Treewidth of the Line Graph of Complete and Complete Multipartite Graphs *

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Abstract

In recent papers by Grohe and Marx, the treewidth of the line graph of the complete graph is a critical example. We determine the exact treewidth of the line graph of the complete graph. By extending these techniques, we determine the exact treewidth of the line graph of a regular complete multipartite graph. For an arbitrary complete multipartite graph, we determine the treewidth of the line graph up to a lower order term.

1 Introduction

The *treewidth* $\mathrm{tw}(G)$ of a graph G is a graph invariant used to measure how "tree-like" G is. It is of particular importance in structural and algorithmic graph theory; see the surveys [1, 6]. $\mathrm{tw}(G)$ is the minimum width of a *tree decomposition* of G, which is defined as follows:

Definition A *tree decomposition* of a graph G is a pair $(T, \{A_x \subseteq V(G) : x \in V(T)\})$ such that:

- T is a tree.
- $\{A_x \subseteq V(G) : x \in V(T)\}$ is a collection of sets of vertices of G, each called a *bag*, indexed by the nodes of T.
- For all $v \in V(G)$, the nodes of T indexing the bags containing v induce a non-empty (connected) subtree of T.

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- For all $vw \in E(G)$, there exists a bag of T containing both v and w.

The *width* of a tree decomposition is the maximum size of a bag of T, minus 1. This minus 1 is added to ensure that every tree has treewidth 1. Similarly, we can define pathwidth $pw(G)$ to be the minimum width of a tree decomposition where the underlying tree is a path.

The line-graph $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$, such that two vertices of $L(G)$ are adjacent when the corresponding edges of G are incident at a vertex.

In recent papers by Marx [4] and Grohe and Marx [3], the treewidth of the line graph of the complete graph is a critical example. For a graph G, let $G^{(q)}$ denote the graph created by replacing each vertex of G with a clique of size q and replacing each edge between two vertices with all of the edges between the new two cliques. Marx [4] shows that if G has large treewidth, then $G^{(q_1)}$ contains the $L(K_n)^{(q_2)}$ as a minor (for appropriate choices of q_1 and q_2). Grohe and Marx [3] show that $\mathrm{tw}(L(K_n)) \geqslant \frac{\sqrt{2}-1}{4}n^2 + O(n)$. In this paper, we determine $\mathrm{tw}(L(K_n))$ exactly. In doing so, our minimum width tree decomposition will be a path, so this result also holds for pathwidth.

Theorem 1.

$$\mathrm{tw}(L(K_n)) = \mathrm{pw}(L(K_n)) = \begin{cases} (\frac{n-1}{2})(\frac{n-1}{2}) + n - 2 & \text{, \textit{if} } n \text{ \textit{is odd}} \\ (\frac{n-2}{2})(\frac{n}{2}) + n - 2 & \text{, \textit{if} } n \text{ \textit{is even}} \end{cases}$$

The complete multipartite graph $K_{n_1,n_2,\ldots,n_k}$ is the graph with k colour classes, of order $n_1, \ldots, n_k$ respectively, containing an edge between every pair of differently coloured vertices. We determine bounds on the treewidth of the line graph of the complete multipartite graph. Again, this is also a bound on the pathwidth.

Theorem 2. *If $k \geqslant 2$ and $n = |V(K_{n_1,\ldots,n_k})|$, then*

$$\begin{aligned} -n(k-1) + \frac{3}{4}k(k-1) - 1 + \frac{1}{2}\left(\sum_{1 \leqslant i < j \leqslant k} n_i n_j\right) \\ \leqslant \mathrm{tw}(K_{n_1,\ldots,n_k}) \leqslant \mathrm{pw}(K_{n_1,\ldots,n_k}) \\ \leqslant \frac{1}{2}n(k+5) + \frac{1}{4}k(k-1) - 4 + \frac{1}{2}\left(\sum_{1 \leqslant i < j \leqslant k} n_i n_j\right). \end{aligned}$$

Theorem 2 implies that when $n_1 = \cdots = n_k = c$, (that is, when our complete multipartite graph is regular) then $\mathrm{tw}(L(K_{c,\ldots,c})) \approx \frac{k^2c^2}{4}$ (ignoring the lower order terms). We improve this result, obtaining an exact answer for the treewidth and pathwidth of the line graph of a regular complete multipartite graph.

Theorem 3. *If $n_1 = n_2 = \cdots = n_k = c \geqslant 1$, then*

$$\mathrm{tw}(L(K_{n_1,\dots,n_k})) = \mathrm{pw}(L(K_{n_1,\dots,n_k})) = \begin{cases} \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} + \frac{k}{4} - \frac{5}{4} & \text{, if } k \text{ odd, } c \text{ odd} \\ \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} - 1 & \text{, if } c \text{ even} \\ \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} + \frac{k}{4} - \frac{3}{2} & \text{, if } k \text{ even, } c \text{ odd} \end{cases}$$

Note that this implies Theorem 1 is a specical case of Theorem 3. In order to prove these results, we use the theory of *brambles* and the Treewidth Duality Theorem, which we present in Section 2. Section 3 presents a framework for proving results about the treewidth of general line graphs, which are of independent interest. Theorem 1 is proved in Section 4. Theorem 2 and Theorem 3 are proved in Section 5 and Section 6.

Finally, note the following conventions: if S is a subgraph of a graph G and $x \in V(G) - V(S)$, then let $S \cup \{x\}$ denote the subgraph of G with vertex set $V(S) \cup \{x\}$ and edge set $E(S) \cup \{xy : y \in S, xy \in E(G)\}$. Similarly, if $u \in V(S)$, let $S - \{u\}$ denote the subgraph with vertex set $V(S) - \{u\}$ and edge set $E(S) - \{uw : w \in S - \{u\}\}$.

2 Brambles and the Treewidth Duality Theorem

A *bramble* of a graph G is a collection $\mathcal{B}$ of connected subgraphs of G such that each pair of subgraphs $X, Y \in \mathcal{B}$ touch, where X and Y *touch* when they either have at least one vertex in common, or there exists an edge in G with one end in $V(X)$ and the other in $V(Y)$. The *order* of a bramble is the size of the smallest hitting set H, where a *hitting set* of a bramble $\mathcal{B}$ is a set of vertices H such that $H \cap V(X) \neq \emptyset$ for all $X \in \mathcal{B}$. For a given graph G, the *bramble number* $\mathrm{bn}(G)$ is the maximum order of a bramble of G. Brambles are important due to the following theorem of Seymour and Thomas [7]:

Theorem 4. *(Treewidth Duality Theorem) For every graph G,* $\mathrm{bn}(G) = \mathrm{tw}(G) + 1$.

In this paper we employ the following standard approach for determining the treewidth and pathwidth of a particular graph G. First we construct a bramble of large order, thus proving a lower bound on $\mathrm{tw}(G)$. Then to prove an upper bound, we construct a path decomposition of small width. A first step in constructing such a path decomposition is to place a minimum hitting set of the bramble in a single bag; when this bag is a bag of maximum size, we have an exact answer for $\mathrm{pw}(G)$ and $\mathrm{tw}(G)$.

3 Line-brambles

Throughout this section, let G be an arbitrary graph. In order to construct a bramble of the line graph $L(G)$, we define the following:

Definition A *line-bramble* $\mathcal{B}$ of G is a collection of connected subgraphs of G satisfying the following properties:

- For all $X \in \mathcal{B}$, $|V(X)| \geqslant 2$.
- For all $X, Y \in \mathcal{B}$, $V(X) \cap V(Y) \neq \emptyset$.

Define a *hitting set* for a line-bramble $\mathcal{B}$ to be a set of edges $H \subseteq E(G)$ that intersects each $X \in \mathcal{B}$. Then define the *order* of $\mathcal{B}$ to be the size of the minimum hitting set H of $\mathcal{B}$.

Lemma 5. *Given a line-bramble $\mathcal{B}$ of G, there is a bramble $\mathcal{B}'$ of $L(G)$ of the same order.*

Proof. Let X be an element of line-bramble $\mathcal{B}$ and let $\mathcal{B}' = \{E(X)|X \in \mathcal{B}\}$ (where here $E(X)$ refers to the subgraph of $L(G)$ induced by the vertices of $E(X)$). Recall X is connected. Now since $|V(X)| \geqslant 2$, X contains an edge. So $E(X)$ induces a non-empty connected subgraph of $L(G)$. Consider $E(X)$ and $E(Y)$ in $\mathcal{B}'$. Thus $V(X) \cap V(Y) \neq \emptyset$. Let v be a vertex in $V(X) \cap V(Y)$. Then there exists some $xv \in E(X)$ and $vy \in E(Y)$, and so in $L(G)$ there is an edge between the vertex xv and the vertex vy. Hence $E(X)$ and $E(Y)$ touch. Thus $\mathcal{B}'$ is a bramble of $L(G)$. All that remains is to ensure $\mathcal{B}$ and $\mathcal{B}'$ have the same order. If H is a minimum hitting set for $\mathcal{B}$, then H is also a set of vertices in $L(G)$ that intersects a vertex in each $E(X) \in \mathcal{B}'$. So H is a hitting set for $\mathcal{B}'$ of the same size. Conversely, if H' is a minimum hitting set of $\mathcal{B}'$, then H' is a set of edges in G that contains an edge in each $X \in \mathcal{B}$. So H' is a hitting set for $\mathcal{B}$. Thus the orders of $\mathcal{B}$ and $\mathcal{B}'$ are equal. $\square$

Hence, in order to determine a lower bound on the bramble number $\text{bn}(L(G))$, it is sufficient to construct a line-bramble of G of large order. We will now define a particular line-bramble for any graph G with $|V(G)| \geqslant 3$.

Definition Given a vertex $v \in V(G)$, the *canonical line-bramble for v of G* is the set of connected subgraphs X of G such that either $|V(X)| > \frac{|V(G)|}{2}$, or $|V(X)| = \frac{|V(G)|}{2}$ and X contains v. Note that if $|V(G)|$ is odd, then no elements of the second type occur.

