Chapter 9 **Wave-Particle Duality**

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Unless otherwise stated:

$$\text{Mass of an Electron, } m_e = 9.11 \times 10^{-31} \text{ kg}$$
$$\text{Mass of a Proton, } m_p = 1.67262 \times 10^{-27} \text{ kg}$$
$$\text{Planck's Constant, } h = 6.626 \times 10^{-34} \text{ J s}$$
$$\text{Speed of Light, } c = 3.00 \times 10^{8} \text{ m/s}$$

1 Discussion Questions

1. Determine:

   (a) the momentum of photons with a frequency of 4.0×10^{14} Hz.

   Solution:
   $$p = \frac{h}{\lambda} = \frac{hf}{c} = 8.8347 \times 10^{-28} \text{ N s}$$

   (b) the momentum of photons with a wavelength of 400 nm.

   Solution:
   $$p = \frac{h}{\lambda} = \frac{h}{400 \times 10^{-9}} = 1.6565 \times 10^{-27} \text{ N s}$$

   (c) A beam of light of wavelength λ is totally reflected at normal incidence by a plane mirror. The intensity of light is such that the photons hit the mirror at a rate n per second. Given that the Planck constant is h, show that the force exerted on the mirror by this beam is $2nh/\lambda$.

   Solution: The change in momentum per photon is $2p = 2h/\lambda$. Thus, the total change in momentum over some time interval t is $2nht/\lambda$. Thus, $F = p/t = 2nh/\lambda$.

2. (a) Determine the de Broglie wavelength of

   i. a bullet of mass 30 g, and moving with speed of 300 m s^{-1}.

   Solution:
   $$p = mv = (0.030)(300) = 9 \text{ Ns} \implies \lambda = \frac{h}{p} = 7.3622 \times 10^{-35} \text{ m}$$

   ii. an electron that has been accelerated from rest through a potential difference of 0.75 kV.

   Solution:
   $$\begin{aligned} E_f &= E_i + \Delta E \\ (m_e c^2)^2 + (pc)^2 &= (m_e c^2 + \Delta E)^2 \\ p &= \sqrt{\left(2m_e c + \frac{\Delta E}{c}\right) \times \frac{\Delta E}{c}} \\ \lambda &= \frac{h}{\sqrt{\left(2m_e c + \frac{q_e V}{c}\right) \times \frac{q_e V}{c}}} \\ &= 4.47948 \times 10^{-11} \text{ m} \end{aligned}$$

(b) In an electron diffraction experiment, the wavelength associated with an electron beam is determined to be 0.14 nm.

i. Find the momentum of an electron in the beam.

Solution:

$$p = \frac{h}{\lambda} = \frac{6.626 \times 10^{-34}}{0.14 \times 10^{-9}} = 4.7329 \times 10^{-24} \text{ N s}$$

ii. If the electrons are initially at rest, through what potential difference should they be accelerated to acquire this momentum.

Solution:

$$\begin{aligned} \Delta E &= \sqrt{(m_e c^2)^2 + (pc)^2} - m_e c^2 \\ V &= \frac{1}{q_e}\left(\sqrt{(m_e c^2)^2 + (pc)^2} - m_e c^2\right) \\ &= 76.834 \text{ V} \end{aligned}$$

3. (a) An electron has kinetic energy 3.00 eV. Find its wavelength.

Solution:

$$p = \sqrt{2m_e E_k} \implies \lambda = \frac{h}{\sqrt{2m_e E_k}} = 7.0853 \times 10^{10} \text{ m}$$

(b) A photon has energy 3.00 eV. Find it's wavelength.

Solution:

$$\lambda = \frac{hc}{E} = 4.14125 \times 10^{-7} \text{ m} \approx 414 \text{ nm}$$

4. The uncertainty in the position and velocity of a particle are 10^{-11} m and 7.9×10^2 m/s respectively. By considering Heisenberg's Uncertainty Principle, suggest what this particle may be.

Solution:

$$\begin{aligned} \Delta x \Delta p &\geq \frac{\hbar}{2} \\ m &\geq \frac{\hbar}{2\Delta x \Delta v} \\ &= 6.67443 \times 10^{-27} \text{ kg} \\ m &\geq 4.0207 \text{ u.} \end{aligned}$$

Thus, it is likely to be an Alpha Particle.

