A second order convergent and linearized difference scheme for the initial-boundary value problem of KdV equation*

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Abstract

In this paper, we construct a three-level linearized difference scheme to solve the initial-boundary value problem of KdV equation using the method of reduction of order. The difference scheme has an invariant. It is also unique solvable. Under the optimal step ratio condition, we prove that the spatio-temporal convergence order of the scheme is both two. At last, two numerical examples demonstrate the effictiveness and convergence order, comparing with a known two-level nonlinear difference scheme.

Keyword: KdV equation; linearized difference scheme; conservation; convergence

1 Introduction

The Korteweg-de Vries (KdV) equation was derived by Korteweg D. J. and de Vries G. in 1985 [6], over 130 years of history. It is one of the most classical mathematical physics equations. The KdV equation has a very important place in nonlinear dispersive waves and has a wide range of applications. Many scholars have studied its solutions from the analytical and numerical point of view. As a matter of fact, the analytical solution of the KdV equation is very difficult to obtain. Therefore, many widely applied numerical methods are used to solve it, such as finite difference method [7, 10, 13, 14, 19], finite element method [1, 9, 15, 16], spectral method [5, 18], meshless method [2–4, 11], etc. Among them, the finite difference method is simple, flexible and general, easy to implement on computers. It is the main numerical method for solving various mathematical and physical problems, especially for equations that depend on time development. Consequently, we will use the finite difference method to solve the KdV equation in this paper.

When we solve the nonlinear evolution equations, we have to consider the corresponding initial value or boundary value problems. There are three main types of problems, which are initial value problems, periodic boundary value problems, and initial-boundary value problems. Now, there have been many studies on the finite difference method for the nonlinear evolution equations. Some of them studied the inital value problems. For this class of problems, the difference schemes were conveniently established by adding zero boundary conditions to the boundary in the actual computation. This actually solved the initial-boundary value problems, or more precisely, the Dirichlet boundary value problems. If the highest order derivative of the

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equation with respect to space variable x is two, then there is little difference in the construction and analysis of the difference schemes, such as Burgers' equation

$$u_t + uu_x = \nu u_{xx}, \quad \nu \text{ is a positive constant.}$$

The reason is that the difference scheme near the boundary is consistent with that on the other inner points. But for equations with higher order of spatial derivatives, the difference can be significant, due to the presence of derivative boundary conditions, such as KdV equation

$$u_t + \gamma uu_x + u_{xxx} = 0, \quad \gamma \text{ is a constant,}$$

whose boundary conditions [17] are

$$u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \quad t \geqslant 0.$$

As a result, it would be more accurate to construct the difference schemes and analyze them directly for the initial-boundary value problems of equations. There have also been studies of periodic boundary problems of the KdV equation using the finite difference method. However, due to the existense of derivative boundary condition, it is difficult to similarly generalize the method for the periodic boundary value problems to the initial-boundary value problems of the KdV equation.

Recently, we established two finite difference schemes for KdV equation with the initial-boundary value conditions in [10]. One was a nonlinear difference scheme and another was linearized. The nonlinear difference scheme was proved to be unconditionally convergent, while the linearized scheme was conditionally convergent. The accuracy of these two difference schemes were first-order in space. Later, in [14], we established a nonlinear difference scheme and proved that its spatial convergence order was two. In the numerical examples, we solved the nonlinear difference scheme using the Newton iterative method, which increased the number of calculations in each time level. In order to improve computational efficiency, we consider establishing a linearized difference scheme remaining the second order spatial conervence.

In this article, we construct a three-level linearized difference scheme for the initial-boundary value problem of KdV equation:

$$\begin{cases} u_t + \gamma uu_x + u_{xxx} = 0, \quad 0 < x < L,\ 0 < t \leqslant T, & \text{(1.1a)} \\ u(x,0) = \varphi(x), \quad 0 < x < L, & \text{(1.1b)} \\ u(0,t) = 0, \quad u(L,t) = 0, \quad u_x(L,t) = 0, \quad 0 \leqslant t \leqslant T, & \text{(1.1c)} \end{cases}$$

where $\varphi(0) = \varphi(L) = \varphi'(L) = 0$, γ is a constant.

The rest of this paper is organized as follows. Some notations are introduced in Section 2. Then we present how to establish the difference scheme and give truncation errors in detail. In Section 3, we will provide the conservation and prove the unique solvability and conditional convergence. In section 4, we will give some numerical simulations to show our theoretical results and compare with the nonlinear difference scheme in [14]. The paper ends with a brief conclusion in Section 5.

2 The difference scheme

In this section, we will use the method of reduction of order to establish a difference scheme for the problem (1.1), and give the truncation errors in detail.

2.1 Notations

Before establishing the difference scheme, we introduce the notations used in this article.

Take two positive integers m and n. Let $h = L/m$, $x_j = jh$, $0 \leqslant j \leqslant m$; $\tau = T/n$, $t_k = k\tau$, $0 \leqslant k \leqslant n$; $\Omega_h = \{x_j \,|\, 0 \leqslant j \leqslant m\}$, $\Omega_\tau = \{t_k \,|\, 0 \leqslant k \leqslant n\}$.

Let

$$\begin{aligned} &\mathcal{U}_h = \{v \,|\, v = \{v_j \,|\, 0 \leqslant j \leqslant m\} \text{ is the grid function on } \Omega_h\}, \\ &\mathring{\mathcal{U}}_h = \{v \,|\, v \in \mathcal{U}_h \text{ and } v_0 = v_m = 0\}. \end{aligned}$$

For any $u, v \in \mathcal{U}_h$, we introduce the following notations:

$$\begin{aligned} &\delta_x v_{j+\frac{1}{2}} = \frac{1}{h}(v_{j+1} - v_j), \quad \delta_x^2 v_j = \frac{1}{h^2}(v_{j+1} - 2v_j + v_{j-1}), \\ &\Delta_x v_j = \frac{1}{2h}(v_{j+1} - v_{j-1}), \quad \delta_x^3 v_{j+\frac{1}{2}} = \frac{1}{h^3}(v_{j+2} - 3v_{j+1} + 3v_j - v_{j-1}), \\ &\psi(u,v)_j = \frac{1}{3}[u_j \Delta_x v_j + \Delta_x (uv)_j], \\ &(u,v) = h\Big(\frac{1}{2}u_0 v_0 + \sum_{j=1}^{m-1} u_j v_j + \frac{1}{2}u_m v_m\Big), \quad \|u\| = \sqrt{(u,u)}, \\ &(\delta_x u, \delta_x v) = h\sum_{j=1}^{m} \big(\delta_x u_{j-\frac{1}{2}}\big)\big(\delta_x v_{j-\frac{1}{2}}\big), \qquad |u|_1 = \sqrt{(\delta_x u, \delta_x u)}. \end{aligned}$$

Let

$$\mathcal{S}_\tau = \{w \,|\, w = (w^0, w^1, \cdots, w^n) \text{ is the grid function on } \Omega_\tau\}.$$

For any $w \in \mathcal{S}_\tau$, we introduce the following notations:

$$\begin{aligned} &w^{k+\frac{1}{2}} = \frac{1}{2}(w^k + w^{k+1}), \qquad w^{\bar{k}} = \frac{1}{2}(w^{k+1} + w^{k-1}), \\ &\delta_t w^{k+\frac{1}{2}} = \frac{1}{\tau}(w^{k+1} - w^k), \quad \Delta_t w^k = \frac{1}{2\tau}(w^{k+1} - w^{k-1}). \end{aligned}$$

It is easy to know that

$$\delta_x^3 v_{j+\frac{1}{2}} = \delta_x^2(\delta_x v_{j+\frac{1}{2}}), \quad \Delta_t w^k = \frac{1}{2}(\delta_t w^{k+\frac{1}{2}} + \delta_t w^{k-\frac{1}{2}}).$$

2.2 Derivation of the difference scheme

Next, we construct the difference scheme by the method of reduction of order.