Lemma 6. *For every graph G with $|V(G)| \geqslant 3$ and for all $v \in V(G)$, the canonical line-bramble for v, denoted $\mathcal{B}$, is a line-bramble of G.*

Proof. By definition, each element of $\mathcal{B}$ is a connected subgraph. Since $|V(G)| \geqslant 3$, each element of $\mathcal{B}$ contains at least two vertices. All that remains to show is that each pair of subgraphs X,Y in $\mathcal{B}$ intersect in at least one vertex. If $|V(X)| = |V(Y)| = \frac{|V(G)|}{2}$, then X and Y intersect at v. Otherwise, without loss of generality, $|V(X)| > \frac{|V(G)|}{2}$ and $|V(Y)| \geqslant \frac{|V(G)|}{2}$. If $V(X) \cap V(Y) = \emptyset$, then $|V(X) \cup V(Y)| = |V(X)| + |V(Y)| > |V(G)|$, which is a contradiction. $\square$

Let $v \in V(G)$ be an arbitrary vertex and let H be a minimum hitting set of $\mathcal{B}$, the canonical line-bramble for v. Consider the graph $G - H$. H is a set of edges, so $V(G - H) = V(G)$. Then each component of $G - H$ contains at most $\frac{|V(G)|}{2}$ vertices, otherwise some component of $G - H$ contains an element of $\mathcal{B}$ that does not contain an edge of H. Similarly, if a component contains $\frac{|V(G)|}{2}$ vertices, it cannot contain the vertex v. Thus, our hitting set H must be large enough to separate G into such components. The next lemma follows directly:

Lemma 7. *For every graph G with $|V(G)| \geqslant 3$ and for all $v \in V(G)$, a set $H \subseteq E(G)$ is a hitting set of $\mathcal{B}$, the canonical line-bramble for v, if and only if every component of $G - H$ has at most $\frac{|V(G)|}{2}$ vertices, and v is not in a component that contains exactly $\frac{|V(G)|}{2}$ vertices.*

Note the similarity between this characterisation and the *bisection width* of a graph (see [2, 5], for example), which is the minimum number of edges between any $A, B \subset V(G)$ where $A \cap B = \emptyset$ and $|A| = \lfloor \frac{|V(G)|}{2} \rfloor$ and $|B| = \lceil \frac{|V(G)|}{2} \rceil$. (Later we show that most of our components have maximum or almost maximum allowable order.)

Lemma 8. *Let G be a graph with $|V(G)| \geqslant 3$, v a vertex of G. and H a minimum hitting set for $\mathcal{B}$, the canonical line-bramble for v. Then no edge of H has both endpoints in the same component of $G - H$.*

Proof. For the sake of a contradiction assume that both endpoints of an edge $e \in H$ are in the same component of $G - H$. Then consider the set $H - e$. By Lemma 7, $H - e$ is a hitting set of $\mathcal{B}$, since the vertex sets of the components of $G - H$ have not changed. But $H - e$ is smaller than the minimum hitting set H, a contradiction. $\square$

4 Line Graph of the Complete Graph

We now prove Theorem 1. Let $G := K_n$. When $n \leqslant 2$, Theorem 1 holds trivially, so we can assume $n \geqslant 3$. Firstly, we shall determine a lower bound by considering the canonical line-bramble for v, denoted $\mathcal{B}$. Given that K_n is regular, we shall choose vertex v of K_n arbitrarily.

If H is a minimum hitting set of $\mathcal{B}$, label the components of $G - H$ as $Q_1, \ldots, Q_p$ such that $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \ldots \geqslant |V(Q_p)|$. We refer to this as labelling the components *descendingly.*

Consider a pair of components (Q_i, Q_j) where $i < j$ and the components are labelled descendingly. We call this a *good pair* if one of the following conditions hold:

1. $|V(Q_i)| < \frac{n}{2} - 1$,
2. n is even, $|V(Q_i)| = \frac{n}{2} - 1$, $V(Q_j) \neq \{v\}$, and $v \notin V(Q_i)$.

Lemma 9. *Let G be the complete graph with $n \geqslant 3$ vertices, v a vertex of G, $\mathcal{B}$ be the canonical line-bramble for v of G, and H a minimum hitting set of $\mathcal{B}$. If $Q_1, \ldots, Q_p$ are the components of $G - H$ labelled descendingly, then $Q_1, \ldots, Q_p$ does not contain a good pair.*

Proof. Say (Q_i, Q_j) is a good pair. Let x be a vertex of Q_j, such that if (Q_i, Q_j) is of the second type, then $x \neq v$. Let H' be the set of edges obtained from H by removing the edges from x to Q_i and adding the edges from x to Q_j. Then the components for $G - H'$ are $Q_1, \ldots, Q_{i-1}, Q_i \cup \{x\}, Q_{i+1}, \ldots, Q_{j-1}, Q_j - \{x\}, Q_{j+1}, \ldots Q_p$. By Lemma 7, to ensure H' is a hitting set, we only need to ensure that $V(Q_i) \cup \{x\}$ is sufficiently small, since all other components are the same as in H, or smaller. If (Q_i, Q_j) is of the first type, then $|V(Q_i) \cup \{x\}| = |V(Q_i)| + 1 < \frac{n}{2}$. If (Q_i, Q_j) is of the second type, $|V(Q_i) \cup \{x\}| = \frac{n}{2}$, but it does not contain v. Thus, by Lemma 7, H' is a hitting set. However, $|H'| = |H| - |V(Q_i)| + |V(Q_j)| - 1 \leqslant |H| - 1$, which contradicts that H is a minimum hitting set. □

Lemma 10. *Let $G, v, \mathcal{B}$ and H be as in Lemma 9. Then $G - H$ has exactly three components.*

Proof. Recall by Lemma 7, we have an upper bound on the order of the components of $G - H$. Firstly, we show that $G - H$ has at least three components. If $G - H$ has only one component, clearly this component is too large. If $G - H$ has two components and n is odd, then one of the components must have more than $\frac{n}{2}$ vertices. If $G - H$ has two components and n is even, it is possible that both components have exactly $\frac{n}{2}$ vertices, however one of these components must contain v. Thus $G - H$ has at least three components. Now, assume $G - H$ has at least four components. We will show that it has a good pair, contradicting Lemma 9. Label the components of $G - H$ descendingly.

If n is odd, we have a good pair of the first type when any two components have less than $\frac{n-1}{2}$ vertices. Thus at least three components have order at least $\frac{n-1}{2}$. Then $|V(G)| \geqslant 3(\frac{n-1}{2}) + 1 > n$ when $n \geqslant 2$, which is a contradiction.

If n is even, we have the first type of good pair whenever two components have less than $\frac{n}{2} - 1$ vertices. Similarly to the previous case, $|V(G)| \geqslant 3(\frac{n}{2} - 1) + 1 > n$, again a contradiction when $n > 4$. If $n = 4$ then each component is a single vertex. Take Q_i, Q_j to be two of these components, neither of which contain the vertex v. Then (Q_i, Q_j) is a good pair of the second type. Hence $G - H$ does not have more than three components, and as such it has exactly three components. □

Lemma 11. *Let $G, v, \mathcal{B}$ and H be as in Lemma 9, and the components of $G - H$ be labelled descendingly. If n is odd then $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and $|V(Q_3)| = 1$. If n is even then $|V(Q_1)| = \frac{n}{2}, |V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$.*

Proof. Lemma 10 shows that $G - H$ has exactly three components.By Lemma 9, (Q_2, Q_3) is not a good pair. Hence $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \frac{n-1}{2}$ when n is odd, and $|V(Q_1)| \geqslant |V(Q_2)| \geqslant$

$\frac{n}{2}-1$ when n is even, or else we have a good pair of the first type. By Lemma 7, when n is odd, $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and so $|V(Q_3)| = 1$. When n is even, however, $\frac{n}{2} - 1 \leqslant |V(Q_1)|, |V(Q_2)| \leqslant \frac{n}{2}$. Since Q_3 is not empty, it follows that $|V(Q_3)| = 1$ or 2. If $|V(Q_3| = 1$, then $|V(Q_1)| = \frac{n}{2}, |V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$, as required. Otherwise, $|V(Q_1)|, |V(Q_2)| = \frac{n}{2}-1$. But then at least one of Q_1, Q_2 does not contain v, and $V(Q_3) \neq \{v\}$. Thus either (Q_1, Q_3) or (Q_2, Q_3) is a good pair of the second type, contradicting Lemma 9. □

Lemma 12. *Let $G, v, \mathcal{B}$ and H be as in Lemma 9. Then $|H| \geqslant (\frac{n-1}{2})(\frac{n-1}{2}) + (n-1)$ when n is odd, and $|H| \geqslant (\frac{n-2}{2})(\frac{n}{2}) + (n-1)$ when n is even.*

Proof. From Lemma 11 we know the order of the components of $G - H$. H contains at least every edge between each pair of components, and since G is complete there is an edge for each pair of vertices. From this it is easy to calculate $|H|$. □

Lemma 12 and the Treewidth Duality Theorem imply:

Corollary 13. *Let G be the complete graph with $n \geqslant 3$ vertices.*

$$\operatorname{tw}(L(K_n)) = \operatorname{bn}(L(K_n)) - 1 \geqslant \begin{cases} (\frac{n-1}{2})(\frac{n-1}{2}) + (n-2) & \text{, if } n \text{ is odd} \\ (\frac{n-2}{2})(\frac{n}{2}) + (n-2) & \text{, if } n \text{ is even.} \end{cases}$$

Now, to obtain an upper bound on $\operatorname{pw}(L(G))$, we construct a path decomposition of $L(G)$. First, label the vertices of G by $1, \ldots, n$. Let T be an n-node path, also labelled by $1, \ldots, n$. The bag A_i for the node labelled i, is defined such that $A_i = \{ij : j \in V(G)\} \cup \{uw : u < i < w\}$. For a given A_i, call the edges in $\{ij : j \in V(G)\}$ *initial edges* and call the edges in $\{uw : u < i < w\}$ *crossover edges*. (Note here these edges of G are really acting as vertices of $L(G)$, but we refer to them as edges for simplicity.)

Lemma 14. *Let G be the complete graph with $n \geqslant 3$ vertices. $(T, \{A_1, \ldots, A_n\})$ is a path decomposition for $L(G)$ of width*

$$\begin{cases} (\frac{n-1}{2})(\frac{n-1}{2}) + (n-2) & \text{, if } n \text{ is odd} \\ (\frac{n-2}{2})(\frac{n}{2}) + (n-2) & \text{, if } n \text{ is even.} \end{cases}$$

Proof. Each edge uw of G appears in A_u and A_w as initial edges. Similarly, all of the edges incident at the vertex u appear in A_u, and the same holds for w. Observe that uw is in A_i if and only if $u \leqslant i \leqslant w$. Thus the nodes indexing the bags containing uw form a connected subtree of T, as required.

Now we determine the size of A_i. A_i contains $n-1$ initial edges and $(i-1)(n-i)$ crossover edges. So $|A_i| = (n-1) + (i-1)(n-i)$. This is maximised when $i = \frac{n+1}{2}$ if n is odd, and when $i = \frac{n}{2}$ or $\frac{n+2}{2}$ if n is even. From this we can calculate the largest bag size, and hence the width of T. □

Lemma 14 gives an upper bound on $\mathrm{pw}(L(K_n))$ and $\mathrm{tw}(L(K_n))$. This, combined with the lower bound in Corollary 13, completes the proof of Theorem 1.