2 Practice Questions

1. A student on a ladder drops small pellets towards a point target on the floor.

   (a) Show that, according to the uncertainty principle, the average miss distance must be at least

   $$\Delta x_f = \left(\frac{2\hbar}{m}\right)^{\frac{1}{2}} \left(\frac{2H}{g}\right)^{\frac{1}{4}}$$

   where H is the initial height of each pellet above the floor and m is the mass of each pellet. Assume that the spread in impact points is given by $\Delta x_f = \Delta x_i + (\Delta v_x)t$.

Solution:
Firstly, note that this is essentially a projectile motion question. Notably, you know S_y, a_y and v_y, so can get an expression for t:

$$S_y = v_y t + \frac{1}{2}a_y t^2 \implies -H = \frac{1}{2}(-g)t^2 \implies t = \sqrt{\frac{2H}{g}}$$

From here, you can get an expression for Δv_f in terms of Δv_i and Δv_x:

$$\Delta x_f = \Delta x_i + (\Delta v_x)t = \Delta x_i + (\Delta v_x)\sqrt{\frac{2H}{g}}$$

By Heisenberg's Uncertainty Principle, we have the following fact too:

$$\Delta x_i \Delta p = \frac{\hbar}{2} \implies \Delta x_i = \frac{\hbar}{2m\Delta v_x}$$

Thus, you substitute Δx_i in terms of Δv_x as follows:

$$\Delta x_f = \frac{\hbar}{2m\Delta v_x} + (\Delta v_x)\sqrt{\frac{2H}{g}}$$

To minimize, we differentiate the above with respect to Δv_x, and let this derivative be 0:

$$\begin{aligned}
\frac{d(\Delta x_f)}{d(\Delta v_x)} &= -\frac{\hbar}{2m(\Delta v_x)^2} + \sqrt{\frac{2H}{g}} = 0 \\
\Delta v_x &= \sqrt{\frac{\hbar}{2m}}\left(\frac{2H}{g}\right)^{-\frac{1}{4}} \\
\Delta x_f &= \sqrt{\frac{\hbar}{2m}}\left(\frac{2H}{g}\right)^{\frac{1}{4}} + \sqrt{\frac{\hbar}{2m}}\left(\frac{2H}{g}\right)^{\frac{1}{4}} \\
&= \sqrt{\frac{2\hbar}{m}}\left(\frac{2H}{g}\right)^{\frac{1}{4}}
\end{aligned}$$

(b) If $H = 2.00$ m and $m = 0.500$ g, what is Δx_f?

> **Solution:**
>
> $$\Delta x_f = \sqrt{\frac{6.626 \times 10^{-34}}{5.00\pi \times 10^{-4}}} \left(\frac{4.00}{9.81}\right)^{\frac{1}{4}} = 5.18995 \times 10^{-16} \text{ m}$$

2. (a) Show that the kinetic energy of a nonrelativistic particle can we written in terms of its momentum as $K = p^2/2m$.

> **Solution:**
>
> $$E_k = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$$

(b) Use the result from (a) to find the minimum kinetic energy of a proton confined within a nucleus having a diameter of 1.00×10^{-15} m.

> **Solution:** By Heisenberg's Uncertainty Principle,
>
> $$\Delta x \Delta p \geq \frac{\hbar}{2} \implies \Delta p \geq \frac{\hbar}{2\Delta x}$$
>
> Notably, $\Delta x = D = 1.00 \times 10^{-15}$ m. Using the result from in (a), we get the following:
>
> $$\begin{aligned} \Delta p &\geq \frac{\hbar}{2\Delta x} \\ E_{\min} &= \frac{1}{2m}\left(\frac{\hbar}{2\Delta x}\right)^2 \\ &= 5.20256 \text{ MeV} \end{aligned}$$

3. An electron ($m_e = 9.11 \times 10^{-31}$ kg) and a bullet ($m = 0.0200$ kg) each have a velocity of magnitude of 500 m/s, accurate to within 0.0100%. Within what limits could we determine the position of the objects along the direction of the velocity.

> **Solution:**
> For Electron:
>
> $$\Delta x = \frac{h}{4\pi\Delta p} = \frac{h}{4\pi m_e(0.05)} = 1.15759 \text{ mm}$$
>
> For Bullet:
>
> $$\Delta x = \frac{h}{4\pi\Delta p} = \frac{h}{4\pi m(0.05)} = 5.27280 \times 10^{-32} \text{ m}$$