Let

$$v = u_x, \quad 0 \leqslant x \leqslant L, \; 0 \leqslant t \leqslant T.$$

Then the problem (1.1) equals to

$$\begin{cases} u_t + \gamma u u_x + v_{xx} = 0, \quad 0 < x < L, \; 0 < t \leqslant T, & \text{(2.1a)} \\ v = u_x, \quad 0 < x < L, \; 0 < t \leqslant T, & \text{(2.1b)} \\ u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant L, & \text{(2.1c)} \\ u(0,t) = 0, \quad u(L,t) = 0, \quad 0 < t \leqslant T, & \text{(2.1d)} \\ v(L,t) = 0, \quad 0 \leqslant t \leqslant T. & \text{(2.1e)} \end{cases}$$

Denote

$$U_j^k = u(x_j, t_k), \quad V_j^k = v(x_j, t_k), \quad 0 \leqslant j \leqslant m, \; 0 \leqslant k \leqslant n.$$

Considering equation (2.1a) at the points $(x_j, t_{\frac{1}{2}})$ and using the Taylor expansion, we have

$$\delta_t U_j^{\frac{1}{2}} + \gamma\psi\big(U^0, U^{\frac{1}{2}}\big)_j + \delta_x^2 V_j^{\frac{1}{2}} = P_j^0, \quad 1 \leqslant j \leqslant m-1. \tag{2.2}$$

Considering equation (2.1a) at the points (x_j, t_k) and using the Taylor expansion, we have

$$\Delta_t U_j^k + \gamma\psi(U^k, U^{\bar{k}})_j + \delta_x^2 V_j^{\bar{k}} = P_j^k, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{2.3}$$

There exists a constant $c_1 > 0$ such that

$$|P_j^0| \leqslant c_1(\tau + h^2), \quad |P_j^k| \leqslant c_1(\tau^2 + h^2), \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1. \tag{2.4}$$

Considering equation (2.1b) at the points $(x_{j+\frac{1}{2}}, t_k)$ and using the Taylor expansion, we get

$$V_{j+\frac{1}{2}}^k = \delta_x U_{j+\frac{1}{2}}^k + Q_j^k, \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n. \tag{2.5}$$

Noticing the initial and boundary value conditions (2.1c)-(2.1e), we have

$$\begin{cases} U_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m, & (2.6)\\ U_0^k = 0, \quad U_m^k = 0, \quad 1 \leqslant k \leqslant n, & (2.7)\\ V_m^k = 0, \quad 0 \leqslant k \leqslant n. & (2.8) \end{cases}$$

Omitting the small terms in (2.2), (2.3) and (2.5) and combining with (2.6)-(2.8), we construct a three-level linearized difference scheme for problem (2.1) as follows

$$\begin{cases} \delta_t u_j^{\frac{1}{2}} + \gamma\psi(u^0, u^{\frac{1}{2}})_j + \delta_x^2 v_j^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-1, & (2.9a)\\ \Delta_t u_j^k + \gamma\psi(u^k, u^{\bar{k}})_j + \delta_x^2 v_j^{\bar{k}} = 0, \quad 1 \leqslant j \leqslant m-1,\ 1 \leqslant k \leqslant n-1, & (2.9b)\\ v_{j+\frac{1}{2}}^k = \delta_x u_{j+\frac{1}{2}}^k, \quad 0 \leqslant j \leqslant m-1,\ 0 \leqslant k \leqslant n, & (2.9c)\\ u_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m, & (2.9d)\\ u_0^k = 0, \quad u_m^k = 0, \quad 1 \leqslant k \leqslant n, & (2.9e)\\ v_m^k = 0, \quad 0 \leqslant k \leqslant n. & (2.9f) \end{cases}$$

For the discrete errors Q_j^k in (2.5), we have the following results.

Lemma 2.1. [14] *Denote*

$$S_j^k = \sum_{l=j+1}^{m-1} (-1)^{l-1-j} \delta_x Q_{l-\frac{1}{2}}^k, \quad 0 \leqslant j \leqslant m-2,\ 0 \leqslant k \leqslant n.$$

There exists a constant $c_2 > 0$ such that, for any $0 \leqslant k \leqslant n$,

$$\begin{aligned} &\big|Q_j^k\big| \leqslant c_2 h^2, \quad 0 \leqslant j \leqslant m-1,\\ &\big|S_{m-2}^k\big| \leqslant c_2 h^3,\\ &\big|S_j^k\big| \leqslant c_2 h^2, \quad 0 \leqslant j \leqslant m-2,\\ &\big|\delta_x S_{j+\frac{1}{2}}^k\big| \leqslant c_2 h^2, \quad 0 \leqslant j \leqslant m-3. \end{aligned}$$

2.3 Calculation of the difference scheme

For ease of calculation, we make a separation of variables for the system of equations (2.9).

Theorem 2.2. *The difference scheme (2.9) is equivalent to the system of equations*

$$\begin{cases} \delta_t u_{j+\frac{1}{2}}^{\frac{1}{2}} + \dfrac{\gamma}{2}\big[\psi(u^0, u^{\frac{1}{2}})_j + \psi(u^0, u^{\frac{1}{2}})_{j+1}\big] + \delta_x^3 u_{j+\frac{1}{2}}^{\frac{1}{2}} = 0, \quad 1 \leqslant j \leqslant m-2, & (2.10a)\\ \delta_t u_{m-1}^{\frac{1}{2}} + \gamma\psi(u^0, u^{\frac{1}{2}})_{m-1} + \dfrac{2}{h^2}\big(\delta_x u_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}}\big) = 0, & (2.10b)\\ \Delta_t u_{j+\frac{1}{2}}^{k} + \dfrac{\gamma}{2}\big[\psi(u^k, u^{\bar{k}})_j + \psi(u^k, u^{\bar{k}})_{j+1}\big] + \delta_x^3 u_{j+\frac{1}{2}}^{\bar{k}} = 0, & \\ \qquad\qquad 1 \leqslant j \leqslant m-2,\ 1 \leqslant k \leqslant n-1, & (2.10c)\\ \Delta_t u_{m-1}^{k} + \gamma\psi(u^k, u^{\bar{k}})_{m-1} + \dfrac{2}{h^2}\big(\delta_x u_{m-\frac{3}{2}}^{\bar{k}} - 3\delta_x u_{m-\frac{1}{2}}^{\bar{k}}\big) = 0, \quad 1 \leqslant k \leqslant n-1, & (2.10d)\\ u_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m, & (2.10e)\\ u_0^k = 0, \quad u_m^k = 0, \quad 1 \leqslant k \leqslant n & (2.10f) \end{cases}$$

and

$$\begin{cases} v_m^k = 0, \quad 0 \leqslant k \leqslant n, & (2.11a)\\ v_j^k = 2\delta_x u_{j+\frac{1}{2}}^k - v_{j+1}^k, \quad j = m-1, m-2, \cdots, 0,\ 0 \leqslant k \leqslant n. & (2.11b) \end{cases}$$

Proof. (I) From (2.9) to (2.10) and (2.11): At first, we notice that (2.9d) and (2.9e) are just (2.10e) and (2.10f), and (2.9f) is (2.11a). In addition, (2.9c) equals to (2.11b), and (2.9a) equals to

$$\begin{cases} \delta_t u_{j+\frac{1}{2}}^{\frac{1}{2}} + \dfrac{\gamma}{2}\big[\psi(u^0, u^{\frac{1}{2}})_j + \psi(u^0, u^{\frac{1}{2}})_{j+1}\big] + \delta_x^2 \dfrac{v_j^{\frac{1}{2}} + v_{j+1}^{\frac{1}{2}}}{2} = 0, \quad 1 \leqslant j \leqslant m-2, & (2.12a)\\ \delta_t u_{m-1}^{\frac{1}{2}} + \gamma\psi(u^0, u^{\frac{1}{2}})_{m-1} + \delta_x^2 v_{m-1}^{\frac{1}{2}} = 0. & (2.12b) \end{cases}$$

From (2.9c), we have

$$v_{j+\frac{1}{2}}^{\frac{1}{2}} = \delta_x u_{j+\frac{1}{2}}^{\frac{1}{2}}, \quad 0 \leqslant j \leqslant m-1. \tag{2.13}$$

Substituting the above equality into (2.12a), we can get (2.10a).

