frac{1}{h} \Big| F^{(l+m)}(|E_j^n|^2)(e_j^n)^m (\overline{E(j,n) + E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} - F^{(l+m)} \\
& \cdot (|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n) + E_{j-1}^n})^m |E(j-1,n+1)|^{2(l-1-k)} |E(j-1,n)|^{2k} \Big| \\
= & \frac{1}{h} \Big| \Big\{ [F^{(l+m)}(|E_j^n|^2) - F^{(l+m)}(|E_{j-1}^n|^2)] (e_j^n)^m (\overline{E(j,n) + E_j^n})^m \\
& \cdot |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2) [(e_j^n)^m - (e_{j-1}^n)^m] (\overline{E(j,n) + E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m [(\overline{E(j,n) + E_j^n})^m - (\overline{E(j-1,n) + E_{j-1}^n})^m] \\
& \cdot |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n) + E_{j-1}^n})^m \\
& \cdot [|E(j,n+1)|^{2(l-1-k)} - |E(j-1,n+1)|^{2(l-1-k)}] \cdot |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n) + E_{j-1}^n})^m \\
& \cdot |E(j-1,n+1)|^{2(l-1-k)} [|E(j,n)|^{2k} - |E(j-1,n)|^{2k}] \Big\} \Big| \\
\le & C |(E_{j-1}^n)_x| \cdot |e_j^n| (2C)^{m-1} \cdot (2C)^m C^{2(l-1)} + C |(e_{j-1}^n)_x| (2C)^{m-1} (2C)^m C^{2(l-1)} \\
& + C(2C)^{m-1} \cdot |e_{j-1}^n| (C + |(E_{j-1}^n)_x|) \cdot 2(2C)^{m-1} \cdot C^{2(l-1)} \\
& + C(2C)^{m-1} \cdot |e_{j-1}^n| (2C)^m C \cdot (2C)^{2(l-1-k)-1} \cdot C^{2k} \\
& + C \cdot (2C)^{m-1} |e_{j-1}^n| (2C)^m C^{2(l-1-k)} \cdot C \cdot (2C)^{2k-1} \\
\le & C(2C)^{2m} \cdot C^{2(l-1)} (|(E_{j-1}^n)_x| \cdot |e_j^n| + |(e_{j-1}^n)_x| + |e_{j-1}^n| \cdot |(E_{j-1}^n)_x| + |e_{j-1}^n|).
\end{aligned}
$$

Other terms in the inequalities (4.5) can be estimated similarly, and substituting these estimates in (4.5) implies that

$$
\begin{aligned}
P_x \le & C(|(e_{j-1}^{n+1})_x| + |(e_{j-1}^n)_x| + |e_j^{n+1}| + |e_{j-1}^{n+1}| + |e_j^n| + |e_{j-1}^n| + \\
& (|e_j^{n+1}| + |e_{j-1}^{n+1}| + |e_j^n| + |e_{j-1}^n|)(|(E_{j-1}^{n+1})_x| + |(E_{j-1}^n)_x|)). \qquad (4.6)
\end{aligned}
$$

Thus,using the inequalities (4.2) and (4.3) we first estimate a simpler term in (4.1):

$$
\begin{aligned}
& \Bigg| \mathrm{Re} \Bigg\{ h \sum_{j=1}^{J} \overline{(E(j,n+1) - E(j,n))} (\eta_j^{n+1} + \eta_j^n) \Bigg[\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} (e_j^{n+1} + e_j^n) \\
& + \left(\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} - \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} \right) (E(j,n+1) + E(j,n)) \Bigg] \Bigg\} \Bigg| \\
\le & \Bigg| h \sum_{j=1}^{J} (\tau \overline{E_t(j,\xi_2)}) (\eta_j^{n+1} + \eta_j^n) \Bigg[\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} (e_j^{n+1} - e_j^n) \\
& + \left(\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} - \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} \right) (E(j,n+1) + E(j,n)) \Bigg] \Bigg| \\
\le & C\tau (\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2), \qquad (4.7)
\end{aligned}
$$

where ξ_2 is located between t^n and t^{n+1}. Then, using the error equation (3.3) and summing by parts, we have

$$\left|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}+e_j^n)\right]\right|$$

$$=\tau\left|\mathrm{Re}\left\{h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right.\right.$$

$$\cdot i[\frac{1}{2}((e_j^{n+1})_{xx}+(e_j^n)_{xx}-R^E$$

$$-\frac{1}{4}(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n))$$

$$\left.\left.+\frac{1}{4}(N_j^n+N_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\right]\right|$$

$$\le\tau\left|\frac{1}{2}h\sum_{j=1}^{J}\left[(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right]_x\right.$$

$$\left.\cdot((e_j^{n+1})_x+(e_j^n)_x)\right|+C\tau(h^2+\tau^2)^2+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2)$$

$$+\tau\left|\frac{1}{4}h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right.$$

$$\cdot\left[(\eta_j^n+\eta_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)+(N(j,n)+N(j,n+1))\right.$$

$$\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)(E_j^{n+1}+E_j^n)$$

$$\left.\left.+(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(e_j^{n+1}+e_j^n)\right]\right|.$$