5 Line-brambles of the Complete Multipartite Graph

We now extend the above result to the line graph $L(G)$ of the complete multipartite graph $G := K_{n_1,\dots,n_k}$, where $k \geqslant 2$. Let $n := |V(G)| = n_1 + \cdots + n_k$. If $n = k$, then $G = K_n$ and Theorem 1 determines $\mathrm{tw}(G)$ exactly, so we may assume $n > k$. Let X_i be the i^{th} colour class of G, with order n_i. Call X_i *odd* or *even* depending on the parity of $|X_i|$. In a similar fashion to Section 4 we shall first find a line-bramble of G.

As we did previously, we shall consider a canonical line-bramble for v denoted $\mathcal{B}$. However, we shall choose vertex v from a colour class of largest order. Note that v has minimum degree. Let H be a hitting set of $\mathcal{B}$, and label the components of $G - H$ by $Q_1, \dots, Q_p$. Denote H and the labelling of its components together as $(H, (Q_1, \dots, Q_p))$. Choose $(H, (Q_1, \dots, Q_p))$ such that the following conditions hold, in order of preference:

(0) $|H|$ is minimised,
(1) $|V(Q_1)|$ is maximised,
(2) $|V(Q_2)|$ is maximised,
$\vdots$
(p) $|V(Q_p)|$ is maximised,
(p+1) v is in the component of highest possible index.

By condition **(0)**, H is a minimum hitting set. Note, as a result of this that $|V(Q_1)| \geqslant |V(Q_2)| \geqslant \dots \geqslant |V(Q_p)|$, otherwise we can keep H and easily find a better choice of labelling. Call a choice of $(H, (Q_1, \dots, Q_p))$ that satisifies these conditions a *good labelling*.

Consider a pair of components (Q_i, Q_j) where $i < j$ and $Q_1, \dots, Q_p$ is from a good labelling. We call this a *good pair* when for all $x \in Q_j$ there exists $y \in Q_i$ such that xy is an edge, and one of the following holds:

1. $|V(Q_i)| < \frac{n}{2} - 1$,
2. n is even, $|V(Q_i)| = \frac{n}{2} - 1$, $v \notin V(Q_i)$ and $V(Q_j) \cap X_s \neq \{v\}$ for all colour classes X_s.

Lemma 15. *Let G be a complete multipartite graph $G := K_{n_1,\dots,n_k}$ such that $k \geqslant 2$ and $n > k$, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, \dots, Q_p))$ a good labelling. Then $Q_1, \dots, Q_p$ does not contain a good pair.*

Proof. Assume (Q_i, Q_j) is a good pair. For each X_s that intersects Q_j, let x_s be some vertex of $Q_j \cap X_s$. If (Q_i, Q_j) is of the second type, choose each $x_s \neq v$. Let H_s be the set of edges created by taking H and removing the edges from x_s to Q_i, then adding the edges from x_s to $(Q_j - X_s)$. Thus we have removed $|V(Q_i)| - |V(Q_i) \cap X_s|$ edges and have added $|V(Q_j)| - |V(Q_j) \cap X_s|$.

Suppose that $|V(Q_j)| - |V(Q_j) \cap X_s| > |V(Q_i)| - |V(Q_i) \cap X_s|$ for each X_s that intersects Q_j. Then

$$\sum_{s: X_s \cap V(Q_j) \neq \emptyset} |V(Q_j)| - |V(Q_j) \cap X_s| > \sum_{s: X_s \cap V(Q_j) \neq \emptyset} |V(Q_i)| - |V(Q_i) \cap X_s|.$$

However, since we are cycling through all colour classes that intersect Q_j,

$$\sum_{s: X_s \cap V(Q_j) \neq \emptyset} |V(Q_j) \cap X_s| = |V(Q_j)|.$$

If there are r such colour classes, then

$$(r-1)|V(Q_j)| > r|V(Q_i)| - \sum_{s: X_s \cap V(Q_j) \neq \emptyset} |V(Q_i) \cap X_s| \geqslant (r-1)|V(Q_i)|.$$

This implies $|V(Q_j)| > |V(Q_i)|$, which is a contradiction of condition **(i)**. Hence, for some s, $|V(Q_j)| - |V(Q_j) \cap X_s| \leqslant |V(Q_i)| - |V(Q_i) \cap X_s|$. Fix such an s.

A component of $G - H_s$ is either one of $Q_1, \ldots, Q_{i-1}, Q_{i+1}, \ldots, Q_{j-1}, Q_{j+1}, \ldots, Q_p$, or $Q_i \cup \{x_s\}$ (which is connected as x_s has a neighbour in Q_i), or strictly contained within Q_j. Since H is a hitting set, to prove H_s is a hitting set it suffices to show that $Q_i \cup \{x_s\}$ is sufficiently small, by Lemma 7. If (Q_i, Q_j) is of the first type, then $|V(Q_i) \cup \{x_s\}| = |V(Q_i)| + 1 < \frac{n}{2}$. So $V(Q_i) \cup \{x_s\}$ is sufficiently small. If (Q_i, Q_j) is of the second type, $|V(Q_i) \cup \{x_s\}| = \frac{n}{2}$, but it does not contain v. Thus H_s is a hitting set. However, $|H_s| = |H| - (|V(Q_i)| - |V(Q_i) \cap X_s|) + (|V(Q_j)| - |V(Q_j) \cap X_s|) \leqslant |H|$. If $|H_s| < |H|$, then condition **(0)** is contradicted. If $|H_s| = |H|$, since $|V(Q_i) \cup \{x_s\}| > |V(Q_i)|$ and only components of higher index have become smaller, H_s is a better choice of minimum hitting set by condition **(i)**, which is a contradiction. □

Lemma 16. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, \ldots, Q_p))$ be as in Lemma 15. $G - H$ has at least three components.*

Proof. By Lemma 7, we have an upper bound on the order of the components of $G - H$. If $G - H$ has only one component, clearly this component is too large. If $G - H$ has two components and n is odd, then one of the components must have more than $\frac{n}{2}$ vertices. If $G - H$ has two components and n is even, it is possible that both components have exactly $\frac{n}{2}$ vertices, however one of these components must contain v. Thus $G - H$ has at least three components. □

If G is a star $K_{1,n-1}$, then $L(G) \cong K_{n-1}$ and $\operatorname{tw}(L(G)) = n - 2$, which satisfies Theorem 2. Now assume that G is not a star. If $(H, (Q_1, \ldots, Q_p))$ is a good labelling where $p \geqslant 4$

and $Q_2, \dots, Q_p$ are all singleton sets and contained within one colour class, then say that $(H, (Q_1, \dots, Q_p))$ is a *rare configuration*.

Lemma 17. *Let G be a complete multipartite graph $G := K_{n_1,\dots,n_k}$ such that $k \geqslant 2, n > k$ and G is not a star, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, \dots, Q_p))$ a good labelling. Then $(H, (Q_1, \dots, Q_p))$ is not a rare configuration.*

Proof. Assume G is a rare configuration, but G is not a star. Let X_s be the colour class of $Q_2, \dots, Q_p$. Since $p \geqslant 4$, we may choose $j \in \{2, \dots, p\}$ such that $V(Q_j) \neq \{v\}$.

Suppose that one of the following conditions hold:

- $|V(Q_1)| < \frac{n}{2} - 1$,
- n is even, $|V(Q_1)| = \frac{n}{2} - 1$ and $v \notin V(Q_1)$.

Q_1 must contain at least two vertices not in X_s since G is not a star or an independent set (as $k \geqslant 2$). So for each $x \in V(Q_2) \cup \cdots \cup V(Q_p)$, there is some $y \in V(Q_1)$ such that $y \notin X_s$, so the edge xy exists. Then (Q_1, Q_j) is a good pair, which contradicts Lemma 15. Thus by Lemma 7,

$$|Q_1| = \begin{cases} \frac{n-1}{2} & \text{, if } n \text{ is odd} \\ \frac{n}{2} - 1 & \text{, if } n \text{ is even and } v \in V(Q_1) \\ \frac{n}{2} & \text{, if } n \text{ is even and } v \notin V(Q_1) \end{cases}$$

Since at least two vertices of Q_1 are not in X_s, we may choose $y \in (V(Q_1) - \{v\}) - X_s$. Say $y \in X_t$. We can assume that $v \in V(Q_1)$ or $v \in V(Q_p)$, since if $v \in V(Q_2) \cup \dots V(Q_{p-1})$, then we can relabel the components $Q_2, \dots, Q_p$ to obtain a choice of $(H, (Q_1, \dots, Q_p)$ which is better with regards to the condition **(p+1)**. Thus let $z \in V(Q_2)$, and so $z \neq v$ since $p \geqslant 4$. Let H' be the set of edges created by taking H and removing the edges from y to $Q_3 \cup \cdots \cup Q_{p-1}$, adding the edges from y to $Q_1 - X_t$, and removing the edges from z to $Q_1 - \{y\}$. Then the components of $G - H'$ are $Q_1 \cup \{z\} - \{y\}$, $\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1}$, Q_p. The component $Q_1 \cup \{z\} - \{y\}$ is connected since $Q_1 - \{y\}$ contains a vertex not in X_s and $z \in X_s$. Similarly, $\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1}$ is connected since $y \in X_t$ and all vertices of $Q_3 \cup \cdots \cup Q_{p-1}$ are in X_s.

By Lemma 7, to show H' is a hitting set, it is sufficient to show that no component of $G - H'$ is too large. Since $|V(Q_1 \cup \{z\} - \{y\})| = |V(Q_1)|$ and $v \neq z$ and H is a hitting set, $Q_1 \cup \{z\} - \{y\}$ is sufficiently small. Similarly Q_p is sufficiently small. However, $|V(\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1})| = p - 2$. As $|V(Q_1)| + \cdots + |V(Q_p)| = n$, it follows that $p - 2 = n - |V(Q_1)| - 1$. In order to show this is sufficiently small, we need to consider the parity of n, which we consider below. Also note,

$$|H'| = |H| - (p - 3) + (|V(Q_1)| - |V(Q_1) \cap X_t|) - (|V(Q_1)| - 1 - |V(Q_1) \cap X_s|).$$

Since $|V(Q_1) \cap X_t| \geqslant 1$ and $|V(Q_1) \cap X_s| \leqslant |V(Q_1)| - 2$, we have $|H'| \leqslant |H| - (p-1) + |V(Q_1)| = |H| + 2|V(Q_1)| - n$. This also depends on the parity of n. Now we consider these separate cases to check the order of $\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1}$ and $|H'|$.

