formulas used by Pi-chuck

Paul TAVENEAUX

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I. Basics

I.1 Original formula

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k}}$$

I.2 π in terms of a and b

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}} \left[13591409 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)!(k!)^3 640320^{3k}} + 545140134 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)!(k!)^3 640320^{3k}} \right]$$

$$\pi = \frac{426880\sqrt{10005}}{1359140a + 545140134b}$$

with

$$a = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)!(k!)^3 640320^{3k}}$$

$$\text{and} \quad b = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)!(k!)^3 640320^{3k}}$$

II. computing $426880\sqrt{10005}$

Remark that $426880\sqrt{10005}$ is the positive root of $f(x) = x^2 - 426880^2 \cdot 10005$

Proof:

$$\begin{aligned}
& x^2 - 426880^2 \cdot 10005 = 0 \\
\iff \quad & x^2 = 426880^2 \cdot 10005 \\
\iff \quad & x = \pm\sqrt{426880^2 \cdot 10005} \\
\iff \quad & x = \pm 426880\sqrt{10005}
\end{aligned}$$

CQFD

II.1 Newton's method

let $a \in \mathbb{R}$ such that $\forall x \in \mathbb{R},\ f(x) = x^2 - a$
the roots of f are quite clearly $\sqrt{a}$ and $-\sqrt{a}$
let α be the positive root of f: $\alpha = \sqrt{a}$ It follows from f's definition that:

$$\begin{aligned}
&\forall x \in \mathbb{R},\ f'(x) = 2x \\
&\forall x \in \mathbb{R},\ f''(x) = 2
\end{aligned}$$

Observe that:

$$\begin{aligned}
\forall(\alpha, x_n) \in \mathbb{R}^2,\ x_n^2 - a + 2x_n(\alpha - x_n) + \frac{1}{2}2(\alpha - x_n)^2 &= x_n^2 - a + 2\alpha x_n - 2x_n^2 + \alpha^2 - 2\alpha x_n + x_n^2 \\
&= x_n^2 - a + 2\alpha x_n - 2x_n^2 + \alpha^2 - 2\alpha x_n + x_n^2 \\
&= \cancel{x_n^2 + x_n^2 - 2x_n^2} - a + \alpha^2 + \cancel{2\alpha x_n - 2\alpha x_n} \\
&= \alpha^2 - a
\end{aligned}$$

$$x_n^2 - a + 2x_n(\alpha - x_n) + \frac{1}{2}2(\alpha - x_n)^2 = f(\alpha) \tag{1}$$

Note that:

$$\begin{aligned}
& f'(x) = 0 \iff x = 0 \\
\implies \forall \epsilon_0 \in]0; \alpha[,\ x \in [\alpha - \epsilon_0; \alpha + \epsilon_0] \implies x > 0 \implies & f'(x) \neq 0
\end{aligned} \tag{2}$$

Let $g \in \mathbb{R}$ be an initial guess of α
And let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined by:

$$\begin{cases} x_0 = g \\ \forall n \in \mathbb{N},\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \end{cases}$$

and let $(\epsilon_n)_{n\in\mathbb{N}}$ be a sequence defined by : $\forall n \in \mathbb{N},\ \epsilon_n = \alpha - x_n$

Claim II.1.1 existance of x_n

$$\begin{aligned}
g = x_0 \in [\alpha - \epsilon_0; \alpha + \epsilon_0] &\implies \forall n \in \mathbb{N},\ f'(x_n) \neq 0 \\
&\implies \forall n \in \mathbb{N},\ x_n \in \mathbb{R}
\end{aligned}$$

Proof: existance of x_n.