Similarly, from (2.9f), we have

$$v_m^{\frac{1}{2}} = 0.$$

Then we can get

$$\delta_x^2 v_{m-1}^{\frac{1}{2}} = \frac{1}{h^2}\big(v_{m-2}^{\frac{1}{2}} - 2v_{m-1}^{\frac{1}{2}} + v_m^{\frac{1}{2}}\big) = \frac{2}{h^2}\big(v_{m-\frac{3}{2}}^{\frac{1}{2}} - 3v_{m-\frac{1}{2}}^{\frac{1}{2}}\big).$$

Substituting the above equality into (2.12b) and using (2.13), we can easily get (2.10b).

By the same token, one can obtain (2.10c) and (2.10d) from (2.9b)-(2.9c) and (2.9f).

(II) From (2.10) and (2.11) to (2.9): It is easy to know that (2.10e) and (2.10f) are (2.9d) and (2.9e), and (2.11a) is (2.9f). From (2.11), we can get (2.9c) and

$$\begin{cases} v_m^{\frac{1}{2}} = 0, & (2.14a)\\ v_{j+\frac{1}{2}}^{\frac{1}{2}} = \delta_x u_{j+\frac{1}{2}}^{\frac{1}{2}}, \quad 0 \leqslant j \leqslant m-1. & (2.14b) \end{cases}$$

From (2.14), we have

$$\frac{2}{h^2}\big(\delta_x u_{m-\frac{3}{2}}^{\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{\frac{1}{2}}\big) = \frac{2}{h^2}\big(v_{m-\frac{3}{2}}^{\frac{1}{2}} - 3v_{m-\frac{1}{2}}^{\frac{1}{2}}\big) = \frac{1}{h^2}\big(v_{m-2}^{\frac{1}{2}} - 2v_{m-1}^{\frac{1}{2}} - 3v_m^{\frac{1}{2}}\big) = \delta_x^2 v_{m-1}^{\frac{1}{2}}.$$

Substituting the above equation into (2.10b), we can get

$$\delta_t u_{m-1}^{\frac{1}{2}} + \gamma\psi(u^0, u^{\frac{1}{2}})_{m-1} + \delta_x^2 v_{m-1}^{\frac{1}{2}} = 0, \tag{2.15}$$

which is (2.9a) for $j=m-1$. Substituting (2.14b) into (2.10a), we have

$$\delta_t u_{j+\frac{1}{2}}^{\frac{1}{2}}+\frac{\gamma}{2}\big[\psi(u^0,u^{\frac{1}{2}})_j+\psi(u^0,u^{\frac{1}{2}})_{j+1}\big]+\delta_x^2\frac{v_j^{\frac{1}{2}}+v_{j+1}^{\frac{1}{2}}}{2}=0,\quad 1\leqslant j\leqslant m-2,$$

which is

$$\frac{1}{2}\Big\{\Big[\delta_t u_{j+1}^{\frac{1}{2}}+\gamma\psi(u^0,u^{\frac{1}{2}})_{j+1}+\delta_x^2 v_{j+1}^{\frac{1}{2}}\Big]+\Big[\delta_t u_j^{\frac{1}{2}}+\gamma\psi(u^0,u^{\frac{1}{2}})_j+\delta_x^2 v_j^{\frac{1}{2}}\Big]\Big\}=0,\quad 1\leqslant j\leqslant m-2.$$

Combining with the above equation and (2.15), it is easy to get that

$$\delta_t u_j^{\frac{1}{2}}+\gamma\psi(u^0,u^{\frac{1}{2}})_j+\delta_x^2 v_j^{\frac{1}{2}}=0,\quad j=m-2,m-3,\cdots,1,$$

which is (2.9a) for $1\leqslant j\leqslant m-2$.

By the same token, one can obtain (2.9b) from (2.10c)-(2.10d) and (2.11).

The proof is completed. □

Difference scheme (2.10) contains only the unknown quantity $\{u_j^k\}$. It is easier to calculate u from (2.10) than from (2.9). Next, we describe in detail how to solve difference scheme (2.10).

Let

$$u^k=\{u_0^k,u_1^k,\cdots,u_m^k\}.$$

From (2.10e), we can get u^0. Then we compute u^1 by (2.10a)-(2.10b) and (2.10f). Let $w=u^{\frac{1}{2}}$. If we get w, we can get u^1 by the expression

$$u^1=2w-u^0.$$

From (2.10a)-(2.10b) and (2.10f), we can get the system of equations about w:

$$\begin{cases}\dfrac{2}{\tau}\big(w_{j+\frac{1}{2}}-u_{j+\frac{1}{2}}^0\big)+\dfrac{\gamma}{2}\big[\psi(u^0,w)_j+\psi(u^0,w)_{j+1}\big]+\delta_x^3 w_{j+\frac{1}{2}}=0,\quad 1\leqslant j\leqslant m-2,\\ \dfrac{2}{\tau}\big(w_{m-1}-u_{m-1}^0\big)+\gamma\psi(u^0,w)_{m-1}+\dfrac{2}{h^2}\big(\delta_x w_{m-\frac{3}{2}}-3\delta_x w_{m-\frac{1}{2}}\big)=0,\\ w_0=0,\quad w_m=0.\end{cases}$$

The value of w can be obtained by solving the above system of quadratic diagonal linear equations using sweep method alike.

Assuming we already know u^k and u^{k-1}, we solve for the value of u^{k+1} below. Let $w=u^{\bar{k}}$. If we get w, we can get u^{k+1} by the expression

$$u^{k+1}=2w-u^{k-1}.$$

From (2.10c)-(2.10d) and (2.10f), we can get the system of equations about w:

$$\begin{cases}\dfrac{2}{\tau}\big(w_{j+\frac{1}{2}}-u_{j+\frac{1}{2}}^k\big)+\dfrac{\gamma}{2}\big[\psi(u^k,w)_j+\psi(u^k,w)_{j+1}\big]+\delta_x^3 w_{j+\frac{1}{2}}=0,\quad 1\leqslant j\leqslant m-2,\\ \dfrac{2}{\tau}\big(w_{m-1}-u_{m-1}^k\big)+\gamma\psi(u^k,w)_{m-1}+\dfrac{2}{h^2}\big(\delta_x w_{m-\frac{3}{2}}-3\delta_x w_{m-\frac{1}{2}}\big)=0,\\ w_0=0,\quad w_m=0.\end{cases}$$

The value of w can be obtained by solving the above system of quadratic diagonal linear equations using sweep method alike.

Furthermore, it is easy to see from Theorem 2.2 that doing an analysis on (2.9) is equivalent to analyzing (2.10).

3 Theoretical analysis

In this section, we will analyze the conservation, unique solvability and convergence of difference scheme (2.9).

3.1 Conservation and unique solvability

First, we give several lemmas that will be used later.

