formulas used by Pi-chuck

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I. Basics

I.1 Original formula

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}} \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!(13591409 + 545140134k)}{(3k)!(k!)^3 640320^{3k}}$$

I.2 π in terms of a and b

$$\frac{1}{\pi} = \frac{1}{426880\sqrt{10005}} \left[13591409 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)!(k!)^3 640320^{3k}} + 545140134 \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)!(k!)^3 640320^{3k}} \right]$$

$$\pi = \frac{426880\sqrt{10005}}{1359140a + 545140134b}$$

with

$$a = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)!}{(3k)!(k!)^3 640320^{3k}}$$

$$\text{and} \quad b = \sum_{k=0}^{\infty} \frac{(-1)^k (6k)! k}{(3k)!(k!)^3 640320^{3k}}$$

II. computing $426880\sqrt{10005}$

Remark that $426880\sqrt{10005}$ is the positive root of $f(x) = x^2 - 426880^2 \cdot 10005$

Preuve: Proof.

$$\begin{aligned} &x^2 - 426880^2 \cdot 10005 = 0 \\ \iff\ & x^2 = 426880^2 \cdot 10005 \\ \iff\ & x = \pm\sqrt{426880^2 \cdot 10005} \\ \iff\ & x = \pm 426880\sqrt{10005} \end{aligned}$$

CQFD

II.1 Newton's method

let f be a smooth function: $f : \mathbb{R} \longrightarrow \mathbb{R}$
let $(x_n)_{n\in\mathbb{N}}$ be a series defined by:

$$\begin{cases} x_0 = \text{a guess sufficiently close to the root} \\ \forall n \in \mathbb{N},\ x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} \end{cases}$$

III. computing the sum

let $(a_n)_{n\in\mathbb{N}}$ be a series such that:

$$\forall n \in \mathbb{N}, a_n = \frac{(-1)^n (6n)!}{(3n)!(n!)^3 640320^{3n}}.$$

and let $(b_n)_{n\in\mathbb{N}}$ be a series defined by:

$$\forall n \in \mathbb{N},\ b_n = n * a_n = \frac{(-1)^n (6n)! n}{(3n)!(n!) n^3 640320^{3n}}.$$

III.1 a and b terms of $(a_n)_{n\in\mathbb{N}}$ $(b_n)_{n\in\mathbb{N}}$

It is pretty clear from $(a_n)_{n\in\mathbb{N}}$'s definition that:

$$a = \sum_{k=0}^{\infty} a_k.$$

also from $(b_n)_{n\in\mathbb{N}}$'s definition:

$$b = \sum_{k=0}^{\infty} b_k = \sum_{k=0}^{\infty} k \cdot a_k.$$

III.2 recursive definition of $(a_n)_{n\in\mathbb{N}}$

Definition III.2.1: recursive definition of $(a_n)_{n\in\mathbb{N}}$

$$\begin{cases} a_0 = 1 \\ \forall n \in \mathbb{N},\ a_{n+1} = -\dfrac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3} \times a_n \end{cases}$$

Preuve: Proof of the recursive definition of $(a_n)_{n\in\mathbb{N}}$. We suppose that $\forall n \in \mathbb{N},\ a_n \neq 0$ and compute:

$$
\begin{aligned}
\frac{a_{n+1}}{a_n} &= \frac{\frac{(-1)^{k+1}(6(k+1))!}{(3(k+1))!((k+1)!)^3 640320^{3(k+1)}}}{\frac{(-1)^k(6k)!}{(3k)!(k!)^3 640320^{3k}}} \\
&= \frac{(-1)^{k+1}(6(k+1))!}{(3(k+1))!((k+1)!)^3 640320^{3(k+1)}} \cdot \frac{(3k)!(k!)^3 640320^{3k}}{(-1)^k(6k)!} \\
&= \frac{-1\cdot(-1)^k(6k+6)!}{(3k+3)!((k+1)\cdot k!)^3 640320^{3k+3}} \cdot \frac{(3k)!(k!)^3 640320^{3k}}{(-1)^k(6k)!} \\
&= -\frac{\cancel{(-1)^k}(6k+6)!\,(3k)!\cancel{(k!)^3}\,\cancel{640320^{3k}}}{(3k+3)!(k+1)^3\cancel{(k!)^3}\,\cancel{640320^{3k}}\,640320^3\,\cancel{(-1)^k}(6k)!} \\
&= -\frac{(6k+6)!(3k)!}{(3k+3)!(k+1)^3 640320^3(6k)!} \\
&= -\frac{(6k+6)(6k+5)(6k+4)(6k+3)(6k+2)(6k+1)\cancel{(6k)!}\,\cancel{(3k)!}}{(3k+3)(3k+2)(3k+1)\cancel{(3k)!}(k+1)^3 640320^3\,\cancel{(6k)!}} \\
&= -\frac{8\cancel{(3k+3)}(6k+5)\cancel{(3k+2)}(6k+3)\cancel{(3k+1)}(6k+1)}{\cancel{(3k+3)}\,\cancel{(3k+2)}\,\cancel{(3k+1)}(k+1)^3 640320^3} \\
&= -\frac{24(6k+5)(2k+1)(6k+1)}{(k+1)^3 640320^3}
\end{aligned}
$$

CQFD