Firstly, say n is odd. In this case $|V(Q_1)| = \frac{n-1}{2}$, so then $|V(\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1})| = p - 2 = n - \frac{n-1}{2} - 1 = \frac{n-1}{2}$, and so $\{y\} \cup Q_3 \cup \cdots \cup Q_{p-1}$ is sufficiently small, and H' is a hitting set. Also, $|H'| \leqslant |H| + 2(\frac{n-1}{2}) - n < |H|$, which contradicts condition **(0)**. Secondly, say n is even and $v \in V(Q_1)$. Then $|V(Q_1)| = \frac{n}{2} - 1$, implying $p - 2 = \frac{n}{2}$, and $|H'| \leqslant |H| - 2$. This contradicts condition **(0)**. Finally, say n is even and $v \notin V(Q_1)$. Then $|V(Q_1)| = \frac{n}{2}$ and $v \in V(Q_p)$. Then $p - 2 = \frac{n}{2} - 1$, and $|H'| \leqslant |H| + 2(\frac{n}{2}) - n = |H|$. However, note that the order of the second largest component of $G - H'$ is $p - 2 = \frac{n}{2} - 1$, whereas for $G - H$ the order of the second largest component is 1. As G is a rare configuration but not a star, $n \geqslant 5$, since $|V(Q_1)| \geqslant 2$ and $p \geqslant 4$, implying $\frac{n}{2} - 1 > 1$. Thus H' is a better choice of minimum hitting set, by condition **(2)**.

Thus, in either case, if G is not a star, but is a rare configuration, then there is a contradiction to one of our conditions on $(H, (Q_1, \ldots, Q_p))$. □

Lemma 18. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, \ldots, Q_p))$ be as in Lemma 17. Then $G - H$ has exactly three components.*

Proof. $G - H$ has at least three components, by Lemma 16. Assume for the sake of a contradiction that $G - H$ has greater than three components. Since $p \geqslant 4$, if all components but Q_1 are singleton sets in the one colour class, then we have a rare configuration. By Lemma 17, this cannot occur. Thus either Q_2 is not a singleton set, or $Q_2, \ldots, Q_p$ are not all in one colour class. Consider a pair (Q_i, Q_j), where $i \in \{1, 2\}$, $i < j$ and if $|V(Q_i)| = 1$ then Q_j and Q_i are not in the same colour class. We can find such a pair for $i = 1$ and for $i = 2$ since this is not a rare configuration. In either case, for all $x \in V(Q_j)$ there exists a $y \in V(Q_i)$ such that xy is an edge, since there is always some $y \in V(Q_i)$ of a different colour class to x. Since (Q_i, Q_j) is not a good pair by Lemma 15, we know $|V(Q_i)|$ is too large. In particular, if n is odd, $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$. However, since each component must contain a vertex and $p \geqslant 4$, the sum of the orders of the components is at least $2(\frac{n-1}{2}) + 2 > n$, which is a contradiction. If n is even and v is in neither Q_1 nor Q_2, then $|V(Q_1)| = |V(Q_2)| = \frac{n}{2}$, which again means the sum of the orders of the components is too large. Finally, if n is even and without loss of generality $v \in V(Q_2)$, then $|V(Q_1)| = \frac{n}{2}$ and $|V(Q_2)| = \frac{n}{2} - 1$, which still gives a contradiction on the orders of the components. Hence $G - H$ has exactly three components. □

Lemma 19. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, \ldots, Q_p))$ be as in Lemma 17. If n is odd, then $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and $|V(Q_3)| = 1$. If n is even, then $|V(Q_1)| = \frac{n}{2}, |V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$.*

Proof. Lemma 18 shows that $G - H$ has three components. Recall that in a good labelling that $|V(Q_1)| \geqslant |V(Q_2)| \geqslant |V(Q_3)|$. If $|V(Q_1)| = 1$, then $n = 3$, and since $\frac{n-1}{2} = 1$, then our statement holds in this case. Thus we can assume $n \geqslant 4$ and $|V(Q_1)| \geqslant 2$. Hence (Q_1, Q_j) is a good pair for $j > 1$ unless Q_1 is too large. If n is odd, then $|V(Q_1)| = \frac{n-1}{2}$. If $|V(Q_2)| = 1$, $\frac{n-1}{2} + 1 + 1 = n$, implying $n = 3$. So $|V(Q_2)| \geqslant 2$, and (Q_2, Q_3) is a good pair unless $|V(Q_2)| = \frac{n-1}{2}$, in which case $|V(Q_3)| = 1$.

If n is even and $v \in V(Q_1)$, then $|V(Q_1)| = \frac{n}{2}-1$. Again, if $|V(Q_2)| = 1$ then $\frac{n}{2}-1+1+1 = n$, implying $n = 2$. So $|V(Q_2)| \geqslant 2$, and (Q_2, Q_3) is a good pair unless $|V(Q_2)| = \frac{n}{2}$, implying $|V(Q_3)| = 1$. (Note here we'd need to relabel the components so they are in descending order of size.) Finally, if n is even and $v \notin V(Q_1)$, then $|V(Q_1)| = \frac{n}{2}$. If $|V(Q_2)| = 1$, then $\frac{n}{2}+1+1 = n$, implying $n = 4$. However, then $|V(Q_3)| = 1$ and our statement holds. If $n \geqslant 5$, then $|V(Q_2)| \geqslant \frac{n}{2} - 1$ else (Q_2, Q_3) is a good pair. Since we must have three components, $|V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$. Either way, our components have the desired size. □

Lemma 20. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, \dots, Q_p))$ be as in Lemma 17. If $v \notin Q_3$, then the vertex in Q_3 is in a different colour class to v.*

Proof. By Lemma 19, $|V(Q_3)| = 1$. Let x be the vertex in Q_3. Assume for the sake of contradiction that x, v are in colour class X_s. If n is odd then $v \in V(Q_1)$ or $V(Q_2)$, but these components have the same order, by Lemma 19. If n is even, $v \in V(Q_2)$, since otherwise v is in a component of order $\frac{n}{2}$, again by Lemma 19. So without loss of generality, $v \in V(Q_2)$. Define the hitting set H' as follows: from H, add the edges from v to Q_2, and then remove the edges from x to Q_2. Since $xv \notin E(G)$, the components of $G - H'$ are $Q_1, (Q_2 - \{v\}) \cup \{x\}$ and $\{v\}$ (as x, v are in the same colour class, $(Q_2 - \{v\}) \cup \{x\}$ is connected). The orders of the components have not changed, and v has not been placed into a component of order $\frac{n}{2}$, so this is a hitting set by Lemma 7. $|H'| = |H| + (|V(Q_2)| - |V(Q_2) \cap X_s|) - (|V(Q_2)| - (|V(Q_2) \cap X_s|)) = |H|$. Since v is now in a component of higher index, this contradicts condition **(p+1)**. □

The previous lemmas give a good idea of the structure of the components of $G - H$. When dealing with the complete graph, this was sufficient. However, in the case of the complete multipartite graph, we also need to know how the components of $G - H$ interact with the colour classes of G. As we might expect, in the optimal case, each colour class is essentially split evenly across the two large components Q_1 and Q_2. In order to show this, however, we need to be careful about the parity of n and the parities of $n_1, \dots, n_k$. Recall that we label the colour classes $X_1, \dots, X_n$. For the following section, we assume that G is a complete multipartite graph such that $k \geqslant 2$ and G is not a star, and as such we have only three components by Lemma 18.

Definition Let $X_i^* := X_i \cap (V(Q_1) \cup V(Q_2))$, and say X_i^* is *even* or *odd* depending on the parity of its order.

Definition
- A colour class X_i is called *balanced* if $|V(Q_1) \cap X_i| = |V(Q_2) \cap X_i|$.

- A colour class X_i is Q_1-*skew* if $|V(Q_1)\cap X_i| \geqslant |V(Q_2)\cap X_i|+1$. When $|V(Q_1)\cap X_i| = |V(Q_2)\cap X_i|+1$, we say X_i is *just-Q_1-skew*.
- A colour class X_i is Q_2-*skew* if $|V(Q_1)\cap X_i|+1 \leqslant |V(Q_2)\cap X_i|$. When $|V(Q_1)\cap X_i|+1 = |V(Q_2)\cap X_i|$, we say X_i is *just-Q_2-skew*.
- (X_i, Xj) is called a *skew pair* if X_i is Q_1-skew and X_j is Q_2-skew.

For simplicity, if X_i is Q_1-skew or Q_2-skew, then we say X_i is *skew*. Similarly if X_i is just-Q_1-skew or just-Q_2-skew, then we say X_i is *just-skew*.

We say G is an *exception* if n is even, and there is a colour class X_s such that $|V(Q_1)\cap X_s| = |V(Q_1)|-1$ and $|V(Q_2)\cap X_s| = |V(Q_2)|-1$.

Lemma 21. *Let G be a complete multipartite graph $G := K_{n_1,\dots,n_k}$ such that $k \geqslant 2$, $n > 4, k$ and G is neither a star nor an exception, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, Q_2, Q_3))$ a good labelling. If (X_i, Xj) is a skew pair, then both X_i and X_j are just-skew.*

Proof. Since no colour class can be both Q_1-skew and Q_2-skew, $i \neq j$. Since $n \geqslant 5$, by Lemma 19, both Q_1 and Q_2 contain at least two vertices, and thus intersect at least two colour classes.

First, we show that both X_i^* and X_j^* contain a vertex other than v. If $X_i^* = \emptyset$, then X_i is not skew. So now assume $X_i^* \neq \emptyset$. Similarly, $X_j^* \neq \emptyset$. If $X_i^* = \{v\}$, then by Lemma 20, $X_i \cap V(Q_3) = \emptyset$, and so $X_i = \{v\}$. But since v is in a largest colour class, every colour class has order one, and as such $k = n$, which contradicts one of our assumptions on n. Thus both X_i^* and X_j^* contain a vertex other than v, and since X_i is Q_1-skew and X_j is Q_2-skew, there are vertices $x \in (V(Q_1)\cap X_i) - \{v\}$ and $y \in (V(Q_2)\cap X_j) - \{v\}$. Then define the hitting set H' as follows: remove the edges from x to $V(Q_2)$ from H, add the edges from x to $V(Q_1)-X_i$, remove the edges from y to $V(Q_1)-\{x\}$, and add the edges from y to $V(Q_2)\cup\{x\}$. Now $G-H'$ has components $(Q_1-\{x\})\cup\{y\}, (Q_2-\{y\})\cup\{x\}$ and Q_3, assuming that $(Q_1-\{x\})\cup\{y\}$ and $(Q_2-\{y\})\cup\{x\}$ are in fact connected (which we now prove).

If $(Q_1-\{x\})\cup\{y\}$ is not connected, then it intersects only one colour class, which must be X_j as $y \in X_j$. Since $x \in X_i$, it follows that $|V(Q_1)\cap X_j| = |V(Q_1)|-1$. Since X_j is Q_2-skew,

$$|V(Q_1)| = |V(Q_1)\cap X_j|+1 \leqslant |V(Q_2)\cap X_j| \leqslant |V(Q_2)|.$$

Since $|V(Q_1)| \geqslant |V(Q_2)|$, we have $|V(Q_1)| = |V(Q_2)|$, and each inequality in the above equation is an equality. In particular, $|V(Q_2)\cap X_j| = |V(Q_2)|$, and thus $V(Q_2) \subseteq X_j$. But Q_2 intersects at least two colour classes, which is a contradiction. Thus $(Q_1-\{x\})\cup\{y\}$ is a connected component of $G-H'$.