Lemma 3.1. [12] *$\forall u \in \mathcal{U}_h$ and $\forall v \in \mathring{\mathcal{U}}_h$, we have*

$$\big(\psi(u,v),v\big)=0.$$

Lemma 3.2. [14] *Let $v \in \mathcal{U}_h$ and $u \in \mathring{\mathcal{U}}_h$ satisfy*

$$v_m=0,\quad v_{j+\frac{1}{2}}=\delta_x u_{j+\frac{1}{2}},\quad 0\leqslant j\leqslant m-1.$$

We have

$$(\delta_x^2 v,u)=\frac{1}{2}\big(v_0\big)^2.$$

For the continuous problem (1.1), the following conservation exists.

Theorem 3.3. [10] *Suppose $u(x,t)$ be the solution of (1.1). Denote*

$$\tilde{E}(t)=\int_0^L u^2(x,t)\mathrm{d}x+\int_0^t u_x^2(0,s)\mathrm{d}s,$$

then we have

$$\tilde{E}(t)=\tilde{E}(0),\quad 0<t\leqslant T.$$

Similarly, difference scheme (2.9) has an invariant as follows.

Theorem 3.4. *Suppose $\{u_j^k, v_j^k \,|\, 0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ be the solution of (2.9). Denote*

$$E^k=\frac{\|u^k\|^2+\|u^{k-1}\|^2}{2}+\tau\Big[\sum_{l=1}^{k-1}(v_0^{\bar{l}})^2+\frac{1}{2}(v_0^{\frac{1}{2}})^2\Big],\quad 1\leqslant k\leqslant n.$$

Then we have

$$E^k=\|u^0\|^2,\quad 1\leqslant k\leqslant n.$$

Proof. (I) Taking an inner product of (2.9a) with $u^{\frac{1}{2}}$, we have

$$\big(\delta_t u^{\frac{1}{2}},u^{\frac{1}{2}}\big)+\gamma\big(\psi(u^0,u^{\frac{1}{2}}),u^{\frac{1}{2}}\big)+\big(\delta_x^2 v^{\frac{1}{2}},u^{\frac{1}{2}}\big)=0.$$

Together with Lemma 3.1 and Lemma 3.2, we can get

$$\frac{1}{\tau}\cdot\frac{\|u^1\|^2-\|u^0\|^2}{2}+\frac{1}{2}\big(v_0^{\frac{1}{2}}\big)^2=0.$$

That is

$$\frac{\|u^1\|^2+\|u^0\|^2}{2}+\frac{\tau}{2}\big(v_0^{\frac{1}{2}}\big)^2=\|u^0\|^2. \tag{3.1}$$

(II) Taking an inner product of (2.9b) with $2u^{\bar{k}}$, we have

$$2\big(\Delta_t u^k,u^{\bar{k}}\big)+2\gamma\big(\psi(u^k,u^{\bar{k}}),u^{\bar{k}}\big)+2\big(\delta_x^2 v^{\bar{k}},u^{\bar{k}}\big)=0,\quad 1\leqslant k\leqslant n-1.$$

Together with Lemma 3.1 and Lemma 3.2, we can get

$$\frac{1}{\tau}\Big(\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}-\frac{\|u^k\|^2+\|u^{k-1}\|^2}{2}\Big)+\big(v_0^{\bar{k}}\big)^2=0,\quad 1\leqslant k\leqslant n-1.$$

That is

$$\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}+\tau\sum_{l=1}^{k}\big(v_0^{\bar{l}}\big)^2=\frac{\|u^1\|^2+\|u^0\|^2}{2},\quad 1\leqslant k\leqslant n-1.$$

Combining with (3.1) and the above equality, we have

$$\frac{\|u^{k+1}\|^2+\|u^k\|^2}{2}+\tau\Big[\sum_{l=1}^{k}(v_0^{\bar{l}})^2+\frac{1}{2}(v_0^{\frac{1}{2}})^2\Big]=\|u^0\|^2,\quad 0\leqslant k\leqslant n-1.$$

This completes this proof. ☐

Next, we prove the unique solvability.

Theorem 3.5. *The difference scheme* (2.9) *has a unique solution.*

Proof. We can get u^0 from (2.9d) and v^0 from (2.9c) and (2.9f).

From (2.9a), (2.9c) and (2.9e)-(2.9f), we can get u^1 and v^1. Consider the system of homogeneous equations

$$\begin{cases}\dfrac{1}{\tau}u_j^1+\dfrac{\gamma}{2}\psi(u^0,u^1)_j+\dfrac{1}{2}\delta_x^2v_j^1=0,\quad 1\leqslant j\leqslant m-1, & (3.2a)\\ v_{j+\frac{1}{2}}^1=\delta_xu_{j+\frac{1}{2}}^1,\quad 0\leqslant j\leqslant m-1, & (3.2b)\\ u_0^1=0,\quad u_m^1=0, & (3.2c)\\ v_m^1=0. & (3.2d)\end{cases}$$

Taking an inner product of (3.2a) with u^1, we have

$$\frac{1}{\tau}\|u^1\|^2+\frac{\gamma}{2}\big(\psi(u^0,u^1),u^1\big)+\frac{1}{2}\big(\delta_x^2v^1,u^1\big)=0.$$

Together with Lemma 3.1 and Lemma 3.2, we get

$$\frac{1}{\tau}\|u^1\|^2+\frac{1}{2}(v_0^1)^2=0.$$

Then we have

$$\|u^1\|=0,$$

which follows

$$u_j^1=0,\quad 0\leqslant j\leqslant m.$$

From (3.2b) and (3.2d), we can get

$$v_j^1=0,\quad 0\leqslant j\leqslant m.$$

That is, (2.9a), (2.9c) and (2.9e)-(2.9f) have the unique solution u^1 and v^1.

Suppose u^k,u^{k-1} and v^k,v^{k-1} are known. From (2.9b)-(2.9c) and (2.9e)-(2.9f), we can get u^{k+1} and v^{k+1}. Consider the system of homogeneous equations

$$\begin{cases}\dfrac{1}{2\tau}u_j^{k+1}+\dfrac{\gamma}{2}\psi(u^k,u^{k+1})_j+\dfrac{1}{2}\delta_x^2v_j^{k+1}=0,\quad 1\leqslant j\leqslant m-1, & (3.3a)\\ v_{j+\frac{1}{2}}^{k+1}=\delta_xu_{j+\frac{1}{2}}^{k+1},\quad 0\leqslant j\leqslant m-1, & (3.3b)\\ u_0^{k+1}=0,\quad u_m^{k+1}=0, & (3.3c)\\ v_m^{k+1}=0. & (3.3d)\end{cases}$$

Taking an inner product of (3.3a) with $2u^{k+1}$, we have

$$\frac{1}{\tau}\|u^{k+1}\|^2 + \gamma\big(\psi(u^k,u^{k+1}),u^{k+1}\big) + \big(\delta_x^2 v^{k+1},u^{k+1}\big) = 0.$$

Together with Lemma 3.1 and Lemma 3.2, we can get

$$\frac{1}{\tau}\|u^{k+1}\|^2 + (v_0^{k+1})^2 = 0.$$

Then we have

$$\|u^{k+1}\| = 0,$$

which follows

$$u_j^{k+1} = 0, \quad 0 \leqslant j \leqslant m.$$

From (3.3b) and (3.3d), we can get

$$v_j^{k+1} = 0, \quad 0 \leqslant j \leqslant m.$$

That is, (2.9b)-(2.9c) and (2.9e)-(2.9f) have the unique solution u^{k+1} and v^{k+1}.