CQFD

let $\forall n \in \mathbb{N},\ P_n$ be a propostion s.t. $\forall n \in \mathbb{N},\ P_n : g = x_0 \in [\alpha - \epsilon_0; \alpha + \epsilon_0] \implies x_n \in [\alpha - \epsilon_0; \alpha + \epsilon_0]$
By induction:
Initialisation: case $n = 0$: $x_0 = g \in [\alpha - \epsilon_0; \alpha + \epsilon_0]$ it immidiatly follows that $x_0 \in [\alpha - \epsilon_0; \alpha + \epsilon_0]$
Induction:
let $n \in \mathbb{N}$ s.t. P_n is verified:

$$\begin{aligned} & g = x_0 \in [\alpha - \epsilon_0; \alpha + \epsilon_0] \implies x_n \in [\alpha - \epsilon_0; \alpha + \epsilon_0] \\ \iff & \alpha - \epsilon_0 \leqslant x_n \leqslant \alpha + \epsilon_0 \\ \iff & \end{aligned}$$

Claim II.1.2 convergence of $(x_n)_{n\in\mathbb{N}}$

$$g \in]0; \alpha[\implies (x_n)_{n\in\mathbb{N}} \text{ converges to } \alpha$$

Proof: convergence of $(x_n)_{n\in\mathbb{N}}$. CQFD

III. computing the sum

let $(a_n)_{n\in\mathbb{N}}$ be a series such that:

$$\forall n \in \mathbb{N}, a_n = \frac{(-1)^n(6n)!}{(3n)!(n!)^3 640320^{3n}}.$$

and let $(b_n)_{n\in\mathbb{N}}$ be a series defined by:

$$\forall n \in \mathbb{N},\ b_n = n * a_n = \frac{(-1)^n(6n)!n}{(3n)!(n!)n^3 640320^{3n}}.$$

III.1 a and b terms of $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$

It is pretty clear from $(a_n)_{n\in\mathbb{N}}$'s definition that:

$$a = \sum_{k=0}^{\infty} a_k.$$

also from $(b_n)_{n\in\mathbb{N}}$'s definition:

$$b = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} k \cdot a_k.$$

III.2 recursive definition of $(a_n)_{n\in\mathbb{N}}$

Definition III.2.1: recursive definition of $(a_n)_{n\in\mathbb{N}}$

$$\begin{cases} a_0 = 1 \\ \forall n \in \mathbb{N},\ a_{n+1} = -\dfrac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3} \times a_n \end{cases}$$

Proof: Proof of the recursive definition of $(a_n)_{n\in\mathbb{N}}$. We suppose that $\forall n \in \mathbb{N},\ a_n \neq 0$ and compute:

$$\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(-1)^{k+1}(6(k+1))!}{(3(k+1))!((k+1)!)^3 640320^{3(k+1)}}}{\frac{(-1)^k(6k)!}{(3k)!(k!)^3 640320^{3k}}} \\
&= \frac{(-1)^{k+1}(6(k+1))!}{(3(k+1))!((k+1)!)^3 640320^{3(k+1)}} \cdot \frac{(3k)!(k!)^3 640320^{3k}}{(-1)^k(6k)!} \\
&= \frac{-1\cdot(-1)^k(6k+6)!}{(3k+3)!((k+1)\cdot k!)^3 640320^{3k+3}} \cdot \frac{(3k)!(k!)^3 640320^{3k}}{(-1)^k(6k)!} \\
&= -\frac{\cancel{(-1)^k}(6k+6)!\,(3k)!\cancel{(k!)^3}\,\cancel{640320^{3k}}}{(3k+3)!(k+1)^3\cancel{(k!)^3}\,\cancel{640320^{3k}}\,640320^3\,\cancel{(-1)^k}(6k)!} \\
&= -\frac{(6k+6)!(3k)!}{(3k+3)!(k+1)^3 640320^3(6k)!} \\
&= -\frac{(6k+6)(6k+5)(6k+4)(6k+3)(6k+2)(6k+1)\cancel{(6k)!}\,\cancel{(3k)!}}{(3k+3)(3k+2)(3k+1)\cancel{(3k)!}(k+1)^3 640320^3\,\cancel{(6k)!}} \\
&= -\frac{8\cancel{(3k+3)}(6k+5)\cancel{(3k+2)}(6k+3)\cancel{(3k+1)}(6k+1)}{\cancel{(3k+3)}\,\cancel{(3k+2)}\,\cancel{(3k+1)}(k+1)^3 640320^3} \\
&= -\frac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3}
\end{aligned}$$

CQFD