If $(Q_2-\{y\})\cup\{x\}$ is not connected, then it intersects only one colour class, which must be X_i as $x \in X_i$. Since $y \in X_j$, it follows that $|V(Q_2)\cap X_i| = |V(Q_2)|-1$. Since X_i is Q_1-skew,

$$|V(Q_1)| \geqslant |V(Q_1)\cap X_i| \geqslant |V(Q_2)\cap X_i|+1 = |V(Q_2)|.$$

By Lemma 19, either $|V(Q_1)| = |V(Q_2)|$ (when n is odd) or $|V(Q_1)| = |V(Q_2)|+1$ (when n is even). If $|V(Q_1)\cap X_i| = |V(Q_1)|$, then $V(Q_1) \subseteq X_i$, contradicting our result that Q_1 intersects at least two colour classes. Otherwise $|V(Q_1) \cap X_i| = |V(Q_1)| - 1$, which can only happen when n is even. In this case, since $|V(Q_1)\cap X_i| = |V(Q_1)|-1$ and $|V(Q_2)\cap X_i| = |V(Q_2)|-1$, G is an exception. This contradiction shows that $(Q_2 - \{y\}) \cup \{x\}$ is a connected component of $G - H'$.

Thus $G - H'$ has components $(Q_1 - \{x\}) \cup \{y\}, (Q_2 - \{y\}) \cup \{x\}$ and Q_3. Hence the orders of the components have not changed. As the vertex v has not changed components, H' is a legitimate hitting set. But since H is the minimum hitting set by condition **(0)**, $|H'| \geqslant |H|$. Hence

$$\begin{aligned}|H'| =&|H| - (|V(Q_2)| - |V(Q_2) \cap X_i|) + (|V(Q_1)| - |V(Q_1) \cap X_i|)\\ &- (|V(Q_1)| - 1 - |V(Q_1) \cap X_j|) + (|V(Q_2)| + 1 - |V(Q_2) \cap X_j|)\\ \geqslant&|H|.\end{aligned}$$

Which implies

$$|V(Q_2) \cap X_i| + |V(Q_1) \cap X_j| \geqslant |V(Q_1) \cap X_i| + |V(Q_2) \cap X_j| - 2.$$

Since X_i is Q_1-skew and X_j is Q_2-skew,

$$|V(Q_1)\cap X_i|+|V(Q_2)\cap X_j|-2 \geqslant |V(Q_2)\cap X_i|+|V(Q_1)\cap X_j| \geqslant |V(Q_1)\cap X_i|+|V(Q_2)\cap X_j|-2.$$

This only holds if every inequality is actually an equality. That is, X_i is just-Q_1-skew and X_j is just-Q_2-skew. ☐

Lemma 22. *Let G be a complete multipartite graph $G := K_{n_1,\dots,n_k}$ such that $k \geqslant 2$, $n > k$ and G is not a star, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, Q_2, Q_3))$ a good labelling. If X_i is skew, then X_i is just-skew.*

Proof. Suppose G is not an exception and $n > 4$. If there exists a Q_1-skew colour class X_s and a Q_2-skew colour class X_t, then either (X_s, X_i) or (X_i, X_t) is a skew pair, and by Lemma 21, X_i is just-skew, as required.

Alternatively, either no colour class is Q_1-skew or no colour class is Q_2-skew. Suppose, for the sake of contradiction, there is a skew colour class X_j that is not just-skew. In the first case, for all ℓ, $|V(Q_1) \cap X_\ell| \leqslant |V(Q_2) \cap X_\ell|$, and $|V(Q_1) \cap X_j| + 2 \leqslant |V(Q_2) \cap X_j|$. Thus

$$|V(Q_1)|+2 = (\sum_{1\leqslant\ell\leqslant k,\ell\neq j}|V(Q_1)\cap X_\ell|)+|V(Q_1)\cap X_j|+2 \leqslant (\sum_{1\leqslant\ell\leqslant k,\ell\neq j}|V(Q_2)\cap X_\ell|)+|V(Q_2)\cap X_j| = |V(Q_2)|.$$

This contradicts $|V(Q_1)| \geqslant |V(Q_2)|$. Similarly, in the second case, $|V(Q_1)| \geqslant |V(Q_2)| + 2$, which contradicts Lemma 19. Thus if $n \geqslant 5$ and G is not an exception, then our statement holds.

Consider the case when G is an exception. Then $|V(Q_1) \cap X_s| = |V(Q_1)| - 1$ and $|V(Q_2) \cap X_s| = |V(Q_2)| - 1$. Since n is even, by Lemma 19, $|V(Q_1)| = |V(Q_2)| + 1$, so X_s is just-skew. There are exactly two other vertices of $Q_1 \cup Q_2$, one in each component, which we label x and y respectively. If x and y are in the same colour class, then that colour class is balanced. Otherwise, x and y are in different colour classes, each of which intersects $Q_1 \cup Q_2$ in one vertex. Such a colour class is just-skew, as required.

Finally, consider the case $n \leqslant 4$. Then $|V(Q_1) \cup V(Q_2)| \leqslant 3$. Thus either $|V(Q_1)| = |V(Q_2)| = 1$, or $|V(Q_1)| = 2$ and $|V(Q_2)| = 1$. If X_i is not just-skew, then X_i contains at least two vertices in some component. Thus, the only possibility to consider is when $|V(Q_1) \cap X_i| = 2$. But then Q_1 is not connected, since both vertices are in the same colour class, which contradicts the fact that Q_1 is a connected component.

Thus X_i is just-skew. □

From Lemma 22 and Lemma 19, we get the following results about $|Q_1 \cap X_i|$ and $|Q_2 \cap X_i|$:

Corollary 23. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, Q_2, Q_3))$ be as in Lemma 22. If a colour class X_i does not intersect Q_3, then*

- *if X_i is balanced, then $|Q_1 \cap X_i| = |Q_2 \cap X_i| = \frac{n_i}{2}$*
- *if X_i is Q_1-skew, then $|Q_1 \cap X_i| = \frac{n_i+1}{2}$ and $|Q_2 \cap X_i| = \frac{n_i-1}{2}$*
- *if X_i is Q_2-skew, then $|Q_1 \cap X_i| = \frac{n_i-1}{2}$ and $|Q_2 \cap X_i| = \frac{n_i+1}{2}$*

Corollary 24. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, Q_2, Q_3))$ be as in Lemma 22. If a colour class X_i does intersect Q_3, then $|V(Q_3) \cap X_i| = 1$ and*

- *if X_i is balanced, then $|Q_1 \cap X_i| = |Q_2 \cap X_i| = \frac{n_i-1}{2}$*
- *if X_i is Q_1-skew, then $|Q_1 \cap X_i| = \frac{n_i}{2}$ and $|Q_2 \cap X_i| = \frac{n_i-2}{2}$*
- *if X_i is Q_2-skew, then $|Q_1 \cap X_i| = \frac{n_i-2}{2}$ and $|Q_2 \cap X_i| = \frac{n_i}{2}$*

Lemma 25. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, Q_2, Q_3))$ be as in Lemma 22. If n is odd, then there is an equal number of Q_1-skew and Q_2-skew colour classes. If n is even, then there is one more Q_1-skew colour class than there are Q_2-skew colour classes.*

Proof. Say there are a Q_1-skew colour classes and b Q_2-skew colour classes. By Lemma 22, if X_i is Q_1-skew, then $|V(Q_1) \cap X_i| = |V(Q_2) \cap X_i| + 1$, and if X_i is Q_2-skew, then $|V(Q_1) \cap X_i| = |V(Q_2) \cap X_i| - 1$. Thus

$$|V(Q_1)| = \sum_{1 \leqslant i \leqslant k} |V(Q_1) \cap X_i| = (\sum_{1 \leqslant i \leqslant k} |V(Q_2) \cap X_i|) + a - b = |V(Q_2)| + a - b.$$

If n is odd, then by Lemma 19, $|V(Q_1)| = |V(Q_2)|$, so $a = b$, as required. When n is even, $|V(Q_1)| = |V(Q_2)| + 1$, so $a = b + 1$. □

From Lemma 18, Lemma 19, Corollary 23 and Corollary 24, we get the following result that summarises this section:

Theorem 26. *Let G be a complete multipartite graph $G := K_{n_1,\ldots,n_k}$ such that $k \geqslant 2, n > k$ and G is not a star, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H,(Q_1,\ldots,Q_p))$ a good labelling. Then $p = 3$. If n is odd, then $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and $|V(Q_3)| = 1$, and if n is even, then $|V(Q_1)| = \frac{n}{2}$, $|V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$. For a colour class X_i,*

$$\lceil \frac{n_i - 2}{2} \rceil \leqslant |V(Q_1) \cap X_i|, |V(Q_2) \cap X_i| \leqslant \lfloor \frac{n_i + 1}{2} \rfloor.$$

Now we can use Theorem 26 to determine a lower bound on $\operatorname{tw}(L(G))$.

Theorem 27. *Let G be a complete multipartite graph $G := K_{n_1,\ldots,n_k}$ where $k \geqslant 2$. Then $\operatorname{tw}(L(G)) + 1 = \operatorname{bn}(L(G)) \geqslant \frac{1}{2}\left(\sum_{1 \leqslant i < j \leqslant k} n_i n_j\right) + \frac{3}{4}k^2 - kn - \frac{3}{4}k + n$.*

Proof. First, consider the case when $k \geqslant 2, n > k$ and G is not a star. Then choose some vertex v in a largest colour class of G, a canonical line-bramble for v denoted $\mathcal{B}$ and a good labelling $(H,(Q_1,\ldots,Q_p))$. It is sufficient to determine a lower bound on $|H|$, since H is a minimum hitting set for $\mathcal{B}$ by condition **(0)**, and since $\mathcal{B}$ forces the existence of a bramble of $L(G)$ of the same order by Lemma 5. Using Theorem 26, we can determine the structure of H. The set H contains all edges with an endpoint in Q_1 and an endpoint in Q_2; simply count these edges. By Theorem 26, $|V(Q_1) \cap X_i|, |V(Q_2) \cap X_i| \geqslant \lceil \frac{n_i}{2} - 1 \rceil$. As $n_i, n_j \geqslant 1$, it follows that $|V(Q_1) \cap X_i||V(Q_2) \cap X_j| \geqslant \left(\frac{n_i}{2} - 1\right)\left(\frac{n_j}{2} - 1\right) - \frac{1}{4}$. So we count the edges from Q_1 to Q_2 as follows:

$$\begin{aligned}
\sum_{i \neq j} |V(Q_1) \cap X_i||V(Q_2) \cap X_j| \geqslant & \sum_{i \neq j} \left(\frac{n_i}{2} - 1\right)\left(\frac{n_j}{2} - 1\right) - \frac{1}{4} \\
= & \frac{1}{4}\left(\sum_{i \neq j} n_i n_j\right) - (k-1)n + \frac{3}{4}k(k-1) \\
= & \frac{1}{2}\left(\sum_{1 \leqslant i < j \leqslant k} n_i n_j\right) + \frac{3}{4}k^2 - kn - \frac{3}{4}k + n.
\end{aligned}$$

This gives the required lower bound on $|H|$ in this case.