This completes the proof. □

3.2 Convergence

Assume that $\{u(x,t),v(x,t)\,|\,(x,t)\in[0,L]\times[0,T]\}$ is the solution of problem (2.1) and $\{u_j^k,v_j^k \mid 0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$ is the solution of difference scheme (2.9) respectively. Denote

$$e_j^k = u(x_j,t_k)-u_j^k, \quad f_j^k = v(x_j,t_k)-v_j^k, \quad 0\leqslant j\leqslant m, 0\leqslant k\leqslant n.$$

Substracting (2.9) from (2.2), (2.3), (2.5)-(2.8), we can get the system of error equations

$$\begin{cases} \delta_t e_j^{\frac{1}{2}} + \gamma\psi\big(u^0,e^{\frac{1}{2}}\big)_j + \delta_x^2 f_j^{\frac{1}{2}} = P_j^0, \quad 1\leqslant j\leqslant m-1, & \text{(3.4a)}\\ \Delta_t e_j^k + \gamma\Big[\psi\big(U^k,U^{\bar{k}}\big)_j - \psi\big(u^k,u^{\bar{k}}\big)_j\Big] + \delta_x^2 f_j^{\bar{k}} = P_j^k, \quad 1\leqslant j\leqslant m-1,\ 1\leqslant k\leqslant n-1, & \text{(3.4b)}\\ f_{j+\frac{1}{2}}^k = \delta_x e_{j+\frac{1}{2}}^k + Q_j^k, \quad 0\leqslant j\leqslant m-1,\ 0\leqslant k\leqslant n, & \text{(3.4c)}\\ e_j^0 = 0, \quad 0\leqslant j\leqslant m, & \text{(3.4d)}\\ e_0^k = 0, \quad e_m^k = 0, \quad 1\leqslant k\leqslant n, & \text{(3.4e)}\\ f_m^k = 0, \quad 0\leqslant k\leqslant n. & \text{(3.4f)} \end{cases}$$

Before providing the convergence result, we present the following two lemmas.

Lemma 3.6. *From the proof of Theorem 4.3 in [10], it follows that the following equality holds*

$$\begin{aligned} &\Big(\psi(U^k,U^{\bar{k}})-\psi(u^k,u^{\bar{k}}),e^{\bar{k}}\Big)\\ =&\frac{1}{12}\Big[\sum_{j=1}^{m-2}U_j^{k+\frac{1}{2}}(e_j^{k+1}e_{j+1}^k - e_j^k e_{j+1}^{k+1}) - \sum_{j=1}^{m-2}U_j^{k-\frac{1}{2}}(e_j^k e_{j+1}^{k-1} - e_j^{k-1}e_{j+1}^k)\Big]\\ &+\frac{h}{3}\sum_{j=1}^{m-1}(\Delta_x U_j^{\bar{k}})e_j^k e_j^{\bar{k}} + \frac{h}{6}\sum_{j=1}^{m-2}(\delta_x U_{j+\frac{1}{2}}^{\bar{k}})e_{j+1}^k e_j^{\bar{k}}\\ &+\frac{1}{12}\sum_{j=1}^{m-2}\Big[(U_i^{\bar{k}}-U_i^{k+\frac{1}{2}})(e_j^{k+1}e_{j+1}^k - e_j^k e_{j+1}^{k+1}) + (U_j^{k-\frac{1}{2}}-U_j^{\bar{k}})(e_j^k e_{j+1}^{k-1} - e_j^{k-1}e_{j+1}^k)\Big]. \end{aligned}$$

Lemma 3.7. *From the proof of Theorem 4.1 in [14], it follows that the following equality holds*

$$(\delta_x^2 f^k, e^k) = \frac{1}{2}(f_0^k)^2 - f_0^k Q_0^k + h f_0^k S_0^k - 2h\sum_{j=0}^{m-2} S_j^k Q_j^k + 2h\sum_{j=1}^{m-2} e_j^k \delta_x S_{j-\frac{1}{2}}^k - 2e_{m-1}^k S_{m-2}^k.$$

Denote

$$c_0 = \max_{(x,t)\in[0,L]\times[0,T]}\{|u(x,t)|, |u_x(x,t)|\}.$$

Theorem 3.8. *Let*

$$\lambda = \frac{c_0|\gamma|\tau}{3h}, \quad c_3 = \frac{|\gamma|c_0 + 6}{2(1-\lambda)},$$

$$c_4 = \exp\{\frac{3c_3T}{2}\}\sqrt{\frac{4c_1^2L + 8c_2^2(1+3L)}{1-\lambda} + \frac{c_1^2L + 2c_2^2 + 8c_2^2L}{c_3(1-\lambda)}},$$

If $\lambda < 1$, we have

$$\|e^k\| \leqslant c_4(\tau^2 + h^2), \quad 0 \leqslant k \leqslant n.$$

Proof. (I) It is easy to know from (3.4d) that

$$\|e^0\| = 0. \tag{3.5}$$

Taking an inner product of (3.4a) with $2e^{\frac{1}{2}}$, we have

$$2\big(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}\big) + 2\gamma\big(\psi(u^0, e^{\frac{1}{2}}), e^{\frac{1}{2}}\big) + 2\big(\delta_x^2 f^{\frac{1}{2}}, e^{\frac{1}{2}}\big) = 2\big(P^0, e^{\frac{1}{2}}\big).$$

That is

$$\frac{1}{\tau}\|e^1\|^2 + 2\big(\delta_x^2 f^{\frac{1}{2}}, e^{\frac{1}{2}}\big) = \big(P^0, e^1\big). \tag{3.6}$$

Combining with (3.4c) and Lemma 3.7, we can get

$$\begin{aligned}
\big(\delta_x^2 f^{\frac{1}{2}}, e^{\frac{1}{2}}\big) &= \frac{1}{2}(f_0^{\frac{1}{2}})^2 - f_0^{\frac{1}{2}} Q_0^{\frac{1}{2}} + h f_0^{\frac{1}{2}} S_0^{\frac{1}{2}} - 2h\sum_{j=0}^{m-2} S_j^{\frac{1}{2}} Q_j^{\frac{1}{2}} + 2h\sum_{j=1}^{m-2} e_j^{\frac{1}{2}} \delta_x S_{j-\frac{1}{2}}^{\frac{1}{2}} - 2e_{m-1}^{\frac{1}{2}} S_{m-2}^{\frac{1}{2}} \\
&= \frac{1}{2}(f_0^{\frac{1}{2}})^2 - f_0^{\frac{1}{2}} Q_0^{\frac{1}{2}} + h f_0^{\frac{1}{2}} S_0^{\frac{1}{2}} - 2h\sum_{j=0}^{m-2} S_j^{\frac{1}{2}} Q_j^{\frac{1}{2}} + h\sum_{j=1}^{m-2} e_j^{1} \delta_x S_{j-\frac{1}{2}}^{\frac{1}{2}} - e_{m-1}^{1} S_{m-2}^{\frac{1}{2}}.
\end{aligned}$$

