

formulas used by Pi-chuck

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December 7, 2023

I. Basics

I.1 Original formula

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! (13591409 + 545140134k)}{(3k)! (k!)^3 640320^{3k}}$$

I.2 π in terms of a and b

$$\begin{aligned} \frac{1}{\pi} &= \frac{1}{426880\sqrt{10005}} \left[13591409 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)! (k!)^3 640320^{3k}} + 545140134 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)! (k!)^3 640320^{3k}} \right] \\ \pi &= \frac{426880\sqrt{10005}}{1359140a + 545140134b} \end{aligned}$$

with

$$\begin{aligned} a &= \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)! (k!)^3 640320^{3k}} \\ \text{and } b &= \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)! (k!)^3 640320^{3k}} \end{aligned}$$

II. computing $426880\sqrt{10005}$

Remark that $426880\sqrt{10005}$ is the positive root of $f(x) = x^2 - 426880^2 \cdot 10005$

Proof:

$$x^2 - 426880^2 \cdot 10005 = 0$$

$$\iff x^2 = 426880^2 \cdot 10005$$

$$\iff x = \pm\sqrt{426880^2 \cdot 10005}$$

$$\iff x = \pm 426880\sqrt{10005}$$

□

II.1 Newton's method

let f be a smooth function: $f : \mathbb{R} \rightarrow \mathbb{R}$

let $(x_n)_{n \in \mathbb{N}}$ be a series defined by:

$$\left\{ \begin{array}{l} x_0 = \text{a guess sufficiently close to the root} \\ \forall n \in \mathbb{N}, x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \end{array} \right.$$

III. computing the sum

let $(a_n)_{n \in \mathbb{N}}$ be a series such that:

$$\forall n \in \mathbb{N}, a_n = \frac{(-1)^k (6k)! k}{(3k)! (k!)^3 640320^{3k}}.$$

and let $(b_n)_{n \in \mathbb{N}}$ be a series defined by:

$$\forall n \in \mathbb{N}, b_n = n * a_n = \frac{(-1)^k (6k)! k}{(3k)! (k!)^3 640320^{3k}}.$$

III.1 a and b terms of the a_k s and b_k s

It is pretty clear from $(a_n)_{n \in \mathbb{N}}$'s definition that:

$$a = \sum_{k=0}^{\infty} a_k.$$

also from $(b_n)_{n \in \mathbb{N}}$'s definition:

$$b = \sum_{k=0}^{\infty} b_k.$$

III.2 recursive definition of the a_n s

$$\begin{cases} a_0 = 0 \\ \forall n \in \mathbb{N}, a_{n+1} = -\frac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3} * a_n \end{cases}$$

Proof of the recursive definition of $(a_n)_{n \in \mathbb{N}}$: We suppose that $\forall n \in \mathbb{N}, a_n \neq 0$ and compute:

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{\frac{(-1)^{k+1} (6(k+1))!}{(3(k+1))! ((k+1)!)^3 640320^{3(k+1)}}}{\frac{(-1)^k (6k)!}{(3k)! (k!)^3 640320^{3k}}} \\ &= \frac{(-1)^{k+1} (6(k+1))!}{(3(k+1))! ((k+1)!)^3 640320^{3(k+1)}} \cdot \frac{(3k)! (k!)^3 640320^{3k}}{(-1)^k (6k)!} \\ &= \frac{-1 \cdot (-1)^k (6k+6)!}{(3k+3)! ((k+1) \cdot k!)^3 640320^{3k+3}} \cdot \frac{(3k)! (k!)^3 640320^{3k}}{(-1)^k (6k)!} \\ &= -\frac{\cancel{(-1)^k} (6k+6)! (3k)! \cancel{(k!)^3} 640320^{3k}}{(3k+3)! (k+1)^3 \cancel{(k!)^3} 640320^{3k} 640320^3 \cancel{(-1)^k} (6k)!} \\ &= -\frac{(6k+6)!(3k)!}{(3k+3)! (k+1)^3 640320^3 (6k)!} \\ &= -\frac{(6k+6)(6k+5)(6k+4)(6k+3)(6k+2)(6k+1) \cancel{(6k)!} \cancel{(3k)!}}{(3k+3)(3k+2)(3k+1) \cancel{(3k)!} (k+1)^3 640320^3 \cancel{(6k)!}} \\ &= -\frac{8 \cancel{(3k+3)} \cancel{(6k+5)} \cancel{(3k+2)} (6k+3) \cancel{(3k+1)} (6k+1)}{\cancel{(3k+3)} \cancel{(3k+2)} \cancel{(3k+1)} (k+1)^3 640320^3} \\ &= -\frac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3} \end{aligned}$$

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