It remains to check the cases when either $n = k$ or G is a star. When $n = k$, G is simply the complete graph, and our lower bound follows by Theorem 1. If G is a star, then $L(G)$ is a complete graph, and the lower bound follows by inspection. $\square$

Using the same techniques as in the above proof, we can also determine an upper bound on $|H|$. We do this now. Note when considering the upper bound, we also need to account for the edges from Q_3 into the components Q_1, Q_2, but there are not many of these edges.

Corollary 28. *Let $G, v, \mathcal{B}$ and $(H, (Q_1, \dots, Q_p))$ be as in Theorem 26. Then* $|H| \leqslant \frac{1}{2}\left(\sum_{1\leqslant i<j\leqslant k} n_i n_j\right) + \frac{1}{2}n(k+1) + \frac{1}{4}k(k-1) - 1.$

Also, our results in this section give a more detailed understanding of H when G is regular.

Theorem 29. *Let G be a complete regular k-partite graph $G := K_{c,\dots,c}$, such that $k \geqslant 2, n > k$, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, \dots, Q_p))$ a good labelling. Then $p = 3$. If n is odd, then $|V(Q_1)| = |V(Q_2)| = \frac{n-1}{2}$ and $|V(Q_3)| = 1$ and*

- *for one colour class X_i, we have $|V(Q_1) \cap X_i| = |V(Q_2) \cap X_i| = \frac{c-1}{2}$ and $|V(Q_3) \cap X_i| = 1$,*
- *for $\frac{k-1}{2}$ other colour classes X_i, we have $|V(Q_1) \cap X_i| = \frac{c+1}{2}$ and $|V(Q_2) \cap X_i| = \frac{c-1}{2}$,*
- *for the remaining $\frac{k-1}{2}$ colour classes X_i, we have $|V(Q_1) \cap X_i| = \frac{c-1}{2}$ and $|V(Q_2) \cap X_i| = \frac{c+1}{2}$.*

If n is even, then $|V(Q_1)| = \frac{n}{2}$, $|V(Q_2)| = \frac{n}{2} - 1$ and $|V(Q_3)| = 1$. If n is even and c is odd, then

- *for one colour class X_i, we have $|V(Q_1) \cap X_i| = |V(Q_2) \cap X_i| = \frac{c-1}{2}$ and $|V(Q_3) \cap X_i| = 1$,*
- *for $\frac{k}{2}$ other colour classes X_i, we have $|V(Q_1) \cap X_i| = \frac{c+1}{2}$ and $|V(Q_2) \cap X_i| = \frac{c-1}{2}$,*
- *for the remaining $\frac{k}{2} - 1$ colour classes X_i, we have $|V(Q_1) \cap X_i| = \frac{c-1}{2}$ and $|V(Q_2) \cap X_i| = \frac{c+1}{2}$.*

Finally, if n is even and c is even, then

- *for one colour class X_i, we have $|V(Q_1) \cap X_i| = \frac{c}{2}$, $|V(Q_2) \cap X_i| = \frac{c}{2} - 1$ and $|V(Q_3) \cap X_i| = 1$,*
- *for the other $k-1$ colour classes X_i, we have $|V(Q_1) \cap X_i| = |V(Q_1) \cap X_i| = \frac{c}{2}$.*

Proof. Since G is regular and $n > k \geqslant 2$, G is not a star. The statements about the number and order of the components of $G - H$ all follow from Lemma 18 and Lemma 19. Since $n = ck$, when n is odd, c is odd and k is odd. When n is even, at least one of c and k are even. Then from Corollary 23, Corollary 24 and Lemma 25, the rest of the theorem follows. $\square$

6 Path decompositions of the Complete Multipartite Graph

We reuse the following notation from the previous section: G is a complete multipartite graph $K_{n_1,\dots,n_k}$, that is not an independent set (that is, $k \geqslant 2$), complete graph (that is, $n > k$) or a star. (Recall we have already proven Theorem 2 for such graphs.) The vertex v of G is chosen from a largest colour class, and $\mathcal{B}$ a canonical line-bramble for v. $(H, (Q_1, \ldots, Q_p))$ is a good labelling, and by Theorem 26 we can assume $p = 3$. $X_1, \ldots, X_k$ are the colour classes of G such that $|X_i| = n_i$.

From the results of the previous section, it is possible to determine the order of a minimum hitting set H. However, first we find a path decomposition of $L(G)$ with width expressed in terms of H, as this will make things easier.

Now we define path decomposition for $L(G)$ as follows. Let T be the underlying path. Since T is a path, it makes sense to refer to a bag *left* or *right* of another bag, depending on the relative positions of the corresponding nodes in T. If a bag is to the right of another bag and the nodes which index them are adjacent in T, then we say it is *directly right*. Similarly define *directly left*. For a vertex u of G, let $\deg_i(u)$ be the number of edges in G incident to u with the other endpoint in the component Q_i.

First, label the vertices of Q_1 by $x_1, \ldots, x_{|V(Q_1)|}$ in some order, which we will specify later. Similarly, label the vertices of Q_2 by $y_1, \ldots, y_{|V(Q_2)|}$, again in an order we will later specify. Finally, by Theorem 26, Q_3 contains a single vertex, which we label z.

Then define the following bags:

- $\gamma := H = \{uw \in E(G) : u, w \text{ are in different components of } G - H\}$,
- for for $1 \leqslant i \leqslant |V(Q_1)|$,

$$\alpha_i := \{x_\ell u \in E(G) : u \in V(Q_1), 1 \leqslant \ell \leqslant i\} \cup \{x_j w \in E(G) : w \in V(G) - V(Q_1), i \leqslant j \leqslant |V(Q_1)|\},$$

- for $1 \leqslant i \leqslant |V(Q_2)|$,

$$\beta_i := \{y_\ell u \in E(G) : u \in V(Q_2), 1 \leqslant \ell \leqslant i\} \cup \{y_j w \in E(G) : w \in V(G) - V(Q_2), i \leqslant j \leqslant |V(Q_2)|\}.$$

Each bag is indexed by a node of T. Left-to-right, the nodes of T index the bags in the following order: $\beta_{|V(Q_2)|}, \ldots, \beta_1, \gamma, \alpha_1, \ldots, \alpha_{|V(Q_1)|}$. Let $\mathcal{X}$ denote the collection of bags. We claim this defines a path decomposition $(T, \mathcal{X})$ for $L(G)$, independent of our ordering of Q_1 and Q_2.

Lemma 30. *Let G be a complete multipartite graph $G := K_{n_1,\ldots,n_k}$ such that $k \geqslant 2$, $n > k$ and G is not a star, v a vertex of G chosen from a largest colour class, $\mathcal{B}$ a canonical line-bramble for v, and $(H, (Q_1, Q_2, Q_3))$ a good labelling. Then $(T, \mathcal{X})$ is a path decomposition of $L(G)$, irrespective of the ordering used on Q_1 and Q_2.*

Proof. Consider $uw \in E(G)$. We require that the nodes indexing the bags containing uw induce a non-empty connected subpath of T. Firstly, assume that u and w are in different components of $G - H$. If $u = x_i$ and $w = y_j$, then $uw \in \beta_j, \ldots, \beta_1, \gamma, \alpha_1, \ldots, \alpha_i$, meaning uw is in precisely this sequence of bags. If $u = x_i$ and $w = z$, then $uw \in \gamma, \alpha_1, \ldots, \alpha_i$. If $u = y_j$ and $w = z$, then $uw \in \beta_j, \ldots, \beta_1, \gamma$.

Secondly, assume that u and w are in the same component of $G-H$, which is either Q_1 or Q_2, since by Theorem 26, $|V(Q_3)|=1$. If $u,w\in V(Q_1)$, then let $u=x_i$ be the vertex of smaller label. Then $uw\in\alpha_i,\dots,\alpha_{|V(Q_1)|}$. If $u,w\in V(Q_2)$, then similarly let $u=y_i$ be the vertex of smaller label. Then $uw\in\beta_{|V(Q_2)|},\dots,\beta_i$. This shows that the nodes indexing the bags containing uw induce a non-empty connected subpath of T.

All that remains is to show that if two edges are incident at a vertex in G (that is, the edges are adjacent in $L(G)$), then there is a bag of $\mathcal{X}$ containing both of them. Now if the shared vertex of the two edges is $x_i\in V(Q_1)$, then by inspection both edges are in α_i. If the shared vertex is $y_j\in V(Q_2)$, then both edges are in β_j. Finally, if the shared vertex is z, then both edges are in γ. □

Now we determine the width of $(T,\mathcal{X})$, which is one less than the order of the largest bag. To do so, we use a specific labelling of $Q_1\cup Q_2$. We do this in two different ways, depending on whether G is regular.

In our first ordering, label the vertices $x_1,\dots,x_{|V(Q_1)|}$ in order of non-decreasing size of the colour class containing x_i, and do the same for $y_1,\dots,y_{|V(Q_2)|}$. We denote this ordering as the *red ordering.*

Lemma 31. *Let $G,v,\mathcal{B},(H,(Q_1,Q_2,Q_3))$ and $(T,\mathcal{X})$ be as in Lemma 30, but assume the ordering on Q_1 and Q_2 is the red ordering. Then $|\alpha_i|\leqslant|\alpha_1|+n-2$, for all $1\leqslant i\leqslant|V(Q_1)|$.*

Proof. We will show that $|\alpha_i|\leqslant|\alpha_{i-1}|+2$ for all i. This implies that $|\alpha_i|\leqslant|\alpha_1|+2(i-1)$. Since $i\leqslant|V(Q_1)|$ and $|V(Q_1)|\leqslant\frac{n}{2}$ by Lemma 7, this is sufficient.

$$\begin{aligned}\alpha_i=&\{x_\ell u,x_jw\in E(G):u\in V(Q_1),w\in V(G)-V(Q_1),1\leqslant\ell\leqslant i,i\leqslant j\leqslant|V(Q_1)|\}\\=&\{x_\ell u\in E(G):u\in V(Q_1),1\leqslant\ell\leqslant i\}\cup\{x_jw\in E(G):w\in V(G)-V(Q_1),i\leqslant j\leqslant|V(Q_1)|\}.\end{aligned}$$