Substituting the above equality into (3.6), combining with Lemma 2.1 and (2.4), we get

$$\begin{aligned}
&\frac{1}{\tau}\|e^1\|^2 + (f_0^{\frac{1}{2}})^2 \\
&= \big(P^0, e^1\big) + 2f_0^{\frac{1}{2}} Q_0^{\frac{1}{2}} - 2h f_0^{\frac{1}{2}} S_0^{\frac{1}{2}} + 4h\sum_{j=0}^{m-2} S_j^{\frac{1}{2}} Q_j^{\frac{1}{2}} - 2h\sum_{j=1}^{m-2} e_j^{1} \delta_x S_{j-\frac{1}{2}}^{\frac{1}{2}} + 2e_{m-1}^{1} S_{m-2}^{\frac{1}{2}} \\
&\leqslant \frac{\tau}{2}\|P^0\|^2 + \frac{1}{2\tau}\|e^1\|^2 + \frac{1}{2}(f_0^{\frac{1}{2}})^2 + 2(Q_0^{\frac{1}{2}})^2 + \frac{1}{2}(f_0^{\frac{1}{2}})^2 + 2h^2(S_0^{\frac{1}{2}})^2 \\
&\quad + 4h\sum_{j=0}^{m-2} |S_j^{\frac{1}{2}}|\cdot|Q_j^{\frac{1}{2}}| + \frac{h}{2}\sum_{j=1}^{m-2}(e_j^1)^2 + 2h\sum_{j=1}^{m-2}(\delta_x S_{j-\frac{1}{2}}^{\frac{1}{2}})^2 + \frac{h}{2}(e_{m-1}^1)^2 + \frac{2}{h}(S_{m-2}^{\frac{1}{2}})^2 \\
&\leqslant \frac{\tau}{2}c_1^2L(\tau + h^2)^2 + \frac{1}{2\tau}\|e^1\|^2 + \frac{1}{2}\|e^1\|^2 + (f_0^{\frac{1}{2}})^2 \\
&\quad + 2c_2^2h^4 + 2c_2^2h^6 + 4c_2^2Lh^4 + 2h\sum_{j=1}^{m-2} c_2^2h^4 + 2c_2^2h^5 \\
&\leqslant \frac{1}{2\tau}\|e^1\|^2 + \frac{1}{2}\|e^1\|^2 + \big(f_0^{\frac{1}{2}}\big)^2 + \frac{\tau}{2}c_1^2L(\tau + h^2)^2 + 2c_2^2(1+3L)h^4.
\end{aligned}$$

That is

$$(1-\tau)\|e^1\|^2 \leqslant c_1^2 L(\tau^2+h^2)^2 + 4c_2^2(1+3L)\tau h^4.$$

When $\tau \leqslant \frac{1}{2}$, we can get

$$\|e^1\|^2 \leqslant 2c_1^2 L(\tau^2+h^2)^2 + 4c_2^2(1+3L)h^4 \leqslant \big[2c_1^2 L + 4c_2^2(1+3L)\big](\tau^2+h^2)^2. \tag{3.7}$$

(II) Taking an inner product of (3.4b) with $2e^{\bar{k}}$, we have

$$2(\Delta_t e^k, e^{\bar{k}}) + 2\gamma\big(\psi(U^k, U^{\bar{k}}) - \psi(u^k, e^{\bar{k}}), e^{\bar{k}}\big) + 2\big(\delta_x^2 f^{\bar{k}}, e^{\bar{k}}\big) = 2\big(P^k, e^{\bar{k}}\big), \quad 1 \leqslant k \leqslant n-1.$$

That is

$$\begin{aligned}
&\frac{1}{\tau}\Big(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} - \frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\Big) + 2\big(\delta_x^2 f^{\bar{k}}, e^{\bar{k}}\big) \\
&= 2\big(P^k, e^{\bar{k}}\big) - 2\gamma\big(\psi(U^k, U^{\bar{k}}) - \psi(u^k, e^{\bar{k}}), e^{\bar{k}}\big), \quad 1 \leqslant k \leqslant n-1.
\end{aligned} \tag{3.8}$$

Combining with (3.4c) and Lemma 3.7, we can get

$$(\delta_x^2 f^{\bar{k}}, e^{\bar{k}}) = \frac{1}{2}(f_0^{\bar{k}})^2 - f_0^{\bar{k}} Q_0^{\bar{k}} + h f_0^{\bar{k}} S_0^{\bar{k}} - 2h\sum_{j=0}^{m-2} S_j^{\bar{k}} Q_j^{\bar{k}} + 2h\sum_{j=1}^{m-2} e_j^{\bar{k}} \delta_x S_{j-\frac{1}{2}}^{\bar{k}} - 2e_{m-1}^{\bar{k}} S_{m-2}^{\bar{k}}.$$

Substituting the above equality into (3.8), combining with (2.4), Lemma 3.6 and Lemma 2.1, denoting

$$G^{k-\frac{1}{2}} = \frac{\gamma\tau}{6}\sum_{j=1}^{m-2} U_j^{k-\frac{1}{2}}(e_j^k e_{j+1}^{k-1} - e_j^{k-1} e_{j+1}^k),$$

then we have

$$\begin{aligned}
&\frac{1}{\tau}\Big[\Big(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} + G^{k+\frac{1}{2}}\Big) - \Big(\frac{\|e^k\|^2+\|e^{k-1}\|^2}{2} + G^{k-\frac{1}{2}}\Big)\Big] + (f_0^{\bar{k}})^2 \\
&= 2(P^k, e^{\bar{k}}) - \gamma\Big\{\frac{2h}{3}\sum_{j=1}^{m-1}(\Delta_x U_j^{\bar{k}}) e_j^k e_j^{\bar{k}} + \frac{h}{3}\sum_{j=1}^{m-2}(\delta_x U_{j+\frac{1}{2}}^{\bar{k}}) e_{j+1}^k e_j^{\bar{k}} \\
&\quad + \frac{1}{6}\sum_{j=1}^{m-2}\Big[(U_i^{\bar{k}} - U_i^{k+\frac{1}{2}})(e_j^{k+1} e_{j+1}^k - e_j^k e_{j+1}^{k+1}) + (U_j^{k-\frac{1}{2}} - U_j^{\bar{k}})(e_j^k e_{j+1}^{k-1} - e_j^{k-1} e_{j+1}^k)\Big]\Big\} \\
&\quad + 2f_0^{\bar{k}} Q_0^{\bar{k}} - 2h f_0^{\bar{k}} S_0^{\bar{k}} + 4h\sum_{j=0}^{m-2} S_j^{\bar{k}} Q_j^{\bar{k}} - 4h\sum_{j=1}^{m-2} e_j^{\bar{k}} \delta_x S_{j-\frac{1}{2}}^{\bar{k}} + 4e_{m-1}^{\bar{k}} S_{m-2}^{\bar{k}} \\
&\leqslant \|P^k\|^2 + \|e^{\bar{k}}\|^2 + |\gamma|\Big(c_0\|e^k\|\cdot\|e^{\bar{k}}\| + \frac{\tau}{3h}c_0\|e^{k+1}\|\cdot\|e^k\| + \frac{\tau}{3h}c_0\|e^k\|\cdot\|e^{k-1}\|\Big) \\
&\quad + \frac{1}{2}(f_0^{\bar{k}})^2 + 2(Q_0^{\bar{k}})^2 + \frac{1}{2}(f_0^{\bar{k}})^2 + 2h^2(S_0^{\bar{k}})^2 \\
&\quad + 4h\sum_{j=0}^{m-2}|S_j^{\bar{k}}|\cdot|Q_j^{\bar{k}}| + h\sum_{j=1}^{m-2}(e_j^{\bar{k}})^2 + 4h\sum_{j=1}^{m-2}(\delta_x S_{j-\frac{1}{2}}^{\bar{k}})^2 + h(e_{m-1}^{\bar{k}})^2 + \frac{4}{h}(S_{m-2}^{\bar{k}})^2 \\
&\leqslant c_1^2 L(\tau^2+h^2)^2 + 2\|e^{\bar{k}}\|^2 + \frac{|\gamma|c_0}{2}(\|e^k\|^2 + \|e^{\bar{k}}\|^2) \\
&\quad + \frac{c_0|\gamma|\tau}{6h}(\|e^{k+1}\|^2 + \|e^k\|^2 + \|e^k\|^2 + \|e^{k-1}\|^2) \\
&\quad + (f_0^{\bar{k}})^2 + 2c_2^2 h^4 + 2c_2^2 h^6 + 4h\sum_{j=0}^{m-2} c_2^2 h^4 + 4h\sum_{j=1}^{m-2} c_2^2 h^4 + 4c_2^2 h^5 \\
&\leqslant \Big(\frac{|\gamma|c_0}{2} + 2 + \lambda\Big)\Big(\frac{\|e^{k+1}\|^2+\|e^k\|^2}{2} + \frac{\|e^k\|^2+\|e^{k-1}\|^2}{2}\Big) \\
&\quad + (f_0^{\bar{k}})^2 + (c_1^2 L + 2c_2^2 + 8c_2^2 L)(\tau^2+h^2)^2, \quad 1 \leqslant k \leqslant n-1,
\end{aligned} \tag{3.9}$$