Furthermore, using inequalities (4.2), (4.3), (4.4) and (4.6), we obtain

$$|\mathrm{Re}[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}$$

$$\cdot\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}-e_j^n)]|$$

$$\le C\tau\left(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2\right.$$

$$\left.+h\sum_{j=1}^{J}|(e_j^{n+1}+e_j^n)((e_j^{n+1})_x+(e_j^n)_x)(|(E_j^{n+1})_x|+|(E_j^n)_x|)|\right)$$

$$+C\tau(h^2+\tau)^2+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2)$$

$$+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)$$

$$\le C\tau(h^2+\tau^2)^2+C\tau(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2$$

$$+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)+C\tau\left[h\sum_{j=1}^{J}(|(e_j^{n+1})_x|^2+|(e_j^n)_x|^2)\right.$$

$$\left.+h\sum_{j=1}^{J}(|(E_j^{n+1})_x|^2+|(E_j^n)_x|^2)(\|e^{n+1}\|_\infty^2+\|e^n\|_\infty^2)\right]$$

$$\le C\tau(h^2+\tau^2)^2+C\tau(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2$$

$$+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2),\tag{4.8}$$

and

$$|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}\right.$$

$$\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)$$

$$\cdot(E(j,n+1)+E(j,n))]|$$

$$=|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\right.$$

$$\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)$$

$$\cdot\overline{(E(j,n+1)+E(j,n))}\cdot i[\frac{1}{2}((e_j^{n+1})_{xx}+(e_j^n)_{xx})-R^E$$

$$-\frac{1}{4}(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}$$

$$
\begin{aligned}
&\cdot (E(j,n+1) + E(j,n)) \\
&+ \frac{1}{4}(N_j^n + N_j^{n+1}) \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1} + E_j^n)\Big]\Big| \\
&\le C\tau(h^2+\tau^2)^2 + C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\
&+ \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)). \qquad (4.9)
\end{aligned}
$$

It follows from (4.7), (4.8) and (4.9) that

$$
\begin{aligned}
|P_1^{n+\frac{1}{2}} - P_2^{n+\frac{1}{2}}| \le\, & C\tau(h^2+\tau^2)^2 + C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\
& + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)). \quad \square
\end{aligned}
$$

Proof of Lemma 5. Using (3.5) and the error equation (3.3), we obtain

$$
\begin{aligned}
&|(R^E \quad , \quad e_j^{n+1} - e_j^n)| = |(O(h^3+\tau^3) \quad , \quad e_j^{n+1} - e_j^n)| \\
&+ \tau \Big| \Big(-\frac{i\tau^2}{24} E_{ttt}(j, n+\frac{1}{2}) - \frac{\tau^2}{8} E_{xxtt}(j, n+\frac{1}{2}) - \frac{\tau^2}{12} E_{xxxx}(j, n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8} E(j,n+\frac{1}{2}) f(|E(j,n+\frac{1}{2})|^2) N_{tt}(j,n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8} N(j,n+\frac{1}{2}) f(|E(j,n+\frac{1}{2})|^2) E_{tt}(j,n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8} N(j,n+\frac{1}{2}) E(j,n+\frac{1}{2}) f'(|E(j,n+\frac{1}{2}|^2) \\
&\cdot \Big(|E(j,n+\frac{1}{2})_{tt} + \frac{\tau^2}{12} N(j,n+\frac{1}{2}) E(j,n+\frac{1}{2}) f''(|E(j,n+\frac{1}{2})|^2)(|E(j,n+\frac{1}{2})|^2)_t \Big)^2 \\
&\qquad , \quad \frac{1}{2}((e_j^{n-1})_{xx} + (e_j^n)_{xx}) - O(h^2+\tau^2) \\
&- \frac{1}{4}(N(j,n) + N(j,n+1)) \frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2}(E(j,n+1) + E(j,n)) \\
&+ \frac{1}{4}(N_j^n + N_j^{n+1}) \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1} + E_j^n) \Big) \Big| \\
&\le C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) \\
&+ C\tau(\tau^4 + h^4 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \tau^2(h^2+\tau^2) + \tau^4 \\
&+ \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \|e^{n+1}\|_2^2 + \|e^n\|_2^2) \\
&\le C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).
\end{aligned}
$$