This is a disjoint union. Let X_s,X_t be the colour classes such that $x_{i-1}\in X_s$ and $x_i\in X_t$, and note that it is possible $s=t$. Then

$$\begin{aligned}|\alpha_i|-|\alpha_{i-1}|=&|\{x_\ell u\in E(G):u\in V(Q_1),1\leqslant\ell\leqslant i\}|\\&-|\{x_\ell u\in E(G):u\in V(Q_1),1\leqslant\ell\leqslant i-1\}|\\&+|\{x_jw\in E(G):w\in V(G)-V(Q_1),i\leqslant j\leqslant|V(Q_1)|\}|\\&-|\{x_jw\in E(G):w\in V(G)-V(Q_1),i-1\leqslant j\leqslant|V(Q_1)|\}|\\\leqslant&\deg_1(x_i)-|\{x_{i-1}w\in E(G):w\in V(G)-V(Q_1)\}|\\=&\deg_1(x_i)-(\deg_G(x_{i-1})-\deg_1(x_{i-1}))\\=&\deg_1(x_i)-(n-n_s-\deg_1(x_{i-1}))\\=&|V(Q_1)|-|V(Q_1\cap X_t)|-(n-n_s-|V(Q_1)|+|V(Q_1\cap X_s)|)\\=&2|V(Q_1)|+n_s-|V(Q_1\cap X_t)|-n-|V(Q_1\cap X_s)|.\end{aligned}$$

Assume for the sake of contradiction that $|\alpha_i| - |\alpha_{i-1}| > 2$. Then:

$$2|V(Q_1)| + n_s > n + |V(Q_1) \cap X_s| + |V(Q_1) \cap X_t| + 2.$$

By the ordering of the vertices in Q_1, $n_t \geqslant n_s$. Then by Theorem 26,

$$|V(Q_1) \cap X_s| + |V(Q_1) \cap X_t| \geqslant \frac{n_s - 2}{2} + \frac{n_t - 2}{2} \geqslant n_s - 2.$$

Hence $2|V(Q_1)| + n_s > n + n_s - 2 + 2$; that is, $2|V(Q_1)| > n$. But $|V(Q_1)| > \frac{n}{2}$ contradicts Lemma 7. □

By symmetry we have:

Lemma 32. *Let $G, v, \mathcal{B}, (H, (Q_1, Q_2, Q_3))$ and $(T, \mathcal{X})$ be as in Lemma 30, but assume the ordering on Q_1 and Q_2 is the red ordering. Then$|\beta_i| \leqslant |\beta_1| + n - 2$, for all $1 \leqslant i \leqslant |V(Q_2)|$.*

Lemma 33. *Let $G, v, \mathcal{B}, (H, (Q_1, Q_2, Q_3))$ and $(T, \mathcal{X})$ be as in Lemma 30. The maximum bag size of $(T, \mathcal{X})$, using the red ordering, is at most $|H| + 2n - 2$.*

Proof. By Lemma 31 and Lemma 32, the maximum size of a bag right of γ is at most $|\alpha_1| + n - 2$, and left of γ it is $|\beta_1| + n - 2$. By inspection, the edges in $\alpha_1 - \gamma$ are all adjacent to x_1. Hence there are at most n of them. Thus $|\alpha_1| \leqslant |\gamma| + n$. Similarly $|\beta_1| \leqslant |\gamma| + n$. Since $\gamma = H$, this is sufficient. □

Given this, we can determine an upper bound on $\mathrm{pw}(L(G))$.

Theorem 34. *Let G be a complete multipartite graph $G := K_{n_1,\ldots,n_k}$ where $k \geqslant 2$. Then $\mathrm{pw}(G) \leqslant \frac{1}{2}\left(\sum_{1 \leqslant i < j \leqslant k} n_i n_j\right) + \frac{1}{4}k^2 + \frac{1}{2}kn - \frac{1}{4}k + \frac{5}{2}n - 4$*

Proof. If $G, v, \mathcal{B}, (H, (Q_1, Q_2, Q_3))$ and $(T, \mathcal{X})$ are as in Lemma 30, then we have a path decomposition of width at most $|H| + 2n - 3$ by Lemma 33. (Note as $k \geqslant 2$, it follows $n \geqslant 2$ and so $2n - 3$ is positive.) Then our result follows from Corollary 28. In the remaining cases, G is either a complete graph or a star, and this result follows by Theorem 1 or inspection, respectively. □

Thus Theorem 2 follows from Theorem 27 and Theorem 34.

When G is regular, that is, $n_1 = \cdots = n_k$, we can get a more accurate bound on the treewidth and pathwidth. Define $c := n_1$ to be the size of each colour class. We need a different ordering of the vertices $x_1, \ldots, x_{|Q_1|}$ and $y_1, \ldots, y_{|Q_2|}$ to obtain our result. In order to do this, we recall the notion of a skew colour class, as defined in Section 5, and the associated results. First consider a colour class X_i that does not intersect Q_3. If X_i is balanced, then say every vertex of X_i is *Type 1*. If X_i is Q_1-skew, then each vertex in $Q_1 \cap X_i$ is *Type 1* and each vertex in

$Q_2 \cap X_i$ is *Type 2*. If X_i is Q_2-skew, then each vertex in $Q_1 \cap X_i$ is *Type 2* and each vertex in $Q_2 \cap X_i$ is *Type 1*. Finally, each vertex in the remaining colour class (that does intersect Q_3) is *Type 3*. Thus each vertex of $V(G) - z$ is either Type 1, 2 or 3. Label the vertices of Q_1 in order $x_1, \ldots, x_{|V(Q_1)|}$ by first labelling Type 1 vertices, then Type 2 vertices, and finally Type 3 vertices. Do the same for $y_1, \ldots, y_{|V(Q_2)|}$. We denote this ordering as the *blue ordering*.

Lemma 35. *Let G be a complete k-partite graph with $n > k$, v a vertex of G, $\mathcal{B}$ a canonical line-bramble for v and $(H, (Q_1, Q_2, Q_3))$ a good labelling. If $k \geqslant 3$, then Q_1 contains at least two Type 1 vertices, and Q_2 contains at least one Type 1 vertex. If $k = 2$ and $c \geqslant 3$, then Q_1 contains at least two Type 1 vertices, and Q_2 contains at least one Type 1 or Type 2 vertex.*

Proof. If X_i is a colour class that does not intersect Q_3, then it intersects both of Q_1 and Q_2—if not, then by Lemma 22, $|X_i| = 1$ and G is the complete graph. Since we are trying to find Type 1 and Type 2 vertices, from now on we only consider colour classes that do not intersect Q_3. If $k \geqslant 5$, then there are at least four colour classes that do not intersect Q_3. From Theorem 29, there are either at least two Q_1-skew and Q_2-skew colour classes, or at least four balanced colour classes. Even if each such colour class intersects each of Q_1 and Q_2 only once, there are still enough colour classes of the correct skew to get all our required Type 1 vertices. Similarly, if $k = 4$ and c is odd, then there are two Q_1-skew colour classes and one Q_2-skew colour class, and if $k = 4$ and c is even, there are three balanced colour classes. This is again sufficient.

If $k = 3$, then by Theorem 29 again, there are enough Q_2-skew or balanced colour classes to ensure that Q_2 has at least one Type 1 vertex. However, if n is odd, there is only one Q_1-skew colour class. In this case, c is odd, and so $c \geqslant 3$. Thus that colour class contains at least two vertices in Q_1. Thus Q_1 has two Type 1 vertices.

Now assume $k = 2$ and $c \geqslant 3$. If c is odd, there is one Q_1-skew colour class, again by Theorem 29. This colour class contains at least two vertices in Q_1 and one in Q_2, which satisfies our requirement, now that Q_2 only requires a Type 2 vertex. If c is even, then there is one balanced colour class. $c \geqslant 3$, so as it is even, $c \geqslant 4$ and each component contains two vertices from this colour class. This is sufficient. ☐

The following lemma strengthens Lemma 31 for the case when G is regular.

Lemma 36. *Let G be a complete k-partite graph with $n > k$, v a vertex of G, $\mathcal{B}$ a canonical line-bramble for v and $(H, (Q_1, Q_2, Q_3))$ a good labelling. Let $(T, \mathcal{X})$ be our path decomposition where Q_1 and Q_2 are ordered by the blue ordering. If $k \geqslant 3$ or $c \geqslant 3$, then $|\alpha_1| \geqslant |\alpha_2| \geqslant \ldots \geqslant |\alpha_{|V(Q_1)|}|$.*

Proof. We will show that $|\alpha_i| \leqslant |\alpha_{i-1}|$ for all i. We can write α_i as the disjoint union

$$\alpha_i = \{x_\ell u \in E(G) : u \in V(Q_1), 1 \leqslant \ell \leqslant i\} \cup \{x_j w \in E(G) : w \in V(G) - V(Q_1), i \leqslant j \leqslant |V(Q_1)|\}.$$

Let X_s, X_t be the colour classes such that $x_{i-1} \in X_s$ and $x_i \in X_t$, and note that it is possible that $s = t$. Define $r := |\{x_i x_f \in E(G) : f < i\}|$. Then

$$\begin{aligned}
|\alpha_i| - |\alpha_{i-1}| =& |\{x_\ell u \in E(G) : u \in V(Q_1), 1 \leqslant \ell \leqslant i\}| \\
& - |\{x_\ell u \in E(G) : u \in V(Q_1), 1 \leqslant \ell \leqslant i-1\}| \\
& + |\{x_j w \in E(G) : w \in V(G) - V(Q_1), i \leqslant j \leqslant |V(Q_1)|\}| \\
& - |\{x_j w \in E(G) : w \in V(G) - V(Q_1), i-1 \leqslant j \leqslant |V(Q_1)|\}| \\
=& \deg_1(x_i) - r - |\{x_{i-1} w \in E(G) | w \in V(G) - V(Q_1)\}| \\
=& \deg_1(x_i) - r - (\deg_G(x_{i-1}) - \deg_1(x_{i-1})) \\
=& \deg_1(x_i) - r - (n - n_s - \deg_1(x_{i-1})) \\
=& |V(Q_1)| - |V(Q_1 \cap X_t)| - r - (n - n_s - |V(Q_1)| + |V(Q_1 \cap X_s)|) \\
=& 2|V(Q_1)| + n_s - r - |V(Q_1 \cap X_t)| - n - |V(Q_1 \cap X_s)|.
\end{aligned}$$

Assume for the sake of contradiction that $|\alpha_i| - |\alpha_{i-1}| > 0$. Then:

$$2|V(Q_1)| + n_s > n + r + |V(Q_1) \cap X_s| + |V(Q_1) \cap X_t|.$$

There are two cases to consider. Firstly, say that both x_{i-1} and x_i are Type 1. So X_s and X_t are both balanced or Q_1-skew, and neither intersects Q_3. Since G is regular, $n_t = n_s$. Then by Corollary 23, $|V(Q_1) \cap X_s| + |V(Q_1) \cap X_t| \geqslant \frac{n_s}{2} + \frac{n_t}{2} = n_s$. Hence $2|V(Q_1)| + n_s > n + n_s + r \geqslant n + n_s$, so $2|V(Q_1)| > n$, which contradicts Lemma 7.