Let
$$E^k = \frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} + G^{k-\frac{1}{2}}.$$
It is easy to know that
$$|G^{k-\frac{1}{2}}| \leqslant \lambda \cdot \frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2}.$$
If $\lambda < 1$, we have
$$(1-\lambda)\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} \leqslant E^k \leqslant (1+\lambda)\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} \leqslant \|e^k\|^2 + \|e^{k-1}\|^2.$$
It follows from (3.9) that
$$\frac{1}{\tau}\big(E^{k+1} - E^k\big) \leqslant c_3\big(E^{k+1} + E^k\big) + (c_1^2 L + 2c_2^2 + 8c_2^2 L)(\tau^2 + h^2)^2, \quad 1 \leqslant k \leqslant n-1.$$
That is
$$(1 - c_3\tau)E^{k+1} \leq (1 + c_3\tau)E^k + (c_1^2 L + 2c_2^2 + 8c_2^2 L)\tau(\tau^2 + h^2)^2, \quad 1 \leqslant k \leqslant n-1.$$
When $c_3\tau \leqslant \frac{1}{3}$, we have
$$E^{k+1} \leqslant (1 + 3c_3\tau)E^k + \Big(\frac{3c_1^2 L}{2} + 3c_2^2 + 12c_2^2 L\Big)\tau(\tau^2 + h^2)^2, \quad 1 \leqslant k \leqslant n-1.$$
Using Gronwall inequality, we can get
$$\begin{aligned} E^k &\leqslant \exp\{3c_3(k-1)\tau\}\Big[E^1 + \Big(\frac{c_1^2 L}{2c_3} + \frac{c_2^2}{c_3} + \frac{4c_2^2 L}{c_3}\Big)(\tau^2 + h^2)^2\Big] \\ &\leqslant \exp\{3c_3 T\}\Big[\|e^1\|^2 + \|e^0\|^2 + \Big(\frac{c_1^2 L}{2c_3} + \frac{c_2^2}{c_3} + \frac{4c_2^2 L}{c_3}\Big)(\tau^2 + h^2)^2\Big], \ 1 \leqslant k \leqslant n. \end{aligned}$$
Substituting (3.5) and (3.7) into the above inequality, we can get
$$E^k \leqslant \exp\{3c_3 T\}\Big(2c_1^2 L + 4c_2^2(1 + 3L) + \frac{c_1^2 L}{2c_3} + \frac{c_2^2}{c_3} + \frac{4c_2^2 L}{c_3}\Big)(\tau^2 + h^2)^2, \quad 1 \leqslant k \leqslant n.$$
It is easy to know that
$$\frac{\|e^k\|^2 + \|e^{k-1}\|^2}{2} \leqslant \frac{c_4^2}{2}(\tau^2 + h^2)^2, \quad 1 \leqslant k \leqslant n.$$
Consequently,
$$\|e^k\| \leqslant c_4(\tau^2 + h^2), \quad 1 \leqslant k \leqslant n.$$
This completes the proof. □

4 Numerical examples

In this section, we present two numerical examples. The numerical results illustrate efficiency of the difference scheme (2.10).

In [14], we presented a two-level nonlinear difference scheme as follows:
$$\begin{cases} \delta_t u_{j+\frac{1}{2}}^{k+\frac{1}{2}} + \dfrac{\gamma}{2}\Big[\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_j + \psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{j+1}\Big] + \delta_x^3 u_{j+\frac{1}{2}}^{k+\frac{1}{2}} = 0, & \\ \qquad\qquad 1 \leqslant j \leqslant m-2, \ 0 \leqslant k \leqslant n-1, & (4.1a) \\ \delta_t u_{m-1}^{k+\frac{1}{2}} + \gamma\psi(u^{k+\frac{1}{2}}, u^{k+\frac{1}{2}})_{m-1} + \dfrac{2}{h^2}\left(\delta_x u_{m-\frac{3}{2}}^{k+\frac{1}{2}} - 3\delta_x u_{m-\frac{1}{2}}^{k+\frac{1}{2}}\right) = 0, \ 0 \leqslant k \leqslant n-1, & (4.1b) \\ u_j^0 = \varphi(x_j), \quad 0 \leqslant j \leqslant m, & (4.1c) \\ u_0^k = 0, \quad u_m^k = 0, \quad 1 \leqslant k \leqslant n, & (4.1d) \end{cases}$$

and solved it by using the Newton iterative method.

Let the numerical solution corresponding to the step size h and τ be $\{u_j^k(h,\tau)\,|\,0\leqslant j\leqslant m, 0\leqslant k\leqslant n\}$.

Denote the error by

$$E(h,\tau)=\max_{1\leqslant k\leqslant n}\left\{\sqrt{h\sum_{j=1}^{m-1}\Big(u_j^k(h,\tau)-u_{2j}^k(\frac{h}{2},\tau)\Big)^2}\right\},$$

$$F(h,\tau)=\big\|u^n(h,\tau)-u^{2n}(h,\frac{\tau}{2})\big\|.$$

When τ is sufficiently small, the spatial convergence order is defined by

$$r_h=\log_2\Big(\frac{E(2h,\tau)}{E(h,\tau)}\Big).$$

When h is sufficiently small, the temporal convergence order is defined by

$$r_\tau=\log_2\Big(\frac{F(h,2\tau)}{F(h,\tau)}\Big).$$

Example 1. *In (1.1), take $T=1,\ L=1,\ \gamma=-6,\ \varphi(x)=x(x-1)^2(x^3-2x^2+2)$. The exact solution is unknown.*

The difference scheme (2.10) will be employed to numerically solve this problem. The numerical accuracy of the difference scheme in space and in time will be verified respectively.

Taking various step size h with the sufficiently small step size $\tau=1/12800$ and varying step size τ with the sufficiently small step size $h=1/12800$, the numerical errors and convergence orders for scheme (2.10) are listed in Table 1 and Table 2. From these tables, we know that the numerical convergence orders of (2.10) can achieve $\mathrm{O}(h^2+\tau^2)$, which are in a good agreement with Theorem 3.8. Furthermore, we can see that the difference scheme (2.10) in this paper is more computationally efficient than the nonlinear difference scheme (4.1). Figure 1 indicates that the energy of the schemes (2.10) is conserved for Example 1.

Table 1: Errors and convergence orders in space of Example 1 ($\tau=1/12800$)

h	Scheme (2.10)			Scheme (4.1)		
	$E(h,\tau)$	r_h	CPU	$E(h,\tau)$	r_h	CPU
1/80	8.250008e-7		0.143s	8.249301e-7		0.416s
1/160	2.062408e-7	2.000	0.189s	2.062284e-7	2.000	0.436s
1/320	5.156014e-8	2.000	0.284s	5.155684e-8	2.000	0.631s
1/640	1.289100e-8	2.000	0.484s	1.288886e-8	2.000	1.063s
1/1280	3.215215e-9	2.003	0.943s	3.210015e-9	2.005	8.395s

Denote

$$H(h,\tau)=\max_{0\leqslant k\leqslant n}\|e^k(h,\tau)\|,$$

and

$$r_h^e=\log_2\Big(\frac{H(2h,\tau)}{H(h,\tau)}\Big),\quad r_\tau^e=\log_2\Big(\frac{H(h,2\tau)}{H(h,\tau)}\Big).$$

Table 2: Errors and convergence orders in time of Example 1 ($h = 1/12800$)

τ	Scheme (2.10)			Scheme (4.1)		
	$F(h,\tau)$	r_τ	CPU	$F(h,\tau)$	r_τ	CPU
1/80	1.410467e-2		0.086s	4.445480e-3		4.601s
1/160	4.362537e-3	1.693	0.161s	1.030215e-3	2.109	9.113s
1/320	1.004853e-3	2.118	0.316s	2.507916e-4	2.038	17.354s
1/640	2.451153e-4	2.035	0.631s	7.133256e-5	1.814	32.648s
1/1280	7.068438e-5	1.794	1.235s	1.842235e-5	1.953	63.290s

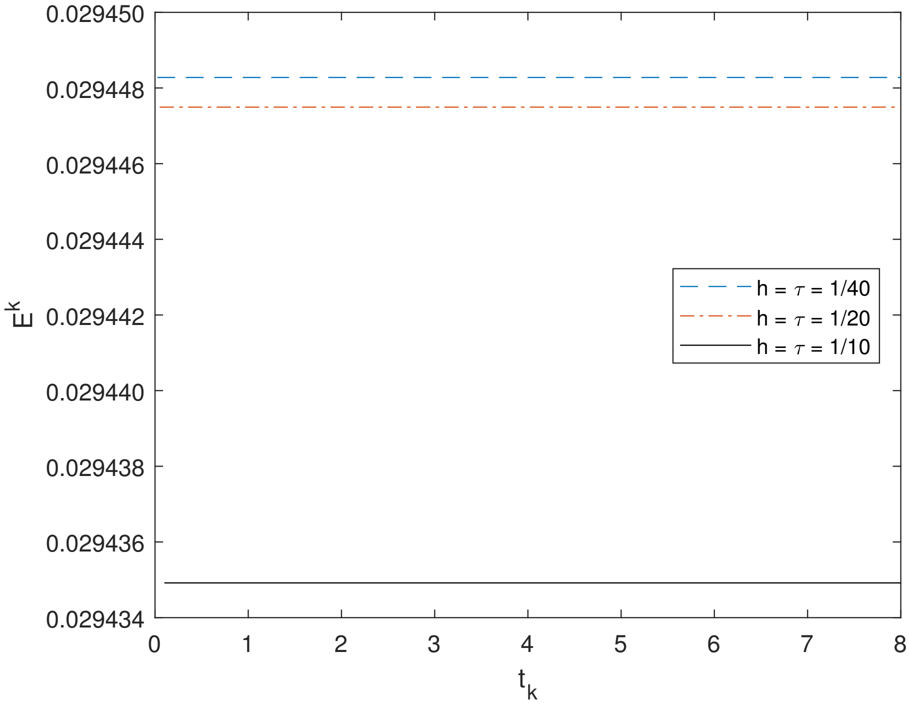

Figure 1: The energy conservation of Example 1

Example 2. [Solitonary wave [8]] *We give a numerical experiment with the exact solution of*

$$u(x,t) = 4\mathrm{sech}^2(x-4t-4), \quad 0 \leqslant x \leqslant 20, 0 \leqslant t \leqslant 1.$$

The corresponding parameter is $\gamma = 3$.

Likewise, we compute the example by difference scheme (2.10) and observe the convergence. Fix the time and space steps as 1/3200, respectively, and calculate the errors and convergence orders in the space and time directions, as shown in Table 3. It can be seen that the numerical results are accordance to the theoretical analysis. In Fig. 2, the numerical and exact solution curves are plotted for $t = 0.25, 0.5, 0.75, 1.0$, respectively. It shows that the numerical solutions are in a very good aggrement with the exact solutions.

Table 3: Errors and convergence orders of Example 2

h	τ	$H(h,\tau)$	r_h^e	h	τ	$H(h,\tau)$	r_τ^e
1/10	1/3200	1.072504e-1		1/3200	1/10	6.082414e-1	
1/20	1/3200	2.720820e-2	1.979	1/3200	1/20	1.643422e-1	1.888
1/40	1/3200	6.855457e-3	1.989	1/3200	1/40	4.195700e-2	1.970
1/80	1/3200	1.975856e-3	1.795	1/3200	1/80	1.057225e-2	1.989

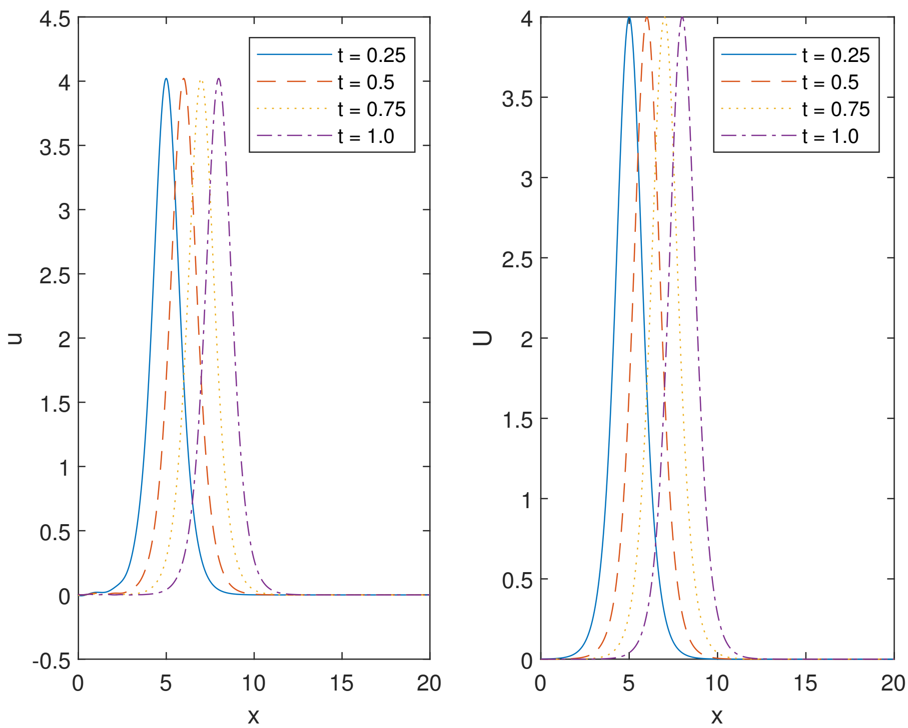

Figure 2: The curve of the numerical solutions of Example 2 ($h = \tau = 1/40$)

In Theorem 3.8, the convergence was proved under the constraint of step size ratio. Indeed, we conjucture from the numerical examples that the restriction of step size ratio may not be needed to ensure that the convergence holds, but we have not yet found a better way to prove unconditional convergence, and will continue our research in future work.

5 Conclusion

In this paper, we consider the numerical solution for the initial-boundary value problem of the KdV equation using the finite difference method. Through the method of reduction of order, we establish a three-level linearized difference scheme and show by numerical simulations that it is more computationally efficient than the two-level nonlinear difference scheme, while remaining the same convergence orders. Using energy analysis method, we prove the conservation, unique solvability and conditional convergence of the difference scheme. This difference scheme may be unconditionally convergent, which has not been proved yet. It will continue to be considered in the future.

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