Secondly, since we ordered our vertices by non-decreasing type, we can assume x_i does not have Type 1. However, by Lemma 35, Q_1 has at least two Type 1 vertices, x_a and x_b. Note if two vertices of Q_1 are in the same colour class, they have the same type, so we know that x_a and x_b are in a different colour class to x_i. Also, $a, b < i$, thus $r \geqslant 2$. Since $n_t = n_s$, by Theorem 26, $|V(Q_1) \cap X_s| + |V(Q_1) \cap X_t| \geqslant \frac{n_s - 2}{2} + \frac{n_t - 2}{2} = n_s - 2$. Hence $2|V(Q_1)| + n_s > n + n_s - 2 + r \geqslant n + n_s$, so $2|V(Q_1)| > n$, which again contradicts Lemma 7. ☐

We must also consider the equivalent argument for bags to the left of γ, as we did in the general case. However, here the arguments are not quite the same.

Lemma 37. *Let $G, v, \mathcal{B}, (H, (Q_1, Q_2, Q_3))$ and $(T, \mathcal{X})$ be as in Lemma 36. If $k \geqslant 3$ or $c \geqslant 3$, then $|\beta_1| \geqslant |\beta_2| \geqslant \ldots \geqslant |\beta_{|V(Q_2)|}|$.*

Proof. We will show that $|\beta_i| \leqslant |\beta_{i-1}|$ for all i. We can write β_i as the disjoint union

$$\beta_i = \{y_\ell u \in E(G) : u \in V(Q_2), 1 \leqslant \ell \leqslant i\} \cup \{y_j w \in E(G) : w \in V(G) - V(Q_2), i \leqslant j \leqslant |V(Q_2)|\}.$$

Let X_s, X_t be the colour classes such that $y_{i-1} \in X_s$ and $y_i \in X_t$, and note that it is possible that $s = t$. Define $r := |\{y_i y_f \in E(G) : f < i\}|$. Then

$$|\beta_i| - |\beta_{i-1}| = |\{y_\ell u \in E(G) : u \in V(Q_2), 1 \leqslant \ell \leqslant i\}|$$

$$
\begin{aligned}
&- |\{y_\ell u \in E(G) : u \in V(Q_2), 1 \leqslant \ell \leqslant i-1\}| \\
&+ |\{y_j w \in E(G) : w \in V(G) - V(Q_2), i \leqslant j \leqslant |V(Q_2)|\}| \\
&- |\{y_j w \in E(G) : w \in V(G) - V(Q_2), i-1 \leqslant j \leqslant |V(Q_2)|\}| \\
=& \deg_2(y_i) - r - |\{y_{i-1} w \in E(G) | w \in V(G) - V(Q_2)\}| \\
=& \deg_2(y_i) - r - (\deg_G(y_{i-1}) - \deg_2(y_{i-1})) \\
=& \deg_2(y_i) - r - (n - n_s - \deg_2(y_{i-1})) \\
=& |V(Q_2)| - |V(Q_2 \cap X_t)| - r - (n - n_s - |V(Q_2)| + |V(Q_2 \cap X_s)|) \\
=& 2|V(Q_2)| + n_s - r - |V(Q_2 \cap X_t)| - n - |V(Q_2 \cap X_s)|.
\end{aligned}
$$

Assume for the sake of contradiction that $|\beta_i| - |\beta_{i-1}| > 0$. Then:

$$2|V(Q_2)| + n_s > n + r + |V(Q_2) \cap X_s| + |V(Q_2) \cap X_t|.$$

There are two cases to consider. Firstly, say that neither of y_i and y_{i-1} have Type 3. So neither X_s nor X_t intersects Q_3. G is regular, so $n_t = n_s$. By Corollary 23, $|V(Q_2) \cap X_s| + |V(Q_2) \cap X_t| \geqslant \frac{n_s-1}{2} + \frac{n_t-1}{2} = n_s - 1$. Hence $2|V(Q_2)| + n_s > n + r + n_s - 1 \geqslant n + n_s - 1$, and so $2|V(Q_2)| > n - 1$. However, Theorem 29 states that $|V(Q_2)| \leqslant \frac{n-1}{2}$, so this is a contradiction.

Secondly, y_i has Type 3. By Lemma 35, Q_2 contains at least one non-Type 3 vertex; this will be of a different colour class to y_i and have a lower numbered index. Hence $r \geqslant 1$. By Theorem 26, $|V(Q_2) \cap X_s| + |V(Q_2) \cap X_t| \geqslant \frac{n_s-2}{2} + \frac{n_t-2}{2} = n_s - 2$, and hence $2|V(Q_2)| + n_s > n + r + n_s - 2 \geqslant n + n_s - 1$. Again, this contradictions Theorem 29. □

Lemma 38. *Let $G, v, \mathcal{B}, (H, (Q_1, Q_2, Q_3))$ and $(T, \mathcal{X})$ be as in Lemma 36. If $k \geqslant 3$ or $c \geqslant 3$, then $|\alpha_1| \leqslant |\gamma|$ and $|\beta_1| \leqslant |\gamma|$.*

Proof. By inspection, $\alpha_1 = \{x_1 u, uw \in E(G) : u \in V(Q_1), w \in V(G) - V(Q_1)\}$. Thus the edges of the form $x_1 u$ are the only edges in α_1 not in γ, and the edges between Q_2 and Q_3 (all of which are adjacent to z) are the only edges in γ not in α_1. Thus $|\alpha_1| + \deg_2(z) - \deg_1(x_1) = |\gamma|$. Suppose for the sake of contradiction that $|\alpha_1| > |\gamma|$. Say $x_1 \in X_s$ and $z \in X_t$. By Lemma 35, x_1 has Type 1, so $s \neq t$. Substituting $\deg_2(z) = |V(Q_2)| - |V(Q_2) \cap X_t|$ and $\deg_1(x_1) = |V(Q_1)| - |V(Q_1) \cap X_s|$ gives

$$|V(Q_1)| - |V(Q_2)| > |V(Q_1) \cap X_s| - |V(Q_2) \cap X_t|.$$

By Theorem 29, $|V(Q_1)| - |V(Q_2)| \leqslant 1$. Similarly, since X_t intersects Q_3, $|V(Q_2) \cap X_t| = \frac{c-1}{2}$ if c is odd, and $|V(Q_2) \cap X_t| = \frac{c-2}{2}$ if c is even. Since $X_s \cap Q_3 = \emptyset$ and x_1 has Type 1, $|V(Q_1) \cap X_s| \geqslant \frac{c}{2}$. Hence $|V(Q_1) \cap X_s| - |V(Q_2) \cap X_t| \geqslant \frac{1}{2}$ if c is odd, or 1 if c is even. However, this value is an integer, so $|V(Q_1) \cap X_s| - |V(Q_2) \cap X_t| \geqslant 1$, implying $|V(Q_1)| - |V(Q_2)| > 1$, which is a contradiction of Theorem 29.

Now we consider $\beta_1 = \{y_1 u, uw \in E(G) : u \in V(Q_2), w \in V(G) - V(Q_2)\}$. Suppose for the sake of contradiction that $|\beta_1| > |\gamma|$. Let $y_1 \in X_s$ and $z \in X_t$. By Lemma 35, x_1 has Type 1 or Type 2, so $s \neq t$. Performing substitutions as we did in the α_1 case gives

$$|V(Q_2)| - |V(Q_1)| > |V(Q_2) \cap X_s| - |V(Q_1) \cap X_t|.$$

Since X_s does not intersect Q_3 and X_t does, by Theorem 29, $|V(Q_2) \cap X_s| \geqslant \frac{c-1}{2}$ and $|V(Q_1) \cap X_t| = \frac{c-1}{2}$ or $\frac{c}{2}$. Thus $|V(Q_2) \cap X_s| - |V(Q_1) \cap X_t| \geqslant 0$ or $-\frac{1}{2}$, but since it is an integer, $|V(Q_2) \cap X_s| - |V(Q_1) \cap X_t| \geqslant 0$, implying $|V(Q_2)| - |V(Q_1)| > 0$, which contradicts Theorem 29. □

By Lemmas 36, 37 and 38, γ is the largest bag in all but a few cases. Recall $\gamma = H$. Hence, together with Theorem 29, we get the following result.

If G is a regular k-partite graph such that $n > k$, and either $k \geqslant 3$ or $c \geqslant 3$, then

$$\operatorname{pw}(L(G)) = \operatorname{tw}(L(G)) = |H| - 1.$$

We now accurately determine $|H|$ when G is regular.

We can determine $|H|$ by calculating the number of edges between Q_1 and Q_2, and the number of edges adjacent to $z \in Q_3$. Theorem 29 gives us all we require. It follows that:

$$|H| = \begin{cases} \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} + \frac{k}{4} - \frac{1}{4} & \text{, if } ck \text{ odd} \\ \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} & \text{, if } c \text{ even} \\ \frac{c^2k^2}{4} - \frac{c^2k}{4} + \frac{ck}{2} - \frac{c}{2} + \frac{k}{4} - \frac{1}{2} & \text{, if } k \text{ even, } c \text{ odd} \end{cases}$$

This gives the exact answer for the treewidth and pathwidth of the line graph of the $(n-c)$-regular complete k-partite graph, when $n > k$ and either $k \geqslant 3$ or $c \geqslant 3$. When $n = k$, we determine the treewidth using by Theorem 1. When $k = 2$ and $c = 2$, G is a 4-cycle and $\operatorname{pw}(K_{2,2}) = \operatorname{tw}(K_{2,2}) = 2$, which satisfies our result by inspection. This proves Theorem 3.

References

[1] H. Bodlaender, *A partial k-arboretum of graphs with bounded treewidth*, Theoret. Comput. Sci. **209** (1998), no. 1-2, 1–45.

[2] J. Díaz, M. Serna, and N. Wormald, *Bounds on the bisection width for random d-regular graphs*, Theoret. Comput. Sci. **382** (2007), no. 2, 120–130.

[3] M. Grohe and D. Marx, *On tree width, bramble size, and expansion*, J. Combinat. Theory, Series B **99** (2009), no. 1, 218–228.

[4] D. Marx, *Can you beat treewidth?*, Theory Comput. **6** (2010), 85–112.

[5] B. Monien and R. Preis, *Upper bounds on the bisection width of 3- and 4-regular graphs*, J. Discrete Algorithms **4** (2006), no. 3, 475–498.

[6] B. Reed, *Tree width and tangles: a new connectivity measure and some applications*, Surveys in combinatorics, London Math. Soc. Lecture Note Ser., vol. 241, Cambridge Univ. Press, 1997, pp. 87–162.

[7] P. Seymour and R. Thomas, *Graph searching and a min-max theorem for tree-width*, J. Combinat. Theory, Series B **58** (1993), no. 1